Jordan groups and homogeneous structures

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

For my family,

those near and far.

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Very well, responded Candide, *but we must cultivate our garden*. [29]

Abstract

A permutation group G acting transitively on a set Ω is a Jordan group if there is a proper subset $\Gamma \subset \Omega$, subject to some non-triviality conditions, such that the pointwise stabiliser in G of $\Omega \setminus \Gamma$ is transitive on Γ. Such sets Γ are called Jordan sets for G. Here we study infinite primitive Jordan groups which are automorphism groups of first order relational structures. We find a model theoretic application in classifying the reducts of an infinite family of semilinearly ordered partial orders, and apply a model-theoretic construction technique to obtain examples of Jordan groups.

The kinds of structures preserved by an infinite primitive Jordan group with a primitive Jordan set were classified by Adeleke and Neumann [4]. In Chapter 2 we apply this classification and further results on the combinatorial behaviour of the families of primitive Jordan sets to obtain an infinite family of non-conjugate, maximal closed subgroups of $\mathrm{Sym}(\omega)$. We obtain a classification of the reducts (up to first order interdefinability) of an infinite family of semilinearly ordered trees.

The classification of Jordan groups was continued by Adeleke and Macpherson [2], and they determined the kinds of structures preserved by infinite primitive Jordan groups, without the assumption that there is a primitive Jordan set. This list includes some exotic kinds of relational structures, including so-called 'limits of betweenness relations'. Non-isomorphic examples of infinite primitive Jordan groups preserving a limit of betweenness relations have been constructed by Adeleke [1] and Bhattacharjee and Macpherson [6]. In Chapter 3 we develop the construction of Bhattacharjee and Macpherson to construct what we believe to be a limit of betweenness relations preserved by the group constructed by Adeleke.

Contents

Chapter 1

Introduction

This thesis is a contribution to the study of Jordan groups, which can be traced back to work of Camille Jordan in the 1870's for finite permutation groups (see P. Neumann [28]). A permutation group G acting transitively on a set Ω is a Jordan group if there is a proper subset $\Gamma \subset \Omega$, subject to some non-triviality conditions, such that the pointwise stabiliser of $\Omega \setminus \Gamma$ is transitive on Γ. Such sets Γ are called Jordan sets for G. An example of a finite, primitive Jordan group is the n–dimensional projective general linear group $PGL(n, F)$, for a F a finite field, acting as automorphisms on the n–dimensional projective space over F. Complements of $n - 1$ – dimensional projective subspaces are proper Jordan sets for $PGL(n, F)$. In [28] (Theorem J1) it is stated as a theorem known to Jordan that a finite primitive group (G, Ω) with a non-trivial Jordan set Γ is at least 2-transitive on Ω. The finite 2-transitive permutation groups are classified (see Cameron [13] for a catalogue), via the Classification of Finite Simple Groups. From this the finite primitive Jordan groups are classified in independent work of Neumann [28] and Kantor [24]. Similar results on finite Jordan groups were applied in model theory by Cherlin, Harrington and Lachlan in [17] to classify the geometries of strictly minimal sets in $\aleph_0$ – stable, $\aleph_0$ – categorical structures.

We are interested in infinite primitive Jordan groups. An example of an infinite, primitive Jordan group is $G = \mathrm{Aut}(\mathbb{Q}, <)$, the automorphism group of the linear ordering on the rationals. The group G naturally acts both faithfully and transitively on $\mathbb{Q}$. As $(\mathbb{Q}, <)$ is dense, an open interval $(a, b) \subseteq \mathbb{Q}$ is a proper Jordan set for G, so G is a Jordan group.

Note that $\mathrm{Aut}(\mathbb{Q}, <)$ is not 2-transitive, as it preserves the linear order. This serves as an example that the study of infinite Jordan groups is quite different from the finite ones. Unlike the finite case, infinite primitive Jordan groups are not necessarily 2-transitive. In this work we study infinite primitive Jordan groups as automorphism groups of countably infinite first order structures.

This approach is motivated by results by Adeleke, Neumann and Macpherson towards the classification of infinite primitive Jordan groups. These results are stated precisely in Section 3 of this chapter. From the results of Adeleke and Neumann [4] we know that if a primitive Jordan group G acting on Ω with a proper primitive Jordan set $\Gamma \subseteq \Omega$ preserves some relational structure on Ω, then G preserves on Ω either some kind of 'linear' relational structure or one of certain 'tree-like' structures. In Section 2 of this chapter we explain what is meant by a 'linear' or 'tree-like' structure. The 'linear' relational structures are one of four derived from a linear ordering, which are present in Cameron's Theorem (proved in [12]) on infinite primitive permutation groups that are highly homogeneous but not highly transitive. The four kinds of 'tree-like' structures are relational structures studied extensively by Adeleke and Neumann in [3], after such structures appeared in work by Cameron in [14] and [15] and are related to certain partial orders studied by Droste in [18]. The theorem of Adeleke and Macpherson in [2] gives a list of the kind of structures which are preserved by a primitive Jordan group (G, Ω), removing the assumption that Ω contains a primitive Jordan set which was required in the theorem of Adeleke–Neumann. In this situation there are exotic possibilities, which are mentioned in Section 3 of this chapter. This includes certain structures which are described as limits of tree-like structures.

In this chapter we give an introduction and background to the various topics required, including stating classification theorems for primitive Jordan groups.

In Chapter 2 we apply results on the classification of infinite primitive Jordan groups and the combinatorics of primitive Jordan sets to study the reducts of relatively 2–transitive semilinear orderings, an infinite family of ω–categorical partial orderings. This is the class of semilinear orderings called 2–homogeneous trees by Droste and classified in [18]. As a result, we obtain an infinite family of non-conjugate maximal closed subgroups of S_∞.

In Chapter 3 we use a generalised form of Fraïssé amalgamation, a construction technique from model theory, to construct a structure which we believe to be a limit of betweenness relations. The automorphism group of this structure is thought to be a primitive Jordan group which preserves a limit of betweenness relations.

In Chapter 4 we discuss possible extensions and some directions for further work.

1.1 Generalities on permutation groups

There are a few notions which are standard in the theory of permutation groups which we will review. A proper introduction to the theory of permutation groups is given in Cameron [10] or Dixon and Mortimer [19]; Cameron's book [11] is a good introductory text. Throughout this thesis we will write (G, Ω) to mean a permutation group G acting on a set Ω. Usually Ω will be a countably infinite set. For $x \in \Omega$ and $g \in G$ the image of x under g will be denoted by $g(x)$ or x^g. We will be explicit about conjugation when we are conjugating.

Definition 1.1.1. Let (G, Ω) be a permutation group. For any x in Ω, the *orbit* of x under the action of G is the subset of Ω written as

$$x^G := \{x^g : g \in G\}.$$

An alternative notation for the same thing is

$$Gx := \{g(x) : g \in G\}.$$

It is implicit in this definition that orbits are always non-empty subsets of Ω.

Definition 1.1.2. We say that a permutation group (G, Ω) is *transitive* if there is only one orbit on Ω. That is, for all x, y in Ω there is a g in G such that $g(x) = y$.

Any action of G on Ω induces an action of G on ordered pairs of elements given by $g(x, y) := (g(x), g(y))$. Extending this to an action by coordinates on n-tuples gives rise to the action (G, Ω^n). Note that in this action on ordered pairs, if $y \neq z$ then (y, z) is not in the orbit of (x, x) for any x; the same for n-tuples. So we will have more concern for the natural action of G on n-tuples of *distinct* elements, the $\overline{x} = (x_1, ..., x_n)$ such that

$$\bigwedge_{i \neq j} x_i \neq x_j.$$

Definition 1.1.3. Given a permutation group (G, Ω), we will refer to the set of *n-tuples of distinct elements* from Ω specifically by the notation $\Omega^n_{\neq}$ and generally regard this as the natural domain of action of G on n-tuples.

Definition 1.1.4. A permutation group (G, Ω) is *k-transitive* if $(G, \Omega^k_{\neq})$ is transitive.

Thus, (G, Ω) is k-transitive if and only if there is only one orbit on k-tuples of distinct elements in the natural coordinate-wise action of G on $\Omega^k_{\neq}$. Note that 2-transitivity is strictly stronger than transitivity and for $m > n$, m-transitivity is strictly stronger than n-transitivity. Sometimes we may say doubly transitive in place of 2-transitive.

Definition 1.1.5. The action (G, Ω) is *highly transitive* if it is k-transitive for every natural number k.

Above we considered the natural action of G on n-tuples of distinct elements. We now turn to the action of G on n-sets. An *n-set* from Ω is a finite subset of Ω of size n. Let $\Omega^{\{n\}} := \{A \subseteq \Omega : |A| = n\}$, the set of n-sets of Ω. The action of (G, Ω) lifts to an action on n-sets given by

$$g(A) := \{g(a) : a \in A\}.$$

Definition 1.1.6. A permutation group (G, Ω) is called *k-homogeneous* if the natural action of G on $\Omega^{\{k\}}$ is transitive. If it is k-homogeneous for all k then we say that (G, Ω) is *highly homogeneous*.

Remark 1.1.7. If (G, Ω) is k-transitive then it is k-homogeneous.

Definition 1.1.8. A *primitive* action (G, Ω) is one which admits no non-trivial, proper congruence relations. In other words, there are no non-trivial, proper equivalence relations on Ω preserved by the action of G.

Definition 1.1.9. A group (G, Ω) is *k-primitive* if and only if it is k-transitive and for any $k - 1$ distinct $a_1, ..., a_{k-1} \in \Omega$, the action of the pointwise stabiliser $G_{(a_1,...,a_{k-1})}$ on $\Omega \setminus \{a_1, ..., a_{k-1}\}$ is primitive.

For some examples, we will consider structures related to the linear ordering of the rationals $(\mathbb{Q}, <)$. They all have automorphism groups which are highly homogeneous.

First consider the automorphism group of the ordering of the rationals $\mathrm{Aut}(\mathbb{Q}, <)$ acting on $\mathbb{Q}$. It is highly homogeneous and it is primitive, but not 2-transitive (and hence not k–primitive or k–transitive for any $k \geq 2$).

A related structure on the rationals is given by the *linear betweenness relation*, written $\mathrm{bet}(x; y, z)$ for $x, y, z \in \mathbb{Q}$. Defined in terms of the linear order, $\mathrm{bet}(x; y, z)$ is a ternary relation which holds when x is between y and z in the linear ordering. That is, for x, y, z in $\mathbb{Q}$,

$$\mathrm{bet}(x; y, z) \iff (y \leq x \leq z) \vee (z \leq x \leq y).$$

We use the semi-colon in $\mathrm{bet}(x; y, z)$ to emphasise the symmetry among the variables in this definition; we have that $\mathrm{bet}(x; y, z) \iff \mathrm{bet}(x; z, y)$. The automorphism group $\mathrm{Aut}(\mathbb{Q}, \mathrm{bet})$ of $(\mathbb{Q}, \mathrm{bet})$ can be thought of as the collection of order preserving and order reversing permutations of $\mathbb{Q}$, with the ordering $(\mathbb{Q}, <)$ in mind. Therefore $\mathrm{Aut}(\mathbb{Q}, <)$ is an index 2 subgroup of $\mathrm{Aut}(\mathbb{Q}, \mathrm{bet})$. Hence, $\mathrm{Aut}(\mathbb{Q}, \mathrm{bet})$ is highly homogeneous and primitive on $\mathbb{Q}$; it is also 2-transitive but not 2-primitive.

Another way to build a new relation on $\mathbb{Q}$ from the linear order is to turn the linear ordering $(\mathbb{Q}, \leq)$ into a *circular ordering* of $\mathbb{Q}$. Consider $\mathbb{Q}$ as a dense subset of the unit circle in $\mathscr{C}$, say the collection of roots of unity. Then for x, y, z in $\mathbb{Q}$ the relation $\mathrm{circ}(x, y, z)$ holds if going around the circle clockwise from x to y to z to x passes through x twice and every point on the circle once. Then for every $x, y, z \in \mathbb{Q}$ the relation $\mathrm{circ}(x, y, z)$ satisfies the following in terms of the linear order,

$$\mathrm{circ}(x, y, z) \iff (x \leq y \leq z) \vee (y \leq z \leq x) \vee (z \leq x \leq y).$$

The automorphism group $\mathrm{Aut}(\mathbb{Q}, \mathrm{circ})$ is highly homogeneous and 2-primitive, but not 3-transitive, on $\mathbb{Q}$ and contains $\mathrm{Aut}(\mathbb{Q}, \leq)$ as the pointwise stabiliser of any $a \in \mathbb{Q}$.

By considering the circular order up to order reversal, we have the *separation relation* on $\mathbb{Q}$. Again consider $\mathbb{Q}$ as a dense subset of the unit circle in $\mathscr{C}$, as in the circular ordering above. For x, y, z, w in $\mathbb{Q}$, define $\mathrm{sep}(x, y; z, w)$ if both of the arcs (one clockwise and one anticlockwise) from x to y pass through $\{z, w\}$. The automorphism group $\mathrm{Aut}(\mathbb{Q}, \mathrm{sep})$ contains $\mathrm{Aut}(\mathbb{Q}, \mathrm{bet})$ as a pointwise stabiliser and $\mathrm{Aut}(\mathbb{Q}, \mathrm{circ})$ as an index two subgroup. Then $\mathrm{Aut}(\mathbb{Q}, \mathrm{sep})$ is highly homogeneous, 2-primitive and 3-transitive but not 4-transitive nor 3-primitive on $\mathbb{Q}$.

It turns out that any highly homogeneous permutation group which is not highly transitive on an infinite set Ω preserves one of these kinds of structure. Note that if G is highly transitive on Ω then the only relational structure it preserves on Ω is trivial, that is the structure $(\Omega, =)$ given by the relation of equality on Ω. Indeed Cameron has classified the relational structures preserved on Ω by permutation groups on Ω which are highly homogeneous but not highly transitive.

Theorem 1.1.10 (Cameron [12])**.** *Let G be a group of permutations of an infinite set Ω and suppose that G is highly homogeneous but not highly transitive on Ω. Then G preserves on Ω either*

1. *A linear order;*
2. *A linear betweenness relation;*
3. *A circular order;*
4. *A separation relation.*

While we have not given the abstract definition of the above relations for arbitrary sets Ω, they can each be built in a similar fashion from a given linear ordering on $(\Omega, <)$. This can be found in Chapter 11 of [7], where there are also lists of axioms for each of these kinds of structure.

In the above examples we have given permutation groups as automorphism groups of certain relational structures. In order to properly talk about automorphism groups of first order structures, we will briefly give the definition of a first–order structure.

Definition 1.1.11. A *first–order language* is a triple $\mathscr{L} = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K})$, where $\{f_i\}_{i\in I}$ is a collection of function symbols (in which each f_i has arity n_i), $\{R_j\}_{j\in J}$ is a collection of relation symbols (in which each R_j has arity m_j) and $\{c_k\}_{k\in K}$ is a collection of constant symbols.

Definition 1.1.12. Given a language $\mathscr{L} = (\{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K})$ a *first–order $\mathscr{L}$–structure* on a set Ω is an $M = (\Omega; \{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K})$ such that for each

1. function symbol f_i we have an n_i–ary function $f_i : M^{n_i} \to M$;

2. relation symbol R_j we have an m_j–ary relation $R_j \subseteq M^{m_j}$;

3. constant symbol c_k we have an interpretation of this constant $c_k \in M$.

The set Ω is called the *domain* of the $\mathscr{L}$–structure M.

Definition 1.1.13. Given an $\mathscr{L}$–structure $M = (\Omega; \{f_i\}_{i\in I}, \{R_j\}_{j\in J}, \{c_k\}_{k\in K})$ on the domain Ω we say that a permutation g of Ω is an *automorphism* of M if

1. for every n_i–ary function f_i, we have $f_i(g(x_1), ..., g(x_{n_i})) = g(f_i(x_1, ..., x_{n_i}))$;

2. for every m_j–ary relation R_j, we have $R_j(g(x_1), ..., g(x_{m_j})) \iff R_j(x_1, ..., x_{m_j})$;

3. g fixes every constant symbol c_k.

The collection of all automorphisms of M is the subgroup $\mathrm{Aut}(M) \leq \mathrm{Sym}(\Omega)$ of the group of all permutations of Ω. When $G \leq \mathrm{Sym}(\Omega)$ is a subgroup of the group of all permutations of Ω such that every $g \in G$ satisfies 1 and 2 above, we say that G *preserves* the functions f_i and that G *preserves* the relations R_j respectively.

Note that when G preserves all of the functions, relations and constants of the structure M, then $G \leq \mathrm{Aut}(M)$ is a subgroup of the automorphism group of M. In particular $\mathrm{Aut}(M)$ preserves all of the functions, relations and constants in M.

When (G, Ω) is the automorphism group of a first–order $\mathscr{L}$–structure on Ω, then there is another homogeneity condition, relative to the $\mathscr{L}$–structure on M, which we will consider.

Definition 1.1.14 (following [3])**.** Let M be an $\mathscr{L}$–structure with domain Ω and let $G = \mathrm{Aut}(M)$ act on Ω as the automorphisms of M. For k a natural number we say the structure M is *relatively k–transitive* if every isomorphism $\varphi : A \to B$ between k–sets $A, B \subseteq \Omega$ extends to an automorphism $g \in G$.

Note that, for any k, the relative k–transitivity of M is dependent on the language $\mathscr{L}$ of M, as the notion of isomorphism depends on the language $\mathscr{L}$. It is quite possible for

M an $\mathscr{L}$–structure and N an $\mathscr{L}'$–structure to be two structures on the same domain Ω such that $\mathrm{Aut}(M) = \mathrm{Aut}(N)$ and M is relatively k–transitive for some k, but N is not relatively k–transitive.

As a ready example, note that $(\mathbb{Q}, <)$ is relatively k–transitive for every k.

Definition 1.1.15. A countably infinite $\mathscr{L}$–structure M is $\aleph_0$-categorical, or ω–categorical, if every countably infinite $\mathscr{L}$–structure N which satisfies the same first order sentences as M is isomorphic to M.

1.1.1 Jordan groups

We are now in a position to give the definition of our main objects of study.

Definition 1.1.16. Let (G, Ω) be a transitive permutation group on an set Ω. A *Jordan set* for G is a subset $\Gamma \subseteq \Omega$ such that $|\Gamma| > 1$ and the pointwise stabiliser $G_{(\Omega \setminus \Gamma)}$ is transitive in its induced action on Γ.

Definition 1.1.17. When Γ is a Jordan set for G we call the pointwise stabiliser $G_{(\Omega \setminus \Gamma)}$ the *Jordan subgroup associated to* Γ.

Note that If (G, Ω) is $k+1$–transitive and $\Sigma \subseteq \Omega$ is any co-finite subset of Ω such that $|\Omega \setminus \Sigma| \leq k$ and $|\Sigma| > 1$, then Σ is a Jordan set for G. This is a direct consequence of the $k+1$–transitivity of (G, Ω). We call such subsets Σ *improper Jordan sets*. A *proper* Jordan set Γ for G is any Jordan set which is not improper.

Definition 1.1.18. Let (G, Ω) be a transitive permutation group on an set Ω. Then (G, Ω) is a *Jordan group* if there is $\Gamma \subseteq \Omega$ which is a proper Jordan set for G.

Consider the automorphism group $\mathrm{Aut}(\mathbb{Q}, <)$ of the linear ordered rationals and let $G := \mathrm{Aut}(Q, <)$ act naturally on $\mathbb{Q}$. Let $\Gamma \subseteq \mathbb{Q}$ be a bounded open interval $\Gamma = (a, b)$ for some $a, b \in \mathbb{Q}$ with $a < b$. Then $G_{(\mathbb{Q} \setminus \Gamma)}$ is transitive on Γ, as $\mathbb{Q}$ is dense and Γ has no endpoints, so as this Γ is infinite and co-infinite, Γ is a proper Jordan set for G. As G is 2–homogeneous, all $\Gamma = (a, b)$ for $a, b \in \mathbb{Q}$ lie in one orbit Γ^G. Moreover, $G_{(\mathbb{Q} \setminus \Gamma)}$

is isomorphic to G and, in particular, $G_{(\mathbb{Q}\setminus\Gamma)}$ is primitive on Γ. That is, Γ is a *primitive* Jordan set for G.

Definition 1.1.19. Suppose G is a primitive Jordan group acting on Ω and Γ a proper Jordan set. If $G_{(\Omega\setminus\Gamma)}$ is primitive in its action on Γ, then Γ is a proper *primitive* Jordan set for G.

1.1.2 Reducts and maximal closed subgroups of $\mathrm{Sym}(\Omega)$.

Let Ω be a countably infinite set. The group of all permutations of Ω, is called the symmetric group on Ω and is denoted by $\mathrm{Sym}(\Omega)$. It is implicit in this name and notation that $\mathrm{Sym}(\Omega)$ acts on Ω. Sometimes $\mathrm{Sym}(\Omega)$ is called S_∞, and we may do so too. There is a natural topology on $\mathrm{Sym}(\Omega)$ given by the *topology of pointwise convergence*. This comes from the product topology on Ω^Ω, where the topology on Ω is discrete. A typical neighbourhood of $g \in \mathrm{Sym}(\Omega)$ is, for a finite subset $A \subset \Omega$,

$$[g]_A := \{h \in \mathrm{Sym}(\Omega) : h \restriction_A = g \restriction_A\}.$$

The topology is generated by open sets of the form $[g]_A$ for $g \in \mathrm{Sym}(\Omega)$ and finite subsets $A \subset \Omega$. This is a separable, metrizable and, in fact, totally disconnected topology endowing $\mathrm{Sym}(\Omega)$ with the structure of a Polish group. A group G is a *Polish group* with respect to the topology τ if τ is a complete, separable and metrizable topology on G such that the functions of group multiplication and taking of inverses is continuous with respect to τ.

Any permutation group of countably infinite degree can be considered as a subgroup of S_∞ and so inherits the Polish group topology from S_∞ by restriction. There is a well known, and extremely useful, characterisation of closed subgroups of $\mathrm{Sym}(\Omega)$. For details see, for example, [10] Theorem 5.8.

Proposition 1.1.20. *Let (G, Ω) be a subgroup of $\mathrm{Sym}(\Omega)$. Then G is the automorphism group of a first-order structure M on Ω if and only if G is a closed subgroup of $\mathrm{Sym}(\Omega)$.*

In the examples above, we considered a structure $(\mathbb{Q}, \mathrm{bet})$ obtained by defining the ternary relation bet on the domain $\mathbb{Q}$ in terms of $<$ in the structure $(\mathbb{Q}, <)$. In an intuitive sense the structure $(\mathbb{Q}, \mathrm{bet})$ is obtained from $(\mathbb{Q}, <)$ by 'forgetting' about the direction of the order $<$. This is an example of a more general relationship that may hold between two structures.

Definition 1.1.21 (See also [9])**.** Let M be an $\mathscr{L}$–structure and N an $\mathscr{L}'$–structure, we say that N is a *definable reduct* of M if

1. N and M have the same domain Ω;
2. each function, relation and constant of $\mathscr{L}'$–structure N is definable (without parameters) in the $\mathscr{L}$–structure M.

The structures M and N are *interdefinable* if N is a definable reduct of M, and M is a definable reduct of N. When N is a definable reduct of M such that M and N are not interdefinable, then N is a *proper* definable reduct of M.

Via Proposition 1.1.20, there is a closely related notion given in terms of automorphism groups of the structures.

Definition 1.1.22 (See also [9])**.** The structure N is a *group reduct* of the structure M if

1. N and M have the same domain Ω;
2. The automorphism group $\mathrm{Aut}(N)$ of N acting on Ω is a closed supergroup $\mathrm{Aut}(N) \geq \mathrm{Aut}(M)$ of the automorphism group $\mathrm{Aut}(M)$ of M acting on Ω.

We say that N is a *proper* group reduct of M if, in addition, $\mathrm{Aut}(N) \gneq \mathrm{Aut}(M)$ properly contains $\mathrm{Aut}(M)$.

We remark that when M, and hence also N, is ω–categorical, then the two notions coincide; N is a (proper) group reduct of M if and only if N is a (proper) definable reduct of M. So when M is ω–categorical we may consider *reducts* of M without ambiguity.

Definition 1.1.23. Suppose Ω is a countably infinite set and G a closed proper subgroup of $\mathrm{Sym}(\Omega)$. We say that G is a *maximal closed* subgroup of $\mathrm{Sym}(\Omega)$ if whenever $H > G$ is a closed subgroup of $\mathrm{Sym}(\Omega)$, then in fact $H = \mathrm{Sym}(\Omega)$.

In this context, Cameron's Theorem 1.1.10 stated above classifies the reducts of $(\mathbb{Q}, <)$ up to interdefinability (see also [26]).

Theorem 1.1.24 (Cameron [12])**.**
Let M be a reduct of the linear ordered rationals $(\mathbb{Q}, <)$. Then M is interdefinable with:

1. *The linear order $(\mathbb{Q}, <)$;*
2. *The linear betweenness relation $(\mathbb{Q}, \mathrm{bet})$ on $\mathbb{Q}$;*
3. *The circular ordering $(\mathbb{Q}, \mathrm{circ})$;*
4. *The separation relation on $(\mathbb{Q}, \mathrm{sep})$ on $\mathbb{Q}$;*
5. *The trivial structure $(\mathbb{Q}, =)$ of $\mathbb{Q}$ considered only as a countably infinite set.*

Corollary 1.1.25 (of Theorem 1.1.24)**.** *Let $H = \mathrm{Aut}(\mathbb{Q}, \mathrm{sep})$ be the group of all permutations of $\mathbb{Q}$ preserving the separation relation* sep *defined in $(\mathbb{Q}, <)$. Then H is a maximal closed subgroup of* $\mathrm{Sym}(\mathbb{Q})$.

1.2 Tree-like structures

Various kinds of tree-like structures are found in many parts of mathematics, in different contexts and for different purposes. Hence the word 'tree' can refer to many different kinds of mathematical structure. As we will not attempt to make a comprehensive survey, we focus on the structures which are relevant to this thesis. They are all relational structures (Ω, R) for various kinds of relation R. In order of introduction, they are *semilinear orders, betweenness relations, C–relations* and *D–relations*. The semilinear orders are fundamental in understanding the others. Semilinear orders are certain partial orderings, in which any two points have a uniquely defined 'path' between them. Betweenness relations are ternary relations that, in a certain semilinear order, retain just the structure given by these 'paths'; a point x of the semilinear order is between y and z if it lies on the path between y and z. A C–relation is a ternary relation on a collection of maximal *chains*, linearly ordered subsets, of a semilinear order. A D–relation is a quaternary relation on the 'ends' of a betweenness relation. All of these statements are made precise in the work of Adeleke and Neumann, published in the memoir [3]. We give some extracts below to cover what we need in this thesis.

A major study of semilinear orders was carried out by Droste in the memoir [18], in which these structures are called trees. In that memoir, Droste sets out many of the features of semilinear orderings which are important to us. In particular, the countably infinite semilinear orders which Droste calls *2–homogeneous trees*, which he classifies in [18], will be of particular importance in Chapter 2.

First we give the definition of a *(lower) semilinearly ordered set*. It is a partially ordered set $(T, \leq)$ such that for each $x \in T$, the restriction of $\leq$ to $\{y : y \leq x\}$, the predecessors of x, is a linear order and such that

$$(\forall x, y)(\exists z)(z \leq x \wedge z \leq y).$$

We will usually use the term semilinear order rather than tree to avoid ambiguity with notions from other areas of mathematics.

A semilinear order is of *positive type* if it contains its meets. That is, given any x, y in T the maximum of $\{z \in T : z \leq x \wedge z \leq y\}$ exists in T, it is called the *meet* of x and

y and is denoted by $\curlywedge(x, y)$ or by $x \curlywedge y$. We will use the notation $a < b$ to mean that $(a \leq b) \wedge (a \neq b)$ and the notation $c \perp d$ means that $(c \nleq d) \wedge (d \nleq c)$.

Definition 1.2.1. Given any (non maximal) point t in a semilinear order $(T, \leq)$, there is an equivalence relation C_t on $\{s \in T : s > t\}$, the vertices above t. The equivalence relation C_t is defined by setting

$$yC_tz \leftrightarrow (\exists w)(t < w \wedge w \leq y \wedge w \leq z).$$

The classes of this equivalence relation are called the *cones at* t. Given $t \in T$ and $x > t$, the cone at t containing x will be denoted by $C_t(x)$.

We will also need a more general notion of cone, which allows us to define cones at 'gaps' of a semilinear order.

Definition 1.2.2 (See [3], Section 4)**.** A subset $\Lambda \subseteq T$ of a semilinear order $(T, \leq)$ is a *lower section* if for all $x \in T$ and $a \in \Lambda$, if $x \leq a$ then $x \in \Lambda$. If Λ is a lower section of $(T, \leq)$ such that for some $t \in T$ we have that $\Lambda = \{x \in T : x \leq t\}$, then Λ is lower section of *positive type*. Otherwise, if Λ is a lower section of $(T, \leq)$ that is not of positive type, then we say that Λ is a lower section of *negative type*.

