Radiation and reaction in scalar quantum electrodynamics

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Abstract. This thesis is a report of work which develops the study of electromagnetic radiation by accelerating charges in the scalar quantum electrodynamic theory. We investigate aspects of this theory in flat spacetime, and in a class of conformally flat and curved spacetimes. In particular, we show that in flat spacetime, the quantum-theoretic prediction for the emission of energy by the particle, in the limit $\hbar \to 0$ and to order e^2 in the coupling constant, may be shown to match the classical calculation. We also calculate the order $\hbar$ correction to this quantity for two specific classes of problem. In the class of conformally flat and curved spacetimes, we compare the change in position due to the radiation reaction with the classical result, and we also consider some of the one-loop corrections to the theory. We show that as $\hbar \to 0$, the conformally flat result and the classical result match, but that in that limit the general spacetime results differ.

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Author's declaration

I declare that the work contained in this thesis is original. Chapter 1 is a review of prior research. Chapters 3, 4, 5, 6, 7, and Appendix A are my original research conducted in collaboration with Dr. Atsushi Higuchi. The work reported in chapters 3 and 4 is published in [1], and the work in chapter 5 is published in [2].

Chapter 1

Introduction

1.1 Historical overview

Electromagnetism has, arguably, been the catalyst for more discoveries in theoretical physics than any other phenomenon, including gravity. This may, in part, be due to the way in which humanity first studied it as two apparently distinct effects: electricity and magnetism. However, the explosion of nineteenth century physics, and in particular the experimental shift from electrostatics to the manipulation of current flows, allowed Maxwell to perceive how the two effects could be unified into one, which we now call electromagnetism.

The classical physics, including electromagnetism, developed during the nineteenth century was remarkably successful. Indeed, it was so successful that one of its principal British exponents, Lord Kelvin, famously commented that 'there [was] nothing new to be discovered in physics.' However, the nineteenth century physics ultimately proved incapable of explaining certain phenomena which had been noted during this great expansion in knowledge: among others, the strikingly regular spectral lines of the hydrogen atom, first noted in 1885 by Balmer; the photoelectric effect, first observed in 1887 by Hertz, which appeared to overthrow the understanding of light as a wave derived from Young's experiment first conducted in 1801; and the spectrum of black body radiation, which had been studied empirically but which classical physics was proving unable to predict, much less to explain.

Perhaps the most striking problem with classical physics related to the phenomenon of radiation by accelerating charges, and was not, in fact, realised until after the development of its resolution. It had been known for some time that electrons were not fixed to atoms, but rather were in motion around a centre — to use the modern name, a nucleus. However, an electron in motion around a nucleus is necessarily accelerating, and accelerating charges lose energy through electromagnetic radiation, an effect which had been first noted theoretically by Larmor in 1897 [3]. In that same paper, Larmor himself noted (and erroneously discounted) the possibility that an orbiting electron will continually lose energy. Such a loss of energy would push the classical electron towards the centre of the atom, causing it to lose yet more energy: eventually, this inspiralling of its electrons would cause the atom to collapse in an extremely short period of time. (The lifetime of the classical Rutherford atom, a slightly later model of the atom motivated by the discovery of the nucleus, can be easily calculated to be of the order of 10^{-11}s.) Needless to say, the continued existence of matter above the subatomic level is no small embarrassment for the classical theory.

The three problems mentioned above were to give rise to the development of quantum physics. The black-body radiation problem was resolved by Max Planck in 1900, when he proposed that electromagnetic energy be emitted by the body in discrete *quanta*. However, Planck appears not to have considered this idea as much more than a mathematical artifice which permitted him to derive a result matching the empirically-measured radiation profile. It was

Albert Einstein who, in a paper published during 1905, his *annus mirabilis*, suggested that Planck's mathematical artifice in fact reflected reality, and that electromagnetic radiation only exists in discrete packets. By making this assumption, and thus overthrowing a century's understanding of light as a simple wave, he was able to explain the second of the three problems: the photoelectric effect. Subsequent development of the quantum theory showed that certain electronic orbits exist at which the electrons do not radiate, thus both stopping a quantum-theory atom from suffering the same fate as the classical one and also explaining the lines in atomic spectra. This developing quantum theory has been resoundingly successful at explaining the physics of the world around us, and although we are rightly wary of repeating Lord Kelvin's mistake, we are confident that quantum physics is more fundamental than classical physics, matching classical theories in the right circumstances but also making many new predictions, which have been experimentally observed to arise.

1.2 Background literature

Our work here shall be focussing on the electromagnetic radiation by accelerating charges, and particularly on the place of this phenomenon in quantum field theory. To that end, we shall briefly review the developments, both in classical theory and quantum theory, which have taken place in the study of this effect.

The understanding of the classical theory continued to develop from Larmor's initial publication in 1897. We may note that the emission of radiation causes an acceleration, and so deduce that there must be a description of this effect as a radiation-reaction force. This force was described by Abraham and Lorentz [4, 5], and is named the Abraham-Lorentz force.

The special theory of relativity — interestingly, also motivated by a problem from classical electromagnetism and initiated as a field of study by Einstein during his *annus mirabilis* — implied that laws expressed under the old Newtonian paradigm were only approximations at low velocities, and therefore a Lorentz-covariant formulation of the emitted energy and the radiation-reaction force were necessary. The Abraham-Lorentz force was made covariant by Dirac in 1938 [6].

The generalisation of the theory of relativity to *local* Lorentz covariance gave rise to the question of the emission of radiation by charged particles moving on a curved spacetime. It was not until the work of DeWitt and Brehme in 1960 [7], with a correction by Hobbs in 1968 [8], that the Abraham-Lorentz-Dirac four-force was written in a generally covariant form. Subsequently, alternative derivations were published by Villaroel [9], and Quinn and Wald [10] who showed that the ALD force may be found from simple axioms; this work has recently been made more rigorous by Gralla, Harte and Wald [11]. Poisson provides sound reviews of both the Lorentz-Dirac [12] and DeWitt-Brehme-Hobbs [13] equations.

As we mentioned earlier, classical physics is a certain limiting behaviour of the quantum theory. It is therefore of interest to investigate effects which can be treated of by both classical and quantum physical theories, to find whether the two agree and also to find whether experimental observations agree with the theoretical predictions. The emission of radiation by accelerating charges is one such area of study.

Indeed, much work along these lines has already been undertaken. The question has been dealt with by several authors [14, 15, 16, 17] as a quantum-mechanical problem, taking the charged particle/body as a quantum particle/body.

However, we do not regard the particle as being the fundamental theoretical object. Instead, in the quantum field-theoretic viewpoint, we treat the charged particle as an excitation of a quantum field. Thus, it has been of interest to consider the possibility of recovering the ALD force, or some implication of it, from quantum electrodynamics. Krivitskiĭ and Tsytovich [18] derived the ALD force from QED by calculating the expectation value of the momentum and taking its time derivative. Independently, Higuchi and Martin have compared the expected position of the charged particle under a linear acceleration in the classical and scalar QED theories, and found them to be equal in the limit $\hbar \to 0$ and to lowest order in α. This work was first done in the non-relativistic régime [19], and subsequently extended to the relativistic régime [20, 21], non-linear motion [22, 23] and the spinor case [24, 25].

Nomura, Sasaki and Yamamoto [26] moved the study of the radiation reaction towards curved spacetime by showing that the radiation from a charged scalar particle in QED in conformally flat spacetime is given by the classical Larmor formula for the corresponding flat spacetime theory in the limit $\hbar \to 0$. This therefore also matches, in part, the finding of Roberts [27], who showed that the DeWitt-Brehme-Hobbs equation is invariant under conformal transformations of the metric.

We shall be following on from the aforementioned work by studying the emission of radiation in scalar quantum electrodynamics (sQED), which has been the theory of choice for much of this work, as it provides a simpler model for the interaction of a massive particle with the electromagnetic field than the spinor theory. In the remainder of this introductory chapter, we shall review the classical and quantum theories, and the model which we shall be using. The general approach is the one shared with the series of papers by Higuchi and Martin.

1.3 Preliminaries

We shall be using the metric $(+ - --)$ throughout, unless explicitly stated otherwise. We shall also generally take units in which $c = 1$, but as necessitated by the semiclassical approximation we shall detail shortly, $\hbar$ will remain explicit in our calculations. Complex conjugates are variously denoted: typically by * for scalar quantities and $\overline{}$ for vector and higher-rank tensor quantities. Hermitian conjugates are typically denoted by †.

1.4 Overview of the work

The thesis shall proceed according to the following scheme. In Ch. 2, we shall consider the theory which underlies the research to be presented in subsequent chapters. Our main tool will be perturbative approach, in which the quantum electrodynamic theory is treated as a formal series expansion in $\hbar$.

In Ch. 3, we shall demonstrate how to derive the Larmor formula from the quantum model we shall be considering, in the case of a background potential depending on time only. In Ch. 4, we shall extend the calculation of the non-relativistic Larmor formula to the next order in $\hbar$, in the separate cases of a potential which varies with time only and with a single spatial co-ordinate only. We show that the results in the two cases differ by a factor of c^2.

In Ch. 5, we shall move from considering the energy emitted by the particle to the change in position induced by the emission, and calculate this position shift for a particle moving on a conformally flat spacetime, where the background potential and conformal factor are both functions of time only. We reduce the theory to a flat spacetime with a mass term which varies only with time, and deduce that the quantum result matches the classical result. We consider the one-loop calculations – the 'vacuum current' and the forward scattering term – for the theory and recover the well-known 'conformal anomaly', in which renormalised quantities do not exhibit an expected invariance between the conformally flat theory and the classically equivalent flat theory with varying mass term. However, we do show that the conformally flat theory results match those of the flat theory, and therefore the classical result.

In Ch. 6, we carry out the calculation of position shift at order $\hbar^0$, and show that there is a discrepancy between our result and the classical one. We speculate that the discrepancy should be bridged by the one-loop, forward scattering contribution to the position shift. We also show that the one-loop vacuum current cancels exactly the analogous classical quantity, as we would expect.

Chapter 2

Background theory

Summary. In this chapter, we shall discuss the mathematical and physical theory underlying the work which will be presented in later chapters.

2.1 The classical theory of electromagnetic radiation by accelerating charges

Since the work presented here has its deepest physical roots in the classical theory of accelerating charges on a flat space-time, it is appropriate to begin this review of the background material there. In addition to the extensions of the most original work from the early twentieth century, we also have the benefit of the presentations found in Jackson [28], Barut [29] and Poisson [13], among many others, which expound more clearly the underlying mathematics. The author is indebted to those sources for the presentation here.

Let us consider a four-current, J^μ, which interacts with an electromagnetic potential, A_α. We may describe the dynamics using a Lagrangian formulation.

The Lagrangian for the system is therefore

$$\mathcal{L}_{\rm EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\alpha A^\alpha, \tag{2.1}$$

where the electromagnetic field strength $F_{\mu\nu}$ is defined by $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. Then by the standard procedure, we obtain Euler-Lagrange equations for this system, which are

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{2.2}$$

The vector potential A_ν has a gauge symmetry: if we define $A'_\nu = A_\nu - \partial_\nu \phi$ for some non-dynamical scalar field ϕ, then it is easy to show that Maxwell's equations, and hence the dynamics, are unaffected. This means that any initial value problem would have as solutions a family of potentials, equivalent up to the application of a scalar field in the way described above. We choose to fix this gauge symmetry by forcing a choice of ϕ through the imposition of the Lorenz gauge, which is to say, the condition that $\partial_\mu A^\mu = 0$. For our purposes here, the benefits of this choice of gauge are twofold: firstly, that it is manifestly covariant; and secondly, that by its application, the equation of motion reduces to

$$\Box A^\mu = J^\mu. \tag{2.3}$$

Clearly, each component of the electromagnetic field independently satisfies the inhomogeneous, massless, Klein-Gordon equation in Cartesian coordinates, and therefore we may construct electromagnetic Green's functions by

$$G_\mu{}^\nu(x - x') = \delta^\nu_\mu G(x - x'), \tag{2.4}$$

where $G(x - x')$ satisfies the equation $\Box_x G(x - x') = \delta^4(x - x')$. We associate the index μ with the point x, which we call a field point, and ν with x', a source point. It is thus evident that

$$A_\mu(x) = \int d^4x' G_{\mu\nu}(x - x') J^\nu(x') \tag{2.5}$$

must be a solution to the Lorenz-gauge equations of motion.

If we suppose that apart from the four-current, the spacetime is a vacuum, and that the only non-zero potential is generated by that current, then we require the Green's function which propagates backward from x. This solution is denoted

$$G^+(x-x') = \frac{\delta(t-t'-\|\mathbf{x}-\mathbf{x}'\|)}{4\pi\|\mathbf{x}-\mathbf{x}'\|} = \frac{1}{2\pi}\theta(t-t')\delta\left((x-x')^2\right) \tag{2.6}$$

and is called the retarded Green's function. The reader will note that this solution is non-zero only on the backward light-cone from x to x'. Then the field that is generated by the four-current is written

$$A_\mu^+ = \int d^4x' G^+_{\mu\nu}(x-x')J^\nu(x'). \tag{2.7}$$

We may also construct an analogous, forward-propagating Green's function, the advanced Green's function:

$$G^-(x-x') = \frac{1}{2\pi}\theta(t'-t)\delta\left((x-x')^2\right), \tag{2.8}$$

which is non-zero only on the forward light-cone from x. The advanced and retarded solutions are related by the identity $G^+(x-x') = G^-(x'-x)$.

Let us suppose that the current is due to a particle of charge e following a trajectory z^μ, parameterised by s. Then the current will be

$$J^\mu(x) = e\int_{-\infty}^{+\infty} ds\frac{dz^\mu}{ds}\delta^4(x-z(s)), \tag{2.9}$$

and therefore the field that this current generates is

$$A_\mu^+(x) = e\int d^4x' ds\, G^+(x-x')\dot{z}_\mu\delta^4(x'-z(s)), \tag{2.10}$$

where the dot refers to differentiation with respect to the parameter s. Carrying out the x' integral, we obtain

$$A_\mu^+(x) = e\int_{-\infty}^{+\infty} ds\, G^+(x-z(s))\dot{z}_\mu. \tag{2.11}$$

We expect that this potential will lead to an electromagnetic field, which in turn exerts a force on the particle. However, there is a difficulty with the field derived from the potential. The derivative of the potential is

$$\partial_\alpha A^+_\beta = e \int_{-\infty}^{+\infty} ds \, \left(\partial_\alpha (x-z)^2\right) \frac{ds}{d\left[(x-z)^2\right]} \frac{d}{ds} G^+ \left(x - z(s)\right) \dot{z}_\beta. \tag{2.12}$$

The partial derivative and the first s derivative may be calculated directly, producing

$$\partial_\alpha A^+_\beta = -e \int_{-\infty}^{+\infty} ds \, \frac{(x_\alpha - z_\alpha)\dot{z}_\beta}{(x^\mu - z^\mu)\dot{z}_\mu} \frac{d}{ds} G^+(x-z), \tag{2.13}$$

and we integrate this by parts. The boundary term will be zero, since the Green's function selects:firstly, only points x which are in the future of $z(s)$ and hence, the boundary term at $s = +\infty$ will vanish; and secondly, only points x which are null-separated from $z(s)$ and hence, the boundary term at $s = -\infty$ will vanish. The delta function within the Green's function constrains the non-zero contributions in the integral to those points on the trajectory which are null-separated from x, one in the future of x and one in its past. The theta function selects the point which lies on the backward light-cone: we denote the value of s at this point by s_+ for consistency. As an implication of the change-of-variables formula for integration, the delta function has a generalised scaling property,

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \tag{2.14}$$

where the x_i are the zeroes of the function g. Applying this, we obtain

$$\partial_\alpha A^+_\beta = \frac{1}{4\pi} \frac{e}{(x^\nu - z^\nu)\dot{z}_\nu} \left[\frac{d}{ds} \frac{(x_\alpha - z_\alpha)\dot{z}_\beta}{(x^\mu - z^\mu)\dot{z}_\mu}\right]_{s=s_+}. \tag{2.15}$$

Clearly, therefore, the electromagnetic field will be infinite at the position of the particle itself. This poses an immediate difficulty, as we are specifically interested in the effect of the field on the particle, which will require the value of the field at the particle's position.

There is a way in which we may avoid this difficulty. The infinite part of the field arises from the Green's function

$$G^{\mathrm{S}}(x-z) = \frac{1}{2}\left(G^{+}(x-z) + G^{-}(x-z)\right). \tag{2.16}$$

We observe that this solution to the inhomogeneous equation $\Box_x G(x-z) = \delta(x-z)$ is symmetric in x and z (for which reason we denote it $G^{\mathrm{S}}(x-z)$), and conclude, following Poisson's argument [13], that it represents incoming and outgoing radiation in equal amounts. Consequently, we argue that it produces no net action on the particle. However, its singularity structure at $x = z(s)$ is the same as $G^{+}(x-z)$. Therefore, we construct the radiative Green's function,

$$G^{\mathrm{R}} = G^{+} - G^{\mathrm{S}} = \frac{1}{2}\left(G^{+} - G^{-}\right), \tag{2.17}$$

and use this to calculate the self-force.

The reader will observe that we may obtain the radiative field from Eq. (2.13), by integrating by parts and replacing G^{+} with G^{R}. We thus obtain

$$F^{\mathrm{R}}_{\alpha\beta} = e\int_{-\infty}^{+\infty} ds\, G^{\mathrm{R}}(x-z)\frac{d}{ds}\left[\frac{(x_\alpha - z_\alpha)\dot{z}_\beta - (x_\beta - z_\beta)\dot{z}_\alpha}{(x^\mu - z^\mu)\dot{z}_\mu}\right], \tag{2.18}$$

where the Green's function $G^{\mathrm{R}}(x-z)$ is

$$G^{\mathrm{R}}(x-z) = \frac{1}{4\pi}\,\mathrm{sgn}\,(x^0 - z^0)\delta\left((x-z)^2\right). \tag{2.19}$$

We are interested in the field on the trajectory of the particle. Let us therefore consider a point $x = z(s_0)$, and transform the integration variable to $u = s - s_0$. Then

$$\begin{aligned} F^{\mathrm{R}}_{\alpha\beta}(z(s_0)) = {} & \frac{e}{4\pi}\int_{-\infty}^{+\infty} du\ \mathrm{sgn}\,(u)\delta\left(u^2\right) \\ & \times \frac{d}{du}\left[\frac{(z_\alpha(s_0) - z_\alpha(s))\dot{z}_\beta(s) - (z_\beta(s_0) - z_\beta(s))\dot{z}_\alpha(s)}{(z^\mu(s_0) - z^\mu(s))\dot{z}_\mu(s)}\right], \end{aligned} \tag{2.20}$$

where the z functions can be expanded as a power series in u. Before considering this, however, we shall consider the signum and delta functions together by using the identity $\operatorname{sgn}(u)\delta\left(u^2\right) = -\delta'(u)$ [29]. We may integrate by parts to move the derivative onto the fraction, and so observe that we now have two derivatives in u. Since we shall be integrating the fraction against $\delta(u)$, we must find all terms of order u^2 in the fraction.

Let us now take the parameter s to be the proper time. We may expand the position as $z(s)-z(s_0) = u\dot{z}(s_0)+\frac{u^2}{2}\ddot{z}(s_0)+\frac{u^3}{6}\dddot{z}(s_0)+\ldots$, and the velocity as $\dot{z}(s)-\dot{z}(s_0) = u\ddot{z}(s_0)+\frac{u^2}{2}\dddot{z}(s_0)+\ldots$. Hence,

$$(z^\mu(s_0) - z^\mu(s))\dot{z}_\mu(s_0) = -u + \mathcal{O}(u^3), \tag{2.21}$$

with the order u^2 term vanishing due to the vanishing dot product between four-velocity and four-acceleration. So the order u^2 term from the fraction in Eq. (2.20) is

$$\frac{1}{3}u^2\left(\dot{z}_\alpha \dddot{z}_\beta - \dddot{z}_\alpha \dot{z}_\beta\right), \tag{2.22}$$

where we have suppressed the arguments on the right-hand side, since everything there is evaluated at s_0. Performing the u integral, which contains only $\delta(u)$, we find

$$F^{\mathrm{R}}_{\alpha\beta}(z(s_0)) = \frac{2}{3}\frac{e}{4\pi}\left(\dot{z}_\alpha \dddot{z}_\beta - \dddot{z}_\alpha \dot{z}_\beta\right), \tag{2.23}$$

and, dotting with $\dot{z}^\beta$, thereby recover the Abraham-Lorentz-Dirac four-force,

$$f^\mu_{\mathrm{LD}} = e\eta^{\mu\alpha}F^{\mathrm{R}}_{\alpha\beta}\dot{z}^\beta = -\frac{2}{3}\frac{e^2}{4\pi}\left(\dddot{z}^\mu + \ddot{z}^2\dot{z}^\mu\right). \tag{2.24}$$

Hence, if we suppose the particle to be accelerated by some external force f^μ as well as the Lorentz-Dirac force, we derive as an equation of motion

$$m\ddot{z}^\mu = f^\mu + m\tau_0\left(\dddot{z}^\mu + \ddot{z}^2\dot{z}^\mu\right), \tag{2.25}$$

where we have defined $\tau_0 = (2/3)(e^2/4\pi m)$. The constant τ_0 has the dimensions of time, and characterises the timescale over which radiative effects are significant. For consistency, we shall re-label the variable s as τ, to emphasise that we have set it to be equal to the proper time.

The solutions to this equation of motion are known to have pathologies. Clearly, we have a trivial solution if there is no external force and the particle is not initially accelerating. However, if there is no external force, then we can dot the equation of motion with $\ddot{z}_\mu$ and observe that we obtain a first-order differential equation for $\ddot{z}^2$, the general solution for which is $\ddot{z}^2 = -C^2 e^{2\tau/\tau_0}$. The magnitude of the acceleration can therefore be seen to increase arbitrarily: the so-called 'runaway solutions'. Such solutions, which result in self-accelerating particles, are manifestly unphysical.

It is possible to resolve this difficulty by converting the equation of motion into integro-differential equations. If we multiply through Eq. (2.25) by a factor $e^{-\tau/\tau_0}$ and re-arrange, we find

$$-\frac{d}{d\tau}\left(\tau_0 e^{-\tau/\tau_0}\ddot{z}_\mu\right) = e^{-\tau/\tau_0}\left(\frac{f^\mu}{m} + \ddot{z}^2\dot{z}^\mu\right). \tag{2.26}$$

If f^μ is bounded for all τ, then we may integrate this and obtain

$$\ddot{z}^\mu(\tau) = \frac{e^{\tau/\tau_0}}{\tau_0}\int^\infty d\tau' e^{-\tau'/\tau_0}\left(\frac{f^\mu(\tau')}{m} + \tau_0\ddot{z}^2(\tau')\dot{z}^\mu(\tau')\right). \tag{2.27}$$

Although every solution of this form satisfies the equation of motion, not every solution of that equation satisfies this integro-differential equation. For example, the unboundedness of the runaway solutions means they do not satisfy this equation. However, having removed one difficulty, we have created another. If we change the integration variable $\tau' = \tau + a\tau_0$, then we find that we have

$$\ddot{z}^\mu(\tau) = \int_0^\infty da\, e^{-a}\left(\frac{f^\mu(\tau + a\tau_0)}{m} + \tau_0\ddot{z}^2(\tau + a\tau_0)\dot{z}^\mu(\tau + a\tau_0)\right). \tag{2.28}$$

It is generally the case that if f^μ is non-zero at any point after time τ then $\ddot{z}^\mu(\tau)$ will be non-zero: one might consider, as an example, a force which is represented by a delta function [6]. Specifically, the acceleration will be non-zero before the force becomes non-zero. In other words, we have removed the self-acceleration problem at the cost of a 'pre-acceleration' problem. The characteristic timescale, τ_0, governs the duration over which a particle will experience a significant 'pre-acceleration', and fortunately is typically very small: for the electron, $\tau_0 \sim 6.3 \times 10^{-24} s$.

These pathologies in the Lorentz-Dirac force arise from the fact that we model the charged particle as a point, when the true model in classical physics should be of an extended body, to which the point particle model is an approximation. It is possible to avoid these pathologies by using a method known as 'reduction of order' [12], in which we consider the problem at scales where the difference between a point particle and a small extended body is negligible. This implies that for a characteristic timescale τ_c over which the acceleration changes, $\tau_0 \ll \tau_c$, and so we may treat the Lorentz-Dirac equation as a series in τ_0/τ_c. If we do this, then we obtain a second-order equation which equivalent to the original Lorentz-Dirac equation to order τ_0/τ_c but without the associated pathologies.

The Lorentz-Dirac force might be thought the natural quantity to consider in our aim, which is to derive a prediction from quantum field theory which can be compared with the classical results. However, 'force' does not have the status in quantum theory that it does in classical theory. In perturbative quantum field theory, the framework we shall be adopting, force is mediated by the exchange of particles and is thus a notion derived from fields, rather than a fundamental concept in its own right. There is no 'force operator', and we cannot calculate the Lorentz-Dirac four-force directly. Instead, we must compare quantum field theoretic effects to those which we classically attribute to a force. Two such effects are a change in the position of the particle (relative to where it would have been without the radiation-reaction force) and the energy carried away by electromagnetic radiation. We shall use each of these at different points in the work which follows, and therefore it is helpful to give the classical calculation of them here.

The energy-momentum lost by the particle is simple to calculate, as

$$\frac{dP^\mu}{d\tau} = \frac{2}{3}\frac{e^2}{4\pi}\left(\dddot{z}^\mu + \ddot{z}^2\dot{z}^\mu\right), \tag{2.29}$$

where P^μ is the momentum of the charged particle. Consequently, if we integrate over such an interval (τ_-, τ_+) that $\ddot{z}^\nu(\tau_-) = \ddot{z}^\nu(\tau_+) = 0$, then

$$\Delta P^\mu = \frac{e^2}{6\pi m^3}\int_{\tau_-}^{\tau_+} \frac{dp^\nu}{d\tau}\frac{dp_\nu}{d\tau} p^\mu d\tau, \tag{2.30}$$

which is the relativistic, four-vector generalisation of the Larmor formula. In this equation, we have used both P^μ and p^μ for the particle's momentum, with ΔP^μ emphasising that this quantity is the final, fixed change in momentum while p^μ varies across the particle's trajectory.

At the level of generality, the change in position due to a force may be found in terms of that force by considering the effects of an external force on a Hamiltonian system. To specialise to the Lorentz-Dirac force would then be possible simply by explicitly substituting that force into the general expression. A fuller version of the argument which follows can be found in a paper by Higuchi and Martin [22]. It can also be found in Ch. 6 of the work here presented, extended to a class of curved spacetimes which contains Minkowski spacetime as a special case.

If we suppose that the acceleration is caused by a potential vector $\mathbf{V}$ and a scalar potential V^0 which are non-zero only in the interval (t_-, t_+), then the accelerating particle is governed by a Hamiltonian,

$$H(\mathbf{p}, \mathbf{x}, t) = \sqrt{\|\mathbf{p} - \mathbf{V}\|^2 + m^2} + V^0, \tag{2.31}$$

and therefore Hamilton's equations hold provided we incorporate the Lorentz-Dirac force as an external force:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \tag{2.32}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} + f_i. \tag{2.33}$$

We write solutions to these equations as ordered pairs of position and momentum, $(\mathbf{x}(t), \mathbf{p}(t))$, and call solutions which satisfy the equations when $f_i = 0$ homogeneous solutions. We may then choose an initial solution, $(\mathbf{x}_0(t), \mathbf{p}_0(t))$, which satisfies the condition $(\mathbf{x}_0(0), \mathbf{p}_0(0)) = (\mathbf{0}, \mathbf{p})$. Perturbations of these solutions which continue to satisfy the homogeneous equations of motion we denote by $(\mathbf{x}_0 + \Delta\mathbf{x}, \mathbf{p}_0 + \Delta\mathbf{p})$. We expand the Hamiltonian in terms of the perturbations in order to obtain their equations of motion.

It is then possible to describe solutions to the inhomogeneous equations (where $f_i \neq 0$) in terms of a family of perturbations on the basic homogeneous solution, $(\mathbf{x}_0 + \delta\mathbf{x}, \mathbf{p}_0 + \delta\mathbf{p})$, by using the equations of motion derived from the homogeneously perturbed Hamiltonian. We index this family of perturbations by (j) for $j \in \{1, 2, 3\}$. Then we are able to find that the inhomogeneous perturbation in the position vector may be written as

$$\delta x^i = \int_{s_-}^{s_+} ds f_j(s) \Delta x^{i(j)}(0; s) \tag{2.34}$$

for a specifically defined, homogeneous perturbation. This expression can be manipulated using certain facts about Hamiltonian motion; we shall not enter into a discussion of them here but note that the required steps are detailed in Ch. 6. Then we use the fact that a change in the momentum, $\Delta\mathbf{p}$, results in a change in the position, $\Delta\mathbf{x}$, to write the change in position due to an external force as

$$\delta x^i = -\int_{t_-}^{t_+} dt\, f_j \left(\frac{\partial x^j}{\partial p_i}\right)_t, \tag{2.35}$$

with the subscript t indicating that we take the partial derivative while holding t constant. The Lorentz-Dirac force is an example of such a force, and we therefore conclude that this expression describes the classical change in position.

2.2 Quantum theory

Before moving to the specific problems which we shall be considering, it is worth reviewing some of the key concepts from quantum field theory which we shall be using. In the following section, we shall consider the canonical quantisation procedure for massive charged scalar and massless vector fields, the quantum state and the vacuum, or ground, state. We shall see a relation between the Green's functions of the previous section and certain quantities which may be calculated in quantum field theory, and note that we may represent these quantities diagrammatically. This diagrammatic representation will be extended as we consider the interactions between the two fields, and we shall consider how the evolution of states may be modelled in this interacting theory. Finally, we shall consider interactions which modify the vacuum state, and interactions which modify the free motion of excitations in each field.

2.2.1 Free fields

We are seeking to compare certain classical predictions with their quantum field theoretical counterparts. As discussed earlier, we shall be using scalar quantum electrodynamics to derive the quantum theoretical results. This theory is an interacting theory of the charged scalar and vector fields, and we therefore shall discuss the quantum theory of the free fields in advance of considering the interaction.

For the charged, *i.e.* complex, scalar field with associated particle mass m, we take the Lagrangian density

$$\mathcal{L}_{\mathrm{S}} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \frac{m^2}{\hbar^2} \phi^\dagger \phi. \tag{2.36}$$

This gives us as equations of motion,

$$\left(\hbar^2 \Box + m^2\right) \phi = 0, \tag{2.37}$$

and its conjugate.

To use the canonical quantisation of the field ϕ, we derive the conjugate field momenta,

$$\pi_\phi = \frac{\partial \mathcal{L}_{\mathrm{S}}}{\partial (\partial_0 \phi)} = \dot{\phi}^\dagger, \tag{2.38}$$

$$\pi_{\phi^\dagger} = \frac{\partial \mathcal{L}_{\mathrm{S}}}{\partial (\partial_0 \phi^\dagger)} = \dot{\phi}, \tag{2.39}$$

and impose the equal-time commutation relations,

$$[\phi(t,\mathbf{x}), \pi_\phi(t,\mathbf{x}')] = i\hbar\delta^3(\mathbf{x}-\mathbf{x}') = \left[\phi^\dagger(t,\mathbf{x}), \pi_{\phi^\dagger}(t,\mathbf{x}')\right]. \tag{2.40}$$

All other commutation relations on the scalar field operators vanish. Since the field ϕ is not Hermitian, we may treat the fields ϕ and $\phi^\dagger$ as independent of each other, and in the momentum-space representation introduce two particles for the fields — the scalar particle and its anti-particle — with annihilation (creation) operators $A^{(\dagger)}$ and $B^{(\dagger)}$ respectively. The main body of the work is concerned with motion under the influence of an external potential, which will replace the (real) differential operator in the Lagrangian above with a complex operator. For that reason, we associate the particle and anti-particle with mode solutions $\Phi_{\mathbf{p}}$ and $\overline{\Phi}_{\mathbf{p}}$ respectively. Since these are free fields, these mode solutions are plane-wave solutions, which gives us

$$\Phi_{\mathbf{p}}(x) = e^{-\frac{i}{\hbar}p\cdot x} = \overline{\Phi}_{\mathbf{p}}(x), \tag{2.41}$$

where ${p_0}^2 = \|\mathbf{p}\|^2 + m^2$. Hence, we may write the two fields ϕ and $\phi^\dagger$ in the momentum-space representation as

$$\phi(x) = \hbar \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left[A(\mathbf{p}) e^{-\frac{i}{\hbar}p\cdot x} + B^\dagger(\mathbf{p}) e^{\frac{i}{\hbar}p\cdot x} \right], \tag{2.42}$$

$$\phi^\dagger(x) = \hbar \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left[A^\dagger(\mathbf{p}) e^{\frac{i}{\hbar}p\cdot x} + B(\mathbf{p}) e^{-\frac{i}{\hbar}p\cdot x} \right]. \tag{2.43}$$

From these expansions, we deduce that the particle creation and annihilation operators obey the commutation relations

$$\left[A(\mathbf{p}), A^\dagger(\mathbf{p}')\right] = \left[B(\mathbf{p}), B^\dagger(\mathbf{p}')\right] = 2p_0(2\pi\hbar)^3\delta^3(\mathbf{p} - \mathbf{p}') \tag{2.44}$$

and its conjugate, with every other commutation relation being zero.

At this point, it is worth considering the nature of quantum states. A state is an element of the Hilbert space on which the field operators act; we denote such elements by $|\psi\rangle$. On a flat space-time, we are able to define a unique state, $|0\rangle$, which satisfies the relations $A(\mathbf{p})\,|0\rangle = B(\mathbf{p})\,|0\rangle = 0$. This state we term the vacuum state, since it may be thought of as being devoid of all particles. States with particles and anti-particles of various momenta are constructed by successive applications of the relevant creation operators. Since these creation operators all commute with each other, we note that the order in which the operators are written does not affect the state.

Briefly taking an heuristic view, we could say that the field operator $\phi^\dagger(x)$ represents the creation of a scalar particle at the position x or, equivalently, the destruction of a scalar anti-particle at the same position. The field operator

$\phi(x')$, likewise, would represent the creation of a scalar anti-particle at x', or equivalently the destruction of a scalar particle at the same position, or in fact both. How we interpret the vacuum expectation value of the product of these two operators depends on the chronological ordering of the two points x and x': if $t < t'$, then we create a particle at x and subsequently destroy a particle at x'; if $t' < t$, then we create an anti-particle at x' and subsequently destroy an anti-particle at x.

The reader will note that the vacuum expectation value, $\langle 0|\, \phi^\dagger(x)\phi(x')\, |0\rangle$, is a solution to the homogeneous Klein-Gordon equation in both x and x'. The same is also true of the vacuum expectation value with the fields $\phi(x')$ and $\phi^\dagger(x)$ reversed. If we construct the operator $\theta(t-t')\phi(x')\phi^\dagger(x) + \theta(t' - t)\phi^\dagger(x)\phi(x')$, then a short calculation shows that the vacuum expectation value of this operator is, contrariwise, a solution to the inhomogeneous Klein-Gordon equation. We shall interpret the meaning of this operator shortly. We thus denote this amplitude $-iG_{\mathrm{F}}(x-x')$, and application of the Fourier expansion from above shows us that

$$G_{\mathrm{F}}(x-x') = \int \frac{d^4p}{(2\pi\hbar)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-\frac{i}{\hbar}p\cdot(x-x')}, \tag{2.45}$$

where we are now considering p_0 as a variable. If we define ${E_{\mathbf{p}}}^2 \equiv \|\mathbf{p}\|^2 + m^2$, then the poles of the integrand are to be found at $\pm(E_{\mathbf{p}} - i\varepsilon)$: the $i\varepsilon$ term has been inserted to displace the poles so that the line of integration does not pass through them, and its sign has been chosen so as to produce the correct theta functions. The intention is to conclude calculations by taking the limit $\varepsilon \to 0$. This displacement of the poles is shown in Fig. 2.1. The reader will note that there are four ways to displace the two poles relative to the line of integration, of which the Feynman prescription is the most useful for our purposes.

If we again consider this heuristically, we can interpret the expectation value we have just calculated as the transition amplitude for a unit of charge

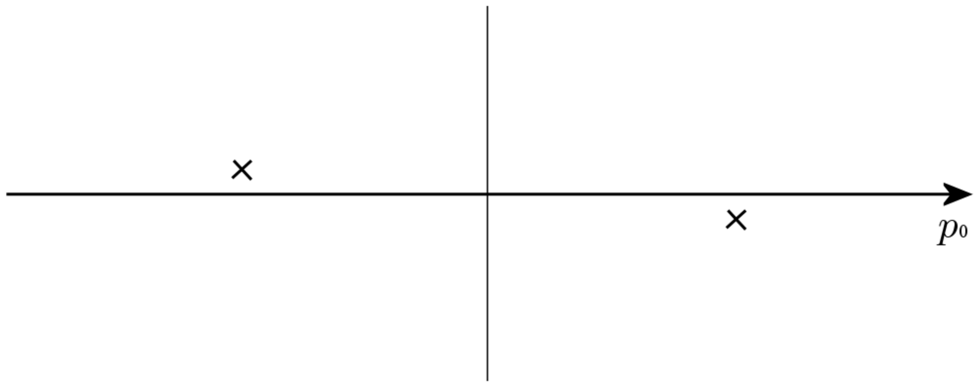


Figure 2.1: The Feynman ε-prescription, displacing the poles of the propagator function above and below the line of integration.

(for sQED, we understand a unit of charge to be $-e$) being transferred between x and x', and we can therefore represent the flow of charge diagrammatically as a directed line, pointing from x to x'. The reader will note that the theta functions in the operator guarantee that the operator corresponding to the earlier point in spacetime appears on the right; we call this arrangement of operators *time-ordering*, and denote it $T\{\phi(x')\phi^\dagger(x)\}$. In the momentum-space representation, we display a flow of charge as a directed line without specified spacetime endpoints; instead, we specify the momentum carried by the field. With the propagation of a particle in this representation, we associate the algebraic expression for the Fourier transform of $G_{\rm F}$,

$$\widetilde{G}_{\rm F}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}. \tag{2.46}$$

We call these functions *propagators* as they represent propagation of particles. Owing to the specific prescription we used to displace the poles, we refer to the propagators derived from this prescription as the Feynman propagators. We shall go on to see that the interactions between the particles allow us to construct diagrams illustrating the possible paths that the state can follow;

Figure 2.2: The free scalar propagator in the spacetime representation. We associate this with the function $G_{\mathrm{F}}(x-x')$. We shall find the momentum-space representation to be the more useful.

the diagrams we thus obtain by the drawing and connection of these lines we call *Feynman diagrams*. Fig. 2.2 shows the spacetime representation of the propagator and the associated algebraic expression.

Another helpful expression involves the commutator of the two fields, not necessarily at equal times. If we take $-i\langle 0|\left[\phi^\dagger(x),\phi(x')\right]|0\rangle$, this can also be shown to be a solution to the Klein-Gordon equation: the advanced-minus-retarded fundamental solution. We obtain the advanced and retarded solutions by multiplying by $\theta(t'-t)$ and $\theta(t-t')$ respectively. These have the same structure as the Feynman propagator given above, but with a different ε-prescription:

$$G^{\pm}(x-x') = \int \frac{d^4p}{(2\pi\hbar)^4}\frac{i}{(p_0\pm i\varepsilon)^2-\|\mathbf{p}\|^2-m^2}e^{-\frac{i}{\hbar}p\cdot(x-x')}. \tag{2.47}$$

For completeness, the Dyson propagator is the acausal propagator produced by displacing the poles to $\pm(E_{\mathbf{p}}+i\varepsilon)$.

At present, we have only been able to consider free particles propagating on the spacetime: after briefly considering the massless vector field, we shall consider the interaction between the two fields, and the Feynman-diagrammatic representation of these interactions.

