Ordinal Analysis of Set Theories; Relativised and Intuitionistic

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Abstract

In the early 1980s, the forum of ordinal analysis switched from analysing subsystems of second order arithmetic and theories of inductive definitions to set theories. The new results were much more uniform and elegant than their predecessors. This thesis uses techniques for the ordinal analysis of set theories developed over the past 30 years to extract some useful information about Kripke Platek set theory, **KP** and some related theories.

First I give a classification of the provably total set functions of **KP**, this result is reminiscent of a classic theorem of ordinal analysis, characterising the provably total recursive functions of Peano Arithmetic, **PA**.

For the remainder of the thesis the focus switches to intuitionistic theories. Firstly, a detailed rendering of the ordinal analysis of intuitionistic Kripke-Platek set theory, **IKP**, is given. This is done in such a way as to demonstrate that **IKP** has the existence property for its verifiable Σ sentences. Combined with the results of [40] this has important implications for constructive set theory.

It was shown in [42] that sometimes the tools of ordinal analysis can be applied in the context of strong set-theoretic axioms such as power set to obtain a characterisation of a theory in terms of provable heights of the cumulative hierarchy. In the final two chapters this machinery is applied to 'scale up' the earlier result about **IKP** to two stronger theories $\mathbf{IKP}(\mathcal{P})$ and $\mathbf{IKP}(\mathcal{E})$. In the case of $\mathbf{IKP}(\mathcal{E})$ this required considerable new technical legwork. These results also have important applications within constructive set theory.

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Contents

Chapter 1

Introduction

Ordinal analysis is a collection of tools and techniques that allow the extraction of certain kinds of information about a formal theory. This thesis is focused on applying these techniques to theories related to Kripke-Platek set theory, **KP**. It could be seen as an attempt at answering the question:

*What extra information can we learn from the ordinal analysis of **KP**?*

Firstly the techniques are used to say something about how **KP** deals with set functions. Next the techniques are transferred to the intuitionistic case and put to good use in creating definable witnesses for existential theorems. Finally these techniques are 'scaled up' to say something useful about two more intuitionistic theories related to **KP**, but of much higher proof theoretic strength than those traditionally analysed in ordinal analyses.

1.1 A brief history of ordinal analysis

Ordinal analysis is the process of characterising a formal theory by the assignment of a transfinite ordinal, which somehow measures its 'proof theoretic strength'. The first example of an ordinal analysis came in the form of Gentzen's consistency proof for arithmetic in 1936 [15]. Gentzen showed that using transfinite induction up to the ordinal

$$\varepsilon_0 = \text{least } \alpha.\ \omega^\alpha = \alpha$$

one may prove the consistency of **PA**. In order to understand the significance of Gentzens result it must be noted that he made use of transfinite induction only for primitive recursive predicates and beyond that only finitistically justifiable arguments. Thus a more accurate (and modern) statement of Gentzen's result could be

$$\mathbf{PRA} + \mathbf{PR\text{-}TI}(\varepsilon_0) \vdash \mathrm{Con}(\mathbf{PA}). \tag{1}$$

It is now fairly commonly accepted that 'finitistic means' can be accurately described by the theory of Primitive Recursive Arithmetic **PRA** ([50]). Gentzen also showed that

$$(2) \qquad \mathbf{PA} \vdash \mathbf{PR\text{-}TI}(\alpha) \quad \text{for any } \alpha < \varepsilon_0.$$

The intuition strongly suggested by (1) and (2) is that the ordinal ε_0 somehow 'measures' the strength of **PA**. Over the years following Gentzen's paper the concept of the *proof theoretic ordinal of a theory* was made rigorous. Ordinal analyses were carried out for ever stronger theories with ever higher corresponding proof theoretic ordinals.

In 1964 Feferman [9] and Schütte [45], [46] independently determined Γ_0 as the 'limit of predicativity', the proof theoretic ordinal of the theory of autonomous ramified progressions. The next major step came from Takeuti who analysed systems of second order arithmetic (first $\Pi^1_1 - CA$ [51] and then $\Delta^1_2 - CA$ [52]). This was the first time an ordinal analysis was obtained for an *impredicative theory*. Next the field began to provide ordinal analyses for theories of iterated inductive definitions (see Bucholz, Pohlers, Sieg and Feferman [5]).

However the landscape of ordinal analysis was dramatically changed in the early 1980s by Jäger [16], [17] and Jäger and Pohlers [19]. They began a switch from analysing subsystems of second order arithmetic to analysing set theories directly. This new field has been termed *admissible proof theory*. The switch was a desirable one since the new methods employed were more transparent and uniform across the analysis of different theories.

KP was of central importance in the new wave of ordinal analysis and has continued to be the base theory over which ever stronger systems have been analysed. The strongest theory that has so far been subjected to an ordinal analysis lies somewhere in the region of $\Pi^1_2 - CA$ or even $\Delta^1_3 - CA$ [33], [38].

1.2 Kripke-Platek set theory

A common justification for the axioms of set theories such as **ZF** is by a description of a universe of sets being created in ordinal stages. We imagine we have created a certain part of the universe V_α and then apply certain set building operations to form $V_{\alpha+1}$. For example if we have a set $x \in V_\alpha$ and $\varphi(y)$ is a formula of set theory we may apply the axiom of separation to form the set

$$\{y \in x \mid \varphi(y)\}.$$

A feature that often goes unnoticed in such an operation is that the formula φ can contain unbounded quantifiers. These quantifiers make reference to a completed universe of sets, of which the new set we are attempting to create is already a member. This kind of definition is called an *impredicative definition* and is philosophically troublesome to some mathematicians. The axioms of replacement and power set give rise to similar concerns.

These worries lead Kripke [22] and Platek [28], in the mid 1960s, to axiomatise a set theory that was compatible with the idea of a growing universe. This standpoint is known as predicativism. The now accepted axioms of **KP** are

Extensionality:	$(\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a) \rightarrow a = b.$
Foundation/Set Induction:	$\forall x[(\forall y \in x)F(y) \rightarrow F(x)] \rightarrow \forall x F(x)$ for any formula F.
Pair:	$\exists z(z = \{a, b\}).$
Union:	$\exists z(z = \cup a).$
Infinity:	$\exists x[x \neq \emptyset \wedge (\forall y \in x)(\exists z \in x)(y \in z)].$
Δ_0-*Separation:*	$\exists y[y = \{x \in a \mid F(x)\}]$ for any Δ_0-formula $F(a)$.
Δ_0-*Collection:*	$(\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$ for any Δ_0-formula G.

A Δ_0 formula is one in which no unbounded quantifiers appear. Note that in [3] infinity is not included in the definition of **KP**, however in proof theory it is now considered convention to include it. Whilst it has been argued that **KP** doesn't fall into the most stringent definition of a predicative theory [9], [10], each of its axioms appear compatible with the idea of a growing universe, making it more philosophically palatable to the predicativists than, for example, **ZF**.

Philosophy aside, **KP** has turned out to be an interesting and rich area of study. One reason for this is that the vast majority of ordinary mathematics and even set theory can be carried out in **KP**. **KP** turns out to be the 'right' theory for extending recursion theory to the ordinals. Moreover, models of **KP**, the so-called *admissible sets* have been a major source of interaction between different areas of logic: recursion theory, model theory and set theory [3].

1.3 Proof theoretic ordinals

The compelling intuition arising from Gentzens result is that the ordinal ε_0 somehow 'measures' the strength of Peano Arithmetic. The immediate thought on how to generalise this measure leads to the following definition of the *proof theoretic ordinal* of a theory T,

$$|T|_{\mathrm{Con}} := \text{least } \alpha. \quad \mathbf{PRA} + \mathbf{TI}(\alpha) \vdash \mathrm{Con}(T). \tag{3}$$

The problem with this definition is that it's not clear how we are to represent ordinals in **PRA**. In fact, it is always possible (see [34]) to cook up an ordering $\prec_T$ on the natural numbers, with order type ω such that

$$\mathbf{PRA} + \mathbf{TI}(\prec_T) \vdash \mathrm{Con}(T).$$

Apparently making a mockery of the measure $|T|_{\mathrm{Con}}$. The ordering $\prec_T$ is highly pathological and 'unnatural', effectively coding up the consistency of T into the definition of the ordering. It has long been suggested ([20], [11], [12]) that if one restricted to 'natural' well orderings, it should be possible to restore the dignity of $|T|_{\mathrm{Con}}$. However, it has proved very difficult to find a rigorous definition of a 'natural' ordinal representation system which excludes all pathological counter-examples [34]. It is thus desirable to distill what is meant by the definition of $|T|_{\mathrm{Con}}$ into a more rigorous mathematical framework, devoid of the word 'natural'.

For simplicity let us assume T is a theory that allows quantification over subsets of $\mathbb{N}$ (e.g. a subsystem of second order arithmetic or a set theory) and that T comprises $\mathbf{ACA}_0$. Suppose $A \subseteq \mathbb{N}$ and $\prec$ is an ordering on A, such that $(A, \prec)$ is definable in the language of T. Let $LO(A, \prec)$ be the formula of T expressing that $(A, \prec)$ is a linear ordering. We define

$$\begin{aligned} WO(A, \prec) :=\, & LO(A, \prec) \wedge \\ & (\forall X \subseteq \mathbb{N})[(\forall u \in A)[(\forall v \prec u)(v \in X) \to u \in X] \to (\forall u \in A)(u \in X)]. \end{aligned}$$

An ordinal α is said to be *provably recursive in* T if there is a well ordering $(A, \prec)$, which is provably recursive in T and of the same order type as α, such that

$$T \vdash WO(A, \prec).$$

We then define

$$|T|_{\sup} := \sup\{\alpha \mid \alpha \text{ is provably recursive in } T\}.$$

It turns out that $|T|_{\sup}$ is a much more robust measure than $|T|_{\mathrm{Con}}$ [34]. The following observation is from [34] p9.

Observation 1.3.1. *"Every ordinal analysis that has so far appeared in the literature has provided a primitive recursive ordinal notation system $(A, \lhd)$ such that T is proof theoretically reducible to* $\mathbf{PA} + \bigcup_{a \in A} \mathbf{TI}(A|_a, \lhd|_a)$*. Moreover, if T is a classical theory then T and* $\mathbf{PA} + \bigcup_{a \in A} \mathbf{TI}(A|_a, \lhd|_a)$ *prove the same arithmetic sentences and if T is an intuitionistic theory then T and* $\mathbf{HA} + \bigcup_{a \in A} \mathbf{TI}(A|_a, \lhd|_a)$ *prove the same arithmetic sentences. Furthermore, $|T|_{sup} = |\lhd|$."*

Chapter 3 provides an ordinal analysis in the sense of 1.3.1 for **IKP**. However, since the publication of [34] a new application of the techniques of ordinal analysis has been pioneered, that of *relativised ordinal analysis*.

Power Kripke-Platek Set theory $\mathbf{KP}(\mathcal{P})$ is formed from **KP** by adding the power set axiom and allowing the power set operation as primitive in the Δ_0 separation and collection schemas. Owing to power set, the proof theoretic strength of $\mathbf{KP}(\mathcal{P})$ dwarfs all theories for which an ordinal analysis (in the sense of 1.3.1) has been carried out to date. Let **PRST** be the weak system of set theory, containing basic operations on sets and the defining axioms for the primitive recursive set functions (see [30]), let **IPRST** stand for **PRST** formulated with intuitionistic logic. Extractable from [42] is a proof theoretic reduction of $\mathbf{KP}(\mathcal{P})$ to a weak system of set theory, e.g. **PRST**, plus transfinite iterations of the power set construction up to but not including the Bachmann-Howard ordinal. When compared with 1.3.1, this looks like a 'scaled up' version of ordinal analysis. In a similar vein chapters 4 and 5 can be seen as giving reductions of $\mathbf{IKP}(\mathcal{P})$ and $\mathbf{IKP}(\mathcal{E})$ to **PRST** plus transfinite iterations of the power set or set-exponentiation operation. Or more precisely, showing that $\mathbf{IKP}(\mathcal{P})$ and $\mathbf{IKP}(\mathcal{E})$ prove the same Σ sentences as **IPRST** together with transfinite iterations of the power set or set exponentiation operation up to but not including the Bachmann-Howard ordinal.

1.4 The existence property

Intuitionistic theories often possess pleasing meta-mathematical properties in comparison to their classical counterparts, such as the disjunction property. For an intuitionistic theory T where quantifiers range over natural numbers, it is often relatively straight forward to show the numerical existence property, i.e. If $T \vdash \exists x A(x)$ then there is some n such that $T \vdash A(n)$ (provided A contains no other free variables). The numerical existence property can also be required of a set theory. A set theory T has the numerical existence property if whenever $T \vdash (\exists x \in \omega)A(x)$, there is some n such that $T \vdash A(n)$. It turns out that most intuitionistic and constructive set theories have the numerical existence property [36], [39]. However, extending this property to unbounded existential set quantifiers poses significant technical challenges and turns out to be impossible in some cases.

Definition 1.4.1. Let T be a theory formulated in a language containing the language of set theory and $A(x)$ be a formula from the language of T with no free variables other than x. T is said to have *the existence property* if whenever $T \vdash \exists x A(x)$ there is a formula $B(x)$ with exactly x free such that

$$T \vdash \exists! x[B(x) \wedge A(x)].$$

T is said to have the *weak existence property* if whenever $T \vdash \exists x A(x)$, there is some formula $C(y)$ with exactly y free, such that

$$T \vdash \exists! y C(y) \wedge \forall x(C(x) \rightarrow \exists y(y \in x)) \wedge \forall y[C(y) \rightarrow (\forall x \in y)A(x)].$$

The weak existence property asks for a definable, inhabited set of witnesses.

Intuitionistic Zermelo-Fraenkel set theory **IZF**, formulated with collection, does not possess the existence property or even the weak existence property [14], [40]. However **IZF** formulated with replacement instead of collection does have the existence property [26]. The comparison of these two results indicates that somehow the collection axiom hinders the defining of witnesses in intuitionistic set theories, this led Beeson ([4] IX.1) to ask

Does any reasonable set theory with collection have the existence property?

Perhaps the most studied form of constructive set theory is *Constructive Zermelo Fraenkel set theory* **CZF** ([1], [2]). It was shown in [49] that **CZF** possesses neither the existence property or the weak existence property. Since **CZF** contains the axiom subset collection, again collection is indicated in the breakdown of the existence property.

Three theories arising in the study of **CZF** are $\mathbf{CZF}^-$, $\mathbf{CZF}^{\mathcal{E}}$ and $\mathbf{CZF}^{\mathcal{P}}$. $\mathbf{CZF}^-$ arises from **CZF** by omitting the subset collection axiom, $\mathbf{CZF}^{\mathcal{E}}$ and $\mathbf{CZF}^{\mathcal{P}}$ then arise from $\mathbf{CZF}^-$ by adding the exponentiation and power set axioms respectively. We have the following easy relationships between the theories

$$\mathbf{CZF}^- \dashv \mathbf{CZF}^{\mathcal{E}} \dashv \mathbf{CZF} \dashv \mathbf{CZF}^{\mathcal{P}}.$$

These implications cannot be reversed. That $\mathbf{CZF} \nvdash \mathbf{CZF}^{\mathcal{P}}$ comes from the fact that $\mathbf{CZF}^{\mathcal{P}}$ has much stronger proof theoretic strength that **CZF** [2], [41]. The fact that $\mathbf{CZF}^{\mathcal{E}} \nvdash \mathbf{CZF}$ was shown in [24]. It was shown in [40] that $\mathbf{CZF}^-$, $\mathbf{CZF}^{\mathcal{E}}$ and $\mathbf{CZF}^{\mathcal{P}}$ all have the weak existence property. Also given in [40] were reductions to three versions of intuitionistic Kripke-Platek set theory, **IKP**, $\mathbf{IKP}(\mathcal{P})$ and $\mathbf{IKP}(\mathcal{E})$, these reductions were given in such a way that if the latter theories possessed the existence property for certain restricted classes of formulae then the corresponding versions of **CZF** would possess the full existence property. It is thus desirable to prove that these three versions of **IKP** have the existence property for Σ, $\Sigma^{\mathcal{P}}$ and $\Sigma^{\mathcal{E}}$ formulae respectively. This is where ordinal analysis enters the stage. The strategy is to embed the three versions of **IKP** into corresponding infinitary systems, then remove problematic inferences (such as collection) from the infinite derivations of existential statements, then show that from these transformed derivations we can extract a witnessing term from the infinitary system. In chapter 3 this programme is carried out in full for **IKP**, thus confirming that $\mathbf{CZF}^-$ has the existence property. In chapters 4 and 5 we define infinitary systems corresponding to $\mathbf{IKP}(\mathcal{P})$ and $\mathbf{IKP}(\mathcal{E})$ respectively. We then remove problematic inferences for derivations of existential statements in these infinitary systems. The final step of extracting witnessing terms from these transformed derivations, thus confirming that $\mathbf{CZF}^{\mathcal{P}}$ and $\mathbf{CZF}^{\mathcal{E}}$ have the existence property, will be carried out in [43].

1.5 Outline of the thesis

The first section of this thesis is concerned with classifying the provably total set-functions of **KP**. A classic result from ordinal analysis is the characterisation of the provably recursive functions of Peano Arithmetic, **PA**, by means of the fast growing hierarchy [7]. Whilst it is possible to formulate the natural numbers within **KP**, the theory speaks primarily about sets. For this reason it is desirable to obtain a characterisation of its provably total set functions. We will show that **KP** proves the totality of a set function precisely when it falls within a hierarchy of set functions based upon a relativised constructible hierarchy.

The third chapter will be concerned with performing an ordinal analysis of Kripke-Platek set theory formulated with intuitionistic logic; **IKP**. This will be carried out in such a way that if **IKP** proves a Σ-sentence A, we can computably extract a term s from the pertaining infinitary system, which witnesses A. This enables us to prove that **IKP** has the existence property for its verifiable Σ sentences. This has important applications within constructive set theory. in particular, when combined with the results of [40], this chapter confirms that $\mathbf{CZF}^-$ has the existence property.

Chapter 4 provides a relativised ordinal analysis for intuitionistic power Kripke-Platek set theory $\mathbf{IKP}(\mathcal{P})$, which comprises **IKP** but where the operation power-set is allowed as primitive in the separation and collection schemas. In particular $\mathbf{IKP}(\mathcal{P})$ proves the power set axiom.

The relativised ordinal analysis for the classical version of the theory, $\mathbf{KP}(\mathcal{P})$, was carried out in [42], the work in this chapter adapts the techniques from that paper to the intuitionistic case. Whilst full cut-elimination cannot be attained, these results allow the classification of the theory in terms of provable heights of the Von-Neumann hierarchy.

The final chapter provides a relativised ordinal analysis for intuitionistic exponentiation Kripke-Platek set theory $\mathbf{IKP}(\mathcal{E})$, which comprises **IKP** and where the operation of set-exponentiation is allowed as primitive in the separation and collection schemas. Given sets a and b, set-exponentiation allows the formation of the set ${}^{a}b$, of all functions from a to b. This work allows us to classify the theory in terms of the provable height of an *exponentiation hierarchy.* This system was much more difficult to analyse than $\mathbf{IKP}(\mathcal{P})$ and posed considerable technical challenges. A particular problem was assigning an ordinal *level* to the formal terms of the infinitary system. Ultimately this turned out to be impossible and had to be dealt with by allowing level declarations in the hypothesis, the level of a term becomes a dynamic property requiring its own derivation in the infinitary system. As far as I know the ideas in this chapter is new.

The results of the final two chapters also have important applications within constructive set theory. In particular, when combined with the results of [40] they provide an important step on the way to proving that the theories $\mathbf{CZF}^{\mathcal{P}}$ and $\mathbf{CZF}^{\mathcal{E}}$ have the full existence property, the final part of this proof will appear given in [43].

Chapter 2

A classification of the provably total set functions of KP

A major application of the techniques of ordinal analysis has been the classification of the provably total recursive functions of a theory. Usually the theories to which this methodology has been applied have been arithmetic theories, in that context it makes most sense to speak about arithmetic functions. The concept of a recursive function on natural numbers and be extended to a more general recursion theory on arbitrary sets. For more details see [25], [27] and [44]. Since **KP** speaks primarily about sets, it is perhaps desirable to obtain a classification of its provably total recursive set functions.

To provide some context we first state a classic result from proof theory, the classification of the *provably total recursive functions* of **PA**. This result probably first appeared in [21], [23] and [48], was considerably simplified by Bucholz and Wainer in [7] and has been carried out in much greater generality by Weiermann in [54]. For the following definitions, suppose we have an ordinal representation system for ordinals below ε_0, together with an assignment of fundamental sequences to the limit ordinal terms. For an ordinal term α, let α_n denote the n-th element of fundamental sequence for α, ie. $\alpha_{n+1} < \alpha_n$ and $\sup_{n<\omega}(\alpha_n) = \alpha$. There are certain technical properties that such an assignment must satisfy, these will not be gone into here, for a detailed presentation see [7].

Definition 2.0.1. For each $\alpha < \varepsilon_0$ we define the function $F_\alpha : \omega \to \omega$ by transfinite recursion as follows

$$F_0(n) := n+1$$
$$F_{\alpha+1}(n) := F_\alpha^{n+1}(n) \; (:= \overbrace{F_\alpha \circ ... \circ F_\alpha}^{n+1}(n))$$
$$F_\alpha(n) := F_{\alpha_n}(n) \quad \text{if } \alpha \text{ is a limit.}$$

This hierarchy is known as the *fast growing hierarchy*. Given unary functions on the natural numbers f and g, we say that f majorises g if theres is some n such that $(\forall m > n)(g(m) < f(m))$. For a recursive function f let $A_f(n, m)$ be the Σ formula expressing that on input n the turing machine for computing f outputs m, to avoid frustrating counter examples let us suppose A_f does this in some 'natural' way.

Theorem 2.0.2. Suppose $f : \omega \to \omega$ is a recursive function. Then

i) If $\mathbf{PA} \vdash \forall x \exists! y A_f(x, y)$ then f is majorised by F_α for some $\alpha < \varepsilon_0$.

ii) $\mathbf{PA} \vdash \forall x \exists! y A_{F_\alpha}(x, y)$ for every $\alpha < \varepsilon_0$.

Proof. This classic result is proved in full in [7]. □

This chapter will be focused on obtaining a similar result for the provably total set functions of **KP**. A similar role to the fast growing hierarchy in Theorem 2.0.2 will be played by the *relativised constructible hierarchy*.

Definition 2.0.3. Let X be any set. We may relativise the constructible hierarchy to X as follows:

$$L_0(X) := TC(\{X\}) \quad \text{the } \textit{transitive closure} \text{ of } \{X\}$$
$$L_{\alpha+1}(X) := \{B \subseteq L_\alpha(X) \;:\; B \text{ is definable over } \langle L_\alpha(X), \in\rangle\}$$
$$L_\theta(X) := \bigcup_{\xi<\theta} L_\xi(X) \quad \text{when } \theta \text{ is a limit.}$$

In section 1. we build an ordinal notation system relativised to an arbitrary well ordering, this will be used for the rest of the chapter. In section 2. we define the infinitary system $\mathbf{RS}_\Omega(X)$, based on the relativised constructible hierarchy and show that we can eliminate cuts for derivations of Σ formulae. In section 3. we embed **KP** into $\mathbf{RS}_\Omega(X)$, allowing us to obtain cut free infinitary derivations of **KP** provable Σ formulae. In section 4. we give a well ordering proof in **KP** for the ordinal notation system given in section 1. Finally we combine the results of this chapter to give a classification of the provably total set functions of **KP**. This result, whilst perhaps known to those who have thought hard about these things, has not appeared in the literature to date.

2.1 A relativised ordinal notation system

The aim of this section is to relativise the construction of the Bachmann-Howard ordinal to contain an arbitrary well ordering $W := (X, \prec)$. We will construct an ordinal representation system that will be primitive recursive given access to an oracle for W. Here the notion of recursive and primitive recursive is extended to arbitrary sets, see [27] or [44] for more detail. The construction of an ordinal representation system for the Bachmann-Howard ordinal is now fairly standard in proof theory, carried out for example in [6]. Intuitively our system will appear similar, only the ordering W will be inserted as an initial segment before new ordinals start being 'named' via the collapsing function.

Before defining the formal terms and the procedure for computing their ordering, it is informative to give definitions for the corresponding ordinals and ordinal functions themselves. To this end we will begin working in **ZFC**, later it will become clear that the necessary ordinals can be expressed as formal terms and comparisons between these terms can be made primitive recursively relative to W.

In what follows **ON** will denote the class of all ordinals. First we require some information about the φ function on ordinals. These definitions and results are well known, see [47].

Definition 2.1.1. For each $\alpha \in \mathbf{ON}$ we define a class of ordinals $Cr(\alpha) \subseteq \mathbf{ON}$ and a class function

$$\varphi_\alpha : \mathbf{ON} \to \mathbf{ON}$$

by transfinite recursion.

i) $Cr(0) := \{\omega^\beta \mid \beta \in \mathbf{ON}\}$ and $\varphi_0(\beta) := \omega^\beta$.

ii) For $\alpha > 0$ $Cr(\alpha) := \{\beta \mid (\forall \gamma < \alpha)(\varphi_\gamma(\beta) = \beta)\}$.

iii) For each $\alpha \in \mathbf{ON}$ $\varphi_\alpha(\cdot)$ is the function enumerating $Cr(\alpha)$.

The convention is to write $\varphi\alpha\beta$ instead of $\varphi_\alpha(\beta)$. An ordinal $\beta \in Cr(0)$ is often referred to as *additive principal*, since for all $\beta_1, \beta_2 < \beta$ we have $\beta_1 + \beta_2 < \beta$.

Theorem 2.1.2.

i) $\varphi\alpha_1\beta_1 = \varphi\alpha_2\beta_2$ if and only if $\begin{cases} & \alpha_1 < \alpha_2 \quad \text{and} \quad \beta_1 = \varphi\alpha_2\beta_2 \\ \text{or} & \alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 = \beta_2 \\ \text{or} & \alpha_2 < \alpha_1 \quad \text{and} \quad \varphi\alpha_1\beta_1 = \beta_2. \end{cases}$

ii) $\varphi\alpha_1\beta_1 < \varphi\alpha_2\beta_2$ if and only if $\begin{cases} & \alpha_1 < \alpha_2 \quad \text{and} \quad \beta_1 < \varphi\alpha_2\beta_2 \\ \text{or} & \alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 < \beta_2 \\ \text{or} & \alpha_2 < \alpha_1 \quad \text{and} \quad \varphi\alpha_1\beta_1 < \beta_2. \end{cases}$

iii) For any additive principal β there are unique ordinals $\beta_1 \leq \beta$ and $\beta_2 < \beta$ such that $\beta = \varphi\beta_1\beta_2$.

Proof. This result is proved in full in [47]. □

Definition 2.1.3. We define $\Gamma_{(\cdot)} : \mathbf{ON} \to \mathbf{ON}$ to be the class function enumerating the ordinals β such that for all $\beta_1, \beta_2 < \beta$ we have $\varphi\beta_1\beta_2 < \beta$. Ordinals of the form Γ_β will be referred to as *strongly critical*.

Now let $\theta \in \mathbf{ON}$ be the unique ordinal corresponding to the order type of the well ordering W.

Definition 2.1.4. Let Ω_θ be the least uncountable cardinal greater than θ. The sets $B_\theta(\alpha) \subseteq \mathbf{ON}$ and ordinals $\psi_\theta(\alpha)$ are defined by transfinite recursion on α as follows:

$$\begin{aligned} B_\theta(\alpha) &:= \text{Closure of } \{0, \Omega\} \cup \{\Gamma_\beta \ : \ \beta \leq \theta\} \text{ under } +,\ \varphi \text{ and } \psi_\theta|_\alpha \\ \psi_\theta(\alpha) &:= \min\{\beta \ : \ \beta \notin B_\theta(\alpha)\} \end{aligned}$$

For the remainder of this section, since θ remains fixed, the subscripts will be dropped from Ω_θ, B_θ and ψ_θ to improve readability. At first glance it may appear strange having the elements from θ inserted into the Γ-numbers. Ultimately we aim to have $+$ and φ as primitive symbols in our notation system, simply having θ as an initial segment here would cause problems with unique representation. Some ordinals could get a name directly from θ and other names by applying $+$ and φ to smaller elements.

Lemma 2.1.5. For each $\alpha \in \mathbf{ON}$:

i) The cardinality of $B(\alpha)$ is $\max\{\aleph_0, |\theta|\}$, where $|\theta|$ denotes the cardinality of θ.

ii) $\psi\alpha < \Omega$.

Proof. i) Let

$$\begin{aligned} B^0(\alpha) :=& \{0, \Omega\} \cup \{\Gamma_\beta \ : \ \beta \leq \theta\} \\ B^{n+1}(\alpha) :=& B^n(\alpha) \cup \{\xi + \eta \ : \ \xi, \eta \in B^n(\alpha)\} \\ & \cup \{\varphi\xi\eta \ : \ \xi, \eta \in B^n(\alpha)\} \\ & \cup \{\psi\xi \ : \ \xi \in B^n(\alpha) \cap \alpha\}. \end{aligned}$$

Observe that $B(\alpha) = \cup_{n<\omega} B^n(\alpha)$, this can be proved by a straightforward induction on n.

If θ is finite then, again by induction on n, we can show that each $B^n(\alpha)$ is also finite. Since $B(\alpha)$ is a countable union of finite sets and $\omega \subseteq B(\alpha)$ it follows that it must have cardinality $\aleph_0$.

Now suppose θ is infinite, so $B(\alpha)$ is the countable union of sets of cardinality $|\theta|$ and thus also has cardinality $|\theta|$.

ii) If $\psi\alpha \geq \Omega$ then $\Omega \subset B(\alpha)$ contradicting i). $\square$

Lemma 2.1.6.

i) If $\gamma \leq \delta$ then $B(\gamma) \subseteq B(\delta)$ and $\psi\gamma \leq \psi\delta$.

ii) If $\gamma \in B(\delta) \cap \delta$ then $\psi\gamma < \psi\delta$.

iii) If $\gamma \leq \delta$ and $[\gamma, \delta) \cap B(\gamma) = \emptyset$ then $B(\gamma) = B(\delta)$.

iv) If ξ is a limit then $B(\xi) = \cup_{\eta<\xi} B(\eta)$.

v) $\psi\gamma$ is a strongly critical and $\psi\gamma \geq \Gamma_{\theta+1}$.

vi) $B(\gamma) \cap \Omega = \psi\gamma$.

vii) If ξ is a limit then $\psi\xi = \sup_{\eta<\xi} \psi\eta$.

viii) $\psi(\gamma + 1) \leq (\psi\gamma)^\Gamma$, where δ^Γ denotes the smallest strongly critical ordinal above δ.

ix) If $\alpha \in B(\alpha)$ then $\psi(\alpha + 1) = (\psi\alpha)^\Gamma$.

x) If $\alpha \notin B(\alpha)$ then $\psi(\alpha + 1) = \psi\alpha$ and $B(\alpha + 1) = B(\alpha)$.

xi) If $\gamma \in B(\gamma)$ and $\delta \in B(\delta)$ then $[\gamma < \delta$ if and only if $\psi\gamma < \psi\delta]$.

Proof. i) Suppose $\gamma \leq \delta$, now note that $B(\delta)$ is closed under $\psi|_\delta$ which includes $\psi|_\gamma$ so $B(\gamma) \subseteq B(\delta)$. From this it immediately follows from the definition that $\psi\gamma \leq \psi\delta$.

ii) From $\gamma \in B(\delta) \cap \delta$ we get $\psi\gamma \in B(\delta)$, thus $\psi\gamma < \psi\delta$ b the definition of $\psi\delta$.

iii) It is enough to show that $B(\gamma)$ is closed under $\psi|_\delta$. Let $\beta \in B(\gamma)$ and $\beta < \delta$, then by assumption $\beta < \gamma$, thus $\psi\beta \in B(\gamma)$.

iv) By i) we have $\cup_{\eta<\xi} B(\eta) \subseteq B(\xi)$. It remains to verify that $Y := \cup_{\eta<\xi} B(\eta)$ is closed under $\psi|_\xi$. So let $\delta \in Y \cap \xi$, since ξ is a limit there is some $\xi_0 < \xi$ such that $\delta \in Y \cap \xi_0$ and there is some $\xi_1 < \xi$ such that $\delta \in B(\xi_1)$. Therefore $\delta \in B(\xi^*) \cap \xi^\star$ where $\xi^* = \max\{\xi_0, \xi_1\}$, thus $\psi\delta \in B(\xi^*) \subseteq Y$.

v) We may write the ordinal $\psi\alpha$ in Cantor normal form, so that $\psi\alpha = \omega^{\alpha_1} + ... + \omega^{\alpha_n}$ with $\alpha_1 \geq ... \geq \alpha_n$. If $n > 1$ then $\alpha_1, ..., \alpha_n < \psi\alpha$ whih implies by the definition of $\psi\alpha$ that $\alpha_1, ..., \alpha_n \in B(\alpha)$. But by closure of $B(\alpha)$ under $+$ and φ we get $\varphi 0\alpha_1 + ... + \varphi 0\alpha_n = \omega^{\alpha_1} + ...\omega^{\alpha_n} \in B(\alpha)$ contradicting $\psi\alpha \notin B(\alpha)$. Thus $\psi\alpha$ is additive principal and it follows from Theroem 2.1.2iii) that we may find ordinals $\gamma \leq \psi\alpha$ and $\delta < \psi\alpha$ such that $\psi\alpha = \varphi\gamma\delta$. If $\delta > 0$ then $\gamma < \psi\alpha$ since $\gamma \leq \varphi\gamma 0 < \varphi\gamma\delta$, but if $\delta, \gamma < \psi\alpha$ then we have $\delta, \gamma \in B(\alpha)$ and hence $\varphi\gamma\delta \in B(\alpha)$ contradicting $\psi\alpha \notin B(\alpha)$. Thus $\psi\alpha = \varphi\gamma 0$, but if $\gamma < \psi\alpha$ then again we get $\varphi\gamma 0 \in B(\alpha)$; a contradiction. So it must be the case that $\psi\alpha = \gamma$, ie. $\psi\alpha$ is additive principal.

For the second part note that $\psi\alpha \neq \Gamma_\beta$ for any $\beta \leq \theta$ since by definition each such $\Gamma_\beta \in B(\alpha)$.

vi) By 2.1.5ii) and the definition of ψ it is clear that $\psi\alpha \subseteq B(\alpha) \cap \Omega$. Now let

$$Y := \psi\alpha \cup \{\delta \geq \Omega \mid \delta \in B(\alpha)\}$$

by v) Y contains $0, \Omega$ and Γ_β for $\beta \leq \theta$, moreover it is closed under $+$ and φ. It remains to show that Y is closed under $\psi|_\alpha$, this follows immediately from ii).

vii) Let ξ be a limit ordinal. Using parts vi), iv) and i) we have

$$\psi\xi = B(\xi) \cap \Omega = (\cup_{\eta<\xi} B(\eta)) \cap \Omega = \cup_{\eta<\xi}(B(\eta) \cap \Omega) = \cup_{\eta<\xi}\psi\eta = \sup_{\eta<\xi}\psi\eta.$$

viii) Let

$$Y := (\psi\alpha)^\Gamma \cup \{\delta \geq \Omega \mid \delta \in B(\alpha)\}.$$

Y is closed under $+$ and φ, also it contans Γ_β for any $\beta \leq \theta$ by v). Moreover it contains $\psi\gamma$ for any $\gamma \leq \alpha$ by i), so it is closed under $\psi|_{(\alpha+1)}$. Therefore Y must contain $B(\alpha + 1)$, and so $\psi(\alpha + 1) \leq (\psi\alpha)^\Gamma$.

ix) From $\alpha \in B(\alpha)$ we get $\alpha \in B(\alpha + 1)$, it then follows from ii) that $\psi\alpha < \psi(\alpha + 1)$. Thus $\psi(\alpha + 1) \leq (\psi\alpha)^\Gamma$ by viii) and $\psi(\alpha + 1) \geq (\psi\alpha)^\Gamma$ from v), so it must be the case that $\psi(\alpha + 1) = (\psi\alpha)^\Gamma$.

x) Suppose $\alpha \notin B(\alpha)$, then $[\alpha, \alpha + 1) \cap B(\alpha) = \emptyset$ so we may apply iii) to give $B(\alpha + 1) = B(\alpha)$ from which $\psi(\alpha + 1) = \psi\alpha$ follows immediately.

xi) Suppose $\gamma \in B(\gamma)$ and $\delta \in B(\delta)$. If $\gamma < \delta$ then from ix) we get $\psi(\gamma + 1) = (\psi\gamma)^\Gamma > \psi\gamma$, but by i) $\psi(\gamma + 1) \leq \psi\delta$.

Now if $\psi\gamma < \psi\delta$ then from the contraposition of i) we get $\gamma < \delta$. $\square$

Definition 2.1.7. We write

i) $\alpha =_{NF} \alpha_1 + ... + \alpha_n$ if $\alpha = \alpha_1 + ... + \alpha_n$, $n > 1$, $\alpha_1, ..., \alpha_n$ are additive principal numbers and $\alpha_1 \geq ... \geq \alpha_n$.

ii) $\alpha =_{NF} \varphi\gamma\delta$ if $\alpha = \varphi\gamma\delta$ and $\gamma, \delta < \varphi\gamma\delta$.

iii) $\alpha =_{NF} \psi\gamma$ if $\alpha = \psi\gamma$ and $\gamma \in B(\gamma)$

Lemma 2.1.8.

i) If $\alpha =_{NF} \alpha_1 + ... + \alpha_n$ then for any $\eta \in \mathbf{ON}$

$$\alpha \in B(\eta) \quad \text{if and only if} \quad \alpha_1, ..., \alpha_n \in B(\eta).$$

ii) If $\alpha =_{NF} \varphi\gamma\delta$ then for any $\eta \in \mathbf{ON}$

$$\alpha \in B(\eta) \quad \text{if and only if} \quad \gamma, \delta \in B(\eta).$$

iii) If $\alpha =_{NF} \psi\gamma$ then for any $\eta \in \mathbf{ON}$

$$\alpha \in B(\eta) \quad \text{if and only if} \quad \gamma \in B(\eta) \cap \eta.$$

Proof. i) Suppose $\alpha =_{NF} \alpha_1 + ... + \alpha_n$, the $\Leftarrow$ direction is clear from the closure of $B(\eta)$ under $+$. For the other direction let

$$AP(\alpha) := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \{\alpha\} & \text{if } \alpha \text{ is additive principal} \\ \{\alpha_1, ..., \alpha_n\} & \text{if } \alpha =_{NF} \alpha_1 + ... + \alpha_n \end{cases}$$

$AP(\alpha)$ stands for the *additive predecessors* of α. Now let

$$Y := \{\gamma \in B(\eta) \mid AP(\gamma) \subseteq B(\eta)\}.$$

Observe that $0, \Omega \in Y$ and $\{\Gamma_\beta \mid \beta \leq \theta\} \subseteq Y$. Now choose any $\gamma, \delta \in Y$, we have $AP(\gamma + \delta) \subseteq AP(\gamma) \cup AP(\delta) \subseteq B(\eta)$, thus Y is closed under $+$. Now $AP(\varphi\gamma\delta) = \{\varphi\gamma\delta\}$ since the range of φ is the additive principal numbers thus Y is closed under φ. Finally $AP(\psi\gamma) = \{\psi\gamma\}$ for any $\gamma \in Y \cap \eta$ so Y is closed under $\psi|_\eta$. It follows that $B(\eta) \subseteq Y$ and thus the other direction is proved.

ii) Again the $\Leftarrow$ direction follows immediately from the closure of $B(\eta)$ under φ. For the other direction we let

$$PP(\alpha) := \begin{cases} \emptyset & \text{if } \alpha = 0 \\ \{\alpha\} & \text{if } \alpha \text{ is strongly critical} \\ \{\gamma, \delta\} & \text{if } \alpha =_{NF} \varphi\gamma\delta \\ \{\alpha_1, ..., \alpha_n\} & \text{if } \alpha =_{NF} \alpha_1 + ... + \alpha_n. \end{cases}$$

for want of a better phrase $PP(\alpha)$ stands for the *predicative predecessors* of α. Now set

$$Y := \{\gamma \in B(\eta) \mid PP(\gamma) \subseteq B(\eta)\}$$

It is easily seen that Y contains 0,Ω and Γ_β for any $\beta \leq \theta$. $PP(\gamma + \delta) \subseteq PP(\gamma) \cup PP(\delta)$ so Y is closed under $+$. $PP(\varphi\gamma\delta) \subseteq \{\gamma, \delta\}$ so Y is closed under φ. Finally $PP(\psi\gamma) = \{\psi\gamma\}$ for any $\gamma < \eta$ by 2.1.6v). It follows that Y must contain $B(\eta)$, which proves the $\Rightarrow$ direction.

iii) Suppose $\alpha =_{NF} \psi\gamma$, the $\Leftarrow$ direction is clear by the closure of $B(\eta)$ under $\psi|_\eta$. For the other direction suppose $\alpha \in B(\eta)$, from this we get $\psi\gamma < \psi\eta$ which gives us $\gamma < \eta$. Now by assumption $\gamma \in B(\gamma)$, and $B(\gamma) \subseteq B(\eta)$ so $\gamma \in B(\eta) \cap \eta$. $\square$

In order to create an ordinal notation system from the ordinal functions described above, we single out a set $R(\theta)$ of ordinals which have a unique canonical description.

Definition 2.1.9. We give an inductive definition of the set $R(\theta)$, and the complexity $G\alpha < \omega$ for every $\alpha \in R(\theta)$

(R1) $0, \Omega \in R(\theta)$ and $G0 := G\Omega := 0$.

(R2) For each $\beta \leq \theta$, $\Gamma_\beta \in R(\theta)$ and $G\Gamma_\beta := 0$.

(R3) If $\alpha =_{NF} \alpha_1 + ... + \alpha_n$ and $\alpha_1, ..., \alpha_n \in R(\theta)$ then $\alpha \in R(\theta)$ and $G\alpha := \max\{G\alpha_1, ..., G\alpha_n\} + 1$.

(R4) If $\gamma, \delta < \Omega$, $\alpha =_{NF} \varphi\gamma\delta$ and $\gamma, \delta \in R(\theta)$ then $\alpha \in R(\theta)$ and $G\alpha := \max\{G\gamma, G\delta\} + 1$.

(R5) If $\gamma \geq \Omega$, $\alpha =_{NF} \varphi 0\gamma$ and $\gamma \in R(\theta)$ then $\alpha \in R(\theta)$ and $G\alpha := G\gamma + 1$

(R6) If $\alpha =_{NF} \psi\gamma$ and $\gamma \in R(\theta)$ then $\alpha \in R(\theta)$ and $G\alpha := G\gamma + 1$

Lemma 2.1.10. Every element $\alpha \in R(\theta)$ is included due to precisely one of the rules (R1)-(R6) and thus the complexity $G\alpha$ is uniquely defined.

Proof. This follows immediately from 2.1.8. $\square$

Our goal is to turn $R(\theta)$ into a formal representation system, the main obstacle to this is that it is not immediately clear how to deal with the constraint $\gamma \in B(\gamma)$ in a computable way. This problem leads to the following definition.

Definition 2.1.11. To each $\alpha \in R(\theta)$ we assign a set $K\alpha$ of ordinal terms by induction on the complexity $G\alpha$:

(K1) $K0 := K\Omega := K\Gamma_\beta := \emptyset$ for all $\beta \leq \theta$.

(K2) If $\alpha =_{NF} \alpha_1 + ... + \alpha_n$ then $K\alpha := K\alpha_1 \cup ... \cup K\alpha_n$.

(K3) If $\alpha =_{NF} \varphi\gamma\delta$ then $K\alpha := K\gamma \cup K\delta$.

(K4) If $\alpha =_{NF} \psi\gamma$ then $K\alpha := \{\gamma\} \cup K\gamma$.

$K\alpha$ consists of the ordinals that occur as arguments of the ψ function in the normal form representation of α. Note that each ordinal in $K\alpha$ belongs to $R(\theta)$ itself and has complexity lower than $G\alpha$.

Lemma 2.1.12. For any $\alpha, \eta \in R(\theta)$

$$\alpha \in B(\eta) \quad \text{if and only if} \quad (\forall \xi \in K\alpha)(\xi < \eta)$$

Proof. The proof is by induction on $G\alpha$. If $G\alpha = 0$ then $\alpha \in B(\eta)$ for any η, and $K\alpha = \emptyset$ by (K1) so the result holds.

Case 1. If $\alpha =_{NF} \alpha_1 + ... + \alpha_n$ then $\alpha \in B(\eta)$ iff $\alpha_1, ..., \alpha_n \in B(\eta)$ by 2.1.8i). Now inductively $\alpha_1, ..., \alpha_n \in B(\eta)$ iff $(\forall \xi \in K\alpha_1 \cup ... \cup K\alpha_n)(\xi < \eta)$, but by (K2) $K\alpha = K\alpha_1 \cup ... \cup K\alpha_n$.

Case 2. If $\alpha =_{NF} \varphi\gamma\delta$ we may argue in a similar fashion to Case 1, using 2.1.8ii) and (K3) instead.

Case 3. If $\alpha =_{NF} \psi\gamma$ then $\alpha \in B(\eta)$ iff $\gamma \in B(\eta) \cap \eta$ by 2.1.8iii). Now by induction hypothesis $\gamma \in B(\eta) \cap \eta$ iff $(\forall \xi \in K\gamma)(\xi < \eta)$ and $\gamma < \eta$, and by (K4) this occurs precisely when $(\forall \xi \in K\alpha)(\xi < \eta)$. $\square$

Recall that θ is the ordinal corresponding to the order type of the well ordering $W = (X, \prec)$. Let

$$\mathcal{L}_W := \{0, \Omega, +, \varphi, \psi\} \cup \{\Gamma_x \, : \, x \in X\} \quad \text{and}$$
$$\mathcal{L}_W^* := \{s \mid s \text{ is a finite string of symbols from } \mathcal{L}_W\}.$$

Now let $T(W) \subseteq \mathcal{L}_W^*$ be the set of strings that correspond to ordinals in $R(\theta)$ expressed in normal form. Owing to Lemma 2.1.10 there is a one to one correspondence between $T(W)$ and $R(\theta)$. The ordering on $T(W)$ induced from the ordering of the ordinals in $R(\theta)$ will be denoted $\prec$. To differentiate between elements of the two sets, greek letters $\alpha, \beta, \gamma, \eta, \xi, ...$ range over ordinals and roman letters $a, b, c, d, e, ...$ range over finite strings from $\mathcal{L}_W^*$.

Theorem 2.1.13. Suppose $W = (X, \prec)$ is an arbitrary well ordering. The set $T(W)$ and the relation $\prec$ on $T(W)$ are primitive recursive in W.

Proof. We need to provide the following two procedures

A) A W-primitive recursive procedure which decides for $a \in \mathcal{L}_W^*$ whether $a \in T(W)$.

B) A W-primitive recursive procedure which decides for non-identical $a, b \in T(W)$ whether $a \prec b$ or $b \prec a$.

We define **A)** and **B)** simultaneously by induction on the term complexity Ga.

For the base stage of **A)** we have $0, \Omega \in T(W)$ and $\Gamma_x \in T(W)$ for all $x \in X$.

For the base stage of **B)** we have $0 \prec \Gamma_x \prec \Omega$ for all $x \in X$ and the terms Γ_x inherit the ordering from W, for which we have access to an oracle.

For the inductive stage of **A)** we require the following 3 things:

A1) A W-primitive recursive procedure that on input $a_1, ..., a_n \in T(W)$ decides whether $a_1 + ... + a_n \in T(W)$.

A2) A W-primitive recursive procedure that on input $a_1, a_2 \in T(W)$ decides whether $\varphi a_1 a_2 \in T(W)$.

A3) A W-primitive recursive procedure that on input $a \in T(W)$ decides whether $\psi a \in T(W)$.

For **A1)** we need to decide if $n > 1$ and if $a_1 \succeq ... \succeq a_n$, which we can do by the induction hypothesis. We also need to decide if $a_1, ..., a_n$ are additive principal; all terms other than those of the form $b_1 + ... + b_m$ $(m > 1)$ and 0 are additive principal.

For **A2)**, First let ORD_W denote the set of $\mathcal{L}_W$ strings which represent an ordinal (not necessarily in normal form), ie. each function symbol has the correct arity. Next we define the set of strings which correspond to the strongly critical ordinals.

$$SC_W := \{\Omega\} \cup \{\Gamma_x \,:\, x \in X\} \cup \{a \in ORD_W \,:\, a \equiv \psi b\}$$

We may decide membership of SC_W in a W-primitive recursive fashion. For the decision procedure we split into cases based upon the form of a_2:

i) If $a_2 \equiv 0$ then $\varphi a_1 a_2 \in T(W)$ whenever $a_1 \notin SC_W$

ii) If $a_2 \in SC_W$ then $\varphi a_1 a_2 \in T(W)$ whenever $a_1 \succeq a_2$ and $a_2 \neq \Omega$.

iii) If $a_2 \succ \Omega$ then $\varphi a_1 a_2 \in T(W)$ exactly when $a_1 = 0$.

iv) If $a_2 \equiv b_1 + ... + b_n \prec \Omega$, with $n > 1$ then $\varphi a_1 a_2 \in T(W)$ regardless of the form of a_1.

iv) If $a_2 \equiv \varphi b_1 b_2 \prec \Omega$ then $\varphi a_1 a_2 \in T(W)$ whenever $a_1 \succeq b_1$.

For a rigourous treatment of the φ function see [47].

The function K from Definition 2.1.11 lifts to a W-primitive recursive function on $T(W)$. Moreover every $b \in Ka$ is a member of $T(W)$ of lower complexity than a. Owing to Lemma 2.1.12, for the decision procedure **A3)** we may first compute Ka, then check whether $(\forall b \in Ka)(b \prec a)$, which we may do by the induction hypothesis.

Finally for the inductive stage of **B)**, given two elements of $T(W)$ we may decide their ordering using the following procedure.

B1) $0 \prec a$ for every $a \neq 0$.

B2) $\Gamma_x \prec \Omega$ for every $x \in X$.

B3) The elements Γ_x inherit the ordering from W.

B4) If $a \in SC_\theta$ or $a \equiv \varphi bc$ then $a_1 + ... + a_n \prec a$ if $a_1 \prec a$.

B5) If $a \in SC_\theta$ then $\varphi bc \prec a$ if $b, c \prec a$.

B6) $\psi b \prec \Omega$ for all b.

B7) $\psi a \succ \Gamma_x$ for all $x \in X$.

B8) $a_1 + ... + a_n \prec b_1 + ... + b_m$ if $n < m$ and $(\forall i \leq n)[a_i \equiv b_i]$
or $\exists i \leq \min(n, m)[\forall j < i(a_j = b_j)$ and $a_i \prec b_i]$.

B9) $\varphi a_1 b_1 \prec \varphi a_2 b_2$ if $a_1 \prec a_2 \ \wedge \ b_1 \prec \varphi a_2 b_2$
or $a_1 = a_2 \ \wedge \ b_1 \prec b_2$
or $a_2 \prec a_1 \ \wedge \ \varphi a_1 b_1 \prec b_2$.

B10) $\psi a \prec \psi b$ if $a \prec b$.

□

2.2 A Tait-style sequent calculus formulation of KP

Definition 2.2.1. The language of **KP** consists of free variables $a_0, a_1, ...,$ bound variables $x_0, x_1, ...,$ the binary predicate symbols $\in$, $\notin$ and the logical symbols $\vee, \wedge, \forall, \exists$ as well as parentheses $), ($.

The atomic formulas are those of the form

$$(a \in b) \quad , \quad (a \notin b)$$

The formulas of KP are defined inductively by:

i) Atomic formulas are formulas.

ii) If A and B are formulas then so are $A \vee B$ and $A \wedge B$.

iii) If $A(b)$ is a formula in which the bound variable x does not occur, then $\forall x A(x)$, $\exists x A(x)$, $(\forall x \in a)A(x)$ and $(\exists x \in a)A(x)$ are all formulas.

Quantifiers of the form $\exists x$ and $\forall x$ will be called unbounded and those of the form $(\exists x \in a)$ and $(\forall x \in a)$ will be referred to as bounded quantifiers.

A formula is said to be Δ_0 if it contains no unbounded quantifiers. A formula is said to be Σ (Π) if it contains no unbounded universal (existential) quantifiers.

The negation $\neg A$ of a formula A is obtained from A by undergoing the following operations:

i) Replacing every occurrence of $\in$,$\notin$ with $\notin$,$\in$ respectively.

ii) Replacing any occurrence of $\wedge, \vee, \forall x, \exists x, (\forall x \in a), (\exists x \in a)$ with $\vee, \wedge, \exists x, \forall x, (\exists x \in a), (\forall x \in a)$ respectively.

Thus the negation of a formula A is in negation normal form. The expression $A \to B$ will be considered shorthand for $\neg A \vee B$.

The expression $a = b$ is to be treated as an abbreviation for $(\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)$.

The derivations of **KP** take place in a Tait-style sequent calculus, finite sets of formulae denoted by Greek capital letters are derived. Intuitively the sequent Γ may be read as the disjunction of formulae occuring in Γ.

The axioms of **KP** are:

Logical axioms:	$\Gamma, A, \neg A$ for any formula A.
Extensionality:	$\Gamma, a = b \wedge B(a) \rightarrow B(b)$ for any formula $B(a)$.
Pair:	$\Gamma, \exists z(a \in z \wedge b \in z)$.
Union:	$\Gamma, \exists z(\forall y \in z)(\forall x \in y)(x \in z)$.
Δ_0-*Separation:*	$\Gamma, \exists y[(\forall x \in y)(x \in a \wedge B(x)) \wedge (\forall x \in a)(B(x) \rightarrow x \in y)]$ for any Δ_0-formula $B(a)$.
Set Induction:	$\Gamma, \forall x[(\forall y \in xF(y) \rightarrow F(x)] \rightarrow \forall xF(x)$ for any formula $F(a)$.
Infinity:	$\Gamma, \exists x[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$.
Δ_0-*Collection:*	$\Gamma, (\forall x \in a)\exists yG(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$ for any Δ_0-formula G.

The rules of inference are

$$(\wedge)\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$(\vee)\ \frac{\Gamma, A}{\Gamma, A \vee B} \qquad \frac{\Gamma, B}{\Gamma, A \vee B}$$

$$(b\exists)\ \frac{\Gamma, a \in b \wedge F(a)}{\Gamma, (\exists x \in b)F(x)} \qquad (\exists)\ \frac{\Gamma, F(a)}{\Gamma, \exists xF(x)}$$

$$(b\forall)\ \frac{\Gamma, a \in b \rightarrow F(a)}{\Gamma, (\forall x \in b)F(x)} \qquad (\forall)\ \frac{\Gamma, F(a)}{\Gamma, \forall xF(x)}$$

$$(\text{Cut})\ \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

In both $(b\forall)$ and $(\forall)$, the variable a must not be present in the conclusion, such a variable is referred to as the *eigenvariable* of the inference.

The *minor formulae* of an inference are those rendered prominently in the premises, the other formulae in the premises will be referred to as *side formulae*. The *principal* formula of an inference is the one rendered prominently in the conclusion. Note that the principal formula can also be a side formula of that inference, when this happens we say that there has been a *contraction*. The rule (Cut) has no principal formula.

As an example of a **KP** derivation, it is informative to show that the bounded and unbounded quantifiers interact with one another as expected.

Lemma 2.2.2. The following are derivable within **KP**:

i) $(\forall x \in b)F(x) \leftrightarrow \forall x(x \in b \rightarrow F(x))$.

ii) $(\exists x \in b)F(x) \leftrightarrow \exists x(x \in b \wedge F(x))$.

Proof. We verify only i) as the proof of ii) is very similar. First note that $a \in b \wedge \neg F(a), a \in b \to F(a)$ is a logical axiom of **KP**, we have the following derivation in **KP**.

$$(\wedge)\,\dfrac{(\vee)\text{ twice}\,\dfrac{(\forall)\,\dfrac{(b\exists)\,\dfrac{a \in b \wedge \neg F(a), a \in b \to F(a)}{(\exists x \in b)\neg F(x), a \in b \to F(a)}}{(\exists x \in b)\neg F(x), \forall x(x \in b \to F(x))}}{(\forall x \in b)F(x) \to \forall x(x \in b \to F(x))} \qquad (\vee)\text{ twice}\,\dfrac{(b\forall)\,\dfrac{(\exists)\,\dfrac{a \in b \wedge \neg F(a), a \in b \to F(a)}{\exists x(x \in b \wedge \neg F(x)), a \in b \to F(a)}}{\exists x(x \in b \wedge \neg F(x)), (\forall x \in b)F(x)}}{\forall x(x \in b \to F(x)) \to (\forall x \in b)F(x)}}{(\forall x \in b)F(x) \leftrightarrow \forall x(x \in b \to F(x))}$$

□

2.3 The infinitary system $\mathbf{RS}_\Omega(X)$

Let X be an arbitrary (well founded) set and let θ be the set-theoretic rank of X (hereby referred to as the $\in$-rank). Henceforth all ordinals are assumed to belong to the ordinal notation system $T(\theta)$ developed in the previous section. The system $\mathbf{RS}_\Omega(X)$ will be an infinitary proof system based on $L_\Omega(X)$; the relativised constructible hierarchy up to Ω.

Definition 2.3.1. We give an inductive definition of the set $\mathcal{T}$ of $\mathbf{RS}_\Omega(X)$ terms, to each term $t \in \mathcal{T}$ we assign an ordinal level $|\,t\,|$

i) For every $u \in TC(\{X\})$, $\bar{u} \in \mathcal{T}$ and $|\,\bar{u}\,| := \Gamma_{\text{rank}(u)}$ [here rank(u) is the $\in$-rank of u and TC denotes the *transitive closure* operator.]

ii) For every $\alpha < \Omega$, $\mathbb{L}_\alpha(X) \in \mathcal{T}$ and $|\,\mathbb{L}_\alpha(X)\,| := \Gamma_{\theta+1} + \alpha$.

iii) If $\alpha < \Omega$, $A(a, b_1, ..., b_n)$ is a formula of **KP** with all free variables displayed and $s_1, ..., s_n$ are terms with levels less than $\Gamma_{\theta+1} + \alpha$ then

$$[x \in \mathbb{L}_\alpha(X) | A(x, s_1, ..., s_n)^{\mathbb{L}_\alpha(X)}]$$

is a term of level $\Gamma_{\theta+1} + \alpha$. Here the superscript $\mathbb{L}_\alpha(X)$ indicates that all unbounded quantifiers occuring in A are replaced by quantifiers bounded by $\mathbb{L}_\alpha(X)$.

The terms of $\mathbf{RS}_\Omega(X)$ are to be viewed as purely formal, syntactic objects. However their names are highly suggestive of the intended interpretation in the relativised constructible hierarchy up to Ω.

Definition 2.3.2. The formulae of $\mathbf{RS}_\Omega(X)$ are of the form $A(s_1, ..., s_n)$, where $A(a_1, ..., a_n)$ is a formula of **KP** with all free variables displayed and $s_1, ..., s_n$ are $\mathbf{RS}_\Omega(X)$ terms.

Formulae of the form $\bar{u} \in \bar{v}$ and $\bar{u} \notin \bar{v}$ will be referred to as *basic*. The properties Δ_0, Σ and Π are inherited from **KP** formulae.

Note that the system $\mathbf{RS}_\Omega(X)$ does not contain free variables

For the remainder of this section we shall refer to $\mathbf{RS}_\Omega(X)$ terms and formulae simply as terms and formulae.

