DISCRETE LAX SYSTEMS AND INTEGRABLE LATTICE EQUATIONS ASSOCIATED WITH ELLIPTIC CURVES

Neslihan Delice

Department of Applied Mathematics

University of Leeds

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The candidate confirms that the work submitted is her own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

- Chapter 3 is based on N. Delice, F.W. Nijhoff and S. Yoo-Kong, On elliptic Lax systems on the lattice and a compound theorem for hyperdeterminants, Journal of Physics A **48** (2015), 035206.

The contribution of the candidate to the work was to perform the majority of the computations and provide the main proofs. The ideas were developed in discussion with Professor Nijhoff. All the essential computations were carried out by myself.

To my parents, Bukriye and Hasan

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Abstract

This thesis deals with discrete Lax systems and integrable lattice equations (i.e., partial difference equations (P∆Es)) associated with elliptic curves. We will be concerned with their derivation and integrability properties, as well as with certain reductions. In particular the construction of a new class of higher-rank elliptic type integrable system forms one of the core results, opening new avenues of investigation.

The primary integrable system of interest is Adler's equation (nowadays often referred to as Q_4), which is a lattice version of the Krichever-Novikov (KN) equation. For this equation we exhibit a new Lax pair, the compatibility of which yields the equation in its so-called 3-leg form and which forms a starting point for the investigation in this thesis. It is this particular Lax pair that is most readily generalized to higher-rank cases, in contrast to other known Lax pairs for Q_4. In fact, the most general class of higher-rank Lax pairs contains not only higher-rank versions of Q_4 but also equations which are conjectured to be related to discrete versions of the Landau-Lifschitz (LL) equations. We will briefly treat the latter, but our main focus will be on the class of higher-rank systems related to Adler's lattice equation.

Furthermore, by considering limits on the solutions, whereby the curve degenerates, we will propose higher-rank analogues of various equations in the well-known ABS list. Finally, we will set up a general scheme that corresponds to isomonodromic deformations on the torus, from which non-autonomous elliptic type difference equations can be derived.

Contents

List of Figures

Chapter 1

Introduction

This thesis is concerned with integrable partial difference equations (lattice equations) associated with elliptic curves. The prime example of such a system is the lattice Krichever-Novikov (KN) equation (or Adler's lattice), which will be studied in detail in chapter 2, and which has a close connection, through the Lagrangian aspects, to the modern theory of elliptic hypergeometric functions.

A general elliptic $N \times N$ matrix Lax scheme is presented, leading to two classes of elliptic lattice systems, one which we interpret as the higher-rank analogue of the Landau-Lifschitz (LL) equations, while the other class we characterize as the higher-rank analogue of Adler's lattice equation. We present the general scheme, but focus mainly on the latter type of models. In the case $N = 2$ a novel Lax representation of Adler's elliptic lattice equation in its so-called 3-leg form is obtained. This Lax pair was presented in [26]. The case of rank $N = 3$ is analyzed using Cayley's hyperdeterminant of format $2 \times 2 \times 2$, yielding a multi-component system of coupled 3-leg quad-equations. Moreover, the elliptic discrete isomonodromic deformation problem, which leads to non-autonomous elliptic lattice equations, has been considered.

In this introductory chapter we collect a number of aspects of the theory that come together in this subject: the elliptic Gamma function and elliptic Beta integral and its relation to the lattice equations under consideration, the general theory of lattice equations integrable in the sense of multidimensional consistency, similarity reductions and isomonodromic deformation problems and the technique of de-autonomization. The results in this chapter are not new but present some of the state-of-the-art ingredients needed for the main topic of the thesis.

The first section of this introduction gives some idea of the theory of elliptic functions and presents some useful formulae which are needed to prove of some relations in this thesis. The next section covered here is an overview of different types of Gamma functions (classic, basic and elliptic) related to three classes of the hypergeometric functions theory and the aspect of elliptic Beta integral interpreted as a star-triangle relation in statistical mechanics. This section of the introduction concentrates on the results given by Spiridonov and Bazhanov et al. [19, 97]. This is followed by a short overview of the integrability of discrete systems, in particular their multi-dimensional consistency. The final topic of this chapter is the theory of isomonodromic deformation problem followed by an outline of the thesis.

1.1 Elliptic functions and their functional relations

An elliptic function is a meromorphic function in the complex plane with two periods ω_1 and ω_2 (ω_1 and ω_2 are only half-periods) such that $\frac{\omega_2}{\omega_1}$ is not real. The theory of elliptic functions has been studied by Abel, Euler, Jacobi, Legendre [1, 49, 61] and others. An important contribution in the subject of elliptic functions has been provided by Weierstrass who introduced what is now called the Weierstrass $\wp$ function [106]. A comprehensive treatise of the theory is given in many textbooks [11, 18, 24, 107] as well as [74]. Many lattice systems covered in the thesis rely on addition formulae for the Weierstrass functions σ, ζ and $\wp$ which will play a central role throughout the thesis. We will first focus on the sigma-function of Weierstrass, $\sigma(z)$, defined by

$$\sigma(z) = z \prod_{(m,n)\neq(0,0)}^{\infty} (1 - \frac{z}{\Omega_{mn}}) \exp\left[\frac{z}{\Omega_{mn}} + \frac{1}{2}\frac{z^2}{\Omega_{mn}^2}\right] , \qquad (1.1)$$

with $\Omega_{mn} = 2m\omega_1 + 2n\omega_2$ and $2\omega_{1,2}$ being a fixed pair of the primitive periods. Alternatively $\sigma(z)$ can be represented in terms of the theta function θ_{11}

$$\sigma(z) = 2\omega_1 \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right)^2 \frac{\theta_{11}(x|\tau)}{\theta'_{11}(0|\tau)} , \quad \tau = \frac{\omega_2}{\omega_1} , \, z = 2\omega_1 x , \qquad (1.2)$$

where $\eta_1 = \zeta(\omega_1)$. We refer to the Appendix A for properties of the theta-functions, from which corresponding properties of the sigma function are inherited. Furthermore, the connections between the standard Weierstrass functions are given

$$\zeta(z) = \frac{d}{dz}(\log \sigma(z)) = \frac{\sigma'(z)}{\sigma(z)} , \qquad \wp(z) = -\frac{d\zeta(z)}{dz} ,$$

where $\sigma(z)$, $\zeta(z)$ are odd functions of z and $\wp(z)$ is an even function. By differentiation (1.1), we have the following expressions:

$$\zeta(z) = \frac{1}{z} + \sum_{(m,n)\neq(0,0)}^{\infty} \left(\frac{1}{z-\Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2}\right) , \qquad (1.3a)$$

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)}^{\infty} \left(\frac{1}{(z-\Omega_{mn})^2} - \frac{1}{\Omega_{mn}^2}\right) . \qquad (1.3b)$$

We note that $\sigma(z)$ is an entire function with its simple zeros at Ω_{mn}. The Weierstrass functions satisfy a number of addition formulas that are functional relations and valid for arbitrary values of their arguments. These functional relations are interconnected. The most fundamental one in the theory is the three-term identity:

$$\sigma(x+a)\sigma(x-a)\sigma(y+b)\sigma(y-b) - \sigma(x+b)\sigma(x-b)\sigma(y+a)\sigma(y-a) = \sigma(x+y)\sigma(x-y)\sigma(a+b)\sigma(a-b) . \qquad (1.4)$$

This addition formula, which is a direct consequence of the parallel formula for the θ-functions (A.7), can be rewritten as

$$\Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y)\,\Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x)\,\Phi_\lambda(y-x) , \qquad (1.5)$$

where the (truncated) Lamé function Φ_κ is given by

$$\Phi_\kappa(x) \equiv \frac{\sigma(x+\kappa)}{\sigma(x)\sigma(\kappa)} \ , \tag{1.6}$$

with some complex numbers κ. A particular limit of (1.5) as $\lambda \to \kappa$ yields the following

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa)+\zeta(x)+\zeta(y)-\zeta(\kappa+x+y)\right] \ , \tag{1.7}$$

which is equivalent to the well-known identity for ζ-function

$$\zeta(x)+\zeta(y)+\zeta(z)-\zeta(x+y+z) = \frac{\sigma(x+y)\,\sigma(y+z)\,\sigma(x+z)}{\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(x+y+z)} \ . \tag{1.8}$$

Furthermore, we have the addition formulae for the Weierstrass $\wp$-function

$$\wp(x)-\wp(y) = \frac{\sigma(x+y)\,\sigma(y-x)}{\sigma^2(x)\,\sigma^2(y)} \quad , \quad \wp'(x) = -\frac{\sigma(2x)}{\sigma^4(x)} \ , \tag{1.9}$$

or:

$$\Phi_\kappa(x)\Phi_{-\kappa}(x) = \wp(x)-\wp(\kappa) \ . \tag{1.10}$$

The generalization of the basic identity (3-term relation for the σ-function (1.4) or the elliptic partial fraction expansion formula (1.5) for the Φ) is:

$$\prod_{i=1}^{n} \Phi_{\kappa_i}(x_i) = \sum_{i=1}^{n} \Phi_{\kappa_1+\ldots+\kappa_n}(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{n} \Phi_{\kappa_j}(x_j-x_i) \ , \tag{1.11}$$

where x_i, κ_i are any non-singular fixed values. This identity has a key role in this thesis. Extending the identity (1.11) (or (1.7)) to $n+1$ variables, including κ_0 and x_0, and subsequently taking the limit $x_0 = x_1 + \varepsilon$, with $\varepsilon \to 0$, we obtain the following identity (after some obvious relabeling of parameters and changes of variables):

$$\begin{gathered}(-1)^{n-1}\Phi_{\kappa_0+\kappa_1+\cdots+\kappa_n}(x_1+\cdots+x_n)\,\frac{\sigma(x_1+\cdots+x_n)}{\prod_{j=1}^{n}\,\sigma(x_j)} \\ \times\left[\zeta(\kappa_0)+\sum_{j=1}^{n}\left(\zeta(\kappa_j)+\zeta(x_j)\right)-\zeta(\kappa_0+\kappa_1+\cdots+\kappa_n+x_1+\cdots+x_n)\right] = \\ \sum_{i=1}^{n}\,\Phi_{\kappa_0+\kappa_1+\cdots+\kappa_n}(x_1+\cdots+\not{x}_i+\cdots+x_n)\,\frac{\sigma(x_1+\cdots+\not{x}_i+\cdots+x_n)\,\sigma^{n-1}(x_i)}{\prod_{\substack{j=1\\ j\neq i}}^{n}\,\sigma(x_i-x_j)}\,\prod_{j=0}^{n}\,\Phi_{\kappa_j}(x_i) \ .\end{gathered} \tag{1.12}$$

Equation (1.12) can be derived from (1.11) by systematic limits, but we omit details of the proof.

1.2 Gamma functions

The history of the classical and basic hypergeometric functions associated with the different types of Gamma functions spans over several centuries. Some introductory overviews on this aspect, and the main results derived in the past by protagonists in the field, are given in $[12,\ 13,\ 35]$, which are the standard reference books for the theory of special functions of hypergeometric type. In this section, we shall follow the treatment given in [101].

The initial important instance of the hypergeometric theory is the Gauss hypergeometric function ${}_2F_1$ related to Euler's classical Gamma function. This function, $\Gamma_e(\xi)$, is defined as an infinite integral representation for $\Re(\xi) > 0$ of the form

$$\Gamma_e(\xi) = \int_0^\infty t^{\xi-1}e^{-t}dt. \tag{1.13}$$

It can be shown from the definition that the Euler Gamma function is analytic for $\Re(\xi) > 0$, has simple poles at $\xi \in \mathbb{Z}_{\leq 0}$ and no zeros. Let us consider the following proposition resulting from the definition of Γ_e.

Proposition 1.2.1 *1. The Euler Gamma funcion $\Gamma_e(\xi)$ satisfies the first order difference equation for $\xi \in \mathbb{Z}_{>0}$*

$$\Gamma_e(\xi + 1) = \xi\Gamma_e(\xi), \tag{1.14}$$

2. It satisfies the Euler's reflection formulas

$$\Gamma_e(\xi)\Gamma_e(1 - \xi) = \frac{\pi}{\sin \pi\xi}. \tag{1.15}$$

Proof

A proof of the first functional equation and the reflection equation for the Gamma function can be found in [107]. □

The trigonometric analogue of the Euler Gamma function is defined as an infinite product form (or q-Gamma function Γ_q)

$$\Gamma_q(\xi) = \frac{(q;q)_\infty}{(q^\xi;q)_\infty(1-q)^{\xi-1}}, \quad (\xi;q)_\infty = \prod_{k=0}^{\infty}(1-\xi q^k). \tag{1.16}$$

In the basic hypergeometric theory we have an extra parameter q with $|q| < 1$ which is fixed. The following relation is satisfied by the q-Gamma function.

Proposition 1.2.2 *Observe that the q-Gamma function satisfies the q-difference equation*

$$\Gamma_q(\xi+1) = \frac{1-q^\xi}{1-q}\Gamma_q(\xi), \tag{1.17}$$

as an analogue of (1.14).

Proof

A proof of the functional equation for Γ_q follows from its definition [35]. □

The q-analogue of the Gauss hypergeometric function, the so-called *basic* hypergeometric function (where the parameter q is referred to as the *base*), is denoted by ${}_2\phi_1(a,b;c,\xi)$ and was introduced by Heine [13].

The elliptic analogue of the other Gamma functions (1.13) and (1.16) is defined by the infinite product [87]

$$\Gamma(\xi;p,q) = \frac{(pq\xi^{-1};p,q)_\infty}{(\xi;p,q)_\infty} \quad \text{where} \quad (\xi;p,q)_\infty = \prod_{i,j=0}^{\infty}(1-\xi p^i q^j), \tag{1.18}$$

for $\xi \in \mathbb{C}\setminus\{0\}$. On the elliptic level there exists two extra parameters $p, q \in \mathbb{C}$ satisfying $|q|$, $|p| < 1$. Furthermore, observe that the product representation of the elliptic Gamma function shows explicitly the poles, zeros and singularity, namely $\Gamma(\xi)$ has simple poles at

ξ equals to $p^{-i}q^{-j}$ for non-negative integers i, j, zeros at $\xi = p^{i+1}q^{j+1}$ for $i, j \in \mathbb{Z}_{\geq 0}$ and essential singularity at ξ equals to zero. The other Gamma functions can be obtained by taking the limit of elliptic parameters and ξ. We shall continue the discussion by giving a following proposition for elliptic Gamma function included.

Proposition 1.2.3 *1. The elliptic Gamma function possesses the reflection property*

$$\Gamma(\xi; p, q)\Gamma(pq\xi^{-1}; p, q) = 1. \tag{1.19}$$

2. *It satisfies the following difference equation*

$$\Gamma(q\xi; p, q) = \frac{1}{(p;p)_\infty}\vartheta(\xi; p)\Gamma(\xi; p, q), \tag{1.20}$$

(a similar relation with $q \leftrightarrow p$) where $\vartheta(\xi; p) = (p;p)_\infty(\xi;p)_\infty(\xi^{-1}p;p)_\infty$ is a multiplicative theta function in the normalization corresponding to the Jacobi's triple product identity.

3. *It satisfies*

$$\Gamma(\xi; p, 0) = \frac{1}{(\xi, p)_\infty}, \tag{1.21}$$

and similar equations obtained by interchanging p and q due to its symmetry;

$$\Gamma(\xi; p, q) = \Gamma(\xi; q, p).$$

Proof

The proof of the reflection property of the elliptic Gamma function is trivial as it can be obtained from its definition. The difference equation for the elliptic Gamma function follows directly from its product representation and the q-difference equations for the infinite product

$$(q\xi; p, q)_\infty = \frac{(\xi; p, q)_\infty}{(\xi; p)_\infty}, \quad (p\xi; p, q)_\infty = \frac{(\xi; p, q)_\infty}{(\xi; q)_\infty}. \tag{1.22}$$

It can be easily shown that the relation between p-shifted factorials and the function (1.18) is composed of setting $q = 0$ in the definition of the elliptic Gamma function. $\square$
The elliptic Gamma function forms a basic ingredient in the theory of elliptic hypergeometric functions. This theory known as the top level of the classical hypergeometric functions was introduced in the work of Frenkel and Turaev [30] in which a connection is established between the elliptic Boltzmann weights, or elliptic 6j-symbols and values of terminating ${}_{12}V_{11}$ elliptic hypergeometric series. Many formulas found in the classic and basic level have been generalized to the elliptic case [95]. For further consideration on the subject we indicatively refer to Spiridonov's overviews [96, 97].

Let τ and σ be complex numbers lying in the right half of the complex plane, although the product formula (1.18) for the elliptic Gamma function is taken as a definition of $\Gamma(\xi)$, it can be interpreted as an infinite series representation

$$\Gamma(e^{-2i(\xi - i\eta)}; p, q) = \exp\left\{ \sum_{k\neq 0}^{\infty} \frac{e^{-2i\xi k}}{k(p^{-k/2} - p^{k/2})(q^{-k/2} - q^{k/2})} \right\}, \tag{1.23}$$

where the parameter η connected to the nomes p and q as

$$e^{-2\eta} = pq, \quad \text{with} \quad p = e^{2\pi i\tau}, \quad q = e^{2\pi i\sigma}.$$

This representation can be obtained easily from the following formula

$$\prod_{j,k=0}^{\infty} (1 - \xi p^j q^k) = \exp\left\{ -\sum_{k=1}^{\infty} \frac{\xi^k}{k(1-p^k)(1-q^k)} \right\}, \quad |q|, |p| < 1. \tag{1.24}$$

Relevant to this thesis is the elliptic Beta integral introduced by Spiridonov [94]. The first integral identity involving the elliptic Gamma function in the elliptic level is known as the elliptic analogue of the Euler's Beta integral in the theory of classical hypergeometric functions. Next, we shall focus on the proof given by Spiridonov of this integral identity from the different perspective by applying a new relation (1.11).

Theorem 1.2.1 (Elliptic Beta Integral [94]) *Let the six complex parameters,* $t = (t_1, t_2, ..., t_6)$ *satisfy* $|\, t_k| < 1$ *and the balancing condition* $Y = \prod_{k=1}^{5} t_k = p\, q/t_6$. *Then*

$$\frac{1}{4\pi i}\int_{\mathbb{T}} \Delta_e(\xi; t_1, t_2, ..., t_5)\,\frac{\mathrm{d}\xi}{\xi} = \frac{1}{(p;p)_\infty (q;q)_\infty}\ , \tag{1.25}$$

where $\mathbb{T}$ *is the positively oriented unit circle and* $\Delta_e(\xi; t_1, t_2, ..., t_5)$ *is defined as*

$$\Delta_e(\xi; t_1, t_2, ..., t_5) = \frac{\prod_{i=1}^{5} \Gamma(t_i\xi, t_i\xi^{-1}, Y\, t_i^{-1}; p, q)}{\Gamma(\xi^2, \xi^{-2}, Y\,\xi, Y\,\xi^{-1}; p, q)\prod_{1\le i<j\le 5}\Gamma(t_it_j; p, q)}, \tag{1.26}$$

where the following convention is used

$$\Gamma(x, y, z; p, q) := \Gamma(x; p, q)\Gamma(y; p, q)\Gamma(z; p, q). \tag{1.27}$$

Proof

The proof follows the one given in [97], but we add a new element to the proof by using the higher degree identity (1.11) for N=3 case

$$\begin{aligned}\Phi_{\kappa_1}(x_1)\Phi_{\kappa_2}(x_2)\Phi_{\kappa_3}(x_3) &= \Phi_{\kappa_1+\kappa_2+\kappa_3}(x_1)\Phi_{\kappa_2}(x_2-x_1)\Phi_{\kappa_3}(x_3-x_1)\\ &+\Phi_{\kappa_1}(x_1-x_2)\Phi_{\kappa_1+\kappa_2+\kappa_3}(x_2)\Phi_{\kappa_3}(x_3-x_2) + \Phi_{\kappa_1}(x_1-x_3)\Phi_{\kappa_2}(x_2-x_3)\Phi_{\kappa_1+\kappa_2+\kappa_3}(x_3).\end{aligned} \tag{1.28}$$

This is the trilinear relation involving six free parameters, each term contains a product of three Φ functions in (1.6). If we use the following relation between Weierstrass σ-function and the Jacobi type theta function, ϑ,

$$\vartheta(e^{2\pi i z}; e^{2\pi i\tau}) = -\frac{i}{2\omega_1} e^{\pi i(z-\tau/4) - 2\omega_1\zeta(\omega_1)z^2}\sigma(2\omega_1 z)\theta'_{11}(0), \quad (\tau = \frac{w_1}{w_2}), \tag{1.29}$$

where $\omega_1,\ \omega_2$ complex variable acts linearly independent in the right half-line and θ_{11} is the Jacobi theta function (see Appendix A), then the relation (1.28) can be rewritten as

$$\begin{aligned}&\vartheta(\alpha y; p)\vartheta(w\beta; p)\vartheta(x\gamma; p)\vartheta(\alpha\beta\gamma; p)\vartheta(wy^{-1}; p)\vartheta(yx^{-1}; p)\vartheta(wx^{-1}; p)\\ &\quad-\vartheta(yx^{-1}\alpha; p)\vartheta(wx^{-1}\beta; p)\vartheta(\alpha\beta\gamma x; p)\vartheta(y; p)\vartheta(w; p)\vartheta(\gamma; p)\vartheta(wy^{-1}; p)\\ &\quad= \frac{w^2}{yx}\vartheta(yw^{-1}\alpha; p)\vartheta(\alpha\beta\gamma w; p)\vartheta(xw^{-1}\gamma; p)\vartheta(\beta; p)\vartheta(y; p)\vartheta(x; p)\vartheta(yx^{-1}; p)\\ &\qquad-\frac{y}{x}\vartheta(\alpha\beta\gamma y; p)\vartheta(xy^{-1}\gamma; p)\vartheta(wy^{-1}\beta; p)\vartheta(\alpha; p)\vartheta(w; p)\vartheta(x; p)\vartheta(wx^{-1}; p)\ ,\end{aligned} \tag{1.30}$$

where $\alpha,\ y,\ w,\ \beta,\ x,\ \gamma$ are arbitrary complex variables. We observe that (1.30) is a four-term identity containing ”six” free parameters, each term including a product of seven theta functions. We now substitute

$$\begin{array}{rclcrcl} y & \to & Y\xi^{-1} & , & \alpha & \to & \xi t_2^{-1}\ , \\ w & \to & Y\xi & , & \beta & \to & \xi^{-1}t_3^{-1}\ , \\ x & \to & Yt_1 & , & \gamma & \to & t_1^{-1}t_4^{-1}, \end{array}$$

into the theta function identity (1.30) to derive the following form

$$\vartheta(t_1\xi;p)\vartheta(t_1\xi^{-1};p)\vartheta(\xi^2;p)\prod_{k=2}^{5}\vartheta(Yt_k^{-1};p)-\vartheta(Y\xi;p)\vartheta(Y\xi^{-1};p)\vartheta(\xi^2;p)\prod_{k=2}^{5}\vartheta(t_1t_k;p)$$
$$=\xi^3t_1\vartheta(t_1Y;p)\vartheta(Y\xi;p)\prod_{k=1}^{5}\vartheta(t_k\xi^{-1};p)-t_1\xi^{-1}\vartheta(Yt_1;p)\vartheta(Y\xi^{-1};p)\prod_{k=1}^{5}\vartheta(t_k\xi;p). \tag{1.31}$$

Multiplying both sides of this equality by $\Delta_e(\xi,t_1,...,t_5)$ in order to obtain the q-difference equation

$$\Delta_e(\xi,qt_1,t_2,...,t_5)-\Delta_e(\xi,t_1,...,t_5)=f(q^{-1}\xi,t_1,...,t_5)-f(\xi,t_1,...,t_5)\ , \tag{1.32}$$

where

$$f(\xi,t_1,...,t_5)=\Delta_e(\xi,t_1,...,t_5)\frac{\prod_{k=1}^5\vartheta(t_k\xi;p)\vartheta(t_1Y;p)}{\prod_{k=2}^5\vartheta(t_1t_k;p)\vartheta(\xi^2;p)\vartheta(Y\xi;p)}\frac{t_1}{\xi}\ . \tag{1.33}$$

Thus, (1.32) is integrated over the variable ξ to obtain zero on the right hand side and by applying the residue theorem to the resulting integrals (as described in [102]), we are led to the equality (1.25). □

The trigonometric limit of the elliptic Beta integral (1.25) where an elliptic nome $p\to 0$ (or $q\to 0$) constructs the Nasrallah-Rahman q-Beta integral $[86]$ which is one parameter generalization of the Askey-Wilson integral [13]. Recently an important connection between the theory of elliptic hypergeometric functions and solvable models of statistical mechanics has been discovered by Bazhanov and Sergeev, [19] demonstrating that the elliptic Beta integral (1.25) provides a new solution of the star-triangle relation with the Boltzmann weight given by the elliptic Gamma function. The equivalence between the elliptic Beta integral and the star-triangle relation given in [19, 96] will be reviewed in the next section by following treatment in [56, 96].

1.2.1 The elliptic Beta integral solution of star-triangle relation

The elliptic Beta integral appears in statistical mechanics as a star-triangle relation

$$\begin{aligned}\int_0^{2\pi} & S(u;p,q)W(\eta-\alpha;\widetilde{u},u)W(\eta-\gamma;\widehat{u},u)W(\alpha+\gamma;\widehat{\widetilde{u}},u)du \\ &= C(\alpha,\gamma;p,q)W(\eta-\alpha-\gamma;\widehat{u},\widetilde{u})W(\gamma;\widetilde{u},\widehat{\widetilde{u}})W(\alpha;\widehat{u},\widehat{\widetilde{u}}),\end{aligned} \tag{1.34}$$

where the Boltzmann weights $W(\alpha)$ and $S(u)$ are given in terms of the elliptic Gamma function (1.23) as

$$W(\alpha;u,\omega)=\frac{\Gamma(e^{-i(u-\omega+i(\alpha-\eta))};p,q)\Gamma(e^{-i(u+\omega+i(\alpha-\eta))};p,q)}{\Gamma(e^{-i(u-\omega-i(\alpha+\eta))};p,q)\Gamma(e^{-i(u+\omega-i(\alpha+\eta))};p,q)}, \tag{1.35}$$

and

$$S(u;p,q)=\frac{(p;p)_\infty(q;q)_\infty}{4\pi\Gamma(e^{2iu};p,q)\Gamma(e^{-2iu};p,q)}. \tag{1.36}$$

Here C depends explicitly on the spectral parameters α and γ as $C(\alpha,\gamma;p,q) = \Gamma(e^{-2\alpha},e^{2(\alpha+\gamma-\eta)},e^{-2\gamma};p,q)$ [19]. Figure 1.1 given below is the star-triangle relation in its graphical form[1]. The integral relation depends on three spectral parameters α, γ, η and three spins variables $\widetilde{u}$, $\widehat{u}$, $\widehat{\widetilde{u}}$ sitting at the white vertices. The integration over the spin variable u is located at the black vertex of the star-shaped on the left-hand side appearing in the Boltzmann weights $S(u)$ and $W(\alpha)$.

[1]The star-triangle relation as depicted in Figure 1.1 has its origin in statistical mechanics, namely as a special relation for Boltzmann weights associated with exactly solved models, see e.g. [17]. Actually, the simplest example of a star-triangle relation can be found in the Kirchhoff laws of electric network theory [57].

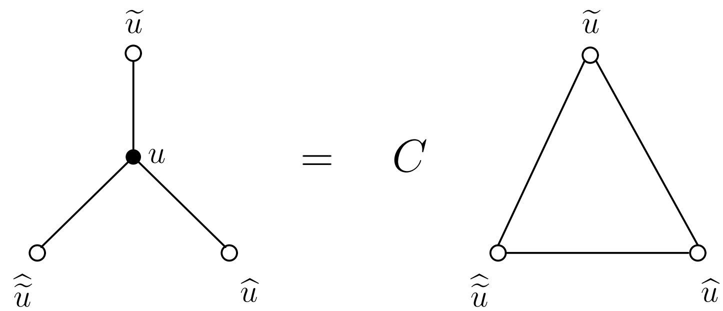


Figure 1.1: The star-triangle relation.

The equivalence between (1.25) and (1.34) can be seen by using the reflection property (1.19) and choosing the following variables for t_k in the theorem 1.2.1

$$t_1 = e^{-\alpha+i\widetilde{u}}, \qquad t_2 = e^{-\alpha-i\widetilde{u}}, \qquad t_3 = e^{-\gamma+i\widehat{u}},$$
$$t_4 = e^{-\gamma-i\widehat{u}}, \qquad t_5 = e^{\alpha+\gamma-\eta+i\widehat{\widehat{u}}}, \qquad t_6 = e^{\alpha+\gamma-\eta-i\widehat{\widehat{u}}},$$

as well as $\xi = e^{iu}$. One can show that the form of star-triangle relation (1.34) does not change when we replace W and C by

$$\overline{W}(\alpha; u, \omega) = K^{-1}(\alpha) W(\alpha; u, \omega), \tag{1.37a}$$
$$\overline{C}(\alpha, \gamma; p, q) = \frac{K(\alpha)\, K(\gamma)\, K(\eta-\alpha-\gamma)}{K(\eta-\alpha)\, K(\eta-\gamma)\, K(\alpha+\gamma)} C(\alpha, \gamma; p, q), \tag{1.37b}$$

for an arbitrary normalization function $K(\alpha)$. To get the expression for $\overline{C}$ which is equal to unity, $\overline{C} = 1$, one can introduce the function

$$K(\alpha) = \exp\left(\sum_{n\in\mathbb{Z}/\{0\}} \frac{p^n q^n e^{2n\alpha}}{n(1-p^n)(1-q^n)(1+p^n q^n)} \right), \tag{1.38}$$

satisfies

$$\frac{K(\alpha)}{K(\eta-\alpha)} \Gamma(e^{-2\alpha}; p, q) = 1 , \qquad K(\alpha)K(-\alpha) = 1. \tag{1.39}$$

The new solution of the star-triangle relation has a bearing on the work in the thesis. Because the Lagrangian form of the discrete integrable equation Q_4 of the ABS classification comes up as a quasi-classical limit of the Boltzmann weights satisfying the star-triangle relation [19]. As a consequence, all the other equations in the ABS list arise as limiting cases [8]. The appearance of the "top" equation Q_4 in the ABS list of affine-linear quadrilateral equations, which was introduced by V. Adler [5] as the permutability condition of the Bäcklund transformations of the KN equation, is remarkable observation.

1.3 Discrete integrable systems

Discrete integrable systems have received a lot of attentions in recent years and contribute to the development of a variety of different fields in mathematics and physics, such as special function theory, numerical analysis, difference geometry and quantum field theory. In particular, they appear in the field of statistical mechanics, for example as a quasi-classical limit of the new solution of the star-triangle relation [19]. Besides the important application to mathematics and physics, discrete systems are also a fast growing field of computer science. Some existing reviews on this subject involve the book [38, 42], and introductory overviews by Nijhoff, [74] and introductory lecture notes taught at the University of Leeds.

The discrete equations appear in the form of difference equations, which are the analogues of differential equations in the continuous theory of integrable systems. Although the theory of difference equations, in its current state, is not as advanced as the theory of differential equations, at the same time in general the former theory is richer as well as more generic. In developing the theory of integrable difference equations, part of the research is focused on the question of what is the proper definition of integrability, and several properties of those difference equations have been proposed as integrability detectors. Many integrable difference systems have been given by discretizing known (integrable) ordinary differential equations (ODEs) and partial differential equations (PDEs). Early examples of discrete integrable equations involving the *korteweg-de Vries* (KdV) equation, *modified KdV* (mKdV) and *sine-Gordon* (sG) equation for instance, were derived by Hirota in [43, 44, 45, 46]. The discretisations appear by taking the exponentiation of a differential operator in Hirota's approach. The "Dutch" school (Capel, Nijhoff, Quispel et al.) derived integrable P∆Es via a Direct Linearization (DL) method, [64, 85], first proposed by Fokas and Ablowitz in 1981, [29] for the specific continuous case of the KdV and the Painlevé II equation.

Another powerful test for integrability in the discrete case is the technique of *singularity confinement*, which was proposed by Grammaticos, Papageorgiou and Ramani in [37] as a proper candidate for a discrete analogue of the Painlevé property. This technique, used to find discrete version of the Painlevé equations, analyzes the initial value problem of a given equation when a singularity of it appears. However, in 1999, Hietarinta and Viallet [40] showed that singularity confinement is a necessary but not sufficient condition for predicting integrability.

Another important integrability test for the discrete system is *3-dimensional consistency* or *Consistency-around-the-cube* (C.A.C) proposed as a feature of integrable P∆Es by Nijhoff et al. in [70]. C.A.C has been used as a tool to investigate and classify lattice equations in [7]. As a consequence of this property one may immediately construct the Lax pairs of the discrete system. We shall focus more closely on the 3D-consistency condition in the next section.

1.3.1 Quadrilateral lattice equations: Multi-dimensional consistency

Two-dimensional lattice equations within the class of *quadrilateral* P∆Es have the following form:

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = 0, \tag{1.40}$$

where the fields $u = u(n, m)$ is the dependent variable, with the shifted variables $\widetilde{u} = u(n+1, m)$, $\widehat{u} = u(n, m+1)$ and $\widehat{\widetilde{u}} = u(n+1, m+1)$ defining the different values of u at the vertices around an elementary plaquette on a rectangular lattice, see Figure 1.2-(a). The spectral parameters α and β are lattice parameters corresponding to lattice direction n, m and attached to the edges of quadrilateral. The fields $u = u(n, m)$ and its shifts are assigned to the vertices of the square lattice. In [7], the lattice equation (1.40) was considered in the classification study of quadrilateral lattices, where has the property of "3D-consistency" or C.A.C. This property was first put forward by Nijhoff and Walker [71] in the study of higher order similarity reductions of integrable PΔEs of KdV type, as a key integrability feature. The CAC property is nowadays regarded as a definition of integrability of 2D lattice equations of the form $Q = 0$, allowing the equations to be consistently embedded in a higher-dimensional lattice. More specifically, applying the equation $Q = 0$ on three elementary plaquette of the cube in Figure 1.2-(b) yields

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = 0, \quad \rightarrow \quad \widehat{\widetilde{u}} = F(u, \widetilde{u}, \widehat{u}; \alpha, \beta), \tag{1.41a}$$

$$Q(u, \widetilde{u}, \overline{u}, \widetilde{\overline{u}}; \alpha, \kappa) = 0, \quad \rightarrow \quad \widetilde{\overline{u}} = F(u, \widetilde{u}, \overline{u}; \beta, \kappa), \tag{1.41b}$$

$$Q(u, \widehat{u}, \overline{u}, \widehat{\overline{u}}; \beta, \kappa) = 0, \quad \rightarrow \quad \widehat{\overline{u}} = F(u, \widehat{u}, \overline{u}; \beta, \kappa), \tag{1.41c}$$

where the given third direction indicated by the shift $^{-}$ denotes a shift in the third independent variable h which is associated with the lattice parameter κ. Substituting the solutions F of (1.41) into the equation $Q = 0$ on the remaining faces of the cube we obtain three separate relations for $\widehat{\widetilde{\overline{u}}} = u(n+1, m+1, h+1)$, namely:

$$Q(\overline{u}, \widetilde{\overline{u}}, \widehat{\overline{u}}, \widehat{\widetilde{\overline{u}}}; \alpha, \beta) = 0, \quad \text{and} \tag{1.42a}$$

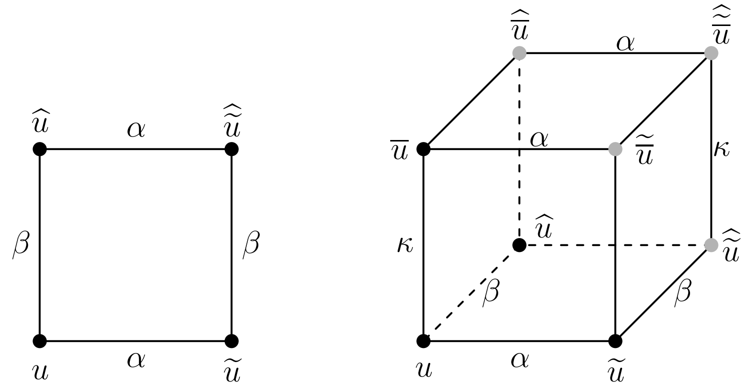
$$Q(\widehat{u}, \widehat{\widetilde{u}}, \widehat{\overline{u}}, \widehat{\widetilde{\overline{u}}}; \alpha, \kappa) = 0, \quad \text{and} \tag{1.42b}$$

$$Q(\widetilde{u}, \widehat{\widetilde{u}}, \widetilde{\overline{u}}, \widehat{\widetilde{\overline{u}}}; \beta, \kappa) = 0. \tag{1.42c}$$

Then the property the C.A.C indicates that these expressions (1.42) produce the same value of $\widehat{\widetilde{\overline{u}}}$, even though there are three separate way to evaluate it. In other words, the final point is independent of the way in which it is calculated.

We shall take the discrete modified KdV equation as an example to illustrate the recipe given in [71]. The lattice mKdV equation can be written:

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha,\ \beta) = \alpha(\widehat{\widetilde{u}}\widetilde{u} - \widehat{u}u) - \beta(\widehat{\widetilde{u}}\widehat{u} - \widetilde{u}u) = 0. \tag{1.43}$$



(a) Face of the square lattice (b) Elementary cube

Figure 1.2: Consistency around the cube.