For the massless vector field, we draw the Lagrangian from Eq. (2.1), setting $J^\alpha=0$ since the field is free:

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2. \tag{2.48}$$

The field is uncharged, so we take A_μ to be real. The equations of motion we derive are

$$\Box A^\mu - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial_\nu A^\nu = 0; \tag{2.49}$$

for the conjugate momenta, we get

$$\pi^\mu = F^{\mu 0} + \frac{1}{\xi} \delta_0^\mu \partial_\nu A^\nu. \tag{2.50}$$

The canonical quantisation procedure is unaffected by our choice of the parameter ξ, provided it remains finite [30]. We choose the Feynman gauge, $\xi = 1$, which reduces the equations of motion to

$$\Box A^\mu = 0. \tag{2.51}$$

We now impose the canonical commutation relations,

$$[A_\mu(t, \mathbf{x}), \pi_{\mu'}(t, \mathbf{x}')] = -i\hbar \eta_{\mu\mu'} \delta^3(\mathbf{x} - \mathbf{x}'). \tag{2.52}$$

Since the field is real, it has only one associated particle. We now expand the field into Fourier modes. The solutions of the equations of motion are plane waves with $k^0 = \|\mathbf{k}\|$, and the field is thus

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left[a_\mu(\mathbf{k}) e^{-ik\cdot x} + a_\mu^\dagger(\mathbf{k}) e^{ik\cdot x}\right]. \tag{2.53}$$

This expansion together with the canonical commutation relations gives us the following non-zero commutation relations for the particle operator:

$$\left[a_\mu(\mathbf{k}), a_{\mu'}^\dagger(\mathbf{k}')\right] = -\eta_{\mu\mu'} 2\hbar k (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}'). \tag{2.54}$$

The free field propagator for the massless vector field may be found by calculating $\langle 0| T\{A_\mu(x) A_{\mu'}(x')\} |0\rangle$; this gives us

$$G^{\mathrm{F}}_{\mu\mu'}(x - x') = -\int \frac{d^4k}{(2\pi)^4} \frac{i\eta_{\mu\mu'}}{k^2 + i\varepsilon} e^{-ik\cdot(x - x')}. \tag{2.55}$$

Turning again to the heuristic understanding we have been using to interpret this propagator, we may say that it represents two components of a photon which travels between x and x': specifically, the component μ at x and μ' at x'. The momentum-space representation of the propagator is

$$\widetilde{G}^{\mathrm{F}}_{\mu\mu'}(k) = \frac{-i\eta_{\mu\mu'}}{k^2 + i\varepsilon}. \tag{2.56}$$

2.2.2 Interacting fields

Having considered the free field theory, we turn to the interacting theory, and specifically to a model of the situation we wish to study. In the classical case we were able to assume some acceleration of the charged particle; in the quantum field case, we shall suppose that the acceleration is the result of some potential field V^μ generated by a current J^μ. We shall restrict the potential by hypothesis to be a function of time only, varying over a bounded interval, $[-T, 0]$, and use the gauge freedom to set $V^0(t)$ equal to a constant, which we can choose to be zero. Consequently, this forces the background field to be divergence-free, which is to say, $\partial_\mu V^\mu = 0$. We can freely choose $V^\mu(t) = 0$ for $t > 0$, but do not require that the same be true for $t < -T$.

We incorporate this potential field into the scalar Lagrangian by making the replacement $\partial_\mu \mapsto D_\mu \equiv \partial_\mu + iV_\mu/\hbar$, and we also have an interaction Lagrangian linking the scalar and massless vector fields:

$$\mathcal{L}_{\text{non}} = \left(D_\mu\phi\right)^\dagger \left(D^\mu\phi\right) + \frac{m^2}{\hbar^2}\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\nu A^\nu\right)^2, \tag{2.57}$$

$$\mathcal{L}_{\text{int}} = -\frac{ie}{\hbar}A^\mu\left[\phi^\dagger \overleftrightarrow{D}_\mu \phi\right] + \frac{e^2}{\hbar^2}A^\mu A_\mu \phi^\dagger\phi, \tag{2.58}$$

where $\overleftrightarrow{D}_\mu = \overrightarrow{D}_\mu - \overleftarrow{D}^\dagger_\mu$. The total Lagrangian for the system is then $\mathcal{L} =$

$\mathcal{L}_{\text{non}} + \mathcal{L}_{\text{int}}$, which simplifies to

$$\mathcal{L} = \left(\left(D_\mu + \frac{ie}{\hbar}A_\mu\right)\phi\right)^\dagger \left(\left(D^\mu + \frac{ie}{\hbar}A^\mu\right)\phi\right) + \frac{m^2}{\hbar^2}\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\left(\partial_\nu A^\nu\right)^2 . \tag{2.59}$$

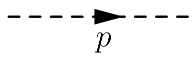
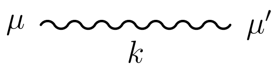
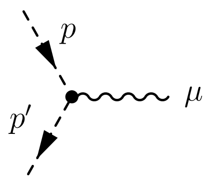
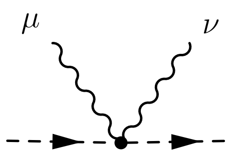
Briefly, let us consider the case where the background potential V^μ has been set to zero, so that we recover standard scalar QED. We may then read off from the interacting Lagrangian the interactions of the scalar and electromagnetic fields. There are two: each has an incoming and an outgoing scalar particle, but one has one photon and the other has two. The algebraic expressions associated with the two momentum-space propagators and these two interactions ('vertices') are noted down in Table 2.1. For a given interacting quantum field theory, these associations between elements of diagrams and algebraic expressions are known as the Feynman rules for that theory. We construct the full algebraic expression by multiplying together the expression for each component of a Feynman diagram, applying momentum conservation and integrating over the 'internal momenta', which are the momenta associated with propagators connected by vertices to other propagators at both ends.

The equations of motion for the non-interacting vector field are identical to the equations of motion for the free field, $\Box A^\mu = 0$: consequently the momentum mode solutions are the same as for the free field, and the four-momentum k^μ still satisfies $k^0 = \|\mathbf{k}\|$.

Let us once again consider the situation with V^μ made as general as our original hypotheses allow. For the non-interacting scalar field, we find the equations of motion on the mode solutions to be

$$\begin{aligned} \hbar^2 D_\mu D^\mu \Phi + m^2\Phi &= 0, \qquad (2.60) \\ \hbar^2 {D_\mu}^\dagger {D^\mu}^\dagger \overline{\Phi} + m^2\overline{\Phi} &= 0. \end{aligned}$$

Table 2.1: The Feynman rules for scalar quantum electrodynamics without a background potential.

Description	Diagrammatic representation	Algebraic representation
Scalar particle		$\dfrac{i}{p^2 - m^2 + i\varepsilon}$
Photon		$\dfrac{i\eta_{\mu\mu'}}{k^2 + i\varepsilon}$
Vertices		$-ie(p_\mu + p'_\mu)$
		$2ie^2\eta_{\mu\nu}$

Since the potential V^μ varies only in t, we may separate the equation of motion into spatial and temporal derivatives, and conclude that $\Phi(p,x), \overline{\Phi}(p,x) \propto e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}$. Hence, we obtain as equations on the time-dependent mode,

$$\hbar^2 {\partial_0}^2 \phi_{\mathbf{p}}(t) + {\sigma_{\mathbf{p}}}^2(t)\phi_{\mathbf{p}}(t) = 0, \tag{2.61}$$

$$\hbar^2 {\partial_0}^2 \overline{\phi}_{\mathbf{p}}(t) + {\overline{\sigma}_{\mathbf{p}}}^2(t)\overline{\phi}_{\mathbf{p}}(t) = 0, \tag{2.62}$$

where ${\sigma_{\mathbf{p}}}^2(t) = \|\mathbf{p} - \mathbf{V}(t)\|^2 + m^2 = {\overline{\sigma}_{-\mathbf{p}}}^2(t)$. Clearly, this implies that the time-varying parts of the particle and anti-particle solutions are related by $\phi_{\mathbf{p}}(t) = \overline{\phi}_{-\mathbf{p}}(t)$. Note that our conditions on V^μ imply that for $t > 0$, the field modes reduce to the usual free field solutions, $\Phi(p,x) = e^{-ip\cdot x/\hbar}$ with $p_0 = \sigma_{\mathbf{p}}(0) = \sqrt{\|\mathbf{p}\|^2 + m^2}$. We note that the canonical commutation relations are constant in time, and therefore we consider the commutation relations at a time $t > 0$. Doing so, we obtain the following non-zero commutation relations for the operators:

$$\left[A(\mathbf{p}), A^\dagger(\mathbf{p}')\right] = \left[B(\mathbf{p}), B^\dagger(\mathbf{p}')\right] = 2p_0(2\pi\hbar)^3\delta^3(\mathbf{p} - \mathbf{p}'). \tag{2.63}$$

We thus expand the scalar field in terms of momentum modes as

$$\phi(x) = \hbar \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left[A(\mathbf{p})\phi_{\mathbf{p}}(t)e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} + B^\dagger(\mathbf{p})\phi^*_{-\mathbf{p}}(t)e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}\right]. \tag{2.64}$$

We shall consider the form of $\phi_{\mathbf{p}}(t)$ later, and turn now to the evolution of the quantum state and the role of the interacting Hamiltonian.

The standard approach to an interacting quantum field theory is perturbation theory. In this approach, we assume that the coupling constant in the interaction term, which in this case is the charge on the scalar (anti-)particle, e, is small. Hence, the interacting Hamiltonian may be considered as a small perturbation on the non-interacting Hamiltonian. We thus find that the interacting Hamiltonian governs the way a state evolves subject to the interaction, and this evolution may be seen as a formal power series in the coupling constant. Development of the perturbation theory may be found in the standard texts, such as Peskin and Schroeder [31] and Itzykson and Zuber [30]. In the latter, the scalar electrodynamic case is treated of directly.

The Hamiltonian density for the scalar electrodynamic theory is found in the canonical manner. From the full Lagrangian, Eq. (2.59), we note that the interaction can be seen as a substitution in the Lagrangian of the complex scalar field, replacing $\partial_\mu \mapsto D_\mu + ieA_\mu/\hbar$. Hence, we consider the interaction Hamiltonian for the scalar field, which means we may define $\mathcal{L}_0 \equiv \mathcal{L}_{\text{non}} - \mathcal{L}_{\text{EM}}$. Then the conjugate fields we derive from $\mathcal{L}_0 + \mathcal{L}_{\text{int}}$ are

$$\pi_\phi = \frac{\partial\left(\mathcal{L}_0 + \mathcal{L}_{\text{int}}\right)}{\partial\left(D_0\phi\right)} = \left(D^0\phi\right)^\dagger - \frac{ie}{\hbar}A^0\phi^\dagger, \tag{2.65}$$

$$\pi_{\phi^\dagger} = \frac{\partial\left(\mathcal{L}_0 + \mathcal{L}_{\text{int}}\right)}{\partial\left(D_0\phi\right)^\dagger} = \left(D^0\phi\right) + \frac{ie}{\hbar}A^0\phi. \tag{2.66}$$

The Hamiltonian density is derived from the Lagrangian by

$$\mathcal{H}(x) = \pi_\phi\dot{\phi} + \pi_{\phi^\dagger}\dot{\phi}^\dagger - \mathcal{L}_0 - \mathcal{L}_{\text{int}}, \tag{2.67}$$

into which we substitute the conjugate fields, obtaining

$$\begin{aligned}\mathcal{H}(x) = \pi_\phi\left(\pi_{\phi^\dagger} - \frac{i}{\hbar}\left(V_0 + eA_0\right)\phi\right) + \pi_{\phi^\dagger}\left(\pi_\phi + \frac{i}{\hbar}\left(V_0 + eA_0\right)\phi^\dagger\right)\\ - \left(D_\mu\phi\right)^\dagger\left(D^\mu\phi\right) + m^2\phi^\dagger\phi + \frac{ie}{\hbar}A^\mu\left[\phi^\dagger\overleftrightarrow{D}_\mu\phi\right] - \frac{e^2}{\hbar^2}A^\mu A_\mu\phi^\dagger\phi.\end{aligned} \tag{2.68}$$

The Hamiltonian then simplifies to $\mathcal{H}(x) = \mathcal{H}_0(x) + \mathcal{H}_{\text{int}}(x)$, where

$$\mathcal{H}_0(x) = \pi_{\phi^\dagger}\pi_\phi + \left(D_i\phi\right)^\dagger\left(D_i\phi\right) + m^2\phi^\dagger\phi - \frac{i}{\hbar}V_0\left(\pi_\phi\phi - \pi_{\phi^\dagger}\phi^\dagger\right), \tag{2.69}$$

$$\mathcal{H}_I(x) = \frac{ie}{\hbar}A^0\left(\pi_\phi\phi - \pi_{\phi^\dagger}\phi^\dagger\right) - \frac{ie}{\hbar}A_i\left(\phi^\dagger\overleftrightarrow{D}_i\phi\right) + \frac{e^2}{\hbar^2}A_iA_i\phi^\dagger\phi. \tag{2.70}$$

Taking $H_0 = \int d^3\mathbf{x}\,\mathcal{H}_0(x)$ to be the Hamiltonian and using equivalence of Hamiltonians under addition of total derivative terms, we obtain as equations of motion

$$\dot{\phi} = \frac{\delta H_0}{\delta \pi_\phi} = \pi_{\phi^\dagger} - \frac{i}{\hbar} V_0 \phi, \tag{2.71}$$

$$\dot{\pi}_\phi = -\frac{\delta H_0}{\delta \phi^\dagger} = D_i D_i \phi - m^2 \phi - \frac{i}{\hbar} V_0 \pi_{\phi^\dagger}, \tag{2.72}$$

and their conjugates. We can re-write these equations

$$\pi_{\phi^\dagger} = D_0 \phi \tag{2.73}$$

$$\left(D_\mu D^\mu + m^2\right)\phi = 0. \tag{2.74}$$

Since in the interaction picture we treat H_0 as the source of Hamilton's equations for ϕ, we can thus let $\pi_{\phi^\dagger} = D_0\phi$; then the interaction Hamiltonian becomes

$$\mathcal{H}_I(x) = \frac{ie}{\hbar} A^\mu \left[\phi^\dagger \overleftrightarrow{D}_\mu \phi\right] + \frac{e^2}{\hbar^2} A_i A_i \phi^\dagger \phi. \tag{2.75}$$

The reader will note that this interaction Hamiltonian is different from what might usually be expected. The presence of a derivative of the scalar field in the interaction Lagrangian is the cause of the difference: if the interaction terms had been simple powers of the fields, then the relationship would have been simply $\mathcal{H}_I = -\mathcal{L}_{\text{int}}$. The interaction Hamiltonian we have derived here differs by $e^2 {A_0}^2 \phi^\dagger \phi / \hbar^2$.

Let us again consider the situation without a background potential V^μ. As we have said, the evolution of states takes place through the interaction Hamiltonian density, $\mathcal{H}_{\text{int}}(x)$; it does so as follows. If we suppose that the fields are in some initial configuration $|i\rangle$ at a time t_0', then a state $|f\rangle$ at some later time t' is given by

$$|f\rangle = T\left\{\exp\left[-i\int_{t_0'}^{t'} dt \int d^3\mathbf{x}\,\mathcal{H}_I(t,\mathbf{x})\right]\right\}|i\rangle\,. \tag{2.76}$$

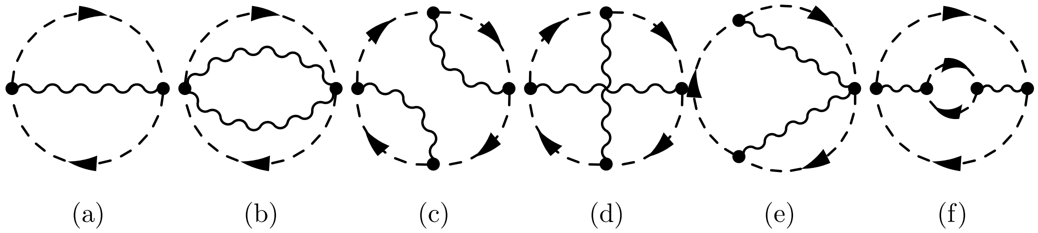
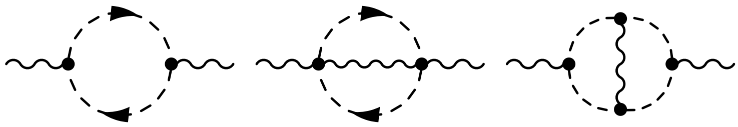


Figure 2.3: Simple vacuum fluctuation diagrams. Example (a) is of order e^2, while (b)—(f) are of order e^4.

If we consider this time-ordered product, it is clear that we can construct terms where the fields from the Hamiltonian interact only with each other, and not with the initial state. That is, self-contained interactions are generated. Indeed, we can see that even were the initial state to be taken as the vacuum state, there would still be terms arising from the Hamiltonian which represent the creation, interaction and subsequent annihilation of particles. These terms are known as vacuum fluctuations, and would appear, diagrammatically, as self-contained loops: Fig. 2.3 shows some simple examples which can occur. It can be shown that these diagrams may be discarded entirely [31].

A final concept we must introduce bears some similarity to these vacuum fluctuations. Instead of considering the evolution of the vacuum, however, we consider the evolution of the particles in the background-free interacting theory. We shall see that what we would consider a free particle in the unperturbed theory is not quite the same in the perturbed theory.

Let us firstly consider the evolution of the state which commences and concludes with a single photon, and add in possible intervening interactions which leave the initial and final states unaffected. Up to order e^4, there are three non-trivial ways of combining copies of the vertices we described in Table 2.1; these are illustrated in Fig. 2.4. It can be shown that the summed effect of these extra diagrams, called loop corrections, is to make the interacting



(a) (b) (c)

Figure 2.4: Loop corrections to the photon propagator up to order e^4. Diagram (a) is a *one-loop* correction while (b) and (c) are *two-loop* corrections.

propagator a scalar multiple of the non-interacting one, with the sum being divergent. The proportionality factor is a certain function of the momentum possessed by the virtual photon, q, which we typically denote by $(1-\Pi(q^2))^{-1}$. For low-q^2 processes, integrating over q (as we do for all internal momenta) will produce the residue of the pole, which we denote $Z_3 \equiv (1-\Pi(0))^{-1}$. Thus for such processes, we may replace the photon propagator $e^2 g_{\mu\nu}/q^2$ by $Z_3 e^2 g_{\mu\nu}/q^2$. Equivalently, we may regard this as a modification of the non-interacting (or bare) coupling constant, which we denote e_0: we then have the *renormalised* charge, which is given by $e^2 = Z_3 {e_0}^2$.

Let us secondly consider the evolution of the state with a single scalar particle. Again, we have ways in which to add in vertices to alter the propagator without altering either the initial or final states, illustrated up to order e^2 in Fig. 2.5. Since there is only one loop in these diagrams, they are called one-loop corrections. Higher order terms in e arise with more loops. We may extract the algebraic form for this correction from either the Feynman diagrams or the exponentiated Hamiltonian, which generates the necessary terms automatically.

As the effect of the renormalisation for the vector field was on the photon propagator or equivalently the unit of charge, so here the effect is to modify

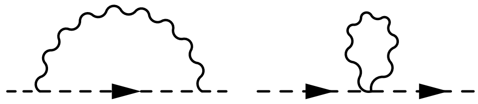


Figure 2.5: Loop corrections to the scalar propagator up to order e^2.

the scalar propagator or equivalently the mass associated with the scalar field; for this latter reason, the summed effect of all such diagrams with a scalar particle entering and emerging is known as the self-energy. Again, this sum is divergent, and we therefore seek some way of removing the difficulty that such a quantity would pose.

By considering the infinite sum of loop corrections, we find repetitions of certain terms, which we can use to show that the sum of these terms reduces to

$$\frac{i}{p^2 - {m_0}^2} + \frac{i}{p^2 - {m_0}^2}\left[-i\Sigma(p)\right]\frac{i}{p^2 - {m_0}^2} + \frac{i}{p^2 - {m_0}^2}\left[-i\Sigma(p)\right]\frac{i}{p^2 - {m_0}^2}\left[-i\Sigma(p)\right]\frac{i}{p^2 - {m_0}^2} + \dots,$$

where $\Sigma(p)$ represents the 'one-particle irreducible' corrections, which is the infinite sum of loop corrections which are not disconnected by the removal of a single propagator, and m_0 represents the 'bare mass' in the Lagrangian. This quantity can be shown, by the binomial theorem, to be equal to

$$\frac{i}{p^2 - {m_0}^2 - \Sigma(p)}. \tag{2.77}$$

We thus conclude that the mass is renormalised by the addition of an extra term, $m^2 = {m_0}^2 + \delta m^2$, where $\delta m^2 = \Sigma(p)|_{p^2=m^2}$[1]. In addition to finding

[1]Clearly this is self-referential as it stands. However, to lowest order in $\hbar$ we may take $\delta m^2 \approx \Sigma(p)|_{p^2={m_0}^2}$.

whether the quantum field theory generates the same predictions as the classical theory, we are interested in whether the one-loop corrections to the mass which arise in the presence of a background field cancel against δm^2. Any difference between the two may give rise to additional quantum corrections in the theory.

2.3 Quantum model

In this section, we shall review the results reported by Higuchi and Martin [20, 21, 22, 23]. The two key findings are that the change in position induced by the background potential matches the prediction of the classical theory, and that the mass counterterm, δm^2, continues to cancel the one-loop corrections to the propagators. This latter is an intricate and complicated calculation which we will not summarise here, although at the appropriate point in the review below we shall note where a failure of cancellation would have a physical effect. We shall provide a sketch of the former, details of which can be found in Martin [25], or in Ch. 6 in a more general form of which the flat metric is a special case.

The initial state for the model is a single scalar particle with a momentum which is distributed around a central momentum $\bar{\mathbf{p}}$ by a function $f(\mathbf{p})$. That is, we set

$$|i\rangle = \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} f(\mathbf{p}) A^\dagger(\mathbf{p}) |0\rangle \,, \tag{2.78}$$

where this is normalised so that $\langle i|i\rangle = 1$. Then to order e^2, the final state satisfies

$$|f\rangle = T \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt' \int d^3\mathbf{x} \mathcal{H}_{\mathrm{I}}(t', \mathbf{x}) \right\}_{\text{conn.}} |i\rangle \,, \tag{2.79}$$

where we have restricted the exponentiated Hamiltonian to those terms which generate connected diagrams. By the unitarity of the time-evolution operator, $\langle f|f\rangle = 1$ as well. We are interested only in those sectors of the final state which contain a single scalar particle, and also only those sectors which contribute to the squared state $\langle f|f\rangle$ and similar quantities up to order e^2. Therefore, we may write the sectors of the state in which we are interested as

$$\begin{aligned} |f\rangle = &\int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\left[1+\frac{i}{\hbar}\mathcal{F}(\mathbf{p})\right] f(\mathbf{p})A^\dagger(\mathbf{p})\,|0\rangle \\ &+\frac{i}{\hbar}\int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})a^\dagger_\mu(\mathbf{k})f(\mathbf{P})A^\dagger(\mathbf{P})\,|0\rangle\,, \end{aligned} \tag{2.80}$$

where $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$, as the scalar particle loses momentum to the photon.

The complex quantity $\mathcal{A}_\mu(\mathbf{p},\mathbf{k})$ is called the emission amplitude, and is related to the probability of finding a photon in the final state. We shall see later that this quantity is of order e.

The complex quantity $\mathcal{F}(\mathbf{p})$ is called the forward-scattering amplitude, since it relates to the one-loop processes depicted in Fig. 2.5. We shall see later that this quantity is of order e^2.

Consider now the equations of motion we derived for the scalar field, Eq. (2.60). We noted that we have solutions which can be separated into factors dependent on each of the co-ordinates, and that applying this gives us solutions $\Phi_\mathbf{p}(x) \propto e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$, leaving us with an equation for $\phi_\mathbf{p}(t)$ which is

$$\hbar^2{\partial_0}^2\phi_\mathbf{p}(t) + {\sigma_\mathbf{p}}^2(t)\phi_\mathbf{p}(t) = 0, \tag{2.81}$$

where $\sigma_\mathbf{p}(t) = \|\mathbf{p}-\mathbf{V}(t)\|^2 + m^2$. The anti-particle mode satisfies $\overline{\Phi}_\mathbf{p}(x) \propto e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$, and thus we find as an equation of motion

$$\hbar^2{\partial_0}^2\overline{\phi}_\mathbf{p}(t) + {\sigma_{-\mathbf{p}}}^2(t)\overline{\phi}_\mathbf{p}(t) = 0. \tag{2.82}$$

We thus conclude that the time-dependent factors of the particle and anti-particle modes are related by

$$\phi_\mathbf{p}(t) = \overline{\phi}_{-\mathbf{p}}(t). \tag{2.83}$$

To find the form of this mode, we apply the technique known as the WKB approximation, by which we treat the solution to the equation of motion as a power series in $\hbar$, and thus suppose that $\log \phi_{\mathbf{p}}(t) = \hbar^{-1} \sum_n \hbar^n s_n(t)$. We solve this for s_n sequentially, obtaining[2]

$$\dot{s}_0 = -\sigma_{\mathbf{p}}(t), \tag{2.84}$$

$$\dot{s}_1 = -\frac{1}{2}\partial_0 \log \sigma_{\mathbf{p}}(t). \tag{2.85}$$

In order to find the expected change in position due to the radiation emission, we must calculate an expectation value for the position of the particle in the case that the background field is removed, which will serve as a baseline. Since there is only one particle in this state, we may use the charge density as a proxy for the particle's position density, and consequently,

$$\left\langle x^i \right\rangle_{\text{in}} = \int d^3\mathbf{x} \left\langle i \right| x^i J^0 \left| i \right\rangle, \tag{2.86}$$

where $J^\mu \equiv \frac{i}{\hbar} g^{\mu\nu} : \phi^\dagger \partial_\nu \phi - (\partial_\nu \phi)^\dagger \phi :$. Applying our definition of the initial state, we find

$$\left\langle x^i \right\rangle_{\text{in}} = \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p}) \overleftrightarrow{\partial_{p_i}} f(\mathbf{p}), \tag{2.87}$$

where $\overleftrightarrow{\partial_{p_i}} \equiv \overrightarrow{\partial_{p_i}} - \overleftarrow{\partial_{p_i}}$.

We may distinguish two sectors within the final state by the number of photons they contain. Hence, we write $|f\rangle = |f_0\rangle + |f_1\rangle$ and see that $\langle f_0 | f_1 \rangle = 0$, owing to the photon creation operator in $|f_1\rangle$ which annihilates the no-photon state $\langle f_0|$. Therefore, we may separate the position expectation value in the final state into two parts: the no-photon and one-photon sectors. Considering first the no-photon sector, we find

$$\left\langle x^i \right\rangle_{\text{fin},0} = \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left[f^*(\mathbf{p}) \overleftrightarrow{\partial_{p_i}} f(\mathbf{p}) \right] \left[1 - \frac{2}{\hbar} \operatorname{Im} \mathcal{F}(\mathbf{p}) \right], \tag{2.88}$$

[2]For $\dot{s}_0$, we obtain $\pm\sigma_{\mathbf{p}}(t)$ and choose the negative sign as $\phi_{\mathbf{p}}(t) \propto e^{-\frac{i}{\hbar} p_0 t}$ in the far future.

while in the one-photon sector, we find

$$\left\langle x^i \right\rangle_{\text{fin},1} = -\frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{P_0}{p_0} \times \left\{ -\left|\mathcal{A}(\mathbf{p},\mathbf{k})\right|^2 \left[f^*(\mathbf{p})\overleftrightarrow{\partial_{p_i}} f(\mathbf{p}) \right] + |f(\mathbf{p})|^2 \left[\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial_{p_i}}\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) \right] \right\}, \tag{2.89}$$

where $P_0 = \sigma_{\mathbf{P}}(0)$ and $|\mathcal{A}(\mathbf{p},\mathbf{k})|^2 \equiv -\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k})$. By studying the implications of the unitarity condition $\langle f|f\rangle = 1$, we deduce that

$$\left\langle x^i \right\rangle_{\text{fin}} = \left\langle x^i \right\rangle_{\text{in}} + \delta x^i_{\text{tree}} + \delta x^i_{\text{loop}}, \tag{2.90}$$

where

$$\delta x^i_{\text{tree}} = -\frac{i}{2}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{P_0}{p_0}|f(\mathbf{p})|^2 \left[\mathcal{A}_\mu(\mathbf{p},\mathbf{k})\overleftrightarrow{\partial_{p_i}}\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) \right], \tag{2.91}$$

$$\delta x^i_{\text{loop}} = -\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}|f(\mathbf{p})|^2 \partial_{p_i} \operatorname{Re}\mathcal{F}(\mathbf{p}). \tag{2.92}$$

The names 'tree' and 'loop' are intended to denote the origin of the two position shifts: the first comes from the tree-level vertex with a single particle entering and a particle-plus-photon leaving, while the second comes from the one-loop corrections to the scalar propagator. It is here that the renormalisation of the mass enters, since any remnant from the renormalisation would cause this position shift to be non-zero. The calculation showing that $\delta x^i_{\text{loop}} = 0$ is long, involved, and found in [25].

In order to calculate the size of the tree-level position shift, we must consider the emission amplitude $\mathcal{A}_\mu(\mathbf{p},\mathbf{k})$. This we do by noting that we have two different definitions of the final state $|f\rangle$: one exact, the other correct to order e^2. The first, exact definition is the evolutionary one,

$$|f\rangle = T\exp\left\{ \frac{i}{\hbar}\int_{-\infty}^{+\infty} dt' \int d^3\mathbf{x}\mathcal{H}_\text{I}(t',\mathbf{x}) \right\}_{\text{conn.}} |i\rangle\,; \tag{2.93}$$

the second, approximate definition includes the order-e term

$$|f_1\rangle = \frac{i}{\hbar}\int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^\mu(\mathbf{p},\mathbf{k})a^\dagger_\mu(\mathbf{k})f(\mathbf{P})A^\dagger(\mathbf{P})\left|0\right\rangle . \tag{2.94}$$

If we consider the terms from the first definition at order e, we see that for any state $|\omega\rangle$,

$$\begin{aligned}
&-\frac{e}{\hbar^2}\left\langle\omega\right|\int d^4x A_\nu :\phi^\dagger \overleftrightarrow{D}^\nu\phi: \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}f(\mathbf{p}')A^\dagger(\mathbf{p})\left|0\right\rangle \\
&\quad = \frac{i}{\hbar}\left\langle\omega\right|\int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^\nu(\mathbf{p},\mathbf{k})a^\dagger_\nu(\mathbf{k})f(\mathbf{P})A^\dagger(\mathbf{P})\left|0\right\rangle .
\end{aligned}\tag{2.95}$$

Hence, if we choose $|\omega\rangle = a^\dagger_\mu(\mathbf{k}')f(\mathbf{p})A^\dagger(\mathbf{p}')\left|0\right\rangle$, we find

$$\mathcal{A}_\mu(\mathbf{p},\mathbf{k}) = -\frac{ie}{\hbar^2}\int \frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3}\left\langle 0\right| A(\mathbf{p}')a_\mu(\mathbf{k})\int d^4x A^\nu :\phi^\dagger \overleftrightarrow{D}_\nu\phi: A^\dagger(\mathbf{p})\left|0\right\rangle . \tag{2.96}$$

Applying the Fourier expansions of the fields and the commutation relations, we find

$$\mathcal{A}_\mu(\mathbf{p},\mathbf{k}) = -ie\hbar\int \frac{d^3\mathbf{p}'}{2p'_0(2\pi\hbar)^3}\int d^4x\, e^{ik\cdot x}\left[\Phi^*_{\mathbf{p}'}(x)\overleftrightarrow{D}_\mu\Phi_{\mathbf{p}}(x)\right]. \tag{2.97}$$

We substitute in the WKB approximation in order to apply the derivative. The momentum of the classical particle, p^μ, satisfies the identity $(p^\mu - V^\mu)(p_\mu - V_\mu) = m^2$; this can be shown to imply that if we define $k\xi \equiv k\cdot x$, then

$$\mathcal{A}_\mu(\mathbf{p},\mathbf{k}) = -e\int d\xi \frac{dx_\mu}{d\xi}e^{ik\xi}, \tag{2.98}$$

where x^μ refers to the trajectory of the equivalent classical particle with momentum $\mathbf{p}$ for $t > 0$, parameterised by ξ. This expression is ill-defined, as the integrand remains finite and non-zero as $\xi \to \pm\infty$. We resolve this difficulty by the insertion of a cut-off factor, $\chi(\xi)$, which satisfies the following conditions:

1. $\chi(\xi)$ equals 1 in the region $[-T, 0]$: that is, while the acceleration is non-zero;

2. $\lim_{\xi \to \pm\infty} \chi(\xi) = 0$; and

3. $\int_{-\infty}^{+\infty} (\chi'(\xi))^2 d\xi$ is bounded.

In App. A, we show that these cut-off functions may be removed for integrals satisfying certain conditions. We obtain the following equations for the emission amplitude, the second derived from the first through integration by parts:

$$\mathcal{A}_\mu(\mathbf{p}, \mathbf{k}) = -e \int d\xi \, \chi(\xi) \frac{dx_\mu}{d\xi} e^{ik\xi} = -\frac{ie}{k} \int d\xi \left(\frac{d^2 x_\mu}{d\xi^2} + \chi'(\xi) \frac{dx_\mu}{d\xi} \right) e^{ik\xi}. \tag{2.99}$$

The first of these definitions can be written, alternatively, as

$$\mathcal{A}_\mu(\mathbf{p}, \mathbf{k}) = -\int d^4x \, j_\mu(\mathbf{p}, x) e^{ik \cdot x}, \tag{2.100}$$

where

$$j_\mu(\mathbf{p}, x) = e \int_{-\infty}^{+\infty} ds \frac{dz_\mu}{ds} \delta^4(x - z(s)) \chi(s) \tag{2.101}$$

is the (cut-off-incorporated) classical current associated with the trajectory $z(s)$ and momentum $\mathbf{p}$.

The reader will recall that we were able to recover the fundamental solutions to the Klein-Gordon equation from vacuum expectation values of products of the fields. Working from these, it is possible to show that we are able to write the emission amplitude, $\mathcal{A}_\mu(\mathbf{p}, \mathbf{k})$, in terms of the positive-frequency part of the retarded electromagnetic field generated by the current $j^\mu(\mathbf{p}, x)$. This will be detailed in Ch. 6. We may apply the Kirchhoff identity, which relates fields to initial data on a Cauchy surface through the retarded Green's function [12], to introduce the radiative field which we used in the classical case, and thus to write

$$\delta x^i_{\text{tree}} = -\int dt \left(\frac{\partial x^j}{\partial p_i} \right)_t f^{\text{R}}_j, \tag{2.102}$$

where f_j^{R} is the electromagnetic force due to the field as defined using the radiative Green's function. The reader will recall that this formula matches exactly the classical calculation we carried out previously.

A similar line of argument will yield the same conclusion for a potential which is dependent on one spatial co-ordinate.

Chapter 3

Energy radiation from an accelerating charged particle

Summary. In this chapter, we shall show that it is possible directly to derive the Larmor formula from the scalar quantum electrodynamic theory presented earlier.

The position shift calculation carried out by Higuchi and Martin and reviewed above verified that the classical and quantum field-theoretic effects matched to zeroth order in $\hbar$. Here we shall calculate the stress-energy tensor of the quantised electromagnetic field on a flat space-time and using this proceed to find the expectation value of the energy content of that field. It will be shown that this result matches the classical Larmor formula.

In Lagrangian field theories, the stress-energy tensor can be defined in a number of different ways. The symmetric stress-energy tensor is defined by the functional derivative,

$$\Theta^{\mu\nu}(x)\delta^4(x-y) = 2\frac{\delta}{\delta g_{\mu\nu}(x)}\mathcal{L}(y). \tag{3.1}$$

Since we are carrying out the functional derivative with respect to the metric, it is necessary to use the curved-space definition of the electromagnetic Lagrangian, which is $\mathcal{L}_{EM} = -\frac{1}{4}\sqrt{-g}g_{\alpha\mu}g_{\beta\nu}F^{\alpha\beta}F^{\mu\nu}$. Then, since $\delta g/\delta g_{\mu\nu} = gg^{\mu\nu}$,

$$\Theta^{\rho\sigma}{}_{EM} = -\sqrt{-g}F^{\mu\beta}F^{\nu}{}_{\beta} - g^{\mu\nu}\mathcal{L}_{EM}. \tag{3.2}$$

The classical energy-momentum four-vector is by definition $\Theta^{0\mu}{}_{EM}$, and at a given time t, the expectation value for the quantised energy-momentum operator is

$$\langle P^{\mu}\rangle_f \equiv \int_{x^0=t} d^3\mathbf{x}\, \langle f| : \Theta^{0\mu} : |f\rangle\,. \tag{3.3}$$

It is useful at this juncture to define a new pair of operators out of the vector field operators,

$$C^{\mu\nu\dagger}(\mathbf{k}) = -k^{\mu}a^{\nu\dagger}(\mathbf{k}) + k^{\nu}a^{\mu\dagger}(\mathbf{k}) \tag{3.4}$$

and its Hermitian conjugate. It satisfies the commutation relations

$$\left[a^{\mu}(\mathbf{k}'), C^{\alpha\beta\dagger}(\mathbf{k})\right] = -\left(k^{\alpha}g^{\mu\beta} - k^{\beta}g^{\mu\alpha}\right)(2\pi)^3 2\hbar k\delta^{(3)}(\mathbf{k}' - \mathbf{k}), \tag{3.5}$$

and acts as the electromagnetic field creation operator, so that

$$F_{\mu\nu} = \int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\left[C_{\mu\nu}(\mathbf{k})e^{-ik\cdot x} + C^{\dagger}_{\mu\nu}(\mathbf{k})e^{ik\cdot x}\right]. \tag{3.6}$$

Inspecting the final state $|f\rangle$ in Eq. (2.80), only the sector with an excitation in the electromagnetic field will contribute to the expectation value of the energy-momentum operator defined in Eq. (3.3); therefore we combine this with Eq. (3.6) to obtain

$$\begin{aligned}\langle P^{\mu}\rangle_f = \frac{1}{\hbar^2}\int &\frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{k}'}{2k'(2\pi)^3}\frac{d^3\mathbf{l}}{2l(2\pi)^3}\frac{d^3\mathbf{l}'}{2l'(2\pi)^3}d^3x \\ &\times \langle 0|\, A(\mathbf{P})f^*(\mathbf{p})\mathcal{A}^*_{\rho}(\mathbf{p},\mathbf{k})a^{\rho}(\mathbf{k}) \\ &\left[-\frac{1}{4}\left(C^{\alpha\beta\dagger}(\mathbf{l})C_{\alpha\beta}(\mathbf{l}')e^{-i(l-l')\cdot x} + C^{\alpha\beta\dagger}(\mathbf{l}')C_{\alpha\beta}(\mathbf{l})e^{i(l-l')\cdot x}\right)g^{0\mu}\right. \\ &\left.+\left(C^{0\nu\dagger}(\mathbf{l}')C^{\mu}{}_{\nu}(\mathbf{l})e^{i(l'-l)\cdot x} + C^{\mu\nu\dagger}(\mathbf{l})C^{0}{}_{\nu}(\mathbf{l}')e^{-i(l'-l)\cdot x}\right)\right] \\ &\times \mathcal{A}^{\sigma}(\mathbf{p}',\mathbf{k}')a^{\dagger}_{\sigma}(\mathbf{k}')f(\mathbf{p}')A^{\dagger}(\mathbf{P}')\,|0\rangle\,. \end{aligned} \tag{3.7}$$

The Ward identity [31] implies that $k^\mu \mathcal{A}_\mu(\mathbf{p},\mathbf{k}) = 0$; applying this together with our commutation relations gives us

$$\langle P^\mu \rangle_f = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{2k(2\pi)^3} \frac{P_0}{p_0} |f(\mathbf{p})|^2 k^\mu |\mathcal{A}(\mathbf{p},\mathbf{k})|^2, \tag{3.8}$$

where $|\mathcal{A}(\mathbf{p},\mathbf{k})|^2 = -\mathcal{A}^*_\alpha(\mathbf{p},\mathbf{k})\mathcal{A}^\alpha(\mathbf{p},\mathbf{k})$. We take the integral to lowest order in $\hbar$: the normalisation of $f(\mathbf{p})$ as well as

$$P_0 = p_0 \left[1 + \frac{\hbar\mathbf{k}\cdot\mathbf{p}}{{p_0}^2}\right] \tag{3.9}$$

combine to produce

$$\langle P^\mu \rangle_f = \int \frac{d^3k}{2k(2\pi)^3} k^\mu \left|\mathcal{A}(\mathbf{p},\mathbf{k})\right|^2. \tag{3.10}$$

From this point on, we have taken $|f(\mathbf{p})|^2$ to be sharply peaked about a certain momentum; henceforth in this chapter, we refer to this peak momentum as $\mathbf{p}$.