For any formula A we define

$$\begin{aligned} k(A) :=& \{|\,t\,| \mid t \text{ occurs in } A, \text{ subterms included}\} \\ & \cup \{\Omega \mid \text{if } A \text{ contains an unbounded quantifier}\}. \end{aligned}$$

If Γ is a finite set of the $\mathbf{RS}_\Omega(X)$ formulae $A_1, ..., A_n$ then we define

$$k(\Gamma) := k(A_1) \cup ... \cup k(A_n).$$

Abbreviations 2.3.3.

i) For $\mathbf{RS}_\Omega(X)$ terms s and t, the expression $s = t$ will be considered as shorthand for

$$(\forall x \in s)(x \in t) \wedge (\forall x \in t)(x \in s).$$

ii) If $|\,s\,| < |\,t\,|$, $A(s,t)$ is an $\mathbf{RS}_\Omega(X)$ formula and $\diamond$ is a propositional connective we define:

$$s \dot{\in} t \diamond A(s,t) := \begin{cases} s \in t \diamond A(s,t) & \text{if } t \equiv \bar{u} \\ A(s,t) & \text{if } t \equiv \mathbb{L}_\alpha(X) \\ B(s) \diamond A(s,t) & \text{if } t \equiv [x \in \mathbb{L}_\alpha(X) \mid B(x)] \end{cases}$$

Our aim will be to remove cuts from certain $\mathbf{RS}_\Omega(X)$ derivations of Σ sentences. In order to do this we need to express a certain kind of uniformity in infinite derivations. The right tool for expressing this uniformity was developed by Bucholz in [8] and is termed *operator control.*

Definition 2.3.4. Let $\mathcal{P}(\mathbf{ON}) := \{Y :\ Y \text{ is a set of ordinals}\}$. A class function

$$\mathcal{H} : \mathcal{P}(\mathbf{ON}) \to \mathcal{P}(\mathbf{ON})$$

is called an Operator if the following conditions are satisfied for $Y, Y' \in \mathcal{P}(\mathbf{ON})$

(H1) $0 \in \mathcal{H}(Y)$ and $\Gamma_\beta \in \mathcal{H}(Y)$ for any $\beta \leq \theta + 1$.

(H2) If $\alpha =_{\mathrm{NF}} \alpha_1 + ... + \alpha_n$ then $\alpha \in \mathcal{H}(Y)$ iff $\alpha_1, ..., \alpha_n \in \mathcal{H}(Y)$.

(H3) If $\alpha =_{\mathrm{NF}} \varphi\alpha_1\alpha_2$ then $\alpha \in \mathcal{H}(Y)$ iff $\alpha_1, \alpha_2 \in \mathcal{H}(Y)$

(H4) $Y \subseteq \mathcal{H}(Y)$

(H5) $Y' \subseteq \mathcal{H}(Y) \Rightarrow \mathcal{H}(Y') \subseteq \mathcal{H}(Y)$

Note that this definition of operator, as with the infinitary system $\mathbf{RS}_\Omega(X)$ is dependent on the set X and its $\in$-rank θ.

Abbreviations 2.3.5. For an operator $\mathcal{H}$:

i) We write $\alpha \in \mathcal{H}$ instead of $\alpha \in \mathcal{H}(\emptyset)$.

ii) Likewise $Y \subseteq \mathcal{H}$ is shorthand for $Y \subseteq \mathcal{H}(\emptyset)$.

iii) For any $\mathbf{RS}_\Omega(X)$ term t, $\mathcal{H}[t](Y) := \mathcal{H}(Y \cup |\, t\,|)$.

iv) If $\mathfrak{X}$ is an $\mathbf{RS}_\Omega(X)$ formula or set of formulae then $\mathcal{H}[\mathfrak{X}](Y) := \mathcal{H}(Y \cup k(\mathfrak{X}))$.

Lemma 2.3.6. Let $\mathcal{H}$ be an operator s an $\mathbf{RS}_\Omega(X)$ term and $\mathfrak{X}$ an $\mathbf{RS}_\Omega(X)$ formula or set of formulae.

i) If $Y \subseteq Y'$ then $\mathcal{H}(Y) \subseteq \mathcal{H}(Y')$.

ii) $\mathcal{H}[s]$ and $\mathcal{H}[\mathfrak{X}]$ are operators.

iii) If $|\, s\,| \in \mathcal{H}$ then $\mathcal{H}[s] = \mathcal{H}$.

iv) If $k(\mathfrak{X}) \subseteq \mathcal{H}$ then $\mathcal{H}[\mathfrak{X}] = \mathcal{H}$.

Proof. These results are easily checked, they are proved in full in [35]. □

Definition 2.3.7. If $\mathcal{H}$ is an operator, α an ordinal and Γ a finite set of $\mathbf{RS}_\Omega(X)$-formulae, we give an inductive definition of the relation $\mathcal{H} \overset{\alpha}{\vdash} \Gamma$ by recursion on α. (*$\mathcal{H}$-controlled derivability in $\boldsymbol{RS}_\Omega(X)$.*) We require always that

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}$$

this condition will not be repeated in the inductive clauses pertaining to the axioms and inference rules below. We have the following axioms:

$$\begin{array}{lll} \mathcal{H} \overset{\alpha}{\vdash} \theta, \bar{u} \in \bar{v} & \text{if} & u, v \in TC(X) \text{ and } u \in v \\ \mathcal{H} \overset{\alpha}{\vdash} \theta, \bar{u} \notin \bar{v} & \text{if} & u, v \in TC(X) \text{ and } u \notin v. \end{array}$$

The following are the inference rules of $\mathbf{RS}_\Omega(X)$, the column on the right gives the requirements on the ordinals, terms and formulae for each rule.

$$(\wedge) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, A \qquad \mathcal{H} \overset{\alpha_1}{\vdash} \Gamma, B}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, A \wedge B} \qquad \alpha_0, \alpha_1 < \alpha$$

$$(\vee) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, C \quad \text{for some } C \in \{A, B\}}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, A \vee B} \qquad \alpha_0 < \alpha$$

$$(\notin)\quad \frac{\mathcal{H}[s] \stackrel{\alpha_s}{\vdash} \Gamma, s \dot{\in} t \rightarrow r \neq s \quad \text{for all } |\,s\,| < |\,t\,|}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma, r \notin t} \qquad \begin{array}{r} \alpha_s < \alpha \\ r \in t \text{ is not basic} \end{array}$$

$$(\in)\quad \frac{\mathcal{H} \stackrel{\alpha_0}{\vdash} \Gamma, s \dot{\in} t \wedge r = s}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma, r \in t} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,s\,| < |\,t\,| \\ |\,s\,| < \Gamma_{\theta+1} + \alpha \\ r \in t \text{ is not basic} \end{array}$$

$$(b\forall)\quad \frac{\mathcal{H}[s] \stackrel{\alpha_s}{\vdash} \Gamma, s \dot{\in} t \rightarrow A(s) \quad \text{for all } |\,s\,| < |\,t\,|}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma, (\forall x \in t)A(x)} \qquad \alpha_s < \alpha$$

$$(b\exists)\quad \frac{\mathcal{H} \stackrel{\alpha_0}{\vdash} \Gamma, s \dot{\in} t \wedge A(s)}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma, (\exists x \in t)A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,s\,| < |\,t\,| \\ |\,s\,| < \Gamma_{\theta+1} + \alpha \end{array}$$

$$(\forall)\quad \frac{\mathcal{H}[s] \stackrel{\alpha_s}{\vdash} \Gamma, A(s) \quad \text{for all } s}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma, \forall x A(x)} \qquad \alpha_s < \alpha$$

$$(\exists)\quad \frac{\mathcal{H} \stackrel{\alpha_0}{\vdash} \Gamma, A(s)}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma, \exists x A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,s\,| < \Gamma_{\theta+1} + \alpha \end{array}$$

$$(\mathrm{Cut})\quad \frac{\mathcal{H} \stackrel{\alpha_0}{\vdash} \Gamma, A \qquad \mathcal{H} \stackrel{\alpha_0}{\vdash} \Gamma, \neg A}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma} \qquad \alpha_0 < \alpha$$

$$(\Sigma\text{-}\mathrm{Ref}_\Omega(X))\quad \frac{\mathcal{H} \stackrel{\alpha_0}{\vdash} \Gamma, A}{\mathcal{H} \stackrel{\alpha}{\vdash} \Gamma, \exists z A^z} \qquad \begin{array}{r} \alpha_0, \Omega < \alpha \\ A \text{ is a } \Sigma \text{ formula} \end{array}$$

A^z results from A by restricting all unbounded quantifiers in A to z. The reason for the condition preventing the derivation of basic formulas in the rules $(\in)$ and $(\notin)$ is to prevent derivations of sequents which are already axioms, as this would cause a hindrance to cut-elimination. The condition that $|\,s\,| < \Gamma_{\theta+1} + \alpha$ in $(\in)$ and $(\exists)$ inferences will allow us to place bounds on the location of witnesses in derivable Σ formulas.

2.4 Cut elimination for $\mathbf{RS}_\Omega(X)$

We need to keep track of the complexity of cuts appearing in a derivation, to this end we define the *rank* of an $\mathbf{RS}_\Omega(X)$ formula.

Definition 2.4.1. The rank of a term or formula is defined by recursion on the construction as follows:

1. $rk(\bar{u}) := \Gamma_{rank(u)}$

2. $rk(\mathbb{L}_\alpha(X)) := \Gamma_{\theta+1} + \omega \cdot \alpha$

3. $rk([x \in \mathbb{L}_\alpha(X)|F(x)]) := \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, rk(F(\bar{\emptyset})) + 2)$

4. $rk(s \in t) := rk(s \notin t) := \max(rk(t) + 1, rk(s) + 6)$

5. $rk((\exists x \in \bar{u})F(x)) := rk((\forall x \in \bar{u})F(x)) := \max(rk(\bar{u}) + 3, rk(F(\bar{\emptyset})) + 2)$.

6. $rk((\exists x \in t)F(x)) := rk((\forall x \in t)F(x)) := \max(rk(t), rk(F(\bar{\emptyset})) + 2)$ if t is not of the form $\bar{u}$.

7. $rk(\exists x F(x)) := rk(\forall x F(x)) := \max(\Omega, rk(F(\bar{\emptyset})) + 1)$

8. $rk(A \wedge B) := rk(A \vee B) := \max(rk(A), rk(B)) + 1$

$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ will be used to denote that $\mathcal{H} \overset{\alpha}{\vdash} \Gamma$ and all cut formulas appearing in the derivation have rank $< \rho$.

Observation 2.4.2. i) For each term t, $rk(t) = \omega \cdot |\, t\,| + n$ for some $n < \omega$.

ii) For each formula A, $rk(A) = \omega \cdot \max(k(A)) + n$ for some $n < \omega$.

iii) $rk(A) < \Omega$ if and only if A is Δ_0.

The next Lemma shows that the rank of a formula A is determined *only* by $\max(k(A))$ and the logical structure of A.

Lemma 2.4.3. For each formula $A(s)$, if $|\, s\,| < \max(k(A(s)))$ then $rk(A(s)) = rk(A(\bar{\emptyset}))$.

Proof. The proof is by induction on the complexity of A.

Case 1. If $A(s) \equiv s \in t$ then by assumption $|\, s\,| < |\, t\,|$, so $rk(A(s)) = rk(t) + 1 = rk(A(\bar{\emptyset}))$.

Case 2. If $A(s) \equiv t \in s$ we may argue in a similar fashion to Case 1.

Case 3. It cannot be the case that $A(s) \equiv s \in s$.

Case 4. If $A(s) \equiv (\exists y \in \bar{u})B(y, s)$ then

$$rk(A(s)) = \max(rk(\bar{u}) + 3, rk(B(\bar{\emptyset}, s)) + 2)$$

and

$$rk(A(\bar{\emptyset})) = \max(rk(\bar{u}) + 3, rk(B(\bar{\emptyset}, \bar{\emptyset})) + 2).$$

4.1 If $|\,\bar{u}\,| > \max(k(B(\bar{\emptyset},\bar{\emptyset})))$ then $|\,s\,| < |\,\bar{u}\,|$ by assumption, so using observation 2.4.2ii) gives us

$$rk(A(s)) = rk(\bar{u}) + 3 = rk(A(\bar{\emptyset})).$$

4.2 If $|\,\bar{u}\,| \leq \max(k(B(\bar{\emptyset},\bar{\emptyset}))$ then $|\,s\,| < \max(k(B(\bar{\emptyset},\bar{\emptyset})))$ by assumption, so by induction hypothesis

$$rk(B(\bar{\emptyset},s)) = rk(B(\bar{\emptyset},\bar{\emptyset}))$$

and hence using Observation 2.4.2ii) gives us

$$rk(A(s)) = rk(B(\bar{\emptyset},\bar{\emptyset})) + 2 = rk(A(\bar{\emptyset})).$$

Case 5. If $A(s) \equiv (\exists y \in t)B(y,s)$ for some t not of the form $\bar{u}$, we may argue in a similar way to case 4.

Case 6. $A(s) \equiv (\exists y \in s)B(y,s)$, now $|\,s\,| < \max(k(A(\bar{\emptyset}))) = \max(k(B(\bar{\emptyset},\bar{\emptyset})))$, so by induction hypothesis

$$rk(B(\bar{\emptyset},s)) = rk(B(\bar{\emptyset},\bar{\emptyset}))$$

and hence using observation 2.4.2 we see that

$$\begin{aligned} rk(A(s)) &= rk(B(\bar{\emptyset},s)) + 2 \\ &= rk(B(\bar{\emptyset},\bar{\emptyset})) + 2 \\ &= rk(A(\bar{\emptyset})). \end{aligned}$$

Case 7. If $A(s) \equiv \exists x B(x,s)$ then by assumption $|\,s\,| < \max(k(A(s))) = \max(k(B(\emptyset,s)))$ so we may apply the induction hypothesis to see that $rk(A(s)) = \max(\Omega, rk(B(\emptyset,s)) + 1) = \max(\Omega, rk(B(\emptyset,\emptyset)) + 1) = rk(A(\emptyset))$.

Case 8. All other cases are either propositional in which case we may just use induction hypothesis directly or are dual to cases already considered. $\square$

Definition 2.4.4. To each non-basic formula A we assign an infinitary disjunction $(\bigvee A_i)_{i\in y}$ or conjunction $(\bigwedge A_i)_{i\in y}$ as follows:

1. $r \in t :\simeq \bigvee(s \dot{\in} t \wedge r = s)_{|\,s\,|<|\,t\,|}$ provided $r \in t$ is not a basic formula.

2. $(\exists x \in t)B(x) :\simeq \bigvee(s \dot{\in} t \wedge B(s))_{|\,s\,|<|\,t\,|}$

3. $\exists x B(x) :\simeq \bigvee(B(s))_{s\in\mathcal{T}}$

4. $B_0 \vee B_1 :\simeq \bigvee(B_i)_{i\in\{0,1\}}$

5. $\neg B :\simeq \bigwedge(\neg B_i)_{i\in y}$ if B is of the form considered in 1.-4.

The idea is that the infinitary conjunction or disjunction lists the premises required to derive A as the principal formula of an $\mathbf{RS}_\Omega(X)$-inference different from (Σ-Ref$_\Omega(X)$) or (Cut).

Lemma 2.4.5. *If $A \simeq (\bigvee A_i)_{i\in y}$ or $A \simeq (\bigwedge A_i)_{i\in y}$ then*

$$\forall i \in y(rk(A_i) < rk(A))$$

Proof. We need only treat the case where $A \simeq (\bigvee A_i)_{i\in y}$ since the other case is dual to this one. We proceed by induction on the complexity of A.

Case 1. Suppose $A \equiv r \in t$ then by assumption either r or t is not of the form $\bar{u}$, we split cases based on the form of t.

1.1 If $t \equiv \bar{u}$ then r is not of the form $\bar{v}$ and $rk(A) = rk(r) + 6$. In this case $A_i \equiv \bar{v} \in \bar{u} \wedge \bar{v} = r$ for some $|\,\bar{v}\,| < |\,\bar{u}\,|$ and we have

$$\begin{aligned} rk(A_i) &= \max(rk(\bar{v} \in \bar{u}), rk(\bar{v} = r)) + 1 \\ &= rk(\bar{v} = r) + 1 \\ &= \max(rk((\forall x \in \bar{v})(x \in r)), rk((\forall x \in r)(x \in \bar{v}))) + 2 \\ &= rk(r) + 5 < rk(r) + 6 = rk(A) \end{aligned}$$

1.2 If $t \equiv \mathbb{L}_\alpha(X)$ then $A_i \equiv s = r$ for some $|\,s\,| < |\,t\,|$. So we have

$$\begin{aligned} rk(A_i) &= rk((\forall x \in s)(x \in r) \wedge (\forall x \in r)(x \in s)) \\ &= \max(rk(s) + 4, rk(r) + 4) \\ &< \max(rk(r) + 1, rk(t) + 6) = rk(A) \end{aligned}$$

1.3 If $t \equiv [x \in \mathbb{L}_\alpha(X) | B(x)]$ then $A_i \equiv B(s) \wedge s = r$ for some $|\,s\,| < |\,t\,|$. So we have

$$rk(A_i) = \max(rk(B(s)) + 1, rk(r = s) + 1).$$

First note that $rk(r = s) + 1 = \max(rk(s) + 5, rk(r) + 5) < rk(A)$. So it remains to verify that $rk(B(s)) + 1 < rk(A)$, for this it is enough to show that $rk(B(s)) < rk(t)$.

1.3.1 If $\max(k(B(s))) \leq |\,s\,|$ then by Observation 2.4.2ii) we have $rk(B(s)) + 1 < \omega \cdot |\,s\,| + \omega \leq rk(t)$.

1.3.2 Otherwise $\max(k(B(s))) > |\,s\,|$ then by Lemma 2.4.3 we have

$$\begin{aligned} rk(B(s)) + 1 &= rk(B(\bar{\emptyset})) + 1 \\ &< \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, rk(B(\bar{\emptyset})) + 2) = rk(t) \end{aligned}$$

Case 2. Suppose $A \equiv (\exists x \in t)B(x)$, we split into cases based on the form of t.

2.1 If $t \equiv \bar{u}$ then $rk(A) := \max(rk(\bar{u}) + 3, rk(B(\bar{\emptyset})) + 2)$. In this case $A_i \equiv \bar{v} \in \bar{u} \wedge B(\bar{v})$ for some $|\,\bar{v}\,| < |\,\bar{u}\,|$, so we have

$$rk(A_i) = \max(rk(\bar{u}) + 2, rk(B(\bar{v})) + 1).$$

Clearly $rk(\bar{u}) + 2 < rk(\bar{u}) + 3$ so it remains to verify that $rk(B(\bar{v})) + 1 < rk(A)$

2.1.1 If $|\bar{v}| \geq \max(k(B(\bar{v})))$ then by Observation 2.4.2i) $rk(B(\bar{v})) + 1 < rk(\bar{u}) < rk(\bar{u}) + 3$.

2.1.2 If $|\bar{v}| < \max(k(B(\bar{v})))$ then by Lemma 2.4.3 $rk(B(\bar{v})) + 1 = rk(B(\bar{\emptyset})) + 1 < rk(B(\bar{\emptyset})) + 2$.

2.2 Now suppose $t \equiv \mathbb{L}_\alpha(X)$, so $rk(A) = \max(rk(t), rk(B(\bar{\emptyset})) + 2)$. In this case $A_i = B(s)$ for some $|\,s\,| < |\,t\,|$.

2.2.1 If $|\,s\,| \geq \max(k(B(s)))$ then $rk(B(s)) < rk(t)$ by Observation 2.4.2.

2.2.2 If $|\,s\,| < \max(k(B(s)))$ then by Lemma 2.4.3 $rk(B(s)) = rk(B(\bar{\emptyset})) < rk(A)$.

2.3. Now suppose $t \equiv [y \in \mathbb{L}_\alpha(X) \mid C(y)]$, so we have

$$\begin{aligned} rk(A) &:= \max(rk(t), rk(B(\bar{\emptyset})) + 2) \\ &= \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, rk(C(\bar{\emptyset})) + 2, rk(B(\bar{\emptyset})) + 2). \end{aligned}$$

In this case $A_i \equiv C(s) \wedge B(s)$ for some $|\,s\,| < |\,t\,|$.

2.3.1 If $|\,s\,| < \max(k(B(s)))$ then $rk(B(s)) + 1 = rk(B(\bar{\emptyset})) + 1 < rk(B(\bar{\emptyset})) + 2$. It remains to show that $rk(C(s)) < rk(A)$.

2.3.1.1 If $\max(k(C(s))) < |\,t\,|$ then $rk(C(s)) + 1 < rk(t)$ by Observation 2.4.2.

2.3.1.2 Now if $\max(k(C(s))) \geq |\,t\,|$ then we may apply Lemma 2.4.3 to give

$$rk(C(s)) + 1 = rk(C(\bar{\emptyset})) + 1 < rk(C(\bar{\emptyset})) + 2 \leq rk(A).$$

2.3.2 If $|\,s\,| \geq \max(k(B(s)))$ then $rk(B(s)) < \Gamma_{\theta+1} + \omega \cdot \alpha$ by Observation 2.4.2. Now we may apply the same argument as in 2.3.1.1 and 2.3.1.2 to yield $rk(C(s)) + 1 < rk(A)$.

Case 3. If $A \equiv \exists x B(x)$ then $rk(A) := \max(\Omega, rk(B(\bar{\emptyset})) + 1)$. In this case $A_i \equiv B(s)$ for some term s.

3.1 If B contains an unbounded quantifier then by Lemma 2.4.3 $rk(B(s)) = rk(B(\bar{\emptyset})) < rk(A)$.

3.2 If B does not contain an unbounded quantifier then $rk(B(s)) < \Omega$ by Observation 2.4.2iii)

Case 4. If $A \equiv B \vee C$ then the result is clear immediately from the definition of $rk(A)$. $\square$

Lemma 2.4.6. **i)** If $\alpha \leq \alpha' \in \mathcal{H}$, $\rho \leq \rho'$, $k(\Gamma') \subseteq \mathcal{H}$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ then $\mathcal{H} \overset{\alpha'}{\underset{\rho'}{\vdash}} \Gamma, \Gamma'$.

ii) If C is a basic formula which holds true in the set X and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \neg C$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$.

iii) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \vee B$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A, B$.

iv) If $A \simeq \bigwedge(A_i)_{i \in y}$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A$ then $(\forall i \in y)\, \mathcal{H}[i] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A_i$.

v) If $\gamma \in \mathcal{H}$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \forall x F(x)$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\forall x \in \mathbb{L}_\gamma(X)) F(x)$.

Proof. All proofs are by induction on α.

i) If Γ is an axiom then Γ, Γ' is also an axiom, and since $\{\alpha'\} \cup k(\Gamma') \subseteq \mathcal{H}$ there is nothing to show.

Now suppose Γ is the result of an inference

$$(\mathrm{I})\ \frac{\ldots \mathcal{H}_i \overset{\alpha_i}{\underset{\rho}{\vdash}} \Gamma_i \ldots}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma}\ (i \in y) \quad \alpha_i < \alpha$$

Using the induction hypothesis we have

$$\ldots \mathcal{H}_i \overset{\alpha_i}{\underset{\rho'}{\vdash}} \Gamma_i, \Gamma' \ldots \quad (i \in y) \quad \alpha_i < \alpha$$

It's worth noting that $k(\Gamma') \subseteq \mathcal{H}_i$, since $\mathcal{H}_i(\emptyset) \supseteq \mathcal{H}(\emptyset)$, this can be observed by looking at each inference rule.

Finally we may apply the inference (I) again to obtain

$$\mathcal{H} \overset{\alpha'}{\underset{\rho'}{\vdash}} \Gamma, \Gamma'$$

as required.

ii) If $\Gamma, \neg C$ is an axiom then so is Γ so there is nothing to show.

Now suppose $\Gamma, \neg C$ was derived as the result of an inference rule (I), then $\neg C$ cannot have been the principal formula since it is basic so we have the premise(s)

$$\mathcal{H}_i \overset{\alpha_i}{\underset{\rho}{\vdash}} \Gamma_i, \neg C \quad \alpha_i < \alpha.$$

Now by induction hypothesis we obtain

$$\mathcal{H}_i \mathrel{\substack{\alpha_i \\ \vdash \\ \rho}} \Gamma_i \quad \alpha_i < \alpha$$

to which we may apply the inference rule (I) to complete the proof.

iii) If $\Gamma, A \vee B$ is an axiom then Γ, A, B is also an axiom. If $A \vee B$ was not the principal formula of the last inference then we can apply the induction hypothesis to its premises and then the same inference again.

Now suppose that $A \vee B$ was the principal formula of the last inference. So we have

$$\mathcal{H} \mathrel{\substack{\alpha_0 \\ \vdash \\ \rho}} \Gamma, C \quad \text{or} \quad \mathcal{H} \mathrel{\substack{\alpha_0 \\ \vdash \\ \rho}} \Gamma, C, A \vee B \quad \text{where } C \in \{A, B\} \text{ and } \alpha_0 < \alpha$$

By i) we may assume that we are in the latter case. By the induction hypothesis, and a contraction, we obtain

$$\mathcal{H} \mathrel{\substack{\alpha_0 \\ \vdash \\ \rho}} \Gamma, A, B$$

Finally using i) yields

$$\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma, A, B\,.$$

iv) If Γ, A is an axiom, then Γ is also an axiom since A cannot be the *active part* of an axiom, so Γ, A_i is an axiom for any $i \in y$. If A was not the principal formula of the last inference then we may apply the induction hypothesis to its premises and then use that inference again.

Now suppose A was the principal formula of the last inference. With the possible use of part i), we may assume we are in the following situation:

$$\mathcal{H}[i] \mathrel{\substack{\alpha_i \\ \vdash \\ \rho}} \Gamma, A, A_i \quad (\forall i \in y) \quad \alpha_i < \alpha.$$

Inductively and via a contraction we obtain

$$\mathcal{H}[i] \mathrel{\substack{\alpha_i \\ \vdash \\ \rho}} \Gamma, A_i\,.$$

Here it is important to note that $\mathcal{H}[i][i] \equiv \mathcal{H}[i]$. To which we may apply part i) to obtain

$$\mathcal{H}[i] \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma, A_i$$

as required.

v) The interesting case is where $\forall x F(x)$ was the principal formula of the last inference. In this case we may assume we are in the following situation:

(1) $$\mathcal{H}[s] \mathrel{\substack{\alpha_s \\ \vdash \\ \rho}} \Gamma, \forall x F(x), F(s) \quad \text{for all terms } s, \text{ with } \alpha_s < \alpha.$$

Using the induction hypothesis yields

$$\mathcal{H}[s] \,|\!\frac{\alpha_s}{\rho}\, \Gamma, (\forall x \in \mathbb{L}_\gamma(X))F(x), F(s) \tag{2}$$

Note that for $|\,s\,| < \Gamma_{\theta+1} + \gamma$ we have $s \,\dot{\in}\, \mathbb{L}_\gamma(X) \to F(s) \equiv F(s)$. So as a subset of (2) we have

$$\mathcal{H}[s] \,|\!\frac{\alpha_s}{\rho}\, \Gamma, (\forall x \in \mathbb{L}_\gamma(X))F(x), s \,\dot{\in}\, \mathbb{L}_\gamma(X) \to F(s) \quad \text{for all } |\,s\,| < \Gamma_{\theta+1} + \gamma, \text{ with } \alpha_s < \alpha.$$

From which one application of $(b\forall)$ gives us the desired result. □

Lemma 2.4.7 (Reduction for $\mathbf{RS}_\Omega(X)$)**.** Suppose $C \equiv \bar{u} \in \bar{v}$ or $C \simeq \bigvee(C_i)_{i\in y}$ and $rk(C) := \rho \neq \Omega$.

$$\text{If} \quad [\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Lambda, \neg C \quad \& \quad \mathcal{H} \,|\!\frac{\beta}{\rho}\, \Gamma, C\,] \quad \text{then} \quad \mathcal{H} \,|\!\frac{\alpha+\beta}{\rho}\, \Lambda, \Gamma$$

Proof. If $C \equiv \bar{u} \in \bar{v}$ then by 2.4.6ii) we have either $\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Lambda$ or $\mathcal{H} \,|\!\frac{\beta}{\rho}\, \Gamma$. Hence using 2.4.6i) we obtain $\mathcal{H} \,|\!\frac{\alpha+\beta}{\rho}\, \Lambda, \Gamma$ as required.

Now suppose $C \simeq \bigvee(C_i)_{i\in y}$, we proceed by induction on β. We have

$$\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Lambda, \neg C \tag{1}$$

$$\mathcal{H} \,|\!\frac{\beta}{\rho}\, \Gamma, C\,. \tag{2}$$

If C was not the principal formula of the last inference in (2), then we may apply the induction hypothesis to the premises of that inference and then the same inference again. Now suppose C was the principal formula of the last inference in (2). If B was the principal formula of the inference (Σ-Ref$_\Omega(X)$), then B is of the form $\exists z F(s_1, ..., s_n)^z$, which implies $rk(B) = \Omega$, therefore the last inference in (2) was not (Σ-Ref$_\Omega(X)$). So we have

$$\mathcal{H} \,|\!\frac{\beta_0}{\rho}\, \Gamma, C, C_{i_0} \quad \text{for some } i_0 \in y,\ \beta_0 < \beta \text{ with } |\,i_0\,| < \Gamma_{\theta+1} + \beta. \tag{3}$$

The induction hypothesis applied to (2) and (3) yields

$$\mathcal{H} \,|\!\frac{\alpha+\beta_0}{\rho}\, \Lambda, \Gamma, C_{i_0}\,. \tag{4}$$

Now applying Lemma 2.4.6iv) to (1) provides

$$\mathcal{H}[i_0] \,|\!\frac{\alpha}{\rho}\, \Lambda, \neg C_{i_0}\,. \tag{5}$$

But $|\,i_0\,| \in \mathcal{H}$ by (4), which means $\mathcal{H}[i_0] = \mathcal{H}$ by Lemma 2.3.6iv), so in fact we have

$$\mathcal{H} \,|\!\frac{\alpha}{\rho}\, \Lambda, \neg C_{i_0}\,. \tag{6}$$

Thus we may apply (Cut) to (4) and (6) (noting that $rk(C_{i_0}) < rk(C) := \rho$ by Lemma 2.4.5) to obtain

$$\mathcal{H} \,|\!\frac{\alpha+\beta}{\rho}\, \Lambda, \Gamma$$

as required. □

Theorem 2.4.8 (Predicative cut elimination for $\mathbf{RS}_\Omega(X)$).
If $\mathcal{H} \overset{\beta}{\underset{\rho+\omega^\alpha}{\vdash}} \Gamma$ and $\Omega \notin [\rho, \rho+\omega^\alpha)$ then $\mathcal{H} \overset{\varphi\alpha\beta}{\underset{\rho}{\vdash}} \Gamma$

Proof. The proof is by main induction on α and subsidiary induction on β. If Γ is an axiom then the result is immediate. If the last inference was anything other that (Cut) we may apply the subsidiary induction hypothesis to its premises and then the same inference again. The crucial case is where the last inference was (Cut), so suppose there is a formula C with $rk(C) < \rho+\omega^\alpha$ such that

(1) $$\mathcal{H} \overset{\beta_0}{\underset{\rho+\omega^\alpha}{\vdash}} \Gamma, C \quad \text{with } \beta_0 < \beta.$$

(2) $$\mathcal{H} \overset{\beta_0}{\underset{\rho+\omega^\alpha}{\vdash}} \Gamma, \neg C \quad \text{with } \beta_0 < \beta.$$

Applying the subsidiary induction hypothesis to (1) and (2) yields

(3) $$\mathcal{H} \overset{\varphi\alpha\beta_0}{\underset{\rho}{\vdash}} \Gamma, C\,.$$

(4) $$\mathcal{H} \overset{\varphi\alpha\beta_0}{\underset{\rho}{\vdash}} \Gamma, \neg C\,.$$

Case 1. If $rk(C) < \rho$ then we may apply (Cut) to (3) and (4), noting that $\varphi\alpha\beta_0+1 < \varphi\alpha\beta \in \mathcal{H}$, to give the desired result.

Case 2. Now suppose $rk(C) \in [\rho, \rho+\omega^\alpha)$, so we may write $rk(C)$ in the following form:

(5) $$rk(C) = \rho + \omega^{\alpha_1} + ... + \omega^{\alpha_n} \quad \text{with } \alpha > \alpha_1 \geq ... \geq \alpha_n.$$

Here $n = 0$ indicates that $rk(C) = \rho$. From (3) we know that $k(C) \subseteq \mathcal{H}$ and thus $rk(C) \in \mathcal{H}$. Now (5) and (H2) and (H3) from Definition 2.3.4 give us $\alpha_1, ..., \alpha_n \in \mathcal{H}$. Since $rk(C) \neq \Omega$ we may apply the Reduction Lemma 2.4.7 to (3) and (4) to obtain

(6) $$\mathcal{H} \overset{\varphi\alpha\beta_0+\varphi\alpha\beta_0}{\underset{\rho+\omega^{\alpha_1}+...+\omega^{\alpha_n}}{\vdash}} \Gamma\,.$$

Now $\varphi\alpha\beta_0 + \varphi\alpha\beta_0 < \varphi\alpha\beta$, so by Lemma 2.4.6i) we have

(7) $$\mathcal{H} \overset{\varphi\alpha\beta}{\underset{\rho+\omega^{\alpha_1}+...+\omega^{\alpha_n}}{\vdash}} \Gamma\,.$$

Applying the main induction hypothesis (since $\alpha_n < \alpha$) to (7) gives

$$\mathcal{H} \overset{\varphi\alpha_n(\varphi\alpha\beta)}{\underset{\rho+\omega^{\alpha_1}+...+\omega^{\alpha_{n-1}}}{\vdash}} \Gamma\,.$$

But since $\varphi\alpha\beta$ is a fixed point of the function $\varphi\alpha_n(\cdot)$ we have

$$\mathcal{H} \overset{\varphi\alpha\beta}{\underset{\rho+\omega^{\alpha_1}+...+\omega^{\alpha_{n-1}}}{\vdash}} \Gamma\,.$$

Now since $\alpha_1, ..., \alpha_{n-1} < \alpha$ we may repeat this application of the main induction hypothesis a further $n-1$ times to obtain

$$\mathcal{H} \overset{\varphi\alpha\beta}{\underset{\rho}{\vdash}} \Gamma$$

as required. $\square$

Lemma 2.4.9 (Boundedness for $\mathbf{RS}_\Omega(X)$)**.** If C is a Σ formula, $\alpha \leq \beta < \Omega$, $\beta \in \mathcal{H}$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C^{\mathbb{L}_\beta(X)}$.

Proof. The proof is by induction on α. If C is basic then $C \equiv C^{\mathbb{L}_\beta(X)}$ so there is nothing to show. If C was not the principal formula of the last inference then we may apply the induction hypothesis to its premises and then the same inference again. Now suppose C was the principal formula of the last inference. The last inference cannot have been (Σ-Ref$_\Omega(X)$) since $\alpha < \Omega$.

Case 1. Suppose $C \simeq \bigwedge(C_i)_{i\in y}$ and $\mathcal{H}[i] \overset{\alpha_i}{\underset{\rho}{\vdash}} \Gamma, C, C_i$ with $\alpha_i < \alpha$. Since C is a Σ formula, there must be some $\eta \in \mathcal{H}(\emptyset) \cap \Omega$ such that $(\forall s \in y)(|\,s\,| < \eta)$. Therefore $C^{\mathbb{L}_\beta(X)} \simeq \bigwedge(C_i^{\mathbb{L}_\beta(X)})_{i\in y}$. Now two applications of the induction hypothesis gives

$$\mathcal{H}[i] \overset{\alpha_i}{\underset{\rho}{\vdash}} \Gamma, C^{\mathbb{L}_\beta(X)}, C_i^{\mathbb{L}_\beta(X)}$$

to which we may apply the appropriate inference to gain the desired result.

Case 2. Now suppose $C \simeq \bigvee(C_i)_{i\in y}$ and $\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, C, C_{i_0}$, with $i_0 \in y$, $|\,i_0\,| < \Gamma_{\theta+1} + \alpha$ and $\alpha_0 < \alpha$. In this case $C^{\mathbb{L}_\beta(X)} \simeq \bigvee(C_i)_{i\in y'}$ where either $y' = y$ or $y' = \{i \in y \mid |\,i\,| < \Gamma_{\theta+1} + \beta\}$. Now by assumption $|\,i_0\,| < \Gamma_{\theta+1} + \alpha < \Gamma_{\theta+1} + \beta$, so $i_0 \in y'$. Thus using the same inference again, or $(b\exists)$ in the case that the last inference was $(\exists)$, we obtain

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C^{\mathbb{L}_\beta(X)}$$

as required. $\square$

Definition 2.4.10. For each $\eta \in T(\theta)$ we define

$$\mathcal{H}_\eta : \mathcal{P}(\mathbf{ON}) \mapsto \mathcal{P}(\mathbf{ON})$$
$$\mathcal{H}_\eta(Y) := \bigcap\{B(\alpha) \mid Y \subseteq B(\alpha) \text{ and } \eta < \alpha\}$$

Lemma 2.4.11. For any η, $\mathcal{H}_\eta$ is an operator.

Proof. We must verify the conditions (H1) - (H5) from Definition 2.3.4.

(H1) Clearly $0 \in \mathcal{H}_\eta(Y)$ and $\{\Gamma_\beta \mid \beta \leq \theta\} \subseteq \mathcal{H}_\eta(Y)$ since these belong in any of the sets $B(\alpha)$. It remains to note that $\mathcal{H}_\eta(Y) \supseteq B(1)$ and since $\Gamma_{\theta+1} = \psi 0 \in B(1)$ we have $\Gamma_{\theta+1} \in \mathcal{H}_\eta(Y)$.

(H2) and (H3) follow immediately from Lemma 2.1.8i) and ii) respectively.

(H4) is clear from the definition. Now for (H5) suppose $Y' \subseteq \mathcal{H}_\eta(Y)$, then $Y' \subseteq B(\alpha)$ for every α such that $\eta < \alpha$ and $Y \subseteq B(\alpha)$. It follows that $\mathcal{H}_\eta(Y') \subseteq \mathcal{H}_\eta(Y)$. $\square$

Lemma 2.4.12. **i)** $\mathcal{H}_\eta(Y)$ is closed under φ and $\psi|_{\eta+1}$.

ii) If $\delta < \eta$ then $\mathcal{H}_\delta(Y) \subseteq \mathcal{H}_\eta(Y)$

iii) If $\delta < \eta$ and $\mathcal{H}_\delta \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ then $\mathcal{H}_\eta \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$

Proof. i) Note that for any X, $\mathcal{H}_\eta(X) = B(\alpha)$ for some $\alpha \geq \eta + 1$.

ii) follows immediately from the definition of $\mathcal{H}_\eta$ and iii) follows easily from ii). $\square$

Lemma 2.4.13. Suppose $\eta \in B(\eta)$ and for any ordinal β let $\hat{\beta} := \eta + \omega^{\Omega+\beta}$.

i) If $\alpha \in \mathcal{H}_\eta$ then $\hat{\alpha}, \psi\hat{\alpha} \in \mathcal{H}_{\hat{\alpha}}$

ii) If $\alpha_0 \in \mathcal{H}_\eta$ and $\alpha_0 < \alpha$ then $\psi\hat{\alpha_0} < \psi\hat{\alpha}$

Proof. i) First note that $\mathcal{H}_\eta(\emptyset) = B(\eta+1)$. Now from $\alpha, \eta \in B(\eta+1)$ we get $\hat{\alpha} \in B(\eta+1)$ and thus $\hat{\alpha} \in B(\hat{\alpha})$. It follows that $\psi\hat{\alpha} \in B(\hat{\alpha}+1) = \mathcal{H}_{\hat{\alpha}}(\emptyset)$.

ii) Suppose that $\alpha_0 \in \mathcal{H}_\eta$ and $\alpha_0 < \alpha$, using the preceding argument we get that $\psi\hat{\alpha_0} \in B(\hat{\alpha_0}+1) \subseteq B(\hat{\alpha})$, thus $\psi\hat{\alpha_0} < \psi\hat{\alpha}$. $\square$

Theorem 2.4.14 (Collapsing for $\mathbf{RS}_\Omega(X)$)**.** Suppose Γ is a set of Σ formulae and $\eta \in B(\eta)$.

$$\text{If} \quad \mathcal{H}_\eta \overset{\alpha}{\underset{\Omega+1}{\vdash}} \Gamma \quad \text{then} \quad \mathcal{H}_{\hat{\alpha}} \overset{\psi\hat{\alpha}}{\underset{\psi\hat{\alpha}}{\vdash}} \Gamma$$

Proof. We proceed by induction on α. First note that from $\alpha \in \mathcal{H}_\eta$ we get $\hat{\alpha}, \psi\hat{\alpha} \in \mathcal{H}_{\hat{\alpha}}$ from Lemma 2.4.13i).

If Γ is an axiom then the result follows by Lemma 2.4.6i). So suppose Γ arose as the result of an inference, we shall distinguish cases according to the last inference of $\mathcal{H}_\eta \overset{\alpha}{\underset{\Omega+1}{\vdash}} \Gamma$.

Case 1. Suppose $A \simeq \bigwedge(A_i)_{i\in y} \in \Gamma$ and $\mathcal{H}_\eta[i] \overset{\alpha_i}{\underset{\Omega+1}{\vdash}} \Gamma, A_i$ with $\alpha_i < \alpha$ for each $i \in y$. Since A is a Σ formula, we must have $\sup\{|i| \mid i \in y\} < \Omega$, therefore as $k(A) \subseteq \mathcal{H}_\eta = B(\eta+1)$ we must have $\sup\{|i| \mid i \in y\} < \psi(\eta+1)$. It follows that for any $i \in y$ $|i| \in \mathcal{H}_\eta$ and thus $\mathcal{H}_\eta[i] = \mathcal{H}_\eta$. This means that we may use the induction hypothesis to give

$$\mathcal{H}_{\hat{\alpha}_i} \overset{\psi\hat{\alpha}_i}{\underset{\psi\hat{\alpha}_i}{\vdash}} \Gamma, A_i \quad \text{for all } i \in y.$$

Now applying Lemma 2.4.12ii) we get

$$\mathcal{H}_{\hat{\alpha}} \overset{\psi\hat{\alpha}_i}{\underset{\psi\hat{\alpha}_i}{\vdash}} \Gamma, A_i \quad \text{for all } i \in y.$$

Upon noting that $\psi\hat{\alpha}_i < \psi\hat{\alpha}$ by 2.4.13ii) we may apply the appropriate inference to obtain

$$\mathcal{H}_{\hat{\alpha}} \mid\frac{\psi\hat{\alpha}}{\psi\hat{\alpha}}\, \Gamma\,.$$

Case 2. Now suppose that $A \simeq \bigvee(A_i)_{i\in y} \in \Gamma$ and $\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1}\, \Gamma, A_{i_0}$ with $i_0 \in y$, $|i_0| \in \mathcal{H}_\eta$ and $\alpha_0 < \alpha$. We may immediately apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha}} \mid\frac{\psi\hat{\alpha}_0}{\psi\hat{\alpha}_0}\, \Gamma, A_{i_0}$$

Now we want to be able to apply the appropriate inference to derive Γ but first we must check that $|\, i_0\,| < \Gamma_{\theta+1} + \psi\hat{\alpha}$. Since $|\, i_0\,| \in \mathcal{H}_\eta = B(\eta+1)$ we have

$$|\, i_0\,| < \psi(\eta+1) < \psi\hat{\alpha} \leq \Gamma_{\theta+1} + \psi\hat{\alpha}.$$

Therefore we may apply the appropriate inference to yield

$$\mathcal{H}_{\hat{\alpha}} \mid\frac{\psi\hat{\alpha}}{\psi\hat{\alpha}}\, \Gamma\,.$$

Case 3. Now suppose the last inference was (Σ-Ref$_\Omega(X)$) so we have $\exists z F^z \in \Gamma$ and $\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1}\, \Gamma, F$ with $\alpha_0 < \alpha$ and F a Σ formula. Applying the induction hypothesis we have

$$\mathcal{H}_{\hat{\alpha}} \mid\frac{\psi\hat{\alpha}_0}{\psi\hat{\alpha}_0}\, \Gamma, F\,.$$

Applying Boundedness 2.4.9 we obtain

$$\mathcal{H}_{\hat{\alpha}} \mid\frac{\psi\hat{\alpha}_0}{\psi\hat{\alpha}_0}\, \Gamma, F^{\mathbb{L}_{\psi\hat{\alpha}_0}(X)}\,.$$

Now by Lemma 2.4.13 $|\,\mathbb{L}_{\psi\hat{\alpha}_0}(X)\,| = \Gamma_{\theta+1} + \psi\hat{\alpha}_0 < \Gamma_{\theta+1} + \psi\hat{\alpha}$, so we may apply ($\exists$) to obtain

$$\mathcal{H}_{\hat{\alpha}} \mid\frac{\psi\hat{\alpha}}{\psi\hat{\alpha}}\, \Gamma, \exists z F^z$$

as required.

Case 4. Finally suppose the last inference was (Cut), so for some A with $rk(A) \leq \Omega$ we have

(1) $$\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1}\, \Gamma, A \quad \text{with } \alpha_0 < \alpha.$$

(2) $$\mathcal{H}_\eta \mid\frac{\alpha_0}{\Omega+1}\, \Gamma, \neg A \quad \text{with } \alpha_0 < \alpha.$$

4.1 If $rk(A) < \Omega$ then A is Δ_0. In this case both A and $\neg A$ are Σ formulae so we may immediately apply the induction hypothesis to both (1) and (2) giving

(3) $$\mathcal{H}_{\hat{\alpha}_0} \mid\frac{\psi\hat{\alpha}_0}{\psi\hat{\alpha}_0}\, \Gamma, A$$

(4) $$\mathcal{H}_{\hat{\alpha}_0} \mid\frac{\psi\hat{\alpha}_0}{\psi\hat{\alpha}_0}\, \Gamma, \neg A\,.$$

Since $k(A) \subseteq \mathcal{H}_\eta(\emptyset) = B(\eta+1)$ and A is Δ_0 it follows from Observation 2.4.2 that $rk(A) \in B(\eta+1) \cap \Omega$. Thus $rk(A) < \psi(\eta+1) < \psi\hat{\alpha}$, so we may apply (Cut) to complete this case.

4.2 Finally suppose $rk(A) = \Omega$. Without loss of generality we may assume that $A \equiv \exists z F(z)$ with F a Δ_0 formula. We may immediately apply the induction hypothesis to (1) giving

(5) $$\mathcal{H}_{\hat{\alpha_0}} \,\Big|\!\frac{\psi\hat{\alpha_0}}{\psi\hat{\alpha_0}}\, \Gamma, A\,.$$

Applying Boundedness 2.4.9 to (5) yields

(6) $$\mathcal{H}_{\hat{\alpha_0}} \,\Big|\!\frac{\psi\hat{\alpha_0}}{\psi\hat{\alpha_0}}\, \Gamma, A^{\mathbb{L}_{\psi\hat{\alpha_0}}(X)}\,.$$

Now using Lemma 2.4.6v) on (2) yields

(7) $$\mathcal{H}_{\hat{\alpha_0}} \,\Big|\!\frac{\alpha_0}{\Omega+1}\, \Gamma, \neg A^{\mathbb{L}_{\psi\hat{\alpha_0}}(X)}\,.$$

Observe that since $\eta, \alpha_0 \in \mathcal{H}_\eta$ we have $\hat{\alpha_0} \in B(\eta+1) \subseteq B(\hat{\alpha_0})$. So since $\Gamma, \neg A^{\mathbb{L}_{\psi\hat{\alpha_0}}(X)}$ is a set of Σ-formulae we may apply the induction hypothesis to (7) giving

(8) $$\mathcal{H}_{\alpha_1} \,\Big|\!\frac{\psi\alpha_1}{\psi\alpha_1}\, \Gamma, \neg A^{\mathbb{L}_{\psi\hat{\alpha_0}}} \quad \text{where } \alpha_1 := \hat{\alpha_0} + \omega^{\Omega+\alpha_0}.$$

Now

$$\alpha_1 = \hat{\alpha_0} + \omega^{\Omega+\alpha_0} = \eta + \omega^{\Omega+\alpha_0} + \omega^{\Omega+\alpha_0} < \eta + \omega^{\Omega+\alpha} := \hat{\alpha}.$$

Owing to Lemma 2.4.13ii) we have $\psi\hat{\alpha_0}, \psi\alpha_1 < \psi\hat{\alpha}$, thus we may apply (Cut) to (6) and (8) giving

$$\mathcal{H}_{\hat{\alpha}} \,\Big|\!\frac{\psi\hat{\alpha}}{\psi\hat{\alpha}}\, \Gamma$$

as required. □

2.5 Embedding KP into $\mathbf{RS}_\Omega(X)$

Definition 2.5.1. i) Given ordinals $\alpha_1, ..., \alpha_n$. The expression $\omega^{\alpha_1}\#...\#\omega^{\alpha_n}$ denotes the ordinal $\omega^{\alpha_{p(1)}} + ... + \omega^{\alpha_{p(n)}}$, where $p : \{1, ..., n\} \mapsto \{1, ..., n\}$ such that $\alpha_{p(1)} \geq ... \geq \alpha_{p(n)}$. More generally $\alpha\#0 := 0\#\alpha := 0$ and $\alpha\#\beta := \omega^{\alpha_1}\#...\#\omega^{\alpha_n}\#\omega^{\beta_1}\#...\#\omega^{\beta_m}$ for $\alpha =_{NF} \omega^{\alpha_1} + ... + \omega^{\alpha_n}$ and $\beta =_{NF} \omega^{\beta_1} + ... + \omega^{\beta_m}$.

ii) If A is any $\mathbf{RS}_\Omega(X)$-formula then $no(A) := \omega^{rk(A)}$.

iii) If $\Gamma = \{A_1, ..., A_n\}$ is a set of $\mathbf{RS}_\Omega(X)$-formulae then $no(\Gamma) := no(A_1)\#...\#no(A_n)$.

iv) $\Vdash \Gamma$ will be used to abbreviate that

$$\mathcal{H}[\Gamma] \,\Big|\!\frac{no(\Gamma)}{0}\, \Gamma \quad \text{holds for any operator } \mathcal{H}$$

v) $\Vdash^\alpha_\rho \Gamma$ will be used to abbreviate that

$$\mathcal{H}[\Gamma] \mid\!\frac{no(\Gamma)\#\alpha}{\rho}\, \Gamma \quad \text{holds for any operator } \mathcal{H}$$

As might be expected $\Vdash^\alpha \Gamma$ and $\Vdash_\rho \Gamma$ stand for $\Vdash^\alpha_0 \Gamma$ and $\Vdash^0_\rho \Gamma$ respectively.

The following lemma shows that under certain conditions we may use $\Vdash$ as a calculus.

Lemma 2.5.2. i) If Γ follows from premises Γ_i by an $\mathbf{RS}_\Omega(X)$ inference other than (Cut) or (Σ-$\text{Ref}_\Omega(X)$) and without contractions then

$$\text{if } \Vdash^\alpha_\rho \Gamma_i \quad \text{then} \quad \Vdash^\alpha_\rho \Gamma$$

ii) If $\Vdash^\alpha_\rho \Gamma, A, B$ then $\Vdash^\alpha_\rho \Gamma, A \vee B$.

Proof. Part i) follows from Lemma 2.4.5. It also needs to be noted that if the last inference was universal with premises $\{\Gamma_i\}_{i\in Y}$, then $\mathcal{H}[\Gamma_i] \subseteq \mathcal{H}[i]$.

For part ii) suppose $\Vdash^\alpha_\rho \Gamma, A, B$, so we have

$$\mathcal{H}[\Gamma] \mid\!\frac{no(\Gamma,A,B)\#\alpha}{\rho}\, \Gamma, A, B\,.$$

Two applications of ($\vee$) and a contraction yields

$$\mathcal{H}[\Gamma] \mid\!\frac{no(\Gamma,A,B)\#\alpha+2}{\rho}\, \Gamma, A \vee B\,.$$

It remains to note that since $\omega^{rk(A\vee B)}$ is additive principal, Lemma 2.4.5 gives us

$$no(\Gamma, A, B)\#\alpha + 2 = no(\Gamma)\#\alpha\#\omega^{rk(A)}\#\omega^{rk(B)} + 2 < no(\Gamma)\#\alpha\#\omega^{rk(A\vee B)} = no(\Gamma, A \vee B)\#\alpha.$$

So we may complete the proof with an application of Lemma 2.4.6i). □

Lemma 2.5.3. Let A be an $\mathbf{RS}_\Omega(X)$ formula and s, t be $\mathbf{RS}_\Omega(X)$ terms.

i) $\Vdash A, \neg A$

ii) $\Vdash s \notin s$

iii) $\Vdash s \subseteq s$ where $s \subseteq s :\equiv (\forall x \in s)(x \in s)$

iv) If $|\,s\,| < |\,t\,|$ then $\Vdash s \mathrel{\dot\in} t \to s \in t$ and $\Vdash \neg(s \mathrel{\dot\in} t), s \in t$

v) $\Vdash s \neq t, t = s$

vi) If $|\,s\,| < |\,t\,|$ and $\Vdash \Gamma, A, B$ then $\Vdash \Gamma, s \mathrel{\dot\in} t \to A, s \mathrel{\dot\in} t \wedge B$

vii) If $|\,s\,| < \Gamma_{\theta+1} + \alpha$ then $\Vdash s \in \mathbb{L}_\alpha(X)$

Proof. i) We use induction of $rk(A)$, and split into cases based upon the form of A.:

Case 1. Suppose $A \equiv \bar{u} \in \bar{v}$. In this case either A or $\neg A$ is an axiom so there is nothing to show.

Case 2. Suppose $A \equiv r \in t$ where $\max(|r|, |t|) \geq \Gamma_{\theta+1}$. By Lemma 2.4.5 and the induction hypothesis we have $\Vdash s \dot{\in} t \wedge r = s, s \dot{\in} t \rightarrow r \neq s$ for all $|s| < |t|$. Thus we have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$\cfrac{\cfrac{\Vdash s \dot{\in} t \wedge r = s, s \dot{\in} t \rightarrow r \neq s}{\Vdash r \in t, s \dot{\in} t \rightarrow r \neq s}\,(\in)}{\Vdash r \in t, r \notin t}\,(\notin)$$

Case 3. Suppose $A \equiv (\exists x \in t)F(x)$. By Lemma 2.4.5 and the induction hypothesis we have $\Vdash s \dot{\in} t \wedge F(s), s \dot{\in} t \rightarrow \neg F(s)$ for all $|s| < |t|$.We have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$\cfrac{\cfrac{\Vdash s \dot{\in} t \wedge F(s), s \dot{\in} t \rightarrow \neg F(s) \quad \text{for all } |s| < |t|}{\Vdash (\exists x \in t)F(x), s \dot{\in} t \rightarrow \neg F(s)}\,(b\exists)}{\Vdash (\exists x \in t)F(x), (\forall x \in t)\neg F(x)}\,(b\forall)$$

Case 4. $A \equiv A_0 \vee A_1$. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$\cfrac{\cfrac{\Vdash A_0, \neg A_0}{\Vdash A_0 \vee A_1, \neg A_0}\,(\vee) \qquad \cfrac{\Vdash A_1, \neg A_1}{\Vdash A_0 \vee A_1, \neg A_1}\,(\vee)}{\Vdash A_0 \vee A_1, \neg A_0 \wedge \neg A_1}\,(\wedge)$$

All other cases may be seen as variations of those above.

ii) We proceed by induction on $rk(s)$. If s is of the form $\bar{u}$ then $s \notin s$ is already an axiom. Inductively we have $\Vdash r \notin r$ for all $|r| < |s|$. Now suppose s is of the form $\mathbb{L}_\alpha(X)$, in this case $r \notin r \equiv r \dot{\in} s \wedge r \notin r$ so we have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Vdash r \dot{\in} s \wedge r \notin r}{\Vdash (\exists x \in s)(x \notin r)}\,(b\exists)}{\Vdash s \neq r}\,(\vee)}{\Vdash r \dot{\in} s \rightarrow s \neq r}\,\text{2.3.3ii)}}{\Vdash s \notin s}\,(\notin)$$

Now suppose s is of the form $[x \in \mathbb{L}_\alpha(X) \,|\, B(x)]$, by i) we have $\Vdash B(r), \neg B(r)$ for any $|r| < |s|$. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Vdash r \notin r \qquad \Vdash B(r), \neg B(r) \quad \text{for any } |r| < |s|}{\Vdash B(r) \wedge r \notin r, \neg B(r)}\,(\wedge)}{\Vdash (\exists x \in s)(x \notin r), \neg B(r)}\,(b\exists)}{\Vdash s \neq r, \neg B(r)}\,(\vee)}{\Vdash B(r) \to s \neq r}\,\text{Lemma 2.5.2ii)}}{\Vdash s \notin s}\,(\notin)
$$

iii) Again we proceed by induction on $rk(s)$. If $s \equiv \bar{u}$ then $\Vdash \bar{v} \notin \bar{u}, \bar{v} \in \bar{u}$ for any $|\bar{v}| < |\bar{u}|$ by part i), so we have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$
\cfrac{\cfrac{\Vdash \bar{v} \notin \bar{u}, \bar{v} \in \bar{u}}{\Vdash \bar{v} \in \bar{u} \to \bar{v} \in \bar{u}}\,\text{Lemma 2.5.2ii)}}{\Vdash (\forall x \in s)(x \in s)}\,(b\forall)
$$

Suppose $s \equiv \mathbb{L}_\alpha(X)$, by the induction hypothesis we have $\Vdash r \subseteq r$ for all $|r| < |s|$. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\Vdash r \subseteq r \qquad \Vdash r \subseteq r}{\Vdash r = r}\,(\wedge)}{\Vdash r \in s}\,(\in)}{\Vdash r \dot{\in} s \to r \in s}\,\text{2.3.3ii)}}{\Vdash (\forall x \in s)(x \in s)}\,(b\forall)
$$

Finally suppose $s \equiv [x \in \mathbb{L}_\alpha(X) \mid B(x)]$, again by the induction hypothesis we have $\Vdash r \subseteq r$ for all $|r| < |s|$. Also by part i) we have $\Vdash \neg B(r), B(r)$ for all such r. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Vdash \neg B(r), r \subseteq r}{\Vdash \neg B(r), r = r}\,(\wedge) \qquad \Vdash \neg B(r), B(r)}{\Vdash \neg B(r), B(r) \wedge r = r}\,(\wedge)}{\Vdash \neg B(r), r \in s}\,(\in)}{\Vdash B(r) \to r \in s}\,\text{Lemma 2.5.2ii)}}{\Vdash (\forall x \in s)(x \in s)}\,(b\forall)
$$

iv) Was shown whilst proving iii).

v) By part i) we have $\Vdash \neg(s \subseteq t), s \subseteq t$ and $\Vdash \neg(t \subseteq s), t \subseteq s$ for all $|s| < |t|$. We have the following template for derivations in $\mathbf{RS}_\Omega(X)$.

$$
\cfrac{\cfrac{\cfrac{\Vdash \neg(s \subseteq t), s \subseteq t}{\Vdash \neg(s \subseteq t) \vee \neg(t \subseteq s), s \subseteq t}\,(\vee) \qquad \cfrac{\Vdash \neg(t \subseteq s), t \subseteq s}{\Vdash \neg(t \subseteq s) \vee \neg(s \subseteq t), t \subseteq s}\,(\vee)}{\Vdash \neg(s \subseteq t) \vee \neg(t \subseteq s), s \subseteq t \wedge t \subseteq s}\,(\wedge)}{\Vdash s \neq t, t = s}\,\text{2.3.3i)}
$$

vi) If $t \equiv \mathbb{L}_\alpha(X)$ then this result is trivial since $s \dot{\in} t \to A := A$ and $s \dot{\in} t \wedge B := B$.

Now if $t \equiv \bar{u}$ then $s \dot{\in} t := s \in t$ and if $t \equiv [x \in \mathbb{L}_\alpha(X) \mid C(x)]$ then $s \dot{\in} t := C(s)$. In either case we have the following template for derivations in $\mathbf{RS}_\Omega(X)$:

$$\dfrac{(\vee)\dfrac{\Vdash \Gamma, A, B}{\Vdash \Gamma, s \dot{\in} t \to A, B} \qquad (\vee)\dfrac{\Vdash \Gamma, \neg(s \dot{\in} t), s \dot{\in} t \quad \text{by i)}}{\Vdash \Gamma, s \dot{\in} t \to A, s \dot{\in} t}}{\Vdash \Gamma, s \dot{\in} t \to A, s \dot{\in} t \wedge B}(\wedge)$$

vii) By part iii) we have $\Vdash s = s$ for all $|s| < \Gamma_{\theta+1} + \alpha$ which means we have $\Vdash s \dot{\in} \mathbb{L}_\alpha(X) \wedge s = s$ for all such s. From which one application of $(\in)$ gives the desired result. $\square$

Lemma 2.5.4 (Extensionality)**.** For any $\mathbf{RS}_\Omega(X)$ formula $A(s_1, ..., s_n)$,

$$\Vdash [s_1 \neq t_1], ..., [s_n \neq t_n], \neg A(s_1, ..., s_n), A(t_1, ..., t_n).$$

Where $[s_i \neq t_i] := \neg(s_i \subseteq t_i), \neg(t_i \subseteq s_i)$.