Solving the equation for $\widehat{\widetilde{u}}$, we have

$$\widehat{\widetilde{u}} = u\,\frac{\alpha\widehat{u} - \beta\widetilde{u}}{\alpha\widetilde{u} - \beta\widehat{u}}, \tag{1.44}$$

and get a similar relation for $\overline{\widetilde{u}}$, $\widehat{\overline{u}}$ in the other pairs of the lattice directions

$$\overline{\widetilde{u}} = u\frac{\alpha\overline{u} - \kappa\widetilde{u}}{\alpha\widetilde{u} - \kappa\overline{u}} \quad , \quad \widehat{\overline{u}} = u\frac{\beta\overline{u} - \kappa\widehat{u}}{\beta\widehat{u} - \kappa\overline{u}}\,. \tag{1.45}$$

Now, if we shift (1.44) in the h-direction, and place the values of $\overline{\widetilde{u}}$, $\widehat{\overline{u}}$ respectively, we obtain

$$\overline{\widehat{\widetilde{u}}} = \frac{\beta(\alpha^2-\kappa^2)\overline{u}\widetilde{u} + \kappa(\beta^2-\alpha^2)\widetilde{u}\widehat{u} + \alpha(\kappa^2-\beta^2)\widehat{u}\overline{u}}{\beta(\alpha^2-\kappa^2)\widehat{u} + \kappa(\beta^2-\alpha^2)\overline{u} + \alpha(\kappa^2-\beta^2)\widetilde{u}}. \tag{1.46}$$

The later expression is invariant under permutations of lattice shifts "$\widetilde{\ }$", "$\widehat{\ }$" and "$\overline{\ }$" together with corresponding lattice parameters. It is obvious that we can obtain the same result for $\overline{\widehat{\widetilde{u}}}$ if we start with the other pairs (1.45) on the cube in Figure 1.2 (b). Hence the mKdV equation (1.43) obeys the C.A.C property.

In 2003 Adler, Bobenko and Suris (ABS) classified all discrete integrable systems, which have the consistency around a cube property, on quad-graphs [7]. All equations of the form Q (1.40), that have the following properties:

1. Q is a first order expression in each of the fields u, $\widetilde{u}$, $\widehat{u}$, $\widehat{\widetilde{u}}$.

2. Q satisfy the D_4 symmetry group of the square.

3. The tetrahedron condition, the value $\widehat{\widetilde{u}}$ is independent of the value u.

With these conditions the different types of equations of the form (1.40) can be reduced to nine models, they are split into three categories, $(A_1 - A_2)$, $(H_1 - H_3)$ and $(Q_1 - Q_4)$. These discrete equations are not independent. In particular, all equation in the $Q-$ list can be obtained as degenerations or limits from Adler's lattice equation [5], Q_4, which is the top level equation in the ABS list of affine-linear quadrilateral equations. For the purpose of this thesis, we will focus on the lattice equation Q_4 in chapter 2, where we shall explore the equation in detail.

Importantly, the equations in the ABS classification can be made manifest through a so-called 3-leg form [22] given by

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) = \varphi(u, \widetilde{u}; \alpha)/\varphi(u, \widehat{u}; \beta) - \psi(u, \widehat{\widetilde{u}}; \alpha, \beta) = 0, \tag{1.47}$$

where the function φ indicates the short leg and ψ indicates the long leg. This is illustrated by Figure 1.3.

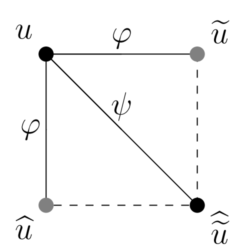


Figure 1.3: 3-leg form of the equation (1.47)

The 3-leg functions φ and ψ in (1.47) give rise to a Lagrange structure for the 3-leg equation via the relations

$$\varphi(u, \widetilde{u}; \alpha) = \frac{\partial}{\partial u}\mathscr{L}(u, \widetilde{u}; \alpha), \quad \psi(u, \widehat{\widetilde{u}}; \alpha, \beta) = \frac{\partial}{\partial u}\Lambda(u, \widehat{\widetilde{u}}; \alpha, \beta)\ , \tag{1.48}$$

which are the defining relations for $\mathscr{L}$ and Λ, defining the relevant action functional [7].

1.4 Similarity reduction and isomonodromic deformation problems

Monodromy is the change of the fundamental solution of a linear differential equation when the argument moves around one of the regular singularities. That change is measured by multiplication of the fundamental solution matrix by a factor from the right given by the monodromy matrix. Furthermore, if the differential equation carries parameters one can deform the differential equation by changing those parameters. *Isomonodromy* means then that the monodromy matrix is effectively invariant under those deformations, hence invariant under such changes of variables.

1.4.1 Derivation of isomonodromic deformation problems from similarity reduction

The first example of an isomonodromic deformation system was first worked out by R. Fuchs [33, 34] who discovered the Painlevé VI equation in 1905 as arising from the deformation of a second order linear ODE. The generalization to matrix differential systems were studied subsequently by Schlesinger [90]. In [2] Ablowitz and Segur showed that Painlevé equations arise from similarity reductions. Let us illustrate the idea by means of the continuous modified KdV (mKdV) equation

$$v_t - 6v^2 v_x + v_{xxx} = 0,$$

where the similarity variable turns out to be of the form

$$\xi = x(3t)^{-1/3} \ ,$$

with the ansatz $v = u(\xi)(3t)^{-1/3}$. One reads, after one integration, the equation

$$u_{\xi\xi} = 2u^3 + \xi u + c,$$

which is P_{II}. Moreover, in [28] Flaschka and Newell did the full isomonodomy theory on the Lax pair arising from those reductions. At the same time Jimbo et al. [51, 52, 53] developed the isomonodromy from a τ-function approach, and they gave the Lax pairs for all (continuous) Painlevé equations. For detailed information on the approach and the historical review, we refer to [3, 25].

The discovery in the 1990s of discrete analogues of the Painlevé equations has been one of the most prominent developments in the field of discrete integrable systems. One of the decisive sources of such non-autonomous nonlinear ordinary difference equations (OΔEs) has been the method of similarity reduction on the lattice, first proposed by Nijhoff and Papageorgiou in 1991 [67]. It lifts to the lattice the above-mentioned approach of obtaining the P_{II} equation from similarity reduction of the mKdV equation, noting that the transition to the lattice is highly nontrivial given that it is not really possible to find a similarity variable in the lattice case. The reduction performed in [70] instead of using a similarity variable employs compatible non-autonomous (and nonlinear) constraints, which allows one to avoid the introduction of a similarity variable. For the quad-equations of the form (1.40), the suitable similarity constraints are given in terms of a configuration forming a cross

$$F(u,\, \widetilde{u},\, \underset{\sim}{u},\, \widehat{u},\, \underset{\frown}{u}) = 0\ . \tag{1.49}$$

Therefore, we have that the system comprising both the lattice equation as well as the constraint can be symbolically represented by the diagram in Figure 1.4. By posing the equation and its similarity constraint on the variable u, we effectively reduce the lattice equation to a nonautonomous OΔE in one independent variable. This can be seen by the fact that all points of the discrete equation can be calculated from a finite set of discrete points (a local initial value problem).

The diagram in Figure 1.5 demonstrates how the similarity constraint and the lattice equation are compatible. Starting from the initial points (indicated at the diagram by $\bullet$) we calculated the other points using (1.40) (values displayed by $\circ$) and (1.49) (values

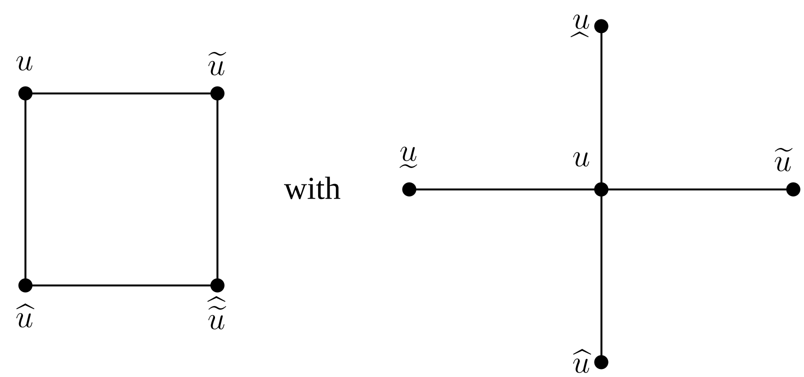


Figure 1.4: The diagram of lattice equation and its similarity constraint

initiated by +). At the eighth step we reach a point at which the evaluation can be multi-valued (symbolized by $\oplus$) which is not acceptable since we are looking for single valued solutions of the discrete equation. The value calculated by means of the lattice equation must coincide with the value computed using the similarity constraint. i.e., both ways of calculating the value at this point must show the same result. This can be verified by direct calculation.

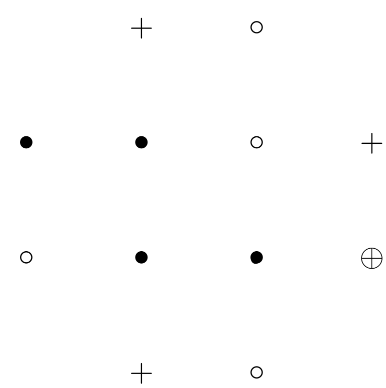


Figure 1.5: Compatibility diagram

If, as a result, for a well-chosen function F of the constraint (1.49), the system of constraint and the equation $Q = 0$ (1.40) are found to be compatible in the sense of the diagram in Figure 1.5, then the configuration of local initial value problem defined on the black points can be iterated throughout the lattice to yield a single-valued solution u

of the system at each point of the lattice. Subsequently, as was shown in [70] for examples in the KdV class of lattice equations, the system can be reduced to an O∆E in any one of the independent variables. As a simple example of this procedure we will give the case of the reduction of a linear P∆E, following the treatment in [42].

Next, we will first investigate the similarity reduction procedure on the linear level where the constraint is compatible with the quad equation.

The linear case

Let us concentrate on the linear quadrilateral equation of the form

$$(\alpha+\beta)(\widehat{u}-\widetilde{u}) = (\alpha-\beta)(\widehat{\widetilde{u}}-u), \tag{1.50}$$

for a dependent variable u and lattice parameters α and β This equation is multidimensionally consistent according to the conventions of the previous section 1.3.1. The similarity constraint for this equation was given first in [70] and reads:

$$\frac{1}{2}n(\widetilde{u}-\underset{\sim}{u}) + \frac{1}{2}m(\widehat{u}-\underset{\frown}{u}) + \mu + \lambda(-1)^{n+m} = 0, \tag{1.51}$$

where $\mu,\ \lambda$ are constant in the lattice variables $n,\ m$ and in the lattice parameters $\alpha,\ \beta$. It can be verified by explicit computation that the constraint (1.51) is compatible with (1.50) in the sense of Figure 1.5, but even in this case the computation is quite tedious so we omit the details. The linear lattice equation (1.50), which is the linearized version of the lattice equation H_1 in the ABS list [7], has a general solution given by the discrete Fourier type integral representation:

$$u(n,m) = \int_\Gamma \left(\frac{\alpha+\kappa}{\alpha-\kappa}\right)^n \left(\frac{\beta+\kappa}{\beta-\kappa}\right)^m \ \frac{\mathrm{d}\kappa}{\kappa}\ , \tag{1.52}$$

where the contour (or curve) Γ is chosen in an appropriate manner. From the integral representation (1.52) the following differential-difference equations in the lattice parameters can be constructed

$$\alpha\frac{\partial u}{\partial\alpha} = -\frac{n}{2}\big(\widetilde{u} - \underset{\sim}{u}\big), \quad \beta\frac{\partial u}{\partial\beta} = -\frac{m}{2}\big(\widehat{u} - \underset{\frown}{u}\big) \tag{1.53}$$

by considering the integral solution (1.52). In addition, these equations are compatible with the lattice equation (1.50), which can be easily verified by direct calculation.

Taking the form of the differential operator (vector field) $\mathbf{X} = \alpha\partial_\alpha + \beta\partial_\beta$ into account we will now impose the scaling invariance $\mathbf{X}Q = 0$ on the solution of the equation $Q = 0$ where $Q := (\alpha+\beta)(\widehat{u} - \widetilde{u}) - (\alpha-\beta)(\widehat{\widetilde{u}} - u)$ is the lattice equation (1.50). This yields

$$\begin{aligned} \mathbf{X}Q &= Q + (\alpha+\beta)X(\widehat{u} - \widetilde{u}) - (\alpha-\beta)\mathbf{X}(\widehat{\widetilde{u}} - u) \\ &= Q + (\alpha+\beta)(\widehat{\eta} - \widetilde{\eta}) - (\alpha-\beta)(\widehat{\widetilde{\eta}} - \eta), \end{aligned} \tag{1.54}$$

where $\eta := (\alpha\partial_\alpha + \beta\partial_\beta)u$. For $\mathbf{X}Q$ to vanish on all solutions of $Q = 0$ we must have $\eta = \widehat{\widetilde{\eta}}$ and $\widehat{\eta} = \widetilde{\eta}$ with the solution

$$\eta = \mu + \lambda(-1)^{n+m}. \tag{1.55}$$

Therefore, we have established that

$$\mathbf{X}u = (\alpha\partial_\alpha + \beta\partial_\beta)u = \mu + \lambda(-1)^{n+m}, \tag{1.56}$$

can be written as the similarity constraint (1.51). It can be seen immediately by using the relation (1.53). We will now proceed with the reduction of deriving an O∆E from the system given above. The idea to get the explicit reduction is to eliminate the tilde-shift (alternatively the hat-shift may be eliminated) by combining primarily on the similarity constraint and the corresponding lattice equation. Writing the constraint as

$$\frac{1}{2}n\,a + \frac{1}{2}m\,b + \eta = 0, \quad \eta = \mu + \lambda(-1)^{n+m} \quad \text{with:} \quad a \equiv (\widetilde{u} - \underset{\sim}{u}),\ b \equiv (\widehat{u} - \underset{\frown}{u}), \tag{1.57}$$

for convenience, we have from the backward shift of the lattice equation in n direction:

$$(\widehat{u} - \underset{\sim}{u}) = t^{-1}(\underset{\sim}{\widehat{u}} - u), \qquad \text{where} \qquad t := (\alpha-\beta)/(\alpha+\beta). \tag{1.58}$$

By combining (1.50) and (1.58) we obtain

$$(\widetilde{u} - \underset{\sim}{u}) = (t^{-1} - t)(\widehat{\widetilde{u}} - u) - t^{-1}(\widehat{\widetilde{u}} - \widehat{\underset{\sim}{u}}) \ , \tag{1.59}$$

or in terms of $a = \widetilde{u} - \underset{\sim}{u}$ this gives the relation:

$$a + t^{-1}\widehat{a} = (t^{-1} - t)x, \tag{1.60}$$

in terms of reduced variable $x := \widehat{\widetilde{u}} - u$. Furthermore, from equation (1.50) we have $b = \widehat{u} - \underset{\wedge}{u} = tx + \underset{\wedge}{x}$. Using now this relation and (1.57) we obtain from (1.60) the following closed-form difference equation in terms of x and its hat-shift only:

$$\lambda(-1)^{n+m}(t^{-1}-1) - \mu(t^{-1}+1) - \frac{m}{2}(tx+\underset{\wedge}{x}) - \frac{m+1}{2}(\widehat{x}+t^{-1}x) = \frac{n}{2}(t^{-1}-t)x, \tag{1.61}$$

in which $t,\ \mu,\ \lambda,\ n$ are parameters of the equation, derived as a reduction of P∆E for u (1.50). This is a second order linear nonautonomous O∆E equation in the independent variable m. The similar relation for n can be obtained equally, where the other discrete variable m becomes just a parameter. Here, we gave a summary of how the similarity reduction procedure on the linear level works in practice. Obviously, pursuing a reduction of the nonlinear systems of P∆E to O∆E on the two-dimensional lattice or the higher-order case is more elaborate [70]. Another advantage of the similarity approach, relevant to later parts of this thesis, is that it also provides a systematic derivation of Lax pairs (monodromy problems) for the lattice equations. This leads to the discrete isomonodromic deformation problem which is derived by implementing the similarity constraint to the Lax pair of the system. Let us show the idea for the linear case.

As already mentioned before, the linear discrete equation (1.50) obeys C.A.C. property of subsection 1.3.1. Therefore this equation can be consistently embedded in a higher dimensional lattice by considering compatible system

$$(\alpha + \beta)(\widehat{u} - \widetilde{u}) = (\alpha - \beta)(\widehat{\widetilde{u}} - u) \ , \tag{1.62a}$$

$$(\alpha + \kappa)(\overline{u} - \widetilde{u}) = (\alpha - \kappa)(\overline{\widetilde{u}} - u) \ , \tag{1.62b}$$

$$(\beta + \kappa)(\overline{u} - \widehat{u}) = (\beta - \kappa)(\overline{\widehat{u}} - u) \ , \tag{1.62c}$$

where $\overline{u}$ defines the shift in the additional direction in the lattice related with a lattice parameter κ as before and by considering the shifted variable $\overline{u} = \varphi$, we obtained an inhomogeneous Lax pair of the form:

$$\widetilde{\varphi} = \Big(\frac{\alpha+\kappa}{\alpha-\kappa}\Big)\varphi - \Big(\frac{\alpha+\kappa}{\alpha-\kappa}\Big)\widetilde{u} + u \quad , \quad \widehat{\varphi} = \Big(\frac{\beta+\kappa}{\beta-\kappa}\Big)\varphi - \Big(\frac{\beta+\kappa}{\beta-\kappa}\Big)\widehat{u} + u \ , \tag{1.63}$$

whose compatibility condition $\widehat{\widetilde{\varphi}} = \widetilde{\widehat{\varphi}}$ arising from shift on the two equations (1.63), leads to the linear equation (1.50).

We will next derive a Lax pair for the system of differential-difference equation ($D\Delta E$) (1.53). This can be achieved in a similar way by using 3D-consistency. Thus, performing the same idea on $D\Delta E$, we get from applying the shift in the additional direction:

$$\alpha\frac{\partial u}{\partial \alpha} = -\frac{n}{2}\big(\widetilde{u} - \underset{\sim}{u}\big) \quad \Rightarrow \quad \alpha\frac{\partial \overline{u}}{\partial \alpha} = -\frac{n}{2}\big(\widetilde{\overline{u}} - \underset{\sim}{\overline{u}}\big) \ . \tag{1.64}$$

Inserting as before $\overline{u} = \varphi$ and using the equation (1.63) shifted in the first direction, we obtain:

$$\alpha\frac{\partial \varphi}{\partial \alpha} = -\frac{n}{2}\big(\widetilde{\varphi} - \underset{\sim}{\varphi}\big) = -\frac{n}{2}\Big(\widetilde{\varphi} - \Big(\frac{\alpha-\kappa}{\alpha+\kappa}\Big)(\varphi - \underset{\sim}{u}) - u\Big), \tag{1.65}$$

a similar equation with n replaced by m, α replaced by β and tilde-shifts with hat shifts. We will consider the derivation of the Lax pair for the similarity reduction. The similarity constraint for the variable u, since it supposes now the existence of a third direction, will adopt the extended form from (1.56)

$$(\alpha\partial_\alpha + \beta\partial_\beta + \kappa\partial_\kappa)u = \mu + \lambda(-1)^{n+m+h}, \tag{1.66}$$

where h related to the additional direction is the independent lattice variable. Applying a bar-shift along this additional direction and inserting $\overline{u} = \varphi$ we get:

$$(\alpha\partial_\alpha + \beta\partial_\beta + \kappa\partial_\kappa)\varphi = \mu - \lambda(-1)^{n+m+h} \ . \tag{1.67}$$

The differential equation in the spectral parameter κ can be obtained immediately inserting the expression acquired for $\partial_\alpha\varphi$ from (1.65) and its β-counterpart, this yields a monodromy problem in terms of the spectral parameter κ

$$\kappa\frac{d}{d\kappa}\varphi = \mu - \lambda(-1)^{n+m+h} - \frac{u}{2}(n+m) + \frac{n}{2}\Big(\frac{\alpha-\kappa}{\alpha+\kappa}\Big)\underset{\sim}{u} + \frac{m}{2}\Big(\frac{\beta-\kappa}{\beta+\kappa}\Big)\underset{\frown}{u}$$
$$-\frac{1}{2}\Big(\frac{\alpha-\kappa}{\alpha+\kappa}n + \frac{\beta-\kappa}{\beta+\kappa}m\Big)\varphi + \frac{n}{2}\widetilde{\varphi} + \frac{m}{2}\widehat{\varphi}\,. \qquad (1.68)$$

It appears to have regular singularities at $\kappa = 0,\ \infty$ and $\kappa = -\alpha,\ -\beta$. The analogy with isomonodromy in this case is that this linear differential equation is compatible with the differential equations in terms of both the lattice parameters (1.65) as well as with the linear difference equations (1.63) for the discrete shifts in the variables n and m. In the following we will derive the relevant monodromy problem for the similarity reductions on the nonlinear level [42]. We will pursue this by analogy with the linear case.

The nonlinear case

We will consider the lattice mKdV equation (1.43). In analogy to (1.53) we introduce differential-difference equation in the lattice parameters:

$$\frac{\partial}{\partial\alpha}\log u = -\frac{n}{\alpha}\frac{\widetilde{u}-\underset{\sim}{u}}{\widetilde{u}+\underset{\sim}{u}}, \qquad (1.69)$$

and similar equation with n replaced by m, α replaced by β and tilde-shift with hat shift. The remarkable fact is that these equations are compatible with the integrable mKdV equation. In analogy to (1.56), we have

$$\mathbf{X}u = (\alpha\partial_\alpha + \beta\partial_\beta)u = (\mu + \lambda(-1)^{n+m})u, \qquad (1.70)$$

which can be obtained by imposing the scaling invariance $\mathbf{X}Q = 0$ on the solutions $Q = 0$ where $Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha,\ \beta) := \alpha(\widehat{\widetilde{u}}\widetilde{u} - \widehat{u}u) - \beta(\widehat{\widetilde{u}}\widehat{u} - \widetilde{u}u)$ is the lattice mKdV quadrilinear function. The discrete version of the above constraint can be obtained straightforwardly using (1.69) and its β-counterpart. This leads to

$$n\frac{\widetilde{u}-\underset{\sim}{u}}{\widetilde{u}+\underset{\sim}{u}} + m\frac{\widehat{u}-\underset{\frown}{u}}{\widehat{u}+\underset{\frown}{u}} = -\mu - \lambda(-1)^{n+m}\,. \qquad (1.71)$$

It can be shown that the equation (1.71) is compatible with the lattice mKdV according to the diagram in Figure 1.5. With the constraints thus obtained, one can use them in conjunction with the original equations, to derive a reduction of the original equation.

We next address the problem of deriving a monodromy problem for the reduction of the lattice mKdV equation, which requires a similar calculation as the one performed in the case of the linear system. In fact, the similarity constraint for the variable u, since it assumes the existence of an additional direction, will adopt the extended form from (1.70).

$$(\alpha\partial_\alpha + \beta\partial_\beta + \kappa\partial\kappa)u = (\mu + \lambda(-1)^{n+m+h})u \ , \tag{1.72}$$

where the lattice parameter κ associated with the additional direction in the lattice is interpreted as a spectral parameter and where h is the independent discrete variable. Next, performing a bar-shift along this direction and inserting $\overline{u} = f/g$ we obtain:

$$\frac{1}{f}(\alpha\partial_\alpha + \beta\partial_\beta + \kappa\partial\kappa)f - \frac{1}{g}(\alpha\partial_\alpha + \beta\partial_\beta + \kappa\partial_\kappa)g = (\mu - \lambda(-1)^{n+m+h}) \ , \tag{1.73}$$

which can be split into two linear equations, leading to the vector similarity constraint:

$$(\alpha\ \partial_\alpha + \beta\partial_\beta + \kappa\partial_\kappa)\chi = \begin{pmatrix} \mu - \lambda(-1)^{n+m+h} + \nu & 0 \\ 0 & \nu \end{pmatrix} \chi \ , \tag{1.74}$$

where $\chi = (f,g)^T$ and ν is arbitrary. Applying the same idea to the differential-difference equation (1.69) we obtain the expressions for $\partial_\alpha\chi$ as

$$\alpha\frac{\partial}{\partial\alpha}\begin{pmatrix} f \\ g \end{pmatrix} = \frac{n}{(\alpha^2-\kappa^2)(\widetilde{u}+\underset{\sim}{u})}\begin{pmatrix} (\alpha^2+\kappa^2)\widetilde{u} + (\alpha^2-\kappa^2)\underset{\sim}{u} & , & 2\alpha\kappa\widetilde{u}\underset{\sim}{u} \\ 2\alpha\kappa & , & 2\alpha^2\underset{\sim}{u} \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} . \tag{1.75}$$

Similarly we have a linear equation for $\partial_\beta\ \chi$ after the replacements $\alpha \to \beta$ and $\widetilde{\ } \to \widehat{\ }$. Inserting these linear equations into (1.74) a differential equation in terms of the spectral parameter κ can be obtained of the form:

$$
\begin{aligned}
\kappa\frac{d}{d\kappa}\chi \;=\; & \begin{pmatrix} n+m+\gamma & 0 \\ 0 & n+m+\gamma-\mu-\lambda(-1)^{n+m} \end{pmatrix}\chi \\
& -\frac{n}{(\alpha^2-\kappa^2)(\widetilde{u}+\underset{\sim}{u})}\begin{pmatrix} (\alpha^2+\kappa^2)\widetilde{u}+(\alpha^2-\kappa^2)\underset{\sim}{u} & , & 2\alpha\kappa\widetilde{u}\underset{\sim}{u} \\ 2\alpha\kappa & , & 2\alpha^2\underset{\sim}{u} \end{pmatrix}\chi \\
& -\frac{m}{(\beta^2-\kappa^2)(\widehat{u}+\underset{\frown}{u})}\begin{pmatrix} (\beta^2+\kappa^2)\widehat{u}+(\beta^2-\kappa^2)\underset{\frown}{u} & , & 2\beta\kappa\widehat{u}\underset{\frown}{u} \\ 2\beta\kappa & , & 2\beta^2\underset{\frown}{u} \end{pmatrix}\chi\,,
\end{aligned}
\tag{1.76}
$$

where γ is a constant. The equation (1.76) has regular singularities at $\kappa = 0, \alpha^2, \beta^2, \infty$. *Monodromy* measures the change in the solution χ as a function of κ^2, when the value of κ^2 moves around one of the regular singularities of the equation in the complex plane. The isomonodromic deformation problem posed by (1.76) in conjunction with (1.75) provides a Lax pair for the Painlevé VI equation, as was shown in [70][2]. At the same time (1.76) in conjunction with the Lax pair for the lattice KdV equation, which is of the form (2.10) with an appropriate choice of matrices, provides an isomonodromic deformation problem for the discrete counterpart of P_{VI} [70]. (We omit the details, as the corresponding computations are quite laborious.)

The similarity reduction technique is not the only way to achieve the monodromy problem for the discrete systems. We will next, relevant to chapter 5 of the thesis, review an alternative method for the derivation of the isomonodromic deformation problem.

1.4.2 Deautonomization of maps

In the previous section we have seen the procedure for the construction of isomonodromic problem from similarity reduction. In this section we shall encounter another approach,

[2]The original Lax pair for P_{VI} was given by R Fuchs in [33, 34], while the first 2×2 matrix Lax pair for that equation was given in [52].

by starting from Lax pairs for a known autonomous integrable map coming from the lattice Gel'fand-Dikii (GD) hierarchy. The GD hierarchy first introduced in [68], where the discrete analogue of the continuous GD hierarchy was derived by using the direct linearization method. For full details of derivation of the integrable mappings the reader is referred to the literature [68].

The approach, which is based on the use of a deautonomizing procedure, has been first introduced by Papageorgiou et al. in [84] in order to construct isomonodromic deformation problems for the lattice Painlevé I-III equations. Here we follow the treatment given in [25, 84]. As noted in [84] the mappings related to the lattice analogue of GD hierarchy are given by a spectral problem of the following form

$$\mathscr{A}(\kappa)\varphi(\kappa) = \tau\varphi(\kappa), \qquad \widehat{\varphi}(\kappa) = \mathscr{B}(\kappa)\varphi(\kappa), \tag{1.77}$$

in which κ is interpreted as the Floquet parameter coming through the periodicity condition of the solution and the spectral parameter τ denotes an eigenvalue of $\mathscr{A}$ the Lax matrices $\mathscr{A}$, $\mathscr{B}$ are given in the form for the mappings of GD type (cf. [25]):

$$\begin{aligned} \mathscr{A}(\kappa) &= \sum_{i=1}^{N} \boldsymbol{\Sigma}_\kappa^i X^{(i)} + X^{(0)} \, , \\ \mathscr{B}(\kappa) &= \boldsymbol{\Sigma}_\kappa Y^{(1)} + Y^{(0)}, \quad N = 2, 3, ... \end{aligned} \tag{1.78}$$

where the shift matrix $\boldsymbol{\Sigma}_\kappa$ takes of the form

$$\boldsymbol{\Sigma}_\kappa = \begin{pmatrix} 0 & 1 & & & \\ \vdots & 0 & 1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 0 & 1 \\ \kappa & \cdots & \cdots & \cdots & 0 \end{pmatrix} ,$$

in which all coefficients $X^{(i)}$, $Y^{(1)}$ and $Y^{(0)}$ are diagonal $2M \times 2M$ (even periods) or $(2M-1)\times(2M-1)$ (odd periods) matrices. The mappings (1.78) are reduction of the discrete KdV equation for $N=2$ and Boussinesq equation for $N=3$ in the GD hierarchy. Next, in order to obtain an the isomonodromic deformation system we deautonomize the equation (1.77) by replacing τ with $\kappa d/d\kappa$. This leads

$$\kappa\frac{d}{d\kappa}\varphi(n;\kappa)=\mathscr{A}(n;\kappa)\varphi(n;\kappa),\qquad \widehat{\varphi}(n;\kappa)=\mathscr{B}(n;\kappa)\varphi(n;\kappa), \tag{1.79}$$

the following relation can be obtained by the compatibility of (1.79)

$$\frac{d}{d\kappa}\mathscr{B}(n;\kappa)=\widehat{\mathscr{A}}(n;\kappa)\mathscr{B}(n;\kappa)-\mathscr{B}(n;\kappa)\mathscr{A}(n;\kappa). \tag{1.80}$$

On the other hand, the monodromy problems are not always of differential type. In another case, the corresponding non-autonomous equations may depend on the lattice variable n exponentially. This led us to reconsider the choice of the deautonomization procedure given above. As shown in the paper [84] the spectral problem (1.77) can be replaced by the q-difference system

$$\varphi(n;q\,\kappa)=\mathscr{A}(n;\kappa)\varphi(n;\kappa),\qquad \widehat{\varphi}(n;\kappa)=\mathscr{B}(n;\kappa)\varphi(n;\kappa), \tag{1.81}$$

rather than to the differential system. Equation (1.81) leads to the compatibility condition

$$\mathscr{B}(n;q\,\kappa)\,\mathscr{A}(n;\kappa)=\widehat{\mathscr{A}}(n;\kappa)\,\mathscr{B}(n;\kappa)\;. \tag{1.82}$$

In the following, let us finish this section by working out an explicit examples of above derivations leading monodromy problems for the discrete Painlevé equations.

$\mathbf{dP_I}$

In this case, for mappings coming from the lattice KdV equation the Lax matrices $\mathscr{A}$ and $\mathscr{B}$ for $M=2$, and $N=2$ are given by

$$\mathscr{A}(n;\kappa)=\begin{pmatrix} f_1 & u_2 & 1\\ \kappa & f_2 & u_3\\ \kappa u_1 & \kappa & f_3\end{pmatrix},\qquad \mathscr{B}(n;\kappa)=\begin{pmatrix} k_1 & 1 & 0\\ 0 & k_2 & 1\\ \kappa & 0 & k_3\end{pmatrix},$$

where f depends on the discrete variable n. From (1.79) we obtain the set of relations

$$\begin{aligned} &k_3u_1 - k_1\widehat{u}_1 + f_1 - \widehat{f}_3 + 1 = 0, && u_1 - \widehat{u}_3 - k_1 + k_2 = 0, \\ &k_1u_2 - k_2\widehat{u}_2 + f_2 - \widehat{f}_1 = 0, \quad , && u_2 - \widehat{u}_1 - k_2 + k_3 = 0, \\ &k_2u_3 - k_3\widehat{u}_3 + f_3 - \widehat{f}_2 = 0, && u_3 - \widehat{u}_2 - k_3 + k_1 = 0, \end{aligned} \tag{1.83a}$$

in addition to

$$(\widehat{f}_i - f_i)k_i = 0, \ i = 1, 2, 3. \tag{1.83b}$$

In order to have some f_i are not constant, we arrange the diagonal entries k_i appropriately, namely by taking $k_2 = k_3 = 0$, $k_1 \neq 0$ it yields $\widehat{f}_1 = f_1$ and $\widehat{f}_3 = f_2 + 1$ having taken into account $\widehat{u}_1 = u_2$. Furthermore, considering Casimir constant $C = u_1 + u_2 + u_3$ we obtain from the relation (1.83a) for $y_n = u_2 = u_2(n)$

$$\begin{aligned} y_{n+1} + y_n + y_{n-1} + \frac{f_2 - f_1}{y_n} = C, \\ f_2 = \frac{1}{2}n + (-1)^n f_0, \quad f_0,\ f_1 = \text{constant}, \end{aligned} \tag{1.84}$$

which is precisely the $\mathrm{dP_I}$ equation. Another choice of the diagonal entries, namely by choosing $k_3 = 0$, $k_1, k_2 \neq 0$, does produce an alternative form of $\mathrm{dP_I}$. Next, the construction of a discrete monodromy problem for $\mathrm{dP_{III}}$ arising from the modified GD class [84] will be considered.

$\mathrm{dP_{III}}$

It was noted in [84] to obtain a Lax pair for $\mathrm{dP_{III}}$ we can consider the case of even dimension $M = 2$ and rank 3, meaning that we have 4×4 matrices in the form

$$\mathscr{A}(n;\kappa) = \begin{pmatrix} f_1 & u_2 & v_3 & 0 \\ 0 & f_2 & u_3 & v_4 \\ \kappa v_1 & 0 & f_3 & u_4 \\ \kappa u_1 & \kappa v_2 & 0 & f_4 \end{pmatrix} \quad , \quad \mathscr{B}(n;\kappa) = \begin{pmatrix} k_1 & w_1 & 0 & 0 \\ 0 & k_2 & 1 & 0 \\ 0 & 0 & k_3 & w_2 \\ \kappa & 0 & 0 & k_4 \end{pmatrix}. \tag{1.85}$$

The set of relations below reveals from the compatibility condition (1.82)

$$\begin{array}{ll} k_1\widehat{v}_1+\widehat{u}_4-w_2u_1-k_3v_1=0, & \widehat{f}_1w_1+k_2\widehat{u}_2-k_1u_2-f_2w_1=0, \\ k_2\widehat{v}_2+\widehat{u}_1w_1-qu_2+k_4v_2=0, & \widehat{f}_3w_2+k_4\widehat{u}_4-k_3u_4-f_4w_2=0, \\ k_3\widehat{v}_3+\widehat{u}_2-w_1u_3-k_1v_3=0, & \widehat{f}_4+k_1\widehat{u}_1-k_4u_1-qf_1=0, \\ k_4\widehat{v}_4+\widehat{u}_3w_2-u_4-k_2v_4=0, & \widehat{f}_2+k_3\widehat{u}_3-k_2u_3-f_3=0, \end{array} \tag{1.86}$$

with

$$\widehat{v}_1=\frac{w_2}{w_1}v_2\ ,\quad \widehat{v}_2=qu_3\ ,\quad \widehat{v}_3=\frac{w_1}{w_2}v_4\ ,\quad \widehat{v}_4=v_1\ . \tag{1.87}$$

Again, we need to tune the diagonal entries k_i similar to the $\mathrm{dP_I}$ case in order to get a nontrivial dependence of the f_i on the discrete variable n. Thus choosing

$$k_2=k_4=0,\ k_1,\ k_3\neq 0\ \Rightarrow\ \widehat{f}_1=f_1,\ \widehat{f}_3=f_3, \tag{1.88}$$

we can derive

$$k_1=w_1\frac{f_1-f_2}{u_2}\ ,\qquad k_3=w_2\frac{f_3-f_4}{u_4}\ , \tag{1.89}$$

using also $\widehat{u}_1=qu_2/w_1,\ \widehat{u}_3=u_4/w_2$ obtained from the first set in (1.86). These two expressions for k_1 and k_3 lead to $\widehat{f}_4=qf_2\ ,\quad \widehat{f}_2=f_4$. Furthermore, from the last relation (1.87) we have

$$v_1v_3=\theta_n=C\lambda^n\ ,\qquad v_2v_4=\theta_{n+1}=C\lambda^{n+1}\ , \tag{1.90}$$

where C is constant. Some variables, which are not being specified by the compatibility relations (1.86), are specified by imposing the following constraint

$$u_1=v_2+f_4,\quad u_2=v_3+f_1,\quad u_3=v_4+f_2,\quad u_4=v_1+f_3\ , \tag{1.91}$$

in addition to $w_1=k_1+1,\quad w_2=k_3+1$. Next, let us introduce the new variable $x_n=v_1$ which implies

$$v_2=\theta_{n+1}/x_{n-1}\ ,\quad v_3=\theta_n/x_n\ ,\quad v_4=x_{n-1},$$

and use the constraints for the remaining two equations in the first set of (1.86) to derive

$$x_{n+1}x_{n-1}=\theta_{n+1}\frac{(x_n+f_3)(\theta_n+f_2x_n)}{(x_n+f_4)(\theta_n+f_1x_n)}\ , \tag{1.92}$$

which is the dP_{III}. The main observation is that the de-autonomization procedure (going from a pure spectral problem to a differential equation in the spectral parameter and thereby making the resulting system of equations non-autonomous) yields appropriate Lax pairs for several of the discrete Painlevé equations. Furthermore, we point out the transition from the differential case to the q-difference case is significant if we want to make the transition to the next (i.e., elliptic) case. In fact, whereas the q-difference case is related to a trigonometric grid, at the elliptic level we will consider difference equations on the torus (namely on the elliptic curve of the parameter κ). It is exactly the later stage, namely the consideration of elliptic isomonodromic deformation problems on the torus that will be the subject of chapter 5.