Taking $\mathcal{A}^\mu(\mathbf{p},\mathbf{k})$ in the second-order derivative form from Eq. (2.99), we also integrate with respect to k to find

$$\langle P^\mu \rangle_f = -\frac{e^2}{16\pi^2} \int d\Omega_\mathbf{k} \int d\xi n^\mu \frac{d^2x^\nu}{d\xi^2}\frac{d^2x_\nu}{d\xi^2}, \tag{3.11}$$

where n^μ is defined by $k^\mu = kn^\mu$ and $d\Omega_\mathbf{k}$ is used to denote integrating over the angular part of the vector $\mathbf{k}$. The terms proportional to $\chi'(\xi)$ vanish as we apply our limit to remove $\chi(\xi)$. Since $\xi \equiv n_\mu x^\mu$, we have

$$\frac{d\xi}{dt} = n_\mu \dot{x}^\mu \tag{3.12}$$

and, applying the chain rule to $x^\mu(t(\xi))$,

$$\frac{d^2x^\mu}{d\xi^2} = \left(\frac{dt}{d\xi}\right)^3 \left(\frac{d\xi}{dt}\frac{d^2x^\mu}{dt^2} - \frac{d^2\xi}{dt^2}\frac{dx^\mu}{dt}\right). \tag{3.13}$$

This gives us the apparently complicated integral

$$\langle P^\mu \rangle_f = -\frac{e^2}{16\pi^2} \int dt \left(\int d\Omega_\mathbf{k} \dot{\xi}^{-5} n^\mu n_\sigma n_\rho\right) \left[\dot{x}^\sigma \dot{x}^\rho \ddot{x}^\nu \ddot{x}_\nu - 2\dot{x}^\sigma \ddot{x}^\rho \ddot{x}^\nu \dot{x}_\nu + \ddot{x}^\sigma \ddot{x}^\rho \dot{x}^\nu \dot{x}_\nu\right]; \tag{3.14}$$

however, the reader will remark that the angular integral, set off in parentheses, is in fact quite simple. It can be rendered simpler still by using the following observation. It is the case that

$$\frac{\partial}{\partial \dot{x}^\alpha}\dot{\xi}^{-2} = -2n_\alpha \dot{\xi}^{-3}, \tag{3.15}$$

where we treat the components of $\dot{x}^\mu$, including $\dot{x}^0$, as independent variables; we set $\dot{x}^0 = 1$ later. Two further derivatives together with an appropriate coefficient, therefore, will give us the integrand we desire.

The basic angular integral, before the application of derivatives with respect to the velocity, is performed by noting that $\dot{\xi} = n_0\dot{x}^0 - \mathbf{n}\cdot\mathbf{v}$ and using the vector $\mathbf{v}$ as the azimuthal axis defining the elevation of $\mathbf{n}$. The result is $\int d\Omega \dot{\xi}^{-2} = 4\pi\gamma^2$, which we differentiate three times to obtain

$$\int d\Omega_{\mathbf{k}} \dot{\xi}^{-5} n^\mu n_\sigma n_\rho = -\frac{4}{3}\pi\left(6\gamma^8 \dot{x}^\mu \dot{x}_\sigma \dot{x}_\rho + \gamma^6\left(\delta^\mu_\sigma \dot{x}_\rho + \dot{x}^\mu g_{\sigma\rho} + \delta^\mu_\rho \dot{x}_\sigma\right)\right). \tag{3.16}$$

We use the fact that $dt/d\tau = \gamma = (\dot{x}_\mu \dot{x}^\mu)^{-\frac{1}{2}}$, and define $\alpha_c = \alpha\hbar$, where $\alpha = e^2/4\pi\hbar$ is the fine structure constant, to obtain

$$\langle P^\mu \rangle_f = -\frac{2\alpha_c}{3}\int dt\, \dot{x}^\mu \gamma^6 \left(\dot{x}^\alpha \dot{x}_\alpha \ddot{x}^\beta \ddot{x}_\beta - (\dot{x}^\alpha \ddot{x}_\alpha)^2\right), \tag{3.17}$$

and so can see that

$$\langle P^\mu \rangle_f = -\frac{2\alpha_c}{3}\int d\tau \frac{dx^\mu}{d\tau}\frac{d^2x^\nu}{d\tau^2}\frac{d^2x_\nu}{d\tau^2}. \tag{3.18}$$

Comparing this with Eq. 2.30, we see that this matches the energy-momentum of the electromagnetic field according to the Larmor formula. The difference in sign arises because here we are considering the energy content of the electromagnetic field, whereas in the earlier equation we were considering the energy-momentum of the particle.

Chapter 4

First-order corrections

Summary. In this chapter, we shall calculate the corrections to first order in $\hbar$ to the Larmor formula, doing this for two different potentials: one varying in t and the other, in z. We shall find that the two situations give qualitatively different results.

Having shown that the Larmor formula may be found directly from the quantum field theory, it would be interesting to investigate any correction which the quantum theory might contribute to the radiated energy. The spectral lines of the hydrogen atom are a demonstration of the difference in predictions between quantum theory and classical theory about the emission of radiation by charged particles. In the previous chapter we showed that, as we might expect, the semiclassical approximation produces the same energy emission to lowest order in $\hbar$. We therefore wish to determine whether any corrections to this energy arise at first order in $\hbar$, and the conditions under which this correction would be significant.

The structure of both sections shall be to extend our previous calculation of the WKB approximation to the necessary order in $\hbar$, and thus to extend the Larmor formula to first order in $\hbar$, in a non-relativistic setting. Two separate categories of correction will be identified, which are then combined and calculated in order to produce the desired first-order correction.

To allow for the possibility of a potential dependent on a single space co-ordinate instead of being dependent on time, we modify the equation for the energy emitted by the scalar particle (Eq. (3.8)) by inserting a Jacobian determinant, $|\partial\mathbf{P}/\partial\mathbf{p}|$, to account for the change of variables in the integral from $\mathbf{P}$ to $\mathbf{p}$:

$$\langle E\rangle_f = \frac{1}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{(2\pi)^3}\left|\frac{\partial\mathbf{P}}{\partial\mathbf{p}}\right|\frac{P_0}{p_0}\left|f(\mathbf{p})\right|^2\left|\mathcal{A}(\mathbf{p},\mathbf{k})\right|^2. \tag{4.1}$$

We now specialise our condition on $|f(\mathbf{p})|^2$ to being infinitely sharply peaked about the specified momentum $\mathbf{p}$. Inspecting this equation, it is clear that any approximation in $\hbar$ will arise from three sources: that Jacobian determinant, the ratio of P_0 to p_0, and the squared emission amplitude. In both the cases we study, we shall see that the first two are easily dealt with, as they cancel exactly against a factor in the squared emission amplitude. It is this latter which requires the calculation of further terms in the $\hbar$-expansion of the wave-function, and it will be from this point that each of the following two sections begins.

4.1 Time-dependent potential

As the reader will recall from work developed previously, if the potential is time-dependent then we are able to suppose that the positive-frequency mode solution to the equation of motion separates into free spatial factors and an as-yet-unknown temporal factor:

$$\Phi_{\mathbf{p}}(x) = \sqrt{p_0}\,\phi_{\mathbf{p}}(t)\exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}\right). \tag{4.2}$$

Writing the temporal factor as

$$\phi_{\mathbf{p}}(t) = \exp\left(\frac{1}{\hbar}\sum_{n=0}^{\infty}\hbar^n s_n(\mathbf{p},t)\right), \tag{4.3}$$

we obtain from the equation of motion, Eq. (2.81), an equation on s_n which is

$$\hbar\sum_{n=0}^{\infty}\hbar^n \ddot{s}_n + \left(\sum_{n=0}^{\infty}\hbar^n \dot{s}_n\right)^2 + {\sigma_{\mathbf{p}}}^2 = 0, \tag{4.4}$$

where $\sigma_{\mathbf{p}}(t)^2 = |\mathbf{p} - \mathbf{V}(t)|^2 + m^2c^2$. This has solutions to order $\hbar^0$ of

$$s_0(\mathbf{p},t) = -i\int^t \sigma_{\mathbf{p}}(t')dt', \text{ and} \tag{4.5}$$

$$s_1(\mathbf{p},t) = -\frac{1}{2}\log\sigma_{\mathbf{p}}(t), \tag{4.6}$$

which gives a normalised wavefunction of

$$\phi_{\mathbf{p}}(t) = \frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)}}\exp\left(-\frac{i}{\hbar}\int_0^t \sigma_{\mathbf{p}}(t')dt'\right). \tag{4.7}$$

One might expect the term at order $\hbar$ in the WKB expansion of $\phi_{\mathbf{p}}(t)$ to contribute to the first-order correction. However, we can read from Eq. (4.4) above that $2\dot{s}_0\dot{s}_2 = -\ddot{s}_1 - {\dot{s}_1}^2$. This implies that s_2 is purely imaginary, and will thus appear in $\phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t)$ as $i\hbar(f_{\mathbf{p}}(t) - f_{\mathbf{P}}(t))$. Since $\mathbf{P}$ will be replaced with $\mathbf{p} - \hbar\mathbf{k}$, the term in the brackets is itself of order $\hbar$, and therefore the first-order WKB correction to the wavefunction is not relevant at order $\hbar$ to the amplitude.

We then substitute the WKB expansion for $\phi_{\mathbf{p}}(t)$ into the equations for the components of the emission amplitude, and obtain the following:

$$\begin{aligned}\mathcal{A}_0(\mathbf{p},\mathbf{k}) &= \frac{-ie\hbar}{2}\sqrt{\frac{p_0}{P_0}}\int dt\, e^{ikt}\frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)\sigma_{\mathbf{P}}(t)}}\\ &\quad\times \exp\left\{-\frac{i}{\hbar}\int_0^t \left(\sigma_{\mathbf{p}}(T)-\sigma_{\mathbf{P}}(T)\right)dT\right\}\\ &\quad\times \left[\frac{1}{2}\frac{d}{dt}\log\left(\frac{\sigma_{\mathbf{P}}(t)}{\sigma_{\mathbf{p}}(t)}\right)-\frac{i}{\hbar}\left(\sigma_{\mathbf{p}}(t)+\sigma_{\mathbf{P}}(t)\right)\right], \qquad (4.8)\\ \mathcal{A}_i(\mathbf{p},\mathbf{k}) &= -\frac{e}{2}\sqrt{\frac{p_0}{P_0}}\int dt\, e^{ikt}\frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)\sigma_{\mathbf{P}}(t)}}\\ &\quad\times \exp\left\{-\frac{i}{\hbar}\int_0^t \left(\sigma_{\mathbf{p}}(T)-\sigma_{\mathbf{P}}(T)\right)dT\right\}\left[2(p_i-V_i(t))-\hbar k_i\right]. \qquad (4.9)\end{aligned}$$

In order to proceed further, we require the function $\sigma_{\mathbf{P}}(t)$ as a series in $\hbar^n$ up to $n=2$. This is easily found to be

$$\sigma_{\mathbf{P}}(t)=\sigma_{\mathbf{p}}(t)\left(1-\frac{\hbar\mathbf{k}\cdot(\mathbf{p}-\mathbf{V}(t))}{\sigma_{\mathbf{p}}(t)^2}+\frac{\hbar^2k^2}{2\sigma_{\mathbf{p}}(t)^2}-\frac{\hbar^2\left(\mathbf{k}\cdot(\mathbf{p}-\mathbf{V}(t))\right)^2}{2\sigma_{\mathbf{p}}(t)^4}\right), \qquad (4.10)$$

which simplifies, by defining $\mathbf{n}=\mathbf{k}/k$ and by application of the equations of motion, to

$$\sigma_{\mathbf{P}}(t)=\sigma_{\mathbf{p}}(t)\left(1-\frac{\hbar k\,\mathbf{n}\cdot\dot{\mathbf{x}}}{\sigma_{\mathbf{p}}(t)}+\frac{\hbar^2k^2}{2\sigma_{\mathbf{p}}(t)^2}\left[1-(\mathbf{n}\cdot\dot{\mathbf{x}})^2\right]\right). \qquad (4.11)$$

It is clear that the ratio $\sigma_{\mathbf{P}}(t)/\sigma_{\mathbf{p}}(t)$ is $1+\hbar A$ for some A of order $\hbar^0$. Thus, the derivative of the logarithm in $\mathcal{A}_0$ above may be discounted as it is intrinsically of order $\hbar$, with another factor $\hbar$ outside the integral. Since the difference $\sigma_{\mathbf{p}}(t)-\sigma_{\mathbf{P}}(t)$ is everywhere accompanied by a factor $\hbar^{-1}$, the reason for expanding $\sigma_{\mathbf{P}}(t)$ to order $\hbar^2$ becomes apparent. Using our expression to second order for that difference, and to first order for $1/\sqrt{\sigma_{\mathbf{P}}(t)}$, we obtain the following expressions for the emission amplitudes:

$$\mathcal{A}_0(\mathbf{p},\mathbf{k}) = -ec\sqrt{\frac{p_0}{P_0}}\int dt\, e^{ik\cdot x}\exp\left\{-\frac{i\hbar k^2 c}{2}\int_0^t\left[1-\frac{(\mathbf{n}\cdot\dot{\mathbf{x}})^2}{c^2}\right]\frac{dT}{\sigma_{\mathbf{p}}(T)}\right\}$$
$$\times\left[1+\frac{\hbar\mathbf{k}\cdot\dot{\mathbf{x}}}{2c\sigma_{\mathbf{p}}(t)}\right]\left[1-\frac{\hbar\mathbf{k}\cdot\dot{\mathbf{x}}}{2c\sigma_{\mathbf{p}}(t)}\right],$$
$$\mathcal{A}_i(\mathbf{p},\mathbf{k}) = -ec\sqrt{\frac{p_0}{P_0}}\int dt\, e^{ik\cdot x}\exp\left\{-\frac{i\hbar k^2 c}{2}\int_0^t\left[1-\frac{(\mathbf{n}\cdot\dot{\mathbf{x}})^2}{c^2}\right]\frac{dT}{\sigma_{\mathbf{p}}(T)}\right\}$$
$$\times\left[\frac{\dot{x}_i}{c}+\frac{\hbar k}{2\sigma_{\mathbf{p}}(t)}\left(\frac{\dot{x}_i(\mathbf{n}\cdot\dot{\mathbf{x}})}{c^2}-n_i\right)\right]. \tag{4.12}$$

We have re-instated factors of c by dimensional analysis in advance of taking the non-relativistic limit. Clearly, the factors multiplying the exponentials in the $\mathcal{A}_0(\mathbf{p},\mathbf{k})$ integrand produce 1 to order $\hbar$, and thus we find that the above expressions give as the squared emission amplitude:

$$|\mathcal{A}(\mathbf{p},\mathbf{k})|^2 = -e^2c^2\frac{p_0}{P_0}\int dt\,dt'\,e^{ik\cdot(x-x')}\exp\left\{-\frac{i\hbar k^2 c}{2}\int_{t'}^t\left[1-\frac{(\mathbf{n}\cdot\dot{\mathbf{x}})^2}{c^2}\right]\frac{dT}{\sigma_{\mathbf{p}}(T)}\right\}$$
$$\times\left[1-\frac{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'}{c^2}-\frac{\hbar}{2}\left(\frac{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'}{c^2}\frac{\mathbf{k}\cdot\dot{\mathbf{x}}}{\sigma_{\mathbf{p}}(t)}-\frac{\mathbf{k}\cdot\dot{\mathbf{x}}'}{\sigma_{\mathbf{p}}(t)}+\frac{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'}{c^2}\frac{\mathbf{k}\cdot\dot{\mathbf{x}}'}{\sigma_{\mathbf{p}}(t')}-\frac{\mathbf{k}\cdot\dot{\mathbf{x}}}{\sigma_{\mathbf{p}}(t')}\right)\right], \tag{4.13}$$

where for the sake of concision we have defined $\mathbf{x}\equiv\mathbf{x}(t)$ and $\mathbf{x}'\equiv\mathbf{x}(t')$.

The correction at order $\hbar$ then comes from two places: firstly from the exponential term in the integrand, and secondly from the bracketed term multiplying it. We shall denote the corrections coming from each of these as ΔE_1 and ΔE_2, respectively, and consider ΔE_1 first.

Considering first ΔE_1, we define $\omega\equiv ck$, and expand the second exponential; we thus obtain

$$\Delta E_1 = \frac{ie^2\hbar}{4c^4}\int\frac{\omega^2 d\omega\, d\Omega_{\mathbf{n}}}{(2\pi)^3}\int dt\,dt'\,e^{i\omega(t-t')}e^{-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}$$
$$\times\,\omega^2(c^2-\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}')\int_{t'}^t\left[1-\frac{(\mathbf{n}\cdot\dot{\mathbf{x}})^2}{c^2}\right]\frac{dT}{\sigma_{\mathbf{p}}(T)}. \tag{4.14}$$

Clearly, the integral over T is not generally tractable: the integral of $\sigma_{\mathbf{p}}(T)^{-1}$ causes one problem, and the integral of $(\mathbf{n}\cdot\dot{\mathbf{x}})^2$ another. Both of these problems are obviated by taking a non-relativistic approximation: it gives us $\sigma_{\mathbf{p}}(t) \approx mc$, and the first term in the integrand dominates the second when $\|\dot{\mathbf{x}}\|/c$ is taken to be small. Thus we are able to replace the troublesome integral with $(t-t')/mc$.

If it were desired to extend the calculation to the relativistic régime, this should be possible by a numerical calculation. First, it would be necessary to carry out the ω integral, regulating it with an $e^{-\epsilon\omega}$ term in order to keep it well-defined. This will allow the $\mathbf{n}$, t and t' integrals to be carried out numerically. In order for this to be done, an explicitly-specified $\mathbf{V}(t)$ is necessary in order to know the velocity, acceleration and $\sigma_{\mathbf{p}}(t)$.

This non-relativistic approximation implies that we must find the correction to lowest order in c^{-1}, or equivalently highest order in c. Therefore, we shall truncate the term $e^{-i\omega\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')/c}$ at order c^{-2}, which is sufficient to capture the term of lowest order in c^{-1}. Thus expanding the exponential, we obtain

$$\Delta E_1 = \frac{ie^2\hbar}{4c^4}\int\frac{\omega^2 d\omega\, d\Omega_{\mathbf{n}}}{(2\pi)^3}\int dt\, dt'\, e^{i\omega(t-t')}\left(1+\frac{i\omega}{c}\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')-\frac{\omega^2}{2c^2}\left(\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')\right)^2\right)$$
$$\times\,\omega^2\left(c^2-\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'\right)\frac{(t-t')}{mc}.$$

Analysing the above integral, we see that there is a term which contributes a single factor of $\mathbf{n}$ where all others are even powers: since the integral $\int d\Omega_{\mathbf{k}} n_i = 0$ (and likewise for all other odd powers of $\mathbf{n}$), we can remove that term immediately, leaving us with

$$\Delta E_1 = \frac{ie^2\hbar}{4mc^5}\int\frac{d\omega\, d\Omega_{\mathbf{n}}}{(2\pi)^3}\int dt\, dt'\, e^{i\omega(t-t')}\omega^4(t-t')$$
$$\times\left(1-\frac{\omega^2}{2c^2}\left(\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')\right)^2\right)\left(c^2-\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'\right). \tag{4.15}$$

We regulate the behaviour of the exponential by adding $i\varepsilon$ to $(t-t')$, and use the identity

$$\int_0^\infty \omega^n e^{i\omega(z+i\varepsilon)}d\omega = \frac{i^{n+1}n!}{(z+i\varepsilon)^{n+1}}, \tag{4.16}$$

to integrate over ω. When this is carried out, we obtain

$$\Delta E_1 = -\frac{e^2\hbar}{4mc^5}\int \frac{d\Omega_{\mathbf{n}}}{(2\pi)^3}\int dt\, dt'(t-t') \times \left[\frac{4!}{(t-t'+i\varepsilon)^5} + \frac{1}{2c^2}\frac{6!}{(t-t'+i\varepsilon)^7}\left(\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')\right)^2\right](c^2 - \dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'). \quad (4.17)$$

This integral is ill-defined, since $t-t'$ may be kept finite while the integrand remains finite for arbitrarily large $|t+t'|$. For this reason, we insert a cut-off factor $\chi(at)\chi(at')$ with $0 < a \le 1$, such that $\chi(at)$ is smooth and compactly supported and such that $\chi(at) = 1$ for $t \in [-T, T]$: *i.e.*, while $V^\mu(t)$ is not constant. Then we find that this integral is a sum of terms of the form $A_1^{(1)}$ and $A_3^{(3)}$ as defined in Eq. (A.1). Therefore, as shown in Appendix A, we can formally integrate by parts with respect to t and t' to reduce the power of $t - t' + i\varepsilon$ in the denominator. Using this, we observe that

$$c^2\int dt\, dt'\,\frac{4!(t-t')}{(t-t'+i\varepsilon)^5} = 0, \quad (4.18)$$

which shows that the highest order non-zero term (in powers of c) in the integrand is of order c^0. Hence,

$$\Delta E_1 = -\frac{e^2\hbar}{4mc^5}\int\frac{d\Omega_{\mathbf{n}}}{(2\pi)^3}\int dt\, dt'\,(t-t')\left[-\frac{4!\,\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'}{(t-t'+i\varepsilon)^5} + \frac{1}{2}\frac{6!(\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}'))^2}{(t-t'+i\varepsilon)^7}\right]. \quad (4.19)$$

The angular integral is now calculated: for the second term in the integrand, we use the identity

$$\int d\Omega_{\mathbf{n}} n_i n_j = 4\pi\left(\frac{1}{3}\delta_{ij}\right). \quad (4.20)$$

Performing this integral and carrying out some simple re-arrangements, we obtain

$$\Delta E_1 = -\frac{e^2\hbar}{2mc^5}\int dt\, dt'\left[-\frac{3!}{(t-t'+i\varepsilon)^4}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}' + \frac{5\times 3!}{(t-t'+i\varepsilon)^6}\left\|\mathbf{x}-\mathbf{x}'\right\|^2\right]. \quad (4.21)$$

These two terms can be combined by integrating the second term by parts twice: once with respect to t, and once with respect to t'. Carrying this through, and continuing to integrate by parts, we finally obtain

$$\Delta E_1 = -\frac{e^2\hbar}{8\pi^2 mc^5}\int \frac{dt\,dt'}{t-t'}\left(\dddot{\mathbf{x}}\cdot\ddot{\mathbf{x}}' - \ddot{\mathbf{x}}\cdot\dddot{\mathbf{x}}'\right). \tag{4.22}$$

We now consider the correction coming from the multiplicative factor in $|\mathcal{A}(\mathbf{p},\mathbf{k})|^2$. Turning back to Eq. (4.13), we extract this correction as

$$\Delta E_2 = -\frac{e^2\hbar}{4}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int dt\,dt'\,e^{ikc(t-t')}\left(1 - i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - \frac{1}{2}\left(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\right)^2\right)$$
$$\times\left(\frac{\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'}{c^2}\mathbf{k}\cdot\left(\frac{\dot{\mathbf{x}}}{\sigma_{\mathbf{p}}(t)} + \frac{\dot{\mathbf{x}}'}{\sigma_{\mathbf{p}}(t)'}\right) - \mathbf{k}\cdot\left(\frac{\dot{\mathbf{x}}'}{\sigma_{\mathbf{p}}(t)} + \frac{\dot{\mathbf{x}}}{\sigma_{\mathbf{p}}(t')}\right)\right). \tag{4.23}$$

Since the terms in the second set of brackets all contain a single factor of $\mathbf{k}$, we only retain the imaginary term from the first set of brackets; the first term in the second set of brackets may be discarded as it is smaller than the second by a factor of c^2; and $\sigma_{\mathbf{p}}(t)$ and $\sigma_{\mathbf{p}}(t')$ may be replaced again by mc. Carrying out the re-scaling of k as we did for ΔE_1, we find

$$\Delta E_2 = -\frac{ie^2\hbar}{2mc^5}\int \frac{d\omega\,d\Omega_{\mathbf{n}}}{(2\pi)^3}\int dt\,dt'\,e^{i\omega(t-t')}\omega^4(\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}'))(\mathbf{n}\cdot(\dot{\mathbf{x}}+\dot{\mathbf{x}}')). \tag{4.24}$$

The integrals over $\Omega_{\mathbf{n}}$ and ω now separate, and produce

$$\Delta E_2 = \frac{e^2\hbar}{24\pi^2 mc^5}\int dt\,dt'\,\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial t'^2}\left(\frac{1}{t-t'+i\varepsilon}\right)(\mathbf{x}-\mathbf{x}')\cdot(\dot{\mathbf{x}}+\dot{\mathbf{x}}'). \tag{4.25}$$

At this point, we should have again to regulate the integral by the use of cut-off factors; however, as earlier this integral is of the form of $A_1^{(1)}$ in Eq. (A.1). Therefore, we may integrate by parts, twice with respect to t and twice with respect to t', and ignore the cut-off factors. Thus we obtain

$$\Delta E_2 = -\frac{e^2\hbar}{24\pi^2 mc^5}\int \frac{dt\,dt'}{t-t'}\left[\dddot{\mathbf{x}}\cdot\ddot{\mathbf{x}}' - \ddot{\mathbf{x}}\cdot\dddot{\mathbf{x}}'\right] = \frac{1}{3}\Delta E_1. \tag{4.26}$$

Our final result for the order $\hbar$ correction to the Larmor formula, therefore, is

$$\Delta E = -\frac{e^2\hbar}{6\pi^2 mc^5}\int \frac{dt\,dt'}{t-t'}\left[\frac{d^3\mathbf{x}}{dt^3}\cdot\frac{d^2\mathbf{x}'}{dt'^2} - \frac{d^2\mathbf{x}}{dt^2}\cdot\frac{d^3\mathbf{x}'}{dt'^3}\right]. \tag{4.27}$$

The reader will observe that this is of lower order in c than the Larmor formula by a factor of c^2.

Example

In order to see the scale of this correction in comparison to the energy emitted, let us consider a simple situation where the acceleration is linear and given by $a(t) = a_0(1 - t^2/{t_0}^2)$ for $|t| < t_0$, and $a(t) = 0$ otherwise. We use the above equation to find

$$\Delta E = \frac{4e^2\hbar {a_0}^2}{3\pi^2 mc^5}. \tag{4.28}$$

On the other hand, the energy radiated according to the Larmor formula can be found from Eq. (3.18) as $E_{\rm em}^{(0)} = 8{a_0}^2 t_0/45\pi c^3$. Hence we have

$$\left|\frac{\Delta E}{E_{\rm em}^{(0)}}\right| = \frac{15\hbar}{2\pi mc^2 t_0}. \tag{4.29}$$

Therefore, the Larmor formula is expected to be a good approximation as long as $t_0 \gg \hbar/mc^2$, which is the time for a light ray to traverse a Compton wavelength of the charged scalar particle. Since the probability distribution for the frequency of the photon emitted is given by the square of the Fourier transform of $a(t)$, the typical energy of the photon emitted will be of order $\hbar/t_0$ (though the probability of emission can be made small by letting a_0 be small). This energy will be comparable to mc^2 if $t_0 \sim \hbar/mc^2$. Then the scattered charged scalar particle will be relativistic, and it is not surprising that the non-relativistic approximation will break down. It is interesting that the classical (non-relativistic) Larmor formula seems to remain a good approximation as long as the scattered state remains non-relativistic even if its momentum may be much different from that of the initial state, in the case where the particle is accelerated by a time-dependent but space-independent vector potential.

4.2 Space-dependent potential

We now turn to the case in which the potential varies in a space co-ordinate, taken to be z. As with the time-dependent case, we assume that the vector potential $V_\mu(z)$ is constant across space and time except for the region where $z \in [-Z, Z]$, $Z > 0$. We assume that $V_\mu(z) = 0$ for $z < -Z$, but allow that the potential may attain a different constant value in the region $z > Z$. We further let $V_z(z) = 0$ for all z by a gauge transformation.

The WKB approximation changes, as now the mode which presents difficulties is the z-dependent factor, not the time-dependent factor. The constant momenta are now p_0 and $\mathbf{p}_\perp = (p_x, p_y)$: note that we define these to be components of the *contravariant* vector $\mathbf{p}$. As a result, the approximate solution to the equation of motion is

$$\Phi_\mathbf{p}(t, \mathbf{x}) = \sqrt{\frac{p}{\kappa_\mathbf{p}(z)}} \exp\left(\frac{i}{\hbar}\int_0^z \kappa_\mathbf{p}(\zeta) d\zeta\right) \exp\left(\frac{i}{\hbar}\left(\mathbf{p}_\perp \cdot \mathbf{x}_\perp - p_0 t\right)\right), \quad (4.30)$$

where the function analogous to $\sigma_\mathbf{p}(t)$ in the time-dependent case is now a varying momentum,

$$\kappa_\mathbf{p}(z) = \sqrt{(p_0 - V_0(z))^2 - |\mathbf{p}_\perp - \mathbf{V}_\perp(z)|^2 - m^2}, \quad (4.31)$$

and the initial momentum $p = \kappa_\mathbf{p}(0) = \sqrt{{p_0}^2 - |\mathbf{p}_\perp|^2 - m^2}$. As with the time-dependent potential, it can be shown that higher-order corrections to the wavefunction do not contribute to the energy emitted at order $\hbar$.

Whereas with the time-dependent potential the Jacobian determinant for changing the momentum variable was unity, here it is

$$\det\left(\frac{\partial \mathbf{P}}{\partial \mathbf{p}}\right) = \frac{dP}{dp}. \quad (4.32)$$

As previously, $P = \kappa_{\mathbf{P}}(0)$, and $P_\mu = p_\mu - \hbar k_\mu$ for $\mu \neq z$. Therefore, in the limit that the momentum distribution is arbitrarily sharply peaked, the energy emitted is given by

$$E_{\text{em}} = -\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{P_0}{p_0}\frac{dp}{dP}\mathcal{A}^*_\mu(\mathbf{p},\mathbf{k})\mathcal{A}^\mu(\mathbf{p},\mathbf{k}). \tag{4.33}$$

Many of the details of the calculation which follows find, as one might expect, direct analogues in the time-dependent case; although occasional mention will be made of these, many will be sufficiently obvious to be left unremarked.

The formula for the emission amplitude, Eq. (2.97), remains the same, and thus after integrating over t, $\mathbf{x}_\perp$ and $\mathbf{p}'$ we find

$$\begin{aligned}\mathcal{A}_0(\mathbf{p},\mathbf{k}) = \frac{e}{2}\int dz\, e^{-ik_z z}\frac{1}{P}\sqrt{\frac{pP}{\kappa_{\mathbf{p}}(z)\kappa_{\mathbf{P}}(z)}}\\ \times(2V_0(z)-(p_0+P_0))\exp\left(\frac{i}{\hbar}\int_0^z(\kappa_{\mathbf{p}}(\zeta)-\kappa_{\mathbf{P}}(\zeta))d\zeta\right),\end{aligned} \tag{4.34}$$

$$\begin{aligned}\mathcal{A}_\perp(\mathbf{p},\mathbf{k}) = \frac{e}{2}\int dz\, e^{-ik_z z}\frac{1}{P}\sqrt{\frac{pP}{\kappa_{\mathbf{p}'}(z)\kappa_{\mathbf{p}'}(z)}}\\ \times(-2\mathbf{V}_\perp(z)+(\mathbf{p}_\perp+\mathbf{P}_\perp))\exp\left(\frac{i}{\hbar}\int_0^z(\kappa_{\mathbf{p}}(\zeta)-\kappa_{\mathbf{P}}(\zeta))d\zeta\right),\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}\mathcal{A}_z(\mathbf{p},\mathbf{k}) = \frac{-ie\hbar}{2}\int dz\, e^{-ik_z z}\frac{1}{P}\sqrt{\frac{pP}{\kappa_{\mathbf{p}}(z)\kappa_{\mathbf{P}}(z)}}\\ \times\left[-\frac{1}{2}\left(\frac{\kappa'_{\mathbf{p}}(z)}{\kappa_{\mathbf{p}}(z)}-\frac{\kappa'_{\mathbf{P}}(z)}{\kappa_{\mathbf{P}}(z)}\right)+\frac{i}{\hbar}(\kappa_{\mathbf{p}}(z)+\kappa_{\mathbf{P}}(z))\right]\\ \times\exp\left(\frac{i}{\hbar}\int_0^z(\kappa_{\mathbf{p}}(\zeta)-\kappa_{\mathbf{P}}(\zeta))d\zeta\right).\end{aligned} \tag{4.36}$$

Up to first order in $\hbar$, *i.e.*, noting that $\hbar(\kappa'_{\mathbf{P}}(z)/\kappa_{\mathbf{P}}(z)-\kappa'_{\mathbf{p}}(z)/\kappa_{\mathbf{p}}(z))\sim\hbar^2$

in Eq. (4.36), we have

$$
\begin{aligned}
|\mathcal{A}(\mathbf{p},\mathbf{k})|^2 = \frac{e^2}{4}\frac{p}{P}\int dz\,dz' &\frac{e^{ik_z(z'-z)}}{\sqrt{\kappa_{\mathbf{p}}(z)\kappa_{\mathbf{P}}(z)\kappa_{\mathbf{p}}(z')\kappa_{\mathbf{P}}(z')}} \\
&\times \exp\left(\frac{i}{\hbar}\int_{z'}^{z}(\kappa_{\mathbf{p}}(\zeta)-\kappa_{\mathbf{P}}(\zeta))d\zeta\right) \\
&\times [(2V_0(z)-(p_0+P_0))(2V_0(z')-(p_0+P_0)) \\
&\quad -(2\mathbf{V}_\perp(z)-(\mathbf{p}_\perp+\mathbf{P}_\perp))\cdot(2\mathbf{V}_\perp(z')-(\mathbf{p}_\perp+\mathbf{P}_\perp)) \\
&\quad -(\kappa_{\mathbf{p}}(z)+\kappa_{\mathbf{P}}(z))(\kappa_{\mathbf{p}}(z')+\kappa_{\mathbf{P}}(z'))]\,. \qquad (4.37)
\end{aligned}
$$

This we can substitute into Eq. (4.33) to find E_{em}.

We note that in E_{em}, we have a factor

$$\frac{P_0}{p_0}\frac{dp}{dP}\frac{p}{P} = 1; \tag{4.38}$$

a fact which may be easily shown by application of the chain rule and the fact that $dP/dP_0 = P_0/P$.

The motion of the corresponding classical particle may be expressed in terms of the momentum $\kappa_{\mathbf{p}}(z)$ as follows:

$$\frac{p_0 - V_0}{\kappa_{\mathbf{p}}(z)} = \left.\frac{dx^0}{dz}\right|_{\mathbf{p}}, \tag{4.39}$$

$$\frac{\mathbf{p}_\perp - \mathbf{V}_\perp}{\kappa_{\mathbf{p}}(z)} = \left.\frac{d\mathbf{x}_\perp}{dz}\right|_{\mathbf{p}}, \tag{4.40}$$

where x^μ is the path of the classical particle under the influence of the potential V_μ and where '$|_{\mathbf{p}}$' indicates that the quantity is evaluated with the initial momentum $\mathbf{p}$. Using these equations, we obtain from Eq. (4.37),

$$
\begin{aligned}
E = -\frac{e^2}{8}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int dz\,dz'\chi(z)\chi(z')e^{ik_z(z'-z)}\exp\left(\frac{i}{\hbar}\int_{z'}^{z}(\kappa_{\mathbf{p}}(\zeta)-\kappa_{\mathbf{P}}(\zeta))\,d\zeta\right) \\
\times\left[\sqrt{\frac{\kappa_{\mathbf{p}}(z)}{\kappa_{\mathbf{P}}(z)}}\left.\frac{dx^\mu}{dz}\right|_{\mathbf{p}} + \sqrt{\frac{\kappa_{\mathbf{P}}(z)}{\kappa_{\mathbf{p}(z)}}}\left.\frac{dx^\mu}{dz}\right|_{\mathbf{P}}\right]\left[\sqrt{\frac{\kappa_{\mathbf{p}}(z')}{\kappa_{\mathbf{P}}(z')}}\left.\frac{dx'_\mu}{dz'}\right|_{\mathbf{p}} + \sqrt{\frac{\kappa_{\mathbf{P}}(z')}{\kappa_{\mathbf{p}(z')}}}\left.\frac{dx'_\mu}{dz'}\right|_{\mathbf{P}}\right].
\end{aligned}
\tag{4.41}
$$

The correction to first order in $\hbar$ will again arise from the exponential and the other factors in the integral: again, we denote these ΔE_1 and ΔE_2 respectively. We shall examine these separately but then combine the intermediate results, as they pair up neatly and cancellations can be made throughout.