Proof. The proof is by induction on $rk(A(s_1, ..., s_n)) \# rk(A(t_1, ..., t_n))$.

Case 1. Suppose $A(s_1, s_2) \equiv s_1 \in s_2$. By the induction hypothesis we have $\Vdash [s_1 \neq t_1], [s \neq t], s_1 \neq s, t_1 = t$ for all $|s| < |s_2|$ and all $|t| < |t_2|$. What follows is a template for derivations in $\mathbf{RS}_\Omega(X)$, for ease of reading the principal formula of each inference is underlined (some lines do not necessarily represent single inferences, but in these cases it is clear how to extend the concept of "principal formula" in a sensible way).

$$\begin{array}{rl}
 & \Vdash [s_1 \neq t_1], [s \neq t], s_1 \neq s, t_1 = t \\
(\vee) & \overline{\Vdash [s_1 \neq t_1], \underline{s \neq t}, s_1 \neq s, t_1 = t} \\
\text{Lemma 2.5.3 vi)} & \overline{\Vdash [s_1 \neq t_1], \underline{t \dot{\in} t_2 \to s \neq t}, s_1 \neq s, \underline{t \dot{\in} t_2 \wedge t_1 = t}} \\
(\in) & \overline{\Vdash [s_1 \neq t_1], t \dot{\in} t_2 \to s \neq t, s_1 \neq s, \underline{t_1 \in t_2}} \\
(\notin) & \overline{\Vdash [s_1 \neq t_1], \underline{s \notin t_2}, s_1 \neq s, t_1 \in t_2} \\
\text{Lemma 2.5.3 vi)} & \overline{\Vdash [s_1 \neq t_1], \underline{s \dot{\in} s_2 \wedge s \notin t_2}, \underline{s \dot{\in} s_2 \to s_1 \neq s}, t_1 \in t_2} \\
(b\exists) & \overline{\Vdash [s_1 \neq t_1], \underline{(\exists x \in s_2)(x \notin t_2)}, s \dot{\in} s_2 \to s_1 \neq s, t_1 \in t_2} \\
(\notin) & \overline{\Vdash [s_1 \neq t_1], (\exists x \in s_2)(x \notin t_2), \underline{s_1 \notin s_2}, t_1 \in t_2} \\
\text{Lemma 2.4.6i)} & \overline{\Vdash [s_1 \neq t_1], \underline{s_2 \neq t_2}, s_1 \notin s_2, t_1 \in t_2}
\end{array}$$

Case 2. Suppose $A(s_1) \equiv s_1 \in s_1$. In this case $\neg A(s_1) \equiv s_1 \notin s_1$ so the result follows from Lemma 2.5.3ii).

Case 3. Suppose $A(s_1, ..., s_n) \equiv (\exists y \in s_i)(B(y, s_1, ..., s_n))$ for some $1 \leq i \leq n$. Inductively we have

$$\Vdash [s_1 \neq t_1], ..., [s_n \neq t_n], \neg B(r, s_1, ..., s_n), B(r, t_1, ..., t_n)$$

for all $|r| < |s_i|$. Now by applying 2.5.3iv) we obtain

$$\Vdash [s_1 \neq t_1], ..., [s_n \neq t_n], r \dot{\in} s_i \to \neg B(r, s_1, ..., s_n), r \dot{\in} s_i \wedge B(r, t_1, ..., t_n)$$

To which we may apply $(b\exists)$ followed by $(b\forall)$ to arrive at the desired conclusion.

Case 4. Suppose $A(s_1,...,s_n) \equiv (\exists x \in r)B(x,s_1,...,s_n)$ for some r not present in $s_1,...,s_n$. From the induction hypothesis we have

$$\Vdash [s_1 \neq t_1],...,[s_n \neq t_n], p \dot{\in} r \to \neg B(p,s_1,...,s_n), p \dot{\in} r \wedge B(p,t_1,...,t_n) \quad \text{for all } |\,p\,| < |\,r\,|.$$

Applying $(b\exists)$ followed by $(b\forall)$ gives us the desired result.

The cases where $A(s_1,...,s_n) \equiv \exists x B(x,s_1,...,s_n)$ or $A(s_1,...,s_n) \equiv B \vee C$ may be treated in a similar manner to case 4. All other cases are dual to one of the ones considered above. $\square$

Lemma 2.5.5 (Set Induction)**.** For any $\mathbf{RS}_\Omega(X)$-formula F:

$$\Vdash^{\omega^{rk(A)}} \forall x[(\forall y \in x)F(y) \to F(x)] \to \forall x F(x)$$

where $A := \forall x[(\forall y \in x)F(y) \to F(x)]$.

Proof. Claim:

$$(*) \qquad \mathcal{H}[A,s] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^{|\,s\,|+1}}{0}\, \neg A, F(s) \quad \text{for any term } s.$$

We begin by verifying (*) using induction on $|\,s\,|$. From the induction hypothesis we know that

$$(1) \qquad \mathcal{H}[A,t] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^{|\,t\,|+1}}{0}\, \neg A, F(t) \quad \text{for all } |\,t\,| < |\,s\,|.$$

By applying $(\vee)$ if necessary to (1) we obtain

$$(2) \qquad \mathcal{H}[A,t,s] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^{|\,t\,|+1}+1}{0}\, \neg A, t \dot{\in} s \to F(t) \quad \text{for all } |\,t\,| < |\,s\,|.$$

To which we may apply $(b\forall)$ yielding

$$(3) \qquad \mathcal{H}[A,s] \,\Big|\!\frac{\eta+2}{0}\, \neg A, (\forall y \in s)F(y) \quad \text{where } \eta := \omega^{rk(A)}\#\omega^{|\,s\,|}.$$

Observe that $no(\neg F(s), F(s)) < \omega^{rk(A)}$, so by Lemma 2.5.3i) we have

$$(4) \qquad \mathcal{H}[A,s] \,\Big|\!\frac{\eta+2}{0}\, \neg F(s), F(s)\,.$$

Applying $(\wedge)$ to (3) and (4) yields

$$(5) \qquad \mathcal{H}[A,s] \,\Big|\!\frac{\eta+3}{0}\, \neg A, (\forall y \in s)F(y) \wedge \neg F(s), F(s)\,.$$

To which we may apply $(\exists)$ to otain

$$(6) \qquad \mathcal{H}[A,s] \,\Big|\!\frac{\eta+4}{0}\, \neg A, \exists x[(\forall y \in x)F(y) \wedge \neg F(x)], F(s)\,.$$

It remains to observe that $\neg A \equiv \exists x[(\forall y \in x)F(y) \wedge \neg F(x)]$ and that $\eta + 4 < \omega^{rk(A)}\#\omega^{|s|+1}$, and hence we may apply Lemma 2.4.6i) to provide

$$\mathcal{H}[A,s] \vdash^{\omega^{rk(A)}\#\omega^{|s|+1}}_{0} \neg A, F(s) \tag{7}$$

so the claim is verified.

Applying $(\forall)$ to (*) gives

$$\mathcal{H}[A] \vdash^{\omega^{rk(A)}\#\Omega}_{0} \neg A, \forall x F(x)\,.$$

Now by two applications of $(\vee)$ we may conclude

$$\mathcal{H}[A] \vdash^{\omega^{rk(A)}\#\Omega+2}_{0} A \rightarrow \forall x F(x)\,.$$

It remains to note that $no(A \rightarrow \forall x F(x)) \geq \omega^{\Omega+1} > \Omega + 2$, so we have

$$\Vdash^{\omega^{rk(A)}}_{0} A \rightarrow (\forall x \in \mathbb{L}_\alpha(X))F(x) \tag{2.1}$$

as required. $\square$

Lemma 2.5.6 (Infinity)**.** Suppose $\omega < \mu < \Omega$, then

$$\Vdash (\exists x \in \mathbb{L}_\mu(X))[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)].$$

Proof. The following gives a template for derivations in $\mathbf{RS}_\Omega(X)$, the idea is that $\mathbb{L}_\omega(X)$ serves as a witness inside $\mathbb{L}_\mu(X)$.

$$\begin{prooftree}
\AxiomC{Lemma 2.5.3vii)}
\UnaryInfC{$\Vdash s \in \mathbb{L}_k(X) \quad \text{for any } |s| < |\mathbb{L}_k(X)| \text{ and } k < \omega.$}
\LeftLabel{2.3.3ii)}
\UnaryInfC{$\Vdash \mathbb{L}_k(X) \dot{\in} \mathbb{L}_\omega(X) \wedge s \in \mathbb{L}_k(X)$}
\LeftLabel{$(b\exists)$}
\UnaryInfC{$\Vdash (\exists z \in \mathbb{L}_\omega(X))(s \in \mathbb{L}_k(X))$}
\LeftLabel{2.3.3ii)}
\UnaryInfC{$\Vdash s \dot{\in} \mathbb{L}_\omega(X) \rightarrow (\exists z \in \mathbb{L}_\omega(X))(s \in z)$}
\LeftLabel{$(b\forall)$}
\UnaryInfC{$\Vdash (\forall y \in \mathbb{L}_\omega(X))(\exists z \in \mathbb{L}_\omega(X))(y \in z)$}
\AxiomC{$\Vdash \mathbb{L}_0(X) \in \mathbb{L}_\omega(X)$}
\LeftLabel{2.3.3ii)}
\UnaryInfC{$\Vdash \mathbb{L}_0(X) \dot{\in} \mathbb{L}_\omega(X) \wedge \mathbb{L}_0(X) \in \mathbb{L}_\omega(X)$}
\LeftLabel{$(b\exists)$}
\UnaryInfC{$\Vdash (\exists z \in \mathbb{L}_\omega(X))(z \in \mathbb{L}_\omega(X))$}
\LeftLabel{$(\wedge)$}
\BinaryInfC{$\Vdash (\forall y \in \mathbb{L}_\omega(X))(\exists z \in \mathbb{L}_\omega(X))(y \in z) \wedge (\exists z \in \mathbb{L}_\omega(X))(z \in \mathbb{L}_\omega(X))$}
\LeftLabel{2.3.3ii)}
\UnaryInfC{$\Vdash \mathbb{L}_\omega(X) \dot{\in} \mathbb{L}_\mu(X) \wedge [(\forall y \in \mathbb{L}_\omega(X))(\exists z \in \mathbb{L}_\omega(X))(y \in z) \wedge (\exists z \in \mathbb{L}_\omega(X))(z \in \mathbb{L}_\omega(X))]$}
\LeftLabel{$(b\exists)$}
\UnaryInfC{$\Vdash (\exists x \in \mathbb{L}_\mu(X))[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$}
\end{prooftree}$$

$\square$

Lemma 2.5.7 (Δ_0-Separation)**.** Suppose $A(a, b_1, ..., b_n)$ be a Δ_0-formula of **KP** with all free variables indicated, μ a limit ordinal and $|s|, |t_0|, ..., |t_n| < \Gamma_{\theta+1} + \mu$.

$$\Vdash (\exists y \in \mathbb{L}_\mu(X))[(\forall x \in y)(x \in s \wedge A(x, t_1, ..., t_n)) \wedge (\forall x \in s)(A(x, t_1, ..., t_n) \rightarrow x \in y)]$$

Proof. Let $\alpha := \max\{|\,s\,|, |\,t_0\,|, ..., |\,t_n\,|\} + 1$ and note that $\alpha < \Gamma_{\theta+1} + \mu$ since μ ia a limit. Now let β be the unique ordinal such that $\alpha = \Gamma_{\theta+1} + \beta$ if such an ordinal exists, if not set $\beta := 0$. Now define

$$t := [z \in \mathbb{L}_\beta(X) \mid z \in s \wedge B(z)]$$

where $B(z) := A(z, t_1, ..., t_n)$. We have the following templates for derivations in $\mathbf{RS}_\Omega(X)$:

$$(b\forall)\,\dfrac{\text{2.3.3ii)}\,\dfrac{\text{Lemma 2.5.2ii)}\,\dfrac{\dfrac{\text{Lemma 2.5.3 i)}}{\Vdash \neg(r \in s \wedge B(r)), r \in s \wedge B(r) \quad \text{for all } |\,r\,| < \alpha}}{\Vdash (r \in s \wedge B(r)) \to r \in s \wedge B(r)}}{\Vdash r \mathrel{\dot{\in}} t \to r \in s \wedge B(r)}}{\Vdash (\forall x \in t)(x \in s \wedge B(r))}$$

In the following derivation r ranges over terms $|\,r\,| < |\,s\,|$.

$$(b\forall)\,\dfrac{\text{Lemma 2.5.2ii)}\,\dfrac{\text{Lemma 2.5.2ii)}\,\dfrac{(\in)\,\dfrac{\text{2.3.3ii)}\,\dfrac{(\wedge)\,\dfrac{(\wedge)\,\dfrac{\dfrac{\text{Lemma 2.5.3 iv)}}{\Vdash \neg(r \mathrel{\dot{\in}} s), r \in s} \qquad \dfrac{\text{Lemma 2.5.3 i)}}{\Vdash \neg B(r), B(r)}}{\Vdash \neg(r \mathrel{\dot{\in}} s), \neg B(r), r \in s \wedge B(r)} \qquad \dfrac{\text{Lemma 2.5.3 iii)}}{\Vdash r = r}}{\Vdash \neg(r \mathrel{\dot{\in}} s), \neg B(r), (r \in s \wedge B(r)) \wedge r = r}}{\Vdash \neg(r \mathrel{\dot{\in}} s), \neg B(r), r \mathrel{\dot{\in}} t \wedge r = r}}{\Vdash \neg(r \mathrel{\dot{\in}} s), \neg B(r), r \in t}}{\Vdash \neg(r \mathrel{\dot{\in}} s), (B(r) \to r \in t)}}{\Vdash r \mathrel{\dot{\in}} s \to (B(r) \to r \in t)}}{\Vdash (\forall x \in s)(B(x) \to x \in t)}$$

Now applying $(\wedge)$ to the two preceding derivations and noting that $|\,t\,| < \Gamma_{\theta+1} + \mu$ gives us

$$\Vdash t \mathrel{\dot{\in}} \mathbb{L}_\mu(X) \wedge [(\forall x \in t)(x \in s \wedge B(r)) \wedge (\forall x \in s)(B(x) \to x \in t)]$$

to which we may apply $(b\exists)$ to obtain

$$\Vdash (\exists y \in \mathbb{L}_\mu(X))[(\forall x \in y)(x \in s \wedge B(x)) \wedge (\forall x \in s)(B(x) \to x \in y)].$$

It should also be checked that

$$t \in \mathcal{H}[(\exists y \in \mathbb{L}_\mu(X))[(\forall x \in y)(x \in s \wedge B(x)) \wedge (\forall x \in s)(B(x) \to x \in y)]]$$

but this is the case since

$$|\,s\,|, |\,t_0\,|, ..., |\,t_n\,| \in k((\exists y \in \mathbb{L}_\mu(X))[(\forall x \in y)(x \in s \wedge B(x)) \wedge (\forall x \in s)(B(x) \to x \in y)])$$

and $|\,t\,| = \max\{\max\{|\,s\,|, |\,t_0\,|, ..., |\,t_n\,|\} + 1, \Gamma_{\theta+1}\}$. □

Lemma 2.5.8 (Pair and Union)**.** Let μ be a limit ordinal and let s, t be $\mathbf{RS}_\Omega(X)$-terms such that $|\,s\,|, |\,t\,| < \Gamma_{\theta+1} + \mu$, then

i) $\Vdash (\exists z \in \mathbb{L}_\mu(X))(s \in z \wedge t \in z)$

ii) $(\exists z \in \mathbb{L}_\mu(X))(\forall y \in s)(\forall x \in y)(x \in z)$

Proof. Let $\alpha := \max\{|\,s\,|, |\,t\,|\} + 1$, now let β be the unique ordinal such that $\alpha = \Gamma_{\theta+1} + \beta$ if such an ordinal exists, otherwise set $\beta := 0$. Now by Lemma 2.5.3vii) we have

$$\Vdash s \in \mathbb{L}_\beta(X) \quad \text{and} \quad \Vdash t \in \mathbb{L}_\beta(X).$$

Now by $(\wedge)$ and noticing that $\beta < \mu$ since μ is a limit, we have

$$\Vdash \mathbb{L}_\beta(X) \mathrel{\dot\in} \mathbb{L}_\mu(X) \wedge (s \in \mathbb{L}_\beta(X) \wedge t \in \mathbb{L}_\beta(X)).$$

To which we may apply $(b\exists)$ to obtain the desired result.

ii) Let β be the unique ordinal such that $|\,s\,| = \Gamma_{\theta+1} + \beta$ if such an ordinal exists, otherwise let $\beta = 0$. By Lemma 2.5.3vii) we have $\Vdash r \in \mathbb{L}_\beta(X)$ for any $|\,r\,| < |\,s\,|$. In the following template for derivations in $\mathbf{RS}_\Omega(X)$, r and t range over terms such that $|\,r\,| < |\,t\,| < |\,s\,|$:

$$\begin{array}{rl}
 & \Vdash r \in \mathbb{L}_\beta(X) \\ \hline
(\vee) \text{ if necessary} & \Vdash r \mathrel{\dot\in} t \rightarrow r \in \mathbb{L}_\beta(X) \\ \hline
(b\forall) & \Vdash (\forall x \in t)(x \in \mathbb{L}_\beta(X)) \\ \hline
(\vee) \text{ if necessary} & \Vdash t \mathrel{\dot\in} s \rightarrow (\forall x \in t)(x \in \mathbb{L}_\beta(X)) \\ \hline
(b\forall) & \Vdash (\forall y \in s)(\forall x \in y)(x \in \mathbb{L}_\beta(X)) \\ \hline
2.3.3\text{ii)} & \Vdash \mathbb{L}_\beta(X) \mathrel{\dot\in} \mathbb{L}_\mu(X) \wedge (\forall y \in s)(\forall x \in y)(x \in \mathbb{L}_\beta(X)) \quad \text{since } \beta < \mu \\ \hline
(b\exists) & \Vdash (\exists z \in \mathbb{L}_\mu(X))(\forall y \in s)(\forall x \in y)(x \in z)
\end{array}$$

□

Lemma 2.5.9 (Δ_0-Collection)**.** Suppose $F(a,b)$ is any Δ_0 formula of **KP**.

$$\Vdash (\forall x \in s)\exists y F(x,y) \rightarrow \exists z (\forall x \in s)(\exists y \in z) F(x,y)$$

Proof. By Lemma 2.5.3i) we have

$$\Vdash \neg(\forall x \in s)\exists y F(x,y), (\forall x \in s)\exists y F(x,y).$$

Applying $(\Sigma\text{-Ref}_\Omega(X))$ yields

$$\mathcal{H}[(\forall x \in s)\exists y F(x,y)] \overset{\alpha+1}{\underset{0}{\vdash}} \neg(\forall x \in s)\exists y F(x,y), \exists z(\forall x \in s)(\exists y \in z)F(x,y)$$

where $\alpha := \omega^{rk((\forall x \in s)\exists y F(x,y))} \# \omega^{rk((\forall x \in s)\exists y F(x,y))}$. Now two applications of $(\vee)$ provides

$$\mathcal{H}[(\forall x \in s)\exists y F(x,y)] \overset{\alpha+3}{\underset{0}{\vdash}} (\forall x \in s)\exists y F(x,y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x,y)\,.$$

It remains to note that

$$\alpha + 3 < \omega^{rk(\forall x \in s)\exists y F(x,y)+1} = no((\forall x \in s)\exists y F(x,y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x,y))$$

so the proof is complete. □

Theorem 2.5.10. If $\mathbf{KP} \vdash \Gamma(a_1, ..., a_n)$ where $\Gamma(a_1, ..., a_n)$ is a finite set of formulae whose free variables are amongst $a_1, ..., a_n$, then there is some $m < \omega$ (which we may compute from the derivation) such that

$$\mathcal{H}[s_1, ..., s_n] \,\big|\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(s_1, ..., s_n)$$

for any operator $\mathcal{H}$ and any $\mathbf{RS}_\Omega(X)$ terms $s_1, ..., s_n$.

Proof. Suppose $\Gamma(a_1, ..., a_n) \equiv \{A_1(a_1, ..., a_n), ..., A_k(a_1, ..., a_n)\}$. Note that for any choice of terms $s_1, ..., s_n$ and each $1 \leq i \leq k$

$$\begin{aligned} rk(A_i(s_1, ..., s_n)) &= \omega \cdot \max(k(A_i(s_1, ..., s_n))) + m_i \quad \text{for some } m_i < \omega \\ &\leq \omega \cdot \Omega + m_i = \Omega + m_i. \end{aligned}$$

Therefore

$$no(A_i(s_1, ..., s_n)) = \omega^{rk(A_i(s_1,...,s_n))} \leq \omega^{\Omega+m_i} = \omega^\Omega \cdot \omega^{m_i} = \Omega \cdot \omega^{m_i}.$$

So letting $m = \max(m_1, ..., m_k) + 1$ we have

$$\begin{aligned} no(\Gamma(s_1, ..., s_n)) &\leq \Omega \cdot \omega^{m_1} \# ... \# \Omega \cdot \omega^{m_n} \\ &= \Omega \cdot (\omega^{m_1} \# ... \# \omega^{m_n}) \\ &\leq \Omega \cdot \omega^m \end{aligned}$$

The proof now proceeds by induction on the **KP** derivation. If $\Gamma(a_1, ..., a_n)$ is an axiom of **KP** then the result follows from 2.5.3i), 2.5.4, 2.5.5, 2.5.6, 2.5.7, 2.5.8 or 2.5.9.

Now suppose that $\Gamma(a_1, ..., a_n)$ arises as the result of an inference rule.

Case 1. Suppose the last inference was $(b\forall)$, so $(\forall x \in a_i)F(x, \bar{a}) \in \Gamma(\bar{a})$ and we are in the following situation in **KP**

$$(b\forall)\ \frac{\Gamma(\bar{a}), c \in a_i \to F(c, \bar{a})}{\Gamma(\bar{a})}$$

where c is different from $a_1, ..., a_n$. Inductively we have some $m < \omega$ such that

(1) $$\mathcal{H}[\bar{s}, r] \,\big|\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), r \in s_i \to F(r, \bar{s}) \quad \text{for all } |\,r\,| < |\,s_i\,|.$$

1.1 If s_i is of the form $\bar{u}$ we may immediately apply $(b\forall)$ to complete this case.

Suppose $s_i \equiv \mathbb{L}_\alpha(X)$ for some α. Applying Lemma 2.4.6iii) to (1) gives

(2) $$\mathcal{H}[\bar{s}, r] \,\big|\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), \neg(r \in s_i), F(r, \bar{s})\,.$$

Since $|\,r\,| < |\,s\,|$, by Lemma 2.5.3vii) we have

(3) $$\Vdash r \in s.$$

Applying (Cut) to (1) and (2) yields

$$\mathcal{H}[\bar{s}, r] \,\Big|\!\frac{\Omega\cdot\omega^m+1}{\Omega+m}\, \Gamma(\bar{s}), F(r, \bar{s})\,. \tag{4}$$

To which we may apply $(b\forall)$ to complete this case.

Suppose $s_i \equiv [x \in \mathbb{L}_\alpha(X) \mid B(x)]$, again we may apply Lemma 2.4.6iii) to (1) to obtain

$$\mathcal{H}[\bar{s}, r] \,\Big|\!\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), \neg(r \in s_i), F(r, \bar{s})\,. \tag{5}$$

Since $|\, r \,| < |\, s \,|$ by Lemma 2.5.3iv) we have

$$\Vdash \neg(r \,\dot{\in}\, s), r \in s. \tag{6}$$

Applying (Cut) to (5) and (6) yields

$$\mathcal{H}[\bar{s}, r] \,\Big|\!\frac{\Omega\cdot\omega^m+1}{\Omega+m}\, \Gamma(\bar{s}), \neg(r \,\dot{\in}\, s_i), F(r, \bar{s})\,. \tag{7}$$

Now two applications of $(\vee)$ provide

$$\mathcal{H}[\bar{s}, r] \,\Big|\!\frac{\Omega\cdot\omega^m+3}{\Omega+m}\, \Gamma(\bar{s}), r \,\dot{\in}\, s_i \to F(r, \bar{s})\,. \tag{8}$$

To which we may apply $(b\forall)$ to complete this case.

Case 2. Suppose the last inference was $(\forall)$ so $\forall x A(x, \bar{a}) \in \Gamma(\bar{a})$ and we are in the following situation in **KP**

$$(\forall)\ \frac{\Gamma(\bar{a}), F(c, \bar{a})}{\Gamma(\bar{a})}$$

where c is different from $a_1, ...a_n$. Inductively we have some $m < \omega$ such that

$$\mathcal{H}[\bar{s}, r] \,\Big|\!\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), F(r, \bar{s}) \quad \text{for all terms } r.$$

We may immediately apply $(\forall)$ to complete this case.

Case 3. Suppose the last inference was $(b\exists)$ so $(\exists x \in s_i)F(x, \bar{s}) \in \Gamma(\bar{s})$ and we are in the following situation in **KP**

$$(b\exists)\ \frac{\Gamma(\bar{a}), c \in a_i \wedge F(c, \bar{a})}{\Gamma(\bar{a})}$$

3.1 Suppose c is different from $a_1, ..., a_n$. Using the induction hypothesis we find some $m < \omega$ such that

$$\mathcal{H}[\bar{s}] \,\Big|\!\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), \bar{\emptyset} \in s_i \wedge F(\bar{\emptyset}, \bar{s})\,. \tag{9}$$

3.1.1 If s_i is of the form $\bar{u}$ we may immediately apply $(b\exists)$ to complete the case.

3.1.2 Suppose s_i is of the form $\mathbb{L}_\alpha(X)$. Applying Lemma 2.4.6iv) to (1) yields

(10) $$\mathcal{H}[\bar{s}] \,\Big|\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), F(\bar{\emptyset}, \bar{s})\,.$$

Noting that in this case $\bar{\emptyset} \mathrel{\dot{\in}} s \wedge F(\bar{\emptyset}, \bar{s}) \equiv F(\bar{\emptyset}, \bar{s})$, we may apply $(b\exists)$ to complete this case.

3.1.3 Suppose s_i is of the form $[x \in \mathbb{L}_\alpha(X) \mid B(x)]$. First we must verify the following claim

(*) $$\Vdash \neg(\bar{\emptyset} \in s_i \wedge F(\bar{\emptyset}, \bar{s})), \bar{\emptyset} \mathrel{\dot{\in}} s_i \wedge F(\bar{\emptyset}, \bar{s}).$$

Note that owing to Lemma 2.5.4 we have $\Vdash [r \neq \bar{\emptyset}], \neg B(r), B(\bar{\emptyset})$ for all $|r| < |s_i|$. In the following template for derivations in $\mathbf{RS}_\Omega(X)$ r ranges over terms $|r| < |s_i|$.

$$\text{Lemma 2.5.2ii)}\;\dfrac{(\wedge)\;\dfrac{(\notin)\;\dfrac{\text{Lemma 2.5.2ii)}\;\dfrac{\text{Lemma 2.5.2ii)}\;\dfrac{\Vdash [r \neq \bar{\emptyset}], \neg B(r), B(\bar{\emptyset})}{\Vdash r \neq \bar{\emptyset}, \neg B(r), B(\bar{\emptyset})}}{\Vdash B(r) \to r \neq \bar{\emptyset}, B(\bar{\emptyset})}}{\Vdash \neg(\bar{\emptyset} \in s_i), B(\bar{\emptyset})} \qquad \dfrac{\text{Lemma 2.5.3i)}}{\Vdash \neg F(\bar{\emptyset}, \bar{s}), F(\bar{\emptyset}, \bar{s})}}{\Vdash \neg(\bar{\emptyset} \in s_i), \neg F(\bar{\emptyset}, \bar{s}), B(\bar{\emptyset}) \wedge F(\bar{\emptyset}, \bar{s})}}{\Vdash \neg(\bar{\emptyset} \in s_i) \vee \neg F(\bar{\emptyset}, \bar{s}), B(\bar{\emptyset}) \wedge F(\bar{\emptyset}, \bar{s})}$$

Now applying (Cut) to (9) and (*) we get

(11) $$\mathcal{H}[\bar{s}] \,\Big|\frac{\Omega\cdot\omega^m+1}{\Omega+m'}\, \Gamma(\bar{s}), \bar{\emptyset} \mathrel{\dot{\in}} s_i \wedge F(\bar{\emptyset}, \bar{s})\,.$$

Note the possible increase in cut rank. We may apply $(b\exists R)$ to (11) to complete this case.

3.2 Suppose c is one of $a_1, ..., a_n$, without loss of generality let us assume $c = a_1$. Applying the induction hypothesis we can compute some $m < \omega$ such that

(12) $$\mathcal{H}[\bar{s}] \,\Big|\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), s_1 \in s_i \wedge F(s_1, \bar{s}).$$

Note that in fact 3.2 subsumes 3.1 since we can conclude (12) from the induction hypothesis regardless of whether or not c is a member of $\bar{a}$. To help with clarity 3.1 is left in the proof above, but in later embeddings we shall dispense with such cases.

If s_1 and s_i are of the form $\bar{u}$ and $\bar{v}$ with $|s_1| < |s_i|$ then we may immediately apply $(b\exists)$ to complete this case. If this is not the case then we verify the following claim

(**) $$\Vdash \neg(s_1 \in s_i \wedge F(s_1, \bar{s})), (\exists x \in s_i) F(x, \bar{s}).$$

To prove (**) we split into cases based on the form of s_i.

3.2.1 Suppose s_i is of the form $\bar{u}$.

3.2.1.1 If s_1 is also of the form $\bar{v}$ [remember that by assumption $|\, s_1 \,| \geq |\, s_i \,|$] then $\neg(s_1 \in s_i), F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})$ is an axiom so we may apply $(\vee)$ twice to complete this case.

3.2.1.2 Now suppose s_1 is not of the form $\bar{v}$. We have following template for derivations in $\mathbf{RS}_\Omega(X)$, here r ranges over terms with $|\, r \,| < |\, s_i \,|$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\text{Lemma 2.5.3i)}}{\Vdash \neg(r \in s_i), r \in s_i} \qquad \cfrac{\text{Lemma 2.5.4}}{\Vdash r \neq s_1, \neg F(s_1, \bar{s}), F(r, \bar{s})}}{\Vdash \neg(r \in s_i), r \neq s_1, \neg F(s_1, \bar{s}), r \in s_i \wedge F(r, \bar{s})}\,(\wedge)}{\Vdash \neg(r \in s_i), r \neq s_1, \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,(b\exists)}{\Vdash r \in s_i \to r \neq s_1, \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,\text{Lemma 2.5.2ii)}}{\Vdash \neg(s_1 \in s_i), \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,(\notin)}{\Vdash \neg(s_1 \in s_i) \vee \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,\text{Lemma 2.5.2ii)}
$$

3.2.2 Now suppose s_i is of the form $\mathbb{L}_\alpha(X)$. In the following template for derivations in $\mathbf{RS}_\Omega(X)$ r ranges over terms with $|\, r \,| < |\, s_i \,|$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\text{Lemma 2.5.4}}{\Vdash r \neq s_1, \neg F(s_1, \bar{s}), F(x, \bar{s})}}{\Vdash r \neq s_1, \neg F(s_1, \bar{s}), r \mathrel{\dot{\in}} s_i \wedge F(x, \bar{s})}\,\text{2.3.3ii)}}{\Vdash r \neq s_1, \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,(b\exists)}{\Vdash r \mathrel{\dot{\in}} s_i \to r \neq s_1, \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,\text{2.3.3ii)}}{\Vdash \neg(s_1 \in s_i), \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,(\notin)}{\Vdash \neg(s_1 \in s_i) \vee \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,\text{Lemma 2.5.2ii)}
$$

3.2.3 Finally suppose s_i is of the form $[x \in \mathbb{L}_\alpha \mid B(x)]$. In the following template for derivations in $\mathbf{RS}_\Omega(X)$ r ranges over terms with $|\, r \,| < |\, s_i \,|$.

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\text{Lemma 2.5.3i)}}{\Vdash \neg B(r), B(r)} \qquad \cfrac{\text{Lemma 2.5.4}}{\Vdash r \neq s, \neg F(s_1, \bar{s}), F(r, \bar{s})}}{\Vdash \neg B(r), r \neq s_1, \neg F(s_1, \bar{s}), B(r) \wedge F(r, \bar{s})}\,(\wedge)}{\Vdash \neg B(r), r \neq s_1, \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,(b\exists)}{\Vdash B(r) \to r \neq s_1, \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,\text{Lemma 2.5.2ii)}}{\Vdash \neg(s_1 \in s_i), \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,(\notin)}{\Vdash \neg(s_1 \in s_i) \vee \neg F(s_1, \bar{s}), (\exists x \in s_i)F(x, \bar{s})}\,\text{Lemma 2.5.2ii)}
$$

This completes the proof of the claim (**). It remains to note that we may apply (Cut) to (**) and (12) to complete Case 3.

Case 4. Suppose the last inference was $(\exists)$ so $\exists x F(x, \bar{s}) \in \Gamma(\bar{s})$ and we are in the following situation in **KP**:

$$
(\exists)\ \frac{\Gamma(\bar{a}), F(c, \bar{a})}{\Gamma(\bar{a})}
$$

Let $p = s_j$ if $c = a_j$ otherwise let $p = \bar{\emptyset}$, from the induction hypothesis we can compute some $m < \omega$ such that

$$\mathcal{H}[\bar{s}] \,\big|\!\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}), F(p, \bar{s})\,.$$

Applying $(\exists)$ completes this case.

Case 5. If the last inference was $(\wedge)$ or $(\vee)$ the result follows immediately by applying the corresponding $\mathbf{RS}_\Omega(X)$ inference to the induction hypotheses.

Case 6. Finally suppose the last inference was (Cut). So we are in the following situation in **KP**

$$(\text{Cut})\;\frac{\Gamma(\bar{a}), B(\bar{a}, \bar{b}) \qquad \Gamma(\bar{a}), \neg B(\bar{a}, \bar{b})}{\Gamma(\bar{a})}$$

Here $\bar{b} := b_1, ..., b_l$ denotes the free variables occurring in B that are different from $a_1, ..., a_n$. Let $\bar{\bar{\emptyset}}$ denote the sequence of l occurrences of $\bar{\emptyset}$. From the induction hypothesis we find m_1 and m_2 such that

$$\mathcal{H}[\bar{s}] \,\big|\!\frac{\Omega\cdot\omega^{m_1}}{\Omega+m_1}\, \Gamma(\bar{s}), B(\bar{s}, \bar{\bar{\emptyset}})$$
$$\mathcal{H}[\bar{s}] \,\big|\!\frac{\Omega\cdot\omega^{m_1}}{\Omega+m_2}\, \Gamma(\bar{s}), \neg B(\bar{s}, \bar{\bar{\emptyset}})$$

To which we may apply (Cut) to complete the proof. □

2.6 A well ordering proof in KP

The aim of this section is to give a well ordering proof in KP for initial segments of formal ordinal terms from $T(\theta)$. First let

$$e_0(\theta) : = \Omega_\theta + 1$$
$$e_{n+1}(\theta) : = \omega^{e_n(\theta)}.$$

Each $e_n(\theta)$ may be seen as a formal term from the representation system $T(\theta)$ from 2.1.13. Although the term is the same, the order type of terms in $T(\theta)$ below $e_n(\theta)$ will be dependent upon θ. We aim to verify that for every $n < \omega$

$$\mathbf{KP} \vdash A_n(\theta) := \exists\alpha\exists f[\text{dom}(f) = \alpha \;\wedge\; \text{range}(f) = \{a \in T(\theta) \mid a \prec \psi_\theta(e_n(\theta)))\}$$
$$\wedge\, \forall\gamma, \delta \in \text{dom}(f)(\gamma < \delta \to f(\gamma) \prec f(\delta))].$$

$A_n(\theta)$ is a Σ-formula of **KP** in which θ is a free variable ranging over ordinals. For the remainder of this section we argue informally in **KP**. The symbols $\alpha, \beta, \gamma, \theta...$ are to be **KP**-variables ranging over ordinals and are ordered by $<$, the symbols $a, b, c, \ldots$ are seen as **KP**-variables ranging over codes of formal terms from $T(\theta)$, these are ordered by $\prec$. For the remainder of this section the variable θ will remain free as we argue in **KP**, for ease of reading we shall simply Ω and ψ instead of Ω_θ and ψ_θ. This proof is an adaptation to the relativised case of a well ordering proof in [35].

Definition 2.6.1. The set Acc_θ is defined by

$$\begin{aligned}\mathrm{Acc}_\theta :=\{a \prec \Omega \mid \exists\alpha\exists f[&\mathrm{dom}(f) = \alpha \wedge \mathrm{range}(f) = \{b : b \preceq a\} \\ &\wedge \forall\gamma, \delta \in \mathrm{dom}(f)(\gamma < \delta \rightarrow f(\gamma) \prec f(\delta))]\}.\end{aligned}$$

Lemma 2.6.2 (Acc_θ-induction)**.** For any **KP**-formula $F(a)$ we have

$$(\forall a \in \mathrm{Acc}_\theta)[(\forall b \prec a)F(b) \rightarrow F(a)] \rightarrow (\forall a \in \mathrm{Acc}_\theta)F(a).$$

Proof. For $a \in \mathrm{Acc}_\theta$ let $o(a)$ and f_a be the unique ordinal and function such that $o(a) = \mathrm{dom}(f_a)$, $\{b : b \preceq a\} = \mathrm{range}(f_a)$ and $\forall\gamma, \delta \in o(a)(\gamma < \delta \rightarrow f_a(\gamma) \prec f_a(\delta))$. Now for a contradiction let us assume that

$$(\forall a \in \mathrm{Acc}_\theta)[(\forall b \prec a)F(b) \rightarrow F(a)] \quad \text{but} \quad \neg F(a_0) \text{ for some } a_0 \in \mathrm{Acc}_\theta$$

Using set induction/foundation we may pick a_0 such that $o(a_0)$ is minimal. (Note that here we must make use of the full set induction schema of **KP** since the formula F is of unbounded complexity) Now for any $b \prec a_0$ we have $o(b) < o(a_0)$, thus by our choice of a_0 we get $F(b)$, thus we have

$$(\forall b \prec a_0)F(b).$$

So by assumption we have $F(a_0)$, contradiction. □

Lemma 2.6.3. Acc_θ has the following closure properties:

i) $b \in \mathrm{Acc}_\theta \wedge a \prec b \quad \rightarrow \quad a \in \mathrm{Acc}_\theta$

ii) $(\forall a \prec b)(a \in \mathrm{Acc}_\theta) \quad \rightarrow \quad b \in \mathrm{Acc}_\theta$

iii) $a, b \in \mathrm{Acc}_\theta \quad \rightarrow \quad a + b \in \mathrm{Acc}_\theta$

iv) $a, b \in \mathrm{Acc}_\theta \quad \rightarrow \quad \varphi ab \in \mathrm{Acc}_\theta$

v) $(\forall\beta \leq \theta)\ \Gamma_\beta \in \mathrm{Acc}_\theta$

Proof. i) Using the notation defined at the start of the proof of Lemma 2.6.2 we may define

$$o(a) := \{\delta \in o(b) \mid f_b(\delta) \preceq a\} \quad \text{and} \quad f_a := f_b|_{o(a)+1}$$

thus witnessing that $a \in \mathrm{Acc}_\theta$.

ii) Let us assume that $(\forall a \prec b)(a \in \mathrm{Acc}_\theta)$, we must verify that $b \in \mathrm{Acc}_\theta$. Using Δ_0-Separation and Infinity we may form the set $\{a \mid a \prec b\}$, therefore $f := \cup_{a \prec b} f_a$ is a set by Δ_0-Collection and Union. Let $\beta := \mathrm{dom}(f)$. Setting $o(b) := \beta + 1$ and $f_b := f \cup \{(\beta, b)\}$ furnishes us with the correct witnesses to confirm that $b \in \mathrm{Acc}_\theta$.

iii) Firstly we must specify what $a+b$ means, since it may not be the case that the string $a+b$ is a term in $T(\theta)$. However, we may define a θ-primitive recursive function $+ : T(\theta) \times T(\theta) \to T(\theta)$ which corresponds to ordinal addition.

Let us assume that $(\forall c \prec b)(a + c \in \mathrm{Acc}_\theta)$, now if we can show that $a + b \in \mathrm{Acc}_\theta$ then the desired result will follow from Acc_θ-induction (2.6.2). Now let $d \prec a + b$, either $d \preceq a$ in which case $d \in \mathrm{Acc}_\theta$ by i) or $d \succ a$ and thus $d = a + c$ for some unique $c \prec b$. Such a c may be determined in a θ-primitive recursive fashion, hence $d \in \mathrm{Acc}_\theta$ by assumption. Thus we have

$$(\forall d \prec a + b)(d \in \mathrm{Acc}_\theta).$$

From which we may use ii) to obtain $a + b \in \mathrm{Acc}_\theta$, completing the proof.

iv) Again a function $\varphi : T(\theta) \times T(\theta) \to T(\theta)$ may be defined in a θ-primitve recursive fashion. It is our aim to show $(\forall x, y \in \mathrm{Acc}_\theta)(\varphi xy \in \mathrm{Acc}_\theta)$, to this end let

$$F(a) := (\forall b \in \mathrm{Acc}_\theta)(\varphi ab \in \mathrm{Acc}_\theta)$$

and assume

(*) $$(\forall z \prec a)F(z)$$

by 2.6.2 it suffices to verify $F(a)$. So let us assume

(**) $$a, b \in \mathrm{Acc}_\theta \quad \text{and} \quad (\forall y \prec b)(\varphi ay \in \mathrm{Acc}_\theta)$$

now we must verify $\varphi ab \in \mathrm{Acc}_\theta$. To do this we prove that

$$d \prec \varphi ab \Rightarrow d \in \mathrm{Acc}_\theta$$

by induction on Gd; the term complexity of d.

1) If d is strongly critical then $d \preceq a$ or $d \preceq b$ in which case $d \in \mathrm{Acc}_\theta$ by (*) or (**).

2) If $d \equiv \varphi d_0 d_1$ then we have the following subcases:

2.1) If $d_0 \prec a$ and $d_1 \prec \varphi ab$ then since $Gd_1 < Gd$ we get $d_1 \in \mathrm{Acc}_\theta$ from the induction hypothesis. So by (*) we get $d \equiv \varphi d_0 d_1 \in \mathrm{Acc}_\theta$

2.2) If $d \equiv \varphi a d_1$ and $d_1 \prec b$ then $d \in \mathrm{Acc}_\theta$ by (**).

2.3 If $a \prec d_0$ and $d \prec b$ then $d \in \mathrm{Acc}_\theta$ since $b \in \mathrm{Acc}_\theta$.

3. If $d \equiv d_1 + ... + d_n$ and $n > 1$ we get $d_1, ..., d_n \in \mathrm{Acc}_\theta$ from the induction hypothesis and thus $d \in \mathrm{Acc}_\theta$ follows from iii).

Thus we have verified that

$$(\forall b \in \mathrm{Acc}_\theta)[(\forall y \prec b)(\varphi ay \in \mathrm{Acc}_\theta) \to \varphi ab \in \mathrm{Acc}_\theta]$$

So, from Acc_θ-induction we get $(\forall b \in \mathrm{Acc}_\theta)(\varphi ab \in \mathrm{Acc}_\theta)$, ie. $F(a)$ completing the proof.

v) We aim to show that

$$(\forall \beta \leq \theta)[(\forall \gamma < \beta)(\Gamma_\gamma \in \mathrm{Acc}_\theta) \to \Gamma_\beta \in \mathrm{Acc}_\theta]$$

from which we may use transfinite induction along θ (since θ is an ordinal) to obtain the desired result.

So suppose $\beta \leq \theta$ and $(\forall \delta < \beta)(\Gamma_\delta \in \mathrm{Acc}_\theta)$. Now suppose $b \prec \Gamma_\beta$, by induction on the term complexity of b we verify that $b \in \mathrm{Acc}_\theta$.

If $b \equiv 0$ we are trivially done by ii) or if $b \equiv \Gamma_\delta$ for some $\delta < \beta$ then we know $b \in \mathrm{Acc}_\theta$ by assumption.

If $b \equiv b_0 + ... + b_n$ or $b \equiv \varphi b_0 b_1$ then we may use parts iii) and iv) and the induction hypothesis since the components b_i have smaller term complexity.

It cannot be the case that $b \equiv \psi b_0$ since $\psi a \succ \Gamma_\theta$ for every a.

Thus using ii) we get that $\Gamma_\beta \in \mathrm{Acc}_\theta$ and the proof is complete. □

Definition 2.6.4. By recursion through the construction of ordinal terms in $T(\theta)$ we define the set $SC_{\prec\Omega}(a)$ which lists the most recent strongly critical ordinal below Ω used in the build up of the ordinal term a:

1) $SC_{\prec\Omega}(0) := SC_{\prec\Omega}(\Omega) := \emptyset$

2) $SC_{\prec\Omega}(a) := \{a\}$ if $a \equiv \Gamma_\beta$ for some $\beta \leq \theta$ or $a \equiv \psi a_0$.

3) $SC_{\prec\Omega}(a_1 + ... + a_n) := \cup_{1\leq i\leq n} SC_{\prec\Omega}(a_i)$

4) $SC_{\prec\Omega}(\varphi a_0 a_1) := SC_{\prec\Omega}(a_0) \cup SC_{\prec\Omega}(a_1)$

5) $SC_{\prec\Omega}(\psi a) := \{\psi a\}$.

Now let

$$M_\theta := \{a \in T(\theta) \mid SC_{\prec\Omega}(a) \subseteq \mathrm{Acc}_\theta\}$$

and

$$a \prec_{M_\theta} b := a, b \in M_\theta \wedge a \prec b.$$

Finally for a definable class U we define the following formula

$$\mathrm{Prog}_{M_\theta}(U) := (\forall y \in M_\theta)[(\forall z \prec_{M_\theta} y)(z \in U) \to (y \in U)]$$

Lemma 2.6.5.

$$\mathrm{Acc}_\theta = M_\theta \cap \Omega := \{a \in M_\theta \mid a \prec \Omega\}$$

Proof. Suppose that $a \in \mathrm{Acc}_\theta$ and observe that $(\forall x \in SC_{\prec\Omega}(a))(x \preceq a)$, thus $SC_{\prec\Omega}(a) \subseteq \mathrm{Acc}_\theta$ by 2.6.3i) thus we have verified that $a \in M_\theta \cap \Omega$.

Now let us suppose that $a \in M_\theta \cap \Omega$, so we know that $SC_{\prec\Omega}(a) \subseteq \mathrm{Acc}_\theta$. By induction on the term complexity Ga we verify that $a \in \mathrm{Acc}_\theta$.

Clearly $0 \in \mathrm{Acc}_\theta$ and if $a \equiv \Gamma_\beta$ for some $\beta \leq \theta$ then $a \in \mathrm{Acc}_\theta$ by Lemma 2.6.3v).

If $a \equiv a_1 + ... + a_n$ then we get $a_1, ..., a_n \in M_\theta \cap \Omega$ since $SC_{\prec\Omega}(a_i) \subseteq SC_{\prec\Omega}(a)$ for each i. Now using the induction hypothesis we get $a_1, ..., a_n \in \mathrm{Acc}_\theta$ and so by Lemma 2.6.3ii) we have $a \in \mathrm{Acc}_\theta$.

If $a \equiv \varphi bc$ then we get $b, c \in M_\theta \cap \Omega$, so using the induction hypothesis we get $b, c \in \mathrm{Acc}_\theta$ and so by Lemma 2.6.3iii) we have $a \in \mathrm{Acc}_\theta$.

If $a \equiv \psi a_0$ then $SC_{\prec\Omega}(a) = \{a\}$ so we have $a \in \mathrm{Acc}_\theta$ by assumption. $\square$

Definition 2.6.6. For a definable class U let

$$U^\delta := \{b \in M_\theta \mid (\forall a \in M_\theta)[M_\theta \cap a \subseteq U \to M_\theta \cap a + \omega^b \subseteq U]\}$$

where $M_\theta \cap a := \{b \in M_\theta \mid b \prec a\}$.

Lemma 2.6.7. $\mathbf{KP} \vdash \mathrm{Prog}_{M_\theta}(U) \to \mathrm{Prog}_{M_\theta}(U^\delta)$

Proof. Assume

(1) $\text{Prog}_{M_\theta}(U)$

(2) $b \in M_\theta$

(3) $(\forall x \prec_{M_\theta} b)(z \in U^\delta)$

Under these assumptons we need to verify that $b \in U^\delta$. Since we already have that $b \in M_\theta$ by (2), it suffices to verify

$$(\forall a \in M_\theta)[M_\theta \cap a \subseteq U \to M_\theta \cap a + \omega^b \subseteq U]$$

to this end we assume that

(4) $a \in M_\theta \quad \text{and} \quad M_\theta \cap a \subseteq U$

Now choose some $d \in M_\theta \cap a + \omega^b$, we must show that $d \in U$ under the assumptions (1)-(4).

If $d \prec a$ then we have $d \in U$ by (4).

If $d = a$ then using (1) and (4) we have $a \in U$.

If $d \succ a$ then since $d \prec a + \omega^b$, we may find $d_1, ..., d_k$ such that

$$d = a + \omega^{d_1} + ... + \omega^{d_k} \quad \text{and} \quad d_k \preceq ... \preceq d_1 \prec b$$

Since $M_\theta \cap a \subseteq U$ we get $M_\theta \cap a + \omega^{d_1} \subseteq U$ from (3).

In a similar fashion using (3) a further $k - 1$ times we obtain

$$M_\theta \cap a + \omega^{d_1} + ... + \omega^{d_k} \subseteq U$$

Finally using one application of $\text{Prog}_{M_\theta}(U)$ (assumption (1)) we have $d \in U$ and thus the proof is complete. ☐

Definition 2.6.8. We define the class X_θ in **KP** as

$$X_\theta := \{a \in M_\theta \mid (\exists x \in Ka)(x \succeq a) \vee \psi a \in \text{Acc}_\theta\}$$

Recall that the function k was defined in Definition 2.1.11 and can be computed in a θ-primitive recursion fashion. The class X_θ may be thought of as those $a \in M_\theta$ for which either ψa is undefined or $\psi a \in \text{Acc}_\theta$.

Lemma 2.6.9. $\mathbf{KP} \vdash \text{Prog}_{M_\theta}(X_\theta)$.

Proof. Assume

(1) $$a \in M_\theta$$

(2) $$(\forall z \prec_{M_\theta} a)(z \in X_\theta)$$

We need to verify that $a \in X_\theta$. If $(\exists x \in Ka)(x \succeq a)$ then we are done, so assume $(\forall x \in Ka)(x \prec a)$ and thus $\psi a \in T(\theta)$ and we must verify that $\psi a \in \mathrm{Acc}_\theta$. To achieve this we verify that

(*) $$b \prec \psi a \;\Rightarrow\; b \in \mathrm{Acc}_\theta$$

from which we would be done by 2.6.3ii). To verify (*) we proceed by induction on Gb, the term complexity of b.

If $b \equiv 0$ or $b \equiv \Gamma_\beta$ for some $\beta \leq \theta$ we are done by 2.6.3v).

If $b \equiv b_0 + ... + b_n$ or $b \equiv \varphi b_0 b_1$ then the result follows by the induction hypothesis and 2.6.3ii) or 2.6.3iii).

So suppose that $b \equiv \psi b_0$. It must be the case that $(\forall x \in Kb_0)(x \prec b_0)$ and $b_0 \prec a$. We must now show that $b_0 \in M_\theta$ in order to use (2) to conclude that $b_0 \in X_\theta$. The claim is that

(**) $$\mathrm{SC}_{\prec\Omega}(b_0) \subseteq \mathrm{Acc}_\theta \quad \text{and thus} \quad b_0 \in M_\theta$$

Suppose $d \in \mathrm{SC}_{\prec\Omega}(b_0)$ then either $d \equiv \Gamma_\beta$ for some $\beta \leq \theta$ in which case $d \in \mathrm{Acc}_\theta$ by 2.6.3v) or $d \equiv \psi d_0 \prec \psi a$ for some d_0. But

$$Gd \leq Gb_0 < Gb$$

and thus $d \in \mathrm{Acc}_\theta$ by induction hypothesis. Thus the claim (**) is verified. Now using (2) we obtain $b_0 \in X_\theta$ which implies $b \equiv \psi b_0 \in \mathrm{Acc}_\theta$. ☐

Lemma 2.6.10. For any $n < \omega$ and any definable class U

$$\mathbf{KP} \vdash \mathrm{Prog}_{M_\theta}(U) \;\rightarrow\; M_\theta \cap e_n(\theta) \subseteq U \wedge e_n(\theta) \in U.$$

Proof. We proceed by induction on n [outside of **KP**].

If $n = 0$ then $\mathrm{Prog}_{M_\theta}(U)$ says that

$$(\forall a \in \mathrm{Acc}_\theta)[(\forall b \prec a)(b \in U) \rightarrow a \in U].$$

So using Acc_θ-induction (Lemma 2.6.2) we obtain $\mathrm{Acc}_\theta \subseteq U$. Hence from 2.6.5 we get $M_\theta \cap \Omega \subseteq U$. Now $\Omega, \Omega + 1 \in M_\theta$ so using $\mathrm{Prog}_{M_\theta}(U)$ a further two times we have $\Omega + 1 := e_o(\theta) \in U$ as required.

Now suppose the result holds up to n; since the induction hypothesis holds for *all* definable classes we have that that

$$\mathbf{KP} \vdash \mathrm{Prog}_{M_\theta}(U^\delta) \to M_\theta \cap e_n(\theta) \subseteq U^\delta \wedge e_n(\theta) \in U^\delta$$

and by Lemma 2.6.7 we have

$$\mathbf{KP} \vdash \mathrm{Prog}_{M_\theta}(U) \to M_\theta \cap e_n(\theta) \subseteq U^\delta \wedge e_n(\theta) \in U^\delta. \tag{1}$$

Now we argue informally in **KP**. Suppose $\mathrm{Prog}_{M_\theta}(U)$, then from (1) we obtain

$$M_\theta \cap e_n(\theta) \subseteq U^\delta \quad \wedge \quad e_n(\theta) \in U^\delta.$$

This says that

$$(\forall b \in M_\theta \cap (e_n(\theta)+1))(\forall a \in M_\theta)[M_\theta \cap a \subseteq U \to M_\theta \cap a + \omega^b \subseteq U].$$

Now if we put $a = 0$ and $b = e_n(\theta)$ (noting that $e_n(\theta) \in M_\theta$) we obtain

$$M_\theta \cap \omega^{e_n(\theta)} \subseteq U$$

from which $\mathrm{Prog}_{M_\theta}(U)$ implies $\omega^{e_n(\theta)} \in U$ as required. □

Theorem 2.6.11. For every $n < \omega$

$$\mathbf{KP} \vdash \psi(e_n(\theta)) \in \mathrm{Acc}_\theta$$

and hence $\mathbf{KP} \vdash A_n(\theta)$.

Proof. By 2.6.9 we have $\mathrm{Prog}_{M_\theta}(X_\theta)$ recalling that

$$X_\theta := \{a \in M_\theta \mid (\exists x \in Ka)(x \succeq a) \vee \psi a \in \mathrm{Acc}_\theta\}.$$

So from 2.6.10 we get $e_n(\theta) \in X$ for any $n < \omega$ and thus $\psi(e_n(\theta)) \in \mathrm{Acc}_\theta$. □

2.7 The provably total set functions of KP

For each $n < \omega$ we define the following recursive set function

$$G_n(X) := L_{\psi(e_n(rk(X)))}(X)$$

For a formula $A(a,b)$ of **KP** let

$$\forall x \exists! y A(x,y) := \forall x \forall y_1 \forall y_2 [A(x,y_1) \wedge A(x,y_2) \to y_1 = y_2] \wedge \forall x \exists y A(x,y).$$

Definition 2.7.1. If T is a theory formulated in the language of set theory, f a set function and $\mathfrak{X}$ a class of formulae. We say that f is $\mathfrak{X}$ definable in T if there is some $\mathfrak{X}$-formula $A_f(a,b)$ with exactly the free variables a, b such that

i) $V \models A_f(x,y) \leftrightarrow f(x) = y$.

ii) $T \vdash \forall x \exists ! y A_f(x,y)$.

Theorem 2.7.2. Suppose f is a set function that is Σ definable in **KP**, then there is some n (which we may compute from the finite derivation) such that

$$V \models \forall x (f(x) \in G_n(x)).$$

Moreover G_m is Σ definable in **KP** for each $m < \omega$.

Proof. Let $A_f(a,b)$ be the Σ formula expressing f such that $\mathbf{KP} \vdash \forall x \exists ! y A_f(x,y)$ and fix an arbitrary set X. Let θ be the rank of X. Applying Theorem 2.5.10 we can compute some $k < \omega$ such that

$$\mathcal{H}_0 \left|\frac{\Omega\cdot\omega^k}{\Omega+k}\right. \forall x \exists ! y A_f(x,y)\,.$$

Applying Lemma 3.4.1iv) twice we get

$$\mathcal{H}_0 \left|\frac{\Omega\cdot\omega^k}{\Omega+k}\right. \exists y A_f(X,y)\,.$$

Applying Theorem 2.4.8 (predicative cut elimination) we get

$$\mathcal{H}_0 \left|\frac{e_{k+1}(\theta)}{\Omega+1}\right. \exists y A_f(X,y)\,.$$

Now by Theorem 2.4.14 (collapsing) we have

$$\mathcal{H}_{e_{k+2}(\theta)} \left|\frac{\psi(e_{k+2}(\theta))}{\psi(e_{k+2}(\theta))}\right. \exists y A_f(X,y)\,.$$

Applying Theorem 2.4.8 (predicative cut elimination) again yields

$$\mathcal{H}_\gamma \left|\frac{\varphi(\psi\gamma)(\psi\gamma)}{0}\right. \exists y A_f(X,y) \quad \text{where } \gamma := e_{k+2}(\theta).$$

Now by Lemma 2.4.9 (boundedness) we obtain

(1) $$\mathcal{H}_\gamma \left|\frac{\alpha}{0}\right. (\exists y \in \mathbb{L}_\alpha) A_f(X,y)^{\mathbb{L}_\alpha} \quad \text{where } \alpha := \varphi(\psi\gamma)(\psi\gamma).$$

Since (1) contains no instances of (Cut) or (Σ-Ref$_\Omega(X)$), it follows by induction on α that

$$L_\alpha(X) \models \exists y A_f(X,y)$$

It remains to note that $L_\alpha(X) \subseteq G_{k+3}(X)$ to complete this direction of the proof.

For the other direction we argue informally in **KP**. Let X be an arbitrary set, we may specify the rank of X in a Δ_0 manner([3] p29). By Theorem 2.6.11 we can find an ordinal of the same order type as $e_n(rk(X))$. We can now generate $L_{e_n(rk(X))}(X)$ by Σ-recursion ([3] p26 theorem 6.4). ∎

The comparison of Theorem 2.0.2 with Theorem 2.7.2 provides a pleasing relation between the arithmetic and set theoretic worlds.

Remark 2.7.3. In fact the first part of 2.7.2 can be carried out *inside* **KP**, i.e. If f is Σ definable in **KP** then we can compute some n such that $\mathbf{KP} \vdash \forall x(\exists! y \in G_n(x))A_f(x,y)$. This is not immediately obvious since it appears we need induction up to $\psi(\varepsilon_{\Omega_\theta+1})$, which we do not have access to in **KP**. The way to get around this is to note that we could, in fact, have managed with an infinitary system based on an ordinal representation built out of $B_\theta(e_m(\theta))$, provided m is high enough, and we may compute how high m needs to be from the finite derivation. We do have access to induction up to $\psi(e_m(\theta))$ in **KP** by Theorem 2.6.11.