1.5 Outline

In this section we will give a short overview of the different chapters of this thesis.

Chapter 2 is concerned with Adler's lattice equation which plays an important role in this thesis. We give a review of its main properties. In particular, starting with alternative forms for Adler's discrete equation based on different choices of the elliptic curve, the connections almost them are presented. The first Lax representation for the Adler's equation is derived by Nijhoff in [72]. The method presented in the article is used to construct Lax pair for the other discrete integrable systems. We introduce a novel Lax representation of Adler's lattice equation obtained from the three-leg form of the discrete KN equation. In addition to chapter the quasi-classical expansion of the star-triangle relation is related to the three-leg form of the Q_4 ABS equation, that was introduced in [19]. we give a short overview of the details of the relation.

Chapter 3 deals with a general elliptic Lax scheme of the higher rank case, which is the generalization of the new Lax pair of Adler's lattice equation introduced in chapter 2. In the case of rank N=3 we show an interesting connection with Caley's hyperdeterminant of format $2 \times 2 \times 2$, and use this connection to construct in explicit form the generalizations of the 3-leg formulae in this case. In fact, along the way we present and use a novel compound theorem for hyperdeterminants, which to our knowledge is a new result in the theory of hyperdeterminants. This chapter has already appeared as part of a joint work of the author with Nijhoff and Yoo-Kong in [26].

Chapter 4 is considered with the rational and hyperbolic (trigonometric) limits of the systems that are given in terms of elliptic functions thereby the connection between Q list in ABS equations is presented. Two continuum limits of the Adler's lattice equation are constructed. We review the derivation of the discrete Ruijsenaars model which is one-step periodic reduction of LL class.

Chapter 5 focuses on elliptic discrete isomonodromic deformation problems (i.e. on Lax pairs on the elliptic curve of the spectral parameter), obtained by de-autonomization of related isospectral problems on the torus. We set up the general scheme and derive the system of compatibility conditions emerging from this novel type of elliptic monodromy problems, and give an initial analysis of the resulting rather complex system of conditions. Finally in chapter 6 we discuss the current study and open problems for the future.

Chapter 2

Adler's equation Q_4 in its various manifestations

There are, to date, several types of integrable discrete systems that are associated with elliptic curves. Such systems include the lattice Landau-Lifschitz (LL) equations constructed in [66] from the lattice version of Sklyanin Lax pair [92], alternatively a projective discretization of the LL [4], resulting from a Darboux transformation of a dressing chain, another lattice version of LL, arising in [6] as a permutability condition of Shabat-Yamilov chain, the elliptic lattice KdV obtained in [73] from the consideration of an infinite matrix scheme with an elliptic Cauchy kernel and an elliptic extension of the lattice Kadomtsev-Petviashvili equation [65], resulting from a direct linearisation method associated with an elliptic Cauchy kernel [50]. Apart from these, there also exists Adler's lattice Krichever-Novikov system (KN) [5], which has various forms, derived from the permutability condition of the B$\ddot{a}$cklund transformations of the KN equation [58]. The various manifestations of Adler's equation are connected and highlighted in this chapter. In the context of what follows later, Adler's equation plays a prominent role so the majority of this chapter is dedicated to a review of its main features.

2.1 Weierstrass form of the Adler system

Adler's discrete equation is an integrable lattice version of the KN equation i.e. of the nonlinear evolution equation

$$u_t = \frac{1}{4}u_{xxx} + \frac{3(r(u) - u_{xx}^2)}{8u_x}\,, \tag{2.1}$$

in which $r(u)$ is a polynomial associated with a Weierstrass elliptic curve

$$\Gamma_W : U^2 = r(u) = 4u^3 - g_2 u - g_3 = 4(u - e_1)(u - e_2)(u + e_1 + e_2). \tag{2.2}$$

Different realizations of the elliptic curve $U^2 = r(u)$ can be taken, but in principle $r(u)$ can be a quartic polynomial in general position. To bring then the curve in standard form, e.g. the Weierstrass form, a Möbius transform of the type

$$u \to \frac{au + b}{cu + d},$$

can be applied yielding the Weierstrass form (2.2). Adler's discrete equation, which was obtained from the permutability condition of the Bäcklund transformations of KN equation (2.1), can be written in the form[1]:

$$\begin{aligned} &A\left[(u-b)(\widehat{u}-b) - (a-b)(c-b)\right]\left[(\widetilde{u}-b)(\widehat{\widetilde{u}}-b) - (a-b)(c-b)\right] \\ &\quad + B\left[(u-a)(\widetilde{u}-a) - (b-a)(c-a)\right]\left[(\widehat{u}-a)(\widehat{\widetilde{u}}-a) - (b-a)(c-a)\right] = \\ &\quad = ABC(a-b)\,, \end{aligned} \tag{2.3}$$

cf. [72], where $u = u(n, m)$ is the dependent variable, with the shifted variables $\widetilde{u} = u(n+1, m)$, $\widehat{u} = u(n, m+1)$ and $\widehat{\widetilde{u}} = u(n+1, m+1)$ defining the different values of u at the vertices around an elementary plaquette, see Figure 1.2-(a). Here $\mathfrak{a}$, $\mathfrak{b}$ are

[1] Note that in the original paper [5] the equation was written in a slightly different form with rather complicated expressions for the coefficients given in terms of the moduli g_2 and g_3 of the Weierstrass curve.

parameters of the lattice equation (2.3) associated with the grid size, and are points, that are, given by $\mathfrak{a} = (a, A)$, $\mathfrak{b} = (b, B)$ together with $\mathfrak{c} = (c, C)$ on a Weierstrass elliptic curve, Γ_W, i.e.

$$A^2 = r(a) \equiv 4a^3 - g_2 a - g_3 \quad , \quad B^2 = r(b) \quad , \quad C^2 = r(c) \; , \tag{2.4}$$

which can be parametrized in terms of Weierstrass $\wp-$function as follows:

$$\begin{aligned}(a, A) &= (\wp(\alpha), \wp'(\alpha)),\\ (b, B) &= (\wp(\beta), \wp'(\beta)),\\ (c, C) &= (\wp(\gamma), \wp'(\gamma)),\end{aligned} \tag{2.5}$$

where α and β are the corresponding uniformising parameters and $\gamma = \beta - \alpha$. The parameters $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are related through the addition formulae on the elliptic curve:

$$\begin{aligned}A(c-b) &= C(a-b) - B(c-a),\\ a+b+c &= \frac{1}{4}\left(\frac{A+B}{a-b}\right)^2 .\end{aligned} \tag{2.6}$$

The collection of the corresponding elliptic functions appeared in chapter 1. We note the following fact about what is possibly the most simple solution of equation (2.3):

Proposition 2.1.1 *A "trivial" solution of the lattice equation* (2.3) *is given by*

$$u = \wp(\xi_0 + n\alpha + m\beta) \quad , \quad \textit{with } \xi_0 \textit{ constant} . \tag{2.7}$$

We call this a trivial solution because it is the counterpart of the zero solution for the lattice potential KdV equation, and as such qualifies as the simplest solution of Q_4. However, the proof that (2.7) is a solution of (2.3) by direct computation is in itself highly nontrivial, and requires the use of several elliptic identities. In particular, it uses expressions of the form

$$(u-b)(\widehat{u}-b) = B[\zeta(\xi) - \zeta(\xi+\beta) + \zeta(2\beta) - \zeta(\beta)] \; ,$$

which itself relies on the identities (1.8) and (1.9) for particular choices of the arguments. Alternatively the proposition is a direct corollary of the 3-leg form of Q_4 which we will present in section 2.1.2.

Remark 2.1.1 One would naively expect that the solution (2.7) can be used as a seed solution for the Bäcklund transform to generate an analogue of soliton solutions for Q_4, where the Bäcklund transform (due to multidimensional consistency) is identical to the equation itself, albeit with a lattice direction associated with the Bäcklund parameter. However, unlike the case of the KdV lattice equation, where the zero solution can be used as a seed solution to generate soliton solutions in this way, in the case of Q_4 the trivial solution does not generate new solutions by Bäcklund transforms (in other word the seed is "non-germinating"). The issue of finding germinating seed solutions was addressed in [14] where the first non trivial solutions of both Q_4 as well as of the continuous counterpart, the KN equation, were constructed for the Jacobi form. General formulae for the analogue of N-soliton solutions were constructed in [15].

2.1.1 C.A.C. Lax pair

The multidimensional consistency property given in section 1.3.1, which means that such equation can be consistently embedded in a multidimensional lattice, and which has been interpreted as a definition of integrability for the discrete system provides also a method to derive Lax pairs for the lattice equations [22, 105]. The method is provided in [72] where the derivation of first Lax pair for the Adler system was presented. The idea is to consider the third direction as auxiliary associated with the spectral parameter κ, replace β by κ and linearise the lattice system in the new variable $\overline{u}$ and its shifts, we have from

(2.3)

$$A\left[(u-k)(\overline{u}-k)-(a-k)(k'-k)\right]\left[(\widetilde{u}-k)(\widetilde{\overline{u}}-k)-(a-k)(k'-k)\right]$$
$$+K\left[(u-a)(\widetilde{u}-a)-(k-a)(k'-a)\right]\left[(\overline{u}-a)(\widetilde{\overline{u}}-a)-(k-a)(k'-a)\right]=$$
$$=AKK'(a-k)\,, \tag{2.8}$$

where[2]

$$k=\wp(\kappa)\qquad,\qquad K=\wp'(\kappa)\quad,$$
$$k'=\wp(\kappa-\alpha)\qquad,\qquad K'=\wp'(\kappa-\alpha)\,.$$

Solving for $\widetilde{\overline{u}}$ and using the addition formula (2.6) leads

$$\widetilde{\overline{u}}=\frac{k_3u\widetilde{u}+k_4u\overline{u}+k_1u\widetilde{u}\overline{u}-k_2\widetilde{u}\overline{u}-k_5(u+\widetilde{u}+\overline{u})-k_6}{k_2u+k_5-k_1u\widetilde{u}-k_4\widetilde{u}-k_1u\overline{u}-k_3\overline{u}+k_0u\widetilde{u}\overline{u}-k_1\widetilde{u}\overline{u}}\,, \tag{2.9}$$

where the coefficients $k_i=k_i(a,\kappa)$ in (2.9) are

$$k_0=A+K\quad,\quad k_1=aK+kA\quad,\quad k_2=a^2K+k^2A\,,$$
$$k_3=-Ak^2-K(a\,(k+k')-k\,k')\quad,\quad k_4=A(a-k)k'-a(A\,k+aK)\,,$$

as well as

$$\begin{aligned}k_5 &= a(k_2+k_3)+k(k_2+k_4)-Ak^3-Ka^3\,,\\ k_6 &= A\left[k^2-(a-k)(k'-k)\right]^2+K\left[a^2-(k-a)(k'-a)\right]^2\\ &\quad +AK\left[A(k-k')+K(a-k')\right]\,.\end{aligned}$$

This Riccati equation can be linearized using the transformation $\overline{u}=\frac{f}{g}$, resulting the following set of equations

$$\begin{cases}\widetilde{f}=\gamma^{-1}\left[(k_5-k_4u-k_1u\widetilde{u}+k_2\widetilde{u})f+(k_5(u+\widetilde{u})+k_6-k_3u\widetilde{u})g\right]\,,\\[2ex] \widetilde{g}=\gamma^{-1}\left[(k_1(u+\widetilde{u})-k_0u\widetilde{u}+k_3)f+(k_1u\widetilde{u}-k_2u+k_4\widetilde{u}-k_5)g\right]\,,\end{cases}$$

[2] The spectral variables $(k,\ K)$ and $(k',\ K')$ on the elliptic curve should obviously not be confused with the standard notation for the moduli and half periods of the Jacobi elliptic functions. The latter will not use in the thesis.

with γ being an arbitrary prefactor. Taking the other set of equation in the same form apart from the obvious replacements: $\widetilde{\ } \to \widehat{\ }$ and $\alpha \to \beta$, we have the following linear system

$$\widetilde{\varphi} \;=\; L_\kappa(\widetilde{u}, u; \alpha)\, \varphi \,, \tag{2.10a}$$

$$\widehat{\varphi} \;=\; M_\kappa(\widehat{u}, u; \beta)\, \varphi \,. \tag{2.10b}$$

The prefactor must be chosen so that the relation for determinants of this equation

$$\det(\widehat{L})\ \det(M) = \det(\widetilde{M})\ \det(L)\,, \tag{2.11}$$

is satisfied, which in this case provides

$$\gamma = (a-k)\Big(KK'\big[(u\widetilde{u} + ua + \widetilde{u}a + \frac{g_2}{4})^2 - (u + \widetilde{u} + a)(4u\widetilde{u}a - g_3)\big]\Big)^{1/2},$$

where $K' = \wp(\kappa - \alpha)$. In the form (2.3) of the equation the Lax matrix reads as follows:

$$L_\kappa \;=\; \frac{1}{\gamma}\begin{pmatrix} k_5 - k_4 u - k_1 u\widetilde{u} + k_2\widetilde{u} & k_5(u+\widetilde{u}) + k_6 - k_3 u\widetilde{u} \\ k_1(u+\widetilde{u}) - k_0 u\widetilde{u} + k_3 & k_1 u\widetilde{u} - k_2 u + k_4\widetilde{u} - k_5 \end{pmatrix}. \tag{2.12}$$

Taking the other part of the Lax matrix M in the same form apart from the following replacements: $\widetilde{\ } \to \widehat{\ }$ and $\alpha \to \beta$. The compatibility relation of the Lax pair (2.10) gives the lattice KN system (2.3).

2.1.2 3-leg form

After its discovery in [5], Adler's lattice equation (2.3) reemerged in [7] as the top equation in the ABS list of affine-linear quadrilateral equations, where it was renamed Q_4. The key integrability characteristic of Adler's equation is its *multidimensional consistency*, [22, 71], which in the case of Adler's system can be made manifest through its so-called 3-leg form, see [7]:

$$\frac{\sigma(\widetilde{\xi} - \xi + \alpha)\,\sigma(\widetilde{\xi} + \xi - \alpha)}{\sigma(\widetilde{\xi} - \xi - \alpha)\,\sigma(\widetilde{\xi} + \xi + \alpha)}\,\frac{\sigma(\widehat{\xi} - \xi - \beta)\,\sigma(\widehat{\xi} + \xi + \beta)}{\sigma(\widehat{\xi} - \xi + \beta)\,\sigma(\widehat{\xi} + \xi - \beta)} = \frac{\sigma(\widehat{\widetilde{\xi}} - \xi - \gamma)\,\sigma(\widehat{\widetilde{\xi}} + \xi + \gamma)}{\sigma(\widehat{\widetilde{\xi}} - \xi + \gamma)\,\sigma(\widehat{\widetilde{\xi}} + \xi - \gamma)}. \tag{2.13}$$

The uniformising variable, $\xi = \xi(n,m)$ in (2.13), is now the dependent variable of the equation, related to the original variable u of the rational form (2.3) of the equation through the identification $u = \wp(\xi)$ and $\gamma = \beta - \alpha$ as before. The equivalence between these two forms can be seen to be a consequence of an interesting identity given in the following elliptic identity:

Proposition 2.1.2 *For arbitrary (complex) variables X, Y, and Z, we have the following identity*

$$\begin{aligned}
&(X-\wp(\xi+\alpha))(Y-\wp(\xi-\beta))(Z-\wp(\xi-\alpha+\beta))\\
&\quad -t^2(X-\wp(\xi-\alpha))(Y-\wp(\xi+\beta))(Z-\wp(\xi+\alpha-\beta))\\
&\quad = s\big\{A\left[(\wp(\xi)-b)(Y-b)-(a-b)(c-b)\right]\left[(X-b)(Z-b)-(a-b)(c-b)\right]\\
&\qquad +B\left[(\wp(\xi)-a)(X-a)-(b-a)(c-a)\right]\left[(Y-a)(Z-a)-(b-a)(c-a)\right]\\
&\qquad -ABC(a-b)\big\},
\end{aligned} \tag{2.14}$$

in which

$$t = \frac{\sigma(\xi-\alpha)\sigma(\xi+\beta)\sigma(\xi+\alpha-\beta)}{\sigma(\xi+\alpha)\sigma(\xi-\beta)\sigma(\xi-\alpha+\beta)}, \quad s = \frac{1-t^2}{(A+B)\wp(\xi)-Ab-aB}. \tag{2.15}$$

and where (a, A), (b, B) and (c, C) are given as before.

Proof

This can be established directly by showing that the coefficients of each monomial $1, X, Y, Z, XY, XZ, YZ$ and XYZ of the identity are equivalent. Expanding the left-hand side of the identity as

$$\begin{aligned}
LHS &:= (1-t^2)XYZ + (t^2\wp(\xi-\alpha)-\wp(\xi+\alpha))YZ + (t^2\wp(\xi+\beta)-\wp(\xi-\beta))XZ\\
&\quad +(t^2\wp(\xi+\alpha-\beta)-\wp(\xi-\alpha+\beta))XY + (\wp(\xi-\beta)\wp(\xi-\alpha+\beta)\\
&\quad -t^2\wp(\xi+\beta)\wp(\xi+\alpha-\beta))X + (\wp(\xi+\alpha)\wp(\xi-\alpha+\beta)-t^2\wp(\xi-\alpha)\wp(\xi+\alpha-\beta))Y\\
&\quad +(\wp(\xi+\alpha)\wp(\xi-\beta)-t^2\wp(\xi-\alpha)\wp(\xi+\beta))Z + t^2\wp(\xi-\alpha)\wp(\xi+\alpha-\beta)\wp(\xi+\beta)\\
&\quad -\wp(\xi+\alpha)\wp(\xi-\beta)\wp(\xi-\alpha+\beta)\ ,
\end{aligned} \tag{2.16}$$

it is obvious that the first term of the first line, $(1-t^2)XYZ$, is equal to the corresponding term on the right hand-side of (2.14) using the definition of s. The rest of the equalities of the corresponding coefficients follow by the same method as explained below. The computations are relatively straightforward, relying on (2.4), elliptic addition formulaes and the Frobenius-Stickelberger formula [31], see Appendix B for more details. First, we make use of this formula in terms of the variables $(\xi, \alpha, -\beta)$

$$\begin{vmatrix} 1 & \wp(\xi) & \wp'(\xi) \\ 1 & \wp(\alpha) & \wp'(\alpha) \\ 1 & \wp(-\beta) & \wp'(-\beta) \end{vmatrix} = \begin{vmatrix} 1 & \wp(\xi) & \wp'(\xi) \\ 1 & a & A \\ 1 & b & -B \end{vmatrix} = 2\frac{\sigma(\xi+\alpha-\beta)\,\sigma(\xi-\alpha)\,\sigma(\alpha+\beta)\,\sigma(\xi+\beta)}{\sigma^3(\xi)\,\sigma^3(\alpha)\,\sigma^3(\beta)},$$

where the Weierstrass $\wp$ is an even function of its argument and consider a similar relation with $(\xi, -\alpha,\ \beta)$. If we divide the former determinant by the latter one, we obtained the following expression for t and s in (2.15)

$$t = \frac{\wp'(\xi)(b-a) - Ab - aB + \wp(\xi)(A+B)}{\wp'(\xi)(b-a) + Ab + aB - \wp(\xi)(A+B)},\ s = \frac{4(a-b)\wp'(\xi)}{(\wp'(\xi)(b-a) + Ab + aB - \wp(\xi)(A+B))^2}\,.$$

Applying the elliptic addition formulae of the form, namely:

$$\wp(\xi) + \wp(\eta) + \wp(\xi \pm \eta) = \frac{1}{4}\left(\frac{\wp'(\xi) \mp \wp'(\eta)}{\wp(\xi) - \wp(\eta)}\right)^2\ , \tag{2.17}$$

on (2.16), we get on the one hand

$$\begin{aligned} LHS = {} & (1-t^2)XYZ + (a + \wp(\xi) - \frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2} + t^2(-a - \wp(\xi) + \frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2}))YZ \\ & + (b + \wp(\xi) - \frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2} + t^2(-b - \wp(\xi) + \frac{(\wp'(\xi)-B)^2}{4(\wp(\xi)-b)^2}))XZ \\ & + (c + \wp(\xi) - \frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2} + t^2(-c - \wp(\xi) + \frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2}))XY \\ & + ((-a - \wp(\xi) + \frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2})(-b - \wp(\xi) + \frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2}) \\ & - t^2(-a - \wp(\xi) + \frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2})(-b - \wp(\xi) + \frac{(\wp'(\xi)-B)^2}{4(\wp(\xi)-b)^2}))Z \\ & + ((-a - \wp(\xi) + \frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2})(-c - \wp(\xi) + \frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2}) - \end{aligned} \tag{2.18}$$

$$
\begin{aligned}
&-t^2(-a-\wp(\xi)+\frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2}))Y\\
&+((-b-\wp(\xi)+\frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2})\\
&-t^2(-b-\wp(\xi)+\frac{(\wp'(\xi)-B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2}))X\\
&+((a+\wp(\xi)-\frac{(\wp'(\xi)-A)^2}{4(\wp(\xi)-a)^2})(-b-\wp(\xi)+\frac{(\wp'(\xi)+B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)\\
&+\frac{(\wp'(\xi)-C)^2}{4(\wp(\xi)-c)^2})+t^2(-a-\wp(\xi)+\frac{(\wp'(\xi)+A)^2}{4(\wp(\xi)-a)^2})(-b-\wp(\xi)\\
&+\frac{(\wp'(\xi)-B)^2}{4(\wp(\xi)-b)^2})(-c-\wp(\xi)+\frac{(\wp'(\xi)+C)^2}{4(\wp(\xi)-c)^2})).
\end{aligned}
\tag{2.19}
$$

The proof is completed by using the relations (2.6) and subsequently (2.4), (2.5) on the coefficients of (2.19) and as well as on the right hand-side of (2.14) repeatedly. □

Identifying $u=\wp(\xi)$, $X=\widetilde{u}=\wp(\widetilde{\xi})$, $Y=\widehat{u}=\wp(\widehat{\xi})$ and $Z=\widehat{\widetilde{u}}=\wp(\widehat{\widetilde{\xi}})$, and using

$$
\wp(\xi)-\wp(\eta)=\frac{\sigma(\eta+\xi)\,\sigma(\eta-\xi)}{\sigma^2(\eta)\sigma^2(\xi)}\quad , \tag{2.20}
$$

it can be readily seen that the elliptic identity (2.14) relates the rational form of Adler's equation in the Weierstrass case (2.3) with the 3-leg (2.13).

The connection between the rational and the elliptic form of the Adler system parallels that of the continuous KN equation, which in its (original) elliptic form reads:

$$
\xi_t=\frac{1}{4}\left(\xi_{xxx}+\frac{3}{2}\,\frac{(1-\xi_{xx}^2)}{\xi_x}-6\wp(2\xi)\,\xi_x^3\right)\ . \tag{2.21}
$$

Equation (2.21) arose in [58, 59, 77] from the study of the finite-gap solutions of the Kadomtsev–Petviashvili equation associated with elliptic curve. In 1984 [93, 98] Sokolov found the Hamiltonian structure and infinite hierarchies of the KN equation. Later work on this equation has been established by Novikov [76] which is bilinear form and algebro-geometric solution scheme but the solution was not explicitly given. The first elliptic solutions were derived together with the discrete analogue in [14].

It is readily seen that we can turn the original equation (rational form) (2.1) into the elliptic form (2.21) of the KN equation by using the identifications $u=\wp(\xi),\ U=\wp'(\xi)$ for the dependent variables and employing the elliptic identities:

$$\wp(2\xi) = \frac{1}{4}\Big(\frac{\wp''(\xi)}{\wp'(\xi)}\Big)^2 - 2\wp(\xi) \quad , \quad \wp''(\xi) = 6\wp^2(\xi) - \frac{g_2}{2} \ , \tag{2.22}$$

in order to express higher derivatives of the $\wp$ in terms of lower derivatives.

2.1.3 Elliptic Lax pair

Lax pairs for the discrete equations are not unique and can be obtained directly from the lattice equations by doing a similar derivation as explained before. We will show that the three-leg form of the Adler system (2.13) allows us to obtain a new Lax pair for Q_4. Applying a *gauge transformation*, we derive an alternative Lax pair for Adler's equation to the one given in (2.12). Again, we consider an auxiliary direction related with the spectral parameter κ on the 3D lattice. Starting from (2.13) by replacing β by κ and using the additional formula (2.20) leads to a fractional linear form in terms of $\overline{\widetilde{u}} = \wp(\ \overline{\widetilde{\xi}}\ )$, $\widetilde{u} = \wp(\widetilde{\xi}\ )$ and $\overline{u} = \wp(\overline{\xi})$:

$$\begin{aligned} F(\xi,\overline{\xi},\widetilde{\xi},\overline{\widetilde{\xi}};\alpha,\kappa) :=& \frac{\sigma^2(\xi-\alpha)(\widetilde{u}-\wp(\xi-\alpha))}{\sigma^2(\xi+\alpha)(\widetilde{u}-\wp(\xi+\alpha))}\frac{\sigma^2(\xi+\kappa)(\overline{u}-\wp(\xi+\kappa))}{\sigma^2(\xi-\kappa)(\overline{u}-\wp(\xi-\kappa))} \\ &-\frac{\sigma^2(\xi+\kappa-\alpha)(\overline{\widetilde{u}}-\wp(\xi+\kappa-\alpha))}{\sigma^2(\xi-\kappa+\alpha)(\overline{\widetilde{u}}-\wp(\xi-\kappa+\alpha))}, \end{aligned}$$

where the overline $^-$ denotes the shift associated with the parameter κ. Going through the same moves as explained in [72], the next step is to solve $\overline{\widetilde{u}}$ from the expression $F(\xi,\overline{\xi},\widetilde{\xi},\overline{\widetilde{\xi}};\alpha,\kappa) = 0$, yielding

$$\overline{\widetilde{u}} = \frac{R^2(\overline{u}-\wp(\xi+\kappa))\wp(\xi-\kappa+\alpha) - \wp(\xi+\kappa-\alpha)(\overline{u}-\wp(\xi-\kappa))}{R^2(\overline{u}-\wp(\xi+\kappa)) - (\overline{u}-\wp(\xi-\kappa))},$$

where

$$R = R(\xi,\widetilde{\xi};\kappa,\alpha) = \frac{\sigma(\xi-\alpha)\sigma(\xi+\kappa)\sigma(\xi-\kappa+\alpha)}{\sigma(\xi+\alpha)\sigma(\xi-\kappa)\sigma(\xi+\kappa-\alpha)}\left(\frac{\widetilde{u}-\wp(\xi-\alpha)}{\widetilde{u}-\wp(\xi+\alpha)}\right)^{1/2}. \tag{2.23}$$

This relation can be linearized in terms of $\widetilde{\overline{u}}$ and $\overline{u}$. Substituting $\overline{u} = f/g, \quad \widetilde{\overline{u}} = \widetilde{f}/\widetilde{g}$ and splitting into two linear equations for f and g leads to:

$$\begin{cases} \widetilde{f} = \gamma^{-1}\Big[\big(R^2\wp(\xi-\kappa+\alpha) - \wp(\xi+\kappa-\alpha)\big)f + \big(\wp(\xi+\kappa-\alpha)\wp(\xi-\kappa) \\ \qquad -R^2\wp(\xi-\kappa+\alpha)\wp(\xi+\kappa)\big)\ g\Big]\ , \\ \\ \widetilde{g} = \gamma^{-1}\Big[\ (R^2-1)\, f + (\wp(\xi-\kappa) - R^2\wp(\xi+\kappa))\, g\Big]\ . \end{cases}$$

These can be given as a matrix system acting on $\psi \equiv (f,g)^T$, where the Lax pair is written as:

$$\widetilde{\psi} = L_{(Q_4)}(\widetilde{\xi},\xi;\alpha)\ \psi, \tag{2.24}$$

together with a similar formula for $\widehat{\psi} = M_{(Q_4)}\psi$ obtained from $F(\xi,\widehat{\xi},\overline{\xi},\widehat{\overline{\xi}};\kappa,\beta) = 0$. From the condition (2.11) for $L_{(Q_4)}$ and $M_{(Q_4)}$, we are led to the choice

$$\gamma^2 = \big(\wp(\xi-\kappa) - \wp(\xi+\kappa)\big)\big(\wp(\xi-\kappa+\alpha) - \wp(\xi+\kappa-\alpha)\big)\ R^2\ . \tag{2.25}$$

The Lax matrix $L_{(Q_4)}$ is then

$$L_{(Q_4)} := \gamma'\ \mathbf{V}(\xi;\kappa-\alpha)^{-1}\begin{pmatrix} R & 0 \\ 0 & R^{-1}\end{pmatrix}\mathbf{V}(\xi;\kappa), \tag{2.26a}$$

where

$$\mathbf{V}(\xi;\kappa) \equiv (u-k)\begin{pmatrix} 1 & -\wp(\xi+\kappa) \\ 1 & -\wp(\xi-\kappa)\end{pmatrix}, \tag{2.26b}$$

with $u = \wp(\xi)$ as always, $k = \wp(\kappa)$ and where γ' is a yet to be specified quantity (it is related to the γ in (2.25)). Next we can apply a gauge transformation of the form:

$$\sigma^{1/2}(2\xi)\sigma^{n/2}(2\alpha)\sigma^{m/2}(2\beta)\chi \equiv \begin{pmatrix} \sigma^2(\xi+\kappa) & 0 \\ 0 & \sigma^2(\xi-\kappa)\end{pmatrix}\mathbf{V}(\xi;\kappa)\psi, \tag{2.27}$$

to derive the following alternative Lax pair for the Adler system:

$$\widetilde{\chi} = L_\kappa\chi = \lambda\begin{pmatrix} \Phi_{2\kappa}(\widetilde{\xi}-\xi-\alpha) & -\Phi_{2\kappa}(\widetilde{\xi}+\xi-\alpha) \\ \Phi_{2\kappa}(-\widetilde{\xi}-\xi-\alpha) & -\Phi_{2\kappa}(-\widetilde{\xi}+\xi-\alpha)\end{pmatrix}\chi\ , \tag{2.28a}$$

$$\widehat{\chi} = M_\kappa \chi = \mu \begin{pmatrix} \Phi_{2\kappa}(\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(\widehat{\xi} + \xi - \beta) \\ \Phi_{2\kappa}(-\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(-\widehat{\xi} + \xi - \beta) \end{pmatrix} \chi, \tag{2.28b}$$

in which Φ_κ denotes the (truncated) Lamé function given in (1.6) and where the explicit form for the coefficients $\lambda = \lambda(\xi, \widetilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \widehat{\xi}; \beta)$ follows from the consistency relation $\widehat{\widetilde{\psi}} = \widetilde{\widehat{\psi}}$ as:

$$\lambda(\xi, \widetilde{\xi}; \alpha) = \left(\frac{\sigma(\widetilde{\xi} + \xi + \alpha)\sigma(\widetilde{\xi} + \xi - \alpha)\sigma(\widetilde{\xi} - \xi - \alpha)\sigma(\widetilde{\xi} - \xi + \alpha)}{\sigma(2\alpha)\sigma(2\xi)\sigma(2\widetilde{\xi})} \right)^{1/2},$$

$$\mu(\xi, \widehat{\xi}; \beta) = \left(\frac{\sigma(\widehat{\xi} + \xi + \beta)\sigma(\widehat{\xi} + \xi - \beta)\sigma(\widehat{\xi} - \xi - \beta)\sigma(\widehat{\xi} - \xi + \beta)}{\sigma(2\beta)\sigma(2\xi)\sigma(2\widehat{\xi})} \right)^{1/2}.$$

Note that we did not need to specify γ' after all, since we have absorbed it into the function λ. Each member of the elliptic Lax pair (2.28) is reminiscent of the time-dependent part of the Lax pair related with the time-discretisation of the 2-particle Ruijsenaars system that was constructed in [69]. The Lax pair (2.28) has already been presented in [104], but not its derivation from a gauge transformation.

2.2 Jacobi form of the Adler system

As we have seen, there are various alternative forms for Adler's discrete equation based on different choices of the underlying elliptic curve. Thus, if one considers (2.3) to be the Weierstrass form of the equation (with parameters on a Weierstrass elliptic curve (2.4)), the equation in Jacobi form (due to Hietarinta, [41]) reads:

$$Q(v, \widetilde{v}, \widehat{v}, \widehat{\widetilde{v}}) = p(v\widetilde{v} + \widehat{v}\widehat{\widetilde{v}}) - q(v\widehat{v} + \widetilde{v}\widehat{\widetilde{v}}) - r(\widetilde{v}\widehat{v} + v\widehat{\widetilde{v}}) + pqr(1 + v\widetilde{v}\widehat{v}\widehat{\widetilde{v}}) = 0. \tag{2.30}$$

Here the dependent variable v is related to u of (2.3) through a fractional linear transformation [41], where (p, P), (q, Q) and (r, R) are now points on a Jacobi type elliptic curve:

$$\Gamma_J : \quad X^2 \equiv x^4 - \gamma x^2 + 1, \qquad \gamma^2 = \mathsf{k} + 1/\mathsf{k}, \tag{2.31}$$

with modulus k of this curve. They can be parametrized in terms of Jacobi elliptic function as follows:

$$\mathfrak{p} = (p, P) = (\sqrt{\mathsf{k}}\,\mathrm{sn}(\alpha;\ \mathsf{k}), \mathrm{sn}'(\alpha;\ \mathsf{k})), \quad \mathfrak{q} = (\mathrm{q}, \mathrm{Q}) = (\sqrt{\mathsf{k}}\,\mathrm{sn}(\beta;\ \mathsf{k}), \mathrm{sn}'(\beta;\ \mathsf{k})),$$
$$\mathfrak{r} = (r, R) = (\sqrt{\mathsf{k}}\,\mathrm{sn}(\alpha - \beta;\ \mathsf{k}), \mathrm{sn}'(\alpha - \beta;\ \mathsf{k}))\ . \tag{2.32}$$

In [9] Adler and Suris pointed out that the Weierstrass form (2.3) and the Jacobi form (2.32) of the Adler equation are equivalent in the sense of Möbius transformation between points on the curves of Γ_W and Γ_J. We will state this link explicitly in the following identity.

Proposition 2.2.1 *For arbitrary variables X, Y, and Z, the following identity holds*

$$\begin{aligned}
&A\Big[(b-a)(c-b)(u-\mathsf{k})(Y-\mathsf{k}) + (d(u+s)-b(u-\mathsf{k}))(d(Y+s)-b(Y-\mathsf{k}))\Big]\\
&\quad\Big[(b-a)(c-b)(X-\mathsf{k})(Z-\mathsf{k}) + (d(X+s)-b(X-\mathsf{k}))(d(Z+s)-b(Z-\mathsf{k}))\Big]\\
&\quad+B\Big[(a-b)(c-a)(u-\mathsf{k})(X-\mathsf{k}) + (d(u+s)-a(u-\mathsf{k}))(d(x+s)-a(X-\mathsf{k}))\Big]\\
&\qquad\Big[(a-b)(c-a)(Y-\mathsf{k})(Z-\mathsf{k}) + (d(Y+s)-a(Y-\mathsf{k}))(d(Z+s)-a(Z-\mathsf{k}))\Big]\\
&\qquad-ABC(a-b)(u-\mathsf{k})(X-\mathsf{k})(Y-\mathsf{k})(Z-\mathsf{k}) = t\Big\{(u-\mathsf{k})(X-\mathsf{k})(Y-\mathsf{k})(Z-\mathsf{k})\\
&\qquad(1-p^2q^2)(p(uX+YZ)-q(uY+XZ)) - (pQ-qP)((XY+uZ)-pq\,(uXYZ+1))\Big\}\ ,
\end{aligned} \tag{2.33}$$

in which

$$t = \frac{81\mathsf{k}^3(-1+\mathsf{k}^4)^5(p+q)e_1^4}{(1+\mathsf{k}^4)^4(p^2+\mathsf{k}^4p^2-2\mathsf{k}^2(1+P))(pQ-qP)(q^2+\mathsf{k}^4q^2-2\mathsf{k}^2(1+Q))}\ ,$$
$$d = \frac{e_1(5\mathsf{k}^4-1)}{2(\mathsf{k}^4+1)}\ , \qquad s = \frac{\mathsf{k}^5-5\mathsf{k}}{5\mathsf{k}^4-1}, \tag{2.34}$$

if one has the following relation between the parameters

$$a = \frac{(5(\mathsf{k}^2p^2-1)-\mathsf{k}^6\,p^2+\mathsf{k}^4(1-5P)+P)e_1}{2(\mathsf{k}^4+1)(-1+\mathsf{k}^2p^2-P)}, \tag{2.35}$$

$$A = \frac{(1-\mathsf{k}^4)(\mathsf{k}^2p^2-1)p}{\mathsf{k}^3(1-\mathsf{k}^2p^2+P)^2}\ , \tag{2.36}$$

and a similar equation with (a, A) replaced by (b, B) and (p, P) replaced by (q, Q). Moreover, the roots e_i $(i = 1, 2)$ in (2.2) *and the modulus* k *given in* (2.31) *are related to each other with bi-rational transformation [9]*

$$\frac{1}{\mathsf{k}^2} + \mathsf{k}^2 = -\frac{6e_1}{2e_2 + e_1} \,, \tag{2.37}$$

where the points (a, A), (b, B), (c, C) and (p, P), (q, Q) are introduced as before.

