UNIVERSAL CONSTRUCTIONS IN ALGEBRAIC AND LOCALLY COVARIANT QUANTUM FIELD THEORY

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Abstract

The present work is concerned with the application of categorical methods in algebraic and locally covariant quantum field theory. Attention is particularly paid to colimits and left Kan extensions, understanding K. Fredenhagen's universal algebra, which is a global (unital) $(C)^*$-algebra associated with a not necessarily up-directed net of local (unital) $(C)^*$-algebras, from the point of view of category theory. The main technical result centres on explicit expressions for the universal algebra and its non-triviality in the case that a net of local unital *-algebras is constructed from linear symplectic spaces via a functorial quantisation prescription. Non-up-directed nets of local (unital) $(C)^*$-algebras typically arise for quantum field theories in a generic curved spacetime with an arbitrary topology. As an example the field strength tensor description of the classical and the quantised free Maxwell field in curved spacetimes is considered. Employing colimits and left Kan extensions, a universal classical and quantum field theory are constructed. Both fail local covariance and dynamical locality but can be reduced to locally covariant and dynamically local theories.

To understand C.J. Isham's twisted quantum fields from the point of view of algebraic and locally covariant quantum field theory, an abstract categorical framework is introduced, which utilises recent ideas of C.J. Fewster on the automorphisms of a locally covariant theory and the group of the global gauge transformations of a theory. The general formalism allows to consider twisted variants of generic locally covariant theories, which need not refer to (quantum) fields at all, on single curved spacetimes. It is argued that the general categorical scheme leads naturally to the classification of the twisted variants of a locally covariant theory by the isomorphism classes of flat smooth principal bundles over the fixed single curved spacetime the twisted variants are considered on. The general categorical scheme and the classification of twisted variants are illustrated by the example of twisted variants of multiple free and minimally coupled real scalar fields of the same mass.

Finally, a new family of pure and quasifree states for the quantised free massive Dirac field on 4-dimensional, oriented and globally hyperbolic ultrastatic slabs with compact spatial section is constructed, arising from a recent description of F. Finster's fermionic projector. These FP-states ("*FP*" for fermionic projector) are tested for the Hadamard property with some negative and some positive results.

Das Rheinische Grundgesetz

Artikel 1: Et es wie't es

Artikel 2: Et kütt wie't kütt

Artikel 3: Et hätt noch immer jot jejange

Artikel 4: (der rheinische Entsorgungsartikel):
Wat fott es es fott

Artikel 4a: Kenne mr nit, bruche mr nit, fott domet!

Artikel 5: (das rheinische Universalgesetz):
Wat soll dä Quatsch!

Artikel 5a: Wer weiß, wofür et jot es

-Konrad Beikircher, *Et kütt wie et kütt. Das rheinische Grundgesetz*, 6th ed., Verlag Kiepenheuer & Witsch, Köln, 2003.

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Author's Declaration

I declare that the work presented in this thesis is entirely due to my own research which I have carried out during the time period from October 2011 to October 2014 at the University of York for I have not used any other resources except where indicated otherwise in this thesis. The topics of this thesis were suggested by my supervisor, Prof. Dr. C.J. Fewster, and the discussions on physical motivation, physical ideas conveyed and on mathematical details have been indispensible.

Parts of this thesis are planned, and partially realised, as joint publications: the content of Chapter 5 on dynamical locality of the free Maxwell field is available as an arXiv-preprint[1] and has been accepted by *Ann. Henri Poincaré* for publication. Also the results of Chapter 7 on pure and quasifree Hadamard states for the quantised free massive Dirac field from the fermionic projector are available as an arXiv-preprint[2]; moreover, a paper on these results has been submitted to *Class. Quantum Grav.* and was provisionally accepted for publication. Finally, a joint paper on twisted quantum fields from the perspective of algebraic and locally covariant quantum field theory (Chapter 6) is planned but exists only in an unpublished draft form at the moment.

The current form of the proof for the open diamond caterpillar covering lemma (Lemma 1.1.6) has been developed with the help of T. Bullock and J. Waldron. The idea of looking into the issue of Lemma 4.6.8 was provided by Dr. A. Schenkel, who also suggested to take the contrapositive in its proof.

[1] [FL14a]: C.J. Fewster and B. Lang, "Dynamical locality of the free Maxwell field" (2014), `http://arxiv.org/abs/1403.7083v3` [math-ph].

[2] [FL14b]: C.J. Fewster and B. Lang, "Pure quasifree states of the Dirac field from the fermionic projector" (2014), `http://arxiv.org/abs/1408.1645v2` [math-ph].

Introduction

In the last few years, the theory of quantum fields has been put to the test most impressively in the most expensive and complex physical experiment in human history, conducted at the Large Hadron Collider (LHC) at the European Organisation for Nuclear Research (CERN). Devoted to finding the Higgs-Boson and thus to confirm and complete the Standard Model of particle physics, a Higgs-Boson-like particle was indeed found, which will most likely be confirmed to be the Higgs-Boson as soon as the LHC resumes operation in the years to come. However, this does not mean at all that the days of quantum field theory are numbered. New challenges, particularly coming from general relativity and cosmology, like dark matter and dark energy, are awaiting.

Despite the exceedingly high accuracy with which the predictions of quantum field theory match the experiments in particle physics, since it is usually staged on flat Minkowski space, it neglects one of the most fundamental findings of the last century. We inevitably live in a curved spacetime due to A. Einstein's famous field equations of general relativity, $\mathrm{Ric} - \frac{1}{2}\,\mathrm{scal}\,g + \Lambda g = -\frac{8\pi G}{c^4}\,T$, where Ric is the Ricci tensor, scal the scalar curvature, g the metric tensor, Λ the cosmological constant, G Newton's gravitational constant, c the speed of light and T the stress-energy-momentum tensor.

Quantum field theory in curved spacetimes (see [Wal84; Wal94] as general references) partially remedies this shortcoming by considering quantum fields in a gravitational background field, which is treated according to the laws of general relativity. Often perceived as the semi-classical approximation to a full quantum theory of gravitation and matter, where the quantum nature of the matter fields and the effects of the gravitational field play a role but the quantum nature of gravity can be neglected, a strong case can be made for the relevance quantum field theory in curved spacetimes. In the same way that the combination of quantum mechanics with special relativity has led to qualitative new features such as the particle-antiparticle dualism, we legitimately may hope for new qualitative features by considering quantum fields in a gravitational background field governed by the laws of general relativity, independently of any kind of quantitative approximation. Quantum field theory in curved spacetimes is more fundamental than ordinary quantum field theory on Minkowski space and we should take its lessons seriously, regardless of what they may be and especially when they seem to contradict our Minkowski space intuition.

Meeting these high expectations, quantum field theory in curved spactimes has helped clarifying conceptual matters in quantum field theory like the role of particles and has provided us with striking insights into the quantum nature of gravity such as (see [Haw75; Unr76; Wal84; FH90; KW91; Wal94] for the details) Hawking radiation (spontaneous particle creation near black holes) and the relation between the mechanical laws of black holes (κ, $\Delta M = \frac{\kappa}{8\pi}\Delta A + \Omega\Delta J$, $\Delta A \geq 0$, $\kappa > 0$)[3] and classical thermodynamics (T, $\Delta E = T\Delta S + W$, $\Delta S \geq 0$, $T > 0$)[4]; further prominent conclusions of quantum field theory in curved spacetimes are the Unruh effect (an accelerated observer feels himself in a thermal bath of particles) and the backreaction of the quantum field on the gravitational field.

To grasp all aspects of quantum field theory in curved spacetimes and not to be tied to the peculiarities of single curved spacetimes, it has turned out to be profitable −if not a necessity− to follow the algebraic approach to quantum field theory. Particularly these days, where phenomenological physical theories and ambitious but highly speculative mathematical entities are predominant in theoretical physics, it seems rather hard to argue the need for a down-to-earth axiomatic (better: general) quantum field theory, e.g. as presented in [SW64; Jos65; BLT75; BLOT90; Haa96; Ara99]. Probably the best reason was given by W. Heisenberg himself and his opinion on this matter can be found at the start of our chapter on general quantum field theory, Chapter 3. However, denying the usefulness of general quantum field theory consequently denies the beautiful insights into the theory of quantum fields which it has permitted us, such as the understanding of the mathematical structure of the correlation functions, the PCT theorem, the connection between spin and statistics, particle collision theory, in particular Haag-Ruelle scattering theory, Haag's theorem, the theorem by Reeh and Schlieder, and many more. Also, one should not forget that the Lehmann-Symanzik-Zimmermann formalism and its widely-used reduction formula have their origin in axiomatic efforts.

Originally formulated on Minkowski space by R. Haag, H. Araki and D. Kastler, algebraic quantum field theory [Ara61; HK64; Hor90; Haa96; Ara99] asserts that the physical content of a quantum field theory is fully accounted for by its net[5] of local observables, that is, a rule $\mathbf{B} \longmapsto \mathfrak{A}(\mathbf{B})$ assigning to each spacetime region $\mathbf{B}$ a C^*-algebra $\mathfrak{A}(\mathbf{B})$ such that certain Haag-Araki-Kastler axioms are fulfilled. $\mathfrak{A}(\mathbf{B})$ is interpreted as the C^*-algebra of all (bounded) local observables associated with the

[3] κ is the surface gravity of the event horizon of the black hole, ΔM the change in mass of the black hole, ΔA the change in the area surface of the event horizon, Ω the angular velocity of the event horizon and ΔJ the change of the angular momentum of the black hole.

[4] T is the temperatur, ΔE the change in energy, ΔS the change in entropy and W the work

[5] Note, some authors prefer to use the more mathematically correct term "*precosheaf*" since it is not said (yet) that the spacetime regions are up-directed by inclusion. However, we continue using the term "*net*" as it is customary in algebraic quantum field theory.

spacetime region $\mathbf{B}$. Traditionally, the spacetimes regions are taken to be bounded open sets ($\Longrightarrow$ compact closure) or open double cones in Minkowski space, which form up-directed nets under inclusion, i.e. for two spacetime regions $\mathbf{B}_1$ and $\mathbf{B}_2$, there is always a third spacetime region $\mathbf{B}_3$ containing them both. This allows for the construction of the C^*-inductive limit $\mathfrak{A}_{\mathrm{qloc}} := \overline{\bigcup_{\mathbf{B}} \mathfrak{A}(\mathbf{B})}$ of the $\mathfrak{A}(\mathbf{B})$, which is referred to as the algebra of quasilocal observables. It is most convenient to define the states of the quantum field theory as normalised (if an identity element exists) positive linear functionals on $\mathfrak{A}_{\mathrm{qloc}}$. Global observables, such as energy, charge or univalence (the superselection rule of spin) are not elements of $\mathfrak{A}_{\mathrm{qloc}}$ but arise in representations by limiting processes. The Gelfand-Naimark-Segal representation theorem implies that the algebraic approach contains the standard formalism of quantum field theory in terms of Hilbert spaces and self-adjoint operators.

The concepts and methods of the algebraic approach have proven extremely fruitful for quantum field theory in curved spacetimes, leading to breakthroughs in the area of quantum energy inequalities [Few00], rigorous perturbative constructions of interacting quantum field theories in curved spacetimes [BFK96; BF00] and to applications in cosmology [DHP09; DHP11; Hac10]. For some other, general aspects of algebraic quantum field theory in curved spacetimes, we mention [HNS84]. S.W. Hawking's original derivation [Haw75] of particle creation near black holes was corroborated in the spirit of the algebraic approach by [FH90] and the essential steps leading to it were clarified. Also applications to the Hawking effect benefited from the ideas of algebraic quantum field theory such as the discussion of the uniqueness and the thermal properties of quasifree states in curved spacetimes with a bifurcate Killing horizon [KW91].

Locally covariant quantum field theory [BFV03] has emerged as the consistent further development of algebraic quantum field theory in curved spacetimes, describing a locally covariant quantum field theory as a functor from a category of curved spacetimes and their embeddings into each other to a category of unital $(C)^*$-algebras and unital *-monomorphisms. A locally covariant quantum field is in this framework a natural transformation between a functor assigning to each spacetime a Hlctvs[6] and a functor assigning to each spacetime a topological unital *-algebra. Formulating algebraic quantum field theory in curved spacetimes within the language of category theory has turned out to be not a mere reformulation of known results but a very fruitful method indeed, as it stresses the physical structures and features in a background independent manner. The categorical viewpoint has led to major progress and great successes in the subject area, such as a general spin-statistics theorem in curved spacetimes [Ver01], quantum energy inequalities [FP06; Few07], superselection theory [BR07; BR09], applications in cosmology [DFP08], analogues of the Reeh-Schlieder theorem in curved

[6]Hlctvs = Hausdorff locally convex topological vector space [Jar81; Rud91; BB03].

spacetimes [San09; Dap11] and, most importantly, helped with the completion of the perturbative construction of interacting quantum field theories in curved spacetimes [HW01; HW02; BDF09]. It also made an impact on the treatment of classical field theory [BFR12] and allows to address fundamental questions such as to what extent a physical theory represents the same physics in all spacetimes (SPASs) [FV12a; FV12b]; locally covariant quantum field theory provides a suitable framework for the discussion of such a profound question.

Having thus set the scenery, what is this thesis about? The main wish of this thesis is to further promote the application of categorical methods in algebraic and locally covariant quantum field theory and to contribute to local-to-global and top-down approaches therein. We hope to accomplish this goal by (a) clarifying K. Fredenhagen's universal algebra [Fre90] from the point of view of category theory and applying it to the field strength tensor description of free Maxwell field in curved spacetimes, and (b) providing an abstract categorical framework to understand C.J. Isham's twisted quantum fields [Ish78b; AI79b] as twisted variants of locally covariant quantum field theories viewed on single curved spacetimes and discussing their classification and properties. In addition, we will (c) construct a family of new Hadamard states for the quantised free massive Dirac field on 4-dimensional, oriented and globally hyperbolic ultrastatic slabs with compact spatial section, utilising a recent description [FR14a] of F. Finster's fermionic projector [Fin98; Fin06].

(a) In a general curved spacetime, it can happen that the net of local observables is not up-directed by inclusion due to the spacetime topology. Hence, the algebra of the quasilocal observables cannot be constructed. In the study of superselection rules for chiral conformal quantum field theories in 2-dimensional Minkowski space (considered as theories on S^1), K. Fredenhagen firstly proposed and used the so-called universal algebra in order to have a convenient global C^*-algebra associated with the full "spacetime" S^1 at his disposal [Fre90]. Soon after its introduction, the universal algebra was employed as a convenient tool for applying the celebrated Doplicher-Haag-Roberts analysis of superselection sectors and particle statistics [DHR69a; DHR69b; DHR71; DHR74; DR90] to such theories [FRS92; GL92; Fre93; DFK04]; it appears that the universal algebra is still valuable in the theory of chiral conformal quantum fields on 2-dimensional Minkowski space, see [CCHW13; CHL13].

We will shed light on the universal algebra from the point of view of category theory: the universal algebra is best understood in terms of colimits and provides thereby the natural generalisation of the notion of the algebra of quasilocal observables; the C^*-inductive limit is nothing short of a colimit.

It was also suggested by K. Fredenhagen to employ the universal algebra in the field strength tensor description of the quantised free Maxwell field; the electromagnetic field plus gauge field theories in general have seen a fair number of developments[7] in algebraic and locally covariant quantum field theory since the start of the new millennium. The results of [Dim92] on the Cauchy problem and the quantisation of the electromagnetic vector potential in 4-dimensional globally hyperbolic spacetimes with compact Cauchy surfaces were generalised by [Pfe09] to smooth differential p-form fields in arbitrary spacetime dimensions, though compact Cauchy surfaces were still assumed for the quantisation. Both C^*-Weyl algebras and unital *-algebras of the smeared quantum field were addressed, smearing with coclosed compactly supported smooth differential 1-forms. Using deformation techniques, [FP03] proved the existence of Hadamard states for the quantised free vector potential on 4-dimensional globally hyperbolic spacetimes with trivial first de Rham cohomolohy group and having compact Cauchy surfaces; an explicit construction for Hadamard states of the quantised free vector potential on asymptotically flat globally hyperbolic spacetimes of dimension 4 was given in [DS13], exploiting a bulk-to-boundary procedure and assuming trivial first or second de Rham cohomology group. Note that both [FP03; DS13] used the unital *-algebra of the smeared quantum field, again smearing with coclosed compactly supported smooth differential 1-forms. In [Dap11], the Reeh-Schlieder property for the quantised free vector potential in terms of C^*-Weyl algebras was investigated and shown to hold in 4-dimensional globally hyperbolic spacetimes with trivial first de Rham cohomolohy group, for bounded causally convex open subsets with non-empty causal complement. However, as we have pointed out, [Dim92; FP03; Pfe09; Dap11; DS13] have in common that they all make additional assumptions on the spacetime topology. To overcome these topological restrictions for the quantised free Maxwell field in terms of the field strength tensor and to study the effects of a non-trivial spacetime topology, K. Fredenhagen's idea was to take the universal algebra as the global field algebra[8] for spacetimes with topologies allowing for field strength tensors that cannot be derived from vector potentials, since standard methods only yielded a non-up-directed net of local field algebras. The construction of a global field algebra via the universal algebra was sketched in [Hol08, Appx.A] and carried out in detail in [DL12].

With the insights gained from viewing the universal algebra as a colimit, we will review and also improve the construction of the universal algebra in [DL12] by highlighting the proper categorical background and supplying enhanced technical lemmas.

[7]Because we will treat the Maxwell field using the exterior calculus of smooth differential forms and not as a gauge field theory, we do not discuss the gauge field theoretic aspects of these recent developments such as [Hol08; HS13; BDS13; BDS14; BDHS14] or the progess in linearised quantum gravity [FH13].

[8]By a "*global field algebra*", we mean a unital *-algebra of the smeared quantum field associated with the full spacetime. By a "*local field algebra*", we mean a unital *-algebra of the smeared quantum field associated with a spacetime region.

In doing so, we will also notice a brand new aspect unfolding, which we hope will contribute to locally covariant quantum field theory. Suppose the following situation: we are given an "*incomplete*" locally covariant quantum field theory, that is, the functor is not defined for all spacetimes considered but only for such spacetimes which are subjected to certain constraints, e.g. topological constraints like contractibility. Is it possible to complete the functor to a locally covariant quantum field theory and, if so, how? Colimits can in principle be used to construct such an extension, however, there may be many extensions or even none at all. Here, category theory provides the notion of a distinguished extension of functors, the so-called left Kan extension. It turns out that if certain colimits actually exist, they already give rise to the left Kan extension. Nevertheless, the left Kan extension can be considered in its own right since its existence is indeed related to colimits but does not rely on them. The question of extending existing functors to obtain locally covariant quantum field theories has not been addressed in the existing literature and we hope that it will be recieved as an interesting question, worth looking into.

Hence, by considering the quantised free Maxwell field in terms of the field strength tensor in curved spacetimes of arbitrary topology, we will precisely find ourselves faced with the task of extending an "*incomplete*" locally covariant quantum field theory to a proper one: standard methods will only yield a functor on spacetimes which obey certain topological restrictions, however, by taking colimits (i.e. the universal algebra), we will construct the left Kan extension and hence obtain a distinguished functor for the quantum field theory which is defined on all spacetimes, regardless of their topology. Unfortunately, this universal F-theory of the quantised free Maxwell field –as we will term the functor obtained in this way– does not turn out to be a locally covariant quantum field theory.

Nevertheless, the universal F-theory exhibits some decent properties such as the validity of the time-slice axiom and causality, which will encourage us to pursue further investigations. For example, we will test the universal F-theory for dynamical locality, a notion originally introduced by [FV12a] in the discussion of SPASs, though it has nowadays the status of a stand-alone notion due to its implications such as additivity, extended locality and a no-go theorem for preferred states in quantum field theory in curved spacetimes. We will find that the universal F-theory is not dynamically local but can be modified to yield a reduced F-theory of the quantised free Maxwell field, which is both locally covariant and dynamically local.

(b) In the second part of this thesis, we will initiate a program to understand and discuss C.J. Isham's twisted quantum fields from the perspective of algebraic and locally covariant quantum field theory. As a precurser to attack quantum gravity and "*quantum topology*", C.J. Isham introduced twisted quantum fields [Ish78b; AI79b] in order to illustrate the effects of the spacetime topology in quantum field theory in curved spacetimes. Twisted quantum fields arise from considering smooth cross-sections in non-trivial smooth vector bundles which satisfy locally the familar field equation and employing them consequently in the quantum description. Since the possibility for non-trivial smooth vector bundles depends on the topology of the smooth base manifold, twisted quantum fields are intimately related to the topology of the underlying spacetime. They can thus be used to probe aspects of the role played by spacetime topology. Also, twisted (quantum) fields provide field configurations which are indeed locally equivalent but globally inequivalent to the standard (quantum) fields employed and supply us for this reason with new toy models for quantum fields in curved spacetimes. Note, a complementary route was taken by [BD79a; BD79b], which used the fundamental group (= first homotopy group) to pull back field theories on curved spacetimes to their respective universal smooth covering manifolds. Different, inequivalent ways of pulling back correspond thereby to inequivalent non-trivial smooth vector bundles and hence to inequivalent twisted quantum field theories.

Twisted quantum fields have many interesting properties, which have been demonstrated in concrete examples to a good level of satisfaction. Most importantly, twisted quantum fields exhibit different (sometimes drastically with a change of sign) renormalised vacuum expectation values for the energy density on ultrastatic spacetimes [Ish78b; DHI79; BD79b; BD99]. Other noteworthy properties (in concrete examples) include the validity of the spin-statistics theorem [Ish78b] and a change to or even a complete suppression of spontaneous symmetry breaking[9] [Ish78b; AI79c]. In the path integral approach to the quantum spinor field, it is pointed out that one has to include the twisted variants in order to obtain a local Lorentz gauge transformation invariant vacuum expectation value generating functional for T-products of stress-energy-momentum tensors [AI79a; BD99].

However, it must have been felt by C.J. Isham that some aspects of twisted quantum fields could benefit from the algebraic approach [Ish78b, Sec.6]: "*Finally, and somewhat speculatively, since we are twisting everything in sight, should the same treatment be applied to the Hilbert spaces themselves in the quantum theory?* [...] *Instinct suggests that the correct handling of a local Haag-Kastler C* algebraic approach might clarify the situation considerably* [...]". Though we will not answer this particular question, we will lay the foundation and outline a general categorical framework, which will allow us to talk more generally about twisted variants of generic locally covariant theories considered on single curved spacetimes which do not refer to twisted (quantum) fields

[9]Consider the Lagrange function $L = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}\mu^2\left(\phi^2 - a^2\right)^2$, $a \neq 0$, which is invariant under the symmetry transformation $\phi \longmapsto -\phi$ and allows for the constant solutions $\phi = \pm a$. However, the constant solutions cannot be smooth cross-sections in a non-trivial smooth vector bundle, i.e. twisted fields. There even are no twisted fields obeying $\phi^2 = a^2$.

necessarily. In this way, the methods of algebraic and locally covariant quantum field theory become available for the investigation of C.J. Isham's twisted quantum fields. Our general categorical framework will be modelled on fibre bundles with a struture group and makes decisive use of C.J. Fewster's recent ideas on the automorphisms and the global gauge group of a locally covariant theory [Few13]. In contrast to C.J. Isham's classification of twisted quantum fields by the isomorphism classes of smooth principal bundles, we will find that the general categorical framework leads to a classification of twisted variants by the isomorphism classes of *flat* smooth principal bundles. In principle, this results in more and even completely new twisted quantum field theories, which have been overlooked so far. As a demonstration for a specific twisted quantum field theory fitting into our abstract categorical framework, we will discuss $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields, that is, twisted variants of the quantum field theory of multiple free and minimally coupled real scalar fields of the same mass.

(c) Finally, we will turn to something completely different from the general theme of this thesis and look into the matter of Hadamard states for the quantised free Dirac field. Algebras of local observables or of locally smeared quantum fields are only half of the battle. In order to establish a connection to a real physical experiment, a theory must produce predictions in terms of numbers which can then be compared with the outcome of the experiment. In the algebraic approach to quantum field theory, this is achieved by states, which are normalised (if an identity element is present) positive linear functionals on the algebras of local observables or of the local smearings of the quantum field. However, not all of these "*mathematical*" states are reasonable from the point of view of physics and criteria to single out the physically sensible states from all possible mathematical states have to be found.

Over the past three decades, the class of the Hadamard states has set the benchmark for physical states of linear quantum field theories in curved spacetimes. Essentially, the Hadamard condition, the first rigorous definition of which was given by [KW91] for the quantised free real scalar field and later for the quantised free Dirac field by [Köh95], governs the singular behaviour[10] of the Wightman two-point distribution, which is associated with a state on the basis that the quantum fields are operator-valued distributions. In particular, the Hadamard condition determines the ultraviolet behaviour, which becomes crucial for renormalised stress-energy-momentum tensors and perturbatively treated interacting quantum field theories in curved spacetimes. Although it is usually known by deformation arguments [FNW81] that Hadamard states must be abundant, it is difficult to give concrete constructions. In the last part of the present thesis, we do exactly this and take up the challenge to construct new

[10] A singular point of a distribution is a point which does not feature an open neighbourhood on which the distribution is given by integration against a test function (more general: test cross-section in a smooth vector bundle).

Hadamard states for the quantised free massive Dirac field.

Utilising a recent description [FR14a] of F. Finster's fermionic projector [Fin98; Fin06], a tool important to split up the solution space of the Dirac equation into positive and negative frequency subspaces, and thus for the discussion of the Dirac sea, we will show how to extract a family of pure and quasifree states for the quantised free massive Dirac field on 4-dimensional, oriented and globally hyperbolic ultrastatic slabs with compact spatial section. For each non-negative integrable function on the real line, we will construct such a state, which we call an FP-state for fermionic projector states. Following the discussions of the so-called SJ-state [AAS12] in [FV12c; FV13], we will show that it can almost always be ruled out that the unsoftened FP-state, which is obtained by considering the characteristic function of the time interval of the slab, is Hadamard. Also, the unsoftened FP-state exhibits almost always undesirable infinite quantum fluctuations, e.g. the normal ordered energy density has almost always infinite quantum fluctuations in the unsoftened FP-state. The significance of the unsoftened FP-state is that it is the closest of all the FP-states to the fermionic projector description and is therefore sometimes considered as the analogue or even forerunner of the SJ-state [FV12c; BF14] for the case of the quantised free Dirac field. On the other hand, we will show that the softened FP-states, which arise from taking non-negative compactly supported smooth functions on the real line and originated from the enterprise to convert the unsoftened FP-state into a Hadamard state in the spirit of [BF14], are always Hadamard states.

An outline of this thesis goes now as follows: in Chapter 1, we will review some basic notions of differential and Lorentzian geometry, and provide some technical lemmas of a topological nature. Chapter 2 is devoted to category theory and universal constructions. We will thoroughly review the important notions of colimits and left Kan extensions and also remind the reader of the categorical concepts on which the dynamical net is built. Aspects of general quantum field theory will be discussed in Chapter 3. We will present the framework of algebraic and locally covariant quantum field theory, and discuss the time-slice axiom, the relative Cauchy evolution and how we quantise classical linear field theories by the means of a quantisation functor. Chapter 4 centres around the universal algebra. We will give its general categorical definition, discuss some criticism and present the main technical theorem of this thesis, which relates the universal algebra to a colimit in a category of symplectic spaces[11] via the quantisation functor. Afterwards, we will apply colimit constructions and left Kan extensions to the free and minimally coupled real scalar field and the free Maxwell field. We will obtain the universal F-theory of the free Maxwell field and address its

[11] By a symplectic space, we always mean a *linear* symplectic space, that is, a real vector space equipped with a non-degenerate skew-symmetric bilinear form, and not a general symplectic manifold.

failure of local covariance. In Chapter 5, we will test the universal F-theory for dynamical locality with a negative result. By going over to the reduced F-theory of the free Maxwell field, we will show that local covariance and dynamical locality are restored. Twisted quantum fields are the topic of Chapter 6. We will first outline the abstract categorical framework and then apply it to locally covariant theories, establishing the classificaton of twisted variants of a locally covariant theory on single curved spacetimes by the isomorphism classes of flat smooth principal bundles. We will exemplify the general scheme by discussing $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields, that is, twisted variants of multiple free and minimally coupled real scalar fields of the same mass. In Chapter 7, we will construct the FP-states and test them for the Hadamard property. Finally, we will summarise our results and point out some missed opportunities.

A friendly word of warning: this is a technical thesis and we will constantly end up establishing practical lemmas rather than big, ground-breaking results. Accordingly, our focus will lie many times on sorting out laborious technicalities and not so much on profound interpretations and discussions. We hope nonetheless that the technical aspects developed in this thesis will be of use to others and are not just ends in themselves.

Chapter 1

Differential Geometry (Some Preliminaries)

> "*IMAGINE SPACECRAFT NAVIGATORS, stoned out of their minds, pitching their laptops and workstations into the recycle bin, to be replaced by lava lamps and incense burners in order to guide today's spacecraft across the Solar System.* [...] *Welcome to the Dune universe, where the Spacing Guild's prescient, spice-saturated Steersmen navigate huge Holtzman-drive-powered Heighliner ships safely through folded space –the only means of interstellar transport throughout the known galaxy.*"

–John C. Smith, "Navigators and the Spacing Guild", *The Science of Dune: An Unauthorized Exploration into the Real Science Behind Frank Herbert's Fictional Universe*, ed. by K.R. Grazier, BenBella Books, 2008.

For the benefit of the reader and also to establish some notation, we recap a few elementary concepts from differential and Lorentzian geometry. This review is by no means exhaustive. As references for the basics of differential geometry serve [AM08; AMR07; GHV72; Lee97; Lee03; Mor01b; Thi88; Thi90; Wal07] and for the aspects of Lorentzian geometry and general relativity, we have consulted [BEE96; BF09; BGP07; HE73; MS08; O'N83; Pen72; SW77a; SW77b; Wal84; Wal12]. For a short but yet insightful introduction to linear differential operators and their principal symbols (and the notations involved), see [BGP07; Wal07; Wal12]. Though a few inevitable notions will be reviewed in the appendix to Chapter 6, we cannot touch on the rich theory of fibre bundles in our small recapitulation unfortunately. This would amount to either us doing a proper job, which is far too excessive and an elaborate introduction to fibre bundles is indeed not the goal of this thesis, or us doing a bad job. Neither of the two options is preferable and we assume the reader to be familiar with the fundamental concepts instead. There are many good textbooks on the subject, with the classics [Ste51; Hus94] leading the way. However, we have extensively worked with [Bär11; Bau09; GHV72; GHV73; KMS99; Mor01b; Mor01a] in particular. Most of the notions we are using in this thesis can of course be found in the references mentioned and we will frequently cite them. For a neat introduction to fibre bundles and their relevance to physics, we recommend the excellent treatise [DM77].

For us, a *smooth manifold* M will always be locally Euclidean, second-countable and Hausdorff topological space, hence paracompact, with a fixed $\mathcal{C}^\infty$-structure; m will usually denote the dimension of a smooth manifold M. $\mathcal{C}^\infty M$ denotes the set of all smooth (real-valued) functions on M, which form a (real) vector space and a $\mathcal{C}^\infty M$-module, and $\mathcal{C}_0^\infty M$ is to denote the set of all compactly supported smooth (real-valued) functions on M; in the same way as $\mathcal{C}^\infty M$, $\mathcal{C}_0^\infty M$ is a (real) vector space and a $\mathcal{C}^\infty M$-module. We denote the *tangent space* of M at a point $x \in M$ by TM_x instead of using the more common notation $T_x M$; the *tangent bundle* of M will be denoted by τ_M. Similarly, the *cotangent space* of M at a point $x \in M$ will be denoted by T^*M_x and the *cotangent bundle* by τ_M^*. The *(r,s)-tensor bundle* of M, $\bigotimes^r \tau_M^* \otimes \bigotimes^s \tau_M$, where[1] $r, s \in \mathbb{N}$, will be denoted by $\tau_M^{(r,s)}$. By convention, $\bigotimes^0 \tau_M^* = \bigotimes^0 \tau_M = \underline{\mathbb{R}}_M$, where $\underline{\mathbb{R}}_M = (M \times \mathbb{R}, M, \mathrm{pr}_1, \mathbb{R})$ is the trivial smooth real vector bundle over M of rank 1. For $p \in \mathbb{N}$, the p-th exterior power of τ_M^*, $\bigwedge^p \tau_M^*$, will be abbreviated by λ_M^p, with the conventions $\lambda_M^0 = \underline{\mathbb{R}}_M$ and $\lambda_M^1 = \tau_M^*$. Similarly, σ_M^p will denote the p-th symmetric power $\bigodot^p \tau_M^*$ of τ_M^*, again with the conventions $\sigma_M^0 = \underline{\mathbb{R}}_M$ and $\sigma_M^1 = \tau_M^*$.

A *smooth vector field* on M is a smooth cross-section in τ_M and the $\mathcal{C}^\infty M$-module of all smooth vector fields on M will be denoted by $\mathscr{X}(M)$ $[= \Gamma^\infty(\tau_M)]$. A *smooth differential p-form* is a smooth cross-section in λ_M^p and the $\mathcal{C}^\infty M$-module of all smooth differential p-forms on M will be denoted by $\Omega^p M$ $[= \Gamma^\infty(\lambda_M^p)]$. We will use the notation $\Omega_0^p M$ $[= \Gamma_0^\infty(\lambda_M^p)]$ to denote the $\mathcal{C}^\infty M$-modules of all compactly supported smooth differential p-forms. Note, $\mathcal{C}^\infty M \cong \Omega^0 M$ and $\mathcal{C}_0^\infty M \cong \Omega_0^0 M$ as $\mathcal{C}^\infty M$-modules, so that we can make the identification whenever convenient. By convention, $\Omega^{-1} M = \{0 \in \mathcal{C}^\infty M\}$ and $\Omega_0^{-1} M = \{0 \in \mathcal{C}_0^\infty M\}$.

If $\psi : M \longrightarrow N$ is a smooth map, the *pullback* via ψ, $\psi^* : \Omega^p N \longrightarrow \Omega^p M$, is defined for $\omega \in \Omega^p N$ by $\psi^*\omega(x; v_1, \ldots, v_p) := \omega(\psi(x); (T_x\psi) v_1, \ldots, (T_x\psi) v_p)$, where $v_1, \ldots, v_p \in TM_x$, $x \in M$ and $T\psi : \tau_M \longrightarrow \tau_N$ is the tangent map induced by ψ. $T_x\psi : TM_x \longrightarrow TN_{\psi(x)}$ is the restriction of $T\psi$ to TM_x and $TN_{\psi(x)}$, $x \in M$. For $f \in \mathcal{C}^\infty N$, the pullback amounts to $\psi^* f = f \circ \psi$. If ψ is a diffeomorphism onto its image, the *pushforward* along ψ, $\psi_* : \Omega_0^p M \longrightarrow \Omega_0^p N$, is given for $\omega \in \Omega_0^p M$ by $\psi_*\omega(x; v_1, \ldots, v_p) := \omega(\psi^{-1}(x); (T_x\psi)^{-1} v_1, \ldots, (T_x\psi)^{-1} v_p)$ for all $v_1, \ldots, v_p \in TN_x$ and for all $x \in \psi(M)$, and by $\psi_*\omega(x; v_1, \ldots, v_p) := 0$ for all $v_1, \ldots, v_p \in TN_x$ and for all $x \in N \smallsetminus \psi(M)$. For $f \in \mathcal{C}^\infty M$, this definition yields[2] $\psi_* f := f \circ \psi\|_M^{-1}$ on $\psi(M)$ and $\psi_* f := 0$ on $N \smallsetminus \psi(M)$.

[1] Please note that for us, zero is a natural number, i.e. $0 \in \mathbb{N}$.

[2] $\psi\|_M : M \xrightarrow{\sim} \psi(M)$ denotes the strong restriction of $\psi : M \longrightarrow N$ to M, i.e. the unique diffeomorphism such that $\iota_{\psi(M)} \circ \psi\|_M = \psi$, where $\iota_{\psi(M)} : \psi(M) \hookrightarrow N$ is the inclusion map. In general, for a map $f : X \longrightarrow Y$, the *strong restriction* of f to a subset $Z \subseteq X$, $f\|_Z$, is the unique map, which is necessarily a surjection, $f\|_Z : Z \twoheadrightarrow f(Z)$ satisfying $\iota_{f(Z)} \circ f\|_Z = f|_Z$, where $\iota_{f(Z)} : f(Z) \hookrightarrow Y$

On any smooth manifold, we have the *exterior derivative* $d : \Omega^p_{(0)}M \longrightarrow \Omega^{p+1}_{(0)}M$ at our disposal, where "(0)" is to indicate that the mapping is valid with and without the subscript "0". The exterior derivative is a linear differential operator of order 1 and defined by

$$d\omega\Big(\mathop{\mathfrak{a}}_{i=0}^{p} X_i\Big) := \sum_{i=0}^{p} (-1)^i X_i \Big(\omega\Big(\mathop{\mathfrak{a}}_{\substack{j=0\\ j\neq i}}^{p} X_j\Big)\Big) + \sum_{\substack{i,j=0\\ i<j}}^{p} (-1)^{i+j}\omega\Big([X_i, X_j], \mathop{\mathfrak{a}}_{\substack{k=0\\ i\neq k\neq j}}^{p} X_i\Big), \tag{1.1}$$

$$X_0, \dots, X_p \in \mathscr{X}(M),\ \omega \in \Omega^p M,$$

where "$\mathfrak{a}$" denotes ordered enumeration. In any smooth chart $\varphi : U \xrightarrow{\sim} W \subseteq \mathbb{R}^m$ of M, the exterior derivative takes the form

$$d\omega|_U = \frac{1}{p!}\,\partial_j\omega_{i_1\dots i_p}\, d\varphi^j \wedge d\varphi^{i_1} \wedge \dots \wedge d\varphi^{i_p}, \qquad \omega \in \Omega^p M. \tag{1.2}$$

The exterior derivative is natural with respect to pullbacks and pushforwards, i.e. if $\psi : M \longrightarrow N$ is a smooth map, $\psi^* d_M = d_N \psi^*$, and if ψ is a diffeomorphism onto its image, $\psi_* d_M = d_N \psi_*$. $\omega \in \Omega^p M$ is called *closed* if and only if $d\omega = 0$; if there is $\theta \in \Omega^{p-1}M$ such that $\omega = d\theta$, ω is called *exact*. We will find it helpful to introduce the vector spaces $\Omega^p_{(0),d}M := \{\omega \in \Omega^p_{(0)}M \mid d\omega = 0\}$, and taking the quotients $H^p_{\mathrm{dR}}M := \Omega^p_d M / d\Omega^{p-1}M$ and $H^p_{\mathrm{dR,c}}M := \Omega^p_{0,d}M / d\Omega^{p-1}_0$, we obtain the *p-th de Rham cohomology group* and the *p-th de Rham cohomology group with compact supports*.

A *semi-Riemannian manifold* is a smooth manifold M equipped with a smooth cross-section $g \in \Gamma^\infty(\tau^*_M \otimes \tau^*_M)$ such that for each $x \in M$, the map $g_x : TM_x \times TM_x \longrightarrow \mathbb{R}$ is a non-degenerate symmetric bilinear form of signature (r, s), where r is the count of all "1" and s is the count of all "−1". If $r = 1$ and $s \geq 1$, we will call (M, g) a *Lorentzian manifold*. Note, g is particularly a smooth bundle metric in the tangent bundle τ_M (see Definition 6.10.12). Any semi-Riemannian manifold (M, g) possesses a canonical linear connection $\nabla : \mathscr{X}(M) \longrightarrow \Omega^1(M; \tau_M)$, the *Levi-Civita connection*, which is metric, torsion-free and also, like every linear connection in a smooth vector bundle, a linear differential operator of order 1. Here, we have introduced the notation for smooth ξ-valued differential p-forms, $\Omega^p(M;\xi) := \Gamma^\infty(\lambda^p_M \otimes \xi)$, where ξ is a smooth vector bundle over M.

Any semi-Riemannian metric g on a smooth manifold M canonically induces smooth bundle metrics $\langle \cdot \mid \cdot \rangle_g$ (also sometimes denoted by g) in $\xi = \tau^{(r,s)}_M, \lambda^p_M, \sigma^p_M$, which take

denotes the inclusion map and $f|_Z : Z \longrightarrow Y$ the restriction of f to Z, i.e. $f|_Z = f \circ \iota_Z$ with the inclusion map $\iota_Z : Z \hookrightarrow X$. Note, in order to emphasise that a map is injective we will use the symbol "$\hookrightarrow$" and to emphasise that a map is surjective we will use "$\twoheadrightarrow$".

in each smooth chart $\varphi : U \xrightarrow{\sim} W \subseteq \mathbb{R}^m$ of M the form

$$\langle \cdot \mid \cdot \rangle_g (x; T, S) = T^{\alpha_1 \dots \alpha_s}_{\beta_1 \dots \beta_r} S^{\mu_1 \dots \mu_s}_{\nu_1 \dots \nu_r} g_{\alpha_1 \mu_1}(x) \dots g_{\alpha_s \mu_s}(x) g^{\beta_1 \nu_1}(x) \dots g^{\beta_r \nu_r}(x) \qquad T, S \in E_x,\ x \in U, \tag{1.3}$$

where $E_x = \bigotimes^r T^*M_x \otimes \bigotimes^s TM_x$, $\bigwedge^p T^*M_x$, $\bigodot^p T^*M_x$. If (M, g) is an orientable semi-Riemannian manifold and an orientation $[\Omega]$ has been chosen, $\langle \cdot \mid \cdot \rangle_g \in \Gamma^\infty(\xi^* \odot \xi^*)$ gives rise to the weakly non-degenerate symmetric bilinear pairings

$$\langle \cdot \mid \cdot \rangle_{2,g} : \Gamma^\infty_0(\xi) \times \Gamma^\infty_0(\xi) \longrightarrow \mathbb{R}, \qquad (\sigma, \tau) \longmapsto \int_M \langle \sigma \mid \tau \rangle_g \operatorname{vol}_{(g,[\Omega])}, \tag{1.4}$$

$$\langle \cdot \mid \cdot \rangle_{2,g} : \Gamma^\infty_0(\xi) \times \Gamma^\infty(\xi) \longrightarrow \mathbb{R}, \qquad (\sigma, \tau) \longmapsto \int_M \langle \sigma \mid \tau \rangle_g \operatorname{vol}_{(g,[\Omega])}, \tag{1.5}$$

and

$$\langle \cdot \mid \cdot \rangle_{2,g} : \Gamma^\infty(\xi) \times \Gamma^\infty_0(\xi) \longrightarrow \mathbb{R}, \qquad (\sigma, \tau) \longmapsto \int_M \langle \sigma \mid \tau \rangle_g \operatorname{vol}_{(g,[\Omega])}. \tag{1.6}$$

Opposed to the *exterior product* or *exterior multiplication* of smooth differential forms, we have furthermore for a semi-Riemannian manifold (M, g) the *interior product* $i : \Omega^p M \times \Omega^q M \longrightarrow \Omega^{p-q} M$, $(\omega, \eta) \longmapsto i_\eta \omega$, $p \geq q$, which is also sometimes called the *interior multi*plication or the *substitution operator*. Suppose $(M, g, [\Omega])$ is an oriented semi-Riemannian manifold; we can then define two more relevant maps. The first one is the *Hodge-*-operator* $* : \Omega^p_{(0)} M \longrightarrow \Omega^{m-p}_{(0)} M$, $\omega \longmapsto i_\omega \operatorname{vol}_{(g,[\Omega])}$. Alternatively, the Hodge-$*$-operator is defined more indirectly by requiring that $*\omega$ is the unique smooth differential $(m-p)$-form such that the identity $\eta \wedge *\omega = \frac{1}{p!} \langle \eta \mid \omega \rangle_g \operatorname{vol}_{(g,[\Omega])}$ holds, where $\omega, \eta \in \Omega^p M$. Hence, $\langle \omega \mid \eta \rangle_{2,g} = \int_M \langle \omega \mid \eta \rangle_g \operatorname{vol}_{(g,[\Omega])} = p! \int_M \omega \wedge *\eta$ for all $\omega, \eta \in \Omega^p M$ such that the integral exists. In each smooth chart $\varphi : U \xrightarrow{\sim} W \subseteq \mathbb{R}^m$ of M, we find for the Hodge-*-operator the local expression

$$*\omega|_U = \frac{\operatorname{sgn} \varphi}{p!\,(m-p)!} \epsilon_{j_1 \dots j_m} \prod_{k=1}^{p} g^{i_k j_k} \omega_{i_1 \dots i_p} \sqrt{|\det g|}\, d\varphi^{j_{p+1}} \wedge \dots \wedge d\varphi^{j_m}, \qquad \omega \in \Omega^p M, \tag{1.7}$$

where $\operatorname{sgn} \varphi = +1$ if φ is a positively oriented smooth chart of $(M, [\Omega])$, $\operatorname{sgn} \varphi = -1$ if φ is a negatively oriented smooth chart of $(M, [\Omega])$ and $\epsilon_{j_1 \dots j_m}$ are the *Levi-Civita symbols*, i.e. $\epsilon_{j_1 \dots j_m} := 1$ if $j_1, \dots, j_m$ is an even permutation of $1, \dots, m$, $\epsilon_{j_1 \dots j_m} := -1$ if $j_1, \dots, j_m$ is an odd permutation of $1, \dots, m$ and $\epsilon_{j_1 \dots j_m} := 0$ otherwise. The Hodge-*-operator is a linear differential operator of order 0 and a $\mathcal{C}^\infty M$-module isomorphism with inverse $*^{-1} = (-1)^{p(m-p)} \frac{|\det g|}{\det g} * : \Omega^p_{(0)} M \longrightarrow \Omega^{m-p}_{(0)} M$. The second map, which we want to introduce, is the _exterior coderivative_ $\delta := (-1)^p *^{-1} d * : \Omega^p_{(0)} M \longrightarrow \Omega^{p-1}_{(0)} M$. It is a linear differential operator of order 1 and takes in any smooth chart $\varphi : U \xrightarrow{\sim} W \subseteq \mathbb{R}^m$ of M the form

$$\delta\omega|_U = -\frac{1}{(p-1)!} \nabla_j \, \omega^j{}_{i_2 \ldots i_p} \, d\varphi^{i_2} \wedge \ldots \wedge d\varphi^{i_p}, \qquad \omega \in \Omega^p M. \tag{1.8}$$

Due to Stokes' theorem, the exterior coderivative is formally adjoint to the exterior derivative in the sense that

$$\int_M d\omega \wedge *\eta = \int_M \omega \wedge *\delta\eta \tag{1.9}$$

whenever $\omega \in \Omega^p M$ and $\eta \in \Omega^{p+1} M$ such that $\operatorname{supp}\omega \cap \operatorname{supp}\eta$ is compact. In the same way as for the exterior derivative, we will find it useful to define the vector spaces $\Omega^p_{(0),\delta} M := \{\omega \in \Omega^p_{(0)} M \mid \delta\omega = 0\}$. $\omega \in \Omega^p M$ is called _coclosed_ if and only if $\delta\omega = 0$, and _coexact_ if and only if there is $\eta \in \Omega^{p+1} M$ with $\omega = \delta\eta$. It is important to mention that the Hodge-$*$-operator and the exterior coderivative are natural with respect to pullbacks via and pushforwards along isometric smooth embeddings $\psi : (M, g, [\Omega]) \longrightarrow (M', g', [\Omega'])$ which preserve the orientation, i.e. $\psi^* *' = *\psi^*$, $\psi_* * = *'\psi_*$, $\psi^*\delta' = \delta\psi^*$ and $\psi_*\delta = \delta'\psi_*$. Also observe that $d, *, \langle \cdot \mid \cdot \rangle_{2,g}$ and δ can be canonically extended to smooth $\mathbb{K}^n$-valued differential p-forms, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $n \in \mathbb{N} \smallsetminus \{0\}$.

Now, suppose that (M, g) is a Lorentzian manifold. For $x \in M$, a tangent vector $v \in TM_x \smallsetminus \{0\}$ is called _timelike_ (resp. _spacelike_, _causal_, _lightlike_ [or _null_]) if and only if $g_x(v, v) > 0$ (resp. < 0, ≥ 0, $= 0$). $0 \in TM_x$ is considered spacelike. A smooth[3] curve $c : I \longrightarrow M$ or a smooth vector field $X \in \mathscr{X}(M)$ is called _timelike_ (resp. _spacelike_, _causal_, _lightlike_ [or _null_]) if and only if $\dot{c}(t)$ or $X(x)$ is timelike (resp. spacelike, causal, lightlike [or null]) for all $t \in I$ or for all $x \in M$. We call a Lorentzian manifold (M, g) _time-orientable_ if and only if there exists a timelike smooth vector field $T \in \mathscr{X}(M)$, which is non-vanishing in particular. The equivalence class $[T] := \{S \in \mathscr{X}(M) \mid S \text{ is timelike and } g_x(T(x), S(x)) > 0 \ \forall x \in M\}$ of a timelike smooth vector field $T \in \mathscr{X}(M)$ is called a _time-orientation_ and to equip (M, g) with a time-orientation is said to _time-orient_ (M, g). Hence, assume that $(M, g, [T])$ is a time-oriented Lorentzian manifold. For $x \in M$, a timelike, causal or lightlike tangent vector $v \in TM_x \smallsetminus \{0\}$ is said to be _future-/past-directed_ or _future-/past-pointing_ if and only if $g_x(T(x), v) > 0$ (resp. < 0). Likewise, a timelike, causal or lightlike smooth

[3] Note, some authors work (for legitimate reasons) with piecewise smooth curves [MS08], *differentiable* curves [Wal84] or piecewise $\mathcal{C}^1$-curves [BGP07; BF09]. We are faithful to the smooth setting just as [SW77a; SW77b; O'N83; BEE96; Wal12].

curve $c : I \longrightarrow M$ or smooth vector field $X \in \mathscr{X}(M)$ is called *future-/past-directed* if and only if $\dot{c}(t)$ or $X(x)$ is future-/past-directed for all $t \in I$ or for all $x \in M$.

We call a connected and time-oriented Lorentzian manifold $(M, g, [T])$ a *spacetime*, hence any spacetime is path connected. A smooth curve from $x \in M$ to $y \in M$ is any smooth curve $c : I \longrightarrow M$ such that $c(t_1) = x$ and $c(t_2) = y$ for some $t_1 < t_2$. For a point $x \in M$ in a spacetime $(M, g, [T])$, the *chronological future/past* are the sets $I^{\pm}(x) := \{y \in M \mid \exists \text{future-/past-directed timelike smooth curve from } x \text{ to } y\}$; the sets $J^{\pm}(x) := \{y \in M \mid y = x \text{ or } \exists \text{future-/past-directed causal smooth curve from } x \text{ to } y\}$ are called the *causal future/past* of $x \in M$. Note that we always have $x \notin I^{\pm}(x)$ and $x \in J^{\pm}(x)$ for $x \in M$. For an arbitrary subset $A \subseteq M$, the chronological resp. causal future/past are the sets $I^{\pm}(A) := \bigcup_{x \in A} I^{\pm}(x)$ resp. $J^{\pm}(A) := \bigcup_{x \in A} J^{\pm}(x)$. Let $p \geq 0$, then a $\mathbb{K}$-valued smooth differential p-form $\omega \in \Omega^p(M; \mathbb{K})$ is called *spacelike compact* if and only if there is a compact subset $K \subseteq M$ such that $\operatorname{supp} \omega \subseteq J(K) := J^+(K) \cup J^-(K)$. The $\mathcal{C}^\infty(M, \mathbb{K})$-module of all spacelike compact $\mathbb{K}$-valued smooth differential p-forms is denoted by $\Omega^p_{\mathrm{sc}}(M; \mathbb{K})$. For an *achronal* subset $A \subseteq M$, i.e. no timelike smooth curve meets A more than once, the *future/past Cauchy development* or *future/past domain of dependence* are defined by $D^{\pm}(A) := \{x \in M \mid \text{every past-/future-inextendible causal smooth curve through } x \text{ intersects } A\}$. The union $D(A) := D^+(A) \cup D^-(A)$ is called the *Cauchy development* or the *domain of dependence* of A.

A spacetime $(M, g, [T])$ is said to be *globally hyperbolic* if and only if the *causality condition* is met, i.e. there are no closed causal smooth curves, and $J^+(x) \cap J^-(y)$ is compact for all $x, y \in M$. Note, in the early literature on globally hyperbolic spacetimes, e.g. [Pen72; HE73; O'N83; BEE96], the *strong causality condition* was required. It was recently shown by [BS07] that it is enough to merely require the causality condition in the definition of global hyperbolicity. A *globally hyperbolic open subset* of a globally hyperbolic spacetime $(M, g, [T])$ is an open subset $O \subseteq M$ such that the causality condition holds on O and for $x, y \in O$, $J^+(x) \cap J^-(y)$ [the causal future and past are taken in $(M, g, [T])$] is compact and contained in O. Thus, O is *causally convex* in $(M, g, [T])$, i.e. every causal smooth curve [in $(M, g, [T])$] with endpoints in O is entirely contained in O. Note that if O is connected, it can thus be regarded as a globally hyperbolic spacetime in its own right if endowed with the structures induced by $(M, g, [T])$, which we will denote by $(O, g|_O, [T|_O])$. A key feature about globally hyperbolic spacetimes are Cauchy surfaces. A subset $\Sigma \subseteq M$ of a spacetime $(M, g, [T])$ is called a *Cauchy hypersurface* or more briefly a *Cauchy surface* if and only if every inextendible timelike smooth curve intersects Σ exactly once. In their celebrated series of papers [BS03; BS05; BS06], A.N. Bernal and M. Sánchez have proven the following important theorem, which strengthens early results by R. Geroch [Ger70a]:

THEOREM 1.1.1. *Let $(M, g, [T])$ be a spacetime. The following statements are* *equivalent:*

(a) $(M, g, [T])$ *is globally hyperbolic.*

(b) *There exists a smooth spacelike Cauchy surface,* Σ*, for* $(M, g, [T])$*, which is particularly a smooth embedded submanifold of* M*.*

Moreover, under (a) *or* (b)*, there is an isometric diffeomorphism*

$$\Phi : (M, g) \xrightarrow{\sim} \left(\mathbb{R} \times \Sigma,\ \beta\, d\,\mathrm{pr}_1 \otimes d\,\mathrm{pr}_1 - h_{\mathrm{pr}_1}\right) \tag{1.10}$$

where $\beta \in \mathcal{C}^\infty(\mathbb{R} \times \Sigma, \mathbb{R}^+)$*,* h_t *is a Riemannian metric on* Σ *depending smoothly on* $t \in \mathbb{R}$ *and each level set* $\Sigma_t := \Phi^{-1}(\mathrm{pr}_1^{-1}(t))$*,* $t \in \mathbb{R}$*, is a smooth spacelike Cauchy surface for* $(M, g, [T])$*. In particular,* $\Sigma = \Phi^{-1}(\mathrm{pr}_1^{-1}(0))$ *and* $\mathrm{pr}_2(\Phi(\sigma)) = \sigma$ *for all* $\sigma \in \Sigma$*.*

We will refer to this theorem as the Bernal-Sánchez splitting theorem. We finish this chapter by providing some additional lemmas of a topological nature. In particular, we show that the contractible globally hyperbolic open subsets of a globally hyperbolic spacetime form a basis for the topology and we prove the open diamond caterpillar covering lemma, which is indispensable for calculating the commutation relations in the universal algebra (see the proof of Proposition 4.5.6).

LEMMA 1.1.2. *Let* $(M, g, [T])$ *be a globally hyperbolic spacetime and* O_1, O_2 *contractible globally hyperbolic open subsets of* $(M, g, [T])$*. For each* $z \in O_1 \cap O_2$*, there is a third contractible globally hyperbolic open subset* O_3 *of* $(M, g, [T])$ *such that* $z \in O_3 \subseteq O_1 \cap O_2$*. In particular, the contractible globally hyperbolic open subsets of* $(M, g, [T])$ *form a basis for the topology of* M*.*

Proof: Since $(M, g, [T])$ is globally hyperbolic, it is strongly causal [BS07] and so the topology on M coincides with the Alexandrov topology ([Pen72, Thm.4.24], [BEE96, Prop.3.11]), which has as its basis the chronological diamonds $I(x, y) := I^+(x) \cap I^-(y)$, $x, y \in M$. Note, for each $x, y \in M$, $I(x, y)$ is a globally hyperbolic open subsets of $(M, g, [T])$ because of [BGP07, Lem.A.5.12] and $J(x', y') \subseteq I(x, y)$ for all $x', y' \in I(x, y)$. In conclusion, $O_1 \cap O_2$ is the union of some chronological diamonds and each $z \in O_1 \cap O_2$ lies in one of them, say $z \in I(x, y) \subseteq O_1 \cap O_2$ for some $x, y \in O_1 \cap O_2$. Because $I(x, y)$ is globally hyperbolic, we can find a smooth spacelike Cauchy surface Σ for $I(x, y)$ containing z. Now, take some contractible open neighbourhood A of z (in the topology of Σ) which is entirely contained in $\Sigma \cap O_1 \cap O_2$ and consider its Cauchy development $O_3 := D(A)$ in $I(x, y)$. Then O_3 is a contractible globally hyperbolic open subset of $I(x, y)$ by [O'N83, Lem.14.43] and hence of $(M, g, [T])$. Furthermore, $z \in O_3 \subseteq O_1 \cap O_2$. ❐

LEMMA 1.1.3. *Let O, W be non-disjoint open sets in a smooth manifold M. Each connected component of $O \cap W$ is open.*

Proof: Let Γ be any connected component of $O \cap W$, $x \in \Gamma$ and $\varphi : U \xrightarrow{\sim} V \subseteq \mathbb{R}^m$ a smooth chart of M containing x. Without the loss of generality, $U \subseteq O \cap W$ or else consider the strong restriction $\varphi\|_{U \cap O \cap W} : U \cap O \cap W \xrightarrow{\sim} \varphi(U \cap O \cap W) \subseteq \mathbb{R}^m$. Because V is open, there exists $\varepsilon > 0$ and an open ball $B_\varepsilon(\varphi(x)) \subseteq V$, which is contractible [in particular, $B_\varepsilon(\varphi(x))$ is connected]. $\varphi^{-1}(B_\varepsilon(\varphi(x))) \subseteq U \subseteq O \cap W$ is open, connected, contains x and thus $\varphi^{-1}(B_\varepsilon(\varphi(x))) \subseteq \Gamma$. This proves that Γ is a neighbourhood for each of its points and consequently, Γ is open. ❐

The following two lemmas will be needed in the proof of the open diamond caterpillar covering lemma:

LEMMA 1.1.4. *Let X be a non-empty topological space.*

(i) *Let $Y, Z \subseteq X$ be non-empty, open and simply connected such that $Y \cap Z \neq \varnothing$ is simply connected and $Y \cup Z = X$. Then, X is simply connected.*

(ii) *Let $n \in \mathbb{N} \setminus \{0\}$ and suppose $\{Y_i \subseteq X \mid i = 1, \ldots, n\}$ is a family of non-empty, simply connected and open subsets of X satisfying $Y_i \cap Y_j = \varnothing$ for $i - 1 \neq j \neq i + 1$, $Y_i \cap Y_{i+1} \neq \varnothing$ is simply connected for $i = 1, \ldots, n-1$, and $\bigcup_{i=1}^n Y_i = X$. Then X is simply connected.*

Proof: (i) follows from the Seifert-van Kampen theorem ([SZ94, Satz 5.3.11], [Hat02, Thm.1.20]), (ii) follows by induction. ❐

LEMMA 1.1.5. *Let M be a smooth manifold.*

(i) *Let $U, V \subseteq M$ be non-empty, open and contractible such that $U \cap V \neq \varnothing$ is contractible and $U \cup V = M$. Then, M is contractible.*

(ii) *Let $n \in \mathbb{N} \setminus \{0\}$ and suppose $\{U_i \subseteq M \mid i = 1, \ldots, n\}$ is a family of non-empty, contractible and open subsets of M satisfying $U_i \cap U_j = \varnothing$ for $i - 1 \neq j \neq i + 1$, $U_i \cap U_{i+1} \neq \varnothing$ is contractible for all $i = 1, \ldots, n-1$, and $\bigcup_{i=1}^n U_i = M$. Then M is contractible.*

Proof: We start with the proof of (i), (ii) follows then by induction. Accordingt to Lemma 1.1.4(i), M is simply connected and by Hurewicz's theorem ([SZ94, Satz 16.8.2], [Hat02, Thm.4.32]) the first singular homology group of M vanishes, $H_1 M = 0$. By the Mayer-Vietoris theorem ([SZ94, Satz 9.4.10], [Lee03, Thm.16.3]), we conclude that $H_p M = 0$ for $p \geq 2$, too. Using again Hurewicz's theorem, we are able to deduce $\pi_p M \cong H_p M = 0$ for all $p \geq 1$ and as a result of this, M is ∞-connected. Since a

CW-complex is ∞-connected if and only if it is contractible ([SZ94, Kor.16.4.10]) and M is homotopy equivalent to a CW-complex by [Hir76, Thm.4.3], M is contractible.❒

The open diamond caterpillar covering lemma now states that a situation as shown in Figure 1.1(b) is always attainable for a globally hyperbolic spacetime $(M, g, [T])$ and any smooth spacelike Cauchy surface Σ for it.

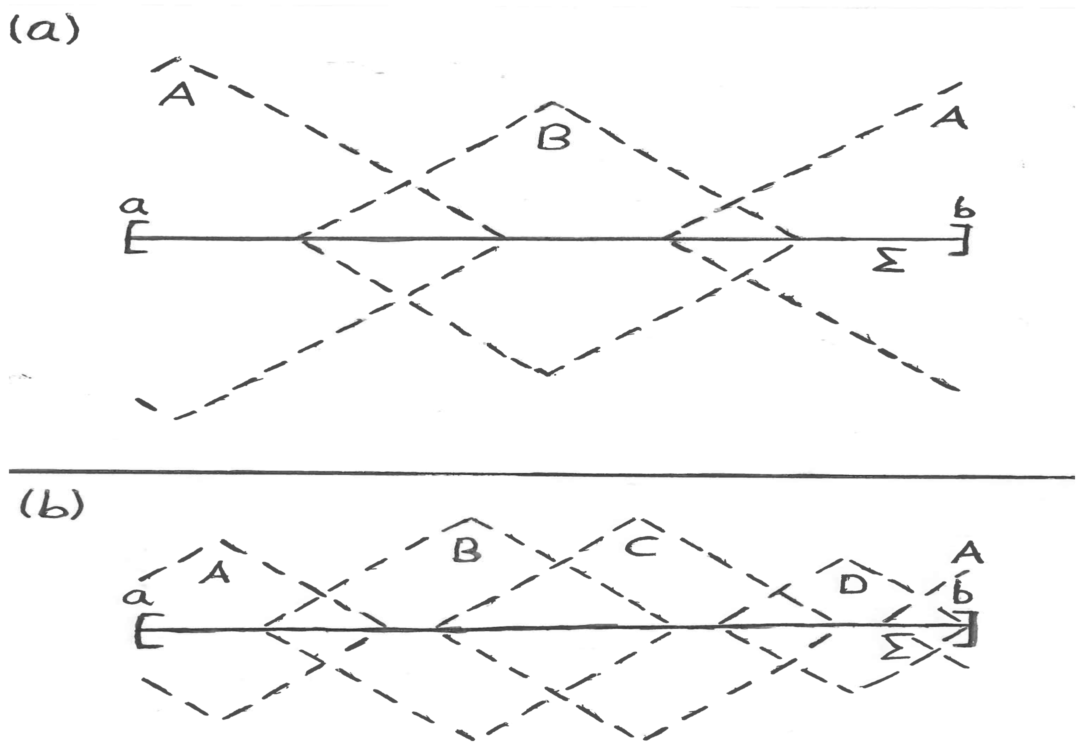


FIGURE 1.1: Illustration of the open diamond caterpillar covering lemma. For simplicity, the topology of $M = \mathbb{R} \times \Sigma$ is taken to be $\mathbb{R} \times S^1$, hence the endpoints a and b have to be identified with each other. Of course, the situation is much more complicated in higher spacetime dimensions. The dashed diamonds A, B, C and D are supposed to be the Cauchy developments of their respective projection to Σ. (a) shows an open cover of Σ, which is not an open diamond caterpillar cover; any globally hyperbolic open subset containing A and B already contains all of Σ and cannot be contractible therefore. (b) shows an open diamond caterpillar cover of Σ; for each pair $X, Y = A, B, C, D$, a third contractible globally hyperbolic open subset containing both X and Y can be found by taking the Cauchy development of a suitable open subset of Σ.

LEMMA 1.1.6. (open diamond caterpillar covering lemma)
Let $\mathbf{M} = (M, g, [T])$ be a globally hyperbolic spacetime and Σ a smooth spacelike Cauchy surface for $\mathbf{M}$. There exists an open cover of Σ by contractible globally hyperbolic open subsets of $\mathbf{M}$ such that for each two members of this cover, we can find a third contractible globally hyperbolic open subset of $\mathbf{M}$, which need not belong to the open cover of Σ, containing them both.

Proof: We recall that any smooth manifold admits a "*good*" open cover (sometimes also called a "*simple*" or "*contractible*" open cover[ing]), that is, each member and all finite intersections of members of the cover are open and contractible, see e.g. [BT82, Thm.5.1], [GHV72, Sec.5.22], [Mor01b, Prop.3.18]. Let us take such a good cover $\{N_i \mid i \in I\}$ for Σ. We claim that the Cauchy developments $\{D(N_i) \mid i \in I\}$ yield a cover of Σ by contractible globally hyperbolic open subsets of $\mathbf{M}$ with the required properties. The Cauchy developments $D(N_i)$ are all globally hyperbolic open subsets of $\mathbf{M}$ by [O'N83, Lem.14.43] and contractible because the N_i are contractible.

Let A and B be any two members of $\{N_i \mid i \in I\}$. If $A = B$, we have nothing to show, and if $A \cap B \neq \varnothing$, we can apply Lemma 1.1.5(i) to $A \cup B$[4] and conclude that $A \cup B$ is a contractible open subset of Σ containing both A and B. Clearly $D(A), D(B) \subseteq D(A \cup B)$ and $D(A \cup B)$ is a contractible globally hyperbolic open subset of $\mathbf{M}$. Now, let $A \cap B = \varnothing$. We will construct a "*chain*" $\{U_i \subseteq \Sigma \mid i = 1, \ldots, n\}$, where $n \in \mathbb{N} \smallsetminus \{0,1\}$, of contractible open subsets $U_i \subseteq \Sigma$ such that $U_1 = A$, $U_n = B$ and the conditions of Lemma 1.1.5(ii) are met, i.e. $U_i \cap U_j = \varnothing$ if $i - 1 \neq j \neq i + 1$ and $U_i \cap U_{i+1} \neq \varnothing$ is contractible for $i = 1, \ldots, n-1$. Then the Cauchy development $D(\bigcup_{i=1}^{n} U_i)$ is a contractible globally hyperbolic open subset of $\mathbf{M}$ containing both $D(A)$ and $D(B)$. Σ is connected, so it is path-connected [Lee03, Prop.1.8(b)] and there exists a path $\gamma : [0,1] \longrightarrow \Sigma$ such that $\gamma(0) = p$ and $\gamma(1) = q$ for some $p \in A$ and $q \in B$. Since $\gamma[0,1]$ is compact (continuous maps map compact sets to compact sets), we find finitely many of the N_i, $i \in I$, which, together with A and B, are sufficient to cover $\gamma[0,1]$. Furthermore, if any of these finitely many sets is entirely contained in another one, we will remove it from the cover altogether. If this happens for A or B, say $A \subseteq A'$ or $B \subseteq B'$, we may proceed with A' instead of A or B' instead of B in the following (any contractible open set of Σ containing A' and B' will also contain A and B). We call the finite open cover thus obtained $\{N_j \mid j = 0, \ldots, n+1\}$, where $n \in \mathbb{N} \smallsetminus \{0\}$, $N_0 := A$, $N_{n+1} := B$. Since the N_j cover $\gamma[0,1]$, which is connected, $\bigcup_{j=0}^{n+1} N_j$ is connected. However, $\bigcup_{j=0}^{n+1} N_j$ may not be contractible just yet. Therefore, we need to discard members of $\{N_j \mid j = 0, \ldots, n+1\}$ until the union of the remaining sets becomes contractible.

We now consider the members of the finite open cover $\{N_j \mid j = 0, \ldots, n+1\}$ for $\gamma[0,1]$ in Σ to be the objects in a small preorder viewed as a small and thin category $\mathcal{J}$. For each $j = 0, \ldots, n+1$, we call the object of $\mathcal{J}$ corresponding to the set N_j quite naturally just j. Whenever $N_j \cap N_k \neq \varnothing$ for $j, k = 0, \ldots, n+1$, we will have one and only one morphism $\mu_{jk} : j \longrightarrow k$. Of course, if $j = k$, we will have the identity morphism $\mu_{jj} = \mathrm{id}_j$. In addition, we have all compositions of morphisms such that the axioms of a category are fulfilled. Our category $\mathcal{J}$ is connected in the sense that for each two

[4] $A \cup B$ is of course endowed with the open smooth submanifold structure induced by Σ.

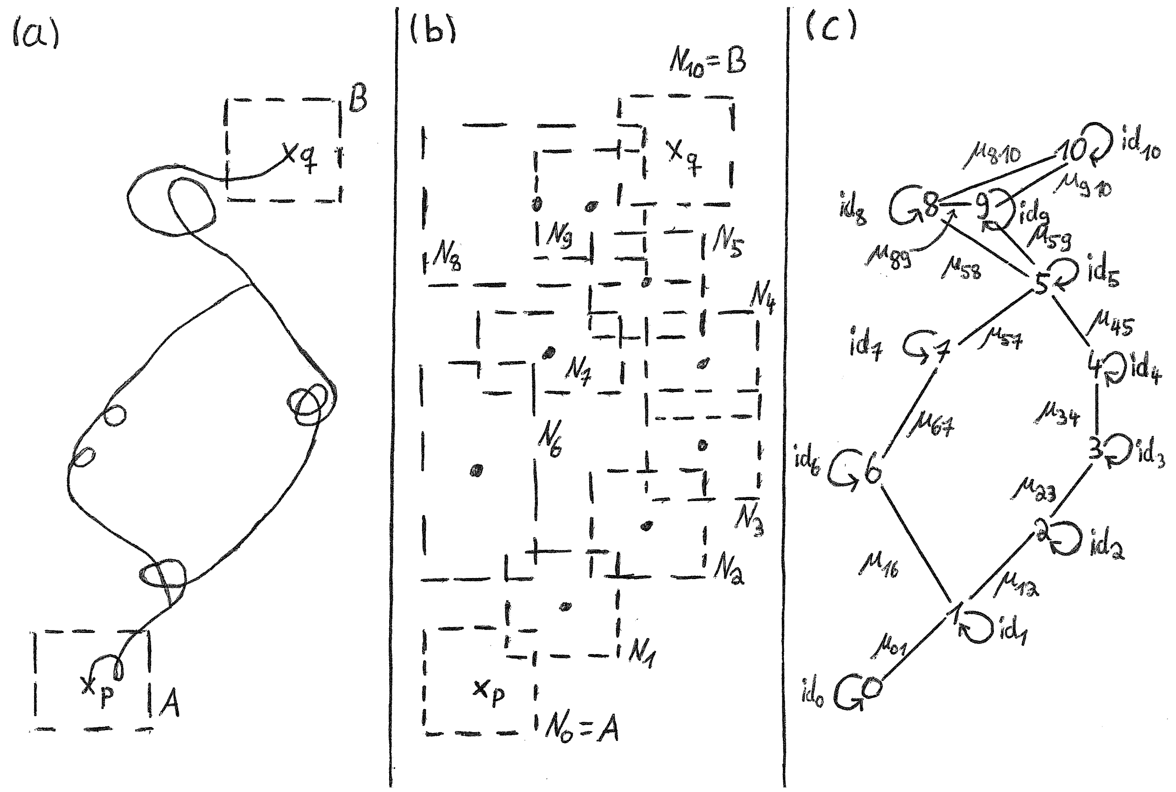


FIGURE 1.2: Visual aid for the proof of Lemma 1.1.6. (a) shows an arbitrary path from $p \in A$ to $q \in B$, (b) some open cover of the path by contractible open subsets with contractible intersection and (c) shows a diagram for the category defined from it.

objects α and ω, there is precisely one morphism $\alpha \longrightarrow \omega$, which can be decomposed into our basic morphisms μ_{jk} in various ways however. If $\mathcal{J}$ was not connected, then $\bigcup_{j=0}^{n+1} N_j$ would not have been connected. Therefore, there exist a unique morphism $0 \longrightarrow n+1$, which can be represented as a finite composition $0 \xrightarrow{\mu_{0j'_1}} j'_1 \xrightarrow{\mu_{j'_1\ldots}} \ldots \xrightarrow{\mu_{\ldots j'_m}} j'_m \xrightarrow{\mu_{j'_m n+1}} n+1$ for some $m \in \mathbb{N} \setminus \{0\}$. From this composition, we throw out all identity morphisms; also the ones that are obtained by composition are thrown out. We are hence left with a compositions of the form $0 \xrightarrow{\mu_{0j_1}} j_1 \xrightarrow{\mu_{j_1\ldots}} \ldots \xrightarrow{\mu_{\ldots j_t}} j_t \xrightarrow{\mu_{j_t n+1}} n+1$ with the followig properties: $1 \leq t \leq n$ and $j_r \neq j_s$ for all $r, s = 1, \ldots, t$ with $r \neq s$. We reduce this composition further by applying the following algorithm, starting with $s = 0$:

- **STEP s:** is $s+1 = \max\{r \in \{s+1, \ldots, t, n+1\} \mid N_{j_r} \cap N_{j_s} \neq \varnothing\}$?
  - ▶ **YES:** proceed to **STEP s+1**.
  - ▶ **NO:** set $s' := \max\{r \in \{s+1, \ldots, t, n+1\} \mid N_{j_r} \cap N_{j_s} \neq \varnothing\}$, replace the current composition under consideration by

$$0 \xrightarrow{\mu_{0j_1}} j_1 \xrightarrow{\mu_{j_1\ldots}} \ldots \xrightarrow{\mu_{\ldots j_s}} j_s \xrightarrow{\mu_{j_s j_{s'}}} j_{s'} \xrightarrow{\mu_{j_{s'}\ldots}} \ldots \xrightarrow{\mu_{\ldots j_t}} j_t \xrightarrow{\mu_{j_t n+1}} n+1 \tag{1.11}$$

and proceed to **STEP s′+1**.

This algorithm ends after finitely many steps and yields a composition of the form $0 \xrightarrow{\mu_{0k_1}} k_1 \xrightarrow{\mu_{k_1\ldots}} \ldots \xrightarrow{\mu_{\ldots k_{t'}}} k_{t'} \xrightarrow{\mu_{k_{t'}n+1}} n+1$ with the properties: $1 \leq t' \leq n$, $k_r \neq k_s$ for all $r, s = 1, \ldots, t'$ with $r \neq s$ and $N_{k_r} \cap N_{k_s} \neq \varnothing$ if and only if $k_r = k_{s-1}, k_{s+1}$, where we set $k_0 := 0$ and $k_{t'+1} = n+1$.

To summarise, we have thus constructed finitely many contractible open subsets $N_{k_0}, \ldots, N_{k_{t'+1}}$ of Σ which meet the requisites of Lemma 1.1.5(ii) and whose union $\bigcup_{r=0}^{t'+1} N_{k_r}$ contains A and B. Accordingly, the Cauchy development $D\left(\bigcup_{r=0}^{t'+1} N_{k_r}\right)$ of $\bigcup_{r=0}^{t'+1} N_{k_r}$ in $\mathbf{M}$ is a contractible globally hyperbolic open subset of $\mathbf{M}$ containing both $D(A)$ and $D(B)$. ❒

Chapter 2

Category Theory

> KIRK: *Khan, you bloodsucker! You're gonna have to do your own dirty work now! Do you hear me? Do you?*
> KHAN: *Kirk. Kirk! You're still alive, my old friend!*
> KIRK: *Still -'old friend!' You've managed to kill just about everyone else, but like a poor marksman, you keep missing the target!*
> KHAN: *Perhaps I no longer need to try, Admiral.*
> KIRK: *Khan...Khan, you've got Genesis, but you don't have me! You were going to kill me, Khan; you're going to have to come down here! You're going to have to come down here!*
> KHAN: *I've done far worse than kill you. I've hurt you. And I wish to go on...hurting you. I shall leave you as you left me. As you left her. Marooned for all eternity in the center of a dead planet...buried alive...buried alive.*
> KIRK: *KHHHHAAAAAAAAAN!!! KHHHHAAAAAAAAAN!!!*

–Star Trek II: The Wrath of Khan (1982)

Without any doubt, physics as well as mathematics have become extremely diverse and specialised since their very beginnings. Despite the great variety of disciplines in each of these two subjects, there is a great number of similar ideas, concepts and constructions used throughout. Prime examples of these similarities are *universal constructions.*

For example, consider the following two well-known constructions in mathematics (see e.g. Section 1.16 and Section 5.4 of [Gre67] or [II, §1, no.3] and [III, §1, no.2] of [Bou89]): let K be a field, X a vector space (resp. algebra) over K and W a linear subspace of (resp. two-sided ideal in) X. We define an equivalence relation $\sim$ on X by $x \sim y :\Longleftrightarrow x - y \in W$ for $x, y \in X$ and consider the equivalence classes $[x]$ for $x \in X$ with respect to $\sim$, which are just the cosets of W in X with respect to the addition, i.e. $[x] = x + W$ for all $x \in X$. On the set $X/W := \{[x] \mid x \in X\}$ there is one and only one structure as a vector space (resp. algebra) over K such that the map $\pi : X \longrightarrow X/W$ defined by $\pi(x) := [x]$ for all $x \in X$ becomes a linear map (resp. algebra homomorphism). This unique structure is given by

$$(2.1)\qquad [x]+[y]=[x+y] \qquad\text{and}\qquad k\,[x]=[kx]\,, \qquad \forall k\in K,\ \forall x,y\in X,$$

and furthermore in the case of algebras,

$$(2.2)\qquad [x]\,[y]=[xy]\,, \qquad \forall x,y\in X.$$

From this it follows $0_{X/W}=[0_X]$ and in the case of unital algebras, $1_{X/W}=[1_X]$ (if $1_X\in W$, then $X/W=\{[0_X]\}$, which is a unital algebra). If X is an associative or commutative algebra, then so is X/W. X/W is called the *quotient* (also sometimes *factor* or *difference*) *vector space* (resp. *algebra*) of X by W and $\pi : X \longrightarrow X/W$, which is surjective, is called the *canonical projection*. The pair $(X/W,\pi)$ has the following integral property (see e.g. [Gre67, Sec.2.3] or [I, §8, no.7, Thm.2] and [II, §1, no.3][1] of [Bou89]), which is called *universal*:

(UQ') Let Y be a vector space (resp. algebra) over K and $f : X \longrightarrow Y$ a linear map (resp. algebra homomorphism) such that $\ker f\subseteq W$. Then there exists one and only one linear map (resp. algebra homomorphism) $[f] : X/W \longrightarrow Y$ that makes the diagram

$$(2.3)\qquad \begin{array}{ccc} X & & \\ {\scriptstyle\pi}\downarrow & \searrow^{f} & \\ X/W & \overset{\exists!\,[f]}{\dashrightarrow} & Y \end{array}$$

commutative, i.e. $[f]\circ\pi=f$.

One can easily take up the position that the compliance with the universal property (UQ') is really all that should matter to the construction of the quotient vector space (resp. algebra) as (UQ') tells us already everything we need to know about its behaviour. In this spirit, the idea behind a universal construction is precisely to characterise a mathematical object by means of a universal property such as (UQ'). In the example above, this point of view leads to: a *quotient* of X by W is any pair (Q,q) consisting of a vector space (resp. algebra) over K and a linear map (resp. algebra homomorphism) $q : X \longrightarrow Q$ such that (Q,q) meets the universal property

[1] Note that it reads in [Bou89, III, §1, no.2]: *More generally, all the results of Chapter I, §8, no.9 are still valid (and also their proofs) when the word "ring" is replaced by "algebra"*. However, given the context it is safe to assume that Chapter I, §8, no.7, was actually meant. Anyway, even if this was not a typo, (UQ') is still fulfilled as can be checked without too much effort.

(UQ'). Q is called a *quotient vector space* (resp. *algebra*) of X by W and q is called the *canonical projection* onto Q.

However, by shifting the focus from an explicit construction to a universal property, we may (and in all cases considered in this thesis will) lose uniqueness in the strongest sense possible, that is, in the sense of equality. A mathematical object which is described by a universal property is not necessarily uniquely determined anymore. Instead, a universal property characterises a mathematical object only up to unique isomorphism, which is the next best thing to uniqueness in the strict sense. There is a standard proof for the uniqueness (up to unique isomorphism) of a mathematical object characterised by a universal property and it will only be given for colimits in the proof of Lemma 2.2.8 in order to avoid needless repetition. We have in the example above: if $(Q, q : X \longrightarrow Q)$ and $(Q', q' : X \longrightarrow Q')$ are two quotients of X by W, then there is one and only one bijective linear map (resp. algebra isomorphism) $f : Q \xrightarrow{\sim} Q'$ such that the diagram

$$\begin{array}{ccc} & X & \\ {}^{q}\swarrow & & \searrow^{q'} \\ Q & \overset{\exists! \sim f}{\dashrightarrow} & Q' \end{array} \tag{2.4}$$

becomes commutative, i.e. $f \circ q = q'$.

We will call any mathematical object which is obtained from given ones by means of a universal property a *universal construction*. Universal constructions will take the centre stage throughout and to detect them and to utilise their universal properties in algebraic and locally covariant quantum field theory is a stated aim of this thesis.

Category theory provides a mathematically precise and universal language to understand universal constructions and other similarities between different areas of mathematics as facets of one and the same categorical notion. The formalism of category theory highlights the underlying structures of mathematical constructs and their common ground, thus interrelating different disciplines of mathematics on a fundamental level and granting a deeper insight into mathematical structures in general. Using the language of category theory can therefore help to clarify and simplify mathematical statements and, sometimes, their proofs as well. Furthermore, category theory can allow us to relate a problem in one branch of mathematics to a problem in another branch, which might be much simpler to solve. However, it also happens that the additional level of abstraction brought in by category theory is not helpful at all in solving a concrete problem.

Although category theory appears everywhere and many physical concepts and ideas can be formulated in a categorical setting, it still came as a bit of a suprise –at least to me– that this fundamental branch of mathematics has proven so fruitful for algebraic quantum field theory in curved spacetimes, in particular by the means of the functorial framework of locally covariant quantum field theory, which will be the topic of Section 3.2.

This gives us abundant reason to study some of the basics of category theory in this thesis. Unfortunately, an introductory presentation of all the categorical notions used in this thesis and illustrating them by examples would be excessive and unreasonable. Fortunately, the literature on category theory is very good so that we can refer the reader to it with a clear conscience. Our main sources for category theory are [AHS04; Bor94; Mac98; Par70]. In particular, the reader can consult these references for the very basic notions of category theory which we will use in this thesis without further review such as: category, large, small, thin, up-directed, terminal object, subcategory, full subcategory, monic, epic, isomorphism, (covariant) functor, faithful, full, natural transformation and natural isomorphism. Following standard conventions, we will use the symbol “ $\hookrightarrow$ ” to denote monics in a category, “ $\twoheadrightarrow$ ” for epics and “ $\xrightarrow{\sim}$ ” for isomorphisms. Other categorical notions such as skeletons of categories, left and right adjoint functors and equivalences of categories will be introduced where needed.

We will always consider the axioms of a category to be realised within set theory. For our purposes, it does not make any difference at all whether we base our category theory on **ZFC** plus a universe of sets[2] (which allows us to speak of proper classes), on the axioms of universes of sets ([Bor94, Sec.1.1], [Mac98, Sec.I.6]), which realise **ZFC** and also allow us to formally speak of proper classes, or on **NGB** [Par70, Appendix] which directly axiomatises classes and defines proper classes[3].

The outline of this chapter goes as follows: in Section 2.1, we supply a list of all categories which will be considered in this thesis. The main topics of Section 2.2 are colimits and left Kan extensions. We will also prove various technical statements, which will be of great use in concrete computations of colimits and left Kan extensions later in this thesis. The categorical notions of equalisers, intersections and unions of subobjects, which are important to the formulation of the dynamical net and dynamical locality in Chapter 5, are presented in Section 2.3. Finally, in the appendix of this chapter, we have collected a few concrete universal constructions, which are used frequently in this thesis. Some of these provide important examples of coequalisers and coproducts.

[2]In **ZFC**, the notion of a proper class, that is, a class which is not a set, stays informal as it is not directly referred to by the axioms. A universe of sets is in this context a model for **ZFC**, i.e. a collection (class) of sets within which the axioms of **ZFC** are realised e.g. the von Neumann universe (also known as the cumulative hierachy) or Gödel's constructible universe. Every collection (class) of sets which itself does not belong to the universe is a proper class.

[3]A class is a set if and only if it is an element of another class, i.e. every set is a class. If a class is not a set, i.e. not an element of another class, it is called a proper class.

2.1 Categories of interest

We collect all categories which will participate in this thesis. They can be basically divided into two kinds: categories of spacetimes and algebraic categories of vector spaces or algebras.

We begin with our spacetime categories:

• **Loc**:[4] $\mathbf{M} = (M, g, [T], [\Omega]) \in \mathbf{Loc}$ if and only if $\mathbf{M}$ is an oriented globally hyperbolic spacetime (in particular: M is connected) of a fixed dimension $m \geq 2$; for $\mathbf{M}, \mathbf{N} \in \mathbf{Loc}$, $\psi \in \mathbf{Loc}(\mathbf{M}, \mathbf{N})$ if and only if $\psi : \mathbf{M} \longrightarrow \mathbf{M}$ is an isometric smooth embedding that preserves the time-orientation and the orientation, and whose image $\psi(M)$ is causally convex in $\mathbf{M}$ (preservation of the causal structure). Most of our results will hold for a general $m \geq 2$, though we will usually think of the physical case $m = 4$. If a result depends on m, we will make this explicit.

For the following topologically restricted subcategories of **Loc**, which are denoted by $\mathbf{Loc}_q$, $\mathrm{loc}^q_{+\mathbf{M}}$, $\mathrm{loc}^q_{-\mathbf{M}}$, loc^q_{+M} and loc^q_{-M}, q is always a subset of the natural numbers without zero (resp. $q = s$ for "*simply connected*", $q = ©$ for "*contractible*"). Taking $q = \varnothing$ is omitting q by definition.

• $\mathbf{Loc}_q$: the full subcategory of **Loc** specified by the rule $\mathbf{M} \in \mathbf{Loc}_q$ if and only if $\mathbf{M} \in \mathbf{Loc}$ and $H^p_{\mathrm{dR}} M = 0$ for all $p \in q$ (resp. M is simply connected, M is contractible).

• $\mathrm{loc}^q_{+\mathbf{M}}$: for $\mathbf{M} \in \mathbf{Loc}$, define the poset (viewed as a category) $\mathrm{loc}^q_{+\mathbf{M}}$ by $O \in \mathrm{loc}^q_{+\mathbf{M}}$ if and only if $O \subseteq M$ is a connected globally hyperbolic open subset of $\mathbf{M}$ (taking $O = M$ is allowed!) and $H^p_{\mathrm{dR}} O = 0$ for all $p \in q$ (resp. O is simply connected, O is contractible);

$$\mathrm{loc}^q_{+\mathbf{M}}(U, V) := \begin{cases} \iota_{UV} : U \hookrightarrow V \text{ (inclusion map)} & \text{if } U \subseteq V \text{ but } U \neq V, \\ \iota_{UV} = \mathrm{id}_U : U \longrightarrow U \text{ (identity map)} & \text{if } U = V, \\ \varnothing & \text{if } U \nsubseteq V, \end{cases} \tag{2.5}$$

$$U, V \in \mathrm{loc}^q_{+\mathbf{M}}.$$

Note, if $H^p_{\mathrm{dR}} M = 0$ for all $p \in q$ (resp. M is simply connected, M is contractible), M becomes the terminal object in $\mathrm{loc}^q_{+\mathbf{M}}$. We can view $\mathrm{loc}^q_{+\mathbf{M}}$ as a subcategory of **Loc** and $\mathbf{Loc}_q$ by identifying $O \in \mathrm{loc}^q_{+\mathbf{M}}$ with $\mathbf{M}|_O = (O, g|_O, [T|_O], [\Omega|_O]) \in \mathbf{Loc}$, which is O equipped with the structures induced by $\mathbf{M}$ and viewed as an oriented globally

[4] We express that X is an object of a category $\mathcal{C}$ by writing $X \in \mathcal{C}$; the hom-sets of $\mathcal{C}$ will be denoted by $\mathcal{C}(X, Y)$. Other common notations are $|\mathcal{C}|$ or $\mathrm{Obj}(\mathcal{C})$ for the class of all $\mathcal{C}$-objects, $\mathrm{arr}_{\mathcal{C}}(X, Y)$, $\mathrm{hom}_{\mathcal{C}}(X, Y)$ or $\mathrm{mor}_{\mathcal{C}}(X, Y)$ for the hom-sets of $\mathcal{C}$ and $\mathrm{Arr}(\mathcal{C})$, $\mathrm{Hom}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ for the class of all $\mathcal{C}$-morphisms.

hyperbolic spacetime in its own right, and regarding the identity and inclusion maps as **Loc**-morphisms. We will adopt this identification from now on without further mention.

• $\mathrm{loc}^q_{-\mathbf{M}}$: for $\mathbf{M} \in \mathbf{Loc}$, define a full subcategory $\mathrm{loc}^q_{-\mathbf{M}}$ of $\mathrm{loc}^q_{+\mathbf{M}}$ by $O \in \mathrm{loc}^q_{-\mathbf{M}}$ if and only if $O \subseteq M$ is a connected globally hyperbolic open subset of $\mathbf{M}$ (taking $O = M$ is now excluded!) and $H^p_{\mathrm{dR}}O = 0$ for all $p \in q$ (resp. O is simply connected, O is contractible). The category $\mathrm{loc}^q_{-\mathbf{M}}$ is a poset but not necessarily up-directed and can also be regarded as a subcategory of **Loc** and $\mathbf{Loc}_q$ by the same identification as before.

• loc^q_{+M}: let M be the underlying smooth manifold of some **Loc**-object $\mathbf{M}$; we define a subcategory loc^q_{+M} of **Loc** and $\mathbf{Loc}_q$ by $\mathbf{N} \in \mathrm{loc}^q_{+M}$ if and only if $\mathbf{N} \in \mathbf{Loc}$, $N \subseteq M$ and $H^p_{\mathrm{dR}}N = 0$ for all $p \in q$ (resp. N is simply connected, N is contractible). For $\mathbf{N}, \mathbf{N}' \in \mathrm{loc}^q_{+M}$, $\psi \in \mathrm{loc}^q_{+M}(\mathbf{N}, \mathbf{N}')$ if and only if $\psi \in \mathbf{Loc}(\mathbf{N}, \mathbf{N}')$ and the underlying set map of ψ is the inclusion map $\iota_{NN'} : N \hookrightarrow N'$ or the identity map $\mathrm{id}_N : N \longrightarrow N$. Thus, loc^q_{+M} is thin and hence a poset (viewed as a category) but not up-directed.

• loc^q_{-M}: let M be the underlying smooth manifold of some **Loc**-object $\mathbf{M}$; we define a full subcategory loc^q_{-M} of loc^q_{+M} by $\mathbf{N} \in \mathrm{loc}^q_{-M}$ if and only if $\mathbf{N} \in \mathbf{Loc}$, $N \subseteq M$, $N \neq M$ and $H^p_{\mathrm{dR}}N = 0$ for all $p \in q$ (resp. N is simply connected, N is contractible).

We now introduce our algebraic categories of algebras and vector spaces.

• **(C)*Alg**: $A \in$ **(C)*Alg** if and only if A is a $(C)^*$-algebra (in particular, A is an associative algebra over $\mathbb{C}$); for $A, B \in$ **(C)*Alg**, $\varphi \in$ **(C)*Alg** (A, B) if and only if $\varphi : A \longrightarrow B$ is a *-homomorphism.

• $\mathbf{(C)^*Alg_1}$: the subcategory of **(C)*Alg** with $A \in \mathbf{(C)^*Alg_1}$ if and only if A is a unital $(C)^*$-algebra; for $A, B \in \mathbf{(C)^*Alg_1}$, $\varphi \in \mathbf{(C)^*Alg_1}(A, B)$ if and only if $\varphi : A \longrightarrow B$ is a unital *-homomorphism.

• $\mathbf{(C)^*Alg^m}$: the subcategory of **(C)*Alg** such that $A \in \mathbf{(C)^*Alg^m}$ if and only if A is a $(C)^*$-algebra; for $A, B \in \mathbf{(C)^*Alg^m}$, $\varphi \in \mathbf{(C)^*Alg^m}(A, B)$ if and only if $\varphi : A \longrightarrow B$ is a *-monomorphism.

• $\mathbf{(C)^*Alg_1^m}$: the subcategory of **(C)*Alg**, $\mathbf{(C)^*Alg_1}$ and $\mathbf{(C)^*Alg^m}$ which is defined by $A \in \mathbf{(C)^*Alg_1^m}$ if and only if A is a unital $(C)^*$-algebra; for $A, B \in \mathbf{(C)^*Alg_1^m}$, $\varphi \in \mathbf{(C)^*Alg_1^m}(A, B)$ if and only if $\varphi : A \longrightarrow B$ is a unital *-monomorphism.

• $\mathbf{Vec}_\mathrm{K}$: let K be a field, then $V \in \mathbf{Vec}_\mathrm{K}$ if and only if V is a vector space over K; for $V, W \in \mathbf{Vec}_\mathrm{K}$, $f \in \mathbf{Vec}_\mathrm{K}(V, W)$ if and only if $f : V \longrightarrow W$ is a linear map.

• **CVec**: $(V,C) \in \mathbf{CVec}$ if and only if (V,C) is a *C-vector space*, i.e. V is a complex vector space and $C : V \longrightarrow V$ is a *C-involution*, i.e. C is complex-conjugate linear and $C \circ C = \mathrm{id}_V$; $f \in \mathbf{CVec}\left((V,C_V),(W,C_W)\right)$ if and only if $f : (V,C_V) \longrightarrow (W,C_W)$ is a *C-homomorphism*, that is, $f : V \longrightarrow W$ is a (complex) linear map and $f \circ C_V = C_W \circ f$ as complex-conjugate linear maps.

• $\mathbf{pSympl}_{\mathbb{R}}$: $(V,\omega) \in \mathbf{pSympl}_{\mathbb{R}}$ if and only if (V,ω) is a *pre-symplectic space*, i.e. V is a real vector space and ω is a *pre-symplectic form*, i.e. a (possibly degenerate) skew-symmetric real bilinear form on V; for $(V,\omega_V),(W,\omega_W) \in \mathbf{pSympl}_{\mathbb{R}}$, we have $f \in \mathbf{pSympl}_{\mathbb{R}}\left((V,\omega_V),(W,\omega_W)\right)$ if and only if $f : V \longrightarrow W$ is a (real) linear map which is *symplectic*, i.e. $\omega_W \circ (f \times f) = \omega_V$.

• $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}}$: the subcategory of $\mathbf{pSympl}_{\mathbb{R}}$ defined by $(V,\omega) \in \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}}$ if and only if $(V,\omega) \in \mathbf{pSympl}_{\mathbb{R}}$; for $(V,\omega_V),(W,\omega_W) \in \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}}$, $f \in \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}}\left((V,\omega_V),(W,\omega_W)\right)$ if and only if $f \in \mathbf{pSympl}_{\mathbb{R}}\left((V,\omega_V),(W,\omega_W)\right)$ and f is injective.

• $\mathbf{Sympl}_{\mathbb{R}}$: the full subcategory of $\mathbf{pSympl}_{\mathbb{R}}$ and subcategory of $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}}$ given by $(V,\omega) \in \mathbf{Sympl}_{\mathbb{R}}$ if and only if (V,ω) is a *symplectic space*, that is, $(V,\omega) \in \mathbf{pSympl}_{\mathbb{R}}$ and ω is a *symplectic form* on V, i.e. ω is a weakly non-degenerate pre-symplectic form.

• $\mathbf{pSympl}_{\mathbb{C}}$: $(V,\omega,C) \in \mathbf{pSympl}_{\mathbb{C}}$ if and only if (V,ω,C) is a *complexified pre-symplectic space*, i.e. (V,C) is a C-vector space and ω is a *com*plexified pre-symplectic form on (V,C), i.e. ω is a (possibly degenerate) skew-symmetric complex bilinear form on V such that $\omega \circ (C \times C) = \overline{} \circ \omega$, where "$\overline{}$" denotes the complex conjugation; for $(V,\omega_V,C_V),(W,\omega_W,C_W) \in \mathbf{pSympl}_{\mathbb{C}}$, $f \in \mathbf{pSympl}_{\mathbb{C}}\left((V,\omega_V,C_V),(W,\omega_W,C_W)\right)$ if and only if $f : (V,C_V) \longrightarrow (W,C_W)$ is a symplectic C-homomorphism.

• $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}$: the subcategory of $\mathbf{pSympl}_{\mathbb{C}}$ given by $(V,\omega,C) \in \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}$ if and only if $(V,\omega,C) \in \mathbf{pSympl}_{\mathbb{C}}$; for $(V,\omega_V,C_V),(W,\omega_W,C_W) \in \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}$, we define that $f \in \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}\left((V,\omega_V,C_V),(W,\omega_W,C_W)\right)$ if and only if, $f \in \mathbf{pSympl}_{\mathbb{C}}\left((V,\omega_V,C_V),(W,\omega_W,C_W)\right)$ and f is injective.

• $\mathbf{Sympl}_{\mathbb{C}}$: the full subcategory of $\mathbf{pSympl}_{\mathbb{C}}$ and subcategory of $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}$ given by $(V,\omega,C) \in \mathbf{Sympl}_{\mathbb{C}}$ if and only if (V,ω,C) is a *complexified sym*plectic space, that is, $(V,\omega,C) \in \mathbf{pSympl}_{\mathbb{C}}$ and ω is a *complexified sym*plectic form on (V,C), i.e. ω is a weakly non-degenerate complexified pre-symplectic form.

The following proposition clarifies the use of the terminology "*complexified*" in the definitions of $\mathbf{pSympl}_{\mathbb{C}}$, $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}$ and $\mathbf{Sympl}_{\mathbb{C}}$. Its importance lies in the application to the computation of colimits, where it will allows us to restrict to the case $\mathbb{K} = \mathbb{C}$ by

Lemma 2.2.14. As a further result, it enables us to prove the existence of quotients and direct sums in **CVec** on abstract categorical grounds, see the appendix. First, a definition:

DEFINITION 2.1.1. Two categories $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent* if and only if there exist functors $F:\mathcal{C}\longrightarrow\mathcal{D}$, $G:\mathcal{D}\longrightarrow\mathcal{C}$ and natural isomorphisms $\eta:\mathrm{Id}_{\mathcal{D}}\overset{\sim}{\dot{\longrightarrow}}F\circ G$, $\varepsilon:G\circ F\overset{\sim}{\dot{\longrightarrow}}\mathrm{Id}_{\mathcal{C}}$. Such functors F and G are called *equivalences*.

PROPOSITION AND DEFINITION 2.1.2. (a) $\mathbf{Vec}_{\mathbb{R}}$ *and* **CVec** *are equivalent categories.* (b) $\mathbf{pSympl}_{\mathbb{R}}$ *and* $\mathbf{pSympl}_{\mathbb{C}}$ *are equivalent categories,* $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}}$ *and* $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}$ *are equivalent categories, and* $\mathbf{Sympl}_{\mathbb{R}}$ *and* $\mathbf{Sympl}_{\mathbb{C}}$ *are equivalent categories. We call the corresponding equivalence* $\mathscr{C}:\mathcal{C}\longrightarrow\mathcal{D}$ *the* complexification functor*, where* $\mathcal{C}=\mathbf{Vec}_{\mathbb{R}},\mathbf{pSympl}_{\mathbb{R}},\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}},\mathbf{Sympl}_{\mathbb{R}}$ *and* $\mathcal{D}=\mathbf{CVec},\mathbf{pSympl}_{\mathbb{C}},\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}$, $\mathbf{Sympl}_{\mathbb{C}}$.

Proof: (a) We are going to construct a full and faithful functor $\mathscr{C}:\mathbf{Vec}_{\mathbb{R}}\longrightarrow\mathbf{CVec}$ such that each C-vector space (X,C) is C-isomorphic to a C-vector space of the form $\mathscr{C}V$, where V is a real vector space. The result follows then from any one of the references [Par70, Sec.2.1, Prop.3], [Bor94, Def.3.4.4], [Mac98, Thm.IV.4.1], [AHS04, Def.3.33]. For any real vector space V, we define $\mathscr{C}V:=V\oplus V$ and equip $\mathscr{C}V$ with the scalar multiplication

$$\mathbb{C}\times\mathscr{C}V\longrightarrow\mathscr{C}V,\qquad(\lambda,(u,v))\longmapsto(u,v)\begin{pmatrix}\mathrm{Re}\,(\lambda)&\mathrm{Im}\,(\lambda)\\-\mathrm{Im}\,(\lambda)&\mathrm{Re}\,(\lambda)\end{pmatrix},\tag{2.6}$$

and the C-involution

$$C_V:\mathscr{C}V\longrightarrow\mathscr{C}V,\qquad(u,v)\longmapsto(u,-v)\,.\tag{2.7}$$

Thus, $(\mathscr{C}V,C_V)$ becomes a C-vector space which we will denote by just $\mathscr{C}V$.

For the definition of the arrow function of $\mathscr{C}$, let $f:V\longrightarrow W$ be a real linear map. A C-homomorphism $\mathscr{C}f:\mathscr{C}V\longrightarrow\mathscr{C}W$ is defined by $\mathscr{C}f\,(u,v):=(f\,(u)\,,f\,(v))$ for $u,v\in V$. Note, $\mathscr{C}$ preserves injectivity and surjectivity according to this definition. Obviously, $\mathscr{C}\,\mathrm{id}_V=\mathrm{id}_{\mathscr{C}V}$ for all real vector spaces V and $\mathscr{C}\,(g\circ f)=\mathscr{C}g\circ\mathscr{C}f$ whenever $f:V\longrightarrow W$ and $g:W\longrightarrow X$ are real linear maps. Hence $\mathscr{C}$ defines a functor $\mathbf{Vec}_{\mathbb{R}}\longrightarrow\mathbf{CVec}$.

It is clear from the definition that $\mathscr{C}f=\mathscr{C}g$ implies $f=g$ for real linear maps $f,g:V\longrightarrow W$. It is also not difficult to see that each C-homomorphism $h:\mathscr{C}V\longrightarrow\mathscr{C}W$ can be written as $\mathscr{C}f$ for some real linear map $f:V\longrightarrow W$: consider the components $h_1,h_2:\mathscr{C}V\longrightarrow W$ of h, which are real linear maps. Linearity of h entails $-h_2\,(u,v)=h_1\,(-v,u)$ and $h_1\,(u,v)=h_2\,(-v,u)$ for all $u,v\in V$. On the other hand, $h\circ C_V=C_W\circ h$ implies

$h_1(u,-v) = h_1(u,v)$ and $h_2(u,-v) = -h_2(u,v)$ for all $u,v \in V$. We see, $h_1(0_V,v) = 0_W$ and $h_2(v,0_V) = 0_W$ for all $v \in V$. Define now a real linear map $f : V \longrightarrow W$ by $f(v) := h_1(v,0_V)$ for all $v \in V$, then $\mathscr{C}f = h$ as

$$\begin{aligned}
\mathscr{C}f(u,v) &= (f(u), f(v)) && (2.8)\\
&= (h_1(u,0_V), h_1(v,0_V)) && (2.9)\\
&= (h_1(u,0_V) + h_1(0_V,v), h_2(0_V,v)) && (2.10)\\
&= (h_1(u,v), h_2(0_V,v) + h_2(u,0_V)) && (2.11)\\
&= (h_1(u,v), h_2(u,v)) \qquad \forall u,v \in V. && (2.12)
\end{aligned}$$

Hence, $\mathscr{C}$ is full and faithful.