Chapter 3

An ordinal analysis of IKP

This chapter provides a detailed rendering of the ordinal analysis of Kripke-Platek set theory formulated with intuitionistic logic, **IKP**. This is done in such a way that not only do we characterise the proof theoretic ordinal of **IKP** (in the sense of [34]), but also so that we are able to extract witness terms from the resulting cut-free derivations of Σ sentences in the infinitary system. This results in a proof that **IKP** has the existence property for Σ sentences, which in conjunction with results in [40] verifies that $\mathbf{CZF}^{-}$ has the full existence property.

This chapter is essentially an application of well known techniques to the intuitionistic case. There are certain technical issues arising in the intuitionistic case that need checking, moreover many of the arguments in this chapter are modular and transfer over to the stronger systems analysed in subsequent chapters with minimal changes.

3.1 A sequent calculus formulation of IKP

Definition 3.1.1. The language of **IKP** consists of free variables $a_0, a_1, ...$, bound variables $x_0, x_1, ...$, the binary predicate symbol $\in$ and the logical symbols $\neg, \vee, \wedge, \rightarrow, \forall, \exists$ as well as parentheses $), ($.

The atomic formulas are those of the form $a \in b$.

The formulas of **IKP** are defined inductively by:

i) All atomic formulas are formulas.

ii) If A and B are formulas then so are $\neg A$, $A \vee B$, $A \wedge B$ and $A \rightarrow B$.

iii) If $A(b)$ is a formula in which the bound variable x does not occur, then $\forall x A(x)$, $\exists x A(x)$, $(\forall x \in a)A(x)$ and $(\exists x \in a)A(x)$ are also formulas.

Quantifiers of the form $\exists x$ and $\forall x$ will be called unbounded and those of the form $(\exists x \in a)$ and $(\forall x \in a)$ will be referred to as bounded quantifiers.

A Δ_0-formula is one in which no unbounded quantifiers appear.

The expression $a = b$ is to be treated as an abbreviation for $(\forall x \in a)(x \in b) \wedge (\forall x \in b)(x \in a)$.

The derivations of **IKP** take place in a two-sided sequent calculus. The sequents derived are *intuitionistic sequents* of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulas and Δ contains at most one formula. The intended meaning of $\Gamma \Rightarrow \Delta$ is that the conjunction of formulas in Γ implies the formula in Δ, or if Δ is empty, a contradiction. The expressions $\Rightarrow \Delta$ and $\Gamma \Rightarrow$ are shorthand for $\emptyset \Rightarrow \Delta$ and $\Gamma \Rightarrow \emptyset$ respectively.
The axioms of **IKP** are:

Logical axioms:	$\Gamma, A, \Rightarrow A$ for every Δ_0 formula A.
Extensionality:	$\Gamma \Rightarrow a = b \wedge B(a) \rightarrow B(b)$ for every Δ_0 formula $B(a)$.
Pair:	$\Gamma \Rightarrow \exists z(a \in z \wedge b \in z)$.
Union:	$\Gamma \Rightarrow \exists z(\forall y \in z)(\forall x \in y)(x \in z)$.
Δ_0-*Separation:*	$\Gamma \Rightarrow \exists y[(\forall x \in y)(x \in a \wedge B(x)) \wedge (\forall x \in a)(B(x) \rightarrow x \in y)]$ for every Δ_0-formula $B(a)$.
Set Induction:	$\Gamma \Rightarrow \forall x[(\forall y \in x F(y) \rightarrow F(x)] \rightarrow \forall x F(x)$ for any formula $F(a)$.
Infinity:	$\Gamma \Rightarrow \exists x[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$.
Δ_0-*Collection:*	$\Gamma \Rightarrow (\forall x \in a)\exists y G(x,y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x,y)$ for any Δ_0-formula G.

The rules of inference are

$$(\wedge L)\ \frac{\Gamma, C \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}\ \text{For } C \in \{A, B\} \qquad (\wedge R)\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$(\vee L)\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta} \qquad (\vee R)\ \frac{\Gamma \Rightarrow C}{\Gamma \Rightarrow A \vee B}\ \text{For } C \in \{A, B\}$$

$$(\neg L)\ \frac{\Gamma \Rightarrow A}{\Gamma, \neg A \Rightarrow} \qquad (\neg R)\ \frac{\Gamma, A \Rightarrow}{\Gamma \Rightarrow \neg A} \qquad (\bot)\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}$$

$$(\rightarrow L)\ \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A}{\Gamma, A \rightarrow B \Rightarrow \Delta} \qquad (\rightarrow R)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$(b\exists L)\ \frac{\Gamma, a \in b \wedge F(a) \Rightarrow \Delta}{\Gamma, (\exists x \in b)F(x) \Rightarrow \Delta} \qquad (b\exists R)\ \frac{\Gamma \Rightarrow a \in b \wedge F(a)}{\Gamma \Rightarrow (\exists x \in b)F(x)}$$

$$(b\forall L)\ \frac{\Gamma, a \in b \to F(a) \Rightarrow \Delta}{\Gamma, (\forall x \in b)F(x) \Rightarrow \Delta} \qquad (b\forall R)\ \frac{\Gamma \Rightarrow a \in b \to F(a)}{\Gamma \Rightarrow (\forall x \in b)F(x)}$$

$$(\exists L)\ \frac{\Gamma, F(a) \Rightarrow \Delta}{\Gamma, \exists x F(x) \Rightarrow \Delta} \qquad (\exists R)\ \frac{\Gamma \Rightarrow F(a)}{\Gamma \Rightarrow \exists x F(x)}$$

$$(\forall L)\ \frac{\Gamma, F(a) \Rightarrow \Delta}{\Gamma, \forall x F(x) \Rightarrow \Delta} \qquad (\forall R)\ \frac{\Gamma \Rightarrow F(a)}{\Gamma \Rightarrow \forall x F(x)}$$

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow A \qquad \Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

In each of the inferences $(b\exists L)$, $(\exists L)$ $(b\forall R)$ and $(\forall R)$ the variable a is forbidden from occurring in the conclusion. Such a variable is known as the *eigenvariable* of the inference.

The *minor formulae* of an inference are those rendered prominently in its premises, the other formulae in the premises will be referred to as *side formulae*. The *principal* formula of an inference is the one rendered prominently in the conclusion. Note that in inferences where the principal formula is on the left, the principal formula can also be a side formula of that inference, when this happens we say that there has been a *contraction*.

3.2 An ordinal notation system

Given below is a very brief description of how to carry out the construction of a primitive recursive ordinal notation system for the Bachmann-Howard ordinal. This construction is very similar to the one carried out in full detail in the previous chapter, only there is no ordering inserted as an initial segment.

Definition 3.2.1. Let Ω be a 'big' ordinal, eg. $\aleph_1$. (In fact we could have chosen ω_1^{CK} as shown in [31].) We define the sets $B^{\Omega}(\alpha)$ and ordinals $\psi_{\Omega}(\alpha)$ by transfinite recursion on α as follows

$$B^{\Omega}(\alpha) \quad = \quad \begin{cases} \textit{closure of } \{0, \Omega\} \textit{ under:} \\ +, (\xi, \eta \mapsto \varphi\xi\eta) \\ (\xi \longmapsto \psi_{\Omega}(\xi))_{\xi<\alpha} \end{cases} \tag{3.1}$$

$$\psi_{\Omega}(\alpha) \quad \simeq \quad \min\{\rho < \Omega : \ \rho \notin B(\alpha)\}. \tag{3.2}$$

It can be shown that $\psi_{\Omega}(\alpha)$ is always defined and thus $\psi_{\Omega}(\alpha) < \Omega$. Moreover, it can also be shown that $B_{\Omega}(\alpha) \cap \Omega = \psi_{\Omega}(\alpha)$.

Let $\varepsilon_{\Omega+1}$ be the least orinal $\eta > \Omega$ such that $\omega^\eta = \eta$. The set $B^\Omega(\varepsilon_{\Omega+1})$ gives rise to a primitive recursive ordinal notation system [6] [35]. The ordinal $\psi_\Omega(\varepsilon_{\Omega+1})$ is known as the Bachmann-Howard ordinal. There are many slight variants in the specific ordinal functions used to build up a notation system for this ordinal, for example rather than 'closing off' under the φ function at each stage, we could have chosen ω-exponentiation, all the systems turn out to be equivalent, in that they eventually 'catch-up' with one another and the specific ordinal functions used can be defined in terms of one another. Here the functions φ and ψ are chosen as primitive since they correspond to the ordinal operations arising from the two main cut elimination theorems of the next section.

3.3 The infinitary system $\mathbf{IRS}_\Omega$

The purpose of this section is to define an intuitionistic style infinitary system $\mathbf{IRS}_\Omega$ within which we will be able to embed **IKP** and then extract useful information about **IKP** derivations.

Henceforth all ordinals will be assumed to belong to the primitive recursive ordinal representation system arising from $B^\Omega(\varepsilon_{\Omega+1})$.

The system is based around the constructible hierarchy up to level Ω.

$$\begin{aligned} L_0 &:= \emptyset \\ L_{\alpha+1} &= \{X \subseteq L_\alpha \mid X \text{ is definable over } L_\alpha \text{ in the language of } \mathbf{IKP} \text{ with parameters}\} \\ L_\lambda &:= \bigcup_{\xi<\lambda} L_\xi \quad \text{if } \lambda \text{ is a limit ordinal} \end{aligned}$$

Definition 3.3.1. We inductively define the terms of $\mathbf{IRS}_\Omega$. To each term t we also assign an ordinal level $|t|$.

i) For each $\alpha < \Omega$, $\mathbb{L}_\alpha$ is a term with $|\mathbb{L}_\alpha| := \alpha$.

ii) If $F(a, b_1, ..., b_n)$ is a formula of **IKP** with all free variables indicated and $s_1, ..., s_n$ are $\mathbf{IRS}_\Omega$ terms with levels less than α, then

$$[x \in \mathbb{L}_\alpha \mid F(x, s_1, ..., s_n)^{\mathbb{L}_\alpha}]$$

is a term of level α. Here $F^{\mathbb{L}_\alpha}$ indicates that all unbounded quantifiers in F are restricted to $\mathbb{L}_\alpha$.

The formulae of $\mathbf{IRS}_\Omega$ are of the form $F(s_1, ..., s_n)$ where $F(a_1, ..., a_n)$ is a fomula of **IKP** with all free variables displayed and $s_1, ..., s_n$ are $\mathbf{IRS}_\Omega$-terms.

Note that the system $\mathbf{IRS}_\Omega$ does not contain free variables. We can think of the universe made up of $\mathbf{IRS}_\Omega$-terms as a formal, syntactical version of L_Ω, unbounded quantifiers in $\mathbf{IRS}_\Omega$-formulas can be thought of as ranging over L_Ω.

For the remainder of this section $\mathbf{IRS}_\Omega$-terms and $\mathbf{IRS}_\Omega$-formulae will simply be referred to as terms and formulae.

A formula is said to be Δ_0 if it contains no unbounded quantifiers.

The Σ-formulae are the smallest collection containing the Δ_0-formulas and containing $A \vee B$, $A \wedge B$, $(\forall x \in s)A$, $(\exists x \in s)A$ and $\exists x A$ whenever it contains A and B. Likewise The Π-formulae are the smallest collection containing the Δ_0-formulas and containing $A \vee B$, $A \wedge B$, $(\forall x \in s)A$, $(\exists x \in s)A$ and $\forall x A$ whenever it contains A and B.

Abbreviation 3.3.2. For $\diamond$ a binary propositional connective, A a formula and s, t terms with $|\,s\,| < |\,t\,|$ we define the following abbreviation:

$$\begin{aligned} s \mathrel{\dot{\in}} t \diamond A :=& A && \text{if } t \text{ is of the form } \mathbb{L}_\alpha \\ :=& B(s) \diamond A && \text{if } t \text{ is of the form } [x \in \mathbb{L}_\alpha \mid B(x)] \end{aligned}$$

Like in **IKP**, derivations in $\mathbf{IRS}_\Omega$ take place in a two sided sequent calculus. Intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ are derived, where Γ and Δ are finite sets of formulae and at most one formula occurs in Δ. $\Gamma, \Delta, \Lambda, ...$ will be used as meta variables ranging over finite sets of formulae.

$\mathbf{IRS}_\Omega$ has no axioms, although note that some of the rules can have an empty set of premises. The inference rules are as follows:

$$(\in L)_\infty \quad \frac{\Gamma,\ p \dot{\in} t \wedge r = p \Rightarrow \Delta \ \text{ for all } |\,p\,| < |\,t\,|}{\Gamma, r \in t \ \ \Rightarrow \Delta}$$

$$(\in R) \quad \frac{\Gamma \Rightarrow s \dot{\in} t \ \wedge \ r = s}{\Gamma \Rightarrow s \in t} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(b\forall L) \quad \frac{\Gamma,\ s \dot{\in} t \rightarrow A(s) \Rightarrow \Delta}{\Gamma, (\forall x \in t)A(x) \ \ \Rightarrow \Delta} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(b\forall R)_\infty \quad \frac{\Gamma \Rightarrow p \dot{\in} t \rightarrow A(p) \ \text{ for all } |\,p\,| < |\,t\,|}{\Gamma \Rightarrow (\forall x \in t)A(x)}$$

$$(b\exists L)_\infty \quad \frac{\Gamma,\ p \dot{\in} t \wedge A(p) \Rightarrow \Delta \ \text{ for all } |\,p\,| < |\,t\,|}{\Gamma,\ (\exists x \in t)A(x) \ \ \Rightarrow \Delta}$$

$$(b\exists R) \quad \frac{\Gamma \Rightarrow s \dot{\in} t \wedge A(s)}{\Gamma \Rightarrow (\exists x \in t)A(x)} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(\forall L) \quad \frac{\Gamma,\ A(s) \Rightarrow \Delta}{\Gamma, \forall x\, A(x) \ \ \Rightarrow \Delta}$$

$$(\forall R)_\infty \quad \frac{\Gamma \Rightarrow A(p) \ \text{ for all } p}{\Gamma \Rightarrow \forall x A(x)}$$

$$(\exists L)_\infty \quad \frac{\Gamma,\ A(p) \Rightarrow \Delta \ \text{ for all } p}{\Gamma,\ \exists x A(x) \ \ \Rightarrow \Delta}$$

$$(\exists R) \quad \frac{\Gamma \Rightarrow A(s)}{\Gamma \Rightarrow \exists x A(x)}$$

$$(\Sigma\text{-Ref}_\Omega) \quad \frac{\Gamma \ \Rightarrow \ A}{\Gamma \Rightarrow \exists z A^z} \quad \text{if } A \text{ is a } \Sigma\text{-formula,}$$

As well as the rules $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\rightarrow L)$, $(\rightarrow R)$ and (Cut) which are defined identically to the rules of the same name in **IKP**.

In general we are unable to remove cuts from $\mathbf{IRS}_\Omega$ derivations, one of the main obstacles to full cut elimination comes from $(\Sigma\text{-Ref}_\Omega)$ since it breaks the symmetry of the other rules. However we can still perform cut elimination on certain derivations, provided they are of a very uniform kind. Luckily, certain embedded proofs from **IKP** will be of this form. In order to express uniformity in infinite proofs we draw on [8], where Bucholz developed a powerful method of describing such uniformity, called *operator control.*

Definition 3.3.3. Let

$$P(ON) = \{X : X \text{ is a set of ordinals}\}.$$

A class function

$$\mathcal{H} : P(ON) \to P(ON)$$

will be called an **operator** if $\mathcal{H}$ satisfies the following conditions for all $X \in P(ON)$:

1. $X \subseteq Y \Rightarrow \mathcal{H}(X) \subseteq \mathcal{H}(Y)$ (monotone)

2. $X \subseteq \mathcal{H}(X)$ (inclusive)

3. $\mathcal{H}(\mathcal{H}(X)) = \mathcal{H}(X)$ (idempotent)

4. $0 \in \mathcal{H}(X)$ and $\Omega \in \mathcal{H}(X)$.

5. If α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then

$$\alpha \in \mathcal{H}(X) \quad \text{iff} \quad \alpha_1, ..., \alpha_n \in \mathcal{H}(X).$$

The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^\sigma$, and decomposition of its members into additive and multiplicative components.

From now on $\alpha \in \mathcal{H}$ and $\{\alpha_1, ..., \alpha_n\} \subseteq \mathcal{H}$ will be considered shorthand for $\alpha \in \mathcal{H}(\emptyset)$ and $\{\alpha_1, ..., \alpha_n\} \subseteq \mathcal{H}(\emptyset)$ respectively.

Definition 3.3.4. If A is a formula let

$$k(A) := \{\alpha \in ON \ : \ \text{the symbol } \mathbb{L}_\alpha \text{ occurs in } A, \text{ subterms included}\}.$$

Likewise we define

$$k(\{A_1, ..., A_n\}) := k(A_1) \cup ... \cup k(A_n) \quad \text{and} \quad k(\Gamma \Rightarrow \Delta) := k(\Gamma) \cup k(\Delta).$$

Now for $\mathcal{H}$ an arbitrary operator, s a term and $\mathfrak{X}$ a formula, set of formulae or a sequent we define

$$\begin{aligned} \mathcal{H}[s](X) :=& \mathcal{H}(X \cup \{|s|\}) \\ \mathcal{H}[\mathfrak{X}](X) :=& \mathcal{H}(X \cup k(\mathfrak{X})) \end{aligned}$$

Lemma 3.3.5. Let $\mathcal{H}$ be an operator, s a term and $\mathfrak{X}$ a formula, set of formulae or sequent.

(i) For any $X, X' \in P(ON)$, if $X' \subseteq X$ then $\mathcal{H}(X') \subseteq \mathcal{H}(X)]$.

(ii) $\mathcal{H}[s]$ and $\mathcal{H}[\mathfrak{X}]$ are operators.

(iii) If $k(\mathfrak{X}) \subseteq \mathcal{H}(\emptyset)$ then $\mathcal{H}[\mathfrak{X}] = \mathcal{H}$.

(iv) If $|\, s \,| \in \mathcal{H}$ then $\mathcal{H}[s] = \mathcal{H}$.

Proof. This result is demonstrated in full in [35]. $\square$

We also need to keep track of the complexity of cuts appearing in derivations.

Definition 3.3.6. The *rank* of a term or formula is determined by

1. $rk(\mathbb{L}_\alpha) := \omega \cdot \alpha$

2. $rk([x \in \mathbb{L}_\alpha \mid F(x)]) := \max\{\omega \cdot \alpha + 1, rk(F(\mathbb{L}_0)) + 2\}$

3. $rk(s \in t) := \max\{rk(s) + 6, rk(t) + 1\}$

4. $rk(A \wedge B) = rk(A \vee B) = rk(A \rightarrow B) := \max\{rk(A) + 1, rk(B) + 1\}$

5. $rk(\neg A) := rk(A) + 1$

6. $rk((\exists x \in t)A(x)) = rk((\forall x \in t)A(x)) := \max\{rk(t), rk(F(\mathbb{L}_0)) + 2\}$

7. $rk(\exists x A(x)) = rk(\forall x A(x)) := \max\{\Omega, rk(F(\mathbb{L}_0)) + 1\}$

Observation 3.3.7. i) $rk(s) = \omega \cdot |s| + n$ for some $n < \omega$.

ii) If A is Δ_0, $rk(A) = \omega \cdot \max(k(A)) + m$ for some $m < \omega$.

iii) If A contains unbounded quantifiers $rk(A) = \Omega + m$ for some $m < \omega$.

iv) $rk(A) < \Omega$ if and only if A is Δ_0.

There is plenty of leeway in defining the actual rank of a formula, basically we need to make sure the following lemma holds.

Lemma 3.3.8. In every rule of $\mathbf{IRS}_\Omega$ other than (Σ-Ref$_\Omega$) and (Cut), the rank of the minor formulae is strictly less than the rank of the principal formula.

Proof. This result is demonstrated for a different set of propositional connectives in [35], the adapted proof to the intuitionistic system is similar. $\square$

Definition 3.3.9 (Operator controlled derivability for $\mathbf{IRS}_\Omega$)**.** Let $\mathcal{H}$ be an operator and $\Gamma \Rightarrow \Delta$ an intuitionistic sequent of $\mathbf{IRS}_\Omega$, we define the relation $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta$ by recursion on α.
We require always that $k(\Gamma \Rightarrow \Delta) \cup \{\alpha\} \subseteq \mathcal{H}$, this condition will not be repeated in the inductive clauses for each of the inference rules of $\mathbf{IRS}_\Omega$ below. The column on the right gives the ordinal requirements for each of the inference rules.

$$(\in L)_\infty \quad \frac{\mathcal{H}[r] \overset{\alpha_r}{\underset{\rho}{\vdash}} \Gamma, r \,\dot{\in}\, t \wedge r = s \Rightarrow \Delta \text{ for all } |r| < |t|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \in t \Rightarrow \Delta} \quad |r| \leq \alpha_r < \alpha$$

$$(\in R) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow r \,\dot{\in}\, t \wedge r = s}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in t} \quad \begin{array}{r} \alpha_0 < \alpha \\ |r| < |t| \\ |r| < \alpha \end{array}$$

$$(b\forall L) \qquad \frac{\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\mid}} \Gamma, s \dot{\in} t \to A(s) \Rightarrow \Delta}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma, (\forall x \in t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,s\,| < |\,t\,| \\ |\,s\,| < \alpha \end{array}$$

$$(b\forall R)_\infty \qquad \frac{\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{\mid}} \Gamma \Rightarrow s \dot{\in} t \to F(s) \text{ for all } |\,s\,| < |\,t\,|}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow (\forall x \in t)F(x)} \qquad |\,s\,| \leq \alpha_s < \alpha$$

$$(b\exists L)_\infty \qquad \frac{\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{\mid}} \Gamma, s \dot{\in} t \wedge F(s) \Rightarrow \Delta \text{ for all } |\,s\,| < |\,t\,|}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma, (\exists x \in t)F(x) \Rightarrow \Delta} \qquad |\,s\,| \leq \alpha_s < \alpha$$

$$(b\exists R) \qquad \frac{\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow s \dot{\in} t \wedge A(s)}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow (\exists x \in t)A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\,s\,| < |\,t\,| \\ |\,s\,| < \alpha \end{array}$$

$$(\forall L) \qquad \frac{\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\mid}} \Gamma, F(s) \Rightarrow \Delta}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma, \forall x F(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |\,s\,| < \alpha \end{array}$$

$$(\forall R)_\infty \qquad \frac{\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{\mid}} \Gamma \Rightarrow F(s) \text{ for all } s}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \forall x F(x)} \qquad |\,s\,| < \alpha_s + 1 < \alpha$$

$$(\exists L)_\infty \qquad \frac{\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{\mid}} \Gamma, F(s) \Rightarrow \Delta \text{ for all } s}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma, \exists x F(x) \Rightarrow \Delta} \qquad |\,s\,| < \alpha_s + 1 < \alpha$$

$$(\exists R) \qquad \frac{\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow F(s)}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma, \Rightarrow \exists x F(x)} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |\,s\,| < \alpha \end{array}$$

$$(\text{Cut}) \qquad \frac{\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\mid}} \Gamma, B \Rightarrow \Delta \qquad \mathcal{H} \stackrel{\alpha_1}{\underset{\rho}{\mid}} \Gamma \Rightarrow B}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ rk(B) < \rho \end{array}$$

$$(\Sigma\text{-Ref}_\Omega) \qquad \frac{\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow A}{\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \exists z\, A^z} \qquad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \text{ is a } \Sigma\text{-formula} \end{array}$$

Lastly if $\Gamma \Rightarrow \Delta$ is the result of a propositional inference of the form $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\to L)$ or $(\to R)$, with premise(s) $\Gamma_i \Rightarrow \Delta_i$ then from $\mathcal{H} \stackrel{\alpha_0}{\underset{\rho}{\mid}} \Gamma_i \Rightarrow \Delta_i$ (for each i) we may conclude $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \Delta$, provided $\alpha_0 < \alpha$.

Lemma 3.3.10 (Weakening and Persistence for $\mathbf{IRS}_\Omega$). **i)** If $\Gamma_0 \subseteq \Gamma$, $k(\Gamma \Rightarrow \Delta) \subseteq \mathcal{H}$, $\alpha_0 \leq \alpha \in \mathcal{H}$, $\rho_0 \leq \rho$ and $\mathcal{H} \stackrel{\alpha_0}{\underset{\rho_0}{\mid}} \Gamma_0 \Rightarrow \Delta$ then

$$\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \Delta$$

ii) If $\beta \geq \gamma \in \mathcal{H}$ and $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{L}_\beta)A(x) \Rightarrow \Delta$ then $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{L}_\gamma)A(x) \Rightarrow \Delta$.

iii) If $\beta \geq \gamma \in \mathcal{H}$ and $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\beta)A(x)$ then $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\gamma)A(x)$

iv) If $\gamma \in \mathcal{H}$ and $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma, \exists x A(x) \Rightarrow \Delta$ then $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{L}_\gamma)A(x) \Rightarrow \Delta$.

v) If $\gamma \in \mathcal{H}$ and $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow \forall x A(x)$ then $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\gamma)A(x)$.

Proof. We show i), ii) and v).

i) is proved by an easy induction on α.

ii) Is also proved using induction on α, suppose $\beta \geq \gamma \in \mathcal{H}(\emptyset)$ and $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{L}_\beta)A(x) \Rightarrow \Delta$. If $(\exists x \in \mathbb{L}_\beta)A(x)$ was not the principal formula of the last inference or the last inference was not $(b\exists L)_\infty$ then we may apply the induction hypotheses to it's premises followed by the same inference again. So suppose $(\exists x \in \mathbb{L}_\beta)A(x)$ was the principal formula of the last inference which was $(b\exists L)_\infty$, so we have

$$\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{L}_\beta)A(x), A(s) \Rightarrow \Delta \quad \text{for all } |s| < \beta, \text{ with } \alpha_s < \alpha.$$

From the induction hypothesis we obtain

$$\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{L}_\gamma)A(x), A(s) \Rightarrow \Delta \quad \text{for all } |s| < \beta, \text{ with } \alpha_s < \alpha$$

but since $\beta \geq \gamma$ this also holds for all $|s| < \gamma$. So by another application of $(b\exists L)_\infty$ we get

$$\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{L}_\gamma)A(x) \Rightarrow \Delta$$

as required.

For v) suppose $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow \forall x A(x)$. The interesting case is where $\forall x A(x)$ was the principal formula of the last inference, which was $(\forall R)_\infty$, in this case we have

$$\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{|}} \Gamma \Rightarrow A(s) \quad \text{for all } s, \text{ with } |\,s\,| < \alpha_s + 1 < \alpha.$$

So taking just the cases where $|\,s\,| < \gamma$ and noting that in these cases $A(s) \equiv s \dot{\in} \mathbb{L}_\gamma \rightarrow A(s)$, we may apply $(b\forall R)$ to obtain

$$\mathcal{H} \stackrel{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow (\forall x \in \mathbb{L}_\gamma)A(x)$$

as required.

The proofs of iii) and iv) may be carried out in a similar manner to those above. $\square$

3.4 Cut elimination for $\mathbf{IRS}_\Omega$

Lemma 3.4.1 (Inversions of $\mathbf{IRS}_\Omega$). **i)** *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \wedge B \Rightarrow \Delta$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A, B \Rightarrow \Delta$.

ii) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A \wedge B$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A$ *and* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow B$.

iii) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \vee B \Rightarrow \Delta$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \Rightarrow \Delta$ *and* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B \Rightarrow \Delta$.

iv) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \to B \Rightarrow \Delta$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, B \Rightarrow \Delta$.

v) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A \to B$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \Rightarrow B$.

vi) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \neg A$ *then* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \Rightarrow$.

vii) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, r \in t \Rightarrow \Delta$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \dot{\in} t \wedge r = s \Rightarrow \Delta$ *for all* $|s| < |t|$.

viii) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \dot{\in} t \wedge A(s) \Rightarrow \Delta$ *for all* $|s| < |t|$.

ix) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow (\forall x \in t)A(x)$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \dot{\in} t \to A(s)$ *for all* $|s| < |t|$.

x) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \exists x A(x) \Rightarrow \Delta$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A(s) \Rightarrow \Delta$ *for all* s.

xi) *If* $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \Rightarrow \forall x A(x)$ *then* $\mathcal{H}[s] \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow A(s)$ *for all* s.

Proof. All proofs are by induction on α, we treat three of the most interesting cases, iv), vi) and x).

iv) Suppose $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, A \to B \Rightarrow \Delta$, If the last inference was not $(\to L)$ or the principal formula of that inference was not $A \to B$ we may apply the induction hypothesis to the premises of that inference, followed by the same inference again. Now suppose $A \to B$ was the principal formula of the last inference, which was $(\to L)$. Thus, with the possible use of weakening, we have

(1) $\qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, B, A \to B \Rightarrow \Delta \qquad$ for some $\alpha_0 < \alpha$.

(2) $\qquad \mathcal{H} \overset{\alpha_1}{\underset{\rho}{\vdash}} \Gamma, A \to B \Rightarrow A \qquad$ for some $\alpha_1 < \alpha$.

Applying the induction hypothesis to (1) yields $\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, B \Rightarrow \Delta$ from which we may obtain the desired result by weakening.

vi) Now suppose $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \neg A$ If $\neg A$ was the principal formula of the last inference which was $(\neg R)$ then we have $\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, A \Rightarrow$ for some $\alpha_0 < \alpha$, from which we may obtain the desired result by weakening. If the last inference was different to $(\neg R)$ we may apply the induction hypothesis to the premises of that inference followed by the same inference again.

x) Finally suppose $\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma, \exists x A(x) \Rightarrow \Delta$. If $\exists x A(x)$ was the principal formula of the last inference which was $(\exists L)_\infty$ then we have

$$\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, \exists x A(x), A(s) \Rightarrow \Delta \quad \text{with } \alpha_s < \alpha \text{ for each } s.$$

Applying the induction hypothesis yields

$$\mathcal{H}[s] \stackrel{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, A(s) \Rightarrow \Delta$$

from which we get the desired result by weakening. If $\exists x A(x)$ was not the principal formula of the last inference or the last inference was not $(\exists L)_\infty$ then we may apply the induction hypothesis to the premises of that inference followed by the same inference again. □

Lemma 3.4.2 (Reduction for $\mathbf{IRS}_\Omega$)**.** Let $\rho := rk(C) \neq \Omega$

$$\text{If} \quad \mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma, C \Rightarrow \Delta \quad \text{and} \quad \mathcal{H} \stackrel{\beta}{\underset{\rho}{\vdash}} \Xi \Rightarrow C \quad \text{then} \quad \mathcal{H} \stackrel{\alpha\#\alpha\#\beta\#\beta}{\underset{\rho}{\vdash}} \Gamma, \Xi \Rightarrow \Delta$$

Proof. The proof is by induction on $\alpha\#\alpha\#\beta\#\beta$. Assume that

(1) $$\rho := rk(C) \neq \Omega$$

(2) $$\mathcal{H} \stackrel{\alpha}{\underset{\rho}{\vdash}} \Gamma, C \Rightarrow \Delta$$

(3) $$\mathcal{H} \stackrel{\beta}{\underset{\rho}{\vdash}} \Xi \Rightarrow C$$

If C was not the principal formula of the last inference in both derivations then we may simply use the induction hypothesis on the premises and then the final inference again.

So suppose C was the principal formula of the last inference in both (2) and (3). Note also that (1) gives us immediately that the last inference in (3) was *not* (Σ-Ref$_\Omega$).

We treat three of the most interesting cases.

Case 1. Suppose $C \equiv r \in t$, thus we have

(4) $$\mathcal{H}[p] \stackrel{\alpha_p}{\underset{\rho}{\vdash}} \Gamma, C, p \,\dot{\in}\, t \wedge r = p \Rightarrow \Delta \quad \text{for all } |\,p\,| < |\,t\,| \text{ with } \alpha_p < \alpha$$

and

(5) $$\mathcal{H} \stackrel{\beta_0}{\underset{\rho}{\vdash}} \Xi \Rightarrow s \,\dot{\in}\, t \wedge r = s \quad \text{for some } |\,s\,| < |\,t\,| \text{ with } \beta_0 < \beta.$$

Now from (5) we know that $|\,s\,| \in \mathcal{H}$ and thus from (4) we have

(6) $$\mathcal{H} \stackrel{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, C, s \,\dot{\in}\, t \wedge r = s \Rightarrow \Delta\,.$$

Applying the induction hypothesis to (6) and (3) yields

$$\mathcal{H} \left|\frac{\alpha_s\#\alpha_s\#\beta\#\beta}{\rho}\right. \Xi, \Gamma, s \mathrel{\dot{\in}} t \wedge r = s \Rightarrow \Delta\,. \tag{7}$$

Finally a (Cut) applied to (5) and (7) yields

$$\mathcal{H} \left|\frac{\alpha\#\alpha\#\beta\#\beta}{\rho}\right. \Xi, \Gamma \Rightarrow \Delta$$

as required.

Case 2. Now suppose $C \equiv (\forall x \in t)F(x)$ so we have

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, C, s \mathrel{\dot{\in}} t \to F(s) \Rightarrow \Delta \quad \text{for some } |\,s\,| < |\,t\,| \text{ with } \alpha_0, |\,s\,| < \alpha \tag{8}$$

and

$$\mathcal{H}[p] \left|\frac{\beta_p}{\rho}\right. \Xi \Rightarrow p \mathrel{\dot{\in}} t \to F(p) \quad \text{for all } |\,p\,| < |\,t\,| \text{ with } \beta_p < \beta. \tag{9}$$

Now (8) gives $s \in \mathcal{H}$ and thus from (9) we have

$$\mathcal{H} \left|\frac{\beta_s}{\rho}\right. \Xi \Rightarrow s \mathrel{\dot{\in}} t \to F(s)\,. \tag{10}$$

Applying the induction hypothesis to (3) and (8) gives

$$\mathcal{H} \left|\frac{\alpha_0\#\alpha_0\#\beta\#\beta}{\rho}\right. \Gamma, \Xi, s \mathrel{\dot{\in}} t \to F(s) \Rightarrow \Delta\,. \tag{11}$$

Finally (Cut) applied to (10) and (11) yields the desired result.

Case 3. Now suppose $C \equiv A \to B$ so we have

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, C \Rightarrow A \quad \text{with } \alpha_0 < \alpha \tag{12}$$

$$\mathcal{H} \left|\frac{\alpha_1}{\rho}\right. \Gamma, C, B \Rightarrow \Delta \quad \text{with } \alpha_1 < \alpha \tag{13}$$

$$\mathcal{H} \left|\frac{\beta_0}{\rho}\right. \Xi, A \Rightarrow B \quad \text{with } \beta_0 < \beta \tag{14}$$

The induction hypothesis applied to (12) and (3) gives

$$\mathcal{H} \left|\frac{\alpha_0\#\alpha_0\#\beta\#\beta}{\rho}\right. \Gamma, \Xi \Rightarrow A\,. \tag{15}$$

Now an appilication of (Cut) to (15) and (14) gives

$$\mathcal{H} \left|\frac{\alpha_0\#\alpha\#\beta\#\beta}{\rho}\right. \Gamma, \Xi \Rightarrow B\,. \tag{16}$$

Inversion (Lemma 3.4.1 iv)) applied to (13) gives

$$\mathcal{H} \left|\frac{\alpha_1}{\rho}\right. \Gamma, B \Rightarrow \Delta\,. \tag{17}$$

Finally a single application of (Cut) to (16) and (17) yields the desired result. $\square$

Theorem 3.4.3 (Predicative Cut Elimination for $\mathbf{IRS}_\Omega$)**.** Suppose $\mathcal{H} \,\big|\!\frac{\alpha}{\rho+\omega^\beta}\, \Gamma \Rightarrow \Delta$, where $\Omega \notin [\rho, \rho+\omega^\beta)$ and $\beta \in \mathcal{H}$, then

$$\mathcal{H} \,\big|\!\frac{\varphi\beta\alpha}{\rho}\, \Gamma \Rightarrow \Delta\,.$$

Provided $\mathcal{H}$ is an operator closed under φ.

Proof. The proof is by main induction on β and subsidiary induction on α.

If the last inference was anything other than (Cut) or was a cut of rank $< \rho$ then we may apply the subsidiary induction hypothesis to the premises and then re-apply the final inference. So suppose the last inference was (Cut) with cut-formula C and $rk(C) \in [\rho, \rho+\omega^\beta)$. So we have

(1) $$\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho+\omega^\beta}\, \Gamma, C \Rightarrow \Delta \quad \text{with } \alpha_0 < \alpha.$$

(2) $$\mathcal{H} \,\big|\!\frac{\alpha_1}{\rho+\omega^\beta}\, \Gamma \Rightarrow C \quad \text{with } \alpha_1 < \alpha.$$

First applying the subsidiary induction hypothesis to (1) and (2) gives

(3) $$\mathcal{H} \,\big|\!\frac{\varphi\beta\alpha_0}{\rho}\, \Gamma, C \Rightarrow \Delta$$

(4) $$\mathcal{H} \,\big|\!\frac{\varphi\beta\alpha_1}{\rho}\, \Gamma, \Rightarrow C\,.$$

Now if $rk(C) = \rho$ then one application of the Reduction Lemma 3.4.2 gives the desired result (once it is noted that $\varphi\beta\alpha_0 \# \varphi\beta\alpha_0 \# \varphi\beta\alpha_1 \# \varphi\beta\alpha_1 < \varphi\beta\alpha$ since $\varphi\beta\alpha$ is additive principal.)

Now let us suppose that $\beta > 0$ and $rk(C) \in (\rho, \rho+\omega^\beta)$. Since $rk(C) < \rho+\omega^\beta$ we can find some $\beta_0 < \beta$ and some $n < \omega$ such that

$$rk(C) < \rho + n\cdot\omega^{\beta_0}.$$

Thus applying (Cut) to (3) and (4) gives

$$\mathcal{H} \,\big|\!\frac{\varphi\beta\alpha}{\rho+n\cdot\omega^{\beta_0}}\, \Gamma \Rightarrow \Delta$$

Now by the main induction hypothesis we obtain

$$\mathcal{H} \,\big|\!\frac{\varphi\beta_0(\varphi\beta\alpha)}{\rho+(n-1)\cdot\omega^{\beta_0}}\, \Gamma \Rightarrow \Delta$$

But by definition $\varphi\beta\alpha$ is a fixed point of the function $\varphi\beta_0(\cdot)$ ie. $\varphi\beta_0(\varphi\beta\alpha) = \varphi\beta\alpha$, so we have

$$\mathcal{H} \,\big|\!\frac{\varphi\beta\alpha}{\rho+(n-1)\cdot\omega^{\beta_0}}\, \Gamma \Rightarrow \Delta$$

From here a further $(n-1)$ applications of the main induction hypothesis yields the desired result. □

Lemma 3.4.4 (Boundedness for $\mathbf{IRS}_\Omega$)**.** If A is a Σ-formula, B is a Π-formula, $\alpha \leq \beta < \Omega$ and $\beta \in \mathcal{H}$ then

i) If $\mathcal{H} \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow A$ then $\mathcal{H} \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow A^{\mathbb{L}_\beta}$.

ii) If $\mathcal{H} \mathrel{|\!\frac{\alpha}{\rho}} \Gamma, B \Rightarrow \Delta$ then $\mathcal{H} \mathrel{|\!\frac{\alpha}{\rho}} \Gamma, B^{\mathbb{L}_\beta} \Rightarrow \Delta$

Proof. Suppose that $\mathcal{H} \mathrel{|\!\frac{\alpha}{\rho}} \Gamma \Rightarrow A$. We proceed by induction on α.

If A was not the principal formula of the last inference then we can simply use the induction hypothesis. If A was the principal formula of the last inference and is of the form $\neg C$, $C \wedge D$, $C \vee D$, $C \rightarrow D$, $(\exists x \in t)C(x)$ or $(\forall x \in t)C(x)$, then again the result follows immediately from the induction hypothesis.

Note that the last inference cannot have been $(\forall R)_\infty$ or (Σ-Ref$_\Omega$) since A is a Σ formula and $\alpha < \Omega$.

So suppose $A \equiv \exists x C(x)$ and

$$\mathcal{H} \mathrel{|\!\frac{\alpha_0}{\rho}} \Gamma \Rightarrow C(s)$$

For some $\alpha_0, |\, s\,| < \alpha$. By induction hypothesis we obtain

$$\mathcal{H} \mathrel{|\!\frac{\alpha_0}{\rho}} \Gamma \Rightarrow C(s)^{\mathbb{L}_\beta} .$$

Which may be written as

$$\mathcal{H} \mathrel{|\!\frac{\alpha_0}{\rho}} \Gamma \Rightarrow s \mathrel{\dot{\in}} \mathbb{L}_\beta \wedge C(s)^{\mathbb{L}_\beta} .$$

Now an application of $(b\exists R)$ yields the desired result.

Part ii) is proved in a similar manner. □

Definition 3.4.5. For each η we define

$$\mathcal{H}_\eta : \mathcal{P}(B^\Omega(\varepsilon_{\Omega+1})) \longrightarrow \mathcal{P}(B^\Omega(\varepsilon_{\Omega+1}))$$
$$\mathcal{H}_\eta(X) := \bigcap \{B^\Omega(\alpha) \; : \; X \subseteq B^\Omega(\alpha) \text{ and } \eta < \alpha\}$$

Lemma 3.4.6. i) $\mathcal{H}_\eta$ is an operator for each η.

ii) $\eta < \eta' \implies \mathcal{H}_\eta(X) \subseteq \mathcal{H}_{\eta'}(X)$

iii) If $\xi \in \mathcal{H}_\eta(X)$ and $\xi < \eta + 1$ then $\psi_\Omega(\xi) \in \mathcal{H}_\eta(X)$

Proof. This is proved in [8]. □

Lemma 3.4.7. Suppose $\eta \in \mathcal{H}_\eta$ and let $\hat{\beta} := \eta + \omega^{\Omega+\beta}$.

i) If $\alpha \in \mathcal{H}_\eta$ then $\hat{\alpha}, \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}$.

ii) If $\alpha_0 \in \mathcal{H}_\eta$ and $\alpha_0 < \alpha$ then $\psi_\Omega(\hat{\alpha_0}) < \psi_\Omega(\hat{\alpha})$.

Proof. i) From $\alpha, \eta \in \mathcal{H}_\eta = B^\Omega(\eta+1)$ we get $\hat{\alpha} \in B^\Omega(\eta+1)$ and hence $\hat{\alpha} \in B^\Omega(\hat{\alpha})$ by 3.4.6ii). Thus $\psi_\Omega(\hat{\alpha}) \in B^\Omega(\hat{\alpha}+1) = \mathcal{H}_{\hat{\alpha}}(\emptyset)$.

ii) Suppose that $\alpha > \alpha_0 \in \mathcal{H}_\eta$. By the argument above we get $\psi_\Omega(\hat{\alpha_o}) \in B^\Omega(\hat{\alpha_0}+1) \subseteq B^\Omega(\hat{\alpha})$, thus $\psi_\Omega(\hat{\alpha_0}) < \psi_\Omega(\hat{\alpha})$. □

Theorem 3.4.8 (Collapsing for $\mathbf{IRS}_\Omega$)**.** Suppose that $\eta \in \mathcal{H}_\eta$, Δ is a set of at most one Σ-formula and Γ a finite set of Π-formulae with $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$ then:

$$\mathcal{H}_\eta \overset{\alpha}{\underset{\Omega+1}{\vdash}} \Gamma \Rightarrow \Delta \quad \text{implies} \quad \mathcal{H}_{\hat{\alpha}} \overset{\psi_\Omega(\hat{\alpha})}{\underset{\psi_\Omega(\hat{\alpha})}{\vdash}} \Gamma \Rightarrow \Delta$$

Proof. We proceed by induction on α. If the last inference was propositional then the assertion follows easily from the induction hypothesis.

Case 1. Suppose the last inference was $(b\forall R)_\infty$, then $\Delta = \{(\forall x \in t)F(x)\}$ and

$$\mathcal{H}_\eta[p] \overset{\alpha_p}{\underset{\Omega+1}{\vdash}} \Gamma \Rightarrow p \mathrel{\dot{\in}} t \to F(p) \quad \text{for all } |\,p\,| < |\,t\,| \text{ with } \alpha_p < \alpha.$$

Since $k(t) \subseteq \mathcal{H}_\eta$, we know that $|\,t\,| \in B(\eta+1)$ and thus $|\,t\,| < \psi_\Omega(\eta+1)$. Thus $k(p) \subseteq \mathcal{H}_\eta$ for all $|\,p\,| < |\,t\,|$, so $\mathcal{H}_\eta[p] = \mathcal{H}_\eta$ for all such p. At this point we would like to use the induction hypothesis, the problem is that $p \mathrel{\dot{\in}} t \to F(p)$ may not be a Σ-formula. Instead we may first use inversion 3.4.1v) to obtain

$$\mathcal{H}_\eta \overset{\alpha_p}{\underset{\Omega+1}{\vdash}} \Gamma, p \mathrel{\dot{\in}} t \Rightarrow F(p)\,.$$

Noting that at worst $p \mathrel{\dot{\in}} t$ contains only bounded quantification, we may now apply the induction hypothesis to give

$$\mathcal{H}_{\hat{\alpha_p}} \overset{\psi_\Omega(\hat{\alpha_p})}{\underset{\psi_\Omega(\hat{\alpha_p})}{\vdash}} \Gamma, p \mathrel{\dot{\in}} t \Rightarrow F(p)\,.$$

Since $\psi_\Omega(\hat{\alpha_p})+1 < \psi_\Omega(\hat{\alpha})$ for all p, we may apply $(\to R)$ and then $(b\forall R)_\infty$ to obtain the desired result.

Case 2. Suppose the last inference was $(b\forall L)$ so $(\forall x \in t)F(x) \in \Gamma$ and

$$\mathcal{H}_\eta \overset{\alpha_0}{\underset{\Omega+1}{\vdash}} \Gamma, s \mathrel{\dot{\in}} t \to F(s) \Rightarrow \Delta \quad \text{for some } |\,s\,| < |\,t\,| \text{ with } \alpha_0 < \alpha.$$

Since $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$, $F(s)$ contains only bounded quantifiers and thus $s \mathrel{\dot{\in}} t \to F(s)$ is itself a Π-formula. So we may apply the induction hypothesis to give

$$\mathcal{H}_{\hat{\alpha_0}} \overset{\psi_\Omega\hat{\alpha_0}}{\underset{\psi_\Omega\hat{\alpha_0}}{\vdash}} \Gamma, s \mathrel{\dot{\in}} t \to F(s) \Rightarrow \Delta$$

from which we obtain the desired result using one applicaton of $(b\forall L)$.

Case 3.$(b\exists L)_\infty$ and $(b\exists R)$ ae similar to cases 1 and 2 but without the worry that the formula in the premise could not be Σ.

Case 4. Suppose the last inference was $(\exists R)$, so $\Delta = \{\exists x F(x)\}$ and

$$\mathcal{H}_\eta \,\big|\frac{\alpha_0}{\Omega+1}\, \Gamma \Rightarrow F(s) \quad \text{for some } |\,s\,| < \alpha \text{ and } \alpha_0 < \alpha.$$

Since $F(s)$ is Σ we may immediately apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha_0}} \,\big|\frac{\psi_\Omega\hat{\alpha_0}}{\psi_\Omega\hat{\alpha_0}}\, \Gamma \Rightarrow F(s)\,.$$

Now since $|\,s\,| \in \mathcal{H}_\eta = B(\eta+1)$ we know that $|\,s\,| < \psi_\Omega(\eta+1) < \psi_\Omega\hat{\alpha}$, so we may apply $(\exists R)$ to obtain the desired result.

Case 5. If the last inference was $(\forall L)$ we may argue in a similar fashion to case 4.

It cannot be the case that the last inference was $(\exists L)$ or $(\forall R)$ since Γ contains only Π formulae and Δ only Σ formulae.

Case 6. Suppose the last inference was (Σ-Ref$_\Omega$), so $\Delta = \{\exists z F^z\}$ for some Σ formula F and

$$\mathcal{H}_\eta \,\big|\frac{\alpha_0}{\Omega+1}\, \Gamma \Rightarrow F\,.$$

The induction hypothesis yields

$$\mathcal{H}_{\hat{\alpha_0}} \,\big|\frac{\psi_\Omega\hat{\alpha_0}}{\psi_\Omega\hat{\alpha_0}}\, \Gamma \Rightarrow F$$

Now applying Boundedness 3.4.4 yields

$$\mathcal{H}_{\hat{\alpha_0}} \,\big|\frac{\psi_\Omega\hat{\alpha_0}}{\psi_\Omega\hat{\alpha_0}}\, \Gamma \Rightarrow F^{\mathbb{L}_{\psi_\Omega(\hat{\alpha_0})}}$$

From which one application of $(\exists R)$ yields the desired result.

Case 7. Finally suppose the last inference was (Cut), then there is a formula C with $rk(C) \le \Omega$ and $\alpha_0 < \alpha$ such that

$$\text{(1)} \qquad \mathcal{H}_\eta \,\big|\frac{\alpha_0}{\Omega+1}\, \Gamma, C \Rightarrow \Delta$$

$$\text{(2)} \qquad \mathcal{H}_\eta \,\big|\frac{\alpha_0}{\Omega+1}\, \Gamma \Rightarrow C$$

7.1 If $rk(C) < \Omega$ then C contains only bounded quantification and as such is both Σ and Π, thus we may apply the induction hypothesis to both (1) and (2) to give

$$\text{(3)} \qquad \mathcal{H}_{\hat{\alpha_0}} \,\big|\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\, \Gamma, C \Rightarrow \Delta$$

$$\text{(4)} \qquad \mathcal{H}_{\hat{\alpha_0}} \,\big|\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\, \Gamma \Rightarrow C\,.$$

Since $k(C) \subseteq \mathcal{H}_\eta$ and $rk(C) < \Omega$, we have $rk(C) < \psi_\Omega(\eta + 1)$, so we may apply (Cut) to (3) and (4) to obtain the desired result.

7.2 If $rk(C) = \Omega$ then $C \equiv \exists x F(x)$ or $C \equiv \forall x F(x)$ with $F(\mathbb{L}_0)$ a Δ_0 formula. The two cases are similar so for simplicity just the case where $C \equiv \exists x F(x)$ is considered.

We can begin by immediately applying the induction hypothesis to (2) since C is a Σ formula, giving

$$\mathcal{H}_{\hat{\alpha_0}} \left|\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\right. \Gamma \Rightarrow C\,.$$

Now applying boundedness 3.4.4 yields

(5) $$\mathcal{H}_{\hat{\alpha_0}} \left|\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\right. \Gamma \Rightarrow C^{\mathbb{L}_{\psi_\Omega(\hat{\alpha_0})}}\,.$$

Since $\psi_\Omega(\hat{\alpha_0}) \in \mathcal{H}_{\hat{\alpha_0}}$ we may apply 3.3.10iii) to (1) to obtain

$$\mathcal{H}_{\hat{\alpha_0}} \left|\frac{\alpha_0}{\Omega+1}\right. \Gamma, (\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x) \Rightarrow \Delta\,.$$

Now $(\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x)$ is bounded and hence Π so by the induction hypothesis we obtain

(6) $$\mathcal{H}_{\hat{\alpha_1}} \left|\frac{\psi_\Omega(\alpha_1}{\psi_\Omega(\alpha_1)}\right. \Gamma, (\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x) \Rightarrow \Delta\,.$$

Where $\alpha_1 := \hat{\alpha_0} + \omega^{\Omega+\alpha_0}$. Since $\alpha_1 < \eta + \omega^{\Omega+\alpha} := \hat{\alpha}$ and $rk((\exists x \in \mathbb{L}_{\psi_\Omega(\hat{\alpha_0})})F(x)) < \psi_\Omega(\alpha)$ we may apply (Cut) to (5) and (6) to complete the proof. □

3.5 Embedding IKP into $\mathbf{IRS}_\Omega$

In this section we show how **IKP** derivations can be carried out in a very uniform manner within $\mathbf{IRS}_\Omega$. First some preparatory definitions. To facilitate independence from Chapter 2, I redefine the commutative sum of α and β, $\alpha\#\beta$.

Definition 3.5.1. i) Given ordinals $\alpha_1, ..., \alpha_n$. The expression $\omega^{\alpha_1}\#...\#\omega^{\alpha_n}$ denotes the ordinal

$$\omega^{\alpha_{p(1)}} + ... + \omega^{\alpha_{p(n)}}$$

where $p : \{1, ..., n\} \mapsto \{1, ..., n\}$ such that $\alpha_{p(1)} \geq ... \geq \alpha_{p(n)}$. More generally $\alpha\#0 := 0\#\alpha := 0$ and if $\alpha =_{NF} \omega^{\alpha_1} + ... + \omega^{\alpha_n}$ and $\beta =_{NF} \omega^{\beta_1} + ... + \omega^{\beta_m}$ then $\alpha\#\beta := \omega^{\alpha_1}\#...\#\omega^{\alpha_n}\#\omega^{\beta_1}\#...\#\omega^{\beta_m}$.

ii) If A is any $\mathbf{IRS}_\Omega$-formula then $no(A) := \omega^{rk(A)}$ and if $\Gamma \Rightarrow \Delta$ is an $\mathbf{IRS}_\Omega$-sequent containing formulas $\{A_1, ..., A_n\}$, then $no(\Gamma \Rightarrow \Delta) := no(A_1)\#...\#no(A_n)$.

iii) $\Vdash \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \,\Big|\!\frac{no(\Gamma\Rightarrow\Delta)}{0}\, \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}$$

iv) $\Vdash^{\xi}_{\rho} \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \,\Big|\!\frac{no(\Gamma\Rightarrow\Delta)\#\xi}{\rho}\, \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}$$

We would like to be able to use $\Vdash$ as a calculus since it dispenses with a lot of superfluous notation, luckily under certain conditions this is possible.

Lemma 3.5.2. i) If $\Gamma \Rightarrow \Delta$ follows from premises $\Gamma_i \Rightarrow \Delta_i$ by an inference other than (Cut) or (Σ-Ref$_\Omega$) and without contractions then

$$\text{if } \Vdash^{\alpha}_{\rho} \Gamma_i \Rightarrow \Delta_i \quad \text{then} \quad \Vdash^{\alpha}_{\rho} \Gamma \Rightarrow \Delta.$$

ii) If $\Vdash^{\alpha}_{\rho} \Gamma, A, B \Rightarrow \Delta$ then $\Vdash^{\alpha}_{\rho} \Gamma, A \wedge B \Rightarrow \Delta$.

Proof. In a similar manner to 2.5.2 the first part follows from the additive principal nature of ordinals of the form ω^{α} and Lemma 3.3.8.

For the second part suppose $\Vdash^{\alpha}_{\rho} \Gamma, A, B \Rightarrow \Delta$ which means we have

$$\mathcal{H}[\Gamma, A, B \Rightarrow \Delta] \,\Big|\!\frac{no(\Gamma\Rightarrow\Delta)\#no(A)\#no(B)\#\alpha}{\rho}\, \Gamma, A, B \Rightarrow \Delta\,.$$

Two applications of $(\wedge L)$ yields

$$\mathcal{H}[\Gamma, A, B \Rightarrow \Delta] \,\Big|\!\frac{no(\Gamma\Rightarrow\Delta)\#no(A)\#no(B)\#\alpha+2}{\rho}\, \Gamma, A \wedge B \Rightarrow \Delta\,.$$

It remains to note that $\mathcal{H}[\Gamma, A, B \Rightarrow \Delta] = \mathcal{H}[\Gamma, A \wedge B \Rightarrow \Delta]$ and

$$no(A)\#no(B) + 2 = \omega^{rk(A)}\#\omega^{rk(B)} + 2 < \omega^{rk(A\wedge B)} = no(A \wedge B)$$

to complete the proof. $\square$

Lemma 3.5.3. For any $\mathbf{IRS}_\Omega$ formulas A, B and terms s, t we have

i) $\Vdash \Gamma, A \Rightarrow A$

ii) $\Vdash s \in s \Rightarrow$

iii) $\Vdash \Rightarrow s \subseteq s$ here $s \subseteq s$ is shorthand for $(\forall x \in s)(x \in s)$.

iv) $\Vdash \Rightarrow s \dot{\in} t \to s \in t$ and $\Vdash s \dot{\in} t \Rightarrow s \in t$, for $|s| < |t|$.

v) $\Vdash s = t \Rightarrow t = s$

vi) If $\Vdash \Gamma, A \Rightarrow B$ then $\Gamma, s \dot{\in} t \wedge A \Rightarrow s \dot{\in} t \wedge B$ for $|s| < |t|$.

vii) If $\Vdash \Gamma, A, B \Rightarrow \Delta$ then $\Vdash \Gamma, s \dot{\in} t \to A, s \dot{\in} t \wedge B \Rightarrow \Delta$ for $|s| < |t|$.

viii) If $|s| < \beta$ then $\Vdash \Rightarrow s \in \mathbb{L}_\beta$

Proof. i) By induction of $rk(A)$. We split into cases based on the form of the formula A.