Proof

The relation (2.33) can be easily seen by direct computation through identities. □

As a direct corollary of Proposition 2.2.1, identifying

$$X = \widetilde{u} \,, \quad Y = \widehat{u} \,, \quad Z = \widehat{\widetilde{u}},$$

we see that the expression in the curly brackets on the right hand side of (2.33) can be written in terms of the following expression

$$Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) := p(uX+YZ)-q(uY+XZ)+\frac{(pQ-qP)}{(p^2\,q^2-1)}((XY+uZ)-pq\,(uXYZ+1)) \,, \tag{2.38}$$

it is not hard to see that the equation $Q(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) = 0$ is, up to some simple computations, equivalent to the Q_4 equation in the form (2.30). It is also straightforward to verify that the relation on the left hand side of (2.33) is equivalent to Adler's equation (2.3) in the sense that the dependent variables are related by a rational transformation, $u \to \frac{\mathsf{k}u+ds}{u-d}$.

Furthermore, Adler's discrete integrable equation is recovered in the quasi-classical limit of star-triangle relation corresponding to the elliptic Beta solution. The model is discovered in [19]. The Lagrangian form of the discrete system (2.3) appears in the quasi-classical expansion of the Boltzmann weights (1.37a) parametrized through the elliptic Gamma function. The latter function contains two elliptic nomes labeled p, q and the Lagrangian of (2.3) is obtained when one of the nomes is real and fixed, while the other one approaches unity

$$p = e^{2i\pi\tau} \quad , \quad q = e^{-2\hbar} \to 1 \quad \text{as} \quad \hbar = -i\pi\sigma \to 0 \tag{2.39}$$

where $\hbar$ plays the role of the Planck constant. Introduce a new function $\lambda(z|\tau)$ as

$$\lambda(z|\tau) = \int_0^z \log \frac{i\theta_{11}(x + \frac{\tau}{2})}{G} dx + \frac{i\pi z^2}{2} + \frac{\pi i \tau z}{4}, \tag{2.40}$$

where θ_{11} stands for the theta function given in [74] and $G(\tau) = G(\frac{\omega_2}{\omega_1}) = \prod_{n=1}^{\infty}(1 - p^n)$. In the limit (2.39) the elliptic Gamma function (1.23) becomes

$$\log \Gamma(z) = \frac{1}{2\hbar} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(e^{2izk} - e^{-2izk})p^{-k/2}p^{kn}}{k^2} + O(\hbar)\;, \tag{2.41}$$

and may be written in terms of the dilogarithm function defined by the power series

$$\log \Gamma(z) = \frac{1}{2\hbar} \sum_{n=0}^{\infty} \mathrm{Li}_2(e^{2izk} p^{\frac{2n+1}{2}}) - \mathrm{Li}_2(e^{-2izk} p^{\frac{2n+1}{2}}) + O(\hbar)\;. \tag{2.42}$$

In the following, using integral representation of the dilogarithm function and making a change of variables yields

$$\log \Gamma(z) = \frac{i}{2\hbar} \int_0^z \log \prod_{n=0}^{\infty} (e^{2ivk} p^{\frac{2n+1}{2}})(e^{-2ivk} p^{\frac{2n+1}{2}}) dv + O(\hbar) = \frac{i}{2\hbar} \lambda(z|\tau) + O(\hbar)\;. \tag{2.43}$$

Therefore, the Boltzmann weight (1.37a) becomes

$$\overline{W}(\alpha; u, v) = \exp\{\frac{1}{2\hbar} \mathscr{L}(\alpha; u, v) + O(\hbar)\} \tag{2.44}$$

where the two point Lagrangian $\mathscr{L}$

$$\mathscr{L}(\alpha; u, v) = \lambda(u - v + i\alpha) - \lambda(u - v + i\alpha) + \lambda(u + v + i\alpha) - \lambda(u + v - i\alpha) - \lambda(2i\alpha|2\tau)\;, \tag{2.45}$$

states a Lagrangian for Q_4 equation [19]. As a consequence the quasi-classical limit of the Boltzmann weight (1.37a) gives the (2.3) equation of the ABS list. In [56] this connection is extended to the rest of the ABS list.

Many interesting results were established for the latter form of the lattice KN equation, notably explicit expressions for the (doubly elliptic) N-soliton solutions, [15] and singular-boundary solutions [16]. It would be interesting to investigate that the Adler's system (2.3) in the Weierstrass form admits some special solutions in terms of elliptic functions. The construction of seed and soliton solutions for the novel system in the Weierstrass form is undertaken at the moment.

2.3 Spin representation

There is another way to represent Adler's equation, which we refer to as "spin representation" and which is connected to the Jacobi form of Adler's equation (2.30). Such a spin representation has been used in connection with the Landau-Lifschitz (LL) equations, cf. e.g.[6]. In continuous level, the original KN equation arises from a spin zero limit of LL equation, [27] although this connection is not pointed out explicitly.

A spin representation of Adler's lattice system is based on the following observation. Introducing (for general N) spin matrices of the form

$$\boldsymbol{S}\cdot\boldsymbol{I} = \boldsymbol{G}\,\boldsymbol{\Omega}\,\boldsymbol{G}^{-1} \tag{2.46}$$

where Ω is a fixed matrix obeying $\mathrm{tr}(\boldsymbol{\Omega}^j) = 0$, $j = 1, \ldots, N-1$, and $\boldsymbol{\Omega}^N = \mathbf{1}$, the latter being the $N \times N$ unit matrix, and where the matrices $\boldsymbol{I}$ represent an appropriate basis in the space of such matrices. The vector $\boldsymbol{S}$ in (2.46), which is the main quantity of interest, can be expressed in terms of the matrix $\boldsymbol{G}$ containing the dynamical variables $u_{i,j}$ in the form:

$$\boldsymbol{G} = (\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_N) \;, \quad \boldsymbol{u}_i = (u_{1,i}, u_{2,i}, \cdots, u_{N,i})^T \;, \tag{2.47}$$

where we can think of the vectors $\boldsymbol{u}_i$ in some projective space like $\mathbb{CP}^N$, implying that we can set (without loss of generality) all first components $u_{1,i} = 1$. In order to expand the obtained matrix, we need a basis in GL_N, which, following [21], we can obtain from the following elementary matrices

$$\boldsymbol{\Omega} = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix} \;, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix} \tag{2.48}$$

and where ω is the N$^{\text{th}}$ root of unity, $\omega = \exp(2\pi i/N)$. These matrices obey the following relations

$$\boldsymbol{I}_{n_1,n_2} := \boldsymbol{\Sigma}^{n_1}\boldsymbol{\Omega}^{n_2} = \omega^{n_1 n_2}\boldsymbol{\Omega}^{n_2}\boldsymbol{\Sigma}^{n_1} \ , \quad \boldsymbol{I}^{\uplus}_{n_1,n_2} = \boldsymbol{I}_{-n_1,-n_2} \ , \tag{2.49}$$

where the $^{\uplus}$ means Hermitian conjugation. We can take as a basis of GL_N the set of matrices $\{\boldsymbol{I}_{n_1,n_2}\,|\,n_1,n_1 \in \mathbb{Z}_N\}$. The aim of this section is to realize the Adler lattice system in terms of appropriately chosen spin vectors that are defined in terms of the above ingredients. The main observation here is that Adler's lattice equation in its Jacobi form (2.30) can be written conveniently in terms of spin vectors.

We have

$$G = \begin{pmatrix} 1 & 1 \\ u & v \end{pmatrix} \ , \quad \boldsymbol{\Omega} = \boldsymbol{\sigma}_3 \ , \quad \boldsymbol{G}\boldsymbol{\sigma}_3\boldsymbol{G}^{-1} = \boldsymbol{S}\cdot\boldsymbol{\sigma} \tag{2.50}$$

in the basis of the standard Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. This leads to the following identification of a spin matrix and (normalised) spin vector

$$\boldsymbol{S}(u,v) = \frac{1}{v-u}\,(uv-1, -i(uv+1), u+v) \ , \quad |\boldsymbol{S}|^2 = \boldsymbol{S}\cdot\boldsymbol{S} = 1 \ , \tag{2.51}$$

which in the case of a real spin vector (when $v = u^*$ the complex conjugate of each other), is the realization of stereographic projection of the complex plane to unit sphere. We have now the following remarkable observation:

Proposition 2.3.1 *Adler's lattice equation in Jacobi form, i.e.* (2.30)*, can be represented in the following spin form:*

$$J_0 + \boldsymbol{S}(v,\widetilde{v})\cdot\boldsymbol{J}\boldsymbol{S}(\widehat{v},\widehat{\widetilde{v}}) = 0 \ , \tag{2.52}$$

in which the coefficient (anisotropy parameters) comprising J_0 and the 3×3 diagonal matrix $\boldsymbol{J} = \mathrm{diag}(J_1, J_2, J_3)$ are given by

$$J_0 = \frac{q-r}{2} \ , \quad J_1 = p\frac{1-qr}{2} \ , \quad J_2 = p\frac{1+qr}{2} \ , \quad J_3 = \frac{q+r}{2} \ , \tag{2.53}$$

with $r = (pQ - qP)/(1-p^2q^2)$.

The proof is by direct computation, writing out the components and identifying the various combinations of terms with the ones occurring in (2.30). Obviously, the particular way (2.52) of writing the equation is not unique: it is subject to the D4 symmetries of the quadrilateral both in how the spin variables depend on the variables v on the vertices and in how the anisotropy parameters depend on the lattice parameters.

This observation suggests that the search for a rational form of higher-rank Adler lattice systems may involve higher spin variables. At this stage it is not yet clear how to construct these variables but it will be subject of future work.

Chapter 3

Elliptic Lax systems on the lattice

3.1 Introduction

In this chapter, we propose a general elliptic Lax scheme of rank N, which is inspired by the novel Lax representation (2.28) for Adler's equation in 3-leg form, derived in the previous chapter 2. This general Lax scheme leads to two distinct classes of systems which we coin as being "of Landau-Lifschitz (LL) type" (or spin-nonzero case) and as "of KN type" (or spin-zero case). We present general results for both classes in section 3.2, some initial results of this section were already presented in [104], but then focus in the remainder of this chapter on the KN class of Lax systems. The latter case requires a separate treatment. In fact, we first study in detail the compatibility conditions for the case $N = 2$, showing by means of this example how Adler's equation emerges, yielding the 3-leg form directly, in contrast to what Lax pair of [72] obtained from consistency-around-the-cube. We next turn to the more typical case $N = 3$, in which case the analysis is markedly more involved. Notably in the rank $N = 3$ case the analysis of the compatibility condition exploits a (to our knowledge novel) *compound theorem* for Cayley's hyperdeterminants of format $2 \times 2 \times 2$, see [23], a result which may have some significance in its own right. We conjecture that the resulting rank 3 lattice system may be regarded as a discrete analogue of a rank 3 Krichever-Novikov type of differential system that was constructed by Mokhov in [63]. Results in this chapter have appeared in the joint paper [26] by the candidate in collaboration with Nijhoff and Yoo-Kong. The general set-up of the elliptic Lax scheme was given in [104], but there the focus was on the LL class of models and the analysis of the KN class was not followed through. Here, in contrast, we will develop the latter aspect more in detail, which requires a totally separate analysis, but for the sake of self-containedness we reiterate the general scheme first.

3.2 General elliptic Lax scheme

Consider the Lax pair of the form:

$$\widetilde{\chi}_\kappa = \mathbf{L}_\kappa \, \chi_\kappa \, , \tag{3.1a}$$

$$\widehat{\chi}_\kappa = \mathbf{M}_\kappa \, \chi_\kappa \, , \tag{3.1b}$$

defining horizontal and vertical shifts of the vector function χ_κ, according to the diagram in Figure 3.1:

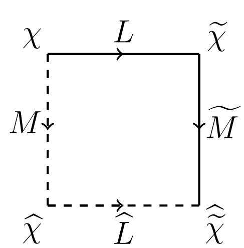


Figure 3.1: Lax compatibility condition (3.4).

where the vectors χ are located at the vertices of the quadrilateral and in which the matrices $\mathbf{L}$ and $\mathbf{M}$ are attached to the edges linking the vertices. The matrices $\mathbf{L}_\kappa$ and

$\mathbf{M}_\kappa$ can be taken of the form:

$$(\mathbf{L}_\kappa)_{i,j} = \Phi_{N\kappa}(\widetilde{\xi}_i - \xi_j - \alpha)h_j \ , \tag{3.2a}$$

$$(\mathbf{M}_\kappa)_{i,j} = \Phi_{N\kappa}(\widehat{\xi}_i - \xi_j - \beta)k_j \ , \tag{3.2b}$$

$$(i,j = 1,\dots,N)$$

where as mentioned earlier, Φ_κ denotes the (truncated) Lamé function

$$\Phi_\kappa(\xi) \equiv \frac{\sigma(\xi+\kappa)}{\sigma(\xi)\sigma(\kappa)}, \tag{3.3}$$

with σ denoting the Weierstrass σ-function. The variables $\xi_i = \xi_i(n,m)$, $(i = 1,\dots,N)$, are the main dependent variables. As before α and β denote the uniformized lattice parameters (as in (2.5)), while κ is the (uniformized) spectral parameter. In (3.2), the coefficients h_j and k_j are functions of the variables ξ_l and their shifts that remain to be determined. The compatibility condition between (3.1a) and (3.1b) is given by the lattice zero-curvature condition:

$$\widehat{\mathbf{L}}_\kappa \mathbf{M}_\kappa = \widetilde{\mathbf{M}}_\kappa \mathbf{L}_\kappa \ . \tag{3.4}$$

Using the addition formula

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa)+\zeta(x)+\zeta(y)-\zeta(\kappa+x+y)\right], \tag{3.5}$$

where $\zeta(x) = \frac{d}{dx}\ln\sigma(x)$ is the Weierstrass zeta function, the consistency relation (3.4) gives rise to

$$\begin{aligned} &\sum_{l=1}^{N} \widehat{h}_l k_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha) + \zeta(\widehat{\xi}_l - \xi_j - \beta) + \zeta(N\kappa) - \zeta(N\kappa + \widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\right] = \\ &= \quad \sum_{l=1}^{N} \widetilde{k}_l h_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) + \zeta(N\kappa) - \zeta(N\kappa + \widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta)\right] \\ &\qquad (i,j = 1,\dots,N) \ . \end{aligned} \tag{3.6}$$

Due to the arbitrariness of the spectral parameter κ the equations (3.6) separate into two parts, namely

$$\left(\sum_{l=1}^{N}\widehat{h}_l\right)k_j=\left(\sum_{l=1}^{N}\widetilde{k}_l\right)h_j\quad,\quad(j=1,\dots,N)\,, \tag{3.7a}$$

$$\left\{\sum_{l=1}^{N}\widehat{h}_l\left[\zeta(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)+\zeta(\widehat{\xi}_l-\xi_j-\beta)\right]\right\}k_j$$
$$=\left\{\sum_{l=1}^{N}\widetilde{k}_l\left[\zeta(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\right]\right\}h_j$$
$$(i,j=1,\dots,N)\,. \tag{3.7b}$$

Now there are two scenarios which we refer to as the "LL type" (or physically, the spin non-zero) case and the "KN type" (spin zero) cases respectively:

1. Discrete LL type case: $\sum_l h_l=\sum_l k_l\neq 0$, in which case we have that the variables h_j, k_j are proportional to each other, $k_j=\rho h_j$, and after summing up (3.7a), we obtain the following conservation law:

$$\frac{\sum_{l=1}^{N}\widehat{h}_l}{\sum_{l=1}^{N}h_l}=\frac{\sum_{l=1}^{N}\widetilde{k}_l}{\sum_{l=1}^{N}k_l}\,, \tag{3.8}$$

   and in which case eqs. (3.7b) reduce to:

$$\sum_{l=1}^{N}\left[\zeta(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)\rho\widehat{h}_l-\zeta(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)\widetilde{k}_l\right]$$
$$=\sum_{l=1}^{N}\left[\zeta(\xi_j-\widehat{\xi}_l+\beta)\rho\widehat{h}_l-\zeta(\xi_j-\widetilde{\xi}_l+\alpha)\widetilde{k}_l\right],\quad(i,j=1,\dots,N)\,. \tag{3.9}$$

   The above system of equations can be reduced under the condition:

$$\widetilde{\Xi}+\widehat{\Xi}=\widehat{\widetilde{\Xi}}+\Xi\quad,\quad\Xi\equiv\sum_{l=1}^{N}\xi_l\,, \tag{3.10}$$

   which is a conservation law for the centre of mass motion. In fact, (3.10) follows from the determinant of the relation (3.4) and using the Frobenius-Stickelberger determinantal formula (B.4).

2. <u>KN type case</u>: $\sum_l h_l = \sum_l k_l = 0$, in which case (3.7a) becomes vacuous. In this case we seek further reductions by the additional constraint $\Xi = \sum_l \xi_l = 0$ (modulo the period lattice of the elliptic functions).

In this section we shall focus primarily on the class of models in # 2, but we shall conclude this section by presenting the general structure of the systems that emerge from the Lax system in both cases, and then in the ensuing sections present an alternative analysis for the Lax system of class # 2 for the cases $N = 2$ and $N = 3$. We proceed with the general analysis of (3.9) by using a trick which was employed in [69], based on an elliptic version of the Lagrange interpolation formula (see Appendix B) in order to identify the variables h_l, k_l. Particularly, consider the following elliptic function, where as a consequence of the conservation law (3.10) for the variables ξ_l the Lagrange interpolation (B.6) of Appendix B is applicable, leading to the following identity:

$$
\begin{aligned}
F(\xi) &= \prod_{l=1}^{N} \frac{\sigma(\xi - \widehat{\widetilde{\xi}}_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widetilde{\xi}_l - \beta)} \\
&= \sum_{l=1}^{N} \left[\zeta(\xi - \widehat{\xi}_l - \alpha) - \zeta(\eta - \widehat{\xi}_l - \alpha)\right] H_l \\
&\quad + \sum_{l=1}^{N} \left[\zeta(\xi - \widetilde{\xi}_l - \beta) - \zeta(\eta - \widetilde{\xi}_l - \beta)\right] K_l,
\end{aligned} \tag{3.11}
$$

which holds for any four sets of variables ξ_l, $\widehat{\xi}_l$, $\widetilde{\xi}_l$, $\widehat{\widetilde{\xi}}_l$ such that (3.10) holds. In (3.11) η can be anyone of the zeroes of $F(\xi)$, i.e. $\widehat{\widetilde{\xi}}_i$ or $\xi_i + \alpha + \beta$, and the coefficients H_l, K_l are given by:

$$
H_l = \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k - \gamma)\right] \prod_{k\neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)}\,, \tag{3.12a}
$$

$$
K_l = \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \gamma)\right] \prod_{k\neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)}\,. \tag{3.12b}
$$

Furthermore, the coefficients obey the *identity*:

$$\sum_{l=1}^{N}(H_l+K_l)=0\,. \tag{3.13}$$

Taking $\xi=\widehat{\widehat{\xi}}_i$, $\eta=\xi_j+\alpha+\beta$ in (3.11) and comparing with (3.7b), we can deduce the following identifications:

$$tH_l=\rho\widehat{h}_l \quad , \quad tK_l=-\widetilde{\rho}\widetilde{h}_l \quad , \quad l=1,\dots,N\,, \tag{3.14}$$

with a function t being an arbitrary proportionality factor. Thus in this case # 1 by eliminating h_l from (3.14) we obtain the set of equations

$$\frac{\widetilde{t}}{\widetilde{\rho}}\widetilde{H}_l+\frac{\widehat{t}}{\widehat{\widehat{\widetilde{\rho}}}}\widehat{K}_l=0 \quad , \quad l=1,\dots,N \tag{3.15}$$

which, by inserting the expressions (3.12) for H_l and K_l, constitute a system of N equations for $N+2$ unknowns ξ_l, ($l=1,\dots,N$), and ρ and t. Rewriting this system (3.15) in explicit form, we obtain the system of N 7-point equations shown in Figure 3.2:

$$\prod_{k=1}^{N}\frac{\sigma(\xi_l-\widetilde{\xi}_k+\alpha)\,\sigma(\xi_l-\underset{\wedge}{\xi}_k-\beta)\,\sigma(\xi_l-\underset{\sim}{\widehat{\xi}}_k+\beta-\alpha)}{\sigma(\xi_l-\widehat{\xi}_k+\beta)\,\sigma(\xi_l-\underset{\sim}{\xi}_k-\alpha)\,\sigma(\xi_l-\underset{\wedge}{\widetilde{\xi}}_k-\beta+\alpha)}=p, \tag{3.16}$$

for $N+1$ variables ξ_i ($i=1,\dots,N$) and $p=-\underset{\sim}{t}\,\underset{\wedge}{\rho}/(\underset{\wedge}{t}\,\rho)$, supplemented with (3.10) which fixes the discrete dynamics of the centre of mass Ξ . In (3.16) the under-accents $\underset{\sim}{\cdot}$ and $\underset{\wedge}{\cdot}$ denote reverse lattice shifts, i.e., $\underset{\sim}{\xi}_i(n,m)=\xi_i(n-1,m)$ and $\underset{\wedge}{\xi}_i(n,m)=\xi_i(n,m-1)$ respectively.

The implicit system of PΔEs arises as Euler-Lagrange equation from the following Lagrangian:

$$\mathscr{L}=\sum_{i,j=1}^{N}\left[f(\xi_i-\widetilde{\xi}_j+\alpha)-f(\xi_i-\widehat{\xi}_j+\beta)-f(\widehat{\xi}_i-\widetilde{\xi}_j+\alpha-\beta)\right]-\ln|p|\,\Xi\,, \tag{3.17}$$

in which the function f is the elliptic dilogarithm $f(x)=\int^x \ln\sigma(\xi)\,d\xi$, with respect to variations of the dependent variables ξ_i ($i=1,2,...,N$). The one-step periodic reduction,

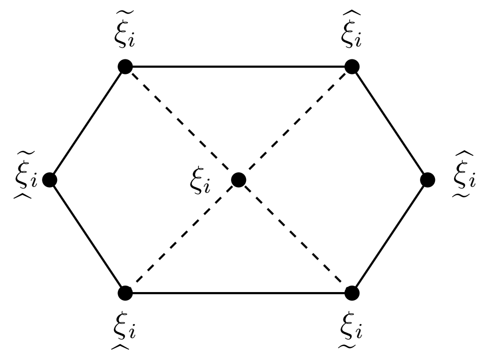


Figure 3.2: The hexagon relation

$\widetilde{\chi}_\kappa = \lambda\chi_\kappa$, leads to an implicit system of OΔEs which amounts to the time-discretization of the Ruijsenaars (relativistic Calogero-Moser) model, given in [69]. We consider the system (3.16) to be "of LL class" although a precise connection with the LL equation remains still to be established. Lattice versions of the LL equation were given in the papers [4, 6, 66]. However, not only the connection of (3.16) with these earlier models remains unclear at this stage, but also the relation between these various discretizations of the LL equation have remained obscure to this date. In the remainder of the thesis we shall concentrate on the case # 2 which, as we show for $N = 2$, leads to Adler's lattice equation in 3-leg form, and for higher rank of N ($N \geq 3$) is expected to lead to higher rank version of Adler's equation. For this case, we shall perform a different kind of analysis.

3.3 Elliptic Lax pairs for 3-leg lattice systems

In this section we shall focus on case # 2 of general elliptic Lax systems introduced in the previous section, corresponding to the "spin-zero" case (where $\sum_{l=1}^{N} h_l = \sum_{l=1}^{N} k_l = 0$). We shall first demonstrate, in the case $N = 2$ of this system, how the 3-leg form of Adler's equation arises in a natural way from this Lax pair. In fact, it turns out that the elaboration of the compatibility conditions for this Lax pair immediately produces the required equations, and is far less laborious than of the consistency-around-the-cube Lax pair of [72] yielding the corresponding rational form of Q_4. Next we shall analyze the much more generic case of $N = 3$, and produce a novel system of elliptic lattice equations, which constitutes the main result of this chapter. We also present the structure of the lattice system arising form the scheme for general N, based on similar ingredients as the ones used in the case # 1 elaborated in the previous section, but subject to slightly different conditions.

3.3.1 Case N=2: Elliptic Lax pair for the Adler 3-leg lattice equation

Let $\xi = \xi_{n,m}$ be a function of the discrete independent variables n, m for which we want to derive a lattice equation from the following Lax pair:

$$\widetilde{\chi} = L_\kappa \chi = \lambda \begin{pmatrix} \Phi_{2\kappa}(\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(\widetilde{\xi} + \xi - \alpha) \\ \Phi_{2\kappa}(-\widetilde{\xi} - \xi - \alpha) & -\Phi_{2\kappa}(-\widetilde{\xi} + \xi - \alpha) \end{pmatrix} \chi \tag{3.18a}$$

$$\widehat{\chi} = M_\kappa \chi = \mu \begin{pmatrix} \Phi_{2\kappa}(\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(\widehat{\xi} + \xi - \beta) \\ \Phi_{2\kappa}(-\widehat{\xi} - \xi - \beta) & -\Phi_{2\kappa}(-\widehat{\xi} + \xi - \beta) \end{pmatrix} \chi\ , \tag{3.18b}$$

in which the coefficients λ are functions $\lambda = \lambda(\xi, \widetilde{\xi}; \alpha)$ and $\mu = \mu(\xi, \widehat{\xi}; \beta)$, respectively. Their explicit form and the derivation of the Lax pair (3.18) were already presented in chapter 2, but λ and μ will actually not be relevant for the determination of the resulting lattice equation, which is Adler's system in 3-leg form. The discrete zero-curvature condition (3.4) can, once again, be analyzed using the addition formula (3.5) for the Lamé function Φ_κ and analyzed entry-by-entry. Applying this to each entry of both the left-hand side and right-hand side of (3.4) we observe that in all four entries a common factor containing the spectral parameter κ will drop out and that we are left with the following four relations:

$$\widehat{\lambda}\mu\left[\zeta(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}-\xi-\beta)-\zeta(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}+\xi+\beta)\right]$$
$$=\widetilde{\mu}\lambda\left[\zeta(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}-\xi-\alpha)-\zeta(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}+\xi+\alpha)\right], \tag{3.19a}$$

$$\widehat{\lambda}\mu\left[\zeta(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}+\xi-\beta)-\zeta(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}-\xi+\beta)\right]$$
$$=\widetilde{\mu}\lambda\left[\zeta(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}+\xi-\alpha)-\zeta(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}+\xi-\alpha)\right], \tag{3.19b}$$

$$\widehat{\lambda}\mu\left[\zeta(-\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}-\xi-\beta)-\zeta(-\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}+\xi+\beta)\right]$$
$$=\widetilde{\mu}\lambda\left[\zeta(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}-\xi-\alpha)-\zeta(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}+\xi+\alpha)\right], \tag{3.19c}$$

$$\widehat{\lambda}\mu\left[\zeta(-\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}+\xi-\beta)-\zeta(-\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)+\zeta(\widehat{\xi}-\xi+\beta)\right]$$
$$=\widetilde{\mu}\lambda\left[\zeta(-\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}+\xi-\alpha)-\zeta(-\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)+\zeta(\widetilde{\xi}-\xi+\alpha)\right], \tag{3.19d}$$

These four relations can be rewritten as:

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\beta-\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)\,\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}$$
$$=\widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\alpha-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)\,\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}, \tag{3.20a}$$

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}-\xi+\beta-\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)\,\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}$$
$$=\widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}-\xi+\alpha-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)\,\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}, \tag{3.20b}$$

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}-\xi-\beta+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)\,\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}$$
$$=\widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}-\xi-\alpha+\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)\,\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}, \tag{3.20c}$$

$$\widehat{\lambda}\mu\frac{\sigma(2\widehat{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\beta+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)\,\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)} = \widetilde{\mu}\lambda\frac{\sigma(2\widetilde{\xi}\,)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\alpha+\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)\,\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}, \tag{3.20d}$$

using the identity

$$\zeta(x)+\zeta(y)+\zeta(z)-\zeta(x+y+z)=\frac{\sigma(x+y)\sigma(x+z)\sigma(y+z)}{\sigma(x)\sigma(y)\sigma(z)\sigma(x+y+z)}\,. \tag{3.21}$$

Eliminating λ and μ, simply by dividing pairwise the relations over each other, we obtain directly the 3-leg formulae. In fact, we obtain two seemingly different-looking equations for ξ, namely:

$$\frac{\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}{\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}\,\frac{\sigma(\widehat{\xi}-\xi-\beta)\,\sigma(\widehat{\xi}+\xi+\beta)}{\sigma(\widehat{\xi}-\xi+\beta)\,\sigma(\widehat{\xi}+\xi-\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}\,, \tag{3.22a}$$

in which as before $\gamma=\beta-\alpha$ and

$$\frac{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\widehat{\xi}-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\widehat{\xi}-\alpha)}\,\frac{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}-\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}-\beta)}{\sigma(\widehat{\widetilde{\xi}}-\widetilde{\xi}+\beta)\,\sigma(\widehat{\widetilde{\xi}}+\widetilde{\xi}+\beta)}=\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\gamma)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\gamma)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\gamma)}\,, \tag{3.22b}$$

but actually these two equations are equivalent. The first equation (3.22a) is identical to (2.13), namely the 3-leg form of the Adler lattice equation given in [7]. The second equation (3.22b) is obtained from the first by interchanging $\xi\leftrightarrow\widehat{\widetilde{\xi}}$, $\alpha\leftrightarrow\beta$, which is a symmetry of the equation. The equivalence between these two forms is made manifest by passing to the rational form (2.3) of the equation, and the latter connection is already given in Proposition 2.1.2. Since the Adler system (2.3) is manifestly invariant under the replacements $u\leftrightarrow\widehat{\widetilde{u}}$, $\alpha\leftrightarrow\beta$ – whilst *not* interchanging $\widetilde{u}$ and $\widehat{u}$ – (this being a particular aspect of the D_4-symmetry of the equation), the 3-leg form (3.22a) is also invariant under the parallel exchange on the level of the uniformising variables: $\xi\leftrightarrow\widehat{\widetilde{\xi}}$, $\alpha\leftrightarrow\beta$. This is the symmetry that connects the two forms (3.22a) and (3.22b), which are hence equivalent.

Remark 3.3.1 The coefficients λ and μ are determined by the condition for which the dynamical equation for the determinants of the Lax matrices L_κ, M_κ needs to be trivially satisfied. Thus, a possible choice for λ and μ is to determine these factors such that $\det(L_\kappa)$ and $\det(M_\kappa)$ are proportional to constants (i.e. independent of ξ), which leads to the following expressions

$$\lambda = \left(\frac{H(u,\widetilde{u},a)}{AU\widetilde{U}}\right)^{1/2} \quad , \quad \mu = \left(\frac{H(u,\widehat{u},b)}{BU\widehat{U}}\right)^{1/2} \ , \tag{3.23}$$

where $u = \wp(\xi)$, $U = r(u) = \wp'(\xi)$, and similary $\widetilde{u} = \wp(\widetilde{\xi}\,)$, $\widetilde{U} = r(\widetilde{u}) = \wp'(\widetilde{\xi}\,)$, and $\widehat{u} = \wp(\widehat{\xi}\,)$, $\widehat{U} = r(\widehat{u}) = \wp'(\widehat{\xi}\,)$. The symmetric triquadratic function H is given by

$$H(u,v,a) \equiv \left(uv + au + av + \frac{g_2}{4}\right)^2 - (4auv - g_3)(u+v+a) \ , \tag{3.24}$$

and which can be obtained in the following form in terms of σ-function

$$\begin{aligned} H(u,v,a) &= (u-v)^2 \left[\frac{1}{4}\left(\frac{U-V}{u-v}\right)^2 - (u+v+a)\right]\left[\frac{1}{4}\left(\frac{U+V}{u-v}\right)^2 - (u+v+a)\right] \\ &= \frac{\sigma(\xi+\eta+\alpha)\,\sigma(\xi+\eta-\alpha)\,\sigma(\xi-\eta+\alpha)\,\sigma(\xi-\eta-\alpha)}{\sigma^4(\xi)\,\sigma^4(\eta)\,\sigma^4(\alpha)} \ , \end{aligned} \tag{3.25}$$

in which $U^2 \equiv r(u)$, $V^2 \equiv r(v)$. Additionally, we have the expression in terms of the polynomial of the curve:

$$\big[r(u) + r(a) - 4(u-a)^2(u+v+a)\big]^2 - 4r(u)\,r(a) = 16(u-a)^2 H(u,v,a) \ . \tag{3.26}$$

We further note at this point that the discriminant of the triquadratic in each argument factorizes:

$$H_v^2 - 2H\,H_{vv} = r(a)r(u) \ . \tag{3.27}$$

In [10] the discriminant properties of affine-linear quadrilaterals and their relation with the corresponding biquadratics and their discriminants, were exploited to tighten the classification result of [7].

Remark 3.3.2 An alternative derivation of the $N = 2$ case can be given using the system of equations (3.12). In this case the variables H_l and K_l admit the following forms

$$H_1 = \frac{\sigma(\widehat{\xi} - \widehat{\widetilde{\xi}} + \alpha)\,\sigma(\widehat{\xi} + \widehat{\widetilde{\xi}} + \alpha)\,\sigma(\widehat{\xi} - \xi - \beta)\,\sigma(\widehat{\xi} + \xi - \beta)}{\sigma(\widehat{\xi} - \widetilde{\xi} - \gamma)\,\sigma(\widehat{\xi} + \widetilde{\xi} - \gamma)\,\sigma(2\widehat{\xi})}\,, \tag{3.28a}$$

$$H_2 = \frac{\sigma(-\widehat{\xi} - \widehat{\widetilde{\xi}} + \alpha)\,\sigma(-\widehat{\xi} + \widehat{\widetilde{\xi}} + \alpha)\,\sigma(-\widehat{\xi} - \xi - \beta)\,\sigma(-\widehat{\xi} + \xi - \beta)}{\sigma(-\widehat{\xi} - \widetilde{\xi} - \gamma)\,\sigma(-\widehat{\xi} + \widetilde{\xi} - \gamma)\,\sigma(-2\widehat{\xi})}\,, \tag{3.28b}$$

$$K_1 = \frac{\sigma(\widetilde{\xi} - \widehat{\widetilde{\xi}} + \beta)\,\sigma(\widetilde{\xi} + \widehat{\widetilde{\xi}} + \beta)\,\sigma(\widetilde{\xi} - \xi - \alpha)\,\sigma(\widetilde{\xi} + \xi - \alpha)}{\sigma(\widetilde{\xi} - \widehat{\xi} + \gamma)\,\sigma(\widetilde{\xi} + \widehat{\xi} + \gamma)\,\sigma(2\widetilde{\xi})}\,, \tag{3.28c}$$

$$K_2 = \frac{\sigma(-\widetilde{\xi} - \widehat{\widetilde{\xi}} + \beta)\,\sigma(-\widetilde{\xi} + \widehat{\widetilde{\xi}} + \beta)\,\sigma(-\widetilde{\xi} - \xi - \alpha)\,\sigma(-\widetilde{\xi} + \xi - \alpha)}{\sigma(-\widetilde{\xi} - \widehat{\xi} + \gamma)\,\sigma(-\widetilde{\xi} + \widehat{\xi} + \gamma)\,\sigma(-2\widetilde{\xi})}\,, \tag{3.28d}$$

if one sets $\xi_1 = -\xi_2 = \xi$. The identity $H_1 + H_2 = 0$ upon inserting the above expressions yields the equation:

$$\left[\frac{\sigma(\widetilde{\xi} + \xi + \alpha)\,\sigma(\widetilde{\xi} - \xi - \alpha)}{\sigma(\widetilde{\xi} + \xi - \alpha)\,\sigma(\widetilde{\xi} - \xi + \alpha)}\right]^{\widehat{}}\ \frac{\sigma(\widehat{\xi} + \xi - \beta)\,\sigma(\widehat{\xi} - \xi - \beta)}{\sigma(\widehat{\xi} + \xi + \beta)\,\sigma(\widehat{\xi} - \xi + \beta)} = \frac{\sigma(\widetilde{\xi} + \widehat{\xi} - \gamma)\,\sigma(\widetilde{\xi} - \widehat{\xi} + \gamma)}{\sigma(\widetilde{\xi} - \widehat{\xi} - \gamma)\,\sigma(\widetilde{\xi} + \widehat{\xi} + \gamma)}\,, \tag{3.29}$$

which is equivalent to the elliptic lattice system (2.3) under the same changes of variables as discussed before. In fact, (3.29) can be obtained from (2.13) by interchanging: $\xi \leftrightarrow \widehat{\xi}$ and $\widehat{\widetilde{\xi}} \leftrightarrow \widetilde{\xi}$. Similarly, the identity $K_1 + K_2 = 0$ upon inserting the expressions (3.28c) and (3.28d) for K_1 and K_2 yields a similar equation to (3.29) which can be obtained from (2.13) by interchanging: $\xi \leftrightarrow \widetilde{\xi}$ and $\widehat{\widetilde{\xi}} \leftrightarrow \widehat{\xi}$. Thus, we recover from the scheme proposed in the previous section the Adler system in the various 3-leg forms based on different vertices of the elementary quadrilateral.