Definition 1.2.3 (See [3], Section 4)**.** Given a lower section Λ of a semilinear order $(T, \leq)$ which is not of the form $\{x : x < a\}$, there is an equivalence relation on $T \setminus \Lambda$ defined by saying that elements $x, y \in T \setminus \Lambda$ are in the same *cone at* Λ if there is some $w \in T \setminus \Lambda$ such that $w \leq x, y$. If there is more than one cone at Λ, then Λ is called a *ramification point*. In this case, the *ramification order* of Λ is the number of cones at Λ.

A lower section Λ can be of positive or negative type, so also as a ramification point Λ has either positive or negative type. If Λ is a lower section of negative type which is not a ramification point, so there is only one cone above Λ, then we will call Λ a *cut of negative type*.

Note that if Λ is a lower section of positive type and $t \in T$ is such that $\Lambda = \{x \in T : x \leq t\}$, then the cones at Λ, as in Definition 1.2.3, coincide exactly with the cones at t, as in Definition 1.2.1. In this situation we say that t is a ramification point if and only if

Λ is, and define t to have the same ramification order as Λ. In this way we can identify lower sections of T of positive type with points of T. We will refer to lower sections of negative type as *points* of negative type, although they are not elements of T. However such points of negative type are elements of the *Dedekind – MacNeille completion* T^+ of T. The Dedekind – MacNeille completion is a general notion of completion for a partial order that we will not go into here. We remark that in our context T^+ contains a unique infimum of $\{x, y\}$ for every x, y in T. When we write $x \curlywedge y$ for x, y in T such that $\{x, y\}$ has no infimum in T, we treat $x \curlywedge y$ as an imaginary element of T^2, which can otherwise be considered as an element of the Dedekind – MacNeille completion T^+.

A subset $\Delta \subseteq T$ is a cone of $(T, \leq)$ if Δ is a cone at Λ for some lower section $\Lambda \subseteq T$. Moreover, Δ is a cone at a ramification point of negative type if such a lower section Λ is of negative type.

Definition 1.2.4 (Following [3], Section 4)**.** A semilinear ordering $(\Omega, <)$ is called *normal* if no cone at a ramification point of negative type has a minimum element.

Definition 1.2.5. Let $(T, \leq)$ be a semilinear order. For $r, s \in T$, we say that the element s is a *successor* of r if

$$r < s \wedge (\forall u \in T)(u < s \rightarrow u \leq r).$$

Given $r \in T$, the set of successors of r will be denoted by

$$\mathrm{succ}_T(r) = \{s \in T : r < s \wedge (\forall u \in T)(u < s \rightarrow u \leq r)\}.$$

Definition 1.2.6. If $(T, \leq)$ is semilinear order and $(A, \leq)$ a substructure of $(T, \leq)$ then A is *convex* in T if for all $a \leq b$ in A, the interval in T between a and b is contained in A,

$$a \leq b \in A \rightarrow \{x \in T : a \leq x \leq b\} \subseteq A.$$

Definition 1.2.7. Let $(T, \leq)$ be a semilinear order. We say that $(T, \leq)$ is *discrete* if for every $a, b \in T$ such that $a < b$ there exists $c \in T$ such that $c \leq b$ and c is a successor of a.

Definition 1.2.8. Given a semilinear order $(T, \leq)$, a *chain* of T is a subset $L \subseteq T$ such that the restriction of $\leq$ to L is a linear order.

In other words a *chain* is a subset L of T, for a semilinear ordering $(T, \leq)$, such that the substructure $(L, \leq)$ of $(T, \leq)$ is a linear ordering. A *maximal chain* is a chain M in $(T, \leq)$ such that any proper subset N of T such that $M \subset N \subseteq T$ is not a chain.

Now we will consider two examples of discrete semilinear orders. The first is called the $\mathbb{N}$–tree. Its name comes from the fact that the maximal branches in this semilinear order are isomorphic to the ordering of the natural numbers reversed. By adding a level above the $\mathbb{N}$–tree we obtain the $\mathbb{N}^{+1}$–tree which we use in Chapter 3.

Definition 1.2.9. Let S be a countably infinite set and $\mathbb{N}$ be the natural numbers (with 0). The $\mathbb{N}$*–tree* is a semilinear ordering $(T, \leq)$ described as follows. The domain T of the $\mathbb{N}$–tree is the union of the set of natural numbers $k \in \mathbb{N}$ together with all finite sequences of the form $v = (k, s_{i_1}, ..., s_{i_n})$ where $n \leq k \in \mathbb{N}$ and $S_{i_n} \in S$ for $1 \leq n \leq k$. The order $\leq$ is defined on T as follows. If $v = k$ and $u = l$ then $u \leq v$ if l is greater than or equal to k in the normal ordering of $\mathbb{N}$. If $v = (k, s_{i_1}, ..., s_{i_n})$ then $u \leq v$ if $u = l$ and l is greater than or equal to k in $\mathbb{N}$, or if $u = (l, s_{j_1}, ..., s_{j_m})$ with $m \leq l \in \mathbb{N}$ and $l = k$ and $m \leq n$ and $s_{j_r} = s_{i_r}$ for $1 \leq r \leq m$.

Definition 1.2.10. The $\mathbb{N}^{+1}$*–tree* is obtained from the $\mathbb{N}$–tree by adding an extra level. For each leaf of the $\mathbb{N}$–tree add a binary splitting above that leaf. The $\mathbb{N}^{+1}$*–tree* is the resulting ordered structure $(V, \leq)$ on an infinite set V.

1.2.1 B–sets and betweenness relations

One feature of semilinear orderings is that, between any two points y and z from a semilinear ordering $(\Omega, \leq)$, there is a naturally defined notion of path. Let $L_y := \{u : u \leq y\}$ and $L_z := \{v : v \leq z\}$ be the chains below y and z respectively, and $y \curlywedge z$ the meet of y and z (which may either be an element of Ω, or an imaginary element, lying the Dedekind–MacNeille completion). What we mean by the natural *path* between y and z in Ω is the set

$$(L_y \bigtriangleup L_z \cup \{y \curlywedge z\}) \cap \Omega;$$

where $L_y \bigtriangleup L_z$ is the symmetric difference between the chain below y and the chain below z. The geometric intuition behind betweenness relations is that a point x is *between* y and

z if x lies on the path between y and z. This is the motivation behind the formal definition given below in terms of axioms worked out in [3].

Definition 1.2.11 (following [3])**.** A *B-relation* is a ternary relation on a non-empty set, X, satisfying the following axioms:

(B1) $B(x; y, z) \to B(x; z, y)$;

(B2) $B(x; y, z) \wedge B(y; x, z) \leftrightarrow x = y$;

(B3) $B(x; y, z) \to B(x; y, w) \vee B(x; w, z)$.

A B*-set* is a non-empty set together with a B-relation. A B–set with only one element will be called a *trivial* B–set. It should be noted that a B-set may be enhanced with various properties. The relation B is called a *betweenness relation* (or sometimes a *true betweenness relation* for emphasis) if

(B4) $\neg B(x; y, z) \to (\exists w)(B(w; x, y) \wedge B(w; x, z))$.

For all B–sets (X, B) used in this thesis, B is betweenness relation (satisfying (B4)).

A B–set (X, B) is *dense* if

(B5) $x \neq y \to (\exists z)(z \neq x \wedge z \neq y \wedge B(z; x, y))$.

Saying that it is of *positive type* means that

(B6) $(\forall x, y, z)(\exists w)(B(w; x, y) \wedge B(w; y, z) \wedge B(w; z, x))$.

Given $\{x, y, z\}$, any w witnessing the relationship required in $(B6)$ is called a *centroid* of $\{x, y, z\}$. In a B-set of positive type, a triple $\{x, y, z\}$ for which B does not hold (in any permutation) has at least one centroid. A B-set of positive type always satisfies $(B4)$, and so is automatically a betweenness relation.

Definition 1.2.12. A *combinatorial tree* is a graph (T, E) with symmetric graph relation E which is simple, connected and with no cycle.

Any finite, connected, unrooted combinatorial tree gives rise to a betweenness relation of positive type. Hence, we will often use graph-terminology when referring to finite connected B-sets. The relation $B(x; y, z)$ holds if the (unique) path from y to z passes through x. Conversely, a finite B-set (X, B) of positive type can be viewed as a connected, unrooted combinatorial tree. Given x and y from X, then x and y are adjacent in the graph if they are distinct and there is no $z \in X \setminus \{x, y\}$ such that $B(z; x, y)$. Given this characterisation, we will refer to the degree of the vertex in the corresponding graph as the *valency* of an element of a finite B-set. A *leaf* is an element of valency 1, a *dyadic* element has valency 2 and an element having valency at least 3 is called a *ramification node*. The set of ramification nodes will be written $\mathrm{ram}(B)$. We will habitually call elements $x \in X$ *nodes* of the B-set (X, B).

Given a betweenness relation on X and a node $a \in X$, there is an equivalence relation K_a on $X \setminus \{a\}$ given by

$$yK_az \Leftrightarrow \neg B(a; y, z).$$

The classes of K_a will be called *branches at* a. If there are at least 3 distinct branches at a, then we call a a *branch point of positive type*. Some authors use the term 'cone at a' for such a set. In order to avoid ambiguity we use the term *branches* when referring to classes of K_a for some a in a B–set (X, B), and reserve the term *cones* for the classes of C_t in a semilinear order $(T, \leq)$ with t in T.

Note that if (X, B) is a finite B-set of positive type then the number of K_a classes at a in X is clearly equal to the valency of a in the corresponding combinatorial tree as described above and a is a ramification node if and only if it is a branch point of positive type. For an arbitrary B-set (X, B), we generalise the notion of valency by defining the *valency* of $a \in X$ to be the number of branches at a.

Definition 1.2.13 (Section 18, [3])**.** Let (Ω, E) be a B–set and $\Lambda \subseteq \Omega$ a subset of Ω. The subset Λ is *convex* for E if and only if $x, z \in \Lambda$ and $E(y; x, z)$ implies $y \in \Lambda$. For $x, z \in \Omega$, the *interval* between x and z is $[x, z] := \{w : E(w; x, z)\}$.

Definition 1.2.14 (Section 18, [3])**.** Suppose (Ω, E) be a B–set and $\Lambda \subseteq \Omega$. We follow [3] in calling Λ a *component* of (Ω, E) if the following conditions are satisfied.

1. Both Λ and $\Omega \setminus \Lambda$ are non-empty and convex;
2. If $x \notin \Lambda, y, z \in \Lambda$ then $|[x, y] \cap [x, z] \cap \Lambda| \geq 1$.

Given a B–set (Ω, E) with E a general betweenness relation on Ω and $a \in \Omega$, the branches at a are components of (Ω, E) (see Theorem 18.3 of [3]).

We will say that an element e of a component Λ is an *endpoint* if for any $y \in \Lambda$ and $z \in \Omega \setminus \Lambda$ we have $e \in [y, z]$.

Let (Ω, E) be a B–set and let $\Lambda \subseteq \Omega$ be a subset of Ω. Points $x, y, z \in \Lambda$ are *collinear* if

$$E(x; y, z) \vee E(y; z, x) \vee E(z; x, y).$$

The B–set (Λ, E) is called *linear* if x, y, z are collinear for all $x, y, z \in \Lambda$. If (Λ, E) is not linear then there are $a, b, c \in \Lambda$ such that $\neg E(a; b, c) \wedge \neg E(b; c, a) \wedge \neg E(c; a, b)$.

Recall that in a semilinear order $(T, <)$ we were able to consider points of negative type (around Definition 1.2.2). We need an analogous notion of negative points in a B–set. First we define *gaps* in a linear B–set.

Definition 1.2.15. Let (Λ, E) be a linear B–set so that E is a linear betweenness relation on Λ. A *cut* of Λ is a partition $\Lambda = L \cup R$ of Λ into disjoint convex subsets L and R. If both L and R have no end-points then $\{L, R\}$ is a *gap* of (Λ, E).

We also need to consider branching points of negative type in a general betweenness relation.

Definition 1.2.16. Let (Ω, E) be a B–set such that E is a general betweenness relation on Ω. A *cut* of Ω is a partition $\Omega = \bigcup_{i \in I} \Omega_i$ into disjoint components Ω_i. A cut of $\Omega = \bigcup_{i \in I} \Omega_i$ such that $|I| \geq 3$ and none of the components Ω_i has an endpoint is a *branch point of negative type*. For such a partition, the Ω_i are *branches* at the branch point of negative type they define.

When (Ω, E) is a B–set with E a general betweenness relation on Ω, then branches at points of negative type are components of (Ω, E) (see Theorem 18.3 of [3]).

Given a B–set (Ω, E) with a general betweenness relation E, the *points of negative type* of Ω are branch points of negative type or gaps of Λ where Λ is a maximal linear subset of Ω.

Definition 1.2.17. Let $(\Omega, \leq)$ be a normal semilinear order. We define the *natural betweenness relation* defined in $(\Omega, \leq)$ to be the relation B on Ω defined from $\leq$ as follows.

$$B(y; x, z) :\Longleftrightarrow$$
$$(x \leq y \leq z) \vee (z \leq y \leq x) \vee (y \leq x \wedge y \perp z) \vee (y \leq z \wedge y \perp x) \vee (y = x \curlywedge z).$$

We remark (see [3], Theorem 17.1) that whenever $(\Omega, \leq)$ is a normal semilinear ordering then (Ω, B) as defined above is a B–set on which B is a betweenness relation.

Definition 1.2.18 (Section 33 in [3])**.** Let $(\Omega, \leq)$ be a semilinear order and E a B–relation on Ω. As in [3] we say that E is *compatible* with the ordering $\leq$ if

(AB1) $y \leq x \leq z \rightarrow E(x; y, z)$;

(AB2) $(y \leq z \wedge E(x; y, z)) \rightarrow (y \leq x \leq z)$.

Theorem 1.2.19 (A special case of Theorem 33.1 from [3])**.** *Let $\mathbb{S} = (\Omega, <)$ be a relatively 2–transitive semilinear order, which is normal, and E a dense general betweenness relation that is compatible with the ordering $(\Omega, \leq)$. Then $(\Omega, E) = (\Omega, B)$ where B is the natural betweenness relation defined in $\mathbb{S}$.*

1.2.2 C–relations

Let $(T, \leq)$ be a semilinear ordering and let $M := \{m_i\}_{i \in I}$ be a collection of maximal chains of $(T, \leq)$, indexed by I. Let C, standing for *chain relation*, be a ternary relation on this collection $M = \{m_i\}$ of maximal chains in $(T, \leq)$ defined for $m_i, m_j, m_k \in M$ by

$$C(m_i; m_j, m_k) \text{ if } (m_i \cap m_j) = (m_i \cap m_k) \subset (m_j \cap m_k).$$

In other words, the chains m_j and m_k are closer to each other then they are to the chain m_i. We call such a structure (M, C) a (proper) C–set.

From [3] we have the following formalisation of this notion of C–set.

Definition 1.2.20 (Section 10 of [3])**.** Let $C(x; y, z)$ be a ternary relation on the set X. The relation C is called a *C–relation* if it satisfies following axioms (C1) – (C4) for every $x, y, z, w \in X$.

(C1) $C(x; y, z) \to C(x; z, y)$;

(C2) $C(x; y, z) \to \neg C(y; x, z)$;

(C3) $C(x; y, z) \to C(x; w, z) \vee C(w; y, z)$;

(C4) $x \neq y \to C(x; y, y)$.

The structure (X, C) is called a C–set when C is a C–relation on X. If, in addition, C satisfies (C5) and (C6), (X, C) is called a *proper* C–set.

(C5) $(\exists v) C(v; x, y)$;

(C6) $x \neq y \to (\exists v)(v \neq y \wedge C(x; y, v))$.

Let (X, C) be a C–set such that, for all $x, y, z \in X$,

(C7) $C(x; y, z) \to (\exists w)(C(w; y, z) \wedge C(x; y, w)$

then (X, C) is a *dense* C–set.

1.2.3 D–relations

In a similar way to a C–relation being considered as a ternary relation on the maximal chains of a semilinear order, a D–relation is a certain relation on a collection of maximal linear subsets of a B–set.

Definition 1.2.21 (Section 22 of [3])**.** Let $D(x, y; z, w)$ be a quaternary relation on the set X. The relation D is called a *D–relation* if it satisfies following axioms (D1) – (D4) for every $x, y, z, w, s \in X$.

(D1) $D(x, y; z, w) \to D(y, x; z, w) \wedge D(x, y; w, z) \wedge D(z, w; x, y)$;

(D2) $D(x, y; z, w) \to \neg D(x, z; y, w)$;

(D3) $D(x, y; z, w) \to D(s, y; z, w) \vee D(x, y; z, s)$;

(D4) $(x \neq z \wedge y \neq z) \to D(x, y; z, z)$;

The structure (X, D) is called a D–set if D is a D–relation on X. If in addition, (X, D) satisfies (D5) below, then it is called a *proper* D–set.

(D5) $(x, y, z \text{ distinct}) \to (\exists v)(v \neq w \wedge D(x, y; z, v)$.

Let (X, D) be a D–set such that, for all $x, y, z, w \in X$,

(D6) $D(x, y; z, w) \to (\exists v)(D(v, y; z, w) \wedge D(x, v; z, w) \wedge D(x, y; v, w) \wedge D(x, y; z, v)$

then (X, D) is a *dense* D–set.

Analogous to the notion of cones in the semilinear order $\mathbb{S}$, there are *sectors* in a D-relation, which we use in order to analyse potential D-relations definable in $\mathbb{S}$.

Definition 1.2.22 (See [3], Section 24)**.** Let (Ω, D) be a D-set and consider a partition of Ω as a disjoint union $\biguplus_{i \in I} \Sigma_i$ of non-empty subsets $\{\Sigma_i\}_{i \in I}$. Let λ be the associated equivalence relation on Ω, with equivalence classes $\{\Sigma_i\}_{i \in I}$. The partition, or the equivalence relation λ, is called a *structural partition* with *sectors* Σ_i if

1. $|I| \geq 3$;

2. $(\forall i \in I)(\omega_1, \omega_2 \in \Sigma_i \wedge \omega_3, \omega_4 \notin \Sigma_i \rightarrow D(\omega_1, \omega_2; \omega_3, \omega_4))$;

3. $\omega_1, \omega_2, \omega_3, \omega_4$ distinct mod $\lambda \rightarrow \neg D(x, y; z, w)$ for all permutations (x, y, z, w) of $(\omega_1, \omega_2, \omega_3, \omega_4)$.

Lemma 1.2.23 ([3], Section 24, paragraph 2)**.** *If $\{\Sigma_i\}_{i \in I}$ is a structural partition of the D-set (Ω, D) and we have $D(\omega_1, \omega_2; \omega_3, \omega_4)$ then there is either some sector Σ_i such that $\omega_1, \omega_2 \in \Sigma_i$ or else some sector Σ_j such that $\omega_3, \omega_4 \in S_j$.*

Theorem 1.2.24 (Theorem 24.2 from [3])**.** *Let a, b, c be distinct elements of the D–set (Ω, D). There is a unique structural partition of Ω with the property that a, b, c lie in different sectors.*

Definition 1.2.25 (As in [3])**.** Let (Ω, D) be a D–set and λ, μ structural partitions of Ω. When Λ is an equivalence class of λ and Σ an equivalence class of μ such that

$$\Lambda \cup \Sigma = \Omega,$$

we say that the structural partitions λ and μ are *linked* and that Λ is the link from λ to μ.

Theorem 1.2.26 (Theorem 25.1 and Corollary 25.2 (1) from [3])**.** *Let (Ω, D) be a D–set and λ, μ distinct structural partitions of Ω. Then λ and μ are linked and there is a unique link from λ to μ. Moreover, a sector Γ of a D–set (Ω, D) uniquely determines its structural partition.*

For such λ and μ the set Λ is called the *link* from λ to μ and we write

$$\text{link}(\lambda, \mu) := \Lambda.$$

Lemma 1.2.27. *Let (Ω, D) be a D–set. If $a, b, c, d \in \Omega$ are such that $D(a, b; c, d)$ then there exist disjoint sectors Γ_1 and Γ_2 such that $a, b \in \Gamma_1$ and $c, d \notin \Gamma_1$ and $a, b \notin \Gamma_2$ and $c, d \in \Gamma_2$.*

Proof. We assume $D(a, b; c, d)$ and that $a, b, c, d \in \Omega$ are distinct. By Theorem 1.2.24 for each of the triples a, b, c and a, c, d there is a unique structural partition separating the elements in the triple. Let λ be the equivalence relation associated to the structural partition separating a, b, c.

We proceed to show that $d \in \lambda(c)$. The equivalence classes of λ form a structural partition (see Definition 1.2.22) of Ω. If $d \in \lambda(a)$ then Definition 1.2.22 (2) then $D(a, d; b, c)$, contradicting (D2), as we have assumed $D(a, b; c, d)$. Similarly if $d \in \lambda(b)$. If $\lambda(d)$ is distinct from $\lambda(a), \lambda(b), \lambda(c)$, then a, b, c, d are distinct mod λ, so by Definition 1.2.22 (3), then we contradict $D(a, b; c, d)$. Hence we must have $d \in \lambda(c)$.

Similarly, we let μ be the equivalence relation associated to the structural partition separating a, c, d. Arguing as in the last paragraph, we have $b \in \mu(a)$.

This part of the proof above is a special case of point (4) under the definition of structural partition in [3] Section 24. The paragraph below finishing the proof of the present lemma does not appear in [3].

If $\lambda(c)$ and $\mu(a)$ are disjoint, we are done. Suppose otherwise, that there is $x \in \lambda(c) \cap \mu(a)$. Let ν be the structural partition separating a, c, x given by Theorem 1.2.24. Of course, $\lambda(c)$ is the only λ class containing x and $\nu(a)$ is the only ν class containing a. Recall that λ separates a, b, c and that all structural partitions are linked (Lemma 1.2.26). So as $\nu(a) \cup \lambda(c) = \Omega$ and $b \notin \lambda(c)$, we must have that $\mathrm{link}(\nu, \lambda) = \nu(a)$ contains b. As $\lambda(c) \cup \nu(a) = \Omega$ and $d \notin \lambda(a)$, we have that $\mathrm{link}(\lambda, \nu) = \lambda(c)$ contains d. Similarly, $\mathrm{link}(\nu, \mu) = \nu(c)$ contains d and $\mathrm{link}(\mu, \nu) = \mu(a)$ contains b. That is $\nu(c)$ is the sector of ν containing c, d and $\nu(a)$ is the sector of ν containing a, b; as they are both sectors of ν, they are disjoint.

If a, b, c, d are not pairwise distinct, then either $a = b$ or $c = d$. If both $a = b$ and $c = d$, let s be any element distinct from a and c, then by Theorem 1.2.24, there is a unique structural partition σ separating a, c, s. Then $\sigma(a)$ and $\sigma(c)$ are disjoint sectors and we are done. Now, without loss of generality, assume that $a = b$ and c, d are distinct. Let τ be the structural partition separating a, c, d provided by Theorem 1.2.24. Then let t be an element distinct from a, c, d such that $t \in \tau(a)$. Then $D(a, t; c, d)$ with a, t, c, d distinct, in which case the proof above gives disjoint sectors Γ_1 and Γ_2 such that $a, t \in \Gamma_1$ and $c, d \notin \Gamma_1$ and $a, t \notin \Gamma_2$ and $c, d \in \Gamma_2$. Hence, as $a = b$ also $b \in \Gamma_1$ and $b \notin \Gamma_2$. □

Theorem 1.2.28 (Theorem 25.3 from [3])**.** *Let Λ be the family of structural partitions of a D–set (Ω, D). Define a ternary relation B_D on Λ by setting, for $\lambda, \mu, \nu \in \Lambda$ structural partitions of Ω,*

$$B_D(\lambda; \nu, \mu) :\Longleftrightarrow \begin{cases} \lambda = \nu & \textit{or} \\ \lambda = \mu & \textit{or} \\ \text{link}(\lambda, \nu) \neq \text{link}(\lambda, \mu). \end{cases}$$

Then (Λ, B_D) is a betweenness relation of positive type on Λ.

Theorem 1.2.29 (Theorem 26.6 from [3])**.** *Let (Ω, D) be a proper D–set and (Λ, B_D) be the B–set with betweenness relation B_D on the set Λ of structural partitions of Ω as defined in Theorem 1.2.28. Then (Λ, B_D) is dense if and only if (Ω, D) is dense.*

Let (Ω, D) be a D–set and (Λ, B_D) be the B–set interpreted on the set Λ of structural partitions of (Ω, D). Take any $\mu, \nu \in \Lambda$ and consider $[\mu, \nu] = \{\lambda : B_D(\lambda; \mu, \nu)\}$ the linear interval with end-points μ, ν. Let $a, b \in \Omega$ be points in the D–relation such that $\mu(a)$ omits b and $\nu(b)$ omits a. Then for any $\lambda \in [\mu, \nu]$, we have $\lambda(a)$ is disjoint from $\lambda(b)$ and that $\lambda(a) \supseteq \mu(a)$ and $\lambda(b) \supseteq \nu(b)$.

Definition 1.2.30 (As in Section 28, [3])**.** Given the context of the preceding paragraph, let (Γ_1, Γ_2) be a cut at a gap of $[\mu, \nu]$. The subsets of Ω defined by

$$\begin{aligned} \Sigma_1 &:= \textstyle\bigcup_{\lambda \in \Gamma_1} \lambda(a), \text{ and} \\ \Sigma_2 &:= \textstyle\bigcup_{\lambda \in \Gamma_2} \lambda(b), \end{aligned}$$

are called *convex halves* of (Ω, D).

Lemma 1.2.31. *If (Ω, D) is a dense D–set, then there is no disjoint pair of sectors Γ_1 and Γ_2 such that $\Gamma_1 \cup \Gamma_2 = \Omega$.*

Proof. Suppose that Γ_1 and Γ_2 are disjoint sectors such that $\Gamma_1 \cup \Gamma_2 = \Omega$. Via Theorem 1.2.26, let λ_1 be the structural partition determined by Γ_1 and λ_2 be the structural partition determined by Γ_2. Note that $\lambda_1 \neq \lambda_2$, as structural partitions have at least 3 parts. If μ is a structural partition of (Ω, D) with a sector Σ properly containing Γ_1, then by Theorem 1.2.26 the partition μ is distinct from λ_1 and linked to λ_1. But $\Sigma \cup \Gamma_1 = \Sigma \neq \Omega$ as $\Sigma \supset \Gamma_1$, and $\Sigma \setminus \Gamma_1$ is non-empty, so Γ_1 cannot be the link from λ_1 to μ. Hence $\Omega \setminus \Sigma$ is contained in a sector of λ_1 and $\Sigma = \mathrm{link}(\mu, \lambda_1)$. But as $\Gamma_1 \cup \Gamma_2 = \Omega$ we have that $\Sigma \cup \Gamma_2 = \Omega$, so also $\Sigma = \mathrm{link}(\mu, \lambda_2)$. Hence, as $\mathrm{link}(\mu, \lambda_1) = \mathrm{link}(\mu, \lambda_2)$, the definition of B_D in Theorem 1.2.28 gives that $\neg B_D(\mu; \lambda_1, \lambda_2)$.

As (Ω, D) is a dense D–set, then by Theorem 1.2.29, the B relation (Λ, B_D) is dense. So there is some structural partition μ such that $B_D(\mu; \lambda_1, \lambda_2)$.

By the previous paragraph and the property that Γ_1 and Γ_2 are disjoint such that $\Gamma_1 \cup \Gamma_2 = \Omega$, this structural partition μ cannot have a sector properly containing Γ_1 or Γ_2. Suppose that Γ_1 (or Γ_2) is a sector of μ. Then by Theorem 1.2.26 we have that $\mu = \lambda_1$ (or $\mu = \lambda_2$). However, this contradicts that (Λ, B_D) is dense, so we are done. □

1.3 Theorems around Jordan groups

In this section we are collecting together the classification results and technical results on Jordan sets which we will use in Chapters 2 and 3; in particular various detailed results on Jordan sets of tree–like structures for use in Chapter 2.

1.3.1 Primitive Jordan groups with primitive Jordan sets

First we note a couple of useful observations. Corollary 1.3.2 is the first step in being able to apply the Theorem 2.1.3 in the study of reducts of structures which contain proper primitive Jordan sets for their automorphism groups.

Theorem 1.3.1 (Theorem 0 of [4])**.** *If G is a primitive Jordan group and if Σ is a primitive cofinite Jordan set then Σ is improper.*

In other words, if Σ is a proper, primitive Jordan set for (G, Ω) an infinite primitive permutation group, then Σ is infinite and co-infinite.