Case 1. If $A \equiv (r \in t)$ then by the induction hypothesis we have

$$\Vdash \Gamma, s \dot{\in} t \wedge r = s \Rightarrow s \dot{\in} t \wedge r = s \quad \text{for all } |s| < |t|.$$

The following is a template for $\mathbf{IRS}_\Omega$ derivations.

$$(\in L)_\infty\ \dfrac{(\in R)\ \dfrac{\Vdash s \dot{\in} t \wedge r = s \Rightarrow s \dot{\in} t \wedge r = s \quad \text{for all } |s| < |t|}{\Vdash s \dot{\in} t \wedge r = s \Rightarrow r \in t \quad \text{for all } |s| < |t|}}{\Vdash r \in t \Rightarrow r \in t}$$

Case 2. If $A \equiv (\exists x \in t)F(x)$ then by the induction hypothesis we have

$$\Vdash s \dot{\in} t \wedge F(s) \Rightarrow s \dot{\in} t \wedge F(s) \quad \text{for all } |s| < |t|.$$

We have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$(b\exists L)_\infty\ \dfrac{(b\exists R)\ \dfrac{\Vdash s \dot{\in} t \wedge F(s) \Rightarrow s \dot{\in} t \wedge F(s) \quad \text{for all } |s| < |t|}{\Vdash s \dot{\in} t \wedge F(s) \Rightarrow (\exists x \in t)F(x) \quad \text{for all } |s| < |t|}}{\Vdash (\exists x \in t)F(x) \Rightarrow (\exists x \in t)F(x)}$$

Case 3. All remaining cases can be proved in a similar fashion to above.

ii) The proof is by induction on $rk(s)$, inductively we get $\Vdash r \in r \Rightarrow$ for all $|r| < |s|$. Now if s is of the form $\mathbb{L}_\alpha$, then $r \in r \equiv r \dot{\in} s \to r \in r$ and we have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$(\in L)_\infty\ \dfrac{(\wedge L)\ \dfrac{(b\forall L)\ \dfrac{\Vdash r \in r \Rightarrow \quad \text{for all } |r| < |s|}{\Vdash (\forall x \in s)(x \in r) \Rightarrow \quad \text{for all } |r| < |s|}}{\Vdash s = r \Rightarrow \quad \text{for all } |r| < |s|}}{\Vdash s \in s \Rightarrow}$$

Now if $s \equiv [x \in \mathbb{L}_\alpha \mid B(x)]$ then we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$(\in L)_\infty\ \dfrac{\text{Lemma 3.5.2ii)}\ \dfrac{(\wedge L)\ \dfrac{(b\forall L)\ \dfrac{(\to L)\ \dfrac{\dfrac{i)}{\Vdash B(r) \Rightarrow B(r) \quad \text{for all } |r| < |s|} \qquad \dfrac{\text{Induction Hypothesis}}{\Vdash r \in r \Rightarrow \quad \text{for all } |r| < |s|}}{\Vdash B(r), B(r) \to r \in r \Rightarrow}}{\Vdash B(r), (\forall x \in s)(x \in r) \Rightarrow}}{\Vdash B(r), r = s \Rightarrow}}{\Vdash B(r) \wedge r = s \Rightarrow}}{\Vdash s \in s \Rightarrow}$$

iii) Again we use induction on $rk(s)$. Inductively we have $\Vdash\Rightarrow r \subseteq r$ for all $|r| < |s|$. If $s \equiv [x \in \mathbb{L}_\alpha \mid B(x)]$ then we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{i)}{\Vdash B(r) \Rightarrow B(r) \quad \text{for all } |r| < |s|} \qquad (\wedge R)\,\dfrac{\dfrac{\text{Induction Hypothesis}}{\Vdash B(r) \Rightarrow r \subseteq r \quad \text{for all } |r| < |s|}}{\Vdash B(r) \Rightarrow r = r}}{\Vdash B(r) \Rightarrow B(r) \wedge r = r}\,(\wedge R)}{\Vdash B(r) \Rightarrow r \in s}\,(\in R)}{\Vdash\Rightarrow r \dot{\in} s \to r \in s}\,(\to R)}{\Vdash\Rightarrow (\forall x \in s)(x \in s)}\,(b\forall R)_\infty
$$

If $s \equiv \mathbb{L}_\alpha$ then we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$
(b\forall R)_\infty\,\dfrac{(\in R)\,\dfrac{(\wedge R)\,\dfrac{\dfrac{\text{Induction Hypothesis}}{\Vdash\Rightarrow r \subseteq r \quad \text{for all } |r| < |s|}}{\Vdash\Rightarrow r = r}}{\Vdash\Rightarrow r \in s}}{\Vdash\Rightarrow (\forall x \in s)(x \in s)}
$$

iv) Was shown whilst proving iii).

v) The following is a template for $\mathbf{IRS}_\Omega$ derivations

$$
(\wedge R)\,\dfrac{(\wedge L)\,\dfrac{\dfrac{i)}{\Vdash s \subseteq t \Rightarrow s \subseteq t}}{\Vdash s = t \Rightarrow s \subseteq t} \qquad (\wedge L)\,\dfrac{\dfrac{i)}{\Vdash t \subseteq s \Rightarrow t \subseteq s}}{\Vdash s = t \Rightarrow t \subseteq s}}{\Vdash s = t \Rightarrow t = s}
$$

vi) Trivial for $t \equiv \mathbb{L}_\alpha$, now if $t \equiv [x \in \mathbb{L}_\alpha \mid C(x)]$ then we have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$
(\wedge R)\,\dfrac{(\wedge L)\,\dfrac{\Vdash \Gamma, A \Rightarrow B}{\Vdash \Gamma, C(s) \wedge A \Rightarrow B} \qquad (\wedge L)\,\dfrac{\Vdash \Gamma, C(s) \Rightarrow C(s)}{\Vdash \Gamma, C(s) \wedge A \Rightarrow C(s)}}{\Vdash \Gamma, C(s) \wedge A \Rightarrow C(s) \wedge B}
$$

vii) This is also trivial for $t \equiv \mathbb{L}_\alpha$ so suppose $t \equiv [x \in \mathbb{L}_\alpha \mid C(x)]$ and we have the following template for $\mathbf{IRS}_\Omega$ derivations.

$$
(\to L)\,\dfrac{(\wedge L)\,\dfrac{\Vdash \Gamma, C(s) \Rightarrow C(s)}{\Vdash \Gamma, C(s) \wedge B \Rightarrow C(s)} \qquad (\wedge L)\,\dfrac{\Vdash \Gamma, A, B \Rightarrow \Delta}{\Vdash \Gamma, A, C(s) \wedge B \Rightarrow \Delta}}{\Vdash \Gamma, C(s) \to A, C(s) \wedge B \Rightarrow \Delta}
$$

viii) Suppose $|s| < \beta$ then we have the following template for derivations in IRS_Ω.

$$
(\in R)\,\dfrac{\dfrac{iii)}{\Vdash\Rightarrow s = s}}{\Vdash\Rightarrow s \in \mathbb{L}_\beta}
$$

□

Lemma 3.5.4. For any terms $s_1, ..., s_n, t_1, ..., t_n$ and any formula $A(s_1, ..., s_n)$ we have

$$\Vdash [s_1 = t_1], ..., [s_n = t_n], A(s_1, ..., s_n) \Rightarrow A(t_1, ..., t_n)$$

Where $[s_i = t_i]$ is shorthand for $s_i \subseteq t_i, t_i \subseteq s_i$.

Proof. We proceed by induction on $rk(A(s_1, ..., s_n))\#rk(A(t_1, ..., t_n))$.

Case 1. Suppose $A(x_1, x_2) \equiv (x_1 \in x_2)$, then for all $|\, s\,| < |\, s_2\,|$ and $|\, t\,| < |\, t_2\,|$ we have the following template for derivations in $\mathbf{IRS}_\Omega$.

$$\begin{array}{r l}
 & \Vdash [s_1 = t_1], [t = s], s_1 = s \Rightarrow t_1 = t \\ \cline{2-2}
\text{Lemma 3.5.2ii)} & \Vdash [s_1 = t_1], t = s, s_1 = s \Rightarrow t_1 = t \\ \cline{2-2}
\text{3.5.3vi)} & \Vdash [s_1 = t_1], t \mathrel{\dot\in} t_2 \wedge t = s, s_1 = s \Rightarrow t \mathrel{\dot\in} t_2 \wedge t_1 = t \\ \cline{2-2}
(\in R) & \Vdash [s_1 = t_1], t \mathrel{\dot\in} t_2 \wedge t = s, s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
(\in L)_\infty & \Vdash [s_1 = t_1], s \in t_2, s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
\text{3.5.3vii)} & \Vdash [s_1 = t_1], s \mathrel{\dot\in} s_2 \to s \in t_2, s \mathrel{\dot\in} s_2 \wedge s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
(\forall L) & \Vdash [s_1 = t_1], (\forall x \in s_2)(x \in t_2), s \mathrel{\dot\in} s_2 \wedge s_1 = s \Rightarrow t_1 \in t_2 \\ \cline{2-2}
(\in L)_\infty & \Vdash [s_1 = t_1], (\forall x \in s_2)(x \in t_2), s_1 \in s_2 \Rightarrow t_1 \in t_2 \\ \cline{2-2}
\text{Lemma 3.3.10i)} & \Vdash [s_1 = t_1], [s_2 = t_2], s_1 \in s_2 \Rightarrow t_1 \in t_2
\end{array}$$

Case 2. If $A(x_1, x_2) \equiv x_1 \in x_1$ then the assertion follows by Lemma 3.5.3ii) and weakening.

Case 3. Suppose $A(x_1, ..., x_n) \equiv (\exists y \in x_i)B(y, x_1, ..., x_n)$, for simplicity let us suppose that $i = 1$. Inductively for all $|\, r\,| < |\, s_1\,|$ we have

$$\begin{array}{r l}
 & \Vdash [s_1 = t_1], ..., [s_n = t_n], r \mathrel{\dot\in} s_1 \wedge B(r, s_1, ..., s_n) \Rightarrow r \mathrel{\dot\in} t_1 \wedge B(r, t_1, ..., t_n) \\ \cline{2-2}
(b\exists R) & \Vdash [s_1 = t_1], ..., [s_n = t_n], r \mathrel{\dot\in} s_1 \wedge B(r, s_1, ..., s_n) \Rightarrow (\exists y \in s_1)B(y, t_1, ..., t_n) \\ \cline{2-2}
(b\exists L)_\infty & \Vdash [s_1 = t_1], ..., [s_n = t_n], (\exists y \in s_1)B(y, t_1, ..., t_n) \Rightarrow (\exists y \in s_1)B(y, t_1, ..., t_n)
\end{array}$$

Case 4. The bounded universal quantification case is dual to the bounded existential one.

Case 5. If $A(x_1, ..., x_n) \equiv \exists y B(y, x_1, ..., x_n)$ then inductively for all terms r we have

$$\Vdash [s_1 = t_1], ..., [s_n = t_n], B(r, s_1, ..., s_n) \Rightarrow B(r, t_1, ..., t_n)$$

subsequently applying $(\exists R)$ followed by $(\exists L)_\infty$ yields the desired result.

Case 6. The unbounded universal quantification case is dual to the unbounded existential one.

Case 7. All propositional cases follow immediately from the induction hypothesis. ◻

Corollary 3.5.5 (Equality)**.** For any $\mathbf{IRS}_\Omega$-formula $A(s_1, ..., s_n)$

$$\Vdash \Rightarrow s_1 = t_1 \wedge ... \wedge s_n = t_n \wedge A(s_1, ..., s_n) \to A(t_1, ..., t_n)$$

Lemma 3.5.6 (Set Induction)**.** For any formula F

$$\Vdash_0^{\omega^{rk(A)}} \Rightarrow \forall x[(\forall y \in x)F(y) \rightarrow F(x)] \rightarrow \forall x F(x).$$

Where $A := \forall x[(\forall y \in x)F(y) \rightarrow F(x)]$.

Proof. First we verify the following claim:

$$(*) \qquad \mathcal{H}[A, s] \overset{\omega^{rk(A)}\#\omega^{|s|+1}}{\underset{0}{\vdash}} A \Rightarrow F(s) \quad \text{for all } s.$$

The claim is verified by induction on $|s|$, inductively suppose that

$$\mathcal{H}[A, t] \overset{\omega^{rk(A)}\#\omega^{|t|+1}}{\underset{0}{\vdash}} A \Rightarrow F(t) \quad \text{holds for all } |t| < |s|.$$

If necessary we may apply $(\rightarrow R)$ to obtain

$$\mathcal{H}[A, t, s] \overset{\omega^{rk(A)}\#\omega^{|t|+1}+1}{\underset{0}{\vdash}} A \Rightarrow t \dot{\in} s \rightarrow F(t)\,.$$

Next applying $(b\forall R)_\infty$ yields

$$\mathcal{H}[A, s] \overset{\omega^{rk(A)}\#\omega^{|s|+2}}{\underset{0}{\vdash}} A \Rightarrow (\forall y \in s)F(y)\,.$$

Also by Lemma 3.5.3i) we have

$$\mathcal{H}[A, s] \overset{\omega^{rk(F(s))}\#\omega^{rk(F(s))}}{\underset{0}{\vdash}} F(s) \Rightarrow F(s)\,.$$

Now one may note that $\omega^{rk(F(s))}\#\omega^{rk(F(s))} \leq \omega^{rk(F(s))+1} \leq \omega^{\max(\Omega, rk(F(\mathbb{L}_0))+3)} = \omega^{rk(A)}$ to see that by weakening we can conclude

$$\mathcal{H}[A, s] \overset{\omega^{rk(A)}\#\omega^{|s|}+2}{\underset{0}{\vdash}} F(s) \Rightarrow F(s)\,.$$

Hence using one application of $(\rightarrow L)$ we get

$$\mathcal{H}[A, s] \overset{\omega^{rk(A)}\#\omega^{|s|}+3}{\underset{0}{\vdash}} A, (\forall y \in s)F(y) \rightarrow F(s) \Rightarrow F(s)\,.$$

Applying $(b\forall L)$ yields

$$\mathcal{H}[A, s] \overset{\omega^{rk(A)}\#\omega^{|s|}+4}{\underset{0}{\vdash}} A \Rightarrow F(s)\,.$$

Thus the claim (*) is verified. A single application of $(\forall R)_\infty$ to (*) furnishes us with

$$\mathcal{H}[A] \overset{\omega^{rk(A)}\#\Omega}{\underset{0}{\vdash}} A \Rightarrow \forall x F(x)\,.$$

Finally applying $(\rightarrow R)$ gives

$$\Vdash_0^{\omega^{rk(A)}} \Rightarrow A \rightarrow \forall x F(x)$$

as required. □

Lemma 3.5.7 (Infinity)**.** For any ordinal $\alpha > \omega$ we have

$$\Vdash\Rightarrow (\exists x \in \mathbb{L}_\alpha)[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$$

Proof. The following is a template for derivations in $\mathbf{IRS}_\Omega$:

$$(b\exists R)\,\dfrac{(\wedge R)\,\dfrac{(b\exists R)\,\dfrac{\dfrac{\text{Lemma 3.5.3 viii)}}{\Vdash\Rightarrow \mathbb{L}_0 \in \mathbb{L}_\omega}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\omega)(z \in \mathbb{L}_\omega)} \qquad (b\forall R)_\infty\,\dfrac{(b\exists R)\,\dfrac{\dfrac{\text{Lemma 3.5.3 viii)}}{\Vdash\Rightarrow s \in \mathbb{L}_\alpha \quad \text{for all } |s| < \alpha < \omega}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\omega)(s \in z) \quad \text{for all } |s| < \omega}}{\Vdash\Rightarrow (\forall y \in \mathbb{L}_\omega)(\exists z \in \mathbb{L}_\omega)(y \in z)}}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\omega)(z \in \mathbb{L}_\omega) \wedge (\forall y \in \mathbb{L}_\omega)(\exists z \in \mathbb{L}_\omega)(y \in z)}}{\Vdash\Rightarrow (\exists x \in \mathbb{L}_\alpha)[(\exists z \in x)(z \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]}$$

□

Lemma 3.5.8 (Δ_0-Separation)**.** Suppose $|s|, |t_1|, ..., |t_n| < \lambda$ where λ is a limit ordinal and $A(a, b_1, ..., b_n)$ is a Δ_0-formula of **IKP** with all free variables displayed, then

$$\Vdash\Rightarrow (\exists y \in \mathbb{L}_\lambda)[(\forall x \in y)(x \in s \wedge A(x, t_1, ..., t_n)) \wedge (\forall x \in s)(A(x, t_1, ..., t_n) \rightarrow x \in y)]$$

Proof. First let $\beta := \max\{|s|, |t_1|, ..., |t_n|\} + 1$ and note that $\beta < \lambda$ since λ is a limit. Now let

$$t := [u \in \mathbb{L}_\beta \mid u \in s \wedge A(u, t_1, ..., t_n)].$$

Let $B(x) := A(x, t_1, ..., t_n)$, in what follows r ranges over terms with $|\,r\,| < |\,t\,|$ and p ranges over terms with $|\,p\,| < |\,s\,|$. We have the following two templates for derivations in $\mathbf{IRS}_\Omega$:

Derivation (1)

$$(b\forall R)_\infty\,\dfrac{(\rightarrow R)\,\dfrac{\dfrac{\text{Lemma 3.5.3i)}}{\Vdash r \in s \wedge B(r) \Rightarrow r \in s \wedge B(r)}}{\Vdash\Rightarrow r \mathrel{\dot\in} t \rightarrow (r \in s \wedge B(r))}}{\Vdash\Rightarrow (\forall x \in t)(x \in s \wedge B(x))}$$

Derivation (2)

$$(b\forall R)_\infty\,\dfrac{(\rightarrow R)\,\dfrac{(\rightarrow R)\,\dfrac{(\in R)\,\dfrac{\dfrac{(\wedge R)\,\dfrac{\dfrac{\text{Lemma 3.5.3iv)}}{\Vdash p \mathrel{\dot\in} s, B(p) \Rightarrow p \in s} \qquad \dfrac{\text{Lemma 3.5.3i)}}{\Vdash p \mathrel{\dot\in} s, B(p) \Rightarrow B(p)}}{\Vdash p \mathrel{\dot\in} s, B(p) \Rightarrow p \in s \wedge B(p)} \qquad (\wedge R)\,\dfrac{\dfrac{\text{Lemma 3.5.3iii)}}{\Vdash\Rightarrow p \subseteq p}}{\Vdash\Rightarrow p = p}}{\Vdash p \mathrel{\dot\in} s, B(p) \Rightarrow (p \in s \wedge B(p)) \wedge p = p}}{\Vdash p \mathrel{\dot\in} s, B(p) \Rightarrow p \in t}}{\Vdash p \mathrel{\dot\in} s \Rightarrow B(p) \rightarrow p \in t}}{\Vdash\Rightarrow p \mathrel{\dot\in} s \rightarrow (B(p) \rightarrow p \in t)}}{\Vdash\Rightarrow (\forall x \in s)(B(x) \rightarrow x \in t)}$$

Now applying $(\wedge R)$ to the conclusions of derivations (1) and (2) we obtain

$$\Vdash\Rightarrow (\forall x \in t)(x \in s \wedge B(x)) \wedge (\forall x \in s)(B(x) \rightarrow x \in t).$$

Finally note that $|t| = \beta < \lambda$ so we may apply $(b\exists R)$ to obtain

$$\Vdash\Rightarrow (\exists y \in \mathbb{L}_\lambda)[(\forall x \in y)(x \in s \wedge B(x)) \wedge (\forall x \in s)(B(x) \rightarrow x \in y)]$$

as required. □

Lemma 3.5.9 (Pair)**.** If λ is a limit ordinal and $|s|, |t| < \lambda$, then

$$\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(s \in z \wedge t \in z)$$

Proof. Let $\delta := \max\{|s|, |t|\} + 1$ and note that $\delta < \lambda$ since λ is a limit. We have the following template for $\mathbf{IRS}_\Omega$ derivations:

$$\dfrac{\dfrac{\dfrac{\text{Lemma 3.5.3viiii)}}{\Vdash\Rightarrow s \in \mathbb{L}_\delta} \qquad \dfrac{\text{Lemma 3.5.3viiii)}}{\Vdash\Rightarrow t \in \mathbb{L}_\delta}}{\Vdash\Rightarrow (s \in \mathbb{L}_\delta \wedge t \in \mathbb{L}_\delta)}\,(\wedge R)}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(s \in z \wedge t \in z)}\,(b\exists R)$$

□

Lemma 3.5.10 (Union)**.** If λ is a limit ordinal and $|s| < \lambda$ then

$$\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(\forall y \in s)(\forall x \in y)(x \in z)$$

Proof. Let $\alpha = |s|$, we have the following template for derivations in $\mathbf{IRS}_\Omega$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 3.5.3viii)}}{\Vdash r \mathrel{\dot{\in}} s, q \mathrel{\dot{\in}} r \Rightarrow q \in \mathbb{L}_\alpha \quad \text{for all } |q| < |r| < \alpha}}{\Vdash r \mathrel{\dot{\in}} s \Rightarrow q \mathrel{\dot{\in}} r \rightarrow q \in \mathbb{L}_\alpha}\,(\rightarrow R)}{\Vdash r \mathrel{\dot{\in}} s \Rightarrow (\forall x \in r)(x \in \mathbb{L}_\alpha)}\,(b\forall R)_\infty}{\Vdash\Rightarrow r \mathrel{\dot{\in}} s \rightarrow (\forall x \in r)(x \in \mathbb{L}_\alpha)}\,(\rightarrow R)}{\Vdash\Rightarrow (\forall y \in s)(\forall x \in y)(x \in \mathbb{L}_\alpha)}\,(b\forall R)_\infty}{\Vdash\Rightarrow (\exists z \in \mathbb{L}_\lambda)(\forall y \in s)(\forall x \in y)(x \in z)}\,(b\exists R)$$

□

Lemma 3.5.11 (Δ_0-Collection)**.** For any Δ_0 formula $F(x, y)$.

$$\Vdash\Rightarrow (\forall x \in s)\exists y F(x, y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y)$$

Proof. Using Lemma 3.5.3 we have

$$\Vdash (\forall x \in s)\exists y F(x,y) \Rightarrow (\forall x \in s)\exists y F(x,y)$$

Now let $\bar{\mathcal{H}} := \mathcal{H}[(\forall x \in s)\exists y F(x,y)]$ and $\alpha := no((\forall x \in s)\exists y F(x,y) \Rightarrow (\forall x \in s)\exists y F(x,y))$, by applying ($\Sigma$-Ref$_\Omega$) we obtain

$$\bar{\mathcal{H}} \left|\frac{\alpha+1}{0}\right. (\forall x \in s)\exists y F(x,y) \Rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x,y)\,.$$

Applying $(\rightarrow R)$ gives

$$\bar{\mathcal{H}} \left|\frac{\alpha+2}{0}\right. \Rightarrow (\forall x \in s)\exists y F(x,y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x,y)\,.$$

It remains to note that

$$\begin{aligned} \alpha + 2 = \alpha =& no((\forall x \in s)\exists y F(x,y) \Rightarrow (\forall x \in s)\exists y F(x,y)) + 2 \\ <& no(\Rightarrow (\forall x \in s)\exists y F(x,y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x,y)) \end{aligned}$$

and $\bar{\mathcal{H}} = \mathcal{H}[\Rightarrow (\forall x \in s)\exists y F(x,y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x,y)]$ to complete the proof. $\square$

Theorem 3.5.12. If **IKP** $\vdash \Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ where $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ is an intuitionistic sequent containing exactly the free variables $\bar{a} := a_1, ..., a_n$, then there is an $m < \omega$ (which we may compute from the **IKP**-derivation) such that

$$\mathcal{H}[\Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})] \left|\frac{\Omega\cdot\omega^m}{\Omega+m}\right. \Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})$$

for any **IRS**$_\Omega$ terms $\bar{s} := s_1, ...s_n$ and any operator $\mathcal{H}$.

Proof. Let A be any **IRS**$_\Omega$ formula, note that by Observation 3.3.7, we have $rk(A) \leq \Omega + l$ for some $l < \omega$. Therefore

$$no(A) = \omega^{rk(A)} \leq \omega^{\Omega+l} = \omega^\Omega \cdot \omega^l = \Omega \cdot \omega^l$$

Thus for any choice of terms $\bar{s}$ we have

$$no(\Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})) \leq \Omega \cdot \omega^m \quad \text{for some } m < \omega.$$

The remainder of the proof is by induction on the derivation **IKP** $\vdash \Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$.

If $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ is an axiom of **IKP** then the assertion follows by Lemmas 3.5.5, 3.5.6, 3.5.7, 3.5.8, 3.5.9, 3.5.10 or 3.5.11. If $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of a propositional inference then we may apply the induction hypothesis to the premises and then the corresponding inference in **IRS**$_\Omega$. In order to cut down on notation we make the following abbreviation, let

$$\bar{\mathcal{H}} := \mathcal{H}[\Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})].$$

Case 1. Suppose that $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of the inference $(b\forall R)$, then $\Delta(\bar{s}) = \{(\forall x \in s_i)F(x)\}$. The induction hypothesis furnishes us with an $k < \omega$ such that

$$\bar{\mathcal{H}}[p] \mid\frac{\Omega\cdot\omega^k}{\Omega+k} \Gamma(\bar{s}) \Rightarrow p \in s_i \to F(p) \quad \text{for all } |\, p\,| < |\, s\,|_i.$$

Now by Lemma 3.4.1v) we have

$$\bar{\mathcal{H}}[p] \mid\frac{\Omega\cdot\omega^k}{\Omega+k} \Gamma(\bar{s}), p \in s_i \Rightarrow F(p)$$

Also by 3.5.3iv) we have

$$\Vdash p \dot{\in} s_i \Rightarrow p \in s_i$$

Applying (Cut) to these two yields

$$\bar{\mathcal{H}}[p] \mid\frac{\Omega\cdot\omega^k+1}{\Omega+k} \Gamma(\bar{s}), p \dot{\in} s_i \Rightarrow F(p)$$

Now by $(\to R)$ we have

$$\bar{\mathcal{H}}[p] \mid\frac{\Omega\cdot\omega^k+2}{\Omega+k} \Gamma(\bar{s}) \Rightarrow p \dot{\in} s_i \to F(p)\,.$$

Hence by $(b\forall R)_\infty$ we have

$$\bar{\mathcal{H}} \mid\frac{\Omega\cdot\omega^{k+1}}{\Omega+k} \Gamma(\bar{s}) \Rightarrow (\forall x \in s_i)F(x)$$

as required.

Case 2. Now suppose that $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of the inference $(b\forall L)$. So $(\forall x \in a_i)F(x) \in \Gamma(\bar{a})$ and we are in the following situation in **IKP**

$$(b\forall L)\ \frac{\Gamma(\bar{a}), c \in a_i \to F(c) \Rightarrow \Delta(\bar{a})}{\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})}$$

If c is not a member of $\bar{a}$ then by the induction hypothesis we have an $m < \omega$ such that

(1) $$\bar{\mathcal{H}} \mid\frac{\Omega\cdot\omega^m}{\Omega+m} \Gamma(\bar{s}), s_1 \in s_i \to F(s_1) \Rightarrow \Delta(\bar{s})\,.$$

Now if c is a member of $\bar{a}$, for simplicity let us suppose that $c = a_1$. Inductively we can find an $m < \omega$ such that (1) is also satisfied. First we verify the following claim:

(2) $$\Vdash \Gamma, (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1)$$

2.1 Suppose s_i is of the form $\mathbb{L}_\alpha$. The claim is verified by the following template for derivations in $\mathbf{IRS}_\Omega$, here r ranges over terms with $|r| < |s_i|$.

$$\begin{array}{r l}
 & \text{Lemma 3.5.4} \\ \hline
(b\forall L) & \Vdash \Gamma, F(r), r \in s_i, r = s_1 \Rightarrow F(s_1) \\ \hline
\text{Lemma 3.5.2ii)} & \Vdash \Gamma, (\forall x \in s_i)F(x), r \in s_i, r = s_1 \Rightarrow F(s_1) \\ \hline
(\in L)_\infty & \Vdash \Gamma, (\forall x \in s_i)F(x), r \in s_i \wedge r = s_1 \Rightarrow F(s_1) \\ \hline
(\to R) & \Vdash \Gamma, (\forall x \in s_i)F(x), s_1 \in s_i \Rightarrow F(s_1) \\ \hline
 & \Vdash \Gamma, (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1)
\end{array}$$

2.2 Now suppose s_i is of the form $[x \in \mathbb{L}_\alpha \mid B(x)]$, we have the following template for derivations in $\mathbf{IRS}_\Omega$, here r and p range over terms with level below $|\, s_i \,|$.

$$
\dfrac{
\text{Lemma 3.5.2ii)}\;\dfrac{
\dfrac{\text{Lemma 3.5.4}}{\Vdash p \,\dot{\in}\, s_i, r = p, r = s_i \Rightarrow r \,\dot{\in}\, s_i}
}{
(\in L)_\infty\;\dfrac{\Vdash p \,\dot{\in}\, s_i \wedge r = p, r = s_i \Rightarrow r \,\dot{\in}\, s_i}{\Vdash r \in s_i, r = s_i \Rightarrow r \,\dot{\in}\, s_i}
}
\qquad
\dfrac{\text{Lemma 3.5.4}}{\Vdash F(r), r \in s_i, r = s_1 \Rightarrow F(s_1)}
}{
(\to L)\;\;(b\forall L)\;\dfrac{\Vdash \Gamma, r \,\dot{\in}\, s_i \to F(r), r \in s_i, r = s_1 \Rightarrow F(s_1)}{
\text{Lemma 3.5.2ii)}\;\dfrac{\Vdash \Gamma, (\forall x \in s_i)F(x), r \in s_i, r = s_1 \Rightarrow F(s_1)}{
(\in L)_\infty\;\dfrac{\Vdash \Gamma, (\forall x \in s_i)F(x), r \in s_i \wedge r = s_1 \Rightarrow F(s_1)}{
(\to R)\;\dfrac{\Vdash \Gamma, (\forall x \in s_i)F(x), s_1 \in s_i \Rightarrow F(s_1)}{\Vdash \Gamma, (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1)}}}}
}
$$

Now that the claim is verified we may apply (Cut) to (1) and (2) to obtain

$$\bar{\mathcal{H}} \,\big|\!\frac{\Omega\cdot\omega^{m'}}{\Omega+m'}\, \Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})$$

where $\Omega + m' := \max\{\Omega + m, rk(s_1 \in s_i \to F(s_1))\}$, which is the desired result.

All other quantification cases are similar to Cases 1 and 2.

Finally suppose $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ was the result of (Cut). So we are in the following situation in **IKP**.

$$\dfrac{\Gamma(\bar{a}), F(\bar{a}, \bar{c}) \Rightarrow \Delta(\bar{a}) \qquad \Gamma(\bar{a}) \Rightarrow F(\bar{a}, \bar{c})}{\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})}$$

Where $\bar{c}$ are the free variables occurring in $F(\bar{a}, \bar{c})$ that are distinct from $\bar{a}$. By the induction hypothesis we can find $m_0, m_1 < \omega$ such that

$$\bar{\mathcal{H}} \,\big|\!\frac{\Gamma\cdot\omega^{m_0}}{\Omega+m_0}\, \Gamma(\bar{s}), F(\bar{s}, \overline{\mathbb{L}_0}) \Rightarrow \Delta(\bar{s})$$
$$\bar{\mathcal{H}} \,\big|\!\frac{\Gamma\cdot\omega^{m_1}}{\Omega+m_1}\, \Gamma(\bar{s}) \Rightarrow F(\bar{s}, \overline{\mathbb{L}_0})\,.$$

Note that $k(F(\bar{s}, \overline{\mathbb{L}_0})) \subseteq \bar{\mathcal{H}}$ so we may apply (Cut) to finish the proof. $\square$

3.6 An ordinal analysis of IKP

Lemma 3.6.1. If A is a Σ-sentence and $\mathbf{IKP} \vdash\, \Rightarrow A$, then there is some $m < \omega$, which we may compute explicitly from the derivation, such that

$$\mathcal{H}_\gamma \,\big|\!\frac{\varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))}{0}\; \Rightarrow\; A \quad \text{where } \gamma := \omega_m(\Omega \cdot \omega^m).$$

Here $\omega_0(\alpha) := \alpha$ and $\omega_{k+1}(\alpha) := \omega^{\omega_k(\alpha)}$.

Proof. Suppose that A is a Σ-sentence and that $\mathbf{IKP} \vdash \Rightarrow A$, then by Theorem 3.5.12 there is some $1 \leq m < \omega$ such that

$$(1) \qquad \mathcal{H}_0 \mathrel{\vert\!\frac{\Omega\cdot\omega^m}{\Omega+m}} \Rightarrow A\,.$$

By applying Predicative Cut Elimination 3.4.3 $(m-1)$ times we obtain

$$(2) \qquad \mathcal{H}_0 \mathrel{\vert\!\frac{\omega_{m-1}(\Omega\cdot\omega^m)}{\Omega+1}} \Rightarrow A\,.$$

Applying Collapsing 3.4.8 to (2) gives

$$(3) \qquad \mathcal{H}_\gamma \mathrel{\vert\!\frac{\psi_\Omega(\gamma)}{\psi_\Omega(\gamma)}} \Rightarrow A \quad \text{where } \gamma := \omega_m(\Omega\cdot\omega^m).$$

Finally by applying Predicative Cut Elimination 3.4.3 again we get

$$\mathcal{H}_\gamma \mathrel{\vert\!\frac{\varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))}{0}} \Rightarrow A$$

completing the proof. $\square$

Theorem 3.6.2. If $A \equiv \exists x C(x)$ is a Σ-sentence such that $\mathbf{IKP} \vdash \Rightarrow A$ then there is an ordinal term $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$, which we may compute from the derivation, such that

$$L_\alpha \models A.$$

Moreover, there is a specific $\mathbf{IRS}_\Omega$ term s, with $|\,s\,| < \alpha$, which we may compute explicitly from the $\mathbf{IKP}$ derivation, such that

$$L_\alpha \models C(s).$$

Proof. Suppose $\mathbf{IKP} \vdash \Rightarrow A$ for some Σ-sentence A, from Lemma 3.6.1 we may compute some $1 \leq m < \omega$ such that

$$\mathcal{H}_\gamma \mathrel{\vert\!\frac{\varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))}{0}} \Rightarrow A \quad \text{where } \gamma := \omega_m(\Omega\cdot\omega^m).$$

Let $\alpha := \varphi(\psi_\Omega(\gamma))(\psi_\Omega(\gamma))$, applying Boundedness 3.4.4 we obtain

$$(2) \qquad \mathcal{H}_\gamma \mathrel{\vert\!\frac{\alpha}{0}} \Rightarrow A^{\mathbb{L}_\alpha}\,.$$

Since the derivation (2) contains no instances of (Cut) or (Σ-Ref$_\Omega$) and the correctness of the remaining rules within L_α is easily verified by induction on the derivation, it may be seen that

$$L_\alpha \models A.$$

For the second part of the theorem note that it must be the case that the final inference in (2) was $(b\exists R)$ and thus by the intuitionistic nature of $\mathbf{IRS}_\Omega$ there must be some s, with $|\,s\,| < \alpha$, such that

$$(3) \qquad \mathcal{H}_\gamma \mathrel{\vert\!\frac{\alpha}{0}} \Rightarrow C(s)^{\mathbb{L}_\alpha}\,.$$

Thus

$$L_\alpha \models C(s). \tag{4}$$

The remainder of the proof is by checking that each part of the embedding and cut elimination of the previous two sections may be carried out effectively, details will appear in [43]. □

Remark 3.6.3. In fact Theorem 3.6.2 can be verified within **IKP**, this is not immediately obvious since we do not have access to induction up to $\psi_\Omega(\varepsilon_{\Omega+1})$. However one may observe that in an infinitary proof of the form (3) above, no terms of level higher than α are used. By carrying out the construction of $\mathbf{IRS}_\Omega$ just using ordinals from $B(\omega_{m+1}(\Omega \cdot \omega^m))$ we get a restricted system, but a system still capable of carrying out the embedding and cut elimination necessary for the particular derivation of the sentence A. This can be done inside **IKP** since we do have access to induction up to $\psi_\Omega(\omega_{m+1}(\Omega \cdot \omega^{m+1}))$. It follows that **IKP** has the set existence property for Σ sentences. More details will be found [43].

Chapter 4

A relativised ordinal analysis of $\mathbf{IKP}(\mathcal{P})$

This chapter provides a relativised ordinal analysis for intuitionistic power Kripke-Platek set theory $\mathbf{IKP}(\mathcal{P})$. The relativised ordinal analysis for the classical version of the theory, $\mathbf{KP}(\mathcal{P})$, was carried out in [42], the work in this chapter adapts the techniques from that paper to the intuitionistic case. We begin by defining an infinitary system $\mathbf{IRS}_\Omega^{\mathcal{P}}$, unlike in $\mathbf{IRS}_\Omega$ the terms in $\mathbf{IRS}_\Omega^{\mathcal{P}}$ can contain sub terms of a higher level, or from higher up the Von-Neumann hierarchy in the intended interpretation. This reflects the impredicativity of the power set operation. Next we prove some cut elimination theorems, allowing us to transform infinite derivations of Σ formulae into infinite derivations with only power-bounded cut formulae. The following section provides an embedding of $\mathbf{IKP}(\mathcal{P})$ into $\mathbf{IRS}_\Omega^{\mathcal{P}}$. The final section collates these results into a relativised ordinal analysis of $\mathbf{IKP}(\mathcal{P})$.

4.1 A sequent calculus formulation of $\mathbf{IKP}(\mathcal{P})$

Definition 4.1.1. The formulas of $\mathbf{IKP}(\mathcal{P})$ are the same as those of $\mathbf{IKP}$ except we also allow *subset bounded quantifiers* of the form

$$(\forall x \subseteq a)A(x) \quad \text{and} \quad (\exists x \subseteq a)A(x).$$

These are treated as quantifiers in their own right, not abbreviations. In contrast, the formula $a \subseteq b$ is still viewed as an abbreviation for the formula $(\forall x \in a)(x \in b)$

Quantifiers $\forall x$, $\exists x$ will still be referred to as unbounded, whereas the other quantifiers (including the subset bounded ones) will be referred to as bounded.

A $\Delta_0^{\mathcal{P}}$-formula of $\mathbf{IKP}(\mathcal{P})$ is one that contains no unbounded quantifiers.

As with **IKP**, the system $\mathbf{IKP}(\mathcal{P})$ derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where at most one formula can occur in Δ.

The axioms of $\mathbf{IKP}(\mathcal{P})$ are the following:

Logical axioms:	$\Gamma, A \Rightarrow A$ for every $\Delta_0^{\mathcal{P}}$–formula A.
Extensionality:	$\Gamma \Rightarrow a{=}b \wedge B(a) \rightarrow B(b)$ for every $\Delta_0^{\mathcal{P}}$-formula $B(a)$.
Pair:	$\Gamma \Rightarrow \exists x[a{\in}x \wedge b{\in}x]$
Union:	$\Gamma \Rightarrow \exists x(\forall y{\in}a)(\forall z{\in}y)(z{\in}x)$
$\Delta_0^{\mathcal{P}}$ *–Separation:*	$\Gamma \Rightarrow \exists y[(\forall x \in y)(x \in a \wedge B(x)) \wedge (\forall x \in a)(B(x) \rightarrow x \in y)]$ for every $\Delta_0^{\mathcal{P}}$-formula $B(a)$.
$\Delta_0^{\mathcal{P}}$ *–Collection:*	$\Gamma \Rightarrow (\forall x \in a)\exists y G(x,y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x,y)$ for every $\Delta_0^{\mathcal{P}}$–formula $G(a,b)$.
Set Induction:	$\Gamma \Rightarrow \forall u\,[(\forall x \in u)\,G(x) \rightarrow G(u)] \rightarrow \forall u\,G(u)$ for every formula $G(b)$.
Infinity:	$\Gamma \Rightarrow \exists x\,[(\exists y \in x)\,y \in x \;\wedge\; (\forall y \in x)(\exists z \in x)\,y \in z]$.
Power Set:	$\Gamma \Rightarrow \exists z\,(\forall x \subseteq a)x \in z$.

The rules of $\mathbf{IKP}(\mathcal{P})$ are the same as those of **IKP** (extended to the new language containing subset bounded quantifiers), together with the following four rules:

$$(pb\exists L)\,\frac{\Gamma, a \subseteq b \wedge F(a) \Rightarrow \Delta}{\Gamma, (\exists x \subseteq b)F(x) \Rightarrow \Delta} \qquad (pb\exists R)\,\frac{\Gamma \Rightarrow a \subseteq b \wedge F(a)}{\Gamma \Rightarrow (\exists x \subseteq b)F(x)}$$

$$(pb\forall L)\,\frac{\Gamma, a \subseteq b \rightarrow F(a) \Rightarrow \Delta}{\Gamma, (\forall x \subseteq b)F(x) \Rightarrow \Delta} \qquad (pb\forall R)\,\frac{\Gamma \Rightarrow a \subseteq b \rightarrow F(a)}{\Gamma \Rightarrow (\forall x \subseteq b)F(x)}$$

As usual it is forbidden for the variable a to occur in the conclusion of the rules $(pb\exists L)$ and $(pb\forall R)$, such a variable is referred to as the eigenvariable of the inference.

4.2 The infinitary system $\mathbf{IRS}_\Omega^{\mathcal{P}}$

The purpose of this section is to introduce an infinitary proof system $\mathbf{IRS}_\Omega^{\mathcal{P}}$. As before all ordinals will be assumed to be members of $B^\Omega(\varepsilon_{\Omega+1})$.

Definition 4.2.1. We define the $\mathbf{IRS}_\Omega^{\mathcal{P}}$ terms. To each $\mathbf{IRS}_\Omega^{\mathcal{P}}$ term t we also assign its ordinal level, $|\,t\,|$.

1. For each $\alpha < \Omega$, $\mathbb{V}_\alpha$ is an $\mathbf{IRS}_\Omega^{\mathcal{P}}$ term with $|\,\mathbb{V}_\alpha\,| = \alpha$.

2. For each $\alpha < \Omega$, we have infinitely many free variables $a_0^\alpha, a_1^\alpha, a_2^\alpha, ...$, with $|\, a_i^\alpha \,| = \alpha$.

3. If $F(x, \bar{y}\,)$ is a $\Delta_0^{\mathcal{P}}$-formula of $\mathbf{IKP}(\mathcal{P})$ (whose free variables are exactly those indicated) and $\bar{s} \equiv s_1, ..., s_n$ are $\mathbf{IRS}_\Omega^{\mathcal{P}}$ terms, then the formal expression $[x \in \mathbb{V}_\alpha \mid F(x, \bar{s}\,)]$ is an $\mathbf{IRS}_\Omega^{\mathcal{P}}$ term with $|\,[x \in \mathbb{V}_\alpha \mid F(x, \bar{s}\,)]\,| := \alpha$.

The $\mathbf{IRS}_\Omega^{\mathcal{P}}$ formulae are of the form $A(s_1, ..., s_n)$, where $A(a_1, ..., a_n)$ is a formula of $\mathbf{IKP}(\mathcal{P})$ with all free variables indicated and $s_1, ..., s_n$ are $\mathbf{IRS}_\Omega^{\mathcal{P}}$ terms.

A formula $A(s_1, ..., s_n)$ of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ is $\Delta_0^{\mathcal{P}}$ if $A(a_1, ..., a_n)$ is a $\Delta_0^{\mathcal{P}}$ formula of $\mathbf{IKP}(\mathcal{P})$.

The $\Sigma^{\mathcal{P}}$ formulae of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ are the smallest collection containing the $\Delta_0^{\mathcal{P}}$ formulae and containing $A \vee B$, $A \wedge B$, $(\forall x \in s)A$, $(\exists x \in s)A$, $(\forall x \subseteq s)A$, $(\exists x \subseteq s)A$ and $\exists x A$ whenever it contains A and B. Likewise The $\Pi^{\mathcal{P}}$-formulae are the smallest collection containing the $\Delta_0^{\mathcal{P}}$ formulae and containing $A \vee B$, $A \wedge B$, $(\forall x \in s)A$, $(\exists x \in s)A$, $(\forall x \subseteq s)A$, $(\exists x \subseteq s)A$ and $\forall x A$ whenever it contains A and B.

The *axioms* of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ are:

(A1) $\Gamma, A \Rightarrow A$ for A in $\Delta_0^{\mathcal{P}}$.

(A2) $\Gamma \Rightarrow t = t$.

(A3) $\Gamma, s_1 = t_1, ..., s_n = t_n, A(s_1, ..., s_n) \Rightarrow A(t_1, ..., t_n)$ for $A(s_1, \ldots, s_n)$ in $\Delta_0^{\mathcal{P}}$.

(A4) $\Gamma \Rightarrow s \in \mathbb{V}_\alpha$ if $|\,s\,| < \alpha$.

(A5) $\Gamma \Rightarrow s \subseteq \mathbb{V}_\alpha$ if $|\,s\,| \leq \alpha$.

(A6) $\Gamma, t \in [x \in \mathbb{V}_\alpha \mid F(x, \bar{s})] \Rightarrow F(t, \bar{s})$ for $F(t, \bar{s})$ is $\Delta_0^{\mathcal{P}}$ and $|\,t\,| < \alpha$.

(A7) $\Gamma, F(t, \bar{s}) \Rightarrow t \in [x \in \mathbb{V}_\alpha \mid F(x, \bar{s})]$ for $F(t, \bar{s})$ is $\Delta_0^{\mathcal{P}}$ and $|\,t\,| < \alpha$.

The *inference rules* of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ are:

$$(b\forall L) \qquad \frac{\Gamma,\ s \in t \rightarrow F(s) \Rightarrow \Delta}{\Gamma, (\forall x \in t)F(x) \;\; \Rightarrow \Delta} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(b\forall R)_\infty \qquad \frac{\Gamma \Rightarrow s \in t \rightarrow F(s) \;\text{ for all } |\,s\,| < |\,t\,|}{\Gamma \Rightarrow (\forall x \in t)F(x)}$$

$$(b\exists L)_\infty \qquad \frac{\Gamma,\ s \in t \wedge F(s) \Rightarrow \Delta \;\text{ for all } |\,s\,| < |\,t\,|}{\Gamma,\ (\exists x \in t)F(x) \;\; \Rightarrow \Delta}$$

$$(b\exists R) \qquad \frac{\Gamma \Rightarrow s \in t \wedge F(s)}{\Gamma \Rightarrow (\exists x \in t)F(x)} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(pb\forall L) \qquad \frac{\Gamma, s \subseteq t \rightarrow F(s) \Rightarrow \Delta}{\Gamma, (\forall x \subseteq t)F(x) \;\; \Rightarrow \Delta} \quad \text{if } |\,s\,| \leq |\,t\,|$$

$$(pb\forall R)_\infty \qquad \frac{\Gamma \Rightarrow s \subseteq t \rightarrow F(s) \;\text{ for all } |\,s\,| \leq |\,t\,|}{\Gamma \Rightarrow (\forall x \subseteq t)F(x)}$$

$$(pb\exists L)_\infty \quad \frac{\Gamma,\ s \subseteq t \wedge F(s) \Rightarrow \Delta \ \text{ for all } |\,s\,| \leq |\,t\,|}{\Gamma, (\exists x \subseteq t)F(x) \ \ \Rightarrow \Delta}$$

$$(pb\exists R) \quad \frac{\Gamma \Rightarrow s \subseteq t \wedge F(s)}{\Gamma \Rightarrow (\exists x \subseteq t)F(x)} \quad \text{if } |\,s\,| \leq |\,t\,|$$

$$(\forall L) \quad \frac{\Gamma, F(s) \Rightarrow \Delta}{\Gamma,\ \forall x\, F(x) \ \ \Rightarrow \Delta}$$

$$(\forall R)_\infty \quad \frac{\Gamma \Rightarrow F(s) \ \text{ for all } s}{\Gamma \Rightarrow \forall x\, F(x)}$$

$$(\exists L)_\infty \quad \frac{\Gamma,\ F(s) \Rightarrow \Delta \ \text{ for all } s}{\Gamma,\ \exists x\, F(x) \ \ \Rightarrow \Delta}$$

$$(\exists R) \quad \frac{\Gamma \Rightarrow F(s)}{\Gamma \Rightarrow \exists x\, F(x)}$$

$$(\in L)_\infty \quad \frac{\Gamma, r \in t \wedge r = s \Rightarrow \Delta \ \text{ for all } |\,r\,| < |\,t\,|}{\Gamma, s \in t \ \ \Rightarrow \Delta}$$

$$(\in R) \quad \frac{\Gamma \Rightarrow r \in t \ \wedge \ r = s}{\Gamma, s \in t} \quad \text{if } |\,r\,| < |\,t\,|$$

$$(\subseteq L)_\infty \quad \frac{\Gamma,\ r \subseteq t \wedge r = s \Rightarrow \Delta \ \text{ for all } |\,r\,| \leq |\,t\,|}{\Gamma,\ s \subseteq t \ \ \Rightarrow \Delta}$$

$$(\subseteq R) \quad \frac{\Gamma \Rightarrow r \subseteq t \ \wedge \ r = s}{\Gamma \Rightarrow s \subseteq t} \quad \text{if } |\,r\,| \leq |\,s\,|$$

$$(\text{Cut}) \quad \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma \Rightarrow A}{\Gamma \ \ \Rightarrow \Delta}$$

$$(\Sigma^{\mathcal{P}}\text{-}Ref) \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \ \exists z\, A^z} \quad \text{if } A \text{ is a } \Sigma^{\mathcal{P}}\text{-formula,}$$

As well as the rules ($\wedge L$), ($\wedge R$), ($\vee L$), ($\vee R$), ($\neg L$), ($\neg R$), ($\bot$), ($\to L$), ($\to R$) from **IKP**. As usual A^z results from A by restricting all unbounded quantifiers to z.

Definition 4.2.2. The *rank* of a formula is determined as follows.

1. $rk(s \in t) := max\{|\,s\,| + 1, |\,t\,| + 1\}$.
2. $rk((\exists x \in t)F(x)) := rk((\forall x \in t)F(x)) := max\{|\,t\,|, rk(F(\mathbb{V}_0)) + 2\}$.
3. $rk((\exists x \subseteq t)F(x)) := rk((\forall x \subseteq t)F(x)) := max\{|\,t\,| + 1, rk(F(\mathbb{V}_0)) + 2\}$.
4. $rk(\exists x\, F(x)) := rk(\forall x\, F(x)) := max\{\Omega, rk(F(\mathbb{V}_0)) + 2\}$.
5. $rk(A \wedge B) := rk(A \vee B) := rk(A \to B) := max\{rk(A), rk(B)\} + 1$.
6. $rk(\neg A) := rk(A) + 1$

Note that the definition of rank for $\mathbf{IRS}_\Omega^\mathcal{P}$ formulae is much less complex than for $\mathbf{IRS}_\Omega$, this is because we are only aiming for partial cut-elimination for this system. In general it will not be possible to remove cuts with $\Delta_0^\mathcal{P}$ cut formulae. Note however that we still have $rk(A) < \Omega$ if and only if A is $\Delta_0^\mathcal{P}$.

We also have the following useful lemma.

Lemma 4.2.3. If A is a formula of $\mathbf{IRS}_\Omega^\mathcal{P}$ with $rk(A) \geq \Omega$ (ie. A contains unbounded quantifiers), and A was the result of an $\mathbf{IRS}_\Omega^\mathcal{P}$ inference other than ($\Sigma^\mathcal{P}$-*Ref*) and (Cut) then the rank of the minor formulae of that inference is strictly less than $rk(A)$.

Definition 4.2.4 (Operator controlled derivability for $\mathbf{IRS}_\Omega^\mathcal{P}$)**.** If $A(s_1, ..., s_n)$ is a formula of $\mathbf{IRS}_\Omega^\mathcal{P}$ then let

$$|\,A(s_1, ...s_n)\,| := \{|\,s_1\,|, ..., |\,s_n\,|\}.$$

Likewise if $\Gamma \Rightarrow \Delta$ is an intuitionistic sequent of $\mathbf{IRS}_\Omega^\mathcal{P}$ containing formulas $A_1, ..., A_n$, we define

$$|\,\Gamma \Rightarrow \Delta\,| := |\,A_1\,| \cup ... \cup |\,A_n\,|.$$

Definition 4.2.5. Let $\mathcal{H}$ be an operator and $\Gamma \Rightarrow \Delta$ an intuitionistic sequent of $\mathbf{IRS}_\Omega^\mathcal{P}$ formulae. We define the relation $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta$ by recursion on α.

If $\Gamma \Rightarrow \Delta$ is an *axiom* and $|\,\Gamma \Rightarrow \Delta\,| \cup \{\alpha\} \subseteq \mathcal{H}$, then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow \Delta$.

We require always that $|\,\Gamma \Rightarrow \Delta\,| \cup \{\alpha\} \subseteq \mathcal{H}$ where $\Gamma \Rightarrow \Delta$ is the sequent in the conclusion, this condition will not be repeated in the inductive clauses pertaining to the inference rules of $\mathbf{IRS}_\Omega^\mathcal{P}$ given below. The column on the right gives the ordinal requirements for each of the inference rules.

$$(\in\! L)_\infty \quad \frac{\mathcal{H}[r] \overset{\alpha_r}{\underset{\rho}{\vdash}} \Gamma, r \in t \wedge r = s \Rightarrow \Delta \text{ for all } |\,r\,| < |\,t\,|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \in t \Rightarrow \Delta} \quad |\,r\,| \leq \alpha_r < \alpha$$

$$(\in\! R) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow r \in t \wedge r = s}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \in t} \quad \begin{array}{r}\alpha_0 < \alpha\\ |\,r\,| < |\,t\,|\\ |\,r\,| < \alpha\end{array}$$

$$(\subseteq\! L)_\infty \quad \frac{\mathcal{H}[r] \overset{\alpha_r}{\underset{\rho}{\vdash}} \Gamma, r \subseteq t \wedge r = s \Rightarrow \Delta \text{ for all } |\,r\,| \leq |\,t\,|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, s \subseteq t \Rightarrow \Delta} \quad |\,r\,| \leq \alpha_r < \alpha$$

$$(\subseteq\! R) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma \Rightarrow r \subseteq t \wedge r = s}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma \Rightarrow s \subseteq t} \quad \begin{array}{r}\alpha_0 < \alpha\\ |\,r\,| \leq |\,t\,|\\ |\,r\,| < \alpha\end{array}$$

$$(b\forall L) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma, s \in t \to A(s) \Rightarrow \Delta}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\forall x \in t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s \,| < |\, t \,| \\ |\, s \,| < \alpha \end{array}$$

$$(b\forall R)_\infty \qquad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma \Rightarrow s \in t \to F(s) \text{ for all } |\, s \,| < |\, t \,|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow (\forall x \in t)F(x)} \qquad |\, s \,| \leq \alpha_s < \alpha$$

$$(b\exists L)_\infty \qquad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma, s \in t \wedge F(s) \Rightarrow \Delta \text{ for all } |\, s \,| < |\, t \,|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\exists x \in t)F(x) \Rightarrow \Delta} \qquad |\, s \,| \leq \alpha_s < \alpha$$

$$(b\exists R) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow s \in t \wedge A(s)}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow (\exists x \in t)A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s \,| < |\, t \,| \\ |\, s \,| < \alpha \end{array}$$

$$(pb\forall L) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma, s \subseteq t \to A(s) \Rightarrow \Delta}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\forall x \subseteq t)A(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s \,| \leq |\, t \,| \\ |\, s \,| < \alpha \end{array}$$

$$(pb\forall R)_\infty \qquad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma \Rightarrow s \subseteq t \to F(s) \text{ for all } |\, s \,| \leq |\, t \,|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow (\forall x \subseteq t)F(x)} \qquad |\, s \,| \leq \alpha_s < \alpha$$

$$(pb\exists L)_\infty \qquad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma, s \subseteq t \wedge F(s) \Rightarrow \Delta \text{ for all } |\, s \,| \leq |\, t \,|}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\exists x \subseteq t)F(x) \Rightarrow \Delta} \qquad |\, s \,| \leq \alpha_s < \alpha$$

$$(pb\exists R) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow s \subseteq t \wedge A(s)}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow (\exists x \subseteq t)A(x)} \qquad \begin{array}{r} \alpha_0 < \alpha \\ |\, s \,| \leq |\, t \,| \\ |\, s \,| < \alpha \end{array}$$

$$(\forall L) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma, F(s) \Rightarrow \Delta}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, \forall x F(x) \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |\, s \,| < \alpha \end{array}$$

$$(\forall R)_\infty \qquad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma \Rightarrow F(s) \text{ for all } s}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \forall x F(x)} \qquad |\, s \,| < \alpha_s + 1 < \alpha$$

$$(\exists L)_\infty \qquad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma, F(s) \Rightarrow \Delta \text{ for all } s}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, \exists x F(x) \Rightarrow \Delta} \qquad |\, s \,| < \alpha_s + 1 < \alpha$$

$$(\exists R) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow F(s)}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, \Rightarrow \exists x F(x)} \qquad \begin{array}{r} \alpha_0 + 1 < \alpha \\ |\, s \,| < \alpha \end{array}$$

$$(\mathrm{Cut}) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma, B \Rightarrow \Delta \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow B}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \Delta} \qquad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(\Sigma^{\mathcal{P}}\text{-}Ref) \qquad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma \Rightarrow A}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \exists z\, A^z} \qquad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \text{ is a } \Sigma^{\mathcal{P}}\text{-formula} \end{array}$$

Lastly if $\Gamma \Rightarrow \Delta$ is the result of a propositional inference of the form $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\rightarrow L)$ or $(\rightarrow R)$, with premise(s) $\Gamma_i \Rightarrow \Delta_i$ then from $\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\mid}} \Gamma_i \Rightarrow \Delta_i$ (for each i) we may conclude $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \Delta$, provided $\alpha_0 < \alpha$.

4.3 Cut elimination for $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$

Lemma 4.3.1 (Weakening and Persistence for $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$)**.**

i) If $\Gamma_0 \subseteq \Gamma$, $|\,\Gamma \Rightarrow \Delta\,| \subseteq \mathcal{H}$, $\alpha_0 \leq \alpha \in \mathcal{H}$, $\rho_0 \leq \rho$ and $\mathcal{H} \overset{\alpha_0}{\underset{\rho_0}{\mid}} \Gamma_0 \Rightarrow \Delta$ then

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \Delta$$

ii) If $\gamma \in \mathcal{H}$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, \exists x A(x) \Rightarrow \Delta$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\exists x \in \mathbb{V}_\gamma)A(x) \Rightarrow \Delta$.

iii) If $\gamma \in \mathcal{H}$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow \forall x A(x)$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow (\forall x \in \mathbb{V}_\gamma)A(x)$.

Proof. All proofs are by induction on α. We show ii), suppose $\gamma \in \mathcal{H}$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, \exists x A(x) \Rightarrow \Delta$. The interesting case is where $\exists x A(x)$ was the principal formula of the last inference which was $(\exists L)_\infty$, in this case we have $\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma, \exists x A(x), A(s) \Rightarrow \Delta$ for each term s with $|\,s\,| < \alpha_s + 1 < \alpha$ (If $\exists x A(x)$ was not a side formula we can use part i) to make it one). By the induction hypothesis we obtain $\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\mid}} \Gamma, (\exists x \in \mathbb{V}_\gamma)A(x), A(s) \Rightarrow \Delta$ for all $|\,s\,| < \gamma$. By $(\wedge L)$ we get

$$\mathcal{H}[s] \overset{\alpha_s+1}{\underset{\rho}{\mid}} \Gamma, (\exists x \in \mathbb{V}_\gamma)A(x), s \in \mathbb{V}_\gamma \wedge A(s) \Rightarrow \Delta.$$

Hence we may apply $(b\exists L)_\infty$ to obtain $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, (\exists x \in \mathbb{V}_\gamma)A(x) \Rightarrow \Delta$ as required. $\square$

Lemma 4.3.2 (Inversions of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$)**.**

i) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, A \wedge B \Rightarrow \Delta$ and $rk(A \wedge B) \geq \Omega$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, A, B \Rightarrow \Delta$.

ii) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow A \wedge B$ and $rk(A \wedge B) \geq \Omega$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow A$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow B$.

iii) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, A \vee B \Rightarrow \Delta$ and $rk(A \vee B) \geq \Omega$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, A \Rightarrow \Delta$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, B \Rightarrow \Delta$.

iv) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, A \rightarrow B \Rightarrow \Delta$ and $rk(A \rightarrow B) \geq \Omega$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, B \Rightarrow \Delta$.

v) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma \Rightarrow A \rightarrow B$ and $rk(A \rightarrow B) \geq \Omega$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{\mid}} \Gamma, A \Rightarrow B$.

vi) If $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma \Rightarrow \neg A$ and $rk(A) \geq \Omega$ then $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma, A \Rightarrow$.

vii) If $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ and $rk(A(\mathbb{V}_0)) \geq \Omega$ then $\mathcal{H}[s] \,\frac{\alpha}{\rho}\, \Gamma, s \in t \wedge A(s) \Rightarrow \Delta$ for all $|s| < |t|$.

viii) If $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma \Rightarrow (\forall x \in t)A(x)$ and $rk(A(\mathbb{V}_0)) \geq \Omega$ then $\mathcal{H}[s] \,\frac{\alpha}{\rho}\, \Gamma \Rightarrow s \in t \rightarrow A(s)$ for all $|s| < |t|$.

ix) If $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma, (\exists x \subseteq t)A(x) \Rightarrow \Delta$ and $rk(A(\mathbb{V}_0)) \geq \Omega$ then $\mathcal{H}[s] \,\frac{\alpha}{\rho}\, \Gamma, s \subseteq t \wedge A(s) \Rightarrow \Delta$ for all $|s| \leq |t|$.

x) If $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma \Rightarrow (\forall x \subseteq t)A(x)$ and $rk(A(\mathbb{V}_0)) \geq \Omega$ then $\mathcal{H}[s] \,\frac{\alpha}{\rho}\, \Gamma \Rightarrow s \subseteq t \rightarrow A(s)$ for all $|s| \leq |t|$.

xi) If $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma, \exists x A(x) \Rightarrow \Delta$ then $\mathcal{H}[s] \,\frac{\alpha}{\rho}\, \Gamma, F(s) \Rightarrow \Delta$ for all s.

xii) If $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma, \Rightarrow \forall x A(x)$ then $\mathcal{H}[s] \,\frac{\alpha}{\rho}\, \Gamma \Rightarrow F(s)$ for all s.