3.3.2 Case N=3:

To generalize the results of the previous subsection to the rank 3 case, we consider the following form of a Lax representation on the lattice:

$$\widetilde{\chi} = \begin{pmatrix} h_1\Phi_{3\kappa}(\widetilde{\xi}_1 - \xi_1 - \alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_1 - \xi_2 - \alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_1 - \xi_3 - \alpha) \\ h_1\Phi_{3\kappa}(\widetilde{\xi}_2 - \xi_1 - \alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_2 - \xi_2 - \alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_2 - \xi_3 - \alpha) \\ h_1\Phi_{3\kappa}(\widetilde{\xi}_3 - \xi_1 - \alpha) & h_2\Phi_{3\kappa}(\widetilde{\xi}_3 - \xi_2 - \alpha) & h_3\Phi_{3\kappa}(\widetilde{\xi}_3 - \xi_3 - \alpha) \end{pmatrix} \chi \, , \tag{3.30a}$$

$$\widehat{\chi} = \begin{pmatrix} k_1\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_1 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_2 - \xi_3 - \beta) \\ k_1\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_1 - \beta) & k_2\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_2 - \beta) & k_3\Phi_{3\kappa}(\widehat{\xi}_3 - \xi_3 - \beta) \end{pmatrix} \chi \, , \tag{3.30b}$$

subject to $\sum_{i=1}^{3} h_i = \sum_{i=1}^{3} k_i = 0$, and where the coefficients h_j, k_j are some functions of the variables ξ_j and their shifts. The compatibility conditions (3.4) of this Lax pair results in a coupled set of Lax equations in terms of the three variables ξ_j as we shall demonstrate by performing a similar type of analysis as in the case $N = 2$, where is understandably more involved.

Eliminating[1] $h_3 = -h_1 - h_2$ and $k_3 = -k_1 - k_2$ we obtain from (3.7b) the following system of equations:

$$\begin{aligned} &\sum_{l=1}^{2} \widehat{h}_l k_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha) + \zeta(\widehat{\xi}_l - \xi_j - \beta) - \zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha) - \zeta(\widehat{\xi}_3 - \xi_j - \beta) \right] \\ &= \sum_{l=1}^{2} \widetilde{k}_l h_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) - \zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta) - \zeta(\widetilde{\xi}_3 - \xi_j - \alpha) \right] \\ &\qquad \forall \; i, j = 1, 2, 3. \end{aligned} \tag{3.31}$$

[1] Instead of h_3 and k_3 we could have eliminated h_1 or h_2 and k_1 or k_2 yielding equivalent results.

and using the addition formula (3.21) we next deduce:

$$\sum_{l=1}^{2}\widehat{h}_l k_j \frac{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\widehat{\xi}_3+\xi_j-\alpha+\beta)\sigma(\widehat{\xi}_l-\widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)\sigma(\widehat{\xi}_l-\xi_j-\beta)\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_3-\alpha)\sigma(\widehat{\xi}_3-\xi_j-\beta)} = \\ = \sum_{l=1}^{2}\widetilde{k}_l h_j \frac{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\widetilde{\xi}_3+\xi_j+\alpha-\beta)\sigma(\widetilde{\xi}_l-\widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)\sigma(\widetilde{\xi}_l-\xi_j-\alpha)\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_3-\beta)\sigma(\widetilde{\xi}_3-\xi_j-\alpha)} \\ \forall\, i,j=1,2,3. \tag{3.32}$$

in order to write (3.32) in a more concise way, we denote the coefficients on the left-hand side and right-hand side of the equation as $A_{ilj}\equiv A_{ilj}(\widehat{\widetilde{\xi}},\widehat{\xi},\xi;\alpha,\beta)$ and $B_{ilj}\equiv B_{ilj}(\widehat{\widetilde{\xi}},\widetilde{\xi},\xi;\alpha,\beta)$ respectively. Noting the common factors h_j/k_j ($j=1,2,3$) in these equations, we next derive the system of six equations

$$\frac{h_j}{k_j}=\frac{A_{11j}\widehat{h}_1+A_{12j}\widehat{h}_2}{B_{11j}\widetilde{k}_1+B_{12j}\widetilde{k}_2}=\frac{A_{21j}\widehat{h}_1+A_{22j}\widehat{h}_2}{B_{21j}\widetilde{k}_1+B_{22j}\widetilde{k}_2}=\frac{A_{31j}\widehat{h}_1+A_{32j}\widehat{h}_2}{B_{31j}\widetilde{k}_1+B_{32j}\widetilde{k}_2} \\ (j=1,2,3)\,. \tag{3.33}$$

We can rewrite the resulting set of relations (3.33) as

$$(A_{11j}B_{21j}-A_{21j}B_{11j})\widehat{h}_1\widetilde{k}_1+(A_{11j}B_{22j}-A_{21j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\ +(A_{12j}B_{21j}-A_{22j}B_{11j})\widehat{h}_2\widetilde{k}_1+(A_{12j}B_{22j}-A_{22j}B_{12j})\widehat{h}_2\widetilde{k}_2=0, \\ (A_{11j}B_{31j}-A_{31j}B_{11j})\widehat{h}_1\widetilde{k}_1+(A_{11j}B_{32j}-A_{31j}B_{12j})\widehat{h}_1\widetilde{k}_2 \\ +(A_{12j}B_{31j}-A_{32j}B_{11j})\widehat{h}_2\widetilde{k}_1+(A_{12j}B_{32j}-A_{32j}B_{12j})\widehat{h}_2\widetilde{k}_2=0, \\ (A_{21j}B_{31j}-A_{31j}B_{21j})\widehat{h}_1\widetilde{k}_1+(A_{21j}B_{32j}-A_{31j}B_{22j})\widehat{h}_1\widetilde{k}_2 \\ +(A_{22j}B_{31j}-A_{32j}B_{21j})\widehat{h}_2\widetilde{k}_1+(A_{22j}B_{32j}-A_{32j}B_{22j})\widehat{h}_2\widetilde{k}_2=0, \\ (j=1,2,3)\,, \tag{3.34}$$

where

$$A_{ilj}=\frac{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\widehat{\xi}_3+\xi_j-\alpha+\beta)\,\sigma(\widehat{\xi}_l-\widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_l-\alpha)\,\sigma(\widehat{\xi}_l-\xi_j-\beta)\,\sigma(\widehat{\widetilde{\xi}}_i-\widehat{\xi}_3-\alpha)\,\sigma(\widehat{\xi}_3-\xi_j-\beta)}\,, \tag{3.35a}$$

$$B_{ilj}=\frac{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\widetilde{\xi}_3+\xi_j+\alpha-\beta)\,\sigma(\widetilde{\xi}_l-\widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_l-\beta)\,\sigma(\widetilde{\xi}_l-\xi_j-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_i-\widetilde{\xi}_3-\beta)\,\sigma(\widetilde{\xi}_3-\xi_j-\alpha)}\,. \tag{3.35b}$$

We observe that these homogeneous bilinear systems for the variables $\widehat{h}_1$, $\widetilde{k}_1$, $\widehat{h}_2$ and $\widetilde{k}_2$ can be resolved by using Cayley's three-dimensional $2 \times 2 \times 2$-hyperdeterminant [23]. Let us recall the general statement (see also [36]):

Definition 3.3.3 *The hyperdeterminant of the* $2\times2\times2$ *hyper-matrix* $A = (a_{ijk})$ $(i, j, k = 0, 1)$ *is given by:*

$$\mathrm{Det}(A) = \left[\det\begin{pmatrix} a_{000} & a_{001} \\ a_{110} & a_{111}\end{pmatrix} + \det\begin{pmatrix} a_{100} & a_{010} \\ a_{101} & a_{011}\end{pmatrix}\right]^2 - 4\det\begin{pmatrix} a_{000} & a_{001} \\ a_{010} & a_{011}\end{pmatrix}\det\begin{pmatrix} a_{100} & a_{101} \\ a_{110} & a_{111}\end{pmatrix}. \tag{3.36}$$

Its main property is the following:

Proposition 3.3.1 *The hyperdeterminant* (3.36) *vanishes identically iff the following set of bilinear equations with six unknowns*

$$\begin{aligned}
&a_{001}x_0y_0 + a_{011}x_0y_1 + a_{101}x_1y_0 + a_{111}x_1y_1 = 0,\\
&a_{000}x_0y_0 + a_{010}x_0y_1 + a_{100}x_1y_0 + a_{110}x_1y_1 = 0,\\
&a_{010}x_0z_0 + a_{011}x_0z_1 + a_{110}x_1z_0 + a_{111}x_1z_1 = 0,\\
&a_{000}x_0z_0 + a_{001}x_0z_1 + a_{100}x_1z_0 + a_{101}x_1z_1 = 0,\\
&a_{100}y_0z_0 + a_{101}y_0z_1 + a_{110}y_1z_0 + a_{111}y_1z_1 = 0,\\
&a_{000}y_0z_0 + a_{001}y_0z_1 + a_{010}y_1z_0 + a_{011}y_1z_1 = 0,
\end{aligned} \tag{3.37}$$

has a non-trivial solution (i.e., none of the vectors $\boldsymbol{x} = (x_0, x_1)$*,* $\boldsymbol{y} = (y_0, y_1)$*,* $\boldsymbol{z} = (z_0, z_1)$ *are equal to the zero vector).*

A proof of this statement can be found in [91]. The cubic hyper-matrix A can be illustrated by the diagram of entries as given in Figure 3.3

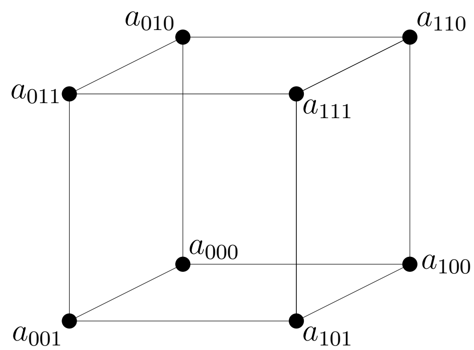


Figure 3.3: Cayley cube

In the case at hand, the components a_{ijk} can be readily identified by comparing (3.34) with the system (3.37) and the variables x_i, y_j with the $\widehat{h}_i$ and $\widetilde{k}_j$, respectively. Noting that these particular coefficients are all 2×2 determinants, it turns out that the following *compound theorem for hyperdeterminants* is directly applicable.

Lemma 3.3.4 (Compound theorem for $2 \times 2 \times 2$ hyper-determinants) *The following identity holds for the compound hyper-determinants of format $2 \times 2 \times 2$:*

$$\left(\begin{vmatrix} \begin{vmatrix} a & a'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} a' & a'' \\ d' & d'' \end{vmatrix} \\ \begin{vmatrix} c & c'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} \end{vmatrix} + \begin{vmatrix} \begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} a & a'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} c' & c'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} c & c'' \\ d & d'' \end{vmatrix} \end{vmatrix} \right)^2$$

$$-4 \begin{vmatrix} \begin{vmatrix} a & a'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} a & a'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} c & c'' \\ b & b'' \end{vmatrix} & \begin{vmatrix} c & c'' \\ d & d'' \end{vmatrix} \end{vmatrix} \cdot \begin{vmatrix} \begin{vmatrix} a' & a'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} a' & a'' \\ d' & d'' \end{vmatrix} \\ \begin{vmatrix} c' & c'' \\ b' & b'' \end{vmatrix} & \begin{vmatrix} c' & c'' \\ d' & d'' \end{vmatrix} \end{vmatrix} = \begin{vmatrix} \begin{vmatrix} a & a'' \\ c & c'' \end{vmatrix} & \begin{vmatrix} b & b'' \\ d & d'' \end{vmatrix} \\ \begin{vmatrix} a' & a'' \\ c' & c'' \end{vmatrix} & \begin{vmatrix} b' & b'' \\ d' & d'' \end{vmatrix} \end{vmatrix}^2 . \tag{3.38}$$

Proof

This can be established by direct computation. Assuming without loss of generality that the entries a'', b'', c'', d'' are all nonzero, we can take out the common product $(a''b''c''d'')^2$ from all terms on the left-hand side. Denoting all the ratios a/a'', a'/a'' by capitals A, A' etc, and noting that the 2×2 determinant $\begin{vmatrix} a/a'' & 1 \\ b/b'' & 1 \end{vmatrix}$ is simply given by $A-B$ (and in a similar way the other determinants occurring in the expression on the left-hand side), then the left-hand side of (3.38) is representable by

$$a''^2\, b''^2\, c''^2\, d''^2 \left[\left(\begin{vmatrix} A-B & A'-D' \\ C-B & C'-D' \end{vmatrix} + \begin{vmatrix} A'-B' & A-D \\ C'-B' & C-D \end{vmatrix} \right)^2 \right.$$
$$\left. -4 \begin{vmatrix} A-B & A-D \\ C-B & C-D \end{vmatrix} \cdot \begin{vmatrix} A'-B' & A'-D' \\ C'-B' & C'-D' \end{vmatrix} \right] .$$

Computing the expression between brackets, we observe that it can be simplified to:

$$((A-C)(B'-D')+(D-B)(C'-A'))^2 - 4(A-C)(B-D)(A'-C')(B'-D') =$$
$$= \begin{vmatrix} A-C & B-D \\ A'-C' & B'-D' \end{vmatrix}^2 ,$$

which leads to the desired result. □

To the best of our knowledge this compound theorem is a new result in the theory of hyper-determinants. It seems intimately linked to the structure of the linear equations (the Lax relations) from which it originate in the present context, and there may be analogues for the case of higher rank hyper-determinants (this is currently under investigation). A connection between hyper-determinants and minors of symmetric matrices was established in [47], but it is not clear whether (and if so how) those results are related to the above proposition. Hyperdeterminants have also appeared in the context of integrable systems as reviewed in [100], where it was pointed out that the vanishing of a $2\times 2\times 2$ Cayley hyperdeterminant can be interpreted as the lattice CKP equation of [55, 89]. However the appearance of the hyperdeterminant in the thesis is of a different nature.

Identifying the coefficients of the system of homogeneous equations (3.34) as entries of a $2 \times 2 \times 2$ hyper-determinant, we observe that the structure of this hyper-determinant is exactly of the form as given in Lemma (3.3.4), and hence we have the following immediate corollary.

Proposition 3.3.2 *Identifying the eight entries* $(a_{ijk})_{i,j,k=0,1}$ *by comparing the first two equations of* (3.37) *with the system of equations* (3.34)*, the hyper-determinant takes the form as given by the compound theorem Lemma* (3.3.4)*, and hence reduces to a perfect square of the form:*

$$\left| \begin{matrix} \begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} & \begin{vmatrix} A_{i'lj} & A_{i'l'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} \\ & \\ \begin{vmatrix} B_{ilj} & B_{il'j} \\ B_{i''lj} & B_{i''l'j} \end{vmatrix} & \begin{vmatrix} B_{i'lj} & B_{i'l'j} \\ B_{i''lj} & B_{i''l'j} \end{vmatrix} \end{matrix} \right|^2 \qquad (j = 1, 2, 3), \tag{3.39}$$

where

$$\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} = \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_{l'})}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_{l'} - \alpha)}$$
$$\times \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\widetilde{\xi}}_{i''})\,\sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j - 2\alpha + \beta)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\xi}_{l'} - \xi_j - \beta)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)}, \tag{3.40}$$

in which we can set $i, i' = 1, 2$*,* $l, l' = 1, 2 \neq 3$*, and where we naturally should take* $i'' = 3$.

Remark 3.3.5 A similar expression for the corresponding determinant in terms of the B_{ilj} as given (3.40) interchanging α and β and the shifts $\widetilde{\ }$ and $\widehat{\ }$.

Proof

The form (3.40) of the relevant 2×2 determinants, using the expressions for the entries

(3.35), is computed as follows. By definition of A_{ilj} given in (3.35) we have

$$\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i'lj} & A_{i'l'j} \end{vmatrix} = \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)}{S(\widehat{\widetilde{\xi}}_i)\ S(\widehat{\widetilde{\xi}}_{i'})\sigma(\widehat{\xi}_l - \xi_j - \beta)\sigma(\widehat{\xi}_{l'} - \xi_j - \beta)\sigma^2(\widehat{\xi}_3 - \xi_j - \beta)}$$
$$\Big[\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \widehat{\xi}_l + \xi_j - \alpha + \beta)\sigma(\widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_3 - \widehat{\xi}_{l'} + \xi_j - \alpha + \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_l - \alpha)$$
$$-\sigma(\widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_3 - \widehat{\xi}_l + \xi_j - \alpha + \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \widehat{\xi}_{l'} + \xi_j - \alpha + \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_{l'} - \alpha)\Big], \tag{3.41}$$

where

$$S(\xi) = \sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widehat{\xi}_{l'} - \alpha)\sigma(\xi - \widehat{\xi}_3 - \alpha).$$

Noting that the difference in the bracket can be simplified by applying the three-term relation for the σ-function in the following form:

$$\sigma(x-a)\sigma(y-b)\sigma(z-b)\sigma(w-a) - \sigma(y-a)\sigma(x-b)\sigma(z-a)\sigma(w-b)$$
$$= \sigma(z+y-a-b)\sigma(x-y)\sigma(x-z)\sigma(b-a), \tag{3.42}$$

in which $x - y = z - w$. Making now the following choice for $x,\ y,\ z,\ w,\ a$ and b in the identity (3.42):

$$x = \widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 + \xi_j - \alpha + \beta \qquad y = \widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_3 + \xi_j - \alpha + \beta$$
$$z = \widehat{\widetilde{\xi}}_i - \alpha \qquad w = \widehat{\widetilde{\xi}}_{i'} - \alpha$$
$$a = \widehat{\xi}_l \qquad b = \widehat{\xi}_{l'}$$

the expression between brackets on the right-hand side of (3.41) simplifies to

$$[\cdots] = \sigma(-\widehat{\xi}_3 + \xi_j + \beta)\ \sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j - 2\alpha + \beta)\ \sigma(\widehat{\widetilde{\xi}}_i - \widehat{\widetilde{\xi}}_{i'})\ \sigma(\widehat{\xi}_{l'} - \widehat{\xi}_l)\ .$$

We substitute the right-hand side of this equation into (3.41) and cancel the first factor against the corresponding one in the prefactor of (3.41). Then using the fact that σ is an odd function, we obtain the desired result given by the determinant in (3.40). In a similar way (or by making the obvious replacements $\alpha \leftrightarrow \beta$ and $\widetilde{\ } \leftrightarrow \widehat{\ }$) the computation of the 2×2 determinant B_{ilj} can be verified. $\square$

We apply now the compound theorem Lemma (3.3.4) to the system of homogeneous equations (3.34). In fact, from that system of equations follows that the ratios $\widehat{h}_i/\widehat{h}_j$ and $\widetilde{k}_i/\widetilde{k}_j$ obey quadratic equations whose discriminant, by virtue of the compound theorem, is a perfect square. Thus, these ratios can be obtained in a rather simple form. We distinguish the two cases: *i)* the hyper-determinant in question, i.e. the determinant (3.39), vanishes, and *ii)* the hyper-determinant is non-zero.

***i)* Case** (3.39)$= 0$

In this case the resulting set of equations is given by the vanishing of the hyper-determinant, i.e. the set of equations:

$$\begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} \begin{vmatrix} B_{i'lj} & B_{i'l'j} \\ B_{i''lj} & B_{i''l'j} \end{vmatrix} = \begin{vmatrix} A_{i'lj} & A_{i'l'j} \\ A_{i''lj} & A_{i''l'j} \end{vmatrix} \begin{vmatrix} B_{ilj} & B_{il'j} \\ B_{i''lj} & B_{i''l'j} \end{vmatrix} . \tag{3.43}$$

Inserting the explicit expression (3.40), and its counterpart in terms of the quantities B_{ilj}, into (3.43) we obtain the relations

$$\frac{\sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j + \beta - 2\alpha)}{\sigma(\widehat{\widetilde{\xi}}_{i'} + \widehat{\widetilde{\xi}}_{i''} - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j + \beta - 2\alpha)} \, \frac{\sigma(\widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_{l'} - \alpha)\sigma(\widehat{\widetilde{\xi}}_{i'} - \widehat{\xi}_3 - \alpha)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)}$$
$$= \frac{\sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_{i''} - \widetilde{\xi}_l - \widetilde{\xi}_{l'} - \widetilde{\xi}_3 + \xi_j + \alpha - 2\beta)}{\sigma(\widehat{\widetilde{\xi}}_{i'} + \widehat{\widetilde{\xi}}_{i''} - \widetilde{\xi}_l - \widetilde{\xi}_{l'} - \widetilde{\xi}_3 + \xi_j + \alpha - 2\beta)} \, \frac{\sigma(\widehat{\widetilde{\xi}}_{i'} - \widetilde{\xi}_l - \beta)\sigma(\widehat{\widetilde{\xi}}_{i'} - \widetilde{\xi}_{l'} - \beta)\sigma(\widehat{\widetilde{\xi}}_{i'} - \widetilde{\xi}_3 - \beta)}{\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_{l'} - \beta)\sigma(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_3 - \beta)}, \quad (j = 1, 2, 3) \tag{3.44}$$

where again we can set $i, i' = 1, 2$, $l, l' = 1, 2 \neq 3$, and where we naturally should take $i'' = 3$. The set of relations (3.44) is a coupled system of three quadrilateral equations (for $j = 1, 2, 3$) of 3-leg type, i.e. in terms of three independent variables which reside in the arguments of the Weierstrass σ-functions[2]. We note that all three equations (for

[2]The same system of equations would have been obtained if, rather than eliminating h_3 and k_3 in its derivation, we would have eliminated one of the other variables among the coefficients h_l and k_l.

$j=1,2,3$) have a common factor, which in the case of a further reduction $\xi_1+\xi_2+\xi_3=0$(mod period lattice) involves only the "long legs" (i.e. the differences over the diagonal). Thus, this system of equations may be too simple to figure as a proper candidate for a higher-rank analogue of the Adler lattice equation.

***ii)* Case** (3.39)$\neq 0$

As a consequence of the compound theorem, Lemma (3.3.4), the hyper-determinant in the case at hand is a perfect square. Thus, going back to the system (3.34), by first eliminating the ratio $\widehat{h}_i/\widehat{h}_j$, we obtain a quadratic for the ratio $\widetilde{k}_i/\widetilde{k}_j$, $(i,j=1,2)$ from which the latter can be solved using the fact that the discriminant of the quadratic (which coincides with the hyper-determinant) is a perfect square. Thus, we obtain rather manageable expressions for the solutions of the mentioned ratios in terms of the 2×2 determinants involving the expressions A_{ilj} and B_{ilj}. The result of this computation is the following:

Proposition 3.3.3 *If the expression* (3.39) *is non-vanishing, we have the following solutions of the system* (3.34) *given in terms of the ratios (i.e., up to a common multiplicative factor)*

$$\text{either}\qquad \frac{\widehat{h}_1}{\widehat{h}_2}=-\frac{A_{32j}}{A_{31j}}\quad\text{together with}\quad \frac{\widetilde{k}_1}{\widetilde{k}_2}=-\frac{B_{32j}}{B_{31j}}\,, \tag{3.45a}$$

$$\text{or}\quad \frac{\widehat{h}_1}{\widehat{h}_2}=-\frac{\begin{vmatrix} B_{11j} & A_{12j} & B_{12j}\\ B_{21j} & A_{22j} & B_{22j}\\ B_{31j} & A_{32j} & B_{32j}\end{vmatrix}}{\begin{vmatrix} B_{11j} & A_{11j} & B_{12j}\\ B_{21j} & A_{21j} & B_{22j}\\ B_{31j} & A_{31j} & B_{32j}\end{vmatrix}}\quad\text{together with}\quad \frac{\widetilde{k}_1}{\widetilde{k}_2}=-\frac{\begin{vmatrix} A_{11j} & A_{12j} & B_{12j}\\ A_{21j} & A_{22j} & B_{22j}\\ A_{31j} & A_{32j} & B_{32j}\end{vmatrix}}{\begin{vmatrix} A_{11j} & A_{12j} & B_{11j}\\ A_{21j} & A_{22j} & B_{21j}\\ A_{31j} & A_{32j} & B_{31j}\end{vmatrix}}\,. \qquad (j=1,2,3) \tag{3.45b}$$

The proof, once again, is by direct computation and involves some determinantal manipulations.

The system of equations resulting from (3.45a), inserting the explicit expressions for the quantities A_{ilj} and B_{ilj} from (3.35) reads as follows

$$\frac{\widehat{h}_1}{\widehat{h}_2} = -\frac{\sigma(\widehat{\widetilde{\xi}}_3 - \widehat{\xi}_2 - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widehat{\xi}_1 - \alpha)\,\sigma(\widehat{\xi}_1 - \xi_j - \beta)\,\sigma(\widehat{\xi}_2 - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_3 - \widehat{\xi}_1 - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widehat{\xi}_2 - \alpha)\,\sigma(\widehat{\xi}_2 - \xi_j - \beta)\,\sigma(\widehat{\xi}_1 - \widehat{\xi}_3)}, \tag{3.46a}$$

$$\frac{\widetilde{k}_1}{\widetilde{k}_2} = -\frac{\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_2 - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_1 - \beta)\,\sigma(\widetilde{\xi}_1 - \xi_j - \alpha)\,\sigma(\widetilde{\xi}_2 - \widetilde{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_1 - \widetilde{\xi}_3 + \xi_j + \alpha - \beta)\,\sigma(\widehat{\widetilde{\xi}}_3 - \widetilde{\xi}_2 - \beta)\,\sigma(\widetilde{\xi}_2 - \xi_j - \alpha)\,\sigma(\widetilde{\xi}_1 - \widetilde{\xi}_3)}. \quad (j = 1, 2, 3) \tag{3.46b}$$

Inserting the expressions of (3.35) into the system of equations (comprising the equations for different values of $j = 1, 2, 3$). The system of equations (3.46) for $j = 1, 2, 3$, we do not consider to be viable because it seems to be overdetermined taking into account the common factors in (3.46a) and (3.46b). Furthermore, neither does it admit the natural solution $\xi_i(n,\, m) = \xi_i(0, 0) + n\alpha + m\beta\,(i = 1, 2, 3)$ nor does it admit the reduction $\xi_1 + \xi_2 + \xi_3 = 0$ (mod period lattice). Thus, we reject this system of equations.

Turning now to the system given by (3.45b) for $j = 1, 2, 3$, this constitutes a more complicated system of quadrilateral elliptic 3-leg type of equations, which can be written as a set of equalities:

$$\frac{\begin{vmatrix} B_{111} & A_{121} & B_{121} \\ B_{211} & A_{221} & B_{221} \\ B_{311} & A_{321} & B_{321} \end{vmatrix}}{\begin{vmatrix} B_{111} & A_{111} & B_{121} \\ B_{211} & A_{211} & B_{221} \\ B_{311} & A_{311} & B_{321} \end{vmatrix}} = \frac{\begin{vmatrix} B_{112} & A_{122} & B_{122} \\ B_{212} & A_{222} & B_{222} \\ B_{312} & A_{322} & B_{322} \end{vmatrix}}{\begin{vmatrix} B_{112} & A_{112} & B_{122} \\ B_{212} & A_{212} & B_{222} \\ B_{312} & A_{312} & B_{322} \end{vmatrix}} = \frac{\begin{vmatrix} B_{113} & A_{123} & B_{123} \\ B_{213} & A_{223} & B_{223} \\ B_{313} & A_{323} & B_{323} \end{vmatrix}}{\begin{vmatrix} B_{113} & A_{113} & B_{123} \\ B_{213} & A_{213} & B_{223} \\ B_{313} & A_{313} & B_{323} \end{vmatrix}}, \tag{3.47a}$$

$$\frac{\begin{vmatrix} A_{111} & A_{121} & B_{121} \\ A_{211} & A_{221} & B_{221} \\ A_{311} & A_{321} & B_{321} \end{vmatrix}}{\begin{vmatrix} A_{111} & A_{121} & B_{111} \\ A_{211} & A_{221} & B_{211} \\ A_{311} & A_{321} & B_{311} \end{vmatrix}} = \frac{\begin{vmatrix} A_{112} & A_{122} & B_{122} \\ A_{212} & A_{222} & B_{222} \\ A_{312} & A_{322} & B_{322} \end{vmatrix}}{\begin{vmatrix} A_{112} & A_{122} & B_{112} \\ A_{212} & A_{222} & B_{212} \\ A_{312} & A_{322} & B_{312} \end{vmatrix}} = \frac{\begin{vmatrix} A_{113} & A_{123} & B_{123} \\ A_{213} & A_{223} & B_{223} \\ A_{313} & A_{323} & B_{323} \end{vmatrix}}{\begin{vmatrix} A_{113} & A_{123} & B_{113} \\ A_{213} & A_{223} & B_{213} \\ A_{313} & A_{323} & B_{313} \end{vmatrix}} , \tag{3.47b}$$

with the determinants expanded by means of the formulae:

$$A_{ilj} = \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \widehat{\xi}_3 + \xi_j - \alpha + \beta)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_3)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)} , \tag{3.48}$$

and

$$\begin{aligned} \begin{vmatrix} A_{ilj} & A_{il'j} \\ A_{klj} & A_{kl'j} \end{vmatrix} &= \frac{\sigma(\widehat{\xi}_l - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_{l'} - \widehat{\xi}_3)\,\sigma(\widehat{\xi}_l - \widehat{\xi}_{l'})}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_{l'} - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_k - \widehat{\xi}_l - \alpha)\sigma(\widehat{\widetilde{\xi}}_k - \widehat{\xi}_{l'} - \alpha)} \\ &\times \frac{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\widetilde{\xi}}_k)\,\sigma(\widehat{\widetilde{\xi}}_i + \widehat{\widetilde{\xi}}_k - \widehat{\xi}_l - \widehat{\xi}_{l'} - \widehat{\xi}_3 + \xi_j - 2\alpha + \beta)}{\sigma(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\widetilde{\xi}}_k - \widehat{\xi}_3 - \alpha)\,\sigma(\widehat{\xi}_l - \xi_j - \beta)\,\sigma(\widehat{\xi}_{l'} - \xi_j - \beta)\,\sigma(\widehat{\xi}_3 - \xi_j - \beta)}, \end{aligned} \tag{3.49}$$

with the B-determinants obtained from these by interchanging $\widetilde{\ }$and $\widehat{\ }$ and α and β. Explicit forms of the equations (3.47a) and (3.47b) can be obtained by expanding the 3×3 determinants along the A- and B-columns respectively using the expression (3.48) and (3.49) and their B-counterparts. We will next give the explicit form of those equations.

3.3.3 Higher-rank N=3 elliptic lattice systems (3.47) in explicit form

To obtain the resulting system for $N = 3$ in explicit form we expand the determinants in (3.47) using the expressions (3.48) and (3.49), namely by expanding the 3×3 determinants along the single column with A-entries (in (3.47a)) and along the column with B-entries (in (3.47b)). Thus everything can be expressed in terms of products of σ-functions. Note, however, that these determinants are not quite of Frobenius (i.e. elliptic Cauchy) type for which we would have pure products. The resulting equations comprise:

$$\begin{aligned}
&\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&=\frac{\sigma(\widehat{\xi}_1-\xi_2-\beta)}{\sigma(\widehat{\xi}_1-\xi_1-\beta)}\frac{\sigma(\widehat{\xi}_2-\xi_1-\beta)}{\sigma(\widehat{\xi}_2-\xi_2-\beta)}\\
&\times\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&\times\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg],
\end{aligned} \tag{3.50a}$$

where

$$S(\xi)=\sigma(\xi-\widetilde{\xi}_1-\beta)\,\sigma(\xi-\widetilde{\xi}_2-\beta)\,\sigma(\xi-\widetilde{\xi}_3-\beta).$$

The second one can be obtained from the first equation (3.50a) by interchanging ξ_2 and ξ_3. Namely,

$$
\begin{aligned}
&\left[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\right.\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&\left.+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\right]\\
&\quad\left[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_3+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_3+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\right.\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_3+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_3+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&\left.+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_3+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_3+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\right]\\
&=\frac{\sigma(\widehat{\xi}_1-\xi_3-\beta)}{\sigma(\widehat{\xi}_1-\xi_1-\beta)}\,\frac{\sigma(\widehat{\xi}_2-\xi_1-\beta)}{\sigma(\widehat{\xi}_2-\xi_3-\beta)}\\
&\times\left[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\right.\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&\left.+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\right]\\
&\times\left[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_3+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_3+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\right.\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_3+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_3+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&\left.+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_3+\alpha-2\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_3+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\right],
\end{aligned}
\tag{3.50b}
$$

where

$$S(\xi)=\sigma(\xi-\widetilde{\xi}_1-\beta)\,\sigma(\xi-\widetilde{\xi}_2-\beta)\,\sigma(\xi-\widetilde{\xi}_3-\beta).$$

Explicit form of the third one arising from (3.47b) can be acquired from the first equality (3.50a) by interchanging $\widetilde{\ }$and $\widehat{\ }$ and α and β. The first one is given as follows:

$$
\begin{aligned}
&\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_2-\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_3-\beta)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_2-\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_3-\beta)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_2)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_2-\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_3-\beta)}\Bigg]\\
&\quad\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_3-\beta)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_3-\beta)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_2)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_1-\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_3-\beta)}\Bigg]\\
&=\frac{\sigma(\widetilde{\xi}_1-\xi_2-\alpha)}{\sigma(\widetilde{\xi}_1-\xi_1-\alpha)}\,\frac{\sigma(\widetilde{\xi}_2-\xi_1-\alpha)}{\sigma(\widetilde{\xi}_2-\xi_2-\alpha)}\\
&\times\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_3-\beta)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_3-\beta)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_2)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_1-\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_3-\beta)}\Bigg]\\
&\times\Bigg[\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_2)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_2-\beta)\,\sigma(\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_3-\beta)}\\
&-\frac{\sigma(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_1)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_2-\beta)\,\sigma(\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_3-\beta)}\\
&+\frac{\sigma(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sigma(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2-2\alpha+\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2-\beta+\alpha)}{K(\widehat{\widetilde{\xi}}_2)\,K(\widehat{\widetilde{\xi}}_3)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_2-\beta)\,\sigma(\widehat{\widetilde{\xi}}_1-\widetilde{\xi}_3-\beta)}\Bigg],
\end{aligned}
\tag{3.50c}
$$

where

$$
K(\xi)=\sigma(\xi-\widehat{\xi}_1-\alpha)\,\sigma(\xi-\widehat{\xi}_2-\alpha)\,\sigma(\xi-\widehat{\xi}_3-\alpha).
$$

We omit the fourth equation arising from (3.47b), which can be obtained from (3.50b) by interchanging $\widetilde{\ }$ and $\widehat{\ }$ and α and β, as we expect that it is implied by the other three equations. So (3.50a)-(3.50c) constitute a coupled system of quadrilateral equations for the three dependent variables $\xi_1(n,m)$, $\xi_2(n,m)$, $\xi_3(n,m)$. For the moment, we do not have a direct proof of that the fourth equation is implied from the equations (3.50a)-(3.50c). The equations are very complicated, not only because of the complexity of each of these equations themselves, but also the fact that they are implicit with the dependent variables sitting in the argument of the elliptic functions. However, we expect the consistency of the system to be valid on the basis of applying random numerical substitutions for initial values in the case of the rational limit. We have done a number of such experiments that give very accurate verifications but we do not know how reliable these numerical results are, as we do not know how robust the numerical algorithms are under the choice of initial conditions.

We further remark that this system as expected allows for a trivial solution of type $\xi_i(n,\,m) = \xi_i(0,0) + n\alpha + m\beta\,(i = 1,2,3)$. An important problem remains the finding of a rational form for the system of equations. This, as well as verifying their reducibility under the additional constraint $\xi_1 + \xi_2 + \xi_3 = 0$(mod period lattice), is currently under investigation. If so, the latter system of equations can be duly regarded as a higher-rank version of Adler's lattice equation in 3-leg form (2.13).

In order to address the problem of finding rational forms we intend to look into the gauge transformation of the 3×3 Lax pair similar to those used in section 2.1.3. This would require discrete extensions of Frobenius-Stickelberger formulae of Appendix B. Thus, we present here a (as far as we are aware) new 3×3 determinantal formula which is expected to play a role in constructing the rational form of the rank 3 generalization of Adler's system from their analogue of the 3-leg forms:

Proposition 3.3.4 *The following identity holds for arbitrary* $\xi_{1,2,3}$ *and* $\kappa_{1,2}$

$$\begin{vmatrix} 1 & \wp(\xi_1+\kappa_1) & \wp(\xi_1+\kappa_2) \\ 1 & \wp(\xi_2+\kappa_1) & \wp(\xi_2+\kappa_2) \\ 1 & \wp(\xi_3+\kappa_1) & \wp(\xi_3+\kappa_2) \end{vmatrix} =$$
$$= \frac{\sigma^2(\kappa_2-\kappa_1)\,\sigma(\xi_1+\xi_2+\xi_3+\kappa_1+2\kappa_2)\,\sigma(\xi_3-\xi_2)\,\sigma(\xi_3-\xi_1)\,\sigma(\xi_2-\xi_1)}{\sigma(\xi_1+\kappa_1)\,\sigma(\xi_2+\kappa_1)\,\sigma(\xi_3+\kappa_1)\,\sigma^2(\xi_1+\kappa_2)\,\sigma^2(\xi_2+\kappa_2)\,\sigma^2(\xi_3+\kappa_2)} \times$$
$$\times\left[\zeta(\xi_1+\kappa_1)+\zeta(\xi_2+\kappa_1)+\zeta(\xi_3+\kappa_1)+2\zeta(\kappa_2-\kappa_1)-\zeta(\xi_1+\xi_2+\xi_3+\kappa_1+2\kappa_2)\right] , \tag{3.51}$$

where $\wp$ *is the Weierstrass elliptic function* (1.3b).