Let us consider the exponential correction first. As with the time-dependent case, we need to find the correction to $\kappa_{\mathbf{P}}(z)$ up to second order in $\hbar$:

$$\kappa_{\mathbf{P}}(z) = \kappa_{\mathbf{p}}(z)\left(1 - \frac{\hbar k}{\kappa_{\mathbf{p}}(z)}\left(\frac{dx^0}{dz} - \frac{\mathbf{n}_\perp}{c}\cdot\frac{d\mathbf{x}_\perp}{dz}\right) + \frac{\hbar^2 k^2}{2\kappa_{\mathbf{p}}(z)^2}\left(-\left(\frac{dx^0}{dz} - \frac{\mathbf{n}_\perp}{c}\cdot\frac{d\mathbf{x}_\perp}{dz}\right)^2 + \frac{{n_z}^2}{c^2}\right)\right). \tag{4.42}$$

Therefore, the correction to the energy emitted coming from the exponential factor is

$$\Delta E_1 = -\frac{ie^2\hbar}{4}\int\frac{dk\,d\Omega_{\mathbf{k}}}{(2\pi)^3}\int dt\,dt' k^4 e^{-ikc(t'-t)+ik\mathbf{n}\cdot(\mathbf{x}'-\mathbf{x})}(c^2 - \dot{\mathbf{x}}\cdot\dot{\mathbf{x}}') \times \int_{t'}^{t}\left(1 - 2\mathbf{n}_\perp\cdot\frac{d\mathbf{x}_\perp}{dT}\right)\left(\frac{dz}{dT}\right)^{-2} dT. \tag{4.43}$$

We make the transformation $\omega = kc$ at this point, and so observe that we need only take the $e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}$ term to second order in k. The reader will note that as previously, the integral over $\Omega_{\mathbf{n}}$ will select only terms in the integrand with an even number of copies of $\mathbf{n}_\perp$. Then we obtain

$$\Delta E_1 = -\frac{ie^2\hbar}{8\pi^2 mc^3}\int d\omega\int dt\,dt'\chi(at)\chi(at')\omega^4 e^{i\omega(t-t'+i\varepsilon)}(c^2 - \dot{\mathbf{x}}\cdot\dot{\mathbf{x}}') \tag{4.44}$$
$$\times\left[\left(1 - \frac{1}{6c^2}\omega^2\left\|\mathbf{x}-\mathbf{x}'\right\|^2\right)\int_{t'}^{t}\dot{z}^{-2}dT - \frac{2}{3c^2}i\omega(\mathbf{x}'_\perp - \mathbf{x}_\perp)\cdot\int_{t'}^{t}\dot{\mathbf{x}}_\perp\dot{z}^{-2}dT\right],$$

again inserting a factor of $i\varepsilon$ to avoid the singularity at $t = t'$ in such a way that the ω integral is convergent. The term of highest order in c is at order c^2: noting that once the ω integral is carried out, this term is of the form of $A_1^{(1)}$, we may therefore integrate by parts in t and t' to obtain:

$$c^2\int d\omega\omega^4\int dt\,dt'\,e^{i\omega(t-t'+i\varepsilon)}\int_{t'}^{t}\dot{z}^{-2}dT = 0. \tag{4.45}$$

This gives us, to lowest order in c^{-1},

$$\Delta E_1 = -\frac{ie^2\hbar}{8\pi^2 mc^3}\int d\omega \int dt\, dt'\chi(at)\chi(at')\omega^4 e^{i\omega(t-t'+i\varepsilon)} \tag{4.46}$$
$$\times\left[\left(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}' + \frac{1}{6}\omega^2\left\|\mathbf{x}-\mathbf{x}'\right\|^2\right)\int_{t'}^{t}\dot{z}^{-2}dT + \frac{2i\omega}{3}(\mathbf{x}'_\perp - \mathbf{x}_\perp)\cdot\int_{t'}^{t}\dot{\mathbf{x}}_\perp\dot{z}^{-2}dT\right].$$

Turning to the factor correction, ΔE_2, clearly $dz/dz = 1$ whether the associated initial momentum is $\mathbf{p}$ or $\mathbf{P}$. When $\mu \neq 3$, however, $dx^\mu/dz|_{\mathbf{P}} = (P^\mu - V^\mu)/\sigma_{\mathbf{P}}(z)$ must be corrected through both $\mathbf{P}$ and $\kappa_{\mathbf{P}}(z)$. Throughout this part of the calculation, we shall use Roman indices to indicate all components except z; then this exact re-writing shows the origins of the correction:

$$\left.\frac{dx^m}{dz}\right|_{\mathbf{P}} = \frac{\kappa_{\mathbf{p}}(z)}{\kappa_{\mathbf{P}}(z)}\left.\frac{dx^m}{dz}\right|_{\mathbf{p}} - \frac{\hbar k^m}{2\kappa_{\mathbf{P}}(z)}. \tag{4.47}$$

Since we are only seeking corrections at first order in $\hbar$, we have that

$$\left.\frac{dx^m}{dz}\right|_{\mathbf{P}} = \left(1 + \frac{\hbar k_n}{2\kappa_{\mathbf{p}}(z)}\frac{dx^n}{dz}\right)\left.\frac{dx^m}{dz}\right|_{\mathbf{p}} - \frac{\hbar k^m}{2\kappa_{\mathbf{p}}(z)}. \tag{4.48}$$

Thus, the correction coming from the factor can be written as follows:

$$E + \Delta E_2 = -\frac{e^2}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int dz\, dz' e^{ik\cdot(x-x')} \tag{4.49}$$
$$\times\left[\sqrt{\frac{\kappa_{\mathbf{p}}(z)}{\kappa_{\mathbf{P}}(z)}}\sqrt{\frac{\kappa_{\mathbf{p}}(z')}{\kappa_{\mathbf{P}}(z')}}\left.\frac{dx^m}{dz}\right|_{\mathbf{p}}\left.\frac{dx'_m}{dz'}\right|_{\mathbf{p}} - \frac{\hbar k^m}{2}\left(\frac{1}{\kappa_{\mathbf{p}}(z')}\left.\frac{dx_m}{dz}\right|_{\mathbf{p}} + \frac{1}{\kappa_{\mathbf{p}}(z)}\left.\frac{dx'_m}{dz'}\right|_{\mathbf{p}}\right) - 1\right];$$

therefore,

$$\Delta E_2 = -\frac{e^2}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\int dz\, dz' e^{ik\cdot(x-x')} \tag{4.50}$$
$$\times\frac{\hbar}{2}\left[\left(\frac{k^n}{\kappa_{\mathbf{p}}(z)}\frac{dx_n}{dz} + \frac{k^n}{\kappa_{\mathbf{p}}(z')}\frac{dx'_n}{dz'}\right)\frac{dx^m}{dz}\frac{dx'_m}{dz'} - \left(\frac{k^n}{\kappa_{\mathbf{p}}(z')}\frac{dx_n}{dz} + \frac{k^n}{\kappa_{\mathbf{p}}(z)}\frac{dx'_n}{dz'}\right)\right].$$

At this point, we make the replacement $\kappa_{\mathbf{p}}(z) \approx m\dot{z}$, expand the summations, and so obtain

$$\Delta E_2 = \frac{e^2\hbar}{32\pi^3 m}\int d^3\mathbf{k}\, dt\, dt'\chi(at)\chi(at')ke^{ik\cdot(x-x')} \tag{4.51}$$
$$\times\left[\left(\frac{c - \mathbf{n}_\perp\cdot\dot{\mathbf{x}}_\perp}{\dot{z}^2} + \frac{c - \mathbf{n}_\perp\cdot\dot{\mathbf{x}}'_\perp}{\dot{z}'^2}\right)\left(c^2 - \dot{\mathbf{x}}_\perp\cdot\dot{\mathbf{x}}'_\perp\right) - \left(c - \mathbf{n}_\perp\cdot\dot{\mathbf{x}}_\perp + c - \mathbf{n}_\perp\cdot\dot{\mathbf{x}}'_\perp\right)\right].$$

Expanding the term $e^{ik\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}')}$, we observe that the angular integral will once again select only terms in the integrand with an even number of copies of $\mathbf{n}$. Making the replacement $\omega = kc$, we find to highest order in c,

$$\Delta E_2 = -\frac{e^2\hbar}{8\pi^2 mc^3}\int d\omega\, dt\, dt'\chi(at)\chi(at')\omega^3 e^{i\omega(t-t'+i\varepsilon)} \tag{4.52}$$
$$\times\left[\left(\dot{\mathbf{x}}_\perp\cdot\dot{\mathbf{x}}'_\perp + \frac{\omega^2}{6}\left\|\mathbf{x}-\mathbf{x}'\right\|^2\right)\left(\dot{z}^{-2}+\dot{z}'^{-2}\right) + \frac{i\omega}{3}\left(\mathbf{x}'_\perp-\mathbf{x}_\perp\right)\cdot\left(\dot{\mathbf{x}}_\perp\dot{z}^{-2}+\dot{\mathbf{x}}'_\perp\dot{z}'^{-2}\right)\right].$$

It is convenient to combine the two integrals ΔE_1 and ΔE_2 at this stage, since all terms in each will pair up to show cancellations and combinations, giving the following expression:

$$\Delta E = -\frac{e^2\hbar}{8\pi^2 mc^3}\int dt\, dt'\chi(at)\chi(at')\int d\omega e^{i\omega(t-t'+i\varepsilon)}\left(I_1+I_2+I_3\right) \tag{4.53}$$

where

$$\begin{aligned}
I_1 &= \omega^3\left(i\omega\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}'\int_{t'}^{t}\dot{z}^{-2}dT + \dot{\mathbf{x}}_\perp\cdot\dot{\mathbf{x}}'_\perp\left(\dot{z}^{-2}+\dot{z}'^{-2}\right)\right),\\
I_2 &= \frac{1}{3}i\omega^4\left(2i\omega(\mathbf{x}_\perp-\mathbf{x}'_\perp)\cdot\int_{t'}^{t}\dot{\mathbf{x}}_\perp\dot{z}^{-2}dT + (\mathbf{x}_\perp-\mathbf{x}'_\perp)\cdot(\dot{\mathbf{x}}_\perp\dot{z}^{-2}+\dot{\mathbf{x}}'_\perp\dot{z}'^{-2})\right),\\
I_3 &= \frac{1}{6}\omega^5\left(i\omega\left\|\mathbf{x}-\mathbf{x}'\right\|^2\int_{t'}^{t}\dot{z}^{-2}dT + \left\|\mathbf{x}-\mathbf{x}'\right\|^2\left(\dot{z}^{-2}+\dot{z}'^{-2}\right)\right).
\end{aligned} \tag{4.54}$$

Each of these integrands may be considered separately according to general principles, before recombining them for the complete result.

Prior to that, however, we must briefly consider the cut-off functions. As with previous calculations, it is desirable that these not appear in the final result: and as with those previous calculations, the integral we have is of the form of $A_1^{(n)}$ in Appendix A. The most complicated of the six is

$$i\int dt\, dt'\chi(at)\chi(at')e^{i\omega(t-t'+i\varepsilon)}\omega^6\left\|\mathbf{x}-\mathbf{x}'\right\|^2\int_{t'}^{t}\dot{z}^{-2}dT; \tag{4.55}$$

here, $n = 3$, $f(t,t') = \dot{x}^\mu\dot{x}'_\mu$, and the three functions $g_i(t)$ are, twice, $x_i(t)$ (to form $\left\|\mathbf{x}-\mathbf{x}'\right\|^2$) and, once, the integral $\int_t \dot{z}^{-2}dT$ (so that $g(t') - g(t) =$

$\int_t^{t'} \dot{z}^{-2} dT$). Therefore, we deduce that after integration by parts, this becomes a sum of terms which either tend to zero or are convergent as the cut-off factors are removed. Hence, it may be formally integrated by parts without the cut-off functions. The other five integrands follow similarly, and therefore the whole integral as a whole is convergent without the cut-off functions which we consequently drop.

To analyse the three pairs in the integrand above, the basic rule of approach is to use the fact that $\omega e^{-i\omega(t-t')} = i\partial_t e^{-i\omega(t-t')} = -i\partial_{t'} e^{-i\omega(t-t')}$, and integrate by parts to move the derivatives from the exponential to the other terms in the integral. We shall choose our derivatives carefully, and use the relation '$\sim$' to denote equivalence of two expressions up to such a replacement of a factor or factors of ω and integration by parts. Therefore, we may say that $\omega \sim \frac{i}{2}(\partial_t - \partial_{t'})$.

We shall illustrate the general strategy using the term with the lowest power in ω, and then state the results of following the similar approach with the other terms. Commencing, then, with that term,

$$\begin{aligned} I_1 &\sim \omega^3 \left[\frac{1}{2} \left(\partial_{t'} - \partial_t \right) \left(\dot{\mathbf{x}} \cdot \dot{\mathbf{x}}' \int_{t'}^{t} \dot{z}^{-2} dT \right) + \dot{\mathbf{x}}_\perp \cdot \dot{\mathbf{x}}'_\perp \left(\dot{z}^{-2} + \dot{z}'^{-2} \right) \right] \\ &= \frac{1}{2} \omega^3 \left[\left(\ddot{\mathbf{x}}' \cdot \dot{\mathbf{x}} - \dot{\mathbf{x}}' \cdot \ddot{\mathbf{x}} \right) \int_{t'}^{t} \dot{z}^{-2} dT + \left(\dot{\mathbf{x}}_\perp \cdot \dot{\mathbf{x}}'_\perp - \dot{z}\dot{z}' \right) \left(\dot{z}^{-2} + \dot{z}'^{-2} \right) \right]. \end{aligned}$$

We now split the terms, so that the derivative from the next factor of ω does not affect any of $\ddot{\mathbf{x}}$, $\ddot{\mathbf{x}}'$, $\dot{z}^{-2}$ or $\dot{z}'^{-2}$:

$$\begin{aligned} I_1 &\sim \frac{1}{2} \omega^2 \Bigg[i\partial_{t'} \left(\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}' \int_{t'}^{t} \dot{z}^{-2} dT \right) + i\partial_t \left(\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}}' \int_{t'}^{t} \dot{z}^{-2} dT \right) \\ &\qquad + i\partial_t (\dot{\mathbf{x}}_\perp \cdot \dot{\mathbf{x}}'_\perp - \dot{z}\dot{z}') \dot{z}'^{-2} - i\partial_{t'} (\dot{\mathbf{x}}_\perp \cdot \dot{\mathbf{x}}'_\perp - \dot{z}\dot{z}') \dot{z}^{-2} \Bigg] \\ &= i\omega^2 \left[\ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}}' \int_{t'}^{t} \dot{z}^{-2} dT - \ddot{z}\dot{z}'^{-1} + \ddot{z}'\dot{z}^{-1} \right]. \end{aligned} \tag{4.56}$$

The general strategy has been to observe that we have two terms in the integrand of I_1 as written at the beginning of the process: the first multiplies the integral, and the second does not. That second term in I_1 comes from the factor part of the correction and is therefore related to a derivative of the first term. Consequently, we might expect combinations and cancellations to arise between derivatives of the first term and the second term. We therefore choose derivatives which will give us such cancellations. In this instance, it was also possible to avoid derivatives of $\dot{z}^{-2}$; in the other two cases, we shall have to allow first derivatives of such terms to arise. We stop once we have obtained a term proportional to ω^2.

Following a similar strategy, the other two integrals become

$$I_2 \sim \frac{1}{3} i\omega^2 \left[\ddot{\mathbf{x}}'_\perp \cdot \dot{\mathbf{x}}_\perp \dot{z}^{-2} - \dot{\mathbf{x}}'_\perp \cdot \ddot{\mathbf{x}}_\perp \dot{z}'^{-2} \right] \tag{4.57}$$

and

$$I_3 \sim -\frac{1}{3} i\omega^2 \left[\ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}}' \int_{t'}^{t} \dot{z}^{-2} dT + \ddot{\mathbf{x}}' \cdot \dot{\mathbf{x}} \dot{z}^{-2} - \dot{\mathbf{x}}' \cdot \ddot{\mathbf{x}} \dot{z}'^{-2} \right] \tag{4.58}$$

Then we see that $I_2 + I_3 = -\frac{1}{3} I_1$, and consequently,

$$I_1 + I_2 + I_3 \sim \frac{2}{3} i\omega^2 \left[\ddot{\mathbf{x}} \cdot \ddot{\mathbf{x}}' \int_{t'}^{t} \dot{z}^{-2} dT + \ddot{z}' \dot{z}^{-1} - \ddot{z} \dot{z}'^{-1} \right]. \tag{4.59}$$

If we were to apply the remaining factors of ω to the derivatives of z, then we would obtain

$$\omega^2 \left[\ddot{z}' \dot{z}^{-1} - \ddot{z} \dot{z}'^{-1} \right] \sim -\dddot{z}' \ddot{z} \dot{z}^{-2} + \ddot{z}' \dddot{z} \dot{z}'^{-2}; \tag{4.60}$$

using our equivalence between ω and derivatives in the time co-ordinates once more, we may write

$$\ddot{z} \dddot{z}' \dot{z}^{-2} \sim -\omega^2 \ddot{z} \ddot{z}' \int_t \dot{z}^{-2} dT + \dddot{z} \dddot{z}' \int_t \dot{z}^{-2} dT, \tag{4.61}$$

and a similar expression for the second term. Therefore,

$$I_1 + I_2 + I_3 \sim \frac{2}{3} i\omega^2 \ddot{\mathbf{x}}_\perp \cdot \ddot{\mathbf{x}}'_\perp \int_{t'}^{t} \dot{z}^{-2} dT + \frac{2}{3} i \,\dddot{z}\, \dddot{z}' \int_{t'}^{t} \dot{z}^{-2} dT. \tag{4.62}$$

Inserting the above expression into Eq. (4.53) and integrating the final factors of ω with $e^{i\omega(t-t'+i\varepsilon)}$, we find

$$\Delta E = \frac{e^2\hbar}{12\pi^2 mc^3} \int dt\, dt' \left[\frac{\dddot{z}\, \dddot{z}'}{t - t' + i\varepsilon} - \frac{2\ddot{\mathbf{x}}_\perp \cdot \ddot{\mathbf{x}}'_\perp}{(t - t' + i\varepsilon)^3} \right] \int_{t'}^{t} \dot{z}^{-2} dT. \tag{4.63}$$

For calculation purposes this result is too singular, a result of the $(t - t' + i\varepsilon)^3$ denominator. Integrating by parts to remove it gives a less concise but more tractable form,

$$\Delta E = \frac{e^2\hbar}{12\pi^2 mc^3} \int \frac{dt\, dt'}{t - t' + i\varepsilon} \Bigg[\dddot{\mathbf{x}} \cdot \dddot{\mathbf{x}}' \int_{t'}^{t} \dot{z}^{-2} dT$$
$$- \frac{1}{2} (\ddot{\mathbf{x}}'_\perp \cdot \dddot{\mathbf{x}}_\perp - \dddot{\mathbf{x}}'_\perp \cdot \ddot{\mathbf{x}}_\perp)(\dot{z}^{-2} + \dot{z}'^{-2}) - \frac{1}{2} \ddot{\mathbf{x}}_\perp \cdot \ddot{\mathbf{x}}'_\perp (\partial_t \dot{z}^{-2} - \partial_{t'} \dot{z}'^{-2}) \Bigg]. \tag{4.64}$$

This correction is of the same order in c as the Larmor formula, although of course it is of higher order in $\hbar$; this is a contrast with Eq. (4.27), which was of lower order in c by a factor of c^2.

Example

To estimate the size of this correction, we consider a charged particle moving at a constant speed v_z in the z-direction, and accelerated in the x-direction with an acceleration given by $a(t) = a_0(1 - t^2/{t_0}^2)$ for $|t| \leq t_0$ and 0 otherwise. It is possible to arrange the vector potential to produce this motion, as by hypothesis, V_x is then determined by $a(t(z))$, $V_y = 0$, and therefore V_0 is determined by v_z and V_x. Then, since v_z is constant, the first term in Eq. (4.64) gives a vanishing contribution, and from the remaining terms, we find

$$\Delta E = -\frac{2e^2\hbar {a_0}^2}{3\pi^2 m {v_z}^2 c^3}, \tag{4.65}$$

and

$$\left|\frac{\Delta E}{E_{\text{em}}^{(0)}}\right| = \frac{15\hbar}{4\pi m {v_z}^2 t_0}. \tag{4.66}$$

Thus, the correction is small, and the Larmor formula is expected to be reliable as long as the kinetic energy associated with the motion in the z-direction is much larger than the energy of the photon emitted, $\hbar/t_0$. This provides a general, loose, lower bound on the velocity $\dot{z}$, which ensures that the case of small $\dot{z}$, which might have raised a concern earlier, need not cause any difficulties.

Chapter 5

Conformally flat space-time

Summary. In this chapter, we consider the radiation reaction of a charged scalar particle of mass m moving in a space-time with metric $g_{\mu\nu} = \Omega^2(t)\eta_{\mu\nu}$. We shall show that at the tree level, this scalar QED theory can be transformed into one with a flat metric and a squared mass term ${M_c}^2(t) \equiv m^2\Omega^2(t) + (\xi - \frac{1}{6})\hbar^2\Omega^2(t)R(t)$. We proceed to demonstrate that this is equivalent to the scalar QED theory of a charged particle of squared mass $M^2(t) = m^2\Omega^2(t)$ moving on a flat space-time, and that therefore the radiation reaction forces agree to leading order in $\hbar$. Considering the one-loop corrections to the theory, we show that a flat-space theory with a general varying mass term gives rise to non-zero corrections, while the one-loop corrections to the conformally-flat theory vanish as $\hbar \to 0$.

5.1 Radiation reaction at tree level

Let us take, for the time being, a general conformally flat metric, $g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu}$, and consider how the classical scalar theory propagating on it may be converted to a flat space-time with a varying mass. We shall then move to the quantum electrodynamic theory and specialise to dependence on t only. We shall find it helpful to appeal to certain facts which may be found in App. D of [32], which treats the case of two metrics related by a conformal transformation.

A charged particle moving on such a space-time experiences an acceleration $a^\mu \equiv u^\alpha \nabla_\alpha u^\mu$, generated by a classical, DeWitt-Brehme-Hobbs, radiation reaction force [13]:

$$f^\mu_{(\mathrm{R})} = \frac{2}{3}\alpha_c \left(\dot{a}^\mu - a^2 u^\mu\right) + \frac{1}{3}\alpha_c \left(-R^\mu{}_\nu u^\nu + u^\mu R_{\alpha\beta} u^\alpha u^\beta\right), \qquad (5.1)$$

where $a^2 = -a^\mu a_\mu$ and we have defined the 'classical fine structure constant', $\alpha_c \equiv \alpha\hbar$. In a general space-time there would be an additional term called the tail term and coming from the propagation of the electromagnetic field within the light-cone. This term is conformally invariant [27]; and since it vanishes on a flat space-time, it also vanishes here. If there is an external electromagnetic field, then the equation of motion for the particle is

$$m a^\mu = F^{\mu\nu}_{\mathrm{ex}} u_\nu + f^\mu_{(\mathrm{R})}. \qquad (5.2)$$

Let us define the flat-space proper time by $\tau_\flat \equiv \Omega^{-1}\tau$, the four-velocity in the corresponding flat-space theory by $u^\mu_\flat \equiv dx^\mu/d\tau_\flat$, the flat-space acceleration by $a^\mu_\flat \equiv u^\alpha_\flat \partial_\alpha u^\mu_\flat = d^2x^\mu/d{\tau_\flat}^2$ and $\dot{a}^\mu_\flat \equiv da^\mu_\flat/d\tau_\flat$. Given these definitions, the relationship between the DeWitt-Brehme-Hobbs force in the space-time with the conformal metric and the equivalent force in the flat space-time is [33]

$$f^\mu_{(\mathrm{R})} = \Omega^{-3} f^\mu_{(\mathrm{R})\flat} \equiv \frac{2}{3}\alpha_c \Omega^{-3}\left(\dot{a}^\mu_\flat - {a_\flat}^2 u^\mu_\flat\right). \qquad (5.3)$$

This may either be shown by a brute force calculation, or by noting (as does [34]) that as Maxwell's equations are conformally invariant, the field tensor is also conformally invariant. Then $f_\mu = F_{\mu\nu}u^\nu = F_{\flat\mu\nu}\Omega u^\nu_\flat$, and so $f^\mu = g^{\mu\nu} f_\nu = \Omega^{-3} f^\mu_\flat$.

If we define a varying mass, $M(x) = m\Omega(x)$, then the equation of motion, Eq. (5.2), becomes

$$\frac{d}{d\tau_\flat}\left[M(x)u_\flat^\mu\right] - \eta^{\mu\nu}\partial_\nu M(x) = e\eta^{\mu\alpha}F^{\rm ex}_{\alpha\beta}u_\flat^\beta + f^\mu_{({\rm R})\flat}. \tag{5.4}$$

Thus, the motion of a charged particle of mass m on a space-time with a conformally-flat metric $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$ is the same as that of a charged particle with mass $m\Omega(x)$ in flat space-time under the influence of the Abraham-Lorentz-Dirac force. It is useful at this point to note that without the ALD force, Eq. (5.4) can be derived from Hamilton's equations, treating position and momentum as functions of time, from the Hamiltonian

$$H(\mathbf{x},\mathbf{p},t) = \sqrt{\|\mathbf{p} - e\mathbf{A}_{\rm ex}(\mathbf{x},t)\|^2 + M^2(\mathbf{x},t)} + eA^0_{\rm ex}(\mathbf{x},t), \tag{5.5}$$

where we have defined $\eta^{\mu\nu}A_{{\rm ex}\,\nu} = (A_{{\rm ex}\,0}, \mathbf{A}_{\rm ex})$.

We also transform the Lagrangian for the theory in the conformally flat space-time into the one in flat space-time. The Lagrangian for scalar QED in a general space-time with a metric $g_{\mu\nu}$ and a background electromagnetic field $A_{{\rm ex}\,\mu}$ is

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + g^{\mu\nu}\left(\mathcal{D}_\mu\phi\right)^\dagger\mathcal{D}_\nu\phi - \left[\left(\frac{m}{\hbar}\right)^2 - \xi R\right]\phi^\dagger\phi\right), \tag{5.6}$$

where $\mathcal{D}_\mu\phi \equiv [\partial_\mu + iV_\mu/\hbar + ieA_\mu/\hbar]\phi$, $V_\mu \equiv eA_{{\rm ex}\,\mu}$ and $F_{\mu\nu} \equiv 2\partial_{[\mu}A_{\nu]}$. Let the metric be conformally flat, *i.e.*, $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$: then $\sqrt{-g} = \Omega^4$ and $R = -6g^{\mu\nu}(\partial_\mu\partial_\nu\log\Omega + \partial_\mu\log\Omega\partial_\nu\log\Omega)$. If we introduce a rescaled scalar field, $\varphi \equiv \Omega\phi$, then we can re-write this Lagrangian, using the fact that Lagrangians are equivalent up to a total derivative. Specifically,

$$\sqrt{-g}g^{\mu\nu}(\partial_\mu\phi)^\dagger(\partial_\nu\phi) = \Omega^4\Omega^{-2}\eta^{\mu\nu}\left[\partial_\mu(\Omega^{-1}\varphi)\right]^\dagger\left[\partial_\nu(\Omega^{-1}\varphi)\right], \tag{5.7}$$

which we expand and re-arrange to produce derivatives of $\log \Omega$ so far as possible. Further re-arrangements lead to

$$= \eta^{\mu\nu}\partial_\mu \log \Omega \partial_\nu \log \Omega \varphi^\dagger \varphi - \eta^{\mu\nu}\partial_\mu \log \Omega \partial_\nu (\varphi^\dagger \varphi) + \eta^{\mu\nu}\partial_\mu \varphi^\dagger \partial_\nu \varphi, \quad (5.8)$$

at which point we use the equivalence of Lagrangians to move the derivative ∂_ν onto the logarithm in the middle term, changing its sign. The first two terms can then be seen to be proportional to the scalar curvature, so that

$$= \frac{1}{6}\Omega^2 R \varphi^\dagger \varphi + \eta^{\mu\nu}\partial_\mu \varphi^\dagger \partial_\nu \varphi. \quad (5.9)$$

Applying this at the relevant point in the Lagrangian, it is simple to see that the rest transforms so that

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \eta^{\mu\nu}\left(\mathcal{D}_\mu \varphi\right)^\dagger \mathcal{D}_\nu \varphi - \frac{{M_c}^2(x)}{\hbar^2}\varphi^\dagger \varphi, \quad (5.10)$$

where

$${M_c}^2(x) = m^2\Omega^2 + \left(\xi - \frac{1}{6}\right)\hbar^2\Omega^2 R. \quad (5.11)$$

The reader will observe that the difference ${M_c}^2(x) - M^2(x)$ is of order $\hbar^2$.

At this point, we impose the condition that the conformal factor Ω and the external electromagnetic field V_μ depend only on time, t. We also let $\Omega(t) \neq 1$ or $V_\mu(t) \neq 0$ only for $-T_1 < t < -T_2$ for some positive constants T_1 and T_2. We also choose the gauge $V_0 = 0$. As a result the background field V_μ satisfies the Lorenz gauge condition, $\eta^{\mu\nu}\partial_\mu V_\nu = 0$. We shall demonstrate that the tree-level motion of the particle in scalar QED with Lagrangian in Eq. (5.10) reproduces the classical motion obeying Eq. (5.4) in the limit $\hbar \to 0$ under these conditions.

Since the vector field A_μ propagates on the light-cone in the conformally-flat space-time, it satisfies the Feynman-gauge free field equation

$$\partial^\alpha \partial_\alpha A_\mu = 0; \quad (5.12)$$

consequently, we may expand the quantised field in the interaction picture in terms of explicit mode functions as

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left[a_\mu(\mathbf{k})e^{-ik\cdot x} + a^\dagger_\mu(\mathbf{k})e^{ik\cdot x}\right]. \tag{5.13}$$

Here, $k = \|\mathbf{k}\|$, and we have the same commutation relations as in the flat space-time case, Eq. (2.54).

The rescaled scalar field φ in the interaction picture may be expanded in terms of solutions to the field equation, which we denote by $\Phi_\mathbf{p}(x)$ and $\overline{\Phi}^*_\mathbf{p}(x)$, the mode functions satisfying the equation of motion. As in the earlier cases, the differential operator generating the equation of motion is not real, and consequently we must distinguish between the two solutions. For $\Phi_\mathbf{p}(x)$, that equation is

$$\left(\hbar^2 D_\mu D^\mu + {M_c}^2(t)\right)\Phi_\mathbf{p}(x) = 0 \tag{5.14}$$

where $D_\mu = \partial_\mu + iV_\mu$; a similar equation holds for $\overline{\Phi}^*_\mathbf{p}(x)$. When $t > -T_2$, we require that our solutions become the flat space-time solutions, *i.e.*, $\Phi_\mathbf{p}(x) = e^{-\frac{i}{\hbar}p\cdot x}$ and $\overline{\Phi}^*_\mathbf{p}(x) = e^{\frac{i}{\hbar}p\cdot x}$, and $p_0 \equiv \sqrt{\|\mathbf{p}\|^2 + m^2}$. Since the background fields are assumed to be smooth, the particle creation effect is non-perturbative in $\hbar$: *i.e.*, it does not occur at any finite order in $\hbar$. Hence $\Phi_\mathbf{p}(x) = e^{-\frac{i}{\hbar}p\cdot x + i\delta}$ for some real number δ for $t < -T_1$ to all orders in $\hbar$ in the WKB approximation, and similarly for $\overline{\Phi}^*_\mathbf{p}(x)$. Note that the momentum may be different in the regions $t < -T_1$ and $t > -T_2$. Then we can expand the scalar field φ as

$$\varphi(x) = \hbar \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left[A(\mathbf{p})\Phi_\mathbf{p}(x) + B^\dagger(\mathbf{p})\overline{\Phi}^*_\mathbf{p}(x)\right]; \tag{5.15}$$

as with the vector field, the non-zero commutation relations are the flat space-time ones.

Since this rescaled scalar field inhabits a flat space-time, the analysis of the position shift follows the same lines as in Ch. 2. That is to say, we can derive the emission amplitude in the same way, and find that as previously, the change in expected position due to the emission derives from two sources: the tree level and the one-loop diagrams. These are

$$\delta_{\text{tree}} x^i = -\frac{i}{2} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \mathcal{A}^{\mu *}(\mathbf{p}, \mathbf{k}) \overleftrightarrow{\partial_{p_i}} \mathcal{A}_\mu(\mathbf{p}, \mathbf{k}), \tag{5.16}$$

$$\delta_{\text{loop}} x^i = -\partial_{p_i} \operatorname{Re} \mathcal{F}(\mathbf{p}). \tag{5.17}$$

It now remains to be shown that a scalar quantum field with a varying mass term generates the same position shift as the equivalent classical theory. The one-loop position shift, however, is not the same as the flat space result (which is zero), although it is zero for the conformally-related theory with a constant mass.

Consider an electromagnetic field coupled to a classical external current j^μ with a Lagrangian,

$$\mathcal{L} = \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e A_\mu j^\mu \right). \tag{5.18}$$

The classical current from a point charge is

$$j^\mu(\mathbf{x}, t) = \frac{dx^\mu}{dt} \delta^3(\mathbf{x} - \mathbf{x}(t)) \tag{5.19}$$

where $\mathbf{x}(t)$ is the position of the particle at time t. The reader will note that this is written differently from the current as we defined it in Ch. 2, although it is easy to see that the two definitions are, in fact, equivalent. In the Feynman gauge, the emission amplitude is given by

$$\mathcal{A}^\mu_{\text{cl}}(\mathbf{k}) = -e \int_{-\infty}^{\infty} d\tau \frac{dx^\mu}{d\tau} e^{ik\cdot x}. \tag{5.20}$$

Higuchi and Martin [22] showed that if the quantum field theory is a Hamiltonian system when $e = 0$ and if the one-photon emission amplitude $\mathcal{A}^\mu(\mathbf{p}, \mathbf{k})$ equals the emission amplitude $\mathcal{A}^\mu_{\text{cl}}(\mathbf{k})$, then in the limit $\hbar \to 0$ the tree-level position shift $\delta_{\rm tree}x^i$ equals the position shift due to the classical ALD force $f^{\mu}_{(\mathrm{R})\flat}$. Eq. (5.5) demonstrates that the first of these conditions is satisfied; we now show that $\mathcal{A}^{\mu}(\mathbf{p},\mathbf{k})$ equals the classical emission amplitude $\mathcal{A}^{\mu}_{\rm cl}(\mathbf{k})$ for the point charge with final momentum $\mathbf{p}$, which is thus sufficient to show that $\delta_{\rm tree}x^i$ is identical to the classical position shift.

Considering the field equation Eq. (5.14), we observe that the deviation from the free-field equation is time-dependent and therefore spatial momentum in the field is conserved. Thus, if we separate variables in the wavefunction $\Phi_{\mathbf{p}}(x)$ and carry out the WKB approximation as in Ch. 2, we find the following:

$$\Phi_{\mathbf{p}}(x) = \sqrt{p_0}\,\phi_{\mathbf{p}}(t)e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}, \tag{5.21}$$

$$\phi_{\mathbf{p}}(t) = \frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)}}\exp\left\{-\frac{i}{\hbar}\int_0^t \sigma_{\mathbf{p}}(t')dt'\right\}\varphi_{\mathbf{p}}(t), \tag{5.22}$$

where $\sigma_{\mathbf{p}}(t) = \sqrt{\|\mathbf{p}-\mathbf{V}(t)\|^2 + {M_c}^2(t)}$. To lowest order in $\hbar$, the function $\varphi_{\mathbf{p}}(t)$ equals 1: it therefore contains the terms of order $\hbar$ and higher.

If we consider the Hamiltonian, Eq. (5.5), and replace $M^2(x) = {M_c}^2(t)$ and $eA_{\mathrm{ex}\,\mu} = V_\mu(t)$, then we find the following equations

$$\sigma_{\mathbf{p}}(t) = M_c(t)\frac{dt}{d\tau}, \tag{5.23}$$

$$\tilde{\mathbf{p}}(t) \equiv \mathbf{p} - \mathbf{V}(t) = M_c(t)\frac{d\mathbf{x}}{d\tau}, \tag{5.24}$$

where we consider $x^\mu(\tau)$ as the worldline of a classical scalar charged particle of mass $M_c(t)$ in a background field $\mathbf{V}(t)$, passing through the space-time origin with a momentum $\mathbf{p}$, and parameterised by the proper time, τ.

If we define $\overleftrightarrow{D^\mu} \equiv \overrightarrow{D^\mu} - \overleftarrow{D^\mu}^{\dagger}$, then the current operator associated with the rescaled scalar field is

$$J^\mu(x) \equiv \frac{i}{\hbar} :\varphi^\dagger(x)\overleftrightarrow{D^\mu}\varphi(x): . \tag{5.25}$$

Pursuing the same line of argument by which we obtained Eq. (2.96), we find

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int \frac{d^3\mathbf{P}}{2P_0(2\pi\hbar)^3}\int d^4x\, \langle\mathbf{P}|\, J^\mu\, |\mathbf{p}\rangle\, e^{ik\cdot x}. \tag{5.26}$$

The commutation relations give us

$$\langle\mathbf{P}|\, J^\mu\, |\mathbf{p}\rangle = i\hbar\Phi^*_{\mathbf{P}}(x)\overleftrightarrow{D}^\mu\Phi_{\mathbf{p}}(x), \tag{5.27}$$

and if we define $\tilde{p}^\mu \equiv (\sigma_{\mathbf{p}}, \mathbf{p})$, then we find to lowest order in $\hbar$,

$$\int d^3\mathbf{x}\, \langle\mathbf{P}|\, J^\mu(x)\, |\mathbf{p}\rangle\, e^{-i\mathbf{k}\cdot\mathbf{x}} = \int d^3\mathbf{x}\left(\tilde{p}^\mu + \tilde{P}^\mu\right)\Phi^*_{\mathbf{P}}(x)\Phi_{\mathbf{p}}(x)e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{5.28}$$

Carrying out the $\mathbf{x}$-integral, we obtain a delta function, $(2\pi\hbar)^3\delta^3(\mathbf{P}-\mathbf{p}+\hbar\mathbf{k})$. Then as $\mathbf{P}$ and $\mathbf{p}$ differ by $\hbar\mathbf{k}$, we are able to approximate $\sigma_{\mathbf{P}}(t) - \sigma_{\mathbf{p}}(t) \approx -\hbar\tilde{\mathbf{p}}\cdot\mathbf{k}/\sigma_{\mathbf{p}}(t)$, and thus

$$\int d^3\mathbf{x}\, \langle\mathbf{P}|\, J^\mu(x)\, |\mathbf{p}\rangle\, e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{2\tilde{p}^\mu p_0}{\sigma_{\mathbf{p}}}\exp\left\{-i\int_0^t \frac{\tilde{\mathbf{p}}\cdot\mathbf{k}}{\sigma_{\mathbf{p}}(t')}dt'\right\}(2\pi\hbar)^3\delta^3(\mathbf{P}-\mathbf{p}+\hbar\mathbf{k}). \tag{5.29}$$

Integrating over $\mathbf{P}$, we find in the limit $\hbar\to 0$

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int dt\frac{\tilde{p}^\mu}{\sigma_{\mathbf{p}}}\exp\left\{-i\int_0^t \frac{\tilde{\mathbf{p}}}{\sigma_{\mathbf{p}}}dt\cdot\mathbf{k}\right\}e^{ikt}, \tag{5.30}$$

and using the equations derived from the Hamiltonian above, we find

$$\mathcal{A}^\mu(\mathbf{p},\mathbf{k}) = -e\int d\tau\frac{dx^\mu}{d\tau}e^{ik\cdot x}. \tag{5.31}$$

The reader will observe, by comparing this with Eq. (5.20), that we have obtained the same expression as $\mathcal{A}^\mu_{\text{cl}}(\mathbf{k})$. Since the difference between the masses used for the two derivations, $M(t)$ and $M_c(t)$, is of order $\hbar^2$, these two expressions are equal in the limit $\hbar\to 0$, and therefore we conclude that the position shift $\delta_{\text{tree}}x^i$ is identical to the corresponding classical position shift in the limit $\hbar\to 0$.

5.2 One-loop corrections for the theory with time-dependent mass

In this section we calculate the one-loop corrections for the charged scalar field with a time-dependent mass $M(t)$, not assumed to arise from a conformal transformation, in an external electromagnetic potential $\mathbf{V}(t)$. Higuchi [19] argued that there is a logarithmic correction to the mass that is of order $\hbar^{-1}$; we confirm this here. We also show that the relation between the external current and the electromagnetic field that it generates is modified.

We recall that at tree level the background field V^μ is generated by an external current J_C^μ; *i.e.*,

$$\partial_\nu(\partial^\nu V^\mu - \partial^\mu V^\nu) = e^2 J_C^\mu. \tag{5.32}$$

Let us denote the original, 'bare' fields and constants of the (flat space-time, varying mass) scalar QED Lagrangian with subscript zeroes. The loop corrections act to multiply these fields and constants by the renormalisation constants, which are defined in the following manner [31]:

$$\begin{aligned} \varphi_0 &= {Z_2}^{1/2}\varphi, \\ A_0^\mu &= {Z_3}^{1/2}A^\mu, \\ e_0 Z_2 {Z_3}^{1/2} &= eZ_1, \\ Z_2 {M_0}^2(t) &= M^2(t) - \delta M^2(t). \end{aligned} \tag{5.33}$$

The Ward-Takahashi identity implies that $Z_1 = Z_2$. Applying these considerations to the scalar Lagrangian, we obtain

$$\mathcal{L} = -\frac{Z_3}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + Z_2(\mathcal{D}_\mu\varphi)^\dagger\mathcal{D}^\mu\varphi - \frac{M^2(t)}{\hbar^2}\varphi^\dagger\varphi + \frac{\delta M^2(t)}{\hbar^2}\varphi^\dagger\varphi - e(J_C^\mu + \Delta J^\mu)A_\mu, \tag{5.34}$$

When considering the one-loop calculations, we shall use the dimensional regularisation procedure, which gives at one loop a multiplicative mass renormalisation, $\delta M^2(t) \propto M^2(t)$. The field strength $\tilde{F}_{\mu\nu}$ is derived from the total electromagnetic field $\tilde{A}_\mu \equiv A_\mu + e^{-1}V_\mu$ in the usual way. We define the derivative operator $\mathcal{D}_\mu$ by $\mathcal{D}_\mu = \partial_\mu + ie\tilde{A}_\mu/\hbar$. The tree-level current J^μ_C is supplemented with the current ΔJ^μ in order to generate the background field V_μ at one-loop order. Thus, Maxwell's equations at one loop are

$$Z_3\partial_\nu(\partial^\nu V^\mu - \partial^\mu V^\nu) = e^2(J^\mu_C + J^\mu_Q), \tag{5.35}$$

where $J^\mu_Q \equiv \langle 0|\, J^\mu \,|0\rangle$, which we call the vacuum current. This equation can be written

$$e^2\Delta J^\mu = e^2 J^\mu_Q - (Z_3 - 1)\partial_\nu(\partial^\nu V^\mu - \partial^\mu V^\nu). \tag{5.36}$$

By using the definition of $\tilde{A}_\mu$ above, substituting it into the Lagrangian and dropping terms which are total derivatives or independent of the fields, we find

$$\mathcal{L} = \frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + Z_2(\mathcal{D}_\mu\varphi)^\dagger\mathcal{D}^\mu\varphi - \frac{M^2(t)}{\hbar^2}\varphi^\dagger\varphi + \frac{\delta M^2(t)}{\hbar^2}\varphi^\dagger\varphi + eA_\mu\Delta J^\mu. \tag{5.37}$$

5.2.1 The supplementary current and Maxwell's equations

Here we shall show that in the semi-classical approximation, the current ΔJ^μ is non-zero at order $\hbar^{-1}$. That is, Maxwell's equations relating the external current to the external electric field are altered at this order. In order to find this correction to Maxwell's equations, we must first calculate the vacuum current $J^\mu_Q = \langle 0|\, J^\mu \,|0\rangle$.