Proof. The proof is by induction on α and many parts are standard for many intuitionistic systems of a similar nature. We show viii) and ix).

viii) Suppose that $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma \Rightarrow (\forall x \in t)A(x)$ and $rk(A(\mathbb{V}_0)) \geq \Omega$. Since A must contain an unbounded quantifier, the sequent $\Gamma \Rightarrow (\forall x \in t)A(x)$ cannot be an axiom. If the last inference was not $(b\forall R)_\infty$ then we may apply the induction hypothesis to the premises of that inference, and then the same inference again. Finally suppose the last inference was $(b\forall R)_\infty$ so we have

$$\mathcal{H}[s] \,\frac{\alpha_s}{\rho}\, \Gamma \Rightarrow s \in t \rightarrow A(s) \quad \text{for all } |\,s\,| < |\,t\,|, \text{ with } \alpha_s < \alpha.$$

Applying weakening completes the proof of this case.

ix) Suppose that $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma, (\exists x \subseteq t)A(x) \Rightarrow \Delta$ and $rk(A(\mathbb{V}_0)) \geq \Omega$. Since $A(x)$ contains an unbounded quantifier $\exists x \subseteq t)A(x)$ cannot be the *active part* of an axiom, thus if $\Gamma, (\exists x \subseteq t)A(x) \Rightarrow \Delta$ is an axiom then so is $\Gamma, s \subseteq t \wedge A(x) \Rightarrow \Delta$ for any $|\,s\,| \leq |\,t\,|$. As in viii) the remaining interesting case is where $(\exists x \subseteq t)A(x)$ was the principal formula of the last inference, which was $(pb\exists L)_\infty$. In this case we have

$$\mathcal{H}[s] \,\frac{\alpha_s}{\rho}\, \Gamma, (\exists x \subseteq t)A(x), s \subseteq t \wedge A(s) \Rightarrow \Delta \quad \text{for all } |\,s\,| \leq |\,t\,| \text{ with } \alpha_s < \alpha.$$

Now applying the induction hypothesis yields $\mathcal{H}[s] \,\frac{\alpha_s}{\rho}\, \Gamma, s \subseteq t \wedge A(s) \Rightarrow \Delta$, to which we may apply weakening to complete the proof of this case. □

Lemma 4.3.3 (Reduction)**.** If $rk(C) := \rho > \Omega$, $\mathcal{H} \,\frac{\alpha}{\rho}\, \Gamma, C \Rightarrow \Delta$ and $\mathcal{H} \,\frac{\beta}{\rho}\, \Xi \Rightarrow C$ then

$$\mathcal{H} \,\frac{\alpha\#\alpha\#\beta\#\beta}{\rho}\, \Gamma, \Xi \Rightarrow \Delta$$

Proof. The proof is by induction on $\alpha\#\alpha\#\beta\#\beta$. The interesting case is where C was the principal formula of both final inferences, notice that in this case the last inference cannot have been ($\Sigma^{\mathcal{P}}$-Ref) since $rk(C) > \Omega$ and the conclusion of an application of ($\Sigma^{\mathcal{P}}$-Ref) always has rank Ω. Thus the rest of the proof follows in the usual way by the symmetry of the rules and Lemmas 4.2.3 and 4.3.2, we treat the case where $C \equiv (\forall x \subseteq t)A(x)$ and C was the principal formula of both last inferences, so we have

(1) $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, C \Rightarrow \Delta$

(2) $\mathcal{H} \overset{\beta}{\underset{\rho}{\vdash}} \Xi \Rightarrow C$

(3) $\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, C, s \subseteq t \to A(s) \Rightarrow \Delta$ with $\alpha_0, |\, s\,| < \alpha$ and $|\, s\,| \leq |\, t\,|$.

(4) $\mathcal{H}[p] \overset{\beta_p}{\underset{\rho}{\vdash}} \Xi \Rightarrow p \subseteq t \to A(p)$ for all $|\, p\,| \leq |\, t\,|$ with $|\, p\,| \leq \alpha_p < \alpha$.

From (3) we know that $s \in \mathcal{H}$, so from (4) we get

(5) $$\mathcal{H} \overset{\beta_s}{\underset{\rho}{\vdash}} \Xi \Rightarrow s \subseteq t \to A(s)\,.$$

Applying the induction hypothesis to (2) and (3) yields

(6) $$\mathcal{H} \overset{\alpha_0\#\alpha_0\#\beta\#\beta}{\underset{\rho}{\vdash}} \Gamma, s \subseteq t \to A(s) \Rightarrow \Delta\,.$$

Finally by applying (Cut) to (5) and (6), whilst noting that by Lemma 4.2.3 $rk(s \subseteq t \to A(s)) < \rho$, we obtain

$$\mathcal{H} \overset{\alpha\#\alpha\#\beta\#\beta}{\underset{\rho}{\vdash}} \Gamma, \Xi \Rightarrow \Delta$$

as required. □

Lemma 4.3.4. If $\mathcal{H} \overset{\alpha}{\underset{\Omega+n+1}{\vdash}} \Gamma \Rightarrow \Delta$ then $\mathcal{H} \overset{\omega^\alpha}{\underset{\Omega+n}{\vdash}} \Gamma \Rightarrow \Delta$ for any $n < \omega$.

Proof. The proof is by induction on α, suppose $\mathcal{H} \overset{\alpha}{\underset{\Omega+n+1}{\vdash}} \Gamma \Rightarrow \Delta$. If $\Gamma \Rightarrow \Delta$ is an axiom there is nothing to show. If $\Gamma \Rightarrow \Delta$ was the result of an inference other that (Cut) or a cut with cut-rank $< \Omega + n$ then we may apply the induction hypothesis to the premises of that inference and then the same inference again. So suppose the last inference was (Cut) with cut-formula C, and that $rk(C) = \Omega + n$. So we have

(1) $\mathcal{H} \overset{\alpha_0}{\underset{\Omega+n+1}{\vdash}} \Gamma, C \Rightarrow \Delta$ with $\alpha_0 < \alpha$.

(2) $\mathcal{H} \overset{\alpha_1}{\underset{\Omega+n+1}{\vdash}} \Gamma \Rightarrow C$ with $\alpha_1 < \alpha$.

Applying the induction hypothesis to (1) and (2) gives

(3) $\mathcal{H} \overset{\omega^{\alpha_0}}{\underset{\Omega+n}{\vdash}} \Gamma, C \Rightarrow \Delta$

(4) $\mathcal{H} \overset{\omega^{\alpha_1}}{\underset{\Omega+n}{\vdash}} \Gamma \Rightarrow C\,.$

Now applying the Reduction Lemma 4.3.3 to (3) and (4) provides us with

$$\mathcal{H} \left|\frac{\omega^{\alpha_0}\#\omega^{\alpha_0}\#\omega^{\alpha_1}\#\omega^{\alpha_1}}{\Omega+n}\right. .$$

It remains to note that $\omega^{\alpha_0}\#\omega^{\alpha_0}\#\omega^{\alpha_1}\#\omega^{\alpha_1} < \omega^\alpha$ since ω^α is additive principal, so we can complete the proof by weakening. □

Theorem 4.3.5 (Partial cut elimination for $\mathbf{IRS}_\Omega^{\mathcal{P}}$)**.** If $\mathcal{H} \left|\frac{\alpha}{\Omega+n+1}\right. \Gamma \Rightarrow \Delta$ then $\mathcal{H} \left|\frac{\omega_n(\alpha)}{\Omega+1}\right. \Gamma \Rightarrow \Delta$ where $\omega_0(\beta) := \beta$ and $\omega_{k+1}(\beta) := \omega^{\omega_k(\beta)}$.

Proof. The proof uses an easy induction on n and the previous Lemma. □

Note that 4.3.5 is much weaker than the full predicative cut elimination result we obtained for $\mathbf{IRS}_\Omega$ (Theorem 3.4.3), this is because in general we cannot eliminate cuts with $\Delta_0^{\mathcal{P}}$ cut-formulae from $\mathbf{IRS}_\Omega^{\mathcal{P}}$ derivations.

Lemma 4.3.6 (Boundedness)**.** If A is a $\Sigma^{\mathcal{P}}$-formula, B is a $\Pi^{\mathcal{P}}$-formula, $\alpha \leq \beta < \Omega$ and $\beta \in \mathcal{H}$ then

i) If $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma \Rightarrow A$ then $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma \Rightarrow A^{\mathbb{V}_\beta}$.

ii) If $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma, B \Rightarrow \Delta$ then $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta} \Rightarrow \Delta$

Proof. The proofs are by induction on α, we show ii), the proof of i) is similar. As with Lemma 3.4.4 the only interesting case is where B was the principal formula of the last inference and B is of the form $\forall x C(x)$. So we have

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B, C(s) \Rightarrow \Delta \quad \text{for some } |s| < \alpha \text{ with } \alpha_0 + 1 < \alpha.$$

Using the induction hypothesis we obtain

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta}, C(s) \Rightarrow \Delta \quad \text{for some } |s| < \alpha \text{ with } \alpha_0 + 1 < \alpha.$$

Now since $\Gamma, B^{\mathbb{V}_\beta} \Rightarrow s \in \mathbb{V}_\beta$ is an axiom, we have $\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta} \Rightarrow s \in \mathbb{V}_\beta$, so by $(\rightarrow L)$ we obtain

$$\mathcal{H} \left|\frac{\alpha_0+1}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta}, s \in \mathbb{V}_\beta \rightarrow C(s) \Rightarrow \Delta \quad \text{for some } |s| < \alpha \text{ with } \alpha_0 + 1 < \alpha.$$

Finally an application of $(b\forall L)$ yields

$$\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma, B^{\mathbb{V}_\beta} \Rightarrow \Delta$$

as required. □

Theorem 4.3.7 (Collapsing)**.** Suppose that $\eta \in \mathcal{H}_\eta$, Δ is a set of at most one $\Sigma^{\mathcal{P}}$-formula and Γ a set of $\Pi^{\mathcal{P}}$-formulae with $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$ then:

$$\mathcal{H}_\eta \mid\!\frac{\alpha}{\Omega+1}\, \Gamma \Rightarrow \Delta \quad \text{implies} \quad \mathcal{H}_{\hat{\alpha}} \mid\!\frac{\psi_\Omega(\hat{\alpha})}{\psi_\Omega(\hat{\alpha})}\, \Gamma \Rightarrow \Delta.$$

Here $\hat{\beta} = \eta + \omega^{\Omega+\beta}$ and the operators $\mathcal{H}_\xi$ are those defined in Definition 3.4.5.

Proof. Note first that from $\eta \in \mathcal{H}_\eta$ and Lemma 3.4.7 we obtain

$$(1) \qquad \hat{\alpha}, \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}.$$

The proof is by induction on α.

Case 0. If $\Gamma \Rightarrow \Delta$ is an axiom then the result follows immediately from (1).

Case 1. If the last inference was propositional then the assertion follows easily by applying the induction hypothesis and then the same inference again.

Case 2. Suppose the last inference was $(pb\forall R)_\infty$, then $\Delta = \{(\forall x \subseteq t)F(x)\}$ and

$$\mathcal{H}_\eta[p] \mid\!\frac{\alpha_p}{\Omega+1}\, \Gamma \Rightarrow p \subseteq t \to F(p) \quad \text{for all } |\,p\,| \leq |\,t\,| \text{ with } \alpha_p < \alpha.$$

Since $|\,t\,| \in \mathcal{H}_\eta(\emptyset) = B^\Omega(\eta+1)$ and $|\,t\,| < \Omega$, we have $|\,t\,| < \psi_\Omega(\eta+1)$, thus $|\,p\,| \in \mathcal{H}_\eta$ for all $|\,p\,| \leq |\,t\,|$. So we have

$$\mathcal{H}_\eta \mid\!\frac{\alpha_p}{\Omega+1}\, \Gamma \Rightarrow p \subseteq t \to F(p)$$

By Lemma 4.3.2 v) we get

$$\mathcal{H}_\eta \mid\!\frac{\alpha_p}{\Omega+1}\, \Gamma, p \subseteq t \Rightarrow F(p)$$

Now since $p \subseteq t$ is $\Delta_0^{\mathcal{P}}$ we may apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha_p}} \mid\!\frac{\psi_\Omega(\hat{\alpha_p})}{\psi_\Omega(\hat{\alpha_p})}\, \Gamma, p \subseteq t \Rightarrow F(p) \quad \text{for all } |\,p\,| \leq |\,t\,| \text{ with } \alpha_p < \alpha.$$

Now noting that $\psi_\Omega(\hat{\alpha_p}) + 1 < \psi_\Omega(\hat{\alpha})$, by applying $(\to R)$ followed by $(pb\forall R)_\infty$ we obtain the desired result. The cases where the last inference was $(b\forall R)_\infty$, $(pb\exists L)_\infty$, $(b\exists L)_\infty$, $(\in L)_\infty$ or $(\subseteq L)_\infty$ are similar.

Case 3. Now suppose the last inference was $(pb\forall L)$, so $(\forall x \subseteq t)F(x) \in \Gamma$ and

$$\mathcal{H}_\eta \mid\!\frac{\alpha_0}{\Omega+1}\, \Gamma, s \subseteq t \to F(s) \Rightarrow \Delta \quad \text{for some } |\,s\,| \leq |\,t\,| \text{ with } \alpha_0 < \alpha.$$

Since $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$ $F(s)$ is a $\Delta_0^{\mathcal{P}}$ formula and thus $s \subseteq t \to F(s)$ is $\Delta_0^{\mathcal{P}}$ as well. So we may apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha_0}} \mid\!\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\, \Gamma, s \subseteq t \to F(s) \Rightarrow \Delta$$

to which we may apply $(pb\forall L)$ to complete this case. The cases where the last inference was $(b\forall L)$, $(pb\exists R)$, $(b\exists R)$, $(\in R)$ or $(\subseteq R)$ are similar.

Case 4. Now suppose the last inference was $(\forall L)$, so $\forall x A(x) \in \Gamma$ and

$$\mathcal{H}_\eta \:\Big|\!\frac{\alpha_0}{\Omega+1}\: \Gamma, F(s) \Rightarrow \Delta \quad \text{for some } |\,s\,| < \alpha \text{ and } \alpha_0 < \alpha.$$

Since $F(s)$ is $\Pi^{\mathcal{P}}$ (in fact $\Delta_0^{\mathcal{P}}$) we may apply the induction hypothesis to obtain

$$\mathcal{H}_{\hat{\alpha_0}} \:\Big|\!\frac{\psi_\Omega(\hat{\alpha_0})}{\psi_\Omega(\hat{\alpha_0})}\: \Gamma, F(s) \Rightarrow \Delta$$

Now since $|\,s\,| \in \mathcal{H}_\eta = B^\Omega(\eta+1)$ we have $|\,s\,| < \psi_\Omega(\eta+1) < \psi_\Omega(\hat{\alpha})$. So we may apply $(\forall L)$ to complete the case. The case where the last inference was $(\exists R)$ is similar.

The rest of the proof is completely analogous to that of Theorem 3.4.8, using boundedness for $\mathbf{IRS}_\Omega^{\mathcal{P}}$ (4.3.6) instead of for $\mathbf{IRS}_\Omega$. $\square$

4.4 Embedding $\mathbf{IKP}(\mathcal{P})$ into $\mathbf{IRS}_\Omega^{\mathcal{P}}$

Definition 4.4.1. As in the embedding section for the case of **IKP**, $\Vdash \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \:\Big|\!\frac{no(\Gamma\Rightarrow\Delta)}{0}\: \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}.$$

Also $\Vdash_\rho^\xi \Gamma \Rightarrow \Delta$ will be used to abbreviate that

$$\mathcal{H}[\Gamma \Rightarrow \Delta] \:\Big|\!\frac{no(\Gamma\Rightarrow\Delta)\#\xi}{\rho}\: \Gamma \Rightarrow \Delta \quad \text{holds for any operator } \mathcal{H}.$$

Only this time we are referring to operator controlled derivability in $\mathbf{IRS}_\Omega^{\mathcal{P}}$.

Lemma 4.4.2. For any formula A

$$\Vdash A \Rightarrow A$$

Proof. We proceed by induction on the complexity of A. If A is $\Delta_0^{\mathcal{P}}$ then this is axiom (A1) of $\mathbf{IRS}_\Omega^{\mathcal{P}}$.

Suppose A is of the form $\exists x F(x)$. Let $\alpha_s = |\,s\,| + no(F(s) \Rightarrow F(s))$ and $\alpha = no(A \Rightarrow A)$, note that $|\,s\,| < \alpha_s + 1 < \alpha_s + 2 < \alpha$ for all s. By the induction hypothesis we have

$$\mathcal{H}[F(s)] \:\Big|\!\frac{\alpha_s}{0}\: F(s) \Rightarrow F(s) \quad \text{for all terms } s \text{ and for an arbitrary operator } \mathcal{H}.$$

Now using weakening if necessary on the operator and $(\exists R)$ we get

$$\mathcal{H}[F(s), s] \:\Big|\!\frac{\alpha_s+1}{0}\: F(s) \Rightarrow \exists x F(x)$$

Finally since $\mathcal{H}[F(s), s](\emptyset) \subseteq \mathcal{H}[\exists x F(x)][s](\emptyset)$ we may apply $(\exists L)_\infty$ to obtain the desired result. The other cases are similar. $\square$

Lemma 4.4.3 (Extensionality)**.** For any formula A and any terms $s_1, ..., s_n, t_1, ..., t_n$

$$\Vdash s_1 = t_1, ..., s_n = t_n, A(s_1, ..., s_n) \Rightarrow A(t_1, ..., t_n).$$

Proof. If A is $\Delta_0^{\mathcal{P}}$ then this is an axiom. The remainder of the proof is by induction on $rk(A(s_1, ..., s_n))$, note that $rk(A(s_1, ..., s_n) = rk(A(t_1, ..., t_n)$ since A is not $\Delta_0^{\mathcal{P}}$.

Case 1. Suppose $A(s_1, ..., s_n) \equiv \exists x B(x, s_1, ..., s_n)$, we know that $rk(B(r, s_1, ..., s_n)) < rk(A(s_1, ..., s_n))$ for all r by Lemma 4.2.3, so by induction hypothesis we have

$$\Vdash s_1 = t_1, ..., s_n = t_n, B(r, s_1, ..., s_n) \Rightarrow B(r, t_1, ..., t_n) \quad \text{for all terms } r.$$

Now successively applying $(\exists R)$ and then $(\exists L)_\infty$ yields the desired result.

Case 2. Now suppose $A(s_1, ..., s_n) \equiv (\exists x \subseteq s_i)B(x, s_1, ..., s_n)$. Since A is not $\Delta_0^{\mathcal{P}}$, B must contain an unbounded quantifier, and thus by Lemma 4.2.3 $\Omega \leq rk(r \subseteq s_i \wedge B(r, s_1, ..., s_n)) < rk(A(s_1, ..., s_n))$ for any $|\,r\,| \leq |\,s_i\,|$, thus by induction hypothesis we have

$$\Vdash s_1 = t_1, ..., s_n = t_n, r \subseteq s_i \wedge B(r, s_1, ..., s_n) \Rightarrow r \subseteq t_i \wedge B(r, t_1, ..., t_n) \quad \text{for all } |\,r\,| \leq |\,s_i\,|.$$

Thus successively applying $(pb\exists R)$ and then $(pb\exists L)_\infty$ yields the desired result. The other cases are similar. □

Lemma 4.4.4 ($\Delta_0^{\mathcal{P}}$-Collection)**.** For any $\Delta_0^{\mathcal{P}}$ formula F

$$\Vdash\Rightarrow (\forall x \in s)\exists y F(x, y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y).$$

Proof. Lemma 4.4.2 provides us with

$$\Vdash (\forall x \in s)\exists y F(x, y) \Rightarrow (\forall x \in s)\exists y F(x, y)$$

Noting that $(\forall x \in s)\exists y F(x, y)$ is a $\Sigma^{\mathcal{P}}$ formula and that $rk((\forall x \in s)\exists y F(x, y)) = \omega^{\Omega+2}$ we may apply ($\Sigma^{\mathcal{P}}$-Ref) to obtain

$$\bar{\mathcal{H}} \,\big|\!\frac{\omega^{\Omega+2}\cdot 2+2}{0}\, (\forall x \in s)\exists y F(x, y) \Rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y)$$

where $\bar{\mathcal{H}} = \mathcal{H}[(\forall x \in s)\exists y F(x, y)]$ and $\mathcal{H}$ is an arbitrary operator. Now applying $(\rightarrow R)$ we get

$$\bar{\mathcal{H}} \,\big|\!\frac{\omega^{\Omega+2}\cdot 2+3}{0}\, \Rightarrow (\forall x \in s)\exists y F(x, y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y).$$

It remains to note that $\omega^{\Omega+2} \cdot 2 + 3 < \omega^{\Omega+3} = no((\Rightarrow \forall x \in s)\exists y F(x, y) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y))$ to see that the result is verified. □

Lemma 4.4.5 (Set Induction)**.** For any formula F

$$\Vdash\Rightarrow \forall x[(\forall y \in x)F(y) \rightarrow F(x)] \rightarrow \forall x F(x).$$

Proof. Let $\mathcal{H}$ be an arbitrary operator and let $A := \forall x[(\forall y \in x)F(y) \rightarrow F(x)]$. First we prove the following

$$\text{Claim:} \quad \mathcal{H}[A,s] \vdash^{\omega^{rk(A)}\#\omega^{|s|+1}}_{0} A \Rightarrow F(s) \quad \text{for all terms } s.$$

The claim is proved by induction on $|s|$. By the induction hypothesis we have

$$\mathcal{H}[A,t] \vdash^{\omega^{rk(A)}\#\omega^{|t|+1}}_{0} A \Rightarrow F(t) \quad \text{for all } |t| < |s|.$$

Using weakening and then $(\rightarrow R)$ we get

$$\mathcal{H}[A,s,t] \vdash^{\omega^{rk(A)}\#\omega^{|t|+1}+1}_{0} A \Rightarrow t \in s \rightarrow F(t) \quad \text{for all } |t| < |s|.$$

Hence by $(b\forall R)_\infty$ we get

$$\mathcal{H}[A,s] \vdash^{\omega^{rk(A)}\#\omega^{|s|}+2}_{0} A \Rightarrow (\forall x \in s)F(x)$$

(the extra $+2$ is needed when $|s|$ is not a limit.) Now let $\eta_s := \omega^{rk(A)}\#\omega^{|s|} + 2$. By Lemma 4.4.2 we have $\mathcal{H}[A,s] \vdash^{\eta_s}_{0} F(s) \Rightarrow F(s)$, so by $(\rightarrow L)$ we get

$$\mathcal{H}[A,s] \vdash^{\eta_s+1}_{0} A, (\forall y \in s)F(y) \rightarrow F(s) \Rightarrow F(s)\,.$$

Finally by applying $(\forall L)$ we get

$$\mathcal{H}[A,s] \vdash^{\eta_s+3}_{0} A \Rightarrow F(s)\,,$$

since $\eta_s + 3 < \omega^{rk(A)}\#\omega^{|s|+1}$ the claim is verified. Now by applying $(\forall R)_\infty$ we deduce from the claim that

$$\mathcal{H}[A] \vdash^{\omega^{rk(A)}+\Omega}_{0} A \Rightarrow \forall x F(x)\,.$$

Hence by $(\rightarrow R)$ we obtain the desired result. $\square$

Lemma 4.4.6 (Infinity)**.** For any operator $\mathcal{H}$ we have

$$\mathcal{H} \vdash^{\omega+2}_{0} \;\Rightarrow \exists x[(\exists y \in x)(y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$$

Proof. First note that for any $|s| < \alpha$ we have $\mathcal{H} \vdash^{0}_{0} s \in \mathbb{V}_\alpha$ by virtue of axiom (A4). Let $|s| = n < \omega$, we have the following derivation in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$:

$$
(\exists R)\,\dfrac{(\wedge R)\,\dfrac{(b\forall R)_\infty\,\dfrac{(\rightarrow R)\,\dfrac{(b\exists R)\,\dfrac{(\wedge R)\,\dfrac{\mathcal{H} \vdash^{0}_{0} \;\Rightarrow \mathbb{V}_{n+1} \in \mathbb{V}_\omega \qquad \mathcal{H} \vdash^{0}_{0} \;\Rightarrow s \in \mathbb{V}_{n+1}}{\mathcal{H} \vdash^{1}_{0} \;\Rightarrow \mathbb{V}_{n+1} \in \mathbb{V}_\omega \wedge s \in \mathbb{V}_{n+1}}}{\mathcal{H} \vdash^{n+2}_{0} \;\Rightarrow (\exists z \in \mathbb{V}_\omega)(s \in z)}}{\mathcal{H} \vdash^{n+3}_{0} \;\Rightarrow s \in \mathbb{V}_\omega \rightarrow (\exists z \in \mathbb{V}_\omega)(s \in z)}}{\mathcal{H} \vdash^{\omega}_{0} \;\Rightarrow (\forall y \in \mathbb{V}_\omega)(\exists z \in \mathbb{V}_\omega)(y \in z)} \qquad (b\exists R)\,\dfrac{(\wedge R)\,\dfrac{\mathcal{H} \vdash^{0}_{0} \;\Rightarrow \mathbb{V}_0 \in \mathbb{V}_\omega}{\mathcal{H} \vdash^{1}_{0} \;\Rightarrow \mathbb{V}_0 \in \mathbb{V}_\omega \wedge \mathbb{V}_0 \in \mathbb{V}_\omega}}{\mathcal{H} \vdash^{2}_{0} \;\Rightarrow (\exists z \in \mathbb{V}_\omega)(z \in \mathbb{V}_\omega)}}{\mathcal{H} \vdash^{\omega+1}_{0} \;\Rightarrow (\forall y \in \mathbb{V}_\omega)(\exists z \in \mathbb{V}_\omega)(y \in z) \wedge (\exists z \in \mathbb{V}_\omega)(z \in \mathbb{V}_\omega)}}{\mathcal{H} \vdash^{\omega+2}_{0} \;\Rightarrow \exists x[(\forall y \in x)(\exists z \in x)(y \in z) \wedge (\exists z \in x)(z \in x)]}
$$

$\square$

Lemma 4.4.7 ($\Delta_0^{\mathcal{P}}$-Separation)**.** If $A(a, b, c_1, ..., c_n)$ is a $\Delta_0^{\mathcal{P}}$-formula of $\mathbf{IKP}(\mathcal{P})$ with all free variables indicated, $r, \bar{s} := s_1, ..., s_n$ are $\mathbf{IRS}_\Omega^{\mathcal{P}}$ terms and $\mathcal{H}$ is an arbitrary operator then:

$$\mathcal{H}[r, \bar{s}] \,|\!\frac{\alpha+7}{\rho}\; \Rightarrow \exists y[(\forall x \in y)(x \in r \wedge A(x, r, \bar{s})) \wedge (\forall x \in r)(A(x, r, \bar{s}) \rightarrow x \in y)]$$

where $\alpha := |\, r \,|$ and $\rho := \max\{|\, r \,|, |\, s_1 \,|, ..., |\, s_n \,|\} + \omega$.

Proof. First we define

$$p := [x \in \mathbb{V}_\alpha \mid x \in r \wedge A(x, r, \bar{s})] \quad \text{and} \quad \bar{\mathcal{H}} := \mathcal{H}[r, \bar{s}].$$

For t any term with $|\, t \,| < \alpha$ the following are derivations in $\mathbf{IRS}_\Omega^{\mathcal{P}}$, first we have:

$$\dfrac{(\wedge R)\,\dfrac{\dfrac{\text{Axiom (A1)}}{\bar{\mathcal{H}} \,|\!\frac{0}{0}\, t \in r \Rightarrow t \in r} \qquad \dfrac{\text{Axiom (A1)}}{\bar{\mathcal{H}} \,|\!\frac{0}{0}\, A(t, r, \bar{s}) \Rightarrow A(t, r, \bar{s})}}{\bar{\mathcal{H}} \,|\!\frac{1}{0}\, t \in r, A(t, r, \bar{s}) \Rightarrow t \in r \wedge A(t, r, \bar{s})} \qquad \dfrac{\text{Axiom (A7)}}{\bar{\mathcal{H}} \,|\!\frac{0}{0}\, t \in r \wedge A(t, r, \bar{s}) \Rightarrow t \in p}}{\dfrac{\bar{\mathcal{H}} \,|\!\frac{2}{\rho}\, t \in r, A(t, r, \bar{s}) \Rightarrow t \in p}{\dfrac{\bar{\mathcal{H}} \,|\!\frac{3}{\rho}\, t \in r \Rightarrow A(t, r, \bar{s}) \rightarrow t \in p}{\dfrac{\bar{\mathcal{H}} \,|\!\frac{4}{\rho}\; \Rightarrow t \in r \rightarrow (A(t, r, \bar{s}) \rightarrow t \in p)}{\bar{\mathcal{H}} \,|\!\frac{\alpha+5}{\rho}\; \Rightarrow (\forall x \in r)(A(x, r, \bar{s}) \rightarrow x \in p)}\,(b\forall R)_\infty}\,(\rightarrow R)}\,(\rightarrow R)}\,(\text{cut})$$

Next we have:

$$(b\forall R)_\infty\,\dfrac{(\rightarrow R)\,\dfrac{\dfrac{\text{Axiom (A6)}}{\bar{\mathcal{H}} \,|\!\frac{0}{0}\, t \in p \Rightarrow t \in r \wedge A(t, r, \bar{s})}}{\bar{\mathcal{H}} \,|\!\frac{1}{0}\; \Rightarrow t \in p \rightarrow t \in r \wedge A(t, r, \bar{s})}}{\bar{\mathcal{H}} \,|\!\frac{\alpha+2}{0}\; \Rightarrow (\forall x \in p)(x \in r \wedge A(x, r, \bar{s}))}$$

Now by applying $(\wedge R)$ followed by $(\exists R)$ to the conclusions of these two derivations we get

$$\bar{\mathcal{H}} \,|\!\frac{\alpha+7}{\rho}\; \Rightarrow \exists y[(\forall x \in y)(x \in r \wedge A(x, r, \bar{s})) \wedge (\forall x \in r)(A(x, r, \bar{s}) \rightarrow x \in y)]$$

as required. $\square$

Lemma 4.4.8 (Pair)**.** For any operator $\mathcal{H}$ and any terms s and t we have

$$\mathcal{H}[s, t] \,|\!\frac{\alpha+2}{0}\; \Rightarrow \exists z(s \in z \wedge t \in z)$$

Where $\alpha := \max(|\, s \,|, |\, t \,|) + 1$.

Proof. The following is a derivation in $\mathbf{IRS}_\Omega^{\mathcal{P}}$:

$$
\dfrac{\dfrac{\text{Axiom (A4)}}{\mathcal{H}[s,t] \mathrel{|}\!\frac{0}{0} \Rightarrow s \in \mathbb{V}_\alpha} \quad \dfrac{\text{Axiom (A4)}}{\mathcal{H}[s,t] \mathrel{|}\!\frac{0}{0} \Rightarrow t \in \mathbb{V}_\alpha}}{\dfrac{\mathcal{H}[s,t] \mathrel{|}\!\frac{1}{0} \Rightarrow s \in \mathbb{V}_\alpha \wedge t \in \mathbb{V}_\alpha}{\mathcal{H}[s,t] \mathrel{|}\!\frac{\alpha+2}{0} \Rightarrow \exists z(s \in z \wedge t \in z)}\,(\exists R)}\,(\wedge R)
$$

□

Lemma 4.4.9 (Union)**.** For any operator $\mathcal{H}$ and any term s we have

$$\mathcal{H}[s] \mathrel{|}\!\frac{\beta+5}{0} \Rightarrow \exists z(\forall y \in s)(\forall x \in y)(x \in z)$$

where $\beta = |\, s \,|$.

Proof. Let r and t be terms such that $|\, r \,| < |\, t \,| < \beta$, we have the following derivation in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$:

$$
\begin{array}{rl}
 & \text{Axiom (A4)} \\ \cline{2-2}
 & \mathcal{H}[s,t,r] \mathrel{|}\!\frac{0}{0} t \in s, r \in t \Rightarrow r \in \mathbb{V}_\beta \\ \cline{2-2}
(\rightarrow R) & \mathcal{H}[s,t,r] \mathrel{|}\!\frac{1}{0} t \in s \Rightarrow r \in t \rightarrow r \in \mathbb{V}_\beta \\ \cline{2-2}
(b\forall R)_\infty & \mathcal{H}[s,t] \mathrel{|}\!\frac{\beta+2}{0} t \in s \Rightarrow (\forall x \in t)(x \in \mathbb{V}_\beta) \\ \cline{2-2}
(\rightarrow R) & \mathcal{H}[s,t] \mathrel{|}\!\frac{\beta+3}{0} \Rightarrow t \in s \rightarrow (\forall x \in t)(x \in \mathbb{V}_\beta) \\ \cline{2-2}
(b\forall R)_\infty & \mathcal{H}[s] \mathrel{|}\!\frac{\beta+4}{0} \Rightarrow (\forall y \in s)(\forall x \in y)(x \in \mathbb{V}_\beta) \\ \cline{2-2}
(\exists R) & \mathcal{H}[s] \mathrel{|}\!\frac{\beta+5}{0} \Rightarrow \exists z(\forall y \in s)(\forall x \in y)(x \in z)
\end{array}
$$

□

Lemma 4.4.10 (Powerset)**.** For any operator $\mathcal{H}$ and any term s we have

$$\mathcal{H}[s] \mathrel{|}\!\frac{\alpha+3}{0} \Rightarrow \exists z(\forall x \subseteq s)(x \in z)$$

where $\alpha = |\, s \,|$.

Proof. Let t be any term with $|\, t \,| < \alpha$, we have the following derivation in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$:

$$
\begin{array}{rl}
 & \text{Axiom (A4)} \\ \cline{2-2}
 & \mathcal{H}[s,t] \mathrel{|}\!\frac{0}{0} t \subseteq s \Rightarrow t \in \mathbb{V}_{\alpha+1} \\ \cline{2-2}
(\rightarrow R) & \mathcal{H}[s,t] \mathrel{|}\!\frac{1}{0} \Rightarrow t \subseteq s \rightarrow t \in \mathbb{V}_{\alpha+1} \\ \cline{2-2}
(pb\forall R)_\infty & \mathcal{H}[s] \mathrel{|}\!\frac{\alpha+2}{0} \Rightarrow (\forall x \subseteq s)(x \in \mathbb{V}_{\alpha+1}) \\ \cline{2-2}
(\exists R) & \mathcal{H}[s] \mathrel{|}\!\frac{\alpha+3}{0} \Rightarrow \exists z(\forall x \subseteq s)(x \in z)
\end{array}
$$

□

Theorem 4.4.11. If $\mathbf{IKP}(\mathcal{P}) \vdash \Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ where $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{a})$ is an intuitionistic sequent containing exactly the free variables $\bar{a} = a_1, ..., a_n$, then there exists an $m < \omega$ (which we may calculate from the derivation) such that

$$\mathcal{H}[\bar{s}] \mid\!\frac{\Omega\cdot\omega^m}{\Omega+m}\, \Gamma(\bar{s}) \Rightarrow \Delta(\bar{s})$$

for any operator $\mathcal{H}$ and any $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$ terms $\bar{s} = s_1, ..., s_n$.

Proof. Note that the rank of $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$ formulas is always $< \Omega + \omega$ and thus the norm of $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$ sequents is always $< \omega^{\Omega+\omega} = \Omega \cdot \omega^{\omega}$. The proof is by induction on the $\mathbf{IKP}(\mathcal{P})$ derivation. If $\Gamma(\bar{a}) \Rightarrow \Delta(\bar{b})$ is an axiom of $\mathbf{IKP}(\mathcal{P})$ then the result follows by one of Lemmas 4.4.2, 4.4.3 4.4.4, 4.4.5, 4.4.6, 4.4.7, 4.4.8, 4.4.9 and 4.4.10. Let $\bar{\mathcal{H}} := \mathcal{H}[\bar{s}]$.

Case 1. Suppose the last inference of the $\mathbf{IKP}(\mathcal{P})$ derivation was $(pb\exists L)$ then $(\exists x \subseteq a_i)F(x) \in \Gamma(\bar{a})$ and from the induction hypothesis we obtain a k such that

$$\bar{\mathcal{H}}[p] \mid\!\frac{\Omega\cdot\omega^k}{\Omega+k}\, \Gamma(\bar{s}), p \subseteq s_i \wedge F(p) \Rightarrow \Delta(\bar{s})$$

for all $|\,p\,| \leq |\,s_i\,|$ (using weakening if necessary). Thus we may apply $(pb\exists L)_{\infty}$ to obtain the desired result.

Case 2. Now suppose the last inference was $(pb\exists R)$ then $\Delta(\bar{a}) = \{(\exists x \subseteq a_i)F(x)\}$ and we are in the following situation in $\mathbf{IKP}(\mathcal{P})$:

$$(pb\exists R)\,\frac{\vdash \Gamma(\bar{a}) \Rightarrow c \subseteq a_i \wedge F(c)}{\vdash \Gamma(\bar{a}) \Rightarrow (\exists x \subseteq a_i)F(x)}$$

2.1 If c is not a member of $\bar{a}$ then by the induction hypothesis we have a $k < \omega$ such that

$$\bar{\mathcal{H}} \mid\!\frac{\Omega\cdot\omega^k}{\Omega+k}\, \Gamma(\bar{s}) \Rightarrow \mathbb{V}_0 \subseteq s_i \wedge F(\mathbb{V}_0)$$

Hence we can apply $(pb\exists R)$ to complete this case.

2.2 Now suppose c is a member of $\bar{a}$ for simplicity let us suppose that $c = a_1$. Inductively we can find a $k < \omega$ such that

$$\text{(1)} \qquad \bar{\mathcal{H}} \mid\!\frac{\Omega\cdot\omega^k}{\Omega+k}\, \Gamma(\bar{s}) \Rightarrow s_1 \subseteq s_i \wedge F(s_1)$$

Next we verify the following

$$\text{(2)} \qquad \text{claim:} \quad \Vdash^{\omega} \Gamma(\bar{s}), s_1 \subseteq s_i \wedge F(s_1) \Rightarrow (\exists x \subseteq s_i)F(x).$$

Owing to axiom (A1) we have

$$\text{(3)} \qquad \bar{\mathcal{H}}[r] \mid\!\frac{0}{0}\, r \subseteq s_i \Rightarrow r \subseteq s_i \quad \text{for all } |\,r\,| \leq |\,s_i\,|.$$

Also by Lemma 4.4.3 we have

(4) $$\Vdash \Gamma[\bar{s}], r = s_1, F(s_1) \Rightarrow F(r) \quad \text{for all } |\, r \,| \leq |\, s_i \,|.$$

Now let $\gamma_r = no(\Gamma[\bar{s}], r = s_1, F(s_1) \Rightarrow F(r))$. Applying $(\wedge R)$ to (3) and (4) provides

$$\bar{\mathcal{H}}[r] \overset{\gamma_r+1}{\underset{0}{|}} \Gamma(\bar{s}), r \subseteq s_i, r = s_1, F(s_1) \Rightarrow r \subseteq s_i \wedge F(r)\,.$$

Using $(pb\exists R)$ we may conclude

$$\bar{\mathcal{H}}[r] \overset{\gamma_r+2}{\underset{0}{|}} \Gamma(\bar{s}), r \subseteq s_i, r = s_1, F(s_1) \Rightarrow (\exists x \subseteq s_i)F(x)\,.$$

Now two applications of $(\wedge L)$ gives us

$$\bar{\mathcal{H}}[r] \overset{\gamma_r+4}{\underset{0}{|}} \Gamma(\bar{s}), r \subseteq s_i \wedge r = s_1, F(s_1) \Rightarrow (\exists x \subseteq s_i)F(x)\,.$$

Now applying $(\subseteq L)_\infty$ provides

$$\bar{\mathcal{H}} \overset{\gamma+5}{\underset{0}{|}} \Gamma(\bar{s}), s_1 \subseteq s_i, F(s_1) \Rightarrow (\exists x \subseteq s_i)F(x)$$

where $\gamma = \sup_{|\, r \,| \leq |\, s_i \,|} \gamma_r$. Finally, by applying $(\wedge L)$ a further two times we can conclude

$$\bar{\mathcal{H}} \overset{\gamma+7}{\underset{0}{|}} \Gamma(\bar{s}), s_1 \subseteq s_i \wedge F(s_1) \Rightarrow (\exists x \subseteq s_i)F(x)\,.$$

Via some ordinal arithmetic it can be observed that

$$\gamma + 7 \leq no(\Gamma(\bar{s}), s_1 \subseteq s_i \wedge F(s_1) \Rightarrow (\exists x \subseteq s_i)F(x))\#\omega,$$

so the claim is verified.

To complete this case we may now apply (Cut) to (1) and (2).

All other cases are similar to those above, or may be treated in a similar manner to Theorem 3.5.12. □

4.5 A relativised ordinal analysis of IKP($\mathcal{P}$)

A major difference to the case of **IKP** is that we don't immediately have the soundness of cut-reduced $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$ derivations of $\Sigma^{\mathcal{P}}$-formulae within the appropriate segment of the Von-Neumann Hierarchy. This is partly due to the fact that we don't have a term for each element of the hierarchy (this can be seen from a simple cardinality argument). In fact we do still have soundness for certain derivations within $V_{\psi_\Omega(\varepsilon_{\Omega+1})}$, which is demonstrated in the next lemma, where we must make essential use of the free variables in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$. First we need the notion of an assignment. Let $VAR_{\mathcal{P}}$ be the set of free variables of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$. A variable assignment is a function

$$v : VAR_{\mathcal{P}} \longrightarrow V_{\psi_{\Omega}(\varepsilon_{\Omega+1})}$$

such that $v(a_i^{\alpha}) \in V_{\alpha+1}$ for each i. v canonically lifts to all terms as follows

$$\begin{aligned} v(\mathbb{V}_{\alpha}) &= V_{\alpha} \\ v(\{x \in \mathbb{V}_{\alpha} \mid F(x, s_1, ..., s_n)\}) &= \{x \in V_{\alpha} \mid F(x, v(s_1), ..., v(s_n))\}. \end{aligned}$$

Moreover it can be seen that $v(s) \in V_{|s|+1}$ and thus $v(s) \in V_{\psi_{\Omega}(\varepsilon_{\Omega+1})}$ for all terms s.

Theorem 4.5.1 (Soundness for $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$)**.** Suppose $\Gamma[s_1, ..., s_n]$ is a finite set of $\Pi^{\mathcal{P}}$ formulae with $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$, $\Delta[s_1, ..., s_n]$ a set containing at most one $\Sigma^{\mathcal{P}}$ formula and

$$\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}] \quad \text{for some operator } \mathcal{H} \text{ and some } \alpha, \rho < \Omega.$$

Then for any assignment v,

$$V_{\psi_{\Omega}(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(s_1), ..., v(s_n)] \to \bigvee \Delta[v(s_1), ..., v(s_n)].$$

Where $\bigwedge \Gamma$ and $\bigvee \Delta$ stand for the conjunction of formulas in Γ and the disjunction of formulas in Δ respectively, by convention $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

Proof. The proof is by induction on α. Note that the derivation $\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ contains no inferences of the form $(\forall R)_{\infty}$, $(\exists L)_{\infty}$ or $(\Sigma^{\mathcal{P}}\text{-}Ref)$ and all cuts have $\Delta_0^{\mathcal{P}}$ cut formulae. All axioms of $\mathbf{IRS}_{\Omega}^{\mathcal{P}}$ can be observed to be sound with respect to the interpretation.

First we treat the case where the last inference was $(pb\forall L)$ so we have

$$\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma[\bar{s}], t \subseteq s_i \to F(t, \bar{s}) \Rightarrow \Delta[\bar{s}] \quad \text{for some } \alpha_0, |t| < \alpha, \text{ with } |t| \leq |s_i|.$$

Since $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$, it follows that $t \subseteq s_i \to F(t, \bar{s})$ is a $\Delta_0^{\mathcal{P}}$ formula. So we may apply the induction hypothesis to obtain

$$V_{\psi_{\Omega}(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \wedge [v(t) \subseteq v(s_i) \to F(v(t), v(\bar{s}))] \to \bigvee \Delta[v(\bar{s})],$$

where $v(\bar{s}) := v(s_1), ..., v(s_n)$. From here the desired result follows by regular logical semantics.

Now suppose the last inference was $(pb\forall R)_{\infty}$, so we have

(1) $$\mathcal{H} \mid\!\frac{\alpha_t}{\rho}\, \Gamma[\bar{s}] \Rightarrow t \subseteq s_i \to F(t, \bar{s}) \quad \text{for all } |t| \leq |s_i| \text{ with } \alpha_t < \alpha.$$

In particular this means we have

(2) $$\mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma[\bar{s}] \Rightarrow a_j^{\beta} \subseteq s_i \to F(a_j^{\beta}, \bar{s}) \quad \text{for some } \alpha_0 < \alpha.$$

Here $\beta := |\, s_i \,|$ and j is chosen such that a_j^β does not occur in any of the terms $s_1, ..., s_n$. If F contains an unbounded quantifier we may use inversion for $\mathbf{IRS}_\Omega^{\mathcal{P}}$ 4.3.2v) to obtain

$$(3) \qquad \mathcal{H} \mathrel{\substack{\alpha_0 \\ \overline{} \\ \rho}} \Gamma[\bar{s}], a_j^\beta \subseteq s_i \Rightarrow F(a_j^\beta, \bar{s}) \quad \text{for some } \alpha_0 < \alpha.$$

So we may apply the induction hypothesis to get

$$(4) \qquad V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})], v(a_j^\beta) \subseteq v(s_i) \rightarrow F(v(a_j^\beta), v(\bar{s}))$$

for all variable assignments v. Thus by the choice of a_j^β we have

$$(5) \qquad V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \rightarrow (\forall x \subseteq v(s_i)) F(x, v(\bar{s}))$$

as required. If F is $\Delta_0^{\mathcal{P}}$ then we may immediately apply the induction hypothesis to (2) to obtain

$$(6) \qquad V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \rightarrow [v(a_j^\beta) \subseteq v(s_i) \rightarrow F(v(a_j^\beta), v(\bar{s}))]$$

for all variable assignments v, again by the choice of a_i^β we obtain the desired result. All other cases may be treated in a similar manner to the two above. $\square$

Lemma 4.5.2. Suppose $\mathbf{IKP}(\mathcal{P}) \vdash \;\Rightarrow A$ for some $\Sigma^{\mathcal{P}}$ sentence A, then there is an $m < \omega$, which we may compute from the derivation, such that

$$\mathcal{H}_\sigma \mathrel{\substack{\psi_\Omega(\sigma) \\ \overline{} \\ \psi_\Omega(\sigma)}} \;\Rightarrow\; A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Proof. Suppose $\mathbf{IKP}(\mathcal{P}) \vdash \;\Rightarrow A$ for some $\Sigma^{\mathcal{P}}$ sentence A, then by Theorem 4.4.11 we can explicitly find some $m < \omega$ such that

$$\mathcal{H}_0 \mathrel{\substack{\Omega \cdot \omega^m \\ \overline{} \\ \Omega + m}} \Rightarrow A\,.$$

Applying Partial cut elimination 4.3.5 we have

$$\mathcal{H}_0 \mathrel{\substack{\omega_{m-1}(\Omega \cdot \omega^m) \\ \overline{} \\ \Omega+1}} \Rightarrow A\,.$$

Now using Collapsing 4.3.7 we obtain

$$\mathcal{H}_\sigma \mathrel{\substack{\psi_\Omega(\sigma) \\ \overline{} \\ \psi_\Omega(\sigma)}} \Rightarrow A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

completing the proof. $\square$

Note that we cannot eliminate *all* cuts from the derivation since we don't have full predicative cut elimination for $\mathbf{IRS}_\Omega^{\mathcal{P}}$ as we do for $\mathbf{IRS}_\Omega$.

Theorem 4.5.3. If A is a $\Sigma^{\mathcal{P}}$-sentence and $\mathbf{IKP}(\mathcal{P}) \vdash \Rightarrow A$ then there is some ordinal term $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$, which we may compute from the derivation, such that

$$V_\alpha \models A.$$

Proof. From Lemma 4.5.2 we obtain some $m < \omega$ such that

$$\text{(1)} \qquad \mathcal{H}_\sigma \left|\frac{\psi_\Omega(\sigma)}{\psi_\Omega(\sigma)}\right. \Rightarrow A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Let $\alpha := \psi_\Omega(\sigma)$. Applying Boundedness 4.3.6 to (1) we obtain

$$\text{(2)} \qquad \mathcal{H}_\sigma \left|\frac{\alpha}{\alpha}\right. \Rightarrow A^{\mathbb{V}_\alpha}.$$

Now applying Theorem 4.5.1 to (2) we obtain

$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A^{V_\alpha}$$

and thus

$$V_\alpha \models A$$

as required. □

Remark 4.5.4. Suppose $A \equiv \exists x C(x)$ is a $\Sigma^{\mathcal{P}}$ sentence and $\mathbf{IKP}(\mathcal{P}) \vdash \Rightarrow A$. As well as the ordinal term α given by Theorem 4.5.3, it is possible to determine (making essential use of the intuitionistic nature of $\mathbf{IRS}_\Omega^{\mathcal{P}}$) a term s, with $|\,s\,| < \alpha$, such that

$$V_\alpha \models C(s).$$

This proof is somewhat more complex than in the case of **IKP** since the proof tree corresponding to (2) above can still contain cuts with $\Delta_0^{\mathcal{P}}$ cut formulae.

Moreover, in order to show that $\mathbf{IKP}(\mathcal{P})$ has the existence property, the embedding and cut elimination for a given finite derivation of a $\Sigma^{\mathcal{P}}$ sentence, needs to be carried out *inside* $\mathbf{IKP}(\mathcal{P})$. In order to do this it needs to be shown that from the finite derivation we can calculate some ordinal term $\gamma < \varepsilon_{\Omega+1}$ such that the embedding and cut elimination for that derivation can still be performed inside $\mathbf{IRS}_\Omega^{\mathcal{P}}$ with the term structure restricted to $B(\gamma)$.

These proofs will appear in [43].

Chapter 5

A relativised ordinal analysis of $\mathbf{IKP}(\mathcal{E})$

This final chapter provides a relativised ordinal analysis for intuitionistic exponentiation Kripke-Platek set theory $\mathbf{IKP}(\mathcal{E})$. Given sets a and b, set-exponentiation allows the formation of the set ${}^a b$, of all functions from a to b. A problem that presents itself in this case is that it is not clear how to formulate a term structure in such a way that we can read off a terms level in the pertinent 'exponentiation hierarchy' from that terms syntactic structure. Instead we work with a term structure similar to that used in $\mathbf{IRS}^{\mathcal{P}}_{\Omega}$, and a terms level becomes a dynamic property *inside* the infinitary system. Making this work in a system for which we can prove all the necessary embedding and cut-elimination theorems turned out to be a major technical hurdle. The end result of the chapter is a characterisation of $\mathbf{IKP}(\mathcal{E})$ in terms of provable height of the exponentiation hierarchy, this machinery will also be used in a later paper by Rathjen [43], to show that $\mathbf{CZF}^{\mathcal{E}}$ has the full existence property.

5.1 A sequent calculus formulation of $\mathbf{IKP}(\mathcal{E})$

Definition 5.1.1. The formulas of $\mathbf{IKP}(\mathcal{E})$ are the same as those of $\mathbf{IKP}$ except we also allow *exponentiation bounded quantifiers* of the form

$$(\forall x \in {}^a b)A(x) \quad \text{and} \quad (\exists x \in {}^a b)A(x).$$

These are treated as quantifiers in their own right, not abbreviations. The formula "$\mathrm{fun}(x,a,b)$" is defined below. It's intuitive meaning is "x is a function from a to b".

$$\begin{aligned}\mathrm{fun}(x,a,b) := x \subseteq a \times b \ \wedge \ &(\forall y \in a)(\exists z \in b)((y,z) \in x)\\ &\wedge (\forall y \in a)(\forall z_1 \in b)(\forall z_2 \in b)[((y,z_1) \in x \wedge (y,z_2) \in x) \to z_1 = z_2]\end{aligned}$$

Quantifiers $\forall x$, $\exists x$ will be referred to as unbounded, whereas the other quantifiers (including the exponentation bounded ones) will be referred to as bounded.

A $\Delta_0^{\mathcal{E}}$-formula of $\mathbf{IKP}(\mathcal{E})$ is one that contains no unbounded quantifiers.

As with **IKP**, the system $\mathbf{IKP}(\mathcal{E})$ derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of formulae and Δ contains at most one formula.

The axioms of $\mathbf{IKP}(\mathcal{E})$ are given by:

Logical axioms:	$\Gamma, A, \Rightarrow A$ for every $\Delta_0^{\mathcal{E}}$–formula A.
Extensionality:	$\Gamma \Rightarrow a{=}b \wedge B(a) \rightarrow B(b)$ for every $\Delta_0^{\mathcal{E}}$-formula $B(a)$.
Pair:	$\Gamma \Rightarrow \exists x[a{\in}x \wedge b{\in}x]$
Union:	$\Gamma \Rightarrow \exists x(\forall y{\in}a)(\forall z{\in}y)(z{\in}x)$
Infinity:	$\Gamma \Rightarrow \exists x\,[(\exists y \in x)\, y \in x \;\wedge\; (\forall y \in x)(\exists z \in x)\, y \in z]$.
$\Delta_0^{\mathcal{E}}$ *–Separation:*	$\Gamma \Rightarrow \exists x((\forall y \in x)(y \in a \wedge A(y)) \wedge (\forall y \in a)(A(y) \rightarrow y \in x))$
	for every $\Delta_0^{\mathcal{E}}$ formula $A(b)$.
$\Delta_0^{\mathcal{E}}$ *–Collection:*	$\Gamma \Rightarrow (\forall x \in a)\exists y B(x,y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)B(x,y)$
	for every $\Delta_0^{\mathcal{E}}$ formula $B(b,c)$.
Set Induction:	$\Gamma \Rightarrow \forall u\,[(\forall x \in u)\, G(x) \;\rightarrow\; G(u)] \;\rightarrow\; \forall u\, G(u)$
	for every formula $G(b)$.
Exponentiation:	$\Gamma \Rightarrow \exists z\,(\forall x \in {}^{a}b)(x \in z)$.

The rules of $\mathbf{IKP}(\mathcal{E})$ are the same as those of **IKP** (extended to the new language containing exponentiation bounded quantifiers), together with the following four rules:

$$(\mathcal{E}b\exists L)\,\frac{\Gamma, \mathrm{fun}(c,a,b) \wedge F(c) \Rightarrow \Delta}{\Gamma, (\exists x \in {}^{a}b)F(x) \Rightarrow \Delta} \qquad (\mathcal{E}b\exists R)\,\frac{\Gamma \Rightarrow \mathrm{fun}(c,a,b) \wedge F(c)}{\Gamma \Rightarrow (\exists x \in {}^{a}b)F(x)}$$

$$(\mathcal{E}b\forall L)\,\frac{\Gamma, \mathrm{fun}(c,a,b) \rightarrow F(c) \Rightarrow \Delta}{\Gamma, (\forall x \in {}^{a}b)F(x) \Rightarrow \Delta} \qquad (\mathcal{E}b\forall R)\,\frac{\Gamma \Rightarrow \mathrm{fun}(c,a,b) \rightarrow F(c)}{\Gamma \Rightarrow (\forall x \in {}^{a}b)F(x)}$$

As usual it is forbidden for the variable a to occur in the conclusion of the rules $(\mathcal{E}b\exists L)$ and $(\mathcal{E}b\forall R)$, such a variable is referred to as the eigenvariable of the inference.

5.2 The infinitary system $\mathbf{IRS}_\Omega^{\mathbb{E}}$

The purpose of this section is to introduce an infinitary system $\mathbf{IRS}_\Omega^{\mathbb{E}}$ within which we will be able to embed $\mathbf{IKP}(\mathcal{E})$. As with the Von-Neumann hierarchy built by iterating the power set operation through the ordinals, one may define an Exponentiation-hierarchy as follows

$$\begin{aligned}
E_0 &:= \emptyset \\
E_1 &:= \{\emptyset\} \\
E_{\alpha+2} &:= \{X \mid X \text{ is definable over } \langle E_{\alpha+1}, \in\rangle \text{ with parameters}\} \\
&\quad \cup \{f \mid \mathrm{fun}(f,a,b) \text{ for some } a,b \in E_\alpha.\} \\
E_\lambda &:= \bigcup_{\beta<\lambda} E_\beta \quad \text{for } \lambda \text{ a limit ordinal.} \\
E_{\lambda+1} &:= \{X \mid X \text{ is definable over } \langle E_{\alpha+1}, \in\rangle \text{ with parameters}\} \quad \text{for } \lambda \text{ a limit ordinal.}
\end{aligned}$$

Lemma 5.2.1. If $y \in E_{\alpha+1}$ and $x \in y$ then $x \in E_\alpha$.

Proof. The proof is by induction on α. If y is a set definable over $\langle E_\alpha, \in\rangle$ with parameters, the members of y, including x, must be members of E_α.

Now suppose $\alpha = \beta+1$ and $y \in E_{\alpha+1}$ is a function $y : p \to q$ for two sets $p, q \in E_\beta$. Since $x \in y$, it follows that x is of the form (x_0, x_1) with $x_0 \in p$ and $x_1 \in q$, we use the standard definition of ordered pair so

$$(x_0, x_1) := \{\{x_0, x_1\}, \{x_0\}\} \tag{1}$$

We must now verify the following claim:

$$\{x_0\}, \{x_1\}, \{x_0, x_1\} \in E_\beta. \tag{*}$$

If $\beta = \gamma + 1$ then by the induction hypothesis applied to $x_0 \in p \in E_\beta$ and $x_1 \in q \in E_\beta$ we get $x_0, x_1 \in E_\gamma$ and thus $\{x_0\}, \{x_1\}, \{x_0, x_1\} \in E_\beta$ as required.

If β is a limit then by the induction hypothesis and the construction of the E hierarchy at limit ordinals, we know that $s_0 \in E_{\beta_0}$ and $s_1 \in E_{\beta_1}$ for some $\beta_0, \beta_1 < \beta$, thus $\{s_0\}, \{s_1\}, \{s_0, s_1\} \in E_{\max(\beta_0,\beta_1)+1}$ which completes the proof of (*).

From (*) and (1) it is clear that $(s_0, s_1) \in E_{\beta+1}$ as required. □

The idea of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ is to build an infinitary system for reasoning about the E hierarchy.

Definition 5.2.2. The terms of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ are defined as follows

1. $\mathbb{E}_\alpha$ is an $\mathbf{IRS}_\Omega^{\mathbb{E}}$ term for each $\alpha < \Omega$.

2. a_i^α is an $\mathbf{IRS}_\Omega^{\mathbb{E}}$ term for each $\alpha < \Omega$ and each $i < \omega$, these terms will be known as free variables.

3. If $F(a, \bar{b})$ is a $\Delta_0^{\mathcal{E}}$ formula of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables indicated, and $t, \bar{s} := s_1, ..., s_n$ are $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms then

$$[x \in t \mid F(x, \bar{s})]$$

is also a term of $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

Observe that $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms do not come with 'levels' as in the other infinitary systems. This is because it is not clear how to immediately read off the location of a given term within the E hierarchy, just from the syntactic information available within that term.

The formulas of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ are of the form $F(s_1, ..., s_n)$, where $F(a_1, ..., a_n)$ is a formula of $\mathbf{IKP}(\mathcal{E})$ with all free variables indicated and $s_1, ..., s_n$ are $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms. The formula $A(s_1, ..., s_n)$ is said to be $\Delta_0^{\mathcal{E}}$ if $A(a_1, ..., a_n)$ is a $\Delta_0^{\mathcal{E}}$ formula of $\mathbf{IKP}(\mathcal{E})$. The $\Sigma^{\mathcal{E}}$ ($\Pi^{\mathcal{E}}$) formulae are the smallest collection containing the $\Delta_0^{\mathcal{E}}$ formulae and closed under $\wedge$, $\vee$, bounded quantification and unbounded existential (universal) quantification.