Proof

A proof of the identity (3.51) is based on the two steps. Firstly, the 3×3 determinant can be rewritten

$$\begin{vmatrix} \wp(\xi_1+\kappa_1)-\wp(\xi_3+\kappa_1) & , & \wp(\xi_1+\kappa_2)-\wp(\xi_3+\kappa_2) \\ \wp(\xi_2+\kappa_1)-\wp(\xi_3+\kappa_1) & , & \wp(\xi_2+\kappa_2)-\wp(\xi_3+\kappa_2) \end{vmatrix} = \frac{\sigma(\xi_3-\xi_1)\sigma(\xi_3-\xi_2)}{\sigma^2(\xi_3+\kappa_1)\sigma^2(\xi_3+\kappa_2)} \times$$
$$\times\left[\frac{\sigma(\xi_1+\xi_3+2\kappa_1)\sigma(\xi_2+\xi_3+2\kappa_2)}{\sigma^2(\xi_1+\kappa_1)\sigma^2(\xi_2+\kappa_2)} - \frac{\sigma(\xi_2+\xi_3+2\kappa_1)\sigma(\xi_1+\xi_3+2\kappa_2)}{\sigma^2(\xi_1+\kappa_2)\sigma^2(\xi_2+\kappa_1)}\right], \tag{3.52}$$

using the addition formula (1.9). Next applying the following higher addition rule:

$$\begin{aligned} &\sigma(\kappa+x)\,\sigma(\lambda+x)\,\sigma(\mu+x)\sigma(\kappa+\lambda+\mu+y)\,\sigma^2(y) \\ &\quad -\sigma(\kappa+y)\,\sigma(\lambda+y)\,\sigma(\mu+y)\sigma(\kappa+\lambda+\mu+x)\,\sigma^2(x) \\ &= \sigma(\kappa)\,\sigma(\lambda)\,\sigma(\mu)\,\sigma(x)\,\sigma(y)\,\sigma(\kappa+\lambda+\mu+x+y)\,\sigma(y-x) \\ &\quad \times\left[\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)+\zeta(y)-\zeta(\kappa+\lambda+\mu+x+y)\right] , \end{aligned} \tag{3.53}$$

and setting

$$\kappa=\lambda=\kappa_2-\kappa_1\,, \qquad \mu=\xi_3+\kappa_1\,,$$
$$x=\xi_1+\kappa_1\,, \qquad y=\xi_2+\kappa_1\,,$$

we obtain the right hand side of (3.51). □

One can consider the above proposition to be discrete versions of the corresponding Frobenius-Stickelberger determinantal identity, namely involving determinants in which the columns are not made out of successive higher derivatives of the $\wp$-function, but are made of shifts in their arguments. Since the right-hand side of (3.51) is not manifestly anti-symmetric with respect to the interchange of κ_1 and κ_2, but the left-hand side is, there must be an additional identity expressing this invariance.

Furthermore, the identity (3.53) is equation (1.12) for $n = 2$ and derives from:

$$\begin{aligned}&\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)+\zeta(y)-\zeta(\kappa+\lambda+\mu+x+y)=\\&=\frac{\Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_{\kappa+\lambda+\mu}(y)-\Phi_\kappa(y)\Phi_\lambda(y)\Phi_\mu(y)\Phi_{\kappa+\lambda+\mu}(x)}{\Phi_{\kappa+\lambda+\mu}(x+y)\left(\wp(x)-\wp(y)\right)}\,.\end{aligned}\tag{3.54}$$

A further generalization of the latter identity (3.54), which plays a key role in the derivation of (3.51), is given by:

$$\begin{aligned}&\Phi_{\kappa+\lambda+\mu+\nu}(x+y+z)\,\frac{\sigma(x+y+z)\,\sigma(x-y)\,\sigma(x-z)\,\sigma(y-z)}{\sigma^2(x)\,\sigma^2(y)\,\sigma^2(z)}\\&\times\left[\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(\nu)+\zeta(x)+\zeta(y)+\zeta(z)-\zeta(\kappa+\lambda+\mu+\nu+x+y+z)\right]=\\=\;&\Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_\nu(x)\left(\wp(z)-\wp(y)\right)\Phi_{\kappa+\lambda+\mu+\nu}(y+z)\\&+\Phi_\kappa(y)\Phi_\lambda(y)\Phi_\mu(y)\Phi_\nu(y)\left(\wp(x)-\wp(z)\right)\Phi_{\kappa+\lambda+\mu+\nu}(x+z)\\&+\Phi_\kappa(z)\Phi_\lambda(z)\Phi_\mu(z)\Phi_\nu(z)\left(\wp(y)-\wp(x)\right)\Phi_{\kappa+\lambda+\mu+\nu}(x+y)\,.\end{aligned}\tag{3.55}$$

which is also obtained from (1.12) for $n = 3$. We will seek in ongoing research to explore novel identities like (3.51) in the search for gauge transformations on the elliptic Lax pairs.

Chapter 4

Degenerations, continuum limits and reductions

In this chapter, we study the rational and hyperbolic limits of Adler's elliptic lattice equation in 3-leg form and the multi-component system of coupled 3-leg quad-equations presented in the previous chapter. These results can be duly regarded as the higher-rank versions of the list of Q equations within the ABS list. Furthermore, we consider the semi-continuum limit, or skew limit, and straight limit of Adler's system in the Weierstrass form. This limit leads to a differential-difference equation which is defined in terms of one continuous and one discrete independent variable. The skew limit of the three-leg form of the Adler system is also investigated. Finally, we will pay attention to the reductions to the elliptic Ruijsenaars-Schneider (RS) system.

4.1 Rational and hyperbolic subcases

In this section we consider the degenerate subcases of the systems derived in the previous chapter 3 obtained by reducing the elliptic curve to the hyperbolic (trigonometric) and rational cases. We consider the cases $N = 2$ and $N = 3$ separately. In the former case we will recover some well-known equations from the ABS list, whilst in the latter case we obtain lattice system which we consider to be of Boussinesq type. The results for the case $N = 2$ have already been presented in [104] where the connection between the ABS discrete equations and the discrete-time elliptic Ruijsenaars-Schneider model has been introduced.

N=2:

In the previous chapter, a general elliptic Lax pair was introduced, leading to the higher-rank analogue of the lattice KN equation. We now consider the equation with 2-particle situation.

4.1.1 Rational case

In the rational limit both periods go to infinity, i.e. $2\omega_1 \to \infty$, $2\omega_2 \to i\infty$, in which case we have the displacement $\sigma(\xi) \to \xi$, yielding to $\Phi_\kappa(\xi) \to \frac{\xi+\kappa}{\kappa\xi}$. The Lax matrices L_κ and M_κ in (3.2) in this case take of the form:

$$L_\kappa = \frac{1}{2\kappa}eh + L_0, \qquad M_\kappa = \frac{1}{2\kappa}ek + M_0 \ , \tag{4.1}$$

where e denotes the (column) vector with 1 in each coordinates $e = (1,1)^T$, h and k are the (row)-vectors with the entries h_j, k_j respectively. In (4.1) L_0 and M_0 are given by

$$L_0 = \sum_{i,j=1}^{2} \frac{h_j}{\widetilde{\xi}_i - \xi_j - \alpha} E_{i,j} \ , \qquad M_0 = \sum_{i,j=1}^{2} \frac{k_j}{\widehat{\xi}_i - \xi_j - \beta} E_{i,j} \ , \tag{4.2}$$

in which the E_{ij} denote the standard elementary matrices, i.e. $(E_{ij})_{mn} = \delta_{i,m}\delta_{j,n}$. From the form of the Lax matrices (4.2), we can then obtain the following relations

$$\begin{aligned} \alpha L_0 - \widetilde{P} L_0 + L_0 P &= -e\, h \ , \\ \beta M_0 - \widehat{P} M_0 + M_0 P &= -e\, k \ , \end{aligned} \tag{4.3}$$

where we have set

$$P = \sum_{k=1}^{2} \xi_k E_{kk}. \tag{4.4}$$

Working on the Lax equation (3.4) and inserting (4.1), we derive

$$\widehat{L_0} M_0 = \widetilde{M_0} L_0 \ , \tag{4.5}$$

together with the relation $\widehat{h}ek_j = \widetilde{k}eh_j$ for $j = 1, 2$. In order to proceed with the general analysis of (4.5) we consider reductions by additional constraints $he = ke = 0$ and $\xi_1 = -\xi_2 = \xi$. Consequently, dividing each entry in the first row of the relation over each other, we find the equation, i.e.

$$\frac{(\widetilde{\xi} - \xi + \alpha)\,(\widetilde{\xi} + \xi - \alpha)}{(\widetilde{\xi} - \xi - \alpha)\,(\widetilde{\xi} + \xi + \alpha)} \frac{(\widehat{\xi} - \xi - \beta)\,(\widehat{\xi} + \xi + \beta)}{(\widehat{\xi} - \xi + \beta)\,(\widehat{\xi} + \xi - \beta)} = \frac{(\widehat{\widetilde{\xi}} - \xi - \gamma)\,(\widehat{\widetilde{\xi}} + \xi + \gamma)}{(\widehat{\widetilde{\xi}} - \xi + \gamma)\,(\widehat{\widetilde{\xi}} + \xi - \gamma)}\,. \tag{4.6}$$

Introducing a new variable $u \equiv \xi^2$ and inserting this to (4.6) we can derive the following relations

$$\begin{aligned} \alpha(\widetilde{u} - \widehat{\widetilde{u}})(u - \widehat{u}) - \beta(\widehat{u} - \widehat{\widetilde{u}})(u - \widetilde{u}) + \beta\alpha(\alpha - \beta)(u + \widehat{u} + \widetilde{u} + \widehat{\widetilde{u}}) \\ = \beta\alpha(\alpha - \beta)(\beta^2 - \alpha\beta + \alpha^2), \end{aligned} \tag{4.7}$$

where we find in particular case for u the Q_2 equation of [7].

4.1.2 Hyperbolic (Trigonometric) case

Let us perform the hyperbolic limit $2\omega_1 \to \infty\ ,\ 2\omega_2 = \frac{1}{2}\pi i$, in which case we can make the substitution $\sigma(\xi) \to \sinh(\xi)$, yielding

$$\Phi_\kappa(\xi) \to \coth(\xi) + \coth(\kappa).$$

In the case Lax matrices (3.2) can be taken the form:

$$L_\kappa = eh\ \cot\kappa + L_0, \qquad M_\kappa = ek\ \coth\kappa + M_0, \tag{4.8}$$

where the Lax matrices L_0 and M_0 are given by

$$L_0 = \sum_{i,j=1}^{2} h_j \coth(\widetilde{\xi}_i - \xi_j - \alpha) E_{i,j} \ , \qquad M_0 = \sum_{i,j=1}^{2} k_j \coth(\widehat{\xi}_i - \xi_j - \beta) E_{i,j} \ . \quad (4.9)$$

From the form of the Lax matrices (4.2), as a result of the dropping down of the terms with the spectral parameter $\coth\kappa$, we can obtain the following relations

$$\begin{aligned} \exp(2\alpha)\exp(2\widetilde{P})L_0 - L_0\exp(2P) &= \exp(2\alpha)\exp(2\widetilde{P})e\,h + e\,h\exp(2P) \ , \\ \exp(2\beta)\exp(2\widehat{P})M_0 - M_0\exp(2P) &= \exp(2\beta)\exp(2\widehat{P})e\,k + e\,k\exp(2P) \ , \end{aligned} \quad (4.10)$$

where P, h, k and e are given as before. We again make the specification (4.5) and assuming the following constraint $he = ke = 0$ again, we subsequently derive the relations

$$\begin{aligned} &\frac{\sinh(\widetilde{\xi}-\xi+\alpha)\sinh(\widetilde{\xi}+\xi-\alpha)\sinh(\widehat{\xi}-\xi-\beta)\sinh(\widehat{\xi}+\xi+\beta)}{\sinh(\widetilde{\xi}-\xi-\alpha)\sinh(\widetilde{\xi}+\xi+\alpha)\sinh(\widehat{\xi}-\xi+\beta)\sinh(\widehat{\xi}+\xi-\beta)} \\ &\qquad = \frac{\sinh(\widehat{\widetilde{\xi}}-\xi+\alpha-\beta)\sinh(\widehat{\widetilde{\xi}}+\xi-\alpha+\beta)}{\sinh(\widehat{\widetilde{\xi}}-\xi-\alpha+\beta)\sinh(\widehat{\widetilde{\xi}}+\xi+\alpha-\beta)} \ , \end{aligned} \quad (4.11)$$

where $\xi = \xi(n,m)$ is the dependent variable of the equation, related the value u of the rational form of $(Q_3)_{\delta=1}$ equation of [7] through the identification $u = \cosh(2\xi)$. The equivalence between two forms can be seen as a consequence of an identity given in the next statement.

Proposition 4.1.1 *The following identity holds for arbitrary variables X, Y, and Z,*

$$\begin{aligned} &\big(X-\cosh(2\xi-2\alpha)\big)\big(Y-\cosh(2\xi+2\beta)\big)\big(Z-\cosh(2\xi-2(\beta-\alpha))\big) \\ &\quad -\big(X-\cosh(2\xi+2\alpha)\big)\big(Y-\cosh(2\xi-2\beta)\big)\big(Z-\cosh(2\xi+2(\beta-\alpha))\big) \\ &\quad = t^{-1}\Big[\alpha(1-\beta^2)\big(\cosh(2\xi)Y+XZ\big)-\beta(1-\alpha^2)\big(\cosh(2\xi)X+YZ\big) \\ &\qquad -(\alpha^2-\beta^2)\Big((YX+\cosh(2\xi)Z)+\frac{(1-\alpha^2)(1-\beta^2)}{4\alpha\beta}\Big)\Big] \ , \end{aligned} \quad (4.12)$$

where

$$t = \frac{2\alpha\beta e^{2\xi}}{e^{4\xi}-1}. \quad (4.13)$$

Proof

It is straightforward calculation; one need to show that the coefficient of each monomial $1, X, Y, Z, XY, XZ, YZ$ and XYZ of the identity are equivalent. It can be readily seen by using the definition of hyperbolic cosine function and the identification $\alpha \to e^{2\alpha}$, $\beta \to e^{2\beta}$ on the right-hand side which completes the proof. □

Identifying $u = \cosh(2\xi)$, $X = \widetilde{u} = \cosh(2\widetilde{\xi}\ )$, $Y = \widehat{u} = \cosh(2\widehat{\xi}\ )$ and $Z = \widehat{\widetilde{u}} = \cosh(2\widehat{\widetilde{\xi}}\ )$, we see that the expression in brackets on the right-hand side of (4.12) can be written in terms of the following quadrilateral expression

$$Q(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}};\alpha,\beta) := \alpha(1-\beta^2)(u\widehat{u}+\widetilde{u}\widehat{\widetilde{u}}) - \beta(1-\alpha^2)(u\widetilde{u}+\widehat{u}\widehat{\widetilde{u}}) \\ -(\alpha^2-\beta^2)\Big((\widehat{u}\widetilde{u}+u\widehat{\widetilde{u}}) + \frac{(1-\alpha^2)(1-\beta^2)}{4\alpha\beta}\Big)\ , \quad (4.14)$$

which the equation $Q(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}};\alpha,\beta) = 0$ is equivalent to the $(Q_3)_{\delta=1}$ equation in the ABS list. Using the identity

$$\sinh(\xi)\sinh(\eta) = \frac{\cosh(\xi+\eta)-\cosh(\xi-\eta)}{2},$$

it is not hard to see that the expression on the left-hand side of (4.12) and the relation (4.11) are equal. We remark that the statement 4.1.1, which is a new identity, can be derived by degeneration (in the hyperbolic limit $\sigma(\xi) \to \sinh(\xi)$) of Proposition 2.1.2. Finally, the hyperbolic limit $2\omega_1 = \frac{1}{2}\pi$, $2\omega_2 \to i\infty$ of the elliptic functions is performed along the similar way after making the substitution

$$\sigma(\xi) \to \sin(\xi)\ .$$

The details will be omitted. Next, let us consider the rational as well as the hyperbolic (trigonometric) limits of the equation (3.50a) with three variables.

N=3:

In chapter 3, we derived from the general elliptic Lax system of rank 3 a coupled system of quadrilateral elliptic 3-leg equations. We now consider the equations in the 3-particle situation and conditions $\sum\limits_{j=1}^{3} h_j = \sum\limits_{j=1}^{3} k_j = 0$. Let us first focus on the rational limit of eqs. (3.50a).

4.1.3 A higher rank analogue of Q_2

By taking the rational limit $\sigma(\xi) \to \xi$ in (3.50a) we obtain

$$
\begin{aligned}
&\Bigg[\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&\quad\Bigg[\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&=\frac{(\widehat{\xi}_1-\xi_2-\beta)\,(\widehat{\xi}_2-\xi_1-\beta)}{(\widehat{\xi}_1-\xi_1-\beta)\,(\widehat{\xi}_2-\xi_2-\beta)}\\
&\times\Bigg[\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&\Bigg[\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg],
\end{aligned}
\tag{4.15}
$$

where

$$S(\xi) = (\xi - \widetilde{\xi}_1 - \beta)\,(\xi - \widetilde{\xi}_2 - \beta)\,(\xi - \widetilde{\xi}_3 - \beta),$$

and coupled to this equation we have two more similar rational equations obtained in the same way from (3.50b) and (3.50c). By analogy with the $N = 2$ case, where the rational limit of Adler's equation was shown to yield the Q_2 equation (after a substitution), we can justifiably consider the above coupled system as constituting a higher-rank version of Q_2. However, in this case the analogue of the substitution used before seems no longer applicable.

4.1.4 A higher rank analogue of $(Q_3)_{\delta=1}$

We can consider the trigonometric limit $\sigma(\xi) \to \sinh(\xi)$ in (3.50a) that becomes

$$\begin{aligned}
&\Bigg[\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&\quad\Bigg[\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\\
&-\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&+\frac{\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\Bigg]\\
&=\frac{\sinh(\widehat{\xi}_1-\xi_2-\beta)}{\sinh(\widehat{\xi}_1-\xi_1-\beta)}\,\frac{\sinh(\widehat{\xi}_2-\xi_1-\beta)}{\sinh(\widehat{\xi}_2-\xi_2-\beta)}\\
&\times\Bigg[\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_1-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}-
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_1-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&\left.+\frac{\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_1+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_1+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_1-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\right]\\
&\left[\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_2)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_2-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_2)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_2-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_3-\widehat{\xi}_3-\alpha)}\right.\\
&-\frac{\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_1+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_1)\,S(\widehat{\widetilde{\xi}}_3)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_2-\alpha)\,(\widehat{\widetilde{\xi}}_2-\widehat{\xi}_3-\alpha)}\\
&\left.+\frac{\sinh(\widehat{\widetilde{\xi}}_2-\widehat{\widetilde{\xi}}_3)\sinh(\widehat{\widetilde{\xi}}_2+\widehat{\widetilde{\xi}}_3-\widetilde{\xi}_1-\widetilde{\xi}_2-\widetilde{\xi}_3+\xi_2+\alpha-2\beta)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\widehat{\xi}_3+\xi_2+\beta-\alpha)}{S(\widehat{\widetilde{\xi}}_2)\,S(\widehat{\widetilde{\xi}}_3)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_2-\alpha)\,\sinh(\widehat{\widetilde{\xi}}_1-\widehat{\xi}_3-\alpha)}\right], \qquad (4.16)
\end{aligned}
$$

where

$$S(\xi)=\sinh(\xi-\widetilde{\xi}_1-\beta)\,\sinh(\xi-\widetilde{\xi}_2-\beta)\,\sinh(\xi-\widetilde{\xi}_3-\beta).$$

The other trigonometric relations coupled to this equation (4.16) are achieved from (3.50b) and (3.50c). By analogy with the $N=2$ case, where the trigonometric limit of Adler's equation revealed the $(Q_3)_{\delta=1}$ equation (after a substitution), we consider the above coupled system as a higher-rank version of $(Q_3)_{\delta=1}$ in 3-leg form. However, the analogue of the substitution used in the previous case is not convenient. Next we will investigate the continuum limit of Adler's lattice equation, leading to associated differential-difference equations.

4.2 Continuum limits

Let us investigate what happens under a continuum limit for Adler's elliptic lattice equation in the Weierstrass form (2.3), bringing us eventually back to the original KN equation. Since the lattice equation includes two discrete variables, n and m, we have to consider the continuum limit in two steps. In the first step, the limit is conducted on only one of the lattice variables (and associated lattice parameter) while keeping the other lattice direction intact. This reduces our equation to drastically different type of intermediate (differential-difference) equation, i.e., an equation with one discrete and one continuous independent variable. In the second step the remaining lattice variable will be continuous. Both cases are obtained by reducing the lattice step associated with the parameters α and β to zero. There are two key continuum limits that are of interest: i) the *straight* limit obtained by taking a limit in one of the discrete directions, ii) the *skew* limit obtained after performing a change of variables on the lattice and involving a combination of two lattice parameters.

The continuum limit for the integrable system of quadrilateral elliptic 3-leg type (3.47), which may be regarded as a higher-rank version of Adler's lattice equation, still remain to be investigated. Each part of the system of equations (3.50a)-(3.50c) is already very complicated and would require computer-aided computations, let alone taking into account that the limit has to be considered for the entire system of equations simultaneously. Thus, doing the systematic continuum limits for those multi-component systems is going to be extremely challenging, and we will not attempt to do those limits here. Instead we will present here the continuum limits of the much simpler case of Adler's lattice equation (both in the rational as well as in the 3-leg form), which will give a good indication of the procedure and of the subtleties involved.

4.2.1 Straight continuum limit

We will consider a particular continuum limit for Adler's elliptic lattice equation by expanding around the branch point of the curve. Let the half-periods of the elliptic functions be given by ω_1 and ω_2, i.e. we have the periodicity condition:

$$\wp(\xi + 2\omega_{1,2}) = \wp(\xi).$$

Introducing a third half period by $\omega_3 = -\omega_1 - \omega_2$, the branch points of the elliptic curve are given by $(e_1, 0)$, $(e_2, 0)$ and $(e_3, 0)$ with $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_2)$, $e_3 = \wp(\omega_3)$, leading to the representation for the curve:

$$A^2 = 4(a-e_1)(a-e_2)(a-e_3).$$

Clearly, $\wp'(\omega_1) = \wp'(\omega_2) = \wp'(\omega_3) = 0$ and the moduli of the curve $g_1, \ , g_2$ can be given in terms of the e_i as:

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3) = -4(g - 3e_1^2) \quad , \quad g_3 = 4e_1e_2e_3 = 4e_1(g - 2e_1^2) \ ,$$

where we have introduced the quantity $g = (e_1 - e_2)(e_1 - e_3) = \frac{1}{2}\wp''(\omega_1)$. Note that if one of the lattice parameters α or β is taken to be a half-period, say $\beta = \omega_1$, implying $(b, B) = (e_1, 0)$ and $c - e_1 = g/(a - e_1)$, then the lattice equation (2.3) leads to:

$$\overline{u} - e_1 = \frac{g}{u - e_1}, \tag{4.1}$$

where we have used the notation for the shift $u \to \overline{u}$ to define the lattice translation associated with the lattice parameter ω_1.

The limit we would like to consider is the one when one of the parameters of the equation, say β, approaches the half-period ω_1, i.e. to consider $\beta = \omega_1 + \delta$ in the limit $\delta \to 0$. The way to do this is to consider the combined shift $u \to \widehat{\overline{u}}$, we take

$$\begin{aligned} \widehat{\overline{u}} &\to u + \sqrt{\delta}\, u_x + \frac{1}{2}\delta u_{xx} + \dots \ , \\ \widehat{\overline{\widetilde{u}}} &\to \widetilde{u} + \sqrt{\delta}\, \widetilde{u}_x + \frac{1}{2}\delta \widetilde{u}_{xx} + \dots \ . \end{aligned}$$

In this expansion we have:

$$\begin{aligned} b &= \wp(\omega_1 + \delta) = e_1 + \delta^2 g + \dots \ , \\ B &= \wp'(\omega_1 + \delta) = 2\delta g + 4\delta^3 e_1 g + \dots \ , \\ c &= \wp(\alpha + \omega_1 - \delta) = \wp(\alpha + \omega_1) - \delta\wp'(\alpha + \omega_1) + \dots \ , \\ C &= -\wp'(\alpha + \omega_1 - \delta) = -\wp'(\alpha + \omega_1) + \delta\wp''(\alpha + \omega_1) + \dots \end{aligned}$$

where we can use

$$\wp(\alpha+\omega_1)=\frac{g}{a-e_1}+e_1 \quad , \quad \wp'(\alpha+\omega_1)=-\frac{gA}{(a-e_1)^2}\ ,$$

and

$$\wp''(\alpha+\omega_1)=\frac{2g}{(a-e_1)^3}\big(A^2-(3a^2+g-3e_1^2)(a-e_1)\big)\ .$$

Expanding to first order in δ we obtain the differential-difference equation:

$$\frac{1}{2}Au_x\widetilde{u}_x=-H(a,u,\widetilde{u})\ , \tag{4.2}$$

with

$$H(a,u,v)=(uv+au+av+3e_1^2-g)^2-4(a+u+v)(auv-e_1g+2e_1^3)\ . \tag{4.3}$$

We note that the equation (4.2) is the formula for the B$\ddot{a}$cklund transformation (BT) of the Krichever-Novikov equation (2.1), which formed the starting point for the construction in [5].

Applying this continuum limit to the Lax pair (2.10) we obtain the following semi-continuous Lax relation

$$\varphi_x=\mathrm{U}_\kappa\varphi\ , \tag{4.4a}$$

with

$$\mathrm{U}_\kappa \;=\; \frac{1}{Ku_x}\begin{pmatrix} \frac{1}{2}g_3-(u+k)(uk-\frac{1}{4}g_2) & g_3(u+k)+(uk+\frac{1}{4}g_2)^2 \\ -(u-k)^2 & -\frac{1}{2}g_3+(u+k)(uk-\frac{1}{4}g_2) \end{pmatrix} \tag{4.4b}$$

which supplements the lattice Lax pair (2.12). The linear equation (4.4) is the spatial part of the Lax pair to the continuous KN equation (2.1), which can be recovered from the original Lax pair given in the paper [58].

4.2.2 Skew continuum limit

The straight limit is not the only way to obtain a semi-discrete lattice equation. Here we consider a particular continuum limit which involves a change of variables on the lattice, namely, $u_{n,m} =: u_{n+m,m}$, and then the shifted variables becomes:

$$\begin{aligned} u_{n+1,m} &\to u_{n+m+1,m} =: \widetilde{u} \ , \\ u_{n,m+1} &\to u_{n+m+1,m+1} =: \widehat{\widetilde{u}} \ , \\ u_{n+1,m+1} &\to u_{n+m+2,m+1} =: \widehat{\widetilde{\widetilde{u}}} \ . \end{aligned} \tag{4.5}$$

Rearranging the discrete variables in (2.3), we have

$$\begin{aligned} A &\Big[(u-b)(\widehat{\widetilde{u}}-b)-(a-b)(c-b)\Big]\Big[(\widetilde{u}-b)(\widehat{\widetilde{\widetilde{u}}}-b)-(a-b)(c-b)\Big] \\ &+B\Big[(u-a)(\widetilde{u}-a)-(b-a)(c-a)\Big]\Big[(\widehat{\widetilde{u}}-a)(\widehat{\widetilde{\widetilde{u}}}-a)-(b-a)(c-a)\Big] = \\ &= ABC(a-b) \ , \end{aligned} \tag{4.6}$$

and taking the limit by transformation

$$\delta = \beta - \alpha \to 0, \quad n \to -\infty, \quad m \to \infty \ , \tag{4.7}$$

such that $m\delta \to t$ finite whilst $n+m$ is to remain fixed. Thus, using the expansions

$$\begin{aligned} b &= \wp(\alpha+\delta) = a + \delta A + \frac{1}{2}\delta^2 A_1 + 2\delta^3 aA + \dots \ , \\ B &= \wp'(\alpha+\delta) = A + \delta A_1 + 6\delta^2 aA + \dots \ , \\ c &= \wp(\delta) = \frac{1}{\delta^2} + O(\delta^2) \ , \\ C &= \wp'(\delta) = -\frac{2}{\delta^3} + O(\delta) \ , \end{aligned}$$

where $A_1 = \wp''(\alpha) = 6a^2 - g_2/2$, we have for the variable u the Taylor expansion:

$$\widehat{u} \to \widetilde{u} + \delta\widetilde{u}_t + \frac{1}{2}\delta^2\widetilde{u}_{tt} + \dots \ , \tag{4.8}$$

and inserting these into the equation (4.6) we obtain the following differential-difference equation:

$$A(\underset{\sim}{v} - \widetilde{v})v_t = A^2(\widetilde{v} + 2v + \underset{\sim}{v} + 6a) - 2(v\widetilde{v} - \frac{1}{2}A_1)(v\underset{\sim}{v} - \frac{1}{2}A_1)\ , \tag{4.9}$$

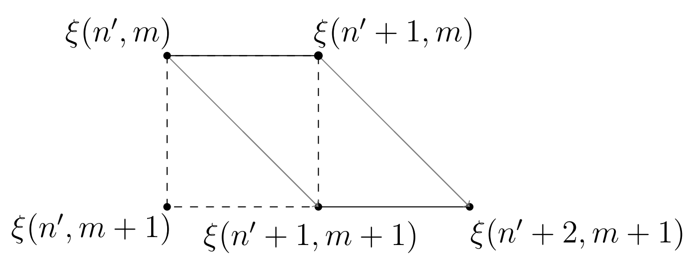
for the variable $v = u - a$. Equation (4.9), which contains one continuous and one discrete variable, is called the mixed lattice KN equation. The Lax pair for the equation (4.9) can be obtained from the Lax pair (2.12) by applying the skew continuum limit.

Continuum limits of 3-leg equations of Adler's equation

We will now consider the same (skew) continuum limits directly on the 3-leg form (2.13) of Adler's equation by performing on a combination of the two lattice directions. Again we will make a change of independent discrete variables as in (4.5). Thus, making the replacements for the dependent variable $\xi(n, m)$ as follows

$$\xi(n+1,m) \to \xi(n'+1,m) := \widetilde{\xi} \quad , \quad \xi(n,m+1) \to \xi(n'+1,m+1) := \widehat{\widetilde{\xi}}\,,$$
$$\xi(n+1,m+1) \to \xi(n'+2,m+1) := \widehat{\widetilde{\widetilde{\xi}}}. \tag{4.10}$$

This can be visualized in the diagram:



Focusing on the limit (4.7) as in the previous case where $m\,\delta \to t$ is finite and $n' = n+m$ is fixed. By this limit and the transformations as given in (4.10), the Adler's equation in

3-leg form (2.13) goes over into the following form

$$\frac{\sigma(\widetilde{\xi}-\xi+\alpha)\,\sigma(\widetilde{\xi}+\xi-\alpha)}{\sigma(\widetilde{\xi}-\xi-\alpha)\,\sigma(\widetilde{\xi}+\xi+\alpha)}\,\frac{\sigma(\widehat{\widetilde{\xi}}-\xi-\delta-\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\xi+\delta+\alpha)}{\sigma(\widehat{\widetilde{\xi}}-\xi+\delta+\alpha)\,\sigma(\widehat{\widetilde{\xi}}+\xi-\delta-\alpha)}$$
$$=\frac{\sigma(\widehat{\widetilde{\widetilde{\xi}}}-\xi-\delta)\,\sigma(\widehat{\widetilde{\widetilde{\xi}}}+\xi+\delta)}{\sigma(\widehat{\widetilde{\widetilde{\xi}}}-\xi+\delta)\,\sigma(\widehat{\widetilde{\widetilde{\xi}}}+\xi-\delta)}\,. \qquad (4.11)$$

The next thing is for a small δ, to apply the Taylor series expansions for a arbitrary quantity y in (4.11):

$$\sigma(\widehat{\widetilde{\xi}}\pm\delta+y)=\sigma(\widetilde{\xi}+y)\Big(1\pm\delta(\dot{\widetilde{\xi}}-1)\zeta(\widetilde{\xi}+y)\Big)+\dots \quad , \qquad (4.12a)$$

and

$$\sigma(\widehat{\widetilde{\widetilde{\xi}}}\pm\delta+y)=\sigma(\widetilde{\widetilde{\xi}}+y)\Big(1\pm\delta(\dot{\widetilde{\widetilde{\xi}}}-1)\zeta(\widetilde{\widetilde{\xi}}+y)\Big)+\dots \quad , \qquad (4.12b)$$

where $\zeta(t)=\frac{d}{dt}\ln\sigma(t)$ is the Weierstrass zeta function and the dot " $\cdot$ " stands for ξ-derivative with respect to a continuous variable t ($\dot{\xi}=\frac{\partial\xi}{\partial t}$). Inserting (4.12) into the equation (4.11), sigma functions σ drop out and then we obtain semi-continuous equation:

$$\frac{\big(1+\delta(\dot{\widetilde{\xi}}-1)\zeta(\widetilde{\xi}-\xi-\alpha)+\dots\big)\big(1+\delta(\dot{\widetilde{\xi}}+1)\zeta(\widetilde{\xi}+\xi+\alpha)+\dots\big)}{\big(1+\delta(\dot{\widetilde{\xi}}+1)\zeta(\widetilde{\xi}-\xi+\alpha)+\dots\big)\big(1+\delta(\dot{\widetilde{\xi}}-1)\zeta(\widetilde{\xi}+\xi-\alpha)+\dots\big)}$$
$$=\frac{\big(1+\delta(\dot{\widetilde{\widetilde{\xi}}}-1)\zeta(\widetilde{\widetilde{\xi}}-\xi)+\dots\big)\,\big(1+\delta(\dot{\widetilde{\widetilde{\xi}}}+1)\zeta(\widetilde{\widetilde{\xi}}+\xi)+\dots\big)}{\big(1+\delta(\dot{\widetilde{\widetilde{\xi}}}+1)\zeta(\widetilde{\widetilde{\xi}}-\xi)+\dots\big)\big(1+\delta(\dot{\widetilde{\widetilde{\xi}}}-1)\zeta(\widetilde{\widetilde{\xi}}+\xi)+\dots\big)},$$

in which one retains the dominant term in the small parameter δ to yield the expression

$$\dot{\widetilde{\xi}}\Big[\zeta(\widetilde{\xi}-\xi-\alpha)+\zeta(\widetilde{\xi}+\xi+\alpha)-\zeta(\widetilde{\xi}-\xi+\alpha)-\zeta(\widetilde{\xi}+\xi-\alpha)\Big]$$
$$=2\zeta(\widetilde{\widetilde{\xi}}+\xi)-2\zeta(\widetilde{\widetilde{\xi}}-\xi)+\zeta(\widetilde{\xi}-\xi+\alpha)-\zeta(\widetilde{\xi}+\xi-\alpha)$$
$$+\zeta(\widetilde{\xi}-\xi-\alpha)-\zeta(\widetilde{\xi}+\xi+\alpha)\,. \qquad (4.13)$$

Applying the following identity to the left-hand side

$$\zeta(x)+\zeta(y)+\zeta(z)-\zeta(x+y+z)=\frac{\sigma(x+y)\sigma(x+z)\sigma(y+z)}{\sigma(x)\sigma(y)\sigma(z)\sigma(x+y+z)}\ , \qquad (4.14)$$

gives an intermediate equation, with one discrete and one continuous variable as follows:

$$\begin{aligned} \dot{\xi} &= \frac{\sigma(\xi+\underset{\sim}{\xi}+\alpha)\sigma(\xi+\underset{\sim}{\xi}-\alpha)\sigma(\xi-\underset{\sim}{\xi}+\alpha)\sigma(\xi-\underset{\sim}{\xi}-\alpha)}{\sigma(2\alpha)\sigma(2\xi)\sigma(2\underset{\sim}{\xi})}\times \\ &\times \Big[2\zeta(\widetilde{\xi}+\underset{\sim}{\xi})-2\zeta(\widetilde{\xi}-\underset{\sim}{\xi})+\zeta(\xi-\underset{\sim}{\xi}+\alpha)-\zeta(\xi+\underset{\sim}{\xi}-\alpha) \\ &\qquad +\zeta(\xi-\underset{\sim}{\xi}-\alpha)-\zeta(\xi+\underset{\sim}{\xi}+\alpha)\Big]\ , \end{aligned} \tag{4.15}$$

which can be cast into the form:

$$A\dot{u} = 2\frac{H(u,\underset{\sim}{u},a)}{\widetilde{u}-\underset{\sim}{u}} + 2(u-a)^2(u+\underset{\sim}{u}+a) - \frac{1}{2}(U^2+A^2)\ . \tag{4.16}$$

However, the alternative 3-leg form (3.22b) gives the continuum limit

$$\begin{aligned} \dot{\xi} &= \frac{\sigma(\xi+\widetilde{\xi}+\alpha)\sigma(\xi+\widetilde{\xi}-\alpha)\sigma(\xi-\widetilde{\xi}+\alpha)\sigma(\xi-\widetilde{\xi}-\alpha)}{\sigma(2\alpha)\sigma(2\xi)\sigma(2\widetilde{\xi})} \\ &\times \big[2\zeta(\underset{\sim}{\xi}-\widetilde{\xi})-2\zeta(\underset{\sim}{\xi}+\widetilde{\xi})+\zeta(\xi+\widetilde{\xi}-\alpha)-\zeta(\xi-\widetilde{\xi}+\alpha) \\ &\qquad +\zeta(\xi+\widetilde{\xi}+\alpha)-\zeta(\xi-\widetilde{\xi}-\alpha)\big]\ , \end{aligned} \tag{4.17}$$

which can be cast into the form:

$$A\dot{u} = 2\frac{H(u,\widetilde{u},a)}{\widetilde{u}-\underset{\sim}{u}} - 2(u-a)^2(u+\widetilde{u}+a) + \frac{1}{2}(U^2+A^2)\ . \tag{4.18}$$

Both eqs. (4.16) and (4.18) are compatible in view of the identity:

$$\frac{H(u,\widetilde{u},a)-H(u,\underset{\sim}{u},a)}{\widetilde{u}-\underset{\sim}{u}} = (u-a)^2(\widetilde{u}+\underset{\sim}{u}+2u+2a) - \frac{1}{2}(U^2+A^2)\ .$$

Thus, we can rewrite (4.16), (4.18) as:

$$A\dot{u} = \frac{H(u,\widetilde{u},a)+H(u,\underset{\sim}{u},a)}{\widetilde{u}-\underset{\sim}{u}} - (u-a)^2(\widetilde{u}-\underset{\sim}{u})\ , \tag{4.19}$$

and this is equivalent to equation (4.9).

