HARMONIC VECTOR FIELDS ON RIEMANNIAN MANIFOLDS

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Abstract. This dissertation investigates harmonic vector fields which are special mappings on Riemannian manifolds with many interesting properties. It aims for a sharp definition of these fields through a focus on several aspects of geometry. Key concepts include the Weitzenböck formula, the divergence theorem, the Euler-Lagrange equation and the Sasaki metric. This particular metric contains horizontal and vertical components which are used to define vertical energy and this, in turn, leads to a definition of harmonic vector fields which are later generalised by the Cheeger-Gromoll metric and the general definition of a harmonic vector field. The dissertation also concentrates on some specific examples of harmonic vector fields such as harmonic unit vector fields, the Hopf vector field, conformal gradient fields on the unit sphere and on the hyperbolic space. The key outcome of this research, presented in the concluding subsections of the dissertation, is the discovery of two new examples that give fresh insight into this important aspect of differential geometry.

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Author's declaration. I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

1. Introduction

This dissertation investigates some special mappings on Riemannian manifolds called harmonic vector fields which have many interesting properties. It first aims to describe a definition for these harmonic vector fields. It then focuses on several examples of harmonic vector fields, such as harmonic unit vector fields, the Hopf vector field and conformal gradient fields on the unit sphere and on the hyperbolic space. It concludes with two new examples of harmonic vector fields from original research carried out for this dissertation throughout 2013 and 2014.

The sections of this dissertation introduce and then expand on several topics related to harmonic vector fields, leading to some new contributions to this area of study. The topics can be summarised as follows.

Section 2 addresses some definitions from topology, metric spaces and differentiable manifolds which are necessary for the rest of this dissertation. These include a manifold, a vector field, a tangent vector, and a metric. Section 3 introduces some basic ideas of Riemannian geometry and vector bundles including tensors and many different types of derivatives. The Weitzenböck formula is proved in a specific case with a corollary afterwards that will be used again in the final section. Section 4 introduces the divergence theorem and more concepts required for the next section. Section 5 defines the energy of a mapping, the Dirichlet property and a harmonic map. Section 6 splits the energy into horizontal and vertical components by using a special type of metric called the Sasaki metric which is calculated by adding its horizontal and vertical components together. The vertical component of the energy, called the vertical energy, is then used for a definition of a harmonic vector field that will be generalised later.

Section 7 is about a specific case of harmonic vector fields called harmonic unit vector fields and Section 8 continues this topic with an investigation of a specific harmonic unit vector field called the Hopf vector field with a proof that it is a harmonic unit vector field. Section 9 studies properties of covariant derivatives, defines the flow of a vector field on a Riemannian manifold and a type of vector field, called a Killing field, that has an interesting identity between the covariant derivatives of any two tangent vectors on the Killing field. Section 10 introduces a 2-parameter family of metrics, which includes the Sasaki metric, called the $h_{p,q}$ metrics, also known as the generalised Cheeger-Gromoll metrics, and applies them to energy and harmonicity for definitions of (p, q)-energy, (p, q)-harmonicity and the general definition of a harmonic vector field.

The last section, section 11, has five subsections about harmonic vector fields on Riemannian space forms. Subsection 11.1 introduces conformal vector fields and gradient fields then considers a proposition about the flow of a vector field in relation to conformality. Subsections 11.2 and 11.3 include two known cases of (p, q)-harmonic vector fields, which are conformal gradient fields on the unit sphere and the hyperbolic space respectively, with proof that they are conformal and (p, q)-harmonic. Results from previous work by M. Benyounes, E. Loubeau and C. Wood are reviewed and expanded in several different ways.

Subsections 11.4 and 11.5 continue the research into harmonic vector fields by including two new cases of harmonic vector fields with proof that they are conformal and (p, q)-harmonic. This original work results from the wide research done for this paper. The new cases are conformal gradient fields on the Euclidean space and conformal extension fields of conformal gradient fields on the unit sphere.

The evolution of the geometry leading to the topic of harmonic vector fields began with J. Eells and J. H. Sampson's paper called *Harmonic mappings of Riemannian manifolds* published in 1964. Eells then wrote one of the earliest books about harmonic maps which was published in 1980, *Selected Topics in Harmonic Maps* with L. Lemaire. This has been followed by various papers and books on differential geometry and harmonic maps written by mathematicians currently researching into this area such as O. Gil-Medrano who wrote *Unit vector fields that are critical points of the volume and of the energy: characterization and examples* in 2005 and M. Benyounes, E. Loubeau and C. Wood who together wrote the papers *Harmonic sections of Riemannian vector bundles, and metrics of Cheeger-Gromoll type* and *Harmonic vector fields on space forms* in 2007 and 2014 respectively.

It is hoped that this dissertation has produced a further contribution to the evolving topic of harmonic vector fields. It has been presented in two ways. Firstly, by defining the equation of a conformal gradient field on the n-dimensional Euclidean space then proving that it is conformal and a gradient field and that it is a harmonic vector field. Secondly, by defining the equation for a conformal extension of conformal gradient fields on the unit sphere then proving that it is conformal and a gradient field and that it is also a harmonic vector field.

2. Recollection of topology, metric spaces and manifolds

This section introduces some important definitions from the areas of differential geometry, metric spaces and topology needed for this dissertation starting with the definition of a topological space which will be used to define a manifold.

Definition 2.1. [16] A *topological space* $T = (X, \mathcal{T})$ consists of a non-empty set X together with a fixed family $\mathcal{T}$ of subsets of X satisfying:
(T1) $X, \emptyset \in \mathcal{T}$,
(T2) the intersection of any two sets in $\mathcal{T}$ is in $\mathcal{T}$,

(T3) The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

An element of $\mathcal{T}$ is called an *open set* of T.

Definition 2.2. [11] Let X be a topological space. A *neighbourhood* $U \subseteq X$ of a point $x \in X$ is any open set which contains the point x.

Definition 2.3. [4] A topological space is called a *Hausdorff topological space* if any two points have non-intersecting neighbourhoods.

Definition 2.4. [16] An *open cover* of an open subset A of a topological space X is a collection of open subsets whose union contains A.

Definition 2.5. [4] An *n-dimensional manifold* M is a Hausdorff topological space that can be locally be identified with the n-dimensional Euclidean space $\mathbb{R}^n$. This means that it can be covered by neighbourhoods which map into open neighbourhoods of $\mathbb{R}^n$. Such a map is called a *chart* or *coordinate system.*

Definition 2.6. [3][1] Let M and N be two manifolds and $\pi : N \to M$ be a mapping. A *section of* π is a map $\sigma : M \to N$ with

$$\pi \circ \sigma = \mathrm{id}_M : M \to M$$

where id_M is the *identity of* M. A mapping diagram is shown below

$$M \underset{\pi}{\overset{\sigma}{\rightleftarrows}} N$$

Definition 2.7. [10] Let $U \subset \mathbb{R}^n$ be an open subset of the n-dimensional Euclidean space $\mathbb{R}^n$ and let $V \subset \mathbb{R}^m$ be an open subset of the m-dimensional Euclidean space $\mathbb{R}^m$. A map $f : U \to V$ is *smooth* if f has continuous partial derivatives of all orders.

Definition 2.8. [4] Let M be any manifold and let $U \subset M$. The *tangent vector* X to the curve c at the point $p = c(a)$ is defined as a map from any function $f : U \to \mathbb{R}$ to a number Xf,

$$\begin{aligned} X : f \mapsto Xf &= \left.\frac{d}{dt}\right|_{t=a} f(c(t)) \\ &= \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i} \end{aligned}$$

where the relationship between the i-th coordinates of X and $x \in M$, denoted by X^i and x^i, is

$$X^i = \left.\frac{d}{dt}\right|_{t=a} x^i(c(t))$$

This may be used to define the *exterior derivative* of the function f to be df given by

$$df(X) := X(f)$$

Since a coordinate basis gives $X(f) = \sum_{i=1}^{n} X^i \frac{\partial f}{\partial x^i}$, then in a coordinate basis the exterior derivative of a function must be given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx$$

Definition 2.9. [3] The *tangent space* $T_x M$ *of a manifold* M *at a point* x is the set of all tangent vectors that can be made into a vector space. The *tangent bundle* is the union of all tangent spaces at every point on the manifold M. It is denoted by

$$TM = \bigcup_{x \in M} T_x M$$

Definition 2.10. A *vector field* σ on an n-dimensional manifold M is a smooth map $\sigma : M \to TM$ such that $\sigma(x) \in T_x M$ for all $x \in M$.

Definition 2.11. [16][13][15] The *distance* between the points $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, denoted by $d(a, b)$, is defined by the the equation

$$d(a, b) = \|b - a\|, \text{ for all } a, b \in \mathbb{R}^n$$

Then for any $r > 0$ let $S_r^{n-1}(a)$ denote the $(n-1)$-*dimensional sphere* of radius r centred at a

$$S_r^{n-1}(a) = \{x \in \mathbb{R}^n : d(x, a) = r\}$$

Let $B_r(a)$ be the *open ball*

$$B_r(a) = \{x \in \mathbb{R}^n : d(x, a) < r\}$$

and let $D_r(a)$ be the *closed ball*

$$D_r(a) = \{x \in \mathbb{R}^n : d(x, a) \leq r\}.$$

If $r = 1$ and $a = 0$ then S_r^{n-1} is the $(n-1)$-*dimensional unit sphere* denoted by S^{n-1}.

An example of a vector field is shown in Figure 2.1 by the vector field $w(x)$ on the 1-dimensional sphere S^1 where x varies as points a, b and c.

Definition 2.12. [16] A subset $S \subset \mathbb{R}^n$ is *bounded* if there exists an $r > 0$ such that $S \subset D_r(0)$. A subset $K \subset \mathbb{R}^n$ is *compact* if K is bounded and closed.

Definition 2.13. [4] A *Riemannian metric of a manifold* M, also known as the *first fundamental form*, is the inner product on the tangent bundle TM of M denoted by $\langle X, Y \rangle$ or $X \cdot Y$ for all $X, Y \in TM$. A manifold equipped with this metric is called a *Riemannian manifold*.

Definition 2.14. [15] An *n-dimensional vector bundle*, denoted by ξ, is a combination of five different elements. Two of them are manifolds

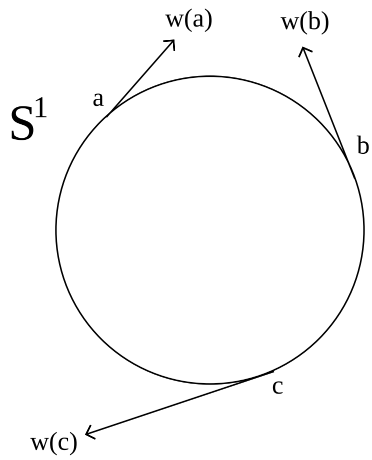


FIGURE 2.1. A vector field w on the unit circle S^1

and three of them are functions. The functions and manifolds are defined as follows

$$\xi = (V, \pi, M, \alpha, \beta)$$

V and M are spaces called the *total space and base space* of ξ respectively, $\pi : V \to M$ is a surjection and α and β are maps

$$\alpha : \bigcup_{p \in M} \pi^{-1}(p) \times \pi^{-1}(p) \to V$$

where p is a point in the base space M and

$$\beta : \mathbb{R} \times V \to V$$

The maps make each inverse image $\pi^{-1}(p) \in V$, known as the *fibre over* p, into an n-dimensional vector space over $\mathbb{R}$ such that the following conditions are true.

$$\alpha\left(\pi^{-1}(p) \times \pi^{-1}(p)\right) \subset \pi^{-1}(p)$$
$$\beta\left(\mathbb{R} \times \pi^{-1}(p)\right) \subset \pi^{-1}(p)$$

For each $p \in M$, there is also a neighbourhood U of p and a homeomorphism $t : \pi^{-1}(U) \to U \times \mathbb{R}^n$.

A vector bundle is sometimes informally referred to by the manifold V or the projection map $\pi : V \to M$.

3. Basic ideas of Riemannian Geometry and vector bundles

Definition 3.1. [5] The *vector space of smooth sections of a vector bundle* $\pi : V \to M$ is defined as the vector space of smooth maps $\sigma : M \to V$ such that each map is a section of π. It is denoted by $\mathcal{C}(V)$.

Definition 3.2. [5] A *linear connection* on a vector bundle $\pi : V \to M$ is a bilinear map ∇ on spaces of sections:

$$\nabla : \mathcal{C}(TM) \times \mathcal{C}(V) \to \mathcal{C}(V)$$

written $\nabla : (X, \sigma) \mapsto \nabla_X \sigma, X \in \mathcal{C}(TM), \sigma \in \mathcal{C}(V)$ and such that for $f \in \mathcal{C}(M)$

$$\begin{aligned} &(i)\ \nabla_{fX}\sigma = f\nabla_X\sigma \\ &(ii)\ \nabla_X(f \cdot \sigma) = Xf \cdot \sigma + f\nabla_X\sigma. \end{aligned} \tag{3.1}$$

$\nabla_X\sigma$ is called the *covariant derivative of σ in the direction of X.*

Definition 3.3. [5] If X and Y are vector fields on M then the vector field $\nabla_X Y - \nabla_Y X$ is called *the Lie bracket of X and Y* and is denoted by

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

Definition 3.4. [5] On the tangent bundle TM, the *torsion of a connection* ∇ is defined by

$$T(X, Y) = -\nabla_X Y + \nabla_Y X + [X, Y]$$

for all $X, Y \in \mathcal{C}(TM)$.

Theorem 3.1 (The Fundamental Theorem of Riemannian Geometry)**.** [5] *If g is a Riemannian metric on TM, the fundamental theorem of Riemannian geometry asserts that there is one and only one connection (the Levi-Civita connection) such that*

$$\nabla g = 0 \text{ and } T = 0$$

Proposition 3.1. [15] *Covariant derivatives satisfy the following four properties on a manifold M. For all tangent vectors $X, Z \in T_xM$ on the tangent space T_xM at a point x, for all vector fields Y on M and for all real numbers $a, b \in \mathbb{R}$*

$$\nabla_{(aX+bZ)}Y = a\nabla_X Y + b\nabla_Z Y \tag{C1}$$

For all vector fields Y, Z on M

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z \tag{C2}$$

For all smooth functions $f : M \to \mathbb{R}$

$$\nabla_X(fY) = f(x)\nabla_X Y + (Xf)Y \tag{C3}$$

For all vector fields Y, Z on M

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{C4}$$

Definition 3.5. [3][14] A *covector* ω at a point x on a manifold M is defined as a linear map of a vector X on T_xM to the 1-dimensional Euclidean space $\mathbb{R}$ denoted by

$$\begin{aligned} \omega : T_xM &\to \mathbb{R} \\ X &\mapsto \omega(X) \end{aligned}$$

The set of all covectors at x is a vector space called the *cotangent space* of T_xM at x denoted by T_x^*M. The *cotangent bundle* is the union of all cotangent spaces at every point on the manifold M. It is denoted by

$$T^*M = \bigcup_{x\in M} T_x^*M$$

Definition 3.6. [3]A *tensor of type* (r, s) is defined as a multilinear map on r covectors and s tangent vectors which can be denoted by

$$\overbrace{T^*M \times ... \times T^*M}^{r \text{ times}} \times \overbrace{TM \times ... \times TM}^{s \text{ times}} \to \mathbb{R}$$

Three examples of tensors are a vector, which is a $(1, 0)$ tensor, a covector, which is a $(0, 1)$ tensor, and a scalar which is a $(0, 0)$ tensor.

Definition 3.7. [5] If $\pi : V \to M$ and $\eta : W \to M$ are two vector bundles the *exterior product* of the total spaces V and W is defined by using the fibres of π and η over all points of M as shown in the following equation

$$V \wedge W = \bigcup_{p\in M} \pi^{-1}(p) \wedge \eta^{-1}(p)$$

The *exterior power* of V, denoted by $\bigwedge^p V$ is the exterior product of V with itself p times. This has the equation

$$\bigwedge^p V = \overbrace{V \wedge ... \wedge V}^{p \text{ times}}$$

Definition 3.8. [10] A *p-form* ω is an alternating tensor of type $(p, 0)$. The set of p-forms can be denoted by $\Omega^p(M)$ which has the equation

$$\Omega^p(M) = \mathcal{C}\left(\bigwedge^p T^*M\right)$$

and the set of p-forms at the point x can be denoted by $\Omega_x^p(M)$ which equals

$$\begin{aligned}
&\Omega_x^p(M) = \bigwedge^p T_x^*M \\
&= \overbrace{T_x^*M \wedge ... \wedge T_x^*M}^{p \text{ times}} \\
&= \left\{ w : \overbrace{T_xM \times ... \times T_xM}^{p \text{ times}} \to \mathbb{R} : \ \omega \text{ is multilinear and alternating} \right\}
\end{aligned}$$

Definition 3.9. Let M be a Riemannian manifold, $x \in M$ and $X, Y, Z \in T_xM$ The second covariant derivative of Z, denoted by $\nabla^2_{X,Y}Z$, is defined by the equation

$$\nabla^2_{X,Y}Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y}Z$$

Definition 3.10. [5][12] The *Riemann tensor* R is a tensor of type $(3,1)$ defined for all $X, Y, Z \in T_xM$ and all $x \in M$ by

$$R(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z \tag{3.2}$$

Recalling Definitions 3.2 and 3.3

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

This is the general definition of R.

Proposition 3.2. [5]*For all $X, Y, Z \in T_xM$ and all $x \in M$ the Riemann tensor satisfies:*

$$R(X,Y)Z = -R(Y,X)Z$$

Proof. Using Equation 3.2

$$\begin{aligned} -R(Y,X)Z &= -(\nabla^2_{Y,X}Z - \nabla^2_{X,Y}Z) \\ &= \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z \\ &= R(X,Y)Z \end{aligned}$$

□

Definition 3.11. The *Ricci tensor at x* is the symmetric bilinear pairing $\text{Ricci} : T_xM \times T_xM \to \mathbb{R}$ defined by

$$\begin{aligned} \text{Ricci}_x(X,Y) &= \sum_{i=1}^{n} g_x(R(X,E_i)E_i, Y) \\ &= \sum_{i=1}^{n} g_x(R(E_i,X)Y, E_i) \end{aligned}$$

where $\{E_i\}_{i=1}^n$ is an orthonormal basis of T_xM.

Definition 3.12. [15] An *endomorphism of a vector space* V is a linear transformation $T : V \to V$. The set of all endomorphisms of V is denoted by $\text{End}(V)$.

Definition 3.13. [5] The *associated Ricci operator $S \in \mathcal{C}(\text{End}(\Omega^p T^*M \times V))$ to a point x and vector-valued p-form field σ* is defined by

$$S_x\sigma(X_1, ..., X_p) = \begin{cases} 0 & \text{if } p = 0 \\ \sum_{k,i}(-1)^k(R(E_i, X_k)\sigma)(E_i, X_1, ..., \hat{X}_k, ..., X_p) & \text{if } p \geq 1 \end{cases}$$

where $\{E_i\}$ is an orthonormal basis of $T_xM, X_k \in \mathcal{C}(TM)$ and $\sigma \in \Omega^p(V)$.

Definition 3.14. [5] The *exterior differential operator* $d : \Omega^p(\xi) \to \Omega^{p+1}(\xi)$ relative to the connection ∇^V is given by

$$\begin{aligned} d\sigma(X_1, .., X_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i+1}\nabla^V_{X_i}(\sigma(X_1, ..., \hat{X}_i, ..., X_{p+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\sigma([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_{p+1}) \end{aligned}$$

Definition 3.15. [5] The *codifferential operator* $d^* : \Omega^p(\xi) \to \Omega^{p-1}(\xi)$ of a p-form $\omega \in \Omega^p(\xi)$ is defined by

$$d^*\omega(X_1, ..., X_{p-1}) = -\sum_i \nabla_{E_i}\omega(E_i, X_1, ..., X_{p-1})$$

Definition 3.16. Let $X = (x_{ij})$ be an $n \times n$ matrix. The *trace of* X is the sum of the diagonal entries of X and is denoted by trace (X). The equation for this is

$$\text{trace}\,(X) = \sum_{i=1}^{n} x_{ii}$$

Definition 3.17. [5] Let ω be a 1-form on the vector bundle V. It is defined to be *metrically dual to the vector field* Z if

$$\omega(Y) = \langle Y, Z\rangle$$

Then the *musical isomorphisms*, denoted by $\flat$ and $\sharp$, are defined such that $\omega = Z^\flat$ and $Z = \omega^\sharp$. Hence

$$g(\omega^\sharp, Y) = \omega(Y) = \left((\omega\,(Y))^\sharp\right)^\flat$$

Definition 3.18. [15] The vector field X metrically dual to the differential df of a smooth mapping on a manifold $f : M \to \mathbb{R}$ is called the *Riemannian gradient vector field of* f and is denoted by ∇f. This can be defined in two ways. Firstly, by using the sharp sign $^\#$ from Definition 3.17, then ∇f is equal to

$$\nabla f = (df)^\#$$

Hence

$$\langle \nabla f, X\rangle = df(X)$$

for all $X \in T_xM$ and all $x \in M$. Secondly, by letting $\{E_i, E_j\}_{i,j=1}^n$ be a basis of T_xM, then the matrix from ∇f is defined as

$$\nabla f = \nabla_{E_i} f(E_j)$$

Lemma 3.1. [5] *Let $\{E_i\}$ be a basis of T_xM, X_j be vectors at x and g^{st} be the inverse of the metric $g(E_s, E_t)$. Then for $\rho \in \Omega^p(\xi)$*

$$(d^*\rho)\,(X_1, ..., X_{p-1}) = -\sum_{s,t} g^{st}\,(\nabla_{E_t}\rho)\,(E_s, X_1, ..., X_{p-1})\,.$$

In particular, if $\rho \in \Omega^1(\xi), d^*\rho = -\,\text{trace}\,\nabla\rho$.

Definition 3.19. [5] The *Hodge-de Rham Laplacian* Δ is defined on V-valued differential forms by

$$\Delta = dd^* + d^*d : \Omega^p(V) \to \Omega^p(V)$$

Definition 3.20. The *rough Laplacian*, $\nabla^*\nabla : \Omega^p(V) \to \Omega^p(V)$ is defined as

$$\nabla^*\nabla\sigma = -\,\text{trace}\,\nabla^2\sigma \text{ for all } \sigma \in \Omega^p(V)$$

Definition 3.21. [15] The *norm* of a vector x in $\mathbb{R}^n$, which is often denoted by both $\|x\|$ or $|x|$, is defined by the square root of the inner product with itself or the square root of the summation of its coordinates squared as shown in the following equations.

$$\begin{aligned}\|x\| = |x| &= \sqrt{\langle x, x\rangle} \\ &= \sqrt{\sum_{i=1}^{n} (x^i)^2}\end{aligned}$$

Definition 3.22. [5] Let $\sigma : M \to N$ be a smooth map and $\{E_i\}$ be an orthonormal basis of T_xM. Its covariant derivative $\nabla_{E_i}\sigma$ can be viewed as a section of the bundle $\mathcal{C}(T^*M \times TM)$, and its *norm* at a point x of M can be denoted by both $\|\nabla\sigma\|$ or $|\nabla\sigma|$. The *Hilbert-Schmidt norm* of $\nabla_{E_i}\sigma$ is defined as

$$\|\nabla\sigma\|^2 = |\nabla\sigma|^2 = \sum_{i=1}^{n} \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle$$

Definition 3.23. [10] The *summation convention* is a convention where, if an index appears in a summation twice, the summation is implicitly summed over without the need to write the sum explicitly.