Next, any C-vector space (X,C) is C-isomorphic to a C-vector space of the form $\mathscr{C}V$ for a real vector space V. In order to see this, define $\operatorname{Re}X := \{w \in X \mid Cw = w\}$ and $\operatorname{Im}X := \{w \in X \mid Cw = -w\}$. As real vector spaces, $X = \operatorname{Re}X \oplus \operatorname{Im}X$ because any vector $w \in X$ can be written as $w = \frac{1}{2}(w + Cw) + \frac{1}{2}(w - Cw)$, where it holds that $C(w + Cw) = w + Cw$ and $C(w - Cw) = -(w - Cw)$. On these grounds, we define $V := \operatorname{Re}X$ and $f : \mathscr{C}V \longrightarrow (X,C)$, $(u,v) \longmapsto u + \mathrm{i}\,v$, is easily seen to be a C-isomorphism.

(b) Let $\mathcal{C}$ stand for one **pSympl**, **pSympl^m** and **Sympl**. As in (a), we are going to construct a full and faithful functor $\mathscr{C} : \mathcal{C}_{\mathbb{R}} \longrightarrow \mathcal{C}_{\mathbb{C}}$ such that each $\mathcal{C}_{\mathbb{C}}$-object (X,ω,C) is $\mathcal{C}_{\mathbb{C}}$-isomorphic to a $\mathcal{C}_{\mathbb{C}}$-object of the form $\mathscr{C}(V,\omega_V)$ for a $\mathcal{C}_{\mathbb{R}}$-object (V,ω_V). To this end, we take the equivalence $\mathscr{C} : \mathbf{Vec}_{\mathbb{R}} \longrightarrow \mathbf{CVec}$ as in (a) and define furthermore for any $\mathcal{C}_{\mathbb{R}}$-object (V,ω_V) a skew-symmetric complex bilinear form $\mathscr{C}\omega_V$ on $\mathscr{C}V$ by

$$\mathscr{C}\omega_V((u,v),(u',v')) := \omega_V(u,u') - \omega_V(v,v') + \mathrm{i}\,\omega_V(u,v') + \mathrm{i}\,\omega_V(v,u'), \quad u,u',v,v' \in V. \tag{2.13}$$

Evidently, $\mathscr{C}\omega_V \circ (C_V \times C_V) = \overline{} \circ \mathscr{C}\omega_V$ and if ω_V is weakly non-degenerate, then so is $\mathscr{C}\omega_V$. In conclusion, the triple $\mathscr{C}(V,\omega_V) := (\mathscr{C}V, \mathscr{C}\omega_V, C_V)$ is a $\mathcal{C}_{\mathbb{C}}$-object and for any $\mathcal{C}_{\mathbb{R}}$-morphism $f : (V,\omega_V) \longrightarrow (W,\omega_W)$, $\mathscr{C}f : \mathscr{C}V \longrightarrow \mathscr{C}W$ is seen to be symplectic (recall from (a) that $\mathscr{C}f$ is injective if f is injective). Hence, we have well-defined a functor $\mathscr{C} : \mathcal{C}_{\mathbb{R}} \longrightarrow \mathcal{C}_{\mathbb{C}}$.

The fullness and faithfullness of $\mathscr{C}$ follow in the same way as in (a) but with some slight additions: let $(V,\omega_V), (W,\omega_W) \in \mathcal{C}_{\mathbb{R}}$ and $h : \mathscr{C}(V,\omega_V) \longrightarrow \mathscr{C}(W,\omega_W)$ a $\mathcal{C}_{\mathbb{C}}$-morphism, then the injectivity of h implies the injectivity of $f : V \longrightarrow W$ defined as in (a) via the first component of h. Assume f was not injective, say $f(v) = 0_W$ for $0_V \neq v \in V$; then $\mathscr{C}f(v,v) = h(v,v) = 0_{\mathscr{C}W}$ $\lightning$. Note that $\mathscr{C}\omega_W \circ (h \times h) = \mathscr{C}\omega_V$ implies $\omega_W(h_1(u,0_V), h_1(v,0_V)) = \omega_V(u,v)$ for all $u,v \in V$, hence f is symplectic.

Consider now any $\mathcal{C}_{\mathbb{C}}$-object (X,ω,C). Restricting ω to the real vector space $\operatorname{Re}X$

as defined in (a) yields a pre-symplectic space $(\operatorname{Re} X, \eta)$ since

$$\eta(u,v) = \omega(u,v) = \omega(Cu,Cv) = \overline{\omega(u,v)} = \overline{\eta(u,v)} \in \mathbb{R} \qquad \forall u,v \in \operatorname{Re} X. \tag{2.14}$$

If ω is weakly non-degenerate, η will be so too. Suppose it was not; then there is a $v \in \operatorname{Re} X$ with $\eta(u,v) = 0_{\mathbb{R}}$ for all $u \in \operatorname{Re} X$, and

$$\omega(u,v) = \omega\left(\frac{1}{2}(u+Cu) - \mathrm{i}\,\frac{\mathrm{i}}{2}(u-Cu), v\right) \tag{2.15}$$

$$= \omega\Big(\frac{1}{2}\underbrace{(u+Cu)}_{\in \operatorname{Re} X}, v\Big) - \mathrm{i}\,\omega\Big(\underbrace{\frac{\mathrm{i}}{2}(u-Cu)}_{\in \operatorname{Re} X}, v\Big) \tag{2.16}$$

$$= \eta\left(\frac{1}{2}(u+Cu), v\right) - \mathrm{i}\,\eta\left(\frac{\mathrm{i}}{2}(u-Cu), v\right) \tag{2.17}$$

$$= 0_{\mathbb{C}} \qquad \forall u \in X \ \lightning. \tag{2.18}$$

It is now not difficult to see that the map $\mathscr{C}(\operatorname{Re} X, \eta) \longrightarrow (X, \omega, C)$, $(u,v) \longmapsto u + \mathrm{i}\, v$, is a symplectic C-isomorphism. ❒

2.2 Colimits and left Kan extensions

We introduce the categorical notions of colimits and left Kan extension, which play chief parts in this thesis, and prove some helpful technical results regarding their computation.

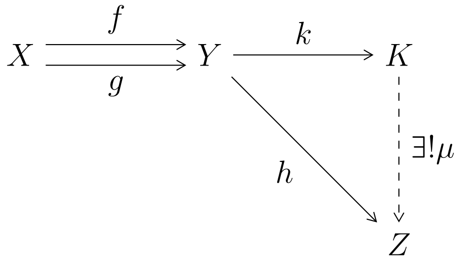
Before we get to the discussion of colimits, we will take a closer look at some of its variants, to be specific: coequalisers and coproducts. This line of action is motivated by the dual statements of [Par70, Sec.2.6, Prop.2], [Bor94, Thm.2.8.1], [Mac98, Thm.V.2.1] and [AHS04, Thm.12.3], where colimits are constructed from coequalisers and coproducts. To see how coequalisers and coproducts can be regarded as colimits, we refer the reader to [Par70, Sec.2.6], [Bor94, Example 2.6.7.d] and [AHS04, Examples 11.28(1) + (2)].

2.2.1 Coequalisers

DEFINITION 2.2.1. Let $\mathcal{C}$ be a category and $f, g : X \longrightarrow Y$ a pair of parallel morphisms. A *coequaliser* or *difference cokernel* for f and g is a morphism $k : Y \longrightarrow K$ which satisfies $k \circ f = k \circ g$ and the following universal property:

(UCoeq) For each morphism $h : Y \longrightarrow Z$ satisfying $h \circ f = h \circ g$, there exists a unique morphism $\mu : K \longrightarrow Z$ such that the diagram

(2.19)



commutes, i.e. h uniquely factorises through k as $h = \mu \circ k$.

If it exists, a coequaliser for a pair of parallel morphisms is always epic and unique up to unique isomorphism by the universal property (UCoeq) (see the dual statements of [Par70, Sec.1.9, Lem.1 + 2] or the dual statements of [Bor94, Prop.2.4.2 + 2.4.3]). We will thus speak of *the* coequaliser. The fact that coequalisers are always epic poses a serious obstruction to the existence of coequalisers in categories whose morphisms are all taken to be monics (see the following counter-examples).

EXAMPLE 2.2.2. (a) In the category $\mathbf{Vec}_{\mathbb{K}}$ (resp. $\mathbf{CVec}$), the coequaliser of a pair of linear maps (resp. C-homomorphisms) $f, g : X \longrightarrow Y$ is the canonical projection $\pi : X \twoheadrightarrow Y/W$ onto the quotient, where $W := \{f(x) - g(x) \mid x \in X\} \subseteq Y$. See the appendix of this chapter for quotients of C-vector spaces.

(b) For the categories $\mathbf{{}^*Alg}$, $\mathbf{{}^*Alg_1}$, $\mathbf{C^*Alg}$ and $\mathbf{C^*Alg_1}$, the coequaliser of a pair of (unital) *-homomorphisms $\varphi, \psi : A \longrightarrow B$ is also given by the canonical projection $\pi : B \twoheadrightarrow B/I$ onto the quotient, where I is the two-sided *-ideal generated by the set $\{\varphi(a) - \psi(a) \mid a \in A\} \subseteq B$. Of course, if C^*-algebras are considered, the norm closure of I has to be taken. For details on quotients of (unital) $(C)^*$-algebras, see again the appendix of this chapter.

COUNTER-EXAMPLE 2.2.3. (a) Suppose $\mathcal{C}$ is any category in which all morphisms are monic and $X \in \mathcal{C}$ has an endomorphism other than the identity, i.e. $\mu \in \operatorname{End} X$ and $\mu \neq \mathrm{id}_X$. Then the pair of parallel morphisms $\mathrm{id}_X, \mu : X \longrightarrow X$ does not have a coequaliser. Indeed, assuming the existence of the coequaliser $k : X \twoheadrightarrow K$ immediately yields a contradiction because of $k \circ \mathrm{id}_X = k \circ \mu$, which implies $\mathrm{id}_X = \mu$ as k is monic $\lightning$.

(b) Counter-Example (a) applies directly to the categories $\mathbf{Sympl}_{\mathbb{K}}$ and $\mathbf{pSympl}_{\mathbb{K}}^{\mathbf{m}}$ by taking X to be any non-trivial (complexified) (pre-)symplectic space (i.e. $\omega_X \neq 0$) and choosing $\mu = -\mathrm{id}_X$. Furthermore, coequalisers do not exist in $\mathbf{pSympl}_{\mathbb{K}}$ in general by the same argument: $k(x) = -k(x)$ for all $x \in X$ implies $k(x) = 0_K$ for all $x \in X$ and hence $\omega_K(k(x), k(y)) = 0_{\mathbb{K}} \neq \omega_X(x, y)$ for some $x, y \in X$, which contradicts k being symplectic.

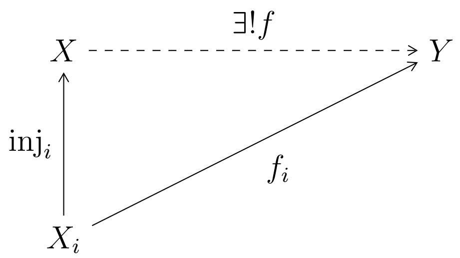
(c) In the categories $\mathbf{{}^*Alg^m}$, $\mathbf{{}^*Alg_1^m}$, $\mathbf{C^*Alg^m}$ and $\mathbf{C^*Alg_1^m}$, let X be the commutative unital C^*-subalgebra of all 2×2-matrices with entries in the complex numbers which is formed by the diagonal 2×2-matrices. The multiplication with complex numbers, the addition and the multiplication of matrices are the standard ones, the *-involution is given by Hermitean conjugation (i.e. complex conjugation and transposition) and the norm is the operator norm. A unital *-homomorphism $\mu : X \longrightarrow X$ which is not equal to the identity is e.g. given by $\mu\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} := \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$ for $a, b \in \mathbb{C}$, and the general argument of (a) applies.

2.2.2 Coproducts

DEFINITION 2.2.4. Let $\mathcal{C}$ be a category, I an arbitrary index set and $\{X_i\}_{i \in I}$ a family of $\mathcal{C}$-objects. A *coproduct* of this family is a pair consisting of a $\mathcal{C}$-object X and a family of $\mathcal{C}$-morphisms $\{\mathrm{inj}_i : X_i \longrightarrow X\}_{i \in I}$, called the *canonical injections*, such that this pair satisfies the universal property:

(U∐) For each $\mathcal{C}$-object Y and family of $\mathcal{C}$-morphisms $\{f_i : X_i \longrightarrow Y\}_{i \in I}$, there is a unique $\mathcal{C}$-morphism $f : X \longrightarrow Y$ satisfying $f \circ \mathrm{inj}_i = f_i$ for all $i \in I$. Diagrammatically speaking, there exists a unique $\mathcal{C}$-morphism $f : X \longrightarrow Y$ making the diagram

(2.20)



commutative for all $i \in I$.

If a coproduct for a family of objects in a category exists, it will be unique up to unique isomorphism due to (U∐) (see the dual statement of [Par70, Sec.1.11, Lem.1], [Bor94, Prop.2.2.2] or the dual statement of [AHS04, Prop.10.22]). In this sense, we will speak of *the* coproduct.

EXAMPLE 2.2.5. The coproduct in the categories $\mathbf{Vec_K}$ and $\mathbf{CVec}$ is just the direct sum; in the categories $\mathbf{{}^*Alg}$, $\mathbf{{}^*Alg_1}$, $\mathbf{C^*Alg}$ and $\mathbf{C^*Alg_1}$, the coproduct is the free product. The notions of the direct sum of C-vector spaces and of the free product of (unital) $(C)^*$-algebras can be found in the appendix to this chapter.

COUNTER-EXAMPLE 2.2.6. (a) Coproducts are not well-behaved in categories whose morphisms are all monics. Suppose $\mathcal{C}$ is such a category and $Y \in \mathcal{C}$ has an endomorphism other than the identity, i.e. $\mu \in \operatorname{End} Y$ and $\mu \neq \mathrm{id}_Y$. Let $I := \{\bullet, *\}$ and $X_\bullet, X_* \in \mathcal{C}$ with $X_\bullet = X_* := Y$. Assume $(X, \{\mathrm{inj}_\bullet : X_\bullet \longrightarrow X, \mathrm{inj}_* : X_* \longrightarrow X\})$ is the coproduct of $X_\bullet$ and X_*, and consider the two $\mathcal{C}$-morphisms $f_\bullet : X_\bullet \longrightarrow Y$ and $f_* : X_* \longrightarrow Y$ defined by $f_\bullet = f_* := \mathrm{id}_Y$. (U$\coprod$) yields a unique $\mathcal{C}$-morphism $f : X \longrightarrow Y$ such that $f \circ \mathrm{inj}_\bullet = f \circ \mathrm{inj}_* = \mathrm{id}_Y$. Because f is monic, $\mathrm{inj}_\bullet = \mathrm{inj}_*$. Now, consider the two $\mathcal{C}$-morphisms $g_\bullet : X_\bullet \longrightarrow Y$ and $g_* : X_* \longrightarrow Y$ defined by $g_\bullet := \mu$ and $g_* := \mathrm{id}_Y$. Again, (U$\coprod$) yields a unique $\mathcal{C}$-morphism $g : X \longrightarrow Y$ such that $g \circ \mathrm{inj}_\bullet = \mu$ and $g \circ \mathrm{inj}_* = \mathrm{id}_Y$. Since $\mathrm{inj}_\bullet = \mathrm{inj}_*$, this implies $\mu = \mathrm{id}_Y$ ↯.

(b) Counter-Example (a) can be easily extended to arbitrary index sets I and particularly applies to the categories $\mathbf{Sympl}_{\mathbb{K}}$, $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$, $\mathbf{{}^*Alg^m}$, $\mathbf{{}^*Alg^m_1}$, $\mathbf{C^*Alg^m}$ and $\mathbf{C^*Alg^m_1}$ by taking Y and $\mu : Y \longrightarrow Y$ as in Counter-Examples 2.2.3(b) + (c).

(c) We give another counter-example, which reveals that the category $\mathbf{pSympl}_{\mathbb{K}}$ does not have all its coproducts. Let $(W, \omega_W, C_W) \in \mathbf{pSympl}_{\mathbb{K}}$ (omit the C-involution if $\mathbb{K} = \mathbb{R}$) be any non-trivial (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic space (i.e. $\omega_W \neq 0$), $I := \{\bullet, *\}$ and $(V_\bullet, \omega_\bullet, C_\bullet) = (V_*, \omega_*, C_*) := (W, \omega_W, C_W)$. We denote their coproduct by $((V, \omega, C), \{\mathrm{inj}_{\bullet/*} : (V_{\bullet/*}, \omega_{\bullet/*}, C_{\bullet/*}) \longrightarrow (V, \omega, C)\})$. Considering the direct sum of (W, ω_W, C_W) with itself, the canonical injections into the $\bullet$ and the $*$ component define $\mathbf{pSympl}_{\mathbb{K}}$-morphisms $f_{\bullet/*} : (V_{\bullet/*}, \omega_{\bullet/*}, C_{\bullet/*}) \longrightarrow (W \oplus W, \omega_W \oplus \omega_W, C_W \oplus C_W)$. Owing to (U$\coprod$), we obtain a uniquely determined $\mathbf{pSympl}_{\mathbb{K}}$-morphism $f : (V, \omega, C) \longrightarrow (W \oplus W, \omega_W \oplus \omega_W, C_W \oplus C_W)$ satisfying the identities $f \circ \mathrm{inj}_{\bullet/*} = f_{\bullet/*}$. As f is symplectic, $\omega(\mathrm{inj}_\bullet(v), \mathrm{inj}_*(w)) = (\omega_W \oplus \omega_W)(f_\bullet(v), f_*(w)) = 0_{\mathbb{K}}$ for all $v \in V_\bullet$ and for all $w \in V_*$. Now define $g_{\bullet/*} : (V_{\bullet/*}, \omega_{\bullet/*}, C_{\bullet/*}) \longrightarrow (W, \omega_W, C_W)$ by $g_\bullet = g_* := \mathrm{id}_{(W, \omega_W, C_W)}$, then (U$\coprod$) provides us with a unique $\mathbf{pSympl}_{\mathbb{K}}$-morphism $g : (V, \omega, C) \longrightarrow (W, \omega_W, C_W)$ such that the identity $g \circ \mathrm{inj}_\bullet = g \circ \mathrm{inj}_* = \mathrm{id}_{(W, \omega_W, C_W)}$ holds. Since g is symplectic and (W, ω_W, C_W) non-trivial, we can find $v, w \in W$ meeting

$$0_{\mathbb{K}} \neq \omega_W(v, w) = \omega_W((g \circ \mathrm{inj}_\bullet)(v), (g \circ \mathrm{inj}_*)(w)) \tag{2.21}$$

$$= \omega(\mathrm{inj}_\bullet(v), \mathrm{inj}_*(w)) \tag{2.22}$$

$$= 0_{\mathbb{K}} \text{ ↯}. \tag{2.23}$$

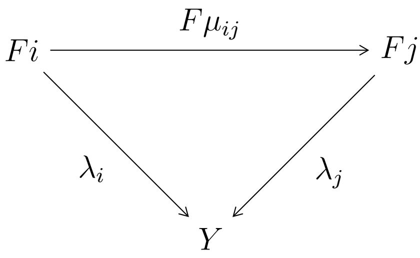
2.2.3 Colimits

Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be a functor and Y a $\mathcal{C}$-object. A *cocone* from the *cobase* F to the *covertex* Y is a natural transformation $\lambda : F \dot{\longrightarrow} \Delta Y$ from F to the *constant functor* $\Delta Y : \mathcal{J} \longrightarrow \mathcal{C}$ on Y, which is defined by $\Delta Y(i) := Y$ for all $i \in \mathcal{J}$ and $\Delta Y(\mu_{ij}) := \mathrm{id}_Y$ for all $\mu_{ij} \in \mathcal{J}(i, j)$, and for all $i, j \in \mathcal{J}$. Since ΔY is the constant functor, λ gives rise to a family $\{\lambda_i : Fi \longrightarrow Y\}_{i \in \mathcal{J}}$ of $\mathcal{C}$-morphisms, which are just the components[5] of λ, in

[5] It is also common to denote the components of a natural transformation $\tau : F \dot{\longrightarrow} G$ between two functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ by $\tau(X)$ for $X \in \mathcal{C}$. We will use both notations in this thesis.

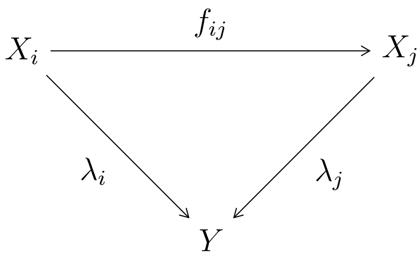
such a way that $\lambda_j \circ F\mu_{ij} = \lambda_i$ for all $\mu_{ij} \in \mathcal{J}(i,j)$ and for all $i,j \in \mathcal{J}$, which is just the naturality property of λ. All these data are conveniently depicted in the commutative diagram

(2.24)



which is actually supposed to stand for a whole family of commutative diagrams, ranging over all $\mu_{ij} \in \mathcal{J}(i,j)$ and over all $i,j \in \mathcal{J}$. Conversely, assume that a $\mathcal{C}$-object Y is given, a possibly class-labelled family of $\mathcal{C}$-objects $\{X_i\}_{i \in I}$, some sets of $\mathcal{C}$-morphisms $M_{ij} \subseteq \mathcal{C}(X_i, X_j)$ (possibly empty) meeting the compatibility condition $g_{jk} \circ f_{ij} \in M_{ik}$ whenever $f_{ij} \in M_{ij}$ and $g_{jk} \in M_{jk}$, and a family of $\mathcal{C}$-morphisms $\{\lambda_i : X_i \longrightarrow Y\}_{i \in I}$ such that the diagram

(2.25)



becomes commutative for all $f_{ij} \in M_{ij}$ and for all $i,j \in I$. Then these data define a category $\mathcal{J}$, a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ and a natural transformation $\lambda : F \dot{\longrightarrow} \Delta Y$:

(2.26)

$$\mathcal{J} := I, \quad \mathcal{J}(i,j) := \begin{cases} i \neq j : & M_{ij} \\ i = j : & M_{ii} \cup \{\mathrm{id}_{X_i}\} \end{cases}, \quad Fi := X_i, \quad Ff_{ij} := f_{ij} \quad \text{and} \quad \lambda(i) := \lambda_i$$

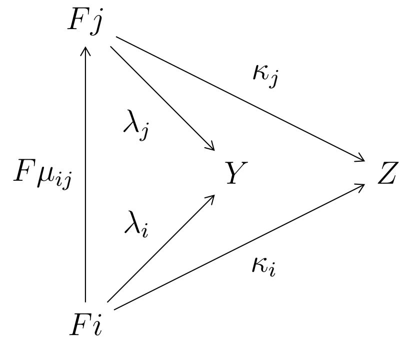
$$\forall f_{ij} \in M_{ij}, \ \forall i,j \in I.$$

Because of this correspondence, we will also call such a family of commutative diagrams a cocone. As we thus see, the notion of a cocone is an excellent way to encode all these data in one short term and we will frequently make use of this convenient manner of speaking.

Given cocones $\lambda : F \dot{\longrightarrow} \Delta Y$ and $\kappa : F \dot{\longrightarrow} \Delta Z$ from a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$, we call the family of the commutative diagrams of the form (again, this one diagram will represent the whole family of commutative diagrams ranging over all $\mu_{ij} \in \mathcal{J}(i,j)$ and over all

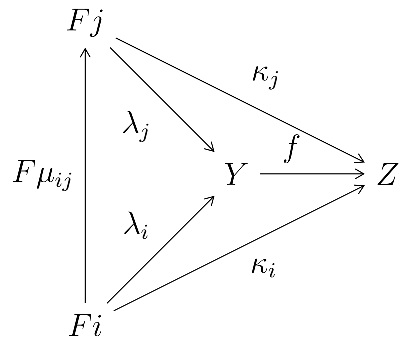
$i, j \in \mathcal{J}$)

(2.27)



a *double cocone* from F to Y and Z, denoted $(\lambda\,;\kappa)$. We say that the double-cocone $(\lambda\,;\kappa)$ *commutes* or is *commutative* if and only if there is a constant factorisation of κ through λ, i.e. there is a $\mathcal{C}$-morphism $f : Y \longrightarrow Z$ such that $\Delta f \circ \lambda = \kappa$, where $\Delta f : \Delta Y \dot{\longrightarrow} \Delta Z$ is the *constant natural transformation* on f defined by $\Delta f_i \coloneqq f$ for all $i \in \mathcal{J}$. Diagrammatically,

(2.28)



is commutative for all $\mu_{ij} \in \mathcal{J}(i, j)$ and for all $i, j \in \mathcal{J}$, i.e. in other words, the identity $f \circ \lambda_i = \kappa_i$ holds for all $i \in \mathcal{J}$.

DEFINITION 2.2.7. Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be a functor. A *colimit* for F consists of a $\mathcal{C}$-object X and a cocone $u : F \dot{\longrightarrow} \Delta X$ from F to X such that this pair meets the universal property:

(UColim) For each $\mathcal{C}$-object Y together with a cocone $\lambda : F \dot{\longrightarrow} \Delta Y$, there is a unique $\mathcal{C}$-morphism $\lambda_u : X \longrightarrow Y$ making the double-cocone $(u\,;\lambda)$ commutative.

If a colimit $(X, u : F \dot{\longrightarrow} \Delta X)$ for a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ is on hand, we will call the $\mathcal{C}$-object X the *universal object* of the colimit and the cocone u the *universal* or *colimiting cocone* of the colimit. The $\mathcal{C}$-morphisms $u_i : Fi \longrightarrow X$ are called the *canonical injections* by convention. Given a $\mathcal{C}$-object Y and a cocone $\lambda : F \dot{\longrightarrow} \Delta Y$, we will call the $\mathcal{C}$-morphism $\lambda_u : X \longrightarrow Y$ which is uniquely determined by (UColim) the *universal $\mathcal{C}$-morphism* associated with λ. Colimits need not exist, but if they do, (UColim) fixes them uniquely up to unique isomorphism. To be more precise:

LEMMA 2.2.8. *Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be a functor. If a colimit for F exists, it will be unique up to unique $\mathcal{C}$-isomorphism in the sense that given two different colimits $(X, u : F \dot{\longrightarrow} \Delta X)$ and $(Y, v : F \dot{\longrightarrow} \Delta Y)$, there is a unique $\mathcal{C}$-isomorphism $X \xrightarrow{\sim} Y$ making the double-cocone $(u\,;v)$ commutative.*

Proof: The following proof shall illustrate the standard argument for the uniqueness (up to unique isomorphism) of universal constructions, serving as an archetype for all other uniqueness proofs of universal constructions. As (X, u) and (Y, v) are colimits for F, u and v are cocones from F in particular. Hence by (UColim), there are uniquely determined $\mathcal{C}$-morphisms $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that f makes $(u\,;v)$ commutative and g makes $(v\,;u)$ commutative, i.e. $\Delta f \circ u = v$ and $\Delta g \circ v = u$. Substituting these two equations into each other yields the two identities $\Delta f \circ \Delta g \circ v = \Delta\,(f \circ g) \circ v = v$ and $\Delta g \circ \Delta f \circ u = \Delta\,(g \circ f) \circ u = u$. But $\Delta\,\mathrm{id}_Y \circ v = v$ and $\Delta\,\mathrm{id}_X \circ u = u$ too! According to (UColim), there can be only one $\mathcal{C}$-morphism making $(u\,;u)$ commutative and there can be only one $\mathcal{C}$-morphism making $(v\,;v)$ commutative. Consequently, $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Thus, f and g are $\mathcal{C}$-isomorphisms and the only ones making the double-cocones $(u\,;v)$ and $(v\,;u)$ commutative. ❐

In the sense of Lemma 2.2.8, we will henceforward speak of *the* colimit for a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$, provided that it actually exists, and denote the universal object by $\varinjlim F$. The entire colimit will be denoted by $\operatorname{colim} F = (\varinjlim F, u : F \dot{\longrightarrow} \Delta \varinjlim F)$. There is a special notion of the case when colimits for functors to a fixed category always exist:

DEFINITION 2.2.9. A category $\mathcal{C}$ is said to be *cocomplete* if and only if the colimit for each functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ from a small category $\mathcal{J}$ exists.

We emphasise that the domain categories are taken to be small in the above defintion. Defining cocompleteness by requiring the existence of the colimit for each functor from all (not necessarily small) categories to $\mathcal{C}$ does not yield a pertinent notion. Such a category $\mathcal{C}$ must be necessarily thin ([Bor94, Prop.2.7.2], [AHS04, Thm.10.32 + Rem.10.33]).

In view of the dual statements of [Par70, Sec.2.6, Prop.2], [Bor94, Thm.2.8.1], [Mac98, Thm.V.2.1], [AHS04, Thm.12.3] and the fact that coequalisers and coproducts can be viewed as special cases of colimits ([Par70, Sec.2.6], [Bor94, Example 2.6.7.d], [AHS04, Examples 11.28(1) + (2)]), we find that Example 2.2.2, Example 2.2.5, Counter-Example 2.2.3(b) + (c) and Counter-Example 2.2.6(b) + (c) imply:

THEOREM 2.2.10. *The categories* $\mathbf{Vec}_{\mathbb{K}}$, $\mathbf{CVec}$, $\mathbf{{}^*Alg}$, $\mathbf{{}^*Alg_1}$, $\mathbf{C^*Alg}$ *and* $\mathbf{C^*Alg_1}$ *are all cocomplete. The categories* $\mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$, $\mathbf{Sympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg^m}$, $\mathbf{{}^*Alg^m_1}$, $\mathbf{C^*Alg^m}$ *and* $\mathbf{C^*Alg^m_1}$ *are all not cocomplete.*

The fact that the categories ***Alg^m**, ***Alg$_1^m$**, **C*Alg^m** and **C*Alg$_1^m$** are not cocomplete is a serious obstruction to the free application of colimits, and as we will see from Definition 4.1.1 K. Fredenhagen's universal algebra, in algebraic and locally covariant quantum field theory. Examples, which will illustrate this issue and also provide less trivial examples for colimits which do not exist in those categories, will be provided by the quantised free Maxwell field in terms of the field strength tensor (see Proposition 5.6.7). We close our discussion of colimits with some helpful results concerning the existence and also the computation of colimits.

LEMMA 2.2.11. *Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be functor, where $\mathcal{J}$ has a terminal object t; in particular, $\mathcal{J}$ is up-directed. Then, the colimit for F exists and*

$$\operatorname{colim} F = (Ft, u : F \dot{\longrightarrow} \Delta Ft)\,, \tag{2.29}$$

where for all $i \in \mathcal{J}$, $u_i := F\mu_{it}$ with the $\mathcal{J}$-morphism $\mu_{it} \in \mathcal{J}(i,t)$.

Proof: The uniqueness of $\mu_{it} \in \mathcal{J}(i,t)$ for all $i \in \mathcal{J}$ entails that u is a well-defined natural transformation. So, let Y be a $\mathcal{C}$-object and $\lambda : F \dot{\longrightarrow} \Delta Y$ a cocone. We claim that $\lambda_t : Ft \longrightarrow Y$ is the universal $\mathcal{C}$-morphism associated with λ. First of all, $\lambda_t \circ u_i = \lambda_t \circ F\mu_{it} = \lambda_i$ for all $i \in \mathcal{J}$ because λ is a cocone from F. Now assume that $\kappa : Ft \longrightarrow Y$ is another $\mathcal{C}$-morphism with the property $\kappa \circ u_i = \lambda_i$ for all $i \in \mathcal{J}$. This identity has to hold for t in particular, i.e. $\kappa \circ u_t = \kappa \circ F\mu_{tt} = \lambda_t$. Since t is terminal, $|\mathcal{J}(t,t)| = 1$, which implies $\mu_{tt} = \mathrm{id}_t$ and so $F\mu_{tt} = \mathrm{id}_{Ft}$. Hence, $\kappa = \lambda_t$. ❐

The next lemma states that naturally isomorphic functor have the same colimit:

LEMMA 2.2.12. *Let $F, G : \mathcal{J} \longrightarrow \mathcal{C}$ be functors and $\eta : F \overset{\dot{\sim}}{\longrightarrow} G$ a natural isomorphism. Then, the colimit for F exists if and only if the colimit for G exists and we have*

$$\operatorname{colim} F \cong (\varinjlim G, v \circ \eta : F \dot{\longrightarrow} \Delta \varinjlim G), \tag{2.30}$$

where $(\varinjlim G, v : G \dot{\longrightarrow} \Delta \varinjlim G) = \operatorname{colim} G$ and

$$\operatorname{colim} G \cong (\varinjlim F, u \circ \eta^{-1} : G \dot{\longrightarrow} \Delta \varinjlim F), \tag{2.31}$$

where $(\varinjlim F, u : F \dot{\longrightarrow} \Delta \varinjlim F) = \operatorname{colim} F$.

Proof: We only show "$\Longleftarrow$". The other direction follows in exactly the same way but F is to be swapped with G, η with η^{-1} and u with v.

Let $\lambda : F \dot{\longrightarrow} \Delta X$ be any cocone from F to a $\mathcal{C}$-object X; then $\lambda \circ \eta^{-1} : G \dot{\longrightarrow} \Delta X$ is a cocone from G to X. (UColim) yields a unique $\mathcal{C}$-morphism $\lambda_v : \varinjlim G \longrightarrow X$

satisfying $\Delta\lambda_v \circ v = \lambda \circ \eta^{-1}$. Clearly, λ_v also meets $\Delta\lambda_v \circ (v \circ \eta) = \lambda$, i.e. λ_v makes the double-cocone $(v \circ \eta\,;\lambda)$ from F to $\varinjlim G$ and X commutative. λ_v is also the unique $\mathcal{C}$-morphism doing so because if $\kappa : \varinjlim G \longrightarrow X$ is another $\mathcal{C}$-morphism with $\Delta\kappa \circ (v \circ \eta) = \lambda$, then $\Delta\kappa \circ v = \lambda \circ \eta^{-1}$ as well and $\kappa = \lambda_v$ by (UColim). ❐

The following lemma highlights the role of equivalences of categories for the computation of colimits. Namely, equivalences of categories preserve colimits, which is due to the fact that they are left and right adjoint functors:

DEFINITION 2.2.13. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjunction* from $\mathcal{D}$ to $\mathcal{C}$ is a triple (F, G, φ) consisting of functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ and a rule φ which assigns to each pair of objects $X \in \mathcal{C}$, $M \in \mathcal{D}$ a bijection of sets $\varphi_{X,M} : \mathcal{D}(FX, M) \overset{\sim}{\longrightarrow} \mathcal{C}(X, GM)$ such that $\varphi_{X,M}(g \circ f) = Gg \circ \varphi_{X,M}(f)$ and $\varphi_{X,M}(f \circ Fh) = \varphi_{X,M}(f) \circ h$ for all $f \in \mathcal{D}(FX, M)$, for all $g \in \mathcal{D}(M, N)$ and for all $h \in \mathcal{C}(Y, X)$, for all $N \in \mathcal{D}$ and for all $Y \in \mathcal{C}$. F is called a *left adjoint* or *left adjunct* for G, and G is called a *right adjoint* or *right adjunct* for F.

LEMMA 2.2.14. *Let $F : \mathcal{J} \longrightarrow \mathcal{C}$ be a functor and $E : \mathcal{C} \longrightarrow \mathcal{D}$ an equivalence of categories. The colimit for F exists if and only if the colimit for $E \circ F$ exists and*

$$\operatorname{colim}(E \circ F) \cong E(\operatorname{colim} F)\,. \tag{2.32}$$

Proof: As E is an equivalence, we can find a functor $I : \mathcal{D} \longrightarrow \mathcal{C}$ such that there are natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{D}} \overset{\dot\sim}{\longrightarrow} E \circ I$ and $\varepsilon : I \circ E \overset{\dot\sim}{\longrightarrow} \mathrm{Id}_{\mathcal{C}}$. Note, E is left and right adjoint to I and I is left and right adjoint to E by [Par70, Sec.2.1, Cor.5], [Bor94, Thm.3.1.5], [Mac98, Thm.IV.1.2]. Consequently, [Par70, Sec.2.7, Thm.3], the dual statement of [Bor94, Prop.3.2.2], the dual statement of [Mac98, Thm.V.5.1] or the dual statement of [AHS04, Prop.18.9] implies that E and I preserve colimits, which immediately shows "$\Longrightarrow$" and (2.32). Assuming that $\operatorname{colim}(E \circ F)$ exists, we conclude the existence of $\operatorname{colim}(I \circ E \circ F) \cong I(\operatorname{colim}(E \circ F))$. Since $I \circ E$ is naturally isomorphic to the identity functor on $\mathcal{C}$, $I \circ E \circ F$ is naturally isomorphic to F. With the help of Lemma 2.2.12, "$\Longleftarrow$" is shown and applying E yields (2.32). ❐

To establish that the colimit for a functor $F : \mathcal{J} \longrightarrow \mathcal{C}$ exists, we will make frequent use of the fact that it is actually enough to investigate F on a special full subcategory $\mathcal{J}$, called a skeleton. More specifically:

DEFINITION 2.2.15. A *skeleton* of a category $\mathcal{C}$ is a full subcategory ☠ such that each $\mathcal{C}$-object is $\mathcal{C}$-isomorphic to one and only one ☠-object, i.e. for all $X \in \mathcal{C}$ exists a unique $A \in$ ☠ such that $\mathrm{iso}_{\mathcal{C}}(X, A) \neq \varnothing$. In particular, no distinct ☠-objects can be (neither ☠- nor $\mathcal{C}$-) isomorphic to each other.

Due to [Mac98, p.93] or [AHS04, Prop.4.14(3)], a category is equivalent to any one of its skeletons and an equivalence is given by the inclusion functor.

LEMMA 2.2.16. *Suppose that $F : \mathcal{J} \longrightarrow \mathcal{C}$ is a functor, $\skull$ a skeleton of $\mathcal{J}$ and $K : \skull \longrightarrow \mathcal{J}$ the inclusion functor. If the colimit for $F \circ K : \skull \longrightarrow \mathcal{C}$ exists, the colimit for F exists and will be given explicitly by*[6]

$$\operatorname{colim} F = \Big(\varinjlim (F \circ K), (v \star E) \circ (F \star \eta) : F \dot{\longrightarrow} \Delta \varinjlim (F \circ K) \circ E\Big), \tag{2.33}$$

where $(\varinjlim (F \circ K), v : F \circ K \dot{\longrightarrow} \Delta \varinjlim (F \circ K))$ is the colimit for $F \circ K$ and $E : \mathcal{J} \longrightarrow \skull$ is any equivalence for which there is a natural isomorphism $\eta : \mathrm{Id}_{\mathcal{J}} \overset{\sim}{\dot{\longrightarrow}} K \circ E$.

Proof: Since K is an equivalence of categories according to [Mac98, Sec.IV.4] and [AHS04, Rem.4.10], an equivalence E and a natural isomorphism η exist. Note that the axiom of choice for classes is used in the construction of E (cf. [Bor94, Proof of Prop.3.4.3], [Mac98, Sec.IV.4]). We figure out the existence and an explicit form of the colimit for $F \circ K \circ E : \mathcal{J} \longrightarrow \mathcal{C}$ first. The lemma follows then from the fact that $F \star \eta : F \overset{\sim}{\dot{\longrightarrow}} F \circ K \circ E$ is a natural isomorphism and from Lemma 2.2.12.

Let $\lambda : F \circ K \circ E \dot{\longrightarrow} \Delta X$ be any cocone from $F \circ K \circ E$ to a $\mathcal{C}$-object X. By one of the references [Par70, Sec.2.1, Prop.3], [Bor94, Prop.3.4.3], [Mac98, Thm.IV.4.1], [AHS04, Def.3.33], E is full and faithful and each $\skull$-object s is $\skull$-isomorphic to a $\skull$-object of the form Ei for some $\mathcal{J}$-object i. By assumption, $\skull$ is a skeleton of $\mathcal{J}$, so if s is the unique $\skull$-object $\mathcal{J}$-isomorphic to the $\mathcal{J}$-object i, $Ei = s = Es$ holds. Furthermore, since E is full, there is a $\mathcal{J}$-morphism $\mu : i \longrightarrow s$ such that $E\mu = \mathrm{id}_s : s \longrightarrow s$. Because λ is a cocone, the diagram

$$\begin{array}{ccc} (F \circ K \circ E)(i) = Fs & \xrightarrow{\quad (F \circ K \circ E)(\mu) = \mathrm{id}_{Fs} \quad} & Fs \\ & {\scriptstyle \lambda_i} \searrow \quad \swarrow {\scriptstyle \lambda_s} & \\ & X & \end{array} \tag{2.34}$$

must commute, i.e. $\lambda_i = \lambda_s$. We define a cocone $\kappa : F \circ K \dot{\longrightarrow} \Delta X$ from $F \circ K$ to X by $\kappa_s := \lambda_s : (F \circ K \circ E)(s) = (F \circ K)(s) \longrightarrow X$ for $s \in \skull$. (UColim) yields a unique $\mathcal{C}$-morphism $\kappa_v : \varinjlim (F \circ K) \longrightarrow X$ satisfying $\Delta\kappa_v \circ v = \kappa$. Surely, κ_v also satisfies

[6] Let $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ and $H, K : \mathcal{B} \longrightarrow \mathcal{C}$ be functors and $\tau : F \dot{\longrightarrow} G$ and $\sigma : H \dot{\longrightarrow} K$ natural transformations. Then we define a natural transformation $F \star \sigma : F \circ H \dot{\longrightarrow} F \circ K$ by $(F \star \sigma)(A) := F(\sigma_A)$ for all $A \in \mathcal{B}$, and a natural transformation $\tau \star H : F \circ H \dot{\longrightarrow} G \circ H$ by $(\tau \star H)(A) := \tau_{HA}$ for all $A \in \mathcal{B}$.

$\Delta\kappa_v \circ (v \star E) = \lambda$ because

$$(\kappa_v \circ (v \star E))(i) = \kappa_v \circ v_{Ei} = \kappa_v \circ v_s = \kappa_s = \lambda_s = \lambda_i \qquad \forall i \in \mathcal{J}, \tag{2.35}$$

where $s = s(i)$ denotes the unique ☠-object $\mathcal{J}$-isomorphic to the $\mathcal{J}$-object i. κ_v is the unique $\mathcal{C}$-morphism with this property because if one lets $\kappa'_v : \varinjlim (F \circ K) \longrightarrow X$ be another one meeting $\Delta\kappa'_v \circ (v \star E) = \lambda$, then surely $\Delta\kappa'_v \circ v = \kappa$ and (UColim) guarantees $\kappa'_v = \kappa_v$. Hence,

$$\operatorname{colim}(F \circ K \circ E) = \Big(\varinjlim (F \circ K), v \star E : F \circ K \circ E \stackrel{\cdot}{\longrightarrow} \varinjlim (F \circ K)\Big). \tag{2.36}$$

As mentioned earlier, $F \circ \eta : F \stackrel{\cdot\sim}{\longrightarrow} F \circ K \circ E$ is a natural isomorphism and the application of Lemma 2.2.12 completes the proof. ❐

2.2.4 Left Kan extensions

DEFINITION 2.2.17. Let $K : \mathcal{B} \longrightarrow \mathcal{C}$ and $F : \mathcal{B} \longrightarrow \mathcal{D}$ be functors. A *left Kan extension* of F along K is a pair constituted by a functor $\operatorname{Lan}_K F : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $u : F \stackrel{\cdot}{\longrightarrow} \operatorname{Lan}_K F \circ K$ with the universal property

(LKan) For each pair (G, τ) consisting of a functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $\tau : F \stackrel{\cdot}{\longrightarrow} G \circ K$, there exists a unique natural transformation $\sigma : \operatorname{Lan}_K F \stackrel{\cdot}{\longrightarrow} G$ satisfying $(\sigma \star K) \circ u = \tau$, i.e. $\sigma_{KA} \circ u_A = \sigma_A$ for all $A \in \mathcal{B}$.

If existent, left Kan extensions are uniquely determined up to unique natural isomorphism (which can be easily deduced from the standard argument for the uniqueness [up to unique isomorphism] of universal constructions, cf. the proof of Lemma 2.2.8). We will hence speak of *the* left Kan extension of a functor $F : \mathcal{B} \longrightarrow \mathcal{D}$ along a functor $K : \mathcal{B} \longrightarrow \mathcal{C}$ from now on (existence provided); we will also frequently use the term *left Kan extension* for the functor $\operatorname{Lan}_K F$ on its own.

Our interest in left Kan extensions can be justified as follows: suppose we have an "*incomplete*" locally covariant theory $F : \mathcal{B} \longrightarrow \mathbf{Phys}$, where $\mathcal{B}$ is a subcategory of $\mathbf{Loc}$, and let $K : \mathcal{B} \longrightarrow \mathbf{Loc}$ be the inclusion functor. Then, if existent, the left Kan extension of F along K, $\operatorname{Lan}_K F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$, can be interpreted as the minimal extension of F to $\mathbf{Loc}$ thanks to (LKan) and the fact that all $\mathbf{Phys}$-morphisms are monic. Also, due to its universal property (LKan), the left Kan extension is to locally covariant quantum field theory what K. Fredenhagen's universal algebra is to algebraic quantum field theory.

In the same way that naturally isomorphic functors have the same colimit, they also have the same left Kan extensions:

LEMMA 2.2.18. *Let $K : \mathcal{B} \longrightarrow \mathcal{C}$ and $F, G : \mathcal{B} \longrightarrow \mathcal{D}$ be functors and $\eta : F \xrightarrow{\dot\sim} G$ a natural isomorphism. The left Kan extension of F along K exists if and only if the left Kan extension of G along K exists. In fact, let $(\operatorname{Lan}_K F, u : F \dot\longrightarrow \operatorname{Lan}_K F \circ K)$ be the left Kan extension of F along K, then the left Kan extension of G along K is explicitly given by*

$$\big(\operatorname{Lan}_K F, u \circ \eta^{-1} : G \dot\longrightarrow \operatorname{Lan}_K F \circ K\big). \tag{2.37}$$

Vice versa, if $(\operatorname{Lan}_K G, v : G \dot\longrightarrow \operatorname{Lan}_K G \circ K)$ is the left Kan extension of G along K, the left Kan extension of F along K is explicitly given by

$$(\operatorname{Lan}_K G, v \circ \eta : F \dot\longrightarrow \operatorname{Lan}_K G \circ K). \tag{2.38}$$

Proof: We only show "$\Longrightarrow$". The other direction follows identically by swapping F with G, η with η^{-1} and u with v.

Let $H : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and $\tau : G \dot\longrightarrow H \circ K$ a natural transformation; then $\tau \circ \eta$ is a natural transformation $F \dot\longrightarrow H \circ K$ and (LKan) yields a uniquely determined natural transformation $\sigma : \operatorname{Lan}_K F \dot\longrightarrow H$ such that $(\sigma \star K) \circ u = \tau \circ \eta$. Clearly, $(\sigma \star K) \circ u \circ \eta^{-1} = \tau$ and σ is the unique natural transformation $\operatorname{Lan}_K F \dot\longrightarrow H$ with this property because let $\rho : \operatorname{Lan}_K F \dot\longrightarrow H$ be another natural transformation satisfying $(\rho \star K) \circ u \circ \eta^{-1} = \tau$; then $(\rho \star K) \circ u = \tau \circ \eta$ as well and (LKan) yields $\rho = \sigma$. ❒

Under certain circumstances, left Kan extensions can be obtained by means of colimits. To this end, we introduce the following notion:

DEFINITION 2.2.19. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and M a $\mathcal{D}$-object. The *comma category* $(F \downarrow M)$ of $\mathcal{C}$-objects F-over M is defined by

$$(F \downarrow M) := \{(X, f) \mid X \in \mathcal{C} \text{ and } f \in \mathcal{D}(FX, M)\} \tag{2.39}$$

and

$$(F \downarrow M)\left((X, f), (Y, g)\right) := \{h \in \mathcal{C}(X, Y) \mid f = g \circ Fh\}, \qquad \forall\, (X, f), (Y, g) \in (F \downarrow M). \tag{2.40}$$

It is more meaningful to use the notation $FX \xrightarrow{f} M$ for $(F \downarrow M)$-objects instead of (X, f), which we want to do from now on. In the same spirit, we denote $(F \downarrow M)$-morphisms $h : (FX \xrightarrow{f} M) \longrightarrow (FY \xrightarrow{g} M)$ by commutative diagrams

$\begin{array}{ccc} FY & \overset{g}{\searrow} & \\ {\scriptstyle Fh}\uparrow & & M \\ FX & \underset{f}{\nearrow} & \end{array}$. With each comma category $(F \downarrow M)$, where $M \in \mathcal{D}$, we automatically get a *projection functor* $P_M : (F \downarrow M) \longrightarrow \mathcal{C}$ which is defined by

$$P_M(FX \xrightarrow{f} M) := X \qquad \forall FX \xrightarrow{f} M \in (K \downarrow M) \tag{2.41}$$

and

$$P_M\left(\begin{array}{ccc} FY & \overset{g}{\searrow} & \\ {\scriptstyle Fh}\uparrow & & M \\ FX & \underset{f}{\nearrow} & \end{array}\right) := (h : X \longrightarrow Y)\,, \tag{2.42}$$

$$\forall \begin{array}{ccc} FY & \overset{g}{\searrow} & \\ {\scriptstyle Fh}\uparrow & & M \\ FX & \underset{f}{\nearrow} & \end{array} \in (F \downarrow M)\,(FX \xrightarrow{f} M, FY \xrightarrow{g} M),$$

$$\forall FX \xrightarrow{f} M, FY \xrightarrow{g} M \in (F \downarrow M).$$

We can now formulate when left Kan extensions can be constructed from colimits (see the proof of [Bor94, Thm.3.7.2] or the dual statement of [Mac98, Cor.X.3.4]):

THEOREM 2.2.20. *Let $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be categories, $\mathcal{B}$ a full subcategory of $\mathcal{C}$, $K : \mathcal{B} \longrightarrow \mathcal{C}$ the inclusion functor and $F : \mathcal{B} \longrightarrow \mathcal{D}$ a functor such that the colimit for the composition $F_X : (K \downarrow X) \xrightarrow{P_X} \mathcal{B} \xrightarrow{F} \mathcal{D}$ exists for all $X \in \mathcal{C}$, say for the sake of further reference $\operatorname{colim} F_X = \left(\varinjlim F_X, u_X : F_X \dot{\longrightarrow} \Delta \varinjlim F_X\right)$. Then the left Kan extension $(\operatorname{Lan}_K F, v : F \dot{\longrightarrow} \operatorname{Lan}_K F \circ K)$ of F along K exists and the object function of $\operatorname{Lan}_K F$ is given by*

$$(\operatorname{Lan}_K F)\,(X) = \varinjlim F_X \qquad \forall X \in \mathcal{C}. \tag{2.43}$$

The arrow function of $\operatorname{Lan}_K F$ is defined by declaring for all $X, Y \in \mathcal{C}$ and for all $f \in \mathcal{C}\,(X, Y)$ that $(\operatorname{Lan}_K F)\,(f)$ is to be the unique $\mathcal{D}$-morphism from $(\operatorname{Lan}_K F)\,(X)$ to $(\operatorname{Lan}_K F)\,(Y)$ satisfying $(\operatorname{Lan}_K F)\,(f) \circ u_X(A \xrightarrow{h} X) = u_Y(A \xrightarrow{f \circ h} Y)$ for all $A \xrightarrow{h} X \in (K \downarrow X)$, which is well-defined thanks to (UColim). Finally, the natural transformation v from F to $\operatorname{Lan}_K F \circ K$ has as its components $v_A = \mathrm{id}_A$ for all $A \in \mathcal{B}$ because of $\varinjlim F_A = FA$.

Any left Kan extension which is of the form as specified in Theorem 2.2.20 is called *pointwise*. For the computation of pointwise left Kan extensions, we will usually enlist the assistance of Lemma 2.2.16, the following lemma and its corollary:

LEMMA 2.2.21. *Consider any full subcategory of* **Loc** *which is of the form* $\mathbf{Loc}_q$ *and let* $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$ *be the inclusion functor (which becomes the identity functor on* **Loc** *if* $\mathbf{Loc}_q$ *is taken to be* **Loc***). Then for each* $\mathbf{M} \in \mathbf{Loc}$*, the comma category* $(K_q \downarrow \mathbf{M})$ *is thin and a skeleton* ☠ *of* $(K_q \downarrow \mathbf{M})$ *is constituted by the* $(K_q \downarrow \mathbf{M})$*-objects*

$$O \xrightarrow{\iota_O} \mathbf{M}, \qquad O \in \begin{cases} \mathrm{loc}^q_{+\mathbf{M}} & \text{if } \mathbf{M} \in \mathbf{Loc}_q \\ \mathrm{loc}^q_{-\mathbf{M}} & \text{if } \mathbf{M} \notin \mathbf{Loc}_q \end{cases} \tag{2.44}$$

and the $(K_q \downarrow \mathbf{M})$*-morphisms*

$$\begin{array}{ccc} V & \xrightarrow{\iota_V} & \\ \iota_{UV}\uparrow & & \mathbf{M} \\ U & \xrightarrow{\iota_U} & \end{array}, \qquad U, V \in \begin{cases} \mathrm{loc}^q_{+\mathbf{M}} & \text{if } \mathbf{M} \in \mathbf{Loc}_q \\ \mathrm{loc}^q_{-\mathbf{M}} & \text{if } \mathbf{M} \notin \mathbf{Loc}_q. \end{cases} \tag{2.45}$$

An equivalence $E : (K_q \downarrow \mathbf{M}) \longrightarrow$ ☠ *and a natural isomorphism* $\eta : \mathrm{Id}_{(K_q \downarrow \mathbf{M})} \xrightarrow{\dot{\sim}} I \circ E$*, where* $I :$ ☠ $\longrightarrow (K_q \downarrow \mathbf{M})$ *denotes the inclusion functor, are given by*

$$E(\mathbf{A} \xrightarrow{f} \mathbf{M}) := f(\mathbf{A}) \xrightarrow{\iota_{f(A)}} \mathbf{M} \qquad \forall \mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_q \downarrow \mathbf{M}) \tag{2.46}$$

and

$$E\left(\begin{array}{ccc} \mathbf{B} & \xrightarrow{g} & \\ h\uparrow & & \mathbf{M} \\ \mathbf{A} & \xrightarrow{f} & \end{array}\right) := \left(\begin{array}{ccc} g(\mathbf{B}) & \xrightarrow{\iota_{g(B)}} & \\ \iota_{f(A)g(B)}\uparrow & & \mathbf{M} \\ f(\mathbf{A}) & \xrightarrow{\iota_{f(A)}} & \end{array}\right) \tag{2.47}$$

$$\forall \left(\begin{array}{ccc} \mathbf{B} & \xrightarrow{g} & \\ h\uparrow & & \mathbf{M} \\ \mathbf{A} & \xrightarrow{f} & \end{array}\right) \in (F \downarrow \mathbf{M})(\mathbf{A} \xrightarrow{f} \mathbf{M}, \mathbf{B} \xrightarrow{g} \mathbf{M}),$$

$$\forall \mathbf{A} \xrightarrow{f} \mathbf{M}, \mathbf{B} \xrightarrow{g} \mathbf{M} \in (F \downarrow \mathbf{M}),$$

plus

$$\eta_{\mathbf{A} \xrightarrow{f} \mathbf{M}} := f\|_A \qquad \forall \mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_q \downarrow \mathbf{M}), \tag{2.48}$$

where $f\|_A : A \xrightarrow{\sim} f(A)$ *is the strong restriction of* $f : A \longrightarrow M$ *to* A*, i.e. the unique diffeomorphism satisfying* $\iota_{f(A)} \circ f\|_A = f$ *with the inclusion map* $\iota_{f(A)} : f(A) \hookrightarrow M$.

Proof: For thinness, suppose $\begin{array}{ccc} \mathbf{B} & \overset{g}{\searrow} & \\ {\scriptstyle h,k}\big\uparrow & & \mathbf{M} \\ \mathbf{A} & \underset{f}{\nearrow} & \end{array}$ is a pair of parallel $(K_q \downarrow \mathbf{M})$-morphisms. Since $f = g \circ h = g \circ k$ and g is injective, we immediately obtain $h = k$. As a consequence, $(K_q \downarrow \mathbf{M})$ is thin and ☠ defines a full subcategory.

Now, let $\mathbf{A} \xrightarrow{f} \mathbf{M}$ be any $(K_q \downarrow \mathbf{M})$-object, then $f(A)$ is a connected globally hyperbolic open subset of $\mathbf{M}$ with $H^p_{\mathrm{dR}} A = 0$ for all $p \in q$, hence $f(A) \in \mathcal{C}$, where $\mathcal{C}$ stands for $\mathrm{loc}^q_{+\mathbf{M}}$ if $\mathbf{M} \in \mathbf{Loc}_q$ and for $\mathrm{loc}^q_{-\mathbf{M}}$ if $\mathbf{M} \notin \mathbf{Loc}_q$. Using the strong restriction $f\|_A$ of f to A, $\mathbf{A} \xrightarrow{f} \mathbf{M}$ is seen to be $(K_q \downarrow \mathbf{M})$-isomorphic to $f(\mathbf{A}) \xrightarrow{\iota_{f(A)}} \mathbf{M}$ via the $(K_q \downarrow \mathbf{M})$-isomorphism $\begin{array}{ccc} f(\mathbf{A}) & \overset{\iota_{f(A)}}{\searrow} & \\ {\scriptstyle f\|_A}\big\uparrow & & \mathbf{M} \\ \mathbf{A} & \underset{f}{\nearrow} & \end{array}$. Assume that $\mathbf{A} \xrightarrow{f} \mathbf{M}$ was also $(K_q \downarrow \mathbf{M})$-isomorphic to another ☠-object $O \xrightarrow{\iota_O} \mathbf{M}$, say via the $(K_q \downarrow \mathbf{M})$-isomorphism $\begin{array}{ccc} O & \overset{\iota_O}{\searrow} & \\ {\scriptstyle g}\big\uparrow & & \mathbf{M} \\ \mathbf{A} & \underset{f}{\nearrow} & \end{array}$; then we would have $f = \iota_O \circ g$, which directly implies $O = \iota_O(O) = \iota_O(g(A)) = f(A)$. Since the strong restriction of f to A is the unique map with the property $\iota_{f(A)} \circ f\|_A = f$, $g = f\|_A$. We thus conclude that ☠ is a skeleton of $(K_q \downarrow \mathbf{M})$. The rest is clear by definition. ❒

COROLLARY 2.2.22. *Under the assumptions of* Lemma 2.2.21, *let* $\mathbf{M} \in \mathbf{Loc}$. *Then* $\mathrm{loc}^q_{+\mathbf{M}}$ *is isomorphic to a skeleton of* $(K_q \downarrow \mathbf{M})$ *if* $\mathbf{M} \in \mathbf{Loc}_q$ *and if* $\mathbf{M} \notin \mathbf{Loc}_q$, $\mathrm{loc}^q_{-\mathbf{M}}$ *is isomorphic to a skeleton of* $(K_q \downarrow \mathbf{M})$.

Proof: If $\mathbf{M} \in \mathbf{Loc}_q$, let $\mathcal{C} = \mathrm{loc}^q_{+\mathbf{M}}$ and if $\mathbf{M} \notin \mathbf{Loc}_q$ let $\mathcal{C} = \mathrm{loc}^q_{-\mathbf{M}}$. Define a functor $F : \mathcal{C} \longrightarrow (K_q \downarrow \mathbf{M})$ by

$$FO := O \xrightarrow{\iota_O} \mathbf{M} \qquad \forall O \in \mathcal{C} \tag{2.49}$$

and

$$F\iota_{UV} := \begin{array}{ccc} V & \overset{\iota_V}{\searrow} & \\ {\scriptstyle \iota_{UV}}\big\uparrow & & \mathbf{M} \\ U & \underset{\iota_U}{\nearrow} & \end{array} \qquad \forall U, V \in \mathcal{C} \text{ such that } U \subseteq V. \tag{2.50}$$

Clearly, F is well-defined as a functor. The image of F is a skeleton of $(K_q \downarrow \mathbf{M})$ by Lemma 2.2.21 and the inverse functor of F is evidently given by the projection functor $P_{\mathbf{M}} : (K_q \downarrow \mathbf{M}) \longrightarrow \mathbf{Loc}_q$ restricted to ☠ of Lemma 2.2.21. ❒

Furthermore to the identifications we have already made, we will also identify for each $\mathbf{M} \in \mathbf{Loc}$, $\mathrm{loc}^q_{+\mathbf{M}}$ (if $\mathbf{M} \in \mathbf{Loc}_q$) [resp. $\mathrm{loc}^q_{-\mathbf{M}}$ if $\mathbf{M} \notin \mathbf{Loc}_q$] with the skeleton ☠ for $(K_q \downarrow \mathbf{M})$ of Lemma 2.2.21.

2.3 Subobjects, equalisers, intersections and unions

We review the categorical notions of subobjects, equalisers, and intersections and unions of subobjects, which are important to the formulation of the dynamical net and dynamical locality in Chapter 5. *We will also provide helpful examples and counter-examples for these notions.*

As an additional source of reference, we mention [FV12a, Appx.B]. In analogy to set theory, linear algebra, etc., the objects of a generic category do not possess any internal structure. Hence, it is not possible to define the direct analogue of subsets, linear subspaces, subalgebras, etc., in category theory on the level of the objects. As it is genuinely the theme in category theory, this must be done on the level of morphisms, that is, by the relations between objects, leading to subobjects:

DEFINITION 2.3.1. Let $\mathcal{C}$ be a category. A *subobject* of an object Y is a monic $m : X \hookrightarrow Y$.

A subobject $m : X \hookrightarrow Y$ is called *smaller* than a subobject $n : Z \hookrightarrow Y$ (and n is called *larger* than m) if and only if there is a morphism $\mu : X \longrightarrow Z$ (which is necessarily unique and monic since m is monic) such that $m = n \circ \mu$.

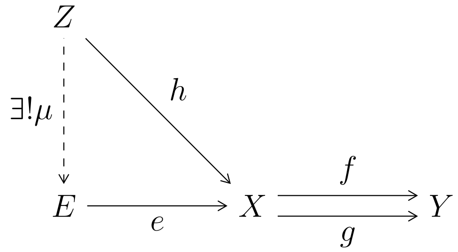
Two subobjects $m, n : X, Z \hookrightarrow Y$ are called *equivalent* if and only if m is larger and smaller than n, i.e. there are morphisms $\mu : X \longrightarrow Z$ and $\nu : Z \longrightarrow X$ such that $m = n \circ \mu$ and $n = m \circ \nu$. Note that μ and ν are necessarily uniquely determined and isomorphisms because μ and ν are monic.

As it will be seen in the examples, equalisers capture the idea of talking about the intersection of images of maps in category theory:

DEFINITION 2.3.2. Let $\mathcal{C}$ be a category and $f, g : X \longrightarrow Y$ a pair of parallel morphisms. A morphism $e : E \longrightarrow X$ is called an *equaliser* or *difference kernel* of f and g if and only if $f \circ e = g \circ e$ and e has the universal property:

(UEq) To each morphism $h : Z \longrightarrow X$ satisfying $f \circ h = g \circ h$, there exists a unique morphism $\mu : Z \longrightarrow E$ such that the diagram

(2.51)



commutes, i.e. h uniquely factorises through e as $h = e \circ \mu$.

If it exists, an equaliser for a pair of parallel morphisms is always monic (and thus a subobject) and uniquely determined up to unique isomorphism by the universal property (UEq) ([Par70, Sec.1.9, Lem.1 + 2], [Bor94, Prop.2.4.2 + 2.4.3]). So, it is justified to speak of *the* equaliser.

COUNTER-EXAMPLE 2.3.3. The category $\mathbf{Sympl}_{\mathbb{K}}$ does not have all its equalisers. To see this, let X be $\mathbb{K}^2$ equipped with the canonical (complexified if $\mathbb{K} = \mathbb{C}$) symplectic form ω, which is defined by $\omega(\vec{w}, \vec{z}) := w^1 - z^1 + w^2 - z^2$ for all $\vec{w} = (w^1, w^2), \vec{z} = (z^1, z^2) \in \mathbb{K}^2$, and the complex conjugation (if $\mathbb{K} = \mathbb{C}$), and let $Y := (\mathbb{K}^2 \oplus \mathbb{K}^2, \omega \oplus \omega, \overline{})$ (omit the complex conjugation if $\mathbb{K} = \mathbb{R}$). Define two morphisms $f, g : X \longrightarrow Y$ by $f(\vec{z}) := (\vec{z}, 0)$ and $g(\vec{z}) := (\vec{z}, (z^1, 0))$ for $\vec{z} \in \mathbb{K}^2$ and assume that their equaliser $e : E \hookrightarrow X$ exists. The identity $f \circ e = g \circ e$ implies $\operatorname{img} e \subseteq \{\vec{z} \in \mathbb{K}^2 \mid z^1 = 0\}$ and since e is monic, E is isomorphic to $\operatorname{img} e$ endowed with the structures induced by X, thus totally degenerate ↯.

We remind the reader of the strong restriction $f\|_Z : Z \twoheadrightarrow f(Z)$ of a map $f : X \longrightarrow Y$ to a subset $Z \subseteq X$. It is the unique map (which is automatically a surjection) such that $\iota_{f(Z)} \circ f|_Z = f$, where $\iota_{f(Z)} : f(Z) \hookrightarrow Y$ denotes the inclusion map.

EXAMPLE 2.3.4. The following categories have all their equalisers: $\mathbf{Vec}_{\mathbb{K}}$, $\mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$, $\mathbf{{}^*Alg}^{\mathbf{m}}_{\mathbf{1}}$, $\mathbf{{}^*Alg_1}$, $\mathbf{C^*Alg}^{\mathbf{m}}_{\mathbf{1}}$ and $\mathbf{C^*Alg_1}$. Let $\mathcal{C}$ be any one of these categories and let $f, g : X \longrightarrow Y$ be a pair of parallel $\mathcal{C}$-morphisms. Clearly, $E := \{x \in X \mid f(x) = g(x)\}$ is a $\mathcal{C}$-object and the the inclusion map $\iota : E \hookrightarrow X$ is a $\mathcal{C}$-morphism with the property $f \circ \iota = g \circ \iota$. The inclusion map is obviously monic since it is injective. If $h : Z \longrightarrow X$ is another $\mathcal{C}$-morphism meeting $f \circ h = g \circ h$, then $\operatorname{img} h \subseteq E$ and h can be splitted as $h = \iota \circ \iota_{\operatorname{img} h\, E} \circ h\|_Z$, where $\iota_{\operatorname{img} h\, E} : \operatorname{img} h \hookrightarrow E$ denotes the inclusion map and $h\|_Z : Z \twoheadrightarrow \operatorname{img} h$ the strong restriction of h to Z. The properties of the strong restriction guarantee that $\iota_{\operatorname{img} h\, E} \circ h\|_Z : Z \longrightarrow E$ is the unique $\mathcal{C}$-morphism meeting $h = \iota \circ \iota_{\operatorname{img} h\, E} \circ h\|_Z$.

Intersections of subobjects are the categorical generalisation of the intersection of subsets, linear subspaces, subalgebras, etc.:

DEFINITION 2.3.5. Let Y be an object in a category $\mathcal{C}$ and $\{m_i : X_i \hookrightarrow Y\}_i$ a possibly class-labelled family of subobjects of Y. An *intersection* of the m_i is a subobject $m : X \hookrightarrow Y$ smaller than all m_i, i.e. m factorises through each m_i via uniquely determined monics $n_i : X \hookrightarrow X_i$ as $m = m_i \circ n_i$, and m satisfies the universal property:

(U⋀) For each morphism $f : Z \longrightarrow Y$ factorising through each m_i, say as $f = m_i \circ f_i$ for morphisms $f_i : Z \longrightarrow X_i$, there is a unique morphism $\hat{f} : Z \longrightarrow X$ such that $f = m \circ \hat{f}$.

$f = m \circ \hat{f}$ particularly implies $n_i \circ \hat{f} = f_i$ for all i. If an intersection of subobjects exists, it will be uniquely determined by its universal property (U⋀) [Par70, Sec.1.12] and we can speak of *the* intersection. The intersection of subobjects $m_i : X_i \hookrightarrow Y$ is also denoted by $\bigwedge_i m_i : \bigwedge_i X_i \hookrightarrow Y$.

COUNTER-EXAMPLE 2.3.6. The category $\mathbf{Sympl}_{\mathbb{K}}$ does not have all its intersections. To be more precise, the category $\mathbf{Sympl}_{\mathbb{K}}$ does not have all its small intersections. Consider $X_1, X_2 := X$ and Y like in Counter-Example 2.3.3 and define monics $m_i : X_i \hookrightarrow Y$ by $m_1(\vec{z}) := (\vec{z}, (z^1, 0))$ and $m_2(\vec{z}) := (\vec{z}, \vec{z})$ for $\vec{z} = (z^1, z^2) \in \mathbb{K}^2$. Assuming that the intersection $m_1 \wedge m_2 : X_1 \wedge X_2 \hookrightarrow Y$ exists, $m_1 \wedge m_2$ factorises through m_1 and m_2 with uniquely determined monics $n_1 : X_1 \wedge X_2 \hookrightarrow X_1$ and $n_2 : X_1 \wedge X_2 \hookrightarrow X_2$. The identity $m_1 \wedge m_2 = m_1 \circ n_1 = m_2 \circ n_2$ implies $\operatorname{img}(m_1 \wedge m_2) \subseteq \operatorname{img} m_1 \cap \operatorname{img} m_2 = \{(z, 0, z, 0) \in \mathbb{K}^4 \mid z \in \mathbb{K}\}$. But since $m_1 \wedge m_2$ is monic, $X_1 \wedge X_2$ is isomorphic to $\operatorname{img}(m_1 \wedge m_2)$ endowed with the structures induced by Y, thus totally degenerate ↯.

EXAMPLE 2.3.7. The following categories have all their small intersections: $\mathbf{Vec}_{\mathbb{K}}$, $\mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$, $\mathbf{*Alg_1}$, $\mathbf{*Alg^m_1}$, $\mathbf{C^*Alg_1}$ and $\mathbf{C^*Alg^m_1}$. Let $\mathcal{C}$ stand for any of these categories and let $\{m_i : X_i \hookrightarrow Y\}_{i \in I}$ be set-labelled family of subobjects for a $\mathcal{C}$-object Y. We can endow $\bigcap_{i \in I} \operatorname{img} m_i$ with the structures induced by Y in order to obtain a $\mathcal{C}$-object X and the inclusion map $\iota : X \hookrightarrow Y$ becomes monic in $\mathcal{C}$. Since the m_i are injective, the strong restrictions $m_i\|_{X_i} : X_i \twoheadrightarrow m_i(X_i)$ are $\mathcal{C}$-isomorphisms. Since all the inclusion maps $\iota_i : X \hookrightarrow m_i(X_i)$ are injective, the $\mathcal{C}$-morphisms $m_i\|^{-1}_{X_i} \circ \iota_i : X \longrightarrow X_i$ are monics and $\iota = m_i \circ m_i\|^{-1}_{X_i} \circ \iota_i$ for all $i \in I$ is met. Let $f : Z \longrightarrow Y$ be a $\mathcal{C}$-morphism that factorises through each m_i with $\mathcal{C}$-morphisms $f_i : Z \longrightarrow X_i$. $f = m_i \circ f_i$ for all $i \in I$ yields $\operatorname{img} f \subseteq \operatorname{img} m_i$ for all $i \in I$ and hence, $\operatorname{img} f \subseteq X$. As a result, we can factorise $f = \iota_{f(Z)} \circ f\|_Z = \iota \circ \iota_{f(Z)X} \circ f\|_Z$ with the inclusion maps $\iota_{f(Z)} : f(Z) \hookrightarrow Y$ and $\iota_{f(Z)X} \circ f\|_Z : Z \longrightarrow X$. Note, $\iota_{f(Z)X} \circ f\|_Z$ is the unique $\mathcal{C}$-morphism $\hat{f} : Z \longrightarrow X$ with the property $f = \iota \circ \hat{f}$ because ι is monic.

The notion of the union of subobjects is not only the literal categorical generalisation of the union of subsets but also the categorical generalisation of the concept of linear subspaces or subalgebras generated by subsets.

DEFINITION 2.3.8. Let Y be an object in a category $\mathcal{C}$ and $\{m_i : X_i \hookrightarrow Y\}_i$ a possibly class-labelled family of subobjects of Y. A *union* of this family is a subobject $m : X \hookrightarrow Y$ larger than all m_i, i.e. each m_i factorises uniquely through m, say as $m_i = m \circ n_i$ for uniquely determined monics $n_i : X_i \hookrightarrow X$, and m satisfies the universal property

(U$\bigvee$) Let $f : Y \longrightarrow Z$ be a morphism and $\mu : W \hookrightarrow Z$ a subobject such that each $f \circ m_i$ factorises through μ, say as $f \circ m_i = \mu \circ f_i$ for morphisms $f_i : X_i \longrightarrow W$; then there exists a unique morphism $\check{f} : X \longrightarrow W$ satisfying $f \circ m = \mu \circ \check{f}$.

$f \circ m = \mu \circ \check{f}$ particularly implies $\check{f} \circ n_i = f_i$ for all i. If a union for a family of subobjects $m_i : X_i \hookrightarrow Y$ exists, it will be uniquely fixed by the universal property (U$\bigvee$) [Par70, Sec.1.12]. We thus speak of *the* union and also denote it by $\bigvee_i m_i : \bigvee_i X_i \hookrightarrow Y$.

COUNTER-EXAMPLE 2.3.9. The category $\mathbf{Sympl}_{\mathbb{K}}$ does not have all its unions. To be more precise, the category $\mathbf{Sympl}_{\mathbb{K}}$ does not have all its small unions. Consider yet again $X_i := X$ and Y of Counter-Example 2.3.3, and define monics $m_i : X_i \hookrightarrow Y$ by $m_i(\vec{z}) := (\vec{z}, (z^1, 0))$ for $\vec{z} = (z^1, z^2) \in \mathbb{K}^2$ and $i = 1, 2$. We assume that their union $m_1 \vee m_2 : X_1 \vee X_2 \hookrightarrow Y$ exists, hence $m_1 \vee m_2$ factorises through the m_i with unique monics $n_i : X_i \hookrightarrow X_1 \vee X_2$. The identities $m_i = (m_1 \vee m_2) \circ n_i$ imply $\operatorname{img} m_1 \cup \operatorname{img} m_2 \subseteq \operatorname{img}(m_1 \vee m_2)$, hence $\operatorname{img}(m_1 \vee m_2) = Y$ because $X_1 \vee X_2$ is supposed to be a $\mathbf{Sympl}_{\mathbb{K}}$-object by assumption. We thus realise that $m_1 \vee m_2$ is an isomorphism. Taking the identity $\mathrm{id}_Y : Y \longrightarrow Y$ and the subobject $\mu : X \hookrightarrow Y$ defined by $\mu := m_1 = m_2$, the morphisms $\mathrm{id}_Y \circ m_i$ factorise through μ via $\mathrm{id}_Y \circ m_1 = m_1 = \mu = \mu \circ \mathrm{id}_X$ and $\mathrm{id}_Y \circ m_2 = m_2 = \mu = \mu \circ \mathrm{id}_X$. By (U$\bigvee$), there must be a unique morphism $\check{f} : X_1 \vee X_2 \longrightarrow X$ with the property $\mathrm{id}_Y \circ m_1 \vee m_2 = \mu \circ \check{f}$ ↯.

EXAMPLE 2.3.10. The categories $\mathbf{Vec}_{\mathbb{K}}$, $\mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{pSympl}_{\mathbb{K}}^{\mathbf{m}}$, $\mathbf{{}^*Alg_1}$, $\mathbf{{}^*Alg_1^m}$, $\mathbf{C^*Alg_1}$ and $\mathbf{C^*Alg_1^m}$ have all their small unions. Let $\mathcal{C}$ be any one of these categories and $\{m_i : X_i \hookrightarrow Y\}_{i \in I}$ a set-labelled family of subobjects for a $\mathcal{C}$-object Y. Consider the $\mathcal{C}$-object X which is generated by the images $\operatorname{img} m_i$, i.e. X is the smallest subobject of Y containing the union $\bigcup_{i \in I} \operatorname{img} m_i$. Surely, the inclusion map $\iota : X \hookrightarrow Y$ satisfies $m_i = \iota_{iY} \circ m_i\|_{X_i} = \iota \circ \iota_{iX} \circ m_i\|_{X_i}$ for all $i \in I$, where $\iota_{iX}, \iota_{iY} : m_i(X_i) \hookrightarrow X, Y$ denote the inclusion maps. Let $f : Y \longrightarrow Z$ be any $\mathcal{C}$-morphism and $\mu : W \hookrightarrow Z$ a subobject such that for each $i \in I$, $f \circ m_i = \mu \circ f_i$ for some $\mathcal{C}$-morphism $f_i : X_i \longrightarrow W$; then $\operatorname{img}(f \circ \iota) \subseteq \operatorname{img} \mu$ and $\mu\|_W^{-1} \circ \iota_{f(X)\mu(W)} \circ f\|_X$ is the unique $\mathcal{C}$-morphism $\check{f} : X \longrightarrow W$ fulfilling $f \circ \iota = \mu \circ \check{f}$ (recall $\iota_{\mu(W)} \circ \mu\|_W = \mu$), where $\iota_{f(X)\mu(W)} : f(X) \hookrightarrow \mu(W)$ and $\iota_{\mu(W)} : \mu(W) \hookrightarrow Z$ denote the inclusion maps.

We give a final, concrete example for unions of subobjects in $\mathbf{Vec}_{\mathbb{K}}$. Though it is just a very special case, it is worth spelling out since it will be applied in the explicit computation of colimits and K. Fredenhagen's universal algebra on multiple occasions.

EXAMPLE 2.3.11. Let $\xi = (E, M, \pi, V)$ and $\eta = (L, M, \varrho, W)$ be smooth vector bundles and $D : \Gamma^{\infty}(\xi) \longrightarrow \Gamma^{\infty}(\eta)$ a linear differental operator. Then, Example 2.3.10 shows that the inclusion map $\iota : D\Gamma_0^{\infty}(\xi) \hookrightarrow \Gamma_0^{\infty}(\eta)$ is the union in $\mathbf{Vec}_{\mathbb{K}}$ of the subobjects $\mathbf{i}_{\eta|_{U_i}*} : D_{U_i}\Gamma_0^{\infty}(\xi|_{U_i}) \hookrightarrow \Gamma_0^{\infty}(\eta)$, where $\{U_i \mid i \in I\}$ is any open cover for M; just use a smooth partition of unity subordinated to $\{U_i \mid i \in I\}$.

Appendix: some universal constructions in some concrete categories

For the benefit of the reader, and for the sake of completeness, we give details of some concrete universal constructions in some of our algebraic categories in this appendix. In particular, we provide details for the universal constructions mentioned in the examples of coproducts and coequalisers, Example 2.2.2 *and* Example 2.2.5. *Concrete universal constructions of particular importance to this thesis are quotients, direct sums, tensor algebras and free products.*

First, we review the following two constructions for complex vector spaces and algebras:

Complex conjugate vector spaces and opposite algebras

To each complex vector space $V = (V, +_V, \cdot_V, 0_V)$, we can always consider its *complex conjugate vector space* $\overline{V} = (V, +_V, \bar{\cdot}_V, 0_V)$, where $\lambda \bar{\cdot}_V v = \overline{\lambda} \cdot_V v$ for all $\lambda \in \mathbb{C}$ and for all $v \in V$. Given any linear map $f : V \longrightarrow W$, there is a unique linear map $\overline{f} : \overline{V} \longrightarrow \overline{W}$ such that the left-hand diagram

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle \mathrm{id}_V}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_W} \\ \overline{V} & \xrightarrow{\ \overline{f}\ } & \overline{W} \end{array} \qquad\qquad \begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle C_V}\downarrow & & \downarrow{\scriptstyle C_W} \\ \overline{V} & \xrightarrow{\ \overline{f}\ } & \overline{W} \end{array} \tag{2.52}$$

commutes, i.e. $\overline{f} \circ \mathrm{id}_V = \mathrm{id}_W \circ f$ as complex-conjugate linear maps. The linear map $\overline{f}$ is defined by $\overline{f}(v) := f(v)$ for all $v \in V$. In this context, a C-involution $C : V \longrightarrow V$ can also be regarded as a bijective linear map $C : V \xrightarrow{\sim} \overline{V}$ (resp. $C : \overline{V} \xrightarrow{\sim} V$) and a linear map $f : V \longrightarrow W$ is a C-homomorphism $(V, C_V) \longrightarrow (W, C_W)$ if and only if $\overline{f} \circ C_V = C_W \circ f$ as linear maps. Diagrammatically, f is a C-homomorphism if and only if the right-hand diagram of complex vector spaces and linear maps in (2.52) is commutative.

To each algebra $A = (A, +_A, \cdot_A, \bullet_A, 0_A)$ over a field K, we can form the <u>*opposite algebra*</u> $A^{\mathrm{opp}} = (A, +_A, \cdot_A, \bullet_A^{\mathrm{opp}}, 0_A)$, where $a \bullet_A^{\mathrm{opp}} b = b \bullet_A a$ for all $a, b \in A$. Any homomorphism $\varphi : A \longrightarrow B$ of algebras gives rise to an algebra homomorphism $\varphi^{\mathrm{opp}} : A^{\mathrm{opp}} \longrightarrow B^{\mathrm{opp}}$ which is uniquely determined by the requirement that the left-hand diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\ A^{\mathrm{opp}} & \xrightarrow{\varphi^{\mathrm{opp}}} & B^{\mathrm{opp}} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\ \overline{A}^{\mathrm{opp}} & \xrightarrow{\overline{\varphi}^{\mathrm{opp}}} & \overline{B}^{\mathrm{opp}} \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle *_A}\downarrow & & \downarrow{\scriptstyle *_B} \\ \overline{A}^{\mathrm{opp}} & \xrightarrow{\overline{\varphi}^{\mathrm{opp}}} & \overline{B}^{\mathrm{opp}} \end{array} \tag{2.53}$$

commutes, i.e. $\varphi^{\mathrm{opp}} \circ \mathrm{id}_A = \mathrm{id}_B \circ \varphi$ as linear maps. The algebra homomorphism φ^{opp} is defined by $\varphi^{\mathrm{opp}}(a) \coloneqq \varphi(a)$ for all $a \in A$.

If A is, more specifically, a complex algebra, we can form the <u>*complex conjugate opposite algebra*</u> $\overline{A}^{\mathrm{opp}}$ and any homomorphism $\varphi : A \longrightarrow B$ of complex algebras yields one and only one algebra homomorphism $\overline{\varphi}^{\mathrm{opp}} : \overline{A}^{\mathrm{opp}} \longrightarrow \overline{B}^{\mathrm{opp}}$ such that the diagram in the middle of (2.53) commutes, i.e. $\overline{\varphi}^{\mathrm{opp}} \circ \mathrm{id}_A = \mathrm{id}_B \circ \varphi$ in the sense of complex-conjugate linear maps. Thus, a *-involution $* : A \longrightarrow A$ on a complex algebra A can also be regarded as an algebra isomorphism $* : A \xrightarrow{\sim} \overline{A}^{\mathrm{opp}}$ (resp. $* : \overline{A}^{\mathrm{opp}} \xrightarrow{\sim} A$); an algebra homomorphism $\varphi : A \longrightarrow B$ between complex algebras A and B with the *-involutions $*_A$ and $*_B$ is a *-homomorphism if and only if $\overline{\varphi}^{\mathrm{opp}} \circ *_A = *_B \circ \varphi$ as algebra homomorphisms. In the language of diagrams, φ is a *-homomorphism if and only if the right-hand side diagram of complex algebras and algebra homomorphisms in (2.53) commutes.

<u>Quotients</u>

<u>**PROPOSITION AND DEFINITION**</u> **2.4.12.** *Let X be a C-vector space [resp. (unital) (C)*-algebra] and $W \subseteq X$ a C-closed linear subspace*[7] *(resp. *-ideal, closed two-sided ideal). Then, there always exists a pair consisting of a C-vector space [resp. (unital) (C)*-algebra] Q and a C-homomorphism [resp. (unital) *-homomorphism] $q : X \longrightarrow Q$ such that the universal property*

(UQ) *If Y is a C-vector space [resp. (unital) (C)*-algebra and $f : X \longrightarrow Y$ a C-homomorphism [resp. (unital) *-homomorphism] such that $\ker f \subseteq W$, then there exists one and only one C-homomorphism [resp. (unital) *-homomorphism] $[f]$:*

[7] A linear subspace $U \subseteq V$ is called <u>*C-closed*</u> in (V, C) if and only if $Cu \in U$ for all $u \in U$.

$Q \longrightarrow Y$ making the diagram

$$
(2.54) \qquad \begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & \\
{\scriptstyle q}\downarrow & \searrow & \\
Q & \overset{\exists!\,[f]}{\dashrightarrow} & Y
\end{array}
$$

commutative, i.e. $[f] \circ q = f$.

is satisfied. The pair (Q, q) *is called a* <u>quotient</u> *of* X *by* W*;* Q *is called a* <u>quotient C-vector space</u> [*resp.* <u>(unital) (C)*-algebra</u>] *of* X *by* W *and* q *is called the* <u>canonical projection</u> *onto* Q.

A quotient of a C-vector space [*resp.* (*unital*) (C)**-algebra*] *X by a C-closed linear subspace* (*resp.* **-ideal, closed two-sided ideal*) *W is unique in the following sense: if* $(Q', q' : X \longrightarrow Q')$ *is another quotient of X by W, then there is a unique C-isomorphism* [*resp.* (*unital*) **-isomorphism*] $f : Q \xrightarrow{\sim} Q'$ *such that* $f \circ q = q'$.

Proof: For X a C-vector space and W a C-closed subspace of X, this follows from Proposition 2.1.2. Though we could also give a direct proof ($\approx$ 2 pages long), we call forward this simple insight from category theory: as in Example 2.2.2(a), the quotient can be understood as a coequaliser in **CVec**, which in return can be understood as a colimit, see e.g. [Par70, Sec.2.6], [Bor94, Example 2.6.7.d] and [AHS04, Examples 11.28(2)]. Hence, the result follows from Proposition 2.1.2 and Lemma 2.2.14. Note, more concretely, the underlying complex vector space V_Q of a quotient Q of X by W turns out to be a complex quotient vector space of V_X by W, where V_X is the underlying complex vector space of X. The C-involution of Q, $C_Q : V_Q \longrightarrow V_Q$, is the unique C-involution on V_Q that makes the canonical projection $q : V_X \longrightarrow V_Q$ a C-homomorphism $q : X \longrightarrow Q$, i.e. $\overline{q} \circ C_X = C_Q \circ q$ as linear maps.

The existence and the universal property of quotient (unital) (C)*-algebras are well-known from the literature, see e.g. [Ric60; Nai72; Dix77a; Sak98; Tak02]. ❒

We understand uniqueness in the sense of Proposition and Definition 2.4.12, speak of *the* quotient of X by W and denote it by $(X/W, \pi : X \twoheadrightarrow X/W)$. There is no harm at all in adopting the same notation for quotients abstractly characterised by the universal property (UQ) as for the concretely constructed quotients in the introduction to this chapter; in fact, we may always think of a quotient as concretely realised like in the introduction to this chapter.

Direct sums

PROPOSITION AND DEFINITION 2.4.13. *Let K be a field, I an arbitrary index set and $\{X_i\}_{i\in I}$ a family of vector spaces over K (resp. C-vector spaces). There always is a vector space S over K (resp. C-vector space) and a family of linear maps (resp. C-homomorphisms) $\{s_i : X_i \longrightarrow S\}_{i\in I}$ such that the universal property*

(U$\oplus$) *Let Y be a vector space over K (resp. C-vector space) and $\{f_i : X_i \longrightarrow Y\}_{i\in I}$ a family of linear maps (resp. C-homomorphisms); then there is one and only one linear map (resp. C-homomorphism) $f : S \longrightarrow Y$ making*

$$\begin{array}{ccc} S & \overset{\exists! f}{\dashrightarrow} & Y \\ {\scriptstyle s_i}\uparrow & \nearrow{\scriptstyle f_i} & \\ X_i & & \end{array} \tag{2.55}$$

commutative for all $i \in I$, i.e. $f \circ s_i = f_i$ for all $i \in I$.

is satisfied. The pair $(S, \{s_i\}_{i\in I})$ is called a direct sum of the X_i; the vector space S over K (resp. C-vector space) on its own is also called a direct sum of the X_i and the linear maps (resp. C-homomorphisms) s_i are called the canonical injections into S.

A direct sum of vector spaces X_i, $i \in I$, is unique in the following sense: whenever there is another direct sum $(S', \{s'_i : X_i \longrightarrow S'\}_{i\in I})$ of the X_i, there is one and only one bijective linear map (resp. C-isomorphism) $f : S \xrightarrow{\sim} S'$ such that $f \circ s_i = s'_i$ for all $i \in I$.

Proof: For vector spaces this follows directly from [Bou89, II, §1, no.6] but see also [Gre67, II, §4]. Now, let $\{X_i\}_{i\in I}$ be a family of C-vector spaces $\{(V_i, C_i)\}_{i\in I}$. Again, a direct proof can be given but it is more efficient to use category theory: as in Example 2.2.5, the direct sum is the coproduct in **CVec**, which can be understood as a colimit, see e.g. [Par70, Sec.2.6], [Bor94, Example 2.6.7.d] and [AHS04, Examples 11.28(1)]. Thereby, Proposition 2.1.2 and Lemma 2.2.14 complete the proof. More explicitly, the underlying vector space V_S of a direct sum S of the X_i is a direct sum of the vector spaces V_i. The C-involution C_S of S is the unique C-involution on V_S that turns all canonical injections $s_i : V_i \longrightarrow V_S$ into C-homomorphisms, i.e. $\overline{s}_i \circ C_i = C_S \circ s_i$ as linear maps for all $i \in I$. ❐

With uniqueness understood in the sense of Proposition and Definition 2.4.13, we will speak of *the* direct sum of the X_i from now on. We will also use the more familar notation $(\bigoplus_{i\in I} X_i, \{\text{inj}^{\oplus}_j : X_j \hookrightarrow \bigoplus_{i\in I} X_i\}_{j\in I})$ for the direct sum and $\{\text{pr}^{\oplus}_j : \bigoplus_{i\in I} X_i \twoheadrightarrow X_j\}_{j\in I}$ for the *canonical projections*. In the case of C-vector spaces $(X_i = (V_i, C_i)$

for each $i \in I$), we will also use the notation $(\bigoplus V_i, C_\oplus)$ for $\bigoplus_{i \in I}(V_i, C_i)$. It is again not harmful in any way to think of the direct sum as being concretely realised in the standard manner like in [Gre67, II, §4] or [Bou89, II, §1, no.6].

Tensor algebras

PROPOSITION AND DEFINITION 2.4.14. *Let K be a field and X a vector space over K (resp. C-vector space). Then, there always exists a pair consisting of an associative unital algebra over K (resp. unital *-algebra), T, and a linear map (resp. C-homomorphism), $t : X \longrightarrow T$, such that the universal property*

(UT) *Let B be an associative unital algebra over K (resp. unital *-algebra) and $f : X \longrightarrow B$ a linear map (resp. C-homomorphism); then there exists one and only one unital algebra homomorphism (resp. unital *-homomorphism) $\varphi : A \longrightarrow B$ such that the following diagram commutes,*

$$\begin{array}{ccc} X & \xrightarrow{\quad t \quad} & T \\ & \searrow^{f} & \Big\downarrow{\scriptstyle \exists! \varphi} \\ & & B \end{array} \tag{2.56}$$

i.e. $\varphi \circ t = f$ as linear maps (resp. C-homomorphisms).

*is satisfied. The pair (T, t) is called a tensor algebra over X; the associative unital algebra over K (resp. unital *-algebra), T, on its own is also called a tensor algebra over X and the linear map (resp. C-homomorphism) t is called the canonical injection into T.*

*A tensor algebra over a vector space X is unique in the following sense: whenever $(T', t' : X \longrightarrow T')$ is another tensor algebra over X, there is a unique unital algebra isomorphism (resp. unital *-isomorphim) $\varphi : T \xrightarrow{\sim} T'$ such that $\varphi \circ t = t'$ in the sense of linear maps (resp. C-homomorphisms).*

Note that (UT) is not formulated in a single category but refers to categories of vector spaces and algebras. Tensor algebras over vector spaces are indeed a very familiar construction, see [Gre78, Chap.3], [Bou89, III, §5, no.1]. Proving the existence of tensor algebras over C-vector spaces is elementary and will therefore be omitted. We note however the following: if $(T, t : (V, C) \longrightarrow T)$ is a tensor algebra over a C-vector space (V, C), then the underlying associative unital algebra over $\mathbb{C}$ of T, A_T, is a tensor algebra over V. The *-involution of T, $*_T$, is the unique *-involution on A_T such that $t : V \longrightarrow A_T$ becomes a C-homomorphism, i.e. $\bar{t} \circ C = *_T \circ t$ as linear maps. It follows already from the universal property (UT) alone (no reference to an explicit construction of T is needed at all) that the canonical injection $t : X \longrightarrow T$ must be injective: equip X with the bilinear map defined by $\bullet_X(x, y) := 0_K$ for all $x, y \in X$, thus turning X into a commutative algebra over K, and adjoin an identity element ([Ric60, Chap.I, §1], [Nai72, §7, Subsec.2, Prop.I], [Tak02, Chap.I, Sec.1]). We obtain a unital algebra X_1 and an algebra monomorphism $\mathrm{inj}_1 : X \hookrightarrow X_1$. Thanks to (UT), there is a unique unital algebra homomorphism $\varphi : T \longrightarrow X_1$ such that $\varphi \circ t = \mathrm{inj}_1$. Since inj_1 is injective, t is injective by [Bou68, II, §3, no.8, Thm.1(c)].

With the concrete constructions given in [Gre78, Chap.3] and [Bou89, III, §5, no.1], it is not difficult to modify Proposition and Definition 2.4.14 to yield the notion, existence and uniqueness of *non-unital tensor algebras*. One simply removes the zeroth summand, i.e. the field over which the vector space is taken, from the direct sum. The universal property met by non-unital tensor algebras is the same as (UT) save for dropping all references to an identity element therein. In the same way, one also obtains non-unital tensor algebras over C-vector spaces, which are *-algebras satisfying (UT) but with all references to an identity element removed.

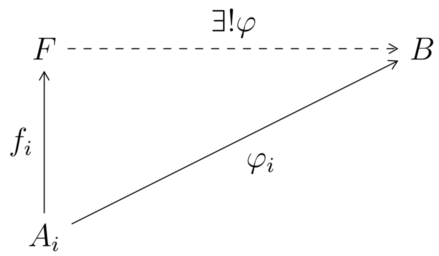
Understanding uniqueness in the sense of Proposition and Definition 2.4.14, we speak of *the* (non-unital) tensor algebra over a vector or C-vector space X and denote it by $(TX, \mathrm{inj}_X^T : X \hookrightarrow TX)$. Since we use the same notation for unital and non-unital tensor algebras, we always make it clear from the context when non-unital tensor algebras are considered.

Free products of algebras

PROPOSITION AND DEFINITION 2.4.15. *Let I be an arbitrary index set and $\{A_i\}_{i\in I}$ a family of (unital) (C)*-algebras. There always exists a pair consisting of a (unital) (C)*-algebra F and a family of (unital) *-homomorphisms $\{f_i : A_i \longrightarrow F\}_{i\in I}$ such that the universal property*

(UF) *Let B be a (unital) (C)*-algebra and $\{\varphi_i : A_i \longrightarrow B\}_{i\in I}$ a family of (unital) *-homomorphisms,. Then there is a unique (unital) *-homomorphism $\varphi : F \longrightarrow B$ such that the diagram*

(2.57)



becomes commutative for all $i \in I$, i.e. $\varphi \circ f_i = \varphi_i$ for all $i \in I$.

is satisfied. The pair $(F, \{f_i\}_{i\in I})$ *is called a* free product *of the* A_i*; the (unital) (C)*-algebra* F *on its own is also called a* free product *of the* A_i *and the (unital) *-homomorphisms* f_i *are called the* canonical injections *into* F*.*

A free product of (unital) (C)-algebras* A_i, $i \in I$*, is unique as follows: given another free product of the* A_i, $(F', \{f'_i : A_i \longrightarrow F'\}_{i\in I})$*, there is a unique (unital) *-isomorphism* $\varphi : F \xrightarrow{\sim} F'$ *such that* $\varphi \circ f_i = f'_i$ *for all* $i \in I$*.*

It is well-known in the literature, see [Avi82; Ped99; KT02; Bla06] and in particular [VDN92], that the free product as described in Proposition and Definiton 2.4.15 exists. Nevertheless, it is instructive and helpful to give a concrete realisation of the free product, which is based on the universal constructions presented thusfar in this appendix.

For (unital) *-algebras A_i, $i \in I$, the free product can be concretely constructed as follows: take the direct sum $(A := \bigoplus_{i\in I} A_i, \{\mathrm{inj}_i^{\oplus} : A_i \hookrightarrow A\}_{i\in I})$, where the A_i are to be regarded as C-vector spaces, and construct the (unital) tensor algebra $(TA, \mathrm{inj}_A^T : A \hookrightarrow TA)$, which will be a (unital) *-algebra. Then, take the quotient of TA by the two-sided *-ideal J which is generated by the two sets

$$\text{(2.58)} \qquad \text{restoring multiplication: } J_\bullet := \left\{ \mathrm{inj}_i^T(a)\,\mathrm{inj}_i^T(b) - \mathrm{inj}_i^T(ab) \;\middle|\; a, b \in A_i,\, i \in I \right\}$$

and

$$\text{(2.59)} \qquad \text{common identity: } J_1 := \left\{ 1_{TA} - \mathrm{inj}_i^T(1_{A_i}) \;\middle|\; i \in I \right\},$$

where we have defined $\mathrm{inj}_i^T := \mathrm{inj}_A^T \circ \mathrm{inj}_i^{\oplus} : A_i \longrightarrow TA$ for each $i \in I$. The underlying set of J is the linear span of $\{arb,\, asb \in TA \mid a, b \in TA,\, r \in J_\bullet,\, s \in J_1\}$ (of course, J_1 is to be omitted if non-unital *-algebras are considered). The pair $(TA/J, \{\pi \circ \mathrm{inj}_i^T : A_i \longrightarrow TA/J\}_{i\in I})$, where $\pi : TA \twoheadrightarrow TA/J$ is the canonical projection onto the quotient, has the universal property (UF).

For (unital) C^*-algebras A_i, $i \in I$, we take a free product $\{f_i : A_i \longrightarrow F\}_{i\in I}$ of the A_i viewed as (unital) *-algebras and equip F with the C^*-norm defined by [Avi82; Ped99; Bla06]:

$$\text{(2.60)} \qquad \|a\| := \sup\left\{ \|D(a)\|_{B(H)} \;\middle|\; D : F \longrightarrow B(H) \text{ is a (unital) *-representation} \right\}, \qquad \forall a \in F.$$

To prove that this well-defines a C^*-norm, we only need to show that the supremum exists and is finite; the rest is seen in a straightforward manner. Suppose a (unital) *-representation $D : F \longrightarrow B(H)$ exists, where H is some Hilbert space. By construction of F, we can write any $a \in F$ (non-canonically and in many different ways) as $a =$

$\sum_{r=1}^{s}\prod_{k=1}^{n} f_{i_k}(a_{i_k})$ for $a_{i_k} \in A_{i_k}$, $i_k \in I$, $n_r \geq 1$ and $s \geq 1$, and compute $\|D(a)\|_{B(H)} \leq \sum_{r=1}^{s}\prod_{k=1}^{n} \|a_{i_k}\|_{i_k}$. This entails that the supremum in (2.60) exists and is finite; other decompositions can be used to improve the upper bound. To show the existence of such a (unital) *-representation, we choose for each $i \in I$ a state $\omega_i : A_i \longrightarrow \mathbb{C}$ and apply the Gelfand-Naimark-Segal construction to obtain a Hilbert space H_i, a (unital) *-representation $D_i : A_i \longrightarrow B(H_i)$ and a cyclic unit vector Ω_i. We define H by the infinite tensor product of Hilbert space $\bigotimes_{i\in I}(H_i, \Omega_i)$ and obtain for each $i \in I$ a (unital) *-representation $\bar{D}_i : A_i \longrightarrow B(H)$ by setting for all $a_i \in A_i$, $\bar{D}_i(a_i)$ to be the continuous extension of $\bigotimes_{j\in I} T_j$, where $T_j = \mathbb{1}_{H_j}$ for all $j \in I \setminus \{i\}$ and $T_i = D(a_i)$. Exploiting (UF), we find a (unital) *-representation $D : F \longrightarrow B(H)$ satisfying $D \circ f_i = \bar{D}_i$ for all $i \in I$.

The norm completion $\overline{F}^{\|\cdot\|}$ of F with respect to the C*-norm given by (2.60) is known to be a (unital) C*-algebra ([Nai72, §16, Sec.1, Prop.I] or [BGP07, Rem.4.1.11]) and the canonical injection $\mathrm{inj}_{\overline{F}^{\|\cdot\|}} : F \hookrightarrow \overline{F}^{\|\cdot\|}$ constitutes a (unital) *-monomorphism. The pair $(\overline{F}^{\|\cdot\|}, \{\mathrm{inj}_{\overline{F}^{\|\cdot\|}} \circ f_i : A_i \longrightarrow \overline{F}^{\|\cdot\|}\}_{i\in I})$ meets the universal property (UF).

We consider the free product of (unital) (C)*-algebras $\{A_i\}_{i\in I}$ to be unique in the sense of Proposition and Definition 2.4.15, thus speak of *the* free product and denote it by $(\bigstar_{i\in I} A_i, \mathrm{inj}_j^{\star} : A_j \longrightarrow \bigstar_{i\in I} A_i)$.

Chapter 3

General Quantum Field Theory

"*Die reine mathematische Spekulation wird unfruchtbar, weil sie aus einem Spiel mit der Fülle der möglichen Formen nicht mehr zurückfindet zu den ganz wenigen Formen, nach denen die Natur wirklich gebildet ist. Und die reine Empirie wird unfruchtbar, weil sie schließlich in endlosen Tabellenwerken ohne inneren Zusammenhang erstickt. Nur aus der Spannung, aus dem Spiel zwischen der Fülle der Tatsachen und den vielleicht dazu passenden mathematischen Formeln können die entscheidenden Fortschritte kommen.*"

-Werner Heisenberg, "Die Bedeutung des Schönen in der exakten Naturwissenschaft", *Quantentheorie und Philosophie: Vorlesungen und Aufsätze*, ed. by J. Busche, Philipp Reclam jun., Stuttgart, 1979: 91-114.

3.1 Algebraic quantum field theory

We present the idea and the formalism of the algebraic approach to quantum field theory.

The idea of an algebraic formulation of quantum theory is probably as old as quantum mechanics itself and originated from the very same epoch-making textbook of J. von Neumann [Neu96] which provided the definitive Hilbert space formalism of quantum mechanics used today. Judging from [JNW34; Neu36], one could even claim that the Hilbert space approach was just intended as an intermediate step towards an algebraic formulation of quantum mechanics. A C^*-algebraic approach to quantum theory had been championed by I.E. Segal for many years, see [Seg47; SM63] and I.E. Segal's work mentioned therein, but it took some time until his ideas were appreciated in quantum field theory. To be precise, it took until [HK64] to put forward weak equivalence of representations of C^*-algebras introduced by [Fel60] as a realistic mathematical notion for "*physical equivalence*", and not unitary equivalence, opening thus the way for a purely abstract C^*-algebraic formulation of quantum field theory. See also [Emc09, Sec.2.1.d] and [Wal94, Sec.4.5].