Corollary 1.3.2. *If (G, Ω) is a primitive Jordan group for which Σ is a proper, primitive Jordan set, and $H \geq G$ is a supergroup, then Σ is a proper, primitive Jordan set for (H, Ω).*

Proof. As the action of $G_{(\Omega\setminus\Sigma)}$ induced on Σ is primitive, given any proper, non trivial equivalence relation ρ on Σ, and $x, y \in \Sigma$ such that $x\rho y$, there is $g \in G_{(\Omega\setminus\Sigma)}$ such that $\neg(g(x)\rho g(y))$. But $g \in H_{(\Omega\setminus\Sigma)}$ as H is a supergroup of G. So $H_{(\Omega\setminus\Sigma)}$ preserves no non-trivial proper equivalence relation on Σ. So Σ is a Jordan set for H. It is a consequence of Theorem 1.3.1 that Σ is infinite and co-infinite, and so Σ is also a proper Jordan set for H. □

Theorem 1.3.3 (Adeleke-Neumann Theorem 3 of [4])**.** *Suppose that (G, Ω) is an infinite, primitive Jordan group with primitive proper Jordan sets. If (G, Ω) is not highly transitive then there is a G-invariant relation R on Ω which is one of*

1. *A dense linear order (R is binary);*
2. *A dense linear betweenness (R is ternary);*
3. *A dense circular order (R is ternary);*
4. *A dense separation relation (R is quaternary);*
5. *A dense semilinear order (R is binary);*
6. *A dense general betweenness relation (R is ternary);*
7. *A C-relation (R is ternary);*
8. *A D-relation (R is quaternary).*

Additionally we make use in Chapter 2 of the finer information given by the following theorem. In this statement the term 'connected region' in the case of a semilinear order or a betweenness relation means what we call a *convex* set for the appropriate structure; while there are analogous notions for C and D–relations, we will not make use of them so have not given the specific definition.

Theorem 1.3.4 (Adeleke-Neumann, Theorem 5.4 of [4])**.** *Suppose that G is primitive on Ω, has a primitive proper Jordan set Σ_0, and is not highly homogeneous. Then there is a G-invariant semilinear order, C-relation, betweenness relation, or D-relation on Ω. Furthermore, in each case, the relation is dense and Σ_0 is a connected region of Ω with respect to the relevant relation.*

The following notion of a connected system, and the following Theorem 1.3.6 on unions of connected systems of Jordan sets is crucial in the proof of Theorem 2.1.3 in [4]. We will use it in the proof of Lemma 2.1.13.

Definition 1.3.5. A family $\mathscr{F}$ of subsets of a set Ω is a *connected system* if for all $\Gamma_1, \Gamma_2 \in \mathscr{F}$ there exist $\Sigma_0, \Sigma_1, ..., \Sigma_k \in \mathscr{F}$ such that $\Sigma_0 = \Gamma_1$ and $\Sigma_k = \Gamma_2$ and $\Sigma_i \cap \Sigma_{i+1} \neq \emptyset$ for $i \in \{0, 1, ..., k-1\}$.

Theorem 1.3.6 (A particular case of Lemma 3.2 from [4]). *Let (G, Ω) be a permutation group acting on Ω and $\mathscr{F}$ be a connected system of primitive Jordan sets for G. Then the union of all sets in $\mathscr{F}$, that is*

$$\bigcup_{\Gamma \in \mathscr{F}} \Gamma,$$

is a primitive Jordan set for G.

Definition 1.3.7. A pair (Γ_1, Γ_2) of subsets $\Gamma_1, \Gamma_2 \subseteq \Omega$ is called a *typical pair* if $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ and $\Gamma_1 \not\subseteq \Gamma_2$ and $\Gamma_2 \not\subseteq \Gamma_1$.

Definition 1.3.8 (following [3]). A collection Σ of subsets of Ω is called *syzygetic* if every typical pair $(\Gamma_1, \Gamma_2) \in \Sigma \times \Sigma$ of sets from Σ covers Ω, that is $\Gamma_1 \cup \Gamma_2 = \Omega$. A subset $\Gamma \subseteq \Omega$ is *syzygetic for G*, where G is a group acting on Ω, if $\Gamma^G := \{\Gamma^g : g \in G\}$ is syzygetic.

Lemma 1.3.9. *If (G, Ω) is a permutation group and $\Gamma \subseteq \Omega$ is syzygetic for G then its complement $\Omega \setminus \Gamma$ is syzygetic for G.*

Lemma 1.3.10 (See Theorem 6.9 of [3]). *Suppose $\mathbb{S} = (\Omega, <)$ be a relatively 2-transitive semilinear order and let $G := \mathrm{Aut}(\mathbb{S})$.*

1. *Let Λ be a lower section of $\mathbb{S}$ which is not of the form $\Lambda_a := \{w : w < a\}$ for any $a \in \Omega$. If Δ is a cone of $\mathbb{S}$ above Λ then Δ is a primitive Jordan set for $G := \mathrm{Aut}(\mathbb{S})$. If Γ is a union of cones at Λ then Γ is a Jordan set for Γ.*

2. *Conversely, if Γ is primitive Jordan set for G then there is a lower section Λ, not of the form $\Lambda_a := \{w : w < a\}$, such that Γ is a cone of $\mathbb{S}$ above Λ.*

Lemma 1.3.11 (See Theorems 5.17 and 5.26 and Lemma 5.22 of [18]). *Let $\mathbb{S} = (\Omega, <)$ be a relatively 2-transitive semilinear order and $G = \mathrm{Aut}(\mathbb{S})$. If Λ is a ramification point and C_1 and C_2 are distinct cones at Λ then there is a $g \in \mathrm{Aut}(\mathbb{S})$ of order 2 such that $C_1^g = C_2$ and $C_2^g = C_1$ which fixes the complement of $C_1 \cup C_2$.*

Lemma 1.3.12 (See Theorem 28.6 from [3]). *Given a D-relation (Ω, D) with automorphism group $H := \mathrm{Aut}(\Omega, D)$, if Γ is a primitive Jordan set for H then Γ is a sector of D or a convex half of D.*

Proof. This is a special case of the classification of proper Jordan sets of a D relation given by [3] Theorem 28.6. □

Lemma 1.3.13 (Corollary 25.2 of [3])**.** *If D is a D-relation on Ω then the family $\mathscr{C}$ of sectors of (Ω, D) is syzygetic.*

Corollary 1.3.14. *Assume D be a D-relation on Ω with automorphism group $G := \mathrm{Aut}(\Omega, D)$. Let $\mathscr{C}$ be the family of sectors of D and Σ be the collection of sectors and convex-halves of (Ω, D). Suppose $\Gamma \in \mathscr{C}$.*

1. *The set of translates Γ^G under $G = \mathrm{Aut}(\Omega, D)$ is syzygetic;*
2. *For any $g \in G$, if (Γ, Γ^g) is typical then $\Gamma \cup \Gamma^g = \Omega$;*
3. *Σ is syzygetic.*

Proof. It is clear from the definition of a syzygetic family of subsets that a subfamily of a syzygetic collection is syzygetic. As the family of all sectors $\mathscr{C}$ is syzygetic (Lemma 1.3.13), then $\Gamma^G \subseteq \mathscr{C}$ is syzygetic. Part (2) is nearly a restatement of (1).

Lemma 1.3.13 states exactly that for any typical pair (Γ, Σ) of sectors of (Ω, D) we have that $\Gamma \cup \Sigma = \Omega$. We need to consider typical pairs (U, V) where either one or both of U and V are convex halves of (Ω, D). First assume that U is a sector and V is a convex half and (U, V) is a typical pair. So V can be written $V = \cup_{i \in I} S_i$ where each S_i is a sector of (Ω, D) and $S_i \subseteq S_j$ for all $i < j$ in I. As $U \cap V \neq \emptyset$ there is some i such that $S_i \cap U \neq \emptyset$, then for all $j \geq i$, $S_j \cap U \neq \emptyset$. Similarly there is some $k \geq i$ such that $S_k \nsubseteq U$. As $U \nsubseteq V$, for all i we have $U \nsubseteq S_i$. So (U, S_k) is a typical pair of sectors and by Lemma 1.3.13, $U \cup S_k = \Omega$. Hence $U \cup S_k \subseteq U \cup V = \Omega$. The case remains is when U and V are convex halves such that (U, V) is a typical pair. Write V as above and $U =: \bigcup_{j \in J} T_j$ where $T_i \subseteq T_j$ for all $i < j$ in J. Similarly to the process above, we find, for large enough $i \in I$ and $j \in J$, a typical pair (S_i, T_j) of sectors and by Lemma 1.3.13 conclude that $S_i \cup T_j = \Omega$ and hence $U \cup V = \Omega$. □

Theorem 1.3.15 (Theorem 18.5 from [3])**.** *The collection of components of a B–set (Ω, E) is syzygetic.*

Theorem 1.3.16 (Theorem 20.2 in [3])**.** *Let (Ω, E) be a B–set and $H := \mathrm{Aut}(\Omega, E)$. Let $\Sigma \subseteq \Omega$ be a proper Jordan set for H in (Ω, E) and let (Σ, E) be the B–set substructure induced from (Ω, E) on Σ. If (Ω, E) is not linear and H is primitive on Ω then either*

1. *both Σ and its complement $\Omega \setminus \Sigma$ are components of (Ω, E); or*
2. *there is a branch point $a \in \Omega$ of positive type such that Σ is a union of branches at the branch point a; or*
3. *there is a branch point α of negative type such that Σ is a union of branches at the branch point α.*

Corollary 1.3.17 (of Theorem 1.3.16)**.** *Let (Ω, E) be a B–set such that E is a dense general betweenness relation and assume $H := \mathrm{Aut}(\Omega, E)$ is 2-transitive on Ω. Then if $\Sigma \subseteq \Omega$ is a proper primitive Jordan set then either,*

1. *both Σ and its complement $\Omega \setminus \Sigma$ are components of (Ω, E); or*
2. *there is a branch point $a \in \Omega$ of positive type such that Σ is a branch at a; or*
3. *there is a branch point α of negative type such that Σ is a branch at α.*

In particular, all of the sets described in the conclusion of this Corollary are of the type described in Definition 1.2.16. Hence if Σ is a proper primitive Jordan set for $\mathrm{Aut}(\Omega, E)$ in the betweenness relation (Ω, E) as above, then Σ is a component of (Ω, E) and, in particular, Σ is convex in E.

Proof. Assume (Ω, E) is a B–set such that E is a dense general betweenness relation and that $H = \mathrm{Aut}(\Omega, E)$ is 2-transitive on Ω.

Let $\Sigma \subseteq \Omega$ be a proper primitive Jordan set for H. Then Σ is a Jordan set for H, which is 2-transitive on Ω and hence primitive on Ω, so by the first part of the previous Theorem 1.3.16, it is of one of the required types described in the conclusion of this corollary. Note that if $X \subseteq \Omega$ is a union of more than one branch, as in either case (2) or (3) of Theorem 1.3.16, it is not a primitive Jordan set for H; the pointwise stabiliser $H_{(\Omega \setminus X)}$ of the complement of X preserves the branch classes which are proper, non-trivial subsets of such an X. □

Corollary 1.3.18. *Let* (Ω, E) *be a B–set with 2-transitive automorphism group* $H := \mathrm{Aut}(\Omega, E)$ *then the collection* Σ *of proper primitive Jordan sets for* H*, is syzygetic.*

Proof. Combine Corollary 1.3.17 with Theorem 1.3.15. □

1.3.2 Primitive Jordan groups without primitive Jordan sets

The moral of Section 1.3.1 is that if G is a primitive Jordan group on Ω such that Ω contains a proper *primitive* Jordan set, then Theorem 2.1.3 of Adeleke and Neumann tells us that G is highly transitive or preserves some familiar tree-like or linear-like relational structure. If we consider primitive Jordan groups (G, Ω) for which there are only imprimitive proper Jordan sets then G might preserve only other types of relational structure. One of these possibilities is that G preserves a kind of incidence geometry on Ω called a Steiner system.

Definition 1.3.19 (See Definition 11.2 of [7])**.** Let k be a natural number greater than 2 and Ω a set. A Steiner k–system on Ω is a collection L of subsets of Ω, called *lines*, such that every line in L has the same size $l > k$ such that

1. There is more than one line;
2. If $a_1, a_2, ..., a_k$ are distinct point of Ω then there is a unique line $l \in L$ such that $a_1, a_2, ..., a_k \in l$.

Other more unfamiliar examples that turn up in the classification theorem of Adeleke and Macpherson are certain limits of tree–like structures or limits of Steiner systems. We state the definitions below.

Definition 1.3.20 (See Definition 2.1.9 in [2] and Definition 2.4 in [6])**.** Let (G, Ω) be a permutation group. We say that G *preserves a limit of betweenness relations* if it is a Jordan group such that there are: a linearly ordered set $(J, \leq)$ with no lower bound in J, a chain $(\Gamma_i : i \in J)$ of subsets of Ω, and a chain $(H_i : i \in J)$ of subgroups of G such that if $i < j$ then $\Gamma_i \supset \Gamma_j$ and $H_i \supset H_j$, and the following hold:

(i) for each $i, H_i = G_{(\Omega \setminus \Gamma_i)}$, H_i is transitive on Γ_i and has a unique non-trivial maximal congruence ρ_i on Γ_i;

(ii) for each $i, (H_i, \Gamma_i/\rho_i)$ is a 2-transitive but not 3-transitive Jordan group preserving a betweenness relation;

(iii) $\bigcup_{i \in J} \Gamma_i = \Omega$;

(iv) $(\bigcup_{i \in J} H_i, \Gamma_i/\rho_i)$ is a 2-primitive but not 3-transitive Jordan group;

(v) if $t \geq s$ then $\rho_t \supseteq \rho_s \restriction_{\Gamma_t}$;

(vi) $\bigcap_{t \in T} \rho_t$ is equality on Ω;

(vii) $(\forall g \in G)(\exists i_0 \in J)(\forall i < i_0)(\exists j \in J)(\Gamma_i^g = \Gamma_j \wedge g^{-1} H_i g = H_j)$;

(viii) for any $\alpha \in \Omega$, G_α preserves a C-relation on $\Omega \setminus \alpha$.

Definition 1.3.21 (See Definition 2.1.9 in [2] and Definition 2.4 in [6])**.** Suppose that (G, Ω) satisfies all the above assumptions with associated chains of subsets $(\Gamma_i : i \in J)$ of Ω and subgroups $(H_i : i \in J)$ of G and equivalence relations $(\rho_i)_{i \in J}$ witnessing the conditions above except (ii), but instead of (ii) satisfying (ii)′ below.

(ii)′ for each $i, (H_i, \Gamma_i/\rho_i)$ is a 2-transitive but not 3-transitive Jordan group preserving a D–relation.

Then we say that (G, Ω) *preserves a limit of D–relations*.

Definition 1.3.22 (Definition 2.1.10 of [2])**.** Let (G, Ω) be an infinite n–transitive but not $(n+1)$–transitive Jordan group, where n is a natural number greater than 3. Then G is said to preserve on Ω a *limit of Steiner systems* if there is a totally ordered index set $(I, \leq)$ with no greatest element, and an increasing chain $(\Pi_i : i \in I)$ of subsets of Ω such that:

(i) $\bigcup_{i \in I} \Pi_i = \Omega$;

(ii) for each $i \in I$ the setwise stabiliser $G_{\{\Pi_i\}}$ of Π_i is $(n-1)$–transitive on Π_i and preserves a non–trivial Steiner $(n-1)$–system on Π_i;

(iii) if $i < j$ then Π_i is a subset of a line of the $G_{\{\Pi_i\}}$–invariant Steiner $(n-1)$–system on Π_i;

(iv) for all $g \in G$ there is $i_0 \in I$, dependent on g, such that for every $i > i_0$ there is $j \in I$ such that $g(\Pi_i) = \Pi_j$ and the image under g of every $(n-1)$–Steiner line on Π_i is an $(n-1)$–Steiner line on Π_j;

(v) for each $i \in I$, the set $\Omega \setminus \Pi_i$ is a Jordan set for (G, Ω).

We are now able to state the main result of Adeleke and Macpherson in [2], a classification of the structures which may be preserved by *any* infinite primitive Jordan group.

Theorem 1.3.23 (Theorem 1.0.1 from [3].)**.** *Let (G, Ω) be an infinite primitive Jordan group. Then either G is highly transitive on Ω or G preserves on Ω one of the following structures:*

(a) a dense linear order;

(b) a dense circular order;

(c) a dense linear betweenness relation;

(d) a dense separation relation;

(e) a dense semilinear order;

(f) a dense general betweenness relation (induced from a semilinear order);

(g) a C–relation;

(h) a D–relation;

(i) a Steiner system;

(j) a limit of betweenness relations;

(k) a limit of D–relations;

(l) a limit of Steiner systems.

In addition, in cases (j), (k) and (l) of the above statement, none of the structures in cases (a)–(i) are preserved by G.

1.4 Fraïssé constructions

The constructions known as Fraïssé constructions from model theory have seen a number of applications in combinatorics and in studying infinite permutation groups. For an introduction to standard Fraïssé constructions we refer the reader to Macpherson's survey of homogeneous structures [26] and Cameron's book [11]. The exposition of a more general form of Fraïssé–Hrushovski constructions given here is similar to that in Section 3 of [5] by Baudisch, Martin-Pizarro and Ziegler. I am grateful to Isabel Müller for bringing that paper to my attention.

Let $\mathscr{K}$ be a class of finite structures with countably many members up to isomorphism and let $\mathscr{E}$ be a countable class of embeddings between them. If an embedding ε is in $\mathscr{E}$ then it may be called an *$\mathscr{E}$–embedding*, or a *strong* embedding.

Definition 1.4.1. The class $(\mathscr{K}, \mathscr{E})$ has the *amalgamation property* if, given $A, B_1, B_2 \in \mathscr{K}$ and embeddings $\varepsilon_i : A \to B_i$ for $i \in \{1, 2\}$ in $\mathscr{E}$, there exists $C \in \mathscr{K}$ and η_1, η_2 in $\mathscr{E}$, where $\eta_i : B_i \to C$, such that, for $i \in \{1, 2\}$, the composition $\eta_i \circ \varepsilon_i$ is an $\mathscr{E}$–embedding of A into C and $\eta_1 \circ \varepsilon_1 = \eta_2 \circ \varepsilon_2$.

We consider direct limits along sequences

$$A_0 \to A_1 \to ... \to A_i \to A_{i+1} \to ...$$

of $\mathscr{L}$–structures $A_i \in \mathscr{K}$, where arrows are $\mathscr{E}$–embeddings.

Definition 1.4.2. A sequence of $\mathscr{E}$–embeddings

$$A_0 \to A_1 \to ... \to A_i \to A_{i+1} \to ...$$

where $A_i \in \mathscr{K}$ for $i \in \mathbb{N}$ is *rich* if, for all $B \in \mathscr{K}$ and strong embeddings ε such that $\varepsilon : A_i \to B$ there is a $j \geq i$ and a strong embedding $\eta : B \to A_j$ such that $\eta \circ \varepsilon$ is the embedding $A_i \to A_j$ given by composition along the sequence.

Definition 1.4.3. The direct limit M of a rich sequence in $(\mathscr{K}, \mathscr{E})$ is called a *Fraïssé limit*.

Definition 1.4.4. If $A_i \in \mathscr{K}$ is an element of some rich sequence of $\mathscr{E}$–embeddings

$$A_0 \to A_1 \to \ldots \to A_i \to A_{i+1} \to \ldots$$

with Fraïssé limit M, we say that A_i *is* $\mathscr{E}$*–embedded in* M.

This theorem is more general than the classical Theorem of Fraïssé from [21]. It generalises a construction of Hrushovski in [22] of which my understanding mostly comes from lectures given by David Evans in Lyon (May 2011), and papers of Evans [20] and Wagner [30]. This Theorem is published in a paper [5] of Baudisch, Martin-Pizarro and Ziegler, citing results of Ziegler in [31].

Theorem 1.4.5 (From [5], citing [31])**.** *If* $(\mathscr{K}, \mathscr{E})$ *is countable and has the amalgamation property then rich sequences exist and the Fraïssé limit* M *is unique up to isomorphism. If* $p : A \to B$ *is an isomorphism between finite* $A, B \in \mathscr{K}$ *that are* $\mathscr{E}$*–embedded in* M*, then* p *extends to an automorphism of* M*.*

Given some embeddings $\varepsilon_1 : A \to B_1$ and $\varepsilon_2 : A \to B_2$, in some special cases we say that *we can identify* $B_1 \setminus A$ *with* $B_2 \setminus A$. This means that the required result of amalgamation, C, can be chosen so to be a copy of B_1 so that η_1 can be chosen to be the identity, and then η_2 to be an isomorphism of B_2 so that $\eta_2(B_2) = B_1$.

Chapter 2

Reducts of semilinear orders

One recent application of the classification of infinite primitive Jordan groups has been in determining the reducts up to first order interdefinability of certain structures. In particular, this strategy may be used when a structure has a primitive Jordan automorphism group by applying the classification results on primitive Jordan groups by Adeleke and Neumann [4] and Adeleke and Macpherson [2] (See Chapter 1.3). This strategy has been used by Bodirsky and Macpherson [9] and Kaplan and Simon [25] in different contexts.

We use this approach in determining the reducts of certain dense semilinear orders. The structures we consider here are countably infinite, *relatively 2-transitive*, lower semilinear orders. They are semilinear orders $\mathbb{S} = (\Omega, <)$ for which any partial isomorphism ρ between two element substructures $A, B \leq \mathbb{S}$ extends to an automorphism $g \in \mathrm{Aut}(\mathbb{S})$ of $(\Omega, <)$. These are called *2-homogeneous trees* by Droste and studied extensively by him in [18]. In Section 6 of [18], Droste classifies such semilinear orders and proves that there are countably many non-isomorphic, relatively 2-transitive lower semilinear orders. Such semilinear orders are either of *positive type*, containing all their meets, or of *negative type* containing no meets of incomparable elements. The results of Droste give that the isomorphism type of a countably infinite, relatively 2-transitive semilinear order is determined by whether it is of positive or negative type, and a countable cardinal (at least 2) stipulating the number of cones at each ramification point (see Theorem 6.21 of [18]).

2.1 Applying the classification of Jordan groups

The starting point for this approach is the observation that if (G, Ω) is a primitive Jordan group (with a primitive Jordan set) then for any supergroup $H \geq G$, (H, Ω) is a primitive Jordan group (with a primitive Jordan set), and all (primitive) Jordan sets for G are (primitive) Jordan sets for H (see Lemma 1.3.2). First we fix $\mathbb{S} = (\Omega, <)$ any of the relatively 2-transitive semiliner orders on a countably infinite set Ω, as classified by Droste in [18]. Note that $G = \mathrm{Aut}(\mathbb{S})$ is a primitive Jordan group on Ω in which cones of $\mathbb{S}$ are primitive Jordan sets for G, so by the observation above, any supergroup H is a primitive Jordan group on Ω with primitive Jordan sets. Then we use the classification theorem of Adeleke and Neumann, along with combinatorial information about Jordan sets, to determine which kinds of structures can be preserved by closed supergroups and then, exactly which structures (up to first order interdefinability) are preserved by such supergroups. We state the main theorem here. The rest of this chapter is dedicated to proving this theorem.

Theorem 2.1.1. *Let* $\mathbb{S} = (\Omega, <)$ *be a countably infinite, relatively 2-transitive lower semilinear order. Let* (H, Ω) *be a closed permutation group such that* $\mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\Omega, <) \subseteq H \subseteq \mathrm{Sym}(\Omega)$. *Then either*

(i) $H = \mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\Omega, <)$*;*

(ii) $H = \mathrm{Aut}(\Omega, B)$, *where* B *is the natural betweenness relation on* $\mathbb{S}$*;*

(iii) $H = \mathrm{Sym}(\Omega)$.

Corollary 2.1.2. *Every* $\mathrm{Aut}(\Omega, B)$ *above is a maximal closed subgroup of* $\mathrm{Sym}(\Omega)$. *There are countably infinitely many such maximal closed subgroups up to conjugacy in* $\mathrm{Sym}(\Omega)$.

It is asked in [9] (Question 5.10) whether there are $2^{\aleph_0}$ maximal closed subgroups up to conjugacy in $\mathrm{Sym}(\Omega)$, for Ω a countably infinite set.

Our starting point is the classification theorem of Adeleke and Neumann, which we recall below.

Theorem 2.1.3 (Adeleke-Neumann [4], Theorem 3)**.** *Suppose that (G, Ω) is an infinite, primitive Jordan group with primitive proper Jordan sets. If (G, Ω) is not highly transitive then there is a G-invariant relation R on Ω which is one of*

1. *A dense linear order (R is binary);*
2. *A dense linear betweenness (R is ternary);*
3. *A dense circular order (R is ternary);*
4. *A dense separation relation (R is quaternary);*
5. *A dense semilinear order (R is binary);*
6. *A dense general betweenness relation (R is ternary);*
7. *A C-relation (R is ternary);*
8. *A D-relation (R is quaternary).*

Theorem 2.1.4 (Adeleke-Neumann [4], Theorem 5.4)**.** *Suppose that G is primitive on Ω, has a primitive proper Jordan set Σ_0, and is not highly homogeneous. Then there is a G-invariant semilinear order, C-relation, betweenness relation, or D-relation on Ω. Furthermore, in each case, the relation is dense and Σ_0 is a connected region of Ω with respect to the relevant relation.*

The structure of the proof of Theorem 2.1.1 is to argue that it is not possible for a relation R of type (1), (2), (3), (4), (7) or (8) in Theorem 2.1.3 to be $\emptyset$–definable in $\mathbb{S}$ and that there is no proper reduct of type (5). We deduce that if Γ is a proper non-trivial reduct of $\mathbb{S}$ with automorphism group (H, Ω), then there is a general betweenness relation R on Ω preserved by H. Finally we show that such a relation R on Ω has the same automorphism group as the natural betweenness relation B defined in the semilinear order $\mathbb{S} = (\Omega, <)$.

Lemma 2.1.5. *Let $\mathbb{S} = (\Omega, <)$ be a relatively 2-transitive semilinear order and let R be a proper reduct of $\mathbb{S}$. Then the automorphism group $H := \mathrm{Aut}(R)$ acting on Ω is 2-transitive.*

Proof. Let $G := \mathrm{Aut}(\mathbb{S})$ act on Ω as the automorphisms of $\mathbb{S}$. As $\mathbb{S}$ is relatively 2-transitive, there are three orbits on pairs of distinct elements. They are:

(A) (x, y) such that $x < y$;

(B) (z, w) such that $z > w$;

(C) (u, v) such that $u \perp v$.

Let $H := \mathrm{Aut}(R)$ be the automorphism group of R acting on Ω. Then $H \geq G$, that is, H is a supergroup of G, as R is $\emptyset$–definable in $\mathbb{S}$. Suppose that for $f \in H$ is such that we have $x^f < y^f$ if and only if $x < y$. Such an f is an automorphism of $\mathbb{S} = (\Omega, <)$ and hence $f \in G$. Therefore, for R to be a proper reduct of $\mathbb{S}$, then there must be some $h \in H \setminus G$ and some pair of distinct elements $a, b \in \Omega$ such that $(a, b)^h$ is in a different G-orbit to that of (a, b). So let $h \in H \setminus G$ and (a, b) be a pair such that $(a, b)^h$ is in a different G-orbit to that of (a, b). We consider, in turn, the six possibilities.

Case 1. Assume $a < b$ and $a^h \perp b^h$. As every incomparable pair is in the same G-orbit, there is some $g \in G$ such that $(a^h, b^h)^g = (b^h, a^h)$. So also $(a, b)^{hgh^{-1}} = (b, a)$ where $b > a$. Hence there is a unique H-orbit on distinct pairs, so H is 2-transitive.

Case 2. Assume $a > b$ and $a^h \perp b^h$. Switching the roles of a and b, we reduce to Case 1 and conclude that H is 2-transitive.

Case 3. Assume $a \perp b$ and $a^h < b^h$. As there is only one G-orbit on incomparable pairs, there is a $g \in G$ such that $(a, b)^g = (b, a)$. Hence $(a, b)^{gh} = (b^h, a^h)$ where $a^{gh} > b^{gh}$. So again there is a unique H-orbit on distinct pairs and H is 2-transitive.

Case 4. Assume $a \perp b$ and $a^h > b^h$. Swapping a and b we reduce to Case 3, concluding that H is 2-transitive.

Case 5. Assume $a < b$ **and** $a^h > b^h$**.** Consider any $c > a$ such that $c \perp b$. If $c^h > b^h$ then H is 2-transitive by considering (c, b) in the context of Case 4. If $c^h < b^h$ then H is 2-transitive by considering (c, b) in Case 3. So we assume that $c^h \perp b^h$. By the semilinearity of $\mathbb{S}$, we have that $c^h \perp a^h$. But as $a < c$ we can consider (a, c) in Case 1 and conclude that H is 2-transitive.

Case 6. Assume $a > b$ **and** $a^h < b^h$**.** Swapping a and b we reduce to Case 5, concluding that H is 2-transitive.