Theorem 3.2. *(Weitzenböck formula in the* 1*-form case)*[5] *Let* $\xi : V \to M$ *be a Riemannian vector bundle over a Riemannian manifold. Then for any* $\sigma \in \Omega^1(\xi)$

$$\Delta\sigma = -\operatorname{trace}\nabla^2\sigma + S(\sigma)$$

Proof. Let $\{E_i\}$ be an orthonormal basis of T_xM extended to a local orthonormal basis of vector fields such that $\nabla_Y E_i = 0$ for all $Y \in T_xM$. Let the vectors $X, Y \in T_xM$ be extended from the point x to a neighbourhood of x so that $\nabla_{E_i}X = 0$ at the point x. When $\sigma \in \Omega^1(\xi)$, $d^*\sigma$ equals,

$$d^*\sigma = -\sum_i \nabla^V_{E_i}\sigma(E_i)$$

$d\sigma(X, Y)$ equals,

$$\begin{aligned}d\sigma(X, Y) &= \nabla^V_X(\sigma(Y)) - \nabla^V_Y(\sigma(X)) - \sigma([X, Y]) \\ &= X(\sigma(Y)) - Y(\sigma(X)) - \sigma([X, Y]) \\ &= (\nabla_X\sigma)(Y) - (\nabla_Y\sigma)(X)\end{aligned}$$

and $(\nabla_X\sigma)(Y)$ and $-(\nabla_Y\sigma)(X)$ equal

$$\begin{aligned}(\nabla_X\sigma)(Y) &= \nabla^V_X(\sigma(Y)) - \sigma(\nabla^M_X Y) \\ -(\nabla_Y\sigma)(X) &= -\nabla^V_Y(\sigma(X)) + \sigma(\nabla^M_Y X).\end{aligned}$$

Using the summation convention

$$\begin{aligned}
-d(d^*\sigma))(X) &= \nabla_X(\nabla_{E_i}\sigma(E_i)) \\
&= (\nabla_X\nabla_{E_i}\sigma)(E_i) - \nabla_{E_i}\sigma(\underbrace{\nabla_X E_i}_{0}) \\
&= (\nabla_X\nabla_{E_i}\sigma)(E_i)
\end{aligned}$$

The Hodge-de Rham Laplacian on V-valued differential forms is completed by calculating $-(d^*(d\sigma))(X)$.

$$\begin{aligned}
-(d^*(d\sigma))(X) &= \sum_i \nabla^V_{E_i}(d\sigma)(E_i, X) \\
&= \nabla_{E_i}(d\sigma(E_i, X)) - d\sigma(\underbrace{\nabla_{E_i}E_i}_{0}, X) - d\sigma(E_i, \underbrace{\nabla_{E_i}X}_{0}) \\
&= \nabla_{E_i}(d\sigma(E_i, X)) \\
&= \nabla_{E_i}\left((\nabla_{E_i}\sigma)(X) - (\nabla_X\sigma)(E_i)\right) \\
&= (\nabla_{E_i}\nabla_{E_i}\sigma)(X) + (\nabla_{E_i}\sigma)(\underbrace{\nabla_{E_i}X}_{0}) - (\nabla_{E_i}\nabla_X\sigma)(E_i) \\
&\quad + (\nabla_X\sigma)(\underbrace{\nabla_{E_i}E_i}_{0}) \\
&= (\nabla_{E_i}\nabla_{E_i}\sigma)(X) - (\nabla_{E_i}\nabla_X\sigma)(E_i)
\end{aligned}$$

Then $-(\Delta\sigma)(X)$ equals

$$\begin{aligned}
-(\Delta\sigma)(X) &= -(d(d^*\sigma))(X) + (-(d^*(d\sigma))(X)) \\
&= (\nabla_X\nabla_{E_i}\sigma)(E_i) - (\nabla_{E_i}\nabla_X\sigma)(E_i) + (\nabla_{E_i}\nabla_{E_i}\sigma)(X)
\end{aligned}$$

The rough Laplacian, $\nabla^*\nabla$, equals

$$\begin{aligned}
\nabla^*\nabla\sigma &= -\operatorname{trace}\nabla^2\sigma \\
&= -\sum_i \nabla^2_{E_i,E_i}\sigma \\
&= -\left(\nabla^V_{E_i}\nabla^V_{E_i}\sigma - \nabla^V_{\underbrace{\nabla^M_{E_i}E_i}_{0}}\sigma\right) \\
&= -\nabla_{E_i}\nabla_{E_i}\sigma
\end{aligned}$$

By the equation for the Riemann tensor in Definition 3.10

$$
\begin{aligned}
(R(X,E_i)\sigma)(E_i) &= (\nabla_X \nabla_{E_i}\sigma)(E_i) - (\nabla_{E_i}\nabla_X\sigma)(E_i) - \nabla_{[X,E_i]}(E_i) \\
&= (\nabla^2_{X,E_i}\sigma - \nabla^2_{E_i,X}\sigma)(E_i) \\
&= \nabla_X\nabla_{E_i}\sigma(E_i) - \nabla_{\underbrace{\cancel{\nabla_X E_i}}_{0}}\sigma(E_i) \\
&\quad - \nabla_{E_i}\nabla_X\sigma(E_i) + \nabla_{\underbrace{\cancel{\nabla_{E_i} X}}_{0}}\sigma(E_i) \\
&= (\nabla_X\nabla_{E_i}\sigma)(E_i) - (\nabla_{E_i}\nabla_X\sigma)(E_i)
\end{aligned}
$$

In this case when $\sigma \in \Omega^1(\xi)$

$$S\left(\sigma\right)\left(X\right) = \sum_i (R(E_i, X)\sigma)(E_i)$$

Using the summation convention and Proposition 3.2

$$-S\left(\sigma\right)\left(X\right) = (R(X,E_i)\sigma)(E_i)$$

Therefore

$$-(\Delta\sigma)(X) = \underbrace{(\nabla_X\nabla_{E_i}\sigma)(E_i) - (\nabla_{E_i}\nabla_X\sigma)(E_i)}_{(R(X,E_i)\sigma)(E_i) = -S(\sigma)(X)} + \underbrace{\nabla_{E_i}\nabla_{E_i}\sigma(X)}_{-\nabla^*\nabla\sigma = \operatorname{trace}\nabla^2\sigma}$$

In conclusion

$$\Delta\sigma = -\operatorname{trace}\nabla^2\sigma + S(\sigma) \qquad \square$$

Corollary 3.1. [5] *Let $\xi : V \to M$ be a Riemannian vector bundle over a Riemannian manifold. For any $\sigma \in \Omega^0(\xi)$ let F equal*

$$F = \frac{1}{2}|\sigma|^2$$

then ΔF equals

$$\Delta F = \langle \nabla^*\nabla\sigma, \sigma\rangle - |\nabla\sigma|^2$$

This corollary will be used later in Section 11 *Harmonic vector fields on Riemannian space forms.*

4. The divergence theorem

Definition 4.1. Let $\{E_i\}$ be an orthonormal basis of T_xM and V be a vector bundle. If σ is a smooth vector bundle-valued 1-form on M, which is a smooth section of the bundle $T^*M \times V$, then the *covariant coderivative* $\nabla^*\sigma$ is the section of V defined by

$$\nabla^*\sigma = -\sum_{i=1}^{n}(\nabla_{E_i}\sigma)(E_i)$$

Let $\nabla : X \mapsto \nabla X$ be the covariant derivative of X where X is a vector field which is a (1,0)-tensor and where ∇X is a (1,1)-tensor which is a TM-valued 1-form. The mapping diagram between ∇ and ∇^* is

$$\mathcal{C}(TM) \underset{\nabla^*}{\overset{\nabla}{\rightleftarrows}} \mathcal{C}(T^*M \otimes TM)$$

Definition 4.2. [10]A *linear functional on a vector space* V is a linear map $\theta : V \to \mathbb{R}$. The *dual space* V^* *of* V is defined as

$$V^* = \{\text{linear functionals on } V\}$$

Theorem 4.1 (Stokes' Theorem)**.** *Let* M *be a compact* n*-dimensional manifold with a boundary. Therefore the boundary of the manifold* ∂M *is a compact* $(n-1)$ *manifold. If* ω *is an* $(n-1)$*-form on* M

$$\int_{\partial M} \omega = \int_M d\omega$$

Definition 4.3. Let (M^n, g) be an n-dimensional Riemannian manifold and let x^i be a local coordinate system. Then the *volume form associated with* g, denoted by $\text{vol}(g)$, is defined by the equation

$$\text{vol}(g) = dx^1 \wedge ... \wedge dx^n \sqrt{\det(g)}$$

where $\det(g)$ is the determinant of g as a matrix.

Definition 4.4. [5] An alternative way to describe the codifferential operator d^* is to define the *Hodge star operator* $* : \Omega^p(\xi) \to \Omega^{n-p}(\xi^*)$ as the unique linear operator satisfying at each point the relation

$$\begin{aligned} \eta \wedge *\psi &= g(\eta, \psi)\text{vol}(g) \\ &= \langle \eta, \psi \rangle \cdot \text{vol}(g) \end{aligned}$$

for all $\eta, \psi \in \Omega^p(\xi)$. Here $\eta \wedge *\psi$ is the real valued m-form defined using the exterior product on forms and the duality pairing between ξ and ξ^*. Relative to the Riemannian structures of V and TM, the codifferential operator $d^* : \Omega^p(\xi) \to \Omega^{p-1}(\xi)$ is characterised as the adjoint of d via the formula

$$\int_M \langle d\eta, \psi \rangle \text{vol}(g) = \int_M \langle \eta, d^*\psi \rangle \text{vol}(g)$$

Example 4.1. Let $\eta, \phi \in \Omega^p(M)$. A special case of the definition above is when $\psi = f \in \Omega^0(M)$ (i.e. : $M \to \mathbb{R}$) and $\eta = 1$

$$\begin{aligned} \eta \wedge *f &= 1 * f \\ &= *f \\ g(\eta, \psi) &= g(1, f) \\ &= 1 \cdot f \\ &= f \\ \text{So } \; *f &= f \;\, \text{vol}(g) \end{aligned}$$

Definition 4.5. [5] Let ω be a 1-form on a vector bundle $\pi : V \to M$, let $\{E_i\}$ be an orthonormal basis of T_xM and let $W = \omega^\sharp$. The *divergence of a vector field* W on a Riemannian manifold is defined by the following equation

$$\text{div } W = \sum_{i=1}^{n} \langle \nabla_{E_i} W, E_i \rangle$$

Theorem 4.2 (Divergence Theorem)**.** *Let Y be a vector field on M, where M is compact. Then*

$$\int_M \text{div}\, Y \,\text{vol}(g) = 0$$

Proof. Let η be a 1-form on M dual to Y. Then the divergence of Y may be characterised as:

$$\text{div } Y = \delta\eta$$

where δ is the exterior coderivative of (M, g), defined by

$$\delta = *d*$$

where d is the exterior derivative, and $*$ is the Hodge star operator. Abbreviating the volume form vol(g) to ω, the Hodge star operator is characterised by the following equation

$$\phi \wedge *\psi = g(\phi, \psi)\omega$$

for all p-forms ϕ, ψ on M. In particular, if f is a smooth function (0-form) on M then

$$*f = f\omega$$

It follows from this, and the involution formula

$$* * \psi = (-1)^{p(n-p)}\psi$$

that

$$(\text{div } Y)\,\omega = *\,(\text{div } Y) = *\delta\eta = * * d * \eta = d\,(*\eta)$$

Now, by Stokes' Theorem

$$\int_M (\text{div } Y)\,\omega = \int_M d\,(*\eta) = \int_{\partial M} *\eta = 0 \qquad \square$$

Proposition 4.1. *Suppose M is compact. Let σ be a section of $\mathcal{E}$, and let α be an $\mathcal{E}$-valued 1-form on M. Then*

$$\int_M \langle \nabla\sigma, \alpha \rangle \,\text{vol}(g) = \int_M \langle \sigma, \nabla^*\alpha \rangle \,\text{vol}(g)$$

and

$$\int_M \langle \alpha, \nabla\sigma \rangle \,\text{vol}(g) = \int_M \langle \nabla^*\alpha, \sigma \rangle \,\text{vol}(g)$$

Proof. By the Hilbert-Schmidt norm

$$\begin{aligned}\int_M \langle \nabla_{E_i}\sigma, \alpha(E_i)\rangle \mathrm{vol}(g) &= \int_M E_i\langle \sigma, \alpha(E_i)\rangle - \langle \sigma, \nabla_{E_i}(\alpha(E_i))\rangle \mathrm{vol}(g) \\ &= \int_M E_i\langle \sigma, \alpha(E_i)\rangle - \langle \sigma, (\nabla_{E_i}\alpha)(E_i) + \alpha(\nabla_{E_i}E_i)\rangle \mathrm{vol}(g) \\ &= \int_M \langle \sigma, \nabla^*\alpha\rangle + E_i\langle \sigma, \alpha(E_i)\rangle - \langle \sigma, \alpha(\nabla_{E_i}E_i)\rangle \mathrm{vol}(g) \\ &= \int_M \langle \sigma, \nabla^*\alpha\rangle \mathrm{vol}(g) \quad \square\end{aligned}$$

5. Harmonic maps and the Euler-Lagrange equation

Definition 5.1. [15] Let $f : U \to V$ be a smooth map between open subsets of n and m-dimensional Euclidean spaces $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. The smooth map f is defined to be a *diffeomorphism* if f is invertible and the inverse map $f^{-1} : V \to U$ is a smooth map. Then the open sets U and V are also defined to be *diffeomorphic* to each other.

Definition 5.2. A subset $D \subset \mathbb{R}^{n+1}$ that is diffeomorphic to an open set $U \subset \mathbb{R}^n$ is defined as a *hypersurface patch*. Let $S \subset \mathbb{R}^{n+1}$ and $V \subset \mathbb{R}^{n+1}$ be open subsets of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Then a hypersurface patch of the form $D = S \cap V$ is defined as a *chart domain*. If each point of S lies in a chart domain then S is a *smooth hypersurface*.

Definition 5.3. [15] A smooth hypersurface $S \in \mathbb{R}^n$ is said to be *orientable* if there exists a smooth *unit normal field on S*. This is a smooth function $\xi : S \to \mathbb{R}^n$ such that, for all $p \in S$, $\xi(p)$ is orthogonal to T_pS and $|\xi(p)| = 1$. If S is orientable then there are precisely two smooth unit normal fields, choice of either of which constitutes an orientation of S.

Definition 5.4. [15] Let $M \subset \mathbb{R}^3$ be a hypersurface, with a unit normal field ξ, and let $\phi : D \to U$ be a smooth chart with local parametrisation $p : U \to D$. It is convenient to label the coordinates $u = (u_1, u_2) \in U$ and then abbreviate the partial derivatives of p as

$$\frac{\partial p}{\partial u_i} = p_i, \ \frac{\partial^2 p}{\partial u_i \partial u_j} = p_{ij}$$

Define three smooth functions $W_i : U \to \mathbb{R}^3$ as

$$W_1 = p_1, \ W_2 = p_2, \ W_3 = \xi \circ p$$

For each $u \in U$ the three vectors $(W_1(u), W_2(u), W_3(u))$ form a basis of $\mathbb{R}^3$ which changes as u varies. The basis (W_1, W_2, W_3) is called the *Weingarten frame of M* with respect to the chart (D, ϕ).

Definition 5.5. [15] Let g_{ij} be the metric

$$g_{ij} = \langle p_i, p_j \rangle = W_i \cdot W_j,\ i, j \in \{1, 2\}$$

The *Christoffel symbols of* M, denoted by Γ_{ij}^k, satisfy the general equation

$$\Gamma_{ij}^k g_{kl} = \frac{1}{2}\left(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l}\right) = \frac{1}{2}\left((g_{il})_j + (g_{jl})_i - (g_{ij})_l\right)$$

Definition 5.6. [5] Let $\phi : (M, g) \to (N, h)$ be a smooth map. Its differential $d\phi$ is a section of the bundle $\Omega^1(\phi TN) = T^*M \times \phi^{-1}TN$, and its *norm at a point* x *of* M is denoted by $|d\phi|$, equipped with the metrics g and h. If (x^i) and (u^α) are local coordinates around x and $\phi(x)$, $|d\phi|^2$ is defined by the equation

$$|d\phi|^2 = g^{ij} h_{\alpha\beta}(\phi)\phi_i^\alpha \phi_j^\beta$$

where $(\phi_i^\alpha) = \left(\frac{\partial \phi^\alpha}{\partial x^i}\right)$ is the local representation of $d\phi$.

Definition 5.7. [5] The *energy density of* ϕ is defined as the function $e(\phi) = \frac{1}{2}|d\phi|^2$ and the *energy of* ϕ is defined to be the number

$$E(\phi) = \int_M e(\phi)\,\mathrm{vol}(g)$$

Example 5.1. In $\mathbb{R}^3$ the energy of f over V is:

$$E(f) = \frac{1}{2}\int\int\int_V \|\nabla f\|^2 dxdydz$$

Definition 5.8. [5]A function b has the *Dirichlet property* on M if b deforms M less than other comparable functions. This inequality is written as

$$E(b) = \int_M \frac{1}{2}|\nabla b|^2\,\mathrm{vol}(g) \leq \int_M \frac{1}{2}|\nabla a|^2\,\mathrm{vol}(g) = E(a)$$

Definition 5.9. [5] For a given v, let ϕ_t be a family of maps such that $\phi_0 = \phi$ and $\frac{\partial \phi_t}{\partial t}\big|_{t=0} = v$. Then $D_v E(\phi)$ is defined as

$$D_v E(\phi) = \left.\frac{dE(\phi_t)}{dt}\right|_{t=0}$$

Definition 5.10. [5] A *map* $\phi : M \to N$ *is harmonic* if and only if it is an extremum of the energy. This means that $D_v E(\phi) = 0$ for all $v \in \mathcal{C}(\phi^{-1}TN)$.

Theorem 5.1. [5] *A map* $\phi : M \to N$ *is harmonic if and only if it satisfies the Euler-Lagrange equation* $\tau(\phi) = 0$*, where* $\tau(\phi) = -d^*d\phi = \mathrm{trace}\,\nabla d\phi$ *is called the tension field of* ϕ*.*

A proof of this theorem above can be seen in [5]. It is also shown in [5] that to express the equation for τ in local coordinates using (x^i) on M and (u^α) on N, let g_{ij} and $h_{\alpha\beta}$ be the metrics and ${}^M\Gamma_{jk}^i$ and ${}^N\Gamma_{\beta\gamma}^\alpha$

be the Christoffel symbols of M and N respectively. Then calculate $\nabla_{\frac{\partial}{\partial x^i}}(d\phi)$

$$\nabla_{\frac{\partial}{\partial x^i}}(d\phi) = \nabla_{\frac{\partial}{\partial x^i}}\left(\phi_j^\alpha dx^j \frac{\partial}{\partial u^\alpha}\right)$$

Apply Definition 3.2 to give

$$\nabla_{\frac{\partial}{\partial x^i}}(d\phi) = \left(\frac{\partial}{\partial x^i}\phi_j^\alpha\right) dx^j \frac{\partial}{\partial u^\alpha} + \phi_j^\alpha \left(\nabla^{T^*M}_{\frac{\partial}{\partial x^i}} dx^j\right)\frac{\partial}{\partial u^\alpha} + \phi_j^\alpha dx^j \nabla^{\phi^{-1}TN}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial u^\alpha}$$

To continue this calculation note that $\nabla^{T^*M}_{\frac{\partial}{\partial x^i}} dx^j = -{}^M\Gamma^j_{ik} dx^k$ and

$$\nabla^{\phi^{-1}TN}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial u^\alpha} = \nabla^{TN}_{\phi_i^\beta \frac{\partial}{\partial u^\beta}} \frac{\partial}{\partial u^\alpha} = \phi_i^\beta\, {}^N\Gamma^\gamma_{\beta\alpha} \frac{\partial}{\partial u^\gamma}$$

Hence

$$(\nabla d\phi)^\alpha_{ij} = \phi^\alpha_{ij} - {}^M\Gamma^k_{ij}\phi^\alpha_k + {}^N\Gamma^\alpha_{\beta\gamma}\phi_i^\beta\phi_j^\gamma$$

Therefore the equation for τ in local coordinates is

$$\tau_\phi^\alpha = g^{ij}(\nabla d\phi)^\alpha_{ij} = -\Delta\phi^\alpha + {}^N\Gamma^\alpha_{\beta\gamma}\phi_i^\beta\phi_j^\gamma g^{ij}$$

6. The Sasaki metric and vertical energy

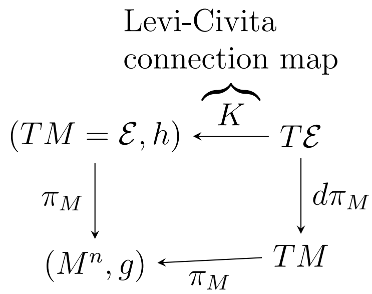
Definition 6.1. [4] Let $\phi : V \to W$ be a linear transformation of vector spaces. The *kernel* of ϕ, denoted by Ker ϕ, is defined by

$$\text{Ker } \phi = \{v \in V | \phi(v) = 0\}$$

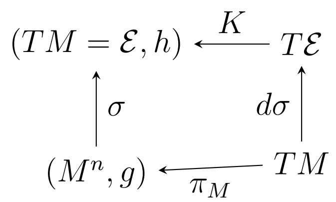
Definition 6.2. [1][9] Let $K : T\mathcal{E} \to TM = \mathcal{E}$ be the connection map for ∇ which is characterised by the equation

$$\nabla_X \sigma = K(d\sigma(X)) \in T_x M$$

where $\sigma : M^n \to TM$ is a section of a vector bundle $\pi_M : TM \to M^n$. A mapping diagram including π_M, K and $d\pi_M$ is shown below



Sections of TM and $T\mathcal{E}$, which are σ and $d\sigma$ respectively, are also shown in the next diagram below.