Under the assumptions that $V^0(t) = 0$ and that $\mathbf{V}$ depends only on time, it is easy to show that $J^0_Q = 0$. This is physically reasonable as a homogeneous electric field cannot generate an inhomogeneous charge distribution.

The spatial components of J_Q^μ are

$$J_Q^i = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{p^i - V^i(t)}{\sigma_\mathbf{p}(t)} |\varphi_\mathbf{p}(t)|^2. \tag{5.38}$$

We wish to calculate $\mathbf{J}_Q$ up to order $\hbar^0$, which necessitates the calculation of $|\varphi_\mathbf{p}(t)|^2$ up to order $\hbar^3$.

Let us define $\Sigma(t) \equiv {\sigma_\mathbf{p}}^2(t)$, and re-write the equation of motion, Eq. (5.14), in terms of the logarithm of the mode function $\phi_\mathbf{p}(t)$:

$$\hbar^2 {\partial_0}^2 \log \phi_\mathbf{p}(t) + \hbar^2 (\partial_0 \log \phi_\mathbf{p}(t))^2 + \Sigma(t) = 0. \tag{5.39}$$

Taking $\phi_\mathbf{p}(t)$ from Eq. (5.22) and replacing ${M_c}^2(t)$ with $M^2(t)$, we obtain

$$\log \phi_\mathbf{p} = -\frac{1}{4} \log \Sigma - \frac{i}{\hbar} \int^t \Sigma^{\frac{1}{2}} dt' + \log \varphi_\mathbf{p}, \tag{5.40}$$

which we substitute into the above equation of motion, finding

$$\frac{1}{i\hbar} \Sigma^{\frac{1}{2}} \frac{d}{dt} \log \varphi_\mathbf{p} = \frac{1}{4} \frac{\dot{\Sigma}}{\Sigma} \frac{d}{dt} \log \varphi_\mathbf{p} - \frac{1}{2} \frac{d^2}{dt^2} \log \varphi_\mathbf{p} - \frac{1}{2} \left(\frac{d}{dt} \log \varphi_\mathbf{p} \right)^2 + \frac{1}{8} \frac{\ddot{\Sigma}}{\Sigma} - \frac{5}{32} \frac{\dot{\Sigma}^2}{\Sigma^2}. \tag{5.41}$$

Plainly, the solution to this equation, $\log \varphi_\mathbf{p}(t)$, may be written as a power series:

$$\log \varphi_\mathbf{p}(t) = \sum_{n=1}^{\infty} (i\hbar)^n \varphi_\mathbf{p}^{(n)}(t). \tag{5.42}$$

Consequently, the odd-order terms in the squared amplitude $|\varphi_\mathbf{p}(t)|^2$ cancel, and thus this may may be written up to order $\hbar^3$ as

$$|\varphi_\mathbf{p}(t)|^2 = 1 - 2\hbar^2 \varphi_\mathbf{p}^{(2)}(t) + \mathcal{O}(\hbar^4). \tag{5.43}$$

We therefore need seek the second-order solution, $\varphi^{(2)}$, and no further in the expansion.

Taking the equation at order $i\hbar$, we find

$$\dot{\varphi}_\mathbf{p}^{(2)} = -\frac{1}{2} \frac{d}{dt} \left(\Sigma^{-\frac{1}{2}} \dot{\varphi}_\mathbf{p}^{(1)} \right), \tag{5.44}$$

and so we must find the first-order solution in order to calculate the second-order solution. That first-order solution is simply read off from the equation of motion:

$$\Sigma^{\frac{1}{2}}\dot{\varphi}_{\mathbf{p}}^{(1)} = \frac{1}{8}\frac{\ddot{\Sigma}}{\Sigma} - \frac{5}{32}\frac{\dot{\Sigma}^2}{\Sigma^2}. \tag{5.45}$$

Hence, we conclude that

$$\varphi_{\mathbf{p}}^{(2)}(t) = \frac{5}{64}\frac{\dot{\Sigma}^2}{\Sigma^3} - \frac{1}{16}\frac{\ddot{\Sigma}}{\Sigma^2}. \tag{5.46}$$

Considering the vacuum current, Eq. (5.38), we observe that the integration variable may be changed to $\tilde{\mathbf{p}} \equiv \mathbf{p} - \mathbf{V}(t)$, so that $d^3\mathbf{p} = d^3\tilde{\mathbf{p}}$. Then we substitute in our equation for $\varphi_{\mathbf{p}}^{(2)}$, noting that the factor of $\tilde{\mathbf{p}}$ in the integrand already will select only those terms in the integrand which have an odd number of factors of $\tilde{\mathbf{p}}$. Therefore,

$$J_Q^i = -\int \frac{d^3\tilde{\mathbf{p}}}{(2\pi)^3\hbar}\frac{\tilde{p}^i}{\sqrt{\|\tilde{\mathbf{p}}\|^2 + M^2(t)}}\left[\frac{5}{8\sigma_{\mathbf{p}}{}^6(t)}\tilde{p}^j\dot{V}^j(t)\frac{d}{dt}M^2(t) - \frac{1}{8\sigma_{\mathbf{p}}{}^4(t)}\tilde{p}^j\ddot{V}^j(t)\right]. \tag{5.47}$$

Let us define the two integrals as J_{Q1}^i and J_{Q2}^i respectively.

In J_{Q1}, we observe that only the factor $\tilde{p}^i\tilde{p}^j$ contributes to the angular part of the integral, and we apply the identity concerning the angular integral,

$$\int d\Omega_{\mathbf{q}}\, q^i q^j = \frac{1}{3}g^{ij}\int d\Omega_{\mathbf{q}}\, q^2, \tag{5.48}$$

to obtain

$$J_{Q1}^i = \frac{5}{24\hbar}\dot{V}^i(t)\frac{d}{dt}M^2(t)\int\frac{d^3\tilde{\mathbf{p}}}{(2\pi)^3}\frac{\|\tilde{\mathbf{p}}\|^2}{(\|\tilde{\mathbf{p}}\|^2 + M^2(t))^{\frac{7}{2}}}. \tag{5.49}$$

This integral is regular, and evaluating it we find

$$J_{Q1}^i = \frac{1}{48\pi^2\hbar}\left(\frac{d}{dt}\log M^2(t)\right)\dot{V}^i. \tag{5.50}$$

The second integral, however, is not regular. We shall use the dimensional regularisation procedure over 3-momentum, setting $D = 4-2\varepsilon$ and introducing the renormalisation scale μ. This gives us

$$\begin{aligned}
e^2 J^i_{Q2} &= -\mu^{4-D} \frac{\ddot{V}^i}{12\hbar} \int \frac{d^{D-1}\tilde{\mathbf{p}}}{(2\pi)^{D-1}} \frac{\|\tilde{\mathbf{p}}\|^2}{\|\tilde{\mathbf{p}}\|^2 + M^2(t)} \\
&= \left(Z_3 - 1 + \frac{e^2}{48\pi^2\hbar} \log \frac{M^2(t)}{m^2} \right) \ddot{V}^i, \qquad (5.51)
\end{aligned}$$

where the renormalisation constant Z_3 is

$$\begin{aligned}
Z_3 &= 1 - \frac{e^2}{48\pi^2\hbar} \Gamma\left(\frac{4-D}{2}\right) \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \\
&= 1 - \frac{e^2}{48\pi^2\hbar} \left(\frac{1}{\varepsilon} - \gamma - \log \frac{m^2}{4\pi\mu^2} \right). \qquad (5.52)
\end{aligned}$$

Using our expressions for J^i_{Q1} and J^i_{Q2} in Eq. (5.36), we find

$$e^2 \Delta J^i = e^2 J^i_{Q1} + \frac{e^2}{48\pi^2\hbar} \ddot{V}^i \log \frac{M^2(t)}{m^2}. \qquad (5.53)$$

Therefore Maxwell's equations $\ddot{\mathbf{V}} = e^2 \mathbf{J}_C$ have quantum corrections at one loop at order $\hbar^{-1}$, and the corrected equations are

$$\ddot{\mathbf{V}} = e^2 \left[1 + \frac{e^2}{48\pi^2\hbar} \log \frac{M^2(t)}{m^2} \right] \mathbf{J}_C + e^2 \mathbf{J}_{Q1}. \qquad (5.54)$$

5.2.2 One-loop corrections to the time-dependent mass

In App. A of [22], it was shown that for a charged scalar particle on a flat space-time, the mass renormalisation term exactly cancels the one-loop contribution up to order $\hbar^{-1}$. Here, we shall show that for the time-dependent mass $M^2(t) = m^2\Omega^2$, there remains a residual term after the mass renormalisation term and terms occurring at one-loop are added together, and that this residual term affects the motion at a lower order in $\hbar^{-1}$ than the ALD force.

The one-loop terms arise from the forward scattering amplitude $\mathcal{F}(\mathbf{p})$, which is given at order e^2 in terms of the interaction Hamiltonian density

$\mathcal{H}_I(x)$ as

$$\frac{2ip_0}{\hbar}(2\pi\hbar)^3\mathcal{F}(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = -\frac{i}{\hbar}\int d^4x \, \langle 0| \, A(\mathbf{p}')\mathcal{H}_I(x)A^\dagger(\mathbf{p}) \, |0\rangle$$
$$-\frac{1}{2\hbar^2}\int d^4x \, d^4x' \, \langle 0| \, A(\mathbf{p}')T[\mathcal{H}_I(x)\mathcal{H}_I(x')]A^\dagger(\mathbf{p}) \, |0\rangle \, , \quad (5.55)$$

where

$$\mathcal{H}_I(x) = eJ^\mu A_\mu + \frac{e^2}{\hbar^2}\sum_{i=1}^{3} A_iA_i \, :\varphi^\dagger\varphi: - \frac{\delta M^2(t)}{\hbar^2} \, :\varphi^\dagger\varphi:, \quad (5.56)$$

and the current J^μ is given by Eq. (5.25).

We consider first the contribution made by the mass counterterm, $\delta M^2(t)$. The relation between the mass and the curved space-time constant mass suggests that $\delta M^2(t) = (\delta m)^2(M^2(t)/m^2)$, where δm^2 is set to the standard value in the on-shell renormalisation. The contribution from this term, which corresponds to diagram (c) in Fig. 5.1, is

$$2ip_0(2\pi\hbar)^3\mathcal{F}^c(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = -\frac{i}{\hbar^2}\frac{\delta m^2}{m^2}\int d^4x \, \langle 0| \, A(\mathbf{p}') \, :\varphi^\dagger\varphi: \, A^\dagger(\mathbf{p}) \, |0\rangle \, M^2(t). \quad (5.57)$$

Expanding the field φ in terms of its modes, we find

$$\mathcal{F}^c(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = -\frac{1}{2p_0\hbar(2\pi\hbar)^2}\frac{\delta m^2}{m^2}\int d^4x\Phi^*_{\mathbf{p}'}(x)\Phi_{\mathbf{p}}(x)M^2(t). \quad (5.58)$$

Integrating over the spatial components, we find

$$\mathcal{F}^c(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = -\int dt\frac{\delta m^2}{m^2}\frac{1}{2\sigma_{\mathbf{p}}(t)}\varphi^*_{\mathbf{p}'}(t)\varphi_{\mathbf{p}}(t)M^2(t)\delta^3(\mathbf{p}-\mathbf{p}'); \quad (5.59)$$

therefore,

$$\mathcal{F}^c(\mathbf{p}) = -\frac{\delta m^2}{m^2}\int dt\frac{|\varphi_{\mathbf{p}}(t)|^2}{2\sigma_{\mathbf{p}}(t)}M^2(t). \quad (5.60)$$

The mass counterterm is

$$\delta m^2 = \frac{e^2}{\hbar}\mu^{4-D}\int\frac{d^Dq}{(2\pi)^Di}\left\{\frac{(p+q)^2}{[q^2-m^2+i\varepsilon][(p-q)^2+i\varepsilon]} - \frac{4}{[(p-q)^2+i\varepsilon]}\right\}; \quad (5.61)$$

dimensionally regularising this with $D = 4 - 2\varepsilon$, we obtain

$$\delta m^2 = -\frac{3e^2m^2}{(4\pi)^2\hbar}\left(\frac{1}{\varepsilon} - \gamma + \frac{7}{3} - \log\frac{m^2}{4\pi\mu^2}\right). \tag{5.62}$$

At this point it is convenient also to carry out the q_0-integral in the first expression for δm^2 and note the result. To do this, we apply the technique of partial fractions to the integrand. We define $K \equiv \|\mathbf{p}-\mathbf{q}\|$; then this gives us

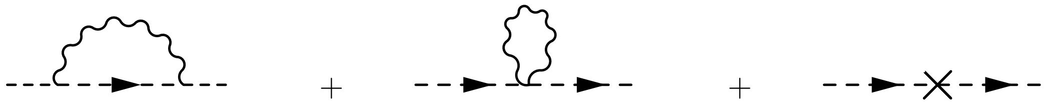
$$\delta m^2 = \frac{e^2}{\hbar}\mu^{4-D}\int\frac{d^Dq}{(2\pi)^D i}\left\{\frac{1}{(q^0)^2 - \sigma_{\mathbf{q}}{}^2(0) + i\varepsilon}\frac{(p^0+\sigma_{\mathbf{q}}(0))^2 - \|\mathbf{p}+\mathbf{q}\|^2}{(p^0-\sigma_{\mathbf{q}}(0))^2 - K^2}\right.$$
$$\left. + \frac{1}{(p^0-q^0)^2 - K^2 + i\varepsilon}\frac{(2p^0+K)^2 - \|\mathbf{p}+\mathbf{q}\|^2}{(p^0+K)^2 - \sigma_{\mathbf{q}}{}^2(0)} - \frac{4}{(q^0-p^0)^2 - K^2 + i\varepsilon}\right\}. \tag{5.63}$$

If we consider the q_0-integral as a contour integral and close the curve in the upper half of the complex plane then the $i\varepsilon$ terms in the denominators will select only the roots $q^0 = -\sigma_{\mathbf{q}}(0) + i\varepsilon$ (with a residue $[-2\sigma_{\mathbf{q}}(0)]^{-1}$) and $q^0 = p^0 - K + i\varepsilon$ (with a residue $[-2K]^{-1}$), in the relevant integrands. In both cases, the poles' contributions to the closing arc cancel, and thus the q^0-integral is simply $2\pi i$ times the residue. Hence, we find

$$\delta m^2 = \frac{e^2}{\hbar}\mu^{4-D}\int\frac{d^{D-1}\mathbf{q}}{(2\pi)^{D-1}}\left\{-\frac{1}{2\sigma_{\mathbf{q}}(0)}\frac{(p^0+\sigma_{\mathbf{q}}(0))^2 - \|\mathbf{p}+\mathbf{q}\|^2}{(p^0-\sigma_{\mathbf{q}}(0))^2 - K^2}\right.$$
$$\left. -\frac{1}{2K}\frac{(2p^0+K)^2 - \|\mathbf{p}+\mathbf{q}\|^2}{(p^0+K)^2 - \sigma_{\mathbf{q}}{}^2(0)} + \frac{2}{K}\right\}. \tag{5.64}$$

We collect terms according to their numerator: the parts of the integrand with a factor of $\|\mathbf{p}+\mathbf{q}\|^2$ simplify, under the technique of partial fractions, to

$$\|\mathbf{p}+\mathbf{q}\|^2\left[\frac{1}{2\sigma_{\mathbf{q}}(0)}\frac{1}{(p^0-\sigma_{\mathbf{q}}(0))^2 - K^2} + \frac{1}{2K}\frac{1}{(p^0-K)^2 - \sigma_{\mathbf{q}}{}^2(0)}\right]$$
$$= -\frac{\|\mathbf{p}+\mathbf{q}\|^2}{4K\sigma_{\mathbf{q}}(0)}\left[\frac{1}{p^0+K+\sigma_{\mathbf{q}}(0)} + \frac{1}{K+\sigma_{\mathbf{q}}(0)-p^0}\right]. \tag{5.65}$$



(a) (b) (c)

Figure 5.1: The Feynman diagrams at order e^2 contributing to the correction to the (time-dependent) mass term.

In like manner, the other terms give us

$$-\frac{(p^0+\sigma_{\mathbf{q}}(0))^2}{2\sigma_{\mathbf{q}}(0)}\frac{1}{(p^0-\sigma_{\mathbf{q}}(0))^2-K^2}-\frac{(2p^0+K)^2}{2K}\frac{1}{(p^0+K)^2-\sigma_{\mathbf{q}}{}^2(0)}$$
$$=-\frac{1}{2K}-\frac{1}{4K\sigma_{\mathbf{q}}(0)}\left[\frac{(p^0+\sigma_{\mathbf{q}}(0))^2}{p-\sigma_{\mathbf{q}}(0)-K}+\frac{(p-\sigma_{\mathbf{q}}(0))^2}{p^0+K+\sigma_{\mathbf{q}}(0)}\right]. \quad (5.66)$$

Combining all these re-arrangements, we find

$$\delta m^2=\frac{e^2}{\hbar}\mu^{4-D}\int\frac{d^{D-1}\mathbf{q}}{(2\pi)^{D-1}}\left\{-\frac{\|\mathbf{p}+\mathbf{q}\|^2}{4K\sigma_{\mathbf{q}}(0)}\left[\frac{1}{\sigma_{\mathbf{q}}(0)+K-p_0}+\frac{1}{\sigma_{\mathbf{q}}(0)+K+p_0}\right]\right.$$
$$\left.+\frac{3}{2K}+\frac{1}{4K\sigma_{\mathbf{q}}(0)}\left[\frac{(p_0-\sigma_{\mathbf{q}}(0))^2}{\sigma_{\mathbf{q}}(0)+K+p_0}+\frac{(p_0+\sigma_{\mathbf{q}}(0))^2}{\sigma_{\mathbf{q}}(0)+K-p_0}\right]\right\}. \quad (5.67)$$

We now consider the contribution made by the second term in the interaction Hamiltonian, $(e^2/\hbar^2)\sum_i A_iA_i :\varphi^\dagger\varphi:$, corresponding to diagram (b) in Fig. 5.1. As with the mass counterterm, this also contributes through the term in Eq. (5.55) with only one copy of the interaction Hamiltonian. Defining this contribution to be $\mathcal{F}^b(\mathbf{p})$, we have

$$\frac{2ip_0}{\hbar}(2\pi\hbar)^3\mathcal{F}^b(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')=-\frac{ie^2}{\hbar^3}\int d^4x\,\langle 0|\,A(\mathbf{p}')\sum_{i=1}^{3}A_iA_i :\varphi^\dagger\varphi\!: A^\dagger(\mathbf{p})\,|0\rangle\,. \quad (5.68)$$

We expand both the vector and scalar fields in terms of modes and note that we may separate the two, so that

$$\mathcal{F}^b(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}')=-\frac{e^2}{2p_0(2\pi\hbar)^3\hbar^2}\int d^4x\,\langle 0|\,A(\mathbf{p}') :\varphi^\dagger\varphi\!: A^\dagger(\mathbf{p})\,|0\rangle\sum_{i=1}^{3}\langle 0|\,A_iA_i\,|0\rangle\,. \quad (5.69)$$

Using the mode expansion for the fields $\varphi(x)$ and $A_\mu(x)$, we find

$$\mathcal{F}^b(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = -\frac{e^2\hbar^2}{2p_0(2\pi\hbar)^3}\int d^4x \Phi^*_{\mathbf{p}'}(x)\Phi_{\mathbf{p}}(x)\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\sum_{i=1}^{3}\eta_{ii} \quad (5.70)$$

The result of the $\mathbf{x}$-integral can be easily deduced from the foregoing considerations of the mass counterterm, with which it shares a great similarity. Carrying this out, and integrating with respect to $\mathbf{p}'$ as well, we find

$$\mathcal{F}^b(\mathbf{p}) = e^2\int dt \frac{|\varphi_{\mathbf{p}}(t)|^2}{2\sigma_{\mathbf{p}}(t)}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{3}{2K}, \quad (5.71)$$

where we have scaled and shifted the integration variable from the wave-vector $\mathbf{k}$ to the momentum $\mathbf{q} \equiv \mathbf{p} - \hbar\mathbf{k}$; then $K = \|\mathbf{p}-\mathbf{q}\|$.

Finally, we consider the first term in the Hamiltonian, corresponding to diagram (a) in Fig. 5.1. The contribution this makes through the first term in Eq. (5.55) vanishes, as the vacuum expectation value of $A_\mu(x)$ is zero. We define the contribution through the second term to be $\mathcal{F}^a(\mathbf{p})$, and note that only the square of the term $eJ^\mu A_\mu$ will contribute up to order e^2. This contribution is

$$\frac{2ip_0}{\hbar}(2\pi\hbar)^3\mathcal{F}^a(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = -\frac{e^2}{2\hbar^2}\int d^4x\, d^4x' \\ \times \langle 0|\, A(\mathbf{p}')T\left[J^\mu(x)A_\mu(x)J^{\mu'}(x')A_{\mu'}(x')\right]A^\dagger(\mathbf{p})\,|0\rangle\,. \quad (5.72)$$

Again, we may separate the fields, so that we find a factor of the form

$$\begin{aligned}\langle 0|\, T\left[A_\mu(x)A_{\mu'}(x')\right]|0\rangle &= \theta(x^0-x'^0)\,\langle 0|\, A_\mu(x)A_{\mu'}(x')\,|0\rangle \\ &\quad + \theta(x'^0-x^0)\,\langle 0|\, A_{\mu'}(x')A_\mu(x)\,|0\rangle \\ &= \hbar\eta_{\mu\mu'}\int\frac{d^3\mathbf{k}}{2k(2\pi)^3}\Big[\theta(x^0-x'^0)e^{-ik\cdot(x-x')} \\ &\quad +\theta(x'^0-x^0)e^{ik\cdot(x-x')}\Big]\,. \end{aligned} \quad (5.73)$$

Since $\theta^2(y) = \theta(y)$ and $\theta(y)\theta(-y) = 0$, we may use this to extract the time-ordering of the scalar current as well. Thus replacing the time-ordered vector field correlation function inside Eq. (5.72) and re-arranging, we find

$$\mathcal{F}^a(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = \frac{ie^2}{4p_0(2\pi\hbar)^3}\int d^4x\, d^4x' \int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\eta_{\mu\mu'} \\ \times \Big[\theta(x^0-x'^0)\langle 0|\, A(\mathbf{p}')J^\mu(x)J^{\mu'}(x')A^\dagger(\mathbf{p})\,|0\rangle\, e^{-ik\cdot(x-x')} \\ +\theta(x'^0-x^0)\langle 0|\, A(\mathbf{p}')J^{\mu'}(x')J^\mu(x)A^\dagger(\mathbf{p})\,|0\rangle\, e^{ik\cdot(x-x')}\Big]. \tag{5.74}$$

Since we integrate over x and x' and $\eta_{\mu\mu'}$ is symmetric in its indices, we may exchange the labelling in the second term of the integrand, which makes that term equal to the first. We may ignore the vacuum diagrams produced by the currents, as the algebraic terms they generate are cancelled by a normalisation condition [31]. Then using the mode expansion for the field $\varphi(x)$, we have

$$\mathcal{F}^a(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = \frac{ie^2}{2p_0(2\pi\hbar)^3}\int d^4x\, d^4x' \\ \times \int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}}{2q_0(2\pi\hbar)^3}\eta_{\mu\mu'}e^{ik\cdot(x-x')}\theta(x^0-x'^0) \\ \times\Big[\Phi^*_{\mathbf{p}'}(x)\Phi^*_{\mathbf{q}}(x')\overleftrightarrow{D}^\mu\overleftrightarrow{D}^{\mu'}\Phi_{\mathbf{q}}(x)\Phi_{\mathbf{p}}(x') + \overline{\Phi}_{\mathbf{q}}(x)\Phi^*_{\mathbf{p}'}(x')\overleftrightarrow{D}^\mu\overleftrightarrow{D}^{\mu'}\Phi_{\mathbf{p}}(x)\overline{\Phi}^*_{\mathbf{q}}(x')\Big]. \tag{5.75}$$

We can turn the anti-particle modes into particle modes by observing that $\overline{\phi}_{\mathbf{p}}(t) = \phi_{-\mathbf{p}}(t)$. Therefore,

$$\mathcal{F}^a(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{p}') = \frac{ie^2}{2p_0(2\pi\hbar)^3}\int d^4x\, d^4x' \int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{q}}{2q_0(2\pi\hbar)^3}\eta_{\mu\mu'} \\ \times\Big[\theta(x^0-x'^0)\Big(\Phi^*_{\mathbf{p}'}(x)\Phi^*_{\mathbf{q}}(x')\overleftrightarrow{D}^\mu\overleftrightarrow{D}^{\mu'}\Phi_{\mathbf{q}}(x)\Phi_{\mathbf{p}}(x')\Big)e^{ik\cdot(x-x')} \\ -\theta(x'^0-x^0)\Big(\Phi_{-\mathbf{q}}(x')\Phi^*_{\mathbf{p}'}(x)\overleftrightarrow{D}^\mu\overleftrightarrow{D}^{\mu'}\Phi_{\mathbf{p}}(x')\Phi^*_{-\mathbf{q}}(x)\Big)e^{-ik\cdot(x-x')}\Big]. \tag{5.76}$$

Applying the spatial derivatives, and integrating over the spatial co-ordinates

$\mathbf{x}$ and $\mathbf{x}'$ as well as the momentum $\mathbf{p}'$, we obtain

$$\mathcal{F}^a(\mathbf{p}) = \frac{ie^2}{4\hbar}\int\frac{d^3\mathbf{q}}{2K(2\pi)^3}\int dtdt'$$
$$\Big[\theta(t-t')\left(\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}_1}(t,t',\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')\right)e^{-iK(t-t')/\hbar}$$
$$+\theta(t'-t)\left(\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}_1}(t,t',\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t')\phi^*_{\mathbf{q}}(t)\right)e^{iK(t-t')/\hbar}\Big], \tag{5.77}$$

where $\mathbf{K} = \mathbf{p}-\mathbf{q}$, $K = \|\mathbf{K}\|$, and

$$\overleftrightarrow{\mathcal{D}_1}(t,t',\mathbf{p},\mathbf{q}) = -\hbar^2\overleftrightarrow{\partial_t}\overleftrightarrow{\partial_{t'}} + [\mathbf{p}+\mathbf{q}-2\mathbf{V}(t)]\cdot[\mathbf{p}+\mathbf{q}-2\mathbf{V}(t')]. \tag{5.78}$$

Now, we seek the terms of lowest non-trivial order in $\hbar$, noting that the WKB expansion can be constructed in the same way as for the theory with a constant mass. Hence, we find that

$$\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\partial_t}\overleftrightarrow{\partial_{t'}}\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t') = \frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)\sigma_{\mathbf{p}}(t')}}\exp\left\{-\frac{i}{\hbar}\int_t^{t'}\sigma_{\mathbf{p}}(T)dT\right\}$$
$$\times\frac{i}{\hbar}\left(-\sigma_{\mathbf{q}}(t)-\sigma_{\mathbf{p}}(t)\right)\frac{i}{\hbar}\left(\sigma_{\mathbf{q}}(t')+\sigma_{\mathbf{p}}(t')\right)\frac{1}{\sqrt{\sigma_{\mathbf{q}}(t)\sigma_{\mathbf{q}}(t')}}\exp\left\{\frac{i}{\hbar}\int_t^{t'}\sigma_{\mathbf{q}}(T)dT\right\}$$
$$=\frac{1}{\hbar^2}\frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)\sigma_{\mathbf{p}}(t')\sigma_{\mathbf{q}}(t)\sigma_{\mathbf{q}}(t')}}\exp\left\{-\frac{i}{\hbar}\int_t^{t'}\left[\sigma_{\mathbf{p}}(T)-\sigma_{\mathbf{q}}(T)\right]dT\right\}$$
$$\times\left(\sigma_{\mathbf{p}}(t)+\sigma_{\mathbf{q}}(t)\right)\left(\sigma_{\mathbf{p}}(t')+\sigma_{\mathbf{q}}(t')\right). \tag{5.79}$$

Hence, the corresponding integrand may be written as

$$\theta(t-t')\frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)\sigma_{\mathbf{p}}(t')\sigma_{\mathbf{q}}(t)\sigma_{\mathbf{q}}(t')}}\exp\left\{-\frac{i}{\hbar}\int_t^{t'}\left[\sigma_{\mathbf{p}}(T)-\sigma_{\mathbf{q}}(T)-K\right]dT\right\}$$
$$\times\left\{-\left(\sigma_{\mathbf{p}}(t)+\sigma_{\mathbf{q}}(t)\right)\left(\sigma_{\mathbf{p}}(t')+\sigma_{\mathbf{q}}(t')\right)+[\mathbf{p}+\mathbf{q}-2\mathbf{V}(t)]\cdot[\mathbf{p}+\mathbf{q}-2\mathbf{V}(t')]\right\}. \tag{5.80}$$

Similarly, the other integrand may be found by noting that the difference between the two amounts to changing $\sigma_{\mathbf{q}} \mapsto -\sigma_{\mathbf{q}}$ and $K \mapsto -K$, and switching the sign of the theta function's argument. Therefore, we have

$$\theta(t'-t)\frac{1}{\sqrt{\sigma_{\mathbf{p}}(t)\sigma_{\mathbf{p}}(t')\sigma_{\mathbf{q}}(t)\sigma_{\mathbf{q}}(t')}}\exp\left\{-\frac{i}{\hbar}\int_t^{t'}[\sigma_{\mathbf{p}}(T)+\sigma_{\mathbf{q}}(T)+K]\,dT\right\}$$
$$\times\left\{-\left(\sigma_{\mathbf{p}}(t)-\sigma_{\mathbf{q}}(t)\right)\left(\sigma_{\mathbf{p}}(t')-\sigma_{\mathbf{q}}(t')\right)+[\mathbf{p}+\mathbf{q}-2\mathbf{V}(t)]\cdot[\mathbf{p}+\mathbf{q}-2\mathbf{V}(t')]\right\}. \tag{5.81}$$

We change the integration variables to $\bar{t}\equiv(t+t')/2$ and $\eta\equiv(t'-t)/\hbar$. The theta function therefore restricts η to be negative (for the first integrand) or positive (for the second). The Jacobian for this transformation is $dt\,dt'=\hbar d\bar{t}\,d\eta$, and so

$$\mathcal{F}^a(\mathbf{p})=ie^2\int\frac{d^3\mathbf{q}}{2K(2\pi)^3}\int d\bar{t}\,[G_-(\mathbf{p},\mathbf{q},\bar{t})+G_+(\mathbf{p},\mathbf{q},\bar{t})], \tag{5.82}$$

where we define

$$G_\pm(\mathbf{p},\mathbf{q},\bar{t})\equiv\pm\int_0^\infty d\eta[f_\pm(\mathbf{p},\mathbf{q},\bar{t})+\mathcal{O}(\hbar^2)]$$
$$\times\exp\left\{\mp i\int_{-\eta/2}^{\eta/2}d\zeta[\pm\sigma_{\mathbf{p}}(\bar{t}+\hbar\zeta)+\sigma_{\mathbf{q}}(\bar{t}+\hbar\zeta)+K]\right\}, \tag{5.83}$$

and

$$f_\pm(\mathbf{p},\mathbf{q},\bar{t})\equiv\frac{1}{4\sigma_{\mathbf{p}}(\bar{t})\sigma_{\mathbf{q}}(\bar{t})}\left\{-[\sigma_{\mathbf{p}}(\bar{t})\mp\sigma_{\mathbf{q}}(\bar{t})]^2+\|\mathbf{p}+\mathbf{q}-2\mathbf{V}(\bar{t})\|^2\right\}. \tag{5.84}$$

The exponential term may be expanded in powers of $\hbar$, as

$$\mp i\int_{-\eta/2}^{\eta/2}d\zeta[\pm\sigma_{\mathbf{p}}(\bar{t}+\hbar\zeta)+\sigma_{\mathbf{q}}(\bar{t}+\hbar\zeta)+K]=\mp i[\pm\sigma_{\mathbf{p}}(\bar{t})+\sigma_{\mathbf{q}}(\bar{t})+K]\eta+\mathcal{O}(\hbar^2). \tag{5.85}$$

Thus, we can ignore the terms of higher order in $\hbar$ and perform the integral over η in our definition of $G_\pm(\mathbf{p},\mathbf{q},\bar{t})$ for the term of lowest order, obtaining

$$\int_0^\infty d\eta\exp\left\{\mp i[\pm\sigma_{\mathbf{p}}(\bar{t})+\sigma_{\mathbf{q}}(\bar{t})+K]\right\}=\frac{\mp i}{\pm\sigma_{\mathbf{p}}(\bar{t})+\sigma_{\mathbf{q}}(\bar{t})+K}. \tag{5.86}$$

Therefore,

$$\mathcal{F}^a(\mathbf{p}) = ie^2 \int \frac{d^3\mathbf{q}}{2K(2\pi)^3} \int d\bar{t} \frac{1}{4\sigma_{\mathbf{p}}(\bar{t})\sigma_{\mathbf{q}}(\bar{t})} \times \left\{ -\frac{i}{-\sigma_{\mathbf{p}}(\bar{t}) + \sigma_{\mathbf{q}}(\bar{t}) + K} \left[-(\sigma_{\mathbf{p}}(\bar{t}) + \sigma_{\mathbf{q}}(\bar{t}))^2 + \|\mathbf{p} + \mathbf{q} - 2\mathbf{V}(\bar{t})\|^2 \right] - \frac{i}{\sigma_{\mathbf{p}}(\bar{t}) + \sigma_{\mathbf{q}}(\bar{t}) + K} \left[-(\sigma_{\mathbf{p}}(\bar{t}) - \sigma_{\mathbf{q}}(\bar{t}))^2 + \|\mathbf{p} + \mathbf{q} - 2\mathbf{V}(\bar{t})\|^2 \right] \right\}. \tag{5.87}$$

If we define $\tilde{\mathbf{q}} \equiv \mathbf{q} - \mathbf{V}(\bar{t})$ and $\tilde{\mathbf{p}} \equiv \mathbf{p} - \mathbf{V}(\bar{t})$, then $\sigma_{\mathbf{q}}(\bar{t}) = \sigma_{\tilde{\mathbf{q}}}(0) \equiv \tilde{q}_0$ and similarly $\sigma_{\mathbf{p}}(\bar{t}) = \sigma_{\tilde{\mathbf{p}}}(0) \equiv \tilde{p}_0$. We may then change the variable of integration from $\mathbf{q}$ to $\tilde{\mathbf{q}}$ and also change $\mathbf{p}$ to $\tilde{\mathbf{p}}$. As a result, we obtain

$$\mathcal{F}^a(\mathbf{p}) = e^2 \int \frac{d^3\tilde{\mathbf{q}}}{(2\pi)^3} \int d\bar{t} \frac{1}{2\sigma_{\mathbf{p}}(\bar{t})} \frac{1}{4K\sigma_{\tilde{\mathbf{q}}}(0)} \times \left\{ \|\tilde{\mathbf{p}} + \tilde{\mathbf{q}}\|^2 \left[\frac{1}{\sigma_{\tilde{\mathbf{q}}}(0) - \tilde{p}_0 + K} + \frac{1}{\sigma_{\tilde{\mathbf{q}}}(0) + \tilde{p}_0 + K} \right] - \left[\frac{(\tilde{p}_0 + \sigma_{\tilde{\mathbf{q}}}(0))^2}{\sigma_{\tilde{\mathbf{q}}}(0) - \tilde{p}_0 + K} + \frac{(\tilde{p}_0 - \sigma_{\tilde{\mathbf{q}}}(0))^2}{\sigma_{\tilde{\mathbf{q}}}(0) + \tilde{p}_0 + K} \right] \right\}. \tag{5.88}$$

As with the theory on a flat space-time with a constant mass term, the $\tilde{\mathbf{q}}$-integral gives rise to infrared divergences in the terms of higher order in $\hbar$: these terms do not contribute to the real part of the forward-scattering amplitude.

Combining our results for $\mathcal{F}^a(\mathbf{p})$ and $\mathcal{F}^b(\mathbf{p})$ and taking only those terms which do not vanish in the limit $\hbar \to 0$ [22], we find a similarity between the terms from the forward-scattering amplitude and the mass renormalisation. We thus conclude that

$$\mathrm{Re}\left[\mathcal{F}^a(\mathbf{p}) + \mathcal{F}^b(\mathbf{p})\right] = -\hbar \int \frac{dt}{2\sigma_{\mathbf{p}}(t)} \Delta M^2(t) + \mathcal{O}(\hbar^2), \tag{5.89}$$

where $\Delta M^2(t)$ is obtained by replacing m^2 with $M^2(t)$ in Eq. (5.67). Therefore,

$$\Delta M^2(t) = -\frac{3e^2}{16\pi^2\hbar} M^2(t) \left(\frac{1}{\varepsilon} - \gamma + \frac{7}{3} - \log \frac{M^2(t)}{4\pi^2\mu^2} \right). \tag{5.90}$$

We observe, therefore, that the mass renormalisation term and the contribution from the rest of the forward-scattering amplitude do not exactly cancel out. Specifically,

$$\Delta M^2(t) - \delta M^2(t) = \frac{3e^2}{16\pi^2\hbar} M^2(t) \log \frac{M^2(t)}{m^2}. \tag{5.91}$$

This non-zero result implies that if it were desired to calculate the effects of one-loop diagrams on the particle's motion, the mass would need to be modified by this quantity. The fact that this correction is of order $\hbar^{-1}$ means that it affects the motion of the particle at that order as shown in Eq. (5.17). Indeed, this effect is at a lower order in $\hbar$ than the ALD force, which is of order $\hbar^0$, and consequently not only might one wish to calculate the one-loop effects, but one would have to in order to have an accurate result to lowest order in $\hbar$.

5.3 One-loop corrections for the conformally flat theory

We have seen that the charged scalar field theory on a conformally flat space-time, with a conformal factor dependent on time only, is classically equivalent to a charged scalar field theory on a flat space-time with a time-dependent mass. Having shown that there exist non-zero one-loop corrections to this classically-equivalent theory with a time-dependent mass on a flat space-time, we shall now show that for the theory on a conformally flat space-time with a constant mass, the one-loop corrections vanish. We shall therefore conclude that the equivalence established earlier, while holding for the classical limit, breaks down at the one-loop level.

In the dimensional regularisation approach, where we continue the dimension of space-time, the conformal transformation $\phi = \Omega^{-1}\varphi$ is modified to $\phi = \Omega^{(2-D)/2}\varphi$; A_μ does not need to be re-scaled as Maxwell's equations are conformally invariant. However, the derivative of the electromagnetic field changes from ordinary to covariant: ${A^\alpha}_{,\beta} \mapsto {A^\alpha}_{;\beta} = (1/\sqrt{-g})\partial_\beta(\sqrt{-g}A^\alpha)$. The classical Lagrangian, Eq. (5.6), is therefore transformed to

$$\mathcal{L} = -\frac{1}{4}\Omega^{D-4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\Omega^{4-D}[\partial_\nu(\Omega^{D-4}A^\nu)]^2 + (\mathcal{D}_\mu\varphi)^\dagger\mathcal{D}^\mu\varphi - \frac{{M_c}^2}{\hbar^2}\varphi^\dagger\varphi, \quad (5.92)$$

with $\mathcal{D}_\mu = \partial_\mu + iV_\mu/\hbar + ieA_\mu/\hbar$, and where the indices are raised and lowered by the flat metric $\eta_{\mu\nu}$. The term playing the role of a mass we defined to be

$${M_c}^2 \equiv m^2\Omega^2 + \left[\xi - \frac{1}{4}\frac{D-2}{D-1}\right]\hbar^2 R\Omega^2. \quad (5.93)$$

Note that in the Lagrangian above, we have chosen to use a gauge-fixing term which would correspond to the Feynman gauge, were $\Omega(t) = 1$.