Remark 4.2.1 In section 4.2.1, the differential-difference equation has been obtained by taking the parameter $\beta \to \omega_1$. To proceed next to the full continuum limit, performed on the remaining parameter α, one can apply on the result (4.9) of the skew limit of Adler system the procedure to obtain the straight limit, i.e. one can take $\alpha = \omega_1 + \delta$ and expand around the half-period ω_1 taking into account the relation (4.1). However, it is quite cumbersome and requires higher-order expansions and subtle changes of variables. The end result will necessarily be the fully continuous KN equation (2.1).

4.3 Reductions

The integrable lattice P∆Es have several types of special solutions. In most cases the process of obtaining these solutions requires the study of reduction of the corresponding P∆Es. We mean that the periodic reduction yields a system of O∆Es. Lattice systems typically admit several types of reductions, e.g.:

1. Periodic reductions (stationary solutions);
2. Non-autonomous scaling-type reductions (often yielding discrete Painlevé equations).

So far little work exists on reductions of either type for elliptic lattice equations. In [76] finite-gap solutions of the continuous KN equation (2.1) was obtained. Another work is that the 2-step periodic reductions of the ABS equations have been studied in [54]. The simplest periodic reduction is the 1-step period one obtained by imposing

$$\widetilde{\boldsymbol{\chi}}_\kappa = \lambda \boldsymbol{\chi}_\kappa \ , \tag{4.20}$$

for which we get an isospectral problem of the form

$$N_\kappa \, \chi_\kappa = \lambda \chi_\kappa \ , \quad \widehat{\chi}_\kappa = M_\kappa \chi_\kappa \ , \tag{4.21}$$

and this is precisely the Lax pair for the discrete-time elliptic Ruijsenaars (RS) model, which is the relativistic variant of the discrete-time elliptic Calogero-Moser (CM) system.

The time-discrete version of the RS system was discovered by Nijhoff, Ragnisco and Kuznetsov in [69] from a reduction of fully discrete Kadomtsev-Petviashvili (KP) equation with three lattice variables. Next, we shall give a brief review of how the discrete system is obtained in [69]. The elliptic Lax matrices have been introduced in the form

$$(N_\kappa)_{i,j} = k_i k_j \ \Phi_\kappa(\xi_i - \xi_j + \beta) \quad , \quad (M_\kappa)_{i,j} = \widehat{k}_i k_j \ \Phi_\kappa(\widehat{\xi}_i - \xi_j + \beta) \qquad (i,j = 1,\dots,N) \tag{4.22}$$

where ξ_i are the position of the particles and β is a parameter of the system associated with the non-relativistic limit. The auxiliary variables k_i do not depend on κ and remain to be determined. As in chapter 1, the hat shift in the dependent variable $\xi_i = \xi_i(n,m)$ will be defined as $\xi_i(n,m+1) = \widehat{\xi}_i$, and $\xi_i(n,m-1) = \underset{\frown}{\xi}_i$. Let us consider first the compatibility $\widehat{N}_\kappa M_\kappa = M_\kappa N_\kappa$, we get from the addition formula (3.5) that

$$\begin{aligned}&\sum_{l=1}^{N} \widehat{k}_l^2 \big[\zeta(\kappa) + \zeta(\widehat{\xi}_i - \widehat{\xi}_l + \beta) + \zeta(\widehat{\xi}_l - \xi_j + \beta) - \zeta(\kappa + 2\beta + \widehat{\xi}_i - \xi_j)\big] \\ &= \sum_{l=1}^{N} k_l^2 \big[\zeta(\kappa) + \zeta(\widehat{\xi}_i - \xi_l + \beta) + \zeta(\xi_l - \xi_j + \beta) - \zeta(\kappa + 2\beta + \widehat{\xi}_i - \xi_j)\big] \ .\end{aligned}$$

Thus by setting $\sum_{l=1}^N \widehat{k}_l^2 = \sum_{l=1}^N k_l^2$, the equations can be separated into a part depending on the spectral parameter κ, and the remainder independent of κ. This leads to the identity

$$\begin{aligned}&\sum_{l=1}^{N} \big[\, \widehat{k}_l^2 \zeta(\widehat{\xi}_i - \widehat{\xi}_l + \beta) - k_l^2 \zeta(\widehat{\xi}_i - \xi_l + \beta)\big] \\ &\qquad\qquad = -\sum_{l=1}^{N} \big[\, \widehat{k}_l^2 \zeta(\widehat{\xi}_l - \xi_j + \beta) - k_l^2 \zeta(\xi_l - \xi_j + \beta)\big] \ ,\end{aligned} \tag{4.23}$$

for all $i,j = 1,\dots,N$. The relation (4.23) can be end up with the form

$$\sum_{l=1}^{N} \big[\, \widehat{k}_l^2 \zeta(\widehat{\xi}_i - \widehat{\xi}_l + \beta) - k_l^2 \zeta(\widehat{\xi}_i - \xi_l + \beta)\big] = q \ , \tag{4.24a}$$

$$\sum_{l=1}^{N} \big[k_l^2 \zeta(\xi_l - \xi_j + \beta) - \widehat{k}_l^2 \zeta(\widehat{\xi}_l - \xi_j + \beta)\big] = q \ , \tag{4.24b}$$

where q does not depend on a particle label. We will assume it to be constant. The equations of motion in terms of the ξ_i can be derived by eliminating the variables k_i from (4.23). In order to do this we will apply the Lagrange interpolation formula (see Appendix B) leading

$$k_l^2 = -q\frac{\prod_{j=1}^{N}\sigma(\xi_l-\xi_j+\beta)\sigma(\xi_l-\widehat{\xi}_j-\beta)}{\prod_{j\neq l}^{N}\sigma(\xi_l-\xi_j)\prod_{j=1}^{N}\sigma(\xi_l-\widehat{\xi}_j)}\,, \tag{4.25a}$$

$$\widehat{k}_l^2 = q\frac{\prod_{j=1}^{N}\sigma(\widehat{\xi}_l-\xi_j+\beta)\sigma(\widehat{\xi}_l-\widehat{\xi}_j-\beta)}{\prod_{j\neq l}^{N}\sigma(\widehat{\xi}_l-\widehat{\xi}_j)\prod_{j=1}^{N}\sigma(\widehat{\xi}_l-\xi_j)}\,, \tag{4.25b}$$

for $l = 1, 2, \ldots, N$. Shifting (4.25b) in the backward direction we get an implicit system of OΔEs

$$\frac{q}{\underset{\frown}{q}}\prod_{l=1:j\neq l}^{N}\frac{\sigma(\xi_l-\xi_j+\beta)}{\sigma(\xi_l-\xi_j-\beta)} = \prod_{l=1}^{N}\frac{\sigma(\xi_l-\widehat{\xi}_j)\sigma(\xi_l-\underset{\frown}{\xi}_j+\beta)}{\sigma(\xi_l-\underset{\frown}{\xi}_j)\sigma(\xi_l-\widehat{\xi}_j-\beta)}\,, \qquad j\in\mathbb{N}\,. \tag{4.26}$$

Thus, taking $q/\underset{\frown}{q}$ to be unity leads to the time-discretization of the Ruijsenaars (relativistic Calogero-Moser) model. The discrete-time RS system in the "$\widetilde{\ }$" direction can be obtained by making the replacement $\widehat{\ }\leftrightarrow\widetilde{\ }$.

The connection between ABS equations and RS system has already presented in [104] where it has been shown that one-step periodic reduction of the system (3.16) to be "of Landau-Lifschitz (LL) class" (or spin-nonzero case) given in chapter 3, $\widetilde{\boldsymbol{\chi}}_\kappa = \lambda\boldsymbol{\chi}_\kappa,$ leads to the discrete-time elliptic RS model (4.26).

The corresponding non-autonomous analogue is obtained by *de-autonomization*, i.e. the replacement

$$\lambda\boldsymbol{\chi}_\kappa \;\rightsquigarrow\; \boldsymbol{\chi}_{\kappa+\tau}\,,$$

i.e. by going over to a non-isospectral problem which in the elliptic case corresponds to a linear difference equation on the torus and the corresponding discrete *isomonodromic deformations*. First examples of such de-autonomizations were considered in [42, 84] and also reviewed in chapter 1.

Chapter 5

Discrete elliptic isomonodromic deformation problems

In this chapter we present a new class of isomonodromic deformation problems which form (in some sense) the nonautonomous counterparts of the Lax pairs studied in chapter 3. Those monodromy problems are obtained by applying the elliptic analogue of the *deautonomization* procedure, outlined in chapter 1 for the difference and q-difference Lax pair associated with discrete Painlevé equations. In the continuous case, there are various elliptic isomonodromic deformation problems known in the literature [60, 99], going back to the work of Okamoto [78, 79, 80, 81, 82, 83], who derived in particular an isomonodromic system for a coupled system of second order ODEs with two free parameters (apart from the moduli of the elliptic curve), which can be thought of as an elliptic generalization of the Painlevé VI equation. Okamoto's work was generalized to an arbitrary order ODE in the paper of Iwasaki [48]. In the discrete case there has been recent work by Yamada and Noumi et al. [75, 103] on Lax pairs for the elliptic discrete Painlevé equation of Sakai [88]. Our approach is different from the latter and we present this new general elliptic isomonodromic Lax scheme in what follows. We show how the compatibility conditions lead to a constitutive set of relations and we perform an initial analysis to derive nonlinear nonautonomous difference equations from the scheme in the simplest nontrivial case.

5.1 General elliptic isomonodromic deformation scheme

In this section, we will show how to set up a novel class of isomonodromic deformation problems on the torus, from the point of view of lattice equations. This follows the structure of the zero-curvature Lax systems treated in chapter 3.

5.1.1 First order scheme

The new system appears as the discrete compatibility condition of a pair of the associated linear problems (Lax pair) defining the shift (translation) of an eigenfunction $\boldsymbol{\chi}_\kappa$ in the n with together the linear difference equation in terms of the spectral parameter,

$$\begin{aligned} \boldsymbol{\chi}_{\kappa+\tau} &= \mathbf{T}_\kappa\, \boldsymbol{\chi}_\kappa \ , &\quad (5.1a)\\ \widetilde{\boldsymbol{\chi}}_\kappa &= \mathbf{L}_\kappa\, \boldsymbol{\chi}_\kappa \ , &\quad (5.1b)\end{aligned}$$

where Lax matrices

$$\begin{aligned} (\mathbf{L}_\kappa)_{i,j} &= H_{i,j}\, \sigma(\kappa)\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha) \ , &\quad (5.2a)\\ (\mathbf{T}_\kappa)_{i,j} &= S_{i,j}\, \sigma(\kappa)\, \Phi_\kappa(\xi_i - \xi_j - \gamma) \ , &\quad (5.2b)\\ & \qquad (i,j = 1,\dots,N) & \end{aligned}$$

in which $H_{i,j}$, $S_{i,j}$ do not depend on κ and remain to be determined. As it turns out γ, and perhaps α and β, will depend explicitly on the discrete variables n, m, while $\xi_i = \xi_i(n,m)$ are the main independent variables. The Ansatz for the Lax pair (5.1) is natural, in view of the fact that the matrices $\mathbf{L}_\kappa$ and $\mathbf{T}_\kappa$ are a natural choice by comparison with the results obtained in [26]. We mention that the extra factor $\sigma(\kappa)$ (in comparison with the Lax matrices in (3.2)) is crucial for the scheme to work, as we shall see.

The compatibility of the system (5.1) gives us

$$\widetilde{\boldsymbol{\chi}}_{\kappa+\tau} = \widetilde{\mathbf{T}}_\kappa \, \mathbf{L}_\kappa \, \boldsymbol{\chi}_\kappa , \tag{5.3a}$$

$$\widetilde{\boldsymbol{\chi}}_{\kappa+\tau} = \mathbf{L}_{\kappa+\tau} \, \mathbf{T}_\kappa \, \boldsymbol{\chi}_\kappa \, . \tag{5.3b}$$

Equating (5.3a) and (5.3b), we derive the Lax equation

$$\mathbf{L}_{\kappa+\tau}\mathbf{T}_\kappa = \widetilde{\mathbf{T}}_\kappa \, \mathbf{L}_\kappa \, . \tag{5.4}$$

Working out the matrix Lax equation (5.4) we obtain

$$\begin{aligned} \sigma(\tau)\Phi_\kappa(\tau) \sum_{l=1}^{N} H_{il} \, S_{lj} \, \Phi_{\kappa+\tau}(\widetilde{\xi}_i - \xi_l - \alpha) \, \Phi_\kappa(\xi_l - \xi_j - \gamma) \\ = \sum_{l=1}^{N} \widetilde{S}_{il} \, H_{lj} \, \Phi_\kappa(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\Phi_\kappa(\widetilde{\xi}_l - \xi_j - \alpha) \, , \\ (\forall i, j = 1, \dots, N) \end{aligned}$$

which can be rewritten in the form

$$\begin{aligned} \sigma(\tau)\Phi_\kappa(\tau) \sum_{l=1}^{N} H_{il} \, S_{lj} \, \big[\Phi_{\kappa+\tau}(\widetilde{\xi}_i - \xi_j - \alpha - \gamma)\Phi_{-\tau}(\xi_l - \xi_j - \gamma) \\ + \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \gamma) \big] \\ = \sum_{l=1}^{N} \widetilde{S}_{il} \, H_{lj} \, \Phi_\kappa(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\Phi_\kappa(\widetilde{\xi}_l - \xi_j - \alpha) \, , \end{aligned}$$

using the addition formulas

$$\Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y) \, \Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x) \, \Phi_\lambda(y-x) \, . \tag{5.5}$$

From the fundamental identity

$$\Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y) \left[\zeta(\kappa) + \zeta(x) + \zeta(y) - \zeta(\kappa + x + y) \right] \, , \tag{5.6}$$

one can basically derive the following relation

$$\sigma(\tau)\sum_{l=1}^{N} H_{il}\, S_{lj}\Bigg[\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Phi_{-\tau}(\xi_l-\xi_j-\gamma)+$$
$$+\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)\Big(\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)+$$
$$+\zeta(\tau)+\zeta(\widetilde{\xi}_i-\xi_j-\alpha-\gamma)\Big)\Bigg]=$$
$$=\sum_{l=1}^{N}\widetilde{S}_{il}\, H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+$$
$$+\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\big]\ .$$
$$(\forall i,j=1,\dots,N) \tag{5.7}$$

Using the relation $\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(x)=\Phi_\kappa(\tau+x)\Phi_\tau(x)$ on the first terms of the first line leaves the term which can go with the third term of line 1 of (5.7) by applying the identity (5.6) once more. Thus, we end up with the form:

$$\sigma(\tau)\sum_{l=1}^{N} H_{il}\, S_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)$$
$$\times\Big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)+\zeta(\widetilde{\xi}_i-\xi_l+\tau-\alpha)-\zeta(\xi_j-\xi_l+\gamma)\Big]$$
$$=\sum_{l=1}^{N}\widetilde{S}_{il}\, H_{lj}\,\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})\Big[\zeta(\kappa)-\zeta(\kappa+\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})+$$
$$+\zeta(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\Big]$$
$$(\forall i,j=1,\dots,N). \tag{5.8}$$

Note that there exists an overall factor $\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\gamma+\tau)$ on the left-hand side and $\Phi_\kappa(\widetilde{\xi}_i-\xi_j-\alpha-\widetilde{\gamma})$ on the right-hand side which can be dropped out by setting

$$\widetilde{\gamma}=\gamma-\tau.$$

Then the remaining terms can be separated into a part depending on the spectral parameter κ, and the remainder independent of κ. This leads to the relations in terms of the variables

$H_{ij},\ S_{ij}$ and ξ_i of the form:

- $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj} = \sum_{l=1}^{N} H_{il} S_{lj} \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \sigma(\tau)\ , \tag{5.9a}$$

- $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha) \Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma}) = \sum_{l=1}^{N} H_{il} S_{lj}\, \Phi_{-\tau}(\xi_l - \xi_j - \gamma), \tag{5.9b}$$

for all $i, j = 1, \ldots, N$. This forms the set of constitutive relations from the Lax equations which no longer depend on the spectral parameter κ. Next we will explicitly disentangle this coupled system that arise from the Lax system in the cases $N = 1$ and $N = 2$. Higher rank for N $(N \geq 3)$ is expected to lead to higher rank version of the discrete equation.

***i)* Case:** $N = 1$

This is the simplest case which can be explicitly solved. Let us now analyze the basic relations of the general scheme in the case $N = 1$ only in order to arrive more explicit equations, showing that the elaboration of the compatibility conditions for the Lax pair immediately produces the ordinary discrete equation. In this case all quantities $H_{i,j}$, $S_{i,j}$ in (5.9) are scalars, leading to the system of equations:

$$\begin{aligned}
&\widetilde{S}_{11} H_{11} = H_{11} S_{11} \Phi_\tau(\widetilde{\xi} - \xi - \alpha)\, \sigma(\tau)\ , \\
&\widetilde{S}_{11} H_{11}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi} - \xi - \alpha) \Phi_{-\tau}(\widetilde{\xi} - \widetilde{\xi} - \widetilde{\gamma}) = H_{11} S_{11}\, \Phi_{-\tau}(\xi - \xi - \gamma).
\end{aligned}$$

Eliminating $\widetilde{S}_{11},\ S_{11}$ and H_{11}, simply by dividing pairwise the relations over each other and using the definition of the Lamé function $\Phi_{\pm\tau}(\xi)$ in (1.6), as well as $\widetilde{\gamma} = \gamma - \tau$, we obtain:

$$\frac{\sigma(\gamma + \tau)\sigma(\gamma - \tau)}{\sigma^2(\gamma)} = \frac{\sigma(\widetilde{\xi} - \xi - \alpha + \tau)\sigma(\widetilde{\xi} - \xi - \alpha - \tau)}{\sigma^2(\widetilde{\xi} - \xi - \alpha)}.$$

Rearranging by using the addition formula (2.20)

$$\frac{\sigma(x + y)\, \sigma(x - y)}{\sigma^2(x)\, \sigma^2(y)} = \wp(y) - \wp(x)\ ,$$

we find that

$$\wp(\widetilde{\xi}-\xi-\alpha) \;=\; \wp(\gamma)\;,$$

which gives a first order difference equation for $\xi_1 =: \xi(n)$, namely

$$\widetilde{\xi}-\xi-\alpha=\pm\gamma(\mathrm{mod\ \ period\ \ lattice})\;.$$

Integrating the latter, using $\gamma=\gamma_0-n\tau$ we get

$$\xi(n)=\xi(0)+(\alpha\pm\gamma_0)n\pm\tfrac{1}{2}n(n-1)\tau\;. \tag{5.11}$$

This indicates that in the simplest case the scheme gives rise to functions obeying the rational version of the equations that are elliptic functions with arguments depending *quadratically* on the discrete independent variable n. The dependence on the square of the discrete variable n seems typical, for Painlevé types equations, and in particular such dependence appears in the parameters of the Painlevé VI [70, 81].

ii) Higher N values

As in the autonomous case we want to eliminate the variables H_{ij}, S_{ij} from the general system given in (5.9) and obtain a closed form system of equations for the dependent variables $\xi_i =: \xi_i(n)$. To write the system (5.9) more concisely we introduce matrices

$$\begin{aligned} A^{\pm}_{ij} &= \sigma(\pm\tau)\Phi_{\pm\tau}(\widetilde{\xi}_i-\xi_j-\alpha),\\ \Gamma^{\pm}_{ij} &= \sigma(\pm\tau)\Phi_{\pm\tau}(\xi_i-\xi_j-\gamma), \end{aligned} \tag{5.12}$$

and the operation of *"glueing"* matrices: for any two matrices $\boldsymbol{A}=(A_{i,j})$, $\boldsymbol{B}=(B_{i,j})$ we introduce the *glued matrix* $[\boldsymbol{AB}]$, given by:

$$([\boldsymbol{AB}])_{i,j} := A_{i,j}B_{i,j}\;,$$

In terms of this notation the above system takes the simple matrix form:

- $\widetilde{\boldsymbol{S}}\cdot\boldsymbol{H}=[\boldsymbol{A}^{+}\boldsymbol{H}]\cdot\boldsymbol{S}$,
- $[\widetilde{\boldsymbol{\Gamma}}^{-}\ \widetilde{\boldsymbol{S}}]\cdot[\boldsymbol{A}^{-}\boldsymbol{H}]=\boldsymbol{H}\cdot[\boldsymbol{\Gamma}^{-}\boldsymbol{S}]\;.$

As in the autonomous case we want the matrix $\mathbf{H}$ to be of rank 1. There are in fact two possibilities that either S is of rank 1 then $\det(\mathbf{A}^-) = 0$ or $[\mathbf{A}^+\mathbf{H}]$ must be of rank 1. Since these cases give rise to a kind of equivalent result, we shall choose one of them, which is the latter, in order to present the analysis here. It follows then from the first equation that $[\mathbf{A}^+\mathbf{H}]$ is of rank 1, since $[\mathbf{A}^-\mathbf{H}]$ is generically not of rank 1, and the second equation then implies that $[\mathbf{\Gamma}^-\mathbf{S}]$ is of rank 1 (and not $\mathbf{S}$ itself!), implying:

$$\det(\mathbf{A}^+) = \det(\Phi_\tau(\widetilde{\xi}_i - \xi_j - \alpha)_{i,j=1,\dots,N} = 0 \quad \Rightarrow \quad \tau + \widetilde{\Xi} - \Xi - N\alpha = 0\ .$$

for $\Xi := \sum_{j=1}^N \xi_j$. (This follows from Frobenius' elliptic Cauchy determinant).

We come to the conclusion from this formula that it make sense to revise the original Lax scheme in order to redefine the coefficient S_{ij} such that $[\Gamma\ S]_{ij} \sim s_i^+\ s_j^-$ is manifestly of rank 1. Thus we need to bring the matrix $\Gamma_{ij} = \sigma(-\tau)\Phi_{-\tau}(\xi_i - \xi_j - \gamma)$ into the original Lax pair (5.2b) by incorporation in the coefficient matrix $\Phi_{-\tau}(\xi_i - \xi_j - \gamma)$. A simple computations and some appropriate scaling yields the revised Lax scheme that will be the starting point in the next section.

5.1.2 Revised scheme

From the implied condition that the matrix $[\mathbf{\Gamma}\boldsymbol{S}]$ must be of rank 1, it is convenient to revise the scheme and absorb the matrix $\mathbf{\Gamma} = (\Gamma_{i,j})$ in the coefficient, leading to an alternative Lax pair of the form:

$$\widetilde{\chi}_\kappa = \mathscr{L}_\kappa\, \chi_\kappa\ , \quad \chi_{\kappa+\tau} = \mathscr{T}_{\kappa+\boldsymbol{\tau}}\, \chi_\kappa\ . \tag{5.13}$$

In (5.13) the revised Lax matrices contain now both rank 1 matrix coefficients, namely they are of the form: $H_{i,j} = h_i^+ h_j^-\ ,\ S_{i,j} = s_i^+ s_j^-$

$$(\mathscr{L}_\kappa)_{i,j} = h_i^+\, \sigma(\kappa)\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha)\, h_j^-\ , \tag{5.14a}$$

$$(\mathscr{T}_\kappa)_{i,j} = s_i^+\, \sigma(\kappa)\, \Phi_\kappa(\xi_i - \xi_j - \gamma)\, s_j^-\ . \qquad (i,j = 1,\dots,N) \tag{5.14b}$$

For this system the calculation proceeds in a similar way as before, and using the addition formulae (5.5) and (5.6) the compatibility yields the following system of equations:

$$\bullet\; h_i^+\Big(\sum_{l=1}^N h_l^- s_l^+\Big)\, s_j^- = \sigma(-\tau)\,\widetilde{s}_i^+ \sum_{l=1}^N \widetilde{s}_l^- h_l^+ \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha)\, h_j^- \;, \tag{5.15a}$$

$$\bullet\; h_i^+ \sum_{l=1}^N h_l^- s_l^+\, \sigma(\tau)\, \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_\tau(\xi_l - \xi_j - \gamma) s_j^- = \widetilde{s}_i^+ \sum_{l=1}^N \widetilde{s}_l^- h_l^+\, \Phi_\tau(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma}) h_j^- \,, \tag{5.15b}$$

(for all $i, j = 1, \ldots, N$), which as before can be cast in the matrix form:

$$\bullet \quad \widetilde{S}\cdot[A^- H] = H\cdot S \;, \tag{5.16a}$$

$$\bullet \quad [A^+ H]\cdot[\Gamma^+ S] = [\widetilde{\Gamma}^+ \widetilde{S}]\cdot H \;, \tag{5.16b}$$

with now the rank 1 matrices $H = h^+(h^-)^T$, $S = s^+(s^-)^T$ and the matrix Γ^+ (instead of Γ^-)

$$\begin{aligned} A_{ij}^{\pm} &= \sigma(\pm\tau)\Phi_{\pm\tau}(\widetilde{\xi}_i - \xi_j - \alpha), \\ \Gamma_{ij}^{+} &= \sigma(\tau)\Phi_\tau(\xi_i - \xi_j - \gamma). \end{aligned} \tag{5.17}$$

Moreover, it can be seen easily from the second relation above that since $[\Gamma\, S]$ is generally not of rank 1 but H is rank 1 matrix then $[A^+\, H]$ must be rank 1. Again, we need to impose that the determinant of the matrix A^+ must equal to zero, i.e. $\mathbf{det}(\boldsymbol{A^+}) = \mathbf{0}$, implying:

$$\widetilde{\Xi} - \Xi = N\alpha - \tau \quad \Rightarrow \quad \Xi(n) = \Xi(0) + (N\alpha - \tau)n \;. \tag{5.18}$$

Next we will consider the lower order values of N, say at $N = 1$ of the revised scheme (5.15).

***i)* Case** $N = 1$

In this case, the compatibility conditions for the revised Lax pair (5.13) gives the first order relation which is almost similar to equation (5.11) obtained in the previous section.

Let us first consider the quantities (5.15) or (5.16), then we are left with the following two relations:

$$h^+h^-s^+s^- = \widetilde{s}^+\widetilde{s}^-h^+h^-\Phi_{-\tau}(\widetilde{\xi} - \xi - \alpha)\,\sigma(-\tau)\,, \tag{5.19a}$$

$$h^+h^-s^+s^-\,\sigma(-\tau)\,\Phi_{-\tau}(\widetilde{\xi} - \xi - \alpha)\Phi_{-\tau}(\widetilde{\xi} - \widetilde{\xi} - \widetilde{\gamma}) = \widetilde{s}^+\widetilde{s}^-h^+h^-\,\Phi_{-\tau}(\xi - \xi - \gamma). \tag{5.19b}$$

Eliminating $h^\pm$ and $s^\pm$, simply by dividing pairwise the relations over each other and using the definition of the function (1.6), as before, this constitutes

$$\frac{\sigma(\widetilde{\gamma}+\tau)\sigma(\tau-\widetilde{\gamma})}{\sigma^2(\gamma)} - \frac{\sigma(\widetilde{\xi}-\xi-\alpha+\tau)\sigma(\widetilde{\xi}-\xi-\alpha-\tau)}{\sigma^2(\widetilde{\xi}-\xi-\alpha)} = 0,$$

which can be rearranged by using the addition formulae (2.20), so we have

$$\wp(\widetilde{\xi} - \xi - \alpha) \;=\; \wp(\widetilde{\gamma}).$$

This gives directly the first order difference equation for $\xi = \xi_{n,m}$ up to modulo the period lattice, that is

$$\widetilde{\xi} - \xi - \alpha \pm \widetilde{\gamma} = 0 \;\Longrightarrow\; \xi = \xi_0 + n\,\alpha \pm n\,\gamma_0 \mp \frac{1}{2}n(n+1)\tau\,, \tag{5.20}$$

where ξ_0 and γ_0 are the integration constants.

***ii)* Case** $N = 2$

To resolve this case, the first identity (5.15a) allows us to identify $h^+ = \rho\widetilde{s}^+$ (for some scalar function ρ), and consequently:

$$s_j^- = \frac{-\sigma(\tau)}{(h^-\cdot s^+)}\sum_{l=1}^{2}\widetilde{s}_l^+\,\widetilde{s}_l^-\;\Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha)\,h_j^-\,.$$

Expressing all the entries of the first and second relation in terms of $s_l^+s_l^- =: S_l$, $s_l^+h_l^- =: H_l$ we get:

$$\Big(1+\frac{H_2}{H_1}\Big)S_1 = A_{11}^-\widetilde{S}_1 + A_{21}^-\widetilde{S}_2\,, \tag{5.21a}$$

$$\Big(\frac{H_1}{H_2}+1\Big)S_2 = A_{12}^-\widetilde{S}_1 + A_{22}^-\widetilde{S}_2\,, \tag{5.21b}$$

Using these new variables the entries of the other matrix relation (5.15b) yields the system:

$$\begin{aligned}\left(A^+_{11}\Gamma^+_{11} + A^+_{12}\Gamma^+_{21}\frac{H_2}{H_1}\right) S_1 &= \left(A^+_{11}\Gamma^+_{12}\frac{H_1}{H_2} + A^+_{12}\Gamma^+_{22}\right) S_2 = \widetilde{\Gamma}^+_{11}\widetilde{S}_1 + \widetilde{\Gamma}^+_{12}\widetilde{S}_2 \;,\\ \left(A^+_{21}\Gamma^+_{11} + A^+_{22}\Gamma^+_{21}\frac{H_2}{H_1}\right) S_1 &= \left(A^+_{21}\Gamma^+_{12}\frac{H_1}{H_2} + A^+_{22}\Gamma^+_{22}\right) S_2 = \widetilde{\Gamma}^+_{21}\widetilde{S}_1 + \widetilde{\Gamma}^+_{22}\widetilde{S}_2 \;.\end{aligned} \tag{5.22}$$

To analyse these further, taking into account that $\mathbf{det}(\boldsymbol{A}^+) = \mathbf{0}$, we rewrite the relations in terms of the ratios $X = H_2/H_1$, $Y = S_2/S_1$ and $Z = \widetilde{S}_1/S_1$, leading to:

$$\begin{aligned}&\frac{Y}{X} = \frac{A^-_{12} + A^-_{22}\widetilde{Y}}{A^-_{11} + A^-_{21}\widetilde{Y}}\;,\\ &Z = \frac{1+X}{A^-_{11} + A^-_{21}\widetilde{Y}}\;,\\ &A^+_{11}\Gamma^+_{11} + A^+_{12}\Gamma^+_{21}X = (A^+_{11}\Gamma^+_{12} + A^+_{12}\Gamma^+_{22}X)Y/X = (\widetilde{\Gamma}^+_{11} + \widetilde{\Gamma}^+_{12}\widetilde{Y})Z\;,\\ &A^+_{11}/A^+_{21} = A^+_{12}/A^+_{22} = (\Gamma^+_{11} + \Gamma^+_{12}\widetilde{Y})/(\Gamma^+_{21} + \Gamma^+_{22}\widetilde{Y})\;.\end{aligned} \tag{5.23}$$

These are in fact four relations for X, Y and $\widetilde{Y}$ with coefficients in terms of ξ. This manageable system (5.23) can be solved by direct computation. Eliminating X, Y and $\widetilde{Y}$ we derive the first order difference equation for $\xi_j(n)$ ($j = 1,2$) given by

$$\begin{aligned}&\Big[\sigma(\alpha+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_1)\,\sigma(-\gamma+\tau+\widetilde{\xi}_1-\widetilde{\xi}_2)\\ &\sigma(2\tau-\alpha-\gamma-\xi_1-\widetilde{\xi}_2)\,\Big(\sigma(-\gamma+\xi_1-\xi_2)\sigma(\alpha-\gamma+\xi_2-\widetilde{\xi}_1)\\ &\sigma(-\alpha-\gamma+\tau-\xi_1+\widetilde{\xi}_1)\,\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_2)-\sigma(-\gamma-\xi_1+\xi_2)\\ &\sigma(\alpha-\gamma+\xi_1-\widetilde{\xi}_1)\,\sigma(-\alpha-\gamma+\tau-\xi_2+\widetilde{\xi}_1)\,\sigma(\alpha+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_2)\Big)+\\ &+\Big(\sigma(-\gamma+\xi_1-\xi_2)\,\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_1)\,\sigma(\alpha-\gamma+\xi_2-\widetilde{\xi}_1)\\ &\sigma(-\alpha-\gamma+\tau-\xi_1+\widetilde{\xi}_1)+\sigma(-\gamma-\xi_1+\xi_2)\,\sigma(\alpha+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha-\gamma+\xi_1-\widetilde{\xi}_1)\\ &\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_1)\,\sigma(-\alpha-\gamma+\tau-\xi_2+\widetilde{\xi}_1)\Big)\sigma(-\alpha-\gamma+2\tau-\xi_1+\widetilde{\xi}_1)\\ &\sigma(\alpha+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_2)\,\sigma(-\gamma+\tau-\widetilde{\xi}_1+\widetilde{\xi}_2)\Big]+\end{aligned}$$

$$+\frac{\sigma(-2\gamma+\tau)\,\sigma(-\gamma+\tau)\,\sigma(\xi_1-\xi_2)\,\sigma(\widetilde{\xi}_1-\xi_2-\alpha+\tau)}{\sigma(\gamma)\,\sigma(3\tau-2\gamma)\,\sigma(\widetilde{\xi}_1-\widetilde{\xi}_2)\,\sigma(\widetilde{\xi}_2-\xi_1-\alpha)}\Big[\sigma(\alpha+\xi_1-\widetilde{\xi}_1)$$
$$\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\tau-\gamma+\widetilde{\xi}_1-\widetilde{\xi}_2)$$
$$\sigma(\widetilde{\xi}_2-\xi_1+2\tau-\alpha-\gamma)-\sigma(\alpha+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\widetilde{\xi}_1-\xi_1+2\tau-\gamma-\alpha)$$
$$\sigma(\alpha+\xi_1-\widetilde{\xi}_2)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\tau-\gamma-\widetilde{\xi}_1+\widetilde{\xi}_2)\Big]\times$$
$$\times\Big[\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_1)\,\sigma(\alpha+\xi_2-\widetilde{\xi}_1)\,\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_2)\,\sigma(\tau-\gamma+\widetilde{\xi}_1-\widetilde{\xi}_2)$$
$$\sigma(\widetilde{\xi}_2-\xi_1+2\tau-\alpha-\gamma)-\sigma(\alpha+\tau+\xi_2-\widetilde{\xi}_1)\,\sigma(\widetilde{\xi}_1-\xi_1+2\tau-\gamma-\alpha)$$
$$\sigma(\alpha+\xi_2-\widetilde{\xi}_2)\,\sigma(\alpha-\gamma+\tau+\xi_1-\widetilde{\xi}_2)\,\sigma(\tau-\gamma-\widetilde{\xi}_1+\widetilde{\xi}_2)\Big]=0\,, \tag{5.24}$$

which is subject to the condition $\xi_1+\xi_2=(2\alpha-\tau)n+\Xi(0)$. This is a first order nonlinear nonautonomous elliptic ordinary difference equation containing three parameters $\Xi(0)$, τ and γ_0. Although (5.24) may be interesting in its own right, we are really seeking a scheme that provides a second order O∆E. In a sense, the scheme of this section forms a parallel to the one for the monodromy problem for P_{VI}, (1.76), albeit with the last term on the right-hand side absent. The compatibility with (1.75) in that case would also produce a first order equation, at most, which is linearisable. Since the scheme (5.1) involves only one lattice direction it constitutes really an analogue to the case of the truncated monodromy problem (1.76) involving only a single lattice shift. Thus, by this analogy, in order to arrive at a higher-order system we expect that we need to involve more than one lattice shift in the elliptic monodromy problem. The alternative choice would be either to consider the full rank matrix case for H or to consider higher rank cases ($N>2$). These alternatives turn out to be very complicated and we will not consider them here but instead in the next section propose a higher-order scheme involving multiple lattice shifts.