The *Sasaki metric* is defined as

$$h(A,B) = g\left(d\pi(A), d\pi(B)\right) + g\left(K(A), K(B)\right)$$

In the mapping diagram above a section of TM is σ and a section of $T\mathcal{E}$ is $d\sigma$ because

$$\pi_M \circ \sigma = \mathrm{id}_M : M \to M$$

and by the chain rule

$$\begin{aligned} d\pi_M \circ d\sigma &= d\left(\mathrm{id}_M\right) \\ &= \mathrm{id}_{TM} : TM \to TM \end{aligned} \tag{6.1}$$

Definition 6.3. [15] At any $x \in M$ and $\epsilon \in \mathcal{E}$ the diagrams in Definition 6.2 can be applied using the tangent spaces $T_\epsilon\mathcal{E}$ and T_xM and the maps $\pi_M(\epsilon) = x$ and $d\pi_M(\epsilon) : T_\epsilon\mathcal{E} \to T_xM$. The *vertical subspace at* ϵ is an n-dimensional subspace V_ϵ of $T_\epsilon\mathcal{E}$ defined as

$$V_\epsilon = \mathrm{Ker}\ d\pi(\epsilon) \subset T_\epsilon\mathcal{E}$$

Definition 6.4. [15] At any $x \in M$ and $\epsilon \in \mathcal{E}$ the diagrams in Definition 6.2 can be applied using the tangent spaces $T_\epsilon\mathcal{E}$ and T_xM and the map $K : T_\epsilon\mathcal{E} \to T_xM$. The *horizontal subspace at* ϵ is an n-dimensional subspace H_ϵ of $T_\epsilon\mathcal{E}$ defined as

$$H_\epsilon = \mathrm{Ker}\ K \subset T_\epsilon\mathcal{E}$$

Proposition 6.1. [15] *Using the subspaces in Definitions 6.3 and 6.4,* $T_\epsilon\mathcal{E} = V_\epsilon \oplus H_\epsilon$ *and* $T\mathcal{E} = V \oplus H$.

Proposition 6.2. *The energy of* σ *can be split into horizontal and vertical components*

$$\begin{aligned} E(\sigma) &= \frac{1}{2}\int_M \|d\sigma\|^2 \,\mathrm{vol}(g) \\ &= \frac{n}{2}\,\mathrm{vol}(M,g) + \frac{1}{2}\int_M \|\nabla\sigma\|^2 \,\mathrm{vol}(g) \end{aligned}$$

Proof. Recall that the energy density of ϕ is the function $e(\phi) = \frac{1}{2}\|d\phi\|^2$ and the energy of ϕ is the number

$$\begin{aligned} E(\phi) &= \int_M e(\phi)\mathrm{vol}(g) \\ &= \frac{1}{2}\int_M \|d\phi\|^2\mathrm{vol}(g) \end{aligned}$$

Let h be the Sasaki Metric and $\sigma : (M,g) \to (TM,h)$ be a vector field on (M,g) which is also compact. By Proposition 6.1, $d\sigma(X)$ can be split into vertical and horizontal components

$$d\sigma(X) = d^V\sigma(X) + d^H\sigma(X)$$

Using the Sasaki metric and the characterisation of K, the horizontal component of $\|d\sigma\|^2$ is

$$\begin{aligned}\|d^H\sigma\|^2 &= h\left(\underbrace{d^H\sigma(E_i)}_{A}, \underbrace{d^H\sigma(E_i)}_{B}\right) \\ &= g\left(d\pi(d^H\sigma(E_i)), d\pi(d^H\sigma(E_i))\right) \\ &\quad + g\left(K(d^H\sigma(E_i)), K(d^H\sigma(E_i))\right) \\ &= g\left(d\pi(d^H\sigma(E_i)), d^H\pi(E_i)\right)\end{aligned}$$

Equation (6.1) leads to

$$\|d^H\sigma\|^2 = g\left(E_i, E_i\right) = n$$

Using the Sasaki metric and the characterisation of K, the vertical component of $\|d\sigma\|^2$ is

$$\begin{aligned}\|d^V\sigma\|^2 &= h\left(\underbrace{d^V\sigma(E_i)}_{A}, \underbrace{d^V\sigma(E_i)}_{B}\right) \\ &= g\left(d\pi(d^V\sigma(E_i)), d\pi(d^V\sigma(E_i))\right) \\ &\quad + g\left(K(d^V\sigma(E_i)), K(d^V\sigma(E_i))\right) \\ &= g\left(K(d^V\sigma(E_i)), K(d^V\sigma(E_i))\right) \\ &= g\left(\nabla_{E_i}\sigma, \nabla_{E_i}\sigma\right) \text{ by Equation (6.2)} \\ &= \|\nabla_{E_i}\sigma\|^2 \\ &= \|\nabla\sigma\|^2\end{aligned}$$

Adding these two components together

$$\begin{aligned}\|d\sigma\|^2 &= h\left(d\sigma(E_i), d\sigma(E_i)\right) \\ &= \overbrace{g(d\pi(d\sigma(E_i)), d\pi(d\sigma(E_i)))}^{h(d^H\sigma(E_i), d^H\sigma(E_i))} + \underbrace{g\left(K(d\sigma(E_i), K(d\sigma(E_i))\right)}_{h(d^V\sigma(E_i), d^V\sigma(E_i))} \\ &= \underbrace{\|d^H\sigma\|^2}_{n} + \underbrace{\|d^V\sigma\|^2}_{\|\nabla\sigma\|^2}\end{aligned}$$

Therefore the energy of σ is

$$\begin{aligned}E(\sigma) &= \frac{1}{2}\int_M \|d\sigma\|^2 \mathrm{vol}(g) \\ &= \frac{1}{2}\int_M \left(\|d^H\sigma\|^2 + \|d^V\sigma\|^2\right) \mathrm{vol}(g) \\ &= \frac{n}{2}\mathrm{vol}(M, g) + \frac{1}{2}\int_M \|\nabla\sigma\|^2 \mathrm{vol}(g)\end{aligned}$$

□

The equation above leads to the following two definitions including a definition of a harmonic vector field which will be generalised in Section 10 called *The generalised Cheeger-Gromoll metric and vertical (p,q)-energy.*

Definition 6.5. [1] The *vertical energy* (or *total bending*) of σ is the number

$$E^V(\sigma) = \frac{1}{2}\int_M \|\nabla\sigma\|^2 \mathrm{vol}(g)$$

Definition 6.6. A vector field σ is a *harmonic vector field* if

$$\left.\frac{d}{dt}\right|_{t=0} E(\sigma_t) = \left.\frac{d}{dt}\right|_{t=0} E^V(\sigma_t) = 0$$

for all smooth variations σ_t of σ through vector fields. A smooth variation σ_t of σ is the map $\Sigma : M \times \mathbb{R} \to TM$ with the conditions that
(1) $\Sigma(x,t) = \sigma_t(x)$,
(2) σ_t is a vector field,
(3) $\sigma_0 = \sigma$.

7. Harmonic unit fields

Definition 7.1. Let (M,g) be a Riemannian manifold and σ be a vector field. The vector field σ is a *unit vector field* if $\|\sigma\| = 1$ and σ is a *harmonic unit vector field* if $\left.\frac{d}{dt}\right|_{t=0} E^V(\sigma_t) = 0$ for all variations σ_t of σ through unit vector fields.

Lemma 7.1. *If σ is a unit vector field then the following equation is true*

$$\langle \nabla^*\nabla\sigma, \sigma\rangle = \|\nabla\sigma\|^2$$

Proof. Using the general equations for the rough Laplacian $\nabla^*\nabla\sigma$ and the trace of X from Definitions 3.16 and 3.20, $-\langle\nabla^*\nabla\sigma,\sigma\rangle$ equals

$$\begin{aligned}
-\langle \nabla^*\nabla\sigma, \sigma\rangle &= -\left\langle -\operatorname{trace}\nabla^2\sigma, \sigma\right\rangle \\
&= \left\langle \sum_{i=1}^{n} \nabla^2_{E_i,E_i}\sigma, \sigma \right\rangle \\
&= \langle \nabla_{E_i}\nabla_{E_i}\sigma - \nabla_{\nabla_{E_i}E_i}\sigma, \sigma\rangle \\
&= E_i\langle \nabla_{E_i}\sigma, \sigma\rangle - \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle - \langle \nabla_{\nabla_{E_i}E_i}\sigma, \sigma\rangle
\end{aligned}$$

Since $\|\sigma\|^2 = 1$ and by the definition of a tangent vector from Definition 2.8,

$$\begin{aligned}
\langle \nabla_X\sigma, \sigma\rangle &= \frac{1}{2}X\langle\sigma,\sigma\rangle \\
&= \frac{1}{2}X\left(\|\sigma\|^2\right) \\
&= \frac{1}{2}X(1) \\
&= 0
\end{aligned}$$

Combining the two equations above leads to the following equation

$$
\begin{aligned}
-\langle \nabla^*\nabla\sigma, \sigma\rangle &= E_i\langle \nabla_{E_i}\sigma, \sigma\rangle - \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle - \langle \nabla_{\nabla_{E_i}E_i}\sigma, \sigma\rangle \\
-\langle \nabla^*\nabla\sigma, \sigma\rangle &= 0 - \|\nabla\sigma\|^2 - 0 \\
-\langle \nabla^*\nabla\sigma, \sigma\rangle &= -\|\nabla\sigma\|^2 \\
\Rightarrow \langle \nabla^*\nabla\sigma, \sigma\rangle &= \|\nabla\sigma\|^2
\end{aligned}
$$

□

A harmonic unit vector field is also known as a harmonic unit field. This alternative term will be used for the rest of this dissertation.

Lemma 7.2. *If ψ is a vector field such that $\langle\psi, \sigma\rangle = 0$ and $\|\sigma\| = 1$, then there exists a smooth variation σ_t of σ with*
(1) $\|\sigma_t\| = 1$ for all t,
(2) $\frac{d}{dt}\big|_{t=0}\sigma_t = \psi$.

Proof. Let $\sigma_t = \frac{\sigma+t\psi}{\|\sigma+t\psi\|}$ for small $|t|$ when σ is a smooth unit vector field. By the quotient rule,

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\sigma_t &= \frac{\left(\|\sigma+t\psi\|_{t=0}\,\frac{d}{dt}\big|_{t=0}(\sigma+t\psi) - \frac{d}{dt}\big|_{t=0}\|\sigma+t\psi\|\,(\sigma+t\psi)_{t=0}\right)}{\|\sigma+t\psi\|^2_{t=0}} \\
&= \psi - \left(\frac{d}{dt}\bigg|_{t=0}\|\sigma+t\psi\|\right)\sigma
\end{aligned}
$$

Differentiating the denominator of σ_t

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\|\sigma+t\psi\| &= \frac{d}{dt}\bigg|_{t=0}\langle\sigma+t\psi, \sigma+t\psi\rangle^{\frac{1}{2}} \\
&= \frac{1}{2}\langle\sigma+t\psi, \sigma+t\psi\rangle^{-\frac{1}{2}}_{t=0}\,\frac{d}{dt}\bigg|_{t=0}\langle\sigma+t\psi, \sigma+t\psi\rangle \\
&= \left\langle \frac{d}{dt}\bigg|_{t=0}(\sigma+t\psi), \sigma\right\rangle \\
&= \langle\psi, \sigma\rangle \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0}\sigma_t &= \psi - \underbrace{\cancel{\left(\frac{d}{dt}\bigg|_{t=0}\|\sigma+t\psi\|\right)}}_{0}\sigma \\
&= \psi.
\end{aligned}
$$

□

Theorem 7.1. *A unit vector field σ is a harmonic unit field if and only if $\nabla^*\nabla\sigma = \|\nabla\sigma\|^2\sigma$.*

Proof. The equation for a harmonic vector field is equivalent to

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} E(\sigma_t) = 0 &\iff \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2}\int_M \|\nabla\sigma_t\|^2 \mathrm{vol}(g) = 0 \\ &\iff \int_M \left.\frac{d}{dt}\right|_{t=0} \|\nabla\sigma_t\|^2 \mathrm{vol}(g) = 0 \end{aligned}$$

where $\|\nabla\sigma_t\|^2$ is the Hilbert-Schmidt norm equalling to

$$\|\nabla\sigma_t\|^2 = \sum_{i=1}^{n} \langle \nabla_{E_i}\sigma_t, \nabla_{E_i}\sigma_t\rangle$$

Now

$$\begin{aligned} \frac{d}{dt}\|\nabla\sigma_t\|^2 &= \frac{d}{dt}\sum_{i=1}^{n} \langle \nabla_{E_i}\sigma_t, \nabla_{E_i}\sigma_t\rangle \\ &= 2\left\langle \left.\frac{d}{dt}\right|_{t=0} \nabla_{E_i}\sigma_t, \nabla_{E_i}\sigma_t \right\rangle \\ &= 2\left\langle \nabla_{E_i}\left(\left.\frac{d}{dt}\right|_{t=0} \sigma_t\right), \nabla_{E_i}\sigma_t \right\rangle \end{aligned}$$

Recall the equation for the vector field ψ from Lemma 7.2 which is $\psi(x) = \left.\frac{d}{dt}\right|_{t=0}\sigma_t$ where $\sigma_t(x)$ is a curve in T_xM and use it in the previous equation for $\frac{d}{dt}\|\nabla\sigma_t\|^2$

$$\begin{aligned} \frac{d}{dt}\|\nabla\sigma_t\|^2 &= 2\left\langle \nabla_{E_i}\left(\left.\frac{d}{dt}\right|_{t=0} \sigma_t\right), \nabla_{E_i}\sigma_t \right\rangle \\ &= 2\left\langle \nabla_{E_i}\psi, \nabla_{E_i}\sigma_t \right\rangle \\ &= 2\left\langle \nabla\psi, \nabla\sigma_t \right\rangle \end{aligned}$$

By the definition of a harmonic vector field and Proposition 4.1

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2}\int_M \|\nabla\sigma_t\|^2 \,\mathrm{vol}(g) &= \int_M \langle \nabla\psi, \nabla\sigma_t\rangle \,\mathrm{vol}(g) \\ \iff \left.\frac{d}{dt}\right|_{t=0} E^V(\sigma_t) &= \int_M \langle \psi, \nabla^*\nabla\sigma_t\rangle \,\mathrm{vol}(g) \end{aligned} \tag{7.1}$$

Recall from Lemma 7.2 that σ_t is a unit vector field. Therefore the following equations are true.

$$\begin{aligned} 1 = \|\sigma_t\|^2 \Rightarrow 0 &= \left.\frac{d}{dt}\right|_{t=0} \langle \sigma_t, \sigma_t\rangle \\ &= 2\left\langle \left.\frac{d}{dt}\right|_{t=0} \sigma_t, \sigma_t \right\rangle \\ &= 2\langle \psi, \sigma_t\rangle \end{aligned}$$

Then by Lemma 7.1

$$\left.\frac{d}{dt}\right|_{t=0} E^V(\sigma_t) = \int_M \langle \psi, \nabla^*\nabla\sigma_t - \langle \nabla^*\nabla\sigma_t, \sigma_t\rangle\sigma_t\rangle \mathrm{vol}(g)$$
$$= \int_M \langle \psi, \nabla^*\nabla\sigma_t - \|\nabla\sigma_t\|^2\sigma_t\rangle \mathrm{vol}(g)$$

If $\nabla^*\nabla\sigma_t = \|\nabla\sigma_t\|^2\sigma_t$

$$\left.\frac{d}{dt}\right|_{t=0} E^V(\sigma_t) = \int_M \langle \psi, \|\nabla\sigma_t\|^2\sigma_t - \|\nabla\sigma_t\|^2\sigma_t\rangle \mathrm{vol}(g)$$
$$= \int_M \langle \psi, 0\rangle \mathrm{vol}(g)$$
$$= 0$$

From the definition of a harmonic vector field recall that $\sigma_0 = \sigma$ so σ is a harmonic unit field if $\nabla^*\nabla\sigma = \|\nabla\sigma\|^2\sigma$. The unit vector σ is a harmonic unit field only if $\nabla^*\nabla\sigma = \|\nabla\sigma\|^2\sigma$ because ψ is arbitrary except that it needs to satisfy $\langle\psi, \sigma\rangle = 0$. Hence σ is a harmonic unit field if and only if $\nabla^*\nabla\sigma = \|\nabla\sigma\|^2\sigma$. □

8. The Hopf vector field

This section continues studying harmonic unit fields and focuses on a specific harmonic unit field called the Hopf vector field. Before it can be defined some more information is needed.

Lemma 8.1. *For any unit vector field* σ, $\langle \nabla^2_{X,Y}\sigma, \sigma\rangle = -\langle \nabla_X\sigma, \nabla_Y\sigma\rangle$.

Proof. By the equation for the second covariant derivative

$$\langle \nabla^2_{X,Y}\sigma, \sigma\rangle = \langle \nabla_X(\nabla_Y\sigma) - \nabla_{\nabla_X Y}\sigma, \sigma\rangle$$
$$= \underbrace{\cancel{X\langle \nabla_Y\sigma, \sigma\rangle}}_{0} - \langle \nabla_Y\sigma, \nabla_X\sigma\rangle - \underbrace{\cancel{\langle \nabla_{\nabla_X Y}\sigma, \sigma\rangle}}_{0}$$
$$= -\langle \nabla_Y\sigma, \nabla_X\sigma\rangle$$

This is true because σ is a unit vector field and for all $Z \in TM$

$$\langle \nabla_Z\sigma, \sigma\rangle = \frac{1}{2}Z\langle\sigma, \sigma\rangle$$
$$= \frac{1}{2}Z\|\sigma\|^2$$
$$= \frac{1}{2}Z(1)$$
$$= 0$$

□

Recall that Theorem 7.1 states that σ is a harmonic unit field if and only if

$$\nabla^*\nabla\sigma = \|\nabla\sigma\|^2\sigma$$

Proposition 8.1. *The vector field σ is a harmonic unit field if and only if $\nabla^*\nabla\sigma = f\sigma$ for some $f : M \to \mathbb{R}$.*

Proof. Theorem 7.1 implies that this proposition is true with $f = \|\nabla\sigma\|^2$.

By Lemma 7.1

$$\begin{aligned}\langle \nabla^*\nabla\sigma, \sigma\rangle &= -\sum_i \langle \nabla^2_{E_i,E_i}\sigma, \sigma\rangle \\ &= \sum_i \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle \\ &= \sum_i \|\nabla_{E_i}\sigma\|^2 \\ &= \|\nabla\sigma\|^2 \text{ by the summation convention}\end{aligned}$$

If $\nabla^*\nabla\sigma = f\sigma$ then

$$\begin{aligned}\langle \nabla^*\nabla\sigma, \sigma\rangle &= \langle f\sigma, \sigma\rangle \\ &= f\langle \sigma, \sigma\rangle \\ &= f \text{ if } \sigma \text{ is a unit field}\end{aligned}$$

$$\text{Hence } f = \|\nabla\sigma\|^2 \text{ if } \sigma \text{ is a unit field}$$

Therefore $\nabla^*\nabla\sigma = f\sigma$ implies that σ is a harmonic unit field and this completes the proof. □

Definition 8.1. The equation for a $(2n+1)$-*sphere* is defined as

$$S^{2n+1} = \{(x_1, ..., x_{2n+2}) : x_1^2 + ... + x_{2n+2}^2 = 1\}$$

This is the same as an n-dimensional unit sphere as defined in Definition 2.11. For example a 1-sphere, S^1, is a unit circle and a 2-sphere, S^2 is a 3-dimensional unit sphere.

Definition 8.2. [15] Let $(G, \circ)$ and $(H, *)$ denote vector spaces G and H equipped with the metrics $\circ$ and $*$ respectively. A function $\theta : G \to H$ is an *isomorphism* if

$$\theta(a \circ b) = \theta(a) * \theta(b) \text{ for all } a, b \in G \tag{8.1}$$

$(G, \circ)$ and $(H, *)$ are *isomorphic*, written $G \cong H$, if there is an isomorphism between them.

There exists an isomorphism $I : \mathbb{R}^{2n+2} \to \mathbb{C}^{n+1}$ such that

$$I(x_1, ..., x_{2n+2}) = (x_1 + ix_2, ..., x_{2n+1} + ix_{2n+2})$$

Then for all $x \in \mathbb{R}^{2n+2}$, $ix \in \mathbb{R}^{2n+2}$ is defined by

$$\begin{aligned}ix &= I^{-1}(iI(x)) \\ &= (-x_2, x_1, -x_4, x_3, ..., -x_{2n+2}, x_{2n+1})\end{aligned}$$

Lemma 8.2. *Let* $x, y \in \mathbb{R}^{2n+2}$ *and* $\cdot$ *be the dot product. The following equation is true*

$$(ix) \cdot y = -x \cdot (iy)$$

Definition 8.3. [8] The *Hopf vector field* $\sigma : S^{2n+1} \to T_x S^{2n+1}$ when $n \in \mathbb{N}$ is defined as

$$\sigma(x) = ix, \text{ for all } x \in S^{2n+1}.$$

Remark 8.1. The equation for the tangent space $T_x S^{2n+1}$ of S^{2n+1} is

$$T_x S^{2n+1} = \{x \in S^{2n+1},\ X \in \mathbb{R}^{2n+2} : x \cdot X = 0\},$$

Let $\sigma(x)$ be the Hopf vector field and therefore $\sigma(x) \in T_x S^{2n+1}$. Then by Lemma 8.2 and the equation for the tangent space $T_x S^{2n+1}$ the following equation is true

$$\begin{aligned} \sigma(x) \cdot x &= (ix) \cdot x \\ &= -x \cdot (ix) \\ &= 0 \end{aligned} \tag{8.2}$$

Definition 8.4. Assume $f : U \to \mathbb{R}^m$ where $U \subset \mathbb{R}^n$ is an open subset. Let $x \in U$, $X \in \mathbb{R}^n$ and X_j denote the j-coordinate of X. The *directional derivative* of f at x in the direction of X is defined as

$$D_X f(x) = \sum_{j=1}^{n} \left. \frac{\partial f_i(x)}{\partial x_j} \right|_x X_j$$

where $f_i : U \to \mathbb{R}$ is the i-th component of f.

Definition 8.5. Using the same functions from the definition above, the *Jacobian matrix* of f at x is defined as

$$J_f(x) = \left(\left. \frac{\partial f_i(x)}{\partial x_j} \right|_x \right) \text{ where } 1 \le i \le m \text{ and } 1 \le j \le n$$

The j-th partial derivative of f at x is denoted by

$$D_j f(x) = \left. \frac{\partial f_i(x)}{\partial x_j} \right|_x$$

Another way to write the equation of the directional derivative is by expanding $X = X_1 E_1 + ... + X_n E_n$, where $X_j = X \cdot E_j$, then

$$D_X f(x) = X_1 D_1 f(x) + ... + X_n D_n f(x)$$

Example 8.1. Let $x = a = (a_1, a_2) \in \mathbb{R}^2$ and $X = h = (h_1, h_2) \in \mathbb{R}^2$. Taking the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ which has the equation

$$f(x_1, x_2) = (x_1 + 2x_2, x_1 x_2, 2x_1 + x_2),$$

It has the partial derivatives

$$D_1 f(a) = (1, a_2, 2), D_2 f(a) = (2, a_1, 1).$$

The Jacobian matrix of f at a is

$$J_f(a) = \begin{pmatrix} 1 & 2 \\ a_2 & a_1 \\ 2 & 1 \end{pmatrix}$$

The directional derivative of f at X is therefore:

$$D_X f(a) : (h_1, h_2) \mapsto (h_1 + 2h_2, a_2 h_1 + a_1 h_2, 2h_1 + h_2).$$

Definition 8.6. [15] *The shape operator (or Weingarten map) A_p of $M \in \mathbb{R}^n$ at p* is the linear map

$$A_p : T_pM \to T_pM; A_p = -d\xi(p)$$

Where ξ is the unit normal field as defined in Definition 5.3.