5.3.1 The supplementary current and Maxwell's equations

Firstly, we shall show that in the limit $\hbar \to 0$, the additional term in the vacuum current entering from the one loop contribution is zero: that is, $\Delta J^\mu = 0$. In the classical theory, we suppose that the background field V^μ is generated by a classical current, J^μ_C, which is given by

$$J^\mu_C = \partial_\nu\left[\Omega^{D-4}\left(\partial^\nu V^\mu - \partial^\mu V^\nu\right)\right]; \quad (5.94)$$

the correction to the current is then

$$e^2\Delta J^\mu = e^2 J^\mu_Q - (Z_3 - 1)\partial_\nu\left[\Omega^{D-4}\left(\partial^\nu V^\mu - \partial^\mu V^\nu\right)\right]. \quad (5.95)$$

Since the currents and the field V^μ depend only on time and since $V^0 = 0$, $\Delta J^0 = 0$ and

$$e^2\Delta\mathbf{J} = e^2\mathbf{J}_{Q1} + e^2\mathbf{J}_{Q2} - (Z_3 - 1)\frac{d}{dt}\left(\Omega^{-2\varepsilon}\dot{\mathbf{V}}\right). \quad (5.96)$$

Applying the derivative to the classical current and using the fact that $\varepsilon(Z_3 - 1) \to -e^2/(48\pi^2\hbar)$ as $\varepsilon \to 0$, we find

$$e^2\Delta\mathbf{J} = e^2\mathbf{J}_{Q1} + e^2\mathbf{J}_{Q2} - \frac{e^2}{48\pi^2\hbar}\left(\frac{d}{dt}\log\Omega^2\right)\dot{\mathbf{V}} - \left(Z_3 - 1 + \frac{e^2}{48\pi^2\hbar}\log\Omega^2\right)\ddot{\mathbf{V}}. \tag{5.97}$$

The currents $\mathbf{J}_{Q1}$ and $\mathbf{J}_{Q2}$ are given by Eqs. (5.50) and (5.51), replacing $M^2(t)$ with ${M_c}^2(t)$: in the limit $\hbar \to 0$, we may in fact replace ${M_c}^2(t)$ with $m^2\Omega^2$, since the difference between the two is of order $\hbar^2$. Then applying those results here, we find

$$e^2\mathbf{J}_{Q1} = \frac{e^2}{48\pi^2\hbar}\left[\frac{d}{dt}\log\left(m^2\Omega^2\right)\right]\dot{\mathbf{V}}, \tag{5.98}$$

$$e^2\mathbf{J}_{Q2} = \left(Z_3 - 1 + \frac{e^2}{48\pi^2\hbar}\log\Omega^2\right)\ddot{\mathbf{V}}. \tag{5.99}$$

Since m is a constant, it is clear that $\Delta\mathbf{J} = \mathbf{0}$, and hence Maxwell's equations do not receive a correction at order e^2 in the limit $\hbar \to 0$.

5.3.2 Vanishing corrections to the mass term

Now we show that there is no correction to the time-dependent mass term in the limit $\hbar \to 0$. In order to proceed with this, it is necessary first to consider further the change generated in the free electromagnetic field equations by the conformal transformation. We obtain the Euler-Lagrange equations from the Lagrangian describing the free electromagnetic field:

$$\mathcal{L}_{\mathrm{EM}} \equiv -\frac{1}{4}\Omega^{D-4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\Omega^{4-D}\left[\partial_\nu(\Omega^{D-4}A^\nu)\right]^2; \tag{5.100}$$

these equations are

$$\partial_\nu\left[F^{\mu\nu} - \partial_\alpha\left(\Omega^{D-4}A^\alpha\right)\eta^{\mu\nu}\right] + \Omega^{4-D}\partial_\nu\left(\Omega^{D-4}A^\alpha\right)\partial^\nu\Omega^{D-4} = 0. \tag{5.101}$$

Re-arranging these, we find

$$\partial_\nu(\Omega^{D-4}\partial^\nu A^\mu) + (D-4)\Omega^{D-4}(\partial^\mu\partial_\nu\log\Omega)A^\nu = 0. \tag{5.102}$$

If we define

$$Q_{\mu\nu} \equiv \Omega^{D-4} \left[2\partial_\mu \partial_\nu \log \Omega - \eta_{\mu\nu} \partial_\alpha \partial^\alpha \log \Omega + \frac{4-D}{2} \eta_{\mu\nu} \partial_\alpha \partial^\alpha \log \Omega \right], \quad (5.103)$$

then the field equations given above are equivalent to

$$\Omega^{\frac{D-4}{2}} \partial_\nu \partial^\nu [\Omega^{\frac{D-4}{2}} A^\mu] - \frac{4-D}{2} Q_{\mu\nu} A^\nu = 0. \quad (5.104)$$

Hence, if we define a Lagrangian

$$\mathcal{L}_{\text{free}} \equiv \mathcal{L}_{\text{EM}} - \frac{4-D}{4} Q_{\mu\nu} A^\mu A^\nu, \quad (5.105)$$

then the Euler-Lagrange equations which we derive from this new Lagrangian are

$$\partial_\nu \partial^\nu \left[\Omega^{\frac{D-4}{2}} A^\mu \right] = 0. \quad (5.106)$$

By regarding the term $Q_{\mu\nu} A^\mu A^\nu$ as an interaction term, we are able to find equations of motion for A_μ which permit us to expand this field in the interaction picture:

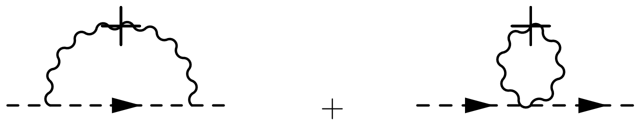
$$A_\mu(x) = \Omega^{\frac{4-D}{2}}(t) \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left[a_\mu(\mathbf{k}) e^{-ik\cdot x} + a_\mu^\dagger(\mathbf{k}) e^{ik\cdot x} \right]. \quad (5.107)$$

The canonical quantisation procedure leads to the standard commutation relations. Observe that the "interaction term" will contribute only in diagrams which are ultraviolet divergent, due to the factor $4 - D = 2\varepsilon$. We shall now show that these diagrams generate a term proportional to $e^2 \hbar Q^\alpha{}_\alpha : \varphi^\dagger \varphi :$ in the one-loop effective Lagrangian.

We let $\hat{Q}_{\mu\nu}(k)$ be the Fourier transform of $Q_{\mu\nu}(x)$, so that

$$Q_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \hat{Q}_{\mu\nu}(k) e^{-ik\cdot x}. \quad (5.108)$$

Then we consider the Feynman diagrams in Fig. 5.2. Using the Feynman rules



(a) (b)

Figure 5.2: The Feynman diagrams contributing to the correction due to the quadratic interaction term. The cross indicates the "interaction term" $-[(4-D)/2]Q_{\mu\nu}A^{\mu}A^{\nu}$.

for scalar electrodynamics in Table 2.1, their sum is

$$\begin{aligned}\hat{\Sigma}(p,k) =& e^2\mu^{4-D}\varepsilon \int \frac{d^D q}{(2\pi)^D i}\left\{\frac{(p^\mu+q^\mu)(p^\nu+k^\nu+q^\nu)\hat{Q}_{\mu\nu}(k)}{(q^2-m^2+i\epsilon)[(p-q)^2+i\epsilon][(p-q+k)^2+i\epsilon]}\right.\\ &\left.-\frac{\eta^{\mu\nu}\hat{Q}_{\mu\nu}(k)}{(q^2+i\epsilon)[(q-k)^2+i\epsilon]}\right\},\end{aligned} \tag{5.109}$$

where we have let $\hbar = 1$. Counting dimensions, we observe that the integrand has a q-dimension of $D-4 \equiv -2\varepsilon$, so it is logarithmically divergent. Hence, to find the pole in ε, we replace all terms of the form $q^2+\ldots$ with $q^2-\lambda^2+i\varepsilon$ for some arbitrary, positive number λ. Consequently, we have

$$\hat{\Sigma}(p,k) = \varepsilon e^2\mu^{2\varepsilon}\int \frac{d^D q}{(2\pi)^D i}\left\{\frac{q^\mu q^\nu \hat{Q}_{\mu\nu}(k)}{(q^2-\lambda^2+i\epsilon)^3} - \frac{\hat{Q}^\alpha{}_\alpha(k)}{(q^2-\lambda^2+i\epsilon)^2}\right\}. \tag{5.110}$$

We Wick-rotate this integral, and then combine the two terms over a single denominator. Applying the dimensional regularisation procedure [31], we obtain

$$\hat{\Sigma}(p,k) = -\frac{3}{64\pi^2}e^2\hat{Q}^\alpha{}_\alpha(k). \tag{5.111}$$

Hence, the contribution to the effective Lagrangian is

$$L_{\text{eff}} = -\frac{3e^2\hbar}{64\pi^2}Q^\alpha{}_\alpha(x) :\varphi^\dagger\varphi:, \tag{5.112}$$

where we have inserted a factor of $\hbar$ by dimensional analysis. Thus, the contribution of the extra mass-like term in Eq. (5.105) to the effective Lagrangian vanishes in the limit $\hbar \to 0$.

Having shown that the additional term is negligible at one-loop to first order in $\hbar$, we may change the electromagnetic Lagrangian from $\mathcal{L}_{\text{EM}}$ to $\mathcal{L}_{\text{free}}$, which leads to the mode expansion given above, and we therefore need only incorporate the factor $\Omega^{(4-D)/2}$ in this mode expansion in order to adapt the calculations in Sec. 5.5.2.2 to this theory. Thus, in Eq. (5.77) the integrand is multiplied by $\Omega^{(4-D)/2}(t)\Omega^{(4-D)/2}(t')$, and in Eq. (5.71) the integrand is multiplied by $\Omega^{4-D}(t)$; these give us

$$\begin{aligned}\mathcal{F}^a(\mathbf{p}) = \frac{ie^2}{4\hbar}\int \frac{d^3\mathbf{q}}{2K(2\pi)^3}\int dt dt'\,\Omega^{(4-D)/2}(t)\Omega^{(4-D)/2}(t') \\ \Big[\theta(t-t')\left(\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}_1}(t,t',\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t)\phi^*_{\mathbf{q}}(t')\right)e^{-iK(t-t')/\hbar} \\ +\theta(t'-t)\left(\phi^*_{\mathbf{p}}(t)\phi_{\mathbf{p}}(t')\overleftrightarrow{\mathcal{D}_1}(t,t',\mathbf{p},\mathbf{q})\phi_{\mathbf{q}}(t')\phi^*_{\mathbf{q}}(t)\right)e^{iK(t-t')/\hbar}\Big], \quad (5.113)\end{aligned}$$

$$\mathcal{F}^b(\mathbf{p}) = \frac{e^2}{\hbar}\int dt\frac{|\varphi_{\mathbf{p}}(t)|^2}{2\sigma_{\mathbf{p}}(t)}\Omega^{4-D}(t)\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{3}{2K}. \quad (5.114)$$

In the earlier section, when considering the equation for $\mathcal{F}^a(\mathbf{p})$, we subsequently undertook a change of variables to $\bar{t} \equiv (t+t')/2$ and $\eta \equiv (t-t')/\hbar$. Applying the same change of variables here turns the additional multiplicative factor into

$$\Omega^{(4-D)/2}(t)\Omega^{(4-D)/2}(t') = \Omega^{4-D}(\bar{t}) + \mathcal{O}(\varepsilon^2\hbar^2\eta^2), \quad (5.115)$$

and since we only have poles of order ε^{-1} at one loop, the latter term does not contribute. Thus in fact the integrands of both $\mathcal{F}^a(\mathbf{p})$ and $\mathcal{F}^b(\mathbf{p})$ are multiplied by $\Omega^{4-D}(t)$ (where we have redefined $\bar{t} = t$ in $\mathcal{F}^a(\mathbf{p})$). Taking the real part of the sum of these terms, we find that

$$\text{Re}\left[\mathcal{F}^a(\mathbf{p}) + \mathcal{F}^b(\mathbf{p})\right] = -\hbar\int\frac{dt}{\sigma_{\mathbf{p}}(t)}\Delta {M_c}^2(t) + \mathcal{O}(\hbar^2). \quad (5.116)$$

We define $\Delta {M_c}^2(t)$ in a similar way to the definition of $\Delta M^2(t)$ in Sec. 5.2.2: we use Eq. (5.67) and replace m^2 with ${M_c}^2(t)$. Note that we must also insert the additional factor of Ω^{4-D} we have just discussed. We combine it with the μ^{4-D} so that this mass renormalisation scale becomes $(\mu\Omega)^{4-D}$. Then we may use an analogy with Eq. (5.62) to conclude that

$$\Delta {M_c}^2(t) = -\frac{3e^2 {M_c}^2(t)}{16\pi^2\hbar}\left(\frac{1}{\varepsilon} - \gamma + \frac{7}{3} - \log\frac{{M_c}^2(t)}{4\pi^2(\mu\Omega)^2}\right). \qquad (5.117)$$

Since ${M_c}^2(t) = m^2\Omega^2 + \hbar^2(\xi - \frac{1}{6})\Omega^2 R$ and we are intending to take the limit $\hbar \to 0$, to order $\hbar^0$ the above equation may be simplified to give

$$\Delta {M_c}^2(t) = -\frac{3e^2}{16\pi^2\hbar} m^2\Omega^2\left(\frac{1}{\varepsilon} - \gamma + \frac{7}{3} - \log\frac{m^2}{4\pi^2\mu^2}\right). \qquad (5.118)$$

Thus, in the limit $\hbar \to 0$, $\Delta {M_c}^2(t) = (\delta m^2)\Omega^2$. As we discussed previously, $\delta M^2(t) = (\delta m^2)M^2(t)/m^2$ for any mass term, and so here, $\delta {M_c}^2(t) = (\delta m^2)\Omega^2$ in the limit $\hbar \to 0$. Hence, the correction to the time-dependent mass $\Delta {M_c}^2(t) - \delta {M_c}^2(t)$, and the corresponding contribution to the position shift $\delta_{\text{loop}} x^i$ in Eq. (5.17), vanish in the limit $\hbar \to 0$.

In conclusion, the equivalence we saw at tree level, between the flat-space theory with a varying mass $M^2(t) = m^2\Omega^2(t)$ and the corresponding conformal theory with a constant mass, breaks down at the one-loop level. This breakdown in equivalence is a manifestation of the conformal anomaly [35], which was first discussed by Capper and Duff [36] as an anomaly regarding the trace of the renormalised stress-energy tensor of the electromagnetic field. As the field is conformally invariant, it was expected that this fact alone should guarantee the invariance of the trace of the dimensionally regularised, electromagnetic stress-energy tensor under conformal transformations. This does not occur. Hence, Fulling and Davies describe the conformal anomaly as a "loss in the renormalized quantum theory of the invariance expected on formal grounds" [37]. Likewise here, we find that the renormalised quantum theory loses an invariance which we would expect on the same, formal grounds.

Chapter 6

Time-dependent metric

Summary. In this chapter, we shall investigate the theory of a minimally-coupled scalar charged particle on a spacetime with a metric which depends only on time. We shall firstly calculate the expected position of the particle at tree level, and argue that a discrepancy between the tree level calculation and the classical calculation implies that the forward scattering terms will make a contribution at order $\hbar^0$ which covers the difference. We shall then show that up to the one-loop level, the quantum vacuum current exactly cancels the classical vacuum current and hence Maxwell's equations are valid without corrections.

6.1 Scalar QED on a curved spacetime

A complex scalar field propagating on a curved spacetime, subject to a background field V^μ and coupled to the electromagnetic field, is governed by the Lagrangian given in Eq. (5.6) which we modify below by supplying the gauge-fixing term $-\frac{1}{2}(\nabla_\mu A^\mu)^2$, so that we have

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + g^{\mu\nu}\left[\left(D_\mu + \frac{ie}{\hbar}A_\mu\right)\phi\right]^\dagger\left[\left(D_\nu + \frac{ie}{\hbar}A_\nu\right)\phi\right]\right.$$
$$\left. - \left[\left(\frac{m}{\hbar}\right)^2 - \xi R\right]\phi^\dagger\phi - \frac{1}{2}(\nabla_\mu A^\mu)^2\right),$$

where $D_\mu\phi = \partial_\mu\phi + iV_\mu(t)\phi/\hbar$. We shall be using the minimally-coupled theory, and hence set $\xi = 0$.

As conditions on the background field, we shall specify that it varies with time and only between $t = -T$ and $t = 0$. As previously, we use a gauge freedom to set $V^0(t) = 0$ for all t. We do not suppose that $V^\mu(-T-\delta)$ and $V^\mu(\delta)$ are equal for positive δ.

For the metric, we shall require that the metric for our spacetime is time-dependent, $g_{\mu\nu}(t)$, and further, we shall require $g_{\mu\nu}(t) = \eta_{\mu\nu}$ for $t < -T$ and $t > 0$. Hence, the region of spacetime which is curved is bounded by space-like hypersurfaces at constant time. We choose a foliation of this spacetime in which $g_{0i} = g_{i0} = 0$. In principle, we could also choose $g_{00} = 1$, but as we expect the tree-level result to be conformally invariant, it is convenient to allow g_{00} to vary from this. We shall define $\Omega^2(t) \equiv g^{00}(t)$. Hence, we write the metric as

$$(g_{\mu\nu}) = \begin{pmatrix} \Omega^{-2} & \mathbf{0}^T \\ \mathbf{0} & -g_{ij} \end{pmatrix}; \tag{6.1}$$

the reader will note that we have chosen the sign of the spatial part of the metric so that its determinant is positive, and that we are using Ω^2 for g^{00}, rather than the conformal factor connecting $g_{\mu\nu}$ with $\eta_{\mu\nu}$, as in Ch. 5.

This metric is not physical, in the sense that it is highly unlikely that a metric of this form can be found which satisfies Einstein's equations. However, it serves as a useful testing ground to see whether the classical and quantum results do indeed match on metrics which are not flat. Moreover, as Martin showed [25], the flat space-time results match for both time-dependent and space-dependent potentials: we have shown in Ch. 4 that the way the potential varies makes a difference at first order in $\hbar$. There may then be an analogy with the case we consider here, so that if it is possible to use an (unphysical) time-dependent metric to simulate a (physical) space-dependent metric, then our result may extend to such 'simulable' metrics too.

We shall proceed to quantise the scalar and vector fields. In order to consider the mode expansion, with the associated creation and annihilation operators, we must select an appropriate vacuum state as a reference. Since the spacetime is flat except in the intermediate region, there exist well-defined vacua in each of the flat regions. However, the intervening time-dependence of the metric means that we cannot suppose that the two vacua are identical. We therefore shall select the vacuum state in the region $t > 0$ as the one which we shall use to define our annihilation and creation operators.

Having made this selection, we then quantise the field $\phi(x)$ by imposing the flat-space commutation relations, Eq. (2.40),

$$[\phi(t, \mathbf{x}), \pi_\phi(t, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}'), \tag{6.2}$$

in the region $t > 0$. Then the standard commutation relations for the creation and annihilation operators are obtained in the usual manner in this region. Since the fields satisfy the equations of motion everywhere, the commutation relations are preserved in the curved region of spacetime and also the flat region with $t < -T$. Hence, we may expand the field ϕ as

$$\phi(x) \equiv \hbar \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left[A(\mathbf{p})\Phi_{\mathbf{p}}(x) + B^\dagger(\mathbf{p})\overline{\Phi}^*_{\mathbf{p}}(x)\right], \tag{6.3}$$

where $p_0 = \sqrt{\|\mathbf{p}\|^2 + m^2}$ and the non-zero commutation relations are

$$\left[A(\mathbf{p}), A^\dagger(\mathbf{p}')\right] = \left[B(\mathbf{p}), B^\dagger(\mathbf{p}')\right] = 2p_0(2\pi\hbar)^3\delta^3(\mathbf{p} - \mathbf{p}'). \tag{6.4}$$

We carry out a similar procedure for the vector field $A_\mu(x)$, imposing the equal-time commutation relations in the region $t > 0$ and propagating them backwards using the fundamental solutions to the equations of motion. However, whereas the scalar field appeared formally identical in its expansion to the flat spacetime case, this vector field does exhibit a degree of difference. We impose a tetrad-like construction on the mode functions, so that the field operators are indexed by frame indices, $(\mathsf{a}, \mathsf{b}, \ldots)$, rather than the spacetime indices, $(\mu, \nu, \ldots)$. Then

$$A_\mu(x) \equiv \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left[\varepsilon^{\mathsf{a}}_\mu(\mathbf{k}, x) a_{\mathsf{a}}(\mathbf{k}) + \overline{\varepsilon^{\mathsf{a}}_\mu(\mathbf{k}, x)} a^\dagger_{\mathsf{a}}(\mathbf{k}) \right], \tag{6.5}$$

where $k = \|\mathbf{k}\|$ and $\varepsilon^{\mathsf{a}}_\mu(\mathbf{k}, x) = \delta^{\mathsf{a}}_\mu$ for $t > 0$. The equation of motion for the free vector field in curved spacetime is

$$\Box A^\mu - R^\mu{}_\nu A^\nu = 0. \tag{6.6}$$

Since the metric is dependent on time only, the field three-momentum is conserved. Hence, we may write

$$\varepsilon^{\mathsf{a}}_\mu(\mathbf{k}, x) = \epsilon^{\mathsf{a}}_\mu(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{6.7}$$

The non-zero commutation relations are then

$$\left[a_{\mathsf{a}}(\mathbf{k}), a^\dagger_{\mathsf{a}'}(\mathbf{k}') \right] = -2\hbar k (2\pi)^3 \eta_{\mathsf{a}\mathsf{a}'} \delta^3(\mathbf{k} - \mathbf{k}'), \tag{6.8}$$

which correspond to the flat spacetime commutation relations.

The equation of motion which governs the modes of the scalar field may be derived from the non-interacting Lagrangian, which is to say, the full Lagrangian with the interaction Lagrangian,

$$\mathcal{L}_\mathrm{I} = -\frac{ie}{\hbar} \sqrt{-g} g^{\mu\nu} A_\mu \left(\phi^\dagger \overleftrightarrow{D_\nu} \phi \right), \tag{6.9}$$

subtracted from it, where we have defined $\overleftrightarrow{D_\nu} \equiv \overrightarrow{D_\nu} - \overleftarrow{D_\nu}^\dagger$. The Euler-Lagrange equation derived from the non-interacting Lagrangian is then

$$\frac{1}{\sqrt{-g}} D_\alpha \left(\sqrt{-g} g^{\alpha\beta} D_\beta \Phi_{\mathbf{p}}(x) \right) + \left(\frac{m}{\hbar} \right)^2 \Phi_{\mathbf{p}}(x) = 0. \tag{6.10}$$

On a general curved spacetime, this equation cannot be solved in closed form. Since we have specified that the metric only varies from the flat metric in a bounded interval of time, the mode solutions in the flat-spacetime regions are the familiar, plane-wave, constant momentum modes, although the intervening variation means that an initial state with a given four-momentum may not evolve into a final state with that same four-momentum. As with the vector field, the scalar field has a conserved three-momentum. Therefore, we may suppose that the mode solutions are separable in the four spacetime coordinates, with the spatial parts of the mode solutions forming the usual plane-wave solutions. Thus each such mode solution may be written

$$\Phi_{\mathbf{p}}(x) = \sqrt{p_0} \phi_{\mathbf{p}}(t) e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{x}}, \tag{6.11}$$

where ${p_0}^2 = \|\mathbf{p}\|^2 + m^2$. It is important to note that the constant momentum is the *co*variant vector p_i, not the *contra*variant vector. The equation of motion for $\phi_{\mathbf{p}}(t)$ is then

$$\hbar^2 \Omega^2 \partial_0 \partial_0 \phi_{\mathbf{p}}(t) + \hbar^2 \Omega^2 \left(\partial_0 \log \sqrt{-g} \Omega(t)^2 \right) \partial_0 \phi_{\mathbf{p}}(t) + \Sigma(t) \phi_{\mathbf{p}}(t) = 0, \tag{6.12}$$

where $\Sigma(t) \equiv {\sigma_{\mathbf{p}}}^2(t) = g^{ij}(p_i - V_i)(p_j - V_j) + m^2$. Thus, $p_0 = \sigma_{\mathbf{p}}(0)$.

The anti-particle mode is treated similarly, with

$$\overline{\Phi}_{\mathbf{p}}(x) = \sqrt{p_0}\, \overline{\phi}_{\mathbf{p}}(t) e^{\frac{i}{\hbar} \mathbf{p} \cdot \mathbf{x}}. \tag{6.13}$$

Then the equation for $\overline{\phi}_{\mathbf{p}}(t)$ is like the one for $\phi_{\mathbf{p}}(t)$, and results in the relation

$$\overline{\phi}_{\mathbf{p}}(t) = \phi_{-\mathbf{p}}(t). \tag{6.14}$$

Therefore, we may consider the particle mode, and use the above relation to transform it into the anti-particle mode. We apply the WKB technique to this mode, seeking the solutions to order $\hbar^0$ and expanding the mode as a series in powers of $\hbar$: we let

$$\phi_{\mathbf{p}}(t) = \exp\left(\frac{1}{\hbar}\sum_{n=0}^{\infty}\hbar^n s_n(t)\right), \tag{6.15}$$

and thus produce from the equation of motion,

$$\hbar\sum_{n=0}^{\infty}\hbar^n \ddot{s}_n + \left(\sum_{n=0}^{\infty}\hbar^n \dot{s}_n\right)^2 + \hbar\partial_0 \log\left(\sqrt{-g}\Omega^2\right)\sum_{n=0}^{\infty}\hbar^n \dot{s}_n + \frac{\Sigma}{\Omega^2} = 0. \tag{6.16}$$

Since all functions are now functions of t only, we have suppressed their arguments. Considering the order $\hbar^0$ term in the above equation, we find $\dot{s}_0 = \pm i\sigma_{\mathbf{p}}/\Omega$; we shall choose the negative sign since $\Phi_{\mathbf{p}}(x) \propto e^{-ip_0 t/\hbar}$ in the future region $t \to \infty$. Hence, $s_0(t) = -i\int^t \sigma_{\mathbf{p}}(t')/\Omega(t')dt'$.

The equation at order $\hbar$ is

$$\ddot{s}_0 + 2\dot{s}_0\dot{s}_1 + \dot{s}_0\partial_0 \log\left(\sqrt{-g}\Omega^2\right) = 0, \tag{6.17}$$

which gives us

$$s_1 = -\frac{1}{2}\log\left(|\dot{s}_0|\sqrt{-g}\Omega^2\right) = \log\frac{1}{\sqrt{\sigma_{\mathbf{p}}\sqrt{-g}\Omega}}. \tag{6.18}$$

Our solution up to order $\hbar^0$ is therefore

$$\phi_{\mathbf{p}}(t) = \frac{1}{\sqrt{\sigma_{\mathbf{p}}\sqrt{-g}\Omega}}\exp\left\{-\frac{i}{\hbar}\int_0^t \frac{\sigma_{\mathbf{p}}(t')}{\Omega(t')}dt'\right\}\varphi_{\mathbf{p}}(t), \tag{6.19}$$

where $\varphi_{\mathbf{p}}(t) \equiv \exp\left(\sum_{n=2}^{\infty}\hbar^{n-1}s_n\right)$ contains all the terms of higher order in $\hbar$, and where we have chosen to use initial conditions at $t = 0$ for the integral.

The reader will note that the equations of motion preserve positive- and negative-frequency modes, and consequently we do not observe particle creation at any order in $\hbar$.

6.2 Position shift

Higuchi and Martin [23, 25] expressed the calculation showing equality between the quantum and classical predictions for the change in position due to the Lorentz-Dirac force in terms of Green's functions derived from the electromagnetic field, set on a flat spacetime. In this section, we show that the position shift (relative to the expected position in the absence of radiation reaction) may be separated into two parts, coming from the tree and one-loop levels. This is largely a review of that earlier work, generalised to the class of spacetimes we are considering in this chapter.

To find the position shift predicted in the quantum theory, we must define the initial and final states and compare the positions which they predict. As in the flat spacetime case, we set the initial state, in the first flat spacetime region, to be

$$|i\rangle = \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} f(\mathbf{p}) A^\dagger(\mathbf{p}) |0\rangle . \tag{6.20}$$

Here, the momentum $\mathbf{p}$ over which we are integrating is what would be the final momentum in the absence of the radiation-reaction force. As usual, the function f is sharply peaked about a given central momentum, and the condition $\langle i|i\rangle = 1$ implies that

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 = 1. \tag{6.21}$$

The evolution of the state $|\mathbf{p}\rangle$ up to order e^2 is then

$$|\mathbf{p}\rangle \mapsto \left[1 + \frac{i}{\hbar}\mathcal{F}(\mathbf{p})\right] |\mathbf{p}\rangle + \frac{i}{\hbar}\int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \eta^{\mathsf{ab}} \mathcal{A}_{\mathsf{a}}(\mathbf{p}, \mathbf{k}) a^\dagger_{\mathsf{b}}(\mathbf{k}) |\mathbf{P}\rangle , \tag{6.22}$$

where $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$ by momentum conservation. As the evolution operator is unitary, the final state is normalised as well, so that $\langle f|f\rangle = 1$. We shall consider the implications of this subsequently.

To commence, we consider the expectation value for the particle were the background potential and variation in the metric to be removed. We consider this expectation value at time $t = 0$. Then we would expect to find the scalar particle at

$$\left\langle x^i \right\rangle_{\text{in}} = \int d^3\mathbf{x}\sqrt{-g}x^i\mathcal{P}_i(x), \tag{6.23}$$

where $\mathcal{P}_\omega(x)$ is the probability density for the excitations of the scalar field in the state $|\omega\rangle$. Since there is only one scalar particle in the final state, this density is equal to the expectation value of the charge density, and so

$$\left\langle x^i \right\rangle_{\text{in}} = \int d^3\mathbf{x}\sqrt{-g}x^i \left\langle i\right| J^0 \left|i\right\rangle . \tag{6.24}$$

Substituting our definitions of the initial state and the charge density, we find

$$\begin{aligned}\left\langle x^i \right\rangle_{\text{in}} &= i\hbar g^{00}\int d^3\mathbf{x}\sqrt{-g}x^i \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \\ &\quad\times f^*(\mathbf{p})\left(\left(\partial_0\Phi_{\mathbf{p}'}\right)\Phi^*_{\mathbf{p}} - \Phi_{\mathbf{p}'}\left(\partial_0\Phi^*_{\mathbf{p}}\right)\right) f(\mathbf{p}'). \qquad (6.25)\\ &= i\hbar g^{00}\int d^3\mathbf{x}\sqrt{-g}x^i \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \\ &\quad\times f^*(\mathbf{p})f(\mathbf{p}')\partial_0\log\left(\frac{\Phi_{\mathbf{p}'}}{\Phi^*_{\mathbf{p}}}\right)\Phi_{\mathbf{p}'}\Phi^*_{\mathbf{p}}. \qquad (6.26)\end{aligned}$$

Using our approximate solution for $\phi_{\mathbf{p}}(t)$, Eq. (6.19), we find

$$\begin{aligned}\left\langle x^i \right\rangle_{\text{in}} &= i\hbar g^{00}\int d^3\mathbf{x}\sqrt{-g}x^i \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \\ &\quad\times f^*(\mathbf{p})f(\mathbf{p}')\left[\frac{1}{2}\partial_0\log\left(\frac{\sigma_{\mathbf{p}}}{\sigma_{\mathbf{p}'}}\right) - \frac{i}{\hbar}\frac{\sigma_{\mathbf{p}'}+\sigma_{\mathbf{p}}}{\Omega}\right] \qquad (6.27)\\ &\quad\times \frac{1}{\Omega^2\sqrt{-g}}\sqrt{\frac{p_0p_0'}{\sigma_{\mathbf{p}}\sigma_{\mathbf{p}'}}}\exp\left\{\frac{i}{\hbar}\int_0^t\frac{\sigma_{\mathbf{p}'}(t')-\sigma_{\mathbf{p}}(t')}{\Omega(t')}dt'\right\}e^{-\frac{i}{\hbar}(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}. \qquad (6.28)\end{aligned}$$

As in Sec. 4.1, we may discard the logarithm since we are only concerned with the term of lowest order in $\hbar$. It is convenient, having now removed all terms with time derivatives, to make use of the fact that we are considering the expectation value at time $t = 0$, which gives $\Omega = 1$ and $\sigma_{\mathbf{p}} = p_0$. In order to carry through the integral over $\mathbf{x}$, we use $x^i = -i\hbar\partial_{p_i} \exp(i\mathbf{p}\cdot\mathbf{x}/\hbar)$ and obtain

$$\left\langle x^i \right\rangle_{\text{in}} = \frac{1}{2}\int d^3\mathbf{x}\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{(2\pi\hbar)^3} \times f^*(\mathbf{p})f(\mathbf{p}')\frac{p_0+p_0'}{\sqrt{p_0p_0'}}\frac{i\hbar}{2}(\partial_{p_i}-\partial_{p_i'})e^{\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}. \tag{6.29}$$

We integrate by parts to move the momentum derivatives from the exponential to the remainder of the integrand, and then integrate with respect to $\mathbf{x}$ and obtain a delta function, $(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')$. Integrating this delta function over $\mathbf{p}'$ then gives us

$$\left\langle x^i \right\rangle_{\text{in}} = \frac{i\hbar}{2}\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} f^*(\mathbf{p})\overleftrightarrow{\partial_{p_i}}f(\mathbf{p}), \tag{6.30}$$

where we define $\overleftrightarrow{\partial_{p_i}} \equiv \overrightarrow{\partial_{p_i}} - \overleftarrow{\partial_{p_i}}$.

We now consider the position expectation value of the particle in the presence of the variations in the metric and the background field, in order to compare this with the initial state position. The final expectation value will be

$$\left\langle x^i \right\rangle_{\text{fin}} = \int d^3\mathbf{x}\sqrt{-g}x^i \left\langle f\right| J^0 \left|f\right\rangle, \tag{6.31}$$

but we note that the state $|f\rangle$ may be split into two sectors: a sector with no electromagnetic excitations, and a sector with one electromagnetic excitation. Considering first the sector with no photons, we have

$$\begin{aligned}\left\langle x^i\right\rangle_{\text{fin},0} = \int d^3\mathbf{x}\sqrt{-g}x^i \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3}\left\langle 0\right| A(\mathbf{p})\left[1-\frac{i}{\hbar}\mathcal{F}^*(\mathbf{p})\right] \\ \times f^*(\mathbf{p})g^{00}\frac{i}{2\hbar}\left(\{\partial_0\phi,\phi^*\}-\{\partial_0\phi,\phi^*\}^\dagger\right)\left[1+\frac{i}{\hbar}\mathcal{F}(\mathbf{p}')\right]f(\mathbf{p}')A^\dagger(\mathbf{p}')\left|0\right\rangle.\end{aligned} \tag{6.32}$$

If we replace $f(\mathbf{p})\left[1+\frac{i}{\hbar}\mathcal{F}(\mathbf{p})\right]$ with $F(\mathbf{p})$, then we may follow the line of argument above for $\langle x^i\rangle_{\text{in}}$, *mutatis mutandis*. This leads us to conclude that

$$\begin{aligned}\left\langle x^i\right\rangle_{\text{fin},0} = \frac{i}{2}\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left(f^*(\mathbf{p})\overleftrightarrow{\partial}_{p_i}f(\mathbf{p})\right)(\hbar-2\,\mathrm{Im}\,\mathcal{F}(\mathbf{p})) \\ -\int\frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\left|f(\mathbf{p})\right|^2\partial_{p_i}\,\mathrm{Re}\,\mathcal{F}(\mathbf{p}),\end{aligned} \tag{6.33}$$

since $\mathcal{F}(\mathbf{p})$ is intrinsically of order e^2.