Although technically succeeding these endeavours, algebraic quantum field theory is much more than a mathematical formulation of quantum field theory. It is a school of thought which founds itself on the principle of locality and the concept of local observables[1]; for this reason, algebraic quantum field theory is also known as *local quantum physics*. It is broadly accepted that the concept of local algebras of observables was conceived with R. Haag's contribution[2] to the international colloquium "*Les problèmes mathématiques de la théorie quantique des champs*" in Lille, June 1957 (see [Haa10a] for an English translation); a first survey of the postulates for quantum field theory in this spirit was given in [HS62], using von Neumann algebras associated with spacetime regions, and [Ara61] was the first in-depth exposition of these ideas, again in terms of von Neumann algebras associated with regions of spacetime. However, it was [HK64] which truly marked the birth of algebraic quantum field theory. As an additional (non-exhaustive) selection of general literature on the subject matter of the algebraic approach, we mention [Hor90; Haa96; Ara99; Emc09]. Helpful introductory remarks can also be found in [BLT75].

Any physical experiment is conducted in a spatially confined laboratory and over a finite period of time. Hence, it should be possible and meaningful to associate each physical quantity determined directly from the experiment with this bounded region of spacetime. This is called the *localisation property*. Furthermore, no signal can travel faster than with the speed of light, which implies that no physical processes taking place in spacelike separated regions of spacetime can influence each other. This is *Einstein's causality principle*. The *principle of locality* is the combination of both, and observables having the specified properties of localisation and causality are called *local observables*.

At the very core of the algebraic approach lies now the metaphysical assumption[3] that since any physical measurement takes place in a finite region of spacetime, the whole theory must be expressible in terms of local observables. Nothing else is needed for a complete description of the physical phenomena. All physically relevant information of a quantum field theory is completely encoded in its net of the local observables,

[1] There have also been attempts [HK64; BLT75] to base the algebraic approach on the notion of operations. An operation is any physical interference with a physical system within a finite time interval (via a physical apparatus) causing a transition of the physical system from an initial state to a final state. A finite or infinite sequence of operations which results in the measurement of a physical quantity yields an observable. Hence, the notion of an operation is more general and more elementary than that of an observable but relies on more general assumptions on the measuring apparatus. Also, properties like locality are more naturally expressed in terms of observables.

[2] `http://www.lqp.uni-goettingen.de/lqp/events/aqft50/haag.pdf`

[3] The reader is admonished not to read this as some kind of philosophical prejudice. "*metaphysical assumption*" has to be understood from an operational point of view here as some axiomatic minimal input.

that is, the correspondence

$$\mathbf{B} \longmapsto \mathfrak{A}(\mathbf{B}) \tag{3.1}$$

which assigns to each spacetime region $\mathbf{B}$ the C^*-algebra $\mathfrak{A}(\mathbf{B})$ of all (bounded[4]) local observables associated with $\mathbf{B}$, i.e. all (bounded) observables which can be measured by means of an experiment in $\mathbf{B}$. Usually, the spacetime regions are taken to be bounded open sets ($\Longrightarrow$ compact closure) or open double cones.

The algebraic approach comes in two kinds, the first one of which is the Haag-Araki theory in terms of "*concrete*" von Neumann algebras [Ara61; HS62; Ara99]. It was only later realised in [HK64] that "*abstract*" C^*-algebras provided the natural setting for understanding superselection sectors and the role of unobservable fields, thus leading to the Haag-Kastler theory. Both the Haag-Araki and the Haag-Kastler theory impose additional conditions on the net of the local observables, so-called Haag-Araki-Kastler axioms. Depending on the context (concrete von Neumann algebras or abstract C^*-algebras), the Haag-Araki-Kastler axioms slightly differ (cf. [Hor90]). We will adopt the Haag-Kastler theory using abstract C^*-algebras, though we will generally speak of the Haag-Araki-Kastler algebraic approach to quantum field theory. A minimal list of Haag-Araki-Kastler axioms imposed on the net of the local observables might be the following one:

(HAK1) Isotony: $\mathbf{B}_1 \subseteq \mathbf{B}_2 \Longrightarrow \mathfrak{A}(\mathbf{B}_1) \subseteq \mathfrak{A}(\mathbf{B}_2)$. Either $\mathfrak{A}(\mathbf{B}_1)$ and $\mathfrak{A}(\mathbf{B}_2)$ have a common identity element or neither of them has an identity element.

(HAK2) The proper orthochronous Poincaré group $\mathrm{P}_+^\uparrow$ acts via automorphisms of the net: $\mathrm{P}_+^\uparrow \ni g \longmapsto \alpha_g$ such that $\alpha_g \mathfrak{A}(\mathbf{B}) = \mathfrak{A}(g\mathbf{B})$.

(HAK3) Locality[5]: $\mathfrak{A}(\mathbf{B}_1)$ commutes[6] with $\mathfrak{A}(\mathbf{B}_2)$ whenever $\mathbf{B}_1$ and $\mathbf{B}_2$ are spacelike separated.

(HAK4) $\mathfrak{A}(\mathbf{B}'') = \mathfrak{A}(\mathbf{B})$, where $\mathbf{B}''$ is the causal completion of $\mathbf{B}$. $\mathbf{B}'$ is the causal complement of $\mathbf{B}$, i.e. the set of all points in spacetime lying spacelike to $\mathbf{B}$.

[4]We may restrict to bounded observables without the loss of generality. Given an unbounded observable represented by an unbounded self-adjoint operator on some Hilbert space, we may consider its family of spectral projectors and bounded functions of it. Given a more general operation represented by a closeable unbounded operator on some Hilbert space, we may make a polar decomposition of its closure and obtain thus a partial isometry and an non-negative self-adjoint operator, which again may be spectrally decomposed.

[5]Other common aliases are: Einstein causality and local commutativity.

[6]This requires a comment. The bounded open sets or the open double cones form an up-directed set under inclusion. Hence, there is some bounded open set or open double cone $\mathbf{B}_3$ such that $\mathbf{B}_1, \mathbf{B}_2 \subseteq \mathbf{B}_3$. By isotony, $\mathfrak{A}(\mathbf{B}_1), \mathfrak{A}(\mathbf{B}_2) \subseteq \mathfrak{A}(\mathbf{B}_3)$. "$\mathfrak{A}(\mathbf{B}_1)$ *commutes with* $\mathfrak{A}(\mathbf{B}_2)$" now means that $[\mathfrak{A}(\mathbf{B}_1), \mathfrak{A}(\mathbf{B}_2)] = \{0_{\mathfrak{A}(\mathbf{B}_3)}\}$ in $\mathfrak{A}(\mathbf{B}_3)$, i.e. $[a,b] = 0_{\mathfrak{A}(\mathbf{B}_3)}$ in $\mathfrak{A}(\mathbf{B}_3)$ for all $a \in \mathfrak{A}(\mathbf{B}_1)$ and for all $b \in \mathfrak{A}(\mathbf{B}_2)$.

(HAK4) is sometimes called the diamond property or causality. It stipulates that the dynamical law of the quantum field theory respects the causal structure of the spacetime (hyperbolic propagation character of the field equations). It is related to [HS62, Postulate 8(b)] and represents a local form of the time-slice axiom (primitive causality) [Hor90, AXIOM IVb].

States are most conveniently described by normalised (if an identity element is present) positive linear functionals on the algebra of quasilocal observables $\mathfrak{A}_{\mathrm{qloc}}$, which is the C^*-inductive limit $\overline{\bigcup_{\mathbf{B}} \mathfrak{A}(\mathbf{B})}$ of the algebras of local observables. The C^*-inductive limit can be taken because of isotony and up-directedness of the bounded open sets or open double cones under inclusion. Also, due to taking uniform limits, the essential character of the local observables is not changed. Note that we can also form the algebra of all local observables $\mathfrak{A}_{\mathrm{loc}} := \bigcup_{\mathbf{B}} \mathfrak{A}(\mathbf{B})$[7].

Though being of a pure abstract and axiomatic nature, the Haag-Araki-Kastler algebraic approach to quantum field theory is also extremely flexible at the same time. To better match a concrete physical problem at hand, the existing axioms may very well be modified or even dropped; new additional axioms may of course be added. For example, regarding (HAK1), we can in principle replace the inclusion of C^*-algebras by more general *-monomorphisms. If we happen to work in curved spacetimes[8], we can modify (HAK2) and replace the action of the proper orthochronous Poincaré group with the group action of the spacetime symmetries; however, since a generic curved spacetime does not exhibit spacetime symmetries at all, we may even drop (HAK2) from our list completely. Considering quantum gauge field theories, we may wish to include new axioms addressing the gauge symmetry. It is also possible (and allowed) to change the ingredients of the net itself (and the interpretation accordingly). In this thesis, we will mostly consider field algebras, i.e. unital *-algebras of smeared quantum fields, instead of C^*-algebras of local observables.

Note that the formalism of the Haag-Araki-Kastler algebraic approach to quantum field theory can do without Lagrangeans, Hamiltonians and does not presuppose any classical theory of which it is the quantisation. In particular, note the absence of quantum fields and particles in the formalism. Indeed, although of utmost importance, quantum fields and particles are not elementary notions of quantum field theory, whose true basic notions are observables and states. Quantum fields are auxiliary constructs serving, among other things, to implement the principle of locality, and are not to be considered as synonyms for elementary particles. Particles themselves should emerge from the formalism of quantum field theory and should not function as the basic ingredient. Furthermore, the genuine subject of study and axiomatisation really is the *net*

[7] The commutativity postulated in (HAK3) can also be regarded in $\mathfrak{A}_{\mathrm{loc}}$ or $\mathfrak{A}_{\mathrm{qloc}}$.

[8] We have not been absolutely clear about this (deliberately): our presentation was only regarding Minkoswki space so far.

of the local observables, i.e. the distribution of the local observables in their spacetime localisations, and not individual local observables and their respective properties.

The great successes of algebraic quantum field theory include the understanding of superselection rules, the role of unobservable fields, particle statistics [HK64; DHR69a; DHR69b; DHR71; DHR74; DR90] and the theory of particle scattering [Haa58; Rue62; AH67]. The algebraic approach also made important contributions to quantum statistical mechanics (thermodynamic limit, KMS-condition for thermal equilibrium states [KMS-states], spontaneous symmetry breaking, dynamical stability and passivity of states, ...), turned out to be helfpful for the discussion of infrared problems in quantum electrodynamics [Buc82] and stimulated the interaction between physics and mathematics, notably with the theory of operator algebras. In the last 25 years, algebraic quantum field theory has proven essential for systematic treatments of quantum fields in curved spacetimes[9], see e.g. [HNS84] for some general aspects. Among the achievements of the algebraic approach to quantum field theory in curved spacetimes are applications to the general discussion of the Hawking effect [FH90; KW91], quantum energy inequalities [Few00], rigorous perturbative constructions of interacting quantum field theories [BFK96; BF00] and applications in cosmology [DHP09; DHP11; Hac10].

3.2 Locally covariant quantum field theory

Locally covariant quantum field theory can be regarded as the consistent further development of algebraic quantum field theory in curved spacetimes and the complete framework was given in the seminal paper [BFV03] by R. Brunetti, K. Fredenhagen and R. Verch, though some of the ideas and concepts had forerunners e.g. [Kay79; Dim80; Kay92] and were used in previous publications [Ver01; HW01; HW02].

Using categorical notions and concepts has proven to be much more than just a mere reformulation of known results. It has considerably advanced the whole subject area of quantum field theory in curved spacetimes and made a significant impact on many of its aspects: [Ver01] (spin-statistics theorem), [FP06; Few07] (quantum energy inequalities), [BR07; BR09] (theory of superselection rules), [DFP08] (cosmology), [San09; Dap11] (Reeh-Schlieder theorem), [BFR12] (classical field theory), [FV12a; FV12b] (SPASs) and, most importantly, [HW01; HW02; BDF09] (perturbative constructions of interacting quantum field theories). It can also be shown [BFV03, Prop.2.3] that the formalism of algebraic quantum field theory can be recovered from the framework of locally covariant quantum field theory.

We will now state the basic definitions of locally covarint quantum field theory:

DEFINITION 3.2.1. A functor $F : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$ is called a *locally covariant*

[9]This fact is due to the aforementioned flexibility of the algebraic approach.

quantum field theory.

DEFINITION 3.2.2. A *locally covariant quantum field* is a natural transformation $\tau : F \dot{\longrightarrow} G$ between a functor $F : \mathbf{Loc} \longrightarrow \mathbf{Hlctvs^m}$, where $\mathbf{Hlctvs^m}$ is the category of all Hausdorff locally convex vector spaces [Jar81; Rud91; BB03] with injective continuous linear maps as morphisms, and a functor $G : \mathbf{Loc} \longrightarrow \mathbf{top^*Alg^m}$, where $\mathbf{top^*Alg^m}$ is the category of all topological *-algebras with continuous *-monomorphisms as morphisms.

Note, insisting on unital *-monomorphisms, that is, injective unital *-homomorphisms is important to the formalism; it fully implements the principle of general covariance of general relativity in a geometrically local fashion.

We provide some more notions of locally covariant quantum field theory.

DEFINITION 3.2.3. We call a locally covariant quantum field theory $F : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$ *causal* if and only if $[F\psi_1(F\mathbf{M}_1), F\psi_2(F\mathbf{M}_2)] = 0_{F\mathbf{N}}$, i.e. $[a,b] = 0_{F\mathbf{N}}$ for all $a \in F\psi_1(F\mathbf{M}_1)$ and for all $b \in F\psi_2(F\mathbf{M}_2)$, whenever we have $\mathbf{Loc}$-morphisms $\psi_i : \mathbf{M}_i \longrightarrow \mathbf{N}$, $i = 1, 2$, such that $\psi_1(M_1)$ and $\psi_2(M_2)$ are spacelike separated in $\mathbf{N}$.

DEFINITION 3.2.4. A $\mathbf{Loc}$-morphism $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ is called a *Cauchy morphism* or simply *Cauchy* if and only if $\psi(M)$ contains a Cauchy surface for $\mathbf{N}$.

For some properties of Cauchy morphisms, see [FV12a, Sec.2.2 + Appx.A.1].

DEFINITION 3.2.5. It is said that a locally covariant quantum field theory $F : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$ obeys the *time-slice axiom* if and only if $F\psi : F\mathbf{M} \longrightarrow F\mathbf{N}$ is a $\mathbf{(C)^*Alg_1^m}$-isomorphism whenever $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ is Cauchy.

The ideas and notions of the functorial framework of algebraic quantum field theory in curved spacetimes presented by [BFV03] are not just restricted to quantum field theory. They can also be fruitfully applied to other physical theories in general. If we, following [FV12a; FV12b], define a variable category **Phys** for the physical systems currently under consideration, where the morphisms are given by the inclusions of physical systems into other physical systems as subsystems, we may generally define:

DEFINITION 3.2.6. A *locally covariant (physical) theory* is a functor $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$.

This allows us to apply the ideas of locally covariant quantum field theory and categorical methods to other branches of physics as well. In this way, the functorial approach is just as flexible as the algebraic one.

3.3 The time-slice axiom

The two most fundamental questions which are pursued in physics are surely

"*How can we explain an observed phenomenon?*"

and

"*What predictions can we make?*".

A good physical theory should ultimately answer both of these two questions; it should always explain observations made and also enable us to make predictions. It is clear from this that predictability must play a crucial part in any physically reasonable theory

In a classical theory which describes the evolution in time of a physical system by some differential equations, the so-called equations of motion, the question of the predictability of the future is equivalent to asking wether or not the initial value problem for the equations of motion is well-posed, that is, asking whether or not the equations of motion can be solved uniquely once sufficient inital data have been provided, and whether or not small perturbations of the initial data change the unique solution drastically. By means of a unique solution, which does not change too much under small alterations of the initial data, the classical theory allows one then to say what configuration the physical system will be in (within sufficiently good bounds) for future times (and/or was in for times in the past)[10] if one is given adequate knowledge of the configuration which the physical system is in at present time (= the initial data up to small perturbations).

In quantum field theory however, the situation is more delicate. Due to their singular behaviour, it is problematic −to say the least− to define quantum fields at a point as operator-valued functions, though they may be defined as sesquilinear forms on a dense domain of the physical Hilbert space [Haa63], and one is forced to formulate them as operator-valued (tempered) distributions [Fri51; SB56; BP57; Wig64; WG65; Jaf67] and [BLT75, Sec.10.4]; we also highly recommend [Wig96]. This is not −as one might think− a result of mathematical abstraction and to "*make the abstract mathematics work*" but a physically reasonable necessity. As discussed in [BR33; BR50] with the help of the electromagnetic field, the measurement of a field quantity at a point of spacetime is in principle impossible and meaningless from the physical point of view; only averages (= smearings) of such quantities over finite spacetime regions have a

[10]One can of course strive for the best-case scenario here, that is, the unique solution is defined for all times in the future and the past. However, this is an ambitious idealisation as unique solutions for equations of motion might only exist locally, i.e. for a finite time-interval. In most cases, a physicist is actually only interested in the configuration of a physical system for the finite time period of an experiment and not for all of the future and/or past, which would also be highly impracticable!

well-defined physical meaning. It is worthwhile noticing that this is one of the many instances where mathematical rigour is in perfect harmony with the actual physical situation and does not display an over-idealisation.

Admittedly, one is now faced with serious obstacles towards the formulation of an initial value problem and its well-posedness on Minkoswki space already. The product of two distributions is not always mathematically well-defined and hence non-linear field equations will cause problems. Also, it is problematic to restrict distributions to closed sets, where we have Cauchy surfaces in mind, so it can become unclear how to impose initial data. In the algebraic approach to quantum field theory, it is even less clear how an initial value problem should be formulated in general. For example, it is not always clear from the start what the local observables associated with a constant time hyperplane $\Sigma_t = \{t\} \times \mathbb{R}^3$ for some $t \in \mathbb{R}$ are or should be.

For these reasons, the time-slice axiom has replaced a well-posed initial value problem in general quantum field theory. Moreover, the time-slice axiom is considerably more potent than a well-posed initial value problem since it is still applicable (i.e. it can still be formulated and makes sense) for quantum field theories which have not been obtained by a Lagrangean or a Hamiltonian formalism or whose field equations are just too complicated to be written down explicitly or where there are no field equations at all.

Foreshadowed in [Wig57] (last paragraph of Sec.6) and also in [Haa10a] (Sec.2, Axiom (IV) 'Causality'[11]), the time-slice axiom was included into the list of postulates for quantum field theory and paid more attention to in [HS62], who called it "*primitive causality*". Note that most of the textbooks on algebraic and general quantum field theory, e.g. [BLT75; BLOT90; Hor90; Haa96], as well as most of the publications we use, e.g. [HS62; Haa72], employ the term "*primitive causality*". However, we will use the synonym "*time-slice axiom*". As shown in [HS62], some models of generalised free fields[12] [Gre61] illustrate that the time-slice axiom is independent from the other postulates for a local quantum field theory. However, not many studies were devoted to the time-slice axiom in the last century. An exception is [Gar75], who showed the equivalence of the time-slice axiom and the diamond property (HAK4) for generalised free fields. Only after the introduction of the generally covariant locality principle in algebraic quantum field theory by [BFV03], more and more attention was paid to the time-slice axiom. Nowadays, it plays the key role for the definition of the relative Cauchy evolution [BFV03; FV12a; FS14a], which we present in Section 3.4, and it is

[11] As R. Haag clearly states, it was not clear at the time (1957) if a connection between the time-slice axiom and the postulate for locality existed.

[12] For a brief orientation, e.g. generalised free scalar fields are free fields whose "*commutator function*" is (up to a sign depending on the conventions chosen) of the form "$\mathrm{i} \int_0^\infty \rho\left(m^2\right) \Delta\left(x-y\,;m\right)\, dm$", where ρ is a positive weight function of not too fast increase and "$\Delta\left(x-y\,;m\right)$" the usual "*commutator function*" for the free scalar field of mass m.

also important for associating an algebra of observables with a Cauchy surface [BF06; Chi08]. In [Chi08; CF09], it was proven that the time-slice axiom holds for the perturbatively treated interacting real scalar quantum field on globally hyperbolic spacetimes and more recently, the time-slice axiom has been utilised in a novel construction of the KMS- and the vacuum state for the interacting real scalar quantum field in Minkowski space [Lin13; Lin14], using a perturbative framework. In this approach, it was also shown that infrared divergences at finite temperature are absent and a previously unknown relation between perturbative algebraic quantum field theory and quantum statistical mechanics was discovered.

We will now state and briefly discuss three versions of the time-slice axiom: the functional analytic, the algebraic and a generalised version of the functorial time-slice axiom given in Definition 3.2.5. The following functional analytic version can be found in [BLT75, Sec.9.3] and with a slight alteration in [SW64, Sec.3-2] and [WG65, Sec.II, Def.1 + 1′]:

DEFINITION 3.3.1. (time-slice axiom; functional analytic)
Let Φ be an operator-valued (Lorentz-tensor or spin-tensor[13]) distribution with components $\Phi_1, \dots, \Phi_n \in \mathscr{S}'(\mathbb{R}^4)$ for $n \in \mathbb{N} \setminus \{0\}$. For any $\mathcal{O} \subseteq \mathbb{R}^4$ open, let $\mathfrak{P}(\mathcal{O})$ denote the algebra of all polynomials in the operators $\Phi_i(f)$, where $i = 1, \dots, n$ and $f \in \mathscr{S}(\mathbb{R}^4)$ with $\operatorname{supp} f \subseteq \mathcal{O}$, and define the time-slice $\mathcal{O}_{t_0,\delta} := \{(t,x,y,z) \in \mathbb{R}^4 \mid t_0 - \delta < t < t_0 + \delta\}$, where $t_0 \in \mathbb{R}$ and $\delta > 0$. Then, $\mathfrak{P}(\mathcal{O}_{t_0,\delta})$ is an irreducible set[14] of (unbounded) operators.

In the functional analytic set-up, the time-slice axiom expresses that the whole quantum field theory, i.e. any operator acting on the physical Hilbert space, can be reconstructed from the polynomials in $\mathfrak{P}(\mathcal{O}_{t_0,\delta})$ for any $t_0 \in \mathbb{R}$ and any $\delta > 0$; more loosely speaking, every operator is a function of the smeared field operators. Hence, every state of the physical quantum field system under consideration can be determined (at least up to a very good approximation) by observations made in a suitably small time interval. This is a very desirable feature since otherwise it would mean that an experimenter would have to make observations at all times in order to determine the configuration of a physical quantum field system -a bad theory as its experimental test would be impracticable.

The functional analytic version of the time-slice axiom can easily be extended to the setting of curved spacetimes which are globally hyperbolic: instead of rapidly

[13] "*Lorentz-tensor*" and "*spin-tensor*" are to indicate that the components of Φ have specified transformation properties under the representation of the proper orthochronous Lorentz group $\mathrm{L}_+^\uparrow$ or its universal covering group $\mathrm{SL}(2;\mathbb{C})$.

[14] A set S of unbounded operators on a Hilbert space H with a common dense domain of definiton D is called *irreducible* if and only if any bounded operator B on H which weakly commutes with each $A \in S$, i.e. $\overline{\langle \varphi \mid BA\psi \rangle} = \langle A^* \varphi \mid B\psi \rangle$ for all $\varphi, \psi \in D$, is a complex multiple of the identity operator. This definition goes back to [Rue62].

decreasing smooth functions $\mathscr{S}(\mathbb{R}^4)$ and tempered distributions $\mathscr{S}'(\mathbb{R}^4)$, which are not available on a generic curved spacetime, one has to work with smooth functions of compact support $\mathscr{D}(M)$ and distributions $\mathscr{D}'(M)$. The time-slice is taken to be an open neighbourhood of a smooth spacelike Cauchy surface.

In algebraic quantum field theory, the time-slice axiom appears in the following form [Haa72, Sec.3]:

DEFINITION 3.3.2. (time-slice axiom; algebraic)
Let $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O})$ be a Haag-Araki-Kastler net of C^*-algebras of local observables, where $\mathcal{O}$ ranges over all regions in Minkowski space[15], and $\mathfrak{A}_{\mathrm{qloc}}$ the C^*-algebra of quasilocal observables, i.e. the C^*-inductive limit of the C^*-algebras of local observables. Suppose $\{\mathcal{O}_i \mid i \in I\}$ is a family of regions in Minkowski space which covers $\Sigma_t := \{t\} \times \mathbb{R}^3$, where $t \in \mathbb{R}$ is fixed. Let $\bigvee_{i \in I} \mathfrak{A}(\mathcal{O}_i)$ be the C^*-algebra generated by the local C^*-algebras $\mathfrak{A}(\mathcal{O}_i)$, $i \in I$. Taking the intersection over all possible open covers of Σ_t by regions in Minkowski space, $\bigcap \bigvee_{i \in I} \mathfrak{A}(\mathcal{O}_i)$, it holds that $\mathfrak{A}_{\mathrm{qloc}} = \bigcap \bigvee_{i \in I} \mathfrak{A}(\mathcal{O}_i)$, and thus $\mathfrak{A}_{\mathrm{qloc}} = \bigvee_{i \in I} \mathfrak{A}(\mathcal{O}_i)$ in particular.

Observe that there are various reformulations of the algebraic time-slice axiom, see e.g. [HS62, Sec.II, Postulate 8(a) + 8(b)], [BLT75, Sec.23.2], [Haa96, (III.1.10)] and [Hor90, Sec.1.2, AXIOM IV + IVa + IVb]; we strongly recommend the last references as it also states versions of the time-slice axiom for the Haag-Araki theory, that is, the concrete operator algebraic approach where the C^*-algebras of local observables are von Neumann algebras in a Hilbert space. Also note that the axiom (HAK4) corresponds to the time-slice axiom and can indeed be regarded as its local form.

The algebraic time-slice axiom is readily carried over to globally hyperbolic spacetimes. As regions one may take e.g. all globally hyperbolic open subsets or, as we will often do in this thesis, all globally hyperbolic open subsets which are contractible, provided a suitable substitute for the C^*-inductive limit, which is the C^*-algebra $\mathfrak{A}_{\mathrm{qloc}}$ of quasilocal observables, is available. However, other choices for regions are possible, depending on the quantum field theory and the particular globally hyperbolic spacetime it is viewed on. The constant time hyperplane $\Sigma_t = \{t\} \times \mathbb{R}^3$, where $t \in \mathbb{R}$, is to be replaced with a smooth spacelike Cauchy surface.

Since we have already stated the time-slice axiom in the functorial framework for quantum field theory by [BFV03] in Definition 3.2.5, we only want to comment briefly on some possible variations of it here. First of all, note that the functorial time-slice axiom is still meaningful if the category $\mathbf{(C)^*Alg_1^m}$ is replaced by the category $\mathbf{(C)^*Alg_1}$. In fact, we can replace the category $\mathbf{(C)^*Alg_1^m}$ by any arbitrary category

[15] We leave a concrete specification of what we mean by a "*region*" in Minkowski space open. We could take, as usual, all bounded open sets or all open double cones. The regions need to form an up-directed poset under inclusion though, in order for the C^*-inductive limit to exist.

$\mathcal{D}$ and the time-slice axiom can still be formulated. Secondly, the domain category **Loc** may very well be replaced by one of its full subcategories of the form $\mathbf{Loc}_q$ or, for $\mathbf{M} \in \mathbf{Loc}_q$, by $(K_q \downarrow \mathbf{M})$ or even by $\mathrm{loc}^q_{+\mathbf{M}}$ or $\mathrm{loc}^q_{-\mathbf{M}}$, where $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$ is the inclusion functor. In order to be able to compute relative Cauchy evolutions for twisted quantum field theories, we will also find it helpful to replace **Loc** with the subcategory category loc_{+M} for $\mathbf{M} \in \mathbf{Loc}$. We summarise:

DEFINITION 3.3.3. (time slice axiom; functorial, generalised)
Let $\mathcal{C}$ stand for one of the categories $\mathbf{Loc}$, $\mathbf{Loc}_q$, $(K_q \downarrow \mathbf{M})$, $\mathrm{loc}^q_{+\mathbf{M}}$, $\mathrm{loc}^q_{-\mathbf{M}}$ or loc^q_{+M}, where $\mathbf{M} \in \mathbf{Loc}$ and $q \subseteq \mathbb{N} \smallsetminus \{0\}$ or $q = s$ or $q =$ ©, and let $\mathcal{D}$ be any category. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is said to obey the *time-slice axiom* if and only if for each Cauchy morphism $f : X \longrightarrow Y$ in $\mathcal{C}$, $Ff : FX \longrightarrow FY$ is a $\mathcal{D}$-isomorphism.

We finish this section by introducing time-slice maps and proving other helpful technical results related to the time-slice axiom. The statement of the following lemma seems to follow from a well-known standard argument as can be found in[KW91, Appx.B]. However, before applying this standard argument, one needs to be a bit more careful since we want to "*squeeze*" the support of some compactly supported smooth cross-section into arbitrary open neighbourhoods $O(\Sigma)$ of a fixed smooth spacelike Cauchy surface Σ. In particular, $O(\Sigma)$ is allowed to be arbitrarily nasty; for example, $O(\Sigma)$ needs not to be a globally hyperbolic open subset or could approach asymptotically Σ as indicated in the figure below (left-hand side above Σ).

LEMMA 3.3.4. *Let* $\mathbf{M} \in \mathbf{Loc}$, Σ *a smooth spacelike Cauchy surface for* $\mathbf{M}$, $O(\Sigma)$ *any open neighbourhood of* Σ *in* M, $\xi = (E, M, \pi, V)$ *a smooth vector bundle and* $D : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi)$ *a normally hyperbolic linear differential operator of metric type. For all* $\sigma \in \Gamma^\infty_0(\xi)$, *there exist* $\sigma_{€}, \sigma_{£} \in \Gamma^\infty_0(\xi)$ *satisfying*

$$\text{(3.2)} \qquad \operatorname{supp} \sigma_{€} \subseteq O(\Sigma) \qquad \textit{and} \qquad \sigma = \sigma_{€} + D\sigma_{£}.$$

Neither $\sigma_{€}$ *nor* $\sigma_{£}$ *are uniquely determined.*

Proof: By applying the Bernal-Sánchez splitting theorem, we may assume $M = \mathbb{R} \times \Sigma$, $g = \beta\, d\mathrm{pr}_1 \otimes d\mathrm{pr}_1 - \mathrm{pr}_2^* h_{\mathrm{pr}_1}$, where $\beta \in \mathcal{C}^\infty(M, \mathbb{R}^+)$, $\{h_t\}_{t \in \mathbb{R}}$ is a smooth 1-parameter family of Riemannian metrics on Σ and $[T] = [\frac{\partial}{\partial \mathrm{pr}_1}]$ without the loss of generality. The overall aim is to find $\varepsilon > 0$ and a compact subset $K \subseteq \Sigma$ (which is also compact in M) such that $Q := (-\varepsilon, \varepsilon) \times K \subseteq O(\Sigma)$ and $[-\varepsilon, \varepsilon] \times K \subseteq O(\Sigma)$ and whenever $p \in J(\operatorname{supp} \sigma)$ fulfils $\mathrm{pr}_1(p) \in (-\varepsilon, \varepsilon)$, then $p \in Q$.

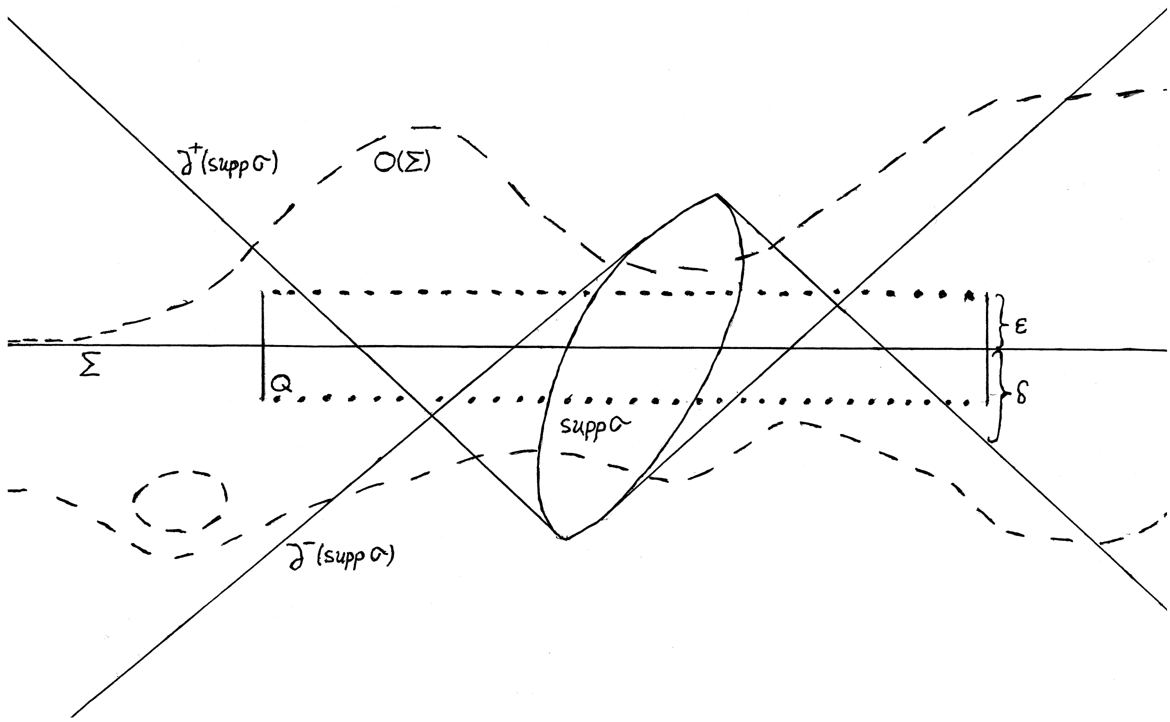


FIGURE 3.1: Visual aid for the proof of Lemma 3.3.4. For simplicity, M is taken to be $\mathbb{R} \times \Sigma$ and $\Sigma = \mathbb{R}$. Observe that $-\varepsilon < \mathrm{pr}_1(p) < \varepsilon$ implies $p \in Q$ for each $p \in J(\operatorname{supp} \sigma)$.

Once we have found such $\varepsilon > 0$ and $K \subseteq \Sigma$ compact, we can apply the standard argument, which we will spell out in detail. We consider $\Sigma_+ := \Sigma_{\varepsilon/2}$, $\Sigma_- := \Sigma_{-\varepsilon/2}$ and let $\{\chi^+, \chi^-\}$ be a smooth partition of unity subordinated to $\{I^+(\Sigma_-), I^-(\Sigma_+)\}$, which is an open cover for M. Denote the unique retarded Green operator for D by G^{ret} and the unique advanced Green operator for D by G^{adv}, which both exist thanks to [BGP07, Cor.3.4.3] or [Wal12, Cor.4.3.7]. Then, we define a smooth cross-section in ξ by

$$\sigma_{€} := \sigma - D\chi^- G^{\mathrm{ret}}\sigma - D\chi^+ G^{\mathrm{adv}}\sigma. \tag{3.3}$$

Let $p \in M$. There are four possibilities: if $\mathrm{pr}_1(p) \geq \frac{\varepsilon}{2}$, then $\chi^+ = 1$, $\chi^- = 0$ and so $\sigma_{€} = \sigma - DG^{\mathrm{adv}}\sigma = \sigma - \sigma = 0$. If $\mathrm{pr}_1(p) \leq -\frac{\varepsilon}{2}$, then $\chi^+ = 0$, $\chi^- = 1$ and we will find $\sigma_{€} = \sigma - DG^{\mathrm{ret}}\sigma = \sigma - \sigma = 0$. If, finally, $\mathrm{pr}_1(p) \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ but $p \notin Q$, then $p \notin J(\operatorname{supp}\sigma)$ and $\sigma_{€} = 0$. From this we see that $\sigma_{€}$ can only be non-zero for $p \in M$ with $\mathrm{pr}_1(p) \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ and so $p \in Q$. Since $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}) \times K \subseteq Q$, $\operatorname{supp}\sigma \subseteq Q \subseteq O(\Sigma)$ is compact. We define another smooth cross-section in ξ by

$$\sigma_{£} := \chi^- G^{\mathrm{ret}}\sigma + \chi^+ G^{\mathrm{adv}}\sigma. \tag{3.4}$$

The compactness of $J^+(\Sigma_-) \cap J^-(\operatorname{supp}\sigma)$ and $J^+(\operatorname{supp}\sigma) \cap J^-(\Sigma_+)$ due to [BGP07, Cor.A.5.4] implies the compactness of $\operatorname{supp}\sigma_{£}$ and we are done.

Hence, we need to find $\varepsilon > 0$ and $K \subseteq \Sigma$ compact with the specified properties. We will achieve this by a constructive procedure. Let $K \subseteq \Sigma$ be any compact subset. In a topological space, a set is open if and only if it is a neighbourhood for each of its points. So we can find for each $q \in K$ an open neighbourhood V_q of $(0,q)$ in M which is of the form $(a_q, b_q) \times U_q$, where $a_q < 0 < b_q$ and $U_q \subseteq \Sigma$ open ($M = \mathbb{R} \times \Sigma$ carries the product topology), and completely contained in $O(\Sigma)$. We can cover K with finitely many of these sets, say $K \subseteq \bigcup_{i=1}^n V_i$, $n > 0$. Take $a := \max\{a_i \mid i = 1, \ldots, n\}$ and $b := \min\{b_i \mid i = 1, \ldots, n\}$, then we see that $K \subseteq (a,b) \times \bigcup_{i=1}^n U_i \subseteq O(\Sigma)$. In other words, we have proven that for any $K \subseteq \Sigma$ compact, there is $t_K > 0$ so that $(t,q) \in O(\Sigma)$ for all $q \in K$ and for all $t \in \mathbb{R}$ satisfying $0 \le |t| < t_K$, i.e. $(-t_K, t_K) \times K \subseteq O(\Sigma)$.

For $t \in \mathbb{R}$, define the sets $K_t^{\pm} := \Sigma_t \cap J^{\pm}(\operatorname{supp}\sigma)$, which are compact by [BGP07, Cor.A.5.4]. Now, let $\delta > 0$ and consider the compact set $K := \operatorname{pr}_2(K_\delta^+) \cup \operatorname{pr}_2(K_{-\delta}^-)$ (continuous maps map compact sets to compact sets). By the above, we find $0 < \varepsilon < \delta$ with the property $Q := (-\varepsilon, \varepsilon) \times K \subseteq O(\Sigma)$. We want to argue that any $p \in J(\operatorname{supp}\sigma)$ with $\operatorname{pr}_1(p) \in (-\varepsilon, \varepsilon)$ automatically lies in Q. Since we surely have $\operatorname{pr}_2(K_t^+) \subseteq \operatorname{pr}_2(K_\delta^+)$ and $\operatorname{pr}_2(K_t^-) \subseteq \operatorname{pr}_2(K_{-\delta}^-)$ for all $t \in (-\varepsilon, \varepsilon)$, $\operatorname{pr}_2(K_t^+ \cup K_t^-) \subseteq \left(K_\delta^+ \cup K_{-\delta}^-\right)$ for all $t \in (-\varepsilon, \varepsilon)$, which shows that $p \in J(\operatorname{supp}\sigma)$ with $\operatorname{pr}_1(p) \in (-\varepsilon, \varepsilon)$ lies in Q. ❒

The dependence of the smooth partition of unity on the smooth cross-section in the proof of Lemma 3.3.4 is a nuisance. However, it can be circumvented at the price of sacrificing arbitrary open neighbourhoods of the smooth spacelike Cauchy surface as follows:

LEMMA 3.3.5. *Let $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ be Cauchy, $\xi = (E, N, \pi, V)$ a smooth vector bundle and $D : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi)$ a normally hyperbolic linear differential operator of metric type. For all $\sigma \in \Gamma_0^\infty(\xi)$, there exist $\sigma_{€}, \sigma_{£} \in \Gamma_0^\infty(\xi)$ with*

$$\operatorname{supp}\sigma_{€} \subseteq \psi(M) \qquad \textit{and} \qquad \sigma = \sigma_{€} + D\sigma_{£}. \tag{3.5}$$

Neither $\sigma_{€}$ nor $\sigma_{£}$ are uniquely determined.

Proof: Fix two smooth spacelike Cauchy surfaces Σ_+ and Σ_- for $\mathbf{N}$ such that $\Sigma_+, \Sigma_- \subseteq \psi(M)$ and Σ_+ lies strictly in the future of Σ_-. This can be achieved using [FV12a, Lem.A.2] and the Bernal-Sánchez splitting theorem. Let $\{\chi^+, \chi^-\}$ be a smooth partition of unity subordinated to the open cover $\{I_{\mathbf{N}}^+(\Sigma_-), I_{\mathbf{N}}^-(\Sigma_+)\}$ of N. For each $\sigma \in \Gamma_0^\infty(\xi)$ define $\sigma_{€}$ by the formula (3.3). Because $\operatorname{supp}\chi^+ \subseteq J_{\mathbf{N}}^+(\Sigma_-)$, it holds $\operatorname{supp}\chi^- \subseteq J_{\mathbf{N}}^-(\Sigma_+)$, $\operatorname{supp} G^{\mathrm{ret/adv}}\sigma \subseteq J_{\mathbf{N}}^{\pm}(\operatorname{supp}\sigma)$ and we find that $\operatorname{supp}\sigma_{€}$ is contained in $\operatorname{supp}\sigma \cup (J_{\mathbf{N}}^+(\Sigma_-) \cap J_{\mathbf{N}}^-(\operatorname{supp}\sigma)) \cup (J_{\mathbf{N}}^-(\Sigma_+) \cap J_{\mathbf{N}}^+(\operatorname{supp}\sigma))$, which is a union of compact sets (use [BGP07, Cor.A.5.4]) and thus compact itself. By construction, $\chi^+ = 1$ and $\chi^- = 0$ in the causal future of Σ_+ in $\mathbf{N}$, and $\chi^+ = 0$ and $\chi^- = 1$ in the causal past of Σ_- in $\mathbf{N}$.

Consequently, $\operatorname{supp}\sigma_{€} \subseteq J^-_{\mathbf{N}}(\Sigma_+) \cap J^+_{\mathbf{N}}(\Sigma_-) \subseteq \psi(M)$ as $\psi(M)$ is causally convex in $\mathbf{N}$. $\sigma_{£}$ is now defined by (3.4) and compactly supported by the same argument given in the proof of Lemma 3.3.4. ❐

The last lemma gives rise to and proves the existence of time-slice maps. As their name suggests, we have introduced time-slice maps in the discussion of the time-slice axiom for specific locally covariant theories, which are based on smooth cross-sections in smooth vector bundles. There, the formalism of time-slice maps will help us to show the validity of the time-slice axiom and to construct inverses explicitly. The specification of inverses usually involves certain choices of representatives of equivalences classes and of smooth spacelike Cauchy surfaces, and time-slice maps will assist us in efficiently dealing with these choices, see e.g. the proof of Proposition 5.3.1.

DEFINITION 3.3.6. Let $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ be Cauchy, $\xi = (E, N, \pi, V)$ a smooth vector bundle and $D : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi)$ a normally hyperbolic linear differential operator of metric type. A *time-slice map* for the triple (ψ, ξ, D) is a linear map $\mathrm{tsm} : \Gamma^\infty_0(\xi) \longrightarrow \Gamma^\infty_0(\xi)$ such that

$$\operatorname{supp}\big((\mathrm{id}_{\Gamma^\infty_0(\xi)} - D \circ \mathrm{tsm})\sigma\big) \subseteq \psi(M) \qquad \forall \sigma \in \Gamma^\infty_0(\xi). \tag{3.6}$$

If a particular time-slice map is understood, we will write

$$\sigma = \sigma_{€} + D\sigma_{£} \tag{3.7}$$

for the corresponding decomposition $\sigma_{£} \coloneqq \mathrm{tsm}\,\sigma$ and $\sigma_{€} \coloneqq \sigma - D\sigma_{£}$. Though we will mostly deal with time-slice maps which are explicitly given by the concrete construction in the proof of Lemma 3.3.5, we can say quite a lot about their properties without ever referring to such a concrete construction.

LEMMA 3.3.7. *Let $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ be Cauchy, $\xi = (E, N, \pi, V)$ a smooth vector bundle, $D : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi)$ a normally hyperbolic linear differential operator of metric type and let $\mathrm{tsm} : \Gamma^\infty_0(\xi) \longrightarrow \Gamma^\infty_0(\xi)$ be a time slice map for (ψ, ξ, D).*

(i) $\mathrm{tsm}(\sigma) \subseteq \psi(M)$ *for all $\sigma \in \Gamma^\infty_0(\xi)$ with* $\operatorname{supp}\sigma \subseteq \psi(M)$.

(ii) *If $\mathrm{tsm}' : \Gamma^\infty_0(\xi) \longrightarrow \Gamma^\infty_0(\xi)$ is another time-slice map for (ψ, ξ, D), we have $\operatorname{supp}((\mathrm{tsm} - \mathrm{tsm}')\sigma) \subseteq \psi(M)$ for all $\sigma \in \Gamma^\infty_0(\xi)$ and thereby, there is $\tau \in \Gamma^\infty_0(\xi)$ such that $\operatorname{supp}\tau \subseteq \psi(M)$ and $\sigma_{€} - \sigma_{€'} = D\tau$.*

Let $\eta = (L, N, \varrho, W)$ be another smooth vector bundle and $P : \Gamma^\infty(\eta) \longrightarrow \Gamma^\infty(\eta)$ a normally hyperbolic linear differential operator of metric type in such a way that D and P are intertwined by a linear differential operator $\partial : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\eta)$, that is, $\partial \circ D = P \circ \partial$. Suppose $\mathrm{tsm}' : \Gamma^\infty_0(\eta) \longrightarrow \Gamma^\infty_0(\eta)$ is a time-slice map for (ψ, η, P).

(iii) $\operatorname{supp}((\partial \operatorname{tsm} - \operatorname{tsm}' \partial)\sigma) \subseteq \psi(M)$ *for all* $\sigma \in \Gamma_0^\infty(\xi)$ *and, accordingly, there is* $\tau \in \Gamma_0^\infty(\eta)$ *such that* $\operatorname{supp}\tau \subseteq \psi(M)$ *and* $(\partial\sigma)_{€'} - \partial\sigma_{€} = P\tau$.

Proof: (i) Take $\sigma \in \Gamma_0^\infty(\xi)$ with $\operatorname{supp}\sigma \subseteq \psi(M)$; then the smooth cross-section $(D \circ \operatorname{tsm})\sigma = \sigma - (\operatorname{id}_{\Gamma_0^\infty(\xi)} - D \circ \operatorname{tsm})\sigma$ is (compactly) supported in $\psi(M)$. As $\operatorname{tsm}\sigma$ is compactly supported, using the retarded Green operator for D, G^{ret}, shows that $G^{\mathrm{ret}}((D \circ \operatorname{tsm})(\sigma)) = \operatorname{tsm}\sigma$. Thus, we have $\operatorname{supp}(\operatorname{tsm}\sigma) \subseteq J_{\mathbf{N}}^+(\operatorname{supp}((D \circ \operatorname{tsm})\sigma)) \subseteq J_{\mathbf{N}}^+(\psi(M))$. In the same way, enlisting the help of the advanced Green operator for D, we can show $\operatorname{supp}(\operatorname{tsm}\sigma) \subseteq J_{\mathbf{N}}^-(\psi(M))$. It follows $\operatorname{supp}(\operatorname{tsm}\sigma) \subseteq J_{\mathbf{N}}^+(\psi(M)) \cap J_{\mathbf{N}}^-(\psi(M)) = \psi(M)$ as $\psi(M)$ is causally convex in $\mathbf{N}$.

(ii) For $\sigma \in \Gamma_0^\infty(\xi)$, the difference $(\operatorname{tsm} - \operatorname{tsm}')\sigma$ is compactly supported and hence $D(\operatorname{tsm} - \operatorname{tsm}')\sigma = (D \circ \operatorname{tsm} - \operatorname{id}_{\Gamma_0^\infty(\xi)} - D \circ \operatorname{tsm}' + \operatorname{id}_{\Gamma_0^\infty(\xi)})\sigma$ has support in $\psi(M)$ by the definition of time-slice maps. Thus $(\operatorname{tsm} - \operatorname{tsm}')\sigma$ is (compactly) supported in the intersection $J_{\mathbf{N}}^+(\psi(M)) \cap J_{\mathbf{N}}^-(\psi(M)) = \psi(M)$ (like in (i), use G^{ret}, G^{adv} and that $\psi(M)$ is causally convex in $\mathbf{N}$). The rest follows from this and (3.7).

(iii) We compute

$$P(\partial \operatorname{tsm}\sigma - \operatorname{tsm}'\partial\sigma) = \partial D \operatorname{tsm}\sigma - P \operatorname{tsm}'\partial\sigma \tag{3.8}$$

$$= \partial(\sigma - \sigma_{€}) - (\partial\sigma - (\partial\sigma)_{€'}) \tag{3.9}$$

$$= (\partial\sigma)_{€'} - \partial\sigma_{€} \qquad \forall \sigma \in \Gamma_0^\infty(\xi). \tag{3.10}$$

As ∂ is a linear differential operator, $\operatorname{supp}\partial\sigma \subseteq \operatorname{supp}\sigma$ and $\operatorname{supp}(\partial \operatorname{tsm}\sigma) \subseteq \operatorname{supp}(\operatorname{tsm}\sigma)$. Hence, $\partial \operatorname{tsm}\sigma - \operatorname{tsm}'\partial\sigma$ is compactly supported. In the same way as in (i), applying the retarded/advanced Green operator for P and using that $\psi(M)$ is causally convex in $\mathbf{N}$ yields that $\partial \operatorname{tsm}\sigma - \operatorname{tsm}'\partial\sigma$ is (compactly) supported in $\psi(M)$. ❐

Let us apply Lemma 3.3.7 to smooth differential forms with a view to the description of the Maxwell field.

COROLLARY 3.3.8. *Let* $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ *be Cauchy,* $\xi = \lambda_N^p$ *the p-th exterior power of the cotangent bundle* τ_N^* *of* N *for* $p \in \mathbb{N}$ *and* $D = \Box_{\mathbf{N}} = -\delta_{\mathbf{N}} d_{\mathbf{N}} - d_{\mathbf{N}}\delta_{\mathbf{N}} : \Omega^p(N;\mathbb{K}) \longrightarrow \Omega^p(M;\mathbb{K})$ *the wave operator for smooth* $\mathbb{K}$*-valued differential p-forms on* N*. Notice,* $d_{\mathbf{N}}\Box_{\mathbf{N}} = \Box_{\mathbf{N}} d_{\mathbf{N}}$ *and* $\delta_{\mathbf{N}}\Box_{\mathbf{N}} = \Box_{\mathbf{N}}\delta_{\mathbf{N}}$*. For any time-slice map* $\operatorname{tsm} : \Omega_0^p(N;\mathbb{K}) \longrightarrow \Omega_0^p(N;\mathbb{K})$ *for* $(\psi, \lambda_N^p, \Box_{\mathbf{N}})$ *and for each* $\omega \in \Omega_0^p(N;\mathbb{K})$,

$$(d_{\mathbf{N}}\omega)_{€} - d_{\mathbf{N}}\omega_{€} = \Box_{\mathbf{N}}\eta \qquad \textit{and} \qquad (\delta_{\mathbf{N}}\omega)_{€} - \delta_{\mathbf{N}}\omega_{€} = \Box_{\mathbf{N}}\theta. \tag{3.11}$$

for some $\eta \in \Omega_0^{p+1}(N;\mathbb{K})$ *and* $\theta \in \Omega_0^{p-1}(N;\mathbb{K})$ *with* $\operatorname{supp}\eta$, $\operatorname{supp}\theta \subseteq \psi(M)$*. Furthermore, if* $\omega \in \Omega_{0,d_{\mathbf{N}}}^p(N;\mathbb{K}) \oplus \Omega_{0,\delta_{\mathbf{N}}}^p(N;\mathbb{K})$*, we have that*

$$\omega_{€} = \omega_{d_{\mathbf{N}}} + \omega_{\delta_{\mathbf{N}}}, \tag{3.12}$$

where $\omega_{d_{\mathbf{N}}} \in \Omega^p_{0,d_{\mathbf{N}}}(N;\mathbb{K})$ *and* $\omega_{\delta_{\mathbf{N}}} \in \Omega^p_{0,\delta_{\mathbf{N}}}(N;\mathbb{K})$ *with* $\operatorname{supp}\omega_{d_{\mathbf{N}}}, \operatorname{supp}\omega_{\delta_{\mathbf{N}}} \subseteq \psi(M)$.

Proof: The first part is a direct consequence of Lemma 3.3.7(iii). Now suppose that $d_{\mathbf{N}}\omega = 0$; then $\operatorname{supp}(d_{\mathbf{M}} \operatorname{tsm}\omega) \subseteq \psi(M)$ by Lemma 3.3.7(iii) and due to (3.7),

$$\omega = \omega_{\in} + \Box_{\mathbf{N}} \operatorname{tsm}\omega \quad \Longleftrightarrow \quad \omega + d_{\mathbf{N}}\delta_{\mathbf{N}} \operatorname{tsm}\omega = \underbrace{\omega_{\in}}_{\text{compactly supported in } \psi(M)} - \overbrace{\delta_{\mathbf{N}} d_{\mathbf{N}} \operatorname{tsm}\omega}^{\text{compactly supported in } \psi(M)}. \tag{3.13}$$

Since the right hand side of the second equation is compactly supported in $\psi(M)$, the left hand side must be so too and (3.12) follows with $\omega_{d_{\mathbf{N}}} := \omega + d_{\mathbf{N}}\delta_{\mathbf{N}} \operatorname{tsm}\omega$ and $\omega_{\delta_{\mathbf{N}}} := \delta_{\mathbf{N}} d_{\mathbf{N}} \operatorname{tsm}\omega$. The case $\delta_{\mathbf{N}}\omega = 0$ is shown analogously. ❒

3.4 The relative Cauchy evolution

The relative Cauchy evolution was first introduced in [BFV03] as a natural form of dynamics for a locally covariant quantum field theory obeying the time-slice axiom. Indeed, as it will become much clearer from Definition 3.4.2, relative Cauchy evolutions act as automorphisms capturing the dynamical reaction of a locally covariant theory to a local perturbation of the background metric, thus exhibiting their dynamical character. It has been observed in [BFV03] (see also [FV12a; FV12b; FS14a]) that the functional derivative of the relative Cauchy evolution with respect to the globally hyperbolic perturbation possesses the significance of a stress-energy-momentum tensor for the locally covariant theory; we will encounter such instances in Section 5.4.2 and Section 6.9. In this respect, the relative Cauchy evolution serves as the replacement for the Lagrangean and the action in locally covariant (quantum field) theory.

In the following years after its introduction, the relative Cauchy evolution has proven important to new approaches towards quantum gravity [BF06], dynamical locality [FV12a; FV12b] and for the computation of the automorphism group of a locally covariant theory [Few13; FS14a], which may function as the global gauge group of the theory.

DEFINITION 3.4.1. Let $\mathbf{M} = (M, g, [T], [\Omega]) \in \mathbf{Loc}$. A *globally hyperbolic perturbation* of $\mathbf{M}$ is a compactly supported symmetric smooth tensor field $h \in \Gamma^\infty_0(\tau^*_M \odot \tau^*_M)$ such that the modification $\mathbf{M}[h] := (M, g+h, [T_{g+h}], [\Omega])$ becomes an object in $\mathbf{Loc}$, where $[T_{g+h}]$ is the unique choice for a time-orientation on $(M, g+h)$ that coincides with $[T]$ outside of $\operatorname{supp} h$ ([FV12a, Sec.3.4],[FV12b, Sec.2]). We will write $H(\mathbf{M})$ for all globally hyperbolic perturbations of $\mathbf{M}$ and $H(\mathbf{M}; O)$ for all globally hyperbolic perturbations of $\mathbf{M}$ whose support is contained in an open subset $O \subseteq M$.

Due to [FV12a, Lem.3.2(a)], each globally hyperbolic perturbation $h \in H(\mathbf{M})$ defines globally hyperbolic open subsets $M^\pm[h] := M \setminus J^\mp_{\mathbf{M}}(\operatorname{supp} h)$ of both $\mathbf{M}$ and $\mathbf{M}[h]$.

Endowing $M^{\pm}[h]$ with the structures induced by $\mathbf{M}$ or $\mathbf{M}[h]$[16], we obtain **Loc**-objects $\mathbf{M}^{\pm}[h] = \mathbf{M}|_{M^{\pm}[h]} = (M^{\pm}[h], g|_{M^{\pm}[h]}, [T]|_{M^{\pm}[h]}, [\Omega]|_{M^{\pm}[h]})$. Moreover, if $\mathbf{M} \in \mathbf{Loc}_q$, then $\mathbf{M}^{\pm}[h] \in \mathbf{Loc}_q$ by [FV12a, Lem.3.2(b) + Prop.2.2]. Owing to [FV12a, Lem.3.2(b)], the inclusion maps $i_{M^{\pm}[h]} : M^{\pm}[h] \hookrightarrow M$ and $j_{M^{\pm}[h]} : M^{\pm}[h] \hookrightarrow M[h]$ become Cauchy morphisms, which we will denote by

$$(3.14) \qquad \imath^{\pm}_{\mathbf{M}}[h] : \mathbf{M}^{\pm}[h] \longrightarrow \mathbf{M} \qquad \text{and} \qquad \jmath^{\pm}_{\mathbf{M}}[h] : \mathbf{M}^{\pm}[h] \longrightarrow \mathbf{M}[h].$$

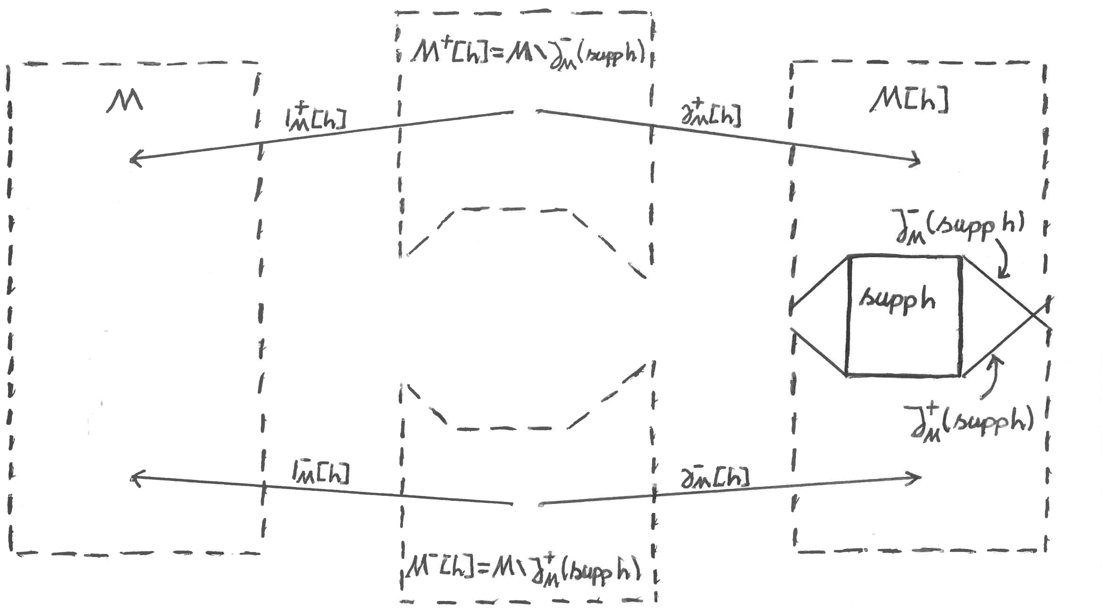


FIGURE 3.2: For the definition of the relative Cauchy evolution. The relative Cauchy evolution is defined by applying a locally covariant theory which obeys the time-slice axiom, starting in the upper part of $\mathbf{M}$ and going clockwise.

DEFINITION 3.4.2. Let $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ be a locally covariant theory or, more generally, any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ as in Definition 3.3.3 obeying the time-slice axiom. The *relative Cauchy evolution* of F induced by $h \in H(\mathbf{M})$ for $\mathbf{M} \in \mathbf{Loc}$ is the $\mathcal{D}$-automorphism $F\mathbf{M} \longrightarrow F\mathbf{M}$ defined by[17]

$$(3.15) \qquad \mathrm{rce}^F_{\mathbf{M}}[h] := F(\imath^-_{\mathbf{M}}[h]) \circ (F(\jmath^-_{\mathbf{M}}[h]))^{-1} \circ F(\jmath^+_{\mathbf{M}}[h]) \circ (F(\imath^+_{\mathbf{M}}[h]))^{-1}.$$

For some more informative properties of relative Cauchy evolutions, which we will not use in this thesis though, we refer the reader to [FV12a, Sec.3.4] and [Fer13a, Sec.2.1.3]. Notice that we have not given the original definition for the relative Cauchy

[16] It does not really matter whether we use $\mathbf{M}$ or $\mathbf{M}[h]$ since $M^{\pm}[h] \cap \operatorname{supp} h = \varnothing$.

[17] As otherwise customary, we are given a choice for the definition of the relative Cauchy evolution. Alternatively to (3.15), we could have also defined $\mathrm{rce}^F_{\mathbf{M}}[h] := F(\imath^+_{\mathbf{M}}[h]) \circ (F(\jmath^+_{\mathbf{M}}[h]))^{-1} \circ F(\jmath^-_{\mathbf{M}}[h]) \circ (F(\imath^-_{\mathbf{M}}[h]))^{-1}$ (starting in the lower part of $\mathbf{M}$ and going anti-clockwise in Figure 3.2), which probably would have been nicer for the sake of interpretation. Anyway, we stick to the standard convention.

evolution of [BFV03] (see also [Fer13a, Sec.2.1.3]), which uses more arbitrary globally hyperbolic open subsets of $\mathbf{M}$ lying strictly in the future and past of $\operatorname{supp} h$, and containing a Cauchy surface for $\mathbf{M}$ each. Our definition is the one of [FV12a], which is equivalent to the definition of [BFV03] (by the results of [FV12a, Sec.3.4]) and more canonical as the constructions involved depend only on the globally hyperbolic perturbation h.

3.5 The quantisation functor

Although algebraic and locally covariant quantum field theory allow us to start with quantum field theories right away, no classical counterpart is needed, they do not suggest how to come up with such theories in the first place. The algebraic and the locally covariant approach provide us with a frame and a language, not with a particular quantum field theory. To get hold of specific quantum field theories, we will help ourselves with the traditional method of quantisation. This is should not be regarded as a draw-back because a correspondence between a classical and a quantum theory via performing a classical limit or a quantisation is desirable many a time.

We will follow a kind of naive procedure insofar as we will start with classical field theories described by symplectic spaces[18], which have served so well in classical mechanics [GS84; Thi88; Arn89; MR99; Wal07; AM08]. Quantum field theories in terms of (non-commutative) unital *-algebras, interpreted as field algebras, i.e. algebras of smeared quantum fields, will then be constructed by a "*correspondence principle*" linearly assigning to each vector in a symplectic space an algebra element such that the the commutator $[\cdot,\cdot]$ becomes $\mathrm{i}\hbar$-times the Poisson bracket $\{\cdot,\cdot\}$, which is expressed by means of the symplectic form. In modern approaches, one would consider Poisson algebras and then use the techniques of deformation quantisation [Lan98; Wal07]. We are fully aware of the obstructions to a complete and consistent quantisation of a classical theory ([Che81], [GS84, Sec.16], [Wal07, Sec.5.2.1], [AM08, Sec.5.4]), and we make thus no claims that our classical and quantum field theories provide complete descriptions. They should rather be regarded as rudimentary theories, capable of development.

Let us now start with some complexified pre-symplectic space (V,ω,C). We consider the tensor algebra $\left(T(V,\omega,C), \mathrm{inj}^T_{(V,\omega,C)} : (V,C) \hookrightarrow T(V,\omega,C)\right)$ over (V,C), see the appendix of Chapter 2, and the two-sided *-ideal $I(V,\omega,C)$ in $T(V,\omega,C)$ generated by the set

$$(3.16)\quad R(V,\omega,C) := \left\{ \frac{1}{\mathrm{i}\hbar}\left[\mathrm{inj}^T_{(V,\omega,C)}(u), \mathrm{inj}^T_{(V,\omega,C)}(v)\right] - \omega(u,v)\cdot 1_{T(V,\omega,C)} \,\middle|\, u,v \in V \right\},$$

[18] By a "*symplectic space*", we always mean a *linear* symplectic space, i.e. a real vector space together with a non-degenerate skew-symmetric bilinear form. General symplectic manifolds are not considered in this thesis.

where by definition

(3.17)
$$\left[\mathrm{inj}^{T}_{(V,\omega,C)}(u),\mathrm{inj}^{T}_{(V,\omega,C)}(v)\right] := \mathrm{inj}^{T}_{(V,\omega,C)}(u)\,\mathrm{inj}^{T}_{(V,\omega,C)}(v) - \mathrm{inj}^{T}_{(V,\omega,C)}(v)\,\mathrm{inj}^{T}_{(V,\omega,C)}(u) \qquad \forall u,v \in V.$$

Hence,

$$I(V,\omega,C) := \left\{ \sum_{i=1}^{n} a_i\, r_i\, b_i \;\middle|\; a_i, b_i \in T(V,\omega,C),\, r_i \in R(V,\omega,C),\, n \in \mathbb{N} \right\}. \tag{3.18}$$

We construct the quotient unital *-algebra of $T(V,\omega,C)$ by $I(V,\omega,C)$ (appendix of Chapter 2),

$$\left(Q(V,\omega,C) := \frac{T(V,\omega,C)}{I(V,\omega,C)}, \pi_{(V,\omega,C)} : T(V,\omega,C) \twoheadrightarrow Q(V,\omega,C) \right), \tag{3.19}$$

and, regarding the unital *-algebra $Q(V,\omega,C)$ as a C-vector space, we define the C-homomorphism

$$\mathrm{qinj}_{(V,\omega,C)} := \pi_{(V,\omega,C)} \circ \mathrm{inj}^{T}_{(V,\omega,C)} : (V,C) \longrightarrow Q(V,\omega,C). \tag{3.20}$$

Observe the striking similarity of this construction so far (and continuing) with the universal enveloping algebra of a Lie algebra [Dix77b, Chap.2], [Bou75, I, §2]), where the tensor algebra over a Lie algebra $\mathfrak{g}$ is considered and divided by the two-sided ideal generated by the elements of the form $x \otimes y - y \otimes x - [x,y]$, $x, y \in \mathfrak{g}$ ($[\cdot,\cdot]$ denotes the Lie bracket).

Now, if $f : (V,\omega_V,C_V) \longrightarrow (W,\omega_W,C_W)$ is a symplectic C-homomorphim, the universal property of the tensor algebra (UT) and the universal property of the quotient unital *-algebra (UQ) yield a unique unital *-homomorphism $Qf : Q(V,\omega_V,C_V) \longrightarrow Q(W,\omega_W,C_W)$ satisfying the property $Qf \circ \mathrm{qinj}_{(V,\omega_V,C_V)} = \mathrm{qinj}_{(W,\omega_W,C_W)} \circ f$ (cf. [Bou89, III, §5, no.2]). Of course, if $f = \mathrm{id}_{(V,\omega,C)}$, then $Qf = \mathrm{id}_{Q(V,\omega,C)}$, and if $g : (W,\omega_W,C_W) \longrightarrow (X,\omega_X,C_X)$ is another symplectic C-homomorphim, then $Q(g \circ f) = Qg \circ Qf$.

PROPOSITION AND DEFINITION 3.5.1. *The rules*

$$\mathbf{pSympl}_{\mathbb{C}} \ni (V,\omega,C) \longmapsto Q(V,\omega,C) \in {}^{*}\mathbf{Alg_1}, \tag{3.21}$$

(3.22)
$$\mathbf{pSympl}_{\mathbb{C}}((V,\omega_V,C_V),(W,\omega_W,C_W)) \ni f \longmapsto Qf \in {}^{*}\mathbf{Alg_1}(Q(V,\omega_V,C_V), Q(W,\omega_W,C_W))$$

define a functor $Q : \mathbf{pSympl}_{\mathbb{C}} \longrightarrow \mathbf{{}^*Alg_1}$, *which we call the* field quantisation *or* infinitesimal Weyl quantisation functor. *Since no other quantisation prescription will be considered in this thesis, we may refer to* Q *just as the* quantisation functor.

It is argued in [FV12b, Sec.5] that for $(V,\omega,C) \in \mathbf{pSympl}_{\mathbb{C}}$, $Q(V,\omega,C)$ is in linear bijection with the symmetric tensor algebra $\bigoplus_{n\in\mathbb{N}} V^{\odot n}$ over V and for a symplectic C-homomorphism $f : (V,\omega_V,C_V) \longrightarrow (W,\omega_W,C_W)$, $Qf : Q(V,\omega_V,C_V) \longrightarrow Q(W,\omega_W,C_W)$ becomes the linear map $\bigoplus_{n\in\mathbb{N}} f^{\odot n} : \bigoplus_{n\in\mathbb{N}} V^{\odot n} \longrightarrow \bigoplus_{n\in\mathbb{N}} W^{\odot n}$ under this linear bijection. In conclusion, if f is injective, Qf will be injective, thus leading to a functor $Q : \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}} \longrightarrow \mathbf{{}^*Alg^m_1}$. Furthermore, [BSZ92, Scholium 7.1] shows that if $(V,\omega,C) \in \mathbf{pSympl}_{\mathbb{C}}$ is symplectic, i.e. ω is weakly non-degenerate, then $Q(V,\omega,C) \in \mathbf{{}^*Alg_1}$ is simple. Hence we also obtain a functor $Q : \mathbf{Sympl}_{\mathbb{C}} \longrightarrow \mathbf{{}^*Alg^m_1}$. We will refer to all of these three functors as the quantisation functor.

LEMMA 3.5.2. *Let* $\mathcal{C} = \mathbf{pSympl}_{\mathbb{C}}, \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{C}}, \mathbf{Sympl}_{\mathbb{C}}$, $F_\omega : \mathcal{C} \longrightarrow \mathbf{CVec}$ *the forgetful functor that forgets about the complexified pre-symplectic form,* $\mathcal{D} = \mathbf{{}^*Alg_1}, \mathbf{{}^*Alg^m_1}$, $F_{\bullet_A} : \mathcal{D} \longrightarrow \mathbf{CVec}$ *the forgetful functor that forgets about the algebra multiplication and* $Q : \mathcal{C} \longrightarrow \mathcal{D}$ *the corresponding quantisation functor. The* C*-homomorphisms* $\mathrm{qinj}_{(V,\omega,C)} : (V,C) \longrightarrow F_{\bullet_A}(Q(V,\omega,C))$, $(V,\omega,C) \in \mathcal{C}$, *define the components of a natural transformation* $\mathrm{qinj} : F_\omega \dot{\longrightarrow} F_{\bullet_A} \circ Q$.

The quantisation functors come along with a useful universal property (the reader is invited to compare this result to the universal property of the universal enveloping algebra of a Lie algebra, see [Dix77b, Lem.2.1.3] or [Bou75, I, §2, Prop.1]):

LEMMA 3.5.3. *Let* (V,ω,C) *be a complexified pre-symplectic space and* B *a unital *-algebra. If we are given a* C*-homomorphism* $f : (V,\omega,C) \longrightarrow B$ *that satisfies*

$$\frac{1}{\mathrm{i}\hbar}\left[f(u), f(v)\right] = \omega(u,v)\cdot 1_B \qquad \forall u,v \in V, \tag{3.23}$$

*then there exists a unique unital *-homomorphism* $\varphi : Q(V,\omega,C) \longrightarrow B$ *such that* $\varphi \circ \mathrm{qinj}_{(V,\omega,C)} = f$ *as* C*-homomorphisms.*

Proof: According to (UT), there exists a uniquely determined unital *-homomorphism $\varphi' : T(V,\omega,C) \longrightarrow B$ satisfying $\varphi' \circ \mathrm{inj}_{(V,\omega,C)} = f$. (3.23) yields

$$\varphi'\left(\frac{1}{\mathrm{i}\hbar}\left[\mathrm{inj}_{(V,\omega,C)}(u), \mathrm{inj}_{(V,\omega,C)}(v)\right] - \omega(u,v)\cdot 1_{T(V,\omega,C)}\right) \tag{3.24}$$

$$= \frac{1}{\mathrm{i}\hbar}\left[f(u), f(v)\right] - \omega(u,v)\cdot 1_B \tag{3.25}$$

$$= 0_B \qquad \forall u,v \in V, \tag{3.26}$$

which implies $I(V,\omega,C) \subseteq \ker\varphi'$. Passing over to the quotient, there exists a unique unital *-homomorphism $\varphi : Q(V,\omega,C) \longrightarrow B$ such that $\varphi \circ \pi_{(V,\omega,C)} = \varphi'$. Consequently $\varphi \circ \mathrm{qinj}_{(V,\omega,C)} = \varphi \circ \pi_{(V,\omega,C)} \circ \mathrm{inj}_{(V,\omega,C)} = \varphi' \circ \mathrm{inj}_{(V,\omega,C)} = f$ and φ must already be the unique unital *-homomorphism $Q(V,\omega,C) \longrightarrow B$ with this property. ❒

COROLLARY 3.5.4. *Let (V,ω,C) be a complexified pre-symplectic space, B a unital *-algebra and suppose there are two unital *-homomorphisms $\varphi,\psi : Q(V,\omega,C) \longrightarrow B$ satisfying $\varphi \circ \mathrm{qinj}_{(V,\omega,C)} = \psi \circ \mathrm{qinj}_{(V,\omega,C)}$ as C-homomorphisms. Then, $\varphi = \psi$.*

Proof: Apply Lemma 3.5.3 to the C-homomorphism $f := \varphi \circ \mathrm{qinj}_{(V,\omega,C)}$. ❒

We can also easily quantise symplectic spaces $(V,\omega) \in \mathbf{Sympl}_{\mathbb{R}}$ and pre-symplectic spaces $(V,\omega) \in \mathbf{pSympl}_{\mathbb{R}}, \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{R}}$. To achieve this, we simply apply the complexification functor (Definition 2.1.2) before we apply the quantisation functor. All in all, we have thus obtained six functors

$$Q : \mathbf{pSympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg_1}, \quad Q : \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}, \quad Q : \mathbf{Sympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}, \quad \mathbb{K} = \mathbb{R},\mathbb{C}. \tag{3.27}$$

to all of which we will refer as the quantisation functor.

Chapter 4

K. Fredenhagen's Universal Algebra

"*One Ring to rule them all, One Ring to find them,*
One Ring to bring them all and in the darkness bind them."

–J.R.R. Tolkien, *The Lord of the Rings, Part 1: The Fellowship of the Ring*, Chapter II, "The Shadow of the Past", HarperCollins, 1997.

In this chapter, we give a thorough discussion of the universal algebra and apply colimit constructions and left Kan extensions in concrete situations; namely, in the classical and the quantum field theory of the free and minimally coupled real scalar field, and in the classical and the quantum field theory of the free Maxwell field in terms of its field strength tensor description. For the purpose of these discussions, we prove the main technical theorem of this thesis, which concerns the colimit preserving ability of the quantisation functor.

We discuss the intention behind the universal algebra and give K. Fredenhagen's original definition as well as the categorical one in Section 4.1. In Section 4.2, we face some of the criticism passed on the universal algebra, namely the possibility of it being trivial (i.e. the zero algebra) and its shortcomings in the representation theory of nets of local (unital) $(C)^*$-algebras and superselection rules. Section 4.3 contains the main technical theorem of this thesis, which links colimits in $\mathbf{pSympl}_{\mathbb{K}}$ to colimits in $\mathbf{{}^*Alg_1}$ via the quantisation functor, using the corresponding colimits in $\mathbf{Vec}_{\mathbb{R}}$ resp. $\mathbf{CVec}$. In the last two sections of this chapter, Section 4.5 and Section 4.6, we construct the universal algebra, colimits and left Kan extensions in the classical and the quantum field theory of the free and minimally coupled real scalar field, and the free Maxwell field, illustrating the technical tools developed in this thesis so far. In these examples, we will make essential use of three lemmas which are purpose-built for the explicit computation of the colimits emerging. We have collected them in Section 4.4.

To avoid spelling out phrases like "*complexified if* $\mathbb{K} = \mathbb{C}$" or "*omit the C-involution if* $\mathbb{K} = \mathbb{R}$" over and over again, we declare that "(*pre-*)*symplectic*" is to be read as "*complexified* (*pre-*)*symplectic*" if $\mathbb{K} = \mathbb{C}$ and that C-involutions are to be ignored if $\mathbb{K} = \mathbb{R}$

throughout. Also, due to repeated occurrence, Q always denotes the quantisation functor (Section 3.5) in one of its variants, $\mathbf{pSympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg_1}$, $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ and $\mathbf{Sympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$. Moreover, $F_\omega : \mathcal{C} \longrightarrow \mathcal{D}$, where $\mathcal{C} = \mathbf{pSympl}_{\mathbb{K}}, \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}, \mathbf{Sympl}_{\mathbb{K}}$ and $\mathcal{D} = \mathbf{Vec}_{\mathbb{R}}, \mathbf{CVec}$, is to denote the corresponding forgetful functor that forgets about the (pre-)symplectic form.

4.1 Origin and definition

We give the original and the catgorical definition of K. Fredenhagen's universal algebra.

As mentioned in the introduction to this thesis, in order to have a global algebra (analogously to the algebra of quasilocal observables) at his disposal, K. Fredenhagen introduced the universal algebra for chiral conformal quantum field theories in 2-dimensional Minkowski space (considered as theories on S^1) in [Fre90]: let $\mathcal{K}$ be the set of all proper open intervals of S^1 whose closure is not S^1, and $\mathcal{A}(I)$, $I \in \mathcal{K}$, von Neumann algebras on some Hilbert space $\mathcal{H}_0$ such that $\mathcal{A}(I) \subseteq \mathcal{A}(J)$ whenever $I \subseteq J$ for $I, J \in \mathcal{K}$ (isotony) and $[\mathcal{A}(I), \mathcal{A}(J)] = \{0_{B(\mathcal{H}_0)}\}$ for $I, J \in \mathcal{K}$ with $I \cap J = \varnothing$ (locality); then a (unital) C^*-algebra $\mathcal{A} = \mathcal{A}(S^1)$ can be characterised by the following universal property. $\mathcal{A}$ contains all $\mathcal{A}(I)$, $I \in \mathcal{K}$, as (unital) C^*-subalgebras and for every family of (unital) representations $\{\pi_I : \mathcal{A}(I) \longrightarrow B(\mathcal{H})\}_{I \in \mathcal{K}}$, where $\mathcal{H}$ is some Hilbert space, with the compatibility condition $\pi_J|_{\mathcal{A}(I)} = \pi_I$ whenever $I \subseteq J$ for $I, J \in \mathcal{K}$, there is a unique (unital) representation $\pi : \mathcal{A} \longrightarrow B(\mathcal{H})$ such that $\mathcal{A}|_{\mathcal{A}(I)} = \pi_I$ for all $I \in \mathcal{K}$.

Though the universal algebra is defined by a universal property, hence its name, it can also be more concretely described by generators and relations [Fre90; Fre93]. Notice, e.g. [Bou89, Chap.III, §2, no.8] and [Bla06, Sec.II.8.3] use the term "*universal (C^*-) algebra*" for a (C^*-)algebra defined by generators and relations.

In the subsequent years after its introduction, the universal algebra has played a helpful role in applying the Doplicher-Haag-Roberts analysis of superselection sectors and particle statistics [DHR69a; DHR69b; DHR71; DHR74; DR90] to chiral conformal quantum field theories in 2-dimensional Minkowski space (viewed as theories on S^1) [FRS92; GL92; Fre93; DFK04] and it is still being used for their representation theory [CCHW13; CHL13].

In the general framework of algebraic quantum field theory, the universal algebra can be considered as a generalisation of the algebra of quasilocal observables to nets of local observables which are not up-directed under the inclusion. However, the universal algebra should be clearly distinguished from the algebra of quasilocal observables: while no new relations can arise in the algebra of quasilocal observables (essentially due to taking uniform limits), new algebraic expressions, which are not apparent in the algebras of local observables, may arise in the universal algebra. Still, both coincide on up-directed nets of algebras of local observables.

This actuality motivated K. Fredenhagen's proposal to deploy the universal algebra in the context of the field algebras, i.e. the unital *-algebras of the smeared quantum field, for the quantised free Maxwell field in terms of the field strength tensor. There, for spacetimes which allow for field strength tensors that cannot be derived from vector potentials, only a non-up-directed net of local field algebras could be obtained in the standard manner. It was hoped that the universal algebra could be used for a sensible global field algebra and, by exploiting the new relations arising, one could rediscover the findings of [Sor79; AS80], that is, a non-trivial centre and superselection rules for topological charges. That this was indeed the case, was confirmed by [Hol08, Appx.A] and [DL12].

There, the more general definiton for the universal algebra of [Fre94, Sec.II.1] was used: let $\mathcal{A}(\mathcal{O})$, $\mathcal{O}$ in a collection of spacetime[1] regions, be unital (C)*-algebras such that there is a family of unital *-monomorphism $\{i_{\mathcal{O}_1\mathcal{O}_2} : \mathcal{A}(\mathcal{O}_1) \longrightarrow \mathcal{A}(\mathcal{O}_2) \mid \mathcal{O}_1, \mathcal{O}_2$ spacetime regions with $\mathcal{O}_1 \subseteq \mathcal{O}_2\}$ satisfying the compatibility condition $i_{\mathcal{O}_2\mathcal{O}_3} \circ i_{\mathcal{O}_1\mathcal{O}_2} = i_{\mathcal{O}_1\mathcal{O}_3}$ for spacetime regions $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_3$; then a global unital (C)*-algebra $\mathcal{A}_\infty$ can be characterised by the following universal property. There are unital *-monomorphisms $\{i_{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \longrightarrow \mathcal{A}_\infty \mid \mathcal{O}$ spacetime region$\}$ such that $i_{\mathcal{O}_2} \circ i_{\mathcal{O}_1\mathcal{O}_2} = i_{\mathcal{O}_1}$ for spacetime regions $\mathcal{O}_1, \mathcal{O}_2$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$ and whenever there is a family of unital *-homomorphisms, $\{\varphi_{\mathcal{O}} : \mathcal{A}(\mathcal{O}) \longrightarrow B \mid \mathcal{O}$ spacetime region$\}$, into another unital (C)*-algebra B, satisfying the compatibility condition $\varphi_{\mathcal{O}_2} \circ i_{\mathcal{O}_1\mathcal{O}_2} = \varphi_{\mathcal{O}_1}$ for spacetime regions $\mathcal{O}_1, \mathcal{O}_2$ with $\mathcal{O}_1 \subseteq \mathcal{O}_2$, there is a unique unital *-homomorphism $\varphi : \mathcal{A}_\infty \longrightarrow B$ such that $\varphi \circ i_{\mathcal{O}} = \varphi_{\mathcal{O}}$ for all spacetime regions $\mathcal{O}$. This definition is indeed a generalisation of the definition of [Fre90] above, which can be easily seen by the Gelfand-Naimark theorem on the existence of isometric representations of any C*-algebra.

Comparing the definition of the universal algebra above and the definition of colimits (Definition 2.2.7), in particular the universal property (UColim), it is now not difficult to recognise the universal algebra as the universal object of the colimit for a functor from a system of spacetime regions and their inclusions into each other to a category of unital (C)*-algebras and unital *-homo/monomorphisms. To do so concretely, we view a net $\mathbf{B} \longmapsto \mathfrak{A}(\mathbf{B})$ of local unital (C)*-algebras, where we consider some $\mathbf{M} \in \mathbf{Loc}$, as a functor $\mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$, where $\mathcal{J}$ is the poset (viewed as a category) of a choice of spacetime regions for $\mathbf{M}$ and their inclusions into each other. However, it will be appropriate to add an additional level of generalisation: we also do not require the unital (C)*-algebras Fi, $i \in \mathcal{J}$, to be different unital (C)*-subalgebras of one unital (C)*-algebra or of each other, and the $F\mu_{ij} : Fi \longrightarrow Fj$, $\mu_{ij} \in \mathcal{J}(i,j)$ and

[1] [Fre94, Sec.II.1] considers Minkowski space but the construction holds for any curved spacetime.

$i, j \in \mathcal{J}$, are allowed to be general unital *-monomorphisms; they are not restricted to be inclusions.

DEFINITION 4.1.1. Let $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$ be a net of local unital $(C)^*$-algebras. Then the *universal [unital (C)*-]algebra* for F is the universal object of the colimit for F viewed as a functor $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1}$.

In order to apply Theorem 2.2.10 with a positive result on the existence of the universal algebra, we have viewed $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$ as a functor $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1}$ in the previous definition. As a corollary of Theorem 2.2.10, it follows now instantly that:

THEOREM 4.1.2. *The universal algebra always exists.*

The universal algebra for a net of local unital $(C)^*$-algebras, $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$, can be constructed concretely as follows, which is motivated by the proofs of the dual statements of [Par70, Sec.2.6, Prop.2], [Bor94, Thm.2.8.1], [Mac98, Thm.V.2.1] and [AHS04, Thm.12.3], where colimits are constructed from coequalisers and coproducts. First, we take the direct sum $(\bigoplus_{i\in\mathcal{J}} Fi, \{\mathrm{inj}_j^{\oplus} : Fj \hookrightarrow \bigoplus_{i\in\mathcal{J}} Fi\}_{j\in\mathcal{J}})$ of the Fi, $i \in \mathcal{J}$, viewed as C-vector spaces and form the tensor algebra $(T(\bigoplus_{i\in\mathcal{J}} Fi), \mathrm{inj}_{\oplus}^T : \bigoplus_{i\in\mathcal{J}} Fi \hookrightarrow T(\bigoplus_{i\in\mathcal{J}} Fi))$, which is a unital *-algebra. We define the C-monomorphism $\mathrm{inj}_j^T := \mathrm{inj}_{\oplus}^T \circ \mathrm{inj}_j^{\oplus} : Fj \longrightarrow T(\bigoplus_{i\in\mathcal{J}} Fi)$ for each $j \in \mathcal{J}$ and form the quotient of $T(\bigoplus_{i\in\mathcal{J}} Fi)$ by the two-sided *-ideal I generated by the sets

$$\text{restoring the multiplication:} \left\{\mathrm{inj}_i^T(a_i)\,\mathrm{inj}_i^T(b_i) - \mathrm{inj}_i^T(a_i b_i) \mid a_i, b_i \in Fi, i \in \mathcal{J}\right\}, \tag{4.1}$$

$$\text{common identity:} \left\{1_A - \mathrm{inj}_i^T(1_{Fi}) \mid i \in \mathcal{J}\right\}, \tag{4.2}$$

$$\text{amalgamation:} \left\{\mathrm{inj}_j^T(F\mu_{ij}(a_i)) - \mathrm{inj}_i^T(a_i) \mid a_i \in Fi,\ \mu_{ij} \in \mathcal{J}(i,j),\ i,j \in \mathcal{J}\right\}. \tag{4.3}$$

Consider now the cocone $u : F \dot{\longrightarrow} \Delta T(\bigoplus_{i\in\mathcal{J}} Fi)/I$ defined by $u_j := \pi \circ \mathrm{inj}_{\oplus}^T \circ \mathrm{inj}_j^{\oplus} : Fj \longrightarrow T(\bigoplus_{i\in\mathcal{J}} Fi)/I$, $j \in \mathcal{J}$, where $\pi : T(\bigoplus_{i\in\mathcal{J}} Fi) \twoheadrightarrow T(\bigoplus_{i\in\mathcal{J}} Fi)/I$ denotes the canonical projection onto the quotient. In the case of unital *-algebras, the colimit for F is the pair $(T(\bigoplus_{i\in\mathcal{J}} Fi)/I, u)$. In the case of unital C^*-algebras, we equip $T(\bigoplus_{i\in\mathcal{J}} Fi)/I$ with the C^*-norm

$$\|a\| := \sup\left\{ \|D(a)\| \;\middle|\; \begin{array}{l} D : T(\bigoplus_{i\in\mathcal{J}} Fi)/I \longrightarrow B(H) \text{ is a unital} \\ \text{*-representation of } T(\bigoplus_{i\in\mathcal{J}} Fi)/I \end{array} \right\}, \tag{4.4}$$

$$a \in T(\textstyle\bigoplus_{i\in\mathcal{J}} Fi)/I,$$

and form the norm-completion $\overline{T(\bigoplus_{i\in\mathcal{J}} Fi)/I}^{\|\cdot\|}$. The colimit for F is given by the pair $\left(\overline{T(\bigoplus_{i\in\mathcal{J}} Fi)/I}^{\|\cdot\|}, \mathrm{inj}_{\overline{T(\bigoplus_{i\in\mathcal{J}} Fi)/I}^{\|\cdot\|}} \circ u\right)$, where $\mathrm{inj}_{\overline{T(\bigoplus_{i\in\mathcal{J}} Fi)/I}^{\|\cdot\|}} : T(\bigoplus_{i\in\mathcal{J}} Fi)/I \hookrightarrow \overline{T(\bigoplus_{i\in\mathcal{J}} Fi)/I}^{\|\cdot\|}$ is the canonical injection into the norm completion and

$\Delta\,\mathrm{inj}_{\overline{T\,(\bigoplus_{i\in\mathcal{J}} Fi)/I}^{\|\cdot\|}} : \Delta T\,(\bigoplus_{i\in\mathcal{J}} Fi)/I \overset{.}{\longrightarrow} \Delta\overline{T\,(\bigoplus_{i\in\mathcal{J}} Fi)/I}^{\|\cdot\|}$ the constant natural transformaton.

Take notice that by utilising universal properties right from the start, we have avoided the rather cumbersome explicit expressions which can be encountered in the proof of [DL12, Thm.3.1]. We would also like to draw the reader's attention to the excellent discussions of the universal algebra in [Fre94, Sec.II.1] and [BFM09, Appx.B].

4.2 Criticism

In this section, we would like to point out some of the weaknesses of the universal algebra and comment on the criticism by [RV12] *and R. Brunetti*[2]*. Most prominent among the points of criticism are the triviality issue and the insufficiency with respect to the representation theory of nets of local (unital) (C)*-algebras and superselection rules.*

Though Theorem 2.2.10 guarantees the existence of the universal algebra, it is not excluded that the universal algebra can turn out to be the trivial algebra, i.e. the zero algebra, which of course would rule out its usefulness to algebraic and locally covariant quantum theory in that instance. Surely, by (UColim), it is always enough to find just one cocone from the functor to a non-zero (unital) (C)*-algebra to establish the non-triviality of the universal algebra, and if the cocone consist of (unital) *-monomorphisms or the local (unital) (C)*-algebras are simple, they are really contained in the universal algebra via (unital) *-monomorphisms. But as the following counter-example reveals, there cannot be any general argument for the non-triviality of the universial algebra, even if all of the local (unital) (C)*-algebras involved are non-trivial (cf. [RV12, Example 5.9]):

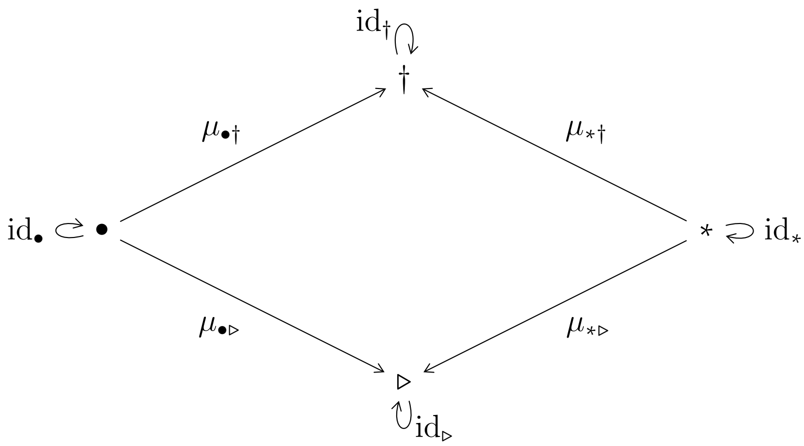
COUNTER-EXAMPLE 4.2.1. Consider the 2×2-matrices with complex entries, $\mathrm{Mat}\,(2\times 2;\mathbb{C})$, equipped with the Hermitean conjugation (complex conjugation and transposition) and the operator norm. This yields a simple ([Gre67, Sec.5.2, Example 2] and [Gre67, Sec.5.12]) unital C*-algebra, which possesses a unital *-automorphism other than the identity:

$$\begin{aligned} &\varphi : \mathrm{Mat}\,(2\times 2;\mathbb{C}) \longrightarrow \mathrm{Mat}\,(2\times 2;\mathbb{C}) \\ &\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{4.5}$$

[2] Private communication. We would also like to thank R. Brunetti for bringing to our attention [BFM09; Fra09].

Let $\mathscr{J}$ be the category defined by the four objects $\bullet, *, \dagger, \triangleright$, and, besides the identity morphisms, by the four morphisms $\mu_{\bullet\dagger} : \bullet \longrightarrow \dagger$, $\mu_{\bullet\triangleright} : \bullet \longrightarrow \triangleright$, $\mu_{*\dagger} : * \longrightarrow \dagger$ and $\mu_{*\triangleright} : * \longrightarrow \triangleright$, see the following diagram:

(4.6)



Then, define a functor $F : \mathscr{J} \longrightarrow \mathbf{(C)^*Alg_1}$ by $F\bullet = F* = F\dagger = F\triangleright := \mathrm{Mat}\,(2\times 2;\mathbb{C})$, $F\mu_{\bullet\dagger} := \varphi$ and $F\mu_{\bullet\triangleright} = F\mu_{*\dagger} = F\mu_{*\triangleright} := \mathrm{id}_{\mathrm{Mat}(2\times 2;\mathbb{C})}$. Thanks to Theorem 2.2.10, $\operatorname{colim} F$ exists, say $\operatorname{colim} F = \left(A, \{\alpha_i : Fi \longrightarrow A\}_{i=\bullet,*,\dagger,\triangleright}\right)$. Assuming that $\alpha_\dagger$ is injective yields $\varphi = \mathrm{id}_{\mathrm{Mat}(2\times 2;\mathbb{C})}$, which is nonsense. So, $\alpha_\dagger$ is not injective and thus has a non-trivial kernel. Since $F\bullet = \mathrm{Mat}\,(2\times 2;\mathbb{C})$ is simple, we conclude that A must be trivial.

This example applies generally to any (unital) $(C)^*$-algebra which is simple and allows for a (unital) *-automorphism other than the identity.

It was recently suggest to us by K. Fredenhagen[3] that a trivial universal algebra might not be a bad thing after all. Since we are not aware of any sensible quantum field theory on a curved spacetime whose universal algebra is the zero algebra, the triviality of the universal algebra may be an indication that a curved spacetime is not suitable for the formulation of a specific quantum field theory on it. In this context, it was also suggested to us by K. Fredenhagen to look into universal algebras of quantum field theories on non-globally hyperbolic spacetimes, which arise from only considering the quantum field theory on globally hyperbolic open subsets. To our knowledge, this has only been done in [Som06]. In this diploma thesis, the free massive real scalar field is considered on the Minkowski half-space, which is non-globally hyperbolic. It is shown that the universal algebra is non-trivial and coincides with the C^*-Weyl algebra resp. CCR-algebra, which is obtained from an ansatz based on the method of images from electrodynamics.

The next major point of criticism of [RV12] and R. Brunetti brought forward against the universal algebra concerns the representation theory of nets of local (unital) $(C)^*$-algebras and the subsequent discussion of superselection sectors. In the axiomatic

[3] Private communication.

framework of algebraic quantum field theory, the key object really is the net of local observables. The algebra of quasilocal observables does not carry any significance other than being a convenient tool and it is not vital to the description of a quantum field theory in the algebraic approach; states can be dealt with by the means of net states [BR09, Sec.2], [RV12, Sec.4.1]: a *net state* ω of a net $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$ of local unital $(C)^*$-algebras assigns to each $i \in \mathcal{J}$ a state (i.e. normalised positive linear functional) $\omega_i : Fi \longrightarrow \mathbb{C}$ such that the local compatibility condition $\omega_j \circ F_{\mu_{ij}} = \omega_i$ is satisfied whenever there is a $\mathcal{J}$-morphism $\mu_{ij} : i \longrightarrow j$.

For the representation theory of the net, Hilbert space representations and generalised net representations are considered [BR09, Sec.2], [RV12, Sec.4.2]: a *Hilbert space representation* of a net $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$ of local unital $(C)^*$-algebras is a family of Hilbert space representations $\{\rho_i : Fi \longrightarrow B(H)\}_{i \in \mathcal{J}}$ meeting the compatibility condition $\rho_j \circ F\mu_{ij} = \rho_i$ for all $\mu_{ij} \in \mathcal{J}(i,j)$ and for all $i, j \in \mathcal{J}$, where $B(H)$ is the unital C^*-algebra of all bounded linear operators on a fixed Hilbert space H. A *generalised representation* is a family $\{\rho_i : Fi \longrightarrow B(H_i)\}_{i \in \mathcal{J}}$ of Hilbert space representations together with a family of injective linear operators $\{A_{\mu_{ij}} : H_i \longrightarrow H_j\}_{\mu_{ij} \in \mathcal{J}(i,j), i,j \in \mathcal{J}}$ satisfying $A_{\mu_{ij}} \circ \rho_i(a) = \rho_j(F\mu_{ij}(a)) \circ A_{\mu_{ij}}$ for all $a \in Fi$, for all $\mu_{ij} \in \mathcal{J}(i,j)$ and for all $i, j \in \mathcal{J}$, and the cocycle condition $A_{\mu_{jk}} \circ A_{\mu_{ij}} = A_{\mu_{ik}}$ whenever there are $\mathcal{J}$-morphisms $\mu_{ij} : i \longrightarrow j$, $\mu_{jk} : j \longrightarrow k$ and $\mu_{ik} : i \longrightarrow k$; generalised net representations arise naturally by applying the Gelfand-Naimark-Segal construction to each state ω_i, $i \in \mathcal{J}$, of a net state ω (see [BR09, Sec.2]).

Having these notions and their techniques at one's disposal begs the question if there really is any need for spending so much time and effort on a global algebra such as the universal algebra. In response, it should be said that the universal algebra is not supposed to be a competitor but should be regarded as a complementary, convenient tool –and one should make use of helpful tools whenever one can profit from them. For example, since a Hilbert space representation of a net of local unital $(C)^*$-algebras $F : \mathcal{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$ is really nothing other than a cocone $\rho : F \dot{\longrightarrow} \Delta B(H)$, they are in a $1:1$-correspondence with the representations of the universal algebra. Of course, the same cannot be said for the generalised notion for representations of nets of local (unital) $(C)^*$-algebras since (UColim) can only apply in some very special cases. In this sense, the universal algebra is a notion too coarse to capture all the aspects of generalised net representations and one has indeed to use finer notions such as the "*enveloping C*-net bundle*" of [RV12]. Nevertheless, in the framework of locally covariant quantum field theory and in most aspects of quantum field theory in curved spacetimes, global (unital) $(C)^*$-algebras associated with the full curved spacetime are required as a matter of fact.

4.3 The main theorem

The topic of this section is the main technical theorem of this thesis; under suitable circumstances, it allows to compute colimits in $\mathbf{pSympl}_{\mathbb{K}}$ *by using the quantisation functor, the corresponding colimits in* $\mathbf{{}^*Alg_1}$ *and the corresponding colimits in* $\mathbf{Vec}_{\mathbb{R}}$ *resp.* $\mathbf{CVec}$.

Before we proceed to the main technical result, we make one important observation.

LEMMA 4.3.1. *The forgetful functor* $F_\omega : \mathcal{C} \longrightarrow \mathcal{D}$ *preserves colimits, i.e. if the colimit for a functor* $F : \mathcal{J} \longrightarrow \mathcal{C}$ *exists, the colimit for* $F_\omega \circ F : \mathcal{J} \longrightarrow \mathcal{D}$ *exists and is given by* $F_\omega(\operatorname{colim} F)$. *Here,* $\mathcal{C}$ *may not taken to be* $\mathbf{Sympl}_{\mathbb{K}}$.

Proof: Assume $F : \mathcal{J} \longrightarrow \mathcal{C}$ is a functor such that the colimit for F exists, which we denote by $\operatorname{colim} F = (\varinjlim F = (V_{\varinjlim}, \omega_{\varinjlim}, C_{\varinjlim}), u : F \dot{\longrightarrow} \Delta \varinjlim F)$. By Theorem 2.2.10, we know that $\operatorname{colim}(F_\omega \circ F) = ((V, C), v : F_\omega \circ F \dot{\longrightarrow} \Delta (V, C))$ exists. Therefore, there is a unique $\mathcal{D}$-morphism $f : (V, C) \longrightarrow (V_{\varinjlim}, C_{\varinjlim})$ satisfying[4] $\Delta f \circ v = F_\omega \star u$ due to (UColim). Note, this implies that all components of v are injective by [Bou68, II, §3, no.8, Thm.1(c)] because all components of $F_\omega \star u$ are injective. Defining a pre-symplectic form on (V, C) via

$$\omega : V \times V \longrightarrow \mathbb{K}, \qquad (x, y) \longmapsto \omega_{\varinjlim}(f(x), f(y)), \tag{4.7}$$

(V, ω, C) becomes a pre-symplectic space (here, a proof for $\mathcal{C} = \mathbf{Sympl}_{\mathbb{K}}$ would not work since we have no means to say whether ω is weakly non-degenerate) as

$$\omega(Cx, Cy) = \omega_{\varinjlim}(f(Cx), f(Cy)) \tag{4.8}$$
$$= \omega_{\varinjlim}(C_{\varinjlim} f(x), C_{\varinjlim} f(y)) \tag{4.9}$$
$$= \overline{\omega_{\varinjlim}(f(x), f(y))} \tag{4.10}$$
$$= \overline{\omega(x, y)} \qquad \forall x, y \in V. \tag{4.11}$$

f is symplectic by the very definition of ω and can hence be regarded as a $\mathbf{pSympl}_{\mathbb{K}}$-morphism. Furthermore, v becomes a cocone from F to (V, ω, C) because

$$\omega(v_i(x_i), v_i(y_i)) = \omega(f(v_i(x_i)), f(v_i(y_i))) \tag{4.12}$$
$$= \omega_{\varinjlim}((f \circ v_i)(x_i), (f \circ v_i)(y_i)) \tag{4.13}$$
$$= \omega_{\varinjlim}(u_i(x_i), u_i(y_i)) \tag{4.14}$$

[4] We remind the reader that for functors $H, K : \mathcal{B} \longrightarrow \mathcal{C}$, $F : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $\sigma : H \dot{\longrightarrow} K$, the natural transformation $F \star \sigma : F \circ H \dot{\longrightarrow} F \circ K$ is defined by $(F \star \sigma)(A) := F(\sigma_A)$ for all $A \in \mathcal{B}$.

$$= \omega_i(x_i, y_i) \qquad \forall x_i, y_i \in F_\omega(Fi),\ \forall i \in \mathcal{J}. \tag{4.15}$$

Thus, $(u\,;v)$ can be viewed as a double-cocone from F to $(V_{\overrightarrow{\lim}}, \omega_{\overrightarrow{\lim}}, C_{\overrightarrow{\lim}})$ and (V,ω,C) and by reason of (UColim), there is a unique $\mathcal{C}$-morphism $g : (V_{\overrightarrow{\lim}}, \omega_{\overrightarrow{\lim}}, C_{\overrightarrow{\lim}}) \longrightarrow (V,\omega,C)$ fulfilling $\Delta g \circ u = v$. On the other hand, we have $\Delta f \circ v = u$, hence $\Delta f \circ \Delta g \circ u = u$ and $\Delta g \circ \Delta f \circ v = v$. Finally, (UColim) implies $f \circ g = \mathrm{id}_{(V_{\overrightarrow{\lim}}, \omega_{\overrightarrow{\lim}}, C_{\overrightarrow{\lim}})}$ and $g \circ f = \mathrm{id}_{(V,\omega,C)}$, which shows our claim after the application of F_ω. ❐

We now present and prove the main technical result of this chapter. Note that Theorem 4.3.2 is a little bit "*von hinten durch die Brust ins Auge*". It started out originally with the intention to obtain a concrete expression for the universal algebra of a functor $F : \mathcal{J} \longrightarrow$ ***Alg₁**, where $\mathcal{J}$ is small and $F = Q \circ G$ with $G : \mathcal{J} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, by applying the quantisation functor Q to a (complexified) (pre-)symplectic space, instead of an abstract characterisation via generators and relations. In this way, we intended also to obtain an argument for the non-triviality of the universal algebra in certain cases. As we quickly realised, such a (complexified) (pre-)symplectic space automatically becomes the universal object of the colimit for G, hence Q preserves the colimit in such instances. However, $\mathbf{pSympl}_{\mathbb{K}}$ is not a cocomplete category according to Theorem 2.2.10, in contrast to $\mathbf{Vec}_{\mathbb{R}}, \mathbf{CVec}$ and $^*\mathbf{Alg_1}$, which are cocomplete. The next logical step was therefore to explicitly construct the colimit for G and thus prove its existence. This was achieved in a quite unusual way, exploiting the result of Lemma 4.3.1. The (complexified) (pre-)symplectic space of the colimit for G is obtained by equipping the (C-)vector space of the colimit for $F_\omega \circ G : \mathcal{J} \longrightarrow \mathcal{D}$ with a (complexified) (pre-)symplectic form defined by the commutator in the universal algebra. Here, the surprising fact is that the classical field theory is constructed from the quantum field theory or at least with significant help from it.