Definition 8.7. [15] *The second fundamental form of M at p* is the bilinear form denoted by $\alpha_p : T_pM \times T_pM \to \mathbb{R}$ and defined as

$$\begin{aligned} \alpha_p(X, Y) &= \langle A_p(X), Y \rangle \\ &= -d\xi(X) \cdot Y \text{ where } \xi \text{ is the unit normal field.} \end{aligned}$$

Remark 8.2. The second fundamental form $\alpha_p(X, Y)$ as defined in the definition above is symmetric.

Definition 8.8. The *Gauss formula* for the covariant derivative $\nabla_X Y$ is defined as

$$\begin{aligned} \nabla_X Y &= D_X Y - \alpha(X, Y)\xi \\ &= D_X Y - (A(X) \cdot Y)\, \xi \\ &= D_X Y + (d\xi(X) \cdot Y)\, \xi \end{aligned}$$

This formula agrees with the more intrinsic definition of the covariant derivative which was written as $\nabla_X \sigma$ in Definition 3.2.

Theorem 8.1. *Let σ be the Hopf vector field on S^{2n+1}. Then $\nabla^*\nabla\sigma = 2n\sigma$. Hence σ is a harmonic unit field.*

Proof. Recall that

$$\nabla^*\nabla\sigma = -\sum_i \nabla^2_{E_i, E_i}\sigma$$

for any orthonormal basis $\{E_i\}$. Therefore $\nabla^2_{X,Y}\sigma$ for all $X, Y \in T_x S^{2n+1}$ needs to be calculated. Taking the equation for the second covariant derivative

$$\nabla^2_{X,Y}\sigma = \nabla_X (\nabla_Y \sigma) - \nabla_{\nabla_X Y}\sigma,$$

then the Gauss formula for the covariant derivative, $\nabla_X Y$ on S^{2n+1}, equals

$$\nabla_X Y = D_X Y - \alpha(X, Y)\xi, \ \forall X \in T_x S^{2n+1}$$

where $D_X Y$ is the directional derivative and $\alpha(X, Y)$ is the second fundamental form. To find the unit normal field note that in this case $S = S^{2n+1}$ is the $(2n + 1)$-dimensional unit sphere therefore the radius

$r = |x|^2 = 1$. This means that $|x| = 1$ and hence the unit normal field of S^{2n+1} is $\xi = x$. Therefore $\alpha(X, Y)$ equals

$$\begin{aligned} \alpha(X,Y) &= A(X)\cdot Y \\ &= -d\xi(X)\cdot Y \\ &= -dx(X)\cdot Y \\ &= -(X\cdot Y) \end{aligned} \tag{8.3}$$

where $A(X)$ is the shape operator and ξ is the unit normal field. Calculate $\nabla_Y \sigma$

$$\begin{aligned} \nabla_Y \sigma &= \nabla_Y (ix) \\ &= D_Y(ix) - \alpha(X,Y) \text{ by the Gauss formula} \\ &= D_Y(ix) + Y\cdot(ix) \text{ by Equation (8.3)} \end{aligned}$$

Using the equation of the directional derivative

$$\begin{aligned} D_Y\sigma(x) &= \sum_{j=1}^{2n+2} \left.\frac{\partial ix}{\partial x_j}\right|_x Y_j \\ &= iY \\ &= \nabla_Y \sigma \end{aligned} \tag{8.4}$$

The Gauss formula for the Hopf vector field equates to

$$\begin{aligned} \nabla_Y \sigma &= D_Y\sigma + (Y\cdot\sigma)(\xi) \\ &= iY + (Y\cdot\sigma)\xi \text{ by Equation (8.4)} \end{aligned}$$

Hence the second covariant derivative for the Hopf vector field is

$$\begin{aligned} \nabla^2_{X,Y}\sigma &= \nabla_X\nabla_Y\sigma - \nabla_{\nabla_X Y}\sigma \\ &= D_X\left(iY + \langle Y,\sigma\rangle\xi\right) + (X\cdot(iY + \underbrace{\cancel{\langle Y,\sigma\rangle\xi}}_{=0}))\xi \\ &\quad - i\left(\nabla_X Y\right) - \left(\left(\nabla_X Y\right)\cdot\sigma\right)\xi \\ &= iD_X Y + \left(X\langle Y,\sigma\rangle\right)\xi + \langle Y,\sigma\rangle D_X\xi + \left(X\cdot(iY)\right)\xi \\ &\quad - i\nabla_X Y - \langle\nabla_X Y,\sigma\rangle\xi \\ &= i\left(\langle X,Y\rangle\xi\right) + \left(\cancel{\langle\nabla_X Y,\sigma\rangle} + \langle Y,\nabla_X\sigma\rangle\right)\xi \\ &\quad + \langle Y,\sigma\rangle D_X x + \left(X\cdot(iY)\right)\xi - \cancel{\langle\nabla_X Y,\sigma\rangle\xi} \\ &= -\langle X,Y\rangle\sigma + \left(Y\cdot\left(iX + (X\cdot\sigma)\,\xi\right)\right)\xi \\ &\quad + \langle Y,\sigma\rangle X + \left(X\cdot(iY)\right)\xi \\ &= \langle Y,\sigma\rangle X - \langle X,Y\rangle\sigma + \cancel{(Y\cdot(iX))\,\xi} + \cancel{(X\cdot(iY))\,\xi} \end{aligned}$$

Therefore

$$\nabla^2_{X,Y}\sigma = \langle Y,\sigma\rangle X - \langle X,Y\rangle\sigma$$

Finally the rough Laplacian of the Hopf vector field equals

$$\begin{aligned}\nabla^*\nabla\sigma &= -\sum_{i}^{2n+1} \nabla^2_{E_i,E_i}\sigma \\ &= -\sum_{i}^{2n+1} \left(\langle E_i,\sigma\rangle E_i - \langle E_i,E_i\rangle\sigma\right) \\ &= (2n+1)\,\sigma - \sigma \\ &= 2n\sigma.\end{aligned}$$

□

9. Killing fields

Definition 9.1. [2] A vector field K on a Riemannian manifold (M, g) is *Killing* if

$$\langle\nabla_X K, Y\rangle + \langle X, \nabla_Y K\rangle = 0 \qquad \text{(Killing's identity)}$$

Definition 9.2. [15] The flow of a vector field σ on a manifold M, denoted by $\phi_t : M \to M$, is defined to be a function on M that satisfies the equation

$$\sigma\left(\phi_t(x)\right) = \frac{d}{dt}\left(\phi_t(x)\right)$$

for all $t \in \mathbb{R}$ and $x \in M$ where $\phi_0 = \mathrm{id}_M$.

Definition 9.3. [16] An isometry $f : X \to Y$ is a bijective map such that

$$d_Y\left(f(x_1), f(x_2)\right) = d_X(x_1, x_2)$$

where d_Y and d_X are metrics.

Example 9.1. Let $\mathbb{R}^2$ and $\mathbb{C}$ denote the 2-dimensional Euclidean space and the complex space respectively. Let (x, y) be the coordinates of a point in $\mathbb{R}^2$ and z be a function $z(x, y) = x + iy \in \mathbb{C}$ which is an isometry from $\mathbb{R}^2$ to $\mathbb{C}$.

Let $\sigma(z) : \mathbb{C} \to \mathbb{C}$ be the vector field

$$\sigma(z) = z^2$$

Define $\phi_t(z)$ to be a mapping $\phi_t : \mathbb{C} \setminus \{\frac{1}{t}\} \to \mathbb{C} \setminus \{\frac{1}{t}\}$

$$\phi_t(z) = \frac{z}{1 - tz}$$

To see if $\phi_t(z)$ is the flow of the vector field σ calculate $\sigma\left(\phi_t(x)\right)$ first

$$\begin{aligned}\sigma\left(\phi_t(z)\right) &= \left(\frac{z}{1 - tz}\right)^2 \\ &= \frac{z^2}{(1 - tz)^2}\end{aligned}$$

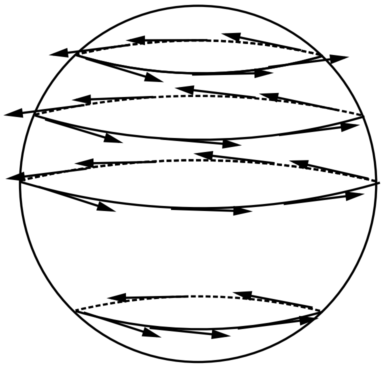


FIGURE 9.1. A Killing field in S^2

Then calculate $\frac{d}{dt}\left(\phi_t(z)\right)$

$$\begin{aligned}\frac{d}{dt}\left(\phi_t(z)\right) &= \frac{d}{dt}\left(\frac{z}{1-tz}\right)\\ &= \frac{(1-tz)d(z)-zd(1-tz)}{(1-tz)^2}\\ &= \frac{(1-tz)0-z(-z)}{(1-tz)^2}\\ &= \frac{z^2}{(1-tz)^2}\\ &= \sigma\left(\phi_t(z)\right)\end{aligned}$$

Therefore $\phi_t(z)$ is the flow of the vector field $\sigma(z)$ on $\mathbb{C}\setminus\{\frac{1}{t}\}$.

Two examples of Killing fields are shown in Figures 9.1 and 9.2 where the arrows depict the flow, $\phi_t : M \to M$, of the unit sphere S^2 and the unit circle S^1 at time t.

Let σ be the Hopf vector field $\sigma(x) = ix$ for all $x \in S^{2n+1}$. The covariant derivative of σ is

$$\nabla_X\sigma = iX + \langle X, \sigma\rangle\xi$$

Using Killing's identity on σ

$$\begin{aligned}\langle\nabla_X\sigma, Y\rangle + \langle X, \nabla_Y\sigma\rangle &= (iX + \langle X,\sigma\rangle\xi)\cdot Y + X\cdot(iY + \langle Y,\sigma\rangle\xi)\\ &= (iX)\cdot Y + X\cdot(iY)\\ &= 0\end{aligned}$$

Therefore σ is a Killing field.

Proposition 9.1. [15]*Recall the equation for the Riemann tensor $R(X,Y)Z$ from section 3*

$$R(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z$$

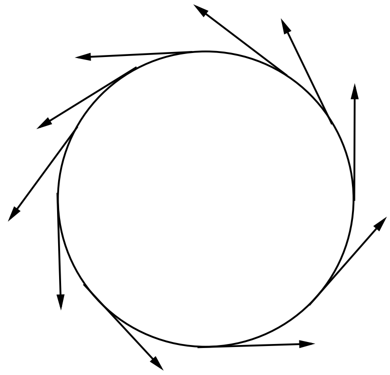


FIGURE 9.2. A Killing field in S^1

The Riemann tensor satisfies the following properties for all $X, Y, Z, W \in T_xS$ *and all* $x \in S$

$$R(X,Y)Z = -R(Y,X)Z \tag{R1}$$

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle \tag{R2}$$

$$\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle \tag{R3}$$

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0 \tag{R4}$$

$$R(X,X)Y = 0 \tag{R5}$$

Remark 9.1. Property (R4) is known as *Bianchi's identity.*

Proposition 9.2. *Let* K *be a Killing field in a Riemannian manifold* (M, g)*. Then*

$$\nabla^2_{X,Y}K = -R(K,X)Y$$

for all $X, Y \in T_xM$ *and all* $x \in M$*.*

Proof. The first step is to prove the following equation

$$\langle \nabla^2_{X,X}K, Z\rangle = -\langle X, \nabla^2_{X,Z}K\rangle \tag{9.1}$$

By Killing's identity

$$\langle \nabla_X K, Y\rangle = -\langle X, \nabla_Y K\rangle$$

Replacing Y with Z

$$\langle \nabla_X K, Z\rangle = -\langle X, \nabla_Z K\rangle$$

Taking the equation for the second covariant derivative

$$\nabla^2_{X,Y}\sigma = \nabla_X(\nabla_Y\sigma) - \nabla_{\nabla_X Y}\sigma \tag{9.2}$$

The second covariant derivatives $\nabla^2_{X,X}K$ and $\nabla^2_{X,Z}K$ are

$$\nabla^2_{X,X}K = \nabla_X(\nabla_X K) - \nabla_{\nabla_X X}K \tag{9.3}$$

and

$$\nabla^2_{X,Z}K = \nabla_X(\nabla_Z K) - \nabla_{\nabla_X Z}K$$

Recall the equation of the Riemannian metric on σ and the second covariant derivative of σ from the proof of Lemma 8.1

$$\langle \nabla^2_{X,Y}\sigma, \sigma\rangle = \langle \nabla_X \left(\nabla_Y \sigma\right) - \nabla_{\nabla_X Y}\sigma, \sigma\rangle \tag{9.4}$$

The Riemannian metric on the second covariant derivative $\nabla^2_{X,X}K$ and Z is

$$\langle \nabla^2_{X,X}K, Z\rangle = \langle \nabla_X \left(\nabla_X K\right) - \nabla_{\nabla_X X}K, Z\rangle$$

The Riemannian metric on the second covariant derivative $\nabla^2_{X,Z}K$ and X is

$$\langle X, \nabla^2_{X,Z}K\rangle = \langle X, \nabla_X \left(\nabla_Z K\right) - \nabla_{\nabla_X Z}K\rangle$$

Extend $X, Z \in T_xM$ to vector fields such that

$$\nabla_Y X = 0 = \nabla_Y Z$$

for all $Y \in T_xM$ so that

$$\nabla_X X = 0$$

To prove Equation (9.1) use Killing's identity and the equation above

$$\begin{aligned}
\langle \nabla_X K, Z\rangle &= -\langle X, \nabla_Z K\rangle \\
\Rightarrow X\langle \nabla_X K, Z\rangle &= -X\langle X, \nabla_Z K\rangle \\
\Rightarrow \langle \nabla_X(\nabla_X K), Z\rangle + \langle \nabla_X K, \nabla_X Z\rangle &= -\langle \nabla_X X, \nabla_Z K\rangle \\
&\quad - \langle X, \nabla_X(\nabla_Z K)\rangle \\
\Rightarrow \langle \nabla_X(\nabla_X K) - \nabla_{\nabla_X X}K, Z\rangle + \langle \nabla_X K, \nabla_X Z\rangle &= -\langle X, \nabla_X(\nabla_Z K)\rangle \\
\Rightarrow \langle \nabla^2_{X,X}K, Z\rangle &= -\langle X, \nabla_X(\nabla_Z K)\rangle \\
&\quad - \langle \nabla_X K, \nabla_X Z\rangle \\
\Rightarrow \langle \nabla^2_{X,X}K, Z\rangle &= -\langle X, \nabla_X(\nabla_Z K)\rangle \\
&\quad + \langle X, \nabla_{\nabla_X Z}K\rangle \\
\Rightarrow \langle \nabla^2_{X,X}K, Z\rangle &= -\langle X, \nabla_X(\nabla_Z K) \\
&\quad - \nabla_{\nabla_X Z}K\rangle \\
\Rightarrow \langle \nabla^2_{X,X}K, Z\rangle &= -\langle X, \nabla^2_{X,Z}K\rangle
\end{aligned}$$

The second step is to prove the following equation

$$\langle \nabla^2_{Z,X}K, X\rangle = 0 \tag{9.5}$$

Using the equation for the second covariant derivative

$$\nabla^2_{Z,X}K = \nabla_Z \left(\nabla_X K\right) - \nabla_{\nabla_Z X}K$$

and by Equation (9.4) the following equation is true

$$\langle \nabla^2_{Z,X}K, X\rangle = \langle \nabla_Z \left(\nabla_X K\right) - \nabla_{\nabla_Z X}K, X\rangle \tag{9.6}$$

To prove Equation (9.5), start with Equation (9.6) and use Proposition 3.1

$$\begin{aligned}\langle\nabla^2_{Z,X}K, X\rangle &= \langle\nabla_Z(\nabla_X K) - \nabla_{\nabla_Z X}K, X\rangle\\ &= \langle\nabla_Z(\nabla_X K) - \nabla_0 K, X\rangle\\ &= \langle\nabla_Z(\nabla_X K), X\rangle\\ &= Z\langle\nabla_X K, X\rangle - \langle\nabla_X K, \nabla_Z X\rangle \text{ by property C3}\\ &= Z\langle\nabla_X K, X\rangle - \langle\nabla_X K, 0\rangle\\ &= Z\langle\nabla_X K, X\rangle\end{aligned}$$

By Killing's identity

$$\begin{aligned}\langle\nabla_X K, X\rangle + \langle X, \nabla_X K\rangle &= 0\\ \Rightarrow 2\langle\nabla_X K, X\rangle &= 0\\ \Rightarrow \langle\nabla_X K, X\rangle &= 0\end{aligned}$$

Hence $\langle\nabla^2_{Z,X}K, X\rangle$ equals

$$\begin{aligned}\langle\nabla^2_{Z,X}K, X\rangle &= Z\langle\nabla_X K, X\rangle\\ &= Z(0)\\ &= 0\end{aligned}$$

The third step is to prove

$$\nabla^2_{X,X}K = -R(K, X)X \tag{9.7}$$

Recall the equation for $R(X, Y)Z$ and property R1 in Proposition 9.1

$$R(X, Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z \tag{9.8}$$

and

$$R(X, Y)Z = -R(Y, X)Z$$

By Equations (9.1) and (9.5)

$$\begin{aligned}\langle\nabla^2_{X,X}K, Z\rangle &= -\langle X, \nabla^2_{X,Z}K\rangle\\ &= 0 - \langle X, \nabla^2_{X,Z}K\rangle\\ &= \langle\nabla^2_{Z,X}K, X\rangle - \langle X, \nabla^2_{X,Z}K\rangle\\ &= \langle\nabla^2_{Z,X}K, X\rangle - \langle\nabla^2_{X,Z}K, X\rangle \text{ by the symmetry of } \langle -, -\rangle\\ &= \langle R(Z, X)K, X\rangle\\ &= -\langle R(Z, X)X, K\rangle \text{ by property R3}\\ &= -\langle R(X, K)Z, X\rangle \text{ by property R2}\\ &= \langle R(X, K)X, Z\rangle \text{ by property R3}\\ &= \langle -R(K, X)X, Z\rangle \text{ by property R1}\end{aligned}$$

$$\Rightarrow \nabla^2_{X,X}K = -R(K, X)X$$

The fourth step is to prove the following equation

$$\nabla^2_{X+Y,X+Y}K + R(K, X+Y)(X+Y) = 2\nabla^2_{X,Y}K + 2R(K,X)Y$$

Using Equations (C1), (C2) and (9.3)

$$\begin{aligned}\nabla^2_{X+Y,X+Y}K &= \nabla_{X+Y}(\nabla_{X+Y}K) - \nabla_{\nabla_{X+Y}X+Y}K \\ &= \nabla_X(\nabla_{X+Y}K) + \nabla_Y(\nabla_{X+Y}K) - \nabla_{\nabla_{X+Y}X+Y}K \\ &= \nabla_X(\nabla_X K) + \nabla_X(\nabla_Y K) + \nabla_Y(\nabla_X K) \\ &\quad + \nabla_Y(\nabla_Y K) - \nabla_{\nabla_{X+Y}X+Y}K \\ &= \nabla_X(\nabla_X K) + \nabla_X(\nabla_Y K) + \nabla_Y(\nabla_X K) \\ &\quad + \nabla_Y(\nabla_Y K) - \nabla_{\nabla_{X+Y}X+\nabla_{X+Y}Y}K \\ &= \nabla_X(\nabla_X K) + \nabla_X(\nabla_Y K) + \nabla_Y(\nabla_X K) \\ &\quad + \nabla_Y(\nabla_Y K) - \nabla_{\nabla_X X+\nabla_Y X+\nabla_X Y+\nabla_Y Y}K \\ &= \nabla_X\nabla_X K + \nabla_X\nabla_Y K + \nabla_Y\nabla_X K + \nabla_Y\nabla_Y K \\ &\quad - \nabla_{\nabla_X X}K - \nabla_{\nabla_X Y}K - \nabla_{\nabla_Y X}K - \nabla_{\nabla_Y Y}K \\ &= \nabla^2_{X,Y}K + \nabla^2_{X,X}K + \nabla^2_{Y,X}K + \nabla^2_{Y,Y}K\end{aligned}$$

By Equations (9.2), (9.7), (9.8), (C1) and (C2)

$$\begin{aligned}R(K, X+Y)(X+Y) &= \nabla^2_{K,X+Y}(X+Y) - \nabla^2_{X+Y,K}(X+Y) \\ &= \nabla_K\left(\nabla_{X+Y}(X+Y)\right) - \nabla_{\nabla_K X+Y}(X+Y) \\ &\quad - \nabla_{X+Y}\left(\nabla_K(X+Y)\right) + \nabla_{\nabla_{X+Y}K}(X+Y) \\ &= \nabla_K\left(\nabla_{X+Y}X + \nabla_{X+Y}Y\right) - \nabla_{\nabla_K X+Y}X - \nabla_{\nabla_K X+Y}Y \\ &\quad - \nabla_{X+Y}\left(\nabla_K X + \nabla_K Y\right) + \nabla_{\nabla_{X+Y}K}X + \nabla_{\nabla_{X+Y}K}Y \\ &= \nabla_K\left(\nabla_X X + \nabla_Y X + \nabla_X Y + \nabla_Y Y\right) \\ &\quad - \nabla_{\nabla_K X+\nabla_K Y}X - \nabla_{\nabla_K X+\nabla_K Y}Y \\ &\quad - \nabla_X\left(\nabla_K X + \nabla_K Y\right) - \nabla_Y\left(\nabla_K X + \nabla_K Y\right) \\ &\quad + \nabla_{\nabla_X K+\nabla_Y K}X + \nabla_{\nabla_X K+\nabla_Y K}Y \\ &= \nabla_K(\nabla_X X) + \nabla_K(\nabla_Y X) + \nabla_K(\nabla_X Y) + \nabla_K(\nabla_Y Y) \\ &\quad - \nabla_{\nabla_K X}X - \nabla_{\nabla_K Y}X - \nabla_{\nabla_K X}Y - \nabla_{\nabla_K Y}Y \\ &\quad - \nabla_X(\nabla_K X) - \nabla_X(\nabla_K Y) - \nabla_Y(\nabla_K X) - \nabla_Y(\nabla_K Y) \\ &\quad + \nabla_{\nabla_X K}X + \nabla_{\nabla_X K}Y + \nabla_{\nabla_Y K}X + \nabla_{\nabla_Y K}Y \\ &= \nabla^2_{K,X}X + \nabla^2_{K,Y}X + \nabla^2_{K,X}Y + \nabla^2_{K,Y}Y \\ &\quad - \nabla^2_{X,K}X - \nabla^2_{Y,K}X - \nabla^2_{X,K}Y - \nabla^2_{Y,K}Y \\ &= R(K,X)X + R(K,Y)X + R(K,X)Y + R(K,Y)Y \\ &= -\nabla^2_{X,X}K - \nabla^2_{Y,Y}K + R(K,Y)X + R(K,X)Y\end{aligned}$$