We must now consider the position expectation value arising from the one-photon sector of the final state. This is

$$\begin{aligned}\left\langle x^i\right\rangle_{\text{fin},1} = \int d^3\mathbf{x}\sqrt{-g}x^i\int\frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3}\frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3}\frac{d^3\mathbf{k}}{2k(2\pi)^3}\frac{d^3\mathbf{k}'}{2k'(2\pi)^3} \\ \times\left\langle 0\right| a_{\mathsf{m}}(\mathbf{k})\mathcal{A}^*_{\mathsf{n}}(\mathbf{p},\mathbf{k})\eta^{\mathsf{mn}}\left(-\frac{i}{\hbar}\right)A(\mathbf{P})f^*(\mathbf{p})g^{00}\frac{i}{2\hbar} \\ \times\left(\{\partial_0\phi,\phi^*\}-\{\partial_0\phi,\phi^*\}^\dagger\right)f(\mathbf{p}')A^\dagger(\mathbf{P}')\frac{i}{\hbar}\eta^{\mathsf{ab}}\mathcal{A}_{\mathsf{a}}(\mathbf{p}',\mathbf{k}')a^\dagger_{\mathsf{b}}(\mathbf{k}')\left|0\right\rangle,\end{aligned} \tag{6.34}$$

where $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$ and likewise for $\mathbf{P}'$. We apply the commutation relation for the electromagnetic field operators, the effect of which is to produce a factor of $\hbar$, to set $\mathbf{k}' = \mathbf{k}$, and to contract the metrics so that we have η^{ma}. Applying the commutation relations for the scalar field as well, we thus find

$$\langle x^i \rangle_{\text{fin},1} = ig^{00}\eta^{\mathsf{aa}'} \int d^3\mathbf{x}\sqrt{-g}x^i \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{2k(2\pi)^3} \frac{d^3\mathbf{k}'}{2k'(2\pi)^3} \times \mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) f^*(\mathbf{p}) \left(\partial_0 \log\left(\frac{\Phi_{\mathbf{P}'}}{\Phi^*_{\mathbf{P}}}\right)\right) \Phi_{\mathbf{P}'}\Phi^*_{\mathbf{P}} f(\mathbf{p}')\mathcal{A}_{\mathsf{a}'}(\mathbf{p}',\mathbf{k}'). \quad (6.35)$$

Then we find that, if we define

$$C_{\mathsf{a}}(\mathbf{p}) \equiv f(\mathbf{p})\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k}), \quad (6.36)$$

we have

$$\langle x^i \rangle_{\text{fin},1} = ig^{00} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \int d^3\mathbf{x}\sqrt{-g}x^i \int \frac{d^3\mathbf{p}}{\sqrt{2p_0}(2\pi\hbar)^3} \frac{d^3\mathbf{p}'}{\sqrt{2p_0'}(2\pi\hbar)^3} \times \eta^{\mathsf{aa}'} C^\dagger_{\mathsf{a}}(\mathbf{p}) C_{\mathsf{a}'}(\mathbf{p}') \left(\partial_0 \log\left(\frac{\Phi_{\mathbf{P}'}}{\Phi^*_{\mathbf{P}}}\right)\right) \Phi_{\mathbf{P}'}\Phi^*_{\mathbf{P}}. \quad (6.37)$$

Clearly, this is very similar to previous expressions with the momenta $\mathbf{p}$ replaced with $\mathbf{P}$. However, the relationship between $\mathbf{P}$ and $\mathbf{p}$ implies that $\Phi_{\mathbf{P}'}\Phi^*_{\mathbf{P}}|_{t=0} = \sqrt{P_0P_0'}e^{-\frac{i}{\hbar}(P_0'-P_0)t}e^{\frac{i}{\hbar}(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}$. As previously, the x^i factor will generate $\frac{i}{\hbar}(\partial_{p_i} - \partial_{p_i'})$. Since the factor $\sqrt{P_0P_0'/(p_0p_0')}$ is symmetric between p and p', when we integrate by parts, the term arrived at by applying the derivative to this factor vanishes as $p \to p'$. Further, we shall subsequently show that $\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k})$ is of order $\hbar^0$, and we know that $\hbar f^* \overleftrightarrow{\partial_{p_i}} f$ is of order $\hbar^0$ as upon integration it will give us $\langle x \rangle_{\text{in}}$. Hence, we have

$$\langle x^i \rangle_{\text{fin},1} = -\frac{i}{2}\eta^{\mathsf{aa}'} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{P_0}{p_0} \left(C^\dagger_{\mathsf{a}}(\mathbf{p}) \overleftrightarrow{\partial_{p_i}} C_{\mathsf{a}'}(\mathbf{p}')\right), \quad (6.38)$$

which we expand to find

$$\langle x^i \rangle_{\text{fin},1} = -\frac{i}{2} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{P_0}{p_0} \times \left\{ - |\mathcal{A}(\mathbf{p},\mathbf{k})|^2 \left[f^*(\mathbf{p}) \overleftrightarrow{\partial_{p_i}} f(\mathbf{p}) \right] + \eta^{\mathsf{aa'}} |f(\mathbf{p})|^2 \left[\mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial_{p_i}} \mathcal{A}_{\mathsf{a'}}(\mathbf{p},\mathbf{k}) \right] \right\}. \tag{6.39}$$

We now consider the unitarity condition $\langle f|f\rangle = 1$. Since there is no cross-term between the zero-photon and one-photon sectors,

$$\begin{aligned}\langle f|f\rangle = &\int \frac{d^3\mathbf{p}}{\sqrt{p_0}(2\pi\hbar)^3} \frac{d^3\mathbf{p}'}{\sqrt{p_0'}(2\pi\hbar)^3} \\ &\times \langle 0| A(\mathbf{p}) \left[1 - \frac{i}{\hbar}\mathcal{F}^*(\mathbf{p}) \right] f^*(\mathbf{p}) f(\mathbf{p}') \left[1 + \frac{i}{\hbar}\mathcal{F}(\mathbf{p}) \right] A^\dagger(\mathbf{p}') |0\rangle \\ &+ \frac{1}{\hbar^2} \int \frac{d^3\mathbf{p}}{\sqrt{p_0}(2\pi\hbar)^3} \frac{d^3\mathbf{p}'}{\sqrt{p_0'}(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{2k(2\pi)^3} \frac{d^3\mathbf{k}'}{2k'(2\pi)^3} \\ &\times \langle 0| f^*(\mathbf{p}) \eta^{\mathsf{ab}} \mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) a_{\mathsf{b}}(\mathbf{k}) A(\mathbf{P}) A^\dagger(\mathbf{P}') \eta^{\mathsf{mn}} a^\dagger_{\mathsf{m}}(\mathbf{k}') \mathcal{A}_{\mathsf{n}}(\mathbf{p}',\mathbf{k}') f(\mathbf{p}') |0\rangle .\end{aligned} \tag{6.40}$$

If we apply the commutation relations to these, then we find

$$\begin{aligned}1 = &\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |f(\mathbf{p})|^2 \left(1 + \frac{2}{\hbar} \operatorname{Im} \mathcal{F}(\mathbf{p}) \right) \\ &- \frac{1}{\hbar} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{d^3\mathbf{k}}{2k(2\pi)^3} \frac{P_0}{p_0} |f(\mathbf{p})|^2 |\mathcal{A}(\mathbf{p},\mathbf{k})|^2 ,\end{aligned} \tag{6.41}$$

where we have dropped the $|\mathcal{F}(\mathbf{p})|^2$ term as $\mathcal{F}(\mathbf{p})$ is of order e^2. Then the condition $\langle i|i\rangle = 1$ can be applied to the first integral to cancel a portion of it. Our conditions on $f(\mathbf{p})$ are consistent with taking the limit $|f(\mathbf{p})|^2 \to (2\pi\hbar)^3\delta^3(\mathbf{p} - \mathbf{p}')$, where $\mathbf{p}'$ is the central momentum about which $f(\mathbf{p})$ is peaked, so that we find

$$\int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \frac{P_0}{p_0} |\mathcal{A}(\mathbf{p},\mathbf{k})|^2 = 2 \operatorname{Im} \mathcal{F}(\mathbf{p}). \tag{6.42}$$

Then if we combine $\langle x^i \rangle_{\text{fin},0} + \langle x^i \rangle_{\text{fin},1}$, we find

$$\begin{aligned}\left\langle x^i \right\rangle_{\text{fin}} = \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left(f^*(\mathbf{p}) \overleftrightarrow{\partial_{p_i}} f(\mathbf{p}) \right) \left(1 - \frac{2}{\hbar} \operatorname{Im} \mathcal{F}(\mathbf{p}) \right) \\ - \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left| f(\mathbf{p}) \right|^2 \partial_{p_i} \operatorname{Re} \mathcal{F}(\mathbf{p}) \\ + \frac{i}{2} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{P_0}{p_0} \left| \mathcal{A}(\mathbf{p},\mathbf{k}) \right|^2 \left[f^*(\mathbf{p}) \overleftrightarrow{\partial_{p_i}} f(\mathbf{p}) \right] \\ - \frac{i}{2} \eta^{\mathsf{a}\mathsf{a}'} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{P_0}{p_0} \left| f(\mathbf{p}) \right|^2 \left[\mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial_{p_i}} \mathcal{A}_{\mathsf{a}'}(\mathbf{p},\mathbf{k}) \right],\end{aligned} \tag{6.43}$$

and can see that our consideration of the normalisation conditions $\langle f|f \rangle = 1$ implies that two of the terms will cancel. Noting that we also have a term of the form $\langle x^i \rangle_{\text{in}}$, we conclude

$$\begin{aligned}\left\langle x^i \right\rangle_{\text{fin}} = \left\langle x^i \right\rangle_{\text{in}} - \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left| f(\mathbf{p}) \right|^2 \partial_{p_i} \operatorname{Re} \mathcal{F}(\mathbf{p}) \\ - \frac{i}{2} \eta^{\mathsf{a}\mathsf{a}'} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{P_0}{p_0} \left| f(\mathbf{p}) \right|^2 \left[\mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial_{p_i}} \mathcal{A}_{\mathsf{a}'}(\mathbf{p},\mathbf{k}) \right].\end{aligned} \tag{6.44}$$

Hence, we define the positions shifts

$$\delta x^i_{\text{tree}} = -\frac{i}{2} \eta^{\mathsf{a}\mathsf{a}'} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{P_0}{p_0} \left| f(\mathbf{p}) \right|^2 \left[\mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) \overleftrightarrow{\partial_{p_i}} \mathcal{A}_{\mathsf{a}'}(\mathbf{p},\mathbf{k}) \right], \tag{6.45}$$

$$\delta x^i_{\text{loop}} = - \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left| f(\mathbf{p}) \right|^2 \partial_{p_i} \operatorname{Re} \mathcal{F}(\mathbf{p}), \tag{6.46}$$

so that

$$\left\langle x^i \right\rangle_{\text{fin}} = \left\langle x^i \right\rangle_{\text{in}} + \delta x^i_{\text{tree}} + \delta x^i_{\text{loop}}. \tag{6.47}$$

6.3 Tree-level position shift

In Ch. 2, we sketched a derivation of the tree-level position shift due to radiation reaction on a flat spacetime. In this section, we calculate the tree-level position shift due to radiation reaction for the class of curved spacetimes we are considering. As in the introductory derivation, our presentation shall follow the contours of the one found in Martin [25].

In order to analyse the tree-level position shift derived above, we must consider the emission amplitude $\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k})$. We may calculate this from the time-dependent evolution of a state as follows. The evolution of the initial state into the final state is described, to the order relevant to our purposes here, by

$$\begin{aligned}\langle\omega| T \left\{\exp\left(-\frac{i}{\hbar}\int d^4x\mathcal{H}_I(x)\right)\right\}_{\text{connected}} |\mathbf{p}\rangle \\ = \left[1+\frac{i}{\hbar}\mathcal{F}(\mathbf{p})\right]\langle\omega|\mathbf{p}\rangle + \frac{i}{\hbar}\int\frac{d^3\mathbf{k}}{2k(2\pi)^3}\eta^{\mathsf{ab}}\langle\omega|\,\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k})a^\dagger_{\mathsf{b}}(\mathbf{k})\,|\mathbf{P}\rangle\,,\end{aligned} \tag{6.48}$$

where T denotes the time-ordered product. We shall omit vacuum polarisation terms, as previously. Hence, we extract the emission amplitude from Eq. (6.22) by taking the inner product with $\langle\omega| = \langle\mathbf{p}'|\,a_{\mathsf{m}}(\mathbf{k}')$ and integrating over $\mathbf{p}'$ with an appropriate measure, so that to order e we have

$$\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k}) = -i\int\frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}\langle\mathbf{p}'|\,a_{\mathsf{a}}(\mathbf{k})T\left\{\exp\left(-\frac{i}{\hbar}\int d^4x\mathcal{H}_I(x)\right)\right\}_{\text{connected}}|\mathbf{p}\rangle\,. \tag{6.49}$$

We find the interacting Hamiltonian density by deriving the full Hamiltonian from the Lagrangian as follows. Since we are interested in the interaction between the scalar and vector fields we shall exclude all terms which do not contain the scalar field. Then the Lagrangian for our current purposes becomes

$$\mathcal{L}_{\text{scal}} = \sqrt{-g}\left(g^{\mu\nu}\left(\mathcal{D}_\mu\phi\right)^\dagger\left(\mathcal{D}_\nu\phi\right) - m^2\phi^\dagger\phi\right), \tag{6.50}$$

where $\mathcal{D}_\mu\phi = D_\mu\phi + ieA_\mu\phi/\hbar$. The Hamiltonian $\mathcal{H}$ which arises from a Lagrangian $\mathcal{L}(\phi_j, \dot{\phi}_j)$ is defined by the equation

$$\mathcal{H} \equiv \sum_i \dot{\phi}_i \pi_{\phi_i} - \mathcal{L}, \tag{6.51}$$

where $\dot{\phi}_i = \partial_0\phi_i$ and $\pi_{\phi_i} = \partial\mathcal{L}/\partial\dot{\phi}_i$. In our case, these conjugate field momenta are

$$\pi_\phi = \sqrt{-g}g^{00}\left(\mathcal{D}_0\phi\right)^\dagger, \tag{6.52}$$
$$\pi_{\phi^\dagger} = \sqrt{-g}g^{00}\left(\mathcal{D}_0\phi\right), \tag{6.53}$$

and hence we derive the Hamiltonian

$$\mathcal{H}(x) = \mathcal{H}_0 + \mathcal{H}_I \tag{6.54}$$

where

$$\begin{aligned}\mathcal{H}_0 = {} & \frac{1}{\sqrt{-g}}g_{00}\pi_\phi\pi_{\phi^\dagger} + \frac{i}{\hbar}V_0\left(\phi^\dagger\pi_{\phi^\dagger} - \phi\pi_\phi\right) \\ & - \sqrt{-g}g^{ij}\left(D_i\phi\right)^\dagger\left(D_j\phi\right) + \sqrt{-g}m^2\phi^\dagger\phi, \end{aligned} \tag{6.55}$$
$$\mathcal{H}_I = \frac{ie}{\hbar}A_0\left(\phi^\dagger\pi_{\phi^\dagger} - \phi\pi_\phi\right) - \frac{ie}{\hbar}\sqrt{-g}g^{ij}A_i\left[\phi^\dagger\overleftrightarrow{D_j}\phi\right] - \frac{e^2}{\hbar^2}\sqrt{-g}g^{ij}A_iA_j\phi^\dagger\phi. \tag{6.56}$$

The first line of the above we define to be the free Hamiltonian, $\mathcal{H}_0(x)$, and the second line, the interacting Hamiltonian, $\mathcal{H}_I(x)$. Then the Hamilton's equations we derive from $H_0 = \int d^3\mathbf{x}\mathcal{H}_0$ are

$$\dot{\phi} = \frac{\delta H_0}{\delta\pi_\phi} = \frac{1}{\sqrt{-g}}g_{00}\pi_{\phi^\dagger} - \frac{i}{\hbar}V_0\phi, \tag{6.57}$$
$$\dot{\pi}_{\phi^\dagger} = -\frac{\delta H_0}{\delta\phi^\dagger} = -\frac{i}{\hbar}V_0\pi_{\phi^\dagger} - \frac{i}{\hbar}\sqrt{-g}g^{ij}V_i\left(D_j\phi\right) - \sqrt{-g}m^2\phi, \tag{6.58}$$

and their conjugates. These equations imply that

$$\pi_{\phi^\dagger} = \sqrt{-g} g^{00} \left(D_0 \phi \right) \tag{6.59}$$

and the conjugate, which allows us to write the interacting Hamiltonian as

$$\mathcal{H}_I(x) = \sqrt{-g} \left(\frac{ie}{\hbar} g^{\mu\nu} A_\mu \left[\phi^\dagger \overleftrightarrow{D_\nu} \phi \right] - \frac{e^2}{\hbar^2} g^{ij} A_i A_j \phi^\dagger \phi \right). \tag{6.60}$$

Returning to Eq. (6.49), we note that we shall be working up to order e^2, which implies that we are only interested in the terms from the exponential with up to two factors of the vector field. However, the additional field operator applied to the exponential eliminates any even multiple of the vector field, so that the only term remaining is

$$\mathcal{A}_{\mathsf{a}}(\mathbf{p}, \mathbf{k}) = -\frac{ie}{\hbar^2} \int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} \langle \mathbf{p}' | \, a_{\mathsf{a}}(\mathbf{k}) \int d^4x \sqrt{-g} g^{\mu\nu} A_\mu : \left[\phi^\dagger \overleftrightarrow{D_\nu} \phi \right] : | \mathbf{p} \rangle \,, \tag{6.61}$$

where we have normal-ordered the scalar fields from the Hamiltonian density to remove the vacuum polarisation diagrams. It is easy to see that the scalar and vector field commutation relations then give us

$$\mathcal{A}_{\mathsf{a}}(\mathbf{p}, \mathbf{k}) = -ie\hbar \int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} \int d^4x \sqrt{-g} g^{\mu\nu} \eta_{\mathsf{a}\mathsf{a}'} \overline{\varepsilon_\mu^{\mathsf{a}'}(\mathbf{k}, x)} \times \left[D_\nu \log \Phi_{\mathbf{p}} - D_\nu^\dagger \log \Phi_{\mathbf{p}'}^* \right] \Phi_{\mathbf{p}'}^*(x) \Phi_{\mathbf{p}}(x). \tag{6.62}$$

We split the summation over ν into spatial and temporal components and apply Eq. (6.19) to the derivatives of the logarithms, taking the term of lowest order in $\hbar$. We therefore find

$$\mathcal{A}_{\mathsf{a}}(\mathbf{p}, \mathbf{k}) = e \int \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} \int d^4x \sqrt{-g} \eta_{\mathsf{a}\mathsf{b}} \overline{\varepsilon_\mu^{\mathsf{b}}(\mathbf{k}, x)} \times \left[-g^{\mu 0} \frac{\sigma_{\mathbf{p}}(t) + \sigma_{\mathbf{p}'}(t)}{\Omega(t)} + g^{\mu i} (p_i + p_i' - 2V_i(t)) \right] \Phi_{\mathbf{p}'}^*(x) \Phi_{\mathbf{p}}(x). \tag{6.63}$$

If we consider the momentum of a classical particle in this spacetime and subject to the background field V^μ,

$$g^{\mu\nu}(p_\mu - V_\mu)(p_\nu - V_\nu) = m^2, \tag{6.64}$$

then we can see that $\sigma_\mathbf{p} = \Omega p_0$, and hence

$$mg_{00}\frac{dt}{d\tau} = p_0 = \frac{\sigma_\mathbf{p}}{\Omega}, \tag{6.65}$$

$$mg_{ij}\frac{dx^j}{d\tau} = p_i - V_i. \tag{6.66}$$

If we denote the four-velocity by u^μ, then we have

$$mu^\mu = \sigma_\mathbf{p}\Omega\frac{dx^\mu}{dt}. \tag{6.67}$$

Hence, the terms in the square brackets from the equation for $\mathcal{A}_\mathsf{a}(\mathbf{p},\mathbf{k})$ above may be replaced to obtain

$$\mathcal{A}_\mathsf{a}(\mathbf{p},\mathbf{k}) = -em\int\frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}\int d^4x\sqrt{-g}\eta_{\mathsf{aa}'}\overline{\varepsilon_\mu^{\mathsf{a}'}(\mathbf{k},x)}\left(u^\mu|_\mathbf{p} + u^\mu|_{\mathbf{p}'}\right)\Phi^*_{\mathbf{p}'}(x)\Phi_\mathbf{p}(x), \tag{6.68}$$

where '$|_\mathbf{p}$' denotes that the quantity is taken at the momentum $\mathbf{p}$.

At this point, we separate the electromagnetic and scalar field modes into spatial and temporal factors, using Eqs. (6.7) and (6.11). Thus we have

$$\mathcal{A}_\mathsf{a}(\mathbf{p},\mathbf{k}) = -em\int\frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3}\int d^4x\sqrt{-g}\eta_{\mathsf{aa}'}\overline{\epsilon_\mu^{\mathsf{a}'}(\mathbf{k},t)}\left(u^\mu|_\mathbf{p} + u^\mu|_{\mathbf{p}'}\right) \times \phi^*_{\mathbf{p}'}(t)\phi_\mathbf{p}(t)e^{\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}'-\hbar\mathbf{k})\cdot\mathbf{x}}. \tag{6.69}$$

We integrate this with respect to $\mathbf{x}$ and so obtain a delta function, $(2\pi\hbar)^3\delta^3(\mathbf{p}'-\mathbf{P})$, where $\mathbf{P} = \mathbf{p} - \hbar\mathbf{k}$. Integrating over $\mathbf{p}'$ as well, we find

$$\mathcal{A}_\mathsf{a}(\mathbf{p},\mathbf{k}) = -\frac{em}{2P_0}\int dt\sqrt{-g}\eta_{\mathsf{aa}'}\overline{\epsilon_\mu^{\mathsf{a}'}(\mathbf{k},t)}\left(u^\mu|_\mathbf{p} + u^\mu|_\mathbf{P}\right)\phi^*_\mathbf{P}(t)\phi_\mathbf{p}(t). \tag{6.70}$$

Since we are considering terms of lowest order in $\hbar$, the u^μ terms may be taken to be $2u^\mu|_{\mathbf{p}}$, and in the scalar field time-dependent modes, we have

$$\phi^*_{\mathbf{P}}(t)\phi_{\mathbf{p}}(t) = \frac{1}{\sqrt{-g}\Omega}\sqrt{\frac{p_0 P_0}{\sigma_{\mathbf{p}}\sigma_{\mathbf{P}}}}\exp\left\{-\frac{i}{\hbar}\int_0^t \frac{\sigma_{\mathbf{p}}(t') - \sigma_{\mathbf{P}}(t')}{\Omega(t')}dt'\right\}. \tag{6.71}$$

Outside the exponential, we may take $\sigma_{\mathbf{P}} \approx \sigma_{\mathbf{p}}$, and in the exponential factor, we treat $\sigma_{\mathbf{p}-\hbar\mathbf{k}}(t')$ as a power series in $\hbar$ in order to write

$$\sigma_{\mathbf{p}}(t') - \sigma_{\mathbf{p}-\hbar\mathbf{k}}(t') \approx -\hbar\left.\frac{d}{d\hbar}\sigma_{\mathbf{p}-\hbar\mathbf{k}}(t')\right|_{\hbar=0}, \tag{6.72}$$

which gives us $\sigma_{\mathbf{p}}(t') - \sigma_{\mathbf{p}-\hbar\mathbf{k}}(t') = \hbar\Omega\mathbf{k}\cdot\dot{\mathbf{x}}$. The dot represents differentiation with respect to the co-ordinate time, t. Hence,

$$\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k}) = -e\sqrt{\frac{p_0}{P_0}}\int dt\eta_{\mathsf{ab}}\overline{\varepsilon^{\mathsf{b}}_\mu(\mathbf{k},x(t))}\left.\frac{dx^\mu}{dt}\right|_{\mathbf{p}}. \tag{6.73}$$

Since δx^i_{tree}, Eq. (6.45), contains factors which are squares in the emission amplitude as well as a factor P_0/p_0, we may drop the prefactor $\sqrt{p_0/P_0}$ from the expression for $\mathcal{A}_{\mathsf{a}}$ above and the P_0/p_0 from δx^i_{tree}. The reader will also note that $P_0/p_0 \approx 1$ in any case.

We may construct the current formed by a charged scalar particle with charge e and momentum $\mathbf{p}$,

$$\sqrt{-g}j^\mu(\mathbf{p},x) = e\left.\frac{dx^\mu}{dt}\right|_{\mathbf{p}}\delta^3(\mathbf{x} - \mathbf{X}_{\mathbf{p}}(t))\chi_\tau(t), \tag{6.74}$$

where $\chi_\tau(t)$ is, as previously, a cut-off function which is 1 while the metric and the background field are varying and which decreases smoothly so that $\chi_\tau(t) = 0$ for $t \le -T-\tau$ and $t \ge \tau$. We substitute this into our expression for $\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k})$ above, to derive an expression for $\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k})$ which includes this classical current,

$$\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k}) = -\int d^4x\sqrt{-g}\eta_{\mathsf{ab}}\overline{\varepsilon^{\mathsf{b}}_\mu(\mathbf{k},x)}j^\mu(\mathbf{p},x). \tag{6.75}$$

Let us consider the classical field which is generated by the classical current defined above, and show that this field arises in connection with δx^i_{tree}. A current $j^\mu(x)$ generates an electromagnetic field propagated by the retarded electromagnetic Green's function:

$$A^+_\mu(x) = \int d^4x' \sqrt{-g'} G^+_{\mu\nu'}(x,x') j^{\nu'}(x'). \tag{6.76}$$

We may find the retarded Green's function from the quantised electromagnetic field as

$$G^+_{\mu\mu'}(x,x') = \frac{i}{\hbar}\theta(x^0 - x^{0'}) \langle 0| \left[A_\mu(x), A_{\mu'}(x')\right] |0\rangle \tag{6.77}$$

$$= i\theta(x^0 - x^{0'})\eta_{\mathsf{aa'}} \times \int \frac{d^3\mathbf{k}}{2k_0(2\pi)^3} \left[\varepsilon^{\mathsf{a}}_\mu(\mathbf{k},x)\overline{\varepsilon^{\mathsf{a'}}_{\mu'}(\mathbf{k},x')} - \overline{\varepsilon^{\mathsf{a}}_\mu(\mathbf{k},x)}\varepsilon^{\mathsf{a'}}_{\mu'}(\mathbf{k},x')\right]. \tag{6.78}$$

Together with the definition of the classical current, these imply that for $t \geq \tau$ we may write the classical retarded field in terms of the quantum emission amplitude,

$$A^+_\mu(t,\mathbf{x}) = i\int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \left(\overline{\varepsilon^{\mathsf{a}}_\mu(\mathbf{k},x)}\mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) - \varepsilon^{\mathsf{a}}_\mu(\mathbf{k},x)\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k})\right). \tag{6.79}$$

If we define the positive- and negative-frequency parts of the field as

$$A^{(+)}_\mu(t,\mathbf{x}) = -i\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k})\varepsilon^{\mathsf{a}}_\mu(\mathbf{k},x), \tag{6.80}$$

$$A^{(-)}_\mu(t,\mathbf{x}) = i\int \frac{d^3\mathbf{k}}{2k(2\pi)^3}\mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k})\overline{\varepsilon^{\mathsf{a}}_\mu(\mathbf{k},x)}, \tag{6.81}$$

then we may write

$$\mathcal{A}^*_{\mathsf{a}}(\mathbf{p},\mathbf{k}) = -2ik\int d^3\mathbf{x}\varepsilon^\mu_{\mathsf{a}}(\mathbf{k},x)A^{(-)}_\mu(t,\mathbf{x}) \tag{6.82}$$

$$\mathcal{A}_{\mathsf{a}}(\mathbf{p},\mathbf{k}) = 2ik\int d^3\mathbf{x}\overline{\varepsilon^\mu_{\mathsf{a}}(\mathbf{k},x)}A^{(+)}_\mu(t,\mathbf{x}). \tag{6.83}$$

Substituting our expressions for $\mathcal{A}_{\mathsf{a}}$ into Eq. (6.45), we find

$$\delta x^i_{\text{tree}} = -\frac{i}{2}\eta^{\mathsf{aa}'} \int \frac{d^3\mathbf{k}}{2k(2\pi)^3} \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left|f(\mathbf{p})\right|^2$$
$$\left(-2ik \int d^3\mathbf{x}' \varepsilon^{\mu'}_{\mathsf{a}}(\mathbf{k}, x') A^{(-)}_{\mu'}(x')\right) \overleftrightarrow{\partial_{p_i}} \left(2ik \int d^3\mathbf{x}'' \overline{\varepsilon^{\mu''}_{\mathsf{a}'}(\mathbf{k}, x'')} A^{(+)}_{\mu''}(x'')\right) \quad (6.84)$$

where $x' = (t, \mathbf{x}')$ and $x'' = (t, \mathbf{x}'')$ and as the reader will recall, $t \geq \tau$ so that the underlying spacetime metric is flat and the background field is zero. Here, then, the vector modes $\varepsilon^{\mathsf{a}}_{\mu}(\mathbf{k}, x) = \delta^{\mathsf{a}}_{\mu} e^{-ik\cdot x}$ and so

$$-2ik = -2i\,\|\mathbf{k}\| = \varepsilon^{\mathsf{a}}_{\mu}(\mathbf{k}, x) \overleftrightarrow{\partial_t}\, \overline{\varepsilon^{\mathsf{a}}_{\mu}(\mathbf{k}, x)}. \quad (6.85)$$

The integral over $\mathbf{k}$ will generate a delta function, $(2\pi)^3\delta^3(\mathbf{x}' - \mathbf{x}'')$, and integrating over either of these will give us

$$\delta x^i_{\text{tree}} = \frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \left|f(\mathbf{p})\right|^2 \int d^3\mathbf{x} g^{\mu\nu} \left[A^{(-)}_{\mu}(x) \overleftrightarrow{\partial_t} \overleftrightarrow{\partial_{p_i}} A^{(+)}_{\nu}(x)\right]. \quad (6.86)$$

The function $f(\mathbf{p})$ is sharply peaked about the particle's initial momentum and is normalised so that the momentum integral is one; hence, provided we understand the momentum of the particle generating the classical field to be this momentum about which $f(\mathbf{p})$ is peaked, we may write

$$\delta x^i_{\text{tree}} = \frac{1}{2} \int d^3\mathbf{x} g^{\mu\nu} \left[A^{(-)}_{\mu}(x) \overleftrightarrow{\partial_t} \overleftrightarrow{\partial_{p_i}} A^{(+)}_{\nu}(x)\right]. \quad (6.87)$$

If we consider a similar expression to the one above, but with $A^{(+)}_{\mu}(x)$ (or the conjugate, $A^{(-)}_{\mu}(x)$) repeated, then within it we find a factor of the form

$$\int d^3\mathbf{x} \varepsilon^{\mathsf{a}}_{\mu}(\mathbf{k}, x) \overleftrightarrow{\partial_t} \varepsilon^{\mathsf{a}'}_{\nu}(\mathbf{k}', x) = 0. \quad (6.88)$$

Consequently, if we consider $A^{\text{ret}}_{\mu}(x) = A^{(+)}_{\mu}(x) + A^{(-)}_{\mu}(x)$, we find

$$\delta x^i_{\text{tree}} = \frac{1}{4} \int d^3\mathbf{x} g^{\mu\nu} A^{\text{ret}}_{\mu}(x) \overleftrightarrow{\partial_t} \overleftrightarrow{\partial_{p_i}} A^{\text{ret}}_{\nu}(x), \quad (6.89)$$

and we may use the symmetry of the metric tensor to conclude that

$$\delta x^i_{\text{tree}} = -\frac{1}{2}\int d^3\mathbf{x} g^{\mu\nu}\left(\partial_{p_i} A^{\text{ret}}_\mu(x)\right)\overleftrightarrow{\partial_t} A^{\text{ret}}_\nu(x). \tag{6.90}$$

We simplify this expression by using the Kirchhoff representation for half the retarded-minus-advanced electromagnetic potential,

$$G^{\text{r-a}}_{\mu'\mu''}(x',x'') = -\frac{1}{2}\int d\Sigma^\nu \left[G^+_{\mu'\mu}(x',x)\nabla_\nu G^{\mu+}_{\mu''}(x,x'') - G^{\mu+}_{\mu''}(x,x'')\nabla_\nu G^+_{\mu'\mu}(x',x)\right]. \tag{6.91}$$

We take $x^0 > \max\left(x^{0\prime}, x^{0\prime\prime}\right)$, since $t \geq \tau$ and therefore the metric is flat. In consequence, we find

$$\delta x^i_{\text{tree}} = \int d^4x' d^4x'' \sqrt{-g(x')}\sqrt{-g(x'')} G^{\text{r-a}}_{\mu'\mu''}(x',x'')\left(\partial_{p_i} j^{\mu'}(x')\right) j^{\mu''}(x'') \tag{6.92}$$

$$= \int d^4x \sqrt{-g(x)}\left(\partial_{p_i} j^\mu(x)\right) A^{\text{r-a}}_\mu(x), \tag{6.93}$$

where $A^{\text{r-a}}_\mu(x)$ is defined analogously to $A^+_\mu(x)$, with the retarded Green's function replaced with the half-retarded-minus-advanced one.

If we consider the definition of the classical current, $j^\mu(x)$, we treat the position function $X(t)$ as a function of $\mathbf{p}$, with $X^0(t) = t$, and so obtain

$$\sqrt{-g(x)}\partial_{p_i} j^\mu = -e \lim_{\Delta p_i \to 0}\frac{1}{\Delta p_i}\left[\frac{dX^\mu}{dt}\delta^3(\mathbf{x}-\mathbf{X}) - \left(\frac{dX^\mu}{dt} + \frac{d\Delta X^\mu}{dt}\right)\delta^3(\mathbf{x}-\mathbf{X}-\Delta\mathbf{X})\right]. \tag{6.94}$$

Hence, if we take the inner product with the half-retarded-minus-advanced electromagnetic potential, we find

$$\int d^3\mathbf{x}\sqrt{-g(x)}\left(\partial_{p_i} j^\mu(x)\right) A^{\text{r-a}}_\mu(x)$$
$$= -e \lim_{\Delta p_i \to 0}\frac{1}{\Delta p_i}\left[\frac{dX^\mu}{dt} A^{\text{r-a}}_\mu(X) - \left(\frac{dX^\mu}{dt} + \frac{d\Delta X^\mu}{dt}\right) A^{\text{r-a}}_\mu(X+\Delta X)\right]. \tag{6.95}$$

We may thus approximate $A^{\text{r}-\text{a}}_\mu(X+\Delta X) \approx A^{\text{r}-\text{a}}_\mu(X) + \Delta X^\alpha \nabla_\alpha A^{\text{r}-\text{a}}_\mu(X)$, and discarding the term which arises at second order in Δp_i, we find

$$\int d^3\mathbf{x}\sqrt{-g(x)}\,(\partial_{p_i} j^\mu(x))\, A^{\text{r}-\text{a}}_\mu(x) \\ = e \lim_{\Delta p_i \to 0} \frac{1}{\Delta p_i}\left[\frac{d\Delta X^\mu}{dt} A^{\text{r}-\text{a}}_\mu(X) + \frac{dX^\mu}{dt}\Delta X^\alpha \nabla_\alpha A^{\text{r}-\text{a}}_\mu(X)\right]. \quad (6.96)$$

The position shift is the integral over time of the quantity above. If we integrate the first term by parts, we find that we have

$$\delta x^i_{\text{tree}} = e\int dt \lim_{\Delta p_i \to 0}\frac{1}{\Delta p_i}\left[-\Delta X^\mu \frac{dX^\alpha}{dt}\nabla_\alpha A^{\text{r}-\text{a}}_\mu(X) + \frac{dX^\mu}{dt}\Delta X^\alpha \nabla_\alpha A^{\text{r}-\text{a}}_\mu(X)\right], \quad (6.97)$$

which, upon an appropriate re-arrangement of the indices, yields

$$\delta x^i_{\text{tree}} = -\int dt \lim_{\Delta p_i \to 0}\frac{1}{\Delta p_i}\frac{dX^\mu}{dt}\Delta X^\alpha F^{\text{r}-\text{a}}_{\alpha\mu} \quad (6.98)$$

$$= -\int dt \left(\frac{\partial X^j}{\partial p_i}\right)_t f^{\text{r}-\text{a}}_j; \quad (6.99)$$

we have changed the spacetime indices to spatial indices since the partial derivative is taken at a constant time. The covector $f^{\text{r}-\text{a}}_j$ is a force derived from the half-retarded-minus-advanced electromagnetic field, $f_j = \dot{x}^\mu F^{\text{r}-\text{a}}_{j\mu}$. Finally in this section, we shall proceed to justify the claim that this quantum position shift matches the classical position shift coming from the same force, and argue that there remains a part to be found from the one-loop contribution to the position shift.

If we consider a classical scalar particle of charge e and mass m on a curved spacetime and subject to a background potential V^μ, both of which we allow to vary with x, then the Hamiltonian for this theory is

$$H = \sqrt{g^{ij}(p_i - V_i)(p_j - V_j) + m^2} + V_0. \quad (6.100)$$

Incorporating an external force, f_i, the equations of motion we derive from this Hamiltonian are

$$\dot{x}_i = \frac{\partial H}{\partial p^i}, \tag{6.101}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} + f_i. \tag{6.102}$$

Then we let $(x_0^i(t), p_{0,k}(t))$ be the solution to the coupled differential equations above subject to $f_i = 0$ and the conditions $(x_0^i(0), p_{0,k}(0)) = (0, p_k)$. We call such a solution an homogeneous solution. This solution is the trajectory of the particle in the absence of the external force which satisfies the final conditions on the particle.

We may write perturbations on this solution which continue to satisfy the equations of motion with $f_i = 0$ as $(x_0^i + \Delta x^i, p_{0,k} + \Delta p_k)$. Expanding the Hamiltonian to second order in these perturbations, we find

$$\begin{aligned} H(x^i + \Delta x^i, p_j + \Delta p_j) &= H(x^i, p_j) + \frac{\partial H}{\partial x^i}\Delta x^i + \frac{\partial H}{\partial p_i}\Delta p_i \\ &+ \frac{1}{2}\left[\frac{\partial^2 H}{\partial x^i \partial x^j}\Delta x^i \Delta x^j + 2\frac{\partial^2 H}{\partial x^i \partial p_i}\Delta x^i \Delta p_j + \frac{\partial^2 H}{\partial p_i \partial p_j}\Delta p_i \Delta p_j\right], \end{aligned} \tag{6.103}$$

which implies that the perturbations are governed by the equations

$$\frac{d}{dt}\Delta x^i = \left(\frac{\partial^2 H}{\partial x^j \partial p_i}\right)_{(x_0^i, p_{0,k})} \Delta x^j + \left(\frac{\partial^2 H}{\partial p_j \partial p_i}\right)_{(x_0^i, p_{0,k})} \Delta p_j, \tag{6.104}$$

$$\frac{d}{dt}\Delta p_i = -\left(\frac{\partial^2 H}{\partial x^j \partial x^i}\right)_{(x_0^i, p_{0,k})} \Delta x^j - \left(\frac{\partial^2 H}{\partial p_j \partial x^i}\right)_{(x_0^i, p_{0,k})} \Delta p_j. \tag{6.105}$$

If $(\Delta x^i, \Delta p_j)$ and $(\Delta X^i, \Delta P_j)$ are solutions, then it is readily verified, using the above equations, that

$$\frac{d}{dt}\left(\Delta x^i \Delta P_i - \Delta X^i \Delta p_i\right) = 0. \tag{6.106}$$

Let us define a set of homogeneous perturbations $(\Delta x^{i(j)}(t;s), \Delta p_k^{(l)}(t;s))$ with $j = 1, 2, 3$ and $s \in (-\infty, \infty)$ by the conditions

$$\Delta x^{i(j)}(s;s) = 0, \tag{6.107}$$

$$\Delta p_i^{(j)}(s;s) = \delta_i^j. \tag{6.108}$$

The homogeneous solution $(x_0^i + \Delta x^{i(j)}(t;s), p_{0,k} + \Delta p_k^{(l)}(t;s))$ is then the solution for the trajectory which crosses $x_0^i(t)$ at $t = s$, but has a momentum which differs by δ_i^j at that point. It is simple to see, from the equations of motion on perturbations given above, that if we define

$$\delta x^i = \int_{-\infty}^{t} ds f_j(s) \Delta x^{i(j)}(t;s), \tag{6.109}$$

$$\delta p_i = \int_{-\infty}^{t} ds f_j(s) \Delta p_i^{(j)}(t;s), \tag{6.110}$$

then the pair $(x_0^i + \delta x^i, p_{0,k} + \delta p_k)$ solves the equations of motion with $f_i \neq 0$; this solution we call the inhomogeneous solution. Note that $(\delta x^i, \delta p_k) = (0,0)$ for $t < -T$.

The classical position shift may therefore be written

$$\delta x_{\mathrm{C}}^i = \int_{-\infty}^{0} ds f_j(s) \Delta x^{i(j)}(0;s). \tag{6.111}$$

The constancy of the symplectic product between perturbations implies that if we choose the two perturbations $\left(\Delta x^{k(i)}(t;s), \Delta p_l^{(i)}(t;s)\right)$ and $\left(\Delta x^{m(j)}(t;u), \Delta p_n^{(j)}(t;u)\right)$, then the symplectic products at $t = s$ and $t = u$ are equal. That is to say,

$$\begin{aligned} \Delta x^{a(i)}(s;s)\Delta p_a^{(j)}(s;u) - \Delta x^{a(j)}(s;u)\Delta p_a^{(i)}(s;s) \\ = \Delta x^{a(i)}(u;s)\Delta p_a^{(j)}(u;u) - \Delta x^{a(j)}(u;u)\Delta p_a^{(i)}(u;s), \end{aligned} \tag{6.112}$$

which implies

$$-\Delta x^{i(j)}(s;u) = \Delta x^{j(i)}(u;s). \tag{6.113}$$

Hence, we may use this identity to re-write

$$\delta x_{\mathrm{C}}^i = -\int_{-\infty}^{0} dt f_j(t) \Delta x^{j(k)}(t;0). \tag{6.114}$$

From the definition of the homogeneous perturbations above, we see that at time $t = 0$, a small change in the momentum, $\Delta p_k^{(l)}$, induces a small change in the position, $\Delta x^i_{(j)}$: that is to say, that

$$\left(\frac{\partial x^i}{\partial p_j}\right)_t = \Delta x^{i(j)}(t;0). \tag{6.115}$$

Hence, we find

$$\delta x^i_{\mathrm{C}} = -\int_{\infty}^{0} dt f_j(t) \left(\frac{\partial x^j}{\partial p_i}\right)_t, \tag{6.116}$$

and conclude that if the external force f_i is identified with the force arising from the half-retarded-minus-advanced field, then this classical position shift and the quantum position shift at tree level are equal.