The role played by Theorem 4.3.2 is hence twofold: whenever it applies, it allows us to determine the existence of the colimit in $\mathbf{pSympl}_{\mathbb{K}}$ and, at the same time, it also helps us establishing the non-triviality of the universal algebra. Since the complexification $\mathscr{C} : \mathbf{pSympl}_{\mathbb{R}} \longrightarrow \mathbf{pSympl}_{\mathbb{C}}$ is an equivalence of categories by Proposition 2.1.2, we can restrict to the case $\mathbb{K} = \mathbb{C}$ thanks to Lemma 2.2.14. We also remind the reader of the forgetful functor $F_{\bullet_A} : {}^*\mathbf{Alg_1} \longrightarrow \mathbf{CVec}$ that forgets about the algebra multiplication. Moreover, recall Section 3.5 and (3.20) in particular.

THEOREM 4.3.2. *Let $\mathcal{J}$ be a small category, $F : \mathcal{J} \longrightarrow \mathbf{pSympl}_{\mathbb{C}}$ a functor and write $Fi = (V_i, \omega_i, C_i)$ for $i \in \mathcal{J}$. Denote the colimit for the functor $F_\omega \circ F : \mathcal{J} \longrightarrow \mathbf{CVec}$ by $((V,C), v : F_\omega \circ F \dot{\longrightarrow} \Delta(V,C))$ and the colimit for the functor $Q \circ F : \mathcal{J} \longrightarrow {}^*\mathbf{Alg_1}$ by $(A, \alpha : Q \circ F \dot{\longrightarrow} \Delta A)$. Both colimits exist due to* Theorem 2.2.10. *Then there is precisely one C-homomorphism $f : (V,C) \longrightarrow A$ such that $f \circ v_i = \alpha_i \circ \mathrm{qinj}_i$ for all $i \in \mathcal{J}$*

as C-homomorphisms, and with the notation introduced, the following statements are equivalent:

(a) $\operatorname{colim} F$ *exists and* $Q(\operatorname{colim} F) \cong \operatorname{colim}(Q \circ F)$.

(b) $\operatorname{colim} F$ *exists* ($\Longrightarrow F_\omega(\operatorname{colim} F) \cong \operatorname{colim}(F_\omega \circ F)$ *according to* Lemma 4.3.1) *and*

$$\frac{1}{\mathrm{i}\hbar}[f(x), f(y)] = \omega(x,y) \cdot 1_A \qquad \forall x, y \in V, \tag{4.16}$$

where ω denotes the complexified pre-symplectic form of $\varinjlim F$.

(c)

$$\frac{1}{\mathrm{i}\hbar}[f(x), f(y)] \in \mathbb{C} \cdot 1_A \qquad \forall x, y \in V. \tag{4.17}$$

(d)

$$\frac{1}{\mathrm{i}\hbar}\left[(\alpha_i \circ \mathrm{qinj}_i)(x_i), (\alpha_j \circ \mathrm{qinj}_j)(y_j)\right] \in \mathbb{C} \cdot 1_A \qquad \forall x_i \in V_i,\ \forall y_j \in V_j,\ \forall i, j \in \mathcal{J}. \tag{4.18}$$

Proof: For each $i \in \mathcal{J}$, $\alpha_i \circ \mathrm{qinj}_i : (V_i, C_i) \longrightarrow A$ is a C-homomorphism. Since $\mathrm{qinj} : F_\omega \circ F \dot{\longrightarrow} F_{\bullet_A} \circ Q \circ F$ is a natural transformation, we have in the sense of C-homomorphisms

$$\alpha_j \circ \mathrm{qinj}_j \circ F\mu_{ij} = \alpha_j \circ Q(F\mu_{ij}) \circ \mathrm{qinj}_i = \alpha_i \circ \mathrm{qinj}_i \qquad \forall \mu_{ij} \in \mathcal{J}(i,j),\ \forall i, j \in \mathcal{J}. \tag{4.19}$$

Hence, (UColim) yields a unique C-homomorphism $f : (V, C) \longrightarrow A$ satisfying the identity $\Delta f \circ v = (F_{\bullet_A} \star \alpha) \circ \mathrm{qinj}$, which is just the cocone notation for $f \circ v_i = \alpha_i \circ \mathrm{qinj}_i$ for all $i \in \mathcal{J}$ in the sense of C-homomorphisms.

"(a) $\Longrightarrow$ (b)": Owing to Lemma 4.3.1, we may take without the loss of generality $F_\omega(\varinjlim F) = (V, C)$ and $v = F_\omega \star u$ as well as $A = Q(V, \omega, C)$ and $\alpha = Q \star u$ by assumption, where ω denotes the complexified pre-symplectic form of $\varinjlim F$. With (UColim), we conclude $f = \mathrm{qinj}_{\varinjlim F} : (V, C) \longrightarrow A$ since it holds that $\mathrm{qinj}_{\varinjlim F} \circ u_i = Qu_i \circ \mathrm{qinj}_i$ for all $i \in \mathcal{J}$ as C-homomorphisms. As $A = Q(\varinjlim F)$, (4.16) holds true.

"(b) $\Longrightarrow$ (c)": trivial. "(c) $\Longrightarrow$ (d)": trivial. "(d) $\Longrightarrow$ (a)" will be shown by the sequence of the following three lemmas, Lemma 4.3.3, Lemma 4.3.4 and Lemma 4.3.5, in which we keep the notation of this theorem. ❐

LEMMA 4.3.3. *There exists a complexified pre-symplectic form ω on (V, C) such that $v : F_\omega \circ F \dot{\longrightarrow} \Delta(V, C)$ becomes a cocone from F to (V, ω, C).*

Proof: We define

$$\omega : V \times V \longrightarrow \mathbb{C}, \qquad (x, y) \longmapsto \frac{1}{\mathrm{i}\hbar}[f(x), f(y)] \text{ (drop } 1_A\text{)}, \tag{4.20}$$

which is easily seen to be a well-defined, bilinear and skew-symmetric because any $x \in V$ can be written, though certainly not uniquely, as[5] $x = \sum_{i \in \mathcal{J}} v_i(x_i)$, where $x_i = 0_i \in V_i$ for almost all $i \in \mathcal{J}$ and (4.18) holds by assumption:

$$\begin{aligned}
\omega(x, y) \cdot 1_A &= \frac{1}{\mathrm{i}\hbar}[f(x), f(y)] && (4.21)\\
&= \frac{1}{\mathrm{i}\hbar}\Big[f\Big(\sum_{i \in \mathcal{J}} v_i(x_i)\Big), f\Big(\sum_{i \in \mathcal{J}} v_i(y_i)\Big)\Big] && (4.22)\\
&= \frac{1}{\mathrm{i}\hbar}\sum_{i,j \in \mathcal{J}}\big[(\alpha_i \circ \mathrm{qinj}_i)(x_i), (\alpha_j \circ \mathrm{qinj}_j)(y_i)\big] \qquad \forall x, y \in V. && (4.23)
\end{aligned}$$

Since

$$\begin{aligned}
\omega(Cx, Cy) \cdot 1_A &= \frac{1}{\mathrm{i}\hbar}[f(Cx), f(Cy)] && (4.24)\\
&= \frac{1}{\mathrm{i}\hbar}\big[f(x)^*, f(y)^*\big] && (4.25)\\
&= -\frac{1}{\mathrm{i}\hbar}[f(x), f(y)]^* && (4.26)\\
&= \overline{\frac{1}{\mathrm{i}\hbar}[f(x), f(y)]} && (4.27)\\
&= \overline{\omega(x, y)} \cdot 1_A \qquad \forall x, y \in V, && (4.28)
\end{aligned}$$

the triple (V, ω, C) becomes a complexified pre-symplectic space. From the first computation and the construction of the quantisation functor, see Section 3.5, it can be seen that

$$\begin{aligned}
\omega(v_i(x_i), v_i(y_i)) \cdot 1_A &= \alpha_i\left(\frac{1}{\mathrm{i}\hbar}[\mathrm{qinj}_i(x_i), \mathrm{qinj}_i(y_i)]\right) && (4.29)\\
&= \alpha_i\big(\omega_i(x_i, y_i) \cdot 1_{Q(Fi)}\big) && (4.30)\\
&= \omega_i(x_i, y_i) \cdot 1_A \qquad \forall x_i, y_i \in V_i,\ \forall i \in \mathcal{J}, && (4.31)
\end{aligned}$$

and so the $v_i : (V_i, C_i) \longrightarrow (V, C)$ become symplectic C-homomorphisms $v_i : Fi \longrightarrow (V, \omega, C)$. This particularly implies that v can also be regarded as a cocone $v : F \dot{\longrightarrow} \Delta(V, \omega, C)$ as claimed. ❒

[5] V can be explicitly constructed as a quotient of the direct sum of the vector spaces V_i, $i \in \mathcal{J}$.

LEMMA 4.3.4. *With the complexified pre-symplectic form defined by* (4.20),

$$\operatorname{colim}(Q \circ F) = \left(Q(V,\omega,C), Q \star v : Q \circ F \dot{\longrightarrow} Q(V,\omega,C)\right). \tag{4.32}$$

Proof: By the universal property of Q (Lemma 3.5.3), there exists a unique unital *-homomorphism $\varphi : Q(V,\omega,C) \longrightarrow A$ such that $\varphi \circ \mathrm{qinj}_{(V,\omega,C)} = f$ as C-homomorphisms. We have

$$\varphi \circ Qv_i \circ \mathrm{qinj}_i = \varphi \circ \mathrm{qinj}_{(V,\omega,C)} \circ v_i = f \circ v_i = \alpha_i \circ \mathrm{qinj}_i \qquad \forall i \in \mathcal{J} \tag{4.33}$$

and Corollary 3.5.4 yields $\varphi \circ Qv_i = \alpha_i$ for all $i \in \mathcal{J}$, which is $\Delta\varphi \circ Q \star v = \alpha$ in cocone notation. By (UColim), there exists a unique unital *-homomorphism $\psi : A \longrightarrow Q(V,\omega,C)$ such that $\Delta\psi \circ \alpha = Q \star v$. Hence, $\Delta(\varphi \circ \psi) \circ \alpha = \alpha$ and $\Delta(\psi \circ \varphi) \circ Q \star v = Q \star v$, which implies $\varphi \circ \psi = \mathrm{id}_A$ and $\psi \circ \varphi = \mathrm{id}_{Q(V,\omega,C)}$. ❒

LEMMA 4.3.5. *The colimit for F exists and with the complexified pre-symplectic form given by* (4.20),

$$\operatorname{colim} F = \left((V,\omega,C), v : F \dot{\longrightarrow} \Delta(V,\omega,C)\right). \tag{4.34}$$

Proof: Thanks to Lemma 4.3.3, we already know that v can be regarded as a well-defined cocone from F to (V,ω,C). It remains to show that this cocone is universal. Therefore, let $\lambda : F \dot{\longrightarrow} (W,\omega_W,C_W)$ be a cocone from F to any complexified pre-symplectic space (W,ω_W,C_W). Forgetting about complexified pre-symplectic forms, (UColim) yields a unique C-homomorphism $\lambda_v : (V,C) \longrightarrow (W,C_W)$ such that $\Delta\lambda_v \circ v = F_\omega \star \lambda$. We have to show that λ_v is symplectic. Since any vector $x \in V$ can be written (non-uniquely) as $x = \sum_{i\in\mathcal{J}} v_i(x_i)$, where $x_i = 0_i \in V_i$ for almost all $i \in \mathcal{J}$, it is enough to show that $\omega_W(\lambda_i(x_i), \lambda_j(y_j)) = \omega(v_i(x_i), v_j(y_j))$ for all $x_i \in V_i$, for all $y_j \in V_j$ and for all $i, j \in \mathcal{J}$. We will read this off by going over to the quantisations of the complexified pre-symplectic spaces involved, using the universal property of the universal algebra.

By Lemma 4.3.4, there exists a unique unital *-homomorphism $\varphi : Q(V,\omega,C) \longrightarrow Q(W,\omega_W,C_W)$ satisfying $\Delta\varphi \circ Q \star v = Q \star \lambda$. Moreover, as C-homomorphisms,

$$\varphi \circ \mathrm{qinj}_{(V,\omega,C)} \circ v_i = \varphi \circ Qv_i \circ \mathrm{qinj}_i = Q\lambda_i \circ \mathrm{qinj}_i = \mathrm{qinj}_{(W,\omega_W,C_W)} \circ \lambda_i \qquad \forall i \in \mathcal{J}. \tag{4.35}$$

With this, we can compute

$$\omega(v_i(x_i), v_j(y_j)) \cdot 1_{Q(W,\omega_W,C_W)} \tag{4.36}$$

$$= \varphi\left(\omega(v_i(x_i), v_j(y_j)) \cdot 1_{Q(V,\omega,C)}\right) \tag{4.37}$$

$$(4.38) \qquad = \varphi\left(\frac{1}{\mathrm{i}\hbar}\left[\left(\mathrm{qinj}_{(V,\omega,C)}\circ v_i\right)(x_i),\left(\mathrm{qinj}_{(V,\omega,C)}\circ v_j\right)(y_j)\right]\right)$$

$$(4.39) \qquad = \frac{1}{\mathrm{i}\hbar}\left[\left(\varphi\circ\mathrm{qinj}_{(V,\omega,C)}\circ v_i\right)(x_i),\left(\varphi\circ\mathrm{qinj}_{(V,\omega,C)}\circ v_j\right)(y_j)\right]$$

$$(4.40) \qquad = \frac{1}{\mathrm{i}\hbar}\left[\left(\mathrm{qinj}_{(W,\omega_W,C_W)}\circ\lambda_i\right)(x_i),\left(\mathrm{qinj}_{(W,\omega_W,C_W)}\circ\lambda_j\right)(y_j)\right]$$

$$(4.41) \qquad = \omega_W\left(\lambda_i(x_i),\lambda_j(y_j)\right)\cdot 1_{Q(W,\omega_W,C_W)}$$

$$\forall x_i \in V_i,\ \forall y_j \in V_j,\ \forall i,j \in \mathcal{J}.$$

In conclusion, λ_v is symplectic and $((V,\omega,C), v : F \dot{\longrightarrow} \Delta(V,\omega,C))$ is the colimit for the functor $F : \mathcal{J} \longrightarrow \mathbf{pSympl}_{\mathbb{C}}$ accordingly. ❒

4.4 Auxiliary lemmas

Lemma 4.3.1 *and* Theorem 4.3.2 *have shown us that the colimit for the functor $F_\omega \circ F : \mathcal{J} \longrightarrow \mathcal{C}$ plays a prominent role for the colimit of a functor $F : \mathcal{J} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, where $\mathcal{J}$ is a small category.* Lemma 4.3.1 *states that we can obtain the colimit for $F_\omega \circ F$ by applying the forgetful functor F_ω to the colimit for F, and in* Lemmas 4.3.3-4.3.5*, we have constructed the colimit for F precisely from the colimit for $F_\omega \circ F$ by endowing it with a (complexified) (pre-)symplectic form stemming from the colimit for the composition of F with the quantisation functor Q, $Q \circ F$, i.e. from the commutation relations in the universal algebra. It becomes thus apparent that we will have to compute colimits in the categories $\mathbf{Vec}_{\mathbb{R}}$ and* **CVec** *at some point in the application of* Theorem 4.3.2*. For this purpose, we derive some helpful results in the following three lemmas concerning vector spaces of compactly supported smooth cross-sections in smooth vector bundles.*

Note the topological restriction we are subjecting ourselves to in this first lemma; without them, there would be almost nothing to show.

LEMMA 4.4.1. *Let $\mathbf{M} \in \mathbf{Loc}$, $\xi = (E, M, \pi, V)$ a smooth $\mathbb{K}$-vector bundle and consider any category of the form $\mathrm{loc}^q_{-\mathbf{M}}$. Define a functor $F : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{K}}$ by*

$$(4.42) \qquad FO := \Gamma^\infty_0\left(\xi|_O\right) \qquad \forall O \in \mathrm{loc}^q_{-\mathbf{M}},$$

and

$$(4.43) \qquad F\iota_{UV} := \mathbf{i}_{\xi|_U \xi|_V *} : \Gamma^\infty_0\left(\xi|_U\right) \hookrightarrow \Gamma^\infty_0\left(\xi|_V\right)$$

$$\forall U, V \in \mathrm{loc}^q_{-\mathbf{M}} \text{ such that } U \subseteq V,$$

where $\mathbf{i}_{\xi|_U \xi|_V *}$ *denotes the pushforward of compactly supported smooth cross-sections along the bundle inclusion* $\mathbf{i}_{\xi|_U \xi|_V} : \xi|_U \longrightarrow \xi|_V$. *Then*

$$\operatorname{colim} F = \left(\Gamma_0^\infty(\xi), \left\{\mathbf{i}_{\xi|_O *} : \Gamma_0^\infty(\xi|_O) \hookrightarrow \Gamma_0^\infty(\xi)\right\}_{O \in \mathrm{loc}^q_{-\mathbf{M}}}\right), \tag{4.44}$$

where $\mathbf{i}_{\xi|_O *}$ *is the pushforward of compactly supported smooth cross-sections along the bundle inclusion* $\mathbf{i}_{\xi|_O} : \xi|_O \longrightarrow \xi$.

Proof: Given a cocone $\lambda : F \dot{\longrightarrow} \Delta W$ to a $\mathbb{K}$-vector space W, we define

$$\lambda_u : \Gamma_0^\infty(\xi) \longrightarrow W, \qquad \sigma \longmapsto \sum_{O \in \mathrm{loc}^q_{-\mathbf{M}}} \lambda_O\left(\mathbf{i}^\#_{\xi|_O} \chi_O \sigma\right), \tag{4.45}$$

where $\mathbf{i}^\#_{\xi|_O} : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi|_O)$ is the pullback via the bundle inclusion $\mathbf{i}_{\xi|_O}$ and $\{\chi_O \mid O \in \mathrm{loc}^q_{-\mathbf{M}}\}$ a smooth partition of unity subordinated to $\{O \mid O \in \mathrm{loc}^q_{-\mathbf{M}}\}$, which is an open cover of M. First of all, we need to show that (4.45) is well-defined, i.e. independent from the smooth partition of unity chosen. To see this, we establish the local compatibility relation

$$\lambda_U\left(\mathbf{i}^\#_{\xi|_U} \sigma\right) = \lambda_V\left(\mathbf{i}^\#_{\xi|_V} \sigma\right) \tag{4.46}$$

for all $\sigma \in \Gamma_0^\infty(\xi)$ with $\operatorname{supp}\sigma \subseteq U \cap V$ whenever $U, V \in \mathrm{loc}^q_{-\mathbf{M}}$ with $U \cap V \neq \varnothing$. Note that (4.46) is not immediately clear because $U \cap V$ need not be an object of $\mathrm{loc}^q_{-\mathbf{M}}$ even if the intersection is connected. Hence, $\lambda_{U\cap V}$ does not always exist as given data and the bundle inclusions $\mathbf{i}_{\xi|_{U\cap V}\xi|_V}, \mathbf{i}_{\xi|_{U\cap V}\xi|_V}, \mathbf{i}_{\xi|_{U\cap V}}$ are not always regarded by the functor F. Also, observe the analogy to [DL12, Lem.3.1].

Now, to show (4.46) for $U, V \in \mathrm{loc}^q_{-\mathbf{M}}$ with $U \cap V \neq \varnothing$ and $\sigma \in \Gamma_0^\infty(\xi)$ with $\operatorname{supp}\sigma \subseteq U \cap V$, we cover $U \cap V$ by $\mathrm{loc}^q_{-\mathbf{M}}$-objects O such that $O \subseteq U \cap V$, which can be done thanks to Lemma 1.1.2. Using a smooth partition of unity $\{\zeta, \zeta_O \mid O \in \mathrm{loc}^q_{-\mathbf{M}}$ such that $O \subseteq U \cap V\}$ subordinated to the open cover $\{M \setminus \operatorname{supp}\sigma, O \mid O \in \mathrm{loc}^q_{-\mathbf{M}}$ such that $O \subseteq U \cap V\}$ of M and the properties of λ as a cocone from F, we break down each side of (4.46) and show that this yields the same result:

$$\lambda_U\left(\mathbf{i}^\#_{\xi|_U} \sigma\right) = \sum_O \lambda_U\left(\mathbf{i}^\#_{\xi|_U} \zeta_O \sigma\right) \tag{4.47}$$

$$= \sum_O \lambda_U\left(\mathbf{i}_{\xi|_O \xi|_U *} \mathbf{i}^\#_{\xi|_O \xi|_U} \mathbf{i}^\#_{\xi|_U} \zeta_O \sigma\right) \tag{4.48}$$

$$= \sum_O \left(\lambda_U \circ F\iota_{OU}\right)\left(\mathbf{i}^\#_{\xi|_O} \zeta_O \sigma\right) \tag{4.49}$$

$$= \sum_O \lambda_O\left(\mathbf{i}^\#_{\xi|_O} \zeta_O \sigma\right) \tag{4.50}$$

$$= \lambda_V\left(\mathbf{i}^\#_{\xi|_V} \sigma\right), \tag{4.51}$$

where $\mathbf{i}^{\#}_{\xi|_O\xi|_U} : \Gamma^\infty(\xi|_U) \longrightarrow \Gamma^\infty(\xi|_O)$ denotes the pullback via the bundle inclusion $\mathbf{i}_{\xi|_O\xi|_U}$. The local compatibility condition (4.46) can now be used to switch between smooth partitions of unities in the definiton (4.45) of λ_u:

$$\sum_{U\in\mathrm{loc}^q_{-\mathbf{M}}} \lambda_U\left(\mathbf{i}^{\#}_{\xi|_U}\chi_U\sigma\right) = \sum_{U,V\in\mathrm{loc}^q_{-\mathbf{M}}} \lambda_U\Big(\mathbf{i}^{\#}_{\xi|_U}\underbrace{\chi_U\kappa_V\sigma}_{\mathrm{supp}\,(\chi_U\kappa_V\sigma)\subseteq U\cap V}\Big) \tag{4.52}$$

$$= \sum_{U,V\in\mathrm{loc}^q_{-\mathbf{M}}} \lambda_V\left(\mathbf{i}^{\#}_{\xi|_V}\chi_U\kappa_V\sigma\right) \tag{4.53}$$

$$= \sum_{V\in\mathrm{loc}^q_{-\mathbf{M}}} \lambda_V\left(\mathbf{i}^{\#}_{\xi|_V}\kappa_V\sigma\right) \qquad \forall\sigma\in\Gamma^\infty_0(\xi), \tag{4.54}$$

where $\{\kappa_O \mid O\in\mathrm{loc}^q_{-\mathbf{M}}\}$ is another smooth partition of unity which is subordinated to $\{O \mid O\in\mathrm{loc}^q_{-\mathbf{M}}\}$. This shows that the definition of λ_u is independent from the partition of unity. The local compatibility relation (4.46) also reveals $\lambda_u\circ\mathbf{i}_{\xi|_O*} = \lambda_O$ for all $O\in\mathrm{loc}^q_{-\mathbf{M}}$:

$$\left(\lambda_u\circ\mathbf{i}_{\xi|_O*}\right)(\sigma) = \lambda_u\left(\mathbf{i}_{\xi|_O*}\sigma\right) \tag{4.55}$$

$$= \sum_{U\in\mathrm{loc}^q_{-\mathbf{M}}} \lambda_U\Big(\mathbf{i}^{\#}_{\xi|_U}\underbrace{\chi_U\mathbf{i}_{\xi|_O*}\sigma}_{\mathrm{supp}\,\chi_U\mathbf{i}_{\xi|_O*}\sigma\subseteq O\cap U}\Big) \tag{4.56}$$

$$= \sum_{U\in\mathrm{loc}^q_{-\mathbf{M}}} \lambda_O\left(\mathbf{i}^{\#}_{\xi|_O}\chi_U\mathbf{i}_{\xi|_O*}\sigma\right) \tag{4.57}$$

$$= \lambda_O\left(\mathbf{i}^{\#}_{\xi|_O}\mathbf{i}_{\xi|_O*}\sigma\right) \tag{4.58}$$

$$= \lambda_O(\sigma) \qquad \forall\sigma\in\Gamma^\infty_0(\xi|_O). \tag{4.59}$$

Also, λ_u is the unique linear map $\Gamma^\infty_0(\xi) \longrightarrow W$ with this property because let $\kappa : \Gamma^\infty_0(\xi) \longrightarrow W$ be another one; then we have

$$\kappa(\sigma) = \kappa\Big(\sum_{O\in\mathrm{loc}^q_{-\mathbf{M}}} \chi_O\sigma\Big) \tag{4.60}$$

$$= \sum_{O\in\mathrm{loc}^q_{-\mathbf{M}}} \kappa\left(\mathbf{i}_{O*}\mathbf{i}^{\#}_{\xi|_O}\chi_O\sigma\right) \tag{4.61}$$

$$= \sum_{O\in\mathrm{loc}^q_{-\mathbf{M}}} \left(\kappa\circ F\iota_O\right)\left(\mathbf{i}^{\#}_{\xi|_O}\chi_O\sigma\right) \tag{4.62}$$

$$= \sum_{O\in\mathrm{loc}^q_{-\mathbf{M}}} \lambda_O\left(\mathbf{i}^{\#}_{\xi|_O}\chi_O\sigma\right) \tag{4.63}$$

$$= \lambda_u(\sigma) \qquad \forall\sigma\in\Gamma^\infty_0(\xi), \tag{4.64}$$

which entails $\kappa = \lambda_u$. ❒

Of course, we cannot expect to work with plain compactly supported smooth cross-sections. As we will see, we have to consider quotients of such vector spaces. The purpose of the following two lemmas is accordingly to deal with quotients.

LEMMA 4.4.2. *Let* $F, G : \mathcal{J} \longrightarrow \mathbf{Vec}_{\mathrm{K}}$ *be functors such that* Gi *is a linear subspace of* Fi *for all* $i \in \mathcal{J}$ *and* $G\mu_{ij} = F\mu_{ij} : Gi \longrightarrow Gj$ *for all* $\mu_{ij} \in \mathcal{J}(i,j)$ *and for all* $i, j \in \mathcal{J}$*, i.e.* $G\mu_{ij}$ *is* $F\mu_{ij}$ *restricted to* Gi *and viewed as a linear map to* Gj*. This of course requires that* $F\mu_{ij}(Gi) \subseteq Gj$*, which is thus assumed. We denote the canonical projections onto the quotients with* $\pi_i : Fi \twoheadrightarrow Fi/Gi$*. By assumption,* $F\mu_{ij}(Gi) \subseteq Gj$ *for all* $\mu_{ij} \in \mathcal{J}(i,j)$ *and for all* $i, j \in \mathcal{J}$*, hence by* (UQ')*, there exist unique linear maps* $[F\mu_{ij}] : Fi/Gi \longrightarrow Fj/Gj$ *satisfying* $[F\mu_{ij}] \circ \pi_i = \pi_j \circ F\mu_{ij}$*. Under these circumstances, a functor* $[F] : \mathcal{J} \longrightarrow \mathbf{Vec}_{\mathrm{K}}$ *is defined by*

$$[F]\, i := Fi/Gi \qquad \forall i \in \mathcal{J} \tag{4.65}$$

and

$$[F]\, \mu_{ij} := [F\mu_{ij}] \qquad \forall \mu_{ij} \in \mathcal{J}(i,j),\ \forall i, j \in \mathcal{J}, \tag{4.66}$$

and the canonical projections π_i *define a natural transformation* $\pi : F \dot{\longrightarrow} [F]$*.*

Proof: This is just a simple application of the the universal property (UQ') of quotient vector spaces. ❒

We now turn to the computation of colimits in the case of quotient vector spaces. Note the appearance of the categorical union, whose additivity property will be used in the proof:

LEMMA 4.4.3. *Let* $F, G : \mathcal{J} \longrightarrow \mathbf{Vec}_{\mathrm{K}}$ *be functors as in* Lemma 4.4.3*. Thanks to* Theorem 2.2.10, $\operatorname{colim} F = (\varinjlim F, u : F \dot{\longrightarrow} \Delta \varinjlim F)$ *exists. Assume that for each* $i \in \mathcal{J}$*,* $m_i := u_i \circ \iota_i : Gi \hookrightarrow \varinjlim F$ *is a subobject, where* $\iota_i : Gi \hookrightarrow Fi$ *is the inclusion map. Then*

$$\operatorname{colim}[F] = \Big(\big[\varinjlim F\big] := \varinjlim F / \bigvee_{i \in \mathcal{J}} Gi,\ [u] : [F] \dot{\longrightarrow} \Delta \big[\varinjlim F\big] \Big) \tag{4.67}$$

where $m : \bigvee_{i \in \mathcal{J}} Gi \hookrightarrow \varinjlim F$ *is the union of the subobjects* $m_i : Gi \hookrightarrow \varinjlim F$ *and* $[u]$ *is the unique natural transformation satisfying* $[u] \circ \pi = \Delta \pi_{\bigvee} \circ u$ $(\pi_{\bigvee} : \varinjlim F \twoheadrightarrow \big[\varinjlim F\big]$ *denotes the canonical projection onto the quotient*$)$*.*

Proof: Owing to Example 2.3.10, $\bigvee_{i \in \mathcal{J}} Gi$ is the linear subspace of $\varinjlim F$ generated by the images $m_i(Gi)$ and m is the inclusion of $\bigvee_{i \in \mathcal{J}} Gi$ into $\varinjlim F$. Thus, for each $i \in \mathcal{J}$, we have that $\ker m_i \subseteq \bigvee_{i \in \mathcal{J}} Gi$ and (UQ') yields a unique linear map $[u_i] : [F]i \longrightarrow \big[\varinjlim F\big]$ satisfying $[u_i] \circ \pi_i = \pi_{\bigvee} \circ u_i$.

Let $\lambda : [F] \dot{\longrightarrow} \Delta W$ be a cocone to a K-vector space W; then $\lambda \circ \pi$ is a cocone from F to W and (UColim) yields a unique linear map $\lambda_u : \varinjlim F \longrightarrow W$ satisfying $\Delta\lambda_u \circ u = \lambda \circ \pi$. By Example 2.3.10, any $v \in \bigvee_{i\in\mathcal{J}} Gi$ can be written as $v = \sum_{i\in\mathcal{J}} m_i(v_i)$ for $v_i \in Gi$ and $v_i \neq 0_{Gi}$ for only finitely many $i \in \mathcal{J}$. Hence,

$$\lambda_u(v) = \lambda_u\Big(\sum_{i\in\mathcal{J}} m_i(v_i)\Big) \tag{4.68}$$

$$= \sum_{i\in\mathcal{J}} \lambda_u((u_i \circ \iota_{GiFi})(v_i)) \tag{4.69}$$

$$= \sum_{i\in\mathcal{J}} (\lambda_u \circ u_i)(\iota_{GiFi}(v_i)) \tag{4.70}$$

$$= \sum_{i\in\mathcal{J}} (\lambda_i \circ \pi_i)(\iota_{GiFi}(v_i)) \tag{4.71}$$

$$= \sum_{i\in\mathcal{J}} \lambda_i[0_{Fi}] \tag{4.72}$$

$$= 0_W \qquad \forall v \in \bigvee_{i\in\mathcal{J}} Gi \tag{4.73}$$

and $\ker\lambda_u \subseteq \bigvee_{i\in\mathcal{J}} Gi$ accordingly. (UQ') yields a unique linear map $[\lambda_u] : [\varinjlim F] \longrightarrow W$ meeting $[\lambda_u] \circ \pi_\vee = \lambda_u$. $[\lambda_u]$ is also the unique linear map satisfying $\Delta[\lambda_u] \circ [u] = \lambda$ because suppose otherwise; then there is $\kappa : [\varinjlim F] \longrightarrow W$ such that $\Delta\kappa \circ [u] = \lambda$. Therefore, $\Delta\kappa \circ [u] \circ \pi = \Delta\kappa \circ \Delta\pi_\vee \circ u = \Delta(\kappa \circ \pi_\vee) \circ u = \lambda \circ \pi$ and $\kappa \circ \pi_\vee = \lambda_u$ by (UColim). Consequently, $\kappa = [\lambda_u]$ by (UQ'). ❒

4.5 The free smooth differential p-form real Klein-Gordon field

As a warm-up to computing universal algebras, and applying colimit constructions and left Kan extensions in the case of the free Maxwell field, we treat the free and minimally coupled real scalar field. Since it does not cause us any more trouble, we provide a treatment in the context of smooth differential p-forms, $p \geq 0$. Indeed, any solution of the free Maxwell equations also satisfies the analogue of the wave equation for smooth differential 2-forms, which partially motivates our course of action. Our other intention is to apply Theorem 4.3.2 *and to show that the universal algebra is the well-known unital *-algebra of the smeared quantum field, which is reassuring. We will deal with real- and complex-valued smooth differential p-forms in one go because there is not much difference between these two cases.*

Fix $\mathbf{M} \in \mathbf{Loc}$ for the moment and let[6] $p \in \mathbb{N}$. The case $p = 0$ constitutes the well-known free and minimally coupled real scalar field. The Lagrangean (smooth $\mathbb{K}$-valued differential m-form) of the free smooth differential p-form real Klein-Gordon

[6] Recall that $0 \in \mathbb{N}$ for us.

field $\phi \in \Omega^p(M;\mathbb{K})$ is

$$\mathcal{L} = \frac{1}{2}\left(d\phi \wedge *d\phi + \delta\phi \wedge *\delta\phi - \mu^2 \phi \wedge *\phi\right) \tag{4.74}$$

with the Euler-Lagrange equation, referred to as the Klein-Gordon equation,

$$(\Box + \mu^2)\,\phi = (-\delta d - d\delta + \mu^2)\,\phi = 0, \tag{4.75}$$

where μ is the reduced mass defined by $\mu := \frac{mc}{\hbar}$, $m \geq 0$ the mass of the field (not to be confused with the fixed spacetime dimension), c the speed of light and $\hbar$ the reduced Planck constant. By computing the principal symbols for the exterior derivative d and the exterior coderivative δ or checking directly in any smooth chart of M, one can see that the Klein-Gordon operator $D := \Box + \mu^2$ is a normally hyperbolic linear differential operator of metric type. Hence, [BGP07, Cor.3.4.3] or [Wal12, Cor.4.3.7] yields unique retarded and advanced Green's operators $G^{\mathrm{ret}} : \Omega^p_0(M;\mathbb{K}) \longrightarrow \Omega^p_{\mathrm{sc}}(M;\mathbb{K})$ and $G^{\mathrm{adv}} : \Omega^p_0(M;\mathbb{K}) \longrightarrow \Omega^p_{\mathrm{sc}}(M;\mathbb{K})$. Building the advanced-minus-retarded Green operator $G := G^{\mathrm{adv}} - G^{\mathrm{ret}}$, we note the following:

LEMMA 4.5.1. (i) *The kernel of D on $\Omega^p_0(M;\mathbb{K})$ is trivial; the kernel of G on $\Omega^p_0(M;\mathbb{K})$ is $D\Omega^p_0(M;\mathbb{K})$ while $\{\phi \in \Omega^p_{\mathrm{sc}}(M;\mathbb{K}) \mid D\phi = 0\}$ is the image of G on $\Omega^p_0(M;\mathbb{K})$. In particular, $\phi \in \Omega^p_{\mathrm{sc}}(M;\mathbb{K})$ solves* (4.75) *if and only if there is $\omega \in \Omega^p_0(M;\mathbb{K})$ such that $\phi = G\omega$.*

For the massless case $m = 0$ ($\Longrightarrow D = \Box = -\delta d - d\delta$), we further have: (ii) *For all $\omega \in \Omega^p_0(M;\mathbb{K})$, $dG\omega = Gd\omega$, $\delta G\omega = G\delta\omega$ and $G\delta d\omega = -Gd\delta\omega$.* (iii) *The kernels of $d\Box$ and $\delta\Box$ on $\Omega^p_0(M;\mathbb{K})$ are $\Omega^p_{0,d}(M;\mathbb{K})$ and $\Omega^p_{0,\delta}(M;\mathbb{K})$, respectively. The kernels of $dG\delta$ and $\delta G d$ are both equal to $\Omega^p_{0,d}(M;\mathbb{K}) \oplus \Omega^p_{0,\delta}(M;\mathbb{K})$.*

Proof: (i) results from [BGP07, Thm.3.4.7] or [Wal12, Thm.4.3.18]. The first two identities of (ii) are proven by [Pfe09, Prop.2.1], which is due to the uniqueness of the retarded and the advanced Green operator. For the third identity, we use (i) and find the identity $G\delta d\omega = G(\delta d + d\delta - d\delta)\omega = G(-\Box - d\delta)\omega = -G\Box\omega - Gd\delta\omega = -Gd\delta\omega$.

(iii): By (i), $d\Box\omega = \Box d\omega = 0$ implies $d\omega = 0$ for all $\omega \in \Omega^p_0(M;\mathbb{K})$ and $\Omega^p_{0,d}(M;\mathbb{K}) \subseteq \ker d\Box$ is clear. In the same way, one shows for all $\omega \in \Omega^p_0(M;\mathbb{K})$ that $\delta\Box\omega = 0$ if and only if $\delta\omega = 0$. If $dG\delta\omega = 0$ for $\omega \in \Omega^p_0(M;\mathbb{K})$, then also $\delta G d\omega = 0$ by (ii) and vice versa. Since $Gd\delta\omega = 0 = G\delta d\omega$ by (ii), (i) yields $d\delta\omega = \Box\alpha$ and $\delta d\omega = \Box\beta$ for some $\alpha, \beta \in \Omega^p_0(M;\mathbb{K})$. Further, $d\Box\alpha = dd\delta\omega = 0$ and $\delta\Box\beta = \delta\delta d\omega = 0$ yield $\alpha \in \Omega^p_{0,d}(M;\mathbb{K})$ and $\beta \in \Omega^p_{0,\delta}(M;\mathbb{K})$. So, $\omega \in \Omega^p_{0,d}(M;\mathbb{K}) + \Omega^p_{0,\delta}(M;\mathbb{K})$ but this sum is surely a direct sum because $\Omega^p_{0,d}(M;\mathbb{K}) \cap \Omega^p_{0,\delta}(M;\mathbb{K}) = 0$ by (i) as $\Box\omega = 0$ for all $\omega \in \Omega^p_0(M;\mathbb{K})$ in the intersection. ❐

As we have indicated in the section on the quantisation functor, Section 3.5, we will only describe a rudimentary theory consisting of basic linear observables. This is due to the problems arising from the attempt to consistently quantise all possible classical observables and also, of course, due to technical simplicity.

Experiments are usually conducted over a finite period of time and in laboratories, which are spatially confined. Also, we take the point of view that measurements are performed on the fields themselves. Accordingly, we will consider the linear functionals on all field configurations $\{\phi \in \Omega^p(M;\mathbb{K}) \mid D\phi = 0\}$ which are of the form $O_\omega : \phi \longmapsto \int_M \phi \wedge *\omega$ for $\omega \in \Omega_0^p(M;\mathbb{K})$. Of course, we want our theory to have as many linear observables as possible in order to be able to differentiate between as many different field configurations as possible. However, taking $\{O_\omega \mid \omega \in \Omega_0^p(M;\mathbb{K})\}$ as our basic linear observables leads to redundant linear observables, which cannot be distinguished on the field configurations considered. To remedy this redundancy, that is, to optimise, we take the linear functionals on $\{\phi \in \Omega^p(M;\mathbb{K}) \mid D\phi = 0\}$ which are of the form $O_{[\omega]} : \phi \longmapsto \int_M \phi \wedge *\omega$ for $[\omega] \in [\Omega_0^p(M;\mathbb{K})]$ as the basic linear observables for the classical field theory. Here, we have defined $[\Omega_0^p(M;\mathbb{K})] := \Omega_0^p(M;\mathbb{K})/D\Omega_0^p(M;\mathbb{K})$.

In order to equip the basic linear observables, which we want to idenitify with $[\Omega_0^p(M;\mathbb{K})]$ from now on, with a symplectic structure, we use Peierls' method ([Pei52], [Haa96, Sec.I.V.]): for $\omega \in \Omega_0^p(M;\mathbb{K})$, we modify (4.74) to yield ($\varepsilon > 0$)

$$\mathcal{L}_\varepsilon = \frac{1}{2}\left(d\phi \wedge *d\phi + \delta\phi \wedge *\delta\phi - \mu^2\phi \wedge *\phi\right) + \varepsilon\phi \wedge *\omega, \tag{4.76}$$

whose Euler-Lagrange equation is

$$D\phi = \varepsilon\omega. \tag{4.77}$$

Suppose $\phi \in \Omega^p(M;\mathbb{K})$ is a solution of (4.75), then $\delta^{\mathrm{ret/adv}}_{\varepsilon O_\omega}\phi := \phi + \varepsilon G^{\mathrm{ret/adv}}\omega$ is the unique solution of (4.77) which coincides with ϕ in the remote past/future. We easily compute the derivative of $\delta^{\mathrm{ret/adv}}_{\varepsilon O_\omega}\phi$ with respect to ε at $\varepsilon = 0$, $\delta^{\mathrm{ret/adv}}_{O_\omega}\phi = G^{\mathrm{ret/adv}}\omega$, and thus find $\delta_{O_\omega}\phi := \delta^{\mathrm{ret}}_{O_\omega}\phi - \delta^{\mathrm{adv}}_{O_\omega}\phi = -G\omega$ (cf. [Haa96, (I.4.3) + (I.4.4)]). Thereby,

$$\{O_\omega, O_\eta\} = \delta_{O_\omega}O_\eta = O_\eta(-G\omega) = \int_M -G\omega \wedge *\eta \qquad \forall \omega, \eta \in \Omega_0^p(M;\mathbb{K}). \tag{4.78}$$

LEMMA 4.5.2. *The map*

$$(\omega,\eta) \longmapsto \int_M \omega \wedge *G\eta, \qquad \omega,\eta \in \Omega_0^p(M;\mathbb{K}), \tag{4.79}$$

is a skew-symmetric bilinear form with radical $D\Omega_0^p(M;\mathbb{K})$.

Proof: Bilinearity is clear and skew-symmetry follows from the general properties of G. Fix $\omega \in \Omega_0^2(M;\mathbb{K})$ and let

$$\int_M \omega \wedge *G\eta = -\int_M G\omega \wedge *\eta = 0 \qquad \forall \eta \in \Omega_0^p(M;\mathbb{K}). \tag{4.80}$$

The non-degeneracy of $\int_M \cdot \wedge * \cdot : \Omega^p(M;\mathbb{K}) \times \Omega_0^p(M;\mathbb{K}) \longrightarrow \mathbb{K}$ implies $G\omega = 0$ and hence $\omega = D\alpha$ for some $\alpha \in \Omega_0^p(M;\mathbb{K})$ by Lemma 4.5.1. The rest follows from the skew-symmetry. ❐

LEMMA 4.5.3. *The tuple*

$$\begin{cases} [\Omega_0^p(M;\mathbb{K})] := \Omega_0^p(M;\mathbb{K})/D\Omega_0^p(M;\mathbb{K}), \\ \mathfrak{u} : [\Omega_0^p(M;\mathbb{K})] \times [\Omega_0^p(M;\mathbb{K})] \longrightarrow \mathbb{K}, \ ([\omega],[\eta]) \longmapsto \int_M \omega \wedge *G\eta, \\ ^{-} : [\Omega_0^p(M;\mathbb{K})] \longrightarrow [\Omega_0^p(M;\mathbb{K})], \ [\omega] \longmapsto [\overline{\omega}] \ \text{(complex conjugation)}, \end{cases} \tag{4.81}$$

is a symplectic space.

If we apply the quantisation functor $Q : \mathbf{Sympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg}_1^{\mathbf{m}}$ to (4.81), we obtain the unital *-algebra of the smeared quantum field for the free smooth differential p-form real Klein-Gordon field; this is the unital *-algebra generated by the elements of the form $\mathbf{\Phi}(\omega)$, $\omega \in \Omega_0^2(M;\mathbb{K})$, which are further subjected to the following relations:

- Linearity: $\mathbf{\Phi}(\lambda\omega + \mu\eta) = \lambda\mathbf{\Phi}(\omega) + \mu\mathbf{\Phi}(\eta)$ for all $\lambda, \mu \in \mathbb{K}$ and for all $\omega, \eta \in \Omega_0^p(M;\mathbb{K})$.
- Hermiticity: $\mathbf{\Phi}(\omega)^* = \mathbf{\Phi}(\overline{\omega})$ for all $\omega \in \Omega_0^p(M;\mathbb{K})$.
- Field equations (in a weak sense): $\mathbf{\Phi}(D\omega) = 0$ for all $\omega \in \Omega_0^p(M;\mathbb{K})$.
- Commutation relations: $[\mathbf{\Phi}(\omega), \mathbf{\Phi}(\eta)] = \mathrm{i}\hbar \int_M \omega \wedge *G\eta \cdot 1_A$ for all $\omega, \eta \in \Omega_0^p(M;\mathbb{K})$.

Note that for $p = 0$, the commutation relations reduce to the familar ones from standard textbooks on quantum field theory, e.g. [BS80; Sch05; Wen03]. We now consider the functorial aspects:

PROPOSITION 4.5.4. *The rules*

$$\mathbf{Loc} \ni \mathbf{M} \longmapsto ([\Omega_0^p(M;\mathbb{K})], \mathfrak{u}_{\mathbf{M}}, ^{-}) \tag{4.82}$$

and

$$\mathbf{Loc}(\mathbf{M},\mathbf{N}) \ni \psi \longmapsto [\psi_*] : ([\Omega_0^p(M;\mathbb{K})], \mathfrak{u}_{\mathbf{M}}, ^{-}) \longrightarrow ([\Omega_0^p(N;\mathbb{K})], \mathfrak{u}_{\mathbf{N}}, ^{-}) \tag{4.83}$$

define a locally covariant theory $\mathcal{P} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ and a locally covariant quantum field theory $\mathfrak{P} := Q \circ \mathcal{P} : \mathbf{Loc} \longrightarrow {}^*\mathbf{Alg}_1^{\mathbf{m}}$.

In (4.83), $[\psi_*] : ([\Omega^p_0(M;\mathbb{K})], \mathfrak{u}_{\mathbf{M}}, {}^-) \longrightarrow ([\Omega^p_0(N;\mathbb{K})], \mathfrak{u}_{\mathbf{N}}, {}^-)$ is defined as follows: we start from the pushforward along ψ, $\psi_* : \Omega^p_0(M;\mathbb{K}) \longrightarrow \Omega^p_0(N;\mathbb{K})$, which is injective, linear, commutes with the complex conjugation if $\mathbb{K} = \mathbb{C}$ and, furthermore, satisfies $\psi_* D_{\mathbf{M}} = D_{\mathbf{N}} \psi_*$. Because of this and (UQ'), there is a unique linear map $[\psi_*] : [\Omega^p_0(M;\mathbb{K})] \longrightarrow [\Omega^p_0(N;\mathbb{K})]$ such that $[\psi_*] \circ \pi_{\mathbf{M}} = \pi_{\mathbf{N}} \circ \psi_*$, where $\pi_{\mathbf{M}|\mathbf{N}} : \Omega^p_0(M|N;\mathbb{K}) \twoheadrightarrow [\Omega^p_0(M|N;\mathbb{K})]$ denote the canonical projections onto the quotients. It is obvious that $[\psi_*]$ is a C-homomorphism if $\mathbb{K} = \mathbb{C}$ and since

$$\begin{aligned}
&(4.84) && \mathfrak{u}_{\mathbf{N}}([\psi_*][\omega], [\psi_*][\eta]) = \mathfrak{u}_{\mathbf{N}}([\psi_*\omega], [\psi_*\eta]) \\
&(4.85) && = \int_N \psi_*\omega \wedge *_{\mathbf{N}} G_{\mathbf{N}} \psi_*\eta \\
&(4.86) && = \int_{\psi(M)} \psi_*\omega \wedge *_{\mathbf{N}} G_{\mathbf{N}} \psi_*\eta \\
&(4.87) && = \int_M \omega \wedge *_{\mathbf{M}} \psi^* G_{\mathbf{N}} \psi_*\eta \\
&(4.88) && = \int_M \omega \wedge *_{\mathbf{M}} G_{\mathbf{M}} \eta \\
&(4.89) && = \mathfrak{u}_{\mathbf{M}}([\omega], [\eta]) \\
& && \forall\, [\omega], [\eta] \in [\Omega^p_0(M;\mathbb{K})],
\end{aligned}$$

$[\psi_*]$ is symplectic and thus a $\mathbf{Sympl}_{\mathbb{K}}$-morphism. Moreover, (UQ') ensures that $[(\psi \circ \varphi)_*] = [\psi_*] \circ [\varphi_*]$ whenever $\varphi : \mathbf{L} \longrightarrow \mathbf{M}$ is another **Loc**-morphism.

We turn now to the computation of colimits and left Kan extensions for the free real Klein-Gordon field in terms of smooth differential p-forms. First, as an auxiliary step, we compute the corresponding colimits in the categories $\mathbf{Vec}_{\mathbb{R}}$ and **CVec**:

PROPOSITION 4.5.5. *Let* $\mathbf{M} \in \mathbf{Loc}$ *and consider any of the categories* $\mathrm{loc}^q_{\multimap\mathbf{M}}$ *and the restriction* $\mathcal{P}^q_{\mathbf{M}} : \mathrm{loc}^q_{\multimap\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ *of* $\mathcal{P}$ *to* $\mathrm{loc}^q_{\multimap\mathbf{M}}$. *Then*

$$\begin{aligned}
(4.90)\quad & \mathrm{colim}\,(F_\omega \circ \mathcal{P}^q_{\mathbf{M}} : \mathrm{loc}^q_{\multimap\mathbf{M}} \longrightarrow \mathcal{C}) \\
& = \Big((F_\omega \circ \mathcal{P})\,\mathbf{M}, \{(F_\omega \circ \mathcal{P})\,\iota_O : (F_\omega \circ \mathcal{P})\,O \longrightarrow (F_\omega \circ \mathcal{P})\,\mathbf{M}\}_{O \in \mathrm{loc}^q_{\multimap\mathbf{M}}} \Big).
\end{aligned}$$

Proof: We start with $\mathbb{K} = \mathbb{R}$. Choose $F : \mathrm{loc}^q_{\multimap\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ as in Lemma 4.4.1 with $\xi = \lambda^p_M$ and define another functor $G : \mathrm{loc}^q_{\multimap\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ by $GO := D_O \Omega^p_0 O$ for $O \in \mathrm{loc}^q_{\multimap\mathbf{M}}$ and $G\iota_{UV} := \iota_{UV*} : D_U \Omega^p_0 U \hookrightarrow D_V \Omega^p_0 V$ for $U, V \in \mathrm{loc}^q_{\multimap\mathbf{M}}$ with $U \subseteq V$. Since $\iota_{UV*} D_U = D_V \iota_{UV*}$ holds for all $U, V \in \mathrm{loc}^q_{\multimap\mathbf{M}}$ such that $U \subseteq V$, the requirements of Lemma 4.4.3 are met and we may apply it. Evidently, $\mathcal{P}^q_{\mathbf{M}} = [F]$ in the terminology of Lemma 4.4.2 and taking Example 2.3.11 (set $\xi = \eta = \lambda^p_M$ and $D = D_{\mathbf{M}}$ therein) into account, we find that the union of the subobjects $\mathfrak{i}_{\lambda^p_O *} : D_O \Omega^p_0 O \hookrightarrow \Omega^p_0 M$ is the inclusion map $\iota : D_{\mathbf{M}}\Omega_0^p M \hookrightarrow \Omega_0^p M$. Hence, (4.90) holds for $\mathbb{K} = \mathbb{R}$. The case $\mathbb{K} = \mathbb{C}$ follows now immediately from Proposition 2.1.2 and Lemma 2.2.14. ❒

PROPOSITION 4.5.6. *Let* $\mathbf{M} \in \mathbf{Loc}$ *and consider for any category* $\mathrm{loc}^q_{-\mathbf{M}}$ *the restrictions* $\mathcal{P}^q_{\mathbf{M}}, \mathfrak{P}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, \mathbf{{}^*Alg^m_1}$ *of* $\mathcal{P}$ *and* $\mathfrak{P}$ *to* $\mathrm{loc}^q_{-\mathbf{M}}$. *Then*

$$\operatorname{colim} \mathcal{P}^q_{\mathbf{M}} = \left(\mathcal{P}\mathbf{M}, \{\mathcal{P}\iota_O : \mathcal{P}O \longrightarrow \mathcal{P}\mathbf{M}\}_{O \in \mathrm{loc}^q_{-\mathbf{M}}}\right), \tag{4.91}$$

$$\operatorname{colim} \mathfrak{P}^q_{\mathbf{M}} = \left(\mathfrak{P}\mathbf{M}, \{\mathfrak{P}\iota_O : \mathfrak{P}O \longrightarrow \mathfrak{P}\mathbf{M}\}_{O \in \mathrm{loc}^q_{-\mathbf{M}}}\right) \tag{4.92}$$

and

$$\operatorname{colim} \mathfrak{P}^q_{\mathbf{M}} \cong Q\left(\operatorname{colim} \mathcal{P}^q_{\mathbf{M}}\right). \tag{4.93}$$

We thus see that, no matter how we topologically restrict the connected globally hyperbolic open subsets of a **Loc**-object **M**, we will always recover our standard classical and quantum field theory on **M** by the colimit construction, in such a way that the quantum field theory is the quantisation of the classical field theory. In particular, the universal algebra will always be the standard unital *-algebra of the smeared quantum field. In more categorical terms, the quantisation functor preserves the colimit in those instances.

Proof: Due to Proposition 2.1.2 and Lemma 2.2.14, we may take $\mathbb{K} = \mathbb{C}$ without the loss of generality and due to Theorems 2.2.10 + 4.3.2, we view $\mathcal{P}^q_{\mathbf{M}}$ and $\mathfrak{P}^q_{\mathbf{M}}$ as functors $\mathcal{P}^q_{\mathbf{M}}, \mathfrak{P}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{pSympl}_{\mathbb{C}}, \mathbf{{}^*Alg_1}$. Then both $\operatorname{colim} \mathfrak{P}^q_{\mathbf{M}} = (A, \alpha : \mathfrak{P}^q_{\mathbf{M}} \dot{\longrightarrow} \Delta A)$, which is the colimit of the quantisation of $\mathcal{P}^q_{\mathbf{M}}$, and $\operatorname{colim}(F_\omega \circ \mathcal{P}^q_{\mathbf{M}})$, which is the corresponding colimit of $\mathcal{P}^q_{\mathbf{M}}$ in **CVec**, exist; we have computed the latter colimit explicitly in Proposition 4.5.5. Like in Theorem 4.3.2, $f : ([\Omega_0^p(M;\mathbb{C})], ^-) \longrightarrow A$ is to denote the unique C-homomorphism satisfying the identity $f \circ [\iota_{O*}] = \alpha_O \circ \mathrm{qinj}_O$ for all $O \in \mathrm{loc}^q_{-\mathbf{M}}$.

We will show that condition (c) of Theorem 4.3.2 holds by explicitly computing the commutator. The general problem is that we a priori do not know the expressions $\left[(\alpha_U \circ \mathrm{qinj}_U)[\omega], (\alpha_V \circ \mathrm{qinj}_V)[\eta]\right]$, where $[\omega] \in [\Omega_0^p(U;\mathbb{C})]$ and $[\eta] \in [\Omega_0^p(V;\mathbb{C})]$, for arbitrary $U, V \in \mathrm{loc}^q_{-\mathbf{M}}$ but only for very specific pairs $U, V \in \mathrm{loc}^q_{-\mathbf{M}}$; otherwise we would be immediately done by a smooth partition of unity argument. Still, we have enough input data to overcome this shortcoming. By "*transferring*" the whole situation to an open diamond caterpillar covering of a smooth spacelike Cauchy surface Σ for **M**, we will be able to compute the commutator for arbitrary pairs $U, V \in \mathrm{loc}^q_{-\mathbf{M}}$ by computing the commutator only for pairs $U, V \in \mathrm{loc}^q_{-\mathbf{M}}$ such that there is $W \in \mathrm{loc}^q_{-\mathbf{M}}$ with $U, V \subseteq W$.

Pick any smooth spacelike Cauchy surface Σ for **M** and let $\{O_i \mid i \in I\}$ be an open diamond caterpillar covering of Σ, that is, an open cover of Σ given by the open diamond caterpillar covering lemma, Lemma 1.1.6. Surely, $O_i \in \mathrm{loc}^q_{-\mathbf{M}}$ for all $i \in I$. For any $\omega \in \Omega^p_0(M;\mathbb{C})$, we can now apply Lemma 3.3.4 and obtain $\omega_€ \in \Omega^p_0(M;\mathbb{C})$ with $\mathrm{supp}\,\omega_€ \subseteq O(\Sigma)$, where $O(\Sigma) := \bigcup_{i\in I} O_i$ is an open neighbourhood of Σ in M, and $\omega_€ := \omega - D_{\mathbf{M}}\omega_£$ for $\omega_£ \in \Omega^p_0(M;\mathbb{C})$ ($\Longrightarrow [\omega_€] = [\omega] \in [\Omega^p_0(M;\mathbb{C})]$). Thus,

$$\big[f[\omega], f[\eta]\big] = \big[f[\omega_€], f[\eta_€]\big] \qquad \forall [\omega],[\eta] \in [\Omega^p_0(M;\mathbb{C})]. \tag{4.94}$$

For $[\omega],[\eta] \in [\Omega^p_0(M;\mathbb{C})]$, take a smooth partition of unity $\{\chi, \chi_i \mid i \in I\}$ subordinated to the open cover $\{M \smallsetminus (\mathrm{supp}\,\omega_€ \cup \mathrm{supp}\,\eta_€), O_i \mid i \in I\}$ of M. Notice, the smooth partition of unity depends on the representatives ω and η. We decompose (4.94) and, using $f \circ [\iota_{O*}] = \alpha_O \circ \mathrm{qinj}_O$ for all $O \in \mathrm{loc}^q_{-\mathbf{M}}$, we find

$$\big[f[\omega], f[\eta]\big] = \Big[f\Big[\sum_{i\in I}\chi_i\omega_€\Big], f\Big[\sum_{j\in I}\chi_j\eta_€\Big]\Big] \tag{4.95}$$

$$= \sum_{i,j\in I}\big[f[\chi_i\omega_€], f[\chi_j\eta_€]\big] \tag{4.96}$$

$$= \sum_{i,j\in I}\big[f[\iota_{i*}\iota_i^*\chi_i\omega_€], f[\iota_{j*}\iota_j^*\chi_j\eta_€]\big] \tag{4.97}$$

$$= \sum_{i,j\in I}\big[(f\circ[\iota_{i*}])[\iota_i^*\chi_i\omega_€], (f\circ[\iota_{j*}])[\iota_j^*\chi_j\eta_€]\big] \tag{4.98}$$

$$= \sum_{i,j\in I}\big[(\alpha_i\circ\mathrm{qinj}_i)[\iota_i^*\chi_i\omega_€], (\alpha_j\circ\mathrm{qinj}_j)[\iota_j^*\chi_j\eta_€]\big]. \tag{4.99}$$

Due to our choice of an open diamond caterpillar covering, we can now compute the commutator for each pair $i,j \in I$ because we can find a contractible globally hyperbolic open subset $O_{n(i,j)}$ of $\mathbf{M}$ which contains both O_i and O_j; for a slightly shorter notation, we will just write n instead of $n(i,j)$. Since α is a cocone from $\mathfrak{P}^q_{\mathbf{M}}$ to A, we have $\alpha_i = \alpha_n \circ \mathfrak{P}^q_{\mathbf{M}}\iota_{in} = \alpha_n \circ \mathfrak{P}\iota_{in}$ and $\alpha_j = \alpha_n \circ \mathfrak{P}\iota_{jn}$. With this, we further compute

$$\big[f[\omega], f[\eta]\big] = \sum_{i,j\in I}\big[(\alpha_n\circ\mathfrak{P}\iota_{in}\circ\mathrm{qinj}_i)[\iota_i^*\chi_i\omega_€], (\alpha_n\circ\mathfrak{P}\iota_{jn}\circ\mathrm{qinj}_j)[\iota_j^*\chi_j\eta_€]\big] \tag{4.100}$$

$$= \sum_{i,j\in I}\big[(\alpha_n\circ(Q\circ\mathcal{P})\,\iota_{in}\circ\mathrm{qinj}_i)[\iota_i^*\chi_i\omega_€], (\alpha_n\circ(Q\circ\mathcal{P})\,\iota_{jn}\circ\mathrm{qinj}_j)[\iota_j^*\chi_j\eta_€]\big] \tag{4.101}$$

$$= \sum_{i,j\in I}\big[(\alpha_n\circ\mathrm{qinj}_n\circ\mathcal{P}\iota_{in})[\iota_i^*\chi_i\omega_€], (\alpha_n\circ\mathrm{qinj}_n\circ\mathcal{P}\iota_{jn})[\iota_j^*\chi_j\eta_€]\big] \tag{4.102}$$

$$= \sum_{i,j\in I}\big[(\alpha_n\circ\mathrm{qinj}_n\circ[\iota_{in*}])[\iota_i^*\chi_i\omega_€], (\alpha_n\circ\mathrm{qinj}_n\circ[\iota_{jn*}])[\iota_j^*\chi_j\eta_€]\big] \tag{4.103}$$

$$= \sum_{i,j\in I}\big[(\alpha_n\circ\mathrm{qinj}_n)[\iota_{in*}\iota_i^*\chi_i\omega_€], (\alpha_n\circ\mathrm{qinj}_n)[\iota_{jn*}\iota_j^*\chi_j\eta_€]\big] \tag{4.104}$$

$$= \sum_{i,j\in I}\big[(\alpha_n\circ\mathrm{qinj}_n)[\iota_n^*\chi_i\omega_€], (\alpha_n\circ\mathrm{qinj}_n)[\iota_n^*\chi_j\eta_€]\big] \tag{4.105}$$

$$= \sum_{i,j\in I} \alpha_n \big[\operatorname{qinj}_n [\iota_n^* \chi_i \omega_{\epsilon}] , \operatorname{qinj}_n [\iota_n^* \chi_j \eta_{\epsilon}] \big]. \tag{4.106}$$

Finally, we recall from the construction of the quantisation functor, see Section 3.5,

$$\big[\operatorname{qinj}_n [\iota_n^* \chi_i \omega_{\epsilon}] , \operatorname{qinj}_n [\iota_n^* \chi_j \eta_{\epsilon}] \big] = \mathrm{i}\,\hbar\, \mathfrak{u}_n \left([\iota_n^* \chi_i \omega_{\epsilon}] , [\iota_n^* \chi_j \eta_{\epsilon}] \right) \cdot 1_{\mathfrak{P}O_n} \tag{4.107}$$

and, since $\mathfrak{P}\iota_n$ is a symplectic C-homomorphism,

$$\mathfrak{u}_n \left([\iota_n^* \chi_i \omega_{\epsilon}] , [\iota_n^* \chi_j \eta_{\epsilon}] \right) = \mathfrak{u}_{\mathbf{M}} \left([\chi_i \omega_{\epsilon}] , [\chi_j \eta_{\epsilon}] \right). \tag{4.108}$$

Putting everything together and summing up, we arrive at

$$\big[f[\omega] , f[\eta] \big] = \mathrm{i}\,\hbar\, \mathfrak{u}_{\mathbf{M}} \left([\omega] , [\eta] \right) \cdot 1_A \in \mathbb{C} \cdot 1_A . \tag{4.109}$$

Hence, condition (c) of Theorem 4.3.2 is met, thereby $\operatorname{colim} \mathcal{P}^q_{\mathbf{M}}$ exists and is given by (4.91) due to Lemma 4.3.5. Moreover, (4.93) holds, which shows (4.92). Since $\mathcal{P}\mathbf{M}$ is a complexified symplectic space and $\mathfrak{P}\mathbf{M}$ a simple unital *-algebra, we have that the colimits exist in $\mathbf{Sympl}_{\mathbb{C}}$ and $\mathbf{{}^*Alg^m_1}$. ❐

Combining Proposition 4.5.6 with Corollary 2.2.22, Lemma 2.2.11, Lemma 2.2.16 and Theorem 2.2.20, we obtain the statement that the standard functors for the free real Klein-Gordon field (in terms of smooth differential p-forms) of Proposition 4.5.4 are the left Kan extensions for all their restrictions to the topologically restricted full subcategories $\mathbf{Loc}_q$ of $\mathbf{Loc}$.

PROPOSITION 4.5.7. *Consider any of the categories* $\mathbf{Loc}_q$*, then* $\mathcal{P}, \mathfrak{P} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, \mathbf{{}^*Alg^m_1}$ *are the left Kan extensions along the inclusion functor* $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$ *of their respective restrictions to* $\mathbf{Loc}_q$*,* $\mathcal{P}_q, \mathfrak{P}_q : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, \mathbf{{}^*Alg^m_1}$*. The natural transformations of the left Kan extensions have the identities as their components.*

4.6 The free Maxwell field

We discuss the free Maxwell field, relating to the universal algebra, colimit constructions and left Kan extensions. Our focus lies for that matter on the field strength tensor description, introducing the universal F-theory of the free Maxwell field. However, we will also address the vector potential description in some detail and briefly present a smooth differential 3-form potential description.

Though a consistent and manifestly covariant construction of the quantised free electromagnetic field has been around since [JP28][7], rigorous investigations were in short supply in the early days of general quantum field theory[8]; it was not even included in the standard textbooks on the subject, e.g. [SW64; Jos65; BLT75]. We can only speculate about the reasons but we suppose that the impossibility to reconcile Maxwell's equations as operator equations with Lorentz-covariance, locality and a positive metric formalism (see [Hor90, Appx.], [SW74] and the references mentioned therein) had some part to play in this.

The first systematic account of the quantised electromagnetic field in general quantum field theory was probably [SW74], which studied the charge superselection rule for quantum electrodynamics using an indefinite metric variant of Wightman-Gårding local quantum field theory. Since the aim of [SW74] was also to discuss gauge, gauge transformation and gauge invariance, the focus was clearly set on the electromagnetic vector potential. Unhappy with the approach taken in [SW74], it was felt by [Bon77; Bon82] that an algebraic outline in the spirit of [Bor62; Uhl62] was to yield a more natural and more transparent formalism for the quantised Maxwell field in general quantum field theory. Using Borchers-Uhlmann algebras, [Bon77; Bon82] developed a description for the non-interacting quantised Maxwell field in terms of both field strength tensor (the "*F-description*") and vector potential (the "*A-description*") on an equal footing and established a "*natural homomorphism*" between the two descriptions. We borrow this terminology from [Bon77; Bon82] and encounter a more general version of the "*natural homomorphism*" in Theorem 4.6.15. Note that [SW74; Bon77; Bon82] worked in Minkowski space and, despite having genuine local quantum field theories, did not explicitly specify commutation relations.

With the remarkable exception of [Lic61], which focussed on the discussion of propagators, the study of the quantised free Maxwell field in cuved spacetimes has always been strongly motivated by the quest for understanding what effects a non-trivial spacetime topology can have, a matter which was already recognised in the classical theory of the free Maxwell field long time ago [MW57]. [Sor79; AS80] are two good examples for this. Both considered the quantised sourceless electromagnetic field strength tensor, [Sor79] on asymptotically flat globally hyperbolic spacetimes and [AS80] on

[7]It is quite amusing to notice that P. Jordan and W. Pauli have also looked into the quantisation of the free electromagnetic vector potential but preferred to give their results for the electromagnetic field strengths: "*Hier sei noch bemerkt, daß für die Viererpotentiale keine einfach formulierbaren relativistisch invarianten V.-R. bestehen, bei denen nur die Δ-Funktion und ihre Ableitungen verwendet werden.*"

[8]The choice of papers in our discussion is very selective and cannot possibly do justice to all contributions. Hence, we briefly want to mention the following important ones: [CGH77; GH85; Gru88; GL00; JL] (C^*-Weyl algebra approach), [Fur95; Fur97; Fur99] (canonical Fock quantisation of the free electromagnetic field on static and ultrastatic spacetimes with compact Cauchy surfaces, and quantisation of free massive vector fields in curved spacetimes). Since we will apply the exterior calculus of smooth differential forms, we will also omit comments on [Hol08; HS13; BDS13; BDS14; BDHS14] (gauge field theoretic aspects of the quantised electromagnetic field in curved spacetimes).

the Schwarzschild-Kruskal spacetime, and discovered that the topology of the underlying spacetime manifests itself in the centre of the quantum algebra, thus giving rise to superselection rules of "*topological charges*". It is also worth noticing that [Sor79] touched on the initial value formulation and formally derived commutation relations from a canonical formalism while [AS80] adopted the Lichnerowicz propagator [Lic61] for the commutation relations of the quantised source-free field strength tensor.

It is fair to say that [Dim92] was the seminal paper for the quantised electromagnetic field in curved spacetimes. [Dim92] considered the source-free electromagnetic vector potential on 4-dimensional globally hyperbolic spacetimes with compact Cauchy surfaces and gave a rigorous and systematic account of the Cauchy problem and of a gauge invariant quantisation procedure. In doing so, [Dim92] focused on the algebraic aspects and gave a description in terms of the smeared quantum field, smearing with coclosed compactly supported smooth differential 1-forms, as well as in terms of the C^*-Weyl algebra. As we have described in the introduction to this thesis, further developments followed in the new millenium [FP03; Pfe09; Dap11; DS13], which, however, also subjected themselves to topological restrictions.

Two approaches to the quantised free Maxwell field in curved spacetimes which do not make such assumption are [DL12; SDH14]. [DL12] works on 4-dimensional globally hyperbolic spacetimes, addresses the Cauchy problem for the non-interacting field strength tensor with source and performs the quantisation of the free Maxwell field in terms of the field strength tensor via the universal algebra of the local unital *-algebras of the smeared quantum field, where smearings of the quantum field with compactly supported smooth differential 2-forms are considered. In doing so, Lichnerowicz's commutation relations are derived and the appearance of non-trivial centres and their consequences are explained. In particular, it was noticed that the principle of local covariance is failed. The focus of [SDH14] lay on the non-interacting vector potential with source in globally hyperbolic spacetimes of arbitrary dimensions and its smooth differential p-form generalisations. The Cauchy problem is presented in a general fashion for distributions and a quantisation in terms of the unital *-algebra of the smeared quantum field, smearing with coclosed compactly supported smooth differential 1-forms, is achieved by ideas from deformation quantisation. The effects of the topology of the underlying spacetime, in particular in view of Aharonov-Bohm type effects, and the failure of local covariance are clarified elaborately.

The approach to the quantised free Maxwell field taken in this thesis is the one of [DL12] with an emphasis on the categorical aspects and, in fact, more is proved here. While [DL12] only considered a contractible setting, we will relax contractibility to less tight topological restrictions. Despite the occuring overlap with [DL12], this is an appreciable change in attitude, valuable to the further developments in our treatise.

Initially, we will take a conservative and modest point of view and only consider the free Maxwell field on **Loc**-objects **M** for which the F- and the A-description coincide, namely $H^1_{\mathrm{dR}}M = H^2_{\mathrm{dR}}M = H^{m-2}_{\mathrm{dR}}M = 0$, and can be expressed by symplectic spaces (classical) and simple unital *-algebras of the smeared quantum field (quantum) in standard, uncontroversial manner. This will yield functors $\mathcal{F} : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ and $\mathfrak{F} : \mathbf{Loc}_q \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$, where $1, 2, m-2 \in q \subseteq \mathbb{N} \setminus \{0\}$ or $q = ©$. We will then deploy the left Kan extension, which will amount to computing colimits, and extend $\mathcal{F}$ and $\mathfrak{F}$ to (as it will turn out) functors $\mathcal{F}_u : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{F}_u : \mathbf{Loc} \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}$. This procedure may genuinely be called a "*top-down*" (because of the left Kan extension) "*local-to-global*" (because of the colimits) approach to the free Maxwell field in locally covariant quantum field theory. Due to the universal constructions used, we will refer to $\mathcal{F}_u$ and $\mathfrak{F}_u$ as the classical and the quantised universal F-theory of the free Maxwell field. As we will see, they do not fulfil the principles of local covariance (Theorem 4.6.9) or dynamical locality (Theorem 5.3.2) due to the injectivity issues caused by non-trivial radicals and centres arising from spacetime topologies with $H^2_{\mathrm{dR}}M \neq 0 \neq H^{m-2}_{\mathrm{dR}}M$. However, these defects will be remedied by the reduced F-theory of the free Maxwell field in Section 5.5, in return for sacrificing the topological sensitivity of the universal F-theory.

Note, although we have always the physical case $m = 4$ for the fixed spacetime dimension in mind, we will formulate and prove our statements for arbitrary fixed spacetime dimension $m \geq 2$ and make it explicit when a result relies on the specific choice of m.

4.6.1 The F-description

Keep $\mathbf{M} \in \mathbf{Loc}$ fixed for the time being; then the free Maxwell equations for the electromagnetic field strength tensor $F \in \Omega^2(M;\mathbb{K})$ are

$$(4.110) \qquad dF = 0 \qquad \text{and} \qquad \delta F = 0.$$

Applying $-\delta$ to the first equation, $-d$ to the second equation and adding both, we see that F also solves the wave equation $\Box F = (-\delta d - d\delta) F = 0$. Hence, as a consequence of Lemma 4.5.1, we have [DL12, Prop.2.2]:

LEMMA 4.6.1. *$F \in \Omega^2_{\mathrm{sc}}(M;\mathbb{K})$ is a solution of (4.110) if and only if $F = G(d\theta + \delta\eta)$ for some $\theta \in \Omega^1_{0,\delta}(M;\mathbb{K})$ and $\eta \in \Omega^3_{0,d}(M;\mathbb{K})$, where $G := G^{\mathrm{adv}} - G^{\mathrm{ret}}$ is the advanced-minus-retarded Green operator for $\Box = -\delta d - d\delta$. If $H^2_{\mathrm{dR}}(M;\mathbb{K}) = 0$ and $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$, $F \in \Omega^2_{\mathrm{sc}}(M;\mathbb{K})$ solves (4.110) if and only if there is $\omega \in \Omega^2_0(M;\mathbb{K})$ such that $F = dG\delta\omega$.*

For the same reasons as in the case of the free smooth differential p-form real Klein-Gordon field, we consider the linear functionals O_ω, $\omega \in \Omega_0^2(M;\mathbb{K})$, on all field configurations $\{F \in \Omega^2(M;\mathbb{K}) \mid dF = 0 \text{ and } \delta F = 0\}$ which are of the form $F \longmapsto \int_M F \wedge *\omega$. Again, we want to have as many linear observables as possible in order to differentiate between as many field configurations as possible but on the other hand, we do not want linear observables which cannot be distinguished on the field configurations considered. Hence, taking account of the field equations, we pass over to the quotient $[\Omega_0^2(M;\mathbb{K})] := \Omega_0^2(M;\mathbb{K}) / (d\Omega_0^1(M;\mathbb{K}) \oplus \delta\Omega_0^3(M;\mathbb{K}))$ and regard the linear functionals $O_{[\omega]}$, $[\omega] \in [\Omega_0^2(M;\mathbb{K})]$, on $\{F \in \Omega^2(M;\mathbb{K}) \mid dF = 0, \delta F = 0\}$ which act via $F \longmapsto \int_M F \wedge *\omega$ as the basic linear observables for the classical theory of the free Maxwell field in terms of the field strength tensor. Recall that $d\Omega_0^1(M;\mathbb{K}) \oplus \delta\Omega_0^3(M;\mathbb{K})$ really is the direct sum: if $\omega \in \Omega_0^2(M;\mathbb{K})$ satisfies $d\omega = 0$ and $\delta\omega = 0$, then surely $\Box\omega = (-\delta d - d\delta)\omega = 0$ which implies $\omega = 0$ by Lemma 4.5.1(i).

We will identify $[\Omega_0^2(M;\mathbb{K})]$ with the basic linear observables from now on and seek to equip $[\Omega_0^2(M;\mathbb{K})]$ with a symplectic structure. In order to use Peierls' method ([Pei52], [Haa96, Sec.I.V.]), we have to assume that $H_{\mathrm{dR}}^2(M;\mathbb{K}) = 0$ because of the Lagrangean formalism. Note, this assumption is purely for technical reasons and does not mean at all that we base our classical or our quantum theory of the free Maxwell field on the vector potential $A \in \Omega^1(M;\mathbb{K})$. Making the assumption $H_{\mathrm{dR}}^2(M;\mathbb{K}) = 0$ to get us started and then using colimits and the left Kan extension to extend our theory to all $\mathbf{M} \in \mathbf{Loc}$, regardless of their topology, our approach to the classical and the quantised free Maxwell field will be completely independent of the vector potential.

Now, for $\omega \in \Omega_0^2(M;\mathbb{K})$, we consider the modified Lagrangean (smooth $\mathbb{K}$-valued differential m-form) $\mathcal{L}_\varepsilon := -\frac{1}{2}F \wedge *F + F \wedge *\omega$ ($\varepsilon > 0$). Rewriting $F = dA$ for $A \in \Omega^1(M;\mathbb{K})$ and applying the variational principle, we find as the Euler-Lagrange equation for $\mathcal{L}_\varepsilon$:

$$\delta F = \varepsilon\delta\omega. \tag{4.111}$$

If $F \in \Omega^2(M;\mathbb{K})$ solves (4.110), $\delta_{\varepsilon O_\omega}^{\mathrm{ret/adv}} F := F - \varepsilon dG^{\mathrm{ret/adv}}\delta\omega$ solves (4.111) and agrees with F in the remote past/future. The derivative with respect to ε at $\varepsilon = 0$ is $\delta_{O_\omega}^{\mathrm{ret/adv}} F = -dG^{\mathrm{ret/adv}}\delta\omega$ and thus $\delta_{O_\omega} F = \delta_{O_\omega}^{\mathrm{ret}} F - \delta_{O_\omega}^{\mathrm{adv}} F = dG\delta\omega$ (cf. [Haa96, (I.4.3) + (I.4.4)]). We come by

$$\{O_\omega, O_\eta\} = \delta_{O_\omega} O_\eta = O_\eta(dG\delta\omega) = \int_M dG\delta\omega \wedge *\eta \qquad \forall \omega, \eta \in \Omega_0^2(M;\mathbb{K}). \tag{4.112}$$

LEMMA 4.6.2. *The map*

$$(\omega, \eta) \longmapsto -\int_M \delta\omega \wedge *G\delta\eta, \qquad \omega, \eta \in \Omega_0^2(M;\mathbb{K}), \tag{4.113}$$

is skew-symmetric, bilinear and has the radical $\Omega^2_{0,d}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta}(M;\mathbb{K})$.

Proof: Bilinearity and skew-symmetry are clear, so we only need to focus on the radical. Fix $\eta \in \Omega^2_0(M;\mathbb{K})$ and let

$$-\int_M \delta\omega \wedge *G\delta\eta = -\int_M \omega \wedge *dG\delta\eta = 0 \qquad \forall\omega \in \Omega^2_0(M;\mathbb{K}). \tag{4.114}$$

The non-degeneracy of $\int_M \cdot \wedge * \cdot : \Omega^2_0(M;\mathbb{K}) \times \Omega^2(M;\mathbb{K}) \longrightarrow \mathbb{K}$ implies $dG\delta\eta = 0$ and $\eta \in \Omega^2_{0,d}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta}(M;\mathbb{K})$ follows from Lemma 4.5.1(iii). One shows $\omega \in \Omega^2_{0,d}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta}(M;\mathbb{K})$ if $-\int_M \delta\omega \wedge *G\delta\eta = 0$ for all $\eta \in \Omega^2_0(M;\mathbb{K})$ by skew-symmetry. ❐

As a simple corollary of Lemma 4.6.2, the following holds:

LEMMA 4.6.3. *The tuple*

$$\begin{cases} [\Omega^2_0(M;\mathbb{K})]] := \Omega^2_0(M;\mathbb{K}) / (d\Omega^1_0(M;\mathbb{K}) \oplus \delta\Omega^3_0(M;\mathbb{K})), \\ \mathfrak{w} : [\Omega^2_0(M;\mathbb{K})] \times [\Omega^2_0(M;\mathbb{K})] \longrightarrow \mathbb{K}, \ ([\omega],[\eta]) \longmapsto -\int_M \delta\omega \wedge *G\delta\eta, \\ ^- : [\Omega^2_0(M;\mathbb{K})] \longrightarrow [\Omega^2_0(M;\mathbb{K})], \ [\omega] \longmapsto [\overline{\omega}] \ \text{(complex conjugation)}, \end{cases} \tag{4.115}$$

is a pre-symplectic space. It is symplectic if $H^2_{\mathrm{dR}}(M;\mathbb{K}) = 0$ *and* $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$.

As announced, we will reject working with degenerate pre-symplectic spaces temporarily and further assume $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$ (in addition to $H^2_{\mathrm{dR}}(M;\mathbb{K}) = 0$). Applying the quantisation functor $Q : \mathbf{Sympl}_{\mathbb{K}} \longrightarrow \mathbf{{}^*Alg^m_1}$, we obtain the unital *-algebra of the smeared quantum field for the free Maxwell field in terms of the field strength tensor, that is, the (simple) unital *-algebra generated by the elements of the form $\mathbf{F}(\omega)$, $\omega \in \Omega^2_0(M;\mathbb{K})$, such that:

- Linearity: $\mathbf{F}(\lambda\omega + \mu\eta) = \lambda\mathbf{F}(\omega) + \mu\mathbf{F}(\eta)$ for all $\lambda,\mu \in \mathbb{K}$ and for all $\omega,\eta \in \Omega^2_0(M;\mathbb{K})$.
- Hermiticity: $\mathbf{F}(\omega)^* = \mathbf{F}(\overline{\omega})$ for all $\omega \in \Omega^2_0(M;\mathbb{K})$.
- Field equations (in a weak sense): $\mathbf{F}(d\theta) = 0$ for all $\theta \in \Omega^1_0(M;\mathbb{K})$ and $\mathbf{F}(\delta\eta) = 0$ for all $\eta \in \Omega^3_0(M;\mathbb{K})$.
- Commutation relations: $[\mathbf{F}(\omega), \mathbf{F}(\eta)] = -\mathrm{i}\hbar \int_M \delta\omega \wedge *G\delta\eta \cdot 1_A$ for all $\omega,\eta \in \Omega^2_0(M;\mathbb{K})$.

The commutation relations are often termed "*Lichnerowicz's commutation relations*", see the remark in [Dim92, Sec.4], [AS80, Sec.3.B] and [Lic61]. We now discuss the functorial aspects.

PROPOSITION 4.6.4. *The rules*

$$\mathbf{Loc}_{\{2,m-2\}} \ni \mathbf{M} \longmapsto \left([\Omega_0^2(M;\mathbb{K})], \mathfrak{w}_{\mathbf{M}}, {}^{-} \right) \tag{4.116}$$

and

$$\mathbf{Loc}_{\{2,m-2\}}(\mathbf{M},\mathbf{N}) \ni \psi \longmapsto [\psi_*] : \left([\Omega_0^2(M;\mathbb{K})], \mathfrak{w}_{\mathbf{M}}, {}^{-} \right) \longrightarrow \left([\Omega_0^2(N;\mathbb{K})], \mathfrak{w}_{\mathbf{N}}, {}^{-} \right), \tag{4.117}$$

define functors $\mathcal{F} : \mathbf{Loc}_{\{2,m-2\}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ *and* $\mathfrak{F} := Q \circ \mathcal{F} : \mathbf{Loc}_{\{2,m-2\}} \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$.

Since $\psi_* d_{\mathbf{M}} = d_{\mathbf{N}} \psi_*$ and $\psi_* \delta_{\mathbf{M}} = \delta_{\mathbf{N}} \psi_*$ for any $\mathbf{Loc}_{\{2,m-2\}}$-morphism $\psi : \mathbf{M} \longrightarrow \mathbf{N}$, $\psi_* : d_{\mathbf{M}} \Omega_0^1(M;\mathbb{K}) \longrightarrow d_{\mathbf{N}} \Omega_0^1(N;\mathbb{K})$ and $\psi_* : \delta_{\mathbf{M}} \Omega_0^3(M;\mathbb{K}) \longrightarrow \delta_{\mathbf{N}} \Omega_0^3(N;\mathbb{K})$ are well-defined. As a result of this, (UQ') yields a a unique linear map $[\psi_*] : [\Omega_0^2(M;\mathbb{K})] \longrightarrow [\Omega_0^2(N;\mathbb{K})]$ such that $[\psi_*] \circ \pi_{\mathbf{M}} = \pi_{\mathbf{N}} \circ \psi_*$, where $\pi_{\mathbf{M}|\mathbf{N}} : \Omega_0^2(M|N;\mathbb{K}) \twoheadrightarrow [\Omega_0^2(M|N;\mathbb{K})]$ denote the canonical projections onto the quotients. $[\psi_*]$ is clearly a C-homomorphism if $\mathbb{K} = \mathbb{C}$ and since

$$\mathfrak{w}_{\mathbf{N}}([\psi_*][\omega], [\psi_*][\eta]) = \mathfrak{w}_{\mathbf{N}}([\psi_*\omega], [\psi_*\eta]) \tag{4.118}$$
$$= -\int_N \delta_{\mathbf{N}} \psi_* \omega \wedge *_{\mathbf{N}} G_{\mathbf{N}} \delta_{\mathbf{N}} \psi_* \eta \tag{4.119}$$
$$= -\int_{\psi(M)} \psi_* \delta_{\mathbf{M}} \omega \wedge *_{\mathbf{N}} G_{\mathbf{N}} \psi_* \delta_{\mathbf{M}} \eta \tag{4.120}$$
$$= -\int_M \delta_{\mathbf{M}} \omega \wedge *_{\mathbf{M}} \psi^* G_{\mathbf{N}} \psi_* \delta_{\mathbf{M}} \eta \tag{4.121}$$
$$= -\int_M \delta_{\mathbf{M}} \omega \wedge *_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} \eta \tag{4.122}$$
$$= \mathfrak{w}_{\mathbf{M}}([\omega], [\eta]) \tag{4.123}$$
$$\forall [\omega], [\eta] \in [\Omega_0^2(M;\mathbb{K})],$$

$[\psi_*]$ is a $\mathbf{Sympl}_{\mathbb{K}}$-morphism. If $\varphi : \mathbf{L} \longrightarrow \mathbf{M}$ is another $\mathbf{Loc}_{\{2,m-2\}}$-morphism, one deduces $[(\psi \circ \varphi)_*] = [\psi_*] \circ [\varphi_*]$ from (UQ').

Having introduced our basic functors $\mathcal{F}$, $\mathfrak{F} : \mathbf{Loc}_{\{2,m-2\}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, ${}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ for the classical and the quantised free Maxwell field in terms of the field strength tensor, we will now turn to the matter of extending them to all of **Loc** by computing colimits and thereby the left Kan extension. For the rest of this section, consider any $q \subseteq \mathbb{N} \smallsetminus \{0\}$ with $2, m-2 \in q$ or let $q = \copyright$. We denote the restrictions of $\mathcal{F}$ and $\mathfrak{F}$ to $\mathbf{Loc}_q$ by $\mathcal{F}_q$, $\mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, ${}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ and for $\mathbf{M} \in \mathbf{Loc}$, $\mathcal{F}_{\mathbf{M}}^q$, $\mathfrak{F}_{\mathbf{M}}^q : \mathrm{loc}_{-\mathbf{M}}^q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, ${}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ are the restrictions of $\mathcal{F}$ and $\mathfrak{F}$ to $\mathrm{loc}_{-\mathbf{M}}^q$. First, we prove an auxiliary lemma, which computes the corresponding colimits in $\mathbf{Vec}_{\mathbb{R}}$ resp. **CVec**.

LEMMA 4.6.5. *For* $\mathbf{M} \in \mathbf{Loc}$,

$$\operatorname{colim}(F_\omega \circ \mathcal{F}^q_{\mathbf{M}}) = \Big(\big([\Omega^2_0(M;\mathbb{K})], {}^- \big), [\iota_*] : F_\omega \circ \mathcal{F}^q_{\mathbf{M}} \dot{\longrightarrow} \Delta\big([\Omega^2_0(M;\mathbb{K})], {}^- \big) \Big), \tag{4.124}$$

where

$$[\Omega^2_0(M;\mathbb{K})] := \Omega^2_0(M;\mathbb{K}) / \big(d_{\mathbf{M}}\Omega^1_0(M;\mathbb{K}) \oplus \delta_{\mathbf{M}}\Omega^3_0(M;\mathbb{K}) \big) \tag{4.125}$$

and

$$[\iota_*]_O := [\iota_{O*}] : (F_\omega \circ \mathcal{F}^q_{\mathbf{M}})\, O \longrightarrow \big([\Omega^2_0(M;\mathbb{K})], {}^- \big) \qquad \forall O \in \mathrm{loc}^q_{-\mathbf{M}}. \tag{4.126}$$

Proof: We only need to consider the case $\mathbb{K} = \mathbb{R}$; $\mathbb{K} = \mathbb{C}$ follows then directly from Proposition 2.1.2 and Lemma 2.2.14. Now, choose $F : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ as in Lemma 4.4.1 with $\xi = \lambda^2_M$ and define another functor $G : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ by $G := d_O\Omega^1_0 O \oplus \delta_O\Omega^3_0 O$ for all $O \in \mathrm{loc}^q_{-\mathbf{M}}$ and $G\iota_{UV} := \iota_{UV*} : d_U\Omega^1_0 U \oplus \delta_U\Omega^3_0 U \hookrightarrow d_V\Omega^1_0 V \oplus \delta_V\Omega^3_0 V$ for all $U, V \in \mathrm{loc}^q_{-\mathbf{M}}$ such that $U \subseteq V$. Since $\iota_{UV*}d_U = d_V\iota_{UV*}$ and $\iota_{UV*}\delta_U = \delta_V\iota_{UV*}$, the requirements of Lemma 4.4.3 are met and we may apply it. Obviously, $\mathcal{F}^q_{\mathbf{M}} = [F]$ in the terminology of Lemma 4.4.2 and by applying Example 2.3.11 twice (1: $\xi = \lambda^1_M$, $\eta = \lambda^2_M$ and $D = d_{\mathbf{M}}$; 2: $\xi = \lambda^3_M$, $\eta = \lambda^2_M$ and $D = \delta_{\mathbf{M}}$), we find with universal property of the direct sum (U$\bigoplus$) that $\iota : d_{\mathbf{M}}\Omega^1_0 M \oplus \delta_{\mathbf{M}}\Omega^3_0 M \hookrightarrow \Omega^2_0 M$ is the union of the subobjects $\iota_{O*} : d_O\Omega^1_0 O \oplus \delta_O\Omega^3_0 O \hookrightarrow \Omega^2_0 M$. ❒

With the result of Lemma 4.6.5, we can now compute the colimits for $\mathcal{F}^q_{\mathbf{M}}$, $\mathfrak{F}^q_{\mathbf{M}}$, and show that the universal algebra is non-trivial, i.e. not the zero algebra. We will achieve this by applying Theorem 4.3.2, as we did in the proof of Proposition 4.5.6:

THEOREM 4.6.6. *Let* $\mathbf{M} \in \mathbf{Loc}$. *We view* $\mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ *as functors* $\mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}, {}^*\mathbf{Alg}_{\mathbf{1}}$ *in this theorem. Then*

$$\operatorname{colim}\mathcal{F}^q_{\mathbf{M}} = \Big(\big([\Omega^2_0(M;\mathbb{K})], \mathfrak{w}_{\mathbf{M}}, {}^- \big), [\iota_*] : \mathcal{F}^q_{\mathbf{M}} \dot{\longrightarrow} \Delta\big([\Omega^2_0(M;\mathbb{K})], \mathfrak{w}_{\mathbf{M}}, {}^- \big) \Big), \tag{4.127}$$

where $\mathfrak{w}_{\mathbf{M}}$ *is the pre-symplectic form on* $([\Omega^2_0(M;\mathbb{K})], {}^-)$ *defined by* $\mathfrak{w}_{\mathbf{M}}([\omega],[\eta]) := -\int_M \delta_{\mathbf{M}}\omega \wedge *_{\mathbf{M}} G_{\mathbf{M}}\delta_{\mathbf{M}}\eta$ *for all* $[\omega],[\eta] \in [\Omega^2_0(M;\mathbb{K})]$. *Moreover,*

$$\operatorname{colim}\mathfrak{F}^q_{\mathbf{M}} \cong Q(\operatorname{colim}\mathcal{F}^q_{\mathbf{M}}). \tag{4.128}$$

Thus, the universal algebra $\mathfrak{F}_u\mathbf{M} = \varinjlim \mathfrak{F}^q_{\mathbf{M}}$ *is described by the unital *-algebra generated by the elements of the form* $\mathbf{F}_u(\omega)$, $\omega \in \Omega^2_0(M;\mathbb{K})$, *such that:*

- Linearity: $\mathbf{F}_u(\lambda\omega + \mu\eta) = \lambda\mathbf{F}_u(\omega) + \mu\mathbf{F}_u(\eta)$ *for all* $\lambda, \mu \in \mathbb{K}$ *and for all* $\omega, \eta \in \Omega_0^2(M;\mathbb{K})$.
- Hermiticity: $\mathbf{F}_u(\omega)^* = \mathbf{F}_u(\overline{\omega})$ *for all* $\omega \in \Omega_0^2(M;\mathbb{K})$.
- Field equations (*in a weak sense*): $\mathbf{F}_u(d\theta) = 0$ *for all* $\theta \in \Omega_0^1(M;\mathbb{K})$ *and* $\mathbf{F}_u(\delta\eta) = 0$ *for all* $\eta \in \Omega_0^3(M;\mathbb{K})$.
- Commutation relations: $[\mathbf{F}_u(\omega), \mathbf{F}_u(\eta)] = -\mathrm{i}\hbar \int_M \delta\omega \wedge *G\delta\eta \cdot 1_{\mathfrak{F}_u\mathbf{M}}$ *for all* $\omega, \eta \in \Omega_0^2(M;\mathbb{K})$.

Proof: The proof is identical to the proof of Proposition 4.5.6 up to (4.107). The step from (4.107) to (4.108) cannot be done here since we do not have a complexified pre-symplectic form $\mathfrak{w}_{\mathbf{M}}$ on $([\Omega_0^2(M;\mathbb{K})], ^{-})$ yet. Instead, we have to compute step by step:

$$\left[f[\omega], f[\eta]\right] = \mathrm{i}\hbar \sum_{i,j\in I} \mathfrak{w}_n\left([\iota_n^*\chi_i\omega_\in], [\iota_n^*\chi_j\eta_\in]\right) \cdot 1_A \tag{4.129}$$

$$= -\mathrm{i}\hbar \sum_{i,j\in I} \left(\int_{O_n} \delta_n\iota_n^*\chi_i\omega_\in \wedge *_n G_n\delta_n\iota_n^*\chi_j\eta_\in\right) \cdot 1_A \tag{4.130}$$

$$= -\mathrm{i}\hbar \sum_{i,j\in I} \left(\int_{O_n} \iota_n^*\delta_{\mathbf{M}}\chi_i\omega_\in \wedge *_n\iota_n^* G_{\mathbf{M}}\iota_{n*}\iota_n^*\delta_{\mathbf{M}}\chi_j\eta_\in\right) \cdot 1_A \tag{4.131}$$

$$= -\mathrm{i}\hbar \sum_{i,j\in I} \left(\int_{\mathbf{M}} \delta_{\mathbf{M}}\chi_i\omega_\in \wedge *_{\mathbf{M}} G_{\mathbf{M}}\delta_{\mathbf{M}}\chi_j\eta_\in\right) \cdot 1_A \tag{4.132}$$

$$= -\mathrm{i}\hbar \left(\int_{\mathbf{M}} \delta_{\mathbf{M}}\omega_\in \wedge *_{\mathbf{M}} G_{\mathbf{M}}\delta_{\mathbf{M}}\eta_\in\right) \cdot 1_A \tag{4.133}$$

$$= -\mathrm{i}\hbar \left(\int_{\mathbf{M}} \delta_{\mathbf{M}}\omega \wedge *_{\mathbf{M}} G_{\mathbf{M}}\delta_{\mathbf{M}}\eta\right) \cdot 1_A \tag{4.134}$$

$$\in \mathbb{C} \cdot 1_A \tag{4.135}$$

Continuing, Theorem 4.3.2(c) is met, thereby $\operatorname{colim} \mathcal{F}_{\mathbf{M}}^q$ exists and is given by (4.127) due to Lemma 4.3.5. Moreover, (4.128) holds. ❐

Note, Theorem 4.6.6 contains and improves the statements of [DL12, Prop.3.1 + 3.2] on the explicit form of the universal algebra and its commutation relations, where $\mathrm{loc}_{\mathbf{M}}^{©}$ was enlarged with the disjoint union of spacelike separated $\mathrm{loc}_{\mathbf{M}}^{©}$-objects and causality [BF06, Sec.2, Axiom 4] was assumed. If we combine Theorem 4.6.6 with Corollary 2.2.22, Lemma 2.2.11, Lemma 2.2.16 and Theorem 2.2.20, we obtain:

THEOREM AND DEFINITION 4.6.7. *In this theorem, we view* $\mathcal{F}_q$, $\mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1^m}$ *as functors* $\mathcal{F}_q$, $\mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1}$. *The left Kan exten-*

sions of $\mathcal{F}_q$ and $\mathfrak{F}_q$ along the inclusion functor $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$ exist and are concretely given by the functors $\mathcal{F}_u, : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$ and $\mathfrak{F}_u : \mathbf{Loc} \longrightarrow \mathbf{{}^*Alg_1}$, where

(4.136)

$$\mathcal{F}_u\mathbf{M} := \begin{cases} [\Omega^2_0(M;\mathbb{K})] := \Omega^2_0(M;\mathbb{K})/d_{\mathbf{M}}\Omega^1_0(M;\mathbb{K}) \oplus \delta_{\mathbf{M}}\Omega^3_0(M;\mathbb{K}), \\ \mathfrak{w}_{\mathbf{M}} : [\Omega^2_0(M;\mathbb{K})] \times [\Omega^2_0(M;\mathbb{K})] \longrightarrow \mathbb{K},\ ([\omega],[\eta]) \longmapsto -\int_M \delta_{\mathbf{M}}\omega \wedge *_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}}\eta, \\ {}^- : [\Omega^2_0(M;\mathbb{K})] \longrightarrow [\Omega^2_0(M;\mathbb{K})],\ [\omega] \longmapsto [\overline{\omega}] \ \text{(complex conjugation)}, \end{cases}$$

$$\forall \mathbf{M} \in \mathbf{Loc},$$

(4.137)
$$\mathcal{F}_u(\psi : \mathbf{M} \longrightarrow \mathbf{N}) := [\psi_*] : \big([\Omega^2_0(M;\mathbb{K})], \mathfrak{w}_{\mathbf{M}}, {}^-\big) \longrightarrow \big([\Omega^2_0(N;\mathbb{K})], \mathfrak{w}_{\mathbf{N}}, {}^-\big),$$
$$\forall \psi \in \mathbf{Loc}(\mathbf{M},\mathbf{N}),\ \forall \mathbf{M},\mathbf{N} \in \mathbf{Loc},$$

and $\mathfrak{F}_u := Q \circ \mathcal{F}$. The natural transformations of the left Kan extensions have the identities as their components. We will refer to $\mathcal{F}_u$ and $\mathfrak{F}_u$ as the <u>classical</u> *and the* <u>quantised universal F-theory</u> *of the free Maxwell field.*

From Theorem 4.6.6 and Theorem 4.6.7, we learn that the categorical formalism of locally covariant quantum field theory itself suggests to consider degenerate presymplectic spaces and non-simple unital *-algebras. We will accept this matter of fact and continue investigating the universal F-theory of the free Maxwell field. Observe that the universal F-theory has a "*topological subtheory*", by which we mean that there are natural transformations

(4.138)
$$\tau : H^{2\oplus m-2}_{\mathrm{dR,c}} \overset{\cdot}{\longrightarrow} F_{\omega,C} \circ \mathcal{F}_u \qquad \text{and} \qquad \sigma : H^{2\oplus m-2}_{\mathrm{dR,c}} \overset{\cdot}{\longrightarrow} F_{\bullet_A,*_A} \circ \mathfrak{F}_u,$$

where $H^{2\oplus m-2}_{\mathrm{dR,c}} : \mathbf{Loc} \longrightarrow \mathbf{Vec}_{\mathbb{K}}$ is the functor defined by rules

(4.139)
$$\mathbf{Loc} \ni \mathbf{M} \longmapsto H^2_{\mathrm{dR,c}}(M;\mathbb{K}) \oplus H^{m-2}_{\mathrm{dR,c}}(M;\mathbb{K})$$

and

(4.140)

$$\mathbf{Loc}(\mathbf{M},\mathbf{N}) \ni \psi \longmapsto \begin{cases} H^2_{\mathrm{dR,c}}(M;\mathbb{K}) \oplus H^{m-2}_{\mathrm{dR,c}}(M;\mathbb{K}) \longrightarrow H^2_{\mathrm{dR,c}}(N;\mathbb{K}) \oplus H^{m-2}_{\mathrm{dR,c}}(N;\mathbb{K}) \\ \qquad [\omega] \oplus [\eta] \longmapsto [\psi_*\omega] \oplus [\psi_*\eta] \end{cases}.$$

$F_{\omega,C} : \mathbf{pSympl}_{\mathbb{K}} \longrightarrow \mathbf{Vec}_{\mathbb{K}}$ is the forgetful functor that forgets about the pre-symplectic form (and about the C-involution if $\mathbb{K} = \mathbb{C}$) and $F_{\bullet_A,*_A} : \mathbf{{}^*Alg_1} \longrightarrow \mathbf{Vec}_{\mathbb{C}}$ is the forgetful functor that forgets about the algebra multiplication and the *-involution. The components of the natural transformations, which are all injective, are given for $\mathbf{M} \in \mathbf{Loc}$

by

$$\tau_{\mathbf{M}} : H^{2\oplus m-2}_{\mathrm{dR,c}}\,\mathbf{M} \longrightarrow F_{\omega,C}\left(\mathcal{F}_u\mathbf{M}\right), \qquad [\omega]\oplus[\eta] \longmapsto [\omega + *_{\mathbf{M}}\eta], \tag{4.141}$$

and

$$\sigma_{\mathbf{M}} : H^{2\oplus m-2}_{\mathrm{dR,c}}\,\mathbf{M} \longrightarrow F_{\bullet_A,*_A}\left(\mathfrak{F}_u\mathbf{M}\right), \qquad [\omega]\oplus[\eta] \longmapsto \mathbf{F}_{\mathbf{M}}\left(\omega + *_{\mathbf{M}}\eta\right). \tag{4.142}$$

LEMMA 4.6.8. *For a* **Loc***-morphism* $\psi : \mathbf{M} \longrightarrow \mathbf{N}$, $\mathcal{F}_u\psi$ *is injective if and only if* $H^{2\oplus m-2}_{\mathrm{dR,c}}\,\psi$ *is injective. If* $H^{2\oplus m-2}_{\mathrm{dR,c}}\,\psi$ *is not injective,* $\mathfrak{F}_u\psi$ *is not injective.*

Proof: "$\Longrightarrow$" is an easy consequence of $F_{\omega,C}\left(\mathcal{F}_u\psi\right)\circ\tau_{\mathbf{M}} = \tau_{\mathbf{N}}\circ H^{2\oplus m-2}_{\mathrm{dR,c}}\,\psi$ and [Bou68, II, §3, no.8, Thm.1(c)]. The statement regarding $\mathfrak{F}_u\psi$ and $H^{2\oplus m-2}_{\mathrm{dR,c}}\,\psi$ is also easily seen from $F_{\bullet_A,*_A}\left(\mathfrak{F}_u\psi\right)\circ\sigma_{\mathbf{M}} = \sigma_{\mathbf{N}}\circ H^{2\oplus m-2}_{\mathrm{dR,c}}\,\psi$ and [Bou68, II, §3, no.8, Thm.1(c)].