By Bianchi's identity and R1

$$\begin{aligned} 0 &= R(K,X)Y + R(X,Y)K + R(Y,K)X \\ &= R(K,X)Y + R(X,Y)K - R(K,Y)X \\ \Rightarrow R(K,Y)X &= R(K,X)Y + R(X,Y)K \\ &= R(K,X)Y + \nabla^2_{X,Y}K - \nabla^2_{Y,X}K \end{aligned}$$

Adding $\nabla^2_{X+Y,X+Y}K$ and $R(K,X+Y)(X+Y)$ together

$$\begin{aligned} \nabla^2_{X+Y,X+Y}K + R(K,X+Y)(X+Y) &= \nabla^2_{X,Y}K + \nabla^2_{X,X}K \\ &\quad + \nabla^2_{Y,X}K + \nabla^2_{Y,Y}K \\ &\quad - \nabla^2_{X,X}K - \nabla^2_{Y,Y}K \\ &\quad + R(K,X)Y \\ &\quad + \nabla^2_{X,Y}K - \nabla^2_{Y,X}K \\ &\quad + R(K,X)Y \\ &= 2\nabla^2_{X,Y}K + 2R(K,X)Y \end{aligned}$$

Applying Equation (9.7) to $X+Y$

$$\nabla^2_{X+Y,X+Y}K = -R(K,X+Y)(X+Y)$$

Therefore

$$\begin{aligned} 2\nabla^2_{X,Y}K + 2R(K,X)Y &= \nabla^2_{X+Y,X+Y}K \\ &\quad + R(K,X+Y)(X+Y) \\ 2\nabla^2_{X,Y}K + 2R(K,X)Y &= -R(K,X+Y)(X+Y) \\ &\quad + R(K,X+Y)(X+Y) \\ 2\nabla^2_{X,Y}K + 2R(K,X)Y &= 0 \\ \Rightarrow 2\nabla^2_{X,Y}K &= -2R(K,X)Y \\ \Rightarrow \nabla^2_{X,Y}K &= -R(K,X)Y \end{aligned}$$

This concludes the proof of this proposition. □

10. The generalised Cheeger-Gromoll metric and vertical (p,q)-energy

Definition 10.1. [1] Using the same notation for the mappings π and K and the manifold M as in Section 6, *The Sasaki metric and vertical energy*, let $\mathcal{E}$ be a vector bundle and let $K : T\mathcal{E} \to \mathcal{E}$ be the connection map for ∇ as defined in Definition 6.2.

$$\begin{array}{ccc} \mathcal{E} & \xleftarrow{K} & T\mathcal{E} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle d\pi} \\ M & \longleftarrow & TM \end{array}$$

Let $\epsilon \in \mathcal{E}$ and $A, B \in T_\epsilon \mathcal{E}$. For any pair of parameters $p, q \in \mathbb{R}$ a *symmetric 2-covariant tensor* $h_{p,q}$ *on* $\mathcal{E}$ is defined as follows

$$h_{p,q}(A,B) = g(d\pi(A), d\pi(B)) + w^p(\epsilon)(\langle KA, KB\rangle + q\langle KA, \epsilon\rangle\langle KB, \epsilon\rangle)$$

where

$$w(\epsilon) = \frac{1}{1+|\epsilon|^2}$$

Remark 10.1. [1] If $(p,q) = (0,0)$ then $h_{p,q}$ is the Sasaki metric

$$h_{0,0}(A,B) = g(d\pi(A), d\pi(B)) + \langle KA, KB\rangle$$

Definition 10.2. [1] If $(p,q) = (1,1)$ then $h_{p,q}$ is the *Cheeger-Gromoll metric*

$$h_{1,1}(A,B) = g(d\pi(A), d\pi(B)) + \frac{1}{1+|\epsilon|^2}(\langle KA, KB\rangle + \langle KA, \epsilon\rangle\langle KB, \epsilon\rangle)$$

In all cases the $h_{p,q}$ metric is known as the *generalised Cheeger-Gromoll metric* and the set of $h_{p,q}$ metrics is known as the *2-parameter family of metrics of Cheeger-Gromoll type.*

If $\{E_i\}$ is an orthonormal basis in M then by the defining equations of the vertical subspace $V_\epsilon = \text{Ker } d\pi(\epsilon)$, the metric $h_{p,q}$ and $\nabla_X \sigma = K(d\sigma(X))$, the vertical component of $\|d\sigma\|^2$ with respect to the metric $h_{p,q}$ is

$$\begin{aligned}
|d^V\sigma|^2 &= h\left(d^V\sigma(E_i), d^V\sigma(E_i)\right) \\
&= g\left(d\pi(d^V\sigma(E_i)), d\pi(d^V\sigma(E_i))\right) \\
&+ w^p(\sigma)\left(\langle K\left(d^V\sigma(E_i)\right), K\left(d^V\sigma(E_i)\right)\rangle + q\langle K\left(d^V\sigma(E_i)\right), \sigma\rangle^2\right) \\
&= w^p(\sigma)\left(\langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle + q\langle \nabla_{E_i}\sigma, \sigma\rangle^2\right) \\
&= w^p(\sigma)\left(|\nabla\sigma|^2 + q|\nabla F|^2\right) \\
&= \frac{1}{(1+|\sigma|^2)^p}\left(|\nabla\sigma|^2 + \frac{1}{4}q|\nabla|\sigma|^2|^2\right)
\end{aligned}$$

where ∇F is the gradient vector of F which has the following equation

$$F = \frac{1}{2}|\sigma|^2$$

Definition 10.3. [1] $E^V_{p,q}$ is the notation for the vertical energy functional with respect to the metric $h_{p,q}$. It is referred to as the *vertical* (p,q)*-energy of* σ and has the equation

$$E^V_{p,q}(\sigma) = \frac{1}{2}\int_M \frac{1}{(1+|\sigma|^2)^p}\left(|\nabla\sigma|^2 + \frac{1}{4}q|\nabla|\sigma|^2|^2\right)\text{vol}(g)$$

for all $\sigma \in \mathcal{C}(\mathcal{E})$. When (p,q) are known this is written as $E^V(\sigma)$.

Lemma 10.1. *If* $(p,q) = (0,0)$ *then the vertical* (p,q)*-energy of* σ *equals the total bending of* σ*.*

Proof. The proof of this statement is shown in the following alignment of equations

$$\begin{aligned} E^V_{0,0}(\sigma) &= \frac{1}{2}\int_M \frac{1}{(1+|\sigma|^2)^p}\left(|\nabla\sigma|^2 + \frac{1}{4}q|\nabla|\sigma|^2|^2\right)\mathrm{vol}(g) \\ &= \frac{1}{2}\int_M \frac{1}{(1+|\sigma|^2)^0}\left(|\nabla\sigma|^2 + \frac{1}{4}0|\nabla|\sigma|^2|^2\right)\mathrm{vol}(g) \\ &= \frac{1}{2}\int_M |\nabla\sigma|^2\mathrm{vol}(g) \\ &= E^V(\sigma) \end{aligned}$$

which is the total bending of σ. □

The equation above leads to the generalised definition of a harmonic vector field as referred to Section 6.

Definition 10.4. [1] A vector field σ is a *harmonic vector field* if σ is *stationary* due to following the equation being true

$$\left.\frac{d}{dt}\right|_{t=0} E^V_{p,q}(\sigma_t) = 0$$

with respect to the metric $h_{p,q}$ on $\mathcal{E}$ for all smooth variations σ_t of σ through sections of $\mathcal{E}$. Then σ is also defined to be a (p,q)*-harmonic section of* $\mathcal{E}$.

The following theorem is important and will be used for the proofs of the later theorems in Section 11. A proof of this theorem can be found in [1].

Theorem 10.1. [1] *A vector field σ is a (p,q)-harmonic section of TM if and only if*

$$T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$$

where

$$T_p(\sigma) = (1+|\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma$$

is a vector field with

$$F = \frac{1}{2}|\sigma|^2$$

and

$$\phi_{p,q}(\sigma) = p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1+|\sigma|^2)\Delta F$$

is an $\mathbb{R}$-valued function.

11. Harmonic vector fields on Riemannian space forms

11.1. Conformal vector fields and the Lie derivative.

Definition 11.1. [4] A vector field σ on a Riemannian manifold (M, g), with $M \subset \mathbb{R}^n$, is *conformal* if and only if

$$g(\nabla_X \sigma, Y) + g(X, \nabla_Y \sigma) = 2\psi \cdot g(X, Y) \text{ for all } X, Y \in \mathbb{R}^n$$

where $\psi \in \mathbb{R}$ and $\psi \neq 0$.

Definition 11.2. [15] For a function (scalar field) ϕ on a manifold N

$$\begin{aligned} \phi : N &\to \mathbb{R} \\ q &\mapsto \phi(q) \end{aligned}$$

Its *pull-back* f^* to a manifold M is defined as

$$\begin{aligned} f^*\phi : M &\to \mathbb{R} \\ p &\mapsto (f^*\phi)(p) = \phi(f(p)) \end{aligned}$$

Let W be a vector field on M. Its *push-forward* f_* to a vector field on N is defined by giving its action on the function ϕ on N in the following equation.

$$(f_* W)\,\phi = W\,(f^*\phi) \tag{11.1}$$

Definition 11.3. [14][15] The *Lie derivative of the function* ϕ at a point p along a vector field σ is defined as

$$\mathscr{L}_\sigma \phi = \lim_{t \to 0} \frac{1}{t}\,[f_t^* \phi - \phi]$$

where $f_t(p)$ for fixed p is the flow of σ. Hence

$$\begin{aligned} (\mathscr{L}_\sigma \phi)(p) &= \lim_{t \to 0} \frac{1}{t}\,[\phi(f_t(p)) - \phi(p)] \\ &= \frac{d}{dt}\phi(f_t(p))\Big|_{t=0} \\ &= \sigma_p \phi \end{aligned}$$

The *Lie derivative of the vector field* W, denoted by $L_\sigma W$, is defined by using similar notation from the Lie derivative L_σ of the function ϕ at the point p along the vector field σ together with the push-foward of the inverse of f_t denoted by $\left(f_t^{-1}\right)_*$. It has the equation

$$\mathscr{L}_\sigma W = \lim_{t \to 0} \frac{1}{t}\,\left[\left(f_t^{-1}\right)_* W\,(f_t\,(p)) - W\,(p)\right]$$

The *Lie derivative of a metric* g *on* $X \in \mathbb{R}^n$ *and* $Y \in \mathbb{R}^n$ is defined by the equation

$$(\mathscr{L}_\sigma g)\,(X, Y) = \frac{d(\phi_t^* g)}{dt}\bigg|_{t=0} (X, Y)$$

Theorem 11.1. [14] *The equation for the Lie derivative of the vector field* W *can be simplified to the following equation.*

$$\mathscr{L}_\sigma W = [\sigma, W]$$

Definition 11.4. Let $\phi : M \to N$ be a smooth mapping of Riemannian manifolds. We say that ϕ is conformal if for all $X, Y \in \mathcal{C}(TM)$:

$$\langle d\phi(X), d\phi(Y)\rangle = \rho(x)\langle X, Y\rangle$$

where $\rho : M \to \mathbb{R}$ is a smooth positive function. The square root of ρ is called the *conformality factor of* ϕ.

Remark 11.1. In the definition above $d\phi_x : T_xM \to T_{\phi(x)}N$ is the differential of ϕ at x.

Proposition 11.1. *Let the flow of a vector field σ on a manifold M be the function $\phi_t : M \to M$ where $t \in \mathbb{R}$. The vector field σ is conformal if each ϕ_t is conformal.*

Proof. For this proof begin with the inner product of $d\phi_t(X)$ and $d\phi_t(Y)$.

$$\begin{aligned}
\langle d\phi_t(X), d\phi_t(Y)\rangle &= \rho_t(x)\langle X, Y\rangle \\
\Rightarrow \left.\frac{d}{dt}\right|_{t=0} \langle d\phi_t(X), d\phi_t(Y)\rangle &= \left.\frac{d\rho_t}{dt}\right|_{t=0} \langle X, Y\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} g(d\phi_t(X), d\phi_t(Y)) \\
&= \left.\frac{d(\phi_t^* g)}{dt}\right|_{t=0} (X, Y) \\
&= (\mathscr{L}_\sigma g)(X, Y) \\
&= \sigma(g(X, Y)) - g(\mathscr{L}_\sigma X, Y) \\
&\quad - g(X, \mathscr{L}_\sigma Y) \\
&= g(\nabla_\sigma X, Y) + g(X, \nabla_\sigma Y) \\
&\quad - g([\sigma, X], Y) - g(X, [\sigma, Y]) \\
&= g\left(\nabla_\sigma X - [\sigma, X], Y\right) \\
&\quad + g(X, \nabla_\sigma Y - [\sigma, Y]) \\
&= g(\nabla_X \sigma, Y) + g(X, \nabla_Y \sigma) \\
&= \langle \nabla_X \sigma, Y\rangle + \langle X, \nabla_Y \sigma\rangle \\
&= 2\psi\langle X, Y\rangle \text{ by Definition 11.1} \\
&= 2\psi \cdot g(X, Y)
\end{aligned}$$

Hence σ is conformal if each ϕ_t is conformal. □

11.2. **Conformal gradient fields on the unit sphere S^n.**

Definition 11.5. Let (M, g) be a Riemannian manifold with a Riemannian metric g and let $X, Y \in T_xM$ be linearly independent tangent vectors of M at a point x. The *sectional curvature* of X and Y, which is denoted by $K(X,Y)$, is defined to have the equation

$$K(X,Y) = \frac{g\left(R(X,Y)Y,X\right)}{\|X\|^2\|Y\|^2 - g(X,Y)^2}$$

where $R(X,Y)Y$ is the Riemann tensor. The Riemannian manifold M has a *constant sectional curvature* if $K(X,Y)$ has a constant value for all tangent vectors X and Y in the tangent bundle TM of the manifold M.

Definition 11.6. A *complete Riemannian manifold M* is a manifold with infinite geodesics.

Definition 11.7. A *Riemannian space form M* is a complete Riemannian manifold with a constant sectional curvature.

This section includes several subsections with examples of conformal (p,q)-harmonic vector fields on Riemannian space forms and all of them have a proof that they are conformal and harmonic. They are also all harmonic sections of the tangent bundle TM of the specific Riemannian space form M. This subsection includes the first example which is a conformal gradient field on the n-dimensional unit sphere $S^n \subset \mathbb{R}^{n+1}$. Before this example a definition is needed for the proof that it is conformal and (p,q)-harmonic.

Definition 11.8. [15] In a Riemannian manifold $M \in \mathbb{R}^n$ the *normal component* a_1 of a vector a in the direction of a vector x is defined by the equation

$$a_1 = \frac{a \cdot x}{\|x\|^2} x$$

The *tangential component* a_2 *of* a in the tangent space T_xM of a manifold M is defined as

$$a_2 = a - a_1 = a - \frac{a \cdot x}{\|x\|^2} x$$

Example figures of normal and tangential components are shown later in this section.

Definition 11.9. A vector field σ on the n-dimensional unit sphere S^n is said to be a *conformal gradient field* if

$$\sigma(x) = a - (a \cdot x)x \text{ for all } x \in S^n \text{ and for some } a \in \mathbb{R}^{n+1} \setminus \{0\}$$

Figure 11.1 gives a diagram of a conformal gradient field on the unit sphere S^2 with dashed grey lines representing the 3 dimensions of $\mathbb{R}^3 \supset S^2$ and black arrows representing the tangent vectors of the conformal gradient field. The dashed grey vector a is known as the *axial vector* because it starts at the origin and goes through the north pole on the vertical axis. The solid grey vector from the origin to the point x represents the radius of the unit sphere S^2 from the origin of the unit sphere to any point $x \in S^2$ and hence $|x|^2 = 1$.

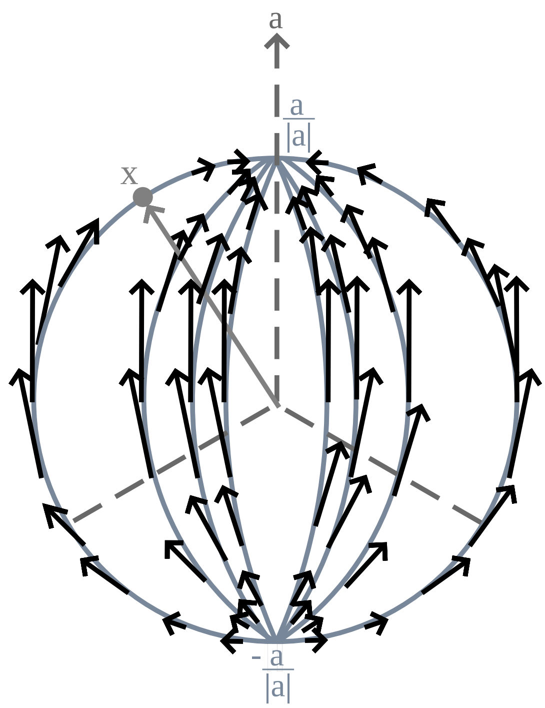


FIGURE 11.1. A diagram of a conformal gradient field on the unit sphere S^2

Proposition 11.2. *The vector field σ defined in Definition 11.9 has the two properties of being a conformal vector field and a gradient field on the unit sphere S^n.*

Proof. To prove this proposition first we show that σ is a gradient field then show that it is conformal. To find the equation for a gradient field on the unit sphere S^n, let $\omega : S^n \to \mathbb{R}$ equal $\omega = a \cdot x$ and calculate $\nabla\omega \cdot X$ which equals

$$\begin{aligned}\nabla\omega \cdot X &= d\,(a \cdot x)\,(X)\\ &= a \cdot dx(X) + x \cdot da(X) \text{ by the chain rule}\\ &= a \cdot X + x \cdot 0\\ &= a \cdot X\end{aligned}$$

Hence $\nabla\omega$ is the tangential component of a. From the definition of a unit sphere S^n it is known that $|x|^2 = 1$ therefore $|x| = 1$ and hence the unit normal field of the unit sphere S^n is $\xi = \pm x$. Using Definition 11.8, Figure 11.2 shows an example of the vectors x and a with the normal component of a in the direction of x and the tangential component of

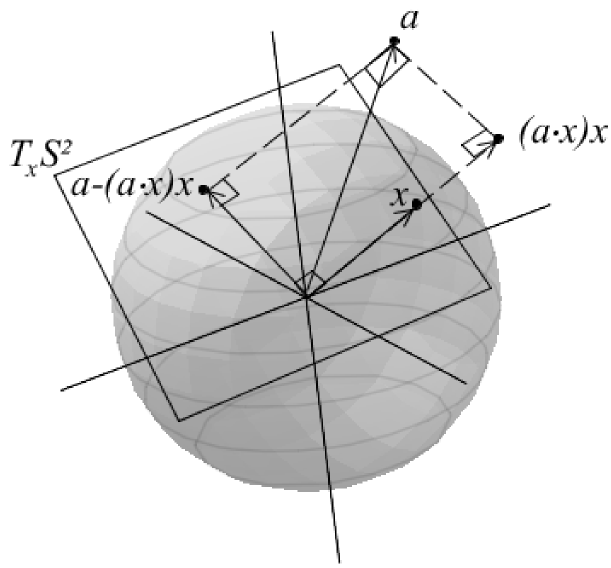


FIGURE 11.2. A diagram of the tangent space T_xS^2 of a point x on the unit sphere S^2 and a vector a with its tangential component in T_xS^2

a in T_xS^2. In $(n+1)$-dimensions, the normal component of a equals

$$\begin{aligned}\frac{a \cdot x}{|x|^2}x &= \frac{a \cdot x}{1}x \text{ because } S^n \text{ is the unit sphere} \\ &= (a \cdot x)x\end{aligned}$$

The tangential component equals

$$\begin{aligned}a - \frac{a \cdot x}{|x|^2}x &= \frac{a \cdot x}{1}x \text{ because } S^n \text{ is the unit sphere} \\ &= a - (a \cdot x)x\end{aligned}$$

Hence $\nabla\omega$ equals

$$\begin{aligned}\nabla\omega &= a - (a \cdot x)x \\ &= \sigma(x)\end{aligned}$$

Therefore $\sigma = \nabla\omega$ so σ is a gradient field of ω. To show that σ is conformal the following equation needs to be true

$$g(\nabla_X\sigma, Y) + g(X, \nabla_Y) = 2\psi \cdot g(X, Y)$$

for some $\psi : M \to \mathbb{R}$. By the Gauss formula and the chain rule $\nabla_X\sigma$ can be split into three components, one including the directional derivative of the vector a, another including the directional derivative of a dot product of an $\mathbb{R}$-valued function and an $\mathbb{R}^n$-valued function and the third including the second fundamental form of σ. This can be shown in the following equation.

$$\nabla_X\sigma = D_Xa - D_X(a \cdot x)x - \alpha(X, \sigma)\xi$$

The first component of $\nabla_X \sigma$ is

$$\begin{aligned} D_X a &= da(X) \\ &= 0 \end{aligned}$$

The second component of $\nabla_X \sigma$ is

$$\begin{aligned} -D_X(a \cdot x)x &= -X(a \cdot x)x - (a \cdot x)D_X x \\ &= -d(a \cdot x)(X)x - (a \cdot x)dx(X) \\ &= -\left(a \cdot X\right)x - (a \cdot x)X \end{aligned}$$

The shape operator $A(X)$ equals

$$\begin{aligned} A(X) &= -d\xi(X) \\ &= -dx(X) \\ &= -X \end{aligned}$$

Hence the third component of $\nabla_X \sigma$ is

$$\begin{aligned} -\alpha(X,\sigma)\xi &= \left(A(X) \cdot \sigma\right) x \\ &= -\left(-\left(X \cdot \sigma\right)(x)\right) \\ &= -\left(-\left(X \cdot \left(a - \left(a \cdot x\right) x\right)\right)(x)\right) \\ &= \left(a \cdot X\right) x - (a \cdot x)x \left(x \cdot X\right) \\ &= \left(a \cdot X\right) x - (a \cdot x)0 \text{ because } x \text{ is orthogonal to X} \\ &= \left(a \cdot X\right) x \end{aligned}$$

Adding the three components together $\nabla_X \sigma$ equals

$$\begin{aligned} \nabla_X \sigma &= -(a \cdot X)x - (a \cdot x)X + (a \cdot X)x \\ &= -(a \cdot x)X \end{aligned}$$

Therefore

$$\begin{aligned} g(\nabla_X \sigma, Y) + g(X, \nabla_Y \sigma) &= -(a \cdot x)(X \cdot Y) - X \cdot (a \cdot x)Y \\ &= -2(a \cdot x)(X \cdot Y) \\ &= \psi \cdot (X \cdot Y) \\ \Rightarrow \psi &= -2(a \cdot x) \end{aligned}$$

Hence σ is conformal. □

Theorem 11.2. [1] *Let $M = S^n \subset \mathbb{R}^{n+1}$ be the n-dimensional unit sphere and let σ be a conformal gradient field on S^n. Then σ is a (p,q)-harmonic section of TM and hence a harmonic vector field if and only if*

$$p = n+1, \;\; q = 2-n, \;\; |a| = \frac{1}{\sqrt{-q}} \;\; \text{and } n \geq 3$$