The axioms of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ are given by

(E1)	$\Gamma, A \Rightarrow A$ for every $\Delta_0^{\mathcal{E}}$–formula A.
(E2)	$\Gamma \Rightarrow t = t$ for every $\mathbf{IRS}_\Omega^{\mathbb{E}}$ term t.
(E3)	$\Gamma, \bar{s}{=}\bar{t}, B(\bar{s}) \Rightarrow B(\bar{t})$ for every $\Delta_0^{\mathcal{E}}$-formula $B(\bar{s})$.
(E4)	$\Gamma \Rightarrow \mathbb{E}_\beta \in \mathbb{E}_\alpha$ for all $\beta < \alpha < \Omega$
(E5)	$\Gamma \Rightarrow a_i^\beta \in \mathbb{E}_\alpha$ for all $i \in \omega$ and $\beta < \alpha < \Omega$
(E6)	$\Gamma, t \in \mathbb{E}_\alpha, s \in t \Rightarrow s \in \mathbb{E}_\alpha$ for all $\alpha < \Omega$
(E7)	$\Gamma, t \in \mathbb{E}_{\alpha+1}, s \in t \Rightarrow s \in \mathbb{E}_\alpha$ for all $\alpha < \Omega$
(E8)	$\Gamma, s \in t, F(s, \bar{p}) \Rightarrow s \in [x \in t \mid F(x, \bar{p})]$
(E9)	$\Gamma, s \in [x \in t \mid F(x, \bar{p})] \Rightarrow s \in t \wedge F(s, \bar{p})$
(E10)	$\Gamma, s \in \mathbb{E}_\alpha, t \in \mathbb{E}_\beta, \mathrm{fun}(p, s, t) \Rightarrow p \in \mathbb{E}_\gamma$ for all $\gamma \geq \max(\alpha, \beta) + 2$.
(E11)	$\Gamma, t \in \mathbb{E}_\beta, \bar{p} \in \mathbb{E}_{\bar{\alpha}} \Rightarrow [x \in t \mid F(x, \bar{p})] \in \mathbb{E}_\gamma$ for all $\gamma \geq \max(\beta, \bar{\alpha})$

Definition 5.2.3. For a formula $A(a_1, ..., a_n)$ of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables $\bar{a} := a_1, ..., a_n$ and any $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms $\bar{s} := s_1, ..., s_n$, we define the $\bar{\beta}$*-rank* $\|A(\bar{s})\|_{\bar{\beta}}$ where $\bar{\beta} := \beta_1, ..., \beta_n$ are any ordinals $< \Omega$. The definition is made by recursion on the build up of the formula A.

i) $\|s \in t\|_{\beta_1, \beta_2} := \max(\beta_1, \beta_2)$

ii) $\|(\exists x \in t)F(x, \bar{s})\|_{\gamma, \bar{\beta}} := \|(\forall x \in t)F(x, \bar{s})\|_{\gamma, \bar{\beta}} := \max(\gamma, \|F(\mathbb{E}_0, \bar{s})\|_{0, \bar{\beta}} + 2)$

iii) $\|(\exists x \in {}^s t)F(x, \bar{p})\|_{\gamma, \delta, \bar{\beta}} := \|(\forall x \in {}^s t)F(x, \bar{p})\|_{\gamma, \delta, \bar{\beta}}$
$:= \max(\gamma + \omega, \delta + \omega, \|F(\mathbb{E}_0, \bar{p})\|_{0, \bar{\beta}} + 2)$

iv) $\|\exists x F(x,\bar{s})\|_{\bar{\beta}} := \|\forall x F(x,\bar{s})\|_{\bar{\beta}} := \max(\Omega, \|F(\mathbb{E}_0,\bar{s})\|_{0,\bar{\beta}}+2)$

v) $\|A \wedge B\|_{\bar{\beta}} := \|A \vee B\|_{\bar{\beta}} := \|A \to B\|_{\bar{\beta}} := \max(\|A\|_{\bar{\beta}}, \|B\|_{\bar{\beta}})+1$

vi) $\|\neg A\|_{\bar{\beta}} := \|A\|_{\bar{\beta}}+1$

We define the *rank* of $A(\bar{s})$ by

$$rk(A(\bar{s})) := \|A(\bar{s})\|_{\bar{0}}$$

Observation 5.2.4.

i) $\|A(\bar{s})\|_{\bar{\beta}} < \Omega$ if and only if A is $\Delta_0^{\mathcal{E}}$

ii) If A contains unbounded quantifiers then $rk(A(\bar{s})) = \|A(\bar{s})\|_{\bar{\beta}}$ for all $\bar{s}$ and $\bar{\beta}$.

Definition 5.2.5 (Operator Controlled Derivability in $\mathbf{IRS}_\Omega^{\mathbb{E}}$)**.** $\mathbf{IRS}_\Omega^{\mathbb{E}}$ derives intuitionistic sequents of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ formulae and Δ contains at most one formula. For $\mathcal{H}$ an operator and α, ρ ordinals we define the relation $\mathcal{H} \,|\!\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \Delta$ by recursion on α.

If $\Gamma \Rightarrow \Delta$ is an axiom and $\alpha \in \mathcal{H}$ then $\mathcal{H} \,|\!\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \Delta$.

It is always required that $\alpha \in \mathcal{H}$, this requirement is not repeated for each inference rule below.

$$(\mathbb{E}\text{-Lim})_\infty \quad \frac{\mathcal{H}[\delta] \,|\!\!\frac{\alpha_\delta}{\rho}\, \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \text{ for all } \delta < \gamma}{\mathcal{H} \,|\!\!\frac{\alpha}{\rho}\, \Gamma, s \in \mathbb{E}_\gamma \Rightarrow \Delta} \quad \begin{array}{r} \gamma \text{ a limit} \\ \alpha_\delta < \alpha \\ \gamma \in \mathcal{H} \end{array}$$

$$(b\forall L) \quad \frac{\begin{array}{l} \mathcal{H} \,|\!\!\frac{\alpha_0}{\rho}\, \Gamma, s \in t \to A(s) \Rightarrow \Delta \\ \mathcal{H} \,|\!\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow t \in \mathbb{E}_\beta \\ \mathcal{H} \,|\!\!\frac{\alpha_2}{\rho}\, \Gamma \Rightarrow s \in \mathbb{E}_\gamma \end{array}}{\mathcal{H} \,|\!\!\frac{\alpha}{\rho}\, \Gamma, (\forall x \in t) A(x) \Rightarrow \Delta} \quad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \beta, \gamma \in \mathcal{H} \\ \gamma < \alpha \\ \gamma \leq \beta \end{array}$$

$$(b\forall R)_\infty \quad \frac{\begin{array}{l} \mathcal{H} \,|\!\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow s \in t \to F(s) \text{ for all } s \\ \mathcal{H} \,|\!\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow t \in \mathbb{E}_\beta \end{array}}{\mathcal{H} \,|\!\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow (\forall x \in t) F(x)} \quad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ \beta \in \mathcal{H} \\ \beta < \alpha \end{array}$$

$$(b\exists L)_\infty \quad \frac{\begin{array}{l} \mathcal{H} \,|\!\!\frac{\alpha_0}{\rho}\, \Gamma, s \in t \wedge F(s) \Rightarrow \Delta \text{ for all } s \\ \mathcal{H} \,|\!\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow t \in \mathbb{E}_\beta \end{array}}{\mathcal{H} \,|\!\!\frac{\alpha}{\rho}\, \Gamma, (\exists x \in t) F(x) \Rightarrow \Delta} \quad \begin{array}{r} \alpha_0, \alpha_1 < \alpha \\ \beta \in \mathcal{H} \\ \beta < \alpha \end{array}$$

$$
(b\exists R)\qquad
\frac{\begin{array}{l}
\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow s \in t \wedge A(s) \\
\mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\beta \\
\mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\gamma
\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\exists x \in t)A(x)}
\qquad
\begin{array}{r}
\alpha_0, \alpha_1, \alpha_2 < \alpha \\
\beta, \gamma \in \mathcal{H} \\
\gamma < \alpha \\
\gamma \leq \beta
\end{array}
$$

$$
(\mathcal{E}b\forall L)\qquad
\frac{\begin{array}{l}
\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, \mathrm{fun}(p,s,t) \to A(p) \Rightarrow \Delta \\
\mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\beta \\
\mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\gamma \\
\mathcal{H} \vdash^{\alpha_3}_{\rho} \Gamma \Rightarrow p \in \mathbb{E}_\delta
\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\forall x \in {}^s t)A(x) \Rightarrow \Delta}
\qquad
\begin{array}{r}
\alpha_0, \alpha_1, \alpha_2, \alpha_3 < \alpha \\
\beta, \gamma, \delta \in \mathcal{H} \\
\delta < \alpha \\
\delta \leq \max(\beta, \gamma) + 2
\end{array}
$$

$$
(\mathcal{E}b\forall R)_\infty\qquad
\frac{\begin{array}{l}
\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow \mathrm{fun}(p,s,t) \to F(p) \text{ for all } p \\
\mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\beta \\
\mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\gamma
\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\forall x \in {}^s t)F(x)}
\qquad
\begin{array}{r}
\alpha_0, \alpha_1, \alpha_2 < \alpha \\
\beta, \gamma \in \mathcal{H} \\
\max(\beta, \gamma) + 2 \leq \alpha
\end{array}
$$

$$
(\mathcal{E}b\exists L)_\infty\qquad
\frac{\begin{array}{l}
\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, \mathrm{fun}(p,s,t) \wedge F(p) \Rightarrow \Delta \text{ for all } p \\
\mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\beta \\
\mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\gamma
\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, (\exists x \in {}^s t)F(x) \Rightarrow \Delta}
\qquad
\begin{array}{r}
\alpha_0, \alpha_1, \alpha_2 < \alpha \\
\beta, \gamma \in \mathcal{H} \\
\max(\beta, \gamma) + 2 \leq \alpha
\end{array}
$$

$$
(\mathcal{E}b\exists R)\qquad
\frac{\begin{array}{l}
\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma \Rightarrow \mathrm{fun}(p,s,t) \wedge A(p) \\
\mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\beta \\
\mathcal{H} \vdash^{\alpha_2}_{\rho} \Gamma \Rightarrow t \in \mathbb{E}_\gamma \\
\mathcal{H} \vdash^{\alpha_3}_{\rho} \Gamma \Rightarrow p \in \mathbb{E}_\delta
\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow (\exists x \in {}^s t)A(x)}
\qquad
\begin{array}{r}
\alpha_0, \alpha_1, \alpha_2, \alpha_3 < \alpha \\
\beta, \gamma, \delta \in \mathcal{H} \\
\delta < \alpha \\
\delta \leq \max(\beta, \gamma) + 2
\end{array}
$$

$$
(\forall L)\qquad
\frac{\begin{array}{l}
\mathcal{H} \vdash^{\alpha_0}_{\rho} \Gamma, F(s) \Rightarrow \Delta \\
\mathcal{H} \vdash^{\alpha_1}_{\rho} \Gamma \Rightarrow s \in \mathbb{E}_\beta
\end{array}}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma, \forall x F(x) \Rightarrow \Delta}
\qquad
\begin{array}{r}
\alpha_0 + 3, \alpha_1 + 3 < \alpha \\
\beta < \alpha \\
\beta \in \mathcal{H}
\end{array}
$$

$$
(\forall R)_\infty\qquad
\frac{\mathcal{H}[\beta] \vdash^{\alpha_\beta}_{\rho} \Gamma, s \in \mathbb{E}_\beta \Rightarrow F(s) \text{ for all } s \text{ and all } \beta < \Omega}{\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma \Rightarrow \forall x F(x)}
\qquad
\beta < \alpha_\beta + 3 < \alpha
$$

$$(\exists L)_\infty \quad \frac{\mathcal{H}[\beta] \,\big|\!\frac{\alpha_\beta}{\rho}\, \Gamma, s \in \mathbb{E}_\beta, F(s) \Rightarrow \Delta \text{ for all } s \text{ and all } \beta < \Omega}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \forall x F(x)} \quad \beta < \alpha_\beta + 3 < \alpha$$

$$(\exists R) \quad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow F(s) \qquad \mathcal{H} \,\big|\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow s \in \mathbb{E}_\beta}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \exists x F(x)} \quad \begin{array}{r} \alpha_0 + 3, \alpha_1 + 3 < \alpha \\ \beta < \alpha \\ \beta \in \mathcal{H} \end{array}$$

$$(\Sigma^{\mathcal{E}}\text{-}Ref) \quad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, A}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, \exists z\, A^z} \quad \begin{array}{r} \alpha_0 + 1, \Omega < \alpha \\ A \text{ is a } \Sigma^{\mathcal{E}}\text{-formula} \end{array}$$

$$(\mathrm{Cut}) \quad \frac{\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma, A(s_1, ..., s_n) \Rightarrow \Delta \qquad \mathcal{H} \,\big|\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow A(s_1, ..., s_n) \qquad \mathcal{H} \,\big|\!\frac{\alpha_2}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \;\; i = 1, ..., n}{\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \Delta} \quad \begin{array}{r} \alpha_0, \alpha_1, \alpha_2 < \alpha \\ \|A(\bar{s})\|_{\bar{\beta}} < \rho \\ \bar{\beta} \in \mathcal{H} \end{array}$$

Lastly if $\Gamma \Rightarrow \Delta$ is the result of a propositional inference of the form $(\wedge L)$, $(\wedge R)$, $(\vee L)$, $(\vee R)$, $(\neg L)$, $(\neg R)$, $(\bot)$, $(\to L)$ or $(\to R)$, with premise(s) $\Gamma_i \Rightarrow \Delta_i$ then from $\mathcal{H} \,\big|\!\frac{\alpha_0}{\rho}\, \Gamma_i \Rightarrow \Delta_i$ (for each i) we may conclude $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow \Delta$, provided $\alpha_0 < \alpha$.

Convention 5.2.6. In cases where terms $\mathbb{E}_\alpha$ and a_i^α occur directly as witnesses in existential rules or in cut formulae we will omit the extra premise declaring the terms location in the $\mathbb{E}$ term hierarchy since

$$\mathbb{E}_\alpha \in \mathbb{E}_{\alpha+1} \quad \text{and} \quad a_i^\alpha \in \mathbb{E}_{\alpha+1}$$

are axioms (E4) and (E5) respectively. It must still be checked that $\alpha \in \mathcal{H}$ however.

5.3 Cut elimination for $\mathbf{IRS}_\Omega^{\mathbb{E}}$

Lemma 5.3.1 (Inversions of $\mathbf{IRS}_\Omega^{\mathbb{E}}$)**.** If $\max(rk(A), rk(B)) \geq \Omega$ then we have the usual propositional inversions for intuitionistic systems:

i) If $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A \wedge B \Rightarrow \Delta$ then $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A, B \Rightarrow \Delta$.

ii) If $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow A \wedge B$ then $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow A$ and $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow B$.

iii) If $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A \vee B \Rightarrow \Delta$ then $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A \Rightarrow \Delta$ and $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, B \Rightarrow \Delta$.

iv) If $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A \to B \Rightarrow \Delta$ then $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, B \Rightarrow \Delta$.

v) If $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow A \to B$ then $\mathcal{H} \,\big|\!\frac{\alpha}{\rho}\, \Gamma, A \Rightarrow B$.

If $rk(A) \geq \Omega$ we have the following additional inversions:

vi) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow \neg A$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, A \Rightarrow$.

vii) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow (\forall x \in t)A(x)$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow s \in t \rightarrow A(s)$ for all terms s.

viii) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, s \in t \wedge A(s) \Rightarrow \Delta$ for all terms s.

ix) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow (\forall x \in {}^s t)A(x)$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow \mathrm{fun}(p, s, t) \rightarrow A(p)$ for all terms p.

x) If $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in {}^s t)A(x) \Rightarrow \Delta$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, \mathrm{fun}(p, s, t) \wedge A(p) \Rightarrow \Delta$ for all terms p.

Finally we have the following persistence properties:

xi) If $\gamma \in \mathcal{H} \cap \Omega$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow \forall x A(x)$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow (\forall x \in \mathbb{E}_\gamma)A(x)$.

xii) If $\gamma \in \mathcal{H} \cap \Omega$ and $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, \exists x A(x) \Rightarrow \Delta$ then $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in \mathbb{E}_\gamma)A(x) \Rightarrow \Delta$.

Proof. All proofs are by induction on α, i) to vi) are standard for intuitionistic systems of this type.

For viii) suppose that $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ and $rk(A(\mathbb{E}_0)) \geq \Omega$. $(\exists x \in t)A(x)$ cannot have been the "active component" of an axiom, so if $\Gamma, (\exists x \in t)A(x) \Rightarrow \Delta$ is an axiom then so is $\Gamma, s \in t \wedge A(s) \Rightarrow \Delta$. Now if $(\exists x \in t)A(x)$ was not the principal formula of the last inference we may apply the induction hypothesis to the premises of that inference followed by the same inference again. Finally if $(\exists x \in t)A(x)$ was the principal formula of the last inference and the last inference was $(b\exists L)_\infty$ so we have

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{|}} \Gamma, (\exists x \in t)A(x), s \in t \wedge A(s) \Rightarrow \Delta \quad \text{for all terms } s \text{ and for some } \alpha_0 < \alpha.$$

Applying the induction hypothesis followed by weakening yields

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma, s \in t \wedge A(s) \Rightarrow \Delta \quad \text{for all terms } s$$

as required. The proofs of vii), xi) and x) are similar.

For xi) suppose $\mathcal{H} \overset{\alpha}{\underset{\rho}{|}} \Gamma \Rightarrow \forall x A(x)$ and $\gamma \in \mathcal{H} \cap \Omega$. $\Gamma \Rightarrow \forall x A(x)$ cannot be an axiom. If the last inference was not $(\forall R)_\infty$ then we may apply the induction hypothesis to its premises and then the same inference again. So suppose the last inference was $(\forall R)_\infty$ in which case we have the premise

$$\mathcal{H}[\delta] \overset{\alpha_\delta}{\underset{\rho}{|}} \Gamma, s \in \mathbb{E}_\delta \Rightarrow A(s) \quad \text{for all } s \text{ and all } \delta < \Omega, \text{ with } \delta < \alpha_\delta + 3 < \alpha.$$

In particular since $\gamma \in \mathcal{H}$ we have

$$\mathcal{H} \overset{\alpha_\gamma}{\underset{\rho}{|}} \Gamma, s \in \mathbb{E}_\gamma \Rightarrow A(s) \quad \text{for all } s \text{ with } \gamma < \alpha_\gamma + 3 < \alpha.$$

So by $(\to R)$ we have

$$\mathcal{H} \mid\!\frac{\alpha_\gamma+1}{\rho}\, \Gamma \Rightarrow s \in \mathbb{E}_\gamma \to A(s) \quad \text{for all } s$$

Now since $\Rightarrow \mathbb{E}_\gamma \in \mathbb{E}_{\gamma+1}$ is an instance of axiom (E4), $\gamma \in \mathcal{H}$ and $\gamma < \alpha$ we may apply $(b\forall R)$ to obtain

$$\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow (\forall x \in \mathbb{E}_\gamma)A(x)$$

as required. The proof of xii) is similar. □

Lemma 5.3.2 (Reduction for $\mathbf{IRS}_\Omega^{\mathbb{E}}$)**.** *Suppose $rk(C(\bar{s})) := \rho > \Omega$ where $C(\bar{a})$ is an $\boldsymbol{IKP}(\mathcal{E})$ formula with all free variables displayed. If*

$$\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow C(\bar{s})$$
$$\mathcal{H} \mid\!\frac{\beta}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow \Delta$$
$$\mathcal{H} \mid\!\frac{\gamma_i}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\eta_i} \quad \textit{with } \eta_i \in \mathcal{H} \cap \Omega \textit{ for each } 1 \leq i \leq n.$$

Then

$$\mathcal{H} \mid\!\frac{\alpha\#\alpha\#\beta\#\beta\#\gamma}{\rho}\, \Gamma \Rightarrow \Delta \quad \textit{where } \gamma := max_{i=1,\ldots,n}(\gamma_i)$$

Proof. The proof is by induction on $\alpha\#\alpha\#\beta\#\beta\#\gamma$. Assume that

(1) $rk(C(\bar{s})) := \rho > \Omega$

(2) $\mathcal{H} \mid\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow C(\bar{s})$

(3) $\mathcal{H} \mid\!\frac{\beta}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow \Delta$

(4) $\mathcal{H} \mid\!\frac{\gamma_i}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\eta_i}$ for each $1 \leq i \leq n$ and for some $\eta_i \in \mathcal{H} \cap \Omega$.

Since $rk(C(\bar{s})) := \rho > \Omega$, C cannot be the 'active part' of an axiom, hence if (2) or (3) are axioms of $\mathbf{IRS}_\Omega^{\mathbb{E}}$ then so is $\Gamma \Rightarrow \Delta$.

If $C(\bar{s})$ was not the principal formula of the last inference in either (2) or (3) then we may apply the induction hypothesis to the premises of that inference and then the same inference again.

So suppose $C(\bar{s})$ was the principal formula of the last inference in both (2) and (3). Since the conclusion of a $(\Sigma^{\mathcal{E}}\text{-}Ref)$ inference always has rank Ω and $rk(C(\bar{s})) := \rho > \Omega$ we may conclude that the last inference of (2) was not $(\Sigma^{\mathcal{E}}\text{-}Ref)$.

Case 1. Suppose $C(\bar{s}) \equiv (\exists x \in s_i)F(x, \bar{s})$, thus we have

$$\begin{array}{lll}
(5) & \mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow r \in s_i \wedge F(r, \bar{s}) & \alpha_0 < \alpha \\
(6) & \mathcal{H} \mid\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_\delta & \alpha_1 < \alpha \text{ and } \delta \in \mathcal{H} \\
(7) & \mathcal{H} \mid\!\frac{\alpha_2}{\rho}\, \Gamma \Rightarrow r \in \mathbb{E}_\xi & \xi, \alpha_2 < \alpha\ ,\ \xi \in \mathcal{H}(\emptyset) \text{ and } \xi \leq \delta \\
(8) & \mathcal{H} \mid\!\frac{\beta_0}{\rho}\, \Gamma, C(\bar{s}), p \in s_i \wedge F(p, \bar{s}) \Rightarrow \Delta & \text{for all } p \text{ and } \beta_0 < \beta \\
(9) & \mathcal{H} \mid\!\frac{\beta_1}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow s_i \in \mathbb{E}_{\delta'} & \delta', \beta_1 < \beta \text{ and } \delta' \in \mathcal{H}(\emptyset)
\end{array}$$

From (8) we obtain

$$\mathcal{H} \mid\!\frac{\beta_0}{\rho}\, \Gamma, C(\bar{s}), r \in s_i \wedge F(r, \bar{s}) \Rightarrow \Delta\,. \tag{10}$$

Applying the induction hypothesis to (2), (4) and (10) yields

$$\mathcal{H} \mid\!\frac{\alpha\#\alpha\#\beta_0\#\beta_0\#\gamma}{\rho}\, \Gamma, r \in s_i \wedge F(r, \bar{s}) \Rightarrow \Delta\,. \tag{11}$$

Note that

$$\begin{aligned}
\Omega < rk(r \in s_i \wedge F(r, \bar{s})) &= rk(F(r, \bar{s})) + 1 \\
&< rk(F(r, \bar{s})) + 2 \\
&= rk(C(\bar{s})) := \rho.
\end{aligned}$$

So we may apply (Cut) to (4),(5),(7) and (11) giving

$$\mathcal{H} \mid\!\frac{\alpha\#\alpha\#\beta\#\beta\#\gamma}{\rho}\, \Gamma \Rightarrow \Delta$$

as required. The case where $C(\bar{s}) \equiv (\forall x \in s_i)F(x, \bar{s})$ is similar.

Now suppose $C(\bar{s}) \equiv (\forall x \in^{s_i} s_j)F(x, \bar{s})$, so we have

$$\begin{array}{lll}
(12) & \mathcal{H} \mid\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow \mathrm{fun}(p, s_i, s_j) \rightarrow F(p, \bar{s}) & \text{for all } p \text{ and } \alpha_0 < \alpha \\
(13) & \mathcal{H} \mid\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow s_i \in \mathbb{E}_\delta & \alpha_1 < \alpha \text{ and } \delta \in \mathcal{H}(\emptyset) \\
(14) & \mathcal{H} \mid\!\frac{\alpha_2}{\rho}\, \Gamma \Rightarrow s_j \in \mathbb{E}_{\delta'} & \alpha_2 < \alpha,\ \delta' \in \mathcal{H}(\emptyset) \text{ and } \max(\delta, \delta') + 2 \leq \alpha \\
(15) & \mathcal{H} \mid\!\frac{\beta_0}{\rho}\, \Gamma, C(\bar{s}), \mathrm{fun}(r, s_i, s_j) \rightarrow F(r, \bar{s}) \Rightarrow \Delta & \beta_0 < \beta \\
(16) & \mathcal{H} \mid\!\frac{\beta_1}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow r \in \mathbb{E}_\xi & \xi < \beta,\ \xi \in \mathcal{H}(\emptyset) \text{ and } \beta_1 < \beta \\
(17) & \mathcal{H} \mid\!\frac{\beta_2}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow s_i \in \mathbb{E}_\zeta & \zeta \in \mathcal{H}(\emptyset) \text{ and } \beta_2 < \beta \\
(18) & \mathcal{H} \mid\!\frac{\beta_3}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow s_j \in \mathbb{E}_{\zeta'} & \zeta' \in \mathcal{H}(\emptyset),\ \beta_3 < \beta \text{ and } \xi \leq \max(\zeta, \zeta') + 2
\end{array}$$

As an instance of (12) we have

$$\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow \mathrm{fun}(r, s_i, s_j) \to F(r, \bar{s})\,. \tag{19}$$

Applying the induction hypothesis to (2), (4) and (15) gives

$$\mathcal{H} \,\Big|\!\frac{\alpha\#\alpha\#\beta_0\#\beta_0\#\gamma}{\rho}\, \Gamma, \mathrm{fun}(r, s_i, s_j) \to F(r, \bar{s}) \Rightarrow \Delta\,. \tag{20}$$

Furthermore the induction hypothesis applied to (2),(4) and (16) gives

$$\mathcal{H} \,\Big|\!\frac{\alpha\#\alpha\#\beta_1\#\beta_1\#\gamma}{\rho}\, \Gamma \Rightarrow r \in \mathbb{E}_\xi\,. \tag{21}$$

Note that

$$\begin{aligned} \Omega < rk(\mathrm{fun}(r, s_i, s_j) \to F(r, \bar{s})) &= rk(F(r, \bar{s})) + 1 \\ &< rk(F(r, \bar{s})) + 2 = rk(C(\bar{s})) \end{aligned}$$

so we may apply (Cut) to (4), (19), (20), (21) to give

$$\mathcal{H} \,\Big|\!\frac{\alpha\#\alpha\#\beta\#\beta\#\gamma}{\rho}\, \Gamma \Rightarrow \Delta\,. \tag{22}$$

as required.

The case where $C(\bar{s}) \equiv (\exists x \in^{s_i} s_j)F(x, \bar{s})$ is similar.

Case 3. Now suppose that $C(\bar{s}) \equiv \forall x F(x, \bar{s})$, so we have

(23) $\quad \mathcal{H}[\delta] \,\Big|\!\frac{\alpha_\delta}{\rho}\, \Gamma, p \in \mathbb{E}_\delta \Rightarrow F(p, \bar{s}) \qquad$ for all p and all $\delta < \Omega$ with $\alpha_\delta + 3 < \alpha$

(24) $\quad \mathcal{H} \,\Big|\!\frac{\beta_0}{\rho}\, \Gamma, C(\bar{s}), F(r, \bar{s}) \Rightarrow \Delta \qquad$ with $\beta_0 + 3 < \beta$

(25) $\quad \mathcal{H} \,\Big|\!\frac{\beta_1}{\rho}\, \Gamma, C(\bar{s}) \Rightarrow r \in \mathbb{E}_\xi \qquad$ with $\xi < \beta$, $\xi \in \mathcal{H}(\emptyset)$ and $\beta_1 + 3 < \beta$.

Since $\xi \in \mathcal{H}(\emptyset)$, from (23) we obtain

$$\mathcal{H} \,\Big|\!\frac{\alpha_\xi}{\rho}\, \Gamma, r \in \mathbb{E}_\xi \Rightarrow F(r, \bar{s}) \tag{26}$$

Applying the induction hypothesis to (2), (4) and (24) gives

$$\mathcal{H} \,\Big|\!\frac{\alpha\#\alpha\#\beta_0\#\beta_0\#\gamma}{\rho}\, \Gamma, F(r, \bar{s}) \Rightarrow \Delta\,. \tag{27}$$

Again applying the induction hypothesis to (2), (4) and (25) gives

$$\mathcal{H} \,\Big|\!\frac{\alpha\#\alpha\#\beta_1\#\beta_1\#\gamma}{\rho}\, \Gamma \Rightarrow r \in \mathbb{E}_\xi\,. \tag{28}$$

Now a (Cut) applied to (26) and (28) yields

$$\mathcal{H} \,\Big|\!\frac{\alpha\#\alpha\#\beta\#\beta_1\#\gamma}{\rho}\, \Gamma \Rightarrow F(r, \bar{s})\,. \tag{29}$$

Note that

$$\Omega \leq rk(F(r,\bar{s})) < rk(F(r,\bar{s})) + 2 = rk(C) = \rho$$

So a (Cut) applied to (4), (27), (28) and (29) yields

$$(30) \qquad \mathcal{H} \mid\!\frac{\alpha\#\alpha\#\beta\#\beta\#\gamma}{\rho}\, \Gamma \Rightarrow \Delta$$

as required.

The case where $C(\bar{s}) \equiv \exists x F(x,\bar{s})$ is similar.

In the cases where $C \equiv A \wedge B, A \vee B, A \to B$ or $\neg A$ we may argue as with other intuitionistic systems of a similar nature. □

Theorem 5.3.3 (Cut Elimination I)**.** If $\mathcal{H} \mid\!\frac{\alpha}{\Omega+n+1}\, \Gamma \Rightarrow \Delta$ then $\mathcal{H} \mid\!\frac{\omega_n(\alpha)}{\Omega+1}\, \Gamma \Rightarrow \Delta$ for all $n < \omega$.

Where $\omega_0(\alpha) = \alpha$ and $\omega_{n+1}(\alpha) = \omega^{\omega_n(\alpha)}$.

Proof. By main induction on n and subsidiary induction on α. The interesting case is where the last inference was (Cut), with cut formula $A(\bar{s})$ such that $rk(A(\bar{s})) = \Omega + n$ and $\bar{s} = s_1, ..., s_m$ are the only terms occurring $A(\bar{s})$. In this case we have

(1) $\quad \mathcal{H} \mid\!\frac{\alpha_0}{\Omega+n+1}\, \Gamma \Rightarrow A(\bar{s}) \qquad$ with $\alpha_0 < \alpha$

(2) $\quad \mathcal{H} \mid\!\frac{\alpha_1}{\Omega+n+1}\, \Gamma, A(\bar{s}) \Rightarrow \Delta \qquad$ with $\alpha_1 < \alpha$

(3) $\quad \mathcal{H} \mid\!\frac{\alpha_2}{\Omega+n+1}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \qquad$ with $\alpha_2 < \alpha$ and $\beta_i \in \mathcal{H}$ for each $i = 1, ..., m$.

Applying the subsidiary induction hypothesis to (1), (2) and (3) gives

(4) $\quad \mathcal{H} \mid\!\frac{\omega^{\alpha_0}}{\Omega+n}\, \Gamma \Rightarrow A(\bar{s}) \qquad$ with $\alpha_0 < \alpha$

(5) $\quad \mathcal{H} \mid\!\frac{\omega^{\alpha_1}}{\Omega+n}\, \Gamma, A(\bar{s}) \Rightarrow \Delta \qquad$ with $\alpha_1 < \alpha$

(6) $\quad \mathcal{H} \mid\!\frac{\omega^{\alpha_2}}{\Omega+n}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i} \qquad$ with $\alpha_2 < \alpha$ and $\beta_i \in \mathcal{H}$ for each $i = 1, ..., m$.

Now applying the Reduction Lemma 5.3.2 to (4), (5) and (6) gives

$$(7) \qquad \mathcal{H} \mid\!\frac{\omega^{\alpha_0}\#\omega^{\alpha_0}\#\omega^{\alpha_1}\#\omega^{\alpha_1}\#\omega^{\alpha_2}}{\Omega+n}\, \Gamma \Rightarrow \Delta\,.$$

Note that $\omega^{\alpha_0}\#\omega^{\alpha_0}\#\omega^{\alpha_1}\#\omega^{\alpha_1}\#\omega^{\alpha_2} < \omega^{\alpha}$ so by weakening we have

$$(7) \qquad \mathcal{H} \mid\!\frac{\omega^{\alpha}}{\Omega+n}\, \Gamma \Rightarrow \Delta\,.$$

Finally applying the main induction hypothesis gives

$$\mathcal{H} \mid\!\frac{\omega_n(\alpha)}{\Omega+1}\, \Gamma \Rightarrow \Delta$$

as required. □

Lemma 5.3.4. If $\gamma \leq \beta < \Omega$ with $\beta, \gamma \in \mathcal{H}(\emptyset)$ and $\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow s \in \mathbb{E}_\gamma$ then

$$\mathcal{H} \,\Big|\!\frac{\alpha+2}{\rho^*}\, \Gamma \Rightarrow s \in \mathbb{E}_\beta$$

where $\rho^* := \max(\rho, \beta + 1)$.

Proof. If $\gamma = \beta$ the result follows by weakening, so suppose $\gamma < \beta$. Assume that

(1) $$\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow s \in \mathbb{E}_\gamma\,.$$

Now as instances of axioms (E4) and (E6) respectively we have

(2) $$\mathcal{H} \,\Big|\!\frac{0}{0}\, \Gamma \Rightarrow \mathbb{E}_\gamma \in \mathbb{E}_\beta$$

(3) $$\mathcal{H} \,\Big|\!\frac{0}{0}\, \Gamma, s \in \mathbb{E}_\gamma, \mathbb{E}_\gamma \in \mathbb{E}_\beta \Rightarrow s \in \mathbb{E}_\beta\,.$$

Applying (Cut) to (2) and (3) yields

(4) $$\mathcal{H} \,\Big|\!\frac{1}{\beta+2}\, \Gamma, s \in \mathbb{E}_\gamma \Rightarrow s \in \mathbb{E}_\beta\,.$$

Now applying a second (Cut) to (1) and (4) supplies us with

$$\mathcal{H} \,\Big|\!\frac{\alpha+2}{\rho^*}\, \Gamma \Rightarrow s \in \mathbb{E}_\beta$$

as required. □

Lemma 5.3.5 (Boundedness)**.** Suppose $\alpha \leq \beta < \Omega$, $\beta \in \mathcal{H}$, A is a $\Sigma^{\mathcal{E}}$-formula and B is a $\Pi^{\mathcal{E}}$ formula then

i) If $\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma \Rightarrow A$ then $\mathcal{H} \,\Big|\!\frac{\alpha}{\rho^*}\, \Gamma \Rightarrow A^{\mathbb{E}_\beta}$.

ii) If $\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, B \Rightarrow \Delta$ then $\mathcal{H} \,\Big|\!\frac{\alpha}{\rho^*}\, \Gamma, B^{\mathbb{E}_\beta} \Rightarrow \Delta$.

where $\rho^* := \max(\rho, \beta + 1)$.

Proof. By induction on α. The interesting case of i) is where $A \equiv \exists x C(x)$ and A was the principal formula of the last inference which was $(\exists R)$. Note that since $\alpha < \Omega$ the last inference cannot have been ($\Sigma^{\mathcal{E}}$-Ref). So we have

(1) $$\mathcal{H} \,\Big|\!\frac{\alpha_0}{\rho}\, \Gamma \Rightarrow C(r) \qquad \text{with } \alpha_0 + 3 < \alpha.$$

(2) $$\mathcal{H} \,\Big|\!\frac{\alpha_1}{\rho}\, \Gamma \Rightarrow r \in \mathbb{E}_\gamma \qquad \text{with } \alpha_1 < \alpha,\ \gamma \in \mathcal{H} \text{ and } \gamma < \alpha.$$

Since $\gamma < \alpha$ we also know that $\gamma < \beta$ so using Lemma 5.3.4 we get

(3) $$\mathcal{H} \,\Big|\!\frac{\alpha_1+2}{\rho^*}\, \Gamma \Rightarrow r \in \mathbb{E}_\beta\,.$$

Now by applying the induction hypothesis to (1) we get

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma \Rightarrow C(r)^{\mathbb{E}_\beta} . \tag{4}$$

$(\wedge R)$ applied to (3) and (4) yields

$$\mathcal{H} \left|\frac{\max(\alpha_0+1,\alpha_1+3)}{\rho^*}\right. \Gamma \Rightarrow r \in \mathbb{E}_\beta \wedge C(r)^{\mathbb{E}_\beta} . \tag{5}$$

Now since $\Gamma \Rightarrow \mathbb{E}_\beta \in \mathbb{E}_{\beta+1}$ is an axiom we may apply $(b\exists R)$ to (2) and (5) giving

$$\mathcal{H} \left|\frac{\alpha}{\rho^*}\right. \Gamma \Rightarrow (\exists x \in \mathbb{E}_\beta)C(x)^{\mathbb{E}_\beta}$$

as required.

Now for ii) the interesting case is where B was the principal formula of the last inference which was $(b\forall L)$, thus $B \equiv \forall x C(x)$. So we have

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B, C(s) \Rightarrow \Delta \qquad \text{with } \alpha_0 < \alpha. \tag{6}$$

$$\mathcal{H} \left|\frac{\alpha_1}{\rho}\right. \Gamma, B \Rightarrow s \in \mathbb{E}_\gamma \qquad \text{with } \alpha_1 + 3 < \alpha,\ \gamma \in \mathcal{H} \text{ and } \gamma < \alpha. \tag{7}$$

Applying the induction hypothesis twice to (6) and once to (7) we get

$$\mathcal{H} \left|\frac{\alpha_0}{\rho}\right. \Gamma, B^{\mathbb{E}_\beta}, C(s)^{\mathbb{E}_\beta} \Rightarrow \Delta \qquad \text{with } \alpha_0 < \alpha. \tag{8}$$

$$\mathcal{H} \left|\frac{\alpha_1}{\rho}\right. \Gamma, B^{\mathbb{E}_\beta} \Rightarrow s \in \mathbb{E}_\gamma \qquad \text{with } \alpha_1 + 3 < \alpha,\ \gamma \in \mathcal{H} \text{ and } \gamma < \alpha. \tag{9}$$

Now since $\gamma < \alpha$ we also know that $\gamma < \beta$ so by applying Lemma 5.3.4 to (9) we get

$$\mathcal{H} \left|\frac{\alpha_1+2}{\rho^*}\right. \Gamma, B^{\mathbb{E}_\beta} \Rightarrow s \in \mathbb{E}_\beta . \tag{10}$$

Applying $(\to L)$ to (8) and (10) supplies us with

$$\mathcal{H} \left|\frac{\max(\alpha_0+1,\alpha_1+3)}{\rho^*}\right. \Gamma, B^{\mathbb{E}_\beta}, s \in \mathbb{E}_\beta \to C(s)^{\mathbb{E}_\beta} \Rightarrow \Delta . \tag{11}$$

Now applying $(b\forall L)$ to (11), (9) and $\Rightarrow \mathbb{E}_\beta \in \mathbb{E}_{\beta+1}$ which is an instance of axiom (E4), we obtain

$$\mathcal{H} \left|\frac{\alpha}{\rho^*}\right. \Gamma, B^{\mathbb{E}_\beta} \Rightarrow \Delta$$

completing the proof. □

Theorem 5.3.6 (Cut Elimination II; Collapsing)**.** Suppose $\eta \in \mathcal{H}_\eta$, Δ is a set of at most one $\Sigma^{\mathcal{E}}$ formula and Γ is a finite set of $\Pi^{\mathcal{E}}$ formulae with $\max_{A\in\Gamma}(rk(A)) \leq \Omega$ then

$$\mathcal{H}_\eta \left|\frac{\alpha}{\Omega+1}\right. \Gamma \Rightarrow \Delta \quad \text{implies} \quad \mathcal{H}_{\hat{\alpha}} \left|\frac{\psi_\Omega(\hat{\alpha})}{\psi_\Omega(\hat{\alpha})}\right. \Gamma \Rightarrow \Delta ,$$

where $\hat{\alpha} := \eta + \omega^\alpha$.

Proof. The proof is by induction on α. Note that since $\eta \in \mathcal{H}_\eta$ we know from Lemma 3.4.7 that

$$\hat{\alpha}, \psi_\Omega(\hat{\alpha}) \in \mathcal{H}_{\hat{\alpha}}.$$

Case 1. If $\Gamma \Rightarrow \Delta$ is an axiom the result follows easily.

Case 2. If $\Gamma \Rightarrow \Delta$ was the result of a propositional inference we may apply the induction hypothesis to the premises of that inference, and then the same inference again.

Case 3. Suppose the last inference was ($\mathbb{E}$-Lim), then $s \in \mathbb{E}_\gamma$ is a formula in Γ for some limit ordinal γ and

$$\mathcal{H}_\eta[\delta] \mathrel{\substack{\alpha_\delta \\ \overline{} \\ \Omega+1}} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \quad \text{for all } \delta < \gamma \text{ with } \alpha_\delta < \alpha.$$

Since $\gamma \in \mathcal{H}_\eta(\emptyset) = B^\Omega(\eta+1)$ and $\gamma < \Omega$ we know that $\gamma < \psi_\Omega(\eta+1)$ and thus $\delta \in \mathcal{H}_\eta$ for all $\delta < \gamma$. So we have

$$\mathcal{H}_\eta \mathrel{\substack{\alpha_\delta \\ \overline{} \\ \Omega+1}} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \quad \text{for all } \delta < \gamma \text{ with } \alpha_\delta < \alpha.$$

Now applying the induction hypothesis provides

$$\mathcal{H}_{\hat{\alpha}} \mathrel{\substack{\psi_\Omega(\hat{\alpha_\delta}) \\ \overline{} \\ \psi_\Omega(\hat{\alpha_\delta})}} \Gamma, s \in \mathbb{E}_\delta \Rightarrow \Delta \quad \text{for all } \delta < \gamma \text{ with } \alpha_\delta < \alpha.$$

Now since $\psi_\Omega(\hat{\alpha_\delta}) < \psi_\Omega(\hat{\alpha})$ we may apply ($\mathbb{E}$-Lim) to get the desired result.

Case 4. Suppose the last inference was $(b\forall L)$, then $(\forall x \in t)F(x) \in \Gamma$ and

(1) $\qquad \mathcal{H}_\eta \mathrel{\substack{\alpha_0 \\ \overline{} \\ \Omega+1}} \Gamma, s \in t \to F(s) \Rightarrow \Delta \qquad$ with $\alpha_0 < \alpha$.

(2) $\qquad \mathcal{H}_\eta \mathrel{\substack{\alpha_1 \\ \overline{} \\ \Omega+1}} \Gamma \Rightarrow t \in \mathbb{E}_\beta \qquad$ $\beta \in \mathcal{H}_\eta(\emptyset)$ and $\alpha_1 < \alpha$.

(3) $\qquad \mathcal{H}_\eta \mathrel{\substack{\alpha_2 \\ \overline{} \\ \Omega+1}} \Gamma \Rightarrow s \in \mathbb{E}_\gamma \qquad$ $\gamma \in \mathcal{H}_\eta(\emptyset)$, $\gamma, \alpha_2 < \alpha$ and $\gamma \leq \beta$.

Since $\max_{A \in \Gamma}(rk(A)) \leq \Omega$, we know that $s \in t \to F(s)$ is a $\Delta_0^{\mathcal{E}}$ formula so we may immediately apply the induction hypothesis to (1), (2) and (3) giving

(4) $\qquad \mathcal{H}_{\hat{\alpha}} \mathrel{\substack{\psi_\Omega(\hat{\alpha_0}) \\ \overline{} \\ \psi_\Omega(\hat{\alpha})}} \Gamma, s \in t \to F(s) \Rightarrow \Delta$

(5) $\qquad \mathcal{H}_{\hat{\alpha}} \mathrel{\substack{\psi_\Omega(\hat{\alpha_1}) \\ \overline{} \\ \psi_\Omega(\hat{\alpha})}} \Gamma \Rightarrow t \in \mathbb{E}_\beta$

(6) $\qquad \mathcal{H}_{\hat{\alpha}} \mathrel{\substack{\psi_\Omega(\hat{\alpha_2}) \\ \overline{} \\ \psi_\Omega(\hat{\alpha})}} \Gamma \Rightarrow s \in \mathbb{E}_\gamma.$

Since $\gamma \in \mathcal{H}_\eta$ we know that $\gamma < \psi_\Omega(\eta+1)$ and thus $\gamma \in \mathcal{H}_{\hat{\alpha}}$ and $\gamma < \psi_\Omega(\hat{\alpha})$. Moreover $\psi_\Omega(\alpha_i) < \psi_\Omega(\alpha)$ for $i = 0, 1, 2$ so we may apply $(b\forall L)$ to complete this case. The case where the last inference was $(b\exists R)$ is treated in a similar manner.

Case 5. Suppose the last inference was $(b\forall R)_\infty$, then $\Delta = \{(\forall x \in t)F(x)\}$ and

(7) $\mathcal{H}_\eta \,\frac{|\alpha_0}{|\Omega+1}\, \Gamma \Rightarrow s \in t \to F(s)$ for all s, with $\alpha_0 < \alpha$.

(8) $\mathcal{H}_\eta \,\frac{|\alpha_1}{|\Omega+1}\, \Gamma \Rightarrow t \in \mathbb{E}_\beta$ with $\alpha_1, \beta < \alpha$ and $\beta \in \mathcal{H}_\eta$.

We may apply Inversion 5.3.1v) to (7) giving

(9) $$\mathcal{H}_\eta \,\frac{|\alpha_0}{|\Omega+1}\, \Gamma, s \in t \Rightarrow F(s)\,.$$

Applying the induction hypothesis to (8) and (9) yields

(10) $$\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_\Omega(\hat{\alpha_1})}{|\psi_\Omega(\hat{\alpha})}\, \Gamma \Rightarrow t \in \mathbb{E}_\beta$$

(11) $$\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_\Omega(\hat{\alpha_0})}{|\psi_\Omega(\hat{\alpha})}\, \Gamma, s \in t \Rightarrow F(s).$$

Note that since $\beta \in \mathcal{H}_\eta$ we know that $\beta < \psi_\Omega(\eta + 1) < \psi_\Omega(\hat{\alpha})$, thus applying $(\to R)$ to (11) followed by $(b\forall R)_\infty$ (noting that $\psi_\Omega(\hat{\alpha_0}) + 1 < \psi_\Omega(\hat{\alpha})$) gives the desired result. The case where the last inference was $(b\exists L)_\infty$ is treated in a similar manner.

Case 6. Now suppose the last inference was $(\mathcal{E}b\exists L)_\infty$, so $(\exists x \in {}^s t)F(x) \in \Gamma$ and

(12) $\mathcal{H}_\eta \,\frac{|\alpha_0}{|\Omega+1}\, \Gamma, \mathrm{fun}(p, s, t) \wedge F(p) \Rightarrow \Delta$ for all p, with $\alpha_0 < \alpha$.

(13) $\mathcal{H}_\eta \,\frac{|\alpha_1}{|\Omega+1}\, \Gamma \Rightarrow s \in \mathbb{E}_\beta$ with $\beta \in \mathcal{H}_\eta$ and $\alpha_1 < \alpha$.

(14) $\mathcal{H}_\eta \,\frac{|\alpha_2}{|\Omega+1}\, \Gamma \Rightarrow t \in \mathbb{E}_\gamma$ with $\alpha_2 < \alpha$, $\gamma \in \mathcal{H}_\eta$ and $\max(\beta, \gamma) + 2 \leq \alpha$.

By assumption $\mathrm{fun}(p, s, t) \wedge F(p)$ is a $\Pi^\mathcal{E}$ [in fact $\Delta_0^\mathcal{E}$] formula so we may apply the induction hypothesis to (12), (13) and (14) giving

(15) $\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_\Omega(\hat{\alpha_0})}{|\psi_\Omega(\hat{\alpha})}\, \Gamma, \mathrm{fun}(p, s, t) \wedge F(p) \Rightarrow \Delta$ for all p.

(16) $$\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_\Omega(\hat{\alpha_1})}{|\psi_\Omega(\hat{\alpha})}\, \Gamma \Rightarrow s \in \mathbb{E}_\beta$$

(17) $$\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_\Omega(\hat{\alpha_2})}{|\psi_\Omega(\hat{\alpha})}\, \Gamma \Rightarrow t \in \mathbb{E}_\gamma\,.$$

Since $\psi_\Omega(\hat{\alpha_i}) < \psi_\Omega(\hat{\alpha})$ for $i = 0, 1, 2$ and $\beta, \gamma \in \mathcal{H}_\eta$ means that $\max(\beta, \gamma) + 2 < \psi_\Omega(\eta + 1) < \psi_\Omega(\hat{\alpha})$ we may apply $(\mathcal{E}b\exists L)_\infty$ to (15), (16) and (17) to complete this case. The case where the last inference was $(\mathcal{E}b\forall R)_\infty$ may be treated in a similar manner.

Case 7. Now suppose the last inference was $(\mathcal{E}b\exists R)$, so $\Delta = \{(\exists x \in {}^s t)F(x)\}$ and we have

(18) $\mathcal{H}_\eta \,\frac{|\alpha_0}{|\Omega+1}\, \Gamma \Rightarrow \mathrm{fun}(p, s, t) \wedge F(p)$ for all p with $\alpha_0 < \alpha$.

(19) $\mathcal{H}_\eta \,\frac{|\alpha_1}{|\Omega+1}\, \Gamma \Rightarrow s \in \mathbb{E}_\beta$ with $\beta \in \mathcal{H}_\eta(\emptyset)$ and $\alpha_1 < \alpha$.

(20) $\mathcal{H}_\eta \,\frac{|\alpha_2}{|\Omega+1}\, \Gamma \Rightarrow t \in \mathbb{E}_\gamma$ with $\gamma \in \mathcal{H}_\eta(\emptyset)$ and $\alpha_2 < \alpha$.

(21) $\mathcal{H}_\eta \,\frac{|\alpha_3}{|\Omega+1}\, \Gamma \Rightarrow p \in \mathbb{E}_\delta$ $\alpha_3, \delta < \alpha$, $\delta \in \mathcal{H}_\eta(\emptyset)$ and $\delta \leq \max(\beta, \gamma) + 2$.

Since $\mathrm{fun}(p,s,t)\wedge F(p)$ is a $\Sigma^{\mathcal{E}}$ formula we can apply the induction hypothesis to (18), (19), (20) and (21) followed by $(\mathcal{E}b\exists R)$, in a similar manner to Case 4. The case where the last inference was $(\mathcal{E}b\forall L)$ can also be treated in a similar manner.

Now suppose the last inference was $(\forall L)$, so $\forall xF(x)\in\Gamma$ and

(22) $$\mathcal{H}_\eta \,\frac{|\alpha_0}{|\Omega+1}\, \Gamma, F(s)\Rightarrow\Delta \qquad \text{with } \alpha_0+3<\alpha.$$

(23) $$\mathcal{H}_\eta \,\frac{|\alpha_1}{|\Omega+1}\, \Gamma\Rightarrow s\in\mathbb{E}_\beta \qquad \beta,\alpha_1+3<\alpha \text{ and } \beta\in\mathcal{H}_\eta(\emptyset).$$

Since $F(s)$ is $\Delta_0^{\mathcal{E}}$ we may immediately apply the induction hypothesis to (22) and (23) giving

(24) $$\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_\Omega(\hat{\alpha_0})}{|\psi_\Omega(\hat{\alpha})}\, \Gamma, F(s)\Rightarrow\Delta$$

(25) $$\mathcal{H}_{\hat{\alpha}} \,\frac{|\psi_\Omega(\hat{\alpha_1})}{|\psi_\Omega(\hat{\alpha})}\, \Gamma\Rightarrow s\in\mathbb{E}_\beta\,.$$

Now since $\beta\in\mathcal{H}_\eta$ we know that $\beta<\psi_\Omega(\eta+1)<\psi_\Omega(\hat{\alpha})$ hence we may apply $(\forall L)$ to (24) and (25) to complete this case. The case where the last inference was $(\exists R)$ can be treated in a similar manner.

Case 9. Now suppose the last inference was $(\Sigma^{\mathcal{E}}\text{-}Ref)$, so $\Delta=\{\exists zA^z\}$ where A is a $\Sigma^{\mathcal{E}}$ formula and

(26) $$\mathcal{H}_\eta \,\frac{|\alpha_0}{|\Omega+1}\, \Gamma\Rightarrow A \qquad \text{with } \alpha_0+1,\Omega<\alpha.$$

We may immediately apply the induction hypothesis to (26) giving

(27) $$\mathcal{H}_{\hat{\alpha_0}} \,\frac{|\psi_\Omega(\hat{\alpha_0})}{|\psi_\Omega(\hat{\alpha_0})}\, \Gamma\Rightarrow A\,.$$

Applying Boundedness 5.3.5i) to (27) provides

(28) $$\mathcal{H}_{\hat{\alpha_0}} \,\frac{|\psi_\Omega(\hat{\alpha_0})}{|\psi_\Omega(\hat{\alpha_0})+2}\, \Gamma\Rightarrow A^{\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})}}\,.$$

Now as an instance of axiom (E4) we have

(29) $$\mathcal{H}_{\hat{\alpha_0}} \,\frac{|0}{|0}\, \Rightarrow\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})}\in\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})+1}\,.$$

Since $\psi_\Omega(\hat{\alpha_0})+1\in\mathcal{H}_{\hat{\alpha}}$ and $\psi_\Omega(\hat{\alpha_0})+1<\psi_\Omega(\hat{\alpha})$ we may apply $(\exists R)$ to (28) and (29) to complete the case.

Now suppose the last inference was (Cut), then we have

(30) $$\mathcal{H}_\eta \,\frac{|\alpha_0}{|\Omega+1}\, \Gamma\Rightarrow A(\bar{s}) \qquad \text{with } \alpha_0<\alpha.$$

(31) $$\mathcal{H}_\eta \,\frac{|\alpha_1}{|\Omega+1}\, \Gamma, A(\bar{s})\Rightarrow\Delta \qquad \text{with } \alpha_1<\alpha.$$

(32) $$\mathcal{H}_\eta \,\frac{|\alpha_2}{|\Omega+1}\, \Gamma\Rightarrow s_i\in\mathbb{E}_{\beta_i} \qquad \text{with } \alpha_2<\alpha,\ \bar{\beta}\in\mathcal{H}_\eta \text{ and } \|A(\bar{s})\|_{\bar{\beta}}\leq\Omega.$$

10.1 If $\|A(\bar{s})\|_{\bar{\beta}} < \Omega$ it follows from $\bar{\beta} \in \mathcal{H}_\eta = B^\Omega(\eta+1)$ that $\|A(\bar{s})\|_{\bar{\beta}} \in B^\Omega(\eta+1)$ and thus $\|A(\bar{s})\|_{\bar{\beta}} < \psi_\Omega(\eta+1) < \psi_\Omega(\hat{\alpha})$. Also A is $\Delta_0^{\mathcal{E}}$, thus we may apply the induction hypothesis to (30), (31) and (32) followed by (Cut) to complete this (sub)case.

10.2 Now suppose $\|A(\bar{s})\|_{\bar{\beta}} = \Omega$. Then either $A \equiv \forall x F(x)$ or $A \equiv \exists x F(x)$ with F a $\Delta_0^{\mathcal{E}}$ formula. The two cases are dual, we assume that the former is the case. Thus A is $\Pi^{\mathcal{E}}$, so we may apply the induction hypothesis to (31) giving

$$\mathcal{H}_{\hat{\alpha_1}} \,\Big|\!\frac{\psi_\Omega(\hat{\alpha_1})}{\psi_\Omega(\hat{\alpha_1})}\, \Gamma, A(\bar{s}) \Rightarrow \Delta \tag{33}$$

Applying Boundedness 5.3.5ii) to (33) yields

$$\mathcal{H}_{\hat{\alpha}} \,\Big|\!\frac{\max(\psi_\Omega(\hat{\alpha_0}),\psi_\Omega(\hat{\alpha_1}))}{\psi_\Omega(\hat{\alpha_1})}\, \Gamma, A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})}} \Rightarrow \Delta\,. \tag{34}$$

Now applying Inversion 5.3.1xi) to (30) gives

$$\mathcal{H}_{\hat{\alpha_0}} \,\Big|\!\frac{\alpha_0}{\Omega+1}\, \Gamma \Rightarrow A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})}}\,. \tag{35}$$

Noting that $A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})}}$ is $\Delta_0^{\mathcal{E}}$ we may apply the induction hypothesis to (35) giving

$$\mathcal{H}_{\alpha^*} \,\Big|\!\frac{\psi_\Omega(\alpha^*)}{\psi_\Omega(\alpha^*)}\, \Gamma \Rightarrow A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})}}\,. \tag{36}$$

where $\alpha^* := \hat{\alpha_0} + \omega^{\Omega+\alpha_0}$. Now applying the induction hypothesis to (32) gives

$$\mathcal{H}_{\hat{\alpha_2}} \,\Big|\!\frac{\psi_\Omega(\hat{\alpha_2})}{\psi_\Omega(\hat{\alpha_2})}\, \Gamma \Rightarrow s_i \in \mathbb{E}_{\beta_i}\,. \tag{37}$$

Now as an instance of axiom (E4) we have

$$\mathcal{H}_{\hat{\alpha}} \,\Big|\!\frac{0}{0}\, \Rightarrow \mathbb{E}_{\psi_\Omega(\hat{\alpha_0})} \in \mathbb{E}_{\psi_\Omega(\hat{\alpha_0})+1}\,. \tag{38}$$

Since $\bar{\beta} \in B^\Omega(\eta+1)$ we get

$$\|A(\bar{s})^{\mathbb{E}_{\psi_\Omega(\hat{\alpha_0})}}\|_{\bar{\beta},\psi_\Omega(\hat{\alpha_0})+1} = \psi_\Omega(\hat{\alpha_0}) + 1 < \psi_\Omega(\hat{\alpha}).$$

It remains to note that

$$\alpha^* = \eta + \omega^{\Omega+\alpha_0} + \omega^{\Omega+\alpha_0} < \eta + \omega^{\Omega+\alpha} = \hat{\alpha}$$

and thus $\psi_\Omega(\alpha^*) < \psi_\Omega(\alpha)$. So we may apply (Cut) to (34),(36),(37) and (38) to conclude

$$\mathcal{H}_{\hat{\alpha}} \,\Big|\!\frac{\psi_\Omega(\hat{\alpha})}{\psi_\Omega(\hat{\alpha})}\, \Gamma \Rightarrow \Delta$$

as required. □

5.4 Embedding $\mathbf{IKP}(\mathcal{E})$ into $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

Definition 5.4.1. If $\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$ is an intuitionistic sequent of $\mathbf{IKP}(\mathcal{E})$ with exactly the free variables $\bar{a} = a_1, ..., a_n$ and containing the formulas $A_1(\bar{a}), ..., A_m(\bar{a})$ then

$$no_{\bar{\beta}}(\Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]) := \omega^{\|A_1\|_{\bar{\beta}}} \# ... \# \omega^{\|A_m\|_{\bar{\beta}}}.$$

For terms $\bar{s} := s_1, ...s_n$ and ordinals $\bar{\beta} := \beta_1, ..., \beta_n$ the expression $\bar{s} \in \mathbb{E}_{\bar{\beta}}$ will be considered shorthand for $s_1 \in \mathbb{E}_{\beta_1}, ..., s_n \in \mathbb{E}_{\beta_n}$

The expression $\Vdash \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ will be considered shorthand for

$$\mathcal{H}[\bar{\beta}] \,\big|\!\frac{no_{\bar{\beta}}(\Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}])}{0}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,.$$

For any operator $\mathcal{H}$ and any ordinals $\bar{\beta} < \Omega$.

The expression $\Vdash^\alpha_\rho \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ will be considered shorthand for

$$\mathcal{H}[\bar{\beta}] \,\big|\!\frac{no_{\bar{\beta}}(\Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]) \# \alpha}{\rho}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,.$$

For any operator $\mathcal{H}$ and any ordinals $\bar{\beta} < \Omega$.

As might be expected $\Vdash^\alpha \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ and $\Vdash_\rho \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ will be considered shorthand for $\Vdash^\alpha_0 \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ and $\Vdash^0_\rho \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ respectively.