Having considered all the possibilities, we conclude that H is 2-transitive on Ω. □

Corollary 2.1.6. *If R is $\emptyset$–definable in $\mathbb{S} = (\Omega, \leq)$ of type (5), a dense semilinear order, then R is the relation $\leq$ of $(\mathbb{S}; \leq)$.*

Proof. This is an immediate corollary of Lemma 2.1.5, as a 2-transitive permutation group does not preserve any non-trivial binary relations. □

Lemma 2.1.7. *There is no relation of type (1) a linear order, (2) linear betweenness, (3) a circular order or (4) a separation relation, $\emptyset$–definable in $\mathbb{S} = (\Omega, \leq)$.*

Proof. Let $G := \mathrm{Aut}(\mathbb{S})$ acting on Ω and $H \leq \mathrm{Aut}(\Omega, R)$ acting on Ω as a subgroup of automorphisms of a reduct (Ω, R) of $\mathbb{S}$. It suffices to show that G preserves no separation relation on Ω. Let $\Gamma \subseteq \Omega$ be a cone of $\mathbb{S}$ and let $a, b \in \Gamma$ such that $a \perp b$. Such a Γ is a proper primitive Jordan set for G and there is a $g \in G_{(\Omega \setminus \Gamma)}$ such that $(a, b)^g = (b, a)$. But for the automorphism group of a separation relation, any Jordan subgroup associated to a proper primitive Jordan set Δ preserves a linear order on Δ. By Lemma 1.3.2, any proper primitive Jordan set for G is a proper primitive Jordan set for $H \geq G$. So Γ is a proper primitive Jordan set for H with distinct $a, b \in \Gamma$ and $g \in G_{(\Omega \setminus \Gamma)} \leq H_{(\Omega \setminus \Gamma)}$ such that $(a, b)^g = (b, a)$. So $H_{(\Omega \setminus \Gamma)}$ does not preserve a linear order on the associated proper primitive Jordan set Γ, and so cannot preserve any separation relation; in other words R cannot be a separation relation on Ω. □

Lemma 2.1.8. *Let (Ω, R) be a reduct of a relatively 2-transitive semilinear order $\mathbb{S} = (\Omega, <)$ such that R is a D–relation.*

1. *If $x, y, z \in \Omega$ such that $B(y; x, z)$ is the natural B–relation defined in $\mathbb{S}$ and U is a sector (or convex half) of R such that U contains x and z but omits y then $y = x \curlywedge z$ and U is the union of all cones of $\mathbb{S}$ above y.*

2. *In particular, if $y \neq x \curlywedge z$ and $x, y, z \in \Omega$ such that $B(y; x, z)$ in the natural B–relation defined in $\mathbb{S}$, then there is no sector (or convex half) U of R such that U contains x and z but omits y.*

Proof. Let $G := \mathrm{Aut}(\mathbb{S}) = \mathrm{Aut}(\Omega, <)$ and let $H := \mathrm{Aut}(\Omega, R)$ be the automorphism group of the reduct (Ω, R) where the relation R is a D-relation definable without parameters in the semilinear order $\mathbb{S}$. Let B be the natural ternary betweenness relation defined in $\mathbb{S} = (\Omega, <)$ as follows.

$$B(y; x, z) :\Longleftrightarrow$$
$$(x \leq y \leq z) \vee (z \leq y \leq x) \vee (y \leq x \wedge y \perp z) \vee (y \leq z \wedge y \perp x) \vee (y = x \curlywedge z).$$

Let x, y, z be distinct elements of Ω and assume that $B(y; x, z)$ in the natural B-relation of $\mathbb{S}$. Let $U \subseteq \Omega$ be a sector (or convex half) of R containing x and z while omitting y.

Case 1. Assume we have $x < y < z$ in $\mathbb{S}$. Let V be the cone $C_y(z)$ of $\mathbb{S} = (\Omega, \leq)$ at y containing z. This V is a proper primitive Jordan set for $\mathrm{Aut}(\mathbb{S})$ and thus a primitive Jordan set for H. By Lemma 1.3.12, V is a member of Σ, the collection of sectors and convex halves of R. Hence by Corollary 1.3.14 part 3, if (U, V) is a typical pair then $U \cup V = \Omega$. But $U \cup V$ omits y, so (U, V) is not a typical pair. As $x \in U \setminus V$ and $z \in U \cap V$ we have $V \subseteq U$.

Next, let W be a cone of $\mathbb{S}$ at y which omits z. Let $g \in \mathrm{Aut}(\mathbb{S})$ fix all elements of Ω outside $\{w : y < w\}$, with $V^g = W$ and $W^g = V$. Note that such a g exists by Lemma 1.3.11. By Corollary 1.3.14, (U, U^g) cannot be typical as its union omits y. If there is some $t \in W \setminus U$ then $t^g \in W^g \setminus U^g = V \setminus U^g \subseteq U \setminus U^g$, as $V \subseteq U$. But we already have $x \in U \cap U^g$ and $t \in U^g \setminus U$ as $W \subseteq U^g$ and certainly $t \neq t^g$. So if $W \setminus U$ is non-empty, then (U, U^g) are typical, contrary to Corollary 1.3.14. So we conclude that $U \supset W$. Hence $w \in U$ for all $w > y$.

Now consider y' such that $x < y' < y < z$ and assume that U omits y'. Let V' be the cone $C_{y'}(z)$ of $(\mathbb{S}, <)$ at y' containing z. Then reading the first paragraph with y' in place of y, we conclude that $V' \subseteq U$. But then $y \in C_{y'}(z) = V'$, contradicting the primary assumption that U omits y. So instead we conclude that U cannot omit any such y'; so $(x, y) \subseteq U$.

Next we consider the possibility that U omits some x' such that $x' < x < y < z$. As the cone $C_x(y)$ at x containing y also contains z and omits x, we have that $(U, C_x(y))$ is typical, as $y \in C_x(y) \setminus U$ and $x \in U \setminus C_x(y)$. But note that $U \cup C_x(y)$ omits x'. As $C_x(y)$ is a primitive Jordan set for $\mathrm{Aut}(\mathbb{S}, \leq)$, it is a primitive Jordan set for H. By Lemma 1.3.12, $C_x(y)$ is a member of Σ, the collection of sectors and convex halves of R. Corollary 1.3.14 part 3 requires that if $(U, C_x(y))$ is a typical pair, then $U \cup C_x(y) = \Omega$. But we have just noticed that $U \cup C_x(y)$ omits x'. So we conclude that $\{w : w < x\} \subseteq U$.

At this stage we know that $\{w : w > y\} \cup \{w : w < y\} \subseteq U$.

Suppose there are $y', z' \in \Omega \setminus U$ such that $y' < z'$ and $y \perp y', z'$. Let x' be some element such that $x' < y$ and $x' < y'$. As $x' < y$, we have that such an x' is in U. By the relative 2-transitivity of $\mathrm{Aut}(\mathbb{S}, <)$, Lemma 1.3.11 and that cones of $\mathbb{S}$ are Jordan sets (Lemma 1.3.10), there is a $g \in \mathrm{Aut}(\mathbb{S}, <)$ such that $(y', z')^g = (y, z)$ and $(y, z)^g = (y', z')$. Moreover, we choose such a g which fixes x'. Then $U \cap U^g$ contains x', $U \setminus U^g$ contains z and $U^g \setminus U$ contains z'. So then (U, U^g) is a typical pair. Yet $U \cup U^g$ omits y and y' in contradiction to Corollary 1.3.14 part 2. We conclude that there is no such pair y', z' omitted by U. So $\Omega \setminus U$ must be an antichain containing y.

As U is a proper Jordan set, we must have $|\Omega \setminus U| > 1$. That is, there must be some $z' \neq y$ which is omitted by U. For such a z' we have $z' \perp y$.

Let $z' \in \Omega \setminus U$ be such an element distinct from y. As $\Omega \setminus U$ is an antichain, for any $w < z'$ and $t > z'$, we have $w, t \in U$. As $\mathrm{Aut}(\mathbb{S}, <)$ is relatively 2-transitive, there is some $g \in \mathrm{Aut}(\mathbb{S}, <)$ such that $(w, z')^g = (z', t)$. Moreover, such a g can be chosen to fix y and z. Then we have $U \cap U^g$ containing z, $U \setminus U^g$ containing t and $U^g \setminus U$ containing

z'. So then (U, U^g) is a typical pair. But $U \cup U^g$ omits y in contradiction to Corollary 1.3.14 part 2. So in fact, there is no such z'.

We conclude that in this case there is no sector or convex half U of R containing x, z and omitting y.

Case 2. Assume we have $z > y$ and $z, y \perp x$ in $\mathbb{S}$. Let V be the (open) cone $C_y(z)$ of $(\mathbb{S}, \leq)$ at y containing z. Note that $z \in U \cap V$. This V is a proper primitive Jordan set for $\mathrm{Aut}(\mathbb{S}, \leq)$ and thus a primitive Jordan set for H. By Lemma 1.3.12, V is a member of Σ, the collection of sectors and convex halves of R. Hence by Corollary 1.3.14 part 3, if (U, V) is a typical pair then $U \cup V = \Omega$. But $U \cup V$ omits y, so (U, V) is not a typical pair. As $x \in U \setminus V$ and $z \in U \cap V$, we have $V \subseteq U$.

Now consider the possibility that there is a y' such that $x \curlywedge y < y' < y < z$ and with U omitting y'. Let V' be the cone $C_{y'}(z)$ of $\mathbb{S} = (\Omega, <)$ at y' containing z. As in the previous paragraph (U, V') cannot be a typical pair, and so $V' \subseteq U$. But $y \in C_{y'}(z) = V'$, contradicting the primary assumption that U omits y. Instead we conclude that U cannot omit any such y'; so $(x \curlywedge y, y) \subseteq U$.

If we now take $w \in (x \curlywedge y, y)$, we have found a triple $w < y < z$ such that U contains w and z and omits y. So by Case 1, we conclude that there is no such sector (or convex half) U of (Ω, R).

If $\mathbb{S}$ is of negative type, it suffices to consider Cases 1 and 2 to exhaust the possible configurations of x, y, z such that $B(y; x, z)$ and we conclude that there is no sector (or convex half) U of R such that U contains x and z but omits y.

However, if $\mathbb{S}$ is of positive type and $B(y; x, z)$ holds then it is possible that $x, z > y$ and $x \perp z$ and $y = x \curlywedge z$. We consider this situation in Case 3 below.

Case 3. Assume we have $x, z > y$ and $x \perp z$ and $y = x \curlywedge z$ in $\mathbb{S}$. Let V be the (open) cone $C_y(z)$ of $(\mathbb{S}, \leq)$ at y containing z. Note that $z \in U \cap V$. This V is a proper primitive Jordan set for $\mathrm{Aut}(\mathbb{S}, \leq)$ and thus a primitive Jordan set for H. By Lemma 1.3.12, V is a member of Σ, the collection of sectors and convex halves of R. Hence by Corollary

1.3.14 part 3, if (U,V) is a typical pair then $U \cup V = \Omega$. But $U \cup V$ omits y, so (U,V) is not a typical pair. As $x \in U \setminus V$ and $z \in U \cap V$ we have $V \subseteq U$. By symmetry, we consider the cone $V' := C_y(x)$ in the same way, concluding that $V \cup V' \subseteq U$. By the same argument, if $t \in U$ and $t > y$ then $C_y(t) \subseteq U$.

Let $w' \in \Omega$ be such that $w' < y < z$ and $w \in \Omega$ such that $w \perp y$. Then $w' \notin U$ by Case 1. If $w \in U$ then, as $w \perp z$, we can reduce to case 2 by considering the configuration of z, y, w. Hence all such w and w' are omitted by U and so $(\{y\} \cup \{w' : w' < y\} \cup \{w : w \perp y\}) \cap U = \emptyset$.

Let $W := \Omega \setminus U$ be the complement of U, so we have $\{y\} \cup \{w' : w' < y\} \cup \{w : w \perp y\} \subseteq W$. As U is syzygetic for H, so by Lemma 1.3.9, W is syzygetic for H. Say W contains some $v > y$, note in particular that $v \neq x$ and $v \neq z$. Take some $g \in \mathrm{Aut}(\mathbb{S})$ fixing $\{u : u \leq y\}$ such that $v^g = z$ and $z^g = v$ and $x^g = x$. Then there is $y \in W \cap W^g$ and $v \in W \setminus W^g$ and $z \in W^g \setminus W$ so (W, W^g) is a typical pair. But $W^g \cup W$ omits x contradicting that W is syzygetic. Hence we conclude that W omits $\{v : y < v\}$.

As W is defined to be the complement of U, this means that $\{v : y < v\} \subseteq U$. As we have already noted, $(\{y\} \cup \{w' : w' < y\} \cup \{w : w \perp y\}) \cap U = \emptyset$. So in fact $U = \{v : y < v\}$. But then U is the disjoint union of cones $C_y(v_i)$ of $\mathbb{S}$ for $i \in I$. □

Lemma 2.1.9. *Let (Ω, R) be a proper reduct of $\mathbb{S} = (\Omega, <)$ such that R is a D-relation. Then $R(x, y; z, w)$ if and only if ($\{x, y\} \cap \{z, w\} = \emptyset$ and ($x = y$ or $z = w$ or there is some cone of $\mathbb{S}$ containing x, y and omitting z, w or vice versa)).*

Proof. Let $G := \mathrm{Aut}(\mathbb{S})$ in its action on Ω and let $H := \mathrm{Aut}(\Omega, R)$ act naturally on Ω. First note that $\mathrm{Aut}(\Omega, R)$ does not preserve a separation relation (Lemma 2.1.7) and is not highly transitive, so by Cameron's Theorem, $\mathrm{Aut}(\Omega, R)$ is not highly homogeneous. Hence Theorem 2.1.4 requires that R is a dense D–relation. So throughout this proof, we may assume that R is a dense D–relation. In particular this means that, by Lemma 1.2.31, there can be no disjoint sectors Γ_1 and Γ_2 of R such that $\Gamma_1 \cup \Gamma_2 = \Omega$.

($\Leftarrow$) If $x = y$ or $w = z$, then $R(x, y; z, w)$ holds by axiom (D1) for a D–relation. So we assume that x, y, z, w are distinct. As cones of $\mathbb{S}$ are primitive Jordan sets for G

(Lemma 1.3.10), they must also be primitive Jordan sets for H (Lemma 1.3.2). Hence by Lemma 1.3.12 any cone Δ of $\mathbb{S}$ is either a sector or a convex half of (Ω, R). If this Δ is a sector of R containing x, y and omitting z, w, then by part 2 of the definition of a sector (Definition 1.2.22), we have $R(x, y; z, w)$. Similarly, if Δ is a sector of R containing z, w and omitting x, y, we have $R(z, w; x, y)$ and by (D1) we have $R(x, y; z, w)$. If Δ is a convex half of R containing x, y and omitting z, w, there is a sector $\Gamma \subseteq \Delta$ containing x, y. Such a Γ omits z, w, and we have $R(x, y; z, w)$.

($\Rightarrow$) Let $H := \mathrm{Aut}(\Omega, R)$ and let Σ be the collection of sectors and convex halves of the D-relation R. Then, by Lemma 1.2.27, $R(x, y; z, w)$ implies that there are disjoint sectors Γ_1 and Γ_2 such that $(x, y \in \Gamma_1 \wedge z, w \notin \Gamma_1) \wedge (z, w \in \Gamma_2 \wedge x, y \notin \Gamma_2)$.

We assume $R(x, y; z, w)$ and that Γ_1 and Γ_2 are disjoint sectors of R such that Γ_1 contains x, y and omits z, w while Γ_2 contains z, w and omits x, y. Certainly $\{x, y\} \cap \{z, w\} = \emptyset$, so if x, y, z, w are not pairwise distinct, then $x = y$ or $z = w$, trivially satisfying the right hand side of the statement. So we may assume that x, y, z, w are distinct.

Case 1. Assume $z \neq y \curlywedge x \neq w$ and $x \neq w \curlywedge z \neq y$. In this case, by Lemmas 1.2.27 and 2.1.8, we have that neither x nor y are between z and w in the natural betweenness relation B of the semilinear order $\mathbb{S}$. Similarly, neither z nor w are between x and y in the natural betweenness relation B.

Subcase (a). Assume $x > z$. As z cannot be between x and y then also $y > z$ and by assumption $z \neq y \curlywedge x$ so $z < y \curlywedge x$. Then, whatever the relationship between x and y, because $x \curlywedge y > z$, we have $x \in C_z(y)$. Yet $C_z(y)$ certainly omits z, so if $C_z(y)$ also omits w then the right hand side of the result is satisfied and we are done. So assume that $w \in C_z(y)$ so $w \geq (w \curlywedge y) > z$.

Now we have that either $x \curlywedge y > w$ or $x \curlywedge y \perp w$ or $x \curlywedge y < w$. If $x \curlywedge y > w$ then $C_w(y)$ is a cone of $\mathbb{S}$ at w containing x, y and omitting z, w. If $x \curlywedge y \perp w$ then let $a := (x \curlywedge y) \curlywedge w$. Then $C_a(y)$ contains x, y and omits z, w. In either of these configurations, the right hand side of the statement is witnessed and then we are done. We rule out the remaining configuration in which $x \curlywedge y < w$ in the following paragraphs.

So suppose that $x \curlywedge y < w$, then as we have already assumed $z < x \curlywedge y$, so in fact $z < x \curlywedge y < w$. As neither x nor y are between w and z and vice versa we have $x \perp y$, so x and y are in distinct cones at $x \curlywedge y$. For the same reason, w is not above x or y and the point w is in a cone at $x \curlywedge y$ which omits x or y (or omits both x and y).

Let $c := x \curlywedge y$.

First assume that $C_c(w)$ omits both x and y. As $x \notin C_c(y)$, we have that x, y and w are in distinct cones at c and $z < c$. Hence x, y, w are incomparable, with the same pairwise meet c and $c > z$. In this case, making use of Lemmas 1.3.10 and Lemma 1.3.11, we let $g \in \mathrm{Aut}(\mathbb{S})$ be an automorphism of $\mathbb{S}$ fixing y and z such that $(w, x)^g = (x, w)$. As $g \in \mathrm{Aut}(\mathbb{S}) \leq H$ certainly $g \in H$. But then $R(x, y; z, w) \wedge R(w, y; z, x)$ contradicting axiom (D2) of a D-relation. So this configuration cannot arise.

Now we assume that $C_c(w)$ omits exactly one of x and y. Without loss of generality, assume that the cone $C_c(w)$ contains y and omits x. So $w \perp x \perp y$ and w is not below y, as w cannot be between x and y and w is not above y, as y cannot be between z and w. In this configuration we have that x, y, w are incomparable, with $y \curlywedge w > x \curlywedge y > z$. But then there is $g \in \mathrm{Aut}(\mathbb{S}) \leq H$ fixing x and z such that $(y, w)^h = (w, y)$. So $R(x, y; z, w) \wedge R(x, w; z, y)$ in contradiction with axiom (D2) of a D-relation.

Subcase (b). Assume that either $y > z$ or $x > w$ or $y > w$. Using the symmetry of the assumptions in Case 1, we use the argument of Subcase (a) after appropriate relabelling of x, y and z, w to produce a cone of $\mathbb{S}$ which witnesses the right hand side of the lemma.

Subcase (c). Assume that either $w > x$ or $z > x$ or $w > y$ or $z > y$. Using the symmetry of the assumptions in Case 1, we use the argument of Subcase (a) after appropriate relabelling of x, y and z, w to produce a cone of $\mathbb{S}$ which witnesses the right hand side of the lemma.

Subcase (d). Assume that $x, y \perp z$ and $x, y \perp w$.

Subsubcase (i). Suppose that $x \curlywedge y = w \curlywedge z$ Let $a := x \curlywedge y = w \curlywedge z$. Note that $a \neq x, y, z, w$ by the assumption from Case 1. It follows under the assumptions of Case 1(d) that x, y, z, w is an antichain.

If x, y, z, w are in distinct cones at a then there is an automorphism $g \in \mathrm{Aut}(\mathbb{S}) \leq H$ of $\mathbb{S}$ such that $(x, y, z, w)^g = (x, w, z, y)$ so that $R(x, y; z, w) \wedge R(x, w; z, y)$ in contradiction with axiom (D2) of a D-relation.

If y and w are in a cone at a omitting x, z then there is again an automorphism $h \in \mathrm{Aut}(\mathbb{S}) \leq H$ of $\mathbb{S}$ such that $(x, y, z, w)^h = (x, w, z, y)$ so that $R(x, y; z, w) \wedge R(x, w; z, y)$ in contradiction with axiom (D2) of a D-relation.

If x and z are in a cone at a omitting y, w then there is an automorphism $f \in \mathrm{Aut}(\mathbb{S}) \leq H$ of $\mathbb{S}$ such that $(x, y, z, w)^f = (z, y, x, w)$ so that $R(x, y; z, w) \wedge R(z, y; x, w)$ in contradiction with axiom (D2) of a D-relation.

If x and w are in a cone at a omitting y, z then there is an automorphism $h \in \mathrm{Aut}(\mathbb{S}) \leq H$ of $\mathbb{S}$ such that $(x, y, z, w)^h = (w, y, z, x)$ so that $R(x, y; z, w) \wedge R(w, y; z, x)$ in contradiction with axiom (D2) of a D-relation.

Finally, if y and z are in a cone at a omitting x, w then there is an automorphism $f \in \mathrm{Aut}(\mathbb{S}) \leq H$ of $\mathbb{S}$ such that $(x, y, z, w)^f = (x, z, y, w)$ so that $R(x, y; z, w) \wedge R(x, z; y, w)$ in contradiction with axiom (D2) of a D-relation.

Subsubcase (ii). Suppose that $x \curlywedge y > w \curlywedge z$. Let $a := x \curlywedge y$ and $b := w \curlywedge z$. The cone $C_b(a)$ contains x and y so if it omits w, z then we are done. The cone $C_b(a)$ cannot contain both w and z, as $w \neq z$ and if $w = b$ (or $z = b$) then $C_b(a)$ omits w (or z). So we assume that $C_b(a)$ contains exactly one of w, z.

Without loss of generality, say $z \in C_b(a)$ and $w \notin C_b(a)$.

If $z < a$ then $C_z(a) \subseteq C_b(a)$ is a cone of $\mathbb{S}$ containing x, y and omitting z, w witnessing the right hand side as required.

Alternatively, if $z \perp a$ then $b < a \curlywedge z$ and there is c such that $a \curlywedge z < c < a$. So $C_c(a) \subseteq C_b(a)$ is a cone of $\mathbb{S}$ containing x, y and omitting z, w witnessing the right hand side of the Lemma as required.

Now suppose that $z > a$. If there is some point $r \in \Omega \setminus \{a\}$ such that $B(r; x, y) \wedge B(r; w, z)$ then, by Lemma 2.1.8, we have $r \in \Gamma_1 \cap \Gamma_2$ contradicting the assumption that Γ_1 and Γ_2 are disjoint. So we may assume that there is no point $r \in \Omega \setminus \{a\}$ such that $B(r; x, y) \wedge B(r; w, z)$. If $x \geq y$ so $y = a := x \curlywedge y$ then we have that $z > y > b := z \curlywedge w$ giving $B(y; w, z)$. So by Lemma 2.1.8 we have $y \in \Gamma_1 \cap \Gamma_2$ contradicting the assumption that Γ_1 and Γ_2 are disjoint. Similarly, if $y \geq x$ then $x = a := x \curlywedge y$ and we reach a contradiction. So we may assume that $y \perp x$ and that there is no point $r \in \Omega \setminus \{a\}$ such that $B(r; x, y) \wedge B(r; w, z)$. The only possibility is that $a = x \curlywedge y$ is not a point of Ω and that x, y, z is an antichain such that a is the meet of any pair. But then there is an $h \in \mathrm{Aut}(\mathbb{S}) \leq H$ fixing w and x such that $(y, z)^h = (z, y)$, whence $R(x, y; z, w) \wedge R(x, z; y, w)$ contradicting (D2). We conclude that no such configuration arises such that $z > a$.

Subsubcase (iii). Suppose that $x \curlywedge y < w \curlywedge z$. Follow the previous paragraph, exchanging the roles of x, y with z, w.

Subsubcase (iv). Suppose that $x \curlywedge y \perp z \curlywedge w$ Let $s := x \curlywedge y$ and $t := z \curlywedge w$ and $u := s \curlywedge t$. Then s and t are in distinct cones of $\mathbb{S}$ at u, and $x, y \in C_u(s)$ and $w, z \notin C_u(s)$, witnessing the right hand side. Hence, under the assumptions of Case 1, we are done.

Case 2. Assume $x \perp y$ and $z = y \curlywedge x$. By Lemma 2.1.8 we have that Γ_1 is the union of all cones above z, so that $\Gamma_1 = \{t : t > z\}$. In particular, by the assumptions on Γ_1, we have that $w \not> z$.

Subcase (a). Assume $w < z$. Let Γ_2 be a sector of R as above, that is Γ_2 is disjoint from Γ_1 and such that $(x, y \in \Gamma_1 \wedge z, w \notin \Gamma_1) \wedge (z, w \in \Gamma_2 \wedge x, y \notin \Gamma_2)$. By Lemma

2.1.8 the sector Γ_2 contains all t such that $B(t; w, z)$, that is the interval of t such that $w < t < z$.

If we suppose that Γ_2 omits some $u < w$, then as $x > z > w > u$ we can take $g \in \mathrm{Aut}(\mathbb{S})$ such that $(x, z, w)^g = (x, w, u)$. Then $x \notin \Gamma_2 \cup \Gamma_2^g$ while $w \in \Gamma_2 \cap \Gamma_2^g$ and $z \in \Gamma_2 \setminus \Gamma_2^g$ and $u \in \Gamma_2^g \setminus \Gamma_2$, contradicting that sectors of R are syzygetic (see Corollary 1.3.14). So we conclude that $\{u : u \leq z\} \subseteq \Gamma_2$.

If we suppose that Γ_2 omits some $v \perp z$ then, by Lemma 2.1.8, Γ_2 omits all $t \geq v$; as v is between t and z in the natural betweenness relation B and $v \neq t \curlywedge z$ for such t. So fix some $t > v$. Take $w' < z$ such that $w' < z \wedge v$ and $g \in \mathrm{Aut}(\mathbb{S})$ fixing w' such that $(v, z)^g = (z, v)$. Then $t \notin \Gamma_2 \cup \Gamma_2^g$, while $w' \in \Gamma_2 \cap \Gamma_2^g$ and $v \in \Gamma_2^g \setminus \Gamma_2$ and $z \in \Gamma_2 \setminus \Gamma_2^g$, contradicting Corollary 1.3.14, that the sectors of R are syzygetic. So we have that $\{u : u \leq z\} \cup \{v : v \perp z\} \subseteq \Gamma_2$.

As Γ_1 and Γ_2 are disjoint by assumption, we now have that $\{u : u \leq z\} \cup \{v : v \perp z\} = \Gamma_2$. That is, Γ_1 and Γ_2 are disjoint, but $\Gamma_1 \cup \Gamma_2 = \Omega$. As R is a dense D–relation (see the first paragraph of the proof of this Lemma) this contradicts Lemma 1.2.31.

Subcase (b). Assume $w \perp z$**.** By Lemma 2.1.8, Γ_2 contains $\{s : w \curlywedge z < s < w\} \cup \{r : w \curlywedge z < r < z\}$. In particular Γ_2 contains z and some $r < z$. As Γ_2 is a sector of the D-relation R, which contains z, r and omits x, y then also $R(x, y; z, r)$. So following the previous paragraph with r in place of w we again conclude that $\{u : u \leq z\} \cup \{v : v \perp z\} = \Gamma_2$. So we have found that Γ_1 and Γ_2 are disjoint, but $\Gamma_1 \cup \Gamma_2 = \Omega$. As R is a dense D–relation this contradicts Lemma 1.2.31.

Case 3. Assume $w \perp z$ **and** $y = w \curlywedge z$**.** The proof that this case does not arise is as in Case 2 with w, z in place of x, y.

The other two cases, in which either $w \perp z$ and $x = w \curlywedge z$ or $x \perp y$ and $w = y \curlywedge x$ are dealt with with the same argument as in Case 2 after appropriate relabelling. □

Lemma 2.1.10. *The relation given in Lemma 2.1.9 is not a proper D-relation. Hence there is no proper D-relation $\emptyset$–definable in $\mathbb{S}$.*

Proof. Following [4], for a D-relation (Ω, D) to be *proper* it is required to satisfy

$$(D5) \text{ For all } x, y, z \in \Omega, (\exists w)(D(x, y; z, w).$$

To see that R does not satisfy $(D5)$, take $x, y, z \in \Omega$ such that $x < z < y$. Note that there is no cone of $\mathbb{S}$ containing x, y and omitting z. So by Lemma 2.1.9, R satisfies $(D5)$ if and only if there is $w \in \Omega$ and a cone U of $\mathbb{S}$ such that U contains z, w and omits x, y. But as $z < y$ any cone of $\mathbb{S}$ containing z also contains y. So R does not satisfy $(D5)$, so it is not a proper D–relation. □

Having eliminated D–relations as possible reducts of $\mathbb{S}$, it is now easy to eliminate C–relations.