Proof. In this example the manifold and Riemannian metric being used are $(M, g) = (S^n, \cdot)$ which are the n-dimensional unit sphere and the dot product. For reference the equation for a conformal gradient field σ on S^n is

$$\begin{aligned}\sigma(x) &= a - (a \cdot x)x \\ &= a - \omega x\end{aligned}$$

and the equation for $\nabla_X \sigma$ is

$$\begin{aligned}\nabla_X \sigma &= -\omega X \\ &= -(a \cdot x)X\end{aligned}$$

By Theorem 10.1, σ is a (p, q)-harmonic section if and only if

$$T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$$

where

$$T_p(\sigma) = (1 + |\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma$$

is a vector field with

$$F = \frac{1}{2}|\sigma|^2$$

and

$$\phi_{p,q}(\sigma) = p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1 + |\sigma|^2)\Delta F$$

is an $\mathbb{R}$-valued function. To calculate $T_p(\sigma)$ and $\phi_{p,q}(\sigma)$ the values F, ∇F, $|\nabla F|^2$, $\nabla_{\nabla F}\sigma$, ΔF, $|\nabla\sigma|^2$, $\nabla^*\nabla\sigma$ and $\nabla^2_{X,Y}\sigma$ need to be found. Starting with the calculation of $\nabla^2_{X,Y}\sigma$

$$\begin{aligned}\nabla^2_{X,Y}\sigma &= \nabla_X\left(\nabla_Y\sigma\right) - \nabla_{\nabla_X Y}\sigma \\ &= \nabla_X\left(-\left(a \cdot x\right)Y\right) - \left(-(a \cdot x)\right)\nabla_X Y \\ &= D_X\left(-\left(a \cdot x\right)Y\right) - \alpha\left(X, \left(-\left(a \cdot x\right)Y\right)\right)\xi + \left(a \cdot x\right)\nabla_X Y \\ &= X\left(-\left(a \cdot x\right)\right)Y + \left(-a \cdot x\right)\nabla_X Y \\ &\quad - \left(-\left(X \cdot -\left(a \cdot x\right)Y\right)\right)x + (a \cdot x)\nabla_X Y \\ &= -(a \cdot X)Y + (X \cdot (a \cdot x)x)Y \\ &= \left(\left(-a + \left(a \cdot x\right)x\right) \cdot X\right)Y \\ &= \left(-\sigma \cdot X\right)Y \\ &= -\left(X \cdot \sigma\right)Y\end{aligned}$$

Then $\nabla^*\nabla\sigma$ equals

$$\begin{aligned}\nabla^*\nabla\sigma &= -\sum_{i=1}^{n}\nabla^2_{E_i,E_i}\sigma \\ &= -\sum_{i=1}^{n}\left(\left(-a + \left(a \cdot x\right)x\right) \cdot E_i\right)E_i \\ &= a - (a \cdot x)x \\ &= \sigma\end{aligned}$$

where $\{E_i\}_{i=1}^{n}$ is an orthonormal basis of $T\mathbb{R}^n$. Recalling the equation for $\nabla_X \sigma$, $|\nabla\sigma|^2$ is

$$\begin{aligned}
|\nabla\sigma|^2 &= \sum_{i=1}^{n} |\nabla_{E_i}\sigma|^2 \\
&= \sum_{i=1}^{n} \| -(a\cdot x)E_i\|^2 \\
&= \sum_{i=1}^{n} \left((a\cdot x)^2 \left(E_i \cdot E_i\right)\right) \\
&= n(a\cdot x)^2 \\
&= n\omega^2
\end{aligned}$$

The rest of the values required include F so that is calculated next by using the equation for $2F$. Let $c = |a|^2$ and $|x|^2 = r^2 = 1$.

$$\begin{aligned}
2F &= |\sigma|^2 \\
&= |a - (a\cdot x)x|^2 \\
&= |a|^2 + (a\cdot x)^2 r^2 - 2(a\cdot x)^2 \\
&= |a|^2 + (a\cdot x)^2 (r^2 - 2) \\
&= c^2 + \omega^2(r^2 - 2) \\
\Rightarrow F &= \frac{1}{2}|a|^2 + \frac{1}{2}(a\cdot x)^2(r^2-2) \\
&= \frac{1}{2}c^2 + \frac{1}{2}\omega^2(r^2-2) \\
&= \frac{1}{2}c^2 + \frac{1}{2}\omega^2(1-2) \\
&= \frac{1}{2}c^2 - \frac{1}{2}\omega^2 \\
\Rightarrow 2F &= c^2 - \omega^2
\end{aligned}$$

Since $\sigma = \nabla\omega$, ∇F can be calculated in this way

$$\begin{aligned}
\nabla F &= \nabla\left(\frac{1}{2}c^2 - \frac{1}{2}\omega^2\right) \\
&= \nabla\frac{1}{2}c^2 - \nabla\frac{1}{2}\omega^2 \\
&= 0 - \omega\nabla\omega \text{ by the chain rule} \\
&= -\omega\sigma
\end{aligned}$$

Using the formula for ∇F in relation to σ from the equation above, $2\nabla_{\nabla F}\sigma$ can be calculated as

$$\begin{aligned} 2\nabla_{\nabla F}\sigma &= -(a\cdot x)2\nabla F \\ &= -2(a\cdot x)\,(-\omega\sigma) \\ &= 2\omega^2\sigma \\ \Rightarrow \nabla_{\nabla F}\sigma &= \omega^2\sigma \end{aligned}$$

ΔF can be found by using Corollary 3.1.

$$\begin{aligned} \Delta F &= -|\nabla\sigma|^2 + (\nabla^*\nabla\sigma)\cdot\sigma \\ &= -n\omega^2 + \sigma\cdot\sigma \\ &= -n\omega^2 + (a-(a\cdot x)x)\cdot(a-(a\cdot x)x) \\ &= -n\omega^2 + |\sigma|^2 \\ &= -n\omega^2 + 2F \\ &= -n\omega^2 + c^2 - \omega^2 \end{aligned}$$

All of the values for $T_p(\sigma)$ have been calculated so they can now be used in the equation for $T_p(\sigma)$.

$$\begin{aligned} T_p(\sigma) &= (1+|\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma \\ &= (1+2F)\sigma + 2p\omega^2\sigma \\ &= (1+2F+2p\omega^2)\sigma \end{aligned}$$

$T_p(\sigma)$ is proportional to σ so no restrictions can be confirmed yet. The last part of $\phi_{p,q}(\sigma)$ required is $|\nabla F|^2$ which equals

$$\begin{aligned} |\nabla F|^2 &= |-\omega\sigma|^2 \\ &= \omega^2|\sigma|^2 \\ &= 2\omega^2 F \\ &= \omega^2(c^2-\omega^2) \\ &= \omega^2c^2 - \omega^4 \end{aligned}$$

Hence $\phi_{p,q}(\sigma)$ equals

$$\begin{aligned} \phi_{p,q}(\sigma) &= p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1+|\sigma|^2)\Delta F \\ &= p(n\omega^2) - pq(2\omega^2F) - q(1+2F)(-n\omega^2+2F) \end{aligned}$$

Still in the case when σ is (p,q)-harmonic, the equation $T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$ can then be simplified to

$$\begin{aligned} T_p(\sigma) &= \phi_{p,q}(\sigma)\sigma \\ \Rightarrow (1+2F+2p\omega^2)\sigma &= (p(n\omega^2) - pq(2\omega^2 F) \\ &\quad - q(1+2F)(-n\omega^2+2F))\sigma \\ \Rightarrow 1+2F+2p\omega^2 &= p(n\omega^2) - pq(2\omega^2 F) \\ &\quad - q(1+2F)(-n\omega^2+2F) \\ \Rightarrow 1+c^2-\omega^2+2p\omega^2 &= p(n\omega^2) - pq\left(\omega^2(c^2-\omega^2)\right) \\ &\quad - q(1+c^2-\omega^2)(-n\omega^2+c^2-\omega^2) \\ \Rightarrow 1+c^2+(2p-1)\omega^2 &= p(n+q)\omega^2 \\ &\quad - q(1+c^2-\omega^2)(c^2-(n-p+1)\omega^2) \end{aligned}$$

This is a polynomial in ω. Since ω is a continuous function on $M = S^n$ this polynomial is zero if and only if the coefficients of the powers of ω vanish. Hence the following equations come from the coefficients of ω^0, ω^2 and ω^4.

From ω^0,

$$\begin{aligned} 1+c^2 &= -q(1+c^2)c^2 \\ \Rightarrow 1 &= -qc^2 \\ \Rightarrow q &= -\frac{1}{c^2} \qquad (11.2) \\ \therefore |a| &= \frac{1}{\sqrt{-q}} \end{aligned}$$

and from ω^2,

$$\begin{aligned} 2p-1 &= p(n+q) - q\left(-c^2 + \left(1+c^2\right)\left(-\left(n-p+1\right)\right)\right) \\ \Rightarrow 2p-1 &= p(n+q) + qc^2 + q(1+c^2)(n-p+1) \end{aligned}$$

and from ω^4,

$$\begin{aligned} 0 &= -q\left(-1\right)\left(-\left(n-p+1\right)\right) \\ 0 &= -q(n-p+1) \\ \Rightarrow 0 &= n-p+1 \\ \Rightarrow p &= n+1 \end{aligned}$$

Using the results of equations from ω^0 and ω^4 in the equation from ω^2 it equals

$$\begin{aligned} 2(n+1) - 1 &= (n+1)(n+q) - 1 + q(1+c^2)(n-n-1+1) \\ \Rightarrow 2(n+1) + 0 &= (n+1)(n+q) + 0 \\ \Rightarrow 2(n+1) &= (n+1)(n+q) \\ \Rightarrow 2(n+1) &= (n+1)n + q(n+1) \\ \Rightarrow (2-n)(n+1) &= q(n+1) \\ \Rightarrow 2-n &= q \\ \therefore n &\geq 3 \text{ by Equation (11.2)} \end{aligned}$$

Hence σ is a (p,q)-harmonic section of TS^n if and only if

$$p = n+1, \;\; q = 2-n, \;\; |a| = \frac{1}{\sqrt{-q}} \text{ and } n \geq 3. \qquad \square$$

11.3. **Conformal gradient fields on the hyperbolic space H^n.** This subsection introduces the hyperbolic space H^n.

Definition 11.10. [2] Let the $(n+1)$-*Lorentzian space* be denoted by $\mathbb{R}^{n,1}$ which equals $\mathbb{R}^{n+1}$ equipped with the *Lorentzian inner product*

$$\langle x, y \rangle = x_1 y_1 + \ldots + x_n y_n - x_{n+1} y_{n+1}$$

for all $x, y \in \mathbb{R}^{n,1}$. Then the *hyperbolic space* H^n is the set

$$H^n = \{x \in \mathbb{R}^{n,1} : \langle x, x \rangle = -1, x_{n+1} > 0\}$$

Let $a \in \mathbb{R}^{n,1}$ be any vector and denote $\mu = \langle a, a \rangle$. The Lorentzian inner product μ may be negative. Let $\omega : H^n \to \mathbb{R}$ be the Lorentzian inner product of $x \in H^n$ and a

$$\omega(x) = \langle a, x \rangle$$

for all $x \in H^n$.

Definition 11.11. A vector field σ on the hyperbolic space H^n is said to be a *conformal gradient field on the hyperbolic space H^n* if

$$\sigma = a + \langle a, x \rangle x \text{ for all } x \in H^n \text{ and } a \in \mathbb{R}^{n,1}$$

Figure 11.3 gives the 2-dimensional hyperbolic space H^2 in the 3-dimensional Euclidean space $\mathbb{R}^3$ with black arrows showing the vectors of a conformal gradient field on this Riemannian manifold.

Definition 11.12. [2] The *light cone* is defined by the set

$$\{x \in \mathbb{R}^{n,1} : \langle x, x \rangle = 0\}$$

There are three different cases in which the vector a could be, as defined as follows. The vector a is *timelike* if $\mu < 0$. The vector a is *spacelike* if $\mu > 0$. The vector a is *lightlike* if $\mu = 0$.

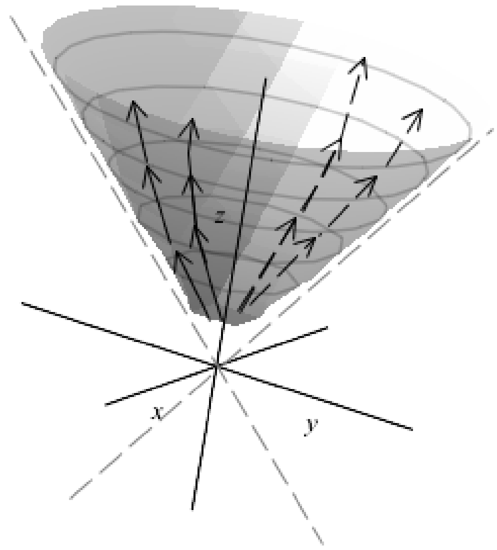


Figure 11.3. A conformal gradient field on the 2-dimensional hyperbolic space H^2

Figure 11.4 gives a diagram of the light cone including the three different cases of the vector a. If a is timelike a is a vector inside the cone as shown by the black arrow going straight through the cone. If a is spacelike a is outside the cone as shown by a diagonal black arrow outside the cone. If a is lightlike a is on the cone and an example of this is shown by the third black arrow on the boundary of the cone.

Proposition 11.3. [2] *The vector field σ defined in Definition 11.11 has the two properties of being a conformal vector field and a gradient field on the hyperbolic space H^n equipped with the Lorentzian inner product, denoted by $\langle -, - \rangle$, where*

$$\begin{aligned}\sigma(x) &= a + \omega(x)x \\ &= a + \langle a, x\rangle x\end{aligned}$$

and σ is the gradient field $\nabla\omega$ of

$$\omega(x) = \langle a, x\rangle$$

Proof. To prove this proposition first show that σ is a gradient field then show that it is conformal. To find the equation for a gradient field on the hyperbolic space H^n, calculate $\langle \nabla\omega, X\rangle$. Since $\omega = \langle a, x\rangle$ is a 1-form in $\mathbb{R}$, $\langle \nabla\omega, X\rangle$ equals

$$\begin{aligned}\langle \nabla\omega, X\rangle &= d\langle a, x\rangle(X) \\ &= \langle a, dx\rangle(X) + \langle x, da\rangle(X) \text{ by the chain rule} \\ &= \langle a, X\rangle + \langle x, 0\rangle \\ &= \langle a, X\rangle\end{aligned}$$

Hence $\nabla\omega$ is the tangential component of a. From the definition of the

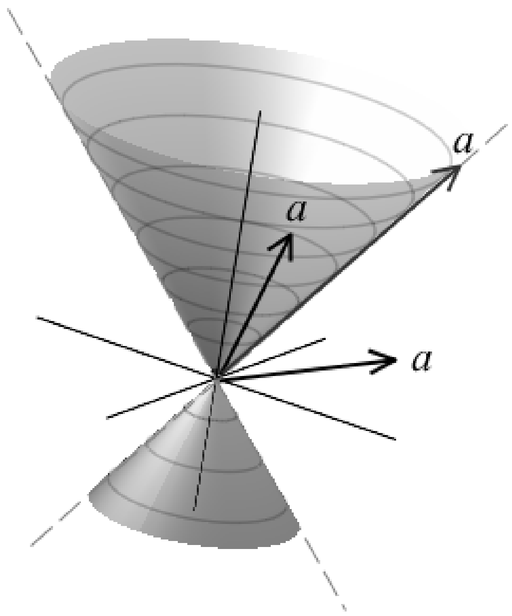


FIGURE 11.4. The light cone with arrows representing the 3 different cases of the vector a

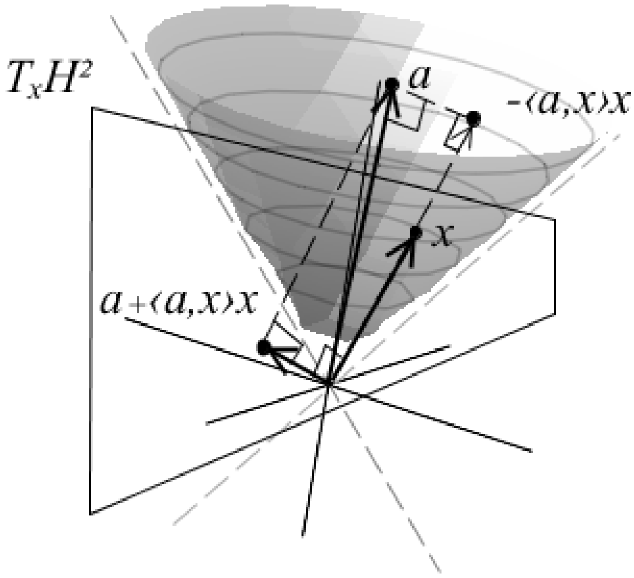


FIGURE 11.5. A diagram of the tangent space T_xH^2 of a point x on the 2-dimensional hyperbolic space H^2 and a vector a with its tangential component in T_xH^2

hyperbolic space it is known that $\langle x, x\rangle = -1$. Using Definition 11.8, Figure 11.5 shows an example of the vectors $x \in H^2$ and $a \in \mathbb{R}^3$ with the normal and tangential components of a. In $(n+1)$-dimensions, the normal component of a in T_xH^n equals

$$\begin{aligned} a_1 &= \frac{\langle a, x\rangle}{\langle x, x\rangle}x \\ &= \frac{\langle a, x\rangle}{-1}x \text{ because } x \text{ is in the hyperbolic space} \\ &= -\langle a, x\rangle x \end{aligned}$$

then the tangential component of a in T_xH^n equals

$$\begin{aligned} a_2 &= a - \frac{\langle a, x\rangle}{\langle x, x\rangle}x \\ &= a - \frac{\langle a, x\rangle}{-1}x \text{ because } x \text{ is in the hyperbolic space} \\ &= a + \langle a, x\rangle x \end{aligned}$$

Hence $\nabla\omega$ equals

$$\begin{aligned} \nabla\omega &= a + \langle a, x\rangle x \\ &= \sigma(x) \end{aligned}$$

Therefore $\sigma = \nabla\omega$ so σ is a gradient field of ω. To show that σ is conformal the following equation needs to be true

$$\langle \nabla_X\sigma, Y\rangle + \langle X, \nabla_Y\rangle = 2\psi\langle X, Y\rangle$$

for some $\psi : M \to \mathbb{R}$. By the chain rule $D_X\sigma$ can be split into two components, one including the directional derivative of the vector a and the other including the directional derivative of a dot product of an $\mathbb{R}$-valued function and an $\mathbb{R}^n$-valued function. This can be shown in the following equation.

$$D_X\sigma = D_Xa + D_X\langle a, x\rangle x$$

The first component of $D_X\sigma$ is

$$\begin{aligned} D_Xa &= da(X) \\ &= 0 \end{aligned}$$

The second component of $D_X\sigma$ is

$$\begin{aligned} D_X\langle a, x\rangle x &= X(\langle a, x\rangle)x + (\langle a, x\rangle)D_Xx \\ &= d(\langle a, x\rangle)(X)x + \langle a, x\rangle dx(X) \\ &= \langle a, X\rangle x + \langle a, x\rangle X \end{aligned}$$

Adding the two components together $D_X\sigma$ equals

$$\begin{aligned} D_X\sigma &= 0 + \langle a, X\rangle x + \langle a, x\rangle X \\ &= \langle a, X\rangle x + \langle a, x\rangle X \end{aligned}$$

By the Gauss formula $\nabla_X \sigma$ is the tangential component of $D_X \sigma$. Hence $\nabla_X \sigma$ equals

$$\begin{aligned}\nabla_X \sigma &= D_X \sigma(x) + \langle D_X \sigma(x), x\rangle x \\ &= \langle a, X\rangle x + \langle a, x\rangle X + \langle \langle a, X\rangle x + \langle a, x\rangle X, x\rangle x \\ &= \langle a, X\rangle x + \langle a, x\rangle X + (\langle \langle a, X\rangle x, x\rangle + \langle \langle a, x\rangle X, x\rangle)\, x \\ &= \langle a, X\rangle x + \langle a, x\rangle X + (\langle a, X\rangle(-1) + \langle a, x\rangle(0))\, x \\ &= \langle a, X\rangle x + \langle a, x\rangle X - \langle a, X\rangle x \\ &= \langle a, x\rangle X\end{aligned}$$

Therefore

$$\begin{aligned}\langle \nabla_X \sigma, Y\rangle + \langle X, \nabla_Y \sigma\rangle &= \langle a, x\rangle\langle X, Y\rangle + \langle X, \langle a, x\rangle\, Y\rangle \\ &= \langle a, x\rangle\langle X, Y\rangle + \langle a, x\rangle\langle X, Y\rangle \\ &= 2\langle a, x\rangle\langle X, Y\rangle \\ &= 2\psi\langle X, Y\rangle \\ \Rightarrow \psi &= \langle a, x\rangle\end{aligned}$$

Hence σ is conformal. □

Theorem 11.3. [2] *Let σ be a conformal gradient field on the hyperbolic space $M = H^n$ equipped with the Lorentzian metric $\langle -, -\rangle$. Let a be any vector in the $(n+1)$-Lorentzian space $\mathbb{R}^{n,1}$ and denote the inner product of a with itself by $\mu = \langle a, a\rangle$. Let x be any vector in the hyperbolic space H^n. Then $\sigma = a + \langle a, x\rangle x$ is a (p, q)-harmonic section of TM and hence a harmonic vector field if and only if any of the following conditions are true*

- *if $\mu > 0$ then $\mu = \frac{1}{n-2}$, $n > 2$, $p = n + 1$ and $q = 2 - n$,*
- *if $\mu < 0$ and $n = 2$, then $\mu = -1$, $p = 3$ and $q = -\frac{1}{2}$,*
- *if $\mu < 0$ and $n > 2$, then $\mu = -1$, $p = n + 1$ and $q = 1 - n + \frac{1}{n}$,*
- *if $\mu < 0$ and $n > 2$, then $\mu = -1$, $p = \frac{1}{2-n}$ and $q = 0$*

Proof. In this example the manifold and Riemannian metric being used are $(M, g) = (H^n, \langle -, -\rangle)$ which are the n-dimensional hyperbolic space and the Lorentzian metric. For reference the equation for σ is

$$\begin{aligned}\sigma(x) &= a + \langle a, x\rangle x \\ &= a + \omega x\end{aligned}$$

and, from the proof of Proposition 11.3, the equation for $\nabla_X \sigma$ is

$$\begin{aligned}\nabla_X \sigma &= \omega X \\ &= \langle a, x\rangle X\end{aligned}$$

By Theorem 10.1, σ is a (p, q)-harmonic section if and only if

$$T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$$

where

$$T_p(\sigma) = (1 + |\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma$$

is a vector field with

$$F = \frac{1}{2}|\sigma|^2$$

and

$$\phi_{p,q}(\sigma) = p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1 + |\sigma|^2)\Delta F$$

is an $\mathbb{R}$-valued function.