"$\Longleftarrow$". Suppose $\mathcal{F}_u\psi$ is not injective; then we can conclude the existence of $0 \neq [\omega] \in F_{\omega,C}\left(\mathcal{F}_u\mathbf{M}\right)$ with $F_{\omega,C}\left(\mathcal{F}_u\psi\right)[\omega] = [\psi_*\omega] = 0 \in F_{\omega,C}\left(\mathcal{F}_u\mathbf{N}\right)$. It follows that $\psi_*\omega = d_{\mathbf{N}}\theta + \delta_{\mathbf{N}}\eta = \tau_{\mathbf{N}}\left([d_{\mathbf{N}}\theta]\oplus[d_{\mathbf{N}} *_{\mathbf{N}} \eta]\right) = \tau_{\mathbf{N}}\left([0]\oplus[0]\right)$ for some $\theta \in \Omega^1_0(N;\mathbb{K})$ and $\eta \in \Omega^3_0(N;\mathbb{K})$. Now, because $\mathcal{F}_u\psi$ is symplectic, $[\omega] \in \operatorname{rad}\boldsymbol{\mathfrak{w}}_{\mathbf{M}}$ and $\omega = \alpha + \beta$, where $\alpha \in \Omega^2_{0,d_{\mathbf{M}}}(M;\mathbb{K}) \smallsetminus d_{\mathbf{M}}\Omega^1_0(M;\mathbb{K})$ and $\beta \in \Omega^2_{0,\delta_{\mathbf{M}}}(M;\mathbb{K}) \smallsetminus \delta_{\mathbf{M}}\Omega^3_0(M;\mathbb{K})$. Hence, $\omega = \alpha + \beta = \tau_{\mathbf{M}}\left([\alpha]\oplus[*^{-1}_{\mathbf{M}}\beta]\right) \neq \tau_{\mathbf{M}}\left([0]\oplus[0]\right)$, which implies $[\alpha]\oplus[*^{-1}_{\mathbf{M}}\beta] \neq [0]\oplus[0]$ since $\tau_{\mathbf{M}}$ is injective. We conclude from $F_{\omega,C}\left(\mathcal{F}_u\psi\right)\circ\tau_{\mathbf{M}} = \tau_{\mathbf{N}}\circ H^{2\oplus m-2}_{\mathrm{dR,c}}\,\psi$ and the injectivity of $\tau_{\mathbf{N}}$ that $H^{2\oplus m-2}_{\mathrm{dR,c}}\,\psi$ cannot be injective. ❒

There are plenty **Loc**-morphisms $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ such that $H^{2\oplus m-2}_{\mathrm{dR,c}}\psi$ is not injective. Let e.g. $\mathbf{N} \in \mathbf{Loc}$ be the Minkowski spacetime (in fact, any $\mathbf{N} \in \mathbf{Loc}_{\{m-2\}}$ will do), $\mathbf{M} \in \mathbf{Loc}$ the Cauchy development in $\mathbf{N}$ of the set $\{0\}\times\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 > 1\}$ and $\psi : \mathbf{N} \longrightarrow \mathbf{M}$ the inclusion map; then by Poincaré duality [GHV72, Chap.V, §4], $H^2_{\mathrm{dR,c}}(M;\mathbb{K}) \neq 0$ and thus there exists $\omega \in \Omega^2_0(M;\mathbb{K})$ satisfying $d_{\mathbf{M}}\omega = 0$ but $[\omega] \neq 0 \in H^2_{\mathrm{dR,c}}(M;\mathbb{K})$. However, $[\psi_*\omega] = 0 \in H^2_{\mathrm{dR,c}}(N;\mathbb{K})$ as $H^2_{\mathrm{dR,c}}(N;\mathbb{K}) = 0$ by assumption, which implies that $H^{2\oplus m-2}_{\mathrm{dR,c}}\psi$ is not injective. Lemma 4.6.8 thus shows:

THEOREM 4.6.9. *The classical and the quantised universal* F*-theory of the free Maxwell field are not locally covariant.*

We close this section with the following proposition, which exhibits symmetry properties of the universal F-theory for fixed spacetime dimension $m = 4$. We skip a proof, which follows in the standard way from the properties of the Hodge-$*$-operator, in particular, recall that the Hodge-$*$-operator intertwines with pullbacks via and pushforwards along **Loc**-morphisms.

PROPOSITION AND DEFINITION 4.6.10. *Let the fixed spacetime dimension be* $m = 4$*. Then for* $\vartheta \in [0, 2\pi)$*, the* electromagnetic duality rotations

$$\begin{aligned} \theta_{\mathbf{M}}(\vartheta) : \Omega^2_0(M;\mathbb{K}) &\longrightarrow \Omega^2_0(M;\mathbb{K}) \\ \omega &\longmapsto \cos(\vartheta)\,\omega + \sin(\vartheta) *_{\mathbf{M}} \omega, \end{aligned} \qquad \mathbf{M} \in \mathbf{Loc}, \tag{4.143}$$

gives rise to an automorphism $\eta_\vartheta : \mathcal{F}_u \xrightarrow{\dot\sim} \mathcal{F}_u$ *with components defined by*

$$\eta_\vartheta(\mathbf{M})[\omega] := [\theta_{\mathbf{M}}(\vartheta)\,\omega] \qquad \forall [\omega] \in \mathcal{F}_u\mathbf{M},\ \forall \mathbf{M} \in \mathbf{Loc}. \tag{4.144}$$

By applying the quantisation functor $Q : \mathbf{pSympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg_1}$*, we also obtain an automorphism* $Q \star \eta_\vartheta : \mathfrak{F}_u \xrightarrow{\dot\sim} \mathfrak{F}_u$*. In the case* $\vartheta = \frac{\pi}{2}$*, which corresponds to interchanging the role of the electric field (resp. electric charge) and the magnetic field (rep. magnetic charge), we say that the classical and the quantised universal* F*-theory are* electromagnetically self-dual*.*

4.6.2 The A-description

It is well-known that if $H^2_{\mathrm{dR}}(M;\mathbb{K}) = 0$, any solution $F \in \Omega^2(M;\mathbb{K})$ of the free Maxwell equations for the field strength tensor (4.110) can be written as $F = dA$ for the electromagnetic vector potential $A \in \Omega^1(M;\mathbb{K})$. Since $dF = ddA = 0$ is automatically fulfilled, (4.110) becomes one single equation

$$\delta dA = 0, \tag{4.145}$$

which is the Euler-Lagrange equation of the Lagrangean (smooth $\mathbb{K}$-valued differential m-form)

$$\mathcal{L} = -\frac{1}{2}\, dA \wedge *dA. \tag{4.146}$$

We now consider (4.145) and (4.146) on any $\mathbf{M} \in \mathbf{Loc}$ in their own right. We will follow [SDH14] for the first bit of our presentation.

So, fix $\mathbf{M} \in \mathbf{Loc}$ unless otherwise is stated. To start with, we consider for $\theta \in \Omega^1_0(M;\mathbb{K})$ the linear functionals O_θ on all field configurations $\{A \in \Omega^1(M;\mathbb{K}) \mid \delta dA = 0\}$ which are given by $A \longmapsto \int_M A \wedge *\theta$. As before, carefully balancing between having enough linear observables to differentiate between different field configurations and having no linear observables which cannot be distinguished by the field configurations considered, we pass to the linear functionals $O_{[\theta]}$, $[\theta] \in [\Omega^1_0(M;\mathbb{K})]$, on $\{A \in \Omega^1(M;\mathbb{K}) \mid \delta dA = 0\}$ which are of the form $A \longmapsto \int_M A \wedge *\theta$, where have taken the field equations into account and defined $[\Omega^1_0(M;\mathbb{K})] := \Omega^1_0(M;\mathbb{K}) / \delta d\Omega^1_0(M;\mathbb{K})$.

Using Peierls' method ([Pei52], [Haa96, Sec.I.V.]), we try to equip $[\Omega_0^1(M;\mathbb{K})]$ with a symplectic structure: for $\theta \in \Omega_0^1(M;\mathbb{K})$, consider the modified Lagrangean (smooth $\mathbb{K}$-valued differential m-form) $(\varepsilon > 0)$

$$\mathcal{L}_\varepsilon = -\frac{1}{2}\, dA \wedge *dA + \varepsilon A \wedge *\theta, \tag{4.147}$$

whose Euler-Lagrange equation is

$$\delta dA = \varepsilon\theta. \tag{4.148}$$

It is clear that (4.148) can only make sense if $\delta\theta = 0$, which forces us to revise our undertaking a bit. We are consequently interested in the linear functionals on $\{A \in \Omega^1(M;\mathbb{K}) \mid \delta dA = 0\}$ which act as $O_{[\theta]} : A \longmapsto \int_M A \wedge *\theta$ but now for $[\theta] \in [\Omega_{0,\delta}^1(M;\mathbb{K})]$, where $[\Omega_{0,\delta}^1(M;\mathbb{K})] := \Omega_{0,\delta}^1(M;\mathbb{K}) / \delta d\Omega_0^1(M;\mathbb{K})$. To find a symplectic structure on $[\Omega_{0,\delta}^1(M;\mathbb{K})]$, which we will identify with the basic linear observables for the classical theory of the free Maxwell field in terms of the vector potential, we consider again (4.147) for $\varepsilon > 0$ but now with $\theta \in \Omega_{0,\delta}^1(M;\mathbb{K})$. Suppose $A \in \Omega^1(M;\mathbb{K})$ is a solution of (4.145), then $\delta_{\varepsilon O_\theta}^{\mathrm{ret/adv}} A := A - \varepsilon G^{\mathrm{ret/adv}}\theta$ is a solution of (4.148) which coincides with A in the remote past/future. We compute the derivative with respect to ε at $\varepsilon = 0$, $\delta_{O_\theta}^{\mathrm{ret/adv}} A = -G^{\mathrm{ret/adv}}\theta$, with ease and find $\delta_{O_\theta} A = \delta_{O_\theta}^{\mathrm{ret}} A - \delta_{O_\theta}^{\mathrm{adv}} A = G\theta$ (cf. [Haa96, (I.4.3) + (I.4.4)]). Thus,

$$\{O_\theta, O_\phi\} = \delta_{O_\theta} O_\phi = O_\phi(G\theta) = \int_M G\theta \wedge *\phi \qquad \forall \theta, \phi \in \Omega_{0,\delta}^1(M;\mathbb{K}). \tag{4.149}$$

LEMMA 4.6.11. *The kernel of $dG = Gd$ on $\Omega_{0,\delta}^1(M;\mathbb{K})$ is $\delta\Omega_{0,d}^2(M;\mathbb{K})$.*

Proof: For $\omega \in \Omega_{0,d}^2(M;\mathbb{K})$, $dG\delta\omega = 0$ by Lemma 4.5.1(ii). Suppose now that $Gd\theta = 0$ for $\theta \in \Omega_{0,\delta}^1(M;\mathbb{K})$; then $d\theta = \Box\omega$ for some $\omega \in \Omega_0^2(M;\mathbb{K})$ by Lemma 4.5.1(i) and $d\omega = 0$ by Lemma 4.5.1(iii). We further notice $\Box\theta = -\delta d\theta = -\delta\Box\omega = -\Box\delta\omega$, hence $\theta = -\delta\omega = \delta(-\omega)$ due to Lemma 4.5.1(i). ❐

LEMMA 4.6.12. *The map*

$$(\theta, \phi) \longmapsto -\int_M \theta \wedge *G\phi, \qquad \theta, \phi \in \Omega_{0,\delta}^1(M;\mathbb{K}), \tag{4.150}$$

is a skew-symmetric bilinear form and $\delta d\Omega_0^1(M;\mathbb{K})$ always lies in its radical. If we have $H_{\mathrm{dR}}^{m-2}(M;\mathbb{K}) = 0$, then the radical is equal to $\delta d\Omega_0^1(M;\mathbb{K})$. If instead $H_{\mathrm{dR}}^1(M;\mathbb{K}) = 0$, the radical is equal to $\delta\Omega_{0,d}^2(M;\mathbb{K})$.

Proof: Bilinearity and skew-symmetry are clear; it is also evident that $\delta d\Omega_0^1(M;\mathbb{K})$ always lies in the radical.

Assuming $H_{\mathrm{dR}}^{m-2}(M;\mathbb{K})=0$, we have $H_{\mathrm{dR,c}}^2(M;\mathbb{K})=0$ by Poincaré duality [GHV72, Chap.5, §4] and thus $\Omega_{0,d}^2(M;\mathbb{K})=d\Omega_0^1(M;\mathbb{K})$. Let $\phi\in\Omega_{0,\delta}^1(M;\mathbb{K})$ and

$$-\int_M \theta\wedge *G\phi = 0 \qquad \forall\theta\in\Omega_{0,\delta}^1(M;\mathbb{K}). \tag{4.151}$$

In particular, this has to hold for all $\theta=\delta\omega$ for some $\omega\in\Omega_0^2(M;\mathbb{K})$:

$$-\int_M \delta\omega\wedge *G\phi = 0 \Longleftrightarrow -\int_M \omega\wedge *dG\phi = -\int_M \omega\wedge *Gd\phi = 0 \qquad \forall\omega\in\Omega_0^2(M;\mathbb{K}). \tag{4.152}$$

The non-degeneracy of $\int_M \cdot\wedge *\cdot : \Omega_0^2(M;\mathbb{K})\times\Omega^2(M;\mathbb{K})\longrightarrow\mathbb{K}$ implies $Gd\phi=0$ and Lemma 4.6.11 shows $\phi=\delta d\alpha$ for some $\alpha\in\Omega_0^1(M;\mathbb{K})$. The same conclusion holds if the roles of θ and ϕ are swapped.

Next, assume $H_{\mathrm{dR}}^1(M;\mathbb{K})=0$, which implies $H_{\mathrm{dR,c}}^{m-1}(M;\mathbb{K})=0$ by Poincaré duality. Consequently, $\Omega_{0,\delta}^1(M;\mathbb{K})=\delta\Omega_0^2(M;\mathbb{K})$ by applying the Hodge-$*$-operator. Taking any $\phi\in\delta\Omega_{0,d}^2(M;\mathbb{K})$, we find

$$-\int_M \theta\wedge *G\phi = -\int_M \delta\omega\wedge *G\delta\eta = -\int_M \omega\wedge *dG\delta\eta = \int_M \omega\wedge *\delta Gd\eta = 0 \qquad \forall\theta\in\Omega_{0,\delta}^1(M;\mathbb{K}). \tag{4.153}$$

Hence, $\delta\Omega_{0,d}^2(M;\mathbb{K})$ lies in the radical. On the other hand, suppose that for $\phi\in\Omega_{0,\delta}^1(M;\mathbb{K})$,

$$-\int_M \theta\wedge *G\phi = -\int_M \delta\omega\wedge *G\delta\eta = -\int_M \omega\wedge *dG\delta\eta = -\int_M \omega\wedge *Gd\delta\eta = 0 \qquad \forall\omega\in\Omega_0^2(M;\mathbb{K}). \tag{4.154}$$

The non-degeneracy of $\int_M \cdot\wedge *\cdot : \Omega_0^2(M;\mathbb{K})\times\Omega^2(M;\mathbb{K})\longrightarrow\mathbb{K}$ yields $Gd\delta\eta=0$ and Lemma 4.5.1(iii) implies $\eta\in\Omega_{0,d}^p(M;\mathbb{K})\oplus\Omega_{0,\delta}^p(M;\mathbb{K})$. Thus, $\phi=\delta\eta\in\delta\Omega_{0,d}^2(M;\mathbb{K})$. The same result holds if θ and ϕ are interchanged. ❒

LEMMA 4.6.13. *The tuple*

$$\begin{cases} [\Omega^1_{0,\delta}(M;\mathbb{K})] := \Omega^1_{0,\delta}(M;\mathbb{K})/\delta d\Omega^1_0(M;\mathbb{K}), \\ \mathfrak{v} : [\Omega^1_{0,\delta}(M;\mathbb{K})] \times [\Omega^1_{0,\delta}(M;\mathbb{K})] \longrightarrow \mathbb{K}, \ ([\theta],[\phi]) \longmapsto -\int_M \theta \wedge *G\phi, \\ ^- : [\Omega^1_{0,\delta}(M;\mathbb{K})] \longrightarrow [\Omega^1_{0,\delta}(M;\mathbb{K})], \ [\theta] \longmapsto [\overline{\theta}] \ \text{(complex conjugation)}, \end{cases} \tag{4.155}$$

is a pre-symplectic space. It is symplectic if $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$.

Applying the quantisation functor $Q : \mathbf{pSympl}_{\mathbb{K}} \longrightarrow \mathbf{{}^*Alg_1}$, we arrive at the unital *-algebra of the smeared quantum field for the free Maxwell field in terms of the vector potential; it is the unital *-algebra generated by the elements of the form $\mathbf{A}(\theta)$, $\theta \in \Omega^1_{0,\delta}(M;\mathbb{K})$, which further satisfy the following conditions:

- Linearity: $\mathbf{A}(\lambda\theta + \mu\phi) = \lambda\,\mathbf{A}(\theta) + \mu\,\mathbf{A}(\phi)$ for all $\lambda,\mu \in \mathbb{K}$ and for all $\theta,\phi \in \Omega^1_{0,\delta}(M;\mathbb{K})$.
- Hermiticity: $\mathbf{A}(\theta)^* = \mathbf{A}(\overline{\theta})$ for all $\theta \in \Omega^1_{0,\delta}(M;\mathbb{K})$.
- Field equations (in a weak sense): $\mathbf{A}(\delta d\theta) = 0$ for all $\theta \in \Omega^1_{0,\delta}(M;\mathbb{K})$.
- Commutation relations: $[\mathbf{A}(\theta),\mathbf{A}(\phi)] = -\mathrm{i}\hbar\int_M \theta \wedge *G\phi \cdot 1_A$ for all $\theta,\phi \in \Omega^1_{0,\delta}(M;\mathbb{K})$.

Observe that the commutation relations are the familiar ones that can be found in standard textbooks on quantum field theory, e.g. [BS80; Sch05; Wen03]. We will now discuss functorial properties. Although we have been reluctant to work with degenerate pre-symplectic spaces before, we will find it now useful to define a functor on all of **Loc** for later comparison with colimits and the left Kan extension.

PROPOSITION 4.6.14. *The rules*

$$\mathbf{Loc} \ni \mathbf{M} \longmapsto \left([\Omega^1_{0,\delta_{\mathbf{M}}}(M;\mathbb{K})], \mathfrak{v}_{\mathbf{M}}, ^-\right) \tag{4.156}$$

and

$$\mathbf{Loc}(\mathbf{M},\mathbf{N}) \ni \psi \longmapsto [\psi_*] : \left([\Omega^1_{0,\delta_{\mathbf{M}}}(M;\mathbb{K})], \mathfrak{v}_{\mathbf{M}}, ^-\right) \longrightarrow \left([\Omega^1_{0,\delta_{\mathbf{M}}}(N;\mathbb{K})], \mathfrak{v}_{\mathbf{N}}, ^-\right), \tag{4.157}$$

define functors $\mathcal{A} : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$ *and* $\mathfrak{A} : \mathbf{Loc} \longrightarrow \mathbf{{}^*Alg_1}$*, where* $\mathfrak{A} := Q \circ \mathcal{A}$*. Neither* $\mathcal{A}$ *nor* $\mathfrak{A}$ *are locally covariant theories.*

$\mathcal{A}$ and $\mathfrak{A}$ are the "*generally covariant*" theories of [SDH14], see in particular [SDH14, Prop.3.3 + (13) + Def.4.5] and set $p = 1$ and $j = 0$ therein. We can now make good on our promise regarding the "*natural homomorphism*" of [Bon77] between the F-description and the A-description.

THEOREM 4.6.15. *Let* $q \subseteq \mathbb{N} \smallsetminus \{0\}$ *such that* $1, 2, m-2 \in q$ *or* $q = \copyright$*, and consider the restrictions* $\mathcal{A}_q, \mathfrak{A}_q : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, \mathbf{{}^*Alg_1^m}$ *and* $\mathcal{F}_q, \mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, \mathbf{{}^*Alg_1^m}$ *of* $\mathcal{A}$*,* $\mathfrak{A}$*,* $\mathcal{F}$ *and* $\mathfrak{F}$ *to* $\mathbf{Loc}_q$*.* $\mathcal{A}_q$ *and* $\mathcal{F}_q$ *are naturally isomorphic, and* $\mathfrak{A}_q$ *and* $\mathfrak{F}_q$ *are naturally isomorphic. A natural isomorphism* $\eta : \mathcal{F}_q \overset{\sim}{\dot{\longrightarrow}} \mathcal{A}_q$ *is explicitly given by* $\eta_{\mathbf{M}} [\omega] := [\delta_{\mathbf{M}} \omega]$ *for all* $[\omega] \in \mathcal{F}_q \mathbf{M}$ *and* $\mathbf{M} \in \mathbf{Loc}_q$*. Hence, an explicit natural isomorphism* $\mathfrak{F}_q \overset{\sim}{\dot{\longrightarrow}} \mathfrak{A}_q$ *is* $Q \star \eta$*.*

Proof: We first show that for each $\mathbf{M} \in \mathbf{Loc}_q$, the map $\eta_{\mathbf{M}} : \mathcal{F}_q \mathbf{M} \longrightarrow \mathcal{A}_q \mathbf{M}$, $[\omega] \longmapsto [\delta_{\mathbf{M}} \omega]$, is a $\mathbf{Sympl}_{\mathbb{K}}$-isomorphism. It is clear that $\eta_{\mathbf{M}}$ is a $\mathbf{Sympl}_{\mathbb{K}}$-morphism. Define $\varepsilon_{\mathbf{M}} : \mathcal{A}_q \mathbf{M} \longrightarrow \mathcal{F}_q \mathbf{M}$ by $\varepsilon_{\mathbf{M}} [\theta] := [\omega]$ such that $[\delta_{\mathbf{M}} \omega] = [\theta]$ for $[\theta] \in \mathcal{A}_q \mathbf{M}$. This is well-defined by our choice of $\mathbf{Loc}_q$ and it is straightforward to see that $\eta_{\mathbf{M}} \circ \varepsilon_{\mathbf{M}} = \mathrm{id}_{\mathcal{A}_q \mathbf{M}}$ and $\varepsilon_{\mathbf{M}} \circ \eta_{\mathbf{M}} = \mathrm{id}_{\mathcal{F}_q \mathbf{M}}$. So, it is left to show that these $\mathbf{Sympl}_{\mathbb{K}}$-isomorphisms define the components of a natural transformation $\eta : \mathcal{F}_q \dot{\longrightarrow} \mathcal{A}_q$. To do so, we compute

$$(\mathcal{A}_q \psi \circ \eta_{\mathbf{M}}) [\omega] = \mathcal{A}_q \psi [\delta_{\mathbf{M}} \omega] = [\psi_* \delta_{\mathbf{M}} \omega] = [\delta_{\mathbf{N}} \psi_* \omega] = \eta_{\mathbf{N}} [\psi_* \omega] = (\eta_{\mathbf{N}} \circ \mathcal{F}_q \psi) [\omega]$$
$$\forall [\omega] \in \mathcal{F}_q \mathbf{M},\ \forall \psi \in \mathbf{Loc}_q (\mathbf{M}, \mathbf{N}),\ \forall \mathbf{M}, \mathbf{N} \in \mathbf{Loc}_q.$$

Concluding, $\eta : \mathcal{F}_q \dot{\longrightarrow} \mathcal{A}_q$ with the components $\eta_{\mathbf{M}}$, $\mathbf{M} \in \mathbf{Loc}_q$, is a natural isomorphism and since functors preserve isomorphisms, so is $Q \star \eta : \mathfrak{F}_q \dot{\longrightarrow} \mathfrak{A}_q$. ❐

Observe, the Minkowski space component of the natural isomorphism in Theorem 4.6.15 is precisely the "*natural homomorphism*" in [Bon77].

Calculating colimits and the left Kan extension appears to be a lot trickier in the A-description. Although we know that colimits and the left Kan extension must exist and are non-trivial by reason of Proposition 4.6.14, which allows us to always construct non-trivial cocones, we did not succeed in finding closed expressions beyond the case where we consider $\mathbf{Loc}_q$ for $q \subseteq \mathbb{N} \smallsetminus \{0\}$ with $1, 2, m-2 \in q$ or $q = \copyright$, and the corresponding subcategories $\mathrm{loc}^q_{-\mathbf{M}}$ for $\mathbf{M} \in \mathbf{Loc}$. Indeed, Theorem 4.6.15 allows us to readily adopt the results of Theorem 4.6.6 (by Lemma 2.2.12) and Theorem 4.6.7 (by Lemma 2.2.18).

Anyway, the reader might also be interested in how these theorems read in the A-description. To make our point, it is more than enough to rephrase Theorem 4.6.6 only. However, to fully appreciate where complications for the computation of colimits and left Kan extensions in the A-description come from, we need to discuss another description of the free Maxwell field, in which the roles of the electric field (resp. electric charge) and the magnetic field (resp. magnetic charge) are interchanged.

4.6.3 The V-description

It is a peculiarity of the free Maxwell field that if $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$, any solution $F \in \Omega^2(M;\mathbb{K})$ of (4.110) can also be written as $F = \delta V$ for $V \in \Omega^3(M;\mathbb{K})$. Now, $\delta F = \delta\delta V = 0$ is automatically fulfilled and (4.110) becomes the single equation

$$d\delta V = 0, \tag{4.158}$$

which can be derived as the Euler-Lagrange equation of the Lagrangean (smooth $\mathbb{K}$-valued differential m-form)

$$\mathcal{L} = -\frac{1}{2}\,\delta V \wedge *\delta V. \tag{4.159}$$

This V-description of the free Maxwell field is due to changing the role of the electric and the magnetic field (resp. charge) and for fixed spacetime dimension $m = 4$; it is precisely related to the A-description by an application of the Hodge-$*$-operator. We will see in Theorem 4.6.21 that computing colimits in the A-description picks up the V-description. First, we regard (4.158) and (4.159) on any $\mathbf{M} \in \mathbf{Loc}$ in their own right.

For $\mathbf{M} \in \mathbf{Loc}$ fixed for the moment, we will pick as the basic linear observables the linear functionals $O_{[\eta]}$, $[\eta] \in [\Omega^3_{0,d}(M;\mathbb{K})] := \Omega^3_{0,d}(M;\mathbb{K})/d\delta\Omega^3_0(M;\mathbb{K})$, on all field configurations $\{V \in \Omega^1(M;\mathbb{K}) \mid d\delta V = 0\}$ which are of the form $V \longmapsto \int_M V \wedge *\eta$. In order to equip them with a symplectic structure, we apply Peierls' method ([Pei52], [Haa96, Sec.I.V.]): for $\eta \in \Omega^3_{0,d}(M;\mathbb{K})$, we consider the modified Lagrangean (smooth $\mathbb{K}$-valued differential m-form) ($\varepsilon > 0$)

$$\mathcal{L}_\varepsilon = -\frac{1}{2}\,\delta V \wedge *\delta V + \varepsilon V \wedge *\eta, \tag{4.160}$$

whose Euler-Lagrange equation is

$$d\delta V = \varepsilon\eta. \tag{4.161}$$

Suppose $V \in \Omega^3(M;\mathbb{K})$ is a solution of (4.158), then $\delta^{\mathrm{ret/adv}}_{\varepsilon O_\eta} V := V - \varepsilon G^{\mathrm{ret/adv}}\eta$ is a solution of (4.161) and agrees with V in the remote past/future. We find for the derivative with respect to ε at $\varepsilon = 0$, $\delta^{\mathrm{ret/adv}}_{O_\eta} V = -G^{\mathrm{ret/adv}}\eta$ and thus $\delta_{O_\eta} V = \delta^{\mathrm{ret}}_{O_\eta} V - \delta^{\mathrm{adv}}_{O_\eta} V = G\eta$ (cf. [Haa96, (I.4.3) + (I.4.4)]). Accordingly,

$$\{O_\eta, O_\varpi\} = \delta_{O_\eta} O_\varpi = O_\varpi(G\eta) = \int_M G\eta \wedge *\varpi \qquad \forall \eta, \varpi \in \Omega^3_{0,d}(M;\mathbb{K}). \tag{4.162}$$

Completely analogous to Lemma 4.6.11, Lemma 4.6.12, Lemma 4.6.13, Proposition 4.6.14 and Theorem 4.6.15, we have the following sequence of lemmas, propositions and theorems:

LEMMA 4.6.16. *The kernel of* $\delta G = G\delta$ *on* $\Omega^3_{0,d}(M;\mathbb{K})$ *is* $d\Omega^2_{0,\delta}(M;\mathbb{K})$.

LEMMA 4.6.17. *The map*

$$(\eta, \varpi) \longmapsto -\int_M \eta \wedge *G\varpi, \qquad \eta, \varpi \in \Omega^3_{0,d}(M;\mathbb{K}), \tag{4.163}$$

is a skew-symmetric bilinear form and $d\delta\Omega^3_0(M;\mathbb{K})$ *lies always in its radical. If we have* $H^2_{\mathrm{dR}}(M;\mathbb{K}) = 0$, *then the radical is equal to* $d\delta\Omega^3_0(M;\mathbb{K})$. *If instead* $H^3_{\mathrm{dR}}(M;\mathbb{K}) = 0$, *the radical is equal to* $d\Omega^2_{0,\delta}(M;\mathbb{K})$.

LEMMA 4.6.18. *The tuple*

$$\begin{cases} [\Omega^3_{0,d}(M;\mathbb{K})] \coloneqq \Omega^3_{0,d}(M;\mathbb{K}) / d\delta\Omega^3_0(M;\mathbb{K}), \\ \mathfrak{h} : [\Omega^3_{0,d}(M;\mathbb{K})] \times [\Omega^3_{0,d}(M;\mathbb{K})] \longrightarrow \mathbb{K}, \ ([\eta], [\varpi]) \longmapsto -\int_M \eta \wedge *G\varpi, \\ {}^{-} : [\Omega^3_{0,d}(M;\mathbb{K})] \longrightarrow [\Omega^3_{0,d}(M;\mathbb{K})], \ [\eta] \longmapsto [\overline{\eta}] \ \text{(complex conjugation)}, \end{cases} \tag{4.164}$$

is a pre-symplectic space. It is symplectic if $H^2_{\mathrm{dR}}(M;\mathbb{K}) = 0$.

Applying the quantisation functor $Q : \mathbf{pSympl}_{\mathbb{K}} \longrightarrow \mathbf{{}^*Alg_1}$ yields the unital *-algebra of the smeared quantum field, i.e. the unital *-algebra generated by the elements of the form $\mathbf{V}(\eta)$, $\eta \in \Omega^3_{0,d}(M;\mathbb{K})$, meeting the requirements:

- Linearity: $\mathbf{V}(\lambda\eta + \mu\varpi) = \lambda\mathbf{V}(\eta) + \mu\mathbf{V}(\varpi)$ for all $\lambda, \mu \in \mathbb{K}$ and for all $\eta, \varpi \in \Omega^3_{0,d}(M;\mathbb{K})$.
- Hermiticity: $\mathbf{V}(\eta)^* = \mathbf{V}(\overline{\eta})$ for all $\eta \in \Omega^3_{0,d}(M;\mathbb{K})$.
- Field equations (in a weak sense): $\mathbf{V}(d\delta\eta) = 0$ for all $\eta \in \Omega^3_{0,d}(M;\mathbb{K})$.
- Commutation relations: $[\mathbf{V}(\eta), \mathbf{V}(\varpi)] = -\mathrm{i}\hbar \int_M \eta \wedge *G\varpi \cdot 1_A$ for all $\eta, \varpi \in \Omega^3_{0,d}(M;\mathbb{K})$.

PROPOSITION 4.6.19. *The rules*

$$\mathbf{Loc} \ni \mathbf{M} \longmapsto \left([\Omega^3_{0,d_{\mathbf{M}}}(M;\mathbb{K})], \mathfrak{h}_{\mathbf{M}}, {}^{-} \right) \tag{4.165}$$

and

$$\mathbf{Loc}(\mathbf{M}, \mathbf{N}) \ni \psi \longmapsto [\psi_*] : \left([\Omega^3_{0,d_{\mathbf{M}}}(M;\mathbb{K})], \mathfrak{h}_{\mathbf{M}}, {}^{-} \right) \longrightarrow \left([\Omega^3_{0,d_{\mathbf{M}}}(N;\mathbb{K})], \mathfrak{h}_{\mathbf{N}}, {}^{-} \right), \tag{4.166}$$

define functors $\mathcal{V} : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$ *and* $\mathfrak{V} : \mathbf{Loc} \longrightarrow \mathbf{{}^*Alg_1}$, *where* $\mathfrak{V} \coloneqq Q \circ \mathcal{V}$. *Neither* $\mathcal{V}$ *nor* $\mathfrak{V}$ *are locally covariant theories.*

THEOREM 4.6.20. *Let $q \subseteq \mathbb{N} \smallsetminus \{0\}$ such that $2, 3, m-2 \in q$ or $q = \copyright$ and let $\mathcal{V}_q$, $\mathfrak{V}_q$: $\mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1^m}$ and $\mathcal{F}_q$, $\mathfrak{F}_q$: $\mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1^m}$ be the restrictions of $\mathcal{V}$, $\mathfrak{V}$, $\mathcal{F}$ and $\mathfrak{F}$ to $\mathbf{Loc}_q$. $\mathcal{V}_q$ and $\mathcal{F}_q$ are naturally isomorphic, and $\mathfrak{V}_q$ and $\mathfrak{F}_q$ are naturally isomorphic. A natural isomorphism $\eta : \mathcal{F}_q \overset{\sim}{\dot{\longrightarrow}} \mathcal{V}_q$ is explicitly defined by $\eta_{\mathbf{M}}[\omega] := [d_{\mathbf{M}}\omega]$ for all $[\omega] \in \mathcal{F}_q\mathbf{M}$ and $\mathbf{M} \in \mathbf{Loc}_q$. Hence, an explicit natural isomorpism $\mathfrak{F}_q \overset{\sim}{\dot{\longrightarrow}} \mathfrak{V}_q$ is $Q \star \eta$, where $Q : \mathbf{Sympl}_{\mathbb{K}} \longrightarrow \mathbf{{}^*Alg_1^m}$ is the quantisation functor.*

4.6.4 Computing colimits in the A-description

We rephrase Theorem 4.6.6 in the A-description. As we have said before, we have not been able to obtain a closed expression for the colimit unless restricting to $q \subseteq \mathbb{N} \smallsetminus \{0\}$ with $1, 2, m-2 \in q$ or $q = \copyright$, where the A- and the F-description coincide.

THEOREM 4.6.21. *Let $q \subseteq \mathbb{N} \smallsetminus \{0\}$ with $1, 2, m-2 \in q$ or $q = \copyright$ and consider the restrictions $\mathcal{A}^q_{\mathbf{M}}, \mathfrak{A}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}, \mathbf{{}^*Alg_1}$ of $\mathcal{A}$ and $\mathfrak{A}$ to $\mathrm{loc}^q_{-\mathbf{M}}$. Then*

$$\operatorname{colim} \mathcal{A}^q_{\mathbf{M}} = \Big(\operatorname{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V}), \, v : \mathcal{A}^q_{M} \dot{\longrightarrow} \Delta\left(\operatorname{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V})\right) \Big), \tag{4.167}$$

where $\operatorname{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V})$ is given by

$$\begin{cases} \operatorname{diag}\left(\frac{\delta_{\mathbf{M}}\Omega^2_0(M;\mathbb{K})}{\delta_{\mathbf{M}} d_{\mathbf{M}}\Omega^2_0(M;\mathbb{K})} \oplus \frac{d_{\mathbf{M}}\Omega^2_0(M;\mathbb{K})}{d_{\mathbf{M}}\delta_{\mathbf{M}}\Omega^2_0(M;\mathbb{K})} \right) := \left\{ [\delta_{\mathbf{M}}\omega] \oplus [d_{\mathbf{M}}\omega] \;\middle|\; \omega \in \Omega^2_0(M;\mathbb{K}) \right\}, \\ (\mathfrak{v} \oplus \mathfrak{h})_{\mathbf{M}} : ([\delta_{\mathbf{M}}\omega] \oplus [d_{\mathbf{M}}\omega], [\delta_{\mathbf{M}}\eta] \oplus [d_{\mathbf{M}}\eta]) \longmapsto -\int_M \delta_{\mathbf{M}}\omega \wedge *_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}}\eta, \\ \overline{} : [\delta_{\mathbf{M}}\omega] \oplus [d_{\mathbf{M}}\omega] \longmapsto [\delta_{\mathbf{M}}\overline{\omega}] \oplus [d_{\mathbf{M}}\overline{\omega}] \;\; \text{(complex conjugation)}, \end{cases} \tag{4.168}$$

and the universal cocone v is defined by the components

$$v_O : \mathcal{A}^q_{\mathbf{M}} O \longrightarrow \operatorname{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V}), \quad [\theta] \longmapsto [\delta_{\mathbf{M}} \iota_{O*}\omega] \oplus [d_{\mathbf{M}} \iota_{O*}\omega] \;\; \text{where } \delta_O\omega = \theta \;\; \forall O \in \mathrm{loc}^q_{-\mathbf{M}}. \tag{4.169}$$

Moreover,

$$\operatorname{colim} \mathfrak{A}^q_{\mathbf{M}} \cong Q\left(\operatorname{colim} \mathcal{A}^q_{\mathbf{M}}\right). \tag{4.170}$$

*The universal algebra $\mathfrak{A}_u\mathbf{M} = \varinjlim \mathfrak{A}^q_{\mathbf{M}}$ is thereby the unital *-algebra generated by the elements of the form $\mathbf{A}_u(\delta_{\mathbf{M}}\omega) \oplus \mathbf{V}_u(d_{\mathbf{M}}\omega)$, $\omega \in \Omega^2_0(M;\mathbb{K})$, satisfying the conditions:*

- Linearity*: for all $\lambda, \mu \in \mathbb{K}$ and for all $\omega, \eta \in \Omega^2_0(M;\mathbb{K})$, we have that*

$$\begin{aligned} &\mathbf{A}_u(\delta_{\mathbf{M}}(\lambda\omega + \mu\eta)) \oplus \mathbf{V}_u(d_{\mathbf{M}}(\lambda\omega + \mu\eta)) \\ &\qquad = \lambda\,\mathbf{A}_u(\delta_{\mathbf{M}}\omega) \oplus \lambda\mathbf{V}_u(d_{\mathbf{M}}\omega) + \mu\,\mathbf{A}_u(\delta_{\mathbf{M}}\eta) \oplus \mu\mathbf{V}_u(d_{\mathbf{M}}\eta) \end{aligned}$$

- Hermiticity: $(\mathbf{A}_u(\delta_{\mathbf{M}}\omega) \oplus \mathbf{V}_u(d_{\mathbf{M}}\omega))^* = \mathbf{A}_u(\delta_{\mathbf{M}}\overline{\omega}) \oplus \mathbf{V}_u(d_{\mathbf{M}}\overline{\omega})$ *for all* $\omega \in \Omega_0^2(M;\mathbb{K})$.

- Field equations (*in a weak sense*): $\mathbf{A}_u(\delta_{\mathbf{M}} d_{\mathbf{M}}\theta) \oplus \mathbf{V}_u(d_{\mathbf{M}} d_{\mathbf{M}}\theta) = 0$ *for all* $\theta \in \Omega_0^1(M;\mathbb{K})$ *and* $\mathbf{A}_u(\delta_{\mathbf{M}}\delta_{\mathbf{M}}\eta) \oplus \mathbf{V}_u(d_{\mathbf{M}}\delta_{\mathbf{M}}\eta) = 0$ *for all* $\eta \in \Omega_0^3(M;\mathbb{K})$.

- Commutation relations: *for all* $\omega, \eta \in \Omega_0^2(M;\mathbb{K})$,
$[\mathbf{A}_u(\delta_{\mathbf{M}}\omega) \oplus \mathbf{V}_u(d_{\mathbf{M}}\omega), \mathbf{A}_u(\delta_{\mathbf{M}}\eta) \oplus \mathbf{V}_u(d_{\mathbf{M}}\eta)] = -\mathrm{i}\,\hbar \int_M \delta_{\mathbf{M}}\omega \wedge *_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}}\eta \cdot 1_{\mathfrak{A}_u\mathbf{M}}$.

Proof: Recall the natural isomorphism of Theorem 4.6.15 and consider from now on its restriction to a natural isomorphism $\eta : \mathcal{F}_{\mathbf{M}}^q \xrightarrow{\dot{\sim}} \mathcal{A}_{\mathbf{M}}^q$. Using η and its inverse $\varepsilon : \mathcal{A}_{\mathbf{M}}^q \xrightarrow{\dot{\sim}} \mathcal{F}_{\mathbf{M}}^q$, every cocone from $\mathcal{A}_{\mathbf{M}}^q$ can be made into a cocone from $\mathcal{F}_{\mathbf{M}}^q$ and vice versa. Thanks to Theorem 4.6.6 and (UColim) in particular, it suffices to show that there is one and only one $\mathbf{pSympl}_{\mathbb{K}}$-isomorphism $\mu : \mathrm{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V}) \xrightarrow{\sim} \mathcal{F}_u\mathbf{M}$ satisfying the identity $\Delta\mu \circ v = [\iota_*] \circ \varepsilon$.

Composing v with η, we obtain a cocone $v \circ \eta$ from $\mathcal{F}_{\mathbf{M}}^q$ to $\mathrm{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V})$. (UColim) yields a unique $\mathbf{pSympl}_{\mathbb{K}}$-morphism $f : \mathcal{F}_u\mathbf{M} \longrightarrow \mathrm{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V})$ such that $\Delta f \circ [\iota_*] = v \circ \eta$. By using a smooth partition of unity argument, it is a simple task to establish that $f[\omega] = [\delta_{\mathbf{M}}\omega] \oplus [d_{\mathbf{M}}\omega]$ for all $[\omega] \in \mathcal{F}_u\mathbf{M}$. From this we can easily find the inverse of f, $g : \mathrm{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V}) \longrightarrow \mathcal{F}_u\mathbf{M}$, $[\delta_{\mathbf{M}}\omega] \oplus [d_{\mathbf{M}}\omega] \longmapsto [\omega]$, which meets $\Delta g \circ v = [\iota_*] \circ \varepsilon$. As f is a $\mathbf{pSympl}_{\mathbb{K}}$-isomorphism, g must be the unique $\mathbf{pSympl}_{\mathbb{K}}$-isomorphism $\mathrm{diag}_{\mathbf{M}}(\mathcal{A} \oplus \mathcal{V}) \xrightarrow{\sim} \mathcal{F}_u\mathbf{M}$ with this property. The rest follows from this. ❒

Of course, if one takes $\mathbf{M} \in \mathbf{Loc}$ such that $H_{\mathrm{dR}}^1(M;\mathbb{K}) = 0$, $H_{\mathrm{dR}}^2(M;\mathbb{K}) = 0$ and $H_{\mathrm{dR}}^{m-2}(M;\mathbb{K}) = 0$ in Theorem 4.6.21, the universal objects of the colimits for $\mathcal{A}_{\mathbf{M}}^q$ and $\mathfrak{A}_{\mathbf{M}}^q$ will coincide with $\mathcal{A}\mathbf{M}$ and $\mathfrak{A}\mathbf{M}$. In general, however, we will get an extra contribution stemming from the V-description of the free Maxwell field, which seems to be responsible for the complications in obtaining closed expressions for the colimits of the restrictions $\mathcal{A}_{\mathbf{M}}^q, \mathfrak{A}_{\mathbf{M}}^q : \mathrm{loc}_{-\mathbf{M}}^q \longrightarrow \mathbf{pSympl}_{\mathbb{K}}, \mathbf{{*}Alg_1}$ of $\mathcal{A}$ and $\mathfrak{A}$ to $\mathrm{loc}_{-\mathbf{M}}^q$, where $q \subseteq \mathbb{N} \setminus \{0\}$ is arbitrary.

Chapter 5

Dynamical Locality of the Free Maxwell Field

In this chapter, we discuss further properties of the classical and the quantised universal F-theory of the free Maxwell field, which were constructed in Section 4.6.1. We have already seen in Theorem 4.6.9 that both $\mathcal{F}_u$, $\mathfrak{F}_u : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{*Alg_1}$ are not locally covariant theories but, as we will see in Proposition 5.3.1, they still obey the time-slice axiom and $\mathfrak{F}_u$ is causal. This allows us to define the relative Cauchy evolutions for $\mathcal{F}_u$ and $\mathfrak{F}_u$, and so to investigate the matter of *dynamical locality*.

Briefly, dynamical locality is an extra condition on locally covariant theories which has emerged from the discussion of how a theory should be formulated such that its physical content is preserved across the various spacetimes it is considered on [Few12; FV12a]; in short *SPASs*, for "*the same physics in all spacetimes*". While the class of all locally covariant theories does not have the *SPASs property*, which is a very reasonable necessary condition all putative notions of SPASs should satisfy, the subclass of all dynamically local locally covariant theories has the SPASs property. However, due to its implications on locally covariant theories such as[1] additivity, extended locality and a no-go theorem for natural states[2], dynamical locality is a notion worth investigating in its own right.

We will quickly see that $\mathcal{F}_u$ and $\mathfrak{F}_u$ are not dynamically local, for the same reasons causing the failure of local covariance. Nevertheless, as we will see as well, $\mathcal{F}_u$ and $\mathfrak{F}_u$ can be easily modified by cutting away any topological sensitivity, thus obtaining dynamically local locally covariant theories $\mathcal{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ and $\mathfrak{R} : \mathbf{Loc} \longrightarrow \mathbf{*Alg_1^m}$. $\mathcal{R}$ will be referred to as the classical reduced F-theory of the free Maxwell field and $\mathfrak{R}$ will be the so-called quantised reduced F-theory of the free Maxwell field.

We give a brief introduction to the same physics in all spacetimes (SPASs) based on [Few12; FV12a] in Section 5.1. In Section 5.2, we give a basic recapitulation of the definitions of the dynamical net and dynamical locality, following [FV12a, Sec.5 + 6]. We will first define the dynamical net very concretely for locally covariant quan-

[1] See [FV12a, Sec.6] for the details.

[2] A <u>*natural state*</u> ω for a locally covariant quantum field theory $F : \mathbf{Loc} \longrightarrow \mathbf{(C)*Alg_1^m}$ is a rule assigning to each $\mathbf{M} \in \mathbf{Loc}$ a state $\omega_{\mathbf{M}} : F\mathbf{M} \longrightarrow \mathbb{C}$ such that $\omega_{\mathbf{N}} \circ F\psi = \omega_{\mathbf{M}}$ for all $\psi \in \mathbf{Loc}(\mathbf{M}, \mathbf{N})$ and for all $\mathbf{M}, \mathbf{N} \in \mathbf{Loc}$.

tum field theories and then abstractly for any locally covariant theory. We argue in Section 5.3 that $\mathcal{F}_u$ and $\mathfrak{F}_u$ both fail dynamical locality and in Section 5.4, we introduce the classical and the quantised reduced F-theory of the free Maxwell field, $\mathcal{R}, \mathfrak{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, \mathbf{{}^*Alg_1^m}$, which are locally covariant. We also show that $\mathcal{R}$ and $\mathfrak{R}$ obey the time-slice axiom and compute their respective relative Cauchy evolutions as well as the stress-energy-momentum tensor of $\mathcal{R}$. In Section 5.5, we prove that $\mathcal{R}$ and $\mathfrak{R}$ are dynamically local. Finally, we give a discussion of the role of the reduced free F-theory in the context of colimits and left Kan extensions in the appendix of this chapter.

5.1 The SPASs property

We discuss the notion of the same physics in all spacetimes, SPASs.

It is indeed a profound question to ask what the physical content of a theory is and a foundational problem of general quantum field theory in curved spacetimes is to understand how a quantum field theory should be formulated such that its physical content is preserved across the various spacetimes it is defined on; i.e. so that it represents the "*same physics in all spacetimes*" (*SPASs*). What is it that makes us say that we have the same quantum Klein-Gordon, Maxwell, Dirac, etc. field on Minkowski, Schwarzschild, Robertson-Walker, deSitter, etc. spacetime?

These questions are not easy to make mathematically precise and it is conceivable that there is more than one satisfactory definition of SPASs or even none at all. In any case, [BFV03] provides a suitable framework to tackle SPASs, namely the categorical framework of locally covariant quantum field theory. There, two locally covariant quantum field theories, which are given by functors $F, G : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$, can considered to be the same quantum field theory if and only if there exists a natural isomorphism $\eta : F \xrightarrow{\dot{\sim}} G$. This point of view can also be adapted to general locally covariant theories $F, G : \mathbf{Loc} \longrightarrow \mathbf{Phys}$. To avoid too restrictive assumptions on a physical theory right from the start, it is advisable to take an indirect approach and not to give a definition of SPASs but some reasonable necessary conditions instead, which should be met by any *good* notion of SPASs. The following two are of such a kind: suppose that we have a collection of physical theories which are to satisfy a particular notion of SPASs; then every theory in this collection should be locally covariant and if two theories in this collection coincide in one spacetime, then they should coincide in all spacetimes. Using the concepts of category theory, these two necessary conditions can be formulated particularly clearly as the SPASs property:

DEFINITION 5.1.1. A collection T of locally covariant theories $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ is said to have the *SPASs property* if and only if whenever $F, G : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ belong to T and there is a partial[3] natural isomorphism $\eta : F \dot{\longrightarrow} G$ between them, then η is a natural isomorphism.

This line of action does not completely resolve the issues raised (we have not specified a concrete definition of SPASs at all) but at least helps to partly solve them. In [FV12a, Sec.4], it was shown that the collection of all locally covariant theories does not have the SPASs property. To single out a class of locally covariant theories which has the SPASs property, [FV12a] introduced the notion of *dynamical locality* as a sufficient condition. Nevertheless, it is worthwhile to investigate dynamical locality in its own right, as already mentioned in the introduction to this chapter. Dynamical locality can be discussed independently from SPASs and has been tested for several quantum field theories already: the free real scalar field is dynamically local in all spacetime dimensions ≥ 2 provided the mass or the curvature coupling is non-zero [FV12b; Fer13a]. The same holds true for the extended theory of Wick polynomials in the massive case with minimal or conformal coupling [Fer13a]. The free Dirac field is also known to be dynamically local in spacetime dimension $= 4$, in the massive and in the massless case [Fer13b]. Due to a rigid gauge symmetry stemming from constant solutions of the field equations, the massless and minimally coupled free real scalar field fails dynamical locality in all spacetime dimensions ≥ 2 [FV12b]. The inhomogeneous and minimally coupled real scalar field has recently been discussed in [FS14b]. There, dynamical locality was established in the massive and the massless case for all spacetime dimensions ≥ 2.

In this chapter, we plan to add the free Maxwell field in terms of the field strength tensor to that list, though we have to change the theory from the universal F-theory to the reduced F-theory for this. In view of the negative result of the massless and minimally coupled free real scalar field regarding dynamical locality, free electromagnetism becomes, as a local gauge field theory, an interesting test case for dynamical locality.

5.2 The dynamical net and dynamical locality

We define the dynamical net and dynamical locality. First, we give the concrete definition for locally coariant quantum field theories and then the abstract categorical definiton for any locally covariant theory.

It is an integral feature of locally covariant quantum field theory that the formalism of algebraic quantum field theory can be retrieved [BFV03, Prop.2.3]: let $F : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$ be a causal locally covariant quantum field theory and fix $\mathbf{M} \in \mathbf{Loc}$. For each $O \in \mathrm{loc}_{-\mathbf{M}}$, we have the inclusion map $\iota_O : O \hookrightarrow M$, the $\mathbf{Loc}$-morphism $\iota_O : O \longrightarrow$

[3] A *partial natural isomorphism* is a natural transformation such that at least one of its components is an isomorphism.

$\mathbf{M}$, the unital $(C)^*$-algebra FO and the unital *-monomorphism $F\iota_O : FO \hookrightarrow F\mathbf{M}$. The image of $F\iota_O$ is the local unital $(C)^*$-algebra associated with O and the assignment $\mathrm{loc}_{-\mathbf{M}} \ni O \longmapsto F\iota_O(FO) \in \mathbf{(C)^*Alg_1^m}$ defines a Haag-Araki-Kastler net of local unital $(C)^*$-algebras. We make the following, preliminary definition:

DEFINITION 5.2.1. Let $F : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$ be a locally covariant quantum field theory and $\mathbf{M} \in \mathbf{Loc}$. The assignment $\mathrm{loc}_{-\mathbf{M}} \ni O \longmapsto F\iota_O(FO) \in \mathbf{(C)^*Alg_1^m}$ is called the *kinematical net*. It is custom to write $F^{\mathrm{kin}}(\mathbf{M}; O)$ for $F\iota_O(FO)$ in this context, where $O \in \mathrm{loc}_{-\mathbf{M}}$.

The adjective "*kinematical*" is well-chosen since the net $\mathrm{loc}_{-\mathbf{M}} \ni O \longmapsto F^{\mathrm{kin}}(\mathbf{M}; O) \in \mathbf{(C)^*Alg_1^m}$ of local unital $(C)^*$-algebras is purely constructed on the basis of F being a functor. The functoriality of F in return corresponds to isotony (HAK1) in algebraic quantum field theory, which is an Haag-Araki-Kastler axiom referring to the kinematics of the quantum field theory.

Invoking the interpretation employed in algebraic quantum field theory, the kinematical net represents a description of the local physics defined by a locally covariant quantum field theory. The *dynamical net*, which we are about to introduce, will provide another description of the local physics which is based on the relative Cauchy evolution, hence the dynamics.

Let $F : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$ be a locally covariant quantum field theory obeying the time-slice axiom (so that the relative Cauchy evolution exists), where we interpret the unital $(C)^*$-algebras $F\mathbf{M}$, $\mathbf{M} \in \mathbf{Loc}$, as algebras of local observables or as algebras of local smearings of the quantum field. Let $\mathbf{M} \in \mathbf{Loc}$; then for a compact subset $K \subseteq M$, any globally hyperbolic perturbation[4] $h \in H(\mathbf{M}; K')$, where[5] $K' := M \smallsetminus J_{\mathbf{M}}(K)$, is a modification of $\mathbf{M}$ in a region which is not causally accessible from K. Consequently, if an observable (resp. smearing of the quantum field) $a \in F\mathbf{M}$ is localised in K, it should be insensitive to such a modification, i.e. $\mathrm{rce}^F_{\mathbf{M}}[h](a) = a$. We use this idea to localise observables (resp. smearing of the quantum field) in K. Namely, we consider an observable (resp. smearing of the quantum field) $a \in F\mathbf{M}$ to be localised in K if and only if it is insensitive to *all* globally hyperbolic perturbations $h \in H(\mathbf{M}; K')$. We define accordingly:

$$F^\bullet(\mathbf{M}; K) = \left\{ a \in F\mathbf{M} \,\middle|\, \mathrm{rce}^F_{\mathbf{M}}[h](a) = a \ \ \forall h \in H(\mathbf{M}; K') \right\}, \tag{5.1}$$

which is a unital $(C)^*$-subalgebra of $F\mathbf{M}$. Finally, to localise observables (resp. smearings of the quantum field) in globally hyperbolic open subsets O of $\mathbf{M}$, we form the

[4]Recall Definition 3.4.1.

[5]Note, [FV12a; FV12b] uses $K^\perp$ to denote $M \smallsetminus J_{\mathbf{M}}(K)$ and K' is defined by $M \smallsetminus \mathrm{cl}\, J_{\mathbf{M}}(K)$. Since $\mathbf{M}$ is globally hyperbolic $J_{\mathbf{M}}(K)$ is closed by [BGP07, Lem.A.5.1] and so $K^\perp = K'$. However, we prefer the notation "$'$", which we have already used in (HAK4) to denote the causal complement.

unital $(C)^*$-subalgebra

$$F^{\mathrm{dyn}}(\mathbf{M};O) := \bigvee_{K\in\mathcal{K}(\mathbf{M};O)} F^{\bullet}(\mathbf{M};K)\,, \tag{5.2}$$

of $F\mathbf{M}$, which is generated by the $F^{\bullet}(\mathbf{M};K)$, where K ranges over a suitable collection $\mathscr{K}(\mathbf{M};O)$ of compact subsets of O. We will take $\mathscr{K}(\mathbf{M};O)$ to be the collection of all compact subsets of O which have a *diamond* neighbourhood whose *base* is contained in O; other choices are possible and sometimes necessary, see [FV12a, Sec.5] and [FV12b, Sec.2], which consider multidiamonds. However, due to our definition of **Loc** and $\mathrm{loc}_{-\mathbf{M}}$, we are only considering *connected* globally hyperbolic spacetimes and *connected* globally hyperbolic open subsets, hence we continue working in the *connected* case. We supplement the following definition ([BR09, Sec.3.1], [FV12a, Def.2.5]):

DEFINITION 5.2.2. Let $\mathbf{M}\in\mathbf{Loc}$. A *Cauchy ball* in a smooth spacelike Cauchy surface Σ for $\mathbf{M}$ is a subset $B\subseteq\Sigma$ contained in a smooth chart $\varphi: U \xrightarrow{\sim} W \subseteq \mathbb{R}^{m-1}$ of Σ such that $\varphi(B)$ is a non-empty open ball in $\mathbb{R}^{m-1}$ whose closure is contained in $\varphi(U)$.

A *diamond* in $\mathbf{M}$ is a relatively compact globally hyperbolic open subset of M which is of the form $D_{\mathbf{M}}(B)$, where $D_{\mathbf{M}}(B)$ is the Cauchy development in $\mathbf{M}$ of some Cauchy ball B for some smooth spacelike Cauchy surface Σ of $\mathbf{M}$. B is called the *base* of the diamond and we say that the diamond is *based* on Σ.

This completes the preliminary, concrete construction of the dynamical net:

DEFINITION 5.2.3. Let $F:\mathbf{Loc}\longrightarrow\mathbf{(C)^*Alg_1^m}$ be a locally covariant quantum field theory obeying the time-slice axiom and pick any $\mathbf{M}\in\mathbf{Loc}$. The *dynamical net* is the assignment $\mathrm{loc}_{-\mathbf{M}}\ni O\longmapsto F^{\mathrm{dyn}}(\mathbf{M};O)\in\mathbf{(C)^*Alg_1^m}$.

DEFINITION 5.2.4. A locally covariant quantum field theory $F:\mathbf{Loc}\longrightarrow\mathbf{(C)^*Alg_1^m}$ is said to obey *dynamical locality* or is called *dynamically local* if and only if it obeys the time-slice axiom and for each $\mathbf{M}\in\mathbf{Loc}$, the kinematical net and the dynamical net coincide.

From an abstract categorical standpoint however, the prelimiary, concrete descriptions of the dynamical and the kinematical net are inadequate. The set-theoretic image of a morphism and elements of objects in a category are not a well-defined category-theoretic notions and we should therefore strive for a formulation of the kinematical net, the dynamical net and dynamical locality which only uses well-defined categorical concepts. Adding this level of abstraction is not chicanery but we will profit immediately from this. For example, we will be able to define the kinematical net, the dynamical net and dynamical locality for any locally covariant theory $F:\mathbf{Loc}\longrightarrow\mathbf{Phys}$ obeying the time-slice axiom. We will also understand rather quickly why the classical and the quantised universal F-theory of the free Maxwell field cannot be dynamically local in the strict sense. The abstract viewpoint of category theory will also lead to an immense simplifications in the discussion of dynamical locality for the quantised reduced F-theory of the free Maxwell field.

We kindly ask the reader to recall the categorical notions of subobjects, equalisers, and intersections and unions of subobjects, which we have presented in Section 2.3. From the point of view of category theory, it is more appropriate to focus attention on subobjects (Definition 2.3.1) rather than on images of morphisms. For the kinematical net, this implies:

DEFINITION 5.2.5. Let $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ be a locally covariant theory and $\mathbf{M} \in \mathbf{Loc}$. The assignment $\mathrm{loc}_{-\mathbf{M}} \ni O \longmapsto F\iota_O \in \mathbf{Phys}(FO, F\mathbf{M})$ of $\mathrm{loc}_{-\mathbf{M}}$-objects to subobjects of $F\mathbf{M}$ is called the *kinematical net*. In this abstract categorical context, we redefine for $O \in \mathrm{loc}_{-\mathbf{M}}$ and $\mathbf{M} \in \mathbf{Loc}$,

$$F^{\mathrm{kin}}(\mathbf{M}; O) := FO \qquad \text{and} \qquad m^{\mathrm{kin}}_{\mathbf{M};O} := F\iota_O : F^{\mathrm{kin}}(\mathbf{M}; O) \hookrightarrow F\mathbf{M}. \tag{5.3}$$

Let $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ now be a locally covariant theory which obeys the time-slice axiom. The construction of the dynamical net consists in three steps, each of which corresponds to the introduction of a categorical notion. The first step, which is finding a categorical expression for "$\mathrm{rce}^F_{\mathbf{M}}[h](a) = a$ *for some* $a \in F\mathbf{M}$", where $\mathbf{M} \in \mathbf{Loc}$, $K \subseteq M$ compact and $h \in H(\mathbf{M}; K')$, is to consider the equaliser (Definition 2.3.2) of $\mathrm{rce}^F_{\mathbf{M}}[h]$ and $\mathrm{id}_{F\mathbf{M}}$:

$$e\left(\mathrm{rce}^F_{\mathbf{M}}[h], \mathrm{id}_{F\mathbf{M}}\right) : E\left(\mathrm{rce}^F_{\mathbf{M}}[h], \mathrm{id}_{F\mathbf{M}}\right) \hookrightarrow F\mathbf{M}. \tag{5.4}$$

Secondly, $F^\bullet(\mathbf{M}; K)$ in (5.1) needs to be defined categorically. This is done by the means of the categorical intersection (Definition 2.3.5)

$$\bigwedge_{h \in H(\mathbf{M};K')} e\left(\mathrm{rce}^F_{\mathbf{M}}[h], \mathrm{id}_{F\mathbf{M}}\right) : \bigwedge_{h \in H(\mathbf{M};K')} E\left(\mathrm{rce}^F_{\mathbf{M}}[h], \mathrm{id}_{F\mathbf{M}}\right) \hookrightarrow F\mathbf{M} \tag{5.5}$$

of the subobjects $e(\mathrm{rce}^F_{\mathbf{M}}[h], \mathrm{id}_{F\mathbf{M}}) : E(\mathrm{rce}^F_{\mathbf{M}}[h], \mathrm{id}_{F\mathbf{M}}) \hookrightarrow F\mathbf{M}$ for $h \in H(\mathbf{M}; K')$, which can also be denoted by

$$m^\bullet_{\mathbf{M};K} : F^\bullet(\mathbf{M}; K) \hookrightarrow F\mathbf{M} \tag{5.6}$$

for the sake of convenience. Thirdly and finally, $F^{\mathrm{dyn}}(\mathbf{M}; O)$ in (5.2) is characterised using the categorical union (Definition 2.3.8)

$$\bigvee_{K\in\mathscr{K}(\mathbf{M};O)} m^{\bullet}_{\mathbf{M};K} : \bigvee_{K\in\mathcal{K}(\mathbf{M};O)} F^{\bullet}(\mathbf{M};K) \hookrightarrow F\mathbf{M} \tag{5.7}$$

of the subobjects $m^{\bullet}_{\mathbf{M};K} : F^{\bullet}(\mathbf{M};K) \hookrightarrow F\mathbf{M}$ over all $K \in \mathscr{K}(\mathbf{M};O)$. Again, we can adopt a more convenient notation:

$$m^{\mathrm{dyn}}_{\mathbf{M};O} : F^{\mathrm{dyn}}(\mathbf{M};O) \hookrightarrow F\mathbf{M}. \tag{5.8}$$

DEFINITION 5.2.6. Let $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ be a locally covariant theory obeying the time-slice axiom and pick any $\mathbf{M} \in \mathbf{Loc}$. The *dynamical net* is the assignment $\mathrm{loc}_{-\mathbf{M}} \ni O \longmapsto (m^{\mathrm{dyn}}_{\mathbf{M};O} : F^{\mathrm{dyn}}(\mathbf{M};O) \hookrightarrow F\mathbf{M}) \in \mathbf{Phys}(F^{\mathrm{dyn}}(\mathbf{M};O), F\mathbf{M})$ of $\mathrm{loc}_{-\mathbf{M}}$-objects to subobjects of $F\mathbf{M}$.

DEFINITION 5.2.7. A locally covariant theory $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ is said to obey *dynamical locality* or is called *dynamically local* if and only if it obeys the time-slice axiom and for each $\mathbf{M} \in \mathbf{Loc}$, the kinematical net and the dynamical net coincide, i.e. the subobjects of $F\mathbf{M}$, $m^{\mathrm{kin}}_{\mathbf{M};O} : F^{\mathrm{kin}}(\mathbf{M};O) \hookrightarrow F\mathbf{M}$ and $m^{\mathrm{dyn}}_{\mathbf{M};O} : F^{\mathrm{dyn}}(\mathbf{M};O) \hookrightarrow F\mathbf{M}$, are equivalent for all $O \in \mathrm{loc}_{-\mathbf{M}}$.

For some properties of the dynamical net, which we will not need in this thesis though, see [FV12a, Sec.5.2].

5.3 The failure of dynamical locality for the universal F-theory

We argue that the classical and the quantised universal F-theory of the free Maxwell field, $\mathcal{F}_u$, $\mathfrak{F}_u : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1}$, both fail dynamical locality in the strict sense and also in a weakened sense. First, we prove that $\mathcal{F}_u$ and $\mathfrak{F}_u$ obey the time-slice axiom and compute their respective relative Cauchy evolutions.

Recall Definition 3.3.6 and the properties of time-slice maps, Lemma 3.3.7. In particular, if $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ is Cauchy and $\mathrm{tsm} : \Omega^2_0(N;\mathbb{K}) \longrightarrow \Omega^2_0(N;\mathbb{K})$ is a time-slice map for $(\psi, \lambda^2_N, \Box_{\mathbf{N}})$, then $\omega_{£} := \mathrm{tsm}\,\omega$ and $\omega_{€} := \omega - \Box_{\mathbf{N}}\omega_{£}$ with $\mathrm{supp}\,\omega_{€} \subseteq \psi(M)$ for $\omega \in \Omega^2_0(N;\mathbb{K})$.

PROPOSITION 5.3.1. *The classical and the quantised universal theory of the free Maxwell field, $\mathcal{F}_u$, $\mathfrak{F}_u : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1}$, obey the time-slice axiom and $\mathfrak{F}_u$ is causal in addition. If $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ is Cauchy, $\mathcal{F}_u\psi : \mathcal{F}_u\mathbf{M} \longrightarrow \mathcal{F}_u\mathbf{N}$ is a $\mathbf{Sympl}_{\mathbb{K}}$-isomorphism whose inverse is explicitly given by*

$$(\mathcal{F}_u\psi)^{-1} : \mathcal{F}_u\mathbf{N} \longrightarrow \mathcal{F}_u\mathbf{M}, \qquad [\omega] \longmapsto [\psi^*\omega_{€}], \tag{5.9}$$

for any time-slice map $\mathrm{tsm} : \Omega_0^2(N;\mathbb{K}) \longrightarrow \Omega_0^2(N;\mathbb{K})$ *for* $(\psi, \lambda_N^2, \Box_{\boldsymbol{N}})$ *and for any representative* $\omega \in \Omega_0^2(M;\mathbb{K})$ *of* $[\omega] \in [\Omega_0^2(M;\mathbb{K})]$.

Proof: It follows from Corollary 3.3.8 that the map $\Xi : \mathcal{F}_u\mathbf{N} \longrightarrow \mathcal{F}_u\mathbf{M}$, $[\omega] \longmapsto [\psi^*\omega_{€}]$, is independent of the representative chosen for $[\omega]$ and Lemma 3.3.7(ii) shows that Ξ is also independent of the individual time-slice map for $(\psi, \lambda_N^2, \Box_{\mathbf{N}})$. We skip a proof that Ξ is linear, symplectic and intertwines with the complex conjugation if $\mathbb{K} = \mathbb{C}$, which is elementary. Next,

(5.10)
$$(\Xi \circ (\mathcal{F}_u\psi))[\omega] = \Xi[\psi_*\omega] = [\psi^*(\psi_*\omega)_{€}] = [\psi^*\psi_*\omega - \psi^*\Box_{\mathbf{N}}(\psi_*\omega)_{£}] = [\psi^*\psi_*\omega] = [\omega] \quad \forall [\omega] \in [\Omega_0^2(M;\mathbb{K})],$$

and

$$((\mathcal{F}_u\psi) \circ \Xi)[\omega] = (\mathcal{F}_u\psi)[\psi^*\omega_{€}] = [\psi_*\psi^*\omega_{€}] = [\omega_{€}] = [\omega] \quad \forall [\omega] \in [\Omega_0^2(N;\mathbb{K})] \tag{5.11}$$

show the rest of our claim regarding the time-slice axiom. It is clear from Theorem 4.6.6 that $\mathfrak{F}_u$ is causal. ❒

With concrete inverses at our disposal, we are now ready to calculate the relative Cauchy evolutions for $\mathcal{F}_u$ and $\mathfrak{F}_u$, induced by globally hyperbolic perturbations $h \in H(\mathbf{M})$ for $\mathbf{M} \in \mathbf{Loc}$. To this end, let $\mathrm{tsm} : \Omega_0^2(M;\mathbb{K}) \longrightarrow \Omega_0^2(M;\mathbb{K})$ be a time-slice map for $(\imath_{\mathbf{M}}^+[h], \lambda_M^2, \Box_{\mathbf{M}})$ and $\mathrm{tsm}' : \Omega_0^2(M;\mathbb{K}) \longrightarrow \Omega_0^2(M;\mathbb{K})$ a time-slice map for $(\jmath_{\mathbf{M}}^-[h], \lambda_M^2, \Box_{\mathbf{M}[h]})$, where $\imath_{\mathbf{M}}^\pm[h] : \mathbf{M}^\pm[h] \longrightarrow \mathbf{M}$ and $\jmath_{\mathbf{M}}^\pm[h] : \mathbf{M}^\pm[h] \longrightarrow \mathbf{M}[h]$ are the Cauchy morphisms defined by the inclusions $i_{M^\pm[h]} : M^\pm[h] \hookrightarrow M$ and $j_{M^\pm[h]} : M^\pm[h] \hookrightarrow M[h]$, and $M^\pm[h] := M \smallsetminus J_{\mathbf{M}}^\mp(\operatorname{supp} h)$. We assume that tsm' is explicitly given as in the proof of Lemma 3.3.5, i.e. with a smooth partition of unity $\{\chi^+, \chi^-\}$ subordinated to the open cover $\{I_{\mathbf{M}[h]}^+(\Sigma_-), I_{\mathbf{M}[h]}^-(\Sigma_+)\}$ of M, where the smooth spacelike Cauchy surfaces Σ_+ and Σ_- for $\mathbf{M}[h]$ are completely contained in $M^-[h]$ such that Σ_+ lies strictly in the future of Σ_- (hence, Σ_+ and Σ_- lie in the causal past of $\operatorname{supp} h$ but do not intersect $\operatorname{supp} h$). Then

$$\mathrm{rce}_{\mathbf{M}}^{\mathcal{F}_u}[h][\omega] = \left(\mathcal{F}_u(\imath_{\mathbf{M}}^-[h]) \circ (\mathcal{F}_u(\jmath_{\mathbf{M}}^-[h]))^{-1} \circ \mathcal{F}_u(\jmath_{\mathbf{M}}^+[h]) \circ (\mathcal{F}_u(\imath_{\mathbf{M}}^+[h]))^{-1}\right)[\omega] \tag{5.12}$$
$$= \left(\mathcal{F}_u(\imath_{\mathbf{M}}^-[h]) \circ (\mathcal{F}_u(\jmath_{\mathbf{M}}^-[h]))^{-1} \circ \mathcal{F}_u(\jmath_{\mathbf{M}}^+[h])\right)[i_{M^+[h]}^*\omega_{€}] \tag{5.13}$$
$$= \left(\mathcal{F}_u(\imath_{\mathbf{M}}^-[h]) \circ (\mathcal{F}_u(\jmath_{\mathbf{M}}^-[h]))^{-1}\right)[j_{M^+[h]*}i_{M^+[h]}^*\omega_{€}] \tag{5.14}$$
$$= \left(\mathcal{F}_u(\imath_{\mathbf{M}}^-[h]) \circ (\mathcal{F}_u(\jmath_{\mathbf{M}}^-[h]))^{-1}\right)[\omega_{€}] \tag{5.15}$$

$$= \mathcal{F}_u\left(\imath^-_{\mathbf{M}}[h]\right)\left[j^*_{M^-[h]}\left(\omega_{\in} - \Box_{\mathbf{M}[h]}\,\underbrace{\chi^- G^{\text{ret}}_{\mathbf{M}[h]}\omega_{\in}}_{\text{empty support}} - \Box_{\mathbf{M}[h]}\,\chi^+ G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}\right)\right] \tag{5.16}$$

$$= \left[i_{M^-[h]*}j^*_{M^-[h]}\left(\omega_{\in} - \Box_{\mathbf{M}[h]}\chi^+ G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}\right)\right] \tag{5.17}$$

$$= [\omega_{\in}] - \left[\,\Box_{\mathbf{M}[h]}\,\chi^+ G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}\right] \tag{5.18}$$

$$= [\omega] - \Big[\left(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}}\right)\underbrace{\chi^+}_{=1\text{ on supp}\,h} G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in} + \Box_{\mathbf{M}}\,\underbrace{\chi^+ G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}}_{\text{compactly supported}}\Big] \tag{5.19}$$

$$= [\omega] - \left[\left(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}}\right)G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}\right] \tag{5.20}$$

$$= [\omega] + \Big[\left(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}}\right)d_{\mathbf{M}}G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in} + d_{\mathbf{M}}\,\underbrace{\left(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}}\right)G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}}_{\text{compactly supported}}\Big] \tag{5.21}$$

$$= [\omega] + \left[\left(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}}\right)G^{\text{adv}}_{\mathbf{M}[h]}d_{\mathbf{M}}\omega_{\in}\right] \tag{5.22}$$

$$= [\omega] + \left[\left(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}}\right)G_{\mathbf{M}[h]}d_{\mathbf{M}}\omega_{\in}\right], \qquad \forall\,[\omega] \in \left[\Omega^2_0(M;\mathbb{K})\right], \tag{5.23}$$

where we have used in the last step that $\operatorname{supp}(G^{\text{ret}}_{\mathbf{M}[h]}d_{\mathbf{M}}\omega_{\in}) \cap \operatorname{supp} h = \varnothing$. For the quantised universal F-theory, we have

$$\operatorname{rce}^{\mathfrak{F}_u}_{\mathbf{M}}[h] = Q\left(\operatorname{rce}^{\mathcal{F}_u}_{\mathbf{M}}[h]\right), \tag{5.24}$$

where $Q : \mathbf{pSympl}_{\mathbb{K}} \longrightarrow \mathbf{{}^*Alg_1}$ is the quantisation functor. Notice that the relative Cauchy evolution can also be written as

$$\operatorname{rce}^{\mathcal{F}_u}_{\mathbf{M}}[h][\omega] = \left[\Box_{\mathbf{M}[h]}\,\chi^- G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}\right] = \left[\Box_{\mathbf{M}}\,\chi^- G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}\right] \qquad \forall\,[\omega] \in \left[\Omega^2_0(M;\mathbb{K})\right], \tag{5.25}$$

which follows from the intermediate steps of the above calculation and from $\operatorname{supp} h \cap \operatorname{supp}\chi^- = \varnothing$. Also note that $\Box_{\mathbf{M}}\chi^- G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}$ is compactly supported but $\chi^- G^{\text{adv}}_{\mathbf{M}[h]}\omega_{\in}$ is not.

In the discussion of the failure of local covariance, we have already encountered arguments which show that $\mathcal{F}_u$ and $\mathfrak{F}_u$ cannot possibly be dynamically local in the strict sense, see the discussion directly before Theorem 4.6.9. To be concrete, let e.g. $\mathbf{N} \in \mathbf{Loc}$ be the Minkowski spacetime[6], $\mathbf{M} \in \mathbf{Loc}$ the Cauchy development in $\mathbf{N}$ of the set $\{0\} \times \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 > 1\}$ and $\psi : \mathbf{N} \longrightarrow \mathbf{M}$ the inclusion map; then it will be impossible for the subobject $f^{\text{dyn}}_{\mathbf{N};M} : \mathcal{F}^{\text{dyn}}_u(\mathbf{N};M) \hookrightarrow \mathcal{F}_u\mathbf{N}$ to be equivalent to the non-monic $f^{\text{kin}}_{\mathbf{N};M} = \mathcal{F}_u\psi : \mathcal{F}^{\text{kin}}_u(\mathbf{N};M) = \mathcal{F}_u\mathbf{M} \longrightarrow \mathcal{F}_u\mathbf{N}$. Indeed, there is no $\mathbf{pSympl}_{\mathbb{K}}$-isomorphism $f : \mathcal{F}^{\text{kin}}_u(\mathbf{N};M) \longrightarrow \mathcal{F}^{\text{dyn}}_u(\mathbf{N};M)$ such that $f^{\text{kin}}_{\mathbf{N};M} = f^{\text{dyn}}_{\mathbf{N};M} \circ f$. The same argument holds of course true for $\mathfrak{F}_u$ because there is no unital *-isomorphism $\varphi : \mathfrak{F}^{\text{kin}}_u(\mathbf{N};M) \longrightarrow \mathfrak{F}^{\text{dyn}}_u(\mathbf{N};M)$ satisfying the identity $\varphi^{\text{kin}}_{\mathbf{N};M} = \varphi^{\text{dyn}}_{\mathbf{N};M} \circ \varphi$, where $\varphi^{\text{dyn}}_{\mathbf{N};M}$:

[6] A similar argument can actually be given for every $\mathbf{N} \in \mathbf{Loc}_{\{2,m-2\}}$

$\mathfrak{F}_u^{\text{dyn}}(\mathbf{N};M) \hookrightarrow \mathfrak{F}_u\mathbf{N}$ is a subobject and $\varphi^{\text{kin}}_{\mathbf{N};M} : \mathfrak{F}_u^{\text{kin}}(\mathbf{N};M) \longrightarrow \mathfrak{F}_u\mathbf{N}$ is non-monic.

Despite this, we want to show that the failure of dynamical locality is more severe and cannot be achieved for $\mathcal{F}_u$ and $\mathfrak{F}_u$ even in a weakened sense. We now take coarser (weaker) kinematical and dynamical nets by only considering globally hyperbolic open subsets which are contractible. Let $\mathbf{M} \in \mathbf{Loc}$ be such that $H^{m-2}_{\text{dR}}(M;\mathbb{K}) \neq 0$. This implies together with Poincaré duality [GHV72, Chap.V, §4] that $H^2_{\text{dR,c}}(M;\mathbb{K}) \neq 0$ and thus the existence of $\omega \in \Omega^2_0(M;\mathbb{K})$ satisfying $d_{\mathbf{M}}\omega = 0$ but $[\omega] \neq 0 \in [\Omega^2_0(M;\mathbb{K})]$ ($\Longrightarrow \mathbf{F}_{\mathbf{M}}(\omega) \neq 0 \in \mathfrak{F}_u\mathbf{M}$). We immediately see

$$\text{rce}^{\mathcal{F}_u}_{\mathbf{M}}[h][\omega] = [\omega] \qquad \text{and} \qquad (\text{rce}^{\mathfrak{F}_u}_{\mathbf{M}}[h])(\mathbf{F}_{\mathbf{M}}(\omega)) = \mathbf{F}_{\mathbf{M}}(\omega) \qquad \forall h \in H(\mathbf{M}). \tag{5.26}$$

Consequently, we have $[\omega] \in \mathcal{F}_u^{\bullet}(\mathbf{M};K)$ and $\mathbf{F}_{\mathbf{M}}(\omega) \in \mathfrak{F}_u^{\bullet}(\mathbf{M};K)$ for all $K \in \mathscr{K}(\mathbf{M};O)$ and for all $O \in \text{loc}^{©}_{-\mathbf{M}}$. This implies $[\omega] \in \mathcal{F}_u^{\text{dyn}}(\mathbf{M};O)$ and $\mathbf{F}_{\mathbf{M}}(\omega) \in \mathfrak{F}_u^{\text{dyn}}(\mathbf{M};O)$ for all $O \in \text{loc}^{©}_{-\mathbf{M}}$. Now, the subobject $f^{\text{dyn}}_{\mathbf{M};O} : \mathcal{F}_u^{\text{dyn}}(\mathbf{M};O) \hookrightarrow \mathcal{F}_u\mathbf{M}$ cannot possibly be equivalent to the subobject $f^{\text{kin}}_{\mathbf{M};O} : \mathcal{F}_u^{\text{kin}}(\mathbf{M};O) \hookrightarrow \mathcal{F}_u\mathbf{M}$ for any $O \in \text{loc}^{©}_{-\mathbf{M}}$ because the (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic form on $\mathcal{F}_u^{\text{dyn}}(\mathbf{M};O)$ is degenerate but that of $\mathcal{F}_u^{\text{kin}}(\mathbf{M};O)$ is weakly non-degenerate. The same is true for $\mathfrak{F}_u$ because $\mathfrak{F}_u^{\text{dyn}}(\mathbf{M};O)$ is not simple[7] for any $O \in \text{loc}^{©}_{-\mathbf{M}}$ but $\mathfrak{F}_u^{\text{kin}}(\mathbf{M};O)$ is simple for all $O \in \text{loc}^{©}_{-\mathbf{M}}$. Hence, for any $O \in \text{loc}^{©}_{-\mathbf{M}}$, the subobject $\varphi^{\text{dyn}}_{\mathbf{M};O} : \mathfrak{F}_u^{\text{dyn}}(\mathbf{M};O) \hookrightarrow \mathfrak{F}_u\mathbf{M}$ cannot be equivalent to the subobject $\varphi^{\text{kin}}_{\mathbf{M};O} : \mathfrak{F}_u^{\text{kin}}(\mathbf{M};O) \hookrightarrow \mathfrak{F}_u\mathbf{M}$. Thus, from the point of view of dynamical locality, the elements $[\omega]$ resp. $\mathbf{F}_{\mathbf{M}}(\omega)$, where $\omega \in \Omega^2_0(M;\mathbb{K})$ is closed but not exact via a compactly supported smooth $\mathbb{K}$-valued differential 1-form, are local everywhere. We conclude:

THEOREM 5.3.2. *The classical and the quantised universal F-theory of the free Maxwell field are not dynamically local, even in the weakened sense obtained by restricting to contractible globally hyperbolic open subsets.*

5.4 The reduced *F*-theory

We saw in the last section that the classical and the quantised universal F-theory of the free Maxwell field, $\mathcal{F}_u, \mathfrak{F}_u : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}, \mathbf{{}^*Alg_1}$, are not dynamically local. However, we were able to clearly identify the cause of this failure, namely the possibility of having non-trivial radicals for $\mathcal{F}_u$ and non-trivial centres for $\mathfrak{F}_u$, which have already spoiled local covariance (see the discussion before Theorem 4.6.9). *So, by ruling these possibilities out, one might expect to establish local covariance and dynamical locality in one*

[7] We will give a neat justification for this in the second paragraph of the appendix of this chapter.

go. The purpose of this section is to verify this.

Removing non-trivial radicals and non-trivial centres, and thus any topological sensitivity of the classical and the quantised universal F-theory of the free Maxwell field, $\mathcal{F}_u,\ \mathfrak{F}_u : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}},\ {}^*\mathbf{Alg_1}$, leads to the classical and the quantised reduced F-theory of the free Maxwell field:

PROPOSITION AND DEFINITION 5.4.1. *The rules (the complex conjugation is to be omitted if* $\mathbb{K} = \mathbb{R}$*)*

$$\mathbf{Loc} \ni \mathbf{M} \longmapsto \big([\![\Omega^2_0(M;\mathbb{K})]\!], \mathbf{r_M}, \bar{\ } \big) \tag{5.27}$$

and

$$\mathbf{Loc}(\mathbf{M},\mathbf{N}) \ni \psi \longmapsto [\![\psi_*]\!] : \big([\![\Omega^2_0(M;\mathbb{K})]\!], \mathbf{r_M}, \bar{\ } \big) \longrightarrow \big([\![\Omega^p_0(N;\mathbb{K})]\!], \mathbf{r_N}, \bar{\ } \big), \tag{5.28}$$

where

$$\begin{cases} [\![\Omega^2_0(M;\mathbb{K})]\!] := \Omega^2_0(M;\mathbb{K}) \,/\, \big(\Omega^2_{0,d_\mathbf{M}}(M;\mathbb{K}) \oplus \Omega^2_{0,\delta_\mathbf{M}}(M;\mathbb{K})\big), \\ \mathbf{r_M} : [\![\Omega^2_0(M;\mathbb{K})]\!] \times [\![\Omega^2_0(M;\mathbb{K})]\!] \longrightarrow \mathbb{K}, \quad ([\![\omega]\!],[\![\eta]\!]) \longmapsto -\int_M \delta_\mathbf{M}\omega \wedge *_\mathbf{M} G_\mathbf{M} \delta_\mathbf{M}\eta, \\ \bar{\ } : [\![\Omega^2_0(M;\mathbb{K})]\!] \longrightarrow [\![\Omega^2_0(M;\mathbb{K})]\!], \quad [\![\omega]\!] \longmapsto [\![\overline{\omega}]\!] \ \text{(complex conjugation)}, \end{cases} \tag{5.29}$$

define a locally covariant theory $\mathcal{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ *and a causal locally covariant quantum field theory* $\mathfrak{R} := Q \circ \mathcal{R} : \mathbf{Loc} \longrightarrow {}^*\mathbf{Alg^m_1}$*, where* $Q : \mathbf{Sympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg^m_1}$ *is the quantisation functor. We call* $\mathcal{R}$ *the* classical *and* $\mathfrak{R}$ *the* quantised reduced F-theory *of the free Maxwell field.*

Proof: It is clear from Lemma 4.6.2 that (5.29) is a (complexified if $\mathbb{K} = \mathbb{C}$) symplectic space for each $\mathbf{M} \in \mathbf{Loc}$. For a $\mathbf{Loc}$-morphism $\psi : \mathbf{M} \longrightarrow \mathbf{N}$, $\psi_* d_\mathbf{M} = d_\mathbf{N} \psi_*$, $\psi_* \delta_\mathbf{M} = \delta_\mathbf{N} \psi_*$ and (UQ') entail that there exist a unique linear map $[\![\psi_*]\!] : [\![\Omega^2_0(M;\mathbb{K})]\!] \longrightarrow [\![\Omega^2_0(N;\mathbb{K})]\!]$ such that $[\![\psi_*]\!] \circ \pi_\mathbf{M} = \pi_\mathbf{N} \circ \psi_*$, where $\pi_{\mathbf{M}|\mathbf{N}} : \Omega^2_0(M|N;\mathbb{K}) \twoheadrightarrow [\![\Omega^2_0(M|N;\mathbb{K})]\!]$ denote the canonical projections onto the quotients. It is evident that $[\![\psi_*]\!]$ is a C-homomorphism if $\mathbb{K} = \mathbb{R}$ and a similar calculation to the one right after Proposition 4.6.4 shows that $[\![\psi_*]\!]$ is symplectic, too. Owing to (UQ'), $[\![(\psi \circ \varphi)_*]\!] = [\![\psi_*]\!] \circ [\![\varphi_*]\!]$ whenever $\varphi : \mathbf{L} \longrightarrow \mathbf{M}$ is another $\mathbf{Loc}$-morphism. Hence, we have well-defined functors. ❒

5.4.1 Time-slice axiom and relative Cauchy evolution

Having established local covariance, we will see now that the classical and the quantised reduced F-theory of the free Maxwell field, $\mathcal{R}, \mathfrak{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}},\ {}^*\mathbf{Alg^m_1}$*, obey the time-slice axiom. We will also compute the relative Cauchy evolutions of* $\mathcal{R}$ *and* $\mathfrak{R}$*. To achieve these goals, we need not* *do much because the difference between $\mathcal{F}_u$ and $\mathcal{R}$, and between $\mathfrak{F}_u$ and $\mathfrak{R}$ is just the (controlled) change of equivalence classes from $[\cdot]$ to $[\![\cdot]\!]$.*

Let $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ be Cauchy and $\mathrm{tsm} : \Omega^2_0(N;\mathbb{K}) \longrightarrow \Omega^2_0(N;\mathbb{K})$ some time-slice map for $(\psi, \lambda^2_N, \Box_{\mathbf{N}})$. By Corollary 3.3.8, we have $\omega_{\in} = \alpha + \beta$ with $\alpha \in \Omega^2_{0,d_{\mathbf{N}}}(N;\mathbb{K})$ and $\beta \in \Omega^2_{0,\delta_{\mathbf{N}}}(N;\mathbb{K})$ such that $\operatorname{supp}\alpha, \operatorname{supp}\beta \subseteq \psi(M)$ whenever $\omega \in \Omega^2_{0,d_{\mathbf{N}}}(N;\mathbb{K}) \oplus \Omega^2_{0,\delta_{\mathbf{N}}}(N;\mathbb{K})$. We may thus adapt Proposition 5.3.1 and the ensuing computation of the relative Cauchy evolution by just replacing $[\cdot]$ with $[\![\cdot]\!]$.

To show that $\mathcal{R}$ and $\mathfrak{R}$ are dynamically local, it will be helpful to establish a connection between the relative Cauchy evolution and the stress-energy-momentum tensor of $\mathcal{R}$. With this purpose in mind, we rearrange $\mathrm{rce}^{\mathcal{R}}_{\mathbf{M}}[h][\![\omega]\!] = [\![\Box_{\mathbf{M}}\chi^- G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}]\!]$, $[\![\omega]\!] \in [\![\Omega^2_0(M;\mathbb{K})]\!]$, into a more convenient expression for $h \in H(\mathbf{M})$ and $\mathbf{M} \in \mathbf{Loc}$, employing a Born expansion as in [FV12b, (B.2)]: since $\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}}$ vanishes outside of $\operatorname{supp} h$, the unique solution of $\Box_{\mathbf{M}}\eta = \omega_{\in} - (\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}$ is obviously $\eta = G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}$. Hence, $G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in} = G^{\mathrm{adv}}_{\mathbf{M}}\omega_{\in} - G^{\mathrm{adv}}_{\mathbf{M}}(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}$ and iteration yields

$$G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in} = G^{\mathrm{adv}}_{\mathbf{M}}\omega_{\in} - G^{\mathrm{adv}}_{\mathbf{M}}(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})\big(G^{\mathrm{adv}}_{\mathbf{M}}\omega_{\in} - G^{\mathrm{adv}}_{\mathbf{M}}(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}\big) \tag{5.30}$$

$$\begin{aligned} &= G^{\mathrm{adv}}_{\mathbf{M}}\omega_{\in} - G^{\mathrm{adv}}_{\mathbf{M}}(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}}\omega_{\in} \\ &\quad + G^{\mathrm{adv}}_{\mathbf{M}}(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}}(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}. \end{aligned} \tag{5.31}$$

For $h = 0 \in H(\mathbf{M})$, the relative Cauchy evolution becomes the identity and hence we obtain $[\![\Box_{\mathbf{M}}\chi^- G^{\mathrm{adv}}_{\mathbf{M}}\omega_{\in}]\!] = [\![\omega]\!] = [\![\omega_{\in}]\!]$ for all $[\![\omega]\!] \in [\![\Omega^2_0(M;\mathbb{K})]\!]$. Substituting the iterated formula for $G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}$ in (5.25) (for $[\![\cdot]\!]$ instead of $[\cdot]$) and using the previous relation, we obtain

$$\begin{aligned} \mathrm{rce}^{\mathcal{R}}_{\mathbf{M}}[h][\![\omega]\!] = [\![&\omega_{\in} - (\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}}\omega_{\in} \\ &+ (\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}}(\Box_{\mathbf{M}[h]} - \Box_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}[h]}\omega_{\in}]\!] \end{aligned} \tag{5.32}$$

$$\begin{aligned} = [\![&\omega + (\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})G_{\mathbf{M}}d_{\mathbf{M}}\omega \\ &+ (\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}}d_{\mathbf{M}}(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})G^{\mathrm{adv}}_{\mathbf{M}[h]}d_{\mathbf{M}}\omega_{\in}]\!] \\ &\forall\, [\![\omega]\!] \in [\![\Omega^2_0(M;\mathbb{K})]\!]. \end{aligned} \tag{5.33}$$

5.4.2 Stress-energy-momentum tensor

Following the reasoning of [FV12b]*, in order to verify dynamical locality for the classical and the quantised reduced F-theory of the free Maxwell field, $\mathcal{R}, \mathfrak{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$, we will characterise the dynamical net of $\mathcal{R}$ on $\mathbf{M} \in \mathbf{Loc}$ using the stress-energy-momentum tensor for $\mathcal{R}$ on $\mathbf{M}$.*

To derive the stress-energy-momentum tensor from the relative Cauchy evolution, we first verify that the relative Cauchy evolution for $\mathcal{R}$ is differentiable in the weak symplectic topology (cf. [FV12b, Sec.3 + Appx.B]) for each $\mathbf{M} \in \mathbf{Loc}$, i.e. for any $h \in \Gamma_0^\infty(\tau_M^* \odot \tau_M^*)$, $\frac{d}{dt}\mathbf{r_M}(\mathrm{rce}_{\mathbf{M}}^{\mathcal{R}}[th]\llbracket\omega\rrbracket, \llbracket\eta\rrbracket)|_{t=0}$ exists for all $\llbracket\omega\rrbracket, \llbracket\eta\rrbracket \in \llbracket\Omega_0^2(M;\mathbb{K})\rrbracket$ and there is a linear map $T_{\mathbf{M}}[h] : (F_{\omega,C} \circ \mathcal{R})\mathbf{M} \longrightarrow (F_{\omega,C} \circ \mathcal{R})\mathbf{M}$ such that

$$\mathbf{r_M}(T_{\mathbf{M}}[h]\llbracket\omega\rrbracket, \llbracket\eta\rrbracket) = \frac{d}{dt}\mathbf{r_M}\left(\mathrm{rce}_{\mathbf{M}}^{\mathcal{R}}[th]\llbracket\omega\rrbracket, \llbracket\eta\rrbracket\right)\Big|_{t=0} \quad \forall \llbracket\omega\rrbracket, \llbracket\eta\rrbracket \in \llbracket\Omega_0^2(M;\mathbb{K})\rrbracket. \tag{5.34}$$

Note, $F_{\omega,C} : \mathbf{Sympl}_{\mathbb{K}} \longrightarrow \mathbf{Vec}_{\mathbb{K}}$ is the forgetful functor that forgets about the (complexified if $\mathbb{K} = \mathbb{C}$) symplectic form and the C-involution if $\mathbb{K} = \mathbb{C}$, and for any $h \in \Gamma_0^\infty(\tau_M^* \odot \tau_M^*)$, there is $\varepsilon > 0$ such that $th \in H(\mathbf{M})$ for all $t \in (-\varepsilon, \varepsilon)$ (cf. [FV12a, Sec.3.4] and [FV12b, Sec.2 + 3]). Using the expression (5.33) and already dropping some terms of order t^2 or higher, we estimate for $\llbracket\omega\rrbracket, \llbracket\eta\rrbracket \in \llbracket\Omega_0^2(M;\mathbb{K})\rrbracket$ up to first order in t (denoted by the symbol "$\approx$"):

$$\mathbf{r_M}\left(\mathrm{rce}_{\mathbf{M}}^{\mathcal{R}}[th]\llbracket\omega\rrbracket - \llbracket\omega\rrbracket, \llbracket\eta\rrbracket\right) \approx \mathbf{r_M}\left(\llbracket(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})G_{\mathbf{M}}d_{\mathbf{M}}\omega\rrbracket, \llbracket\eta\rrbracket\right) \tag{5.35}$$

$$\approx -\int_M \delta_{\mathbf{M}}(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})G_{\mathbf{M}}d_{\mathbf{M}}\omega \wedge *_{\mathbf{M}}G_{\mathbf{M}}\delta_{\mathbf{M}}\eta \tag{5.36}$$

$$\approx -\int_M (\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})\varpi \wedge *_{\mathbf{M}}d_{\mathbf{M}}G_{\mathbf{M}}\delta_{\mathbf{M}}\eta \tag{5.37}$$

$$\approx -\int_M (\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})\varpi \wedge *_{\mathbf{M}}F_{\llbracket\eta\rrbracket}, \tag{5.38}$$

where $\varpi := G_{\mathbf{M}}d_{\mathbf{M}}\omega$ and $F_{\llbracket\eta\rrbracket} := d_{\mathbf{M}}G_{\mathbf{M}}\delta_{\mathbf{M}}\eta$. To avoid a heavy calculation in smooth charts of M, it is advisable to use abstract index notation, see [Wal84, Sec.2.4 + 3.1]. In abstract index notation, we have that

$$(d_{\mathbf{M}}\omega)_{a_1\ldots a_{p+1}} = \sum_{i=1}^{p+1}(-1)^{i+1}\nabla_{a_i}\omega_{a_1\ldots a_{i-1}a_{i+1}\ldots a_{p+1}}, \tag{5.39}$$

$$(\delta_{\mathbf{M}}\omega)_{a_1\ldots a_{p-1}} = -\nabla_{a_0}\omega^{a_0}{}_{a_1\ldots a_{p-1}}, \tag{5.40}$$

$$\omega \wedge *_{\mathbf{M}}\eta = \frac{1}{p!}\omega_{a_1\ldots a_p}\eta^{a_1\ldots a_p}\,\mathrm{vol}_{\mathbf{M}} \quad \forall \omega = \omega_{a_1\ldots a_p}, \eta = \eta_{a_1\ldots a_p} \in \Omega^p(M;\mathbb{K}),\ p \geq 0, \tag{5.41}$$

and

$$\langle T \mid S\rangle_g = T_{a_1\ldots a_r}S^{a_1\ldots a_r}, \tag{5.42}$$

$$\langle T \mid S\rangle_{2,g} = \int_M T_{a_1\ldots a_r}S^{a_1\ldots a_r}\,\mathrm{vol}_{\mathbf{M}} \quad \forall T = T_{a_1\ldots a_r}, S = S_{a_1\ldots a_r} \in \Gamma^\infty\left(\tau_M^{(r,0)};\mathbb{K}\right),\ r \geq 0. \tag{5.43}$$

From [FR04, (229) + (231)], we gather up to first order in t:

$$(\delta_{\mathbf{M}[th]} - \delta_{\mathbf{M}})\varpi = \left((\delta_{\mathbf{M}[th]} - \delta_{\mathbf{M}})\varpi\right)_{ab} \tag{5.44}$$

$$\approx t\left(\nabla_c\left(h^{cd}\varpi_{dab}\right) - \frac{1}{2}\left(\nabla_d h^c_c\right)\varpi^d{}_{ab} + \left(\nabla_c h_{da}\right)\varpi^{cd}{}_b - \left(\nabla_c h_{db}\right)\varpi^{cd}{}_a\right) \tag{5.45}$$

where ∇ stands for the Levi-Civita connection with respect to g. From this we can already conclude that

$$T_{\mathbf{M}}\,[\![\omega]\!] = [\![\upsilon]\!], \tag{5.46}$$

where $\upsilon \in \Omega^2_0(M;\mathbb{K})$ is defined by

$$\begin{aligned}\upsilon_{ab} &:= \nabla_c\left(h^{cd}\left(G_{\mathbf{M}}d_{\mathbf{M}}\omega\right)_{dab}\right) - \frac{1}{2}\left(\nabla_d h^c_c\right)\left(G_{\mathbf{M}}d_{\mathbf{M}}\omega\right)^d{}_{ab} \\ &\quad + \left(\nabla_c h_{da}\right)\left(G_{\mathbf{M}}d_{\mathbf{M}}\omega\right)^{cb}{}_b - \left(\nabla_c h_{db}\right)\left(G_{\mathbf{M}}d_{\mathbf{M}}\omega\right)^{cd}{}_a .\end{aligned} \tag{5.47}$$

Using the divergence theorem

$$\int_M \delta_{\mathbf{M}}\theta\,\mathrm{vol}_{\mathbf{M}} = \int_M \delta_{\mathbf{M}}\theta \wedge *_{\mathbf{M}} *^{-1}_{\mathbf{M}}\,\mathrm{vol}_{\mathbf{M}} = \int_M \theta \wedge *_{\mathbf{M}} d_{\mathbf{M}} 1 = 0 \qquad \forall\theta \in \Omega^1_0(M;\mathbb{K}), \tag{5.48}$$

we find

$$\int_M \nabla_c\left(h^{cd}\varpi_{dab}\right)F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}} = \int_M \Big[\nabla_c\underbrace{\left(h^{cd}\varpi_{dab}F^{ab}_{[\![\eta]\!]}\right)}_{=:\theta^c} - h^{cd}\varpi_{dab}\nabla_c F^{ab}_{[\![\eta]\!]}\Big]\,\mathrm{vol}_{\mathbf{M}} \tag{5.49}$$

$$= -\int_M h^{cd}\varpi_{dab}\nabla_c F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}}, \tag{5.50}$$

$$\int_M \left(\nabla_d h^c_c\right)\varpi^d{}_{ab}F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}} = \int_M \Big[\nabla_d\underbrace{\left(h^c_c\,\varpi^d{}_{ab}F^{ab}_{[\![\eta]\!]}\right)}_{=:\theta^d} - h^c_c\,\nabla_d\left(\varpi^d{}_{ab}F^{ab}_{[\![\eta]\!]}\right)\Big]\,\mathrm{vol}_{\mathbf{M}} \tag{5.51}$$

$$= -\int_M h^c_c\underbrace{\nabla_d\left(\varpi^d{}_{ab}\right)}_{=-(\delta_{\mathbf{M}}\varpi)_{ab}}F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}} - \int_M h^c_c\underbrace{\varpi^d{}_{ab}\nabla_d F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}}}_{=3!\,\varpi\wedge *_{\mathbf{M}}d_{\mathbf{M}}F_{[\![\eta]\!]}=0} \tag{5.52}$$

and

$$\int_M \left(\nabla_c h_{da}\right)\varpi^{cd}{}_b F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}} = \int_M \Big[\nabla_c\underbrace{\left(h_{da}\varpi^{cd}{}_b F^{ab}_{[\![\eta]\!]}\right)}_{=:\theta^c} - h_{da}\nabla_c\left(\varpi^{cd}{}_b F^{ab}_{[\![\eta]\!]}\right)\Big]\,\mathrm{vol}_{\mathbf{M}} \tag{5.53}$$

$$= -\int_M h_{da}\underbrace{\nabla_c\left(\varpi^{cd}{}_b\right)}_{=-(\delta_{\mathbf{M}}\varpi)^d{}_b}F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}} - \int_M h_{da}\varpi^{cd}{}_b\nabla_c F^{ab}_{[\![\eta]\!]}\,\mathrm{vol}_{\mathbf{M}} . \tag{5.54}$$

Introducing $F_{[\![\omega]\!]} \coloneqq -\delta_{\mathbf{M}}\varpi = -\delta_{\mathbf{M}}G_{\mathbf{M}}d_{\mathbf{M}}\omega = d_{\mathbf{M}}G_{\mathbf{M}}\delta_{\mathbf{M}}\omega$ and putting the terms together, we arrive (in first order of t) at

$$\text{(5.55)}\quad \mathbf{r_M}\left([\![\left(\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}}\right)G_{\mathbf{M}}d_{\mathbf{M}}\omega]\!] - [\![\omega]\!], [\![\eta]\!]\right) \approx t\int\limits_M h_{cd}\left(\frac{1}{4}\,g^{cd}\,(F_{[\![\omega]\!]})_{ab}F^{ab}_{[\![\eta]\!]} - g_{ab}F^{ca}_{[\![\omega]\!]}F^{db}_{[\![\eta]\!]}\right)\mathrm{vol}_{\mathbf{M}}$$

and thereby

$$\text{(5.56)}\quad \frac{d}{dt}\mathbf{r_M}\left(\mathrm{rce}^{\mathcal{R}}_{\mathbf{M}}[th]\,[\![\omega]\!], [\![\eta]\!]\right)\Big|_{t=0} = \int\limits_M h_{cd}\left(\frac{1}{4}\,g^{cd}\left(F_{[\![\omega]\!]}\right)_{ab}F^{ab}_{[\![\eta]\!]} - g_{ab}F^{ca}_{[\![\omega]\!]}F^{db}_{[\![\eta]\!]}\right)\mathrm{vol}_{\mathbf{M}}$$

$$\text{(5.57)}\quad = \int\limits_M h_{cd}\,T^{cd}_{\mathbf{M}}\left([\![\omega]\!], [\![\eta]\!]\right)\mathrm{vol}_{\mathbf{M}}$$

$$\text{(5.58)}\quad = \left\langle h \,\middle|\, T_{\mathbf{M}}\left([\![\omega]\!], [\![\eta]\!]\right)\right\rangle_{2,g},$$

where we have introduced the polarised stress-energy-momentum tensor $T_{\mathbf{M}}\left([\![\omega]\!], [\![\eta]\!]\right) \in \Gamma^{\infty}\left(\tau^{*}_{M}\odot\,\tau^{*}_{M};\mathbb{K}\right)$ for the classical reduced F-theory on $\mathbf{M}$ for $[\![\omega]\!], [\![\eta]\!] \in [\![\Omega^2_0\left(M;\mathbb{K}\right)]\!]$ by

$$\text{(5.59)}\quad T^{ab}_{\mathbf{M}}\left([\![\omega]\!], [\![\eta]\!]\right) \coloneqq \frac{1}{4}\,g^{ab}\,(F_{[\![\omega]\!]})_{cd}F^{cd}_{[\![\eta]\!]} - g_{cd}F^{ac}_{[\![\omega]\!]}F^{bd}_{[\![\eta]\!]}$$

and $F_{[\![\omega]\!]} \coloneqq d_{\mathbf{M}}G_{\mathbf{M}}\delta_{\mathbf{M}}\omega$ for any representative $\omega \in \Omega^2_0\left(M;\mathbb{K}\right)$ of $[\![\omega]\!] \in [\![\Omega^2_0\left(M;\mathbb{K}\right)]\!]$. Take notice that the very same expression is obtained for the classical universal F-theory if $[\![\,\cdot\,]\!]$ is replaced with $[\cdot]$.