Lemma 2.1.11. *Let $\mathbb{S} = (\Omega, <)$ be a relatively 2-transitive semilinear order with automorphism group $G = \mathrm{Aut}(\mathbb{S})$. Let R be a proper reduct of $\mathbb{S}$ on the domain Ω and let $H = \mathrm{Aut}(R)$ be the automorphism group of R acting on Ω. There is no dense proper C-relation on Ω preserved by H.*

Proof. First note that, as G is a primitive Jordan group with primitive Jordan sets in Ω, so is H. To prove the contrapositive, suppose that R is a proper C-relation on Ω preserved by H. By Theorem 2.1.4 we may assume that (Ω, R) is a dense proper C–set, that is axioms (C1)–(C7) hold for R. From R we make a proper D-relation L on Ω, which will contradict Lemma 2.1.10. We take our inspiration from [3] Theorem 23.5 and define

$$L(x, y; z, w) :\iff ((R(x; z, w) \wedge R(y; z, w)) \vee (R(z; x, y) \wedge R(w; x, y)).$$

We now prove that L is a dense proper D–relation on Ω. Although this is similar to [3] Theorem 23.5, we are working with slightly different conditions on the C–relation and we need a different conclusion.

Axioms (D1) to (D4) for L follow from (C1) to (C4) for R making use of the symmetries in the definition of L as in the proof of [3] Theorem 23.5. So (Ω, L) is a D–set. To see (D5), and that (Ω, L) is a proper D–set, we take x, y, z distinct in Ω, we need to establish that there exists $v \in \Omega$ such that $L(x, y; z, v)$. As R is a proper C-relation we consider the internal semilinear order interpretable in (Ω, C). By considering possible configurations of x, y, z as maximal chains in that semilinear order, by using (C5) and (C6) we see that there is a suitable chain v satisfying $L(x, y; z, v)$.

□

Next, we investigate betweenness relations on Ω, aiming to show that the only betweenness relation arising in a proper reduct of $\mathbb{S}$ is the natural one.

Lemma 2.1.12. *Let (Ω, E) be a betweenness relation with 2-transitive automorphism group $H := \mathrm{Aut}(\Omega, E)$. Assume that a, b, c are distinct and $E(a; b, c)$. Then there is no proper Jordan set for H containing a and omitting b and c.*

Proof. Assume that a, b, c are distinct and $E(a; b, c)$ and Γ is a proper Jordan set containing a and omitting b, c. Then the interval between b and c, excluding the end-points is the set $(b, c) := \{x : E(x; b, c) \wedge x \neq b \wedge x \neq c\}$ is definable over b, c. If $y \in (b, c)$ and $z \notin (b, c)$ there is no automorphism $h \in H$ fixing b and c pointwise such that $y^h = z$. As we have assumed that $a \in \Gamma$ and $a \in (b, c)$, the Jordan set Γ cannot contain anything outside (b, c), so $\Gamma \subseteq (b, c)$. But as Γ is a proper Jordan set, it contains some $a' \neq a$. Then take $d \in \Omega \setminus \Gamma$ such that $E(a; b, d) \wedge \neg E(a; c, d)$ and $\neg E(a'; b, d) \wedge E(a'; c, d)$ so that $a \in (b, d) \cap (b, c)$ and $a' \in (c, d) \cap (b, c)$. Such a d exists as by 2-transitivity, there is a branch point δ between a and a' such that b, c, d lie in different sectors at this branch point α. Now there is no $h \in H_{(b,c,d)}$ fixing b, c, d pointwise such that $a^h = a'$, contradicting the assumption that Γ is a Jordan set. □

Lemma 2.1.13. *Let $\mathbb{S} = (\Omega, <)$ be a relatively 2-transitive semilinear order and (Ω, E) a reduct of $\mathbb{S}$ such that E is a dense general betweenness relation on Ω. Let $G := \mathrm{Aut}(\mathbb{S})$ and $H := \mathrm{Aut}(\Omega, E)$ and let $\Lambda \subseteq \Omega$ be a component at δ where either δ is an element of Ω. Then Λ is a primitive Jordan set for H.*

Proof. Take $\mathbb{S}$, (Ω, E), G, H, Λ and Σ as in the statement of the Lemma.

For any $x, y \in \Omega$ such that $x < y$, there is a cone Δ of $\mathbb{S}$ containing y and omitting x. By Lemma 1.3.10, the set Δ is a primitive Jordan set for G and hence for H (Lemma

1.3.2). By Lemma 2.1.5, H is 2-transitive on Ω. Hence, for *any* distinct x, y in Ω, there is a primitive Jordan set for H (an H-translate of Δ) containing y and omitting x.

Let I be a convex linear subset of Ω such that I is maximal subject to lying in a component at δ. By fixing δ, the betweenness relation E induces the structure of a dense linear order $\prec$ on I. As $(I, \prec)$ is dense and by the maximality on I, the point δ is the infimum of $(I, \prec)$. Take a sequence $(y_n)_{n\in\mathbb{N}}$ of $y_n \in I$ such that $\delta \prec y_{n+1} \prec y_n$ for every $n \in \mathbb{N}$ which converges to δ in the sense that, for all $y_n \in I$ there is $m > n$ such that $y_m \in I \setminus \{y_n\}$ with $B(y_m; y_n, \delta)$. For each $n \in \mathbb{N}$ let U_n be a primitive Jordan set for H which contains y_{n-1} and omits y_n. Considering the possibilities given by Corollary 1.3.17 we have that $U_{n+1} \supseteq U_n$ for every $n \in \mathbb{N}$. So $\{U_n\}_{n\in\mathbb{N}}$ make up a connected system of primitive Jordan sets (Definition 1.3.5). Because $(I, \prec)$ is dense we have that $\Lambda = \ bigcup_{n\in\mathbb{N}} U_n$. In other words, Λ is the union of a connected system of primitive Jordan sets, and by Theorem 1.3.6 it is a primitive Jordan set. □

Lemma 2.1.14. *Let (Ω, E) be a proper reduct of $\mathbb{S} = (\Omega, \leq)$ such that E is a B-relation on Ω. Then E is compatible with $\leq$ in the sense of Definition 1.2.18. That is, we have:*

(AB1) $y \leq x \leq z \rightarrow E(x; y, z)$*;*

(AB2) $(y \leq z \wedge E(x; y, z)) \rightarrow (y \leq x \leq z)$*.*

Proof. Let $G := \mathrm{Aut}(\mathbb{S})$ and $H := \mathrm{Aut}(\Omega, E)$.

To prove (AB1) we assume that $y \leq x \leq z$. If $x = y$ or $x = z$ then $E(x; y, z)$ holds trivially. So we may assume that $x \neq y$ and $x \neq z$, so then $y < x < z$.

Let a be such that $x < a < z$, and let $C_a(z)$ be the cone of $\mathbb{S}$ at a containing z, note that $C_a(z)$ omits x, y. Then as $C_a(z)$ is a primitive Jordan set for G, it is a primitive Jordan set for H, containing z and omitting x, y. So by Lemma 2.1.12, we have $\neg E(z; x, y)$.

Let $b \in \Omega$ be such that $y < b < x$. The cone $C_b(x)$ is a cone of $\mathbb{S}$ containing x, z and omitting y, so as it is a primitive Jordan set for G, it is a primitive Jordan set for H. By Corollary 1.3.17, any primitive Jordan set for H is convex in E. So we have $\neg E(y; x, z)$.

We now assume that

$$\neg E(y;x,z) \wedge \neg E(z;x,y) \wedge \neg E(x;y,z). \tag{2.1.1}$$

Therefore, in the betweenness relation (Ω, E) the elements x, y, z are in distinct branches at some branch point δ, which may be of positive or negative type. For $t \in \{x, y, z\}$, let U_t be the branch at δ containing t and omitting $\{x, y, z\} \setminus \{t\}$. By Lemma 2.1.13, U_x, U_y and U_z are all primitive Jordan sets for H. We follow a similar line of argument to that used in Lemma 2.1.8, especially Case 1: we seek a contradiction in order to rule out this configuration. Recall that the collection of primitive Jordan sets for H is syzygetic (see Theorem 1.3.15).

Let W be the branch of (Ω, E) at x containing y and z. From Lemma 2.1.13 we know that

$$W \text{is a proper primitive Jordan set for } H. \tag{2.1.2}$$

Let $U := C_x(z)$ be the cone of $\mathbb{S}$ at x containing z. Then U is a proper primitive Jordan set for H. Note that $z \in W \cap U$ and $x \notin U \cup W$ and $y \in W \setminus U$, so by Theorem 1.3.15 we must have that $U \subseteq W$.

Suppose that W omits some $t > x$. From the last paragraph, we know that $t \notin C_x(z)$. From the relative 2-transitivity of $G \leq H$ and Lemma 1.3.11, we can find $h \in G \leq H$ fixing x and y and flipping the cones U and $C_x(t)$ such that $(z,t)^h = (t,z)$. Note that $y \in W \cap W^h$ and $z \in W \setminus W^h$ and $t \in W^h \setminus W$ but that $x \notin W \cup W^h$, in contradiction with Theorem 1.3.15. So we conclude that W contains $\{t \in \Omega : t > x\}$.

Now suppose that W omits some $s \in \Omega$ such that $y < s < x$. Fix some $t \in \Omega$ such that $x < t < z$. Let $g \in G \leq H$ be some automorphism of $\mathbb{S}$ such that $(y,s,x)^g = (s,x,t)$. Clearly $z^g > x^g = t > x$ and as we have chosen $t > x$, we know from the conclusion of the previous paragraph that both $z^g, t \in W$. But now we have that $x \notin W \cup W^g$ while $z^g \in W \cap W^g$ and $t \in W \setminus W^g$ and $s \in W^g \setminus W$. This contradicts Theorem 1.3.15, so we conclude that W contains $\{s \in \Omega : y < s < x\}$.

Now suppose that W omits some $r < y$. Let $f \in G \leq H$ be an automorphism of $\mathbb{S}$ fixing r such that $(y,x)^f = (x,z)$. Hence $z^f > x^f = z > x$ and so $z^f \in W$. Note that

$r \notin W \cup W^f$ while $z^f \in W \cap W^f$ and $z \in W \setminus W^f$ and $x \in W^f \setminus W$. This contradicts Theorem 1.3.15, so we conclude that W contains $\{r \in \Omega : r \leq y\}$. We now have that W contains $\{t \in \Omega : x < t\} \cup \{s \in \Omega : s < x\}$.

Suppose that W omits some $r' \perp x$ such that $r' \curlywedge y < y$. Let $h \in G \leq H$ be an automorphism of $\mathbb{S}$ fixing r' such that $(y, x)^h = (x, z)$. As $z^h > x^h = z > x$ we know that $z^h \in W$. Note that $r' \notin W \cup W^h$ while $z^h \in W \cap W^h$ and $z \in W \setminus W^h$ and $x \in W^h \setminus W$. This contradicts Theorem 1.3.15, so we conclude that W contains $\{r \in \Omega : r \curlywedge y < y\}$.

It remains to consider the case in which W omits some v such that $v \perp x$ and $y \leq (x \curlywedge v) < x$. Suppose that W omits a pair of such elements u, v which are comparable and distinct. Without loss of generality say $u < v$, so we have $u, v \perp x$ and $y \leq (u \curlywedge x) = (v \curlywedge x) < x$. Let $g \in G \leq H$ be an automorphism of $\mathbb{S}$ fixing y such that $(u, v, x, z)^g = (x, z, u, v)$. But then $x, u \notin W \cup W^g$ while $y \in W \cap W^g$ and $z \in W \setminus W^g$ and $v \in W^g \setminus W$ in contradiction with Theorem 1.3.15. So there are no comparable pairs of elements omitted by W. Therefore the complement $\Omega \setminus W$ of W is an antichain in $\mathbb{S}$ and for all $v \in \Omega \setminus W$, we have $y \leq (v \curlywedge x) < x$.

Suppose now that there is some $v \in \Omega \setminus W$ such that $v \perp x$ and $y \leq (v \curlywedge x) < x$. Let $d := v \curlywedge x$ and let w be an element such that $d < w < v$. As w is comparable to v we know that $w \in W$. Note that $y \leq d$, the cones $C_d(v)$ and $C_d(x)$ at d are distinct and that $w \in C_d(v)$. So let $h \in G \leq H$ be an automorphism of $\mathbb{S}$ fixing y and flipping the cones $C_d(v)$ and $C_d(x)$ such that $(w, x, v)^h = (x, v, z)$. Then we have that $v \notin W \cup W^h$, while $y \in W \cap W^h$ and $z \in W \setminus W^h$ and $x \in W^h \setminus W$ in contradiction with Theorem 1.3.15. Therefore we have deduced that x is the only element in $\Omega \setminus W$. But then, as H is 2-transitive on Ω, the subset W is not a proper Jordan set for H; its complement has cardinality $|\Omega \setminus W| = 1$. This contradicts (2.1.2).

From this analysis, we conclude that the configuration required by assuming (2.1.1) is impossible and so that assumption is false. Having exhausted the other possibilities, we conclude $E(x; y, z)$ as required by (AB1).

For (AB2) we assume $(y \leq z \wedge E(x; y, z))$. If $y = z$, then under the assumption that $E(x; y, z)$, the axiom (B2) implies that $x = y = z$ and we are done. Hence we may assume that $y < z$.

If $x > z$ then let $a \in \Omega$ such that $z < a < x$ and let $C_a(x)$ be the cone at a of $\mathbb{S}$ containing x. Then $C_a(x)$ contains x and omits y and z. But $C_a(x)$ is a primitive Jordan set for G and hence for H. In fact, $C_z(x)$ is a primitive Jordan set for H containing x and omitting y,z such that $E(x; y, z)$ in contradiction to Lemma 2.1.12.

Now assume $x \perp z$. Let b be such that $b \perp z$ and $x \curlywedge z < b < x$. Let U be the cone of $\mathbb{S}$ at b containing x, then U omits y, z. As U is a primitive Jordan set for G it is a primitive Jordan set for H. But U contains x and omits y, z and $E(x; y, z)$, in contradiction to Lemma 2.1.12.

So we assume that $x < z$. Suppose $x < y$ and then let c be such that $x < c < y$. Then the cone $C_c(y)$ contains y and z while omitting x. But as $C_c(y)$ is a Jordan set for G it is a Jordan set for H. But as $E(x; y, z)$, this contradicts that $C_c(y)$ is convex for E, as all primitive Jordan sets for H are convex for E by Corollary 1.3.17.

Hence we must have that $y < x < z$. □

Lemma 2.1.15. *If* (Ω, E) *is proper reduct of* $\mathbb{S}$ *for which* E *is of type* (6)*, a dense general betweenness relation, then* $\mathrm{Aut}(\Omega, E) = \mathrm{Aut}(\Omega, B)$ *where* B *is the natural betweenness relation defined in* $\mathbb{S}$*.*

Proof. Assuming that (Ω, E) is proper reduct of $\mathbb{S} = (\Omega, \leq)$ and E is a dense general betweenness relation Lemma 2.1.14 proves that E is compatible with the ordering $\leq$. Then using Theorem 1.2.19, we conclude that $\mathrm{Aut}(\Omega, E) = \mathrm{Aut}(\Omega, B)$ as required. □

Chapter 3

A tree of betweenness relations

After the classification theorem for primitive Jordan groups of Adeleke and Macpherson [2], efforts have been made to construct primitive Jordan groups which preserve the unfamiliar ‘limit’ structures which appear in that classification, while preserving none of the more familiar structures. The two known examples of a primitive Jordan group preserving a limit of betweenness relations have been constructed by Adeleke [1] and by Bhattacharjee and Macpherson [6]. These two examples are certainly non-isomorphic: Bhattacharjee and Macpherson’s example is the automorphism group of an $\aleph_0$–categorical relational structure, whereas it is known that Adeleke’s example is not oligomorphic. The goal of this chapter is to adapt the methods in [6] to obtain a group like that of [1].

The key idea in [6] is to define a class of finite structures called ‘trees of B–sets’; a carefully arranged collection of B–sets in some sense parametrised by finite semilinear orders. They prove that it is possible to amalgamate the finite structures in this class and, using a generalised form of Fraïssé’s Theorem, they obtain a Fraïssé limit. The automorphism group of their limit structure is their example. In this limit structure it is possible to recover a countably infinite, $\aleph_0$–branching semilinear order of positive type with dense maximal chains which could be considered the structure tree for their construction.

In contrast, Adeleke in [1] constructs his example as a direct limit of groups constructed in ω many steps, with no explicit reference to an invariant structure. We adapt the ideas in [6] to construct 'trees of B–sets' indexed by a semilinear order with discrete levels (which we call the $\mathbb{N}^{+1}$–tree, as described by Definition 1.2.10), in order to construct, using an appropriate adaptation of Fraïssé's theorem, a limit structure which we believe is preserved by Adeleke's group.

3.1 Trees of B-sets

This is the combinatorial description and is based on that given in [6], though there are differences. The main difference is that we work with a class of certain finite subtrees T^A of the tree T which will be fixed. The section after this will detail how to view this class of structures in a first order language. First, we give the definition of a *finite tree of B-sets*.

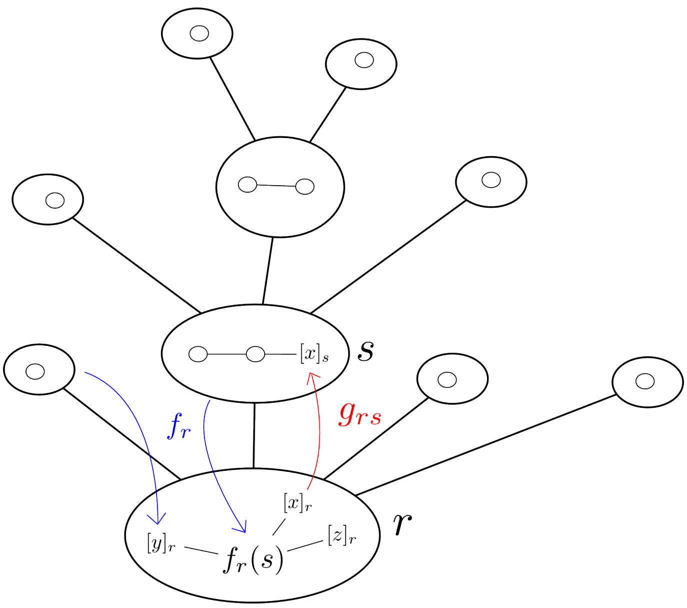


Figure 3.1: An illustrated example of a tree of B-sets over T.

Throughout this section, we fix T to be the $\mathbb{N}^{+1}$–tree $(T; <)$ as given in Definition 1.2.10. We enrich T with $\curlywedge$, the infimum function on $(T; <)$, and work in the enriched structure $(T; \curlywedge, <)$. We call T the *structure tree* for the following structures. A finite tree of B-sets over T, call it A, consists of a finite, convex (Definition 1.2.6) subtree $(T^A, \curlywedge, <)$ of T, together with a finite set M^A on which a structure is coded via a careful arrangement of finite B-sets $\{(B(t), B_t)\}_{t \in T^A}$. So for each vertex t in the tree T^A, there is a finite B-relation B_t with respective domain $B(t)$. The nodes of each $B(t)$ will be identified with certain subsets of M^A; these nodes are set up to be equivalence classes of an equivalence relation defined by Definition 3.1.4 and Lemma 3.1.5. The notation $[a]_t$ stands for the subset of M^A represented as a node of $B(t)$ and containing the element $a \in M^A$. Being finite and meet closed, the tree T^A has a unique minimal element, the *root*. When $r \in T^A$ is the root of T^A, we require that the nodes $[x]_r$ of the root B–set $B(r)$ are in bijection with the singleton subsets of M^A. Informally we identify M^A with $B(r)$, so in some sense M^A becomes a B–set with extra structure.

We will sometime say that $T^A \subseteq T$ is the subtree of T which is *populated* by A.

Recall from Section 1.2 that given a node a of a B-set, the equivalence relation K_a is that of B-set branches around a. Above a vertex t in the tree T^A, there is also the set of cones above t, each cone being an equivalence class of the equivalence relation C_t. As T^A is discrete, it suffices to consider the immediate successors $\mathrm{succ}(t)$ of a vertex t in T^A and note that $\mathrm{succ}(t) \subseteq \{s \in T^A : s > t\}$ provides a natural set of representatives for the set of cones in T^A above t; each distinct successor of t representing a distinct cone.

The constraints on the arrangement of B-sets on the tree are given by two families of functions, one between the tree and B-sets and another between pairs of B-sets.

For each vertex $t \in T^A$ such that $|B(t)| > 1$, there is a function

$$f_t : \{s \in T^A : s > t\} \to B(t).$$

We require that for each $t \in T^A$, f_t is a surjection onto the nodes in $B(t)$ and we require that the induced map f_t^{-1} is a bijection from the nodes of $B(t)$ to the set of cones in T^A above t. With that in mind, f_t^{-1} can be considered as a bijection between the successors, in T^A, of t and the nodes of the B-set $B(t)$. Hence, a condition required between the tree T^A and the system of B-sets is that the number of cones above any vertex $t \in T^A$ is equal to the cardinality of $B(t)$, unless $|B(t)| = 1$. In fact, $|B(t)| = 1$ if and only if t is maximal in T^A.

The relationships between B-sets of comparable vertices in T^A will be governed by a family $\{g_{ts}\}_{t<s}$, where $t, s \in T^A$. Here we define the g_{ts} for $s \in \mathrm{succ}_A(t)$, and later this is extended to any $s > t$ in T^A by composition. Given t and a successor s, g_{ts} is a surjection from $B(t) \setminus \{f_t(s)\}$ to $B(s)$; we require that each fibre $g_{ts}^{-1}(a)$ of any $a \in B(s)$ is a branch of B_t at $f_t(s)$ and moreover, that branches of B_t around $f_t(s)$ are in bijection with the elements of $B(s)$ via g_{ts}^{-1}. In particular, when $s \in \mathrm{succ}(t)$ the cardinality of $B(s)$ is equal to the number of branches around $f_t(s)$ in B_t.

Definition 3.1.1. A *finite tree of B–sets*, A consists of a finite semilinear ordered set T^A, a finite set M^A along with all of the B–sets and the functions between them to the requirements described above.

Let A be a finite tree of B–sets and take $v < s < t$ consecutive in T^A. Then g_{st} is defined on $B(s) \setminus \{f_s(t)\}$ and, one step below, g_{vs} is defined on $B(v) \setminus \{f_v(s)\}$. Now, as g_{vs} is a surjection onto $B(s)$, we can consider the composition $g_{st} \circ g_{vs}$ on the domain $B(v) \setminus \{f_v(s) \cup g_{vs}^{-1}(f_s(t))\}$.

Definition 3.1.2. For any $v < t \in T^A$ of a finite tree of B–sets A, there is a unique strictly increasing chain of $s_i \in T^A$, $\{s_0, s_1, ..., s_n\}$ such that $s_0 = v$, $s_n = t$ and $s_{i+1} \in \mathrm{succ}(s_i)$. We define g_{vt} to be the composition

$$g_{vt} := g_{s_{n-1}t} \circ ... \circ g_{s_{i-1}s_i} \circ ... \circ g_{vs_1}.$$

As such, it is defined on

$$g_{vs_1}^{-1} \circ ... \circ g_{s_{i-1}s_i}^{-1} \circ ... \circ g_{s_{n-1}t}^{-1}(B(t)) \subseteq B(v).$$

Lemma 3.1.3. *Let A be a finite tree of B–sets and take $v < s < t$ in T^A. Then*

$$g_{vt} = g_{st} \circ g_{vs},$$

and each $g_{vt}^{-1}(a)$ is a union of branches around $f_v(t) \in B(v)$.

Proof. This is clear from definitions. □

Definition 3.1.4. Given a finite tree of B–sets A, in which r is the root of T^A, we have already defined the notation $[a]_r$ for $a \in M^A$ as denoting the singleton $\{a\}$, so $[a]_s \in B(s)$. In the definition of a tree of B–sets above, we required that we may identify $\{[a]_r\}_{a \in M^A}$ with $B(r)$, the domain of the root B–set. For any $s > r$, and $a \in M^A$ such that $a \in \mathrm{Dom}(g_{rs})$, we define

$$[a]_s := g_{rs}([a]_r).$$

Lemma 3.1.5. *Following from the definition above, for any $s > v \in T^A$, $[.]_s$ induces an equivalence relation ρ_{vs} on $B(v)$. For $a, b \in B(v)$ then $a\rho_{vs}b$ if $[a]_s = [b]_s$ or if $a, b \in B(v) \setminus$ Im(g_{vs}^{-1}). The subset of M^A represented by $[b]_s$ is $g_{vs}^{-1}([b]_s)$.*

As an example, if s is a successor of r (in the tree T^A) and $b \in M^A$, then $[b]_s$ is a branch of the B–set $(B(r), B_r)$ at the node $f_r(b)$, an element of $B(r)$.

Soon, we aim to present trees of B–sets in a different way, as first order structures in a 2–sorted language. To do so, we need a relation which captures the structure of the B–sets on the structure tree. The meaning of this relation is given in the following definition.

Definition 3.1.6. To encode the structure of a finite tree of B–sets in a relation, we shall define a relation $L(v; x, y, z)$ on $T \times M^3$. We define $L(t; a, b, c)$ to hold if and only if, one of the following occurs in A:

1. In the B-set at t, we have $B_t([a]_t; [b]_t, [c]_t)$;
2. For some $s \geq t$ at which $[a]_s, [b]_s, [c]_s$ are non-empty and distinct, we have $B_s([a]_s; [b]_s, [c]_s)$.

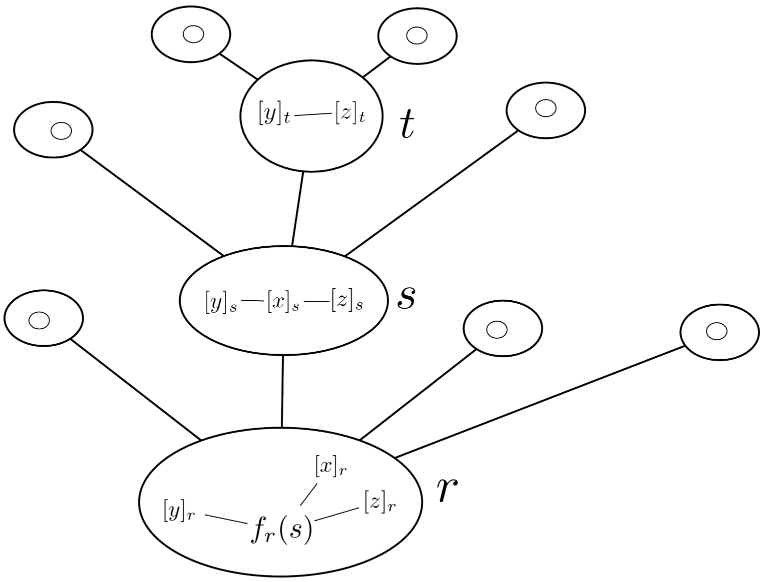


Figure 3.2: An instance of $L(r; x, y, z)$.

Definition 3.1.7. An *arboreal isomorphism* between two finite trees of B-sets A and A' over the common structure tree T consists of a tree automorphism $\tau \in \mathrm{Aut}(T;<)$ and for each $s \in T^A$, a B-set isomorphism $\varphi_s : B(s) \to B(\tau(s))$ such that

1. the tree automorphism τ restricted to T^A is an order isomorphism
$$\tau \restriction_{T^A} : T^A \to T^{A'};$$

2. for each s in T^A and all $t \geq s$ in T^A the B-set isomorphism φ_s and the tree automorphism τ commute with the all the relevant bijections
$$f_s : \{u \in T^A : u > s\} \to B(u)\}$$
and
$$f_{\tau(s)} : \{v \in T^A : v > \tau(s)\} \to B(\tau(s))\}$$
and the collection of bijections $\{g_{st}\}_{t \geq s}$, each with domain $B(s) \setminus \{f_t(s)\}$, in the following manner:
$$g_{\tau(s)\tau(t)} \circ \varphi_s \quad = \varphi_t \circ g_{st}; \tag{3.1.1}$$
$$f_{\tau(s)} \circ \tau \quad = \varphi_s \circ f_s. \tag{3.1.2}$$

Definition 3.1.8. Given a countable tree T, we shall write $\mathscr{A}_T$ for the class of all finite trees of B-sets over T considered up to the notion of arboreal isomorphism above.

The formalism we need to work in is that of first order structures. We begin by giving a desirable language in which to parse the $\mathbb{N}^{+1}$-tree. Let $(T, <, \curlywedge)$ be the $\mathbb{N}^{+1}$-tree with $\curlywedge$ interpreted as the meet function. Let $D_0(t)$ be a predicate for the leaves of T. For each $n \in \mathbb{N}$, let $D_n(t)$ be a predicate on T saying that t is the predecessor of some s such that $D_{n-1}(s)$. These D_n so defined are disjoint and, for each n, the predicate D_n picks out the vertices of T at depth n below the leaves.