Again, to calculate $T_p(\sigma)$ and $\phi_{p,q}(\sigma)$ the values F, ∇F, $|\nabla F|^2$, $\nabla_{\nabla F}\sigma$, ΔF, $|\nabla\sigma|^2$, $\nabla^*\nabla\sigma$ and $\nabla^2_{X,Y}\sigma$ need to be found. Starting with the calculation of $\nabla^2_{X,Y}\sigma$

$$\begin{aligned}
\nabla^2_{X,Y}\sigma &= \nabla_X\left(\nabla_Y\sigma\right) - \nabla_{\nabla_X Y}\sigma \\
&= \nabla_X\left(\langle a, x\rangle Y\right) - \langle a, x\rangle\nabla_X Y \\
&= D_X\left(\langle a, x\rangle Y\right) - \alpha\left(X, \langle a, x\rangle Y\right)\xi - \langle a, x\rangle\nabla_X Y \\
&= X\left(\langle a, x\rangle\right)Y + \langle a, x\rangle\nabla_X Y \\
&\quad - \left(-\langle X, \langle a, x\rangle Y\rangle\right)(x) - \langle a, x\rangle\nabla_X Y \\
&= \langle a, X\rangle Y + \langle X, \langle a, x\rangle x\rangle Y \\
&= \langle a + \langle a, x\rangle x, X\rangle Y \\
&= \langle\sigma, X\rangle Y
\end{aligned}$$

So $\nabla^*\nabla\sigma$ equals

$$\begin{aligned}
\nabla^*\nabla\sigma &= -\sum_{i=1}^{n}\langle a + \langle a, x\rangle x, E_i\rangle E_i \\
&= -a - \langle a, x\rangle x \\
&= -\sigma
\end{aligned}$$

where $\{E_i\}_{i=1}^n$ is an orthonormal basis of $T\mathbb{R}^n$. Recalling the equation for $\nabla_X\sigma$, $|\nabla\sigma|^2$ is

$$\begin{aligned}
|\nabla\sigma|^2 &= \sum_{i=1}^{n}|\nabla_{E_i}\sigma|^2 \\
&= \sum_{i=1}^{n}\|\langle a, x\rangle E_i\|^2 \\
&= \sum_{i=1}^{n}\left(\langle a, x\rangle^2\langle E_i, E_i\rangle\right) \\
&= n\langle a, x\rangle^2 \\
&= n\omega^2
\end{aligned}$$

The rest of the values required include F so that is calculated next by using the equation for $2F$. Since μ is the inner product of $a \in \mathbb{R}^{n,1}$ with itself it can also be denoted as $\mu = |a|^2$. From the definition of H^n, the

Lorentzian inner product of $x \in H^n$ with itself is $\langle x, x\rangle = |x|^2 = r^2 = -1$.

$$\begin{aligned}
2F &= |\sigma|^2 \\
&= |a + \langle a, x\rangle x|^2 \\
&= |a|^2 + \langle a, x\rangle^2 r^2 + 2\langle a, x\rangle^2 \\
&= |a|^2 + \langle a, x\rangle^2 (r^2 + 2) \\
&= \mu + \omega^2 (r^2 + 2) \\
\Rightarrow F &= \frac{1}{2}|a|^2 + \frac{1}{2}\langle a, x\rangle^2 (r^2 + 2) \\
&= \frac{1}{2}\mu + \frac{1}{2}\omega^2 (r^2 + 2) \\
&= \frac{1}{2}\mu + \frac{1}{2}\omega^2 (-1 + 2) \\
&= \frac{1}{2}\mu + \frac{1}{2}\omega^2 \\
\Rightarrow 2F &= \mu + \omega^2
\end{aligned}$$

Since $\sigma = \nabla\omega$, ∇F can be calculated in this way

$$\begin{aligned}
\nabla F &= \nabla\left(\frac{1}{2}\mu^2 + \frac{1}{2}\omega^2\right) \\
&= \nabla\frac{1}{2}\mu^2 + \nabla\frac{1}{2}\omega^2 \\
&= 0 + \frac{1}{2}2\omega\nabla\omega \text{ by the chain rule} \\
&= \omega\sigma
\end{aligned}$$

Using the formula for ∇F in relation to σ from the equation above, $2\nabla_{\nabla F}\sigma$ can be calculated as

$$\begin{aligned}
2\nabla_{\nabla F}\sigma &= \langle a, x\rangle 2\nabla F \\
&= 2\langle a, x\rangle\,(\omega\sigma) \\
&= 2\omega^2\sigma \\
\Rightarrow \nabla_{\nabla F}\sigma &= \omega^2\sigma
\end{aligned}$$

ΔF can be found by using Corollary 3.1.

$$\begin{aligned}
\Delta F &= -|\nabla\sigma|^2 - \langle\nabla^*\nabla\sigma, \sigma\rangle \\
&= -n\omega^2 - \langle\sigma, \sigma\rangle \\
&= -n\langle a, x\rangle^2 - \langle(a + \langle a, x\rangle x), (a + \langle a \cdot x\rangle x)\rangle \\
&= -n\langle a, x\rangle^2 - |\sigma|^2 \\
&= -n\langle a, x\rangle^2 - 2F \\
&= -n\omega^2 - \mu - \omega^2
\end{aligned}$$

All of the values for $T_p(\sigma)$ have been calculated so they can now be used in the equation for $T_p(\sigma)$.

$$\begin{aligned} T_p(\sigma) &= (1+|\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma \\ &= -(1+2F)\sigma + 2p\omega^2\sigma \\ &= \left(-(1+2F)+2p\omega^2\right)\sigma \end{aligned}$$

$T_p(\sigma)$ is proportional to σ so no restrictions can be confirmed yet. The last part of $\phi_{p,q}(\sigma)$ required is $|\nabla F|^2$ which equals

$$\begin{aligned} |\nabla F|^2 &= |\omega\sigma|^2 \\ &= \omega^2|\sigma|^2 \\ &= 2\omega^2 F \\ &= \omega^2(\mu+\omega^2) \\ &= \omega^2\mu + \omega^4 \end{aligned}$$

Hence $\phi_{p,q}(\sigma)$ equals

$$\begin{aligned} \phi_{p,q}(\sigma) &= p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1+|\sigma|^2)\Delta F \\ &= p(n\omega^2) - pq(2\omega^2 F) - q(1+2F)(-n\omega^2 - 2F) \end{aligned}$$

Still in the case when σ is (p,q)-harmonic, the equation $T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$ can then be simplified to

$$\begin{aligned} T_p(\sigma) &= \phi_{p,q}(\sigma)\sigma \\ \Rightarrow (-(1+2F)+2p\omega^2)\sigma &= (p(n\omega^2) - pq(2\omega^2 F) \\ &\quad - q(1+2F)(-n\omega^2 - 2F))\sigma \\ \Rightarrow -(1+2F)+2p\omega^2 &= p(n\omega^2) - pq(2\omega^2 F) \\ &\quad - q(1+2F)(-n\omega^2 - 2F) \\ \Rightarrow -(1+2F)+p(4F-2\mu) &= p(n(2F-\mu)) - pq((4F-2\mu)F) \\ &\quad - q(1+2F)(-n(2F-\mu)-2F) \\ \Rightarrow -1-2F+4Fp-2\mu p &= n2Fp - np\mu - pq4F^2 + 2pq\mu F \\ &\quad + qn2F - qn\mu + q2F + qn4F^2 - qn\mu 2F \\ &\quad + q4F^2 \\ \Rightarrow 0 &= (1 - nq\mu - pn\mu + 2p\mu)(2F)^0 \\ &\quad + (-2p+1+nq+pq\mu+nq+q-qn\mu)2F \\ &\quad + (-pq+qn+q)(2F)^2 \end{aligned}$$

This is a polynomial in $2F$. Since $2F$ is a continuous function on $M = H^n$ this polynomial is zero if and only if the coefficients of the powers of $2F$ vanish. Hence the following equations come from the coefficients of $(2F)^0$, $2F$ and $(2F)^2$.

From $(2F)^0$,

$$\begin{aligned} 1 - nq\mu - pn\mu + 2p\mu &= 0 \\ \Rightarrow ((2-n)p - nq)\mu &= -1 \end{aligned} \tag{11.3}$$

and from $2F$,

$$\begin{aligned} -2p + 1 + nq + pq\mu + nq + q - qn\mu &= 0 \\ \Rightarrow p(n-2) + (n+1)q + (p-n)q\mu &= -1 \end{aligned} \tag{11.4}$$

and from $(2F)^2$,

$$\begin{aligned} -pq + qn + q &= 0 \\ \Rightarrow (n+1-p)q &= 0 \end{aligned}$$

To prove the four conditions, analyse the resulting equations from $(2F)^0$, $2F$ and $(2F)^2$. If $q = 0$, Equation (11.3) equals

$$\begin{aligned} ((2-n)p - n0)\mu &= -1 \\ \Rightarrow ((2-n)p)\mu &= -1 \\ \Rightarrow p &= \frac{1}{2-n} \text{ and } \mu = -1 \\ \Rightarrow n &> 2 \end{aligned}$$

Hence the fourth condition of the theorem is true. If $p = n + 1$ and $\mu = -1$ then Equation (11.3) equals

$$\begin{aligned} ((2-n)(n+1) - nq)(-1) &= -1 \\ \Rightarrow (2-n)(n+1) - nq &= 1 \\ \Rightarrow nq &= 1 + n - n^2 \\ \Rightarrow q &= \frac{1}{n} + 1 - n \end{aligned}$$

Hence the third condition is true. Let $p = n+1$ so that the subtraction of Equation (11.3) from Equation (11.4) equals

$$\begin{aligned} (n-2)(n+1)+(n+1)q+(n+1-n)qu-((2-n)(n+1)-nq)\,\mu &= -1+1 \\ \Rightarrow (n-2)(n+1) + (n+1)q + q\mu - ((2-n)(n+1) - nq)\,\mu &= 0 \\ \Rightarrow (n-2)(n+1)(1+\mu) + nq(1+\mu) + q(1+\mu) &= 0 \\ \Rightarrow (n-2+q)(1+\mu)(n+1) &= 0 \\ \Rightarrow (n-2+q)(1+\mu) &= 0 \end{aligned}$$

From this equation the rest of the conditions can be proved. Let $q = 2 - n$

$$\begin{aligned} (n-2+2-n)(1+\mu) &= 0(1+\mu) \\ \Rightarrow 0(1+\mu) &= 0 \end{aligned}$$

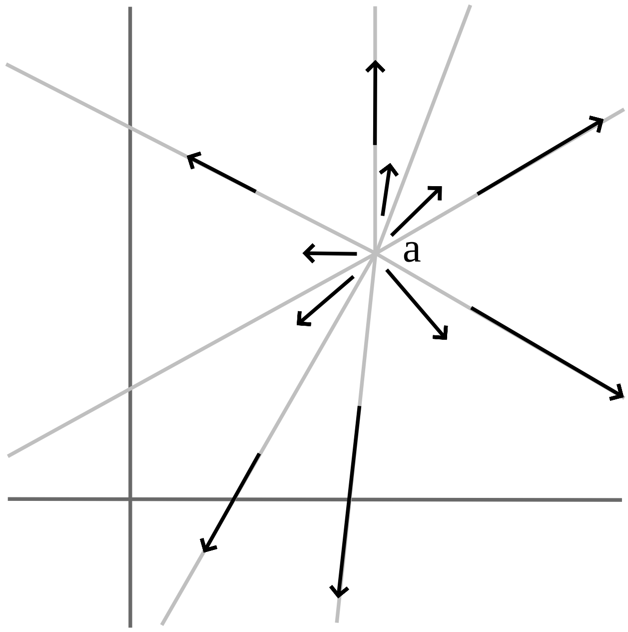


FIGURE 11.6. A conformal gradient field σ on the Euclidean space $\mathbb{R}^2$

So the equation from the subtraction is true if $q = 2-n$. Then Equation (11.3) equals

$$\begin{aligned}((2-n)(n+1) - n(2-n))\mu &= -1 \\ \Rightarrow (2-n)\mu &= -1 \\ \Rightarrow \mu &= \begin{cases} \frac{1}{n-2} & \text{if } n > 2 \\ -1 & \text{if } n = 2 \end{cases}\end{aligned}$$

If $n = 2$ then $q = 0$ and $p = 3$. Hence the first and second conditions are true. One can also show that the four cases are the only possibilities. □

11.4. **Conformal gradient fields on the Euclidean space $\mathbb{R}^n$.** Here is another example of a harmonic vector field which is a (p, q)-harmonic section of the tangent bundle $T\mathbb{R}^n$ of the n-dimensional Euclidean space $\mathbb{R}^n$.

Definition 11.13. A vector field σ on the Euclidean space $\mathbb{R}^n$ is said to be a conformal gradient field if

$$\sigma(x) = k(x-a) \text{ for some } k \in \mathbb{R}, \text{ where } k \neq 0 \text{ and } x, a \in \mathbb{R}^n.$$

Figure 11.6 gives an example diagram of a conformal gradient field σ on the 2-dimensional Euclidean space $\mathbb{R}^2$. In this diagram the point a is the centre of the vector field.

Proposition 11.4. *The vector field $\sigma = k(x-a)$ defined in Definition 11.13 has the two properties of being a conformal vector field and a gradient field on the n-dimensional Euclidean space $\mathbb{R}^n$ equipped with the dot product, denoted by any of the two equal functions $- \cdot - = g(-,-)$, where σ is the gradient field $\nabla\omega$ of*

$$f = \frac{1}{2}k\|x-a\|^2$$

Proof. To show that σ is conformal the following equation needs to be true

$$g(\nabla_X \sigma, Y) + g(X, \nabla_Y) = 2\psi \cdot g(X,Y)$$

for some $\psi : M \to \mathbb{R}$. In the case of $M = \mathbb{R}^n$

$$\begin{aligned}\nabla_X \sigma &= D_X \sigma \\ &= d\sigma(X) \\ &= kX\end{aligned}$$

Therefore

$$\begin{aligned}g(\nabla_X \sigma, Y) + g(X, \nabla_Y \sigma) &= kX \cdot Y + X \cdot kY \\ &= 2k(X \cdot Y) \\ &= 2\psi \cdot g(X,Y) \\ \text{where } \psi &= k\end{aligned}$$

Hence σ is conformal. To show that σ is a gradient field and prove that $\sigma = \nabla f$, note that $\nabla f = \text{grad } f$ which is characterised by the equation

$$g(\nabla f, X) = df(X)$$

so

$$\begin{aligned}\nabla f &= \sum_{i=1}^{n} g(\nabla f, E_i)E_i \\ &= \sum_{i=1}^{n} df(E_i)E_i\end{aligned}$$

Starting with $2df(X)$ to find $df(X)$ and then find ∇f

$$\begin{aligned}2df(X) &= dk\left(\|x-a\|^2\right)(X) \\ &= k\left(d\left(|a|^2\right)(X) - d\left(2(a\cdot x)\right)(X) + d\left(|x|^2\right)(X)\right) \\ &= k(0 - d\left(2(a\cdot x)\right)(X) \\ &\quad + d\left(|x|^2\right)(X)) \text{ by the chain rule and because } da = 0 \\ &= k\left(-2d(a\cdot x)(X) + d\left(|x|^2\right)(X)\right)\end{aligned}$$

Let $|x|^2 = r^2$. To calculate $d(r^2)(X)$ the Jacobian matrix of r^2 can be used. This is

$$\begin{aligned} J_{r^2}(x) &= \left(\frac{\partial x_i^2}{\partial x_j}\bigg|_x \right) \text{ where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n \\ &= (2x_1, ..., 2x_n) \end{aligned}$$

Using Definition 8.4, $d(r^2)(X)$ equals

$$\begin{aligned} d(r^2)(X) &= D_X r^2 \\ &= \sum_{j=1}^{n} \frac{\partial f_i(x)}{\partial x_j}\bigg|_x X_j \\ &= J_{r^2}(x) \cdot (X) \\ &= (2x_1, ..., 2x_n) \cdot (X_1, ..., X_n) \\ &= 2x \cdot X \end{aligned}$$

For $d(a \cdot x)(X)$ recall that

$$d(f \cdot g)(X) = f \cdot dg(X) + g \cdot df(X)$$

Therefore

$$\begin{aligned} d(a \cdot x)(X) &= a \cdot dx(X) + x \cdot da(X) \\ &= a \cdot X + x \cdot 0 \\ &= a \cdot X \end{aligned}$$

Hence $2df(X)$ equals

$$\begin{aligned} 2df(X) &= k\left(d(r^2)(X) - 2d(a \cdot x)(X)\right) \\ &= 2k(x-a) \cdot X \\ \Rightarrow df(X) &= k(x-a) \cdot X \\ &= \sigma \cdot X \\ &= g\left(\nabla f, X\right) \\ &= \nabla f \cdot X \\ \Rightarrow \sigma &= \nabla f \end{aligned}$$

Therefore σ is the gradient field of f. Since σ is conformal and a gradient field, σ is a conformal gradient field on the Euclidean space $\mathbb{R}^n$. □

Theorem 11.4. *Let σ be a conformal gradient field on $M = \mathbb{R}^n$. Then σ is a (p,q)-harmonic section of TM and hence a harmonic vector field if and only if one of the following conditions is true*

- $p = n$ *and* $q = 2 - n$,
- $q = 0$ *and* $p = 0$,
- $q = 0$ *and* $n = 2$.

Proof. For this proof the manifold and Riemannian metric being used are $(M, g) = (\mathbb{R}^n, \cdot)$ which are the n-dimensional Euclidean space and the dot product. The equation for σ is

$$\sigma(x) = k(x - a) \text{ where } k \in \mathbb{R} \text{ and } x, a \in \mathbb{R}^n$$

As stated in Theorem 10.1, σ is a (p, q)-harmonic section if and only if

$$T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$$

where

$$T_p(\sigma) = (1 + |\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma$$

is a vector field with

$$F = \frac{1}{2}|\sigma|^2$$

and

$$\phi_{p,q}(\sigma) = p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1 + |\sigma|^2)\Delta F$$

is an $\mathbb{R}$-valued function. To calculate $T_p(\sigma)$ and $\phi_{p,q}(\sigma)$ the values F, ∇F, $|\nabla F|^2$, $\nabla_{\nabla F}\sigma$, ΔF, $|\nabla\sigma|^2$, $\nabla^*\nabla\sigma$ and $\nabla^2_{X,Y}\sigma$ need to be found. Recalling the equation for $\nabla_X\sigma$, $|\nabla\sigma|^2$ equals

$$\begin{aligned}
|\nabla\sigma|^2 &= \sum_{i=1}^{n} |\nabla_{E_i}\sigma|^2 \\
&= \sum_{i=1}^{n} \|kE_i\|^2 \\
&= \sum_{i=1}^{n} k^2 \\
&= nk^2
\end{aligned}$$

where $\{E_i\}_{i=1}^n$ is an orthonormal basis of $T\mathbb{R}^n$. The equation for F can be found by using the equation for $2F$.

$$\begin{aligned}
2F &= |\sigma|^2 \\
&= |k(x - a)|^2 \\
&= \sigma(x) \cdot \sigma(x) \\
&= k^2(x - a) \cdot (x - a) \\
&= k^2\left(|x|^2 - 2a \cdot x + |a|^2\right) \\
&= k^2\left(r^2 - 2a \cdot x + |a|^2\right) \\
\Rightarrow F &= \frac{1}{2}k^2\left(r^2 - 2a \cdot x + |a|^2\right)
\end{aligned}$$

$\nabla F = \text{grad } F$ is characterised by the equation

$$g(\nabla f, X) = dF(X)$$

so

$$\begin{aligned}\nabla F &= \sum_{i=1}^{n} g(\nabla F, E_i)E_i \\ &= \sum_{i=1}^{n} dF(E_i)E_i\end{aligned}$$

Starting with $2dF(X)$ to find $dF(X)$

$$\begin{aligned}2dF(X) &= d\left(k^2\left(r^2 - 2a\cdot x + |a|^2\right)\right)(X) \\ &= k^2\left(d(r^2)(X) - 2d(a\cdot x)(X) + d(|a|^2)(X)\right) \\ &= k^2\left(d(r^2)(X) - 2d(a\cdot x)(X)\right) \text{ because } da = 0\end{aligned}$$

The equations for $d(r^2)(X)$ and $d(a\cdot x)(X)$ in the proof of Proposition 11.4 lead to the equation for $dF(X)$.

$$\begin{aligned}2dF(X) &= k^2\left(d(r^2)(X) - 2d(a\cdot x)(X)\right) \\ &= k^2\left(2x\cdot X - 2a\cdot X\right) \\ &= 2k^2(x-a)\cdot X \\ \Rightarrow dF(X) &= k^2(x-a)\cdot X \\ &= \nabla F\cdot X \\ \Rightarrow \nabla F &= k^2(x-a) \\ &= k\sigma\end{aligned}$$

so $|\nabla F|^2$ equals

$$\begin{aligned}|\nabla F|^2 &= |k\sigma|^2 \\ &= k^2|\sigma|^2 \\ &= k^2(2F) \\ &= 2k^2F \\ &= k^2\left(k^2\left(r^2 - 2a\cdot x + |a|^2\right)\right) \\ &= k^4\left(r^2 - 2a\cdot x + |a|^2\right)\end{aligned}$$

Using Equation (3.1)(i) from Definition 3.2, $\nabla_{\nabla F}\sigma$ is

$$\begin{aligned}\nabla_{\nabla F}\sigma &= \nabla_{k\sigma}\sigma \\ &= k\nabla_{\sigma}\sigma \\ &= k\left(k\sigma\right) \\ &= k^2\sigma\end{aligned}$$

By the definitions of the associated Ricci operator S, the covariant coderivative ∇^*, the Weitzenböck formula and the rough Laplacian the equation for ΔF is

$$\begin{aligned}\Delta F &= \nabla^*\nabla F + S(F)\\ &= \nabla^*(\nabla F) + 0 \text{ because } F \text{ is a 0-form}\\ &= -\sum_{i=1}^{n} \left(\nabla_{E_i}\left(\nabla F\right)\right)\cdot E_i\\ &= -\sum_{i=1}^{n} \left(\nabla_{E_i}\left(k\sigma\right)\right)\cdot E_i\\ &= -\sum_{i=1}^{n} \left(k\nabla_{E_i}\sigma\right)\cdot E_i\\ &= -\sum_{i=1}^{n} k\left(kE_i\right)\cdot E_i\\ &= -nk^2\end{aligned}$$

To find $\nabla^*\nabla\sigma = -\sum_{i=1}^{n}\nabla^2_{E_i,E_i}\sigma$, the equation for $\nabla^2_{X,Y}\sigma$ needs to be calculated first.

$$\begin{aligned}\nabla^2_{X,Y}\sigma &= \nabla_X(\nabla_Y\sigma) - \nabla_{\nabla_X Y}\sigma\\ &= \nabla_X(kY) - k\nabla_X Y\\ &= k\nabla_X Y - k\nabla_X Y\\ &= 0\end{aligned}$$

Therefore

$$\begin{aligned}\nabla^*\nabla\sigma &= -\sum_{i=1}^{n}\nabla^2_{E_i,E_i}\sigma\\ &= 0\end{aligned}$$

Hence all the equations needed for $T_p(\sigma)$ and $\phi_{p,q}\sigma$ have been calculated. By Proposition 11.4, σ can be defined as $\sigma = \nabla f$ where $f = \frac{1}{2}\|x-a\|^2$. The mapping $f : \mathbb{R}^n \to \mathbb{R}$ can be simplified to

$$\begin{aligned}f &= \frac{1}{2}\|x-a\|^2\\ &= \frac{1}{2k}|\sigma|^2\\ \Rightarrow 2kf &= |\sigma|^2\end{aligned}$$

This equation can be used in the equality of $T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$ when σ is (p,q)-harmonic. Applying the previous formulae to the equations for

$\phi_{p,q}(\sigma)$ and $T_p(\sigma)$, the equation of $\phi_{p,q}(\sigma)$ in terms of f is

$$\begin{aligned}\phi_{p,q}(\sigma) &= p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1+|\sigma|^2)\Delta F \\ &= pnk^2 - pqk^2|\sigma|^2 - q(1+|\sigma|^2)(-nk^2) \\ &= pnk^2 - pqk^4(r^2 + |a|^2 - 2a\cdot x) \\ &\quad - q(1+k^2(r^2+|a|^2-2a\cdot x))(-nk^2) \\ &= pnk^2 - pqk^4r^2 - pqk^4|a|^2 + 2pqk^4 a\cdot x \\ &\quad + qnk^2 + qnk^4r^2 + qnk^4|a|^2 - 2qnk^4 a\cdot x \\ &= nk^2\left((p+q) + (qn-pq)|\sigma|^2\right) \\ &= nk^2\left((p+q) + 2(qn-pq)kf\right)\end{aligned}$$

and the equation for $T_p(\sigma)$ in terms of σ is

$$\begin{aligned}T_p(\sigma) &= (1+|\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma \\ &= (1+|\sigma|^2)0 + 2pk^2\sigma \\ &= 2pk^2\sigma\end{aligned}$$

Since $T_p(\sigma)$ is porportional to σ, the equation $T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$ can be simplified to

$$\begin{aligned}T_p(\sigma) &= \phi_{p,q}(\sigma)\sigma \\ \Rightarrow 2pk^2\sigma &= \left(nk^2\left((p+q)+2(qn-pq)kf\right)\right)\sigma \\ \Rightarrow 2p &= (p+q)n + 2(qn-pq)kf \\ \Rightarrow 2pf^0 &= (p+q)nf^0 + 2(qn-pq)kf^1 \\ \Rightarrow 0f^0 &= ((n-2)p+nq)f^0 + 2(qn-pq)kf^1 \\ \Rightarrow 0f^0 &= ((n-2)p+nq)f^0 + 2(n-p)qkf^1\end{aligned}$$

This is a polynomial in f. Since f is a continuous function on $M = \mathbb{R}^n$ this polynomial is zero if and only if the coefficients of the powers of f vanish. Hence the following equations come from the coefficients of f^0 and f^1.