5.5 The reduced F-theory is dynamically local

We will now prove that the classical and the quantised reduced F-theory of the free Maxwell field, $\mathcal{R}, \mathfrak{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, {}^{*}\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$, obey dynamical locality. In doing so, we will follow basically the argumentation of [FV12b]*; the main technical point of difference is that besides the massless Klein-Gordon equation (for smooth differential 2-forms), we also need to take into account the free Maxwell equations* (4.110).

In the classical case, we can work with concrete (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic spaces and avoid referring to the underlying categorical notions such as subobjects for the most part. In the quantum case however, we will profit immensely from the abstract categorical point of view. Once we have shown that dynamical locality holds for classical reduced F-theory of the free Maxwell field, $\mathcal{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, we can conclude that quantised reduced F-theory of the free Maxwell field, $\mathfrak{R} : \mathbf{Loc} \longrightarrow {}^{*}\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$, which is $Q \circ \mathcal{R}$ with the quantisation functor $Q : \mathbf{Sympl}_{\mathbb{K}} \longrightarrow {}^{*}\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$, obeys dynamical locality by applying [FV12b, Thm.5.3]. This reference states that the quantisation of a dynamically local theory is dynamically local if the following items [FV12b, p.1688] are fulfilled, which we have already adopted to our setting:

($\mathscr{L}1$) The relative Cauchy evolution of $\mathcal{R}$ is differentiable in the weak symplectic topology as in (5.34), and the resulting linear maps $T_{\mathbf{M}}[h]$, where $h \in H(\mathbf{M})$ and $\mathbf{M} \in \mathbf{Loc}$, obey

$$\mathfrak{r}_{\mathbf{M}}\left(T_{\mathbf{M}}[h]\llbracket\omega\rrbracket, \llbracket\overline{\omega}\rrbracket\right) = \int_M h_{ab} T^{ab}_{\mathbf{M}}\left(\llbracket\omega\rrbracket, \llbracket\overline{\omega}\rrbracket\right) \mathrm{vol}_{\mathbf{M}}, \tag{5.60}$$

$$\forall \llbracket\omega\rrbracket \in \llbracket\Omega^2_0(M;\mathbb{K})\rrbracket,\ \forall h = h_{ab} \in H(\mathbf{M};O),\ \forall O \in \mathrm{loc}_{-\mathbf{M}},\ \forall \mathbf{M} \in \mathbf{Loc}.$$

($\mathscr{L}2$) For each $O \in \mathrm{loc}_{-\mathbf{M}}$ containing the support of $h \in \Gamma^\infty_0(\tau^*_M \odot \tau^*_M)$, $\mathrm{img}\, T_{\mathbf{M}}[h]$ can be identified with a subset of $\mathcal{R}^{\mathrm{kin}}(\mathbf{M};O)$.

($\mathscr{L}3$) $\mathcal{R}$ obeys extended locality, i.e. $\mathrm{img}\, r^{\mathrm{kin}}_{\mathbf{M};U} \cap \mathrm{img}\, r^{\mathrm{kin}}_{\mathbf{M};V} = 0 \in \mathcal{R}\mathbf{M}$ for spacelike separated $U, V \in \mathrm{loc}_{-\mathbf{M}}$ and $\mathbf{M} \in \mathbf{Loc}$.

($\mathscr{L}4$) For $\mathbf{M} \in \mathbf{Loc}$ and $K \subseteq M$ compact: $\mathcal{R}^\bullet(\mathbf{M};K) = \bigcap_{\substack{h \in \Gamma^\infty_0(\tau^*_M \odot \tau^*_M) \\ \mathrm{supp}\, h \subseteq K'}} \ker T_{\mathbf{M}}[h]$.

We start now with the classical reduced F-theory $\mathcal{R}$ and characterise its dynamical net using the stress-energy-momentum tensor, which we have computed in the last section. In order for equalisers, unions and intersections to exist, we regard $\mathcal{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ as a functor $\mathcal{R} : \mathbf{Loc} \longrightarrow \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$. For $\mathbf{M} \in \mathbf{Loc}$, we can associate to each $\llbracket\omega\rrbracket \in \llbracket\Omega^2_0(M;\mathbb{K})\rrbracket$ a solution of (4.110) which has compact support on smooth spacelike Cauchy surfaces for $\mathbf{M}$ by setting $F_{\llbracket\omega\rrbracket} := d_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} \omega$ for any representative $\omega \in \Omega^2_0(M;\mathbb{K})$. Accordingly, in the classical reduced F-theory, we are restricting our attention to solutions of the free Maxwell equations which are of the form $d_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} \omega$ for $\omega \in \Omega^2_0(M;\mathbb{K})$. Recall from Lemma 4.6.1 that this is precisely the case if $\mathbf{M} \in \mathbf{Loc}_{\{2,m-2\}}$. Since the free Maxwell equations (4.110) possess a well-posed Cauchy problem [DL12, Prop.2.1],

$$d_{\mathbf{M}[h]} G_{\mathbf{M}[h]} \delta_{\mathbf{M}[h]} \mathcal{R}\left(\jmath^+_{\mathbf{M}}[h]\right)\left(\left(\mathcal{R}\left(\imath^+_{\mathbf{M}}[h]\right)\right)^{-1} \llbracket\omega\rrbracket\right) = d_{\mathbf{M}[h]} G_{\mathbf{M}[h]} \delta_{\mathbf{M}[h]} \omega_{€}$$

is the unique solution of the free Maxwell equations on $\mathbf{M}[h]$ which coincides with $F_{\llbracket\omega\rrbracket}$ on $M^+[h]$ (cf. [FV12b, Sec.3]). The agreement is seen by restriction and $\omega_{€} = \omega - \Box_{\mathbf{M}} \omega_{£}$, the uniqueness follows from the well-posedness of the Cauchy problem. If $\eta \in \Omega^2_0(M;\mathbb{K})$ is a representative of $\mathrm{rce}^{\mathcal{R}}_{\mathbf{M}}[h]\llbracket\omega\rrbracket$, then $d_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} \eta$ is the unique solution of (4.110) on $\mathbf{M}$ agreeing with $d_{\mathbf{M}[h]} G_{\mathbf{M}[h]} \delta_{\mathbf{M}[h]} \omega_{€}$ on $M^-[h]$, which follows by the explicit formulas for the relative Cauchy evolution of $\mathcal{R}$. This interpretation of the relative Cauchy evolution will become very helpful now:

LEMMA 5.5.1. *Let K be any compact subset of $\mathbf{M} \in \mathbf{Loc}$. Then*

(5.61)
$$\mathcal{R}^{\bullet}(\mathbf{M}; K) = \{[\![\omega]\!] \in \mathcal{R}\mathbf{M} \mid \operatorname{supp} T_{\mathbf{M}}([\![\omega]\!], [\![\overline{\omega}]\!]) \subseteq J_{\mathbf{M}}(K)\} = \bigcap_{\substack{h \in \Gamma_0^\infty(\tau_M^* \odot \tau_M^*) \\ \operatorname{supp} h \subseteq K'}} \ker T_{\mathbf{M}}[h]$$

and also $\mathcal{R}^{\bullet}(\mathbf{M}; K) = \left\{[\![\omega]\!] \in \mathcal{R}\mathbf{M} \mid \operatorname{supp} F_{[\![\omega]\!]} \subseteq J_{\mathbf{M}}(K)\right\}$.

Proof: Labelling the sets in their order of appearance from the left to the right by I, II and III respectively, we will prove that $I \subseteq III \subseteq II \subseteq I$.

Starting with $I \subseteq III$, suppose $[\![\omega]\!] \in \mathcal{R}^{\bullet}(\mathbf{M}; K)$. For $h \in \Gamma_0^\infty(\tau_M^* \odot \tau_M^*)$ with support in K', we can find $\varepsilon > 0$ such that $th \in H(\mathbf{M}; K')$ for all $t \in (-\varepsilon, \varepsilon)$ and as $\operatorname{rce}_{\mathbf{M}}^{\mathcal{R}}[th][\![\omega]\!] = [\![\omega]\!]$ for all $t \in (-\varepsilon, \varepsilon)$, we have $\frac{d}{dt}\mathfrak{r}_{\mathbf{M}}(\operatorname{rce}_{\mathbf{M}}^{\mathcal{R}}[th][\![\omega]\!], [\![\eta]\!])|_{t=0} = 0$ for all $[\![\eta]\!] \in \mathcal{R}\mathbf{M}$. Hence, also $\mathfrak{r}_{\mathbf{M}}(T_{\mathbf{M}}[h][\![\omega]\!], [\![\eta]\!]) = 0$ for all $[\![\eta]\!] \in \mathcal{R}\mathbf{M}$ and by weak non-degeneracy of $\mathfrak{r}_{\mathbf{M}}$, $[\![\omega]\!] \in \ker T_{\mathbf{M}}[h]$; as h was arbitrary, $I \subseteq III$.

For $III \subseteq II$, if $[\![\omega]\!] \in III$, then we have in particular $\mathfrak{r}_{\mathbf{M}}(T_{\mathbf{M}}[h][\![\omega]\!], [\![\overline{\omega}]\!]) = \langle h \mid T_{\mathbf{M}}([\![\omega]\!], [\![\overline{\omega}]\!])\rangle_{2,g} = 0$ for all $h \in \Gamma_0^\infty(\tau_M^* \odot \tau_M^*)$ with support $\operatorname{supp} h \subseteq K'$, so $\operatorname{supp} T_{\mathbf{M}}([\![\omega]\!], [\![\overline{\omega}]\!]) \subseteq J_{\mathbf{M}}(K)$ as required.

Finally, to prove $II \subseteq I$, we note that $\operatorname{supp} T_{\mathbf{M}}([\![\omega]\!], [\![\overline{\omega}]\!]) \subseteq J_{\mathbf{M}}(K)$ implies that $\operatorname{supp} F_{[\![\omega]\!]} \subseteq J_{\mathbf{M}}(K)$ because the energy density, which is the sum of the moduli squared of the off-diagonal components of $F_{[\![\omega]\!]}$ (in some frame), must vanish at each point $x \notin J_{\mathbf{M}}(K)$. Accordingly, $F_{[\![\omega]\!]}$ is a solution of Maxwell's equations in $\mathbf{M}[h]$ for every $h \in H(\mathbf{M}; K')$. Hence, by the well-posedness of the Cauchy problem [DL12, Prop.2.1], $F_{[\![\omega]\!]}$ is the unique solution on $\mathbf{M}[h]$ that coincides with $F_{[\![\omega]\!]}$ on $M^+[h]$ and also the unique solution on $\mathbf{M}$ that coincides with $F_{[\![\omega]\!]}$ on $M^-[h]$. Thus, $[\![\omega]\!]$ and $\operatorname{rce}_{\mathbf{M}}^{\mathcal{R}}[h][\![\omega]\!]$ give rise to the same solution of the free Maxwell equations on $\mathbf{M}$, which implies $\operatorname{rce}_{\mathbf{M}}^{\mathcal{R}}[h][\![\omega]\!] = [\![\omega]\!]$ and consequently $[\![\omega]\!] \in \mathcal{R}^{\bullet}(\mathbf{M}; K)$. The final statement is immediate from the argument just given. ❒

LEMMA 5.5.2. *For all $O \in \operatorname{loc}_{-\mathbf{M}}$, we have $\mathcal{R}^{\mathrm{kin}}(\mathbf{M}; O) \subseteq \mathcal{R}^{\mathrm{dyn}}(\mathbf{M}; O)$.*

Proof: Let $[\![\omega]\!] \in \mathcal{R}^{\mathrm{kin}}(\mathbf{M}; O)$ with $\omega \in \Omega_0^2(M; \mathbb{K})$, $\operatorname{supp}\omega \subseteq O$, a representative of $[\![\omega]\!]$. Choosing for each $x \in \operatorname{supp}\omega$ a Cauchy ball B_x containing x and taking the Cauchy developments, we find an open cover $\{D_{\mathbf{M}}(B_x) \mid x \in \operatorname{supp}\omega\}$ of $\operatorname{supp}\omega$ in M. Since $\operatorname{supp}\omega$ is compact, already finitely many of these sets are enough to cover $\operatorname{supp}\omega$, say $\operatorname{supp}\omega \subseteq \bigcup_{i=0}^n D_{\mathbf{M}}(B_i)$ with[8] $n \in \mathbb{N}$. Let $\{\chi, \chi^i \mid i = 0, \dots, n\}$ be a smooth partition of unity subordinated to the open cover $\{M \setminus \operatorname{supp}\omega, D_{\mathbf{M}}(B_i) \mid i = 0, \dots, n\}$ of M. Define $\omega_i \coloneqq \chi^i \omega \in \Omega_0^2(M; \mathbb{K})$, where $\operatorname{supp}\omega_i \subseteq D_{\mathbf{M}}(B_i) \cap O$; then $\omega = \sum_{i=0}^n \omega_i$. By construction, $\operatorname{supp}\omega_i \in \mathscr{K}(\mathbf{M}; O)$. As $\operatorname{supp} T_{\mathbf{M}}([\![\omega_i]\!], [\![\overline{\omega}_i]\!]) \subseteq J_{\mathbf{M}}(\operatorname{supp}\omega_i)$, Lemma 5.5.1 yields $[\![\omega_i]\!] \in \mathcal{R}^{\bullet}(\mathbf{M}; \operatorname{supp}\omega_i)$ and hence, $[\![\omega]\!] = \sum_{i=0}^n [\![\omega_i]\!] \in \mathcal{R}^{\mathrm{dyn}}(\mathbf{M}; O)$ because $\mathcal{R}^{\mathrm{dyn}}(\mathbf{M}; O)$ is the

[8] "$\mathbb{N}$" denotes the set of all natural numbers including zero.

smallest (compexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic subspace of $\mathcal{R}\mathbf{M}$ containing $\mathcal{R}^{\bullet}(\mathbf{M}; K)$ for all $K \in \mathscr{K}(\mathbf{M}; O)$.

The following lemma can be considered as an analogue to [FV12b, Lem.3.1.] and is integral to the proof that the kinematical and the dynamical net coincide:

LEMMA 5.5.3. *Let* $\mathbf{M} \in \mathbf{Loc}$ *and* $K \subseteq O \in \mathrm{loc}_{-\mathbf{M}}$ *compact. There exists* $\chi \in \mathcal{C}^{\infty} M$ *such that every solution* $F \in \Omega^2(M; \mathbb{K})$ *of the free Maxwell equations with* $\operatorname{supp} F \subseteq J_{\mathbf{M}}(K)$ *can be written as*

$$F = G_{\boldsymbol{M}} \Box_{\boldsymbol{M}} \chi F, \tag{5.62}$$

where $\Box_{\mathbf{M}} \chi F \in \Omega^2_0(M; \mathbb{K})$, $\delta_{\mathbf{M}} \chi F \in \Omega^1_0(M; \mathbb{K})$ *and* $d_{\mathbf{M}} \chi F \in \Omega^3_0(M; \mathbb{K})$ *are compactly supported in* O.

Proof: Since K is compact in O, we can find a smooth spacelike Cauchy surface Υ_+ for O which lies strictly in the future of K and another smooth spacelike Cauchy surface Υ_- for O which lies strictly in the past of both Υ^+ and K. This can be achieved by using the Bernal-Sáchnez splitting theorem. The two sets $K_+ := J^+_O(K) \cap \Upsilon_+$ and $K_- := J^-_O(K) \cap \Upsilon_-$ are compact in O by [BGP07, Cor.A.5.4] and hence compact in M. Since O is causally convex in M, $K_+ = J^+_{\mathbf{M}}(K) \cap \Upsilon_+$ and $K_- = J^-_{\mathbf{M}}(K) \cap \Upsilon_-$. We define $K_0 := J_{\mathbf{M}}(K) \cap J^+_{\mathbf{M}}(K_+) \cap J^-_{\mathbf{M}}(K_-)$ which is compact by reason of [BGP07, Lem.A.5.7] and as closed subsets of compacts sets are compact in Hausdorff spaces.

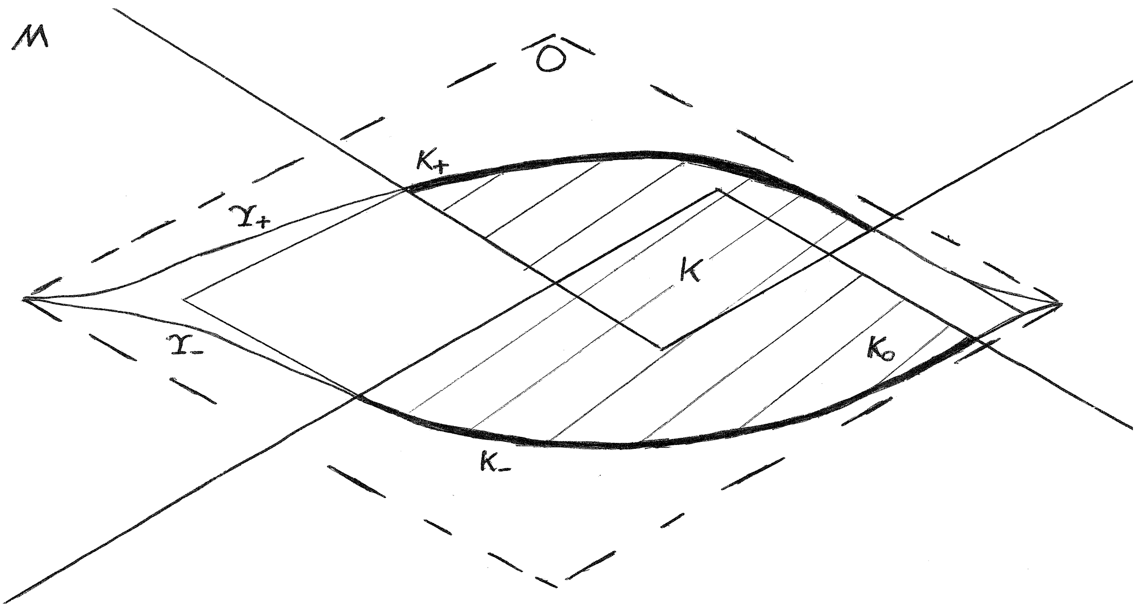


FIGURE 5.1: Visual aid for the proof of Lemma 5.5.3. $O \subseteq M$ is the dashed diamond, Υ_+ and Υ_- give rise to the eye-shaped form inside O, K is the shaded solid parallelogram inside O, the lines representing K_+ and K_- are drawn in bold and K_0 is given by the shaded area.

$K_0 \subseteq O$ because let $x \in K_0$; then there is a future-directed, causal smooth curve from K_- to x and a past-directed, causal smooth curve from K^+ to x. Joining these two causal smooth curves, we have found a causal smooth curve from K_- to K^+ going through x. Since O is causally convex, this causal smooth curve must entirely lie in O and so must x. K_+ and K_- are also compact in M, hence using the Bernal-Sáchnez splitting theorem again, we can find smooth spacelike Cauchy surfaces Σ_+ and Σ_- lying strictly between them such that Σ_+ lies strictly in the future of Σ_-. Let $\{\chi^+, \chi^-\}$ be a smooth partition of unity subordinated to $\{I^+_{\mathbf{M}}(\Sigma_-), I^-_{\mathbf{M}}(\Sigma_+)\}$, which is an open cover for M. It holds $\operatorname{supp}\chi^- F \subseteq J_{\mathbf{M}}(K) \cap J^-_{\mathbf{M}}(\Sigma_+)$, $\chi^- = 0$ on $J^+_{\mathbf{M}}(K^+)$ and $\chi^- = 1$ on $J^-_{\mathbf{M}}(K_-)$. Now assume $x \notin K_0$; then either $F(x) = 0$ or $F(x) \neq 0$ and either $\chi^-(x) = 0$ or $\chi^-(x) = 1$. Hence, $d_{\mathbf{M}}\chi^- F = 0$ and $\delta_{\mathbf{M}}\chi^- F = 0$ outside of K_0 and are thereby compactly supported in $K_0 \subseteq O$. It immediately follows that $\Box_{\mathbf{M}}\chi^- F$ is compactly supported in O too.

As $\chi^- F$ is compactly supported to the future and $\chi^+ F$ compactly supported to the past, $G^{\mathrm{adv}}_{\mathbf{M}} \Box_{\mathbf{M}} \chi^- F = \chi^- F$ and $G^{\mathrm{ret}}_{\mathbf{M}} \Box_{\mathbf{M}} \chi^+ F = \chi^+ F$. Since $\chi^+ = 1 - \chi^-$ and $\Box_{\mathbf{M}} F = 0$, adding these two equations yields $G_{\mathbf{M}} \Box_{\mathbf{M}} \chi^- F = F$. Defining $\chi := \chi^-$ concludes the proof of this lemma. ❒

Recall from Example 2.3.10 that for $\mathbf{M} \in \mathbf{Loc}$ and $O \in \mathrm{loc}_{-\mathbf{M}}$, $\mathcal{R}^{\mathrm{dyn}}(\mathbf{M}; O)$ is the (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic subspace of $\mathcal{R}\mathbf{M}$ which is generated by the union $\bigcup_{K \in \mathscr{K}(\mathbf{M};O)} \mathcal{R}^{\bullet}(\mathbf{M}; K)$.

LEMMA 5.5.4. *For all $O \in \mathrm{loc}_{-\mathbf{M}}$, we have $\mathcal{R}^{\mathrm{dyn}}(\mathbf{M}; O) \subseteq \mathcal{R}^{\mathrm{kin}}(\mathbf{M}; O)$.*

Proof: We start by showing that for each $K \in \mathscr{K}(\mathbf{M}; O)$, $[\![\omega]\!] \in \mathcal{R}^{\bullet}(\mathbf{M}; K)$ has a representative $\eta \in \Omega^2_0(M; \mathbb{K})$ with $\operatorname{supp}\eta \subseteq O$. By Lemma 5.5.1, we have $\operatorname{supp} d_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} \omega \subseteq J_{\mathbf{M}}(K)$ for any representative $\omega \in \Omega^2_0(M; \mathbb{K})$ of $[\![\omega]\!]$. Now, by definition of $\mathscr{K}(\mathbf{M}; O)$, K has a diamond neighbourhood $D_{\mathbf{M}}(B)$ based in a smooth spacelike Cauchy surfaces for $\mathbf{M}$ such that $B \subseteq O$. Note that $D_{\mathbf{M}}(B)$ might not be entirely contained in O. The Cauchy development, $D_O(B)$ is a globally hyperbolic open subset of O and $\mathbf{M}$, which is furthermore contractible. Because O is causally convex in $\mathbf{M}$, $D_O(B) = D_{\mathbf{M}}(B) \cap O$ and $K \subseteq D_O(B)$ thereby. We can now apply Lemma 5.5.3 to $D_O(B)$ and find that $F := d_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} \omega = G_{\mathbf{M}} \Box_{\mathbf{M}} \chi F = -G_{\mathbf{M}} \delta_{\mathbf{M}} d_{\mathbf{M}} \chi \mathrm{F} - G_{\mathbf{M}} d_{\mathbf{M}} \delta_{\mathbf{M}} \chi F$, where $d_{\mathbf{M}} \chi F \in \Omega^3_0(M; \mathbb{K})$ and $\delta_{\mathbf{M}} \chi F \in \Omega^1_0(M; \mathbb{K})$ are compactly supported in $D_O(B)$. Since $D_O(B)$ is contractible, there are $\eta_1, \eta_2 \in \Omega^2_0(M; \mathbb{K})$ such that $\operatorname{supp}\eta_1, \operatorname{supp}\eta_2 \subseteq D_O(B)$ and $d_{\mathbf{M}} \chi F = d_{\mathbf{M}} \eta_1$ and $\delta_{\mathbf{M}} \chi F = \delta_{\mathbf{M}} \eta_2$. Thus we find the identity $d_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} \omega = d_{\mathbf{M}} G_{\mathbf{M}} \delta_{\mathbf{M}} (\eta_1 - \eta_2)$, which shows $[\![\omega]\!] = [\![\eta_1 - \eta_2]\!]$. Accordingly, $\eta := \eta_1 - \eta_2 \in \Omega^2_0(M; \mathbb{K})$ is a representative of $[\![\omega]\!]$ which is compactly supported in O (because η is compactly supported in $D_O(B) \subseteq O$). ❒

The following statement is now immediate from Lemma 5.5.2 and Lemma 5.5.4:

THEOREM 5.5.5. *The classical reduced F-theory of the free Maxwell field is dynamically local.*

As a corollary to Theorem 5.5.5, we obtain:

THEOREM 5.5.6. *The quantised reduced F-theory of the free Maxwell field is dynamically local.*

Proof: Since $\mathfrak{R} : \mathbf{Loc} \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ is given by the composition of $\mathcal{R} : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}^{\mathbf{m}}$ with the quantisation functor $Q : \mathbf{pSympl}_{\mathbb{K}}^{\mathbf{m}} \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$, we need to check ($\mathscr{L}1$–$\mathscr{L}4$) as stated at the beginning of this section. ($\mathscr{L}1$) is obvious from (5.56), ($\mathscr{L}2$) follows from (5.46) and (5.47), ($\mathscr{L}3$) is clear by the definition of $\mathcal{R}$, see Definition 5.4.1, and ($\mathscr{L}4$) is proven by Lemma 5.5.1. Hence, [FV12b, Thm.5.3] applies and completes the proof of this theorem. ❒

Appendix: the reduced F-theory vs. colimits and left Kan extensions

Recall the functors $\mathcal{F}, \mathfrak{F} : \mathbf{Loc}_{\{2,m-2\}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ of Proposition 4.6.4, *let $q \subseteq \mathbb{N} \smallsetminus \{0\}$ such that $2, m-2 \in q$ and consider the restrictions $\mathcal{F}_q, \mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ and $\mathcal{F}_{\mathbf{M}}^q, \mathfrak{F}_{\mathbf{M}}^q : \mathrm{loc}_{-\mathbf{M}}^q \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ of $\mathcal{F}$ and $\mathfrak{F}$ to $\mathbf{Loc}_q$ and $\mathrm{loc}_{-\mathbf{M}}^q$, $\mathbf{M} \in \mathbf{Loc}$. For the existence of the colimits, hence for the computation of pointwise left Kan extensions thus the classical and the quantised universal F-theory $\mathcal{F}_u, \mathfrak{F}_u : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}, {}^*\mathbf{Alg}_{\mathbf{1}}$, we viewed $\mathcal{F}$ and $\mathfrak{F}$ as functors $\mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$ and $\mathbf{Loc} \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}$. In this appendix, we want to argue the necessity of this step and discuss the classical and the quantised reduced F-theory, $\mathcal{R}, \mathfrak{R} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$, in the context of colimits and left Kan extensions.*

First, note that (LKan) yields uniquely determined natural transformations

$$\rho : \mathcal{F}_u \stackrel{\cdot}{\longrightarrow} \mathcal{R} \qquad \text{and} \qquad \pi := Q \star \rho : \mathfrak{F}_u \stackrel{\cdot}{\longrightarrow} \mathfrak{R}, \tag{5.63}$$

whose components are all surjections. Q denotes the quantisation functor $\mathbf{Sympl}_{\mathbb{K}} \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ and $Q \star \rho$ is the natural transformation $\mathfrak{F}_u \stackrel{\cdot}{\longrightarrow} \mathfrak{R}$ with the components $\pi_{\mathbf{M}} := Q\rho_{\mathbf{M}}$ for $\mathbf{M} \in \mathbf{Loc}$. ρ is the natural transformation defined by

$$\rho_{\mathbf{M}} : \mathcal{F}_u\mathbf{M} \longrightarrow \mathcal{R}\mathbf{M}, \qquad [\omega] \longmapsto [\![\omega]\!], \qquad \mathbf{M} \in \mathbf{Loc}. \tag{5.64}$$

Since $\Omega^2_{0,d_{\mathbf{M}}}(M;\mathbb{K}) = d_{\mathbf{M}}\Omega^1_0(M;\mathbb{K})$ if $H^2_{\mathrm{dR,c}}(M;\mathbb{K}) \cong H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$ and $\Omega^2_{0,\delta_{\mathbf{M}}}(M;\mathbb{K}) = \delta_{\mathbf{M}}\Omega^3_0(M;\mathbb{K})$ if $H^{m-2}_{\mathrm{dR,c}}(M;\mathbb{K}) \cong H^2_{\mathrm{dR}}(M;\mathbb{K}) = 0$, the components $\rho_{\mathbf{M}}$ and $\pi_{\mathbf{M}}$ will be the identity map whenever $\mathbf{M} \in \mathbf{Loc}_{\{2,m-2\}}$. If $\mathbf{M} \in \mathbf{Loc}$ such that $H^2_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$ or

$H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$, then $\rho_{\mathbf{M}}$ and $\pi_{\mathbf{M}}$ are not injective, which implies that $\mathfrak{F}^{\mathrm{dyn}}_u(\mathbf{M};O)$ cannot be simple for $O \in \mathrm{loc}^{©}_{-\mathbf{M}}$ because the composition of unital *-homomorphisms $\mathfrak{F}^{\mathrm{dyn}}_u(\mathbf{M};O) \xrightarrow{\varphi^{\mathrm{dyn}}_{\mathbf{M};O}} \mathfrak{F}_u\mathbf{M} \xrightarrow{\pi_{\mathbf{M}}} \mathfrak{R}\mathbf{M}$ is not injective. We now have the following negative result on the existence of colimits for $\mathcal{F}^q_{\mathbf{M}}$ and $\mathfrak{F}^q_{\mathbf{M}}$ in $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$ and $\mathbf{{}^*Alg^m_1}$:

PROPOSITION 5.6.7. *Let* $\mathbf{M} \in \mathbf{Loc}$ *with* $H^2_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$ *or* $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$. *Then the colimits for* $\mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}, \mathbf{{}^*Alg^m_1}$ *do not exist.*

Proof: Let $\mathcal{C} = \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}, \mathbf{{}^*Alg^m_1}$, $\mathcal{D} = \mathbf{pSympl}_{\mathbb{K}}, \mathbf{{}^*Alg_1}$, $F = \mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}}$, $G = \mathcal{F}_u, \mathfrak{F}_u$, $v = [\iota_*], Q \star [\iota_*]$, $R = \mathcal{R}, \mathfrak{R}$ and suppose that the colimit for $F : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathcal{C}$ exists, say $\mathrm{colim}\, F = (\varinjlim F, u : F \dot{\longrightarrow} \Delta \varinjlim F)$. u can also be regarded as a cocone in $\mathcal{D}$ and Theorem 4.6.6 yields a unique $\mathcal{D}$-morphism $f : G\mathbf{M} \longrightarrow \varinjlim F$ such that $\Delta f \circ v = u$. On the other hand, since all components of v are injective, there is a unique $\mathcal{C}$-morphism $g : \varinjlim F \longrightarrow G\mathbf{M}$ satisfying $\Delta g \circ u = v$ due to (UColim). Since this results in $\Delta(g \circ f) \circ v = v$ and $\Delta(f \circ g) \circ u = u$, (UColim) allows us to conclude $g \circ f = \mathrm{id}_{G\mathbf{M}}$ and $f \circ g = \mathrm{id}_{\varinjlim F}$.

We now consider the cocone in $\mathcal{C}$ of Proposition 5.6.8, $\lambda : F \dot{\longrightarrow} \Delta R\mathbf{M}$, where $\lambda = [\![\iota_*]\!], Q \star [\![\iota_*]\!]$. (UColim) supplies us with a unique $\mathcal{C}$-morphism $\lambda_u : \varinjlim F \longrightarrow R\mathbf{M}$ satisfying $\Delta\lambda_u \circ u = \lambda$. On the other hand, λ is also a cocone in $\mathcal{D}$ and by Theorem 4.6.6, there exists a unique $\mathcal{D}$-morphism $\lambda'_v : G\mathbf{M} \longrightarrow R\mathbf{M}$ such that $\Delta\lambda'_v \circ v = \lambda$. Surely, $\lambda'_v = \rho_{\mathbf{M}}, \pi_{\mathbf{M}}$ and is thus not injective by assumption. However, owing to (UColim), $\lambda'_v = \lambda_u \circ f$ and $\lambda_u = \lambda'_v \circ g$ ↯. ❐

The non-existence of the colimits for $\mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}, \mathbf{{}^*Alg^m_1}$ does not rule out the existence of the left Kan extensions for $\mathcal{F}_q, \mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$, $\mathbf{{}^*Alg^m_1}$ along the inclusion functor $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$. Also, we did not exclude the existence of the colimit for $\mathcal{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$ in Proposition 5.6.7. Indeed, the proof given does not work in this case. Unfortunately, we will not be able to come to a definite conclusion regarding these problems in this thesis; we just do not know. Anyway, the content of the following proposition is that if we assume that the colimits for $\mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathcal{C}, \mathbf{{}^*Alg^m_1}$ exist, $\mathcal{C}$ stands for $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$ or $\mathbf{Sympl}_{\mathbb{K}}$, they turn out to be given by $\mathcal{R}\mathbf{M}$ and $\mathfrak{R}\mathbf{M}$, making (categorically) a good case for the reduced F-theory of the free Maxwell field:

PROPOSITION 5.6.8. *Assuming that the colimits for* $\mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathcal{C}, \mathbf{{}^*Alg^m_1}$ *exist, where* $\mathcal{C}$ *stands for* $\mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$ *or* $\mathbf{Sympl}_{\mathbb{K}}$, *we have*

$$\mathrm{colim}\, \mathcal{F}^q_{\mathbf{M}} = (\mathcal{R}\mathbf{M}, [\![\iota_*]\!] : \mathcal{F}^q_{\mathbf{M}} \dot{\longrightarrow} \Delta\mathcal{R}\mathbf{M}) \tag{5.65}$$

and

$$\operatorname{colim}\mathfrak{F}^q_{\mathbf{M}} = (\mathfrak{R}\mathbf{M},\, Q \star [\![\iota_*]\!] : \mathfrak{F}^q_{\mathbf{M}} \dot{\longrightarrow} \Delta\mathfrak{R}\mathbf{M}), \tag{5.66}$$

where $[\![\iota_*]\!]$ *is defined by* $[\![\iota_*]\!]_O := [\![\iota_{O*}]\!]$ *for all* $O \in \mathrm{loc}^q_{-\mathbf{M}}$.

Proof: Let $\mathcal{D} = \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}, \mathbf{Sympl}_{\mathbb{K}}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$, $F = \mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}}$ and $v = [\![\iota_*]\!], Q\star[\![\iota_*]\!]$. Now, suppose that $\operatorname{colim} F = (\varinjlim F, u : F \dot{\longrightarrow} \Delta \varinjlim F)$ exists; then (UColim) yields a unique $\mathcal{D}$-morphism $f : \varinjlim F \longrightarrow R\mathbf{M}$ such that $\Delta f \circ u = v$. On the other hand, Theorem 4.6.6 yields two unique $\mathcal{D}$-morphisms $g, h : \mathcal{F}_u\mathbf{M} \longrightarrow \varinjlim F, R\mathbf{M}$ satisfying $\Delta g \circ [\iota_*] = u$ and $\Delta h \circ [\iota_*] = v$. Since $\Delta (f \circ g) \circ [\iota_*] = v$, (UColim) implies $h = f \circ g$. Of course, $h = \rho_{\mathbf{M}}, \pi_{\mathbf{M}}$ and is thus surjective, which implies that f is surjective [Bou68, II, §3, no.8, Thm.1(d)]. However, f is also injective and constitutes thereby a $\mathcal{D}$-isomorphism. We claim that f^{-1} is the unique $\mathcal{D}$-morphism with the property $\Delta f^{-1} \circ v = u$. Suppose $k : R\mathbf{M} \longrightarrow \varinjlim F$ was another one; then $\Delta (k \circ f) \circ u = u$ and (UColim) determines $k \circ f = \mathrm{id}_{\varinjlim F}$. As f is a $\mathcal{D}$-isomorphism, we get $k = f^{-1}$. All in all, (5.65) and (5.66) follow. ❐

This and the next proposition make a good case for the classical and the quantised reduced F-theory as distinguished extensions of $\mathcal{F}_q, \mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathcal{C}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ ($\mathcal{C} = \mathbf{Sympl}_{\mathbb{K}}, \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$).

PROPOSITION 5.6.9. *Let* $\mathcal{C} = \mathbf{Sympl}_{\mathbb{K}}, \mathbf{pSympl}^{\mathbf{m}}_{\mathbb{K}}$. *Under the assumption that the left Kan extensions for* $\mathcal{F}_q, \mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathcal{C}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ *along the inclusion functor* $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$ *exist, we have*

$$(\mathrm{Lan}_{K_q} \mathcal{F}_q, u : \mathcal{F}_q \dot{\longrightarrow} \mathrm{Lan}_{K_q} \mathcal{F}_q \circ K_q) = (\mathcal{R}, \mathrm{id} : \mathcal{F}_q \dot{\longrightarrow} \mathcal{R} \circ K_q) \tag{5.67}$$

and

$$(\mathrm{Lan}_{K_q} \mathfrak{F}_q, u : \mathfrak{F}_q \dot{\longrightarrow} \mathrm{Lan}_{K_q} \mathfrak{F}_q \circ K_q) = (\mathfrak{R}, \mathrm{id} : \mathfrak{F}_q \dot{\longrightarrow} \mathfrak{R} \circ K_q). \tag{5.68}$$

Proof: Let $F = \mathcal{F}_q, \mathfrak{F}_q$, $R = \mathcal{R}, \mathfrak{R}$, $v = \rho, \pi$ and suppose that $\mathrm{Lan}_{K_q} F$ exists; by (LKan), we obtain a unique natural transformation $\sigma : \mathrm{Lan}_{K_q} F \dot{\longrightarrow} R$ such that $(\sigma \star K_q) \circ u = \mathrm{id}$. Owing to Theorem 4.6.7, we get two unique natural transformations $\tau, \upsilon : F \dot{\longrightarrow} \mathrm{Lan}_{K_q} F, R$ satisfying $(\tau \star K_q) \circ \mathrm{id} = u$ and $(\upsilon \star K_q) \circ \mathrm{id} = \mathrm{id}$. Also by (LKan), $\upsilon = \sigma \circ \tau$, which implies that all components of σ are surjective. Since all components of σ are already injective, we can invert σ and claim that σ^{-1} is the unique natural transformation $R \dot{\longrightarrow} \mathrm{Lan}_{K_q} F$ with the property $(\sigma^{-1} \star K_q) \circ \mathrm{id} = u$. Suppose $\eta : R \dot{\longrightarrow} \mathrm{Lan}_{K_q} F$ was another one; then $((\eta \circ \sigma) \star K) \circ u = u$. Consequently, $\eta \circ \sigma = \mathrm{id} : \mathrm{Lan}_{K_q} F \dot{\longrightarrow} \mathrm{Lan}_{K_q} F$ and thus $\eta = \sigma^{-1}$. As a result, (5.67) and (5.68). ❐

Chapter 6

C.J. Isham's Twisted Quantum Fields

"*It's alright, there's a change in the story*
It's alright, there's a change in the plan
A twist in the tale"

–Deep Purple, "A Twist In The Tale", *The Battle Rages On...*, 1993.

Consider $\mathbf{M} \in \mathbf{Loc}$, where we assume for simplicity that the fixed spacetime dimension is $= 2$ and $M \cong \mathbb{R} \times S^1$. Then the common answer to the question of what is meant by the classical free real scalar field on $\mathbf{M}$ is usually a smooth function $\phi \in \mathcal{C}^\infty M$ such that the homogeneous Klein-Gordon equation (4.75), $D\phi = 0$, is satisfied[1]. However, this is not the only way to think about it. The smooth function ϕ assigns to each point $x \in M$ a real value $\phi(x) \in \mathbb{R}$, hence we can attach the real line $\mathbb{R}$ to each point $x \in M$, view the graph $\{(x, \phi(x)) \in M \times \mathbb{R} \mid x \in M\}$ of ϕ and say that ϕ takes the value $\phi(x)$ in the real line over the point $x \in M$. Now we are really considering ϕ as a smooth cross-section in the trivial smooth real vector bundle over M of rank 1, $\underline{\mathbb{R}}_M = (M \times \mathbb{R}, M, \mathrm{pr}_1, \mathbb{R})$. Of course, ϕ still satisfies the homogeneous Klein-Gordon equation, i.e. $D\phi(x) = (x, D\phi(x)) = (x, 0)$ for all $x \in M$.

This is not our only option. We can also "*twist*" the copies of $\mathbb{R}$ attached to each line $\mathbb{R} \times \{e^{i\vartheta}\} \subseteq M$, $\vartheta \in [0, 2\pi)$, and view ϕ as a smooth function on M taking its values in the product $\mathbb{R} \times N$ with the Möbius strip N. Hence, we could view ϕ as a smooth cross-section σ in the *non-trivial* smooth real vector bundle $\xi = (\mathbb{R} \times N, M, \pi, \mathbb{R})$.

The structure group of ξ is $\mathrm{GL}(1;\mathbb{R})$ and due to the method of reduction (see [Ste51, §9.4], [Hus94, Chap.6], [Bau09, Sec.2.5]), the structure group of ξ may taken to be $\mathrm{O}(1) = \mathbb{Z}_2 = \{1, -1\}$ without the loss of generality. It is enough to cover $\mathbb{R} \times S^1$ with two open subsets $U := \mathbb{R} \times I$ and $V := \mathbb{R} \times J$, where I and J are two proper open intervals of S^1, over which there are local trivialisations $\theta_U : \xi|_U \xrightarrow{\sim!} \underline{\mathbb{R}}_U$ and $\theta_V : \xi|_V \xrightarrow{\sim!} \underline{\mathbb{R}}_V$, and which intersect in two disjoint open subsets $\mathbb{R} \times K$ and $\mathbb{R} \times L$. As smooth transition

[1] Note that C.J. Isham motivates twisted fields in [Ish78b; AI79b] considering the free real scalar field on smooth spacelike Cauchy surfaces, probably in view of the canonical quantisation method which he used.

function $g_{UV} : U \cap V \longrightarrow \mathbb{Z}_2$, we may take $g_{UV}(x) = 1$ for $x \in \mathbb{R} \times K$ and $g_{UV}(x) = -1$ for $x \in \mathbb{R} \times L$. Observe that this yields a non-trivial smooth $\mathbb{Z}_2$-cocycle.

Viewing ϕ as a smooth cross-section σ in ξ, we have that $\sigma_U = \theta_U^{\#}\phi_U$, $\sigma_V = \theta_V^{\#}\phi_V$, where $f_W := f|_W$ for $f = \sigma, \phi$ and $W = U, V$; for the transition of smooth vector bundle charts, we hence find $\sigma_U|_{U\cap V}(x) = \sigma_V|_{U\cap V}(x)$ for all $x \in \mathbb{R} \times K$ and $\sigma_U|_{U\cap V}(x) = -\sigma_V|_{U\cap V}(x)$ for all $x \in \mathbb{R} \times L$, compared to $\phi_U|_{U\cap V}(x) = \phi_V|_{U\cap V}(x)$ for all $x \in U \cap V$ for the untwisted field. We see that the description of the classical free real scalar field in terms of the non-trivial smooth vector bundle ξ is locally equivalent to description using the trivial smooth real vector bundle $\underline{\mathbb{R}}_M$ but globally inequivalent. Still, the smooth cross-section σ satisfies a (global) "*twisted*" version of the homogeneous Klein-Gordon equation: $\widetilde{D}\sigma = 0$, where $\theta_{U*}\widetilde{D}|_U\sigma_U = D|_U\phi_U$ and $\theta_{V*}\widetilde{D}|_V\sigma_V = D|_V\phi_V$.

The situation which we have just described is genuinely what C.J. Isham [Ish78b; AI79b] refers to as twisted fields, that is, smooth cross-sections in non-trivial smooth vector bundles which locally satisfiy the standard field equations but are different globally due to smooth transition functions which form a non-trivial smooth cocycle. Twisted quantum fields are then obtained by smearing the quantum field, which is an operator-valued distribution, with smooth cross-sections in the non-trivial smooth vector bundle. Hence, twisted quantum fields are obtained by using smooth cross-sections in non-trivial smooth vector bundles in the quantisation description of classical field theories.

Accordingly, by proceeding in the same way as in Section 4.5, where we have discussed the free real Klein-Gordon field (set $p = 0$ in that section), but using smooth cross-sections in the non-trivial smooth vector bundle ξ, we obtain the classical and the twisted quantum field theory in the sense of C.J. Isham for the example above: the twisted classical field theory is given in terms of the symplectic space

$$(6.1)\quad \begin{cases} [\Gamma_0^\infty(\xi)] := \Gamma_0^\infty(\xi)/\widetilde{D}\Gamma_0^\infty(\xi), \\ \widetilde{\mathfrak{u}} : [\Gamma_0^\infty(\xi)] \times [\Gamma_0^\infty(\xi)] \longrightarrow \mathbb{R}, \ ([\sigma],[\sigma']) \longmapsto \langle \sigma \mid \widetilde{G}\sigma' \rangle_{2,\xi} = \int_M \langle \sigma \mid \widetilde{G}\sigma' \rangle_\xi \,\mathrm{vol}_{\mathbf{M}}, \end{cases}$$

where $\widetilde{G}$ is the advanced-minus-retarded Green operator for $\widetilde{D}$ and $\langle \cdot \mid \cdot \rangle_\xi \in \Gamma^\infty(\xi^* \odot \xi^*)$ is the smooth bundle metric which is locally given by $\langle \sigma(x) \mid \sigma'(x) \rangle_{E_x} = \phi(x)\phi'(x)$ for all $x \in M$, where σ (resp. σ') is $\phi \in \mathcal{C}^\infty M$ (resp. $\phi' \in \mathcal{C}^\infty M$) viewed as a smooth cross-section in ξ. The twisted quantum field theory is hence given by the unital *-algebra which is generated by the elements of the form $\widetilde{\mathbf{\Phi}}(\sigma)$, $\sigma \in \Gamma_0^\infty(\xi)$, satisfying the following relations:

- Linearity: $\widetilde{\mathbf{\Phi}}(\lambda\sigma + \mu\tau) = \lambda\widetilde{\mathbf{\Phi}}(\sigma) + \mu\widetilde{\mathbf{\Phi}}(\tau)$ for all $\lambda, \mu \in \mathbb{R}$ and for all $\sigma, \tau \in \Gamma_0^\infty(\xi)$.

- Hermiticity: $\widetilde{\mathbf{\Phi}}(\sigma)^* = \widetilde{\mathbf{\Phi}}(\sigma)$ for all $\sigma \in \Gamma_0^\infty(\xi)$.

- Field equations (in a weak sense): $\widetilde{\mathbf{\Phi}}\left(\widetilde{D}\sigma\right) = 0$ for all $\sigma \in \Gamma_0^\infty(\xi)$.
- Commutation relations: $\left[\widetilde{\mathbf{\Phi}}(\sigma), \widetilde{\mathbf{\Phi}}(\tau)\right] = \mathrm{i}\,\hbar\,\langle\sigma \mid \widetilde{G}\tau\rangle_{2,\xi} \cdot 1_A$ for all $\sigma, \tau \in \Gamma_0^\infty(\xi)$.

Having clarified what twisted (quantum) fields in the sense of C.J. Isham are and how they come about[2], it is clear that they will not exist for curved spacetimes which do not allow non-trivial smooth vector bundles. Since the existence of non-trivial smooth vector bundles is tied to the topology of the curved spacetime considered, twisted quantum fields allow us to probe aspects of the role played by the spacetime topology. They also provide us with new field configurations, which are locally equivalent but globally inequivalent to the standard ones, and yield new toy models for quantum field theory in curved spacetimes in this way. It is argued that it is important to consider these new field configurations, in the same way that one needs to take into account inequivalent spin connections in the path integral approach to the quantum spinor field [AI79a; BD99].

Twisted (quantum) fields have numerous interesting properties, which have been established for some concrete examples very satisfactorily. Most noteworthy, twisted quantum fields have different renormalised vacuum expectation values for the energy density on ultrastatic spacetimes [Ish78b; DHI79; BD79b; BD99]. This difference is sometimes very striking with a change of the sign. Other properties, which have been verified in concrete examples, are the validity of the spin-statistics theorem and a change or even a complete suppression of spontaneous symmetry breaking [Ish78b; AI79c]: $\phi \longmapsto -\phi$ is a symmetry transformation for the Lagrange function $L = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}\mu^2\left(\phi^2 - a^2\right)^2$, $a \neq 0$, which allows for solutions with $\phi^2 = a^2$, in particular $\phi = \pm a$. These solutions, however, cannot be smooth cross-sections in a non-trivial smooth vector bundle, i.e. twisted fields, because smooth cross-sections in a non-trivial smooth vector bundle must be zero in some fibre. This observation is of particular interest to the issue of dynamical locality for the massless and minimally coupled free real scalar field. As it was established in [FV12b], dynamical locality does not hold precisely because of constant solutions. So, one might wonder if the twisted varaints of the massless and minimally coupled free real scalar fields can be made dynamically local in some sense. We come back to this in Section 6.9.

In this chapter, we outline a categorical framework for understanding C.J. Isham's twisted quantum fields from the perspective of algebraic and locally covariant quantum field theory. This categorical framework is completely abstract and general, and not limited to the traditional understanding of twisted (quantum) fields. It will quite

[2]Despite the danger of overly repeating ourselves: twisted fields arise from smooth cross-sections in non-trivial smooth vector bundles which locally satisfy the standard field equations; twisted quantum fields are obtained by using smooth cross-sections in non-trivial smooth vector bundles in the quantum description.

generally allow us to talk about twisted variants of generic locally covariant theories (though considered on single curved spacetimes), not referring to (quantum) fields at all.

We begin with a preliminary discussion of C.J. Isham's twisted quantum fields from the point of view of algebraic quantum field theory in Section 6.1. In that discussion, where we argue that a twisted quantum field theory is a controlled change of the net of the local (unital) $(C)^*$-algebras, we provide some motivation for the general scheme and its ingredients, emphasising on the decisive role of the global gauge group and the observables. In Section 6.2, we state the abstract categorical scheme for twisted variants of locally covariant theories and explain it in detail. Key to this general description is the idea of C.J. Fewster [Few13] that the automorphisms of a locally covariant theory or a suitable subcollection thereof may function as the global gauge group, thus allowing us to identify the observables and to gain control over possible twists. Since twisted (quantum) fields arise in C.J. Isham's description from non-trivial fibre bundles with a structure group, it should not come as a surprise that the general scheme is reminiscent of the definition of fibre bundles with a structure group as found in [Ste51, §2]. Focusing on a local description of the abstract categorical scheme via transition functions, we investigate cohomological aspects, equivalence and existence of twisted variants in Section 6.3. Essentially, we recover analogues of the familiar statements for fibre bundles with a structure group (e.g. [Ste51, §§2-3]).

Adopting the general scheme to the context of generic locally covariant theories is the subject matter of Section 6.4. That section culminates in the classification theorem (Theorem 6.4.2) and in the construction theorem (Theorem 6.4.3). We find that we are naturally led to a classification of twisted variants for locally covariant theories by the isomorphism classes of *flat* smooth principal bundles. This is a relevant difference to C.J. Isham's original classification via isomorphism classes of smooth principal bundles since more and even entirely new twisted quantum field theories can arise, not appearing in C.J. Isham's classification. We provide some examples for our classification at the end of Section 6.4, which are all motivated from twisted (quantum) field theories for further studies, e.g. $\mathrm{O}(n)$- and shift-twisted free and minimally coupled real scalar fields, the $\mathrm{SL}(2;\mathbb{C})$-twisted free Dirac field and the $\mathrm{U}(1)$-twisted free Maxwell field. As the main illustration for the abstract theory, we present in detail the example of $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields in the course of Sections 6.5-6.9.

Section 6.5 reviews the theory of multiple free and non-minimally coupled real scalar fields of the same mass, and we identify the structures relevant to the discussion of twisted variants in Section 6.6. In Section 6.7, we concretely construct the theory of $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields in the spirit of [Ish78b;

AI79b]. We further show that this concrete construction fits in the general framework. In doing so, we also lay the basis for performing colimit constructions and left Kan extensions in Section 6.9, where we discuss further properties of the $\mathrm{O}(n)$-twisted variants. Since a non-trivial consistent twisting is essentially tied to the specific smooth manifold which underlies the curved spacetime, the $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields cannot constitute a locally covariant (quantum field) theory. Nevertheless, we are still able to compute relative Cauchy evolutions, classical stress-energy-momentum tensors and to address the dynamical net in a sensible way in Section 6.9. However, regarding dynamical locality, we do not come to a conclusion in this thesis for lack of time. Lastly, in the appendix to this chapter, we collect some notions from the theory of smooth vector bundles and smooth principal bundles used throughout this chapter.

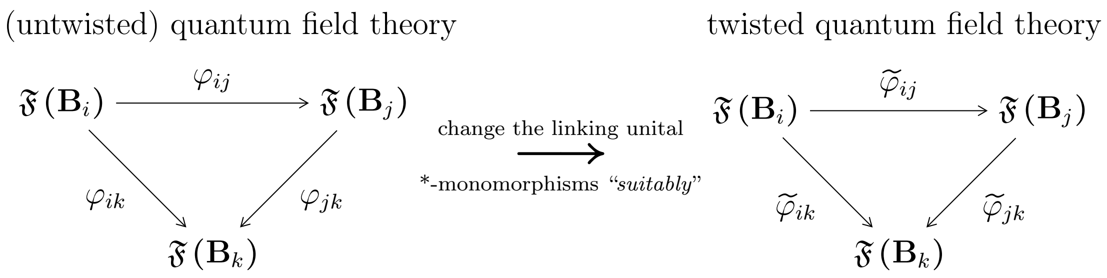
6.1 Fields, observables and gauge transformations

As a preliminary discussion and motivation for our general abstract scheme, we try to understand C.J. Isham's twisted quantum fields [Ish78b; AI79b] *from the point of view of algebraic quantum field theory. Our assumptions are based on the general properties exhibited by the examples in the literature and, in particular, by the example provided in the introduction to this chapter. Furthermore, the discussion is also influenced by the example of* $\mathrm{O}(n)$*-twisted free and minimally coupled real scalar fields of the same mass* (Sections 6.5-6.9). *Recall that classical twisted fields in the sense of C.J. Isham are cross-sections in a non-trivial smooth vector bundle which locally satisfy the standard field equations. Similarly, twisted quantum fields are obtained by using smooth cross-sections in non-trivial smooth vector bundles in the quantum description of the standard field theory.*

Let us assume that we are given a twisted quantum field theory on some $\mathbf{M} \in \mathbf{Loc}$ in the form of a unital *-algebra $\widetilde{\mathfrak{F}}(\mathbf{M})$ which results from considering smooth cross-sections in a non-trivial smooth vector bundle $\xi = (E, M, \pi, V)$, just like in the introduction to this chapter. Of course, the topology of M is such that non-trivial smooth vector bundles over M with typical fibre V exist and there is a corresponding standard quantum field theory in terms of a unital *-algebra $\mathfrak{F}(\mathbf{M})$, which is derived from smooth cross-sections in the trivial smooth vector bundle $\underline{V}_M$. We also assume that the standard and the twisted quantum field theory considered can be localised, that is, for a suitable choice of spacetime regions for $\mathbf{M}$, we have nets of local unital *-algebras $\mathbf{B} \longmapsto \mathfrak{F}(\mathbf{B})$ and $\mathbf{B} \longmapsto \widetilde{\mathfrak{F}}(\mathbf{B})$, which can be consistently included into $\mathfrak{F}(\mathbf{M})$ and $\widetilde{\mathfrak{F}}(\mathbf{M})$, respectively. For the clarity of the underlying idea, we view the inclusions

$\mathfrak{F}(\mathbf{B}_i) \subseteq \mathfrak{F}(\mathbf{B}_j)$ and $\widetilde{\mathfrak{F}}(\mathbf{B}_i) \subseteq \widetilde{\mathfrak{F}}(\mathbf{B}_j)$ for spacetime regions $\mathbf{B}_i \subseteq \mathbf{B}_j$ as general unital *-monomorphisms $\varphi_{ij} := \mathfrak{F}\mu_{ij} : \mathfrak{F}(\mathbf{B}_i) \longrightarrow \mathfrak{F}(\mathbf{B}_j)$ and $\widetilde{\varphi}_{ij} := \widetilde{\mathfrak{F}}\mu_{ij} : \widetilde{\mathfrak{F}}(\mathbf{B}_i) \longrightarrow \widetilde{\mathfrak{F}}(\mathbf{B}_j)$. Because ξ is locally trivial, it is locally isomorphic (as a smooth vector bundle) to $\underline{V}_M$. This should be reflected in the quantum field theory, that is, assuming that our spacetime regions have been chosen even more suitably, $\widetilde{\mathfrak{F}}(\mathbf{B}) \cong \mathfrak{F}(\mathbf{B})$ for all spacetime regions $\mathbf{B}$.

Taking the point of view of algebraic quantum field theory, the significant difference between the standard and the twisted quantum field theory cannot lie in the individual algebras, which are isomorphic. The twist must occur in the local unital *-monomorphisms which link the local unital *-algebras! We therefore redefine $\widetilde{\mathfrak{F}}(\mathbf{B}) = \mathfrak{F}(\mathbf{B})$ for all spacetime regions $\mathbf{B}$ and summarise diagrammatically:



Note, we could have also started with the twisted quantum field theory and changed its linking unital *-monomorphisms to obtain the untwisted standard quantum field theory in this scheme.

The two fundamental questions to ask at this point concern the existence and the classification of twisted variants: how do we change the linking unital *-monomorphisms suitably (existence) and how do we know that we truly have obtained a twisted variant (classification)? We obtain possible answers to these two questions by entering the realm of locally covariant quantum field theory and utilising ideas stemming from C.J. Fewster's original motivation to look into twisted quantum fields; namely if twisted quantum fields are at all related to the topological superselection sectors discovered in [BR09]. We will not establish a connection in this thesis though.

In the famous Doplicher-Haag-Roberts analysis of superselection sectors [DHR69a; DHR69b; DHR71; DHR74; DR90], whose methods were carried over to curved spacetimes by [BR07; BR09], the "*observable algebra*" $\mathfrak{A}(\mathbf{B})$ is distinguished inside the "*field algebra*" $\mathfrak{F}(\mathbf{B})$ as the fixed points under the global gauge group G, i.e. the group of gauge transformations of the first kind, which acts via unital *-automorphisms of the global field algebra $\mathfrak{F}(\mathbf{M})$ in a local fashion: $g(\mathfrak{F}(\mathbf{B})) = \mathfrak{F}(\mathbf{B})$ for all spacetime regions $\mathbf{B}$, where $g \in G$. Precisely the strong restrictions of these unital *-automorphisms are allowed to be used in changing the linking unital *-monomorphisms, that is, $\{g\|_{\mathbf{B}} : \mathfrak{F}(\mathbf{B}) \xrightarrow{\sim} \mathfrak{F}(\mathbf{B}) \mid \mathbf{B} \text{ spacetime region}, g \in G\}$. Hence a quantum field theory and a twisted variant of it have the same observable content. We thus deliberately impose the observables as an additional structure which should be preserved by the suitable change of the linking unital *-monomorphisms. In this way, we also prevent too arbitrary notions of twisted variants.

As in Section 4.1, we now view the nets of local unital *-algebras $\mathbf{B} \longmapsto \mathfrak{F}(\mathbf{B})$ and $\mathbf{B} \longmapsto \widetilde{\mathfrak{F}}(\mathbf{B})$ as functors $\mathfrak{F}, \widetilde{\mathfrak{F}} : \mathscr{J}_{\mathbf{M}} \longrightarrow \mathbf{*Alg_1^m}$; the category $\mathscr{J}_{\mathbf{M}}$ is thereby formed from the spacetime regions $\mathbf{B}$ and their inclusions into each other. Recall that we have redefined $\widetilde{\mathfrak{F}}(\mathbf{B}) := \mathfrak{F}(\mathbf{B})$ for all spacetime regions $\mathbf{B} \in \mathscr{J}_{\mathbf{M}}$. Having identified the observables $\mathfrak{A}(\mathbf{B}) \subseteq \mathfrak{F}(\mathbf{B})$ (by what we have said before, $\mathfrak{A}(\mathbf{B}) \subseteq \widetilde{\mathfrak{F}}(\mathbf{B})$ too) for all spacetime regions $\mathbf{B} \in \mathscr{J}_{\mathbf{M}}$ using the global gauge group G, we say that $\widetilde{\mathfrak{F}}$ is truly a twisted variant of $\mathfrak{F}$ if and only if there is no natural isomorphism $\eta : \widetilde{\mathfrak{F}} \xrightarrow{\dot{\sim}} \mathfrak{F}$ which leaves the observables fixed, i.e. such that $\eta_{\mathbf{B}}\|_{\mathfrak{A}(\mathbf{B})} = \mathrm{id}_{\mathfrak{A}(\mathbf{B})}$ for all $\mathbf{B} \in \mathscr{J}_{\mathbf{M}}$.

To get hold of a global gauge group G, we will considerably benefit from [Few13], which discusses the role of the automorphisms of a locally covariant theory and asserts that the automorphisms themselves or a suitable subcollection thereof may function as the global gauge group of the theory. We will continue this thought in Section 6.2 and Section 6.4.

The points raised in this preliminary discussion will be precisely captured by our abstract categorical framework for twisted variants.

6.2 The general scheme

We define a (U,G)-functor and discuss its definition in detail.

DEFINITION 6.2.1. Let $U : \mathcal{M} \longrightarrow \mathcal{C}$ be a functor and G the collection of automorphisms of U, $\mathrm{Aut}\, U$, or a suitable subcollection thereof such that G acts faithfully on U, i.e. $G \ni g \longmapsto g_A \in \mathrm{Aut}(UA)$ is injective for each $\mathcal{M}$-object A. We will refer to U as the *prototype functor* of which we want to consider twisted variants and to G as the *global gauge group*. A *(U,G)-functor* is a pair consisting of a functor $T : \mathcal{M} \longrightarrow \mathcal{C}$ and a rule $\mathfrak{G}$ that assigns to each $\mathcal{M}$-morphism $f : A \longrightarrow B$ a $\mathcal{C}$-isomorphism $\mathfrak{G}f : TA \longrightarrow UA$, such that the following conditions are
met:

(1) for each non-empty hom-set $\mathcal{M}(A,B)$, there are $g(A;B), g(B;A) \in G$ such that for every composition of $\mathcal{M}$-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$,

$$g(B;C)_A = \mathfrak{G}(g \circ f) \circ (\mathfrak{G}f)^{-1}$$
$$g(C;B)_A = g(B;C)_A^{-1}; \tag{6.2}$$

(2) for every composition $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathcal{M}$-morphisms,

$$\mathfrak{G}g \circ Tf = Uf \circ \mathfrak{G}(g \circ f)\,; \tag{6.3}$$

(3) for all $\mathcal{M}$-objects A,

$$TA = UA. \tag{6.4}$$

There is always at least one (U,G)-functor, namely the trivial one (U,id), where $\mathrm{id}\, f := \mathrm{id}_{UA}$ for all $f \in \mathcal{M}(A,B)$ and for all $A, B \in \mathcal{M}$. We make no claims that there always exists a non-trivial (U,G)-functor.

Please take notice that this is not the general scheme originally suggested by C.J. Fewster, who assigns the elements of the global gauge group differently. In analogy to the local description of fibre bundles with a structure group, using the automorphisms of a functor or a suitable subcollection thereof as the global gauge group allows us to assign only one group element with each intersection of open subsets of the base space. In order to deal with multiple connected intersections, that is, to avoid assigning only one group element with a multiple connected intersection of open subsets of the space base, C.J. Fewster introduces the notion of *common wedges*. A non-empty ordered collection of (not necessarily distinct) $\mathcal{M}$-objects $(A_1, \ldots, A_n)$, $n \in \mathbb{N} \setminus \{0\}$, is said to have a *common wedge* if and only if there are $\mathcal{M}$-morphisms $f_i : B \longrightarrow A_i$, $i = 1, \ldots, n$, with a common domain; we denote this by $B \xrightarrow{f_1,\ldots,f_n} A_1, \ldots, A_n$. A non-empty ordered collection of (not necessarily distinct) $\mathcal{M}$-objects $(A_1, \ldots, A_n)$, $n \in \mathbb{N} \setminus \{0\}$, is called *admissible* if and only if each non-empty subcollection of $(A_1, \ldots, A_n)$ admits a product[3]. Under some additional assumptions on the category $\mathcal{M}$, C.J. Fewster assigns to each admissible pair of $\mathcal{M}$-objects (B, C) an element $g(B\,; C) \in G$ such that for every common wedge $A \xrightarrow{f,g} B, C$, it holds that $g(B\,;C)_A = \mathfrak{G}g \circ (\mathfrak{G}f)^{-1}$.

REMARK 6.2.2. (i) As mentioned in the introduction, the definition of a (U,G)-functor is reminiscent of the definition and properties of (locally constant resp. flat) fibre bundles with a structure group. In this spirit, we supply a guideline to this analogy, of which we hope it will be helpful to the reader. However, the reader is well-advised not to take the analogy to literally since we can consider categories $\mathcal{M}$ for

[3] A *product* of a set-labelled family of objects $\{X_i\}_{i\in I}$ in a category is the dual notion of the coproduct (Definition 2.2.4), i.e. an object X and a family of morphisms $\{\mathrm{pr}_i : X \longrightarrow X_i\}_{i\in I}$ such that the universal property (U$\prod$) holds: for each object Y and family of morphisms $\{f_i : Y \longrightarrow X_i\}_{i\in I}$, there is one and only one morphism $f : Y \longrightarrow X$ such that $\mathrm{pr}_i \circ f = f_i$ for all $i \in I$.

which the analogy completely breaks down:

$$\begin{aligned}
\mathcal{M} &\triangleq \text{“}\textit{base space}\text{”},\\
A \in \mathcal{M} &\triangleq \text{“}\textit{simply connected open subset of } \mathcal{M}\text{”},\\
G &\triangleq \text{“}\textit{structure group}\text{”},\\
U &\triangleq \text{“}\textit{trivial fibre bundle over } \mathcal{M}\text{”},\\
(U, \mathrm{id}) &\triangleq \text{“}\textit{trivial } G\textit{-fibre bundle over } \mathcal{M}\text{”},\\
T &\triangleq \text{“}\textit{fibre bundle over } \mathcal{M}\text{”},
\end{aligned}$$

for $A \in \mathcal{M}$,

$$\begin{matrix} UA \\ TA \end{matrix} \triangleq \text{“}\textit{restriction of the total space of } \begin{matrix} U \\ T \end{matrix} \textit{ to } A\text{”},$$

for $f \in \mathcal{M}(A, B)$,

$$\mathfrak{G}f \triangleq \text{“}\textit{local trivialisation for } T \textit{ over } A\text{”},$$

for $A, B \in \mathcal{M}$,

$$\mathcal{M}(A, B) \neq \varnothing \triangleq \text{“}A \subseteq B\text{”},$$

for any non-empty hom-set $\mathcal{M}(A, B)$,

$$\begin{matrix} g(A;B) \\ g(B;A) \end{matrix} \triangleq \text{“}\textit{transition function from } \begin{matrix} A \\ B \end{matrix} \textit{ to } \begin{matrix} B \\ A \end{matrix}\text{”},$$

for any non-empty hom-sets $\mathcal{M}(A, B)$ and $\mathcal{M}(B, C)$,

$$\begin{matrix} g(B;C)_A \\ g(C;B)_A \end{matrix} \triangleq \text{“}\textit{restriction of } \begin{matrix} g(B;C) \\ g(C;B) \end{matrix} \textit{ to } A\text{”}$$

and

$$\begin{aligned}
\mathfrak{G} &\triangleq \text{“}\textit{locally constant } G\textit{-fibre bundle atlas for } T\text{”},\\
(T, \mathfrak{G}) &\triangleq \text{“}\textit{locally constant (resp. flat) } G\textit{-fibre bundle over } \mathcal{M}\text{”}.
\end{aligned}$$

(ii) In abstract category theory, the objects of a category do not possess any internal structure. Hence, we can only assign one element of G with each transition of local trivialisations for T. As a result, we have “*transition constants*” rather than transition functions. This is the locally constant (resp. flat) aspect of our set-up.

(iii) Because of (ii) and the classification of flat smooth principal bundles ([Mor01a, Thm.2.9] and [Mor01b, Thm.6.60]; cf. also [Ste51, Thm.13.9]), we think of $\mathcal{M}$-objects as simply connected open subsets of the base space $\mathcal{M}$.

(iv) Pay attention to the fact that we associate the local trivialisations for T with the $\mathcal{M}$-morphisms and not with the $\mathcal{M}$-objects, which is a deviation from the usual description of fibre bundles. However, this is exactly the right thing to do in the abstract categorical setting. In order to specify transitions of local trivialisations for T and their associated transitions functions, which are essential to the description of fibre bundles, we need to say when and how two $\mathcal{M}$-objects intersect and what restrictions to these intersections are. This is precisely implemented by attaching the local trivialisations for T to the $\mathcal{M}$-morphisms: since $\mathcal{M}(A,B) \neq \varnothing$ is understood to mean "$A \subseteq B$", we regard for any $\mathcal{M}$-morphism $f : A \longrightarrow B$, $\mathfrak{G}f$ as a local trivialisation for T over A which has been obtained by restriction of a local trivialisation for T over B. Trying to trivialise over $\mathcal{M}$-objects ($\mathfrak{G}$ assigns to each $\mathcal{M}$-object A a $\mathcal{C}$-isomorphism $\mathfrak{G}A : TA \longrightarrow UA$), we would not have been able to sensibly define transitions of local trivialisations for T without further thought.

(v) The requirement of a faithful G-action on U is essential for imposing a G-structure on T; it is also crucial for Proposition 6.3.1, Lemma 6.3.3, Proposition 6.3.4 and Theorem 6.3.5. It allows us to relate elements of G in a unique manner to transitions of local trivialisations for T as their transition functions, which is precisely the content of the condition (1). Without faithfulness, $\mathfrak{G}(g \circ f) \circ (\mathfrak{G}f)^{-1}$ could be the component of more than one element of G, which would spoil any chances of viewing G as a structure group for T.

(vi) Condition (2) looks curious but is in truth a naturality condition in disguise. It entails that T is locally trivial in the functorial sense, i.e. naturally isomorphic to U locally. For each $\mathcal{M}$-object A, consider the category $(\mathcal{M} \downarrow A)$ of $\mathcal{M}$-morphisms with codomain A, where a $(\mathcal{M} \downarrow A)$-morphism $h : (B \xrightarrow{f} A) \longrightarrow (C \xrightarrow{g} A)$ is just an $\mathcal{M}$-morphism $h : B \longrightarrow C$ such that $g \circ h = f$ (cf. Definition 2.2.19). Then (2) states that the composite functors $T|_{(\mathcal{M}\downarrow A)} : (\mathcal{M} \downarrow A) \xrightarrow{P_A} \mathcal{M} \xrightarrow{T} \mathcal{C}$ and $U|_{(\mathcal{M}\downarrow A)} : (\mathcal{M} \downarrow A) \xrightarrow{P_A} \mathcal{M} \xrightarrow{U} \mathcal{C}$ are naturally isomorphic and a natural isomorphism $\eta : T|_{(\mathcal{M}\downarrow A)} \xrightarrow{\dot\sim} U|_{(\mathcal{M}\downarrow A)}$ is defined by $\eta_{B \xrightarrow{f} A} := \mathfrak{G}f$ for $B \xrightarrow{f} A \in (\mathcal{M} \downarrow X)$. Here, $P_A : (\mathcal{M} \downarrow A) \longrightarrow \mathcal{M}$ denotes the projection functor defined by $P_A(B \xrightarrow{f} A) := B$ for $(\mathcal{M} \downarrow A)$-objects and by $P_A\big(h : (B \xrightarrow{f} A) \longrightarrow (C \xrightarrow{g} A)\big) := (h : B \longrightarrow C)$ for $(\mathcal{M} \downarrow A)$-morphisms.

(vii) Condition (3) is not essential to the definiton of a (U,G)-functor and can be omitted if so desired, in which case we still have that TA and UA are $\mathcal{C}$-isomorphic. We have included it because it conveys the important message that "*the twist is in the linking morphisms*", i.e. the twist is an affair of morphisms and not of objects.

6.3 Cohomology and existence

Fibre bundles[4] *with a structure group allow for a local descripton in terms of the cocycles formed by their transition functions. Using these cocycles, one can determine the existence and the isomorphism classes of fibre bundles with a structure group. Our aim this section is to do the analogue for (U, G)-functors; that is, we determine the notion of isomorphism of (U, G)-functors and classify them up to this notion of isomorphism. Indeed,* Definition 6.3.2 *stems from* [Ste51, Sec.2.5]*,* Lemma 6.3.3 *and* Proposition 6.3.4 *are the analogues of* [Ste51, Lem.2.8 + Lem.2.10]*. Ultimately, we are led to the notion of* twisted variants *of prototype functors,* Definition 6.3.6.

Throughout this section, let $U : \mathcal{M} \longrightarrow \mathcal{C}$ be the prototype functor on which we have a faithful G-action, where G is taken to be the automorphisms $\operatorname{Aut} U$ of U or a suitable subcollection thereof. Recall that faithful means that $G \ni g \longmapsto g_A \in \operatorname{Aut}(UA)$ is injective for each $\mathcal{M}$-object A. First, we note that the transition functions of a (U, G)-functor satisfy the cocycle condition:

PROPOSITION 6.3.1. *Let $(T, \mathfrak{G})$ be a (U, G)-functor, then we have*

$$g(A; A) = e_G \qquad \forall A \in \mathcal{M}, \tag{6.5}$$

$$g(B; A) = g(A; B)^{-1} \qquad \forall \mathcal{M}(A, B) \neq \varnothing \tag{6.6}$$

and

$$g(A_{\sigma(1)}; A_{\sigma(3)}) = g(A_{\sigma(2)}; A_{\sigma(3)})\, g(A_{\sigma(1)}; A_{\sigma(2)}) \tag{6.7}$$

$$\forall \sigma \in S_3 \text{ whenever } \mathcal{M}(A_1, A_2) \neq \varnothing \neq \mathcal{M}(A_2, A_3).$$

Recall that we tend to think of $\mathcal{M}(A, B) \neq \varnothing$ as "$A \subseteq B$" and accordingly of $\mathcal{M}(A_1, A_2) \neq \varnothing \neq \mathcal{M}(A_2, A_3)$ as "$A_1 \subseteq A_2 \subseteq A_3$".

Proof: All three identities follow directly from (6.2) and the fact that G acts faithfully on U. For example, let $f \in \mathcal{M}(A, B) \neq \varnothing$, then consider the composition of $\mathcal{M}$-morphisms $A \xrightarrow{\mathrm{id}_A} A \xrightarrow{f} B$. By (6.2), $g(B; A)_A = g(A; B)^{-1}_A$. Since $G \ni g \longmapsto g_A \in \operatorname{Aut}(UA)$ is injective, we conclude $g(B; A) = g(A; B)^{-1}$. ❒

The following definition introduces the concept of mapping transformations of [Ste51, §2.5] and states when we call (U, G)-functors isomorphic:

[4] Recall that we always consider fibre bundles to be locally trivial in this thesis.

DEFINITION 6.3.2. Two (U, G)-functors $(T, \mathfrak{G})$ and $(T', \mathfrak{G}')$ are called *(U, G)-isomorphic* if and only if there exists a natural isomorphism $\eta : T \overset{\cdot}{\xrightarrow{\sim}} T'$ and for each non-empty hom-set $\mathcal{M}(A, B)$, there is $\bar{g}(A; B) \in G$, called *mapping transformation*, such that for all compositions of $\mathcal{M}$-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$,

$$\bar{g}(B; C)_A = \mathfrak{G}'(g \circ f) \circ \eta_A \circ (\mathfrak{G} f)^{-1}. \tag{6.8}$$

We now turn to the local description of (U, G)-isomorphism, relating to the transition functions of (U, G)-functors, and state when two (U, G)-functors are (U, G)-isomorphic in terms of their transition functions:

LEMMA 6.3.3. *Two (U, G)-functors $(T, \mathfrak{G})$ and $(T', \mathfrak{G}')$ are (U, G)-isomorphic if and only if for every non-empty hom-set $\mathcal{M}(A, B)$, there is $\bar{g}(A; B) \in G$ such that the identity*

$$\begin{aligned} \bar{g}(A_{\sigma(2)}; A_{\sigma(3)}) g(A_{\sigma(1)}; A_{\sigma(2)}) &\\ = \bar{g}(A_{\sigma(1)}; A_{\sigma(3)}) &\\ &= g'(A_{\sigma(2)}; A_{\sigma(3)}) \, \bar{g}(A_{\sigma(1)}; A_{\sigma(2)}) \end{aligned} \tag{6.9}$$

holds for every $\sigma \in S_3$ whenever $\mathcal{M}(A_1, A_2) \neq \varnothing \neq \mathcal{M}(A_2, A_3)$.

Proof: "$\Longrightarrow$" follows directly from (6.2), (6.8) and the faithfulness of the action of G on U.

For "$\Longleftarrow$", we define for each $\mathcal{M}$-object A a $\mathcal{C}$-isomorphism $\eta_A : TA \xrightarrow{\sim} T'A$ by $\eta_A := (\mathfrak{G}' \mathrm{id}_A)^{-1} \circ \bar{g}(A; A)_A \circ \mathfrak{G} \mathrm{id}_A$. We first show that (6.8) is met, from which we will conclude that the η_A are the components of a natural isomorphism. Taking any composition $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathcal{M}$-morphisms, we compute using (6.2) and (6.9) that

$$\mathfrak{G}'(g \circ f) \circ \eta_A \circ (\mathfrak{G} f)^{-1} = \mathfrak{G}'(g \circ f) \circ (\mathfrak{G}' \mathrm{id}_A)^{-1} \circ \bar{g}(A; A)_A \circ \mathfrak{G} \mathrm{id}_A \circ (\mathfrak{G} f)^{-1} \tag{6.10}$$
$$= g'(A; C)_A \, \bar{g}(A; A)_A \, g(B; A)_A \tag{6.11}$$
$$= g'(A; C)_A \, \bar{g}(B; A)_A \tag{6.12}$$
$$= \bar{g}(B; C)_A. \tag{6.13}$$

Now, let $f : A \longrightarrow B$ be any $\mathcal{M}$-morphism. We have already established that $\mathfrak{G}'(g \circ f) \circ \eta_A = \bar{g}(B; C)_A \circ \mathfrak{G} f$ for every composition of $\mathcal{M}$-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$. Using this identity for the composition of $\mathcal{M}$-morphisms $A \xrightarrow{f} B \xrightarrow{\mathrm{id}_B} B$, the naturality of $\bar{g}(B; C)$ and (6.3), we obtain

$$Uf \circ \mathfrak{G}' f \circ \eta_A = Uf \circ \bar{g}(B; B)_A \circ \mathfrak{G} f \tag{6.14}$$

$$(6.15) \qquad = \bar{g}(B;B)_B \circ Uf \circ \mathfrak{G}f$$
$$(6.16) \qquad = \bar{g}(B;B)_B \circ \mathfrak{G}\,\mathrm{id}_B \circ Tf$$
$$(6.17) \qquad = \mathfrak{G}'\,\mathrm{id}_B \circ (\mathfrak{G}'\,\mathrm{id}_B)^{-1} \circ \bar{g}(B;B)_B \circ \mathfrak{G}\,\mathrm{id}_B \circ Tf$$
$$(6.18) \qquad = \mathfrak{G}'\,\mathrm{id}_B \circ \eta_B \circ Tf.$$

On the other hand, $Uf \circ \mathfrak{G}'f \circ \eta_B = \mathfrak{G}'\,\mathrm{id}_B \circ Tf \circ \eta_B$ by (6.3) and since $\mathfrak{G}'\,\mathrm{id}_B$ is a $\mathcal{C}$-isomorphism, we conclude $Tf \circ \eta_A = \eta_B \circ Tf$. Thus, the $\mathcal{C}$-isomorphisms η_A form the components of a natural isomorphism $\eta : T \xrightarrow{\dot{\sim}} T'$. ❐

The following proposition shows the cohomological aspect of (U,G)-isomorphism on the level of the transiton functions:

PROPOSITION 6.3.4. *Two (U,G)-functors $(T,\mathfrak{G})$ and $(T',\mathfrak{G}')$ are (U,G)-isomorphic if and only if their transition functions are cohomologous, i.e. there is a rule r assigning to each $A \in \mathcal{M}$ an element $r(A) \in G$ such that it holds $g'(A;B) = r(B)^{-1} g(A;B)\, r(A)$ whenever $\mathcal{M}(A,B) \neq \varnothing$.*

Proof: "$\Longrightarrow$". Let $A \xrightarrow{f} B \xrightarrow{g} C$ be any composition of $\mathcal{M}$-morphisms. Then we find with the help of (6.13) and (6.2) that

$$(6.19) \qquad \bar{g}(C;C)_A\, g(B;C)_A\, \bar{g}(B;B)_A^{-1} = g'(B;C)_A.$$

As G acts faithfully on U, $g'(B;C) = \bar{g}(C;C)\, g(B;C)\, \bar{g}(B;B)^{-1}$ follows.

"$\Longleftarrow$". By assumption, there exists a rule r that assigns to each $\mathcal{M}$-object A an element $r(A) \in G$ such that $g'(A;B) = r(B)^{-1} g(A;B)\, r(A)$ whenever $\mathcal{M}(A,B) \neq \varnothing$. For any non-empty hom-set $\mathcal{M}(A,B)$, we define $\bar{g}(A;B) \in G$ by setting $\bar{g}(A;B) := r(B)^{-1} g(A;B) = g'(A;B)\, r(A)^{-1}$ and observe for any composition of $\mathcal{M}$-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$

$$(6.20) \qquad \bar{g}(B;C)\, g(A;B) = r(C)^{-1} g(B;C)\, g(A;B) = r(C)^{-1} g(A;C)$$
$$(6.21) \qquad = \bar{g}(A;C)$$

and

$$(6.22) \qquad g'(B;C)\, \bar{g}(A;B) = g'(B;C)\, g'(A;B)\, r(A)^{-1} = g'(A;C)\, r(A)^{-1}$$
$$(6.23) \qquad = \bar{g}(A;C).$$

Hence, $(T,\mathfrak{G})$ and $(T',\mathfrak{G}')$ are (U,G)-isomorphic by Lemma 6.3.3. ❐

So far, we have established (U,G)-isomorphism in terms of the transition functions of a (U, G)-functor and we have shown uniqueness up to cohomology. We now formulate and prove an existence theorem for (U, G)-functors starting from the transition functions, which is analogous to [Ste51, Thm.3.2]:

THEOREM 6.3.5. *Let* $k := \{g(A;B), g(B;A) \in G \mid \mathcal{M}(A,B) \neq \varnothing\}$ *be a collection of automorphisms of* U *satisfying the cocycle conditions of* Proposition 6.3.1. *Then there is a* (U,G)*-functor* $(T, \mathfrak{G})$ *whose collection of transition functions is precisely* k*. If* $(T', \mathfrak{G}')$ *is another* (U,G)*-functor with this property, then* $(T, \mathfrak{G})$ *and* $(T', \mathfrak{G}')$ *are* (U,G)*-isomorphic.*

Proof: Proposition 6.3.4 shows the statement on (U, G)-isomorphism, so we only have to deal with existence. Define $TA := UA$ and $Tf := g(A;B)_B \circ Uf$ for all $f \in \mathcal{M}(A,B)$ and for all $A, B \in \mathcal{M}$. Obviously, $T\,\mathrm{id}_A = g(A;A)_A \circ U\,\mathrm{id}_A = \mathrm{id}_{UA}$; for any composition of $\mathcal{M}$-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$,

$$T(g \circ f) = g(A;C)_C \circ U(g \circ f) \tag{6.24}$$
$$= g(B;C)_C \circ g(A;B)_C \circ Ug \circ Uf \tag{6.25}$$
$$= g(B;C)_C \circ Ug \circ g(A;B)_B \circ Uf \tag{6.26}$$
$$= Tg \circ Tf, \tag{6.27}$$

where we have used that $g(A;B)$ is an automorphism of U, in particular a natural transformation $U \dot{\longrightarrow} U$. Hence, $T : \mathcal{M} \longrightarrow \mathcal{C}$ is a well-defined functor and obeys the condition (3) of Definition 6.2.1, too.

For every $\mathcal{M}$-morphism $f : A \longrightarrow B$, we set $\mathfrak{G}f := g(A;B)_A$ and need to verify the conditions (1) and (2) of Definition 6.2.1. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a composition of $\mathcal{M}$-morphisms; then

$$\mathfrak{G}(g \circ f) \circ (\mathfrak{G}f)^{-1} = g(A;C)_A \circ g(A;B)_A^{-1} = g(A;C)_A \circ g(B;A)_A \tag{6.28}$$
$$= g(B;C)_A \tag{6.29}$$

and the condition (1) is checked. To show the condition (2), let $A \xrightarrow{f} B \xrightarrow{g} C$ again be a composition of $\mathcal{M}$-morphisms. We compute

$$\mathfrak{G}g \circ Tf = g(B;C)_B \circ g(A;B)_B \circ Uf \tag{6.30}$$
$$= g(A;C)_B \circ Uf \tag{6.31}$$
$$= Uf \circ g(A;C)_A \tag{6.32}$$
$$= Uf \circ \mathfrak{G}(g \circ f), \tag{6.33}$$

where we have again exploited that $g(A;C)$ is an automorphisms of U and a natural transformation $U \dot{\longrightarrow} U$ in particular. In conclusion, the condition (2) of Definition 6.2.1 holds and $(T, \mathfrak{G})$ is a (U, G)-functor. ❐

DEFINITION 6.3.6. We call every (U, G)-functor which is not (U, G)-isomorphic to the trivial (U, G)-functor (U, id) a *twisted variant* of U.

6.4 Twisted variants of locally covariant theories and their classification

We adapt the general scheme to locally covariant theories. We classify and show the existence of twisted variants, and also provide some concrete examples of their classification.

Let $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ be a locally covariant theory and, for the moment, assume that G is a Lie group. It is problematic –to say the least– to consistently choose a smooth G-cocycle for *every* $\mathbf{M} \in \mathbf{Loc}$ in such a way that the collection of all these choices can be regarded as non-trivial since the existence of non-trivial smooth G-cocycles is tied to the topology of the underlying smooth manifold. To illustrate one of the problems that can arise, consider the following: let $\mathbf{M} \in \mathbf{Loc}_{©}$, i.e. $\mathbf{M} \in \mathbf{Loc}$ and M is contractible, and let $O \subseteq M$ be a connected globally hyperbolic open subset of $\mathbf{M}$ such that there is a non-trivial smooth G-cocycle c_O for the open cover of O given by all connected globally hyperbolic open subsets of O, which are also connected globally hyperbolic open subsets of $\mathbf{M}$ as O is equipped with the structures induced by $\mathbf{M}$; in the light of Lemma 1.1.2, considering all connected globally hyperbolic open subsets of a $\mathbf{Loc}$-object is a reasonable choice for an open cover. Now, due to local covariance and consistency, the non-trivial smooth G-cocycle should be preserved by viewing O as smoothly embedded in $\mathbf{M}$ via the $\mathbf{Loc}$-morphism given by the inclusion map $\iota_O : O \hookrightarrow M$; i.e. c_O should be the restriction of a smooth G-cocycle $c_{\mathbf{M}}$ for the open cover of M which is given by all connected globally hyperbolic open subsets of $\mathbf{M}$. However, this is impossible. Since M is contractible, any smooth G-cocycle for any open cover of M is trivial necessarily and so are its restrictions to O.

Because of such issues, we do not try and consider the locally a covariant theory $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ as a prototype functor for the general scheme of Section 6.2. Instead, we consider twisted variants for F over individual $\mathbf{Loc}$-objects, as it has been done originally by C.J. Isham for twisted quantum fields in [Ish78b; AI79b], though we stress that F is a generic locally covariant theory and not necessarily refers to (quantum) fields.

With some hindsight, we restrict F to the comma category $(K_s \downarrow \mathbf{M})$ (recall Definition 2.2.19) for $\mathbf{M} \in \mathbf{Loc}$, which yields the functor

$$U : (K_s \downarrow \mathbf{M}) \xrightarrow{P_{\mathbf{M}}} \mathbf{Loc}_s \xrightarrow{K_s} \mathbf{Loc} \xrightarrow{F} \mathbf{Phys}, \tag{6.34}$$

where $P_{\mathbf{M}} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Loc}_s$ denotes the projection functor (see the paragraph after Definition 2.2.19). The functor U will serve as the prototype functor, over which we want to consider twisted variants, for the general scheme of Section 6.2. At this point, it is useful to remind the reader of our notation for $(K_s \downarrow \mathbf{M})$-objects and $(K_s \downarrow \mathbf{M})$-morphisms (see again the paragraph after Definition 2.2.19): a $(K_s \downarrow \mathbf{M})$-object is a $\mathbf{Loc}$-morphism $f : \mathbf{A} \longrightarrow \mathbf{M}$, where $\mathbf{A} \in \mathbf{Loc}_s$, and will be denoted by $\mathbf{A} \xrightarrow{f} \mathbf{M}$; a $(K_s \downarrow \mathbf{M})$-morphism from a $(K_s \downarrow \mathbf{M})$-object $\mathbf{A} \xrightarrow{f} \mathbf{M}$ to a $(K_s \downarrow \mathbf{M})$-object $\mathbf{B} \xrightarrow{g} \mathbf{M}$ is a $\mathbf{Loc}_s$-morphism $h : \mathbf{A} \longrightarrow \mathbf{B}$ such that the identity $g \circ h = f$ holds. We denote this situation more catchy as a commutative diagram $\begin{array}{ccc} \mathbf{B} & \overset{g}{\searrow} & \\ {\scriptstyle h}\uparrow & & \mathbf{M} \\ \mathbf{A} & \underset{f}{\nearrow} & \end{array}$. It can thus be said that U is essentially F considered only on simply connected $\mathbf{Loc}$-objects for which there is a $\mathbf{Loc}$-morphism to $\mathbf{M}$, i.e. only on simply connected $\mathbf{Loc}$-objects which can be identified with simply connected globally hyperbolic open subsets of $\mathbf{M}$.

To help the reader keep track of how we are naturally led to the classification of twisted variants by the classification of flat smooth principal bundles, which culminates in Theorem 6.4.2, we provide the following figure, where an arrow means "*leads to*".

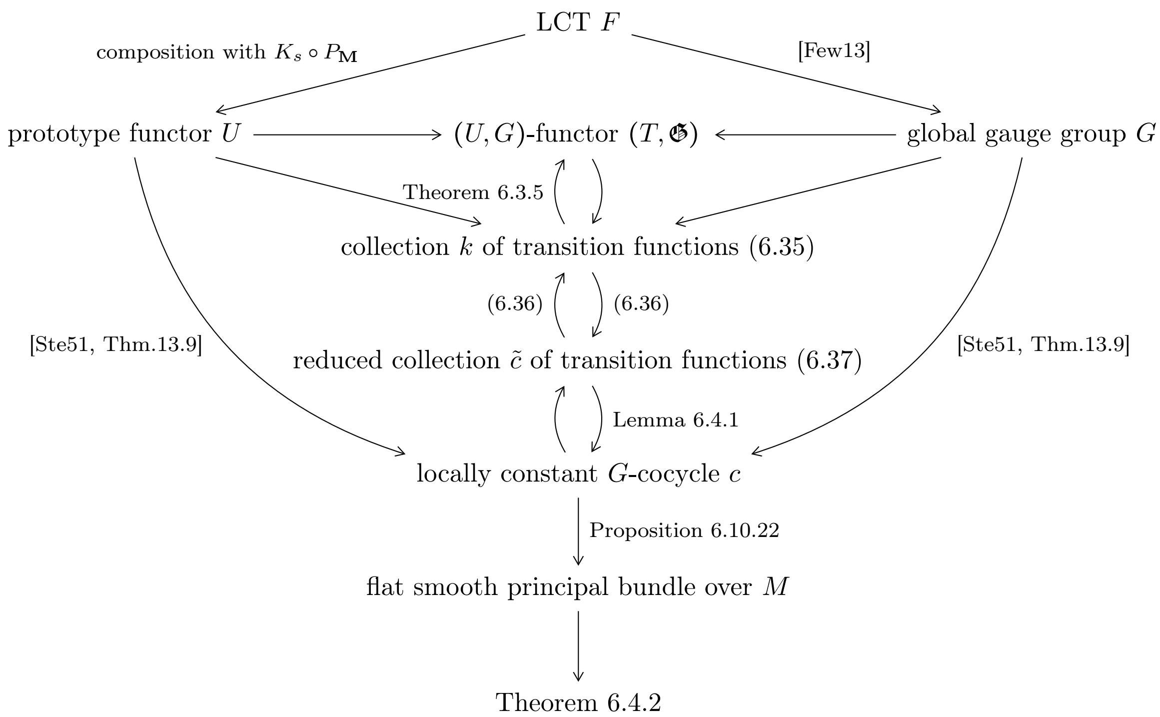


FIGURE 6.1: Schematic outline for the classification and the construction of twisted variants of locally covariant theories.

The figure is also to help the reader keeping track of the steps in the explicit construction of twisted variants for locally covariant theories, which is the content of

Theorem 6.4.3 and its proof.

In the spirit of [DHR69a; DHR69b; Few13], we will think of $F\mathbf{M}$ as the "*field algebra*" for $\mathbf{M} \in \mathbf{Loc}$, though to emphasise again, F is a generic locally covariant theory and does not necessarily refer to (quantum) fields; owing to [Few13], we can use the automorphisms $\operatorname{Aut} F$ or a suitable subcollection thereof as the global gauge group and determine the "*observable algebra*" $A\mathbf{M}$ as being the fixed points of $F\mathbf{M}$ under all selected automorphisms of F. This also defines for all $\mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_s \downarrow \mathbf{M})$ what we consider as the observables in $U(\mathbf{A} \xrightarrow{f} \mathbf{M}) = F\mathbf{A}$ which are to be let alone by twists (recall the discussion in Section 6.1). Any automorphism $\eta : F \overset{\cdot}{\xrightarrow{\sim}} F$ gives rise to an automorphism $\varepsilon : U \overset{\cdot}{\xrightarrow{\sim}} U$ by setting $\varepsilon_{\mathbf{A}\xrightarrow{f}\mathbf{M}} := \eta_{\mathbf{A}}$ for all $\mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_s \downarrow \mathbf{M})$. We take this collection of automorphisms for U or a suitable subcollection thereof as the global gauge group G for U. This ensures that we really are regarding only global gauge transformations, which are the automorphisms of F (and not of U).

Let $(T, \mathfrak{G})$ be a (U, G)-functor and let

$$k := \{g\,(\mathbf{A}\,;\mathbf{B} \xrightarrow{f;h} \mathbf{M}),\, g\,(\mathbf{B}\,;\mathbf{A} \xrightarrow{h;f} \mathbf{M}) \in G \mid (K_s \downarrow \mathbf{M})\,(\mathbf{A},\mathbf{B} \xrightarrow{f,h} \mathbf{M}) \neq \varnothing\} \tag{6.35}$$

be the collection of all transitions functions of $\mathfrak{G}$, where we have written $\mathbf{A}\,;\mathbf{B} \xrightarrow{f;h} \mathbf{M}$ instead of $\mathbf{A} \xrightarrow{f} \mathbf{M}\,;\mathbf{B} \xrightarrow{h} \mathbf{M}$ and likewise for the comma replacing the semicolon. Recall that $(K_s \downarrow \mathbf{M})$ is thin with skeleton $\mathrm{loc}^s_{-\mathbf{M}}$ (Lemma 2.2.21 and Corollary 2.2.22). In the light of Proposition 6.3.4,

$$\begin{aligned} g\,(\mathbf{A}\,;\mathbf{B} \xrightarrow{f;h} \mathbf{M}) = {} & \underbrace{g\,(h\,(\mathbf{B})\,;\mathbf{B} \xrightarrow{\iota_{h(B)};h} \mathbf{M})}_{= r\,(\mathbf{B}\xrightarrow{h}\mathbf{M})^{-1}}\; g\,(f\,(\mathbf{A})\,;h\,(\mathbf{B}) \xrightarrow{\iota_{f(A)};\iota_{h(B)}} \mathbf{M}) \\ & \times \underbrace{g\,(\mathbf{A};f\,(\mathbf{A}) \xrightarrow{f;\iota_{f(A)}} \mathbf{M})}_{= r\,(\mathbf{A}\xrightarrow{f}\mathbf{M})} \\ & \forall\,(K_s \downarrow \mathbf{M})\,(\mathbf{A},\mathbf{B} \xrightarrow{f,h} \mathbf{M}) \neq \varnothing \end{aligned} \tag{6.36}$$

entails that the isomorphism class of $(T, \mathfrak{G})$ is already completely determined by the collection of transition functions

$$\tilde{c} := \{g\,(U;V \xrightarrow{\iota_U;\iota_V} \mathbf{M}),\, g\,(V;U \xrightarrow{\iota_V;\iota_V} \mathbf{M}) \in G \mid U, V \in \mathrm{loc}^s_{-\mathbf{M}} \text{ with } U \subseteq V \text{ or } V \subseteq U\}. \tag{6.37}$$

According to the general scheme of Section 6.2, only one $g\,(U;V) \in G$ is associated with each $\mathrm{loc}^s_{-\mathbf{M}}$-morphism

$$\begin{array}{ccc} V & \xrightarrow{\iota_V} & \\ \uparrow{\scriptstyle \iota_{UV}} & & \mathbf{M} \\ U & \xrightarrow{\iota_U} & \end{array}$$

, in such a way that the cocycle conditions of

Proposition 6.3.1 are fulfilled. Here, we have further abbreviated $g\,(U;V \xrightarrow{\iota_U;\iota_V} \mathbf{M})$ by $g\,(U;V)$. $\tilde{c}$ is, however, not a locally constant G-cocycle for the open cover $\{U \in \mathrm{loc}^s_{-\mathbf{M}}\}$ of M since we are only given transition constants for intersections $U \cap V$ with $U \subseteq V$ or $V \subseteq U$, $U, V \in \mathrm{loc}^s_{-\mathbf{M}}$. The following lemma shows that this provides already enough data to specify a unique locally constant G-cocycle for $\{U \in \mathrm{loc}^s_{-\mathbf{M}}\}$. We recall that the simply connected globally hyperbolic open subsets of $\mathbf{M}$ form a basis for the topology of M due to Lemma 1.1.2 and we also assume that G is a (finite-dimensional) Lie group. However, the proof given will also work for infinite-dimensional Lie groups or for topological groups.

LEMMA 6.4.1. *Let M be a smooth manifold and G a Lie group. Suppose $\mathcal{B}$ is a basis for the topology of M and $\tilde{c} := \{g\,(B\,;B') : B \cap B' \longrightarrow G \mid B, B' \in \mathcal{B}$ such that $B \subseteq B'$ or $B' \subseteq B\}$ is a collection of smooth functions meeting the cocycle conditions of* Proposition 6.3.1. *Then there is a unique way to extend this collection to a smooth G-cocycle c for $\mathcal{B}$. If $\tilde{c}' := \{g'\,(B\,;B') : B \cap B' \longrightarrow G \mid B, B' \in \mathcal{B}$ such that $B \subseteq B'$ or $B' \subseteq B\}$ is another collection of smooth functions satisfying the cocycle conditions of* Proposition 6.3.1 *and is cohomologous to $\tilde{c}$, then c' will be cohomologous to c.*

Proof: Take any $B, B' \in \mathcal{B}$ satisfying $B \cap B' \neq \varnothing$ and define for each $A \in \mathcal{B}$ such that $A \subseteq B \cap B'$ a smooth function $g\,(B\,;A\,;B') : A \longrightarrow G$ by $g\,(B\,;A\,;B') := g\,(A\,;B')\,g\,(B\,;A)$. Note, this immediately yields $g\,(B'\,;A\,;B) = g\,(B\,;A\,;B')^{-1}$. Let $A, A' \in \mathcal{B}$ be such that $A, A' \subseteq B \cap B'$ and $A \cap A' \neq \varnothing$. Since

$$\begin{aligned}
&(6.38) && (g\,(B\,;A\,;B')\,|_{A\cap A'})\,|_C = g\,(B\,;A\,;B')\,|_C \\
&(6.39) && \qquad = g\,(A\,;B')\,|_C\, g\,(B\,;A)\,|_C \\
&(6.40) && \qquad = g\,(A\,;B')\,|_C\, g\,(C\,;A)\, g\,(A\,;C)\, g\,(B\,;A)\,|_C \\
&(6.41) && \qquad = g\,(C\,;B')\, g\,(B\,;C) \\
&(6.42) && \qquad = (g\,(B\,;A'\,;B')\,|_{A\cap A'})\,|_C \\
& && \qquad\qquad \forall C \in \mathcal{B} \text{ such that } C \subseteq A \cap A',
\end{aligned}$$

it follows that $g\,(B\,;A\,;B')\,|_{A\cap A'} = g\,(B\,;A'\,;B')\,|_{A\cap A'}$ because $\mathcal{B}$ is a basis for the topology of M. Thanks to [Lee03, Lem.2.1], there exists a unique smooth map $g\,(B\,;B') : B \cap B' \longrightarrow G$ such that $g\,(B\,;B')\,|_A = g\,(B\,;A\,;B')$ for all $A \in \mathcal{B}$ with $A \subseteq B \cap B'$. If $B \subseteq B'$ or $B' \subseteq B$, then we have of course $g\,(B\,;B \cap B'\,;B') = g\,(B\,;B')$. For $B, B' \in \mathcal{B}$ such that $B \cap B' \neq \varnothing$, $g\,(B'\,;B) = g\,(B\,;B')^{-1}$ follows directly from $g\,(B'\,;A\,;B) = g\,(B\,;A\,;B')^{-1}$ for all $A \subseteq B \cap B'$.