Lemma 3.1.9. *When the $\mathbb{N}^{+1}$–tree, T, is parsed in the language*
$$\mathscr{L}_T := \{<, \curlywedge, \{D_n\}_{n \in \mathbb{N}}\},$$

every isomorphism between finitely generated substructures of T extends to an automorphism of T. Moreover,

$$(\mathrm{Aut}(T, <, \curlywedge), T) = (\mathrm{Aut}(T, \mathscr{L}_T), T),$$

that is, the $\mathscr{L}_T$–automorphisms of T are exactly the automorphisms of $(T, <, \curlywedge)$.

Proof. It is clear that any automorphism of $(T, <, \curlywedge)$ is an automorphism of $(T, \mathscr{L}_T)$. We can show directly that in the language $\{<, \curlywedge, \{D_n\}_{n\in\mathbb{N}}\}$ every isomorphism between finitely generated substructures A and B of T extends to an automorphism of T by induction on the depth $D_k(r)$ of the root of A. □

Let T be the $\mathbb{N}^{+1}$–tree and $\mathscr{L}$ be the two–sorted language

$$\mathscr{L} := \{M, T; \curlywedge(s,t), <(s,t), \{D_n(t)\}_{n\in\mathbb{N}}, L(t;x,y,z), L(x,y,z)\}.$$

The function symbol $\curlywedge : T \times T \to T$ is interpreted as the poset infimum function, the binary relation $<$ is the poset relation on the tree, and each $D_n(t)$ is a unary predicate on T expressing that the vertex t is depth n below a leaf of T. The relation $L(t;x,y,z) \subseteq T \times M^3$ is a quaternary relation. The ternary relation $L(x,y,z) \subseteq M^3$ holds if, for some t, the relation $L(t;x,y,z)$ holds.

Given $A \in \mathscr{A}_T$, a finite tree of B–sets over T, recall that T^A is a finite, convex (Definition 1.2.6) subtree of T of positive type. As the subtree T^A is also of positive type, we have additionally that for all a, b in T^A, the maximum of $\{x \in T : x \leq a \wedge x \leq b\}$ is in T^A.

Let $\tilde{A}$ be a finite $\mathscr{L}$–structure, such that the structure induced on T^A is the finite structure tree $(T^A, \curlywedge, <, \{D_n\}_{n\in\mathbb{N}})$, which is convex in T. The set $M^{\tilde{A}} = M^A$ carries the structure induced by $\tilde{A}$ on the M sort. For t in T^A and a, b, c from M^A, we declare that $\tilde{A} \models L(t;a,b,c)$ if and only if, one of the following holds in A:

1. In the B-set at t, we have $B_t([a]_t; [b]_t, [c]_t)$;
2. For some $s \geq t$ at which $[a]_s, [b]_s, [c]_s$ are non-empty and distinct, we have $B_s([a]_s; [b]_s, [c]_s)$.

Definition 3.1.10. The ternary relation $L(a;b,c)$ holds in $\tilde{A}$ if $a,b,c \in M^{\tilde{A}}$ are distinct and at the root $r \in T^{\tilde{A}} = T^{A}$ we have $L(r;a,b,c)$ and $[a]_r, [b]_r, [c]_r$ are distinct nodes of $B(r)$.

Let $\mathscr{C}_T$ be the class of finite $\mathscr{L}$–structures arising in this way from finite trees of B–sets.

A simple, but useful fact to note from the definition of $L(t;a,b,c)$ is the following.

Lemma 3.1.11. $((\exists c)L(t;a,b,c) \wedge L(t;b,a,c)) \leftrightarrow [a]_t = [b]_t$.

This fact can be used as the $\mathscr{L}$ definition of $[.]_t$ in the $\mathscr{L}$–structures $\mathscr{C}_T$. As a convention we say that the B–set at t is *trivial* if in the B–set at t in T^A is such that there is only one such node $[a]_t$.

Definition 3.1.12. Let $A \in \mathscr{A}_T$ (or similarly $A \in \mathscr{C}_T$), we say that $L(a;b,c)$ is witnessed at the vertex s if and only if

$$L(s;a,b,c) \wedge (s = \mathrm{Max}\{t \in T^A : L(t;a,b,c) \wedge [a]_t, [b]_t, [c]_t \text{ non-empty and disjoint}\}).$$

Lemma 3.1.13. *Given two trees of B–sets, A and B in $\mathscr{A}_T$, for every arboreal isomorphism (in the sense of Definition 3.1.7) from A to B there is a corresponding $\mathscr{L}$–structure isomorphism between $\tilde{A}$ and $\tilde{B}$.*

Conversely, given two $\mathscr{L}$–structures $\tilde{A}$ and $\tilde{B}$ in $\mathscr{C}_T$, arising from trees of B–sets A and B respectively, for every $\mathscr{L}$–isomorphism from $\tilde{A}$ to $\tilde{B}$ there is an arboreal isomorphism (in the sense of Definition 3.1.7) from A to B.

Proof. For the forward direction, assume that A and B in $\mathscr{A}_T$ are arboreally isomorphic. Let $\tau : T \to T$ be a tree isomorphism and $\{\varphi_s\}_{s \in T^A}$ a collection of B–set isomorphisms witnessing this. Then the restriction $\tau \restriction_{T^A}$, of τ to the finite subset T^A, is an isomorphism from $T^A = T^{\tilde{A}}$ to $T^B = T^{\tilde{B}}$ which preserves all of $\mathscr{L}_T := \{\curlywedge, <, D_n\}$. It remains to show that this extends to an $\mathscr{L}$-isomorphism $\beta : \tilde{A} \to \tilde{B}$. On the tree sort, T^A, we define $\beta \restriction_T$ to be $\tau \restriction_{T^A}$; that is, for every $t \in T^A$,

$$\beta \restriction_T (t) := \tau \restriction_{T^A} (t).$$

Let r be the root of T^A, clearly $\tau(r)$ is the root of T^B. We are given the B–set isomorphism $\varphi_r : B(r) \to B(\tau(r))$ as part of the system $\{\varphi_s\}_{s\in T^A}$. By definition, the elements of $B(r)$ and $B(\tau(r))$ are all singletons of M^B. We define β on M^A pointwise by, for each $a \in M^A$, setting $\beta \restriction_M (a)$ to be the unique element of $\varphi_r(a)$. It remains to check that, for all $t \in T^A$ and $a, b, c \in M^A$,

$$\tilde{A} \models L(t; a, b, c) \Rightarrow \tilde{B} \models L(\tau(t); \beta(a), \beta(b), \beta(c)).$$

This is a matter of following through definitions.

Conversely, we assume that $\tilde{A}$ and $\tilde{B}$ from $\mathscr{C}_T$ are isomorphic as $\mathscr{L}$–structures and that A and B, respectively, are trees of B-sets that give rise to them. Let $\beta : A \mapsto B$ be the $\mathscr{L}$–isomorphism. By Lemma 3.1.9, the partial $\mathscr{L}_T$-isomorphism $\beta \restriction_{T^A}: T^A \mapsto T^B$ extends to τ an automorphism of T. To obtain an arboreal isomorphism we need to define the collection $\{\varphi_s\}_{s\in T^A}$ of bijections $\varphi_s : B(s) \to B(\tau(s))$. For each $s \in T^A$, define φ_s at each $a \in M^A$ by

$$\varphi_s([a]_s) := [\beta(a)]_{\tau(s)}.$$

These φ_s are B–set isomorphisms, as β preserves $L(s; x, y, z)$, which by definition of L requires that the structure of each B–set is preserved between $B(s)$ and $B(\beta(s))$. To see that the relationships required by equation 3.1.1 hold, we calculate that for each $[a]_s$ in $B(s)$ we have, for $t > s$,

$$\begin{aligned} \varphi_t \circ g_{st}([a]_s) = \varphi_t([a]_t) = [\beta(a)]_{\tau(t)} = & \quad g_{\tau(s)\tau(t)}\left([\beta(a)]_{\tau(s)}\right) \\ = & \quad g_{\tau(s)\tau(t)} \circ \varphi_s\left([\beta(a)]_s\right). \end{aligned}$$

To see the relationships of equation 3.1.2, note that they clearly hold for the leaves l of T^A. Also if equation 3.1.2 holds for $s \models D_n$ then, by assumptions on β, it holds for $t \models D_{n+1}$ the predecessor of s; so by induction on the (finite) depth n, we conclude that equation 3.1.2 holds for all of the φ_t. □

3.2 Amalgamation in $\mathscr{C}_T$

In this section, we use a version of Fraïssé amalgamation, applied to the class of finite trees of B-sets parsed as L–structures, to obtain a new structure whose automorphism group preserves a limit of betweenness relations. We refer to Chapter 1, Section 4 for the details of the amalgamation property for strong embeddings and the generalised version of Fraïssé's Theorem 1.4.5 which we need for this construction.

We have already identified a class of $\mathscr{L}$–structures $\mathscr{C}_T$, the members of which can be described by respective counterparts in the class $\mathscr{A}_T$ (by Lemma 3.1.13). We now describe certain 'one-point' extensions in $\mathscr{C}_T$; embeddings between trees of B–sets A and E where $|M^E| = |M^A| + 1$. It is clear in the description that these 'one-point' extensions of the sort M also involve extensions on the sort T of at least one vertex, and often more. After defining this class of basic extensions, we show that embeddings realising these basic extensions are $\mathscr{L}$–embeddings, and define our class $\mathscr{E}$ of strong embeddings to be those which realise finite sequences of basic extensions.

Let $A \in \mathscr{C}_T$ be a finite $\mathscr{L}$–structure which, by Lemma 3.1.13 can be pictured via the corresponding tree of B–sets over T. There are various possible arrangements of $E \in \mathscr{C}_T$ such that $|M^E| = |M^A| + 1$ and where A is the induced substructure of E obtained by restriction of M^E to M^A and T^E to T^A. Such a pair $A \subseteq E$ is called a *one–point extension* over A. The element e is the sole member of $M^E \setminus M^A$. We define the following kinds of one–point extensions: σ, ε, ν. We call these *basic extensions*. To describe each extension, we specify how the tree T^E relates to T^A, and how the quaternary relation $L(t; x, y, z)$ in E is related to that in A (using parameters from A). Given such an extension of A witnessed by $E \supseteq A$, the corresponding *embedding* is the inclusion, $A \hookrightarrow E$. In the following we fix the convention that $M^E = M^A \cup \{e\}$.

$\rightarrow_\sigma$ A *star extension*. Intuitively in this extension, we are adding a new root r in T^E below the root s of T^A such that the B–set $B(r)$ at r is a 'star' of leaves around a single ramification point $f_r(s)$ whose leaves correspond (via g_{rs}) to the elements of $B(s)$. More formally, we mean that the root r of T^E is the predecessor of the root s of T^A, so for some n, we have $D_{n+1}(r) \wedge D_n(s)$. Take an enumeration $\{a_i\}_{i \leq k}$

of M^A. In T^E, the root r has $|M^E| = |M^A| + 1 = k + 1$ successors $\{t_i\}_{i \le k} \cup \{s\}$, including s, the root of T^A. The domain of T^A is the subset $\{v \in T^E : v \ge s\} \subseteq T^E$. We choose the enumeration $\{t_i\}$ of $\mathrm{succ}(s) \subseteq T^E$ so that, in the tree of B–sets version of E, $t_i = f^{-1}(a_i)$ for each i. As n is such that $D_{n+1}(r) \wedge D_n(s)$, we also have $\bigwedge_{i \le k} D_n(t_i)$. At the root r, we have:

(i) $L^E(r; e, e, e)$;

(ii) $L^E(r; e, e, a_j) \wedge L^E(r; e, a_j, e)$;

(iii) $L^E(r; a_i, a_j, e) \wedge L^E(r; a_i, e, a_j) \iff i = j$;

(iv) $L^E(r; e, a_i, a_j) \iff i \ne j$;

(v) $L^E(r; a_i, a_j, a_k) \iff [L^A(s; a_i, a_j, a_k) \wedge i, j, k \text{ distinct}\,] \vee [i = j \vee i = k]$.

The relations (i) and (ii) and (iii) are saying that e is in a node of the root B–set which is different to the nodes containing each of the a_j. The relations in (iv) say that the node containing e is between nodes containing distinct a_i in the root B–set. Item (v) is stating, as usual, that the instances of L between the a_i at the root r are either between distinct a_i and inherited from the successor s or that a_i is equal to one of the a_j, a_k. For the vertices in $\{t_i\}_{i \le k}$, we have $L^E(t_i; x, y, z) \iff x, y, z \ne a_i$; in the tree of B–sets, the B–set at t_i has just one node containing points of M^E except the vertex a_i. For any vertex $t \ge s$, we have $L^E(t; x, y, z) \iff L^A(t; x, y, z)$. This is illustrated in Figure 3.3. For the sake of completeness, we will consider a basic extension of the empty structure on $\emptyset$ to an E with $|T^E| = 1 = |M^E|$ to be a star extension.

Otherwise, we may make a *root extension*, where r, the root of T^E, is also the root of T^A. In the following fix n such that $D_n(r)$, so $n := \mathrm{Max}_{k \ge 0}\{(\exists t \in T^A) D_k(t)\}$, and m the level of the highest leaf of T^A, that is $m := \mathrm{Min}_{k \le n}(\exists t \in T^A) D_k(t)$. In the tree of B–sets description of A, the new node $[e]_r$ is added as a singleton to the B–set $(B(r), B_r)$. Doing this may add requirements higher in T^E, which are satisfiable in $\mathscr{C}_T$ by Lemma 3.2.1. In this situation, there are two cases we describe.

$\to_\varepsilon$ If $0 < m < n$, the B–set at r is non-trivial, meaning there are at least two nodes in $(B(r), B_r)$ and hence an edge. In this case, we allow the following kind of

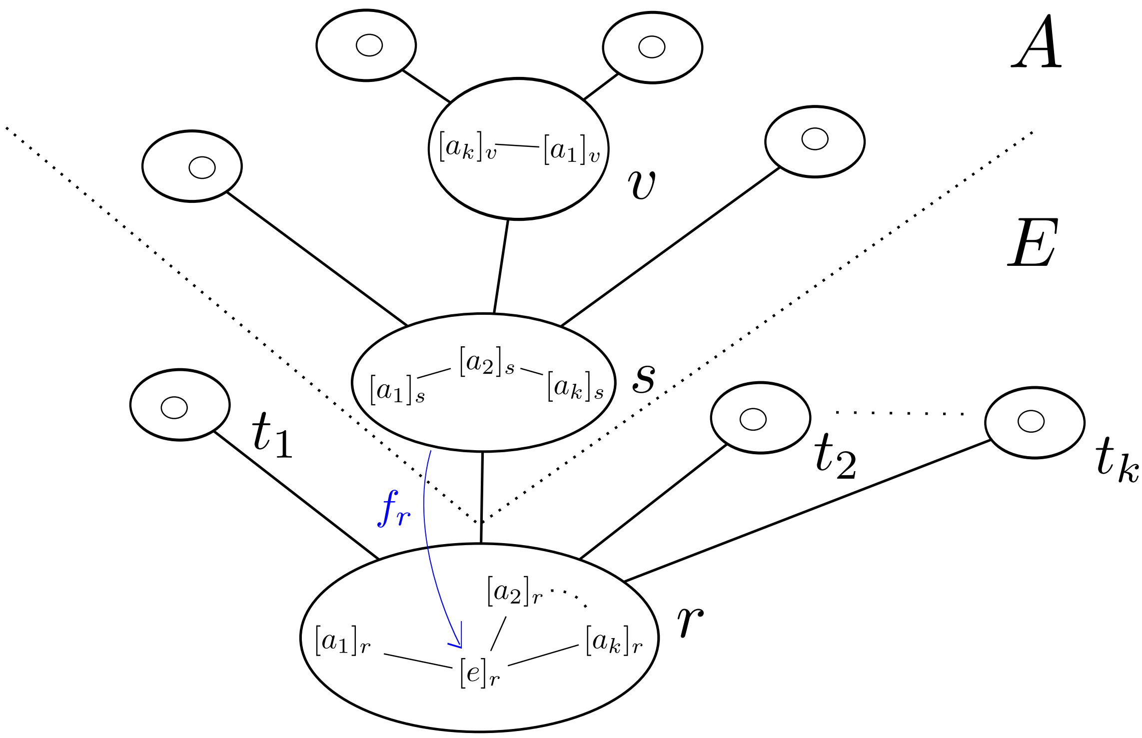
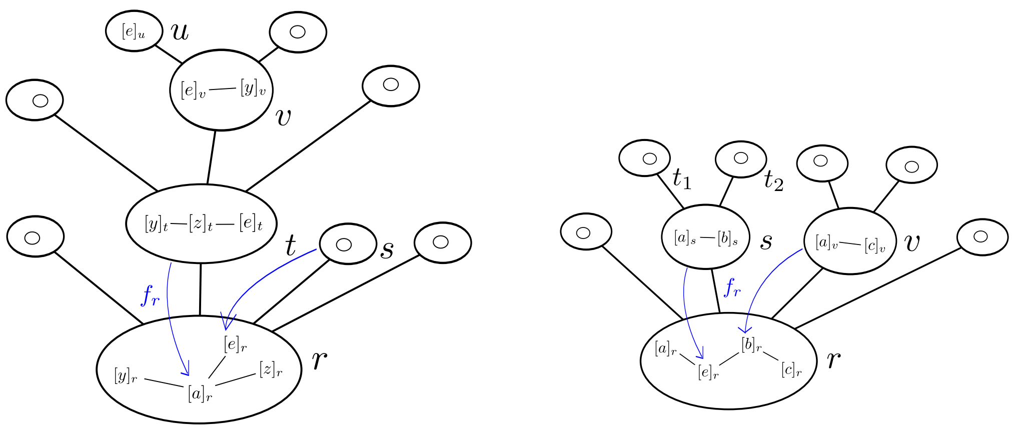


Figure 3.3: A star extension, $A \rightarrow_\sigma E$.

extension. We take an edge in $(B(r), B_r)$ in A, between two nodes $[a]_r$ and $[b]_r$. We replace this edge with the new dyadic node $[e]_r$ and two new edges, $\{[a]_r, [e]_r\}$ and $\{[e]_r, [b]_r\}$. We also add a new successor s of r in T^E, that is, $D_{n-1}(s)$ holds and $n - 1 \geq m > 0$. In E, the B–set at s contains two nodes, $[a]_s$ and $[b]_s$ corresponding (via the function f_s) to the two branches of $(B(r), B_r)$ at $[e]_r$. Here $T^E = T^A \cup \{s, t_1, t_s\}$ arranged such that above s in T^E there are two vertices t_1, t_2 such that $D_{n-2}(t_1)$ and $D_{n-2}(t_2)$ and both with trivial B–sets. Note that $n - 2 \geq m - 1 \geq 0$ by assumption, so such t_1 and t_2 exist in T. See Figure 3.4b.

$\rightarrow_\nu$ If $n > 1$, we have the following kind of extension. The new node $[e]_r$ is added as a leaf to an existing node $[a]_r \in B(r)$. Consequently, we add a new successor s of r, so $D_{n-1}(s)$, with a trivial B–set. The B–set $B(t)$ at $t := f_r^{-1}([a]_r)$ gains another node, $g_{rt}([e]_r)$. This part of the definition is inductive: adding the node $[e]_t = g_{rt}([e]_r)$ to the B-set $B(t)$ is treated as a one–point extension of the substructure of A induced on $\{v \in T^A : v \geq t\}$. See Figure 3.4a. If $n = m = 1$ we also have a version of the ν-extension. That is, we have $D_1(r)$ for r the only vertex in T^A. The new node $[e]_r$ is added as a leaf to the existing node $[a]_r \in B(r)$. Then $T^E = T^A \cup \{s_1, s_2\} = \{r, s_1, s_2\}$ where σ_1, σ_2 are successors of the root r such that $s_1 = f_r^{-1}([a]_r)$ and $s_2 = f_r^{-1}([e]_r)$. In $B(s_1)$ there is one node $[e]_{s_1} = g_{rs_1}([e]_r)$ and in $B(s_2)$ there is one node $[a]_{s_2} = g_{rs_2}([a]_r)$.



(a) Adding a leaf to a node, $A \to_\nu E$. (b) Adding a node on an edge, $A \to_\varepsilon E$.

Figure 3.4: Two kinds of root extension.

As promised above, we show that the basic extensions described above are each consistent.

Lemma 3.2.1. *Let $A \in \mathscr{C}_T$ and α be a basic extension of A. Then there is an $E \in \mathscr{C}_T$ such that α is realised by an embedding of A into E and $|M^E \setminus M^A| = 1$.*

Proof. We consider each kind of extension in turn.

α is a σ–extension. As T has no root, and every vertex has an immediate predecessor, and all vertices have infinitely many immediate successors, it is always possible to extend T^A in the manner required by a σ extension. It is then clear from the description of σ that A can be extended.

α **is an** ε**–extension.** For there to be an edge in the B–set at the root of A, we must have $m - n \geq 1$. In the definition of an ε extension, we also have that $m \geq 1$ and $n \geq 2$. An ε extension requires populating two more vertices of T above the root of T^A. Under the assumptions on n and m, in particular that $n \geq 2$, it is clear that there is room to do this for any relevant A.

α **is a** ν**–extension.** If we have $n = m = 1$, then $|M^A| = 1$ and T^A has just one vertex, r, at which the B–set has just one node. Let E be such that T^E has 3 vertices, r, t_1 and t_2 such that $r < t_1 \wedge r < t_2 \wedge t_1 \perp t_2$ with $D_0(t_1) \wedge D_0(t_2) \wedge D_1(r)$. The domain M^E has 2 elements a, b, so the B-set at r has two distinct nodes, the B-sets at t_1 and t_2 are trivial. Then A embeds into E as the structure induced on the element a in the desired way.

Otherwise $n > 1$, which we will now assume.

We consider first the case $n = 2$. Say $|M^A| = k$ and that the B-set at the root r of A is a linear B-set of length k, so $B(r)$ viewed as a finite connected graph is linear with k elements. Enumerate $B(r)$ as $[a_0]_r, [a_1]_r, ..., [a_k]_r$ so that $[a_0]_r$ and $[a_k]_r$ are the end points of the line. The successor s of r in T^A given by $s = f_r^{-1}([a_k]_r)$ has a trivial B-set and we have $D_1(s)$. Let E have a linear B-set at r of length $k + 1$, enumerate this $[a_0]_r, [a_1]_r, ...[a_k]_r, [b]_r$ so that $[a_0]_r$ and $[b]_r$ are the end points. As $[a_k]_r$ is a dyadic node in the structure E the successor s of r in T^A given by $s = f_r^{-1}([a_k]_r)$ has 2 nodes $[a_0]_s, [b]_s$ and s has 2 successors u and v at depth 0, $D_0(u) \wedge D_0(v)$. Taking A to be embedded as the substructure of E induced on $a_0, ..., a_k$ we see that A embeds into E witnessing a ν extension at a_k. In doing so we also satisfy a ν-extension of $\{a_0\}$ with the element b_1 to a structure on $\{a_0, b_1\}$; where s is the appropriate root for the purpose of this extension. for any size of A.

Now we assume that $n > 2$ and ν-extensions at any depth l such that $1 < l < n$ can be satisfied. So we are extending an A with root r with $D_n(r)$. Adding a leaf to any node $[a]_r$ requires populating a new successor of r, at depth $n - 1$, and adding a node to the B–set at the vertex $f^{-1}([a]_r)$. Populating the new successor can clearly be done. To add a node to the required B–set, we can use a ν–extension by the induction hypothesis, as the vertex $f^{-1}([a]_r)$ is at depth $n - 1$ and $1 < n - 1 < n$.

□

Definition 3.2.2. Given some $A \in \mathscr{C}_T$, let $\mathscr{E}_1(A)$ be the set of embeddings realising basic extensions over A, together with isomorphisms of A. Let $\mathscr{E}_1$ be the set of embeddings realising a basic extension of some $A \in \mathscr{C}_T$, that is

$$\mathscr{E}_1 = \bigcup_{A \in \mathscr{C}_T} \mathscr{E}_1(A).$$

The following lemma establishes that all one–point extensions of the home sort are basic extensions; so are realised by an embedding in $\mathscr{E}_1(A)$.

Lemma 3.2.3. *For all $E \in \mathscr{C}_T$, if $A \in \mathscr{C}_T$ is a substructure of E such that $|M^E \backslash M^A| = 1$, then the one–point extension $A \subseteq E$ is witnessed by an embedding in $\mathscr{E}_1(A)$, realising one of the basic extensions above.*

Proof. Take $E \in \mathscr{C}_T$ and consider the possibilities of removing a point b from M^E to obtain a subset M^A. In order to obtain a substructure A of E we will also remove vertices of the tree T^E related to b, that is the vertices in the pre-image of $[b]_r$ via f_r, the set $f_r^{-1}([b]_r) \subseteq T^E$. We first divide into three cases based on the type of $[b]_r$ in the root B–set $(B(r), B_r)$.

(i) $[b]_r$ is a leaf of B_r and $D_n(r)$ with $n > 1$, in which case $A \subseteq E$ is a ν–extension, so realised in $\mathscr{E}_1$;

(ii) $[b]_r$ is one of the two leaves of B_r and $D_1(r)$, and T^A is the single vertex r, in which case $A \subseteq E$ is a ν–extension, so realised in $\mathscr{E}_1$;

(iii) $[b]_r$ is one of the two leaves of B_r and $D_1(r)$, and T^A a single vertex $s > r$ with $D_0(s)$, in which case $A \subseteq E$ is a σ–extension, so realised in $\mathscr{E}_1$;

(iv) $[b]_r$ is a dyadic node of B_r, and $D_n(r)$ for $r > 1$, in which case $A \subseteq E$ is an ε–extension;

(v) $[b]_r$ is a ramification node of B_r. If the root B–set of E, $(B(r), B_r)$, is a star with $[b]_r$ the ramification node, then $A \subseteq E$ is a σ–extension. Otherwise, if $(B(r), B_r)$

is not a star, we show there is no induced $\mathscr{L}$–structure on $M^E \setminus \{b\}$; that is, this situation can never arise as a one–point extension $A \subseteq E$ between structures in $\mathscr{C}_T$.

Under these conditions, we can find w, x, y, z having the configuration depicted in Figure 3.5a in the root B–set of B. We focus our attention on formulas concerning $\{x, y, z, w, b\} \subseteq E$ and remain aware that $\{x, y, z, w\} \subseteq A$ but that $b \notin A$. Note that at the root r of T^E we have $L(r; z, x, w) \wedge L(r; z, y, w) \wedge [x]_r \neq [y]_r$, so r must be populated in A too. In E there are populated vertices $s, u, v \in \operatorname{succ}(r)$ where $s = f_r^{-1}(b), u = f_r^{-1}(x)$ and $v = f_r^{-1}(z)$. We use Figure 3.5b as a guide. Note that in the B–set at u we have $[y]_u = [z]_u = [w]_u$ and at the B–set at v we have $[x]_v = [y]_v$. As $[b]_r$ is the centroid (ramification point) of the nodes $[x]_r, [y]_r, [z]_r$ and $s = f_r^{-1}(b)$, for some $t \geq s$ in E, we have exactly one of

$$B_t([z]_t; [x]_t, [y]_t), B_t([x]_t; [y]_t, [z]_t), B_t([y]_t; [z]_t, [x]_t).$$

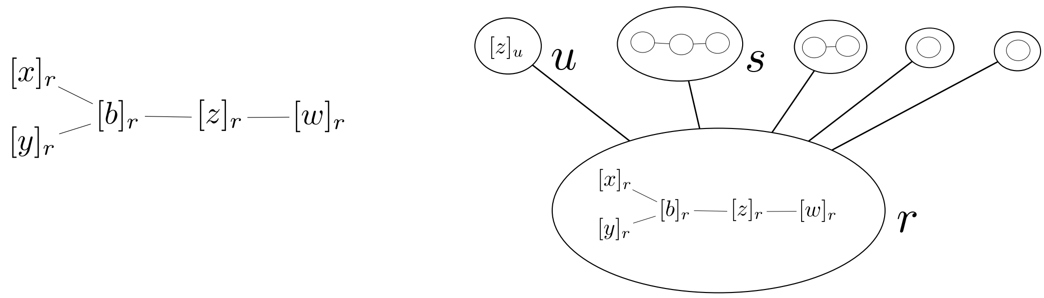
Which one depends on the structure of E, so we divide into the corresponding 3 cases.

Case 1: $(\exists t \in T^E)(t \geq s \wedge B_t([z]_t; [x]_t, [y]_t))$. So $E \models L(r; z, x, y)$ and $A \models L(r; z, x, y)$. But as $b \notin A$, so $s \notin A$ and $\{t \in T^E | t \geq s\} \cap A = \emptyset$. Hence $L^A(r; z, x, y)$ must be witnessed in the B–set at the root r of A. So in A we have $B_r([z]_r; [x]_r, [y]_r)$. But the B–set at $v = f_r^{-1}(z)$ demands that $[x]_v = [y]_v$, requiring that, in A as well as E, the nodes $[x]_r$ and $[y]_r$ are in the same branch at $[z]_r$ in the root B–set $(B(r), r)$. This contradicts that $B_r([z]_r; [x]_r, [y]_r)$ holds in A.