5.2 Higher order scheme

In order to derive higher-order OΔEs we extend the isomonodromic problem to a higher order one as follows:

$$\begin{aligned} \boldsymbol{\chi}_{\kappa+\tau} &= \mathbf{T'}_\kappa\, \boldsymbol{\chi}_\kappa \ , \qquad (5.25) \\ (\mathbf{T'}_\kappa)_{i,j} : &= \sigma^2(\kappa) \sum_{l'=1}^{N} S_{i,j}^{(l')} \Phi_\kappa(\xi_i - \eta_{l'}) \Phi_\kappa(\eta_{l'} - \xi_j - \gamma) \ , \quad (i,j = 1, \dots, N) \ , \end{aligned}$$

where the η_l variables as well as the extended coefficients $S_{i,j}^{(l')}$ remain to be determined. We consider this difference equation on the torus in conjunction with the lattice Lax system

$$\widetilde{\boldsymbol{\chi}}_\kappa = \mathbf{L}_\kappa\, \boldsymbol{\chi}_\kappa \ , \quad (\mathbf{L}_\kappa)_{i,j} = H_{i,j}\, \sigma(\kappa)\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha) \ , \qquad (5.26a)$$

$$\widehat{\boldsymbol{\chi}}_\kappa = \mathbf{M}_\kappa\, \boldsymbol{\chi}_\kappa \ , \quad (\mathbf{M}_\kappa)_{i,j} = K_{i,j}\, \sigma(\kappa)\, \Phi_\kappa(\widehat{\xi}_i - \xi_j - \beta) \ , \qquad (5.26b)$$

where as before we like to take the coefficient matrices $\mathbf{H}$ and $\mathbf{K}$ of rank 1 and independent of the spectral variable κ.

We can think of the scheme above as an elliptic de-autonomization of a higher-order periodic reduction on the lattice. 2-step periodic reduction: $\boldsymbol{\chi} \to \overline{\boldsymbol{\chi}} \to \widehat{\widehat{\boldsymbol{\chi}}} = \lambda\boldsymbol{\chi}$ followed by de-autonomization: $\lambda\boldsymbol{\chi} \rightsquigarrow \boldsymbol{\chi}_{\kappa+\tau}$ However, now we want to keep the midpoint unspecified associated with some value η for ξ.

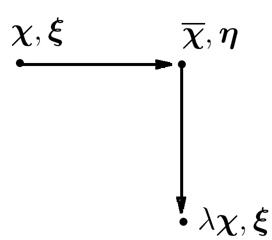


Figure 5.1: 2-step periodic reduction.

This system leads to the system of compatibility conditions:

$$\mathbf{L}_{\kappa+\tau}\,\mathbf{T}'_\kappa = \widetilde{\mathbf{T}}'_\kappa\,\mathbf{L}_\kappa \ , \tag{5.27a}$$

$$\mathbf{M}_{\kappa+\tau}\,\mathbf{T}'_\kappa = \widehat{\mathbf{T}}'_\kappa\,\mathbf{M}_\kappa \ , \tag{5.27b}$$

$$\widehat{\mathbf{L}}_\kappa\,\mathbf{M}_\kappa = \widetilde{\mathbf{M}}_\kappa\,\mathbf{L}_\kappa \ . \tag{5.27c}$$

To do this most effectively we need a new elliptic identity, generalizing (1.7),

$$\begin{aligned}\frac{\Phi_\kappa(x)\,\Phi_\kappa(y)\,\Phi_\kappa(z)}{\Phi_\kappa(x+y+z)} &= \tfrac{1}{2}\Big[\big(\zeta(\kappa)+\zeta(x)+\zeta(y)+\zeta(z)-\zeta(\kappa+x+y+z)\big)^2 \\ &\quad +\wp(\kappa)-\big(\wp(x)+\wp(y)+\wp(z)+\wp(\kappa+x+y+z)\big)\Big] \ ,\end{aligned} \tag{5.28}$$

$\forall\,\kappa,x,y,z$. The consistency condition $\mathbf{L}_{\kappa+\tau}\,\mathbf{T}'_\kappa = \widetilde{\mathbf{T}}'_\kappa\,\mathbf{L}_\kappa$ leads

$$\begin{aligned}&\sigma(\tau)\Phi_\kappa(\tau)\sum_{l,l'=1}^N H_{il}\,S_{lj}^{(l')}\,\Phi_{\kappa+\tau}(\widetilde{\xi}_i-\xi_l-\alpha)\,\Phi_\kappa(\xi_l-\eta_{l'})\,\Phi_\kappa(\eta_{l'}-\xi_j-\gamma) \\ &\qquad = \sum_{l,l'=1}^N \widetilde{S}_{il}^{(l')}\,H_{lj}\Phi_\kappa(\widetilde{\xi}_i-\widetilde{\eta}_{l'})\,\Phi_\kappa(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\widetilde{\gamma})\,\Phi_\kappa(\widetilde{\xi}_l-\xi_j-\alpha)\ ,\end{aligned} \tag{5.29}$$

or equivalently

$$\begin{aligned}&\sigma(\tau)\sum_{l,l'=1}^N H_{il}\,S_{lj}^{(l')}\,\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\,\Phi_\kappa(\tau+\widetilde{\xi}_i-\xi_l-\alpha)\,\Phi_\kappa(\xi_l-\eta_{l'})\,\Phi_\kappa(\eta_{l'}-\xi_j-\gamma) \\ &\qquad = \sum_{l,l'=1}^N \widetilde{S}_{il}^{(l')}\,H_{lj}\Phi_\kappa(\widetilde{\xi}_i-\widetilde{\eta}_{l'})\,\Phi_\kappa(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\widetilde{\gamma})\,\Phi_\kappa(\widetilde{\xi}_l-\xi_j-\alpha)\ .\end{aligned} \tag{5.30}$$

The latter was derived by using the expression $\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(x) = \Phi_\tau(x)\,\Phi_\kappa(\tau+x)$ on the first two Φ terms of (5.29). Furthermore, applying the above identity (5.28) we end up with the form

$$\sum_{l,l'=1}^{N} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau)\, H_{il}\, S_{lj}^{(l')}$$
$$\times\Big[\Big(\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) + \zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma)\Big)^2$$
$$+\wp(\kappa) - \Big(\wp(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) + \wp(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \wp(\xi_l - \eta_{l'}) + \wp(\eta_{l'} - \xi_j - \gamma)\Big)\Big]$$
$$= \sum_{l,l'=1}^{N} \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma})\, \widetilde{S}_{il}^{(l')}\, H_{lj}$$
$$\times\Big[\Big(\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \zeta(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) + \zeta(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma}) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha)\Big)^2$$
$$+\wp(\kappa) - \Big(\wp(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \wp(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) + \wp(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma}) + \wp(\widetilde{\xi}_l - \xi_j - \alpha)\Big)\Big]\ .$$

There is a common factor $\Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau)$ on the left-hand side, and a common factor $\Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma})$ on the right-hand side, which can once again be identified if we set $\widetilde{\gamma} = \gamma - \tau$, so that they cancel. The remaining terms separate in accordance with their different dependence on κ. Thus, we have terms containing only the external indices i and j, which yield

$$\sum_{l,l'=1}^{N} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, H_{il}\, S_{lj}^{(l')} = \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\ . \tag{5.31}$$

The linear terms in $\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_i - \alpha - \gamma + \tau)$ lead to

$$\sum_{l,l'=1}^{N} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, H_{il}\, S_{lj}^{(l')}\Big(\zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma)\Big)$$
$$= \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\, \Big(\zeta(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) + \zeta(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma + \tau) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha)\Big)\ . \tag{5.32}$$

Finally, the terms, which do not depend on κ, give rise to

$$\sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\,\sigma(\tau)\Big[\Big(\zeta(\tau+\widetilde{\xi}_i-\xi_l-\alpha)+\zeta(\xi_l-\eta_{l'})+\zeta(\eta_{l'}-\xi_j-\gamma)\Big)^2 -\wp(\tau+\widetilde{\xi}_i-\xi_l-\alpha)-\wp(\xi_l-\eta_{l'})-\wp(\eta_{l'}-\xi_j-\gamma)\Big]$$
$$= \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\,\Big[\Big(\zeta(\widetilde{\xi}_i-\widetilde{\eta}_{l'})+\zeta(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\gamma+\tau)+\zeta(\widetilde{\xi}_l-\xi_j-\alpha)\Big)^2 -\wp(\widetilde{\xi}_i-\widetilde{\eta}_{l'})-\wp(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\gamma+\tau)-\wp(\widetilde{\xi}_l-\xi_j-\alpha)\Big]\,. \tag{5.33}$$

Therefore, we have obtained the following constitutive relations:

- $$\sum_{l,l'=1}^{N} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i-\xi_l-\alpha)\, H_{il}\, S_{lj}^{(l')} = \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\,, \tag{5.34a}$$
- $$\sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \frac{\sigma(\widetilde{\xi}_i-\eta_{l'}+\tau-\alpha)\sigma(\xi_l-\xi_j-\gamma)\sigma(\widetilde{\xi}_i-\xi_l-\xi_j+\eta_{l'}-\alpha-\widetilde{\gamma})}{\sigma(\widetilde{\xi}_i-\xi_l-\alpha+\tau)\sigma(\xi_l-\eta_{l'})\sigma(\eta_{l'}-\xi_j-\gamma)} = \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\, \frac{\sigma(\widetilde{\xi}_i-\widetilde{\xi}_l-\widetilde{\gamma})\sigma(\widetilde{\eta}_{l'}-\xi_j-\widetilde{\gamma}-\alpha)\sigma(\widetilde{\xi}_i+\widetilde{\xi}_l-\xi_j-\widetilde{\eta}_{l'}-\alpha)}{\sigma(\widetilde{\xi}_i-\widetilde{\eta}_{l'})\sigma(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\widetilde{\gamma})\sigma(\widetilde{\xi}_l-\xi_j-\alpha)}\,, \tag{5.34b}$$
- $$\sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \Phi_{-\tau}(\xi_l-\eta_{l'})\, \Phi_{-\tau}(\eta_{l'}-\xi_j-\gamma) = \sigma(-\tau)\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\, \Phi_{-\tau}(\widetilde{\xi}_i-\widetilde{\eta}_{l'})\, \Phi_{-\tau}(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\widetilde{\gamma})\, \Phi_{-\tau}(\widetilde{\xi}_l-\xi_j-\alpha)\,. \tag{5.34c}$$

where $\widetilde{\gamma}=\gamma-\tau$. The second relation is obviously derived by using (1.8) on both side of (5.32), whereas (5.34c) can be obtained by applying the identity (5.28) on (5.33). The general scheme is of relations derived is rather complicated. The first and last relation can be written in the form:

- $$\widetilde{\mathbf{S}}\cdot\mathbf{H} = [\mathbf{A}^+\mathbf{H}]\cdot\mathbf{S}\,, \tag{5.35a}$$
- $$[\widetilde{\boldsymbol{\Delta}}^-\widetilde{\mathbf{S}}\,\widetilde{\boldsymbol{\Gamma}}^-]\cdot[\mathbf{A}^-\mathbf{H}] = \mathbf{H}\cdot[\boldsymbol{\Delta}^-\mathbf{S}\,\boldsymbol{\Gamma}^-]\,, \tag{5.35b}$$

where we have used the same notation as before, with the matrix $\mathbf{S}$ as the matrix with entries $(\mathbf{S})_{i,j} = \sum_{l'} S_{ij}^{(l')}$, and where the "doubly glued" matrix $[\mathbf{\Delta}^- \mathbf{S}\, \mathbf{\Gamma}^-]$ is the matrix with entries:

$$\sum_{l'} S_{ij}^{(l')}\, \Phi_{-\tau}(\xi_i - \eta_{l'})\, \Phi_{-\tau}(\eta_{l'} - \xi_j - \gamma) =: [\mathbf{\Delta}^- \mathbf{S}\, \mathbf{\Gamma}^-]_{ij}\ .$$

The equation (5.34b) is the most complicated to write in a matrix form. In order to achieve this we actually first go back to (5.32) and using also (5.31) add extra terms in the summand to obtain the equality

$$\begin{aligned}\sum_{l,l'=1}^{N} & H_{il}\, S_{lj}^{(l')} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\Big(\zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \xi_j - \widetilde{\gamma}) + \zeta(-\tau) \\ & -\zeta(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma) + \zeta(\tau) - \zeta(\xi_l - \xi_j - \widetilde{\gamma})\Big) \\ = & \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\Big(\zeta(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) + \zeta(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma + \tau) + \zeta(-\tau) - \zeta(\widetilde{\xi}_i - \widetilde{\xi}_l - \gamma) \\ & + \zeta(\widetilde{\xi}_i - \widetilde{\xi}_l - \gamma) + \zeta(\tau) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) - \zeta(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma})\Big)\ . \end{aligned} \tag{5.36}$$

Applying the identity (1.7) on each quadruple of ζ terms in the summands, we obtain

$$\begin{aligned}\sum_{l,l'=1}^{N} & H_{il}\, S_{lj}^{(l')}\Big[-\sigma^2(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\frac{\Phi_\tau(\xi_l - \eta_{l'})\Phi_\tau(\eta_{l'} - \xi_j - \gamma)}{\Phi_\tau(\xi_l - \xi_j - \gamma)} \\ & + \frac{\Phi_\tau(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma})}{\Phi_\tau(\xi_l - \xi_j - \gamma)}\Big] \\ = & \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\Big[\frac{\sigma(-\tau)\Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\eta}_{l'})\Phi_{-\tau}(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma})}{\Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})} \\ & - \frac{\sigma(-\tau)\Phi_{-\tau}(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma})\sigma(\tau)\Phi_\tau(\widetilde{\xi}_l - \xi_j - \alpha)}{\Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\sigma(-\tau)}\Big]\ . \end{aligned} \tag{5.37}$$

Thus, "the middle relation" (5.34b) can be written more concisely as follows

$$\begin{aligned}\big[[\widetilde{\mathbf{\Delta}}^- \widetilde{\mathbf{S}}\, \widetilde{\mathbf{\Gamma}}^-]/\widetilde{\mathbf{\Gamma}}^-\big] \cdot \mathbf{H} - \big[\mathbf{C}^-\big([\widetilde{\mathbf{S}}/\widetilde{\mathbf{\Gamma}}^-] \cdot [\mathbf{A}^+\mathbf{H}]\big)\big] \\ = \big[\mathbf{C}^+\big(\mathbf{H} \cdot [\mathbf{S}/\Gamma^+]\big)\big] - [\mathbf{A}^+\mathbf{H}] \cdot \big[[\mathbf{\Delta}^+ \mathbf{S}\, \mathbf{\Gamma}^+]/\mathbf{\Gamma}^+\big], \end{aligned} \tag{5.38}$$

where

$$\mathbf{C}_{ij}^{\pm} = \sigma(\pm\tau)\Phi_{\pm\tau}(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}), \tag{5.39}$$

and where we have introduced the operation

$$[\mathbf{X}/\mathbf{\Gamma}^{\pm}] = \frac{\mathbf{X}_{ij}}{\Phi_{\pm\tau}(\xi_i - \xi_j - \gamma)\sigma(\pm\tau)}.$$

Although the notation is somewhat ad-hoc, it may prove useful in determining the ranks of the matrices.

We will discuss the strategy to analyze the case that $N = 2$. As before, we want to take $\mathbf{H}$ of rank 1, in which case it follows from (5.35a) that either $\mathbf{S}$ is of rank 1, or $\det(\mathbf{A}^{+})$ vanishes. Focusing on the latter option, then from the third relation, (5.35b), we conclude that the matrix $[\widetilde{\mathbf{\Delta}}^{-}\widetilde{\mathbf{S}}\,\widetilde{\mathbf{\Gamma}}^{-}]$ must be of rank 1. Next these rank conditions can be implemented on the matrix form (5.38) with the aim to eliminate $\mathbf{H}$, $\mathbf{S}$. Furthermore, we have to solve for the yet undetermined quantities $\eta_{l'}$, where from the diagram 5.1 it is suggestive to expect a solution for $\eta_{l'}$ of the form either $\eta_{l'} = \widetilde{\xi}_{l'} + \beta$ or $\eta_{l'} = \widehat{\xi}_{l'} + \alpha$ (both choices being compatible because of (5.27c)). At the same time we must set $\gamma = \alpha + \beta$, and to account for the nonautonomicity we need to assume that $\beta = \beta(m) = \beta(0) - m\tau$, $\alpha = \alpha(n) = \alpha(0) - n\tau$. These are the natural assumptions, under which we expect the scheme given by the three matrix relations, (5.35a), (5.35b) and (5.38), to be resolvable and to lead to a second order nonautonomous OΔE for ξ_1 subject to the condition (5.18).

Remark 5.2.1 As a byproduct, the autonomous limit of the higher-order reduced Lax system of this section, we can consider the 2-step higher-time flow of the RS model of [69]. This would have a Lax pair of the form which is obtained by setting instead of (5.25) the spectral problem:

$$\lambda\boldsymbol{\chi}_{\kappa} = \mathbf{T}'_{\kappa}\boldsymbol{\chi}_{\kappa}\ , \tag{5.40}$$

supplemented by (5.26a) and where $\tau = 0$ and γ is constant. In that stationary case, (5.27a) becomes

$$\mathbf{L}_{\kappa}\,\mathbf{T}'_{\kappa} = \widetilde{\mathbf{T}}'_{\kappa}\,\mathbf{L}_{\kappa}\ . \tag{5.41}$$

The compatibility of (5.41) follows similar analysis as the one for nonautonomous case, making use of (5.28) and the result is the following set of constitutive relations:

- $$\sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')} = \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\ , \tag{5.42a}$$

- $$\begin{aligned}&\sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \frac{\sigma(\widetilde{\xi}_i-\eta_{l'}-\alpha)\sigma(\xi_l-\xi_j-\gamma)\sigma(\widetilde{\xi}_i-\xi_l-\xi_j+\eta_{l'}-\alpha-\gamma)}{\sigma(\widetilde{\xi}_i-\xi_l-\alpha)\sigma(\xi_l-\eta_{l'})\sigma(\eta_{l'}-\xi_j-\gamma)}\\ &=\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\, \frac{\sigma(\widetilde{\xi}_i-\widetilde{\xi}_l-\gamma)\sigma(\widetilde{\eta}_{l'}-\xi_j-\gamma-\alpha)\sigma(\widetilde{\xi}_i+\widetilde{\xi}_l-\xi_j-\widetilde{\eta}_{l'}-\alpha)}{\sigma(\widetilde{\xi}_i-\widetilde{\eta}_{l'})\sigma(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\gamma)\sigma(\widetilde{\xi}_l-\xi_j-\alpha)}\ ,\end{aligned} \tag{5.42b}$$

- $$\begin{aligned}&\sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \Phi_{\kappa_0}(\widetilde{\xi}_i-\xi_l-\alpha)\, \Phi_{\kappa_0}(\xi_l-\eta_{l'})\, \Phi_{\kappa_0}(\eta_{l'}-\xi_j-\gamma)\\ &=\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\, \Phi_{\kappa_0}(\widetilde{\xi}_i-\widetilde{\eta}_{l'})\, \Phi_{\kappa_0}(\widetilde{\eta}_{l'}-\widetilde{\xi}_l-\gamma)\, \Phi_{\kappa_0}(\widetilde{\xi}_l-\xi_j-\alpha)\ ,\end{aligned} \tag{5.42c}$$

where in the latter we can fix κ_0 to be any non-singular fixed value. We can also obtain the first two (5.42a), (5.42b) from the limit $\tau \to 0$ of (5.34), while (5.42c) needs a separate analysis. Equations (5.42) represent a system of constitutive relations for a higher-rank RS flows and in what we consider the higher order to be a hierarchy of RS flows. This system should be made explicit by solving for the coefficients $\mathbf{H}$, $\mathbf{S}$ as well as the intermediate variable $\eta_{l'}$.

5.3 Discussion

In this chapter we have proposed the general structure of an elliptic isomonodromic system and obtained a constitutive set of relations from the compatibility conditions. In contrast to existing elliptic isomonodromy deformation systems on the torus for discrete Painlevé type equations, the one proposed here can be readily extended to any rank, and as such we would expect it to contain higher order discrete Painlevé equations of the type of the Garnier systems. We naturally expect that there are elliptic Painlevé type equations coming out of the scheme given by the matrix relations (5.35a), (5.35b) and (5.38) for $N = 2$. The full analysis of these equations still needs to be performed and this is left as the subject of future study.

Chapter 6

Conclusions

6.1 Summary of results

This thesis deals with a novel class of elliptic Lax systems on the lattice and corresponding nonlinear lattice systems. In particular, we are concerned with a lattice version of the famous Krichever-Novikov equation and its higher-rank case.

Chapter 1 was mainly a review, but contains also a few novel elements, such as the use of the identity (1.28) in the proof of the elliptic Beta integral, as well as the new identities (1.11) and (1.12) which we have not encountered in the vast literature on elliptic functions. In chapter 2 we pull together some mostly known facts about Adler's lattice equation, but there are also some new insights, such as the compound identity (2.14) connecting the 3-leg and rational form of Adler's lattice equation, as well as the spin representation of the Jacobi form. However, the main results in the thesis are found in chapters 3-5 which deal with the novel Lax systems.

In chapter 3 a general class of higher-rank elliptic Lax representations for systems of PΔEs on the 2D lattice has been proposed and investigated. Distinguishing between what we called spin-zero (generalizations of Adler's lattice equation) and spin-nonzero (generalized Landau-Lifschitz (LL) type) models, we gave the general structure of the resulting equations (from the compatibility conditions) for the latter, but concentrated mainly on the former case for $N = 2$ and $N = 3$. For $N = 2$ it has shown in [104] that the Lax systems leads indeed to Adler's lattice equation in its 3-leg form (for the Weierstrass class) and we have analyzed how these results generalize to the case $N = 3$ (as a representative example for the higher-rank case). The case of rank $N = 3$ is analyzed using Cayley's hyperdeterminant of format $2 \times 2 \times 2$, yielding a multi-component system of coupled 3-leg quad-equations. This chapter also contains a new result which we refer to as Compound theorem for $2 \times 2 \times 2$ hyper-determinants given in Lemma (3.3.4). In our view, the significance of the results of this chapter is not only to add a new class of elliptic type of integrable systems to our already substantial zoo of such systems, but to depart from the rather restrictive confinement of 2×2 systems to which all ABS type systems, [7], belong. To obtain good insights in the essential structures behind (discrete and continuous) integrable systems, such departures into the multi-component cases are necessary.

In chapter 5, the reductions to iso-spectral or isomonodromic problems were discussed. The latter reductions, achieved by means of deautonomization of isospectral problems on the torus, lead to systems of nonautonomous elliptic O∆Es, which are expected to yield elliptic discrete Painlevé equations and possibly higher-order analogues. We set up the general scheme and made some initial analysis, but there is more to be done to obtain a closed form of the equations. Our approach, of systematically deriving Lax pairs (or monodromy problems) from a general perspective allows for a natural extension to higher rank and higher order forms and as such is in contrast to existing elliptic monodromy problems [75, 103], proposed as Lax pairs for the famous elliptic Painlevé equation of Sakai [88]. It remains an open question whether or not the elliptic discrete Painlevé equation can be detected in our scheme. Nevertheless, our approach provides an alternative scheme for obtaining in principle such nonautonomous O∆Es. It would be interesting to compare our Lax systems to the existing ones in the literature, which is not quite trivial because the latter ones tend to employ multiplicative forms of the elliptic functions, such as the ones discussed in section 1.1.

6.2 Future work

The higher-rank lattice system, which we have proposed in chapter 3, as far as we are aware, forms the first integrable lattice system generalizing the famous Q_4 equation. As it stands the rank 3 system is the analogue of 3-leg form of Adler's equation with the dependent variable appearing in the argument of elliptic functions. It is highly desirable to find its rational form analogous to the rational form of the Adler equation. In that form further properties, such as multidimensional consistency, symmetries and the construction of solutions (such a solution solutions) can be studied. For the moment these goals are hampered by the sheer complexity of the system, and would require various machineries, such as use of generalized Frobenius-Stickelberger formulae. A possible outcome would be to establish a connection with a differential system obtained by O. Mokhov in the 1980s, [63], arising from third order commuting differential operators defining rank 3 vector bundles over an elliptic curve, cf. [62]. This is the only system that is comparable with our system at the continuous level.

Another direction is to consider the Landau-Lifshitz class of models, whose (higher-order) periodic reductions are expected to yield higher-order time discretizations of the Ruijsenaars-Schneider model of [69]. In the thesis we concentrated mostly on what we called the spin-zero case, whereas some results concerning periodic reductions of the spin non-zero case was already obtained in [104]. As a direction for the future, establishing connections with the recently found master-solution of the quantum Yang-Baxter equations, [19] and its multi-spin generalization [20], may be of interest.

Recently, isomonodromic deformation problems for Sakai's elliptic discrete Painlevé equation [88] have been considered by several authors [75, 103]. The completion of the scheme proposed in chapter 5 would provide an alternative approach to such elliptic monodromy problems, with a potential to find natural extensions to higher rank and higher order of the $2 \times 2 \times 2$. It would also be interesting to further explore elliptic discrete integrable systems in higher dimensions, such as the elliptic lattice KP equation constructed recently in [50], which is essentially a system in 3+1 dimensions.

Appendix A

Jacobi theta functions and proof of the higher degree identity (1.11)

Here we give a brief summary of some relevant formulae for the theory of theta functions and a proof of the new elliptic identity in (1.11). Many textbooks on this material exist, but we prefer the ones by Akhiezer, [11], Whittaker and Watson [107] and the relevant chapter in [18], whilst Hancock [39] is a good general reference. This Appendix follows closely the Notes [74] which provide a more constructive, rather than algebra-geometric approach to the functions.

A.1 Formulae for Jacobi theta functions

The Jacobi theta functions constitutes a fundamental part of the theory of elliptic functions. The definitions are given with modulus τ as infinite series

$$\theta_{ab}(x|\tau) = \sum_{n\in\mathbb{Z}} \exp\left[\pi i\tau(n+\frac{a}{2})^2 + 2\pi i(n+\frac{a}{2})(x+\frac{b}{2})\right], \tag{A.1}$$

where the parameters a, b are sometimes referred to as the *caracteristics* of the theta functions, and we can have them take the values in $\mathbb{Z}_2$. The series for $\theta_{a,b}$ converges uniformly for all discs $|x| \leq R$ in the complex plane, for arbitrary real $R > 0$, whenever the (fixed modulus) τ has a strictly positive imaginary part. The following quasi-periodicity conditions, satisfied by the Jacobi theta functions, follow from the definitions:

$$\theta_{ab}(x+1|\tau) = e^{\pi i a}\theta_{ab}(x|\tau)\ ,\ \theta_{ab}(x+\tau|\tau) = e^{-\pi i(\tau+2x+b)}\theta_{ab}(x|\tau). \tag{A.2}$$

The $\theta_{11}(x)$ function is odd, $\theta_{11}(-x|\tau) = -\theta_{11}(x|\tau)$, and $\theta_{00}(x),\ \theta_{01}(x),\ \theta_{10}(x)$ are all even functions:

$$\theta_{00}(-x|\tau) = \theta_{00}(x|\tau)\ , \quad \theta_{01}(-x|\tau) = \theta_{01}(x|\tau)\ , \quad \theta_{10}(-x|\tau) = \theta_{10}(x|\tau). \tag{A.3}$$

The $\theta_{11}(x)$ function is related to theta function of rational characteristic: $\vartheta(z;q) = (q;q)_\infty(z;q)_\infty(z^{-1}q;q)_\infty$ by the triple product relation

$$\theta_{11}(x|\tau) = -ie^{-\pi i(x-\frac{\tau}{4})}\vartheta(e^{2\pi i x};\ e^{2\pi i\tau})\ , \tag{A.4}$$

where we take $z = e^{2\pi i x}$ and $p = e^{2\pi i\tau}$. Furthermore, the multiplication of two theta functions can be given in the general formula:

$$\theta_{ab}(x|\tau)\,\theta_{a'b'}(y|\tau) = \theta_{AB}(x+y|2\tau)\theta_{A'B'}(x-y|2\tau) + \theta_{A+1,B}(x+y|2\tau)\theta_{A'+1,B'}(x-y|2\tau) \tag{A.5}$$

where we have the characteristics:

$$A = \frac{a+a'}{2},\ B = b+b',\ A' = \frac{a-a'}{2},\ B' = b-b' \tag{A.6}$$

for $a, b, a', b' \in \mathbb{Z}_2$. From these *bilinear relations* between theta functions of modulus τ and of modulus 2τ we can, by elimination, obtain many quartic relations between the θ-functions of different characteristic (but of the same modulus), see e.g. [107], but most of these quartic relations are not very insightful. In contrast for θ_{11}, which is the only odd theta function, we have closed-form relation:

$$\begin{aligned}\theta_{11}(x+a)\theta_{11}(x-a)\theta_{11}(y+b)\theta_{11}(y-b) + \theta_{11}(x+b)\theta_{11}(x-b)\theta_{11}(a+y)\theta_{11}(a-y)\\ +\theta_{11}(x+y)\theta_{11}(x-y)\theta_{11}(b+a)\theta_{11}(b-a) = 0\ .\end{aligned} \tag{A.7}$$

It is easily seen that the theta function relation (A.7), which plays a key role in the theory of elliptic functions, is identical to the sigma equation (1.4).

A.2 Proof of the higher degree identity (1.11)

The proof of the higher order elliptic identity given in (1.11) can be achieved directly by simple iteration. The generalization of the basic identity (3-term relation for the σ-function (1.7) or the elliptic partial fraction expansion formula) is:

$$\prod_{i=1}^{n} \Phi_{\kappa_i}(x_i) = \sum_{i=1}^{n} \Phi_{\kappa_1+\cdots+\kappa_n}(x_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \Phi_{\kappa_j}(x_j - x_i)\ , \tag{A.8}$$

which can be easily proven by induction as follows:

Case I: The statement holds when n is equal to 2. It is a simple matter to prove using the three-term relation (1.4). Firstly we make a change of variables. Let

$$\begin{aligned} x &= x_1 + \frac{\kappa_1 - x_2}{2}\ , \\ y &= \kappa_2 + \frac{x_2 + \kappa_1}{2}\ , \\ a &= \frac{\kappa_1 + x_2}{2}\ , \\ b &= \frac{x_2 - \kappa_1}{2}\ . \end{aligned}$$

Then the three-term relation becomes

$$\begin{aligned} \sigma(x_1+\kappa_1)\sigma(x_1-x_2)\sigma(\kappa_2+x_2)\sigma(\kappa_2+\kappa_1) &= \sigma(x_1)\sigma(x_1+\kappa_1-x_2)\sigma(\kappa_2+\kappa_1+x_2)\sigma(\kappa_2) \\ &\quad +\sigma(x_1+\kappa_2+\kappa_1)\sigma(x_1-\kappa_2-x_2)\sigma(x_2)\sigma(\kappa_1)\ . \end{aligned}$$

If we divide the above relation by:

$$\sigma(x_1)\sigma(x_2)\sigma(x_1-x_2)\sigma(\kappa_2+\kappa_1)\sigma(\kappa_2)\sigma(\kappa_1)\ ,$$

we obtain the following identity

$$\Phi_{\kappa_1}(x_1)\Phi_{\kappa_2}(x_2) = \Phi_{\kappa_1}(x_1-x_2)\Phi_{\kappa_1+\kappa_2}(x_2) + \Phi_{\kappa_1+\kappa_2}(x_1)\Phi_{\kappa_2}(x_2-x_1)\ . \tag{A.9}$$

Therefore, the first case can be verified.

Case II: Assume the statement holds for some n (some unspecified value of n). It must be shown that also holds for $n+1$:

$$\prod_{i=1}^{n+1} \Phi_{\kappa_i}(x_i) = \sum_{i=1}^{n+1} \Phi_{\sum_{i=1}^{n+1} \kappa_i}(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \Phi_{\kappa_j}(x_j - x_i) \ . \tag{A.10}$$

It is a simple matter to prove this relation using the identities for Φ_κ function. Firstly, applying the induction hypothesis on the left-hand side of (A.10)

$$\prod_{i=1}^{n+1} \Phi_{\kappa_i}(x_i) = \sum_{i=1}^{n} \Phi_{\Lambda}(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{n} \Phi_{\kappa_j}(x_j - x_i) \Phi_{\kappa_{n+1}}(x_{n+1}) \ , \ \Lambda = \sum_{i=1}^{n} \kappa_i \ , \tag{A.11}$$

and using (A.9) from the case I between the first and last term, we have

$$\begin{aligned}
\prod_{i=1}^{n+1} \Phi_{\kappa_i}(x_i) = \sum_{i=1}^{n} \big\{ & \Phi_{\Lambda+\kappa_{n+1}}(x_i) \Phi_{\kappa_{n+1}}(x_{n+1} - x_i) \\
& + \Phi_{\Lambda}(x_i - x_{n+1}) \Phi_{\Lambda+\kappa_{n+1}}(x_{n+1}) \big\} \prod_{\substack{j=1 \\ j\neq i}}^{n} \Phi_{\kappa_j}(x_j - x_i) \\
= \sum_{i=1}^{n} & \Phi_{\Lambda+\kappa_{n+1}}(x_i) \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \Phi_{\kappa_j}(x_j - x_i) \\
& + \Phi_{\Lambda+\kappa_{n+1}}(x_{n+1}) \sum_{i=1}^{n} \Phi_{\Lambda}(x_i - x_{n+1}) \prod_{\substack{j=1 \\ j\neq i}}^{n} \Phi_{\kappa_j}(x_j - x_i) \ ,
\end{aligned} \tag{A.12}$$

where $\Lambda = \sum_{i=1}^{n} \kappa_i$. Clearly, arranging the terms by using the induction hypothesis, we get the right hand side of (A.10). It has been verified that indeed it holds when $n+1$.

Appendix B

The Frobenius-Stickelberger type identities

Here we collect some results related to the elliptic determinantal formulae of Frobenius and Frobenius-Stickelberger type (i.e. elliptic Cauchy and Vandermonde determinants). The Frobenius-Stickelberger formula, [31] is given by

$$\begin{aligned} &\begin{vmatrix} 1 & \wp(x_1) & \wp'(x_1) & \cdots & \wp^{(n-2)}(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) & \cdots & \wp^{(n-2)}(x_2) \\ 1 & \wp(x_3) & \wp'(x_3) & \cdots & \wp^{(n-2)}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(x_n) & \wp'(x_n) & \cdots & \wp^{(n-2)}(x_n) \end{vmatrix} \\ = \quad & (-1)^{(n-1)(n-2)/2} \; 1!2!3!...(n-1)! \; \frac{\sigma(x_1 + x_2 + ... + x_n) \prod_{i<j=1}^{n} \sigma(x_i - x_j)}{\prod_{i=1}^{n} \sigma^n(x_i)} \end{aligned} \tag{B.1}$$

Denoting the Frobenius-Stickelberger *matrix* $\mathscr{P}(x_1,\ldots,x_n) = \mathscr{P}(\boldsymbol{x})$ by:

$$\mathscr{P}(\boldsymbol{x}) = \begin{pmatrix} 1 & \wp(x_1) & \wp'(x_1) & \cdots & \wp^{(n-2)}(x_1) \\ 1 & \wp(x_2) & \wp'(x_2) & \cdots & \wp^{(n-2)}(x_2) \\ 1 & \wp(x_3) & \wp'(x_3) & \cdots & \wp^{(n-2)}(x_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(x_n) & \wp'(x_n) & \cdots & \wp^{(n-2)}(x_n) \end{pmatrix} \tag{B.2}$$

we have from Cramer's rule the following factorization formula:

$$\left[\mathscr{P}(\boldsymbol{x})\cdot\mathscr{P}(\boldsymbol{y})^{-1}\right]_{i,j} = \frac{1}{\sigma^n(x_i)}\,\Phi_\Sigma(x_i-y_j)\sigma^n(y_j)\frac{\prod_{l=1}^n \sigma(x_i-y_l)}{\prod_{l\neq j}\sigma(y_j-y_l)}\,, \tag{B.3}$$

in which $\Sigma \equiv \Sigma_{l=1}^n\, y_l$. As a consequence we obtain from this the Frobenius-Stickelberger determinantal formula, [32]

$$\det\left(\Phi_\kappa(x_i-y_j)\right)_{i,j=1,\ldots,N} = \frac{\sigma(\kappa+\Sigma)}{\sigma(\kappa)}\,\frac{\prod_{i<j}\,\sigma(x_i-x_j)\,\sigma(y_j-y_i)}{\prod_{i,j}\,\sigma(x_i-y_j)}\,,\quad \Sigma := \sum_{i=1}^N (x_i-y_i)\,. \tag{B.4}$$

Conversely, the Frobenius-Stickelberger formula (B.1) can be obtained from the Frobenius formula by a set of degenerate limits. The elliptic Lagrange interpolation formulae

$$\prod_{i=1}^N \frac{\sigma(\xi-x_i)}{\sigma(\xi-y_i)} = \sum_{i=1}^N \Phi_{-\Sigma}(\xi-y_i)\frac{\prod_{j=1}^N \sigma(y_i-x_j)}{\prod_{\substack{j=1\\j\neq i}}^N \sigma(y_i-y_j)}\,, \tag{B.5}$$

which holds if $\Sigma \neq 0$, and if $\Sigma = 0$:

$$\prod_{i=1}^N \frac{\sigma(\xi-x_i)}{\sigma(\xi-y_i)} = \sum_{i=1}^N \left[\zeta(\xi-y_i)-\zeta(x-y_i)\right]\frac{\prod_{j=1}^N \sigma(y_i-x_j)}{\prod_{\substack{j=1\\j\neq i}}^N \sigma(y_i-y_j)}\,, \tag{B.6}$$

where x denotes any of the zeroes x_i, ($i = 1,\ldots,N$) on the left-hand side. Both (B.5) and (B.6) can be obtained from the Frobenius formula [32] by row-or column expansions (adding an extra row and column to the Frobenius matrix, say with $x_0 = \xi$ and $y_0 = \eta$, and then expanding along that row or column) and (B.6) can subsequently be obtained from a limiting case of the latter.

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