It remains to show that the cocycle condition is met. So, let $B, B', B'' \in \mathcal{B}$ such that $B \cap B' \cap B'' \neq \varnothing$. Since

$$g(B';B'')|_A\, g(B;B')|_A = g(B';A;B'')\, g(B;A;B') \tag{6.43}$$

$$= g(A;B'')\, g(B';A)\, g(A;B')\, g(B;A) \tag{6.44}$$

$$= g(A;B'')\, g(B;A) \tag{6.45}$$

$$= g(B;A;B'') \tag{6.46}$$

$$= g(B;B'')|_A \tag{6.47}$$

$$\forall A \in \mathcal{B} \text{ such that } A \subseteq B \cap B' \cap B'',$$

we find $g(B';B'')\, g(B;B') = g(B;B'')$ on $B \cap B' \cap B''$ due to $\mathcal{B}$ being a basis for the topology of M.

The next two items to check concern the uniqueness of the smooth G-cocycle c just obtained and whether c' is cohomologous to c if $\tilde{c}'$ is cohomologous to $\tilde{c}$, i.e. for each $B \in \mathcal{B}$ there is a smooth function $r(B) : B \longrightarrow G$ such that $g'(B;B') = r^{-1}(B')|_{B\cap B'}\, g(B;B')\, r(B)|_{B\cap B'}$ for all $g'(B;B') \in \tilde{c}'$. Both, however, follow immediately from the construction and the restriction properties. ❒

Hence, $\tilde{c}$ canonically extends to a locally constant G-cocycle for $\{U \in \mathrm{loc}^s_{\mathbf{-M}}\}$ and by Proposition 6.10.22, it induces a flat smooth principal G-bundle, where we continue to assume that the global gauge group G for the prototype functor U forms a Lie group. Putting everything together, we arrive at the classification for twisted variants of locally covariant theories:

THEOREM 6.4.2. *Let $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ be a locally covariant theory, pick $\mathbf{M} \in \mathbf{Loc}$ and let $U : (K_s \downarrow \mathbf{M}) \xrightarrow{P_{\mathbf{M}}} \mathbf{Loc}_s \xrightarrow{K_s} \mathbf{Loc} \xrightarrow{F} \mathbf{Phys}$ be the prototype functor over which we want to consider twisted variants, where $K_s : \mathbf{Loc}_s \longrightarrow \mathbf{Loc}$ is the inclusion and $P_{\mathbf{M}} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Loc}_s$ the projection functor. Let G be a collection of automorphisms of U which are of the form $\varepsilon : U \stackrel{\sim}{\dot{\longrightarrow}} U$, $\varepsilon_{\mathbf{A} \xrightarrow{f} \mathbf{M}} = \eta_{\mathbf{A}}$ for all $\mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_s \downarrow \mathbf{M})$, where $\eta \in \mathrm{Aut}\, F$. We assume that G forms a Lie group. Then the isomorphism classes of twisted variants $(T, \mathfrak{G})$ of U are in a $1:1$-correspondence with the isomorphism classes of flat smooth principal G-bundles over M. By* [Mor01a, Thm.2.9] *and* [Mor01b, Thm.6.60]*, the isomorphism classes of flat smooth principal G-bundles over M are in a $1:1$-correspondence with the conjugacy classes of group homomorphisms $\pi_1(M) \longrightarrow G$.*

As Theorem 6.4.2 reveals, there are no twisted variants for $U : (K \downarrow \mathbf{M}) \longrightarrow \mathbf{Phys}$ if the underlying smooth manifold of $\mathbf{M} \in \mathbf{Loc}$ has vanishing fundamental group (= first homotopy group), $\pi_1(M) = e$. For $\pi_1(M) \neq e$, there is a simple constructive description for twisted variants, to which the explicit statement of [Ste51, Thm.13.9] is important: *Let X be an arcwise connected*[5], *arcwise locally connected*[6] *and semi-locally 1-connected*[7] *topological space and G a totally disconnected topological group, i.e. G is equipped with the discrete topology. Then the equivalence classes of (continuous) principal bundles over X with structure group G are in $1:1$ correspondence with the equivalence classes (under inner automorphisms of G) of homomorphisms of $\pi_1(X)$ into G.*

THEOREM 6.4.3. (twisted variants construction lemma)
Under the assumptions of Theorem 6.4.2*, there are twisted variants $(T,\mathfrak{G})$ of U.*

Proof: By [Ste51, Thm.13.9], we can find a continuous principal G-bundle over M, where G carries the discrete topology. Hence, taking a continuous principal G-bundle atlas and considering its continuous G-cocycle of transistion functions, we obtain a locally constant G-cocycle $c = \{g_{UV} : U \cap V \longrightarrow G \mid U,V \in \mathrm{loc}^s_{-\mathbf{M}}$ such that $U \cap V \neq \varnothing\}$ for the open cover $\{U \in \mathrm{loc}^s_{-\mathbf{M}}\}$ of M, which we reduce to a collection $\tilde{c} = \{g_{UV}, g_{VU} \mid U,V \in \mathrm{loc}^s_{-\mathbf{M}}$ such that $U \subseteq V$ or $V \subseteq U\}$. We define a collection k of transition functions satisfying the cocycle conditions of Proposition 6.3.1 by $g(\mathbf{A};\mathbf{B} \xrightarrow{f;h} \mathbf{M}) \coloneqq g_{f(A)h(B)}$ and $g(\mathbf{B};\mathbf{A} \xrightarrow{h;f} \mathbf{M}) \coloneqq g_{h(B)f(A)}$ for all non-empty hom-sets $(K_s \downarrow \mathbf{M})(\mathbf{A},\mathbf{B} \xrightarrow{f,h} \mathbf{M})$. The application of Theorem 6.3.5 yields a (U,G)-functor $(T,\mathfrak{G})$. If c is taken to be trivial, we recover U. If c is non-trivial, $(T,\mathfrak{G})$ is a twisted variant of U and if c' is another locally constant G-cocycle for $\{U \in \mathrm{loc}^s_{-\mathbf{M}}\}$ which is cohomologous to c, $(T',\mathfrak{G}')$ is (U,G)-isomorphic to $(T,\mathfrak{G})$. ❐

To conclude this section, we carry out classifications of twisted variants for some $\mathbf{M} \in \mathbf{Loc}$ with $\pi_1(M) \neq e$ and some Lie groups G. To be more specific, we will consider $\mathbf{M} \in \mathbf{Loc}$ such that $M \cong \mathbb{R} \times \mathbb{R}^2 \times S^1$ ($\Longrightarrow \pi_1(M) \cong \pi(S^1) \cong \mathbb{Z}$) and $M \cong \mathbb{R} \times \mathbb{R} \times S^1 \times S^1$ ($\Longrightarrow \pi_1(M) \cong \mathbb{Z} \times \mathbb{Z}$). The choices of the Lie group G are motivated by specific field theories. Note however that the specific locally covariant theory $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$ considered is irrelevant for the classification of twisted variants.

- $M \cong \mathbb{R} \times \mathbb{R}^2 \times S^1$, $G = \mathrm{O}(n)$, $n = 1,2,\ldots$ ($\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields)

We need to classify all group homomorphisms $\varphi : \mathbb{Z} \longrightarrow \mathrm{O}(n)$ up to inner automorphism of $\mathrm{O}(n)$. Because φ is a group homomorphism, $\varphi(0) = E_n$ (E_n denotes

[5] An *arc* in a topological space X is a continuous map $\alpha : [0,1] \longrightarrow X$ which is a homeomorphism onto its image. By [Lee03, Prop.1.8(b)], any connected smooth manifold is path connected which implies together with [Wil70, Cor.31.6] that any connected smooth manifold is arcwise connected.

[6] [Lee03, Lem.1.6] shows that any smooth manifold is arcwise locally connected.

[7] A topological space X is called *semi-locally 1-connected* if and only if for each point $p \in X$ there exists an open neighbourhood U such that each closed curve in U is homotopic to a constant in X, leaving its endpoints fixed.

the identity $n \times n$-matrix) and $\varphi(z) = \varphi(1)^z$ for all $z \in \mathbb{Z}$. Hence, any group homomorphism $\varphi : \mathbb{Z} \longrightarrow \mathrm{O}(n)$ is uniquely determined by $\varphi(1) \in \mathrm{O}(n)$. Now, up to inner automorphism, any matrix in $\mathrm{O}(n)$ is of the form

$$\begin{pmatrix} E_k & & & & \\ & -E_l & & & \\ & & R(\vartheta_1) & & \\ & & & \ddots & \\ & & & & R(\vartheta_m) \end{pmatrix}, \tag{6.48}$$

where $k, l, m = 0, 1, 2, \ldots$ such that $n = k + l + 2m$, $R(\vartheta) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$, $0 < \vartheta_1 \le \ldots \le \vartheta_m < \pi$ and all other entries are zero. Hence the isomorphism classes of twisted variants on $\mathbf{M}$ are labelled by $k \in \{0, \ldots, n\}$ and $0 < \vartheta_1 \le \ldots \le \vartheta_m < \pi$ such that $k + 2m \le n$.

• $M \cong \mathbb{R} \times \mathbb{R} \times S^1 \times S^1$, $G = \mathrm{O}(n)$, $n = 1, 2, \ldots$ ($\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields)

We need to classify all group homomorphisms $\varphi : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathrm{O}(n)$ up to inner automorphism of $\mathrm{O}(n)$. We have $\varphi(0,0) = E_n$, $\varphi(z_1, z_2) = \varphi((z_1, 0) + (0, z_2)) = \varphi(1,0)^{z_1} \varphi(0,1)^{z_2}$ for $(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z}$ and so, φ is uniquely determined by $\varphi(1,0)$ and $\varphi(0,1)$. In conclusion, the isomorphism classes of twisted variants on $\mathbf{M}$ are labelled by twice the labels as previously.

• $M \cong \mathbb{R} \times \mathbb{R}^2 \times S^1$, $G = \mathbb{R}^n$, $n = 1, 2, \ldots$ (shift-twisted free and minimally coupled real scalar fields)

$\mathbb{R}^n$ is viewed as an Abelian group with the addition being the group multiplication. The neutral element is of course $0_{\mathbb{R}^n}$. We seek to classify all group homomorphisms $\varphi : \mathbb{Z} \longrightarrow \mathbb{R}^n$ up to inner automorphism of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Abelian, every conjugacy class consists of precisely one group homomorphism $\varphi : \mathbb{Z} \longrightarrow \mathbb{R}^n$. Since φ is a group homomorphism, $\varphi(0_{\mathbb{Z}}) = 0_{\mathbb{R}^n}$ and $\varphi(z) = \varphi(1_{\mathbb{Z}})^z$ for all $z \in \mathbb{Z}$. Hence, any group homomorphism $\varphi : \mathbb{Z} \longrightarrow \mathbb{R}^n$ and thus isomorphism class of twisted variants on $\mathbf{M}$ is uniquely determined the value $\varphi(1_{\mathbb{Z}}) \in \mathbb{R}^n$.

• $M \cong \mathbb{R} \times \mathbb{R}^2 \times S^1$, $G = \mathrm{SL}(2; \mathbb{C})$ (twisted free Dirac field)

We are looking at the classification of group homomorphisms $\varphi : \mathbb{Z} \longrightarrow \mathrm{SL}(2; \mathbb{C})$ up to inner automorphism of $\mathrm{SL}(2; \mathbb{C})$. As φ is a group homomorphism, $\varphi(0) = E_2$ and $\varphi(z) = \varphi(1)^z$ for all $z \in \mathbb{Z}$. So, φ is uniquely determined by its value $\varphi(1)$. It is well-known from the theory of the Jordan normal form of a matrix that any $A \in \mathrm{SL}(2; \mathbb{C})$

is equivalent under inner automorphism of $\mathrm{SL}(2;\mathbb{C})$ to one of the matrices

$$(6.49)\qquad \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} \text{ for } \alpha\in\mathbb{C}\smallsetminus\{0\}\,,\qquad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\qquad\text{or}\qquad \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}.$$

Hence, picking one of these matrices is picking one isomorphism class of twisted variants on $\mathbf{M}$.

This result is of particular interest because according to C.J. Isham's classification using the isomorphism classes of smooth principal $\mathrm{SL}(2;\mathbb{C})$-bundles, twisted (quantum) Dirac fields on $\mathbf{M}$ do not exist. Any "*twisted*" structures for the free Dirac field were credited to inequivalent spin-frame projections before [Ish78a; DHI79]. Unfortunately, we do not find the time in this thesis to investigate this entirely new possibility of twisted (quantum) fields any further.

- $M \cong \mathbb{R}\times\mathbb{R}^2\times S^1$, $G = \mathrm{U}(1)$ (twisted free Maxwell field)

$\mathrm{U}(1)$ is Abelian and so each conjugacy class of group homomorphisms $\varphi:\mathbb{Z}\longrightarrow\mathrm{U}(1)$ consists of exactly one element. φ is uniquely determined by its properties as a group homomorphism, $\varphi(0_{\mathbb{Z}}) = 1_{\mathbb{C}}$ and $\varphi(z) = \varphi(1_{\mathbb{Z}})^z$ for all $z\in\mathbb{Z}$. Hence, we use the familiar parametrisation $\{e^{\mathrm{i}\theta}\mid\theta\in[0,2\pi)\}$ of $\mathrm{U}(1)$ to label all isomorphism classes of twisted variants on $\mathbf{M}$.

6.5 Multiple free and minimally coupled real scalar fields of the same mass

We consider $n = 1,2,3,\dots$ free and minimally coupled real scalar fields of the same mass ≥ 0 and review their locally covariant (quantum field) theory. The case of $n = 1$ free and minimally coupled real scalar field of the mass ≥ 0 was covered in Section 4.5 (*set $p = 0$ therein*). *This section is a requisite preliminary for both twisted quantum fields in the sense of C.J. Isham* [Ish78b; AI79b] (*also recall the example provided at the beginning of this chapter*) *and for our general scheme* (Section 6.2 + 6.4). *On the basis of the locally covariant (quantum field) theory of n free and minimally coupled real scalar fields of the same mass, we will identify the global gauge group and the prototype functor in* Section 6.6 *which is vital for the construction of* $\mathrm{O}(n)$*-twisted free and minimally coupled real scalar fields in* Section 6.7, *again for both in the sense of C.J. Isham and our general scheme.*

Fix $\mathbf{M}\in\mathbf{Loc}$ for the moment and let $n\in\mathbb{N}\smallsetminus\{0\}$. For the purpose of twisted variants, it is more convenient to work with smooth vector bundles and smooth cross-sections instead of $\mathbb{R}^n$-valued smooth functions. We thus consider the trivial smooth $\mathbb{R}$-vector bundle of rank n, $\underline{\mathbb{R}^n}_M = (M\times\mathbb{R}^n, M, \mathrm{pr}_1, \mathbb{R}^n)$. Since $\underline{\mathbb{R}^n}_M$ is trivial and the direct sum $\bigoplus_{i=1}^{n} \underline{\mathbb{R}}_M$ of n copies of the trivial smooth $\mathbb{R}$-vector bundle of rank 1, $\underline{\mathbb{R}}_M = (M \times \mathbb{R}, M, \mathrm{pr}_1, \mathbb{R})$, we have the isomorphisms of $\mathcal{C}^\infty M$-modules (see [GHV72, Sec.2.14, Example 1 + 2]):

$$\Gamma^\infty \left(\underline{\mathbb{R}^n}_M\right)) \cong \bigoplus_{i=1}^{n} \Gamma^\infty \left(\underline{\mathbb{R}}_M\right) \cong \bigoplus_{i=1}^{n} \mathcal{C}^\infty M \cong \mathcal{C}^\infty \left(M, \mathbb{R}^n\right). \tag{6.50}$$

This will allow us to readily reformulate the theory of n free and minimally coupled real scalar fields of the same mass into the language of smooth vector bundles and smooth cross-sections. We start with the important linear differential operators which give rise to the homogeneous Klein-Gordon equation for smooth cross-sections in $\underline{\mathbb{R}^n}_M$.

Any smooth cross-section $f \in \Gamma^\infty \left(\underline{\mathbb{R}^n}_M\right)$ can be written as $f = (\mathrm{id}_M, \vec{f}\,)$, where $\vec{f} = (f^1, \dots, f^n)$ with smooth functions $f^1, \dots, f^n \in \mathcal{C}^\infty M$. The Levi-Civita connection on $\mathbf{M}$, $\nabla : \mathscr{X}(M) \longrightarrow \Omega^1 (M; \tau_M)$, induces a linear connection in $\underline{\mathbb{R}}_M$ by $\nabla f = (\mathrm{id}_M, \nabla f) = (\mathrm{id}_M, df)$ and extends to $\underline{\mathbb{R}^n}_M$ componentwise (cf. [GHV73, Sec.7.12, Example 3]), i.e.

$$\begin{aligned} \nabla : \Gamma^\infty \left(\underline{\mathbb{R}^n}_M\right) &\longrightarrow \Omega^1 \left(M; \underline{\mathbb{R}^n}_M\right) \\ f &\longmapsto \left(\mathrm{id}_M, d\vec{f}\,\right) = \left(\mathrm{id}_M, df^1, \dots, df^n\right). \end{aligned} \tag{6.51}$$

The covariant exterior derivative with respect to ∇ is precisely the exterior derivative applied componentwise ($p = 0, 1, 2, \dots$),

$$\begin{aligned} \mathrm{d}^\nabla \colon \Omega^p \left(M; \underline{\mathbb{R}^n}_M\right) &\longrightarrow \Omega^{p+1} \left(M; \underline{\mathbb{R}^n}_M\right) \\ \omega = (\mathrm{id}_M, \vec{\omega}) &\longmapsto (\mathrm{id}_M, d\vec{\omega}) = \left(\mathrm{id}_M, d\omega^1, \dots, d\omega^n\right), \end{aligned} \tag{6.52}$$

where $\vec{\omega} = (\omega^1, \dots, \omega^n)$ with smooth differential p-forms $\omega^1, \dots, \omega^n \in \Omega^p M$. For this reason, we write d instead of d^∇. Similarly, for $p = 0, 1, 2, \dots$, the Hodge-$*$-operator

$$\begin{aligned} * : \Omega^p \left(M; \underline{\mathbb{R}^n}_M\right) &\longrightarrow \Omega^{m-p} \left(M; \underline{\mathbb{R}^n}_M\right) \\ \omega &\longmapsto (\mathrm{id}_M, *\vec{\omega}) = \left(\mathrm{id}_M, *\omega^1, \dots, *\omega^n\right), \end{aligned} \tag{6.53}$$

and the covariant exterior coderivative with respect to ∇,

$$\begin{aligned} \delta = \delta^\nabla \colon \Omega^p \left(M; \underline{\mathbb{R}^n}_M\right) &\longrightarrow \Omega^{p-1} \left(M; \underline{\mathbb{R}^n}_M\right) \\ \omega &\longmapsto (\mathrm{id}_M, \delta\vec{\omega}) = \left(\mathrm{id}_M, \delta\omega^1, \dots, \delta\omega^n\right), \end{aligned} \tag{6.54}$$

are given by their usual action in each component. We thereby define the Klein-Gordon operator for n free and minimally coupled real scalar fields of the same mass by

$$\begin{aligned} D : \Gamma^\infty \left(\underline{\mathbb{R}^n}_M\right) &\longrightarrow \Gamma^\infty \left(\underline{\mathbb{R}^n}_M\right) \\ f &\longmapsto \left(\mathrm{id}_M, D\vec{f}\right) = \left(\mathrm{id}_M, Df^1, \dots, Df^n\right), \end{aligned} \tag{6.55}$$

which is just applying the Klein-Gordon operator $D = \Box + \mu^2 = -\delta d + \mu^2 : \mathcal{C}^\infty M \longrightarrow \mathcal{C}^\infty M$ ($\mu = \frac{mc}{\hbar}$ is the reduced mass, where $m \geq$ is the mass of the fields, c the speed of light and $\hbar$ the reduced Planck's constant) for $n = 1$ free and minimally coupled real scalar field of the mass $m \geq 0$ in each component. The homogeneous Klein-Gordon equation for smooth cross-sections in $\underline{\mathbb{R}^n}_M$ is thus the homogeneous Klein-Gordon equation $D\phi^i = 0$ for smooth functions $\phi^i \in \mathcal{C}^\infty M$, $i = 1, \dots, n$, in each component:

$$D\phi = \left(\mathrm{id}_M, D\vec{\phi}\right) = \left(\mathrm{id}_M, D\phi^1, \dots, D\phi^n\right) = (\mathrm{id}_M, 0) = 0, \qquad \phi \in \Gamma^\infty\left(\underline{\mathbb{R}^n}_M\right). \tag{6.56}$$

$D : \Gamma^\infty\left(\underline{\mathbb{R}^n}_M\right) \longrightarrow \Gamma^\infty\left(\underline{\mathbb{R}^n}_M\right)$ is a normally hyperbolic linear differential operator of metric type and by [BGP07, Cor.3.4.3] or [Wal12, Cor.4.3.7], there are unique retarded and advanced Green's operators, which act componentwise via the unique retarded and advanced Green operators $G^{\mathrm{ret/adv}} : \mathcal{C}_0^\infty M \longrightarrow \mathcal{C}_{\mathrm{sc}}^\infty M$ for $D : \mathcal{C}^\infty M \longrightarrow \mathcal{C}^\infty M$:

$$\begin{aligned} G^{\mathrm{ret/adv}} : \Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right) &\longrightarrow \Gamma_{\mathrm{sc}}^\infty\left(\underline{\mathbb{R}^n}_M\right) \\ f &\longmapsto \left(\mathrm{id}_M, G^{\mathrm{ret/adv}}\vec{f}\right) = \left(\mathrm{id}_M, G^{\mathrm{ret/adv}}f^1, \dots, G^{\mathrm{ret/adv}}f^n\right). \end{aligned} \tag{6.57}$$

In the same way as in Lemma 4.5.1(i), [BGP07, Thm.3.4.7] or [Wal12, Thm.4.3.18] yields that $\ker D$ is trivial on $\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$ while $\ker G = D\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$ on $\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$, where $G := G^{\mathrm{adv}} - G^{\mathrm{ret}}$. The image of G on $\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$ is given by $\{\phi \in \Gamma_{\mathrm{sc}}^\infty\left(\underline{\mathbb{R}^n}_M\right) \mid D\phi = 0\}$. $\phi \in \Gamma_{\mathrm{sc}}^\infty\left(\underline{\mathbb{R}^n}_M\right)$ satisfies $D\phi = 0$ if and only if $\phi = Gf$ for some $f \in \Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$.

For the definition of a symplectic form on $\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$, we consider the standard Riemannian bundle metric $\langle\cdot \mid \cdot\rangle_{\mathrm{Eucl}}$ in $\underline{\mathbb{R}^n}_M$, which is given by

$$\begin{aligned} \langle\cdot \mid \cdot\rangle_{\mathrm{Eucl}} : M &\longrightarrow M \times (\mathbb{R}^n)^* \times M \times (\mathbb{R}^n)^* \\ x &\longmapsto \begin{cases} \langle\cdot \mid \cdot\rangle_{\mathrm{Eucl}}(x) : \{x\} \times \mathbb{R}^n \times \{x\} \times \mathbb{R}^n \longrightarrow \mathbb{R} \\ \qquad (x, \vec{u}, x, \vec{v}) \longmapsto \vec{u} \cdot \vec{v} = \sum_{i=1}^n u^i v^i. \end{cases} \end{aligned} \tag{6.58}$$

Then we obtain a skew-symmetric bilinear form $\mathfrak{u} : \Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right) \times \Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right) \longrightarrow \mathbb{R}$ with radical $D\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$ by

$$(f, g) \longmapsto \langle f \mid Gg\rangle_{2,\mathrm{Eucl}} = \int_M \langle f \mid Gg\rangle_{\mathrm{Eucl}} \,\mathrm{vol}_{\mathbf{M}} = \sum_{i=1}^n \int_M f^i\, Gg^i \,\mathrm{vol}_{\mathbf{M}}. \tag{6.59}$$

Applying the quantisation functor $Q : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{{}^*Alg_1^m}$ to the symplectic space $\left(\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)/D\Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right), \mathfrak{u}\right)$ thus obtained, we get a simple unital *-algebra which is generated by the elements of the form $\mathbf{\Phi}(f)$, $f \in \Gamma_0^\infty\left(\underline{\mathbb{R}^n}_M\right)$, which are subject to the conditions:

- Linearity: $\mathbf{\Phi}(\lambda f + \mu g) = \lambda\mathbf{\Phi}(f) + \mu\mathbf{\Phi}(g)$ for all $\lambda, \mu \in \mathbb{R}$ and for all $f, g \in$

$\Gamma_0^\infty(\underline{\mathbb{R}^n}_M)$.

- Hermiticity: $\mathbf{\Phi}(f)^* = \mathbf{\Phi}(f)$ for all $f \in \Gamma_0^\infty(\underline{\mathbb{R}^n}_M)$.
- Field equations (in a weak sense): $\mathbf{\Phi}(Df) = 0$ for all $f \in \Gamma_0^\infty(\underline{\mathbb{R}^n}_M)$.
- Commutation relations: $[\mathbf{\Phi}(f), \mathbf{\Phi}(g)] = \mathrm{i}\hbar \langle f \mid Gg\rangle_{2,\mathrm{Eucl}} \cdot 1_A$ for all $f, g \in \Gamma_0^\infty(\underline{\mathbb{R}^n}_M)$.

Regarding the categorical aspects of these constructions, we note that we obtain functors

$$\begin{aligned} \mathcal{F} : \mathbf{Loc} &\longrightarrow \mathbf{Sympl}_{\mathbb{R}} \\ \mathbf{M} &\longmapsto ([\Gamma_0^\infty(\underline{\mathbb{R}^n}_M)] \coloneqq \Gamma_0^\infty(\underline{\mathbb{R}^n}_M)/D_{\mathbf{M}}\Gamma_0^\infty(\underline{\mathbb{R}^n}_M), \mathfrak{u}_{\mathbf{M}}) \\ (\psi : \mathbf{M} \longrightarrow \mathbf{N}) &\longmapsto [\Psi, \psi]_* : ([\Gamma_0^\infty(\underline{\mathbb{R}^n}_M)], \mathfrak{u}_{\mathbf{M}}) \longrightarrow ([\Gamma_0^\infty(\underline{\mathbb{R}^n}_N)], \mathfrak{u}_{\mathbf{N}}) \end{aligned} \tag{6.60}$$

and $\mathfrak{F} \coloneqq Q \circ \mathcal{F} : \mathbf{Loc} \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$. $\mathcal{F}$ is a locally covariant theory and $\mathfrak{F}$ is a causal locally covariant quantum field theory.

We elaborate on the definition of the arrow function of $\mathcal{F}$ and $\mathfrak{F}$. Let $\psi : \mathbf{M} \longrightarrow \mathbf{N}$ be a **Loc**-morphism; then we define a smooth vector bundle monomorphism $(\Psi, \psi) : \underline{\mathbb{R}^n}_M \longrightarrow \underline{\mathbb{R}^n}_N$ by $\Psi \coloneqq \psi \times \mathrm{id}_{\mathbb{R}^n} : M \times \mathbb{R}^n \longrightarrow N \times \mathbb{R}^n$. The pushforward of compactly supported smooth cross-sections in $\underline{\mathbb{R}^n}_M$ along (Ψ, ψ), $(\Psi, \psi)_* : \Gamma_0^\infty(\underline{\mathbb{R}^n}_M) \longrightarrow \Gamma_0^\infty(\underline{\mathbb{R}^n}_N)$, is an injective linear map and evidently given by $(\Psi, \psi)_* f = (\mathrm{id}_N, \psi_* \vec{f}) = (\mathrm{id}_N, \psi_* f^1, \dots, \psi_* f^n)$ for $f \in \Gamma_0^\infty(\underline{\mathbb{R}^n}_M)$, hence, nothing else but the pushforward of compactly supported smooth functions along ψ applied in each component. We conclude that $(\Psi, \psi)_*$ intertwines $D_{\mathbf{M}}$ with $D_{\mathbf{N}}$. Accordingly, there is a unique linear map $[\Psi, \psi]_* : [\Gamma_0^\infty(\underline{\mathbb{R}^n}_M)] \longrightarrow [\Gamma_0^\infty(\underline{\mathbb{R}^n}_N)]$ by (UQ') such that $[\Psi, \psi]_* \circ \pi_{\mathbf{M}} = \pi_{\mathbf{N}} \circ (\Psi, \psi)_*$, where $\pi_{\mathbf{M}|\mathbf{N}} : \Gamma_0^\infty(\underline{\mathbb{R}^n}_{M|N}) \longrightarrow\!\!\!\!\!\rightarrow [\Gamma_0^\infty(\underline{\mathbb{R}^n}_{M|N})]$ are the canonical projections onto the quotients. We further note that $[\Psi, \psi]_*$ is injective and symplectic.

In the same way as we had Proposition 4.5.6 and Proposition 4.5.7 concerning colimits and left Kan extensions for $n = 1$ free and non-minimally coupled real scalar field of the mass $m \geq 0$, we have the following two propositions. The first one states that, regardless of how we topologically restrict the connected globally hyperbolic open subsets of a **Loc**-object, we will always recover the standard classical and quantum field theory for $n \geq 1$ free and minimally coupled real scalar fields of the same mass $m \geq 0$ by the colimit, in such a way that the quantum field theory is the quantisation of the classical field theory. In the language of category theory, the quantisation functor preserves the corresponding colimits. This implies that the universal algebra will always be the standard unital *-algebra of the smeared quantum field, which we have stated above. Moreover, the second proposition expresses the fact that the standard functors for $n \geq 1$ free and non-minimally coupled real scalar fields of the same mass $m \geq 0$ as specified above are the left Kan extensions for all their respective restrictions to the topologically restricted full subcategories $\mathbf{Loc}_q$ of $\mathbf{Loc}$, where $q \subseteq \mathbb{N} \setminus \{0\}$ or $q = s$ or $q = ©$.

PROPOSITION 6.5.1. *Consider any full subcategory of* $\mathbf{Loc}$ *of the form* $\mathbf{Loc}_q$*, let* $\mathbf{M} \in \mathbf{Loc}$ *and consider the restrictions* $\mathcal{F}^q_{\mathbf{M}}, \mathfrak{F}^q_{\mathbf{M}} : \mathrm{loc}^q_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ *of* $\mathcal{F}$ *and* $\mathfrak{F}$ *to* $\mathrm{loc}^q_{-\mathbf{M}}$*. Then*

$$\operatorname{colim} \mathcal{F}^q_{\mathbf{M}} = \Big(\mathcal{F}\mathbf{M}, \{\mathcal{F}\iota_O : \mathcal{F}^q_{\mathbf{M}} O \longrightarrow \mathcal{F}\mathbf{M}\}_{O \in \mathrm{loc}^q_{-\mathbf{M}}} \Big), \tag{6.61}$$

$$\operatorname{colim} \mathfrak{F}^q_{\mathbf{M}} = \Big(\mathfrak{F}\mathbf{M}, \{\mathfrak{F}\iota_O : \mathfrak{F}^q_{\mathbf{M}} O \longrightarrow \mathfrak{F}\mathbf{M}\}_{O \in \mathrm{loc}^q_{-\mathbf{M}}} \Big) \tag{6.62}$$

and

$$\operatorname{colim} \mathfrak{F}^q_{\mathbf{M}} \cong Q\left(\operatorname{colim} \mathcal{F}^q_{\mathbf{M}}\right). \tag{6.63}$$

PROPOSITION 6.5.2. *Consider any full subcategory of* $\mathbf{Loc}$ *of the form* $\mathbf{Loc}_q$*. Then* $\mathcal{F}$ *and* $\mathfrak{F}$ *are the left Kan extensions along the inclusion functor* $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$ *of their respective restrictions to* $\mathbf{Loc}_q$*,* $\mathcal{F}_q, \mathfrak{F}_q : \mathbf{Loc}_q \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$*. The natural transformations of the left Kan extensions have the respective identities as their components.*

6.6 Global Gauge Group and Prototype Functor

We specify the global gauge group for the classical and the quantum field theory of multiple free and minimally coupled real scalar fields of the same mass, $\mathcal{F}, \mathfrak{F} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$*, which will turn out to be* $\mathrm{O}(n)$*, and state the prototype functor over which we want to consider* $\mathrm{O}(n)$*-twisted variants.*

The automorphisms of $\mathcal{F}, \mathfrak{F} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ have been computed in [Few13] and are as follows: for mass ≥ 0 and fixed spacetime dimension ≥ 2, $\operatorname{Aut}\mathcal{F} \cong \mathrm{O}(n)$ [Few13, Thm.5.2]; to be more precise, any $\eta \in \operatorname{Aut}\mathcal{F}$ is of the form $\eta_{\mathbf{M}}[f] = [Rf]$ for $f \in \Gamma^\infty_0(\underline{\mathbb{R}^n}_M)$ and for all $\mathbf{M} \in \mathbf{Loc}$, where $R \in \mathrm{O}(n)$ is fixed and does not vary with $\mathbf{M} \in \mathbf{Loc}$ [Few13, Lem.5.1]. Pay attention to the fact that [Few13] employs a different functor for the classical field theory of $n \geq 1$ free and mnimally coupled real scalar fields of the same mass ≥ 0, which is, however, naturally isomorphic to the complexification $\mathscr{C} \circ \mathcal{F} : \mathbf{Loc} \longrightarrow \mathbf{Sympl}_{\mathbb{C}}$ (see Proposition 2.1.2; as an equivalence, $\mathscr{C}$ is full and faithful). This allows us to apply the results of [Few13] without hesitation.

For $\mathfrak{F}$, the situation is a bit more delicate: for the computation of the automorphisms, $\mathfrak{F}$ had be considered together with a suitable state space (see [Few13, Sec.5]);

then one finds for mass > 0 and fixed spacetime dimension ≥ 2 that $\mathrm{Aut}\,\mathfrak{F} \cong \mathrm{O}(n)$ ($\mathfrak{F}$ standing for the pair consisting of $\mathfrak{F}$ and the suitable state space now), where any $\eta \in \mathrm{Aut}\,\mathfrak{F}$ is of the form $\eta_{\mathbf{M}}\,\mathbf{\Phi}_{\mathbf{M}}(f) = \mathbf{\Phi}_{\mathbf{M}}(Rf)$ for all $f \in \Gamma_0^\infty(\underline{\mathbb{R}^n}_M)$ and for all $\mathbf{M} \in \mathbf{Loc}$, where $R \in \mathrm{O}(n)$ [Few13, Thm.5.4]. For mass = 0 and fixed spacetime dimension ≥ 3, we have the semi-direct product $\mathrm{Aut}\,\mathfrak{F} \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$ [Few13, Thm.5.4].

In what follows, we concentrate on the $\mathrm{O}(n)$-aspect, that is, we pick $\mathrm{O}(n)$ as our global gauge group G for $\mathcal{F}$ and $\mathfrak{F}$ ($\mathfrak{F}$ is now regarded on its own, without a state space) and consider the cases with mass ≥ 0 and fixed dimension ≥ 2 of the **Loc**-objects or mass = 0 and fixed spacetime dimension ≥ 3. As argued in Section 6.4, for $\mathbf{M} \in \mathbf{Loc}$, our prototype functors will be

$$\mathcal{U} : (K_s \downarrow \mathbf{M}) \xrightarrow{P_{\mathbf{M}}} \mathbf{Loc}_s \xrightarrow{K_s} \mathbf{Loc} \xrightarrow{\mathcal{F}} \mathbf{Sympl}_{\mathbb{R}} \tag{6.64}$$

and

$$\mathfrak{U} : (K_s \downarrow \mathbf{M}) \xrightarrow{P_{\mathbf{M}}} \mathbf{Loc}_s \xrightarrow{K_s} \mathbf{Loc} \xrightarrow{\mathfrak{F}} {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}\,. \tag{6.65}$$

The global gauge group $\mathrm{O}(n)$ acts as follows on $\mathcal{U}$ (resp. $\mathfrak{U}$): for $R \in \mathrm{O}(n)$ and $\mathbf{A} \xrightarrow{f} \mathbf{M} \in (K \downarrow \mathbf{M})$, we have that $R_{\mathbf{A}\xrightarrow{f}\mathbf{M}}[f] := [Rf]$ for all $[f] \in [\Gamma_0^\infty(\underline{\mathbb{R}^n}_A)]$ (resp. $R_{\mathbf{A}\xrightarrow{f}\mathbf{M}}\mathbf{\Phi}_{\mathbf{A}}(f) := \mathbf{\Phi}_{\mathbf{A}}(Rf)$ for all $f \in \Gamma_0^\infty(\underline{\mathbb{R}^n}_A)$). Since each global gauge transformation is thus uniquely determined by a single element of $\mathrm{O}(n)$, we have a faithful action of $\mathrm{O}(n)$ on $\mathcal{U}$ and $\mathfrak{U}$, i.e. the maps $\mathrm{O}(n) \ni R \longmapsto R \in \mathrm{Aut}(\mathcal{U})$ and $\mathrm{O}(n) \ni R \longmapsto R \in \mathrm{Aut}(\mathfrak{U})$ are injective. As our classification (Theorem 6.4.2) dictates, $\mathcal{U}$ and $\mathfrak{U}$ will only allow for twisted variants if $\mathbf{M} \in \mathbf{Loc}$ such that $\pi_1(M) \neq e$. Hence, it makes sense to exclude $\mathbf{M} \in \mathbf{Loc}$ with $\pi_1(M) = e$ from our considerations now.

6.7 O(n)-twisted free and minimally coupled real scalar fields

An immediate advantage of our abstract general scheme is that we can readily get twisted variants by applying Theorem 6.4.3 *to our prototype functors* $\mathcal{U}, \mathfrak{U} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$, $\mathbf{M} \in \mathbf{Loc}$ *with* $\pi_1(M) \neq e$. *Anyway, as stated in the introduction to this chapter, we want to give a constructive description of* $\mathrm{O}(n)$*-twisted free and minimally coupled real scalar fields of the same mass in the spirit of C.J. Isham* [Ish78b; AI79b]. *That is, we want to obtain the twisted classical field theory via smooth cross-sections in a non-trivial smooth vector bundle which locally satisfy the homogeneous Klein-Gordon equation* (6.56) *but are globally different due to a non-trivial* $\mathrm{O}(n)$*-cocycle of transition functions. Similarly, we want to construct the twisted quantum field theory by using smooth cross-sections in non-trivial* *smooth vector bundles for the quantum description. In doing this, we can demonstrate that C.J. Isham's twisted quantum fields fit into our general framework of* Section 6.2 *and into its adaptation to locally covariant theories in* Section 6.4. *Furthermore, this also provides a good basis for the investigation of further properties of* O(n)*-twisted free and minimally coupled real scalar fields, and for the application of colimit constructions and the universal algebra.*

We fix $\mathbf{M} \in \mathbf{Loc}$ such that $\pi_1(M) \neq e$ and take any non-trivial locally constant O(n)-cocycle for the open cover of M which is given by all simply connected open subsets of M,

$$c := \left\{ R_{UV} : U \cap V \longrightarrow \mathrm{O}(n) \;\middle|\; \begin{array}{l} U, V \subseteq M \text{ open and simply con-} \\ \text{nected such that } U \cap V \neq \varnothing \end{array} \right\}. \tag{6.66}$$

Let $\mathscr{P} = (P, M, \pi, \mathrm{O}(n), \mathfrak{p})$ be the smooth principal O(n)-bundle over M which is constructed from c by the smooth principal bundle reconstruction lemma, Theorem 6.10.17, and $\mathfrak{P}$ a smooth principal G-bundle atlas for $\mathscr{P}$ whose smooth O(n)-cocycle of transition functions is precisely c. By Proposition 6.10.22, $\mathscr{P}$ becomes flat in a canonical fashion. Choosing the defining matrix representation $\rho : \mathrm{O}(n) \longrightarrow \mathrm{GL}(n; \mathbb{R})$, we construct the smooth vector bundle $\xi = (E, M, \varrho, \mathbb{R}^n)$ with typical fibre $\mathbb{R}^n$ associated with $\mathscr{P}$ (Proposition and Definiton 6.10.18). From $\mathfrak{P}$, we obtain a smooth vector bundle atlas $\mathfrak{G}$ for ξ whose smooth cocycle of transition functions is also precisely c. We denote the members of $\mathfrak{G}$ by $\theta_U : \xi|_U \xrightarrow{\sim!} \underline{\mathbb{R}^n}_U$, where $U \subseteq M$ open and simply connected, and note that we can thus turn ξ into a smooth vector bundle with structure group O(n), $\xi_{\mathrm{O}(n)} = (\xi, \mathrm{O}(n), \mathfrak{l}, <\mathfrak{G}>)$.

For the construction of an O(n)-twisted Levi-Civita connection, which is locally the pullback of the Levi-Civita connection in $\underline{\mathbb{R}^n}_M$, (6.51), the following lemma is noteworthy:

LEMMA 6.7.1. *Let* $\phi : \xi|_U \xrightarrow{\sim!} \underline{\mathbb{R}^n}_U$ *and* $\psi : \xi|_V \xrightarrow{\sim!} \underline{\mathbb{R}^n}_V$ *be smooth local trivialisations for* ξ *such that* $U \cap V \neq \varnothing$, $f \in \Gamma^\infty(\underline{\mathbb{R}^n}_{U\cap V})$ *and recall that* $f = \left(\mathrm{id}_{U\cap V}, \vec{f}\,\right)$ *for* $\vec{f} = (f^1, \ldots, f^n)$ *with* $f^1, \ldots, f^n \in \mathcal{C}^\infty(U \cap V)$. *Then*

$$(\psi\|_{U\cap V})_* (\phi\|_{U\cap V})^{\#} f = \left(\mathrm{id}_{U\cap V}, A_{\phi\psi}\vec{f}\,\right), \tag{6.67}$$

where $A_{\phi\psi} : U \cap V \longrightarrow \mathrm{GL}(n; \mathbb{R})$ *is the smooth transition function from* ϕ *to* ψ. *If* $A_{\phi\psi}$

is locally constant, then

$$\nabla_{U\cap V}\left(\psi\|_{U\cap V}\right)_*\left(\phi\|_{U\cap V}\right)^{\#} f = \left(\psi\|_{U\cap V}\right)_*\left(\phi\|_{U\cap V}\right)^{\#}\nabla_{U\cap V} f \qquad \forall f\in\Gamma^\infty\left(\underline{\mathbb{R}^n}_{U\cap V}\right). \tag{6.68}$$

Proof: (6.67) is an easy consequence of the definitions for pullbacks and pushforwards:

$$\left(\left(\psi\|_{U\cap V}\right)_*\left(\phi\|_{U\cap V}\right)^{\#} f\right)(x) = \psi_x\left(\phi_x^{-1}\left(f(x)\right)\right) = \left(x, A_{\phi\psi}(x)\vec{f}(x)\right) \qquad \forall x\in U\cap V. \tag{6.69}$$

(6.68) follows from the product rule of the Levi-Civita connection $\nabla_{U\cap V}$ in $\underline{\mathbb{R}^n}_{U\cap V}$. ❐

The O(n)-twisted Levi-Civita connection $\widetilde{\nabla}:\Gamma^\infty(\xi)\longrightarrow\Omega^1(M;\xi)$ is now going to be defined by the linear connection in ξ specified by the following proposition:

PROPOSITION 6.7.2. *There exists a linear connection* $\widetilde{\nabla}:\Gamma^\infty(\xi)\longrightarrow\Omega^1(M;\xi)$ *uniquely determined by* $\widetilde{\nabla}_U\theta_U^{\#}=\theta_U^{\#}\nabla_U$ *for all simply connected open subsets* $U\subseteq M$, *where* $\widetilde{\nabla}_U := \widetilde{\nabla}|_U:\Gamma^\infty(\xi|_U)\longrightarrow\Omega^1(U;\xi|_U)$.

Proof: Using Lemma 6.7.1, we check

$$\begin{aligned}
&\left(\theta_U^{\#}\nabla_U\theta_{U*}\sigma|_U\right)|_{U\cap V} = \left(\theta_U^{\#}\|_{U\cap V}\right)\nabla_{U\cap V}\left(\theta_{U*}\|_{U\cap V}\right)\sigma|_{U\cap V} && (6.70)\\
&= \left(\theta_V^{\#}\|_{U\cap V}\right)\left(\theta_{V*}\|_{U\cap V}\right)\left(\theta_U^{\#}\|_{U\cap V}\right)\nabla_{U\cap V}\left(\theta_{U*}\|_{U\cap V}\right)\sigma|_{U\cap V} && (6.71)\\
&= \left(\theta_V^{\#}\|_{U\cap V}\right)\nabla_{U\cap V}\left(\theta_{V*}\|_{U\cap V}\right)\left(\theta_U^{\#}\|_{U\cap V}\right)\left(\theta_{U*}\|_{U\cap V}\right)\sigma|_{U\cap V} && (6.72)\\
&= \left(\theta_V^{\#}\|_{U\cap V}\right)\nabla_{U\cap V}\left(\theta_{V*}\|_{U\cap V}\right)\sigma|_{U\cap V} && (6.73)\\
&= \left(\theta_V^{\#}\nabla_V\theta_{V*}\sigma|_V\right)|_{U\cap V} && (6.74)
\end{aligned}$$

$$\forall U,V\subseteq M \text{ open, simply connected and } U\cap V\neq\varnothing,\ \forall\sigma\in\Gamma^\infty(\xi).$$

Hence, by the smooth cross-section gluing lemma, Lemma 6.10.4, we can define $\widetilde{\nabla}$ for all $\sigma\in\Gamma^\infty(\xi)$ as the unique smooth ξ-valued differential 1-form $\widetilde{\nabla}\sigma$ satisfying $(\widetilde{\nabla}\sigma)|_U=\theta_U^{\#}\nabla_U\theta_{U*}\sigma|_U$ for all simply connected open subsets $U\subseteq M$. Obviously, $\widetilde{\nabla}$ has the required restriction property and the product rule is easily seen. ❐

With the O(n)-twisted Levi-Civita connection at our disposal, we obtain for $p=0,1,2,\ldots$, the O(n)-twisted covariant exterior derivative $\mathrm{d}^{\widetilde{\nabla}}:\Omega^p(M;\xi)\longrightarrow\Omega^{p+1}(M;\xi)$ and the O(n)-twisted covariant exterior coderivative $\delta^{\widetilde{\nabla}}:\Omega^p(M;\xi)\longrightarrow\Omega^{p-1}(M;\xi)$ in the usual way (see the appendix to this chapter). From $\mathrm{d}^{\widetilde{\nabla}}$ and $\delta^{\widetilde{\nabla}}$, we assemble the

O(n)-twisted covariant wave operator

$$\widetilde{\Box} := -\delta^{\widetilde{\nabla}} \mathrm{d}^{\widetilde{\nabla}} : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi) \tag{6.75}$$

and the O(n)-twisted Klein-Gordon operator

$$\widetilde{D} := \widetilde{\Box} + \mu^2 : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi), \tag{6.76}$$

where μ is as before in Section 6.5 the reduced mass. The homogeneous O(n)-twisted Klein-Gordon equation is now simply

$$\widetilde{D}\sigma = 0, \qquad \sigma \in \Gamma^\infty(\xi). \tag{6.77}$$

The linear differential operators $\mathrm{d}^{\widetilde{\nabla}}$, $\delta^{\widetilde{\nabla}}$, $\widetilde{\Box}$ and $\widetilde{D}$ inherit the restriction property from $\widetilde{\nabla}$ (cf. [GHV73, Sec.7.15]), i.e.

$$\mathrm{d}_U^{\widetilde{\nabla}} \theta_U^{\#} = \theta_U^{\#} d_U, \qquad \delta_U^{\widetilde{\nabla}} \theta_U^{\#} = \theta_U^{\#} \delta_U, \qquad \widetilde{\Box}_U \theta_U^{\#} = \theta_U^{\#} \Box_U \qquad \text{and} \qquad \widetilde{D}_U \theta_U^{\#} = \theta_U^{\#} D_U \tag{6.78}$$
$$\forall U \subseteq M \text{ open and simply connected,}$$

where we have written $\mathrm{d}_U^{\widetilde{\nabla}} := \mathrm{d}^{\widetilde{\nabla}}|_U : \Omega^p(U;\xi|_U) \longrightarrow \Omega^{p+1}(U;\xi|_U)$, $\delta_U^{\widetilde{\nabla}} := \delta^{\widetilde{\nabla}}|_U : \Omega^p(U;\xi|_U) \longrightarrow \Omega^{p-1}(U;\xi|_U)$ for $p = 0,1,2,\ldots$, $\widetilde{\Box}_U := \widetilde{\Box}|_U : \Gamma^\infty(\xi|_U) \longrightarrow \Gamma^\infty(\xi\_U)$ and $\widetilde{D}_U := \widetilde{D}|_U : \Gamma^\infty(\xi|_U) \longrightarrow \Gamma^\infty(\xi|_U)$. From this or from Lemma 6.10.8 and Lemma 6.10.11, we immediately deduce:

COROLLARY 6.7.3. $\widetilde{D} : \Gamma^\infty(\xi) \longrightarrow \Gamma^\infty(\xi)$ *and* $\widetilde{D}_U := \widetilde{D}|_U : \Gamma^\infty(\xi|_U) \longrightarrow \Gamma^\infty(\xi|_U)$, *where* $U \subseteq M$ *open and simply connected, are normally hyperbolic linear differential operators of metric type.*

As $\mathbf{M}$ is globally hyperbolic, [BGP07, Cor.3.4.3] or [Wal12, Cor.4.3.7] guarantees the existence of unique retarded and advanced Green's operators $\widetilde{G}^{\mathrm{ret/adv}} : \Gamma_0^\infty(\xi) \longrightarrow \Gamma_{\mathrm{sc}}^\infty(\xi)$ for $\widetilde{D}$ and $\widetilde{G}_U^{\mathrm{ret/adv}} : \Gamma_0^\infty(\xi|_U) \longrightarrow \Gamma_{\mathrm{sc}}^\infty(\xi|_U)$ for $\widetilde{D}_U$ for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$. Due to the restriction property of $\widetilde{D}$, the identities $\widetilde{G}^{\mathrm{ret/adv}}|_U = \widetilde{G}_U^{\mathrm{ret/adv}}$, which is $(\widetilde{G}^{\mathrm{ret/adv}}\sigma)|_U = \widetilde{G}_U^{\mathrm{ret/adv}}\sigma|_U$ for all $\sigma \in \Gamma_0^\infty(\xi|_U)$, and $\widetilde{G}_U^{\mathrm{ret/adv}}\theta_U^{\#} = \theta_U^{\#} G_U^{\mathrm{ret/adv}}$ hold for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$.

Recall the standard Riemannian bundle metric $\langle\cdot\mid\cdot\rangle_{\mathrm{Eucl}}$ in $\underline{\mathbb{R}^n}_M$ from (6.58). Since $\langle\cdot\mid\cdot\rangle_{\mathrm{Eucl}}$ is the lift of the ρ-invariant standard inner product on $\mathbb{R}^n$, $\langle\cdot\mid\cdot\rangle_{\mathrm{Eucl}} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$, $\langle\vec{u}\mid\vec{v}\rangle_{\mathrm{Eucl}} = \vec{u}\cdot\vec{v} = \sum_{i=1}^k u^i v^i$, Proposition 6.10.19 yields unique smooth Riemannian bundle metrics $\langle\cdot\mid\cdot\rangle_{\xi|_U}$ in $\xi|_U$ satisfying $\langle\cdot\mid\cdot\rangle_{\xi|_U} = \phi^*\langle\cdot\mid\cdot\rangle_{\mathrm{Eucl}_U} = \langle\phi_*\cdot\mid\phi_*\cdot\rangle_{\mathrm{Eucl}_U}$ for all $\phi : \xi_{\mathrm{O}(n)}|_U \xrightarrow{\sim!} \underline{\mathbb{R}^n_{\mathrm{O}(n)}}_U \in <\mathfrak{G}>$ and a unique smooth Riemannian bundle metric $\langle\cdot\mid\cdot\rangle_\xi$ in ξ satisfying the identity $\langle\cdot\mid\cdot\rangle_\xi|_U = \langle\cdot\mid\cdot\rangle_{\xi|_U}$ for all simply connected open subsets $U \subseteq M$.

In the following lemma, we establish the metric compatibility of the O(n)-twisted Levi-Civita connection $\widetilde{\nabla} : \Gamma^\infty(\xi) \longrightarrow \Omega^1(M;\xi)$:

LEMMA 6.7.4. *$\widetilde{\nabla}$ is metric with respect to $\langle \cdot \mid \cdot \rangle_\xi$ and $\widetilde{\nabla}_U$ is metric with respect to $\langle \cdot \mid \cdot \rangle_{\xi|_U}$ for all simply connected open subsets $U \subseteq M$.*

Proof: Since the Levi-Civita connection is metric, we compute

$$\begin{aligned}
&\left(\langle \widetilde{\nabla}_X \sigma \mid \tau \rangle_\xi + \langle \sigma \mid \widetilde{\nabla}_X \tau \rangle_\xi\right)|_U \\
&= \left\langle (\widetilde{\nabla}_X \sigma)|_U \mid \tau|_U \right\rangle_{\xi|_U} + \left\langle \sigma|_U \mid (\widetilde{\nabla}_X \tau)|_U \right\rangle_{\xi|_U} && (6.79)\\
&= \left\langle \theta_{U*} \widetilde{\nabla}_{X|_U} \sigma|_U \mid \theta_{U*} \tau|_U \right\rangle_{\mathrm{Eucl}_U} + \left\langle \theta_{U*} \sigma|_U \mid \theta_{U*} \widetilde{\nabla}_{X|_U} \tau|_U \right\rangle_{\mathrm{Eucl}_U} && (6.80)\\
&= \left\langle \nabla_{X|_U} \theta_{U*} \sigma|_U \mid \theta_{U*} \tau|_U \right\rangle_{\mathrm{Eucl}_U} + \left\langle \theta_{U*} \sigma|_U \mid \nabla_{X|_U} \theta_{U*} \tau|_U \right\rangle_{\mathrm{Eucl}_U} && (6.81)\\
&= X|_U \left\langle \theta_{U*} \sigma|_U \mid \theta_{U*} \tau|_U \right\rangle_{\mathrm{Eucl}_U} && (6.82)\\
&= X|_U \left\langle \sigma|_U \mid \tau|_U \right\rangle_{\xi|_U} && (6.83)\\
&= \left(X \langle \sigma \mid \tau \rangle_\xi\right)|_U && (6.84)
\end{aligned}$$

$\forall U \subseteq M$ open and simply connected, $\forall X \in \mathscr{X}(M)$, $\forall \sigma, \tau \in \Gamma^\infty(\xi)$.

This shows our claim. ❐

As an important consequence of the metric compatibility of $\widetilde{\nabla}$, (6.179) in Proposition 6.10.15 holds, from which we can immediatly deduce:

COROLLARY 6.7.5. *From* Lemma 6.7.4 *and* Proposition 6.10.15 *it follows that*

$$\langle \widetilde{D}\sigma \mid \tau \rangle_{2,\xi} = \langle \sigma \mid \widetilde{D}\tau \rangle_{2,\xi} \qquad \forall \sigma \in \Gamma_0^\infty(\xi),\ \forall \tau \in \Gamma^\infty(\xi) \quad [\textit{resp. } \forall \sigma \in \Gamma^\infty(\xi),\ \forall \tau \in \Gamma_0^\infty(\xi)], \tag{6.85}$$

and for all simply connected open subsets $U \subseteq M$,

$$\langle \widetilde{D}_U\sigma \mid \tau \rangle_{2,\xi|_U} = \langle \sigma \mid \widetilde{D}_U\tau \rangle_{2,\xi|_U} \qquad \forall \sigma \in \Gamma_0^\infty(\xi|_U),\ \forall \tau \in \Gamma^\infty(\xi|_U) \quad [\textit{resp. } \forall \sigma \in \Gamma^\infty(\xi|_U),\ \forall \tau \in \Gamma_0^\infty(\xi|_U)]. \tag{6.86}$$

In view of constructing a symplectic space for the classical field theory from the solutions of the homogeneous O(n)-twisted Klein-Gordon equation (6.77), we prove the following statement:

LEMMA 6.7.6.

$$\langle \sigma \mid \widetilde{G}\tau \rangle_{2,\xi} = -\langle \widetilde{G}\sigma \mid \tau \rangle_{2,\xi} \qquad \forall \sigma, \tau \in \Gamma_0^\infty(\xi) \tag{6.87}$$

and for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$,

$$\langle \sigma \mid \widetilde{G}_U \tau \rangle_{2,\,\xi|_U} = -\,\langle \widetilde{G}_U \sigma \mid \tau \rangle_{2,\,\xi|_U} \qquad \forall \sigma, \tau \in \Gamma_0^\infty\left(\xi|_U\right). \tag{6.88}$$

Proof: We only prove (6.88); (6.87) follows from a standard smooth partition of unity argument. Hence, let $U \in \mathrm{loc}^s_{-\mathbf{M}}$:

$$\langle \sigma \mid \widetilde{G}_U \tau \rangle_{2,\,\xi|_U} = \int_U \langle \sigma \mid \widetilde{G}_U \tau \rangle_{\xi|_U} \,\mathrm{vol}_U \tag{6.89}$$

$$= \int_U \langle \theta_{U*}\sigma \mid G_U \theta_{U*}\tau \rangle_{\mathrm{Eucl}_U} \,\mathrm{vol}_U \tag{6.90}$$

$$= \mathfrak{u}_U\left(\theta_{U*}\sigma, G_U \theta_{U*}\tau\right) \tag{6.91}$$

$$= -\mathfrak{u}_U\left(G_U\theta_{U*}\sigma, \theta_{U*}\tau\right) \tag{6.92}$$

$$= -\,\langle \widetilde{G} \sigma \mid \tau \rangle_{2,\,\xi|_U} \qquad \forall \sigma, \tau \in \Gamma_0^\infty\left(\xi|_U\right), \tag{6.93}$$

where $\mathfrak{u}_U$ is the symplectic form of the symplectic space $\mathcal{F}U$, see (6.59) and (6.60).❐

We now introduce symplectic spaces for the classical field theory of $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields:

PROPOSITION 6.7.7. *The maps*

$$\widetilde{\mathfrak{u}} : \Gamma_0^\infty(\xi) \times \Gamma_0^\infty(\xi) \longrightarrow \mathbb{R}, \qquad (\sigma, \tau) \longmapsto \langle \sigma \mid \widetilde{G}\tau \rangle_{2,\,\xi}, \tag{6.94}$$

and for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$,

$$\widetilde{\mathfrak{u}}_U : \Gamma_0^\infty\left(\xi|_U\right) \times \Gamma_0^\infty\left(\xi|_U\right) \longrightarrow \mathbb{R}, \qquad (\sigma, \tau) \longmapsto \langle \sigma \mid \widetilde{G}_U\tau \rangle_{2,\,\xi|_U}, \tag{6.95}$$

are skew-symmetric bilinear forms having radicals $\widetilde{D}\Gamma_0^\infty(\xi)$ *and* $\widetilde{D}_U\Gamma_0^\infty\left(\xi|_U\right)$. *Hence,*

$$\begin{cases} \left[\Gamma_0^\infty(\xi)\right] := \Gamma_0^\infty(\xi) / \widetilde{D}\Gamma_0^\infty(\xi), \\ \widetilde{\mathfrak{u}} : \left[\Gamma_0^\infty(\xi)\right] \times \left[\Gamma_0^\infty(\xi)\right] \longrightarrow \mathbb{R}, \ \left([\sigma], [\tau]\right) \longmapsto \langle \sigma \mid \widetilde{G}\tau \rangle_{2,\,\xi}, \end{cases} \tag{6.96}$$

and for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$,

$$\begin{cases} \left[\Gamma_0^\infty\left(\xi|_U\right)\right] := \Gamma_0^\infty\left(\xi|_U\right) / \widetilde{D}_U\Gamma_0^\infty\left(\xi|_U\right), \\ \widetilde{\mathfrak{u}}_U : \left[\Gamma_0^\infty\left(\xi|_U\right)\right] \times \left[\Gamma_0^\infty\left(\xi|_U\right)\right] \longrightarrow \mathbb{R}, \ \left([\sigma], [\tau]\right) \longmapsto \langle \sigma \mid \widetilde{G}_U\tau \rangle_{2,\,\xi|_U}, \end{cases} \tag{6.97}$$

are symplectic spaces.

Proof: Since the bilinear pairing $\langle \cdot \mid \cdot \rangle_{2,\,\xi} : \Gamma_0^\infty(\xi) \times \Gamma^\infty(\xi) \longrightarrow \mathbb{R}$ is weakly non-degenerate, $\langle \sigma \mid \widetilde{G}\tau \rangle_{2,\,\xi} = 0$ for all $\sigma \in \Gamma_0^\infty(\xi)$ implies $\widetilde{G}\tau = 0$ and thus $\tau = \widetilde{D}\upsilon$ for some $\upsilon \in \Gamma_0^\infty(\xi)$ by [BGP07, Thm.3.4.7] or [Wal12, Thm.4.3.18]. Similarly, because the bilinear pairing $\langle \cdot \mid \cdot \rangle_{2,\xi} : \Gamma^\infty(\xi) \times \Gamma_0^\infty(\xi) \longrightarrow \mathbb{R}$ is weakly non-degenerate, $\langle \sigma \mid \widetilde{G}\tau \rangle_{2,\xi} = -\langle \widetilde{G}\sigma \mid \tau \rangle_{2,\xi} = 0$ for all $\tau \in \Gamma_0^\infty(\xi)$ implies $\widetilde{G}\sigma = 0$. Hence, $\sigma = \widetilde{D}\upsilon$ for some $\upsilon \in \Gamma_0^\infty(\xi)$. The same arguments apply if we restrict to $U \in \mathrm{loc}^s_{-\mathbf{M}}$. ❒

With this, we have achieved our goal to construct $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields of the same mass in the spirit of C.J. Isham. The twisted classical field theory is given by the symplectic space (6.96) and applying the quantisation functor $Q : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$ to it, we obtain the twisted quantum field theory in terms of the simple unital *-algebra which is generated by the elements of the form $\widetilde{\mathbf{\Phi}}(\sigma)$, $\sigma \in \Gamma_0^\infty(\xi)$, which satisfy the conditions:

- Linearity: $\widetilde{\mathbf{\Phi}}(\lambda\sigma + \mu\tau) = \lambda\widetilde{\mathbf{\Phi}}(\sigma) + \mu\widetilde{\mathbf{\Phi}}(\tau)$ for all $\lambda, \mu \in \mathbb{R}$ and for all $\sigma, \tau \in \Gamma_0^\infty(\xi)$.
- Hermiticity: $\widetilde{\mathbf{\Phi}}(\sigma)^* = \widetilde{\mathbf{\Phi}}(\sigma)$ for all $\sigma \in \Gamma_0^\infty(\xi)$.
- Field equations (in a weak sense): $\widetilde{\mathbf{\Phi}}\big(\widetilde{D}\sigma\big) = 0$ for all $\sigma \in \Gamma_0^\infty(\xi)$.
- Commutation relations: $\big[\widetilde{\mathbf{\Phi}}(\sigma), \widetilde{\mathbf{\Phi}}(\tau)\big] = \mathrm{i}\,\hbar\, \langle \sigma \mid \widetilde{G}\tau \rangle_{2,\xi} \cdot 1_A$ for all $\sigma, \tau \in \Gamma_0^\infty(\xi)$.

Our next objective is to show that this fits perfectly into our general scheme for twisted variants of locally covariant theories. To this end, we have to exhibit the functoriality of our construction so far. In doing this, we will profit from our emphasis on the local description and statements for $U \in \mathrm{loc}^s_{-\mathbf{M}}$.

6.8 Connection to the general scheme

We show that the $\mathrm{O}(n)$*-twisted free and minimally coupled real scalar fields constructed in the spirit of C.J. Isham fit into our abstract categorical scheme for twisted variants of locally covariant theories.*

We continue working under the assumptions of Section 6.7 and with its results. In order to make contact with our general scheme (Section 6.2 + 6.4), we need to expose the (local) functorial properties of the construction of the $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields, which is the reason why we have always stated the results of Section 6.7 also in terms of $U \in \mathrm{loc}^s_{-\mathbf{M}}$. Considering $\mathrm{loc}^s_{-\mathbf{M}}$, we will form a functor $\mathcal{T}'' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ and, applying the quantisation functor $Q : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$, a functor $\mathfrak{T}'' : \mathrm{loc}^s_q \longrightarrow {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$, which can in principle already be regarded [up to their domain category, which is a skeleton of $(K_s \downarrow \mathbf{M})$] as twisted variants of $\mathcal{U}$ and $\mathfrak{U}$. However, in order to implement condition (3) of Definition 6.2.1, we will rather consider functors $\mathcal{T}' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ and $\mathfrak{T}' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$, which are naturally isomorphic to $\mathcal{T}''$ and $\mathfrak{T}''$ and satisfy $\mathcal{T}'U = \mathcal{U}U$ and $\mathfrak{T}'U = \mathfrak{U}U$ for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$. Finally, we will extend $\mathcal{T}'$ and $\mathfrak{T}'$ to functors $\mathcal{T} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ and $\mathfrak{T} : (K_s \downarrow \mathbf{M}) \longrightarrow$ $\mathbf{{}^*Alg_1^m}$, of which we will show that they are twisted variants of $\mathcal{U}$ and $\mathfrak{U}$ according to our general scheme. In doing all this, we will also collect some results for the computation of colimits in Section 6.9, which will recover the symplectic space (6.96) and the resulting simple unital *-algebra as the global structures for the twisted variants of our prototype functors $\mathcal{U} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ and $\mathfrak{U} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{{}^*Alg_1^m}$, $\mathbf{M} \in \mathbf{Loc}$, in our abstract scheme.

As the first step to the construction of $\mathcal{T}''$ and $\mathfrak{T}''$, we show that a $\mathrm{loc}^s_{-\mathbf{M}}$-morphism $\iota_{UV} : U \longrightarrow V$ gives rise to a $\mathbf{Sympl}_{\mathbb{R}}$-morphim $([\Gamma_0^\infty(\xi|_U)], \widetilde{u}_U) \longrightarrow ([\Gamma_0^\infty(\xi|_V)], \widetilde{u}_V)$ in a functorial way via the bundle inclusion $\mathbf{i}_{\xi|_U \xi|_V} : \xi|_U \longrightarrow \xi|_V$. The pushforward $\mathbf{i}_{\xi|_U \xi|_V *} : \Gamma_0^\infty(\xi|_U) \longrightarrow \Gamma_0^\infty(\xi|_V)$ along the bundle inclusion is an injective linear map satisfying $\mathbf{i}_{\xi|_U \xi|_V *} \widetilde{D}_U = \widetilde{D}_V \, \mathbf{i}_{\xi|_U \xi|_V *}$, which is seen from $\widetilde{D}_U = \mathbf{i}^\#_{\xi|_U \xi|_V} \widetilde{D}_V \, \mathbf{i}_{\xi|_U \xi|_V *}$, i.e.

$$
\begin{aligned}
&(6.98) && \mathbf{i}^\#_{\xi|_U \xi|_V} \widetilde{D}_V \, \mathbf{i}_{\xi|_U \xi|_V *}\sigma = \left(\widetilde{D}_V \, \mathbf{i}_{\xi|_U \xi|_V *}\sigma\right)|_U \\
&(6.99) && = \left(\left(\widetilde{D} \, \mathbf{i}_{\xi|_U *}\sigma\right)|_V\right)|_U \\
&(6.100) && = \left(\widetilde{D} \, \mathbf{i}_{\xi|_U *}\sigma\right)|_U \\
&(6.101) && = \widetilde{D}_U \sigma \qquad \forall \sigma \in \Gamma_0^\infty(\xi|_U),
\end{aligned}
$$

and the fact that pullback $\mathbf{i}^\#_{\xi|_U \xi|_V} : \Gamma^\infty(\xi|_V) \longrightarrow \Gamma^\infty(\xi|_U)$ induces the left inverse of $\mathbf{i}_{\xi|_U \xi|_V *}$. Accordingly, $\mathbf{i}_{\xi|_U \xi|_V *}$ gives rise to a unique linear map $[\mathbf{i}_{\xi|_U \xi|_V *}] : [\Gamma_0^\infty(\xi|_U)] \longrightarrow [\Gamma_0^\infty(\xi|_V)]$ such that $[\mathbf{i}_{\xi|_U \xi|_V *}] \circ \pi_{\xi|_U} = \pi_{\xi|_V} \circ \mathbf{i}_{\xi|_U \xi|_V *}$, where $\pi_{\xi|_{U|V}} : \Gamma_0^\infty(\xi|_{U|V}) \longrightarrow\!\!\!\!\rightarrow [\Gamma_0^\infty(\xi|_{U|V})]$ denote the canonical projections onto the quotients. Furthermore, this unique linear map is injective and symplectic:

$$
\begin{aligned}
&(6.102) && \widetilde{\mathbf{u}}_V\left([\mathbf{i}_{\xi|_U \xi|_V *}][\sigma], [\mathbf{i}_{\xi|_U \xi|_V *}][\tau]\right) = \widetilde{\mathbf{u}}_V\left([\mathbf{i}_{\xi|_U \xi|_V *}\sigma], [\mathbf{i}_{\xi|_U \xi|_V *}\tau]\right) \\
&(6.103) && = \left\langle \mathbf{i}_{\xi|_U \xi|_V *}\sigma \;\middle|\; \widetilde{G}_V \, \mathbf{i}_{\xi|_U \xi|_V *}\tau \right\rangle_{2, \xi|_V} \\
&(6.104) && = \int_V \left\langle \mathbf{i}_{\xi|_U \xi|_V *}\sigma \;\middle|\; \widetilde{G}_V \, \mathbf{i}_{\xi|_U \xi|_V *}\tau \right\rangle_{\xi|_V} \mathrm{vol}_V \\
&(6.105) && = \int_U \left\langle \mathbf{i}_{\xi|_U \xi|_V *}\sigma \;\middle|\; \widetilde{G}_V \, \mathbf{i}_{\xi|_U \xi|_V *}\tau \right\rangle_{\xi|_V} |_U \, \mathrm{vol}_U \\
&(6.106) && = \int_U \left\langle \sigma \;\middle|\; (\widetilde{G}_V \, \mathbf{i}_{\xi|_U \xi|_V *}\tau)|_U \right\rangle_{\xi|_U} \mathrm{vol}_U \\
&(6.107) && = \int_U \langle \sigma \mid \widetilde{G}_U \tau \rangle_{\xi|_U} \mathrm{vol}_U \\
&(6.108) && = \langle \sigma \mid \widetilde{G}_U \tau \rangle_{2, \xi|_U} \\
&(6.109) && = \widetilde{\mathbf{u}}_U([\sigma], [\tau]) \qquad \forall [\sigma], [\tau] \in [\Gamma_0^\infty(\xi|_U)].
\end{aligned}
$$

A simple application of (UQ') shows that if $\iota_{VW} : V \longrightarrow W$ is another $\mathrm{loc}^s_{-\mathbf{M}}$-morphism, then $[\mathfrak{i}_{\xi|_V \xi|_W *}] \circ [\mathfrak{i}_{\xi|_U \xi|_V *}] = [\mathfrak{i}_{\xi|_U \xi|_W *}]$. We have thus proven:

PROPOSITION 6.8.1. *The rules*

$$\mathcal{T}''U := \left([\Gamma^\infty(\xi|_U)], \widetilde{\mathfrak{u}}_U\right) \qquad \forall U \in \mathrm{loc}^s_{-\mathbf{M}}, \tag{6.110}$$

and

$$\mathcal{T}''\iota_{UV} := [\mathfrak{i}_{\xi|_U \xi|_V *}] \qquad \forall U, V \in \mathrm{loc}^s_{-\mathbf{M}} \text{ such that } U \subseteq V, \tag{6.111}$$

define a functor $\mathcal{T}'' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ *and, applying the quantisation functor* Q : $\mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{{}^*Alg^m_1}$*, a functor* $\mathfrak{T}'' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{{}^*Alg^m_1}$.

As we have said, $\mathcal{T}''$ and $\mathfrak{T}''$ are only intermediate functors on the search for twisted variants of $\mathcal{U}$ and $\mathfrak{U}$ since we want to incorporate the condition (3) of Definition 6.2.1. We will consider naturally isomorphic functors $\mathcal{T}' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ and $\mathfrak{T}' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{{}^*Alg^m_1}$ instead. Before we do, we collect some result concerning the construction of colimits for later purposes, see Section 6.9. To be specific, we compute the corresponding colimit for $\mathcal{T}'' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ in the category $\mathbf{Vec}_{\mathbb{R}}$:

LEMMA 6.8.2. *Let* $F_\omega : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ *be the forgetful functor that forgets about the symplectic form.*

$$\begin{aligned}\operatorname{colim}&\left(F_\omega \circ \mathcal{T}'' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}\right) \\ &= \left([\Gamma^\infty_0(\xi)], \left\{[\mathfrak{i}_{\xi|_U *}] : (F_\omega \circ \mathcal{T}'')U \longrightarrow [\Gamma^\infty_0(\xi)]\right\}_{U \in \mathrm{loc}^s_{-\mathbf{M}}}\right).\end{aligned} \tag{6.112}$$

Proof: Let $F : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ be the functor of Lemma 4.4.1 with $q = s$ and define a functor $G : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ by $GU := \widetilde{D}_U \Gamma^\infty(\xi|_U)$ for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$ and $G\iota_{UV} := \mathfrak{i}_{\xi|_U \xi|_V *} : \widetilde{D}_U \Gamma^\infty(\xi|_U) \longrightarrow \widetilde{D}_V \Gamma^\infty(\xi|_V)$ for all $U, V \in \mathrm{loc}^s_{-\mathbf{M}}$ with $U \subseteq V$. Then surely the requisites of Lemma 4.4.2 are fulfilled and it holds that $\mathcal{T}'' = [F]$. Hence, (6.112) follows from Lemma 4.4.1, Lemma 4.4.3 and Example 2.3.11. ❐

We now define functors $\mathcal{T}'$, $\mathfrak{T}'$: $\mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$, $\mathbf{{}^*Alg^m_1}$, which are naturally isomorphic to $\mathcal{T}''$, $\mathfrak{T}''$: $\mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$, $\mathbf{{}^*Alg^m_1}$ and satisfy $\mathcal{T}'U = \mathcal{U}'U$ and $\mathfrak{T}'U = \mathfrak{U}'U$ for all $U \in \mathrm{loc}^s_{-\mathbf{M}}$:

PROPOSITION 6.8.3. *The functor* $\mathcal{T}' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ *defined by*

$$\mathcal{T}'U := \mathcal{U}U \qquad \forall U \in \mathrm{loc}^s_{-\mathbf{M}} \tag{6.113}$$

and

$$\mathcal{T}'\iota_{UV} := \theta_{V*} \circ \mathcal{T}''\iota_{UV} \circ \theta_U^{\#} \qquad \forall U, V \in \mathrm{loc}^s_{-\mathbf{M}} \text{ such that } U \subseteq V \tag{6.114}$$

is naturally isomorphic to $\mathcal{T}''$. *A natural isomorphism* $\Theta : \mathcal{T}' \xrightarrow{\dot\sim} \mathcal{T}''$ *is defined by* $\Theta_U := \left[\theta_U^{\#}\right]$ *for all* $U \in \mathrm{loc}^s_{-\mathbf{M}}$, *where* $\left[\theta_U^{\#}\right]$ *is determined by the unique linear map* $\left[\theta_U^{\#}\right] : [\Gamma^\infty(\underline{\mathbb{R}^n}_U)] \longrightarrow [\Gamma^\infty(\xi|_U)]$ *satisfying* $\left[\theta_U^{\#}\right] \circ \pi_U = \pi_{\xi|_U} \circ \theta_U^{\#}$. *As a consequence,* $\mathfrak{T}' := Q \circ \mathcal{T}' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ *is naturally isomorphic to* $\mathfrak{T}''$ *and a natural isomorphism is given by* $Q \star \Theta : Q \circ \mathcal{T}' \xrightarrow{\dot\sim} Q \circ \mathcal{T}''$.

For the purpose of Section 6.9, we give the analogue of Lemma 6.8.2 for $\mathcal{T}'$:

COROLLARY 6.8.4. *Let* $F_\omega : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ *be the forgetful functor that forgets about the symplectic form.*

$$\begin{aligned} &\mathrm{colim}\left(F_\omega \circ \mathcal{T}' : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}\right) \\ &\qquad = \Big([\Gamma^\infty_0(\xi)], \left\{ [\mathbf{i}_{\xi|_U *}] \circ \Theta_U : (F_\omega \circ \mathcal{T}')\,U \longrightarrow [\Gamma^\infty_0(\xi)] \right\}_{U \in \mathrm{loc}^s_{-\mathbf{M}}} \Big). \end{aligned} \tag{6.115}$$

Proof: Apply Lemma 2.2.12. ❐

In order for $\mathcal{T}'$ and $\mathfrak{T}'$ to become twisted variants of $\mathcal{U}$ and $\mathfrak{U}$, we need to extend them from their domain category $\mathrm{loc}^s_{-\mathbf{M}}$ to the category $(K_s \downarrow \mathbf{M})$. We recall Lemma 2.2.21 and Corollary 2.2.22 for the followig proposition:

PROPOSITION 6.8.5. *A functor* $\mathcal{T} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ *is defined by*

$$\mathcal{T}(\mathbf{A} \xrightarrow{f} \mathbf{M}) := \mathcal{U}(\mathbf{A} \xrightarrow{f} \mathbf{M}) \qquad \forall\, \mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_s \downarrow \mathbf{M}) \tag{6.116}$$

and by

$$\mathcal{T}\left(\begin{array}{ccc} \mathbf{B} & \overset{g}{\searrow} & \\ h \uparrow & & \mathbf{M} \\ \mathbf{A} & \underset{f}{\nearrow} & \end{array}\right) := \mathcal{U}\left(\begin{array}{ccc} g(\mathbf{B}) & \overset{\iota_{g(B)}}{\searrow} & \\ g\|_B \uparrow & & \mathbf{M} \\ \mathbf{B} & \underset{g}{\nearrow} & \end{array}\right)^{-1} \circ \mathcal{T}'\iota_{f(A)g(B)} \circ \mathcal{U}\left(\begin{array}{ccc} f(\mathbf{A}) & \overset{\iota_{f(A)}}{\searrow} & \\ f\|_A \uparrow & & \mathbf{M} \\ \mathbf{A} & \underset{f}{\nearrow} & \end{array}\right) \tag{6.117}$$

$$\forall \begin{array}{ccc} \mathbf{B} & \overset{g}{\searrow} & \\ h \uparrow & & \mathbf{M} \\ \mathbf{A} & \underset{f}{\nearrow} & \end{array} \in (K_s \downarrow \mathbf{M})\big(\mathbf{A}, \mathbf{B} \xrightarrow{f,g} \mathbf{M}\big),\ \forall\, \mathbf{A}, \mathbf{B} \xrightarrow{f,g} \mathbf{M} \in (K_s \downarrow \mathbf{M}).$$

A functor $\mathfrak{T} : (K_s \downarrow \mathbf{M}) \longrightarrow {}^*\mathbf{Alg}^{\mathbf{m}}_{\mathbf{1}}$ *is defined by* $\mathfrak{T} := Q \circ \mathcal{T}$.

In view of Section 6.9, we collect the analogues of Lemma 6.8.2 and Corollary 6.8.4 for $\mathcal{T}$, i.e. we compute the corresponding colimit of $\mathcal{T}$ in $\mathbf{Vec}_{\mathbb{R}}$:

COROLLARY 6.8.6. *Let $F_\omega : \mathbf{Sympl}_\mathbb{R} \longrightarrow \mathbf{Vec}_\mathbb{R}$ be the forgetful functor that forgets about the symplectic form.*

$$\text{(6.118)}\quad \operatorname{colim}\left(F_\omega \circ \mathcal{T} \colon (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Vec}_\mathbb{R}\right) = \Big(\left[\Gamma_0^\infty(\xi)\right], u : (F_\omega \circ \mathcal{T}) \stackrel{.}{\longrightarrow} \Delta\left[\Gamma_0^\infty(\xi)\right]\Big),$$

where the universal cocone u is given by

$$\text{(6.119)}\qquad u_{\mathbf{A}\xrightarrow{f}\mathbf{M}} = \left[\mathfrak{i}_{\xi|_{f(A)}*}\right] \circ \Theta_{f(A)} \circ (F_\omega \circ \mathcal{T})\, \eta_{\mathbf{A}\xrightarrow{f}\mathbf{M}}, \qquad \mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_s \downarrow \mathbf{M}).$$

Here, $\eta : \mathrm{Id}_{(K_s \downarrow \mathbf{M})} \stackrel{\sim}{\stackrel{.}{\longrightarrow}} I \circ E$ denotes the natural isomorphism of Lemma 2.2.21 *with the inclusion functor $I : \mathrm{loc}^s_{-\mathbf{M}} \longrightarrow (K_s \downarrow \mathbf{M})$ and the equivalence $E : (K_s \downarrow \mathbf{M}) \longrightarrow \mathrm{loc}^s_{-\mathbf{M}}$.*

We now show that our candidate functors $\mathcal{T}, \mathfrak{T} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_\mathbb{R}, {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$ of Proposition 6.8.5, which were constructed from the non-trivial locally constant $\mathrm{O}(n)$-cocycle c, (6.66), are twisted variants of $\mathcal{U}, \mathfrak{U} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_\mathbb{R}, {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$ according to Definition 6.3.6, that is:

THEOREM 6.8.7. *$\mathcal{T} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_\mathbb{R}$ (resp. $\mathfrak{T} : (K_s \downarrow \mathbf{M}) \longrightarrow {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$) is a $(\mathcal{U}, \mathrm{O}(n))$-functor [resp. $(\mathfrak{U}, \mathrm{O}(n))$-functor] and a twisted variant of $\mathcal{U} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_\mathbb{R}$ (resp. $\mathfrak{U} : (K_s \downarrow \mathbf{M}) \longrightarrow {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$), which both correspond to the non-trivial locally constant $\mathrm{O}(n)$-cocycle c in* (6.66).

The theorem is shown by the following lemma, whose proof is evident:

LEMMA 6.8.8. *Let $\mathcal{C} = \mathbf{Sympl}_\mathbb{R}, {}^*\mathbf{Alg}_\mathbf{1}^\mathbf{m}$, $\mathscr{U} = \mathcal{U}, \mathfrak{U}$, $\mathscr{T} = \mathcal{T}, \mathfrak{T}$ and $\mathscr{G} = \mathcal{G}, \mathfrak{G}$. For each $(K_s \downarrow \mathbf{M})$-morphism* $\begin{array}{ccc} \mathbf{B} & \xrightarrow{g} & \\ h\uparrow & & \mathbf{M} \\ \mathbf{A} & \xrightarrow{f} & \end{array} : (\mathbf{A} \xrightarrow{f} \mathbf{M}) \longrightarrow (\mathbf{B} \xrightarrow{g} \mathbf{M})$, *there is a $\mathcal{C}$-isomorphism* $\mathscr{G}\left(\begin{array}{ccc} \mathbf{B} & \xrightarrow{g} & \\ h\uparrow & & \mathbf{M} \\ \mathbf{A} & \xrightarrow{f} & \end{array}\right) : \mathscr{T}(\mathbf{A} \xrightarrow{f} \mathbf{M}) \stackrel{\sim}{\longrightarrow} \mathscr{U}(\mathbf{A} \xrightarrow{f} \mathbf{M})$ *such that the identity*

$$\text{(6.120)}\quad \mathscr{G}\left(\begin{array}{ccc} \mathbf{C} & \xrightarrow{i} & \\ j\uparrow & & \mathbf{M} \\ \mathbf{B} & \xrightarrow{g} & \end{array}\right) \circ \mathscr{T}\left(\begin{array}{ccc} \mathbf{B} & \xrightarrow{g} & \\ h\uparrow & & \mathbf{M} \\ \mathbf{A} & \xrightarrow{f} & \end{array}\right)$$
$$= \mathscr{U}\left(\begin{array}{ccc} \mathbf{B} & \xrightarrow{g} & \\ h\uparrow & & \mathbf{M} \\ \mathbf{A} & \xrightarrow{f} & \end{array}\right) \circ \mathscr{G}\left(\begin{array}{ccc} \mathbf{C} & \xrightarrow{i} & \\ j\circ h\uparrow & & \mathbf{M} \\ \mathbf{A} & \xrightarrow{f} & \end{array}\right)$$

holds for every composition of $(K_s \downarrow \mathbf{M})$*-morphisms* $\begin{array}{ccc} \mathbf{C} & \searrow^{i} & \\ {\scriptstyle j}\uparrow & & \mathbf{M} \\ \mathbf{B} & \nearrow_{g} & \end{array} \circ \begin{array}{ccc} \mathbf{B} & \searrow^{g} & \\ {\scriptstyle h}\uparrow & & \mathbf{M} \\ \mathbf{A} & \nearrow_{f} & \end{array}$. *For* $\mathscr{G} = \mathcal{G}$*, this symplectic, bijective and linear map is given by* $\mathcal{U}\left(\begin{array}{ccc} f(\mathbf{A}) & \searrow^{\iota_{f(A)}} & \\ {\scriptstyle f\|_A}\uparrow & & \mathbf{M} \\ \mathbf{A} & \nearrow_{f} & \end{array}\right)^{-1} \circ$ $[(\theta_{g(B)}\|_{f(A)})_*] \circ [\theta^{\#}_{f(A)}] \circ \mathcal{U}\left(\begin{array}{ccc} f(\mathbf{A}) & \searrow^{\iota_{f(A)}} & \\ {\scriptstyle f\|_A}\uparrow & & \mathbf{M} \\ \mathbf{A} & \nearrow_{f} & \end{array}\right)$ *and for* $\mathscr{G} = \mathfrak{G}$*, this unital *-isomorphism is given by* $Q\left(\mathcal{G}\left(\begin{array}{ccc} \mathbf{B} & \searrow^{g} & \\ {\scriptstyle h}\uparrow & & \mathbf{M} \\ \mathbf{A} & \nearrow_{f} & \end{array}\right)\right)$. *Furthermore, with these* $\mathcal{C}$*-isomorphisms, if we denote for every non-empty hom-set* $(K_s \downarrow \mathbf{M})(\mathbf{A}, \mathbf{B} \xrightarrow{f,g} \mathbf{M})$ *the automorphisms of* $\mathscr{U}$ *associated with* $R_{f(A)g(B)}, R_{g(B)f(A)} \in \mathrm{O}(n)$ *by* $g(A;B), g(B;A) : \mathscr{U} \xrightarrow{\dot{\sim}} \mathscr{U}$*, then for each composition of* $(K_s \downarrow \mathbf{M})$*-morphisms* $\begin{array}{ccc} \mathbf{C} & \searrow^{i} & \\ {\scriptstyle j}\uparrow & & \mathbf{M} \\ \mathbf{B} & \nearrow_{g} & \end{array} \circ \begin{array}{ccc} \mathbf{B} & \searrow^{g} & \\ {\scriptstyle h}\uparrow & & \mathbf{M} \\ \mathbf{A} & \nearrow_{f} & \end{array}$*, the identities*

(6.121) $$g(B;C)_A = \mathscr{G}\left(\begin{array}{ccc} \mathbf{C} & \searrow^{i} & \\ {\scriptstyle j\circ h}\uparrow & & \mathbf{M} \\ \mathbf{A} & \nearrow_{f} & \end{array}\right) \circ \mathscr{G}\left(\begin{array}{ccc} \mathbf{B} & \searrow^{g} & \\ {\scriptstyle h}\uparrow & & \mathbf{M} \\ \mathbf{A} & \nearrow_{f} & \end{array}\right)^{-1}$$

and

(6.122) $$g(C;B) = g(B;C)^{-1}$$

hold.

Pay attention to the fact for $\mathbf{A}, \mathbf{B} \xrightarrow{f,g} \mathbf{M}$ and $h : \mathbf{A} \longrightarrow \mathbf{B}$, the identity $g \circ h = f$ implies $g\|_B^{-1} \circ \iota_{f(A)g(B)} = h \circ f\|_A^{-1}$:

(6.123) $$g \circ h = f \Longleftrightarrow \iota_{g(B)} \circ g\|_B \circ h = \iota_{f(A)} f\|_A = \iota_{g(B)} \circ \iota_{f(A)g(B)} \circ f\|_A$$

(6.124) $$\Longrightarrow g\|_B \circ h = \iota_{f(A)g(B)} \circ f\|_A$$

(6.125) $$\Longrightarrow g\|_B \circ h \circ f\|_A^{-1} = \iota_{f(A)g(B)}.$$

Also observe the important identity $\left([(\theta_{g(B)}\|_{f(A)})_*] \circ [\theta^{\#}_{f(A)}]\right)[\sigma] = [R_{f(A)g(B)}\sigma]$ for all

$[\sigma] \in [\Gamma_0^\infty(\underline{\mathbb{R}^n}_{f(A)})]$.

6.9 Further properties of O(n)-twisted free and min. coupled real scalar fields

We investigate further properties of O(n)*-twisted free and minimally coupled real scalar fields such as the time-slice axiom, the relative Cauchy evolution, the stress-energy-momentum tensor and the issue of dynamical locality. Note that, due to the lack of time, we will not finish our discussion of dynamical locality.*

We continue to work with the assumptions, notations and results of Sections 6.5-6.8. Also recall that we have said in Section 6.6 that either of the following two cases is on hand: mass > 0 and fixed spacetime dimension ≥ 2 or mass = 0 and fixed spacetime dimension ≥ 3. Our first result addresses global aspects. Indeed, since it was excluded that $\mathbf{M} \in \mathbf{Loc}$ is simply connected, no symplectic space resp. unital *-algebra is associated with $\mathbf{M}$ by $\mathcal{T} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ resp. $\mathfrak{T}\,(K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{{}^*Alg_1^m}$. The following proposition asserts that the symplectic space (6.96) and the simple unital *-algebra resulting from the application of the quantisation functor $Q : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{{}^*Alg_1^m}$, which were considered in the twisted classical and twisted quantum field theory of $n \geq 1$ free and minimally coupled real scalar fields of the same mass ≥ 0 in the sense of C.J. Isham, are the universal objects of the colimits for $\mathcal{T}$ and $\mathfrak{T}$, hence preferred choices for a global symplectic space and a global unital *-algebra associated with $\mathbf{M}$:

THEOREM 6.9.1. *Let* $\mathcal{T}, \mathfrak{T} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, \mathbf{{}^*Alg_1^m}$, $\mathbf{M} \in \mathbf{Loc}$ *such that* $\pi_1(M) \neq e$, *be the twisted variants of* $\mathcal{U}, \mathfrak{U} : (K_s \downarrow \mathbf{M}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, \mathbf{{}^*Alg_1^m}$ *discussed in the previous section. Then*

$$\operatorname{colim} \mathcal{T} = \Big(([\Gamma_0^\infty(\xi)], \mathfrak{u}),\, u : \mathcal{T} \dot{\longrightarrow} \Delta([\Gamma_0^\infty(\xi)], \mathfrak{u}) \Big), \tag{6.126}$$

where u *is defined by* (6.119), *and*

$$\operatorname{colim} \mathfrak{T} \cong Q(\operatorname{colim} \mathcal{T}). \tag{6.127}$$

Proof: Thanks to Lemma 2.2.16 and Corollary 2.2.22, we only need to consider the restriction of $\mathcal{T}$ and $\mathfrak{T}$ to $\mathrm{loc}^s_{\mathbf{M}}$ in the computation of the colimits. Due to Proposition 2.1.2 and Lemma 2.2.14, we may also go over to the complexification. The rest is identical to the proof given for Proposition 4.5.6. ❐

With a global symplectic space ($\varinjlim \mathcal{T}$) and a global unital *-algebra ($\varinjlim \mathfrak{T}$) associated with $\mathbf{M}$, we would like now to address matters for O(n)-twisted free and minimally coupled real scalar fields such as the time-slice axiom, the relative Cauchy evolution, the stress-energy-momentum tensor and the dynamical net. To do so, we need to consider $\mathbf{M} = (M, g, [T], [\Omega])$ and $\mathbf{M}[h] = (M, g + h, [T_{g+h}], [\Omega])$ for each globally hyperbolic perturbation $h \in H(\mathbf{M})$ of g (recall Definition 3.4.1) at the same time, which is currently not the case.

We can effortlessly repeat our analysis of Section 6.7 + 6.8 for each $\mathbf{N} \in \mathbf{Loc}$ with the same underlying smooth manifold M as $\mathbf{M}$ and with the same non-trivial locally constant $\mathrm{O}(n)$-cocycle c, (6.66), and obtain the twisted variants

$$\mathcal{T}_{\mathbf{N}} : (K_s \downarrow \mathbf{N}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}} \qquad \text{and} \qquad \mathfrak{T}_{\mathbf{N}} : (K_s \downarrow \mathbf{N}) \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}} \tag{6.128}$$

of

$$\mathcal{U}_{\mathbf{N}}, \mathfrak{U}_{\mathbf{N}} : (K_s \downarrow \mathbf{N}) \xrightarrow{P_{\mathbf{N}}} \mathbf{Loc}_s \xrightarrow{K_s} \mathbf{Loc} \xrightarrow{\mathcal{F}, \mathfrak{F}} \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}. \tag{6.129}$$

Our aim is to combine all of the functors $\mathcal{T}_{\mathbf{N}}$ into a single functor $\bar{\mathcal{T}}$ and all of the functors $\mathfrak{T}_{\mathbf{N}}$ into a single functor $\bar{\mathfrak{T}}$, which will then enable us to discuss the time-slice axiom, the relative Cauchy evolution, the stress-energy-momentum tensor and the dynamical net.

We therefore fix the underlying smooth manifold M of $\mathbf{M} \in \mathbf{Loc}$ with $\pi_1(M) \neq e$ and consider the categories loc_M, loc_{+M}, loc^s_{-M} and loc^s_{+M} (see Section 2.1). We define functors $\hat{\mathcal{T}} : \mathrm{loc}^s_{-M} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}$ and $\hat{\mathfrak{T}} : \mathrm{loc}^s_{-M} \longrightarrow {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ via the rules

$$\hat{\mathcal{T}}\mathbf{U} \coloneqq \mathcal{F}\mathbf{U} = ([\Gamma_0^\infty(\underline{\mathbb{R}^n}_U)], \mathfrak{u}_{\mathbf{U}}) \qquad \forall \mathbf{U} \in \mathrm{loc}^s_{-M}, \tag{6.130}$$

$$\begin{aligned} \hat{\mathcal{T}}\psi_{\mathbf{UV}} : ([\Gamma_0^\infty(\underline{\mathbb{R}^n}_U)], \mathfrak{u}_{\mathbf{U}}) &\longrightarrow ([\Gamma_0^\infty(\underline{\mathbb{R}^n}_V)], \mathfrak{u}_{\mathbf{V}}) \\ [f] &\longmapsto [\mathrm{id}_V, \iota_{UV*} R_{UV}\vec{f}\,] \\ &\forall \mathbf{U}, \mathbf{V} \in \mathrm{loc}^s_{-M} \text{ such that } \mathrm{loc}^s_{-M}(\mathbf{U}, \mathbf{V}) \neq \varnothing \end{aligned} \tag{6.131}$$

and $\hat{\mathfrak{T}} \coloneqq Q \circ \hat{\mathcal{T}}$. We intend to extend $\hat{\mathcal{T}}$ and $\hat{\mathfrak{T}}$ to loc_{+M} via pointwise left Kan extension along the inclusion functor $k_s : \mathrm{loc}^s_{-M} \longrightarrow \mathrm{loc}_{+M}$, which will yield our desired functors $\bar{\mathcal{T}}$ and $\bar{\mathfrak{T}}$.

Closely examining the comma categories $(k_s \downarrow \mathbf{N})$, we realise that they are (to be more precise: isomorphic to) the more familiar category $\mathrm{loc}^s_{-\mathbf{N}}$ (if N is not simply connected) or $\mathrm{loc}^s_{+\mathbf{N}}$ (if N is simply connected), which is a skeleton of $(K_s \downarrow \mathbf{N})$ by Corollary 2.2.22. Hence, the composite functors $\hat{\mathcal{T}}_{\mathbf{N}}, \hat{\mathfrak{T}}_{\mathbf{N}} : (k_s \downarrow \mathbf{N}) \xrightarrow{p_{\mathbf{N}}} \mathrm{loc}^s_{-M} \xrightarrow{\hat{\mathcal{T}}, \hat{\mathfrak{T}}} \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$, where $p_{\mathbf{N}} : (K_s \downarrow \mathbf{N}) \longrightarrow \mathrm{loc}^s_{-M}$ denotes the projection functor (see the paragraph after Definition 2.2.19), are the same functors as (again, to be more precise: naturally isomorphic to) the restrictions of $\mathcal{T}_{\mathbf{N}}, \mathfrak{T}_{\mathbf{N}} : (K_s \downarrow \mathbf{N}) \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, {}^*\mathbf{Alg}_{\mathbf{1}}^{\mathbf{m}}$ to

$\mathrm{loc}^s_{\mathbf{N}}$. Owing to Proposition 6.9.1 (covers the cases where N is not simply connected) and Lemma 2.2.11 (covers the cases where N is simply connected), we can apply Theorem 2.2.20 and obtain the pointwise left Kan extensions of $\hat{\mathcal{T}}$ and $\hat{\mathfrak{T}}$ along k_s, which will be denoted by $\bar{\mathcal{T}}, \bar{\mathfrak{T}} : \mathrm{loc}_{+M} \longrightarrow \mathbf{Sympl}_{\mathbb{R}}, \mathbf{{}^*Alg^m_1}$. Of course, $\bar{\mathfrak{T}} = Q \circ \bar{\mathcal{T}}$.

We turn now to the discussion of the time-slice aixiom:

PROPOSITION 6.9.2. *$\bar{\mathcal{T}}$ and $\bar{\mathfrak{T}}$ obey the time-slice axiom and $\bar{\mathfrak{T}}$ is causal. If a loc_{+M}-morphism $\psi_{\mathbf{UV}} : \mathbf{U} \longrightarrow \mathbf{V}$ is Cauchy for U, V simply connected, the inverse of $\bar{\mathcal{T}}\psi_{\mathbf{UV}}$ is explicitly given by*

$$\bar{\mathcal{T}}\mathbf{V} \ni [f] \longmapsto \left[\mathrm{id}_U, \iota^*_{UV} R_{VU} \vec{f}_{\in}\right] \in \bar{\mathcal{T}}\mathbf{U} \tag{6.132}$$

for any time-slice map $\mathrm{tsm} : \Gamma^\infty_0(\underline{\mathbb{R}^n}_V) \longrightarrow \Gamma^\infty_0(\underline{\mathbb{R}^n}_V)$ *for* $(\psi_{\mathbf{UV}}, \underline{\mathbb{R}^n}_V, D_{\mathbf{V}})$ *and for any representative* $f \in \Gamma^\infty_0(\underline{\mathbb{R}^n}_V)$ *of* $[f] \in [\Gamma^\infty_0(\underline{\mathbb{R}^n}_V)]$. *If a* loc_{+M}*-morphism* $\psi : \mathbf{N} \longrightarrow \mathbf{N}'$ *is Cauchy for N, N' not simply connected, the inverse of $\bar{\mathcal{T}}\psi$ is explicitly given by*

$$\bar{\mathcal{T}}\mathbf{N}' \ni [\sigma] \longmapsto \bar{\mathcal{T}}\psi\|_N^{-1}\left[\mathbf{i}^{\#}_{\xi|_N \xi|_{N'}} \sigma_{\in}\right] \in \bar{\mathcal{T}}\mathbf{N} \tag{6.133}$$

for any time-slice map $\mathrm{tsm} : \Gamma^\infty_0(\xi|_{N'}) \longrightarrow \Gamma^\infty_0(\xi|_{N'})$ *for* $(\psi, \xi|_{N'}, \widetilde{D}_{\mathbf{N}'})$ *and for any representative* $\sigma \in \Gamma^\infty_0(\xi|_{N'})$ *of* $[\sigma] \in [\Gamma^\infty_0(\xi|_{N'})]$.

Proof: This is analogue to the proof of Proposition 5.3.1. ❒

With the time-slice axiom established, the computation of the relative Cauchy evolutions for $\bar{\mathcal{T}}$ and $\bar{\mathfrak{T}}$ follows along the same lines as in Section 5.3 and Section 5.5. We may thus skip the intermediate steps and only state the important expressions. So, let $h \in H(\mathbf{M})$, $\mathrm{tsm} : \Gamma^\infty_0(\xi) \longrightarrow \Gamma^\infty_0(\xi)$ a time-slice map for $(\imath^+_{\mathbf{M}}[h], \xi, \widetilde{D}_{\mathbf{M}})$ and $\mathrm{tsm}' : \Gamma^\infty_0(\xi) \longrightarrow \Gamma^\infty_0(\xi)$ a time-slice map for $(\jmath^-_{\mathbf{M}}[h], \xi, \widetilde{D}_{\mathbf{M}[h]})$. We also assume that tsm' is explicitly constructed like in the proof of Lemma 3.3.5, i.e. with a smooth partition of unity $\{\chi^+, \chi^-\}$ subordinated to the open cover $\{I^+_{\mathbf{M}[h]}(\Sigma_-), I^-_{\mathbf{M}[h]}(\Sigma_+)\}$ of M, where the smooth spacelike Cauchy surfaces Σ_+ and Σ_- for $\mathbf{M}[h]$ are completely contained in $M^-[h]$ such that Σ_+ lies strictly in the future of Σ_- (hence, Σ_+ and Σ_- lie in the causal past of $\mathrm{supp}\, h$ but do not intersect $\mathrm{supp}\, h$). We find

$$\mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[h][\sigma] = \left[\widetilde{D}_{\mathbf{M}}\, \chi^-\, \widetilde{G}^{\mathrm{adv}}_{\mathbf{M}[h]}\, \sigma_{\in}\right] \qquad \forall\, [\sigma] \in [\Gamma^\infty_0(\xi)] \tag{6.134}$$

and applying a Born expansion as in Section 5.4.1,

$$\text{(6.135)}\qquad \begin{aligned} \mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[h][\sigma] = \Big[\sigma &+ \big(\delta^{\widetilde{\nabla}}_{\mathbf{M}[h]} - \delta^{\widetilde{\nabla}}_{\mathbf{M}}\big)\, \mathrm{d}^{\widetilde{\nabla}}_{\mathbf{M}}\, \widetilde{G}_{\mathbf{M}}\sigma \\ &+ \big(\widetilde{D}_{\mathbf{M}[h]} - \widetilde{D}_{\mathbf{M}}\big)\, \widetilde{G}^{\mathrm{adv}}_{\mathbf{M}} \big(\widetilde{D}_{\mathbf{M}[h]} - \widetilde{D}_{\mathbf{M}}\big)\, \widetilde{G}^{\mathrm{adv}}_{\mathbf{M}[h]}\, \sigma \epsilon\Big] \end{aligned}$$

$$\forall\, [\sigma] \in [\Gamma^\infty_0(\xi)].$$

Of course, $\mathrm{rce}^{\bar{\mathfrak{T}}}_{\mathbf{M}}[h] = Q\,(\mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[h])$.