Case 2: $(\exists t \in T^E)(t \geq s \wedge B_t([x]_t; [y]_t, [z]_t))$. In this case $E \models L(r; x, y, z)$, so as above $A \models L(r; x, z, y)$ which must be witnessed in the B–set at the root r of A. But in A, as in E, the B–set at u witnesses $[y]_u = [z]_u$ where $u = f_r^{-1}(x)$; this contradicts the assumption that $B_r([x]_r; [y]_r, [z]_r)$ holds in A.

Case 3: $(\exists t \in T^E)(t \geq s \wedge B_t([y]_t; [z]_t, [x]_t))$. As in Case 2, switching x and y in the argument and using the vertex $l = f_r^{-1}([y]_r)$ in place of u.

□



(a) (b)

Figure 3.5

As all basic extensions of $A \in \mathscr{C}_T$ are realised in $\mathscr{C}_T$, we can consider finite sequences $(\alpha_i)_{i \leq m}$ of basic extensions. In this sequence, we mean that α_1 is a basic extension over an initial finite structure A which is realised by an embedding $\psi_1 \in \mathscr{E}_1(A)$ into a larger finite structure A_1. Then α_i is a basic extension over a realisation A_{i-1} of α_{i-1}, itself being realised by an embedding $\psi_i \in \mathscr{E}_i(A_{i-1})$.

Corollary 3.2.4. *Any finite sequence $(\alpha_i)_{i \leq m}$ of basic extensions of length m can be realised by an embedding φ between finite structures A and E in $\mathscr{C}_T$.*

Proof. Let A be the structure being extended by the first extension α_1. By induction on m, the above Lemma 3.2.3 allows us to realise each basic extension α_i, each defined over a structure A_{i-1}, with an embedding ψ_i of A_{i-1} into an A_i such that $|A_i \setminus A_{i-1}| = 1$. In particular 3.2.3 establishes the corollary for sequences of length 1. For the induction argument, assume that the corollary holds for sequences of length strictly less than m. Then there is an embedding $\varphi_{m-1} = \psi_{m-1}...\psi_1$ of A into some A_{m-1} which realises the sequence of extensions $(\alpha_i)_{i \leq m-1}$. However, α_m is a basic extension over A_{m-1}, so by 3.2.3, is realised by some embedding ψ_m of A_{m-1} into A_m. Then $\varphi_m := \psi_m...\psi_1$ is an embedding of A into A_m as desired. □

Definition 3.2.5. Let $\mathscr{E}_n$ be the class of embeddings realising sequences $(\alpha_i)_{i \leq n}$ of basic extensions of length $\leq n$. Let $\mathscr{E}$ be the class of embeddings, between members of $\mathscr{C}_T$, realising finite sequences of basic extensions of members of $\mathscr{C}_T$.

Corollary 3.2.6. *The class $\mathscr{E}$ is the closure of $\mathscr{E}_1$ under composition.*

Proof. This follows from the definition of $\mathscr{E}$ and Corollary 3.2.4. □

Lemma 3.2.7. *If $\varphi : A \to E$ is an embedding of finite structures A, E in $\mathscr{C}_T$, then there is a finite sequence $(A_i)_{i \leq n}$ of structures from $\mathscr{C}_T$ and embeddings $\psi_i \in \mathscr{E}_1$ such that $A_0 = A$, $A_n = E$, for each i, the embedding ψ_i from A_i into A_{i+1} realises a basic extension of A_i and*

$$\varphi = \psi_n ... \psi_0.$$

Proof. Lemma 3.2.3 is the special case of the required statement restricted to embeddings $\varphi : A \to E$ between finite structures A and E for which $|M^E \setminus M^A| = 1$. We make the inductive assumption that the Lemma holds for embeddings $\varphi : A \to E$ where $|M^E \setminus M^A| < n$. We continue under the assumption that $|M^A \setminus M^E| = n$. We want to find some $b \in M^E \setminus M^A$ such that $E' = A \cup \{b\}$ is a substructure of E which realises a basic extension of A. Then $|M^{E'} \setminus M^E| < n$ so there is $\theta = \psi_n ... \psi_1$ embedding E' into E where $\psi_n, ..., \psi_0$ are in $\mathscr{E}_1$; setting ψ_0 from $\mathscr{E}_1(A)$ embedding A into E' we have $\varphi = \theta\psi_0$ as required.

So we have that $T^A \subseteq T^E$. If the root r of T^E is also the root of T^A, then by the proof of Lemma 3.2.3 there is a b in E which realises either an ε extension or a ν extension of A. This b suffices for our argument.

Otherwise the root r of T^E is strictly below the root s of T^A. If $T^E \cap \{t \in T^E | t \geq s\} \supsetneq T^A$ then let E' be the structure induced by restriction of E to $T^E \cap \{t \in T^E | t \geq s\}$ with corresponding domain $M^{E'} \subseteq M^E$. Such E' is a structure strictly extending A and clearly $|M^{E'} \setminus M^A| < n$ and $|M^E \setminus M^{E'}| < n$. So we are done by induction. Otherwise, if $T^E \cap \{t \in T^E | t \geq s\} = T^A$ then we take $E' \subset E$ realising a σ extension of A. □

Lemma 3.2.8. *The class of structures $\mathscr{C}_T$ has the amalgamation property with respect to embeddings from $\mathscr{E}_1$.*

Proof. Let ψ_1 and ψ_2 be embeddings from $\mathscr{E}_1$ of A into E_1 and E_2 respectively. We will assume that E_1 and E_2 both strictly contain A, that is neither ψ_1 nor ψ_2 is an isomorphism, as otherwise the amalgamation is trivial. Let α_1 and α_2 be the basic extensions over A realised by ψ_1 and ψ_2 respectively. We will refer to the new points of the extensions as $b_1 \in M^{E_1} \setminus M^A$ and $b_2 \in M^{E_2} \setminus M^A$. If E_1 and E_2 are isomorphic over A then we may identify b_1 and b_2, thus identifying E_1 and E_2. In this case amalgamation is completed with identities.

We are proving that there is a $C \in \mathscr{C}_T$ and $\eta, \theta \in \mathscr{E}_1$ such that $E_1 \to_\eta C$ and $E_2 \to_\theta C$ satisfy $\eta.\psi_1 = \theta.\psi_2$.

Case 1: α_1 and α_2 are σ extensions. There is only one way to realise a σ extension over A. So in this case, we identify b_1 and b_2 (see Figure 3.6).

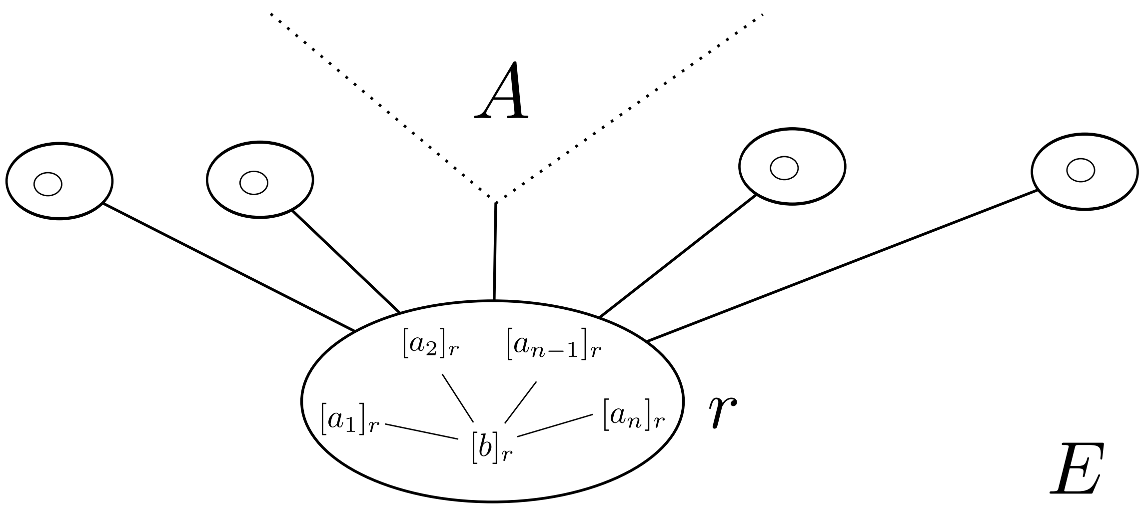


Figure 3.6: Amalgamating two σ–extensions $(b_1 = b_2 = b)$.

Case 2: α_1 and α_2 are ε extensions. First assume that each extension is done via different edges in the root B–set of A, $(B(r), B_r)$, so b_1 and b_2 cannot be identified. Let $\eta : E_1 \to C$ witness the extension of E_1 by α_2, where α_2 should strictly speaking be considered as an extension of $\psi_1(A) \subseteq E_1$. This is possible as there are infinitely many successors in T of the same type above r, the root of T^A. Then note that $\eta.\psi_1$ embeds A into $C \in \mathscr{C}_T$. Similarly $\theta.\psi_2$ embeds A into C where θ witnesses the extension of $\psi_2(A) \subseteq E_2$ by $\alpha_1 \in \mathscr{E}_1$. These embeddings into C witnesses both the extension of A by $\alpha_2 \circ \alpha_1$ and $\alpha_1 \circ \alpha_2$ (see Figure 3.7).

Otherwise, if the extensions α_1, α_2 operate on the same edge of $(B(r), B_r)$ then, as there is only one type of successor of the root r of T^A in T, α_1, α_2 describe the same 1–type over A so we may identify b_1 and b_2.

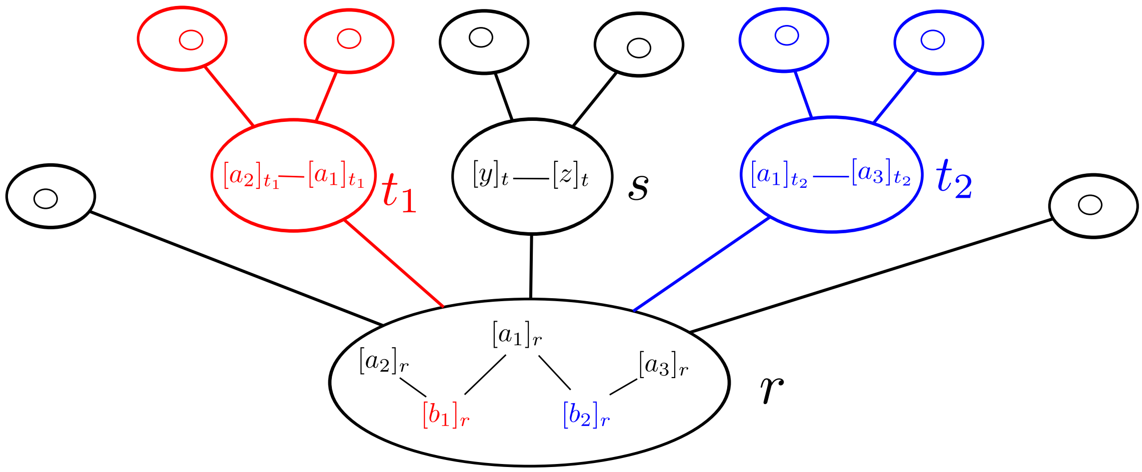


Figure 3.7: Amalgamating two ε–extensions.

Case 3: α_1 is an ε–extension and α_2 is a σ–extension. Let ψ_1 be the embedding $A \to E_1$ and let ψ_2 be the embedding $A \to E_2$. We let C be the extension of $\psi_1(A) \subseteq E_1$ by α_2, being witnessed by $\eta : E_1 \to C$; which is easy to do as T has no root. We need to check that the embedding $\theta : E_2 \to C$ is in $\mathscr{E}_1$. Consider the basic extension β of E_2 which adds a new node $[b_1]_r$ as a leaf to the ramification node $[b_2]_r$ at the root B–set $(B(r), B_r)$, which is a star, and where the restriction of β to $\psi_2(A)$ is the same as the extension α_1 on $E_2 \setminus \{b_2\} = A$ (see Figure 3.8).

Case 4: α_1 and α_2 are ν–extensions. Let ν_1 be the embedding $A \to E_1$ and let ν_2 be the embedding $A \to E_2$. First we describe this amalgamation assuming $|M^A| = 1$, so T^A has one vertex, r and the B–set $(B(r), B_r)$ has one node, $[a]_r$. The two extensions α_1 and α_2 of A must both add a new node as a leaf at $[a]_r$. Let C be the extension α_1 of $\nu_2(A) \subseteq E_2$, which witnesses the amalgamation as required.

Next, we assume that amalgams of this type exist in $(\mathscr{C}_T, \mathscr{E}_1)$ for A such that $|A| < n$. We then prove the claim for $A \in \mathscr{A}_T$ such that $|A| = n$. If we can identify b_1 and b_2, we

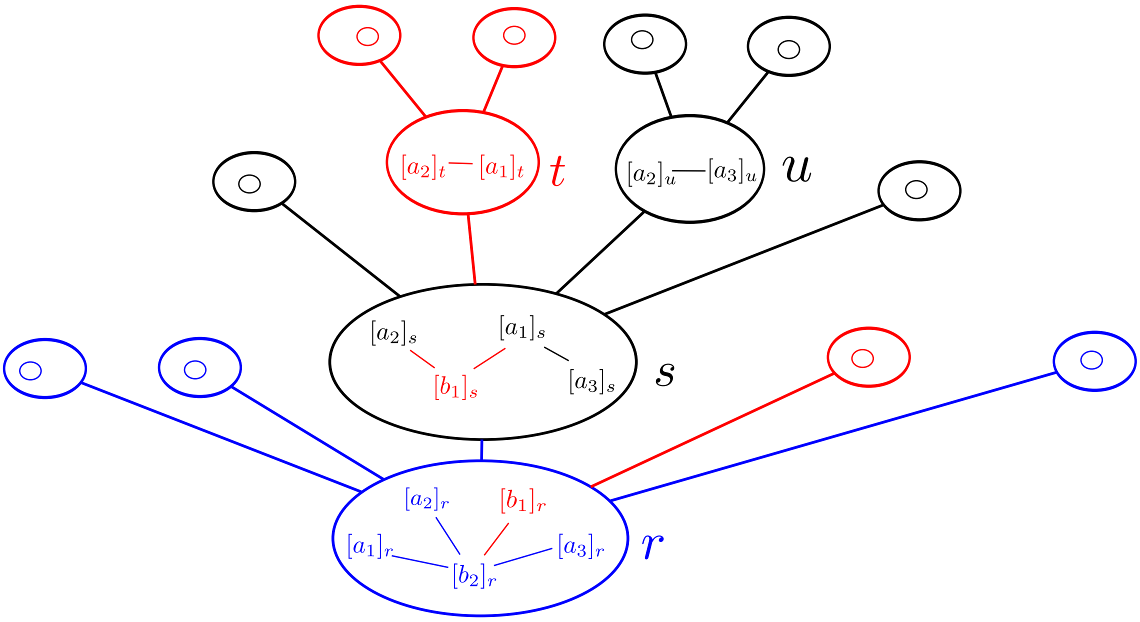


Figure 3.8: Amalgamating an ε extension with an σ extension.

do. Assume otherwise, so the extensions α_1 and α_2 do differ. One possibility is that they differ in the root B–set $(B(r), B_r)$, in that the $[b_i]_r$, for $i = 1, 2$, are added to different nodes of $(B(r), B_r)$. In that case we let C and embedding θ witness the extension α_2 of $\nu_1(A) \subseteq E_1$ by ν_2. Otherwise, the $[b_i]_r$, for $i = 1, 2$, are added as leaves to the same node $[a]_r$ of $(B(r), B_r)$ resulting in two nodes being added to the B–set at the vertex $f_r^{-1}([a]_r)$; that can be considered as an amalgamation over $A \setminus \{a\}$, which can be completed by the inductive assumption on $|A|$. See Figure 3.9 as a guide to the typical case.

Case 5: α_1 is a ν extension and α_2 is an ε extension. Let ν_1 be the embedding $A \to E_1$ and ε_2 the embedding $A \to E_2$. We take C witnessing the extension α_2 of $\nu_1(A) \subseteq E_1$. Then b_1 and b_2 will not be identified, as they have different types over A. As there are infinitely many successors of r of the required type, we find in C the successor associated to b_1 is distinct from that associated to b_2. In this case C suffices to witnesses the extension α_1 of $\varepsilon_2(A) \subseteq E_2$.

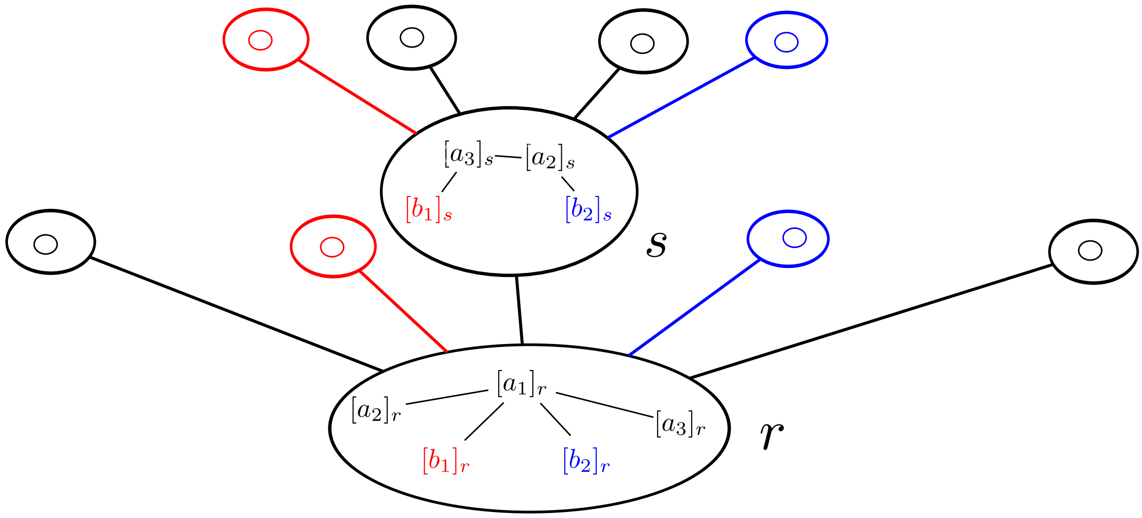


Figure 3.9: Amalgamating two ν extensions.

Case 6: α_1 is a ν extension and α_2 is a σ extension. Let ν_1 be the embedding $A \to E_1$ and let σ_2 be the embedding $A \to E_2$. We let C and embedding η witness the extension α_2 of $\nu_1(A) \subseteq E_1$. We need to show that there is an embedding $\theta : E_2 \to C$ in E_1 witnessing a basic extension of E_2. Consider a ν extension β of E_2 which adds a new node $[b_1]_r$ at the ramification node $[b_2]_r$ at the root B–set $(B(r), B_r)$ of E_2 and which witnesses α_1 on $E_2 \setminus \{b_2\} = A$.

□

Corollary 3.2.9. *The class of structures $\mathscr{C}_T$ has the amalgamation property with respect to embeddings from $\mathscr{E}$.*

Theorem 3.2.10. *The class $(\mathscr{C}_T, \mathscr{E})$ has, up to isomorphism, a unique (generalised) Fraïssé limit M_T. For any $A \in \mathscr{C}_T$ any $\mathscr{E}$-embedding of A into M_T extends to an automorphism of M_T. The T–sort of M_T is the whole of T.*

Proof. As noted in Corollary 3.2.6, $\mathscr{E}$ is the closure under composition of $\mathscr{E}_1$ and by Lemma 3.2.8 we have the amalgamation property with respect to the countable class of embeddings $\mathscr{E}_1$. Hence we have the amalgamation property with respect to $\mathscr{E}$. So $(\mathscr{C}_T, \mathscr{E})$ satisfies the conditions of Theorem 1.4.5. □

As M_T is the Fraïssé limit of $(\mathscr{C}_T, \mathscr{E})$, every member of $\mathscr{C}_T$ is $\mathscr{E}$–embedded into M_T.

Definition 3.2.11. If $A \subseteq M_T$ is a finite, $\mathscr{E}$–embedded substructure of M_T, we write $A \leq M_T$. Moreover, for $A \in \mathscr{C}_T$ and $E \in \mathscr{C}_T \cup \{M_T\}$, we reserve the substructure notation $A \leq E$ only for cases in which A is $\mathscr{E}$–embedded in E.

3.3 Recovering the tree of B–sets.

The aim of this section is to describe how the whole combinatorial structure of the tree of betweenness relations we have constructed is encoded in the action of the automorphism group $G_0 := \mathrm{Aut}(M; L(x; y, z))$. That is, we interpret the structure

$$M_T = (M, T; \wedge(s,t), < (s,t), \{D_n(t)\}_{n\in\mathbb{N}}, L(t; x, y, z), L(x, y, z))$$

in the reduct

$$M_T^0 = (M; L(x, y, z))\,.$$

The following is adapted from a similar interpretation in the work of Bhattacharjee and Macpherson, and appears in [6], although their example is different. We adapt their interpretation to our current setting and much of their work is used here to prove analogous results.

Definition 3.3.1.

To save on notation, we make the following definitions

$$L(x; y, z/w) \leftrightarrow L(x; y, z) \wedge L(x; y, w);$$

$$L(x; y, u/v/w) \leftrightarrow L(x; y, u) \wedge L(x; y, v) \wedge L(x; y, w).$$

Definition 3.3.2. We will use for a 4-ary relation $L'(x; y, z; w)$ which says that $L(x; y, z)$ holds and is witnessed at some $t \in T$, where the B–set B_t *omits* w. The B–set B_t omits w if for all $[a]_t \in B(t)$, $w \notin [a]_t$. Formally, the relation L' is defined from L by

$$\begin{aligned} L'(x; y, z; w) \;\leftrightarrow\;\; & L(x; y, z) \wedge L(w; x, y) \wedge L(w; y, z) \wedge L(w; z, x) \wedge \\ & \neg(\exists r)(L(r; y, x/z/w) \wedge L(r; z, x/w) \wedge L(w; x, r)). \end{aligned}$$

The typical situation for which $L'(x; y, z; w)$ occurs is depicted in Figure 3.10. At s, the B–set induced on x, y, z, w is a star with $[w]_s$ at the centre. The L relation $L(x; y, z)$ is witnessed in some B–set B_t where $t > s$ and $t \in f_s^{-1}([w]_s)$, so w is omitted from B_s.

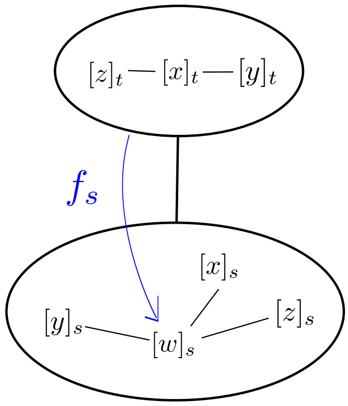


Figure 3.10: The relation $L'(x; y, z; w)$.

Definition 3.3.3. We also define a 6-ary relation P which holds on (x, y, z, u, v, w) whenever $L(x; y, z) \wedge L(u; v, w)$ and are witnessed in the same B–sets.

$$\begin{aligned} P(x; y, z : u; v, w) \leftrightarrow \quad & L(x; y, z) \wedge L(u; v, w) \wedge \\ & (\forall r)(L'(x; y, z; r) \leftrightarrow L'(u; v, w; r)) \end{aligned}$$

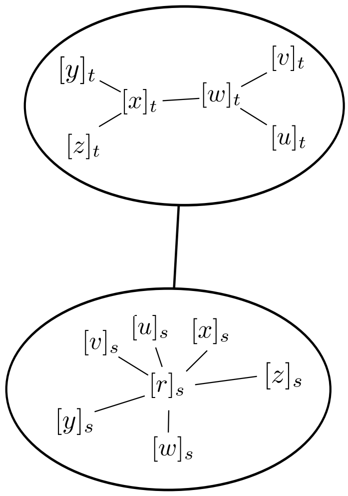


Figure 3.11: An instance of the relation $P(x; y, z : w; u, v)$.

Lemma 3.3.4. *If $A, C \in \mathscr{C}_T$ are finite such that, strongly embedded in M_T^0 both containing $\{x, y, z, u, v, w\}$ such that x, y, z are distinct in A. If $A \leq C$, then*

$$A \models P(x; y, z : u; v, w) \Leftrightarrow C \models P(x; y, z : u; v, w). \tag{3.3.3}$$

Proof. We prove this in the same way that Bhattacharjee and Macpherson prove an analogous statement (Lemma 5.1) in [6]. As the definition of P in terms of L is universal (Definition 3.3.3), its validity is preserved by substructure. So the right-to-left direction is clear. For the right-to-left direction, we recall Lemma 3.2.7. As a consequence of Lemma 3.2.7, if the statement 3.3.3 above holds for situations when C is a basic extension of A, then the result follows. Then it remains to check that 3.3.3 holds for cases when C is a basic extension of A as given in the list before Lemma 3.2.1. $\square$

Corollary 3.3.5. *The structure $M_T^0 \models P(x; y, z : u; v, w)$ if and only if there is a finite $A \in \mathscr{C}_T$, strongly embedded substructure $A \leq M_T^0$ containing $\{x, y, z, u, v, w\}$ such that $A \models P(x; y, z : u; v, w)$.*

Proof as in [6], remarks after Lemma 5.1. It is clear that if $M_T^0 \models P(x; y, z : u; v, w)$ then there is a finite strongly embedded substructure $A \leq M_T^0$ containing $\{x, y, z, u, v, w\}$ lying in $\mathscr{C}_T$ and such that $A \models P(x; y, z : u; v, w)$. Conversely, by Lemma 3.3.4, there is such an A if and only if for every finite, strongly embedded $C \in \mathscr{C}_T$ such that C contains $\{x, y, z, u, v, w\}$, we have $C \models P(x; y, z : u; v, w)$. By the construction of M_T, Theorem 3.2.10, if it holds for all such C, then it holds in M_T. But as P is defined just in terms of L, it holds in M_T^0. $\square$

Definition 3.3.6. Take $x, y, z \in M$ such that $L(x; y, z)$.

(i) We define sets which will consist of all the points $w \in M$ which appear in the betweenness relation witnessing $L(x; y, z)$;

$$S_{xyz} := \{w : P(x; y, z : w; y, z) \vee P(x; y, z : x; w, y) \vee P(x; y, z : x; w, z)\}.$$

(ii) Letting $L^* := \{(u, v, w) \in M^3 : L(u; v, w)\}$, we will have that P defines an equivalence relation on L^* with equivalence classes containing the triples which define the same set S_{xyz}. That is,

$$P(u; v, w : a; b, c) \leftrightarrow S_{uvw} = S_{abc}.$$

We use $\langle u, v, w \rangle$ as notation for the P–class containing (u, v, w).

(iii) Define a partial order $\leq$ on P–classes by reverse set inclusion:

$$\langle u, v, w\rangle \ \leq \ \langle a, b, c\rangle \ \leftrightarrow S_{uvw} \supseteq S_{abc}.$$

We intend that this recovers the partial order of the structure tree (see Proposition 3.3.7 (c)), with the P–classes $\langle u, v, w\rangle$ corresponding to vertices of T . From this point on, we may consider the P–class of the tuple (x, y, z) to be the vertex $t \in T$. We now define

$$\Gamma_t := S_{xyz}.$$

(iv) The classes S_{xyz} collect together as subsets of M the elements of M appearing in some node of each betweenness relation. We now define subsets of M which are the nodes of B_t. Define an equivalence relation (see Prop. 3.3.7 (e)) E_{xyz} on S_{xyz}. Set $uE_{xyz}v$ if and only if:

$$\begin{aligned}(\forall r, s \in S_{xyz}) \quad & [(P(x; y, z : u; r, s) \leftrightarrow P(x; y, z : v; r, s)) \\ & \wedge(P(x; y, z : r; u, s) \leftrightarrow P(x; y, z : r; v, s)).]\end{aligned}$$

Where $\langle x, y, z\rangle = t$, the class of E_{xyz} containing $a \in M$ will be denoted by $[a]_t$.

(v) We then define $R_{xyz} := S_{xyz}/E_{xyz}$. The elements of R_{xyz} are interpreted as the nodes $B(t)$ of the betweenness relation at t witnessing $L(x; y, z)$.