Lemma 5.4.2. For any formula $A(\bar{a})$ of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables displayed and any $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms $\bar{s} = s_1, ..., s_n$

$$\Vdash_\Omega A(\bar{s}) \Rightarrow A(\bar{s})$$

Proof. By induction on the construction of the formula A. If A is $\Delta_0^{\mathcal{E}}$ then this is an instance of axiom (E1).
Suppose $A(\bar{s}) \equiv \forall x F(x, \bar{s})$. For each $\gamma < \Omega$ we define

$$\alpha^\gamma := \gamma + no_{\gamma,\bar{\beta}}(F(t, \bar{s}) \Rightarrow F(t, \bar{s})),$$

note that

$$\gamma < \alpha^\gamma < \alpha^\gamma + 8 < no_{\bar{\beta}}(A(\bar{s}) \Rightarrow A(\bar{s})).$$

By axiom (E1) we have

$$\text{(1)} \qquad \mathcal{H}[\gamma, \bar{\beta}] \,\big|\!\frac{0}{0}\, t \in \mathbb{E}_\gamma \Rightarrow t \in \mathbb{E}_\gamma \quad \text{for all } t \text{ and all } \gamma < \Omega.$$

Now from the induction hypothesis we have

$$\text{(2)} \qquad \mathcal{H}[\gamma, \bar{\beta}] \,\big|\!\frac{\alpha^\gamma}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_\gamma, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t \text{ and all } \gamma < \Omega.$$

It is worth noting that this use of the induction hypothesis is where we really need cuts of $\bar{\beta}$-rank arbitrarily high in Ω. Applying $(\forall L)$ to (1) and (2) yields

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash_{\Omega}^{\alpha^{\gamma}+4} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\gamma}, A(\bar{s}) \Rightarrow F(t, \bar{s})$$

to which we may apply $(\forall R)_{\infty}$ to get the desired result.

Case 2. Now suppose $A \equiv (\forall x \in s_i)F(x, \bar{s})$. From the induction hypothesis we have

$$\text{(3)} \qquad \mathcal{H}[\delta, \bar{\beta}] \vdash_{\Omega}^{\omega^{\|F(t,\bar{s})\|_{\delta,\bar{\beta}}} \cdot 2} t \in \mathbb{E}_{\delta}, \bar{s} \in \mathbb{E}_{\bar{\beta}}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t \text{ and all } \delta < \Omega.$$

In particular when $\delta = \beta_i$ in (3) we have

$$\text{(4)} \qquad \mathcal{H}[\delta, \bar{\beta}] \vdash_{\Omega}^{\alpha_0} t \in \mathbb{E}_{\beta_i}, \bar{s} \in \mathbb{E}_{\bar{\beta}}, F(t, \bar{s}) \Rightarrow F(t, \bar{s})$$

where $\alpha_0 := \omega^{\|F(t,\bar{s})\|_{\beta_i,\bar{\beta}}} \cdot 2$ Now as an instance of axiom (E6) we have

$$\text{(5)} \qquad \mathcal{H}[\bar{\beta}] \vdash_{0}^{0} s_i \in \mathbb{E}_{\beta_i}, t \in s_i \Rightarrow t \in \mathbb{E}_{\beta_i}$$

Now applying (Cut) to (4) and (5) yields

$$\text{(6)} \qquad \mathcal{H}[\bar{\beta}] \vdash_{\Omega}^{\alpha_0+1} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in s_i, F(t, \bar{s}) \Rightarrow F(t, \bar{s})\,.$$

As an instance of axiom (E1) we have

$$\text{(7)} \qquad \mathcal{H}[\bar{\beta}] \vdash_{0}^{0} t \in s_i \Rightarrow t \in s_i\,.$$

Applying $(\rightarrow L)$ to (6) and (7) yields

$$\text{(8)} \qquad \mathcal{H}[\bar{\beta}] \vdash_{\Omega}^{\alpha_0+2} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in s_i, t \in s_i \rightarrow F(t, \bar{s}) \Rightarrow F(t, \bar{s})\,.$$

An application of $(b\forall L)$ to (5) and (8) provides

$$\mathcal{H}[\bar{\beta}] \vdash_{\Omega}^{\alpha_0+3} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in s_i, (\forall x \in s_i)F(x, \bar{s}) \Rightarrow F(t, \bar{s})\,.$$

Finally using $(\rightarrow R)$ followed by $(b\forall R)_{\infty}$ and noting that $\alpha_0 + 5 < no_{\bar{\beta}}(A(\bar{s}) \Rightarrow A(\bar{s}))$ we get the desired result.

Case 3. Suppose that $A \equiv (\exists x \in^{s_i} s_j)F(x, \bar{s})$. From the induction hypothesis we know that

$$\text{(9)} \qquad \mathcal{H}[\bar{\beta}, \delta] \vdash_{\Omega}^{\omega^{\|F(t,\bar{s})\|_{\delta,\bar{\beta}}} \cdot 2} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\delta}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t \text{ and all } \delta < \Omega.$$

In particular when $\delta = \gamma := \max(\beta_i, \beta_j) + 2$ we have

$$\text{(10)} \qquad \mathcal{H}[\bar{\beta}] \vdash_{\Omega}^{\alpha_0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\gamma}, F(t, \bar{s}) \Rightarrow F(t, \bar{s}) \quad \text{for all } t,$$

where $\alpha_0 := \omega^{\|F(t,\bar{s})\|_{\bar{\beta},\gamma}} \cdot 2$. Now as an instance of axiom (E10) we have

$$(11) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{0}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(t, s_i, s_j) \Rightarrow t \in \mathbb{E}_{\gamma}\,.$$

Applying (Cut) to (10) and (11) gives

$$(12) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\alpha_0+1}{\Omega}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(t, s_i, s_j), F(t, \bar{s}) \Rightarrow F(t, \bar{s})\,.$$

As an instance of axiom (E1) we have

$$(13) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{0}{0}\; \mathrm{fun}(t, s_i, s_j) \Rightarrow \mathrm{fun}(t, s_i, s_j)$$

Applying $(\wedge R)$ to (12) and (13) gives

$$(14) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\alpha_0+2}{\Omega}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(t, s_i, s_j), F(t, \bar{s}) \Rightarrow \mathrm{fun}(t, s_i, s_j) \wedge F(t, \bar{s})\,.$$

Now applying $(\mathcal{E}b\exists R)$ to (11) and (14) yields

$$(15) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\alpha_0+3}{\Omega}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(t, s_i, s_j), F(t, \bar{s}) \Rightarrow (\exists x \in {}^{s_i}s_j)F(x, \bar{s})\,.$$

Two applications of $(\wedge L)$ gives

$$(15) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\alpha_0+5}{\Omega}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(t, s_i, s_j) \wedge F(t, \bar{s}) \Rightarrow (\exists x \in {}^{s_i}s_j)F(x, \bar{s})\,.$$

Finally using $(\mathcal{E}b\exists L)_{\infty}$ gives

$$(15) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\alpha_0+6}{\Omega}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, (\exists x \in {}^{s_i}s_j)F(x, \bar{s}) \Rightarrow (\exists x \in {}^{s_i}s_j)F(x, \bar{s})\,.$$

It remains to note that $\alpha_0 + 6 < no_{\bar{\beta}}(A(\bar{s}) \Rightarrow A(\bar{s}))$ to complete this case.

All other cases are either propositional, for which the proof is standard or may be regarded as dual to one of the three above. $\square$

Lemma 5.4.3 (Extensionality)**.** For any formula $A(\bar{a})$ of $\mathbf{IKP}(\mathcal{E})$ (not necessarily with all free variables displayed) and any $\mathbf{IRS}_{\Omega}^{\mathbb{E}}$ terms $\bar{s} := s_1, ..., s_n, \bar{t} := t_1, ..., t_n$ we have

$$\Vdash_{\Omega} \bar{s} = \bar{t}, A(\bar{s}) \Rightarrow A(\bar{t})$$

where $\bar{s} = \bar{t}$ is shorthand for $s_1 = t_1, ..., s_n = t_n$.

Proof. If $A(\bar{s})$ is $\Delta_0^{\mathcal{E}}$ then this is an instance of axiom (E3). The remainder of the proof is by induction on $rk(A(\bar{s}))$, note that since A is assumed to contain an unbounded quantifier

$$rk(A) = \|A(\bar{s})\|_{\bar{\beta}} \geq \Omega \quad \text{for any ordinals } \bar{\beta} < \Omega.$$

Case 1. Suppose $A(\bar{s}) \equiv \forall x F(x, \bar{s})$. By the induction hypothesis we have

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \,\Big|\!\frac{no_{\bar{\beta},\bar{\gamma},\delta}(\bar{s}=\bar{t}, F(r,\bar{s}) \Rightarrow F(r,\bar{t}))}{\Omega}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, r \in \mathbb{E}_{\delta}, \bar{s} = \bar{t}, F(r, \bar{s}) \Rightarrow F(r, \bar{t})$$

for all r and all $\delta < \Omega$. For ease of reading we suppress the other terms possibly occurring in $F(r, \bar{s})$ and the assumptions about their locations in the $\mathbb{E}$ hierarchy since these do not affect the proof. By virtue of axiom (E1) we have

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \,\big|\!\frac{0}{0}\, r \in \mathbb{E}_\delta \Rightarrow r \in \mathbb{E}_\delta\,.$$

Hence we may apply $(\forall L)$ to obtain

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \,\big|\!\frac{\alpha_\delta}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, t \in \mathbb{E}_{\bar{\gamma}}, \bar{s} = \bar{t}, r \in \mathbb{E}_\delta, \forall x F(x, \bar{s}) \Rightarrow F(r, \bar{t})$$

where $\alpha_\delta := \delta + no_{\bar{\beta},\bar{\gamma},\delta}(\bar{s} = \bar{t}, F(r, \bar{s}) \Rightarrow F(r, \bar{t})) + 1$. Note that

$$\alpha_\delta + 3 < no_{\bar{\beta},\bar{\gamma}}(\bar{s} = \bar{t}, A(\bar{s}) \Rightarrow A(\bar{t})) =: \alpha.$$

Hence we may apply $(\forall R)_\infty$ to obtain

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}] \,\big|\!\frac{\alpha}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, \bar{s} = \bar{t}, A(\bar{s}) \Rightarrow A(\bar{t})$$

as required.

Case 2. Now suppose $A(\bar{s}) \equiv (\forall x \in {}^{s_i} s_j) F(x, \bar{s})$. Using the induction hypothesis we have

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \,\big|\!\frac{\alpha_0}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, r \in \mathbb{E}_\delta, \bar{s} = \bar{t}, F(r, \bar{s}) \Rightarrow F(r, \bar{t}) \tag{1}$$

for any term r and any $\delta < \Omega$, where $\alpha_0 = no_{\bar{\beta},\bar{\gamma},\delta}(\bar{s} = \bar{t}, F(r, \bar{s}) \Rightarrow F(r, \bar{t}))$. At this point we set $\delta = \max(\beta_i, \beta_j) + 2$, note that $\delta \in \mathcal{H}[\bar{\beta}, \bar{\gamma}]$. By virtue of axiom (E1) we have

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}] \,\big|\!\frac{0}{0}\, \mathrm{fun}(r, s_i, s_j) \Rightarrow \mathrm{fun}(r, s_i, s_j). \tag{2}$$

Hence by $(\to L)$ we get

$$\begin{aligned} \mathcal{H}[\bar{\beta}, \bar{\gamma}] \,\big|\!\frac{\alpha_0+1}{\Omega}\, & \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, r \in \mathbb{E}_\delta, \bar{s} = \bar{t}, \\ & \mathrm{fun}(r, s_i, s_j) \to F(r, \bar{s}), \mathrm{fun}(r, s_i, s_j) \Rightarrow F(r, \bar{t}). \end{aligned} \tag{3}$$

As an instance of axiom (E10) we have

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}] \,\big|\!\frac{0}{0}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(r, s_i, s_j) \Rightarrow r \in \mathbb{E}_\delta\,. \tag{4}$$

An application of (Cut) to (3) and (4) yields

$$\begin{aligned} \mathcal{H}[\bar{\beta}, \bar{\gamma}] \,\big|\!\frac{\alpha_0+2}{\Omega}\, & \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, \bar{s} = \bar{t}, \\ & \mathrm{fun}(r, s_i, s_j) \to F(r, \bar{s}), \mathrm{fun}(r, s_i, s_j) \Rightarrow F(r, \bar{t}). \end{aligned} \tag{5}$$

Now applying $(\mathcal{E}b\forall L)$ to (4) and (5) gives

$$\mathcal{H}[\bar{\beta}, \bar{\gamma}] \,\big|\!\frac{\alpha_0+3}{\Omega}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, \bar{s} = \bar{t}, (\forall x \in {}^{s_i} s_j) F(x, \bar{s}), \mathrm{fun}(r, s_i, s_j) \Rightarrow F(r, \bar{t}). \tag{6}$$

Note that $\alpha_0 \geq \Omega$ since F is not $\Delta_0^{\mathcal{E}}$, so we don't have to worry about the condition $\delta < \alpha_0 + 3$. Now as an instance of axiom $(E3)$ we have

$$(7) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}] \mathop{|\!\frac{0}{\;\;}}_{0} \bar{s} = \bar{t}, \mathrm{fun}(r, t_i, t_j) \Rightarrow \mathrm{fun}(r, s_i, s_j)\,.$$

Also axiom (E10) gives rise to

$$(8) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}] \mathop{|\!\frac{0}{\;\;}}_{0} \bar{t} \in \mathbb{E}_{\bar{\gamma}}, \mathrm{fun}(r, t_i, t_j) \Rightarrow r \in \mathbb{E}_\eta \quad \text{where } \eta = \max(\gamma_i, \gamma_j) + 2.$$

Applying (Cut) to (6),(7) and (8) gives

$$(9) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}] \mathop{|\!\frac{\alpha_0+4}{\quad}}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, \bar{s} = \bar{t}, (\forall x \in {}^{s_i} s_j) F(x, \bar{s}), \mathrm{fun}(r, t_i, t_j) \Rightarrow F(r, \bar{t}).$$

Now $(\rightarrow R)$ gives

$$(10) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}] \mathop{|\!\frac{\alpha_0+5}{\quad}}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, \bar{s} = \bar{t}, (\forall x \in {}^{s_i} s_j) F(x, \bar{s}) \Rightarrow \mathrm{fun}(r, t_i, t_j) \rightarrow F(r, \bar{t}).$$

Finally we may apply $(\mathcal{E}b\forall R)_\infty$, noting that $\alpha_0 + 6 < no_{\bar{\beta},\bar{\gamma}}(\bar{s} = \bar{t}, A(\bar{s}) \Rightarrow A(\bar{t}))$ to complete this case.

Note that it could also be the case that $A(\bar{s}) \equiv (\forall x \in {}^p q) F(x, \bar{s})$ where p and/or q is not a member of $\bar{s}$. The following case is an example of this kind of thing.

Case 3. Suppose $A(\bar{s}) \equiv (\exists x \in p) F(x, \bar{s}, p)$, where p is not present in $\bar{s}$. By the induction hypothesis we have

$$(11) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \mathop{|\!\frac{\alpha_0}{\;\;}}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, p \in \mathbb{E}_\delta, r \in \mathbb{E}_\delta, \bar{s} = \bar{t}, F(r, \bar{s}, p) \Rightarrow F(r, \bar{t}, p)$$

where $\alpha_0 := no_{\bar{\beta},\bar{\gamma},\delta,\delta}(\bar{s} = \bar{t}, F(r, \bar{s}, p) \Rightarrow F(r, \bar{t}, p))$. As an instance of axiom (E1) we have

$$(12) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \mathop{|\!\frac{0}{\;\;}}_{0} r \in p \Rightarrow r \in p\,.$$

Applying $(\wedge R)$ to (11) and (12) yields

$$(13) \quad \mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \mathop{|\!\frac{\alpha_0+1}{\quad}}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, p \in \mathbb{E}_\delta, r \in \mathbb{E}_\delta, \bar{s} = \bar{t}, F(r, \bar{s}, p), r \in p \Rightarrow r \in p \wedge F(r, \bar{t}, p)\,.$$

As an instance of axiom (E6) we have

$$(14) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \mathop{|\!\frac{0}{\;\;}}_{0} p \in \mathbb{E}_\delta, r \in p \Rightarrow r \in \mathbb{E}_\delta\,.$$

(Cut) applied to (12) and (13) gives

$$(15) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \mathop{|\!\frac{\alpha_0+2}{\quad}}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, p \in \mathbb{E}_\delta, \bar{s} = \bar{t}, F(r, \bar{s}, p), r \in p \Rightarrow r \in p \wedge F(r, \bar{t}, p)\,.$$

Now $(b\exists R)$ gives

$$(16) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \mathop{|\!\frac{\alpha_0+3}{\quad}}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, p \in \mathbb{E}_\delta, \bar{s} = \bar{t}, F(r, \bar{s}, p), r \in p \Rightarrow A(\bar{s})\,.$$

Two applications of $(\wedge L)$ gives

$$(17) \qquad \mathcal{H}[\bar{\beta}, \bar{\gamma}, \delta] \mathop{|\!\!\!-}^{\alpha_0+5}_{\Omega} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \bar{t} \in \mathbb{E}_{\bar{\gamma}}, p \in \mathbb{E}_{\delta}, \bar{s} = \bar{t}, r \in p \wedge F(r, \bar{s}, p) \Rightarrow A(\bar{s})\,.$$

To which we may apply $(b\exists L)$ to complete this case.

All other cases are similar to one of those above. □

Lemma 5.4.4 (Set induction)**.** For any formula $F(a)$ of $\mathbf{IKP}(\mathcal{E})$ we have

$$\Vdash_{\Omega} \Rightarrow \forall x[(\forall y \in x)F(y) \to F(x)] \to \forall x F(x).$$

Proof. Let $\mathcal{H}$ be an arbitrary operator and let

$$A := \forall x[(\forall y \in x)F(y) \to F(x)].$$

Let $\bar{p}$ be the terms other than s that occur in $F(s)$, sub-terms not included. Let $\bar{\mathcal{H}} := \mathcal{H}[\bar{\beta}]$ where $\bar{\beta}$ is an arbitrary choice of ordinals $< \Omega$. In the remainder of the proof we shall just write $\bar{\mathcal{H}} \mathop{|\!\!\!-}^{\alpha}_{\rho} \Gamma \Rightarrow \Delta$ instead of $\mathcal{H}[\bar{\beta}] \mathop{|\!\!\!-}^{\alpha}_{\rho} \bar{p} \in \mathbb{E}_{\bar{\beta}}, \Gamma \Rightarrow \Delta$, since $\bar{p} \in \mathbb{E}_{\bar{\beta}}$ will always remain a side formula in the derivation.

Claim:

$$(*) \qquad \bar{\mathcal{H}}[\gamma] \mathop{|\!\!\!-}^{\omega^{rk(A)}\#\omega^{\gamma+1}}_{\Omega} A, s \in \mathbb{E}_{\gamma} \Rightarrow F(s) \quad \text{for all } \gamma < \Omega \text{ and all terms } s.$$

Note that since A contains an unbounded quantifier $rk(A) = no_{\bar{\beta}}(A)$. We prove the claim by induction on γ. Thus the induction hypothesis supplies us with

$$(1) \qquad \bar{\mathcal{H}}[\delta] \mathop{|\!\!\!-}^{\omega^{rk(A)}\#\omega^{\delta+1}}_{\Omega} A, t \in \mathbb{E}_{\delta} \Rightarrow F(t) \quad \text{for all } \delta < \gamma \text{ and all terms } t.$$

So by weakening we have

$$(2) \qquad \bar{\mathcal{H}}[\gamma, \delta] \mathop{|\!\!\!-}^{\omega^{rk(A)}\#\omega^{\delta+1}}_{\Omega} A, s \in \mathbb{E}_{\gamma}, t \in s, t \in \mathbb{E}_{\delta} \Rightarrow F(t)\,.$$

Case 1. Suppose $\gamma = \gamma_0 + 1$, so a special case of (2) becomes

$$(3) \qquad \bar{\mathcal{H}}[\gamma] \mathop{|\!\!\!-}^{\omega^{rk(A)}\#\omega^{\gamma}}_{\Omega} A, s \in \mathbb{E}_{\gamma}, t \in s, t \in \mathbb{E}_{\gamma_0} \Rightarrow F(t)\,.$$

As an instance of axiom (E7) we have

$$(4) \qquad \bar{\mathcal{H}}[\gamma] \mathop{|\!\!\!-}^{0}_{0} s \in \mathbb{E}_{\gamma}, t \in s \Rightarrow t \in \mathbb{E}_{\gamma_0}\,.$$

Applying (Cut) to (3) and (4) yields

$$(5) \qquad \bar{\mathcal{H}}[\gamma] \mathop{|\!\!\!-}^{\omega^{rk(A)}\#\omega^{\gamma}+1}_{\Omega} A, s \in \mathbb{E}_{\gamma}, t \in s \Rightarrow F(t)\,.$$

$(\to R)$ followed by $(b\forall R)_\infty$ provides

$$\text{(6)} \qquad \bar{\mathcal{H}}[\gamma] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^\gamma+3}{\Omega}\; A, s \in \mathbb{E}_\gamma \Rightarrow (\forall x \in s)F(x)\,.$$

Now from Lemma 5.4.2 we have

$$\text{(7)} \qquad \bar{\mathcal{H}}[\gamma] \,\Big|\!\frac{no_{\bar{\beta},\gamma}(F(s)\Rightarrow F(s))}{\Omega}\; s \in \mathbb{E}_\gamma, F(s) \Rightarrow F(s)\,.$$

Since $no_{\bar{\beta},\gamma}(F(s) \Rightarrow F(s)) < \omega^{rk(A)}$, by $(\to L)$ we get

$$\text{(8)} \qquad \bar{\mathcal{H}}[\gamma] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^\gamma+4}{\Omega}\; A, s \in \mathbb{E}_\gamma, (\forall x \in s)F(x) \to F(s) \Rightarrow F(s)\,.$$

To which we may apply $(\forall L)$ giving

$$\text{(9)} \qquad \bar{\mathcal{H}}[\gamma] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^\gamma+1}{\Omega}\; A, s \in \mathbb{E}_\gamma \Rightarrow F(s)$$

as required.

Case 2. Now suppose γ is a limit ordinal. Applying ($\mathbb{E}$-Lim) to (2) provides us with

$$\text{(10)} \qquad \bar{\mathcal{H}}[\gamma] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^\gamma}{\Omega}\; A, s \in \mathbb{E}_\gamma, t \in s, t \in \mathbb{E}_\gamma \Rightarrow F(t)\,.$$

As an instance of axiom (E6) we have

$$\text{(11)} \qquad \bar{\mathcal{H}}[\gamma] \,\Big|\!\frac{0}{0}\; s \in \mathbb{E}_\gamma, t \in s \Rightarrow t \in \mathbb{E}_\gamma\,.$$

An application of (Cut) to (10) and (11) yields

$$\text{(12)} \qquad \bar{\mathcal{H}}[\gamma] \,\Big|\!\frac{\omega^{rk(A)}\#\omega^\gamma+1}{\Omega}\; A, s \in \mathbb{E}_\gamma, t \in s \Rightarrow F(t)\,.$$

The remainder of this case can proceed exactly as in Case 1 from (5) onwards. Thus the claim (*) is verified.

Finally applying $(\forall R)_\infty$ to (*) gives

$$\bar{\mathcal{H}} \,\Big|\!\frac{\omega^{rk(A)}\#\Omega}{\Omega}\; A \Rightarrow \forall x F(x)\,.$$

Finally noting that $\omega^{rk(A)}\#\Omega < no_{\bar{\beta}}(A \to \forall x F(x))$ we may apply $(\to R)$ to complete the proof. $\square$

Lemma 5.4.5 (Infinity)**.** For any operator $\mathcal{H}$ we have

$$\mathcal{H} \,\Big|\!\frac{\omega+4}{\omega}\; \Rightarrow \exists x[(\exists y \in x)(y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]\,.$$

Proof. Firstly note that by Definiton 3.3.3 $1, \omega \in \mathcal{H}$. We have the following derivation trees in $\mathbf{IRS}^{\mathbb{E}}_{\Omega}$.

$$
\begin{array}{rc}
 & \text{Axiom (E4)} \\ \hline
 & \mathcal{H} \vdash^{0}_{0} \Rightarrow \mathbb{E}_0 \in \mathbb{E}_\omega \\ \hline
(\wedge R) & \mathcal{H} \vdash^{1}_{0} \Rightarrow \mathbb{E}_0 \in \mathbb{E}_\omega \wedge \mathbb{E}_0 \in \mathbb{E}_\omega \\ \hline
(b\exists R) & \mathcal{H} \vdash^{2}_{0} \Rightarrow (\exists y \in \mathbb{E}_\omega)(y \in \mathbb{E}_\omega)
\end{array}
$$

$$
\dfrac{
\dfrac{\dfrac{\text{Axiom (E6)}}{\mathcal{H} \vdash^{0}_{0} s \in \mathbb{E}_n, \mathbb{E}_n \in \mathbb{E}_{n+1} \Rightarrow s \in \mathbb{E}_{n+1}} \quad \dfrac{\text{Axiom (E4)}}{\mathcal{H} \vdash^{0}_{0} \Rightarrow \mathbb{E}_n \in \mathbb{E}_{n+1}}}{\mathcal{H} \vdash^{1}_{n+3} s \in \mathbb{E}_n \Rightarrow s \in \mathbb{E}_{n+1}}\,(\text{Cut})
\qquad
\dfrac{\text{Axiom (E4)}}{\mathcal{H} \vdash^{0}_{0} \Rightarrow \mathbb{E}_{n+1} \in \mathbb{E}_\omega}
}{
\begin{array}{rc}
 & \mathcal{H} \vdash^{2}_{n+3} s \in \mathbb{E}_n \Rightarrow \mathbb{E}_{n+1} \in \mathbb{E}_\omega \wedge s \in \mathbb{E}_{n+1} \\ \hline
(b\exists R) & \mathcal{H} \vdash^{n+3}_{n+3} s \in \mathbb{E}_n \Rightarrow (\exists z \in \mathbb{E}_\omega)(s \in z) \\ \hline
(\mathbb{E}\text{-Lim}) & \mathcal{H} \vdash^{\omega}_{\omega} s \in \mathbb{E}_\omega \Rightarrow (\exists z \in \mathbb{E}_\omega)(s \in z) \\ \hline
(\rightarrow R) & \mathcal{H} \vdash^{\omega+1}_{\omega} \Rightarrow s \in \mathbb{E}_\omega \rightarrow (\exists z \in \mathbb{E}_\omega)(s \in z) \\ \hline
(b\forall R)_\infty & \mathcal{H} \vdash^{\omega+2}_{\omega} \Rightarrow (\forall y \in \mathbb{E}_\omega)(\exists z \in \mathbb{E}_\omega)(y \in z)
\end{array}
}\,(\wedge R)
$$

Applying $(\wedge R)$ followed by $(b\exists R)$ to the conclusions of the two proof trees above yields the required result. □

Lemma 5.4.6 ($\Delta^{\mathcal{E}}_0$-Separation)**.** For any $\Delta^{\mathcal{E}}_0$ formula $A(a, \bar{b})$ of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables $a, \bar{b} = b_1, ..., b_n$, any $\mathbf{IRS}^{\mathbb{E}}_{\Omega}$ terms $r, s_1, ..., s_n$ and any operator $\mathcal{H}$:

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash^{\alpha+7}_{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow \exists x[(\forall y \in x)(y \in r \wedge A(y, \bar{s})) \wedge (\forall y \in r)(A(y, \bar{s}) \rightarrow y \in x)]$$

where $\alpha = \max(\bar{\beta}, \gamma)$.

Proof. First let

$$p := [x \in r \mid A(x, \bar{s})].$$

As an instance of axiom (E11) we have

$$\mathcal{H}[\gamma, \bar{\beta}] \vdash^{0}_{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow p \in \mathbb{E}_\alpha \,. \tag{1}$$

Moreover we have the following derivations in $\mathbf{IRS}^{\mathbb{E}}_{\Omega}$:

$$
\dfrac{
\begin{array}{rc}
 & \text{Axiom (E9)} \\ \hline
 & \mathcal{H} \vdash^{0}_{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma, t \in p \Rightarrow t \in r \wedge A(t, \bar{s}) \\ \hline
(\rightarrow R) & \mathcal{H} \vdash^{1}_{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow t \in p \rightarrow t \in r \wedge A(t, \bar{s})
\end{array}
\qquad (1)
}{
\mathcal{H} \vdash^{\alpha+2}_{0} \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow (\forall y \in p)(y \in r \wedge A(y, \bar{s}))
}\,(b\forall R)_\infty
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom (E8)}}{\mathcal{H} \mid\!\frac{0}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma, t \in r, A(t,\bar{s}) \Rightarrow t \in p}}{\mathcal{H} \mid\!\frac{1}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma, t \in r \Rightarrow A(t,\bar{s}) \to t \in p}\;(\to R)}{\mathcal{H} \mid\!\frac{2}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow t \in r \to (A(t,\bar{s}) \to t \in p)}\;(\to R)}{\mathcal{H} \mid\!\frac{\gamma+3}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, r \in \mathbb{E}_\gamma \Rightarrow (\forall y \in r)(A(y,\bar{s}) \to y \in p)}\;(b\forall R)_\infty
$$

Now applying $(\wedge R)$ to (1) and the conclusions of the two proof trees above, followed by an application of $(\exists R)$ yields the desired result. □

Lemma 5.4.7 (Pair)**.** For any operator $\mathcal{H}$, and $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms s, t and any ordinals $\beta, \gamma < \Omega$:

$$\mathcal{H}[\beta,\gamma] \mid\!\frac{\alpha+6}{\alpha+2}\; s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \exists z(s \in z \wedge t \in z)$$

where $\alpha := \max(\beta, \gamma)$.

Proof. If $\beta = \gamma$ the proof is straightforward, without loss of generality let us assume $\beta > \gamma$. As instances of axioms (E6) and (E4) we have

(1) $$\mathcal{H}[\beta,\gamma] \mid\!\frac{0}{0}\; t \in \mathbb{E}_\gamma, \mathbb{E}_\gamma \in \mathbb{E}_\beta \Rightarrow t \in \mathbb{E}_\beta$$

(2) $$\mathcal{H}[\beta,\gamma] \mid\!\frac{0}{0}\; \Rightarrow \mathbb{E}_\gamma \in \mathbb{E}_\beta\,.$$

Applying (Cut) gives

(3) $$\mathcal{H}[\beta,\gamma] \mid\!\frac{1}{\beta+2}\; t \in \mathbb{E}_\gamma \Rightarrow t \in \mathbb{E}_\beta\,.$$

By axiom (E1) we have

(4) $$\mathcal{H}[\beta,\gamma] \mid\!\frac{0}{0}\; s \in \mathbb{E}_\beta \Rightarrow s \in \mathbb{E}_\beta\,.$$

Applying $(\wedge R)$ to (3) and (4) provides

(5) $$\mathcal{H}[\beta,\gamma] \mid\!\frac{2}{\beta+2}\; s \in \mathbb{E}_\beta, t \in \mathbb{E}_\beta \Rightarrow s \in \mathbb{E}_\beta \wedge t \in \mathbb{E}_\beta\,,$$

to which we may apply $(\exists R)$ giving

$$\mathcal{H}[\beta,\gamma] \mid\!\frac{\beta+6}{\beta+2}\; s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \exists z(s \in z \wedge t \in z)\,.$$

as required. □

Lemma 5.4.8 (Union)**.** For any operator $\mathcal{H}$, $\mathbf{IRS}_\Omega^{\mathbb{E}}$ term s and any $\beta < \Omega$ we have

$$\mathcal{H}[\beta] \mid\!\frac{\beta+9}{\beta+2}\; s \in \mathbb{E}_\beta \Rightarrow \exists z[(\forall y \in s)(\forall x \in y)(x \in z)]\,.$$

Proof. We have the following template for derivations in $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

$$
\text{(Cut)}\ \dfrac{\dfrac{\text{Axiom (E6)}}{\mathcal{H}[\beta]\ \frac{0}{0}\ t \in \mathbb{E}_\beta, r \in t \Rightarrow r \in \mathbb{E}_\beta} \qquad \dfrac{\text{Axiom (E6)}}{\mathcal{H}[\beta]\ \frac{0}{0}\ s \in \mathbb{E}_\beta, t \in s \Rightarrow t \in \mathbb{E}_\beta}}{\text{(}\exists R\text{)}\ \dfrac{(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{(b\forall R)_\infty\ \dfrac{(\to R)\ \dfrac{\mathcal{H}[\beta]\ \frac{1}{\beta+2}\ s \in \mathbb{E}_\beta, t \in s, r \in t \Rightarrow r \in \mathbb{E}_\beta}{\mathcal{H}[\beta]\ \frac{2}{\beta+2}\ s \in \mathbb{E}_\beta, t \in s \Rightarrow r \in t \to r \in \mathbb{E}_\beta}}{\mathcal{H}[\beta]\ \frac{\beta+3}{\beta+2}\ s \in \mathbb{E}_\beta, t \in s \Rightarrow (\forall x \in t)(x \in \mathbb{E}_\beta)}}{\mathcal{H}[\beta]\ \frac{\beta+4}{\beta+2}\ s \in \mathbb{E}_\beta \Rightarrow t \in s \to (\forall x \in t)(x \in \mathbb{E}_\beta)}}{\mathcal{H}[\beta]\ \frac{\beta+5}{\beta+2}\ s \in \mathbb{E}_\beta \Rightarrow (\forall y \in s)(\forall x \in y)(x \in \mathbb{E}_\beta)}}{\mathcal{H}[\beta]\ \frac{\beta+9}{\beta+2}\ s \in \mathbb{E}_\beta \Rightarrow \exists z(\forall y \in s)(\forall x \in y)(x \in z).}}
$$

□

Lemma 5.4.9 ($\Delta_0^{\mathcal{E}}$-Collection)**.** Let $F(a, b, \bar{c})$ be any $\Delta_0^{\mathcal{E}}$ formula of $\mathbf{IKP}(\mathcal{E})$ containing exactly the free variables displayed then for any $\bar{s} = s_1, ..., s_n$

$$\Vdash_\Omega \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s}) \to \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s}).$$

Proof. Since F is $\Delta_0^{\mathcal{E}}$ we have

$$no_{\bar{\beta}}((\forall x \in s_i)\exists y F(x, y, \bar{s})) = \omega^{\Omega+2}.$$

Hence by Lemma 5.4.2 we have

$$\mathcal{H}[\bar{\beta}]\ \frac{\omega^{\Omega+2}\cdot 2}{\Omega}\ \bar{s} \in \mathbb{E}_{\bar{\beta}}, (\forall x \in s_i)\exists y F(x, y, \bar{s}) \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s})\,.$$

Applying ($\Sigma^{\mathcal{E}}$-Ref) gives

$$\mathcal{H}[\bar{\beta}]\ \frac{\omega^{\Omega+2}\cdot 2+2}{\Omega}\ \bar{s} \in \mathbb{E}_{\bar{\beta}}, (\forall x \in s_i)\exists y F(x, y, \bar{s}) \Rightarrow \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s})\,.$$

By $(\to R)$ we get

$$\mathcal{H}[\bar{\beta}]\ \frac{\omega^{\Omega+2}\cdot 2+3}{\Omega}\ \bar{s} \in \mathbb{E}_{\bar{\beta}} \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s}) \to \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s})\,.$$

Finally since $\omega^{\Omega+2}\cdot 2 + 3 < \omega^{\Omega+3}$ we may conclude

$$\Vdash_\Omega \Rightarrow (\forall x \in s_i)\exists y F(x, y, \bar{s}) \to \exists z(\forall x \in s_i)(\exists y \in z)F(x, y, \bar{s})$$

as required. □

Lemma 5.4.10 (Exponentiation)**.** For any terms s, t any $\beta, \gamma < \Omega$ and any operator $\mathcal{H}$

$$\mathcal{H}[\beta, \gamma]\ \frac{\delta+4}{\delta+3}\ s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \exists z(\forall x \in {}^s t)(x \in z)$$

where $\delta := \max(\beta, \gamma) + 2$.

Proof. First let

$$p := [x \in \mathbb{E}_\delta \mid \mathrm{fun}(x, s, t)].$$

As an instance of axiom (E10) we have

$$\text{(1)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{0}{0}} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma, \mathrm{fun}(q, s, t) \Rightarrow q \in \mathbb{E}_\delta \quad \text{for all } q.$$

Also axiom (E8) provides

$$\text{(2)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{0}{0}} q \in \mathbb{E}_\delta, \mathrm{fun}(q, s, t) \Rightarrow q \in p \quad \text{for all } q.$$

Applying (Cut) to (1) and (2) provides

$$\text{(3)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{1}{\delta+2}} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma, \mathrm{fun}(q, s, t) \Rightarrow q \in p \quad \text{for all } q.$$

Now by $(\rightarrow R)$ we have

$$\text{(4)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{2}{\delta+2}} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \mathrm{fun}(q, s, t) \rightarrow q \in p \quad \text{for all } q.$$

Thus we may use $(\mathcal{E}b\forall R)_\infty$ giving

$$\text{(5)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{\delta+1}{\delta+2}} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow (\forall x \in {}^s t)(x \in p) \quad \text{for all } q.$$

As instances of axioms (E11) and (E4) we also have

$$\text{(6)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{0}{0}} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma, \mathbb{E}_\delta \in \mathbb{E}_{\delta+1} \Rightarrow p \in \mathbb{E}_{\delta+1}$$

$$\text{(7)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{0}{0}} \Rightarrow \mathbb{E}_\delta \in \mathbb{E}_{\delta+1}.$$

We may apply (Cut) to (6) and (7) to obtain

$$\text{(8)} \qquad \mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{1}{\delta+3}} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow p \in \mathbb{E}_{\delta+1}.$$

Finally by applying $(\exists R)$ to (5) and (8) we get

$$\mathcal{H}[\beta, \gamma] \mathrel{|\!\!\frac{\delta+4}{\delta+3}} s \in \mathbb{E}_\beta, t \in \mathbb{E}_\gamma \Rightarrow \exists z(\forall x \in {}^s t)(x \in z)$$

as required. □

Theorem 5.4.11. If $\mathbf{IKP}(\mathcal{E}) \vdash \Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$ with $\bar{a}$ the only free variables occurring in the intuitionistic sequent $\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$. Then there is a $k < \omega$ such that for any $\mathbf{IRS}^{\mathbb{E}}_\Omega$ terms $\bar{s}$, any $\bar{\beta} < \Omega$ and any operator $\mathcal{H}$

$$\mathcal{H}[\bar{\beta}] \mathrel{|\!\!\frac{\Omega\cdot\omega^k}{\Omega+k}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,.$$

Proof. The proof is by induction on the $\mathbf{IKP}(\mathcal{E})$ derivation. If $\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]$ is an axiom of $\mathbf{IKP}(\mathcal{E})$ then the result follows by one of lemmas 5.4.2, 5.4.3, 5.4.4, 5.4.5, 5.4.6, 5.4.7, 5.4.8, 5.4.9 and 5.4.10.

Case 1. Suppose the last inference was $(\mathcal{E}b\exists L)$, then $(\exists x \in {}^{a_i}a_j)F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(\mathcal{E}b\exists L)\ \frac{\Gamma[\bar{a}], \mathrm{fun}(b, a_i, a_j) \wedge F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]}$$

where b does not occur in $\bar{a}$. By the induction hypothesis we have a k_0 such that

$$(1) \qquad \mathcal{H}[\bar{\beta}, \gamma] \,\Big|\!\frac{\Omega\cdot\omega^{k_0}}{\Omega+k_0}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_{\gamma}, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j) \wedge F(p) \Rightarrow \Delta[\bar{s}]$$

for all p and all $\gamma < \Omega$. Let us choose the special case of (1) where $\gamma := \max(\beta_i, \beta_j) + 2$ and note that for this choice of γ, $\mathcal{H}[\bar{\beta}, \gamma] = \mathcal{H}[\bar{\beta}]$. Now $\mathrm{fun}(p, s_i, s_j) \Rightarrow \mathrm{fun}(p, s_i, s_j)$ is an axiom due to (E1) and by Lemma 5.4.2 we have $\Vdash_\Omega F(p) \Rightarrow F(p)$ so applying $(\wedge R)$ gives

$$(2) \qquad \Vdash_\Omega \mathrm{fun}(p, s_i, s_j), F(p) \Rightarrow \mathrm{fun}(p, s_i, s_j) \wedge F(p).$$

Applying (Cut) to (1) and (2) provides

$$(3) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_1}}{\Omega+k_1}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_{\gamma}, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j), F(p) \Rightarrow \Delta[\bar{s}]\,.$$

Now as an instance of axiom (E10) we have

$$(4) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{0}{0}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(p, s_i, s_j) \Rightarrow p \in \mathbb{E}_{\gamma}\,.$$

So (Cut) to (3) and (4) gives

$$(5) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_1}+1}{\Omega+k_1}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j), F(p) \Rightarrow \Delta[\bar{s}]\,.$$

To which we may apply $(\wedge L)$ twice followed by $(\mathcal{E}b\exists L)_\infty$ to complete the case.

Case 2. Suppose the last inference was $(\mathcal{E}b\exists R)$ then $\Delta[\bar{a}] = \{(\exists x \in {}^{a_i}a_j)F(x)\}$ and the final inference looks like

$$(\mathcal{E}b\exists R)\ \frac{\Gamma[\bar{a}] \Rightarrow \mathrm{fun}(b, a_i, a_j) \wedge F(b)}{\Gamma[\bar{a}] \Rightarrow (\exists x \in {}^{a_i}a_j)F(x)}$$

Suppose b is a member of $\bar{a}$, without loss of generality let us suppose that $b \equiv a_1$, so by the induction hypothesis we have a $k_0 < \omega$ such that

$$(8) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_0}}{\Omega+k_0}\, \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \mathrm{fun}(s_1, s_i, s_j) \wedge F(s_1).$$

If b is not a member of $\bar{a}$ we can also conclude (8) by the induction hypothesis. As an instance of axiom (E1) we have $\mathrm{fun}(s_1, s_i, s_j) \Rightarrow \mathrm{fun}(s_1, s_i, s_j)$ to which we may apply $(\wedge L)$ giving

$$(9) \qquad \mathcal{H}[\bar{\beta}] \,\Big|\!\frac{1}{0}\, \mathrm{fun}(s_1, s_i, s_j) \wedge F(s_1) \Rightarrow \mathrm{fun}(s_1, s_i, s_j)\,.$$

Now applying (Cut) to (8) and (9) yields

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_0}+1}{\Omega+k_0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \mathrm{fun}(s_1, s_i, s_j) \tag{10}$$

Axiom (E10) gives us

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{0}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(s_1, s_i, s_j) \Rightarrow s_1 \in \mathbb{E}_\delta \quad \text{where } \delta := \max(\beta_i, \beta_j) + 2. \tag{11}$$

So applying (Cut) to (10) and (11) gives

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_0}+1}{\Omega+k_0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow s_1 \in \mathbb{E}_\delta. \tag{12}$$

Finally we may apply $(\mathcal{E}b\exists R)$ to (8) and (12) to complete this case.

Case 3. Now suppose the last inference was $(\mathcal{E}b\forall L)$, so $(\forall x \in {}^{a_i}a_j)F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(\mathcal{E}b\forall L)\ \frac{\Gamma[\bar{a}], \mathrm{fun}(b, a_i, a_j) \to F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}].}$$

If b is present in $\bar{a}$, without loss of generality let us suppose $b \equiv a_1$, regardless of whether b is present in $\bar{a}$, by the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{\Omega\cdot\omega^{k_0}}{\Omega+k_0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_\gamma, \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j) \to F(p) \Rightarrow \Delta[\bar{s}]\,. \tag{13}$$

The problem here is that β_1 may be greater than $\max(\beta_i, \beta_j)+2$ meaning we cannot immediately apply $(\mathcal{E}b\forall L)$, moreover unlike in case 2 it is not possible to derive $\bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}] \Rightarrow \mathrm{fun}(s_1, s_i, s_j)$. Instead we verify the following claim:

$$\Vdash_\Omega \Gamma[\bar{s}], (\forall x \in {}^{s_i}s_j)F(x) \Rightarrow \mathrm{fun}(s_1, s_i, s_j) \to F(s_1) \tag{*}$$

To prove the claim we first note that as an instance of axiom (E10) we have

$$\mathcal{H}[\bar{\beta}] \,\Big|\!\frac{0}{0}\; \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathrm{fun}(s_1, s_i, s_j) \Rightarrow s_1 \in \mathbb{E}_\delta \quad \text{where } \delta := \max(\beta_i, \beta_j) + 2. \tag{14}$$

Then we have the following template for derivations in $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{(E1)}}{\Vdash \mathrm{fun}(s_1, s_i, s_j) \Rightarrow \mathrm{fun}(s_1, s_i, s_j)} \qquad \dfrac{\text{Lemma 5.4.2}}{\Vdash_\Omega F(s_1) \Rightarrow F(s_1)}}{\Vdash_\Omega \mathrm{fun}(s_1, s_i, s_j) \to F(s_1), \mathrm{fun}(s_1, s_i, s_j) \Rightarrow F(s_1)}\,(\to L) \qquad (14)}{\Vdash_\Omega (\forall x \in {}^{s_i}s_j)F(x), \mathrm{fun}(s_1, s_i, s_j) \Rightarrow F(s_1)}\,(\mathcal{E}b\forall L)}{\Vdash_\Omega (\forall x \in {}^{s_i}s_j)F(x) \Rightarrow \mathrm{fun}(s_1, s_i, s_j) \to F(s_1)}\,(\to R)$$

Thus the claim is verified. Now we may complete the case by applying (Cut) to (13) and (*).

Case 4. Now suppose the last inference was $(b\forall L)$, so $(\forall x \in a_i)F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(b\forall L)\ \frac{\Gamma[\bar{a}], b \in a_i \to F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}].}$$

If b does occur in $\bar{a}$, without loss of generality we may assume $b \equiv a_1$. Regardless of whether b is present in $\bar{a}$, by the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}] \mathrel{\Big|\!\!\frac{\Omega\cdot\omega^{k_0}}{\Omega+k_0}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}], s_1 \in s_i \to F(s_1) \Rightarrow \Delta[\bar{s}]. \tag{15}$$

Claim:

$$\Vdash_\Omega (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1). \tag{**}$$

To prove the claim we first note that by axiom (E6) we have

$$\mathcal{H}[\bar{\beta}] \mathrel{\Big|\!\!\frac{0}{0}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, s_1 \in s_i \Rightarrow s_1 \in \mathbb{E}_{\beta_i} \tag{16}$$

Then we have the following template for derivations in $\mathbf{IRS}_\Omega^{\mathbb{E}}$.

$$\cfrac{\cfrac{\cfrac{\cfrac{\text{(E1)}}{\Vdash s_1 \in s_j \Rightarrow s_1 \in s_j} \qquad \cfrac{\text{Lemma 5.4.2}}{\Vdash_\Omega F(s_1) \Rightarrow F(s_1)}}{\Vdash_\Omega s_1 \in s_j \to F(s_1), s_1 \in s_j \Rightarrow F(s_1)}\,(\to L) \qquad (16)}{\Vdash_\Omega (\forall x \in s_i)F(x), s_1 \in s_j \Rightarrow F(s_1)}\,(b\forall L)}{\Vdash_\Omega (\forall x \in s_i)F(x) \Rightarrow s_1 \in s_i \to F(s_1)}\,(\to R)$$

Finally we may apply (Cut) to (15) and (**) to complete this case.

Case 5. Now suppose the last inference was $(\forall L)$, so $\forall x F(x) \in \Gamma[\bar{a}]$ and the final inference looks like

$$(\forall L)\ \frac{\Gamma[\bar{a}], F(b) \Rightarrow \Delta[\bar{a}]}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}].}$$

If b is a member of $\bar{a}$, without loss of generality let us assume $b \equiv a_1$. By the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}] \mathrel{\Big|\!\!\frac{\Omega\cdot\omega^{k_0}+1}{\Omega+k_0}} \bar{s} \in \mathbb{E}_{\bar{\beta}}, \Gamma[\bar{s}], F(s_1) \Rightarrow \Delta[\bar{s}]. \tag{19}$$

If b is not a member of $\bar{a}$ we can in fact still conclude (19) from the induction hypothesis. Now as an instance of axiom (E1) we have

$$\mathcal{H}[\bar{\beta}] \mathrel{\Big|\!\!\frac{0}{0}} \bar{s} \in \mathbb{E}_{\bar{\beta}} \Rightarrow s_1 \in \mathbb{E}_{\beta_1}\ . \tag{20}$$

So applying $(\forall L)$ gives the desired result.

Case 6. Now suppose the last inference was $(\forall R)$, then $\{\forall x F(x)\} \equiv \Delta[\bar{a}]$ and the final inference looks like

$$(\forall L)\ \frac{\Gamma[\bar{a}] \Rightarrow F(b)}{\Gamma[\bar{a}] \Rightarrow \forall x F(x)}$$

with b not present in $\bar{a}$. By the induction hypothesis we have a $k_0 < \omega$ such that

$$\mathcal{H}[\bar{\beta}, \gamma] \mid\!\frac{\Omega\cdot\omega^{k_0}}{\Omega+k_0}\ \bar{s} \in \mathbb{E}_{\bar{\beta}}, p \in \mathbb{E}_{\gamma}, \Gamma[\bar{s}] \Rightarrow F(p)$$

for all p and all $\gamma < \Omega$. Applying $(\forall R)_\infty$ gives the desired result.

Case 7. Suppose the last inference was (Cut) then the derivation looks like

$$\frac{\Gamma[\bar{a}], B(\bar{a}, \bar{b}) \Rightarrow \Delta[\bar{a}] \qquad \Gamma[\bar{a}] \Rightarrow B(\bar{a}, \bar{b})}{\Gamma[\bar{a}] \Rightarrow \Delta[\bar{a}]}$$

where each member of $\bar{b}$ is distinct from the members of $\bar{a}$. By the induction hypothesis we get $k_0, k_1 \in \omega$ such that

$$\mathcal{H}[\bar{\beta}] \mid\!\frac{\Omega\cdot\omega^{k_0}}{\Omega+k_0}\ \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathbb{E}_0 \in \mathbb{E}_1, \Gamma[\bar{s}], B(\bar{s}, \bar{\mathbb{E}_0}) \Rightarrow \Delta[\bar{s}] \tag{21}$$

$$\mathcal{H}[\bar{\beta}] \mid\!\frac{\Omega\cdot\omega^{k_1}}{\Omega+k_1}\ \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathbb{E}_0 \in \mathbb{E}_1, \Gamma[\bar{s}] \Rightarrow B(\bar{s}, \bar{\mathbb{E}_0})\,. \tag{22}$$

Now since $\Rightarrow \mathbb{E}_0 \in \mathbb{E}_1$ is an instance of axiom (E4) and $\bar{s} \in \mathbb{E}_{\bar{\beta}} \Rightarrow s_i \in \mathbb{E}_{\beta_i}$ is an instance of axiom (E1) we may apply (Cut) to (21) and (22) giving

$$\mathcal{H}[\bar{\beta}] \mid\!\frac{\Omega\cdot\omega^{k}}{\Omega+k}\ \bar{s} \in \mathbb{E}_{\bar{\beta}}, \mathbb{E}_0 \in \mathbb{E}_1, \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]\,. \tag{23}$$

Finally applying (Cut) to (23) and $\mathcal{H}[\bar{\beta}] \mid\!\frac{0}{0}\ \mathbb{E}_0 \in \mathbb{E}_1$ we can complete this case.

All other cases can be treated in a similar manner to one of those above. □

5.5 A relativised ordinal analysis of IKP($\mathcal{E}$)

Analogously to with $\mathbf{IRS}_\Omega^{\mathcal{P}}$ we will prove a soundness theorem for certain $\mathbf{IRS}_\Omega^{\mathbb{E}}$ derivable sequents in $E_{\psi_\Omega(\varepsilon_{\Omega+1})}$. Again we need the notion of an assignment. Let $VAR_{\mathcal{E}}$ be the set of free variables of $\mathbf{IRS}_\Omega^{\mathbb{E}}$, an assignment is a map

$$v : VAR_{\mathcal{E}} \longrightarrow E_{\psi_\Omega(\varepsilon_{\Omega+1})}$$

such that $v(a_i^\alpha) \in E_{\alpha+1}$ for all $i < \omega$ and ordinals α. Again an assignment canonically lifts to all $\mathbf{IRS}_\Omega^{\mathbb{E}}$ terms by setting

$$\begin{aligned} v(\mathbb{E}_\alpha) &= E_\alpha \\ v([x \in t \mid F(x, s_1, ..., s_n)]) &= \{x \in v(t) \mid F(x, v(s_1), ..., v(s_n))\}. \end{aligned}$$

The difference between here and the case of $\mathbf{IRS}_\Omega^{\mathcal{P}}$ is that for a given term t, it is no longer possible to ascertain the location of $v(t)$ within the E-hierarchy solely by looking at the syntactic structure of t. It is however possible to place an upper bound on that location using the following function

$$m(\mathbb{E}_\alpha) := \alpha$$
$$m(a_i^\alpha) := \alpha$$
$$m([x \in t \mid F(x, s_1, ..., s_n)]) := \max(m(t), m(s_1), ..., m(s_n)) + 1.$$

It can be observed that $v(s) \in E_{m(s)+1}$ for any s, however in general $m(s)$ is only an upper bound on a terms position in the E-hierarchy.

Theorem 5.5.1 (Soundness for $\mathbf{IRS}_\Omega^{\mathbb{E}}$)**.** Suppose $\Gamma[s_1, ..., s_n]$ is a finite set of $\Pi^{\mathcal{E}}$ formulae with $\max\{rk(A) \mid A \in \Gamma\} \leq \Omega$, $\Delta[s_1, ..., s_n]$ a set containing at most one $\Sigma^{\mathcal{E}}$ formula and

$$\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}] \quad \text{for some operator } \mathcal{H} \text{ and some } \alpha, \rho < \Omega.$$

Then for any assignment v,

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(s_1), ..., v(s_n)] \to \bigvee \Delta[v(s_1), ..., v(s_n)].$$

Where $\bigwedge \Gamma$ and $\bigvee \Delta$ stand for the conjunction of formulae in Γ and the disjunction of formulae in Δ respectively, by convention $\bigwedge \emptyset := \top$ and $\bigvee \emptyset := \bot$.

Proof. The proof is by induction on α. Note that the derivation $\mathcal{H} \mathrel{\substack{\alpha \\ \vdash \\ \rho}} \Gamma[\bar{s}] \Rightarrow \Delta[\bar{s}]$ contains no inferences of the form $(\forall R)_\infty$, $(\exists L)_\infty$ or $(\Sigma^{\mathcal{E}}\text{-}Ref)$ and all cuts have $\Delta_0^{\mathcal{E}}$ cut formulae.

All axioms apart from (E6) and (E7) are clearly sound under the interpretation, the soundness of (E6) and (E7) follows from Lemma 5.2.1.

Now suppose the last inference was $(\mathcal{E}b\exists R)$, so amongst other premises we have

$$\mathcal{H} \mathrel{\substack{\alpha_0 \\ \vdash \\ \rho}} \Gamma[\bar{s}] \Rightarrow \text{fun}(t, s_i, s_j) \wedge A(t, \bar{s}) \quad \text{for some } \alpha_0 < \alpha.$$

Applying the induction hypothesis yields

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \to [\text{fun}(v(t), v(s_i), v(s_j)) \wedge A(v(t), \bar{s})] \quad \text{where } v(\bar{s}) := v(s_1), ..., v(s_n).$$

Suppose $\Gamma[v(\bar{s})]$ holds in $E_{\psi_\Omega(\varepsilon_{\Omega+1})}$, so we have

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \text{fun}(v(t), v(s_i), v(s_j)) \wedge A(v(t), v(\bar{s})).$$

It remains to note that the function space ${}^{v(s_i)}v(s_j)$ is a member of $E_{\psi_\Omega(\varepsilon_{\Omega+1})}$ and thus

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models (\exists x \in {}^{v(s_i)}v(s_j))A(x, v(\bar{s})).$$

as required.

Now suppose the last inference was $(\mathcal{E}b\exists L)_\infty$, thus amongst other premises we have

$$(2) \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma[\bar{s}], \mathrm{fun}(p, s_i, s_j) \wedge A(p, \bar{s}) \Rightarrow \Delta[\bar{s}] \quad \text{for all terms } p \text{ and some } \alpha_0 < \alpha.$$

For the remainder of this case fix an arbitrary valuation v_0. Let $\beta_0 := m(s_i)$, $\beta_1 := m(s_j)$ and $\beta := \max(\beta_0, \beta_1) + 2$. Choose k such that a_k^β does not occur in any of the terms in $\bar{s}$. As a special case of (2) we have

$$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma[\bar{s}], \mathrm{fun}(a_k^\beta, s_i, s_j) \wedge A(a_k^\beta, \bar{s}) \Rightarrow \Delta[\bar{s}]\,.$$

Applying the induction hypothesis we get

$$(3) \qquad E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v(\bar{s})] \wedge [\mathrm{fun}(v(a_k^\beta), v(s_i), v(s_j)) \wedge A(v(a_k^\beta), v(\bar{s}))] \rightarrow \bigvee \Delta[v(\bar{s})]$$

for all valuations v. In particular (3) holds true for all valuations v which coincide with v_0 on $\bar{s}$. By the choice of a_k^β it follows that

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models \bigwedge \Gamma[v_0(\bar{s})] \rightarrow \bigvee \Delta[v_0(\bar{s})]$$

as required.

All other cases may be treated in a similar manner to those above, using similar reasoning to Theorem 4.5.1. $\square$

Lemma 5.5.2. Suppose $\mathbf{IKP}(\mathcal{E}) \vdash \Rightarrow A$ for some $\Sigma^{\mathcal{E}}$ sentence A, then there exists an $n < \omega$, which we may compute from the derivation, such that

$$\mathcal{H}_\sigma \overset{\psi_\Omega(\sigma)}{\underset{\psi_\Omega(\sigma)}{\vdash}} \Rightarrow A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Proof. Suppose $\mathbf{IKP}(\mathcal{E}) \vdash \Rightarrow A$, then by Theorem 5.4.11 we can explicitly calculate some $1 \leq m < \omega$ such that

$$\mathcal{H}_0 \overset{\Omega\cdot\omega^m}{\underset{\Omega+m}{\vdash}} \Rightarrow A$$

Applying partial cut elimination for $\mathbf{IRS}_\Omega^{\mathbb{E}}$ 5.3.3 we get

$$\mathcal{H}_0 \overset{\omega_{m-1}(\Omega\cdot\omega^m)}{\underset{\Omega+1}{\vdash}} \Rightarrow A\,.$$

Finally by applying collapsing for $\mathbf{IRS}_\Omega^{\mathbb{E}}$ 5.3.6 we get

$$\mathcal{H}_{\omega_m(\Omega\cdot\omega^m)} \overset{\psi_\Omega(\omega_m(\Omega\cdot\omega^m))}{\underset{\psi_\Omega(\omega_m(\Omega\cdot\omega^m))}{\vdash}} \Rightarrow A$$

as required. $\square$

Theorem 5.5.3. If A is a $\Sigma^{\mathcal{E}}$-sentence and $\mathbf{IKP}(\mathcal{E}) \vdash \Rightarrow A$ then there is an ordinal term $\alpha < \psi_\Omega(\varepsilon_{\Omega+1})$, which we may compute from the derivation, such that

$$E_\alpha \models A.$$

Proof. By Lemma 5.5.2 we can determine some $m < \omega$ such that

$$\mathcal{H}_\sigma \,\Big|\frac{\psi_\Omega(\sigma)}{\psi_\Omega(\sigma)}\, \Rightarrow A \quad \text{where } \sigma := \omega_m(\Omega \cdot \omega^m).$$

Let $\alpha := \psi_\Omega(\sigma)$. Applying boundedness 5.3.5 we get

$$\mathcal{H} \,\Big|\frac{\alpha}{\alpha}\, \Rightarrow A^{\mathbb{E}_\alpha}.$$

Now Theorem 5.5.1 yields

$$E_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A^{\mathbb{E}_\alpha}.$$

It follows that

$$E_\alpha \models A$$

as required. $\square$

Remark 5.5.4. Suppose $A \equiv \exists x C(x)$ is a $\Sigma^{\mathcal{E}}$ sentence and $\mathbf{IKP}(\mathcal{E}) \vdash \Rightarrow A$. As in the case of $\mathbf{IKP}(\mathcal{P})$, as well as the ordinal term α given by Theorem 5.5.3, it is possible to compute a specific $\mathbf{IRS}_\Omega^{\mathbb{E}}$ term s such that $E_\alpha \models C(s)$. Moreover this process can be carried out inside $\mathbf{IKP}(\mathcal{E})$. These results will be verified in [43].

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