This is not, however, the entirety of the field, nor is the force we described the entirety of the Lorentz-Dirac force. In a curved spacetime, the radiative Green's functions do not equal half the retarded-minus-advanced electromagnetic Green's functions. (The reader is encouraged to consult Poisson's review [13] for fuller details of the considerations which follow here. The original work on which our argument here chiefly depends was published by Detweiler and Whiting [38].) In order to describe the radiative Green's functions and their causal structure, it is first necessary to reprise briefly the Hadamard construction of the retarded and advanced Green's functions, which gives us the following structure:

$$G^{\pm}_{\mu\mu'}(x,x') = U_{\mu\mu'}(x,x')\delta_{\pm}(\sigma(x,x')) + V_{\mu\mu'}(x,x')\theta_{\pm}(\sigma(x,x')). \tag{6.117}$$

The function $\sigma(x,x')$ is Synge's world function, which is half the squared geodesic separation between x and x'; the function $\delta_{\pm}$ is the delta function selecting x on the forward $(+)$ or backward $(-)$ light-cone from x'; and $\theta_{\pm}$ is likewise the theta function selecting x in the chronological future $(+)$ or past $(-)$ of x'. We then define the radiative Green's functions as follows, and apply the Hadamard construction to obtain

$$G^{\mathrm{R}}_{\mu\mu'}(x,x') = \frac{1}{2}\left(G^{+}_{\mu\mu'}(x,x') - G^{-}_{\mu\mu'}(x,x') + V_{\mu\mu'}(x,x')\right) \tag{6.118}$$

$$= \frac{1}{2}U_{\mu\mu'}(x,x')\left[\delta_{+}(\sigma) - \delta_{-}(\sigma)\right] + V_{\mu\mu'}(x,x')\left[\theta_{+}(\sigma) + \frac{1}{2}\theta(-\sigma)\right]. \tag{6.119}$$

From this we may read the causal structure of the Green's functions: there are parts which lie along the forward and backward light-cones as in the flat spacetime case, but two additional parts arise on a curved spacetime, one when x is in the causal future of x' (which is called the tail part) and the other when x and x' are space-like separated. Since we define the radiative electromagnetic potential, from which we derive the classical Lorentz-Dirac force, by

$$A^{\mathrm{R}}_{\mu}(x) = \int d^4x' G^{\mathrm{R}}_{\mu\mu'}(x,x') j^{\mu'}(x'), \tag{6.120}$$

we consequently conclude that we may write the classical radiation reaction four-force, schematically, as

$$ma_{\mu} = \frac{e}{2}\left(F^{\mathrm{lc},+}_{\mu\nu} - F^{\mathrm{lc},-}_{\mu\nu}\right)u^{\nu} + eF^{\mathrm{tail},+}_{\mu\nu}u^{\nu} + \frac{e}{2}F^{\mathrm{sp.}}_{\mu\nu}u^{\nu}, \tag{6.121}$$

and that therefore the classical position shift, which is linear in the reaction force, will exhibit a similar structure.

The position shift we have thus far derived may be written, using the same schema, as

$$\delta\mathbf{x}_{\mathrm{tree}} = \frac{1}{2}\left(\delta\mathbf{x}^{\mathrm{lc},+} - \delta\mathbf{x}^{\mathrm{lc},-}\right) + \frac{1}{2}\left(\delta\mathbf{x}^{\mathrm{tail},+} - \delta\mathbf{x}^{\mathrm{tail},-}\right). \tag{6.122}$$

We do not expect *a priori* that the one-loop contribution to the position shift, $\delta\mathbf{x}_{\mathrm{loop}}$ will equal zero. Indeed, we may conclude from the foregoing considerations that if the classical and quantum position shift predictions are to match, it must contribute something which is schematically,

$$\delta\mathbf{x}_{\mathrm{loop}} = \frac{1}{2}\left(\delta\mathbf{x}^{\mathrm{tail},+} + \delta\mathbf{x}^{\mathrm{tail},-} + \delta\mathbf{x}^{\mathrm{sp.}}\right); \tag{6.123}$$

the quantity $\delta\mathbf{x}^{\text{sp.}}$ having been included for completeness, since we expect it to be zero. Preliminary investigations have indicated that this calculation will be considerably more complicated and protracted than the flat spacetime version, and we shall not be carrying out this calculation here. However, we shall proceed to present the vacuum current calculation and to show that there is no correction to Maxwell's equations arising at this order.

6.4 Vacuum current

The reader will recall that the background potential $V^\mu(t)$ is generated by a classical current and gives rise to a field $W_{\alpha\beta} = 2\nabla_{[\alpha}V_{\beta]}$. As in Ch. 5, the tree-level current J^μ_C may be supplemented with a one-loop level current, ΔJ^μ, if the vacuum quantum current, $\langle J^\mu\rangle_0$, is not equal to the renormalised classical current, $(Z_3-1)J^\mu_C$, where Z_3 is defined as in Eq. (5.52). We therefore investigate the possibility that a vacuum current arises at one-loop level in this theory.

The field equation for this current is

$$J^\mu_C = g^{\mu\alpha}g^{\nu\beta}\nabla_\nu\left(\nabla_\alpha V_\beta - \nabla_\beta V_\alpha\right). \tag{6.124}$$

Since $V^0(t) = 0$, only the spatial components of this current will be non-zero. We also note that $\partial_i V^\mu = 0$, and so we have

$$J^i_C = \Omega^2\left(g^{ij}\nabla_0\partial_0 V_j - g^{ij}\Gamma^k{}_{0j}\partial_0 V_k + g^{ij}\Gamma^a{}_{0a}\partial_0 V_j\right). \tag{6.125}$$

We now calculate the quantum expectation value for the vacuum current, $J^\mu_Q \equiv \langle J^\mu\rangle$. The current operator in curved spacetime is

$$J^\mu = -\frac{i}{2\hbar}g^{\mu\nu}\left(\{D_\nu\phi,\phi^*\} - \{D_\nu\phi,\phi^*\}^\dagger\right), \tag{6.126}$$

and so the expectation value of the current relative to the final flat-spacetime vacuum is

$$\begin{aligned}\langle J^\mu \rangle = & -\frac{i\hbar}{2} g^{\mu\nu} \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} \\ & \times \langle 0| \Big[\Big(A(\mathbf{p}) D_\nu \Phi_\mathbf{p}(x) + B^\dagger(\mathbf{p}) D_\nu \overline{\Phi}^*_\mathbf{p}(x) \Big) \left(A^\dagger(\mathbf{p}') \Phi^*_{\mathbf{p}'}(x) + B(\mathbf{p}') \overline{\Phi}_{\mathbf{p}'}(x) \right) \\ & \quad + \left(A^\dagger(\mathbf{p}') \Phi^*_{\mathbf{p}'}(x) + B(\mathbf{p}') \overline{\Phi}_{\mathbf{p}'}(x) \right) \Big(A(\mathbf{p}) D_\nu \Phi_\mathbf{p}(x) + B^\dagger(\mathbf{p}) D_\nu \overline{\Phi}^*_\mathbf{p}(x) \Big) \\ & - \Big(A^\dagger(\mathbf{p}) (D_\nu \Phi_\mathbf{p}(x))^\dagger + B(\mathbf{p}) (D_\nu \overline{\Phi}^*_\mathbf{p}(x))^\dagger \Big) \Big(A(\mathbf{p}') \Phi_{\mathbf{p}'}(x) + B^\dagger(\mathbf{p}') \overline{\Phi}^*_{\mathbf{p}'}(x) \Big) \\ - & \Big(A(\mathbf{p}') \Phi_{\mathbf{p}'}(x) + B^\dagger(\mathbf{p}') \overline{\Phi}^*_{\mathbf{p}'}(x) \Big) \Big(A^\dagger(\mathbf{p}) (D_\nu \Phi_\mathbf{p}(x))^\dagger + B(\mathbf{p}) (D_\nu \overline{\Phi}^*_\mathbf{p}(x))^\dagger \Big) \Big] |0\rangle \, .\end{aligned} \tag{6.127}$$

From this, only four terms do not annihilate the vacuum, which if we re-label some of the momenta we may combine thus:

$$\begin{aligned}\langle J^\mu \rangle = & \frac{i\hbar}{2} \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \frac{d^3\mathbf{p}'}{2p_0'(2\pi\hbar)^3} \\ \times & \langle 0| \Big[A(\mathbf{p}) A^\dagger(\mathbf{p}') \Big(\Phi^*_{\mathbf{p}'}(x) \overleftrightarrow{D}^\mu \Phi_\mathbf{p}(x) \Big) + B(\mathbf{p}) B^\dagger(\mathbf{p}') \Big(\overline{\Phi}_\mathbf{p}(x) \overleftrightarrow{D}^\mu \overline{\Phi}^*_{\mathbf{p}'}(x) \Big) \Big] |0\rangle \, .\end{aligned} \tag{6.128}$$

The reader will observe that we may exchange the anti-particle mode functions for particle mode functions by using the definitions of the modes and the identity in Eq. (6.14), so that when we apply the relevant commutation relations, we find

$$\langle J^\mu \rangle = \frac{i}{\hbar} \int \frac{d^3\mathbf{p}}{2p_0(2\pi\hbar)^3} \left(\phi_\mathbf{p}(t) e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \overleftrightarrow{D}^\mu \phi^*_\mathbf{p}(t) e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} - \phi_\mathbf{p}(t) e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \overleftrightarrow{D}^\mu \phi^*_\mathbf{p}(t) e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} \right) . \tag{6.129}$$

We therefore see that for $\mu = 0$, the terms in the integrand cancel and we obtain $\langle J^0 \rangle = 0$, while for spatial values of μ, the terms reinforce. Thus we consider the spatial components only, and find

$$\left\langle J^i \right\rangle = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{p^i - V^i}{\sigma_\mathbf{p}\sqrt{-g}\Omega} \left|\varphi_\mathbf{p}\right|^2 . \tag{6.130}$$

In order to find this vacuum current up to order $\hbar^0$, we must, as in Ch. 5, consider the WKB expansion to third order in $\hbar$.

Substituting the solution to order $\hbar^0$, Eq. (6.19), back into Eq. (6.16), we obtain the equation of motion for the terms of higher order in $\hbar$,

$$\sum_{n=2}^{\infty}\hbar^{n-1}\ddot{s}_n+\left(\sum_{n=2}^{\infty}\hbar^{n-1}\dot{s}_n\right)^2+\left(\frac{\dot{\Omega}}{\Omega}-\frac{1}{2}\frac{\dot{\Sigma}}{\Sigma}-\frac{2i}{\hbar}\frac{\Sigma^{\frac{1}{2}}}{\Omega}\right)\sum_{n=2}^{\infty}\hbar^{n-1}\dot{s}_n$$
$$+\left(\frac{5}{16}\left(\frac{\dot{\Sigma}}{\Sigma}\right)^2-\frac{1}{4}\frac{\ddot{\Sigma}}{\Sigma}-\frac{1}{4}\frac{\dot{\Omega}}{\Omega}\frac{\dot{\Sigma}}{\Sigma}-\frac{1}{4}\left(\frac{\dot{\Omega}}{\Omega}\right)^2-\frac{1}{2}\frac{\ddot{\Omega}}{\Omega}+\frac{3\dot{g}^2}{16g^2}-\frac{\ddot{g}}{4g}-\frac{\dot{g}\dot{\Omega}}{2g\Omega}\right)=0. \tag{6.131}$$

To make this simpler to handle, let us define

$$u=\frac{\Sigma^{\frac{1}{2}}}{\Omega}, \tag{6.132}$$

$$v=\frac{5}{16}\left(\frac{\dot{\Sigma}}{\Sigma}\right)^2-\frac{1}{4}\frac{\ddot{\Sigma}}{\Sigma}-\frac{1}{4}\frac{\dot{\Omega}}{\Omega}\frac{\dot{\Sigma}}{\Sigma}-\frac{1}{4}\left(\frac{\dot{\Omega}}{\Omega}\right)^2-\frac{1}{2}\frac{\ddot{\Omega}}{\Omega}+\frac{3\dot{g}^2}{16g^2}-\frac{\ddot{g}}{4g}-\frac{\dot{g}\dot{\Omega}}{2g\Omega}. \tag{6.133}$$

Then the equation of motion can be shown easily to simplify to

$$\sum_{n=2}^{\infty}\ddot{s}_n+\left(\sum_{n=2}^{\infty}\hbar^{n-1}\dot{s}_n\right)^2-\left(\partial_0\log u+\frac{2iu}{\hbar}\right)\sum_{n=2}^{\infty}\hbar^{n-1}\dot{s}_n+v=0. \tag{6.134}$$

Taking the above equation at order $\hbar^0$, we have

$$\dot{s}_2=\frac{iv}{2u}. \tag{6.135}$$

At order $\hbar^1$, this equation gives us

$$\ddot{s}_2-\partial_0\log u\dot{s}_2-2iu\dot{s}_3=0. \tag{6.136}$$

On re-arranging, we find

$$2i\dot{s}_3=\frac{\ddot{s}_2}{u}-\frac{\dot{u}}{u}\dot{s}_2 \tag{6.137}$$

and hence conclude that

$$\dot{s}_3 = \frac{1}{4}\partial_0 \left(\frac{v}{u^2} \right). \tag{6.138}$$

Expanding out these terms according to our substitutions above, and integrating, we therefore find

$$s_3 = -\frac{\Omega^2}{\Sigma}\left(\frac{5}{64}\left(\frac{\dot{\Sigma}}{\Sigma}\right)^2 - \frac{1}{16}\frac{\ddot{\Sigma}}{\Sigma} - \frac{1}{16}\frac{\dot{\Omega}}{\Omega}\frac{\dot{\Sigma}}{\Sigma} - \frac{1}{16}\left(\frac{\dot{\Omega}}{\Omega}\right)^2 - \frac{1}{8}\frac{\ddot{\Omega}}{\Omega} + \frac{3\dot{g}^2}{64g^2} - \frac{\ddot{g}}{16g} - \frac{\dot{g}\dot{\Omega}}{8g\Omega} \right). \tag{6.139}$$

Since we can easily see that s_4 will be imaginary, we conclude that it will not contribute to $|\varphi_{\mathbf{p}}|^2$ and so we do not need to calculate this directly. Likewise, s_2 will not not contribute, and so $|\varphi_{\mathbf{p}}|^2 = 1 + 2\hbar^2 s_3$.

We may carry out the change of variables $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{V}$, which implies that $d^3\mathbf{p} = d^3\tilde{\mathbf{p}}$. We note that in the integral over momentum, terms which are odd in $\tilde{p}_i$ will not contribute. Therefore, contributions can only arise from s_3, and specifically from terms in s_3 which involve derivatives of Σ. Hence,

$$\left\langle J^i \right\rangle = \frac{2\Omega}{\hbar\sqrt{-g}} g^{ij} \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{\tilde{p}_j}{\Sigma^{\frac{3}{2}}} \left(\frac{5}{64}\frac{\dot{\Sigma}^2}{\Sigma^2} - \frac{1}{16}\frac{\ddot{\Sigma}}{\Sigma} - \frac{1}{16}\frac{\dot{\Omega}}{\Omega}\frac{\dot{\Sigma}}{\Sigma} \right). \tag{6.140}$$

Not every portion of the derivatives of Σ will contribute, though, as some parts will be even in $\tilde{\mathbf{p}}$. The derivatives are

$$\frac{\mathrm{d}}{\mathrm{dt}}\Sigma = \nabla_0\Sigma = 2g^{ij}\tilde{p}_i\dot{\tilde{p}}_j, \tag{6.141}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{dt}^2}\Sigma &= \nabla_0\partial_0\Sigma + \Gamma^0{}_{00}\partial_0\Sigma \\ &= 2g^{ij}\dot{\tilde{p}}_i\dot{\tilde{p}}_j + 2g^{ij}\tilde{p}_i\ddot{\tilde{p}}_j + \partial_0 \log\Omega^2 \cdot g^{ij}\tilde{p}_i\dot{\tilde{p}}_j, \end{aligned} \tag{6.142}$$

where the dot refers to the covariant derivative with respect to the time coordinate, not the partial derivative. For the purposes of calculation, we let $\tilde{p}_0 = 0$. Removing terms which are even in $\tilde{\mathbf{p}}$ and noting a cancellation between the last term in the bracket and a part of the second, we are left with

$$\langle J^i \rangle = -\frac{2\Omega}{\hbar\sqrt{-g}} g^{ij} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\tilde{p}_j}{\Sigma^{\frac{5}{2}}} \left[\frac{5}{16} \frac{g^{kl} g^{mn} \dot{\tilde{p}}_k \dot{\tilde{p}}_m \tilde{p}_l \tilde{p}_n}{\Sigma} - \frac{1}{8} \left(g^{kl} \dot{\tilde{p}}_k \dot{\tilde{p}}_l + g^{kl} \ddot{\tilde{p}}_k \tilde{p}_l \right) \right]. \tag{6.143}$$

We may therefore deduce the following two equations:

$$\dot{\tilde{p}}_i = -\partial_0 V_i - \Gamma^j{}_{i0} \tilde{p}_j; \tag{6.144}$$

$$\ddot{\tilde{p}}_i = -\nabla_0 \partial_0 V_i + \Gamma^j{}_{0i} \partial_0 V_j - \partial_0 \Gamma^j{}_{0i} \cdot \tilde{p}_j + \Gamma^0{}_{00} \Gamma^j{}_{0i} \tilde{p}_j + \Gamma^j{}_{0i} \Gamma^k{}_{0j} \tilde{p}_k, \tag{6.145}$$

where $\nabla_0 \partial_0 V_i$ is defined as though $\partial_0 V_i$ were the $0i$ component of a covariant, rank two tensor. Using these, we remove some more terms which are odd in $\tilde{p}$, and so obtain

$$\begin{aligned} \langle J^i \rangle = -\frac{2\Omega}{\hbar\sqrt{|g|}} g^{ij} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \bigg[& \frac{5}{16} g^{kl} g^{mn} \frac{\tilde{p}_j \tilde{p}_l \tilde{p}_a \tilde{p}_n}{\Sigma^{\frac{7}{2}}} \left(\Gamma^a{}_{0k} \partial_0 V_m + \Gamma^a{}_{0m} \partial_0 V_k \right) \\ & -\frac{1}{8} g^{kl} \frac{\tilde{p}_m \tilde{p}_j}{\Sigma^{\frac{5}{2}}} \left(\Gamma^m{}_{0k} \partial_0 V_l + \Gamma^m{}_{0l} \partial_0 V_k \right) - \frac{1}{8} g^{kl} \frac{\tilde{p}_j \tilde{p}_l}{\Sigma^{\frac{5}{2}}} \left(-\nabla_0 \partial_0 V_k + \Gamma^a{}_{0k} \partial_0 V_a \right) \bigg]. \end{aligned} \tag{6.146}$$

Now, we shift the integral over p^i so that $\tilde{p}_i \mapsto p_i$, with the effect that $\Sigma \mapsto p^2 + m^2$ (understanding that $p^2 = g^{ij} p_i p_j$ is a function of t through the variation in the metric). This gives us the following divergent integrals:

$$I_{ij} = \int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{p_i p_j}{(p^2 + m^2)^{\frac{5}{2}}}; \tag{6.147}$$

$$I_{ijkl} = \int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{p_i p_j p_k p_l}{(p^2 + m^2)^{\frac{7}{2}}}, \tag{6.148}$$

where we define $D - 1 = 3 - 2\varepsilon$. As in the flat spacetime case, we may replace $p_i p_j \mapsto \frac{1}{D-1} g_{ij} p^2$ in the first integral [31]. In the second integral, we similarly replace $p_i p_j p_k p_l \mapsto \frac{1}{(D-1)(D+1)} (g_{ij} g_{kl} + g_{ik} g_{jl} + g_{il} g_{jk}) p^4$. Hence, extracting the index structure from each integral, we obtain, respectively,

$$I_2 = \int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{p^2}{(p^2 + m^2)^{\frac{5}{2}}}, \tag{6.149}$$

$$I_4 = \int \frac{d^{D-1}\mathbf{p}}{(2\pi)^{D-1}} \frac{p^4}{(p^2 + m^2)^{\frac{7}{2}}}. \tag{6.150}$$

In order to apply dimensional regularisation to this integral, we must change the integration variable so that the space over which the integration takes place is Euclidean. Therefore, let us define a kind of square root for the metric: a positive, real, symmetric matrix E which satisfies $g_{ij} = (E^2)_{ij}$. Then we can use this matrix to define a new (Euclidean) momentum, $\mathbf{p}_E \equiv E^{-1}\mathbf{p}$.

If we consider the integrals above, then we see that we must find p^2 and $d^{D-1}\mathbf{p}$ in terms of this new momentum. Taking p^2, we see that $p^2 = g^{ij}p_ip_j$, and we replace the metric with its matrix expression, so that $p^2 = \mathbf{p}^T(E^2)^{-1}\mathbf{p}$, understanding $\mathbf{p}$ to be a vector. It is simple to see from this point that we may combine the matrices with the momenta in such a way that we find $p^2 = {p_E}^2$. Since the matrix E^{-1} is clearly the Jacobian for the transformation, this gives us $d^{D-1}\mathbf{p} = (\det E)d^{D-1}\mathbf{p}_E = \sqrt{\det{(g_{ij})}}d^{D-1}\mathbf{p}_E$, and so the integrals become

$$I_2 = \sqrt{\det{(g_{ij})}}\int \frac{d^{D-1}\mathbf{p}_E}{(2\pi)^{D-1}}\frac{{p_E}^2}{({p_E}^2+m^2)^{\frac{5}{2}}}; \tag{6.151}$$

$$I_4 = \sqrt{\det{(g_{ij})}}\int \frac{d^{D-1}\mathbf{p}_E}{(2\pi)^{D-1}}\frac{{p_E}^4}{({p_E}^2+m^2)^{\frac{7}{2}}}. \tag{6.152}$$

These integrals are now dimensionally regularisable and give us

$$I_2 = \sqrt{\det{(g_{ij})}}\frac{D-1}{12\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\varepsilon}{2}}\Gamma\left(\frac{\varepsilon}{2}\right); \tag{6.153}$$

$$I_4 = \sqrt{\det{(g_{ij})}}\frac{(D-1)(D+1)}{60\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\varepsilon}{2}}\Gamma\left(\frac{\varepsilon}{2}\right) \tag{6.154}$$

$$= \frac{D+1}{5}I_2. \tag{6.155}$$

The reader will recall that the integrals

$$I_{ij} = \frac{1}{D-1}g_{ij}I_2, \tag{6.156}$$

$$I_{ijkl} = \frac{1}{(D-1)(D+1)}(g_{ij}g_{kl}+g_{ik}g_{jl}+g_{il}g_{jk})I_4. \tag{6.157}$$

Clearly, therefore, the factors of $D-1$ and $D+1$ cancel, and thus we find

$$I_{ij} = -4(Z_3 - 1)g_{ij}\sqrt{\det(g_{ij})}; \tag{6.158}$$

$$I_{ijkl} = -\frac{4}{5}(Z_3 - 1)(g_{ij}g_{kl} + g_{ik}g_{jl} + g_{il}g_{jk})\sqrt{\det(g_{ij})}, \tag{6.159}$$

where Z_3 is defined in Eq. (5.52).

We can simplify the determinants by using $\sqrt{|g|} = \Omega^{-1}\sqrt{\det(g_{ij})}$, which allows us to write $\langle J^i \rangle$ as

$$\begin{aligned} \left\langle J^i \right\rangle &= 2\Omega^2(Z_3 - 1)g^{ij} \\ &\times \left[\frac{1}{4}g^{kl}g^{mn}(g_{jl}g_{an} + g_{ja}g_{ln} + g_{jn}g_{la})\left(\Gamma^a{}_{0k}\partial_0 V_m + \Gamma^a{}_{0m}\partial_0 V_k\right)\right. \\ -\frac{1}{2}g^{kl}g_{mj}&\left(\Gamma^m{}_{0k}\partial_0 V_l + \Gamma^m{}_{0l}\partial_0 V_k\right) - \left.\frac{1}{2}g^{kl}g_{jl}\left(-\nabla_0\partial_0 V_k + \Gamma^a{}_{0k}\partial_0 V_a\right)\right]. \end{aligned} \tag{6.160}$$

Working through the index contractions, and using the fact that under the conditions placed on the metric $g^{ij}\Gamma^k{}_{0j} = \Gamma^i{}_{0j}g^{jk}$, we find

$$\left\langle J^i \right\rangle = \Omega^2(Z_3 - 1)\left[\Gamma^m{}_{0m}g^{ij}\partial_0 V_j + g^{ij}\nabla_0\partial_0 V_j - \Gamma^i{}_{0j}g^{jk}\partial_0 V_k\right]. \tag{6.161}$$

From the above equation and Eq. (6.125), we can see that $(Z_3 - 1)J^i_C = J^i_Q$, and therefore there is no need for a supplementary vacuum current and no corrections to Maxwell's equations arise at the one-loop level in this theory. The reader will note that this result specialises to the conformally flat result we found in the preceding chapter.

Chapter 7

Conclusions and outlook

Summary. In this chapter we shall summarise the findings presented heretofore, and discuss possible directions in which the work could be extended.

In the preceding chapters, we have extended the study of the radiation of accelerating charges in quantum field theory in two different directions. The context for the work has been the classical problem, long known and well-studied, and its quantum field-theoretic counterpart, which is a continuing field of research. The work by Higuchi and Martin [20, 21, 22, 23] is the starting-point for our studies here, as they demonstrated that both scalar and spinor quantum theories predict the correct, *i.e.* classical, change in the trajectory due to the radiation effect at tree level.

As we are taking the semiclassical approximation, it is natural to expect that the quantum theory should reproduce the classical results to order $\hbar^0$. This is indeed what has been found in previous work, and in Ch. 3 we showed that the Larmor formula itself may be directly recovered through the quantised, symmetric, stress-energy tensor.

We do not expect that the classical and quantum results will match exactly, but rather that the quantum-theoretic calculation will match the classical result at order $\hbar^0$, with the potential for quantum theory to contribute correcting terms at higher orders in $\hbar$. Consequently, in Ch. 4, we investigated the corrections to the Larmor formula in two different cases: a background potential varying only in time, and a background potential varying only in the z coordinate. The calculation proved sufficiently complicated that we needed to take a non-relativistic limit, but having done so, we were able to find that the results differed qualitatively between the two cases. In the time-dependent case, we found, from Eq. (4.27), that

$$\Delta E = -\frac{e^2\hbar}{6\pi^2 mc^5}\int \frac{dt\,dt'}{t-t'}\left[\frac{d^3\mathbf{x}}{dt^3}\cdot\frac{d^2\mathbf{x}'}{dt'^2} - \frac{d^2\mathbf{x}}{dt^2}\cdot\frac{d^3\mathbf{x}'}{dt'^3}\right].$$

In the z-dependent case, we found a result which was larger by a factor c^2, Eq. (4.63):

$$\Delta E = \frac{e^2\hbar}{12\pi^2 mc^3}\int dt\,dt'\left[\frac{\dddot{z}\,\dddot{z}'}{t-t'-i\varepsilon} - \frac{2\ddot{\mathbf{x}}_\perp\cdot\ddot{\mathbf{x}}'_\perp}{(t-t'-i\varepsilon)^3}\right]\int_t^{t'}\dot{z}^{-2}dT.$$

Applying these two formulae to a simple example, we were thus able to deduce that that the first-order quantum correction in this semiclassical approximation arises only when the relevant energy of the scalar particle is not significantly larger than the energy of the emitted photon. In the t-dependent case, the relevant energy is the mass-energy, mc^2, while in the z-dependent case, the relevant energy is the z-term of the kinetic energy, $\frac{1}{2}m{v_z}^2$.

Another interesting extension of the question of radiation emission by charged particles is to curved spacetimes. It had been shown by Roberts [27] that the classical radiation effect is conformally invariant, and consequently, it was reasonable to begin by investigating the quantum theory in a conformally-flat spacetime. In Ch. 5, we took the metric to be $g_{\mu\nu} = \Omega^2(t)\eta_{\mu\nu}$, and subjected the particle to a time-varying background potential. We showed that we could turn the theory with this metric into a theory on Minkowski spacetime at a cost of creating a time-varying mass term, ${M_c}^2(t) = m^2\Omega^2 + (\xi - \frac{1}{6})\hbar^2\Omega^2 R$;

this caused no great obstacle to recovering the classical result from the quantum field theory. The results we obtained were equivalent to the results we would obtain from a varying mass term $M^2(t) = m^2\Omega^2$, as the difference between the two is of order $\hbar^2$.

The more interesting phenomenon arose at the one-loop level. Here, we showed that a general time-dependent mass, $M(t)$, gives rise to correction terms which affect both Maxwell's equations for the background potential and also the mass renormalisation term. This difference, being of order $\hbar^{-1}$, would result in a measurable effect which was greater than the Abraham-Lorentz-Dirac force.

When we considered the conformally-flat theory, however, we found that these non-zero loop corrections vanished. Thus, although we had established an equivalence at tree level between the conformally-flat theory and the flat theory with time-dependent mass $M_c(t)$, the same does not hold for the one-loop calculation. We also noted that this breakdown of equivalence was a manifestation of the more general and well-known 'conformal anomaly'.

In Ch. 6, we pressed this one stage further and took a time-varying metric. By the use of an appropriate foliation, we were able to write this metric in a simpler form, which was more conducive to the calculations we needed to carry out. Following a similar line of argument to that produced by Martin [25], we were able to show that the tree-level position shift does not quite match the classical prediction.

Specifically, we obtained the position shift stemming from the half-retarded-minus-advanced electromagnetic field from the tree-level calculation. This term propagates on and within both light-cones, with an anti-symmetry between the forward and backward contributions. The classical result, however, derives from the radiative electromagnetic field, which does not possess the same anti-symmetry. The causal structure between the two is thus different, and we concluded that we would expect the one-loop level position shift to make a contribution at order $\hbar^0$ such that the discrepancy between the tree-level and classical calculations is bridged.

Looking to the potential for further work along these lines, we note that this work has focussed on the complex scalar field. While this provides a useful toy model of the behaviour of a spinor, the spinor theory and the scalar theory have some significant differences. For example, the scalar quantum electrodynamic theory contains an additional vertex: the interaction between two scalar particles and two photons. Also, the spinor theory exhibits a much richer structure: among other examples, a non-zero (indeed, non-integral) spin. Therefore, it would be interesting to investigate whether the results presented here are matched by the spinor quantum electrodynamic calculations.

Considering the results themselves, the $\hbar$ correction was found in two cases which involved variation in only one co-ordinate. Clearly, we would wish to have results which cover all possible background potentials. It would also be desirable to extend the calculation so that a fully relativistic result could be calculated.

The result we found is, in principle, a prediction which could be tested empirically, although real-world examples would require, as we have already said, the spinor theory. As we noted in Ch. 4, the hydrogen atom owes its spectral lines fully to quantum mechanics as classical mechanics cannot replicate this result at all. If we were to investigate the behaviour of an electron loosely bound to a considerably larger nucleus, which we call a Rydberg atom, this should provide a scenario in which the semiclassical approximation may be applied and a quantum correction at order $\hbar$ be found to the energy emitted by an electron changing levels. The resulting calculations could then be subjected to experimental test.

The work carried out in the time-dependent spacetime also requires completion. The vacuum current has been found and the tree-level calculation performed, but mass renormalisation remains to be tackled. We expect that this will close the gap which currently exists between the tree-level calculation and the classical prediction.

However, even once completed, this is not an entirely general spacetime. By restricting our case to a time-dependent metric, we were able to use the translation invariance to infer conservation of three-momentum, and so to simplify the problem we faced. It is clearly desirable that the results here be extended to a fully general metric with a fully general background potential.

Appendix A

Cut-off independence of integrals in Ch. 4

In this Appendix we show that formal integration by parts used in Ch. 4 to find the quantum corrections is justified. Take the interval $I \equiv [-T, T]$ with $T > 0$, such that the acceleration $\ddot{x}^\mu(t) \neq 0$ only if $t \in I$. Then let $f(t, t')$ be a smooth function which satisfies the following condition. The derivative $\partial_t f(t, t')$ (resp. $\partial_{t'} f(t, t')$) is supported on a subset of $I \times \mathbb{R}$ (resp. $\mathbb{R} \times I$); this implies that f and its first derivatives are bounded. We also let $g_i(t)$, for $i = 1, 2, \ldots, n$, be smooth functions such that the support of $g_i''(t)$ is a subset of I; again, this implies that $g_i'(t)$ are bounded. Finally, we introduce our cut-off factor, $\chi(t)$, which is compactly supported with $\chi(t) = 1$ for $t \in I$. We use $\chi(at)$, $0 < a \leq 1$, as our cut-off factor, and take the limit $a \to 0$ in the end. Note that this definition guarantees that

$$\lim_{a \to 0} \int_{-\infty}^{+\infty} (\chi'(at))^2 dt = 0,$$

which had been necessary to derive the Larmor formula in Ch. 3.

Then we consider the following integrals:

$$A_1^{(n)} = \int dtdt' \frac{f(t,t')}{(t-t'+i\epsilon)^{n+4}} \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi(at)\chi(at'), \tag{A.1}$$

$$A_2 = \int dtdt' \frac{\partial_t f(t,t')}{(t-t'+i\epsilon)^{n+3}} \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi(at)\chi(at'), \tag{A.2}$$

$$A_3^{(n)} = a \int dtdt' \frac{f(t,t')}{(t-t'+i\epsilon)^{n+3}} \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi'(at)\chi(at'), \tag{A.3}$$

$$A_4 = a \int dtdt' \frac{\partial_{t'} f(t,t')}{(t-t'+i\epsilon)^{n+2}} \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi'(at)\chi(at'), \tag{A.4}$$

$$A_5 = a^2 \int dtdt' \frac{f(t,t')}{(t-t'+i\epsilon)^{n+2}} \left\{ \prod_{i=1}^{n} [g_i(t) - g_i(t')] \right\} \chi'(at)\chi'(at'). \tag{A.5}$$

Our aim is to demonstrate that integrals of the form of A_1 may be written as a sum of: firstly, integrals with undifferentiated cut-off factors which are convergent without those cut-offs; and secondly, integrals which tend to zero as $a \to 0$. This implies that we may formally integrate the first category of integrals by parts, discounting the cut-off factors, until the integral is convergent. Hence, we shall find that our method in previous chapters is justified.

Turning first to A_2, the support of $\partial_t f(t,t')$ implies that the t-integral is over the interval I, so $\chi(at) = 1$. We have already noted that $\partial_t f(t,t')$ is bounded. Hence, the t-integral is convergent. As g_i' is compactly supported, it must be that there are constants $\alpha_i^{\pm}, \beta_i^{\pm}$ such that $g_i(t) = \alpha_i^{\pm} t + \beta_i^{\pm}$ as $t \to \pm\infty$. Thus,

$$\frac{\prod_{i=1}^{n}(g_i(t) - g_i(t'))}{(t-t'+i\varepsilon)^{n+3}} \to \frac{\prod_{i=1}^{n} \alpha_i^{\pm}}{(t-t'+i\varepsilon)^3} \tag{A.6}$$

as $t' \to \pm\infty$. Hence, it is clear that the t'-integral is also convergent.

Turning our attention to the integral A_4 we observe that the t'-derivative on f restricts the integral to I, which forces $\chi(at') = 1$. The factor $\chi'(at)$ restricts the t-integral to $\mathbb{R} \setminus I$, which in turn forces $\partial_{t'} f(t,t') = F(t')$ and $g_i(t) = \alpha_i^{\pm} t + \beta_i^{\pm}$ (dependent on whether t is greater or less than zero). Then the t'-integral is over a bounded region and the integrand is finite, so this integral is convergent. The t-integral is also manifestly convergent for both positive and negative t, and hence $A_4 \to 0$ as $a \to 0$.

A similar argument can be applied to the case of A_5. The derivatives on the cut-offs restrict the two integrals to four disjoint regions of the tt'-plane: $(-\infty, -T) \times (-\infty, -T)$, $(-\infty, -T) \times (T, \infty)$, $(T, \infty) \times (-\infty, -T)$, and $(T, \infty) \times (T, \infty)$. Then we note that in these regions, f takes a constant value and the g_i are linear functions. Thus, the integral over the regions with $\operatorname{sgn} t = \operatorname{sgn} t'$ (which we denote '$\pm\pm$') may be written

$$A_5|_{\pm\pm} = a^2 f_{\pm\pm} \prod_{i=1}^{n} \alpha_i^{\pm} \int_{\pm T}^{\pm\infty} dt \int_{\pm T}^{\pm\infty} dt' \frac{\chi'(at)\chi'(at')}{(t - t' + i\varepsilon)^2}; \tag{A.7}$$

the integrals where $\operatorname{sgn} t = -\operatorname{sgn} t'$ are then

$$A_5|_{\pm\mp} = a^2 f_{\pm\mp} \int_{\pm T}^{\pm\infty} dt \int_{\mp\infty}^{\mp T} dt' \frac{\chi'(at)\chi'(at')}{(t - t' + i\varepsilon)^{n+2}} \prod_{i=1}^{n} \left(\alpha_i^{\pm} t + \beta_i^{\pm} - \alpha_i^{\mp} t' - \beta_i^{\mp}\right). \tag{A.8}$$

In all cases, the integrals are finite, and hence $A_5 \to 0$ as $a \to 0$.

For the integral $A_3^{(n)}$, we can integrate this by parts with respect to t'. We attach an extra label to the integral to make clear how many terms are in the product; as will be seen, we can reduce the number of terms by the procedure of integration by parts, which proceeds as follows:

$$\begin{aligned}
A_3^{(n)} &= a \int dt\, dt' \frac{f(t,t')}{(t - t' + i\varepsilon)^{n+3}} \prod_{i=1}^{n} (g_i(t) - g_i(t'))\chi'(at)\chi(at) \\
&= -\frac{a}{n+2} \int dt\, dt' \frac{1}{(t - t' + i\varepsilon)^{n+2}} \frac{\partial}{\partial t'} \left[f(t,t') \prod_{i=1}^{n} (g_i(t) - g_i(t'))\chi(at') \right] \chi'(at) \\
&= -\frac{A_4}{n+2} - \frac{A_5}{n+2} \qquad\qquad (\text{A.9}) \\
&\quad + \frac{a}{n+2} \int dt\, dt' \frac{f(t,t')}{(t - t' + i\varepsilon)^{n+2}} \sum_{k=1}^{n} \left[g_k'(t') \prod_{i \neq k} (g_i(t) - g_i(t')) \right] \chi'(at)\chi(at').
\end{aligned}$$

Now, the first and second of these terms tend to zero as $a \to 0$ as already shown; the last, as suggested earlier, is a sum of terms of the form $A_3^{(n-1)}$ because the partial derivatives of $f(t,t')g_k'(t')$ with respect to t and t' have support in $I \times \mathbb{R}$ and $\mathbb{R} \times I$ respectively. Therefore, by repeated application of this reduction procedure, all terms of the form $A_3^{(n)}$ will have the same behaviour in the limit $a \to 0$ as the term $A_3^{(0)}$. This term is

$$\begin{aligned} A_3^{(0)} &= -\frac{a}{2}\int dt\,dt' \frac{1}{(t-t'+i\varepsilon)^2}\frac{\partial}{\partial t'}\left[f(t,t')\chi(at')\right]\chi'(at) \\ &= -\frac{1}{2}A_4 - \frac{1}{2}A_5, \end{aligned} \tag{A.10}$$

and we have already seen that terms of these forms vanish in the limit $a \to 0$.

Finally, we are able to turn to the first integral in the list, which is the form of the ones we encounter in our calculations. Again, we integrate by parts:

$$\begin{aligned} A_1^{(n)} &= \frac{1}{n+3}\int dt\,dt' \frac{1}{(t-t'+i\varepsilon)^{n+3}}\frac{\partial}{\partial t}\left[f(t,t')\prod_{i=1}^{n}(g_i(t)-g_i(t'))\chi(at)\right]\chi(at') \\ &= \frac{A_2}{n+3} + \frac{A_3^{(n)}}{n+3} \\ &\quad + \frac{1}{n+3}\int dt\,dt' \frac{f(t,t')}{(t-t'+i\varepsilon)^{n+3}}\sum_{k=1}^{n}\left[g_k'(t)\prod_{i\neq k}(g_i(t)-g_i(t'))\right]\chi(at)\chi(at'). \end{aligned} \tag{A.11}$$

The $A_3^{(n)}$ term has already been shown to vanish in the limit, and the A_2 term is independent of the cut-off. The last term is a sum of terms of the form $A_1^{(n-1)}$. Again, therefore, we can continue to integrate by parts until we reach terms of the form $A_1^{(0)}$, to which we can now turn. This integral is

$$\begin{aligned} A_1^{(0)} &= \int dt\,dt' \frac{f(t,t')}{(t-t'+i\varepsilon)^4}\chi(at)\chi(at') \\ &= \frac{1}{3}A_2^{(0)} + \frac{1}{3}A_3^{(0)}, \end{aligned} \tag{A.12}$$

and as we have already established, the latter vanishes in the limit while the former is independent of the cut-off.

Thus we have shown that we may integrate any integral of the form A_1 with respect to t (or t'), disregarding the cut-off factor until the integral is rendered convergent without it.

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