(vi) Now to recover the B_t relation on R_{xyz}. For $\langle x, y, z\rangle = t$ and $[u]_t, [v]_t, [w]_t \in R_{xyz}$ classes of E_{xyz}, define

$$B_t([u]_t; [v]_t, [w]_t) \leftrightarrow [u]_t = [v]_t \vee [u]_t = [w]_t \vee P(x; y, z : u; v, w).$$

Proposition 3.3.7. *(a) If P and L^* are defined as in Definition 3.3.3, then P induces an equivalence relation on L^*;*

(b) For triples x, y, z and u, v, w from L^*, then

$$S_{xyz} = S_{uvw} \iff M^0_T \models P(x; y, z : u; v, w);$$

(c) If the binary relation $\leq$ on L^* is defined as in Definition 3.3.6 (iii) then it is a semilinear order of positive type on L^* whose elements are P–classes;

(d) *Moreover, if $\leq$ is defined as in Definition 3.3.6 (iii), then $(L^*/P, \leq)$ is isomorphic to $(T^{<1}, \leq)$ as a semilinear order where $T^{<1} = T \setminus \{t \in T : D_0(t) \vee D_1(t)\}$ is the subtree of T consisting only of elements of depth at least two in T;*

(e) *If E_{xyz} is defined as in Definition 3.3.6 (iv), then it is an equivalence relation on S_{xyz};*

(f) *The relation B_t, for $t = \langle x, y, z\rangle$, given by Definition 3.3.6 (vi) is well defined and B_t is a betweenness relation of positive type on $R_{xyz} = S_{xyz}/E_{xyz}$.*

Proof of (a). Reflexivity and symmetry are clear from the definition. To deal with transitivity, consider $(x, y, z), (u, v, w)$ and (a, b, c) from L^* such that

$$P(x; y, z : u; v, w) \wedge P(u; v, w : a; b, c).$$

Expanding via the definition of P, we have

$$\begin{aligned} & L(x; y, z) \ \wedge\ L(u; v, w) \ \wedge L(a; b, c) \\ \wedge \quad & (\forall r)(L'(x; y, z; r) \leftrightarrow L'(u; v, w; r)) \\ \wedge \quad & (\forall s)(L'(a; b, c; s) \leftrightarrow L'(u; v, w; s)). \end{aligned}$$

Clearly implying $P(x; y, z : a; b, c)$, as required. □

Proof of (b).

($\Leftarrow$) We follow closely the proof of Proposition 5.3 (i) from [6]. Assume that (x, y, z) and (u, v, w) are triples in L^* such that $P(x; y, z : u; v, w)$. Now we suppose that $r \in S_{xyz}$ and we proceed to show that $r \in S_{uvw}$. Take some strongly embedded, finite structure $A \leq M^0_T$ containing $\{x, y, z, u, v, w, r\}$. The formula defining P is universal in terms of L. So as A is a substructure of M^0_T, and $M^0_T \models P(x; y, z : u; v, w)$, so also $A \models P(x; y, z : u; v, w)$. As A is a finite, strongly embedded structure, we can consider its structure as a finite tree of B–sets. So in this sense we read $P(x; y, z : u; v, w)$ as saying that in A, the relations $L(x; y, z)$ and $L(u; v, w)$ hold and are witnessed at the same vertex $t \in T^A$. As $r \in S_{xyz}$, then $[r]_t$ is in the same B–set of the structure A. So we can take a finite structure $C \leq A$, such that the vertex t is the root of T^C and C contains

$\{x, y, z, u, v, w, r\}$. As $L(u; v, w)$ holds in C, by viewing C as a finite tree of B–sets, the betweenness relation B_t at the root of C witnesses $B_t(\{u\}; \{v\}, \{w\})$, so as axiom (B3) holds for the betweenness relation B_t we have $B_t(\{u\}; \{r\}, \{w\}) \vee B_t(\{u\}; \{v\}, \{r\})$. Hence,

$$P(u; v, w : u; r, w) \vee P(u; v, w : u; v, r) \vee P(u; v, w : r; v, w).$$

By Corollary 3.3.5, as this holds in C it also holds in M_T^0. So checking Definition 3.3.6 (i), we conclude that $r \in S_{uvw}$.

($\Rightarrow$) Again we follow the arguments in [6], here of Proposition 5.3 (ii). As we assume $S_{xyz} = S_{uvw}$, we may assume $L(x; y, z)$ and $L(u; v, w)$. So if $M_T^0 \models \neg P(x; y, z : u; v, w)$ then, considering Definition 3.3.3, there is r witnessing $\neg P(x; y, z : u; v, w)$. We may assume that we have $L'(x; y, z; r) \wedge \neg L'(u; v, w; r)$. For this to hold in M_T^0, it must hold in some finite structure containing $\{x, y, z, u, v, w, r\}$ coming from a finite tree of B–sets A. Hence in this finite tree of B–sets, A, we have that at some vertex t of T^A the nodes $[x]_t, [y]_t, [z]_t$ are in distinct branches around $[r]_t$, but that vertex t of T^A, the triple $[u]_t, [v]_t, [w]_t$ are not in distinct branches around $[r]_t$. So $r \notin S_{xyz}$, as the B–set witnessing $L(x; y, z)$ excludes $[r]_t$. But, as $[u]_t, [v]_t, [w]_t$ are not in distinct branches around $[r]_t$, either $L(u; v, w)$ is witnessed at t, in which case $r \in S_{uvw}$, or else r shares a node with one of u, v, w in the B–set in which $L(u; v, w)$ is witnessed, whence $r \in S_{uvw}$. We conclude that any such r witnessing $\neg P(x; y, z : u; v, w)$ is such that $r \in S_{uvw} \setminus S_{xyz}$, in contradiction to the assumption that $S_{xyz} = S_{uvw}$. □

Proof of (c). Firstly, as sets are partially ordered by set inclusion, it is clear that $\leq$ is a partial order from its Definition 3.3.6 (iii).

Following the proof in [6] (see there Proposition 5.3 (ii)), to check semilinearity it suffices to show that given two sets S_{xyz} and S_{uvw} with neither containing the other, there is some S_{abc} containing both (satisfying (S4)) but that there is no $\{l, m, n\}$ such that S_{lmn} is contained in both S_{xyz} and S_{uvw}. To obtain such an S_{abc}, consider a finite $\mathscr{L}$–structure $A \in \mathscr{C}_T$ and a triple (a, b, c), such that $\{x, y, z, u, v, w, a, b, c\} \subseteq M^A$ and

$A \models L(a;b,c) \wedge L(x;y,z) \wedge L(u;v,w) \wedge \neg P(x;y,z : u;v,w)$ in which $L(a;b,c)$ is witnessed at the root B–set of A. By construction (see Proposition 3.2.10) this A is $\mathscr{E}$–embedded in M_T. Taking S_{abc} to be the subset of M defined in Definition 3.3.6 (i), then $S_{abc} \supseteq S_{xyz}$ and $S_{abc} \supseteq S_{uvw}$. To show that there is no $\{l,m,n\}$ such that $S_{lmn} \subseteq S_{xyz} \cap S_{uvw}$. Assume the contrary, that there is such a triple (l,m,n). As neither S_{xyz} nor S_{uvw} contains the other, there is $r \in S_{xyz} \setminus S_{uvw}$ and $s \in S_{uvw} \setminus S_{xyz}$. We also have a,b,c as in the above, so $S_{abc} \supseteq S_{xyz} \cup S_{uvw}$. Under these assumptions, there would be a finite, strongly embedded $A \leq M_T$ containing $\{a,b,c,r,x,y,z,s,u,v,w,l,m,n\}$ witnessing that $S^A_{lmn} \subseteq S^A_{xyz} \subseteq S^A_{abc}$ and $S^A_{lmn} \subseteq S^A_{uvw} \subseteq S^A_{abc}$. Hence we would have, in T^A, distinct vertices $\langle x,y,z\rangle \neq \langle u,v,w\rangle$ and $\langle l,m,n\rangle$ such that $\langle x,y,z\rangle \perp \langle u,v,w\rangle$ with $\langle x,y,z\rangle, \langle u,v,w\rangle \leq \langle l,m,n\rangle$, contradicting the semilinearity of $(T^A, \leq^A)$. Hence there can be no such $A \leq M_T$, and hence no (l,m,n) or S_{lmn}.

Again following the proof in [6] (of Proposition 5.3 (ii)) we consider finite, strongly embedded substructures to see that $(L^*/P, \leq)$ is of positive type. Take $\langle x,y,z\rangle$ and $\langle u,v,w\rangle$ incomparable in $(L^*/P, \leq)$ and some finite, strongly embedded $A \leq M$ containing $\{x,y,z,u,v,w\}$. As $A \in \mathscr{C}_T$ the vertices $\langle x,y,z\rangle$ and $\langle u,v,w\rangle$ have a greatest lowest bound in T^A, say $\langle a,b,c\rangle$, as T^A is of positive type. In every finite $B \geq A$ in which A is strongly embedded, $\langle a,b,c\rangle$ is the greatest lowest bound of $\langle x,y,z\rangle$ and $\langle u,v,w\rangle$, and hence also in M_T.

□

Proof of (d). Let $\theta : L^*/P \to T$ be defined by $\theta(\langle x,y,z\rangle) := t$ if every finite, strongly embedded $A \leq M_T$ containing $\{x,y,z\}$ witnesses $L(x;y,z)$ at $t \in T^A \subseteq T$. Note that if this is witnessed in some finite, strongly embedded $A \leq M_T$, then it is witnessed in every finite, strongly embedded $A \leq B \leq M_T$. So θ is well defined. It is an embedding from $(L^*/P, \leq)$ to $(T, \leq)$, as it is an embedding for all finite, strongly embedded $A \leq M$. To see that it is surjective, consider any $s \in T$. Then there is a $B \leq M_T$ such that $s \in T^B$. As it is a member of $\mathscr{C}_T$, such a B contains a triple of distinct elements (u,v,w) such that $L(u;v,w)$ is witnessed at $s \in T^B$. So s is in the image of θ. Hence θ is an isomorphism between $(L^*/P, \leq)$ and $(T, \leq)$. □

Proof of (e). Reflexivity and symmetry are clear from the Definition 3.3.6 (iv). Transitivity follows upon expanding $(uE_{xyz}v) \wedge (vE_{xyz}w)$ using the Definition 3.3.6 (iv). So we have,

$$(\forall r, s \in S_{xyz}) \qquad [(P(x; y, z : u; r, s) \leftrightarrow P(x; y, z : v; r, s)) \\ \wedge(P(x; y, z : r; u, s) \leftrightarrow P(x; y, z : r; v, s))],$$

as well as,

$$(\forall a, b \in S_{xyz}) \qquad [(P(x; y, z : v; a, b) \leftrightarrow P(x; y, z : w; a, b)) \\ \wedge(P(x; y, z : a; v, b) \leftrightarrow P(x; y, z : a; w, b))].$$

By combining these statements, we have that,

$$(\forall r, s \in S_{xyz}) \qquad [(P(x; y, z : u; r, s) \leftrightarrow P(x; y, z : v; r, s) \leftrightarrow P(x; y, z : w; r, s)) \\ \wedge(P(x; y, z : r; u, s) \leftrightarrow P(x; y, z : r; v, s) \leftrightarrow P(x; y, z : r; w, s))].$$

From which we conclude $(uE_{xyz}w)$. □

Proof of (f). As in [6] (of Proposition 5.3 (iv)), we show that B_t is well defined by proving that for all $u' \in [u]_t, v' \in [v]_t$ and $w' \in [w]_t$ we have that $L(u; v, w) \leftrightarrow L(u'; v', w')$. The nodes $[u]_t, [v]_t$ and $[w]_t$ are classes of the equivalence relationship E_{xyz}, where $\langle x, y, z\rangle$, defined as in Definition 3.3.6 (iv) in terms P. We follow the equivalences

$$P(u; v, w : x; y, z) \leftrightarrow P(u'; v, w : x; y, z) \leftrightarrow P(u'; v', w : x; y, z) \leftrightarrow P(u'; v', w' : x; y, z),$$

from which we conclude that $L(u; v, w) \leftrightarrow L(u'; v', w')$. Following the proof in [6] (of Proposition 5.3 (iv)) we check axioms for a betweenness relation. Axioms (B1) and (B2) follow immediately from the symmetry of Definition 3.3.6 (vi). We proceed to prove (B3), which for convenience, we state here in the relevant form in the current context; for all $[u]_t, [v]_t, [w]_t$ and $[r]_t \in R_{xyz}$,

$$B_t([u]_t; [v]_t, [w]_t) \rightarrow B_t([u]_t; [v]_t, [r]_t) \vee B_t([u]_t; [r]_t, [w]_t). \qquad (3.3.4)$$

Assume we have $B_t([u]_t; [v]_t, [w]_t)$. If either $[u]_t = [v]_t$ or $[u]_t = [w]_t$ then (B3) follows easily, so we assume $[u]_t \neq [v]_t$ and $[u]_t \neq [w]_t$. In that case, directly from Definition 3.3.6 (vi) and Definition 3.3.3, we have $P(x; y, z : u; v, w)$ and $L(u; v, w)$. Whence from part (b) of this Lemma, $S_{xyz} = S_{uvw}$ and we have $t = \langle x, y, z\rangle = \langle u, v, w\rangle$. So then we consider some $r \in S_{uvw}$. First assume that $r \notin [u]_t$, so we witness the negation of Definition 3.3.6 (iv) in every finite, strongly embedded substructure containing $\{u, v, w, x, y, z, r\}$. So in these finite substructures, considering the negation of Definition 3.3.3, we have

$$P(u; v, w : u; r, w) \vee P(u; v, w : u; v, r).$$

Back to Definition 3.3.6 (vi), as $[r]_t \neq [u]_t$ and $[u]_t = [v]_t$ and $[u]_t = [w]_t$, we have, covering either case in the last line,

$$B_t([u]_t; [r]_t, [w]_t) \vee B_t([u]_t; [v]_t, [r]_t),$$

as required by $(B3)$, in the right hand side of statement (3.3.4). On the other hand, if $r \in [u]_t$, so that $[r]_t \neq [u]_t$, then the implication in statement (3.3.4) holds, which is clear in Definition 3.3.6 (vi).

To show that these B–sets are of positive type, we need them to satisfy (B6). In the present context, that is, fixing $t = \langle x, y, z\rangle$:

$$(\forall [u]_t, [v]_t, [w]_t \in R_{xyz})(\exists [r]_t)(B_t([r]_t; [u]_t, [v]_t) \rightarrow B_t([r]_t; [v]_t, [w]_t) \vee B_t([r]_t; [w]_t, [u]_t)). \tag{3.3.5}$$

However, in the construction, this is a requirement for every structure in $\mathscr{C}_T$, thus for any finite, strongly embedded substructure of M_T that contains $\{x, y, z, u, v, w\}$. So as every finite, strongly embedded substructure of M_T has a witness r such that $(B_t([r]_t; [u]_t, [v]_t) \rightarrow B_t([r]_t; [v]_t, [w]_t) \vee B_t([r]_t; [w]_t, [u]_t))$, by the construction in Chapter 4, in particular Theorem 3.2.10, r is also a witness of this statement in the limit structure M_T. As pointed out in [6] (proof of Proposition 5.3 (iv)) and [3] (first page of Section 15), the positive type condition (B6) together with (B1),(B2) and (B3) imply (B4), so B_t is a betweenness relation of positive type on R_{xyz}. □

3.4 Summary of claims about M_T

The work in this chapter was motivated by the task of producing a relational structure which is invariant under the Jordan group preserving a limit of betweenness relations constructed by Adeleke in [1] in the style of the Fraïssé style construction used by [6]. One major difference to that of [6] is that the structure they construct is $\aleph_0$–categorical, but Adeleke's group in [1] is not oligomorphic, so a preserved structure cannot be $\aleph_0$–categorical.

We provide this as a synopsis of the intended results of this chapter. I am fairly certain that the following assertions are correct, and proved by small adaptations of the arguments in [5]. However, due to shortage of time they cannot be claimed at this stage. In this chapter we have constructed a 2-sorted Fraïssé limit of finite trees of B–sets. By fixing the semilinear order $(T, <)$ on one sort to be the so-called $\mathbb{N}^{+1}$–tree in advance, and stipulating that the finite trees of B–sets carry appropriate finite substructures of T in that sort, in the 2-sorted limit it is the $\mathbb{N}^{+1}$–tree which is the induced semilinear ordering on that sort. The picture we have is that the limit structure can be considered as an infinite tree of B–sets, with each vertex of the tree populated by a B–set. The vertices in the top level, the $t \in T$ with $D_0(t)$, are populated by B–sets of size 1. Below that, for the $s \in T$ such that $D_1(s)$, the B–sets at s are populated with B–sets of size 2, up to isomorphism there is only one such B–set. For the $u \in T$ such that $D_2(u)$, the B–sets at u have infinitely many nodes and are dense linear B-sets. For all other $v \in T$, such that $D_n(v)$ for $n \geq 3$, all the B–sets are isomorphic, we see them as infinite dense $\aleph_0$–branching B–sets of positive type. The functions f_t and g_{st} from the finite structures carry through to M_T as direct limits and, in essence, the structure on M_T is controlled by these functions.

In Section 3.3, we were able to recover essential parts of the structure of M_T from the reduct of M_T down to just the main sort M and the ternary relation L on M; this reduct is called M_T^0.

As in [6], our Jordan group G is the group of automorphisms of (M, L). Adapting the arguments in [6] we see that an example of a proper Jordan set is a non-empty node $[a]_t$, for $a \in M$ and $t \in T$ such that $[a]_t$ is a non–empty node of the B–set at $t \in T$. Strictly speaking, the vertex t is recovered as a triple of distinct elements of M. As such, it turns out that G is primitive, as there are no equivalence relations preserved on elements of M.

To satisfy the definition of a group preserving a limit of betweenness relations (Definition 1.3.20), we select an infinite chain from T to serve as the infinite chain in the definition. As in [6], our group G does not preserve any of the more familiar structures preserved by primitive Jordan groups.

The sense in which this seems to produce a group like Adeleke's example in [1] is that his construction works as an iterated process. We see this iterated process in our construction by considering larger and larger finite portions of the $\mathbb{N}^{+1}$–tree with consecutively lower roots. To capture each of Adeleke's iterations, we consider upward closed finite subtrees of the structure tree with the induced structure from M_T such that the root of the finite tree in the next structure in the process is immediately below the root of the previous one; at each stage we have an infinite structure and can consider the limit. Adeleke takes a limit of the groups obtained at each stage to construct his example. We believe that his limit of groups preserves the limit of our approximating structures.

Chapter 4

Extensions and discussion

The aim with the construction in Chapter 3 is to provide a flexible method of construction which can unify the techniques used in [1] and [6]. The flexibility lies in the choice of semilinear order which is fixed in advance on the tree sort for the construction. As we saw in Chapter 3 (for the example on the $\mathbb{N}^{+1}$–tree), the underlying tree can be recovered from the reduct of the limit structure to the structure induced on the M sort by the relation L. So structures obtained by fixing non-isomorphic trees in advance of the construction should be non-isomorphic, giving rise to a rich supply of examples of Jordan groups preserving a limit of betweenness relations.

4.1 More trees of betweenness relations

Our first task in generalising the construction in Chapter 3 is to build a source of semilinear orders to play the role that the $\mathbb{N}^{+1}$–tree did in that construction. In this section we sketch out a conjectured program for future work. This program should, if realised, produce an infinite class of suitable structure trees.

First we fix a countable linear order $(I; \prec)$, enriching the language of a semilinear order with depth predicates ordered as $(I; \prec)$ and building an infinitely branching semilinear order with depths from I.

Fix a countable linear order $(I; \prec)$.

Let $\mathscr{L}_I := \{<, V, C, \{D_i\}_{i\in I}\}$ be a language consisting of a binary relation $<$, ternary relations V and C and countably many unary predicates $\{D_i\}_{i\in I}$.

Let $\mathscr{C}_I$ be the class of finite structures $(A; <^A, V^A, C^A, \{D_i^A\}_{i\in I})$, considered up to isomorphism, satisfying the following. Note that $x \perp y$ is notational shorthand for $\neg(x < y) \wedge \neg(y < x) \wedge (y \neq x)$.

(K1) $(A; <^A)$ is a partial order such that for all x the set $\{y : y < x\}$ is linearly ordered by $<^A$;

(K2) The relation $V^A(x; y, z)$ says that

$$(x < y \wedge x < z \wedge y \perp z) \wedge (\forall w > x)((w \perp y) \vee (w \perp z));$$

(K3) The relation $C^A(x; y, z)$ says that

$$(x \perp y \perp z) \wedge (\forall w < x)((w \perp y \wedge w \perp z) \vee (w < y \wedge w < z));$$

(K4) An axiom says that $C^A(x; y, z)$ says

$$(\forall r)\neg(V^A(r; y, z) \wedge V^A(r; z, x) \wedge V^A(r; z, x));$$

(K5) For all $a \in A$ there is a unique $i \in I$ such that $D_i^A(a)$;

(K6) If $a < b$ and $i, j \in I$ such that $D_i^A(a)$ and $D_j^A(b)$, then $i < j$.

From this point, we usually drop the superscript (as in A above), as the interpretation of the language is taken in the structure which should be clear from context.

Conjectured Theorem 4.1.1. *The class $\mathscr{C}_I$ is an amalgamation class, so by Fraïssé's theorem, there is a unique, countably infinite $\mathscr{L}_I$-structure T_I which is homogeneous and has age $\mathscr{C}_I$.*

Proof. We check the conditions of Fraïssé's Theorem.

There are only countably many isomorphism types in $\mathscr{C}_I$. Each finite structure A only makes use of finitely many levels, say J^A, from the countable linear order $(I,<)$, so there are only countably many choices of such finite J^A. Given a finite selection J of levels from I, there are only finitely many semilinear orders occupying just those levels. Hence there are only countably many isomorphism types for finite A in $\mathscr{C}_I$.

$\mathscr{C}_I$ has the hereditary property. The properties above characterising the class $\mathscr{C}_I$ are universal, which are preserved under taking induced substructures.

$\mathscr{C}_I$ has the amalgamation property. See Conjectured Lemma 4.1.5 below.

□

Definition 4.1.2. For $A \in \mathscr{C}_I$, say A is $\mathscr{E}$–closed if the following hold.

1. If $a \perp b$ then $(\exists c)(c < a \wedge c < b)$;
2. If $b > a$ and $c > a$ and $b \perp c$ then $V(a;b,c)$ or $(\exists d > a)V(d;b,c)$;
3. If $a \perp b \perp c \perp a$ then $C(a;b,c) \vee C(b;c,a) \vee C(c;a,b)$ or $(\exists d)(V(d;a,b) \wedge V(d;b,c) \wedge V(d;c,a))$;
4. If $a < b$ such that $D_n(a) \wedge D_m(b)$ with $n,m \in I$ where there are only finitely many $i \in I$ with $n \prec i \prec m$, then $(\exists c_n, ..., c_i, ..., c_m)$ such that, for all such i, $D_i(c_i)$ and $a = c_n < c_i < c_{i+1} < c_m = b$.

Lemma 4.1.3. *If $A \in \mathscr{C}_I$ is $\mathscr{E}$–closed then*

1. $x \perp y$ *if and only if* $(\exists w)V(w;x,y)$*;*
2. $V(x;y,z)$ *if and only if* $y \perp z$ *and* $x = sup\{w : w < y \wedge w < z\}$*;*
3. $C(x;y,z)$ *if and only if* $x \perp y \perp z \perp x$ *and* $(\exists w)(w < y \wedge w < z \wedge w \perp x)$.

Lemma 4.1.4. *Let A, B_1, B_2 in $\mathscr{C}_I$ be an amalgamation problem where A, B_1 and B_2 are $\mathscr{E}$–closed and $|B_1 \setminus A| = |B_2 \setminus A| = 1$. Then we can solve the amalgamation on $B_1 \cup B_2$.*

Proof. Let $b_1 \in B_1 \setminus A$ and $b_2 \in B_2 \setminus A$. We describe how b_1 and b_2 relate. As B_1 and B_2 are in $\mathscr{C}_I$, we have that there are n and m in $\mathscr{C}_I$ such that $D_n(b_1)$ and $D_m(b_2)$. As the D_i are unary predicates, they remain in the amalgam $B_1 \cup B_2$. The following cases may not be mutually exclusive, so we treat them in order of preference. First considering the conditions of case 1, then if they are not satisfied for the case in question, moving on to later cases in turn.

Case 1: $(\exists a \in A)(b_1 < a \wedge b_2 < a)$**.** Let a be minimal in A such that $(b_1 < a \wedge b_2 < a)$. By (K1), we may only have $b_1 \leq b_2$ or $b_2 \leq b_1$. If $n \in I$ is such that $D_n(b_1)$ and $D_n(b_2)$, we identify b_1 and b_2. Otherwise, there is $n \neq m$ in I such that $D_n(b_1)$ and $D_m(b_2)$. Without loss of generality, say that in this case we have $n \prec m$. By (K6) this forces $b_1 < b_2 < a$. Certainly there is no instance of the relation C involving both b_1 and b_2, as b_1 and b_2 are comparable. It remains to determine what happens with the relation V. Certainly, by (K2), $\neg V(x; b_1, b_2)$ for all x in A, as b_1 and b_2 are comparable by $<$ and $\neg V(b_2; b_1, x)$ for all x in A as $b_2 \not< b_1$. In fact, by (K2), we may only have $V(b_1; b_2, c)$ for some $c \in A$ if $b_1 < c$ and $b_2 \perp c$. But then, as $a > b_2$, also $c \perp a$. But a and c are in A and $a \perp c$, as A is $\mathscr{E}$–closed, already there exists d in A such that $V(d; a, c)$. So we conclude $\neg V(b_1; b_2, c)$ for all c in A.

Case 2: $(\exists a \in A)(b_1 < a \wedge a < b_2)$**.** We must be in a situation such that $D_n(b_1)$ and $D_r(a)$ and $D_m(b_2)$ with $n \prec r \prec m$ and by transitivity of $<$, we have $b_1 < b_2$. Similarly to Case 1, we have no instances of C or V involving both b_1 and b_2.

Case 3: $(\exists a \in A)(b_2 < a \wedge a < b_1)$**.** This is included to make the case listing clear. It is just the same as Case 2 with the roles of b_1 and b_2 reversed.

Assuming none of the above situations arise, we argue that we may reduce to Case 4 below. Because, if A is non-empty and $b_i \perp a$ for all a in A, then B_i is not $\mathscr{E}$–closed. So for $i = 1, 2$ there is $a_i \in A$ such that $b_i > a_i$. As A is $\mathscr{E}$–closed there is an $a \in A$ such that $a \leq a_1$ and $a \leq a_2$.

Case 4: $(\exists a \in A)(a < b_1 \wedge a < b_2)$**.** We take a maximal in A such that $(a < b_1 \wedge a < b_2)$. Keeping note of the assignments in the following analysis, note that the following designations make $B_1 \cup B_2$ into a structure in $\mathscr{C}_I$. If there is an $x \in A$ such that $a < x < b_1$ and $x \nless b_2$ then in fact $x \perp b_2$, or else if $x > b_2$ we would be in Case 3, whence we conclude that $b_1 \perp b_2$. Then we have $V(a; b_1, b_2)$ if and only if $V(a; x, b_2)$. For $y \notin B_x(b_1)$ we have

- $C(y; b_1, b_2)$ if and only if $C(y; x, b_2)$;
- $C(b_1; y, b_2)$ if and only if $C(x; y, b_2)$;
- $C(b_2; y, b_1)$ if and only if $C(b_2; y, x)$.

For $y' \in B_x(b_1)$ we have either $y > b_1$ or $y < b_1$ or $C(b_2; b_1, y)$. Similarly if there is a $z \in A$ such that $a < z < b_2$ and $z \nless b_1$, then $z \perp b_1$ hence $b_2 \perp b_1$ and we follow the lines above with z in place of x and switching the roles of b_1 and b_2. If there is $v \in A$ such that $a < b_1 < v$ and $b_2 \perp v$ then $b_1 \perp b_2$. Then we have $V(a; b_1, b_2)$ if and only if $V(a; v, b_2)$. For $y \perp b_1$, we have

- $C(y; b_1, b_2)$ if and only if $C(y; v, b_2)$;
- $C(b_1; y, b_2)$ if and only if $C(v; y, b_2)$;
- $C(b_2; y, b_1)$ if and only if $C(b_2; y, v)$.

Similarly if there is $w \in A$ such that $a < b_2 < w$ and $b_1 \perp w$ then $b_1 \perp b_2$ and the V and C relations are determined as in the previous lines concerning v. Otherwise, there is no such x, z, v or w and we may set b_1 to be comparable to b_2. In fact, say $n, m \in I$ are such that $D_n(b_1)$ and $D_m(b_2)$. Then set

- $b_1 < b_2$ if and only if $n \prec m$;
- $b_1 = b_2$ if and only if $n = m$;
- $b_1 > b_2$ if and only if $m \prec n$.

□

Conjectured Lemma 4.1.5. *$\mathscr{C}_I$ has the amalgamation property.*

Conjectured Proposition 4.1.6. *There is an infinite substructure C of T such that $(C;<)$ is a maximal chain of T and, as a linear order, is isomorphic to $(I,\prec)$.*

Conjectured Theorem 4.1.7. *If $(I;\prec)$ and $(J;\prec)$ are non-isomorphic, countable linear orders, then the semilinear orders T_I and T_J as constructed above are not isomorphic*

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