From f^0

$$(n-2)p + nq = 0 \tag{11.5}$$

and from f^1

$$\begin{aligned}2(n-p)qk &= 0 \\ \Rightarrow 2(n-p)q &= 0 \\ \Rightarrow (n-p)q &= 0\end{aligned}$$

Equation (11.5) leads to two cases for σ to be (p,q)-harmonic. If $p = n$ then

$$\begin{aligned}(n-2)n + nq &= 0 \\ \Rightarrow (n-2) + q &= 0 \\ \Rightarrow q &= 2-n\end{aligned}$$

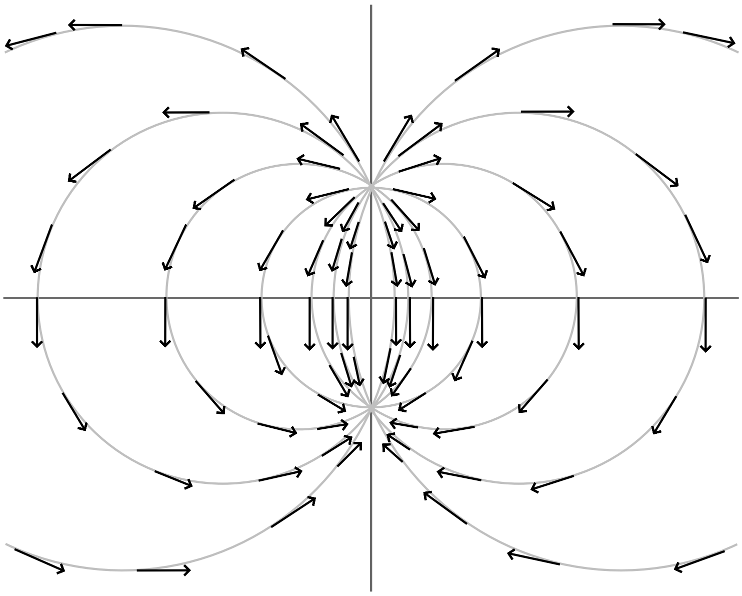


FIGURE 11.7. A conformal extension field

If $q = 0$ then

$$\begin{aligned}(n-2)p + n0 &= 0\\ \Rightarrow (n-2)p &= 0\\ \therefore \text{ either } p &= 0\\ \text{or } n &= 2\end{aligned}$$

This completes the proof of this theorem. □

11.5. **Conformal extension fields.**

Definition 11.14. [4] Let $D \subset \mathbb{R}^n$ be any subset of the n-dimensional Euclidean space and f be a mapping $f : D \to \mathbb{R}^m$. An *extension field of f at $a \in D$* is a mapping $\tilde{f} : U \to \mathbb{R}^m$ from a neighbourhood U of a which equals f on the intersection of U and D denoted by $U \cap D$. This relationship between $\tilde{f}$ and f is denoted by the equation

$$\tilde{f}\Big|_{U\cap D} = f|_{U\cap D}$$

Definition 11.15. Let $\tilde{\sigma}(x) = a - (a \cdot x)x$ be a conformal gradient field on $S^{n-1} \subset \mathbb{R}^n$. Hence $\tilde{\sigma}(x)$ is a mapping $\tilde{\sigma}(x) : S^{n-1} \to \mathbb{R}^n$. A vector field σ on the Euclidean space $\mathbb{R}^n$ is a conformal extension field of $\tilde{\sigma}(x)$ if

$$\sigma(x) = \left(\frac{1}{2}(r^2 + 1)\right) a - (a \cdot x)x \text{ where } r = |x|$$

Figure 11.7 gives a diagram showing a conformal extension field.

Proposition 11.5. *Let $\tilde{\sigma}(x)$ be a conformal gradient field on $S^{n-1} \subset \mathbb{R}^n$. Hence the equation for $\tilde{\sigma}(x)$ is*

$$\tilde{\sigma}(x) = a - (a \cdot x)x$$

The vector field σ on the Euclidean space $\mathbb{R}^n$ defined in Definition 11.15 has the two properties of being a conformal vector field and an extension field of $\tilde{\sigma}(x)$ where the equation for σ is

$$\sigma(x) = \left(\frac{1}{2}(r^2+1)\right)a - (a \cdot x)x$$

Proof. To show that σ is conformal the following equation needs to be true

$$g(\nabla_X \sigma, Y) + g(X, \nabla_Y) = 2\psi \cdot g(X,Y)$$

for some $\psi : M \to \mathbb{R}$. In the case of $M = \mathbb{R}^n$

$$\begin{aligned}\nabla_X \sigma &= D_X \sigma \\ &= d\sigma(X)\end{aligned}$$

Recall Equation (3.1)(ii) which can also be stated as

$$\nabla_X(fY) = (Xf)\,Y + f\nabla_X Y \text{ where } f \in \mathbb{R} \text{ and } Y \in \mathbb{R}^n$$

Hence $\nabla_X \sigma$ can be split into two components both including a dot product of $\mathbb{R}$-valued function and an $\mathbb{R}^n$-valued function. This is shown in the following equation.

$$\nabla_X \sigma = \nabla_X \left(\frac{1}{2}(r^2+1)\right)a - \nabla_X (a \cdot x)x$$

The first component of $\nabla_X \sigma$ is

$$\begin{aligned}\nabla_X \left(\frac{1}{2}(r^2+1)\right)a &= X\left(\frac{1}{2}(r^2+1)\right)a - \left(\frac{1}{2}(r^2+1)\right)\nabla_X a \\ &= X\left(\frac{1}{2}(r^2+1)\right)a - \left(\frac{1}{2}(r^2+1)\right)0 \\ &= X\left(\frac{1}{2}r^2\right)a \\ &= \frac{1}{2}d(r^2)(X)a \\ &= \frac{1}{2}\left(2X \cdot x\right)a \\ &= (X \cdot x)a\end{aligned}$$

The second component of $\nabla_X \sigma$ is

$$\begin{aligned}-\nabla_X (a \cdot x)x &= -X(a \cdot x)x - (a \cdot x)\nabla_X x \\ &= -\left(d(a \cdot x)(X)\right)x - (a \cdot x)dx(X) \\ &= -\left(a \cdot X\right)x - (a \cdot x)X\end{aligned}$$

Adding the two components together $\nabla_X \sigma$ equals

$$\nabla_X \sigma = (x \cdot X)a - (a \cdot X)x - (a \cdot x)X \tag{11.6}$$

Therefore

$$\begin{aligned} g(\nabla_X \sigma, Y) + g(X, \nabla_Y \sigma) &= (x \cdot X)(a \cdot Y) - (a \cdot X)(x \cdot Y) \\ &\quad - (a \cdot x)(X \cdot Y) + (x \cdot Y)(a \cdot X) \\ &\quad - (a \cdot Y)(x \cdot X) - (a \cdot x)(X \cdot Y) \\ &= -2(a \cdot x)(X \cdot Y) \\ &= 2\psi \cdot g(X, Y) \\ \Rightarrow \psi &= -(a \cdot x) \end{aligned}$$

Hence σ is conformal. To prove that σ is an extension field of $\tilde{\sigma}$, let x be a unit vector starting at the origin. Then $x \in S^{n-1} \subset \mathbb{R}^n$ and $|x|^2 = r^2 = 1$. So $\sigma(x)$ equals

$$\begin{aligned} \sigma(x) &= \left(\frac{1}{2}(r^2 + 1)\right) a - (a \cdot x)x \\ &= \left(\frac{1}{2}(1 + 1)\right) a - (a \cdot x)x \\ &= \left(\frac{2}{2}\right) a - (a \cdot x)x \\ &= a - (a \cdot x)x \\ &= \tilde{\sigma}(x) \end{aligned}$$

Hence

$$\tilde{\sigma}(x)|_{\mathbb{R}^n \cap S^{n-1}} = \sigma(x)|_{\mathbb{R}^n \cap S^{n-1}}$$

Therefore $\sigma(x)$ is an extension field of $\tilde{\sigma}(x)$. In conclusion, $\sigma(x)$ is a conformal extension field of $\tilde{\sigma}(x)$. □

Theorem 11.5. *Let σ be a conformal extension field of $\tilde{\sigma}$ as denoted in Proposition 11.5. Then σ is a (p, q)-harmonic section of TM and hence a harmonic vector field if and only if $n = 2$ and $q = 0$.*

Proof. In this case the manifold and Riemannian metric being used are $(M, g) = (\mathbb{R}^n, \cdot)$ which are the n-dimensional Euclidean space and the dot product. For reference, the equation for σ is

$$\sigma(x) = \frac{1}{2}(r^2 + 1)a - (a \cdot x)x \text{ where } r \in \mathbb{R} \text{ and } x, a \in \mathbb{R}^n$$

The vector field σ is a (p, q)-harmonic section if and only if

$$T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$$

where

$$T_p(\sigma) = (1 + |\sigma|^2)\nabla^* \nabla \sigma + 2p\nabla_{\nabla F} \sigma$$

is a vector field with

$$F = \frac{1}{2}|\sigma|^2$$

and

$$\phi_{p,q}(\sigma) = p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1+|\sigma|^2)\Delta F$$

is an $\mathbb{R}$-valued function. To calculate $T_p(\sigma)$ and $\phi_{p,q}(\sigma)$ the values F, ∇F, $|\nabla F|^2$, $\nabla_{\nabla F}\sigma$, ΔF, $|\nabla\sigma|^2$, $\nabla^*\nabla\sigma$ and $\nabla^2_{X,Y}\sigma$ need to be found, starting with the calculation of $\nabla^2_{X,Y}\sigma$. Using Equation (11.6) for $\nabla_Y\sigma$, $\nabla^2_{X,Y}\sigma$ equals

$$\begin{aligned}
\nabla^2_{X,Y}\sigma &= \nabla_X(\nabla_Y\sigma) - \nabla_{\nabla_X Y}\sigma \\
&= \nabla_X\left((x\cdot Y)a - (a\cdot Y)x - (a\cdot x)Y\right) \\
&\quad - (x\cdot\nabla_X Y)a + (a\cdot\nabla_X Y)x + (a\cdot x)\nabla_X Y \\
&= (X\cdot Y)a + (x\cdot\nabla_X Y)a \\
&\quad - (a\cdot\nabla_X Y)x - (a\cdot Y)X \\
&\quad - (a\cdot X)Y - (a\cdot x)\nabla_X Y \\
&\quad - (x\cdot\nabla_X Y)a + (a\cdot\nabla_X Y)x + (a\cdot x)\nabla_X Y \\
&= (X\cdot Y)a - (a\cdot Y)X - (a\cdot X)Y
\end{aligned}$$

Hence $\nabla^*\nabla\sigma$ equals

$$\begin{aligned}
\nabla^*\nabla\sigma &= -\left(\sum_{i=1}^{n}\nabla^2_{E_i,E_i}\sigma\right) \\
&= -\left(\sum_{i=1}^{n}(E_i\cdot E_i)a - (a\cdot E_i)E_i - (a\cdot E_i)E_i\right) \\
&= 2a - na \\
&= (2-n)a
\end{aligned}$$

Again let $|x|^2 = r^2$. By Equation (11.6), $|\nabla\sigma|^2$ is

$$\begin{aligned}
|\nabla\sigma|^2 &= \sum_{i=1}^{n} |\nabla_{E_i}\sigma|^2 \\
&= \sum_{i=1}^{n} \|\, ((x\cdot E_i)a - (a\cdot E_i)x - (a\cdot x)E_i)\,\|^2 \\
&= \sum_{i=1}^{n} ((x\cdot E_i)^2|a|^2 + 2(-(a\cdot x)E_i)((x\cdot E_i)a) \\
&\qquad + (a\cdot E_i)^2|x|^2 + 2((x\cdot E_i)a)(-(a\cdot E_i)x) \\
&\qquad + (a\cdot x)^2(E_i\cdot E_i) + 2(-(a\cdot E_i)x)(-(a\cdot x)E_i)) \\
&= \sum_{i=1}^{n} \big((x\cdot E_i)^2|a|^2 + (a\cdot E_i)^2r^2 - 2(x\cdot E_i)(a\cdot E_i)(a\cdot x)\big) \\
&\qquad + n(a\cdot x)^2 \\
&= n(a\cdot x)^2 - 2(a\cdot x)^2 + |a|^2r^2 + |a|^2r^2 \\
&= (n-2)(a\cdot x)^2 + 2|a|^2r^2
\end{aligned}$$

The rest of the values required include F so that is calculated next by using the equation for $2F$.

$$\begin{aligned}
2F &= |\sigma|^2 \\
&= \left|\frac{1}{2}(r^2+1)a - (a\cdot x)x\right|^2 \\
&= \frac{1}{4}(r^2+1)^2|a|^2 + (a\cdot x)^2r^2 - (r^2+1)(a\cdot x)^2 \\
&= \frac{1}{4}(r^2+1)^2|a|^2 - (a\cdot x)^2 \\
\Rightarrow F &= \frac{1}{8}(r^2+1)^2|a|^2 - \frac{1}{2}(a\cdot x)^2
\end{aligned}$$

$\nabla F = \text{grad } F$ is characterised by the equation

$$g(\nabla f, X) = dF(X)$$

so

$$\begin{aligned}
\nabla F &= \sum_{i=1}^{n} g(\nabla F, E_i)E_i \\
&= \sum_{i=1}^{n} dF(E_i)E_i
\end{aligned}$$

Starting with $2dF(X)$ to find $dF(X)$ and then find ∇F

$$
\begin{aligned}
2dF(X) &= d2F(X) \\
&= d\left(\frac{1}{4}(r^2+1)^2|a|^2 - (a\cdot x)^2\right)(X) \\
&= \frac{1}{4}2(r^2+1)d(r^2+1)(X)|a|^2 - 2(a\cdot x)d(a\cdot x)(X) \\
&= \frac{1}{2}(r^2+1)d(r^2)(X)|a|^2 - 2(a\cdot x)d(a\cdot x)(X) \\
&= \frac{1}{2}(r^2+1)2x\cdot X|a|^2 - 2(a\cdot x)a\cdot X \\
&= |a|^2(r^2+1)x\cdot X - 2(a\cdot x)a\cdot X \\
\Rightarrow dF(X) &= \frac{1}{2}|a|^2(r^2+1)x\cdot X - (a\cdot x)a\cdot X \\
&= \left(\frac{1}{2}|a|^2(r^2+1)x - (a\cdot x)a\right)\cdot X \\
&= \nabla F\cdot X \\
\Rightarrow \nabla F &= \frac{1}{2}|a|^2(r^2+1)x - (a\cdot x)a
\end{aligned}
$$

Now that ∇F has been found, $2\nabla_{\nabla F}\sigma$ can be calculated

$$
\begin{aligned}
2\nabla_{\nabla F}\sigma &= \nabla_{2\nabla F}\sigma \\
&= (x\cdot 2\nabla F)a - (a\cdot 2\nabla F)x - (a\cdot x)2\nabla F \\
&= |a|^2(r^2+1)r^2a - 2(a\cdot x)^2a \\
&\quad - |a|^2(r^2+1)(a\cdot x)x + 2|a|^2(a\cdot x)x \\
&\quad - |a|^2(r^2+1)(a\cdot x)x + 2(a\cdot x)^2a \\
&= |a|^2r^2(r^2+1)a - 2|a|^2r^2(a\cdot x)x \\
&= 2|a|^2r^2\sigma
\end{aligned}
$$

All of the values for $T_p(\sigma)$ have been calculated so they can now be used in the equation for $T_p(\sigma)$.

$$
\begin{aligned}
T_p(\sigma) &= (1+|\sigma|^2)\nabla^*\nabla\sigma + 2p\nabla_{\nabla F}\sigma \\
&= (1+|\sigma|^2)(2-n)a + 2p|a|^2r^2\sigma
\end{aligned}
$$

$T_p(\sigma)$ needs to be divisible by σ if σ is (p,q)-harmonic so $n = 2$. By Corollary 3.1, ΔF equals

$$\begin{aligned}
\Delta F &= -|\nabla\sigma|^2 + (\nabla^*\nabla\sigma)\cdot\sigma \\
&= -\left((n-2)(a\cdot x)^2 + 2|a|^2r^2\right) + (2-n)a\cdot\left(\frac{1}{2}(r^2+1)a - (a\cdot x)x\right) \\
&= -(n-2)(a\cdot x)^2 - 2|a|^2r^2 + (2-n)\left(\frac{1}{2}(r^2+1)|a|^2 - (a\cdot x)^2\right) \\
&= (2-n+n-2)(a\cdot x)^2 + \left(-2+\frac{2}{2}-\frac{1}{2}n\right)|a|^2r^2 + (2-n)\frac{1}{2}|a|^2 \\
&= -\frac{1}{2}(n+2)|a|^2r^2 + \frac{1}{2}(2-n)|a|^2
\end{aligned}$$

The last part of $\phi_{p,q}(\sigma)$ required is $|\nabla F|^2$ which equals

$$\begin{aligned}
|\nabla F|^2 &= \left|\frac{1}{2}|a|^2(r^2+1)x - (a\cdot x)a\right|^2 \\
&= \left(\frac{1}{2}\right)^2|a|^4(r^2+1)^2r^2 + (a\cdot x)^2|a|^2 - 2\left(\frac{1}{2}(r^2+1)|a|^2(a\cdot x)^2\right) \\
&= \frac{1}{4}|a|^4(r^2+1)^2r^2 + (a\cdot x)^2|a|^2 - (r^2+1)|a|^2(a\cdot x)^2 \\
&= \frac{1}{4}|a|^4(r^2+1)^2r^2 - r^2|a|^2(a\cdot x)^2 \\
&= 2|a|^2r^2\left(\frac{1}{8}(r^2+1)^2|a|^2 - \frac{1}{2}(a\cdot x)^2\right) \\
&= 2|a|^2r^2F
\end{aligned}$$

As proved in the equation for $T_p(\sigma)$, σ is (p,q)-harmonic if and only if $n = 2$. Therefore $\phi_{p,q}(\sigma)$ equates to

$$\begin{aligned}
\phi_{p,q}(\sigma) &= p|\nabla\sigma|^2 - pq|\nabla F|^2 - q(1+|\sigma|^2)\Delta F \\
&= p\left((n-2)(a\cdot x)^2 + 2|a|^2r^2\right) - pq\left(2|a|^2r^2F\right) \\
&\quad - q(1+|\sigma|^2)\left(-\frac{1}{2}(n+2)|a|^2r^2 + \frac{1}{2}(2-n)|a|^2\right) \\
&= 2|a|^2r^2p - 2pq|a|^2r^2F + 2q(1+|\sigma|^2)|a|^2r^2 \\
&= 2|a|^2r^2(p - pqF + q(1+2F))
\end{aligned}$$

Still in the case when $n = 2$, the equation $T_p(\sigma) = \phi_{p,q}(\sigma)\sigma$ can then be simplified to

$$\begin{aligned} T_p(\sigma) &= \phi_{p,q}(\sigma)\sigma \\ \Rightarrow 2p|a|^2r^2\sigma &= \left(2|a|^2r^2(p - pqF + q(1+2F))\right)\sigma \\ \Rightarrow p &= p - pqF + q(1+2F) \\ \Rightarrow 0 &= -pqF + q(1+2F) \\ \Rightarrow 0 &= q + q(2-p)F \end{aligned}$$

This is a polynomial in F. Since f is a continuous function on $M = \mathbb{R}^n$ this polynomial is zero if and only if the coefficients of the powers of F vanish. Hence the following equations come from the coefficients of F^0 and F^1.

From F^0

$$q = 0 \tag{11.7}$$

and from F^1

$$q(2-p) = 0 \tag{11.8}$$

Due to Equation (11.7), q must always equal 0 therefore Equation (11.8) equals

$$\begin{aligned} q(2-p) &= 0(2-p) \\ &= 0 \end{aligned} \tag{11.9}$$

which is true for Equation (11.8). Hence p can be any real number. Therefore a conformal extension field σ of $\tilde{\sigma}$ is a (p,q)-harmonic section of $T\mathbb{R}^n$ and hence a harmonic vector field if and if only if $n = 2$ and $q = 0$. □

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