On this basis, it is not too different from Section 5.4.2 to compute the stress-energy-momentum tensor of $\bar{\mathcal{T}}$. So, let $h \in \Gamma^\infty_0(\tau^*_M \odot \tau^*_M)$ and $\varepsilon > 0$ such that $th \in H(\mathbf{M})$ for all $t \in (-\varepsilon, \varepsilon)$. Already dropping some terms of order t^2 or higher, we calculate

$$\text{(6.136)}\qquad \widetilde{\mathfrak{u}}_{\mathbf{M}}\big(\mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[th][\sigma] - [\sigma], [\tau]\big) \approx \widetilde{\mathfrak{u}}_{\mathbf{M}}\Big(\big[(\delta^{\widetilde{\nabla}}_{\mathbf{M}[h]} - \delta^{\widetilde{\nabla}}_{\mathbf{M}})\, \widetilde{G}_{\mathbf{M}}\, \mathrm{d}^{\widetilde{\nabla}}_{\mathbf{M}}\sigma\big], [\tau]\Big)$$

$$\forall\, [\sigma], [\tau] \in [\Gamma^\infty_0(\xi)]$$

up to first order in t. It is enough to understand what happens if $[\sigma]$ or $[\tau]$ (without the loss of generality, we choose $[\sigma]$) has a representative with compact support in a simply connected globally hyperbolic open subset U of $\mathbf{M}$. For general $[\sigma]$, $[\tau]$, we can use a smooth partition of unity argument.

$$\text{(6.137)}\quad \widetilde{\mathfrak{u}}_{\mathbf{M}}\Big(\big[(\delta^{\widetilde{\nabla}}_{\mathbf{M}[h]} - \delta^{\widetilde{\nabla}}_{\mathbf{M}})\, \mathrm{d}^{\widetilde{\nabla}}_{\mathbf{M}}\, \widetilde{G}_{\mathbf{M}}\sigma\big], [\tau]\Big) = \int_M \big\langle (\delta^{\widetilde{\nabla}}_{\mathbf{M}[h]} - \delta^{\widetilde{\nabla}}_{\mathbf{M}})\, \mathrm{d}^{\widetilde{\nabla}}_{\mathbf{M}}\, \widetilde{G}_{\mathbf{M}}\sigma \,\big|\, \widetilde{G}_{\mathbf{M}}\tau\big\rangle_\xi \,\mathrm{vol}_{\mathbf{M}}$$

$$\text{(6.138)}\quad = \int_U \big\langle (\delta^{\widetilde{\nabla}}_{\mathbf{M}[h]} - \delta^{\widetilde{\nabla}}_{\mathbf{M}})|_U\, \mathrm{d}^{\widetilde{\nabla}}_U\, \widetilde{G}_U\sigma \,\big|\, \widetilde{G}_U\tau|_U\big\rangle_{\xi|_U} \,\mathrm{vol}_U$$

$$\text{(6.139)}\quad = \int_U \big\langle (\delta^{\widetilde{\nabla}}_{\mathbf{M}[h]} - \delta^{\widetilde{\nabla}}_{\mathbf{M}})|_U\, \mathrm{d}^{\widetilde{\nabla}}_U\, \widetilde{\phi}_{[\sigma]}|_U \,\big|\, \widetilde{\phi}_{[\tau]}|_U\big\rangle_{\xi|_U} \,\mathrm{vol}_U$$

$$\text{(6.140)}\quad = \int_U \big\langle \theta_{U*}(\delta^{\widetilde{\nabla}}_{\mathbf{M}[h]} - \delta^{\widetilde{\nabla}}_{\mathbf{M}})|_U\, \mathrm{d}^{\widetilde{\nabla}}_U\, \widetilde{\phi}_{[\sigma]}|_U \,\big|\, \theta_{U*}\,\widetilde{\phi}_{[\tau]}|_U\big\rangle_{\mathrm{Eucl}_U} \,\mathrm{vol}_U$$

$$\text{(6.141)}\quad = \int_U \big\langle (\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})|_U\, d_U\phi \,\big|\, \psi\big\rangle_{\mathrm{Eucl}_U} \,\mathrm{vol}_U$$

$$\text{(6.142)}\quad = \sum_{i=1}^n \int_U (\delta_{\mathbf{M}[h]} - \delta_{\mathbf{M}})|_U\, d_U\phi^i \wedge *_U\psi^i,$$

where $\widetilde{\phi}_{[\sigma]} := \widetilde{G}_{\mathbf{M}}\sigma$, $\widetilde{\phi}_{[\tau]} := \widetilde{G}_{\mathbf{M}}\tau$, $\phi := \theta_{U*}\,\widetilde{\phi}_{[\sigma]}|_U$ and $\psi := \theta_{U*}\,\widetilde{\phi}_{[\tau]}|_U$. For the following calculations, it is advisable to use abstract index notation again (see [Wal84, Sec.2.4 + 3.1]) and to recall (5.39)-(5.43). From [FR04, (229) + (231)], we conclude up to first order in t:

$$\text{(6.143)}\qquad \big(\delta_{\mathbf{M}[th]} - \delta_{\mathbf{M}}\big)|_U\, d_U\phi^i \approx t\Big(\nabla_a\big(h^{ab}\,\nabla_b\phi^i\big) - \frac{1}{2}\,(\nabla_b h^a_a)\,\nabla^b\phi^i\Big),$$

where ∇ stands for the Levi-Civita connection on U. From this, we can already infer the existence of a linear map $\widetilde{T}_{\mathbf{M}}(h) : [\Gamma_0^\infty(\xi)] \longrightarrow [\Gamma_0^\infty(\xi)]$ satisfying

$$\widetilde{\mathfrak{u}}_{\mathbf{M}}\left(\widetilde{T}_{\mathbf{M}}(h)[\sigma],[\tau]\right) = \frac{d}{dt}\widetilde{\mathfrak{u}}_{\mathbf{M}}\left(\mathrm{rce}_{\mathbf{M}}^{\bar{\mathcal{T}}}[th][\sigma],[\tau]\right)\Big|_{t=0} \quad \forall\,[\sigma],[\tau] \in [\Gamma_0^\infty(\xi)]. \tag{6.144}$$

$\widetilde{T}_{\mathbf{M}}(h)$ is given by

$$\widetilde{T}_{\mathbf{M}}(h)[\sigma] = \nabla_a\left(h^{ab}\,\widetilde{\nabla}_b\widetilde{G}_{\mathbf{M}}\sigma\right) - \frac{1}{2}\,\nabla_a\,h_b^b\,\widetilde{\nabla}^a\widetilde{G}_{\mathbf{M}}\sigma \qquad \forall\,[\sigma] \in [\Gamma_0^\infty(\xi)], \tag{6.145}$$

where ∇ denotes now the Levi-Civita connection on $\mathbf{M}$ and $\widetilde{\nabla}$ the O(n)-twisted Levi-Civita connection. We further compute for any $i = 1,\ldots,n$:

$$\int_U \nabla_a\left(h^{ab}\nabla_b\phi^i\right)\psi^i\,\mathrm{vol}_U = \int_U \nabla_a\underbrace{\left(h^{ab}\left(\nabla_b\phi^i\right)\psi^i\right)}_{=:\,\omega^a}\mathrm{vol}_U - \int_U h^{ab}\left(\nabla_b\phi^i\right)\nabla_a\psi^i\,\mathrm{vol}_U \tag{6.146}$$

and

$$\int_U (\nabla_a h_b^b)(\nabla^a\phi^i)\psi^i\,\mathrm{vol}_U = \int_U \nabla_a\underbrace{\left(h_b^b\left(\nabla^a\phi^i\right)\psi^i\right)}_{=:\,\omega^a}\mathrm{vol}_U - \int_U h_b^b\,\nabla_a((\nabla^a\phi^i)\psi^i)\,\mathrm{vol}_U \tag{6.147}$$

$$= -\int_U h_b^b\left(\nabla_a\nabla^a\phi^i\right)\psi^i\,\mathrm{vol}_U - \int_U h_b^b\left(\nabla^a\phi^i\right)\nabla_a\psi^i\,\mathrm{vol}_U \tag{6.148}$$

$$= \int_U h_b^b\,\mu^2\,\phi^i\psi^i\,\mathrm{vol}_U - \int_U h_b^b\left(\nabla^a\phi^i\right)\nabla_a\psi^i\,\mathrm{vol}_U\,. \tag{6.149}$$

In these computations, we have used the divergence theorem (5.48) and the fact that ϕ^i is a solution of the homogeneous Klein-Gordon equation, i.e. $\left(\nabla_a\nabla^a + \mu^2\right)\phi^i = 0$. We symmetrise (6.146) and add the result to (6.149). Substituting (6.143) into (6.142) yields thereby

$$\widetilde{\mathfrak{u}}_{\mathbf{M}}\left(\left[(\delta_{\mathbf{M}[h]}^{\widetilde{\nabla}} - \delta_{\mathbf{M}}^{\widetilde{\nabla}})\,\mathrm{d}_{\mathbf{M}}^{\widetilde{\nabla}}\,\widetilde{G}_{\mathbf{M}}\sigma\right],[\tau]\right) \tag{6.150}$$

$$= t\sum_{i=1}^{n}\int_U h^{ab}\left(-\frac{1}{2}\,\nabla_a\phi^i\nabla_b\psi^i - \frac{1}{2}\,\nabla_b\phi^i\nabla_a\psi^i + \nabla^a\phi^i\nabla_a\psi^i g_{ab} - \mu^2\phi^i\psi^i g_{ab}\right)\mathrm{vol}_U \tag{6.151}$$

$$\begin{aligned} = -t\int_M h^{ab}\Big(&\frac{1}{2}\,\langle\widetilde{\nabla}_a\,\widetilde{\phi}_{[\sigma]}\mid\widetilde{\nabla}_b\,\widetilde{\phi}_{[\tau]}\rangle_\xi + \frac{1}{2}\,\langle\widetilde{\nabla}_b\,\widetilde{\phi}_{[\sigma]}\mid\widetilde{\nabla}_a\,\widetilde{\phi}_{[\tau]}\rangle_\xi \\ &- \langle\widetilde{\nabla}^a\,\widetilde{\phi}_{[\sigma]}\mid\widetilde{\nabla}_a\,\widetilde{\phi}_{[\tau]}\rangle_\xi\,g_{ab} + \mu^2\,\langle\,\widetilde{\phi}_{[\sigma]}\mid\,\widetilde{\phi}_{[\tau]}\rangle_\xi\,g_{ab}\Big)\,\mathrm{vol}_{\mathbf{M}} \end{aligned} \tag{6.152}$$

$$= -t\left\langle h\;\middle|\;\widetilde{T}_{\mathbf{M}}([\sigma],[\tau])\right\rangle_{2,g}, \tag{6.153}$$

where we have introduced the polarised stress-energy-momentum tensor $\widetilde{T}_{\mathbf{M}}([\sigma],[\tau]) \in \Gamma^\infty(\tau_M^* \odot \tau_M^*)$ of $\bar{\mathcal{T}}$ on $\mathbf{M}$ for $[\sigma],[\tau] \in [\Gamma_0^\infty(\xi)]$, by making the definitions $\widetilde{\phi}_{[\sigma]} := \widetilde{G}_{\mathbf{M}}\sigma$,

$\widetilde{\phi}_{[\tau]} := \widetilde{G}_{\mathbf{M}}\tau$ and

$$\begin{aligned}(6.154)\qquad \widetilde{T}_{\mathbf{M}}([\sigma],[\tau]) := &\frac{1}{2}\langle\widetilde{\nabla}\widetilde{\phi}_{[\sigma]} \mid \widetilde{\nabla}\widetilde{\phi}_{[\tau]}\rangle_{\xi} + \frac{1}{2}\langle\widetilde{\nabla}\widetilde{\phi}_{[\tau]} \mid \widetilde{\nabla}\widetilde{\phi}_{[\sigma]}\rangle_{\xi} \\ &- \langle\widetilde{\nabla}\widetilde{\phi}_{[\sigma]} \mid \widetilde{\nabla}\widetilde{\phi}_{[\tau]}\rangle_{\lambda^1_M\otimes\xi}\, g + \mu^2\langle\widetilde{\phi}_{[\sigma]} \mid \widetilde{\phi}_{[\tau]}\rangle_{\xi}\, g.\end{aligned}$$

It is now a simple task to calculate

$$(6.155)\qquad \frac{d}{dt}\widetilde{\mathfrak{u}}_{\mathbf{M}}\left(\mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[th][\sigma],[\tau]\right)\Big|_{t=0} = \widetilde{\mathfrak{u}}_{\mathbf{M}}(T_{\mathbf{M}}(h)[\sigma],[\tau])$$

$$(6.156)\qquad = -\langle h \mid T_{\mathbf{M}}([\sigma],[\tau])\rangle_{2,g}$$

$$\forall[\sigma],[\tau]\in[\Gamma^\infty_0(\xi)],\ \forall h\in\Gamma^\infty_0(\tau^*_M\odot\tau^*_M).$$

We can now turn to the investigation of the dynamical nets for $\bar{\mathcal{T}}$ and $\bar{\mathfrak{T}}$, following the same approach as in Section 5.5 for the reduced F-theory of the free Maxwell field. Unfortunately, we only break the first ground and have to leave a completion of this analysis for the future. In the same way as Lemma 5.5.1, we show that the dynamical net for $\bar{\mathcal{T}}$ can be charaterised by using the stress-energy-momentum tensor of the theory:

LEMMA 6.9.3. *Let K be any compact subset of $\mathbf{M}$ and define $I := \bar{\mathcal{T}}^\bullet(\mathbf{M};K)$, $II := \{[\sigma]\in\bar{\mathcal{T}}\mathbf{M} \mid \operatorname{supp}\widetilde{T}_{\mathbf{M}}([\sigma],[\sigma]) \subseteq J_{\mathbf{M}}(K)\}$ and $III := \bigcap_{\substack{h\in\Gamma^\infty_0(\tau^*_M\odot\tau^*_M)\\ \operatorname{supp}h\subseteq K'}}\ker\widetilde{T}_{\mathbf{M}}(h)$. Then we have $I = II = III$ and also $\bar{\mathcal{T}}^\bullet(\mathbf{M};K) = \{[\sigma]\in\bar{\mathcal{T}}\mathbf{M} \mid \operatorname{supp}\widetilde{\nabla}\widetilde{\phi}_{[\sigma]} \subseteq J_{\mathbf{M}}(K)\}$.*

Proof: We show $I \subseteq III \subseteq II \subseteq I$ and begin with $I \subseteq III$. Suppose $[\sigma]\in I$. For $h\in\Gamma^\infty_0(\tau^*_M\odot\tau^*_M)$ with support in K', we find $\varepsilon > 0$ such that $th\in H(\mathbf{M};K')$ for all $t\in(-\varepsilon,\varepsilon)$ and as $\mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[th][\sigma] = [\sigma]$ for all $t\in(-\varepsilon,\varepsilon)$, $\frac{d}{dt}\widetilde{\mathfrak{u}}_{\mathbf{M}}(\mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[th][\sigma],[\tau])|_{t=0} = 0$ for all $[\tau]\in\bar{\mathcal{T}}\mathbf{M}$. Hence, $\widetilde{\mathfrak{u}}_{\mathbf{M}}(\widetilde{T}_{\mathbf{M}}(h)[\sigma],[\tau]) = 0$ for all $[\tau]\in\bar{\mathcal{T}}\mathbf{M}$ and by the weak non-degeneracy of $\widetilde{\mathfrak{u}}_{\mathbf{M}}$, $[\sigma]\in\ker\widetilde{T}_{\mathbf{M}}(h)$; as h was arbitrarily chosen, $I\subseteq III$.

For $III\subseteq II$, take $[\sigma]\in III$. Then $\widetilde{\mathfrak{u}}_{\mathbf{M}}(\widetilde{T}_{\mathbf{M}}(h)[\sigma],[\sigma]) = -\langle h \mid \widetilde{T}_{\mathbf{M}}([\sigma],[\sigma])\rangle_{2,g} = 0$ for all $h\in\Gamma^\infty_0(\tau^*_M\odot\tau^*_M)$ with support $\operatorname{supp}h\subseteq K'$. By the weak non-degeneracy of $\langle\cdot\mid\cdot\rangle_{2,g|_{K'}}$, we can thus conclude $\operatorname{supp}\widetilde{T}_{\mathbf{M}}([\sigma],[\sigma])\subseteq J_{\mathbf{M}}(K)$ as required.

Finally, to prove $II\subseteq I$, we note that $\operatorname{supp}\widetilde{T}_{\mathbf{M}}([\sigma],[\sigma])\subseteq J_{\mathbf{M}}(K)$ implies that $\operatorname{supp}\widetilde{\nabla}\widetilde{\phi}_{[\sigma]}\subseteq J_{\mathbf{M}}(K)$, which can be seen by picking for each $x\in K'$ a Lorentz frame $(e_0,\ldots,e_m)$ for TM_x, where m denotes the (fixed) spacetime dimension here, and looking at $\widetilde{T}_{\mathbf{M}}([\sigma],[\sigma])(x;e_0,e_0)$. This also shows that $\operatorname{supp}\widetilde{\phi}_{[\sigma]}\subseteq J_{\mathbf{M}}(K)$ if the mass of the fields $m > 0$ ($\Longrightarrow \mu > 0$). Accordingly, $\widetilde{\phi}_{[\sigma]}$ is a solution of $\widetilde{D}_{\mathbf{M}[h]}\widetilde{\phi} = 0$ for every $h\in H(\mathbf{M};K')$. Hence, by the well-posedness of the Cauchy problem ([BGP07, Thm.3.2.11 + 3.2.12] or [Wal12, Thm.4.2.16 + 4.2.19]), $\widetilde{\phi}_{[\sigma]}$ is the unique solution on $\mathbf{M}[h]$ that coincides with $\widetilde{\phi}_{[\sigma]}$ on $M^+[h]$ and also the unique solution on $\mathbf{M}$ that coincides with $\widetilde{\phi}_{[\sigma]}$ on $M^-[h]$. Thus, $[\sigma]$ and $\mathrm{rce}^{\bar{\mathcal{T}}}_{\mathbf{M}}[h][\sigma]$ give rise to the same solution of $\widetilde{D}_{\mathbf{M}}\widetilde{\phi} = 0$, which implies $\mathrm{rce}_{\mathbf{M}}^{\bar{\mathcal{T}}}[h][\sigma] = [\sigma]$ and consequently $[\sigma] \in I$. The final statement follows from the arguments just given. ❐

Appendix: some notions from the theory of smooth fibre bundles

We review some important notions and results concernig smooth fibre bundles, which we have used in the course of this chapter.

Let $\{X_i \mid i \in I\}$ be a cover for a set X, i.e. $X_i \subseteq X$ for all $i \in I$ and $\bigcup_{i\in I} X_i = X$, where I is some arbitrary index set. To keep our notation short and sharp, we shall write

$$X_{ij} := X_i \cap X_j, \qquad (i,j) \in I \times I, \tag{6.157}$$
$$X_{ijk} := X_i \cap X_j \cap X_k, \qquad (i,j,k) \in I \times I \times I$$

and

$$I^x := \{(i,j) \in I \times I \mid X_{ij} \neq \varnothing\}, \tag{6.158}$$
$$I^{xx} := \{(i,j,k) \in I \times I \times I \mid X_{ijk} \neq \varnothing\}.$$

We will also write $ij \in I^x$ instead of $(i,j) \in I^x$ and $ijk \in I^{xx}$ instead of $(i,j,k) \in I^{xx}$.

Though it seems plausible that the following lemma is well-known and should be findable in any textbook on fibre bundles, it is not stated in any of our references explicitly. Hence, we give a proof:

LEMMA 6.10.4. (smooth cross-section gluing lemma)
Let $\xi = (B, M, \pi, F)$ be a smooth fibre bundle and $\{U_i \mid i \in I\}$ an open cover of M. Given a family of smooth local cross-sections $\{\sigma_i \in \Gamma^\infty(\xi|_{U_i}) \mid i \in I\}$ such that $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}}$ whenever $ij \in I^x$, there is a unique smooth global cross-section $\sigma \in \Gamma^\infty(\xi)$ satifying $\sigma|_{U_i} = \sigma_i$ for all $i \in I$.

Proof: We start with existence. Consider the smooth maps

$$\iota_{B_{U_i}} \circ \sigma_i : U_i \longrightarrow B, \qquad i \in I. \tag{6.159}$$

By assumption

$$\iota_{B_{U_i}} \circ \sigma_i \circ \iota_{U_{ij}U_i} = \iota_{B_{U_i}} \circ \iota_{B_{U_{ij}}B_{U_i}} \circ \sigma_i|_{U_{ij}} \tag{6.160}$$
$$= \iota_{B_{U_{ij}}} \sigma_i|_{U_{ij}} \tag{6.161}$$
$$= \iota_{B_{U_{ij}}} \sigma_j|_{U_{ij}} \tag{6.162}$$
$$= \iota_{B_{U_j}} \circ \sigma_j \circ \iota_{U_{ij}U_j} \qquad \forall ij \in I^x. \tag{6.163}$$

Due to [Lee03, Lem.2.1], there exists a unique smooth map $\sigma : M \longrightarrow B$ with the property $\sigma \circ \iota_{U_i} = \iota_{B_{U_i}} \circ \sigma_i$ for all $i \in I$. Let $x \in M$, then there is an $i \in I$ with $x \in U_i$ and

$$\pi(\sigma(x)) = \pi((\sigma \circ \iota_{U_i})(x)) = \pi\left(\left(\iota_{B_{U_i}} \circ \sigma_i\right)(x)\right) = \pi(\sigma_i(x)) = x. \tag{6.164}$$

This shows that σ is a smooth cross-section in ξ. Furthermore,

$$\iota_{B_{U_i}} \circ \sigma|_{U_i} = \sigma \circ \iota_{U_i} = \iota_{B_{U_i}} \circ \sigma_i \qquad \forall i \in I \tag{6.165}$$

and since $\iota_{B_{U_i}}$ is injective for all $i \in I$, we obtain $\sigma|_{U_i} = \sigma_i$ for all $i \in I$.

To see uniqueness, let $\tau \in \Gamma^\infty(\xi)$ be another smooth cross-section in ξ satisfying $\tau|_{U_i} = \sigma_i$ for all $i \in I$. Let $x \in M$ be arbitrary and $i \in I$ such that $x \in U_i$.

$$\tau(x) = \tau(\iota_{U_i}(x)) = (\tau \circ \iota_{U_i})(x) = \tau|_{U_i}(x) = \sigma_i(x) = \sigma(x), \tag{6.166}$$

which shows $\tau = \sigma$. ❐

Linear connections and covariant friends

DEFINITION 6.10.5. A *linear connection* in a smooth $\mathbb{K}$-vector bundle ξ is a $\mathbb{K}$-linear map

$$\nabla : \Gamma^\infty(\xi) \longrightarrow \Omega^1(M;\xi) \tag{6.167}$$

satisfying the product rule

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma, \qquad f \in \mathcal{C}^\infty(M,\mathbb{K}),\, \sigma \in \Gamma^\infty(\xi). \tag{6.168}$$

The following is a well-known result for linear connections (see e.g. [BGP07, Example A.4.6] or [Wal07, Bsp. A.5.6]); a proof will therefore be omitted.

LEMMA 6.10.6. *Any linear connection ∇ in a smooth vector bundle $\xi = (E, M, \pi, V)$ is a linear differential operator of order 1 with principal symbol given by $\sigma_\nabla(\xi)(v) = \xi \otimes v$ for $v \in E_x$, $\xi \in T^*M_x$ and $x \in M$.*

DEFINITION 6.10.7. Let ∇ be a linear connection in a smooth vector bundle $\xi = (E, M, \pi, V)$. The *covariant exterior derivative* with respect to ∇,

$$\mathrm{d}^\nabla : \Omega^p(M;\xi) \longrightarrow \Omega^{p+1}(M;\xi), \qquad p = 0, 1, 2, \ldots, \tag{6.169}$$

is defined by the formula

$$\text{(6.170)}\quad \mathrm{d}^{\nabla}\omega\Big(\underset{i=0}{\overset{p}{\mathfrak{a}}} X_i\Big) := \sum_{i=0}^{p} (-1)^i\ \nabla_{X_i}\omega\Big(\underset{\substack{j=0\\ j\neq i}}{\overset{p}{\mathfrak{a}}} X_j\Big) + \sum_{\substack{i,j=0\\ i<j}}^{p} (-1)^{i+j}\omega\Big([X_i, X_j],\ \underset{\substack{k=0\\ i\neq k\neq j}}{\overset{p}{\mathfrak{a}}} X_i\Big),$$

$$X_0, \ldots, X_p \in \mathscr{X}(M),\ \omega \in \Omega^p(M;\xi),$$

where " $\mathfrak{a}$ " denotes ordered enumeration.

The covariant exterior derivative satisfies the product rule

$$\text{(6.171)}\qquad \mathrm{d}^{\nabla}(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p\,\omega \wedge \mathrm{d}^{\nabla}\eta,$$

$$\omega \in \Omega^p(M;\mathbb{K}),\ \eta \in \Omega^q(M;\xi),\ p,q = 0,1,2,\ldots,$$

and in particular

$$\text{(6.172)}\qquad \mathrm{d}^{\nabla}(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p\,\omega \wedge \nabla\sigma,$$

$$\omega \in \Omega^p(M;\mathbb{R}),\ \sigma \in \Gamma^{\infty}(\xi),\ p = 0,1,2,\ldots,$$

see [GHV73, Sec.7.14]. Lemma 6.10.6, (6.170) and the product rule (6.172) imply (cf. [BGP07, Example A.4.5]):

LEMMA 6.10.8. *Assume that ∇ is a linear connection in a smooth vector bundle $\xi = (E, M, \pi, V)$. The covariant exterior derivative with respect to ∇, d^{∇}, is a linear differential operator of order 1 with principal symbol given by $\sigma_{\mathrm{d}^{\nabla}}(\xi)(v) = \xi \wedge v$ for $v \in \Lambda^p(T^*M_x) \otimes E_x$, $\xi \in T^*M_x$, $x \in M$ and $p = 0,1,2,\ldots$.*

DEFINITION 6.10.9. Let $(M, g, [\Omega])$ be an oriented semi-Riemannian manifold of dimension m and $\xi = (E, M, \pi, V)$ a smooth $\mathbb{K}$-vector bundle. We define the *Hodge-$*$-operator* for ξ-valued smooth differential p-forms, $p = 0,1,2,\ldots$, by the means of the canonical $\mathcal{C}^{\infty}M$-module isomorphism $\Omega^p(M;\xi) \cong \Omega^p M \otimes \Gamma^{\infty}(\xi)$ as $* \otimes \mathrm{id}_{\Gamma^{\infty}(\xi)}$, where $* : \Omega^p M \longrightarrow \Omega^{m-p}M$ is the ordinary Hodge-$*$-operator. However, we will also use just $*$ to denote the Hodge-$*$-operator for ξ-valued smooth differential p-forms.

It is clear that the Hodge-$*$-operator is a $\mathcal{C}^{\infty}(M,\mathbb{K})$-module isomorphism with inverse $*^{-1} = (-1)^{p(m-p)}\frac{\det g}{|\det g|}\,*$ and a linear differential operator of order 0 with principal symbol given by $\sigma_*(\xi) = *$ for all $\xi \in T^*M$ (cf. [BGP07, Example A.4.7]).

Combining the covariant exterior derivative and the Hodge-$*$-operator, we define the covariant exterior coderivative:

DEFINITION 6.10.10. Let ξ be a smooth vector bundle over an oriented semi-Riemannian manifold $(M, g, [\Omega])$ and ∇ a linear connection in ξ. Then the *covariant* *exterior coderivative* with respect to ∇,

$$\delta^{\nabla}: \Omega^{p}(M;\xi) \longrightarrow \Omega^{p-1}(M;\xi), \qquad p = 0,1,2,\ldots, \tag{6.173}$$

is defined by $\delta^{\nabla} := (-1)^{p} *^{-1} \mathrm{d}^{\nabla} *$.

Using the product rule for the covariant exterior derivative (6.172) and the definition of the Hodge-$*$-operator, one quickly derives the product rule for the covariant exterior coderivative,

$$\delta^{\nabla}(\omega \otimes \sigma) = \delta\omega \otimes \sigma + (-1)^{m} *^{-1} (*\omega \wedge \nabla\sigma), \tag{6.174}$$
$$\omega \in \Omega^{p}(M;\mathbb{R}),\ \sigma \in \Gamma^{\infty}(\xi),\ p = 0,1,2,\ldots.$$

Lemma 6.10.8, [BGP07, Rem.A.4.8] and the product rule readily show:

LEMMA 6.10.11. *Assume that ∇ be a linear connection in a smooth vector bundle $\xi = (E, M, \pi, V)$ over an oriented semi-Riemannian manifold $(M, g, [\Omega])$. Then the covariant exterior coderivative with respect to ∇, δ^{∇}, is a linear differential operator of order 1 with principal symbol given by $\sigma_{\delta^{\nabla}}(\xi)(v) = -\xi(v^{\#})$ for $v \in \Lambda^{p}(T^{*}M_{x}) \otimes E_{x}$, $\xi \in T^{*}M_{x}$ and $x \in M$.*

DEFINITION 6.10.12. Let $\xi = (E, M, \pi, V)$ be a smooth $\mathbb{K}$-vector bundle. A *smooth bundle metric* in ξ is a smooth cross-section $\langle\cdot \mid \cdot\rangle_{\xi} \in \Gamma^{\infty}(\overline{\xi}^{*} \otimes \xi^{*})$ such that for all $x \in M$, $\langle\cdot \mid \cdot\rangle_{E_x} := \langle\cdot \mid \cdot\rangle_{\xi}(x) : \overline{E}_{x} \times E_{x} \longrightarrow \mathbb{K}$ is a non-degenerate and symmetric ($\mathbb{K} = \mathbb{R}$) resp. Hermitean ($\mathbb{K} = \mathbb{C}$) sesquilinear form on E_x. If, in addition, $\langle\cdot \mid \cdot\rangle_{E_x}$ is positive definite for all $x \in M$, we will call the smooth bundle metric $\langle\cdot \mid \cdot\rangle_{\xi}$ *Riemannian* ($\mathbb{K} = \mathbb{R}$) resp. *Hermitean* ($\mathbb{K} = \mathbb{C}$). Take notice that we have used $\overline{E}_{x}$ to denote the complex conjugate vector space of E_x (appendix of Chapter 2) and $\overline{\xi}$ for the complex conjugate smooth vector bundle of ξ ([Mor01b, Sec.5.5(d)], [Bau09, Sec.2.4]).

If $\langle\cdot \mid \cdot\rangle_{\xi}$ is a smooth bundle metric in a smooth $\mathbb{K}$-vector bundle $\xi = (E, M, \pi, V)$, we can use any smooth cross-sections $\sigma \in \Gamma^{\infty}(\overline{\xi})$ and $\tau \in \Gamma^{\infty}(\xi)$ to build a smooth function $\langle\sigma \mid \tau\rangle_{\xi} : M \longrightarrow \mathbb{K}$ by setting $\langle\sigma \mid \tau\rangle_{\xi}(x) := \langle\sigma(x) \mid \tau(x)\rangle_{E_x}$ for all $x \in M$. If σ or τ are compactly supported, then so is $\langle\sigma \mid \tau\rangle_{\xi}$.

For all $U \subseteq M$ open, the restriction $\langle\cdot \mid \cdot\rangle_{\xi}|_{U} \in \Gamma^{\infty}(\overline{\xi}^{*} \otimes \xi^{*}|_{U})$ of $\langle\cdot \mid \cdot\rangle_{\xi}$ to U is a smooth bundle metric in $\xi|_{U}$. Recall that $\langle\cdot \mid \cdot\rangle_{\xi}|_{U}$ is defined to be the unique smooth cross-section in $\overline{\xi}^{*} \otimes \xi^{*}|_{U}$ fulfilling the equation $\iota_{\overline{E}\otimes E|_{U}} \circ \langle\cdot \mid \cdot\rangle_{\xi}|_{U} = \langle\cdot \mid \cdot\rangle_{\xi} \circ \iota_{U}$, where $\iota_{\overline{E}\otimes E|_{U}} : \overline{E} \otimes E|_{U} \hookrightarrow \overline{E} \otimes E$ and $\iota_{U} : U \hookrightarrow M$ denote the inclusion maps. With this in mind, we have for any smooth cross-sections $\sigma \in \Gamma^{\infty}(\overline{\xi})$ and $\tau \in \Gamma^{\infty}(\xi)$ the restriction property $\langle\sigma \mid \tau\rangle_{\xi}|_{U} = \langle\sigma|_{U} \mid \tau|_{U}\rangle_{\xi}|_{U}$.

LEMMA 6.10.13. *Let $(M, g, [\Omega])$ be an oriented semi-Riemannian manifold, $\xi = (E, M, \pi, V)$ a smooth $\mathbb{K}$-vector bundle and $\langle \cdot \mid \cdot \rangle_\xi$ a smooth bundle metric in ξ. Then*

$$(6.175) \qquad \langle \cdot \mid \cdot \rangle_{2,\xi} : \Gamma_0^\infty\left(\overline{\xi}\right) \times \Gamma_0^\infty(\xi) \longrightarrow \mathbb{K}, \qquad (\sigma, \tau) \longmapsto \int_M \langle \sigma \mid \tau \rangle_\xi \operatorname{vol}_{(g,[\Omega])},$$

is a weakly non-degenerate and symmetric ($\mathbb{K} = \mathbb{R}$) resp. Hermitean ($\mathbb{K} = \mathbb{C}$) sesquilinear form on $\Gamma_0^\infty(\xi)$;

$$(6.176) \qquad \langle \cdot \mid \cdot \rangle_{2,\xi} : \Gamma_0^\infty\left(\overline{\xi}\right) \times \Gamma^\infty(\xi) \longrightarrow \mathbb{K}, \qquad (\sigma, \tau) \longmapsto \int_M \langle \sigma \mid \tau \rangle_\xi \operatorname{vol}_{(g,[\Omega])},$$

and

$$(6.177) \qquad \langle \cdot \mid \cdot \rangle_{2,\xi} : \Gamma^\infty\left(\overline{\xi}\right) \times \Gamma_0^\infty(\xi) \longrightarrow \mathbb{K}, \qquad (\sigma, \tau) \longmapsto \int_M \langle \sigma \mid \tau \rangle_\xi \operatorname{vol}_{(g,[\Omega])},$$

are weakly non-degenerate sesquilinear pairings such that $\langle \sigma \mid \tau \rangle_{2,\xi} = \overline{\langle \tau \mid \sigma \rangle_{2,\xi}}$ for all $\sigma \in \Gamma_0^\infty\left(\overline{\xi}\right)$ [resp. $\sigma \in \Gamma^\infty\left(\overline{\xi}\right)$] and for all $\tau \in \Gamma^\infty(\xi)$ [resp. $\tau \in \Gamma_0^\infty(\xi)$].

DEFINITION 6.10.14. Assume that ∇ is a linear connection in a smooth vector bunde $\xi = (E, M, \pi, V)$. Then ∇ is called metric with respect to a smooth bundle metric $\langle \cdot \mid \cdot \rangle_\xi$ in ξ if and only if

$$(6.178) \qquad X \langle \sigma \mid \tau \rangle_\xi = \langle \nabla_X \sigma, \tau \rangle_\xi + \langle \sigma, \nabla_X \tau \rangle_\xi \qquad \forall X \in \mathscr{X}(M),\ \forall \sigma, \tau \in \Gamma^\infty(\xi).$$

If we introduce for $\sigma, \tau \in \Gamma^\infty(\xi)$ the smooth differential 1-forms $\langle \nabla \sigma \mid \tau \rangle_\xi$, $\langle \sigma \mid \nabla \tau \rangle_\xi \in \Omega^1(M; \mathbb{K})$ via $\langle \nabla \sigma \mid \tau \rangle_\xi(x; v) := \langle \nabla \sigma(x; v) \mid \tau(x) \rangle_{E_x}$ and $\langle \sigma \mid \nabla \tau \rangle_\xi(x; v) := \langle \sigma(x), \nabla \tau(x; v) \rangle_{E_x}$ for all $v \in T_x M$ and for all $x \in M$, (6.178) reads $d \langle \sigma, \tau \rangle_\xi = \langle \nabla \sigma \mid \tau \rangle_\xi + \langle \sigma \mid \nabla \tau \rangle_\xi$.

Taking any smooth bundle metric $\langle \cdot \mid \cdot \rangle_\xi$ in a smooth vector bundle $\xi = (E, M, \pi, V)$ of rank k over an oriented semi-Riemannian manifold $(M, g, [\Omega])$, we can define for all $p = 0, 1, 2, \ldots$, a smooth bundle metric $\langle \cdot \mid \cdot \rangle_{\lambda^p \otimes \xi}$ in $\lambda^p \otimes \xi$ via $\langle u \mid v \rangle_{\Lambda^p(T^* M_x) \otimes E_x} := \langle u^i \mid v^j \rangle_g \langle E_i(x) \mid E_j(x) \rangle_{E_x}$ for all $u, v \in \Lambda^p(T^* M_x) \otimes E_x$ and for all $x \in M$, where $E_1, \ldots, E_k$ is any smooth local framing for ξ over an open neighbourhood U of x. With this in mind we now have [Bau09, Satz 7.3]:

PROPOSITION 6.10.15. *Let $(M, g, [\Omega])$ be an oriented semi-Riemannian manifold, $\xi = (E, M, \pi, V)$ a smooth $\mathbb{K}$-vector bundle, $\langle \cdot \mid \cdot \rangle_\xi$ a smooth bundle metric in ξ and ∇*

a metric linear connection. Then

$$\langle \mathrm{d}^{\nabla}\omega \mid \eta \rangle_{2,\,\lambda^{p+1}\otimes\xi} = \langle \omega \mid \delta^{\nabla}\eta \rangle_{2,\,\lambda^{p}\otimes\xi} \qquad (6.179)$$
$$\forall \omega \in \Omega^p_0\left(M;\overline{\xi}\,\right),\ \forall \eta \in \Omega^{p+1}\left(M;\xi\right),\ p = 0,1,2,\ldots$$
$$[\textit{resp.}\ \forall \omega \in \Omega^p\left(M;\overline{\xi}\,\right),\ \forall \eta \in \Omega^{p+1}_0\left(M;\xi\right)].$$

Smooth principal bundles

EXAMPLE 6.10.16. Let M be a smooth manifold and G a Lie group. The tuple $\underline{G}_M = (M \times G, M, \mathrm{pr}_1, G, \mathfrak{r})$, where $\mathfrak{r} : M \times G \times G \longrightarrow G$, $(x,g,h) = (x, gh)$, is a smooth principal G-bundle, called the *trivial smooth principal G-bundle over M*.

The following theorem can be found in [KMS99, Thm.6.4], [Mor01b, Prop.6.4 + 6.6], [Bau09, Satz 2.8], [Bär11, Sec.2.2]:

THEOREM 6.10.17. (smooth principal bundle reconstruction lemma)
Let G a Lie group, M a smooth manifold, $\{U_\alpha \mid \alpha \in A\}$ an open cover for M and suppose that $c := \{g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G \mid \alpha\beta \in A^x\}$ is a smooth G-cocycle for $\{U_\alpha \mid \alpha \in A\}$. Then there exist a smooth principal G-bundle $\mathscr{P} = (P, M, \pi, G, \mathfrak{p})$ and a smooth principal G-bundle atlas $\mathfrak{P} = \{\phi_\alpha : \mathscr{P}|_{U_\alpha} \xrightarrow{\sim !} \underline{G}_{U_\alpha} \mid \alpha \in A\}$ for $\mathscr{P}$ whose smooth G-cocycle of transition functions is precisely c. If $\mathscr{R} = (R, M, \varrho, G, \mathfrak{q})$ is another smooth principal G-bundle over M with a smooth principal G-bundle atlas $\mathfrak{R} = \{\psi_\alpha : \mathscr{R}|_{U_\alpha} \xrightarrow{\sim !} \underline{G}_{U_\alpha} \mid \alpha \in A\}$, whose smooth G-cocycle of transition functions is also c, then $\mathscr{P}$ and $\mathscr{R}$ are strongly isomorphic.

The following proposition is taken from [GHV73, Sec.5.3 + 5.6], [KMS99, Thm.10.7], [Bau09, Sec.2.3] and [Bär11, Sec.2.2], and regards the associated smooth fibre bundle construction.

PROPOSITION AND DEFINITION 6.10.18. *Assumet that $\mathscr{P} = (P, M, \pi, G, \mathfrak{p})$ is a smooth principal bundle, F a smooth manifold on which G acts smoothly from the left via $\mathfrak{l} : G \times F \longrightarrow F$ and consider the smooth right action on the smooth product manifold $P \times F$ defined by*

$$\mathfrak{q} : P \times F \times G \longrightarrow P \times F, \qquad (z, v, g) \longmapsto \left(z \cdot_{\mathfrak{p}} g, g^{-1} \bullet_{\mathfrak{l}} v\right). \qquad (6.180)$$

Then there is a unique smooth manifold structure on the set of all orbits

$$P \times_G F = \{(z, v) \cdot_{\mathfrak{q}} G \mid (z, v) \in P \times F\} \qquad (6.181)$$

such that $\xi[\mathscr{P}] = (P \times_G F, M, \varrho, F)$ becomes a smooth fibre bundle, where the bundle projection $\varrho : P \times_G F \longrightarrow M$ is defined to be the unique smooth map that makes the *diagram*

$$\begin{array}{ccc} P \times F & \xrightarrow{[\cdot]} & P \times_G F \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \varrho} \\ P & \xrightarrow{\pi} & M \end{array} \tag{6.182}$$

commutative. Every smooth principal G-bundle atlas $\mathfrak{P} = \{\phi^P_\alpha : \mathscr{P}|_{U_\alpha} \xrightarrow{\sim!} \underline{G}_{U_\alpha}\}_{\alpha\in A}$ *of* $\mathscr{P}$ *gives rise to a smooth G-fibre bundle atlas* $\mathfrak{G} = \{\phi_\alpha : \xi[\mathscr{P}]|_{U_\alpha} \xrightarrow{\sim!} \underline{F}_{U_\alpha}\}_{\alpha\in A}$ *for* $(\xi[\mathscr{P}], G, \mathfrak{l})$ *which has exactly the same smooth G-cocycle of transition functions as* $\mathfrak{P}$*. Thus,* $\xi_G[\mathscr{P}] = (\xi[\mathscr{P}], G, \mathfrak{l}, < \mathfrak{G} >)$ *becomes a smooth G-fibre bundle. Every smooth principal G-bundle atlas of* $\mathscr{P}$ *determines the same smooth G-structure on* $(\xi[\mathscr{P}], G, \mathfrak{l})$*. We will refer to* $\xi_G[\mathscr{P}]$ *as the* smooth fibre bundle with typical fibre F and structure group G associated with $\mathscr{P}$ *or the* smooth G-fibre bundle with typical fibre F associated with $\mathscr{P}$.

If F *is a (finite-dimensional) vector space,* $\rho : G \longrightarrow \mathrm{GL}(F)$ *a representation and* G *acts smoothly from the left on* F *via* ρ*, i.e.* $g \bullet_{\mathfrak{l}} v = \rho(g)v$ *for all* $g \in G$ *and for all* $v \in F$*, then* $\xi[\mathscr{P}]$ *is also smooth vector bundle and* $\mathfrak{G}$ *a smooth vector bundle atlas for* $\xi[\mathscr{P}]$*. In this case, we will call* $\xi_G[\mathscr{P}]$ *the* *smooth vector bundle with typical fibre F and structure group G associated with* $\mathscr{P}$.

PROPOSITION 6.10.19. *Let* $\mathscr{P} = (P, M, \pi, G, \mathfrak{p})$ *be a smooth principal G-bundle,* V *a finite-dimensional vector space over* $\mathbb{K}$*,* $\rho : G \longrightarrow \mathrm{GL}(V)$ *a representation, let G act smoothly from the left on V via* ρ *and denote the smooth vector bundle with typical fibre V and structure group G associated with* $\mathscr{P}$ *by* $\xi_G[\mathscr{P}] = (\xi[\mathscr{P}], G, \mathfrak{l}, \mathcal{G})$*, where* $\xi[\mathscr{P}] = (E, M, \varrho, V)$*. Suppose* $\langle\cdot|\cdot\rangle_V : \overline{V} \times V \longrightarrow \mathbb{K}$ *is a* ρ*-invariant and symmetric* ($\mathbb{K} = \mathbb{R}$) *resp. Hermitean* ($\mathbb{K} = \mathbb{C}$) *sesquilinear form on V and lift* $\langle\cdot|\cdot\rangle_V$ *to a smooth bundle metric* $\langle\cdot|\cdot\rangle_{\underline{V}_M}$ *in* $\underline{V}_M$ *by*

$$\begin{aligned} \langle\cdot|\cdot\rangle_{\underline{V}_M} : M &\longrightarrow \overline{\underline{V}}^*_M \otimes \underline{V}^*_M \\ x &\longmapsto \begin{cases} \langle\cdot|\cdot\rangle_{\underline{V}_M}(x) : (\{x\}\times\overline{V}) \times (\{x\}\times V) \longrightarrow \mathbb{K} \\ \qquad ((x,u),(x,v)) \longmapsto \langle u|v\rangle_V . \end{cases} \end{aligned} \tag{6.183}$$

For $U \subseteq M$ *open, denote the restriction of* $\langle\cdot|\cdot\rangle_{\underline{V}_M}$ *to U by* $\langle\cdot|\cdot\rangle_{\underline{V}_U}$*. Then* $\langle\cdot|\cdot\rangle_V$ *induces for all open subsets* $U \subseteq M$ *over which there exists a smooth local trivialisation a unique smooth bundle metric* $\langle\cdot|\cdot\rangle_{\xi[\mathscr{P}]|_U}$ *in* $\xi[\mathscr{P}]|_U$ *such that*[8] $\langle\cdot|\cdot\rangle_{\xi[\mathscr{P}]|_U} = \phi^*\langle\cdot|\cdot\rangle_{\underline{V}_U}$

[8] Observe, for any strong smooth vector bundle isomorphism $\phi : \xi \xrightarrow{\sim!} \eta$ and smooth bundle metric $\langle\cdot|\cdot\rangle_\eta$ in η, $\phi^*\langle\cdot|\cdot\rangle_\eta(\sigma,\tau) = \langle\overline{\phi}_*\sigma|\phi_*\tau\rangle_\eta$ for all $\sigma \in \Gamma^\infty(\overline{\xi})$ and for all $\tau \in \Gamma^\infty(\xi)$, cf. [GHV72,

for all $\phi : \xi_G[\mathscr{P}]|_U \xrightarrow{\sim!} \underline{V_G}_U \in \mathcal{G}$ *and a unique smooth bundle metric* $\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}$ *in* $\xi[\mathscr{P}]$ *such that* $\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}|_U = \langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]|_U}$ *for all* $U \subseteq M$ *over which there exists a smooth local trivialisation. For all* $x \in M$, $\langle\cdot\,|\,\cdot\rangle_{E_x}$ *has the same signature as* $\langle\cdot\,|\,\cdot\rangle_V$.

Proof: We simply define a smooth cross-section $\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}$ in $\overline{\xi[\mathscr{P}]}^* \otimes \xi[\mathscr{P}]^*$ by $\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}|_U := \phi^*\langle\cdot\,|\,\cdot\rangle_{\underline{V}_U}$ for any $\phi : \xi_G[\mathscr{P}]|_U \xrightarrow{\sim!} \underline{V_G}_U \in \mathcal{G}$. Since we have the local compatibility condition

$$
\begin{aligned}
&(6.184) \quad \left(\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}|_U\right)|_{U\cap V} = \left(\phi^*\langle\cdot\,|\,\cdot\rangle_{\underline{V}_U}\right)|_{U\cap V}\\
&(6.185) \quad = \left\langle \left(\overline{\phi}\,\|_{U\cap V}\right)_* \cdot \,\middle|\, (\phi\|_{U\cap V})_* \cdot \right\rangle_{\underline{V}_{U\cap V}}\\
&(6.186) \quad = \left\langle \left(\overline{\phi}\,\|_{U\cap V}\right)_* \left(\overline{\psi}\,\|_{U\cap V}\right)^{\#} \left(\overline{\psi}\,\|_{U\cap V}\right)_* \cdot \,\middle|\, (\phi\|_{U\cap V})_* (\psi\|_{U\cap V})^{\#} (\psi\|_{U\cap V})_* \cdot \right\rangle_{\underline{V}_{U\cap V}}\\
&(6.187) \quad = \left\langle \rho(g_{\psi\phi}) \left(\overline{\psi}\,\|_{U\cap V}\right)_* \cdot \,\middle|\, \rho(g_{\psi\phi}) (\psi\|_{U\cap V})_* \cdot \right\rangle_{\underline{V}_{U\cap V}}\\
&(6.188) \quad = \left\langle \left(\overline{\psi}\,\|_{U\cap V}\right)_* \cdot \,\middle|\, (\psi\|_{U\cap V})_* \cdot \right\rangle_{\underline{V}_{U\cap V}}\\
&(6.189) \quad = \left(\psi^*\langle\cdot\,|\,\cdot\rangle_{\underline{V}_V}\right)|_{U\cap V}\\
&(6.190) \quad = \left(\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}|_V\right)|_{U\cap V}\\
&\qquad\qquad \forall \phi : \xi_G[\mathscr{P}]|_U \xrightarrow{\sim!} \underline{V_G}_U,\ \psi : \xi_G[\mathscr{P}]|_V \xrightarrow{\sim!} \underline{V_G}_V \in \mathcal{G} \text{ with } U \cap V \neq \varnothing,
\end{aligned}
$$

the smooth cross-section gluing lemma, Lemma 6.10.4, yields that $\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}$ is well-defined and the unique smooth cross-section in $\overline{\xi[\mathscr{P}]}^* \otimes \xi[\mathscr{P}]^*$ with the property $\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}|_U = \phi^*\langle\cdot\,|\,\cdot\rangle_{\underline{V}_U}$ for all $\phi : \xi_G[\mathscr{P}]|_U \xrightarrow{\sim!} \underline{V_G}_U \in \mathcal{G}$. With $\langle\cdot\,|\,\cdot\rangle_{\xi[\mathscr{P}]}$ characterised in this manner, $\phi_x : E_x \xrightarrow{\sim} \{x\} \times V$ is isometric with respect to $\langle\cdot\,|\,\cdot\rangle_{E_x}$ and $\langle\cdot\,|\,\cdot\rangle_V$ for all $x \in U$ and for any $\phi : \xi_G[\mathscr{P}]|_U \xrightarrow{\sim!} \underline{V_G}_U \in \mathcal{G}$ by definition. Hence, for all $x \in M$, $\langle\cdot\,|\,\cdot\rangle_{E_x}$ shares the same signature with $\langle\cdot\,|\,\cdot\rangle_V$. ❒

DEFINITION 6.10.20. Let $\mathscr{P} = (P, M, \pi, G, \mathfrak{p})$ be a smooth principal bundle. A *principal connection* in $\mathscr{P}$ is a strong smooth vector bundle homomorphism $V : \tau_P \xrightarrow{!} \tau_P$ satisfying the following three conditions:

(PC1) $V^2 = V$.

(PC2) $\operatorname{img} V_z = TvP_z$ for all $z \in P$, where TvP_z is the vertical subspace of TP_z.

(PC3) V is equivariant, i.e. $T\mathfrak{p}(\cdot, g) \circ V = V \circ T\mathfrak{p}(\cdot, g)$ for all $g \in G$, where $T\mathfrak{p}(\cdot, g) : \tau_P \longrightarrow \tau_P$ is the tangent map induced by the smooth map $\mathfrak{p}(\cdot, g) : P \longrightarrow P$.

Any principal connection $V : \tau_P \xrightarrow{!} \tau_P$ in a smooth principal bundle $\mathscr{P} = (P, M, \pi, G, \mathfrak{p})$ induces (and is induced by) a $\mathfrak{g}$-valued smooth differential 1-form, the so-called *connection 1-form* $\omega \in \Omega^1(P; \mathfrak{g})$. ω is given by $\omega(z; v) = (T_{e_G}\mathfrak{p}(\cdot, z))^{-1}(V_z v)$ for all

Sec.2.9 + 2.15] and [Bau09, Sec.2.4].

$v \in TP_z$ and for all $z \in P$, where $T\mathfrak{p}(\cdot, z) : TG \longrightarrow TP$ is the tangent map induced by the smooth map $\mathfrak{p}(\cdot, z) : G \longrightarrow P$. The *curvature 2-form* $\Omega \in \Omega^2(P; \mathfrak{g})$ of the principal connection V is the $\mathfrak{g}$-valued smooth differential 2-form defined by $\Omega(z; u, v) := d\omega(z; (\mathrm{id}_{TP_z} - V_z) u, (\mathrm{id}_{TP_z} - V_z) v)$ for all $u, v \in TP_z$ and for all $z \in P$, where $d : \Omega^1(P; \mathfrak{g}) \longrightarrow \Omega^2(P; \mathfrak{g})$ is the exterior derivative for $\mathfrak{g}$-valued smooth differential 1-forms. V is called *flat* if and only if its curvature 2-form Ω vanishes identically.

EXAMPLE 6.10.21. Let G be a Lie group, $\underline{G}_M$ the trivial smooth principal G-bundle over a smooth manifold M and denote the canonical strong smooth vector bundle isomorphism $\tau_{M\times G} \xrightarrow{\sim !} \tau_M \oplus \tau_G$ by Ξ. A flat principal connection in $\underline{G}_M$ is given by $\bar{V} := \Xi^{-1} \circ \mathrm{inj}^{\oplus}_{\tau_G} \circ T\,\mathrm{pr}_2 \circ \Xi : \tau_{M\times G} \xrightarrow{!} \tau_{M\times G}$, to which we will refer as the *standard flat principal connection* in $\underline{G}_M$.

Let $(\mathscr{P}, V)$ and $(\mathscr{R}, W)$ be flat smooth principal bundles. An *isomorphism of flat smooth principal bundles*, $(\Phi, \varphi, f) : (\mathscr{P}, V) \xrightarrow{\sim} (\mathscr{R}, W)$ is an isomorphism of smooth principal bundles $(\Phi, \varphi, f) : \mathscr{P} \xrightarrow{\sim} \mathscr{R}$ such that $T\Phi \circ V = W \circ T\Phi$. A flat smooth principal bundle $(\mathscr{P}, V)$, where $\mathscr{P} = (P, M, \pi, G, \mathfrak{p})$, is called *trivial* if and only if it is strongly isomorphic to the trivial smooth principal G-bundle over M, $\underline{G}_M$, equipped with the standard flat principal connection $\bar{V}$.

Though we believe that the following proposition, which deals with the construction of a flat smooth principal bundle from a locally constant cocycle, is well-known and standard, we could not find the explicit statement anywhere in the literature:

PROPOSITION 6.10.22. *Let M be a smooth manifold, $\{U_\alpha \mid \alpha \in A\}$ an open cover for M, G a Lie group and $c := \{g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G \mid \alpha\beta \in A^x\}$ a locally constant G-cocycle. Then there are a flat smooth principal G-bundle $(\mathscr{P} = (P, M, \pi, G, \mathfrak{p}), V)$ and an atlas of smooth local trivialisations $\mathfrak{P} = \{\phi_\alpha : (\mathscr{P}|_{U_\alpha}, V\|_{U_\alpha}) \xrightarrow{\sim !} (\underline{G}_{U_\alpha}, \bar{V}_\alpha) \mid \alpha \in A\}$ such that the smooth G-cocycle of transition functions of $\mathfrak{P}$ is precisely c. Moreover, $(\mathscr{P}, V)$ is trivial as a flat smooth principal G-bundle if and only if there exists a family of locally constant functions $\{r_\alpha : U_\alpha \longrightarrow G \mid \alpha \in A\}$ satisfying $e_G = r_\beta(x)^{-1} g_{\alpha\beta}(x) r_\alpha(x)$ for all $x \in U_{\alpha\beta}$ and for all $\alpha\beta \in A^x$.*

Proof: By the smooth principal bundle reconstruction lemma, Theorem 6.10.17, we obtain a smooth principal G-bundle $\mathscr{P} = (P, M, \pi, G, \mathfrak{p})$ and a smooth principal G-bundle atlas $\mathfrak{P} = \{\phi_\alpha : \mathscr{P}|_{U_\alpha} \xrightarrow{\sim !} \underline{G}_{U_\alpha} \mid \alpha \in A\}$, whose smooth G-cocycle of transition functions is precisely c. For each $\alpha \in A$, we define a flat principal connection $V_\alpha : \tau_{P_{U_\alpha}} \xrightarrow{!} \tau_{P_{U_\alpha}}$ by the pullback $V_\alpha := T\phi_\alpha^{-1} \circ \bar{V}_\alpha \circ T\phi_\alpha$. Let $\mathfrak{i}_\alpha : \tau_{P_{U_\alpha}} \longrightarrow \tau_P$ denote the bundle inclusion for all $\alpha \in A$ and $\mathfrak{i}_{\alpha\beta} : \tau_{P_{U_{\alpha\beta}}} \longrightarrow \tau_P$ and $\mathfrak{i}_{\alpha\beta,\alpha} : \tau_{P_{U_{\alpha\beta}}} \longrightarrow \tau_{P_{U_\alpha}}$ the bundle inclusions for all $\alpha\beta \in A^x$; then

$$(6.191)\quad \left(\mathfrak{i}_\alpha \circ V_\alpha\right)|_{P_{U_{\alpha\beta}}} = \mathfrak{i}_\alpha \circ T\phi_\alpha^{-1} \circ \bar{V}_\alpha \circ T\phi_\alpha \circ \mathfrak{i}_{\alpha\beta,\alpha}$$

$$= \mathfrak{i}_\alpha \circ T\phi_\alpha^{-1} \circ \bar{V}_\alpha \circ \mathfrak{i}_{\alpha\beta,\alpha} \circ T\phi_\alpha\|_{U_{\alpha\beta}} \tag{6.192}$$

$$= \mathfrak{i}_\alpha \circ T\phi_\alpha^{-1} \circ \mathfrak{i}_{\alpha\beta,\alpha} \circ \bar{V}\|_{U_{\alpha\beta}} \circ T\left(\phi_\alpha\|_{U_{\alpha\beta}} \circ \phi_j^{-1}\|_{U_{\alpha\beta}} \circ \phi_\beta\|_{U_{\alpha\beta}}\right) \tag{6.193}$$

$$= \mathfrak{i}_\alpha \circ \mathfrak{i}_{\alpha\beta,\alpha} \circ T\left(\phi_\beta^{-1}\|_{U_{\alpha\beta}} \circ \phi_\beta\|_{U_{\alpha\beta}} \circ \phi_\alpha^{-1}\|_{U_{\alpha\beta}}\right) \circ \bar{V}\|_{U_{\alpha\beta}} \circ T\phi_\beta\|_{U_{\alpha\beta}} \tag{6.194}$$

$$= \mathfrak{i}_{\alpha\beta} \circ T\phi_\beta^{-1}\|_{U_{\alpha\beta}} \circ \bar{V}\|_{U_{\alpha\beta}} \circ T\phi_\beta\|_{U_{\alpha\beta}} \tag{6.195}$$

$$= \left(\mathfrak{i}_\beta \circ V_\beta\right)|_{P_{U_{\alpha\beta}}} \qquad \forall \alpha\beta \in A^x. \tag{6.196}$$

The smooth vector bundle homomorphism gluing lemma, which is a mild generalisation of [Lee03, Lem.2.1], entails that there is a unique flat principal connection $V : \tau_P \xrightarrow{!} \tau_P$ such that $V\|_{U_\alpha} = V_\alpha$ for all $\alpha \in A$ and $\mathfrak{P}$ is a flat smooth principal G-bundle atlas.

Now suppose that there are locally constant functions r_α satisfying the identity $e_G = r_\beta(x)^{-1} g_{\alpha\beta}(x) r_\alpha(x)$ for all $x \in U_{\alpha\beta}$ and for all $\alpha\beta \in A^x$. Then, by the smooth principal bundle reconstruction lemma, $\mathscr{P}$ is trivial as a smooth principal G-bundle and a global trivialisation is given by the unique strong smooth principal G-bundle isomorphism $\Phi : \mathscr{P} \xrightarrow{\sim !} \underline{G}_M$ such that $\Phi|_{U_\alpha} = \mathfrak{i}_{\underline{G}_{U_\alpha}\underline{G}_M} \circ (\mathrm{pr}_1, (r_\alpha^{-1} \circ \mathrm{pr}_1)\,\mathrm{pr}_2) \circ \phi_\alpha$ for all $\alpha \in A$, where $\mathfrak{i}_{\underline{G}_{U_\alpha}\underline{G}_M} : \underline{G}_{U_\alpha} \longrightarrow \underline{G}_M$ denotes the bundle inclusion. Φ is well-defined thanks to the smooth principal bundle homomorphism gluing lemma, which is also a mild generalisation of [Lee03, Lem.2.1], and by construction, $T\Phi \circ V = \bar{V} \circ T\Phi$.

Let us assume now that there is a strong smooth principal G-bundle isomorphism $\Phi : \mathscr{P} \xrightarrow{\sim !} \underline{G}_M$ such that $T\Phi \circ V = \bar{V} \circ T\Phi$. Hence, for each $\alpha \in A$, we can define a smooth local trivialisation for $(\mathscr{P}, V)$ via $\Phi\|_{U_\alpha} : \mathscr{P}|_{U_\alpha} \xrightarrow{\sim !} \underline{G}_{U_\alpha}$. In particular, $V\|_{P_{U_\alpha}} = V_\alpha = T\Phi\|_{U_\alpha}^{-1} \circ \bar{V}_\alpha \circ T\Phi\|_{U_\alpha}$ for all $\alpha \in A$. By considering the transition function from $\Phi\|_{U_\alpha}$ to ϕ, we obtain smooth functions $r_\alpha : U_\alpha \longrightarrow G$ such that $e_G = r_\beta(x)^{-1} g_{\alpha\beta}(x) r_\alpha(x)$ for all $x \in U_{\alpha\beta}$ and for all $\alpha\beta \in A^x$. Substituting $V_\alpha = T\phi_\alpha^{-1} \circ \bar{V}_\alpha \circ T\phi_\alpha$ in $T\Phi\|_{U_\alpha} \circ V_\alpha = \bar{V}_\alpha \circ T\Phi\|_{U_\alpha}$ yields $T(\Phi\|_{U_\alpha} \circ \phi_\alpha^{-1}) \circ \bar{V}_\alpha = \bar{V}_\alpha \circ T(\Phi\|_{U_\alpha} \circ \phi_\alpha^{-1})$ for all $\alpha \in A$. Without the loss of generality, there are smooth charts $\varphi_\alpha : U_\alpha \xrightarrow{\sim} W_\alpha \subseteq \mathbb{R}^m$ of M; otherwise, we can find smooth charts of M around each point $x \in M$ which are completely contained in one of the U_α and further restrict. We also pick some smooth chart $\psi : U \xrightarrow{\sim} W \subseteq \mathbb{R}^n$ of G. For $(x,g) \in U_\alpha \times U$, we can write any $v \in T(M \times G)_{(x,g)}$ as $v = v_h + v_v = \sum_{j=1}^m v^j \frac{\partial}{\partial(\varphi_i \times \psi)^j}\Big|_{(x,g)} + \sum_{k=m+1}^{n+m} v^k \frac{\partial}{\partial(\varphi_i \times \psi)^k}\Big|_{(x,g)}$. The restricted identity $T(\Phi\|_{U_\alpha} \circ \phi_\alpha^{-1}) \circ \bar{V}_\alpha = \bar{V}_\alpha \circ T(\Phi\|_{U_\alpha} \circ \phi_\alpha^{-1})$ implies now that $T(\Phi\|_{U_\alpha} \circ \phi_\alpha^{-1})\, v_h \left[(\varphi \times \psi)^k\right] = v_h\left[(x,g) \longmapsto \psi^k(r_\alpha^{-1}(x)\,g)\right] = 0$ for $k = m+1, \ldots, m+n$, for all $v \in T(M \times G)_{(x,g)}$ and for all $(x,g) \in U_\alpha \times U$. This can surely only be the case if r_α^{-1} (and thus r_α) is locally constant on U_α. ❒

Chapter 7

On Pure and Quasifree States for the Quantised Free Massive Dirac Field

So far, we have only dealt with algebraic aspects of algebraic and locally covariant quantum field theory in this thesis. Indeed, (unital) (C)*-algebras of local observables or local smearings of quantum fields are the main objects of study and axiomatisation in algebraic quantum field theory but to make at all contact with actual physical experiments, a theory needs to make predictions in terms of numbers, which can be compared with the outcome of measurements. This is achieved by states, which are normalised (if an identity element is present) positive linear functionals on the algebras of local observables resp. local smearings of quantum fields. However, not all of these "*mathematical*" states are physically sensible and it is thus important to distinguish the physically reasonable states from the unphysical ones.

Over the past decades, the *Hadamard condition* has emerged as *the* criterion for physical states in linear quantum field theories in curved spacetimes. Essentially, the Hadamard condition is a condition on the Wightman two-point distribution which is associated with a state and fixes its singular structure in terms of the spacetime metric and the wave equation obeyed by the quantum field via the *Hadamard recursion relations* (see e.g. [DB60]). Physically speaking, the Hadamard condition determines the ultraviolet behaviour, i.e. the behaviour at small distances or, equivalently, for large momenta. *Hadamard states* thus become significant for renormalised stress-energy-momentum tensors, perturbative constructions of interacting quantum field theories and finite quantum fluctuations. Though it is usually known by deformation arguments [FNW81] that Hadamard states are abundant, it is notoriously difficult to give explicit constructions.

It is quite laborious and involved to state the Hadamard condition, hence we refrain from doing so and refer the reader to the literature. The rigorous definition of Hadamard states was first given in [KW91] for the quantised free real scalar field in curved spacetimes. An equivalent, more modern way to characterise Hadamard states for linear quantum field theories in curved spacetimes is due to the works [Rad92; Rad96; RV96], utilising the powerful techniques of *microlocal analysis* and *wave front sets*. The wave front set is a refined notion of the *singular support* of a distribution, which indicates the localisation of singularities, and the *frequency set* of a distribution, which indicates the directions in which singularities occur. For microlocal analysis in general and the wave front set of a distribution, the reader is invited to consult [Hör71; DH72; Hör90; Dui96; FJ98; Ver99; BF09] and in particular [BDH14].

The Hadamard condition for the quantised free Dirac field in curved spacetimes was given in [Köh95] (see also [Kra99; Kra00; SV01]), extending the rigorous definition of Hadamard states for the quantised free real scalar field in [KW91] by using an ansatz of [NO84]. For the definition of Hadamard states for the quantised free Dirac field in terms of wave front sets, we further refer the reader to [SV00; SV01; Hol01; FV02; San08; Hac10].

The goal of this chapter is to present a construction for a whole family of Hadamard states for the quantised free massive Dirac field on 4-dimensional, oriented and globally hyperbolic ultrastatic slabs with compact spatial section, which have not been considered previously in the literature. The addition of "*massive*" means here that the mass of the field is assumed to be > 0; this assumption on the mass is to avoid zero modes. The starting point for our construction of these new Hadamard states is a recent description in [FR14a] of F. Finster's fermionic projector [Fin98; Fin06], a notion introduced for the discussion of the Dirac sea where it provides a splitting of the solution space of the Dirac equation into "*positive*" and "*negative frequency solutions*". Utilising this description, we obtain a pure and quasifree state, the *unsoftened FP-state* ("*FP*" for fermionic projector), in a canonical way. We show that this state can almost always ruled out to be Hadamard and exhibits infinite quantum fluctuations, e.g. the normal ordered energy density has infinite quantum fluctuations in the unsoftened FP-state. Note, there is no mention of states in [FR14a] at all; neither is there in [FR14b].

Our reasoning follows similar lines of [FV12c; FV13], which showed that the so-called *SJ-state* for the free and minimally coupled real scalar quantum field in curved spacetimes, "*SJ*" stands for R.D. Sorkin and S. Johnston, can almost always ruled out to be Hadamard, does not give rise to a natural state[1] and exhibits infinite quantum fluctuations. The SJ-state has been recently put forward by [AAS12] as a physically reasonable and distinguished state for the free and minimally coupled real scalar quantum field in curved spacetimes. Furthermore, the construction of the SJ-state only uses intrinsic properties of the globally hyperbolic spacetime it is considered on, namely the advanced-minus-retarded Green operator of the Klein-Gordon operator and spectral theory of bounded linear operators on L^2-spaces. This, however, seemed to be in con-

[1] Recall that a <u>*natural state*</u> ω for a locally covariant quantum field theory $F : \mathbf{Loc} \longrightarrow \mathbf{(C)^*Alg_1^m}$ is a rule $\omega : \mathbf{Loc} \ni \mathbf{M} \longmapsto \omega_{\mathbf{M}}$, where the $\omega_{\mathbf{M}} : F\mathbf{M} \longrightarrow \mathbb{C}$ are states, such that $\omega_{\mathbf{N}} \circ F\psi = \omega_{\mathbf{M}}$ for all $\psi \in \mathbf{Loc}(\mathbf{M}, \mathbf{N})$ and for all $\mathbf{M}, \mathbf{N} \in \mathbf{Loc}$. The SJ-states fail to yield a natural state because for $\mathbf{M}, \mathbf{N} \in \mathbf{Loc}$ such that $M \subseteq N$, $M \neq N$ and the inclusion map $\iota : M \hookrightarrow N$ is a $\mathbf{Loc}$-morphism, the Wightman two-point distributions, which are associated with the corresponding SJ-states, can differ on smooth functions with compact support in M.

flict with the no-go theorem for natural states [FV12a, Thm.6.13] by C.J. Fewster and R. Verch; consequently, it was investigated in [FV12c; FV13] if the SJ-state featured unphysical properties and if the no-go theorem was flawed. Note that it is remarked in [BF14; FV13] that the fermionic projector is the analogue or even the forerunner of the SJ-state in the case of the quantised free Dirac field and it is only fair to say that these remarks actually have given rise to the investigation in the present chapter.

In [BF14], it was shown how to modify the SJ-state in order to always yield a Hadamard state, coming at the price of uncanonically introducing a compactly supported smooth function. Hence, the background covariant character is spoiled. Inspired by this result, we also modify the unsoftened FP-state in the spirit of [BF14], though somewhat different in detail. We are thus led to a whole family of FP-states, labelled by non-negative integrable functions on the real line. We argue that the *softened FP-states*, which are obtained by using compactly supported smooth functions and are thereby the analogue of the Brum-Fredenhagen-modified SJ-states, are always Hadamard.

As we have mentioned before briefly, the definition of Hadamard states is very complex and it is thus quite hard and laborious to directly verify that a given state satisfies the Hadamard condition. In this chapter, we bypass a direct check by applying the *comparism test* for Hadamard states: since, by definition, Wightman two-point distributions associated with Hadamard states have the same singular structure, they only differ by a term which corresponds to an integration against a smooth function or, more generally, smooth cross-section in a smooth vector bundle. This allows us to determine whether or not a given state is Hadamard by examining its difference with a reference state which is known to be Hadamard. If the difference of the associated Wightman two-point distributions is given by integration against a smooth function, a Hadamard state is on hand and if the difference is singular, the given state is not a Hadamard state.

In the first section of this chapter, Section 7.1, we review the free massive Dirac equation for spinors and cospinors on general 4-dimensional, oriented and globally hyperbolic spacetimes and in Section 7.2, we point out some particularities for 4-dimensional, oriented and globally hyperbolic ultrastatic spacetimes and slabs with compact spatial section. In Section 7.3, we construct solution spaces for the free massive Dirac equations on 4-dimensional, oriented and globally hyperbolic ultrastatic spacetimes and slabs with compact spatial section. These solution spaces are employed in Section 7.4 for an ensuing CAR-quantisation in terms of the completion of the self-dual CAR-algebra. Using the methods of [Ara70], we construct a pure and quasifree state in Section 7.4, which will be our reference Hadamard state. In Section 7.5, we show how the unsoftened FP-state can be extracted from the description of the fermionic projector in [FR14a] on 4-dimensional, oriented and globally hyperbolic ultrastatic slabs with compact spatial section. Since it does not complicate our discussion and formulas at all, we deal with the construction of the unsoftened FP-state and its modifications, in particular the softened FP-states, all at once. We show in Section 7.6 that the unsoftened FP-state can almost always ruled out to be a Hadamard state and in Section 7.7, we prove that the softened FP-states are always Hadamard. Finally, we argue in Section 7.8 that the normal ordered energy density has infinite quantum fluctuations in the unsoftened FP-state. Some calculations, which we regard as too bulky and too distracting from the main body, are collected in the appendix to this chapter.

7.1 The free massive Dirac equations on globally hyperbolic spacetimes

We collect the necessary notions to formulate the free massive Dirac equation for spinor and cospinor fields on general globally hyperbolic spacetimes.

For a detailed discussion of spin structures, (co)spinors, spin connections and the free Dirac equations, which is impossible for us to provide here within reasonable bounds, we refer the reader to the pertinent literature [Ger68; Ger70b; Ish78a; Dim82; FV02; San08; San10; Fer13a]. We will simply collect the results needed for our purposes.

Let $\mathbf{M} = (M, g, [T], [\Omega])$ be an oriented globally hyperbolic spacetime of dimension 4, equipped with a fixed *smooth global Lorentz framing* $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$; that is, $\varepsilon_\mu \in \mathscr{X}(M)$, $\mu = 0,1,2,3$, such that $(\varepsilon_0(x), \dots, \varepsilon_3(x))$ is a time-oriented, oriented and orthonormal basis of TM_x for each $x \in M$. The (algebraic) dual basis of ε_μ will be denoted by $\varepsilon^\mu \in \Omega^1 M$, so that $\varepsilon^\mu(\varepsilon_\nu) = \delta^\mu{}_\nu$, and of course $g = \eta_{\mu\nu} \varepsilon^\mu \otimes \varepsilon^\nu$, where $\eta_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$ is the Minkowski metric. Since smooth principal $\mathrm{SL}(2;\mathbb{C})$-bundles over 4-dimensional globally hyperbolic spacetimes are automatically trivial [Ish78a], we may view *spinor fields* as $\mathbb{C}^4$-valued smooth functions, i.e. as elements in $\mathcal{C}^\infty(M, \mathbb{C}^4)$, while smooth $(\mathbb{C}^4)^*$-valued functions, i.e. elements in $\mathcal{C}^\infty(M, (\mathbb{C}^4)^*)$, are *cospinor fields*. Elements of $\mathbb{C}^4$ (resp. $(\mathbb{C}^4)^*$) are regarded as column (resp. row) vectors. Also note that $\mathcal{C}^\infty(M, \mathbb{C}^4)$ and $\mathcal{C}^\infty(M, (\mathbb{C}^4)^*)$ can be canonically identified with the $C^\infty(M, \mathbb{C})$-modules of smooth cross-sections $\Gamma^\infty(\underline{\mathbb{C}^4}_M)$ and $\Gamma^\infty((\underline{\mathbb{C}^4}_M)^*)$, where $\underline{\mathbb{C}^4}_M := (M \times \mathbb{C}^4, M, \mathrm{pr}_1, \mathbb{C}^4)$ is the trivial smooth complex vector bundle over M of rank 4 and $(\underline{\mathbb{C}^4}_M)^*$, which is given by $(M \times (\mathbb{C}^4)^*, M, \mathrm{pr}_1, (\mathbb{C}^4)^*)$, its dual.

We choose the Pauli realisation [BLT75, (7.31)] for the *γ-matrices*,

$$(7.1) \qquad \gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = -\mathrm{i} \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix},$$

with the *Pauli matrices*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.2}$$

In addition to the *Clifford relations* $\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, we note the identities $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^*$, $(\gamma^0)^* = \gamma^0$ and $(\gamma^i)^* = -\gamma^i$ ("$*$" denotes *Hermitean conjugation*, i.e. complex conjugation and transposition), which we will use throughout without further mention.

The *free massive Dirac equations* for spinor fields $\psi \in \mathcal{C}^\infty(M, \mathbb{C}^4)$ and cospinor fields $\varphi \in \mathcal{C}^\infty(M, (\mathbb{C}^4)^*)$ are now

$$D^{\mathrm{sp}}\psi = (-\mathrm{i}\,\nabla\!\!\!/^{\mathrm{sp}} + \mu)\,\psi = \left(-\mathrm{i}\,\gamma^\mu\nabla^{\mathrm{sp}}_{\varepsilon_\mu} + \mu\right)\psi = 0 \tag{7.3}$$

and

$$D^{\mathrm{cosp}}\varphi = (\mathrm{i}\,\nabla\!\!\!/^{\mathrm{cosp}} + \mu)\,\varphi = \mathrm{i}\left(\nabla^{\mathrm{cosp}}_{\varepsilon_\mu}\varphi\right)\gamma^\mu + \varphi\mu = 0, \tag{7.4}$$

where $\mu := \frac{mc}{\hbar}$ is the reduced mass with $m > 0$ the mass of the field, c the speed of light and $\hbar$ the reduced Planck constant. $\nabla^{\mathrm{sp}} : \mathcal{C}^\infty(M;\mathbb{C}^4) \longrightarrow \Omega^1(M;\mathbb{C}^4)$ and $\nabla^{\mathrm{cosp}} : \mathcal{C}^\infty(M;(\mathbb{C}^4)^*) \longrightarrow \Omega^1(M;(\mathbb{C}^4)^*)$ are called the *spin connections* and are given by

$$\nabla^{\mathrm{sp}}\psi = d\psi^A(\varepsilon_\mu)\,\varepsilon^\mu \otimes e_A + \varepsilon^\mu \otimes \Gamma_\mu\,\psi, \qquad \psi \in \mathcal{C}^\infty(M, \mathbb{C}^4), \tag{7.5}$$

and

$$\nabla^{\mathrm{cosp}}\varphi = d\varphi_A(\varepsilon_\mu)\,\varepsilon^\mu \otimes e^A - \varepsilon^\mu \otimes \varphi\,\Gamma_\mu, \qquad \varphi \in \mathcal{C}^\infty(M, (\mathbb{C}^4)^*), \tag{7.6}$$

where $\Gamma_\mu = \frac{1}{4}\Gamma^\lambda_{\mu\nu}\gamma_\lambda\gamma^\nu$, $\Gamma^\lambda_{\mu\nu}\varepsilon_\lambda = \nabla_{\varepsilon_\mu}\varepsilon_\nu$ with $\nabla : \mathscr{X}(M) \longrightarrow \Omega^1(M;\tau_M)$ the the Levi-Civita connection, e_A is the standard basis for $\mathbb{C}^4$ and e^A the corresponding (algebraic) dual basis of $(\mathbb{C}^4)^*$. Using Koszul's formula [O'N83, Thm.3.11], one can easily show $\Gamma^\nu_{\mu\nu} = 0$ (no summation!).

As usual, the *Dirac adjoint* is the complex-conjugate linear bijection

$$\dagger : \mathcal{C}^\infty(M, \mathbb{C}^4) \longrightarrow \mathcal{C}^\infty(M, (\mathbb{C}^4)^*), \qquad \psi \longmapsto \psi^*\gamma^0. \tag{7.7}$$

Since it will be clear from the context whether we apply the Dirac adjoint to a spinor field or its inverse to a cospinor field, we will write $\psi^\dagger$ and $\varphi^\dagger$ for both the Dirac adjoints of $\psi \in \mathcal{C}^\infty(M, \mathbb{C}^4)$ and $\varphi \in \mathcal{C}^\infty(M, (\mathbb{C}^4)^*)$. Observe that $(\nabla\!\!\!/^{\mathrm{sp}}\psi)^\dagger = \nabla\!\!\!/^{\mathrm{cosp}}\psi^\dagger$ for all $\psi \in \mathcal{C}^\infty(M, \mathbb{C}^4)$ and $(\nabla\!\!\!/^{\mathrm{cosp}}\varphi)^\dagger = \nabla\!\!\!/^{\mathrm{sp}}\varphi^\dagger$ for all $\varphi \in \mathcal{C}^\infty(M, (\mathbb{C}^4)^*)$, which readily yields $(D^{\mathrm{sp}}\psi)^\dagger = D^{\mathrm{cosp}}\psi^\dagger$ and $(D^{\mathrm{cosp}}\varphi)^\dagger = D^{\mathrm{sp}}\varphi^\dagger$.

Owing to the global hyperbolicity of $\mathbf{M}$, the equations (7.3) and (7.4) have well-posed Cauchy problems (see e.g. [Dim82, Thm.2.3] or [Müh11, Thm.2]), and unique retarded and advanced Green's operators (see e.g. [Dim82, Thm.2.1] or [Müh11, Thm.1]). We denote the unique retarded and advanced Green operators for spinors (resp. cospinors) by $S^{\mathrm{ret}}, S^{\mathrm{adv}}$ (resp. $C^{\mathrm{ret}}, C^{\mathrm{adv}}$).

7.2 The free massive Dirac equations on ultrastatic spacetimes and slabs

We investigate some special properties of the Dirac operators for spinor and cospinor fields on ultrastatic spacetimes and slabs.

A spacetime $(M, g, [T])$ is called *ultrastatic* if it is of the smooth product form $M = \mathbb{R} \times \Sigma$ with metric $g = d\,\mathrm{pr}_1 \otimes d\,\mathrm{pr}_1 - \mathrm{pr}_2^*\, h$, where h is a Riemannian metric on Σ. Naturally, we will always take the time-orientation $[T]$ defined by $\frac{\partial}{\partial\,\mathrm{pr}_1}$. If M is the smooth product manifold $(a,b) \times \Sigma$, where $a, b \in \mathbb{R}$ with $a < b$, and h is a Riemannian metric on Σ, the spacetime $((a,b) \times \Sigma, d\,\mathrm{pr}_1 \otimes d\,\mathrm{pr}_1 - \mathrm{pr}_2^*\, h, [T])$ is said to be an *ultrastatic slab*. By [Kay78, Prop.5.2], an ultrastatic spacetime or slab is globally hyperbolic if and only if (Σ, h) is a complete Riemannian manifold, as is certainly the case by the Hopf-Rinow theorem if Σ is taken to be compact [O'N83, Cor.5.23]. Note that, in the terminology of [FR14a], our ultrastatic slabs have "*finite lifetime*".

Let $(\Sigma, h, [\omega])$ be an oriented, connected and compact Riemannian manifold of dimension 3. Hence, Σ is parallelisable [Sti35], that is, the tangent bundle τ_Σ of Σ is trivial. This ensures the existence of oriented smooth global framings for τ_Σ and by Gram-Schmidt, the existence of oriented and orthonormal smooth global framings for τ_Σ. Fix such a one, say (X_1, X_2, X_3), where $X_i \in \mathscr{X}(\Sigma)$, $i = 1,2,3$; then we define smooth vector fields $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathscr{X}((a,b) \times \Sigma)$ on the smooth product manifold $(a,b) \times \Sigma$, where $-\infty \le a < b \le \infty$, by setting $\varepsilon_i(t, x; f) \coloneqq X_i(x; f(t, \cdot))$ for all $t \in (a,b)$, for all $x \in \Sigma$ and for all $f \in \mathcal{C}^\infty((a,b) \times \Sigma)$.

The quadruple $\mathbf{M} = \left((a,b) \times \Sigma, d\,\mathrm{pr}_1 \otimes d\,\mathrm{pr}_1 - \mathrm{pr}_2^*\, h, \left[\frac{\partial}{\partial\,\mathrm{pr}_1}\right], [d\,\mathrm{pr}_1 \wedge \mathrm{pr}_2^*\, \omega]\right)$ is a 4-dimensional, oriented and globally hyperbolic ultrastatic spacetime or slab with compact spatial section and $(\varepsilon_0 \coloneqq \frac{\partial}{\partial\,\mathrm{pr}_1}, \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is a smooth global Lorentz framing by construction. We will use precisely this smooth global Lorentz framing in the definition of the spin connections (7.5) and (7.6), and in the free massive Dirac equations (7.3) and (7.4). From now on, we will always consider 4-dimensional, oriented and globally hyperbolic ultrastatic spacetimes or slabs with spin connections defined in the way just described.

Using the Koszul formula (see again [O'N83, Thm.3.11]), one may show that $\Gamma^\lambda_{\mu\nu}$

vanishes if μ, ν or λ is zero, which implies $\Gamma_0 = 0$ and $\Gamma_i = \frac{1}{4}\Gamma^k_{ij}\gamma_k\gamma^j$ in (7.5) and (7.6). Furthermore, Γ^k_{ij} does not depend on $t \in (a,b)$ by construction and can be regarded as a smooth function on Σ. Using the fact that the tensor product $\mathcal{C}^\infty((a,b),\mathbb{C}) \otimes \mathcal{C}^\infty(\Sigma,\mathbb{C}^4)$ can be identified with a dense linear subspace of $\mathcal{C}^\infty((a,b)\times\Sigma,\mathbb{C}^4)$ in a continuous way and similarly, $\mathcal{C}^\infty((a,b),\mathbb{C}) \otimes \mathcal{C}^\infty(\Sigma,(\mathbb{C}^4)^*)$ can be continuously identified with a dense linear subspace of $\mathcal{C}^\infty((a,b)\times\Sigma,(\mathbb{C}^4)^*)$, this all implies that the Dirac operators for spinors and cospinors can be written in split form:

$$D^{\mathrm{sp}} = -\mathrm{i}\frac{\partial}{\partial t}\otimes\gamma^0 + \mathbb{1}\otimes\gamma^0 H^{\mathrm{sp}} \qquad \text{and} \qquad D^{\mathrm{cosp}} = \mathrm{i}\frac{\partial}{\partial t}\otimes\gamma^0 + \mathbb{1}\otimes H^{\mathrm{cosp}}(\cdot)\,\gamma^0, \tag{7.8}$$

where $\mathbb{1}$ denotes the identity on $\mathcal{C}^\infty((a,b),\mathbb{C})$; $H^{\mathrm{sp}} : \mathcal{C}^\infty(\Sigma,\mathbb{C}^4) \longrightarrow \mathcal{C}^\infty(\Sigma,\mathbb{C}^4)$ and $H^{\mathrm{cosp}} : \mathcal{C}^\infty(\Sigma,(\mathbb{C}^4)^*) \longrightarrow \mathcal{C}^\infty(\Sigma,(\mathbb{C}^4)^*)$ are linear differential operators of order 1 defined by

$$H^{\mathrm{sp}}\chi := -\mathrm{i}\,\gamma^0\gamma^i\left(X_i(\chi^A)e_A + \Gamma_i(t,\cdot)\,\psi\right) + \mu\gamma^0\chi, \qquad \chi \in \mathcal{C}^\infty(\Sigma,\mathbb{C}^4), \tag{7.9}$$

and

$$H^{\mathrm{cosp}}\zeta := \mathrm{i}\left(X_i(\zeta_A)\,e^A - \zeta\,\Gamma_i(t,\cdot)\right)\gamma^i\gamma^0 + \mu\zeta\gamma^0, \qquad \zeta \in \mathcal{C}^\infty(\Sigma,(\mathbb{C}^4)^*). \tag{7.10}$$

Recall that it does not matter which $t \in (a,b)$ is taken because of the time-independence of Γ_i.

LEMMA 7.2.1. *H^{sp} and H^{cosp} are elliptic.*

Proof: We only prove the statment for H^{cosp}; the proof for H^{sp} is analogous. From (7.10), the principal symbol of H^{cosp} is seen to be $\sigma_{H^{\mathrm{cosp}}}(\xi) = \mathrm{i}\,X_i(x;\xi)\,\gamma^i\gamma^0$ for $\xi \in T^*\Sigma_x$, $x \in \Sigma$. Expanding for all $x \in M$, $\xi \in T^*\Sigma_x$ in the (algebraic) dual bases of the X_i, we obtain $\sigma_{H^{\mathrm{cosp}}}(\xi) = \mathrm{i}\,\xi_i\gamma^i\gamma^0$ for all $\xi \in T^*\Sigma$. One easily computes the determinant $\det(\xi_i\gamma^i\gamma^0) = (\xi_1^2+\xi_2^2+\xi_3^2)^2$, which shows that $\sigma_{H^{\mathrm{cosp}}}(\xi)$ is an isomorphism of complex vector spaces for all $\xi \in T^*\Sigma$ unless $\xi = 0 \in T^*\Sigma_x$ for $x \in \Sigma$. ❒

Using the standard inner products in $\mathbb{C}^4$ and $(\mathbb{C}^4)^*$, we obtain canonical pairings for smooth $\mathbb{C}^4$-valued and $(\mathbb{C}^4)^*$-valued functions. To be more specific, we have smooth functions (recall from before that " $*$ " denotes the Hermitean conjugate) $\chi^*\kappa$ and $\xi\zeta^*$ for $\chi,\kappa \in \mathcal{C}^\infty(\Sigma,\mathbb{C}^4)$ and $\zeta,\xi \in \mathcal{C}^\infty(\Sigma,(\mathbb{C}^4)^*)$. H^{sp} and H^{cosp} have now the following important property:

LEMMA 7.2.2. *H^{sp} and H^{cosp} are metric compatible in the sense that*

$$\left(H^{\mathrm{sp}}\chi\right)^*\kappa - \chi^*\left(H^{\mathrm{sp}}\kappa\right) = \mathrm{i}\,d\left(\chi^*\gamma^0\gamma^i\kappa\right)(X_i) \qquad \forall\chi,\kappa \in \mathcal{C}^\infty(\Sigma,\mathbb{C}^4) \tag{7.11}$$

and

$$\xi\left(H^{\mathrm{cosp}}\zeta\right)^* - \left(H^{\mathrm{cosp}}\xi\right)\zeta^* = \mathrm{i}\, d\left(\xi\gamma^0\gamma^i\zeta^*\right)(X_i) \qquad \forall \zeta,\xi \in \mathcal{C}^\infty\left(\Sigma,(\mathbb{C}^4)^*\right). \tag{7.12}$$

Proof: Again, we only show our claim for H^{cosp} because the proof for H^{sp} is analogous. We directly compute (important note: $\overline{\zeta}_A$ denotes the complex conjugate of $\zeta_A \in \mathbb{C}$; throughout this thesis the symbol "$^{-}$" does *not* denote the Dirac adjoint)

$$\xi\left(H^{\mathrm{cosp}}\zeta\right)^* - \left(H^{\mathrm{cosp}}\xi\right)\zeta^* = \mathrm{i}\,\xi\,\gamma^0\gamma^i \sum_{A=1}^{4} X_i\left(\overline{\zeta}_A\right)e_A - \mathrm{i}\,X_i\left(\xi_B\right)e^B\gamma^i\gamma^0\,\zeta^* \tag{7.13}$$

$$= \mathrm{i}\,\xi\,\gamma^0\gamma^i \sum_{A=1}^{4} X_i\left(\overline{\zeta}_A\right)e_A + \mathrm{i}\,X_i\left(\xi_B\right)e^B\gamma^0\gamma^i\,\zeta^* \tag{7.14}$$

$$= \mathrm{i}\sum_{A=1}^{4} \xi_B\,X_i\left(\overline{\zeta}_A\right)e^B\gamma^0\gamma^i e_A + X_i\left(\xi_B\right)\overline{\zeta}_A\,e^B\gamma^0\gamma^i e_A \tag{7.15}$$

$$= \mathrm{i}\sum_{A=1}^{4} X_i\left(\xi_B\,\overline{\zeta}_A\right)e^B\gamma^0\gamma^i e_A \tag{7.16}$$

$$= \mathrm{i}\,X_i\left(\xi\gamma^0\gamma^i\zeta^*\right) \tag{7.17}$$

$$= \mathrm{i}\,d\left(\xi\gamma^0\gamma^i\zeta^*\right)(X_i) \qquad \forall \zeta,\xi \in \mathcal{C}^\infty\left(\Sigma,(\mathbb{C}^4)^*\right), \tag{7.18}$$

which shows the metric compatibility. ❒

By Stokes' theorem, the metric compatibility of H^{sp} and H^{cosp} has the following important consequence:

COROLLARY 7.2.3. *In the sense of* [LM89, Chap.III, §5], *H^{sp} and H^{cosp} are self-adjoint with respect to the standard L^2-inner products*

$$\langle\cdot\,|\,\cdot\rangle_\Sigma : \mathcal{C}^\infty\left(\Sigma,\mathbb{C}^4\right)\times\mathcal{C}^\infty\left(\Sigma,\mathbb{C}^4\right)\longrightarrow\mathbb{C}, \qquad (\chi,\kappa)\longmapsto\int_\Sigma \chi^*\kappa\,\mathrm{vol}_h, \tag{7.19}$$

and

$$\langle\cdot\,|\,\cdot\rangle_\Sigma : \mathcal{C}^\infty\left(\Sigma,(\mathbb{C}^4)^*\right)\times\mathcal{C}^\infty\left(\Sigma,(\mathbb{C}^4)^*\right)\longrightarrow\mathbb{C}, \qquad (\zeta,\xi)\longmapsto\int_\Sigma \xi\zeta^*\,\mathrm{vol}_h\,. \tag{7.20}$$

Given these results, we may apply [LM89, Thm.III.5.8] to conclude that the eigenvalues of H^{sp} and H^{cosp} are real, have finite multiplicity, are countably many, say[2] $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$, are unbounded in magnitude, and that their corresponding eigenfunctions are smooth.

Once normalised with respect to $\langle\cdot\,|\,\cdot\rangle_\Sigma$, we denote the smooth eigenfunctions by $\chi_n \in \mathcal{C}^\infty\left(\Sigma,\mathbb{C}^4\right)$ and $\zeta_n \in \mathcal{C}^\infty\left(\Sigma,(\mathbb{C}^4)^*\right)$, where $n\in\mathbb{N}$, and their L^2-equivalence classes

[2] For us, "0" is a natural number and therefore included in the set $\mathbb{N}$ of natural numbers.

furnish orthonormal bases for $L^2(\Sigma, \mathbb{C}^4; \mathrm{vol}_h)$ and $L^2(\Sigma, (\mathbb{C}^4)^*; \mathrm{vol}_h)$. As we have that $(H^{\mathrm{sp}}\chi)^\dagger = H^{\mathrm{cosp}}\chi^\dagger$ for all $\chi \in \mathcal{C}^\infty(\Sigma, \mathbb{C}^4)$ and $(H^{\mathrm{cosp}}\zeta)^\dagger = H^{\mathrm{sp}}\zeta^\dagger$ for all $\zeta \in \mathcal{C}^\infty(\Sigma, (\mathbb{C}^4)^*)$, each eigenvalue of H^{sp} is also an eigenvalue of H^{cosp} and vice versa. The Dirac adjoint of each eigenfunction for H^{sp} to a certain eigenvalue is also an eigenfunction for H^{cosp} to that very same eigenvalue and vice versa. Hence, without the loss of generality we may assume $\lambda_n = \nu_n$ and $\chi_n^\dagger = \zeta_n$ for all $n \in \mathbb{N}$. Furthermore, from $\{\gamma^\mu, \gamma^5\} = 0$ and $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ it can be seen that if $H^{\mathrm{sp}}\chi = \lambda\chi$ and $H^{\mathrm{cosp}}\zeta = \lambda\zeta$ for $\lambda \in \mathbb{R}$, $\chi \in \mathcal{C}^\infty(\Sigma, \mathbb{C}^4)$ and $\zeta \in \mathcal{C}^\infty(\Sigma, (\mathbb{C}^4)^*)$, then $H^{\mathrm{sp}}\gamma^6\chi = -\lambda\gamma^6\chi$ and $H^{\mathrm{cosp}}\zeta(\gamma^6)^\top = -\lambda\zeta(\gamma^6)^\top$, where we have introduced

$$\gamma^6 := -\mathrm{i}\,\gamma^5\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix} \tag{7.21}$$

and "⊤" denotes transposition. So, for each eigenvalue $\lambda \in \mathbb{R}$ of H^{sp} and H^{cosp}, $-\lambda$ is also an eigenvalue of H^{sp} and H^{cosp}. Without the loss of generality, we may thus arrange the eigenvalues and smooth eigenfunctions of H^{sp} and H^{cosp} for our comfort as follows ($z \in \mathbb{Z}' := \mathbb{Z} \setminus \{0\}$):

$$\ldots \le \lambda_{-3} \le \lambda_{-2} \le \lambda_{-1} \le -\mu < 0 < \mu \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots \quad \text{and} \quad \lambda_{-z} = -\lambda_z; \tag{7.22}$$

$$H^{\mathrm{sp}}\chi_z = \lambda_z\chi_z, \qquad H^{\mathrm{cosp}}\zeta_z = \lambda_z\zeta_z, \qquad \chi_z^\dagger = \zeta_z \qquad \text{and} \qquad \zeta_z^\dagger = \chi_z. \tag{7.23}$$

We may also further assume, and this property will useful in Section 7.5, that

$$\langle \chi_w \mid \gamma^0\chi_z\rangle_\Sigma = \langle \zeta_w \mid \zeta_z\gamma^0\rangle_\Sigma = \begin{cases} \frac{\mu}{\lambda_z} & \text{if } z = w \\ \sqrt{1 - \frac{\mu^2}{\lambda_z^2}} & \text{if } z = -w \\ 0 & \text{if } z \neq \pm w \end{cases} \qquad \forall w, z \in \mathbb{Z}'. \tag{7.24}$$

This assumption can be justified as follows. Let $\zeta, \xi \in \mathcal{C}^\infty(\Sigma, (\mathbb{C}^4)^*)$ be normalised with respect to $\langle \cdot \mid \cdot \rangle_\Sigma$, $H^{\mathrm{cosp}}\zeta = \lambda\zeta$ and $H^{\mathrm{cosp}}\xi = \nu\xi$ for $\lambda, \nu \in \mathbb{R}$. It follows from $\{H^{\mathrm{cosp}}, \gamma^0\} = 2\mu$, that is $(H^{\mathrm{cosp}}\zeta)\gamma^0 + H^{\mathrm{cosp}}(\zeta\gamma^0) = 2\mu\zeta$ for all $\zeta \in \mathcal{C}^\infty(\Sigma, (\mathbb{C}^4)^*)$, and Corollary 7.2.3 that

$$(\lambda + \nu)\langle \zeta \mid \xi\gamma^0\rangle_\Sigma = \langle \lambda\zeta \mid \xi\gamma^0\rangle_\Sigma + \langle \zeta \mid \nu\xi\gamma^0\rangle_\Sigma \tag{7.25}$$

$$= \langle H^{\mathrm{cosp}}\zeta \mid \xi\gamma^0\rangle_\Sigma + \langle \zeta \mid (H^{\mathrm{cosp}}\xi)\gamma^0\rangle_\Sigma \tag{7.26}$$

$$= \langle \zeta \mid H^{\mathrm{cosp}}(\xi\gamma^0)\rangle_\Sigma + \langle \zeta \mid (H^{\mathrm{cosp}}\xi)\gamma^0\rangle_\Sigma \tag{7.27}$$

$$= \langle \zeta \mid \{H^{\mathrm{cosp}}, \gamma^0\}\xi\rangle_\Sigma \tag{7.28}$$

$$= 2\mu\langle \zeta \mid \xi\rangle_\Sigma, \tag{7.29}$$

which particularly implies that 0 is not an eigenvalue of H^{sp} or H^{cosp} as long as $m > 0$ ($\Longrightarrow \mu > 0$). Suppose it was; then we find a normalised eigenvector η to the eigenvalue 0 and $0 = (0+0)\langle \eta \mid \eta\gamma^0\rangle_\Sigma = 2\mu\langle \eta \mid \eta\rangle_\Sigma = 2\mu$, which is a clear-cut contradiction. We further conclude

$$\langle \zeta \mid \xi\gamma^0\rangle_\Sigma = \begin{cases} \frac{\mu}{\nu}\langle \zeta \mid \xi\rangle_\Sigma & \text{if } \lambda = \nu \\ 0 & \text{if } \lambda \neq \pm\nu \end{cases}, \tag{7.30}$$

which is not exhaustive as the case $\lambda = -\nu$ is left open. This allows us to assert the mass gap $(-\mu, \mu)$: let η be an eigenfunction of H^{cosp} to the eigenvalue ν of H^{cosp}, then we have by Cauchy-Schwarz $\left|\frac{\mu}{\nu}\right| \|\eta\|_\Sigma^2 = |\langle \eta \mid \eta\gamma^0\rangle_\Sigma| \leq \|\eta\|_\Sigma \|\eta\gamma^0\|_\Sigma = \|\eta\|_\Sigma^2$.

Now, pick an eigenvalue $\nu \in \mathbb{R}$ of H^{cosp}, let n_ν be the multiplicity of ν and assume that $\zeta_{\nu_1}, \dots, \zeta_{\nu_{n_\nu}}$ forms an orthonormal basis for the corresponding eigenspace in $L^2(\Sigma, (\mathbb{C}^4)^*; \mathrm{vol}_h)$. Then, $\zeta_{\nu_1}(\gamma^6)^\top, \dots, \zeta_{\nu_{n_\nu}}(\gamma^6)^\top$ clearly gives rise to an orthonormal basis for the eigenspace of the eigenvalue $-\nu$ of H^{cosp} in $L^2(\Sigma, (\mathbb{C}^4)^*; \mathrm{vol}_h)$. Since the eigenfunctions of H^{cosp} give rise to an orthonormal basis of $L^2(\Sigma, (\mathbb{C}^4)^*; \mathrm{vol}_h)$ and $\|\zeta_{\nu_i}\gamma^0\|^2 = \|\zeta_{\nu_i}\|^2 = 1$, we find with the help of (7.30) that

$$\sum_{i=1}^{n_\nu} |\langle \zeta_{\nu_i}(\gamma^6)^\top \mid \zeta_{\nu_j}\gamma^0\rangle_\Sigma|^2 = 1 - \frac{\mu^2}{\nu^2} \geq 0. \tag{7.31}$$

Hence, if $\nu = \pm\mu$, $\langle \zeta_{\nu_i}(\gamma^6)^\top \mid \zeta_{\nu_j}\gamma^0\rangle_\Sigma = \sqrt{1-\frac{\mu^2}{\nu^2}}\,\delta_{ij} = 0$ and we may continue working with $\zeta_{\nu_1}(\gamma^6)^\top, \dots, \zeta_{\nu_{n_\nu}}(\gamma^6)^\top$. If $\nu \neq \pm\mu$, we define

$$\tilde{\zeta}_{\nu_i} := \frac{1}{\sqrt{1-\frac{\mu^2}{\nu^2}}} \sum_{j=1}^{n_\nu} \langle \zeta_{\nu_j}(\gamma^6)^\top \mid \zeta_{\nu_i}\gamma^0\rangle_\Sigma\, \zeta_{\nu_j}(\gamma^6)^\top = \frac{1}{\sqrt{1-\frac{\mu^2}{\nu^2}}}\left(\zeta_{\nu_i}\gamma^0 - \frac{\mu}{\nu}\zeta_{\nu_i}\right), \quad i = 1, \dots, n_\nu. \tag{7.32}$$

The second identity holds because of (7.30):

$$\sum_{j=1}^{n_\nu} \langle \zeta_{\nu_j}(\gamma^6)^\top \mid \zeta_{\nu_i}\gamma^0\rangle_\Sigma\, \zeta_{\nu_j}(\gamma^6)^\top \tag{7.33}$$

$$= \sum_{j=1}^{n_\nu} \langle \zeta_{\nu_j}(\gamma^6)^\top \mid \zeta_{\nu_i}\gamma^0\rangle_\Sigma\, \zeta_{\nu_j}(\gamma^6)^\top + \sum_{j=1}^{n_\nu} \langle \zeta_{\nu_j} \mid \zeta_{\nu_i}\gamma^0\rangle_\Sigma\, \zeta_{\nu_j} - \sum_{j=1}^{n_\nu} \langle \zeta_{\nu_j} \mid \zeta_{\nu_i}\gamma^0\rangle_\Sigma\, \zeta_{\nu_j} \tag{7.34}$$

$$= \zeta_{\nu_i}\gamma^0 - \frac{\mu}{\nu}\zeta_{\nu_i}. \tag{7.35}$$

It is now easily seen that $\tilde{\zeta}_{\nu_1}, \dots, \tilde{\zeta}_{\nu_{n_\nu}}$ satisfy the desired properties, that is, $H^{\mathrm{cosp}}\tilde{\zeta}_{\nu_i} = -\nu\,\tilde{\zeta}_{\nu_i}$, $\langle \tilde{\zeta}_{\nu_i} \mid \tilde{\zeta}_{\nu_j}\rangle_\Sigma = \delta_{ij}$ and $\langle \tilde{\zeta}_{\nu_i} \mid \zeta_{\nu_j}\gamma^0\rangle_\Sigma = \sqrt{1-\frac{\mu^2}{\nu^2}}\,\delta_{ij}$. The spinor case is analogous.

7.3 Solutions to the free massive Dirac equations

We use the results of Section 7.2 *to solve the free massive Dirac equations* (7.3) *and* (7.4)*, and to get adequate control of the solution spaces. We consider* 4*-dimensional, oriented and globally hyperbolic ultrastatic spacetimes or slabs* $\mathbf{M}$ *with compact spatial section* Σ *and spin connections as in* Section 7.2. *Our interest lies in all smooth solutions with smooth Cauchy data* (*recall that* Σ *is assumed to be compact*). *Thus, by* [BG12, Thm.3.5]*, any solution of interest can be written in terms of the advanced-minus-retarded Green operators* $S := S^{\mathrm{adv}} - S^{\mathrm{ret}}$ *and* $C := C^{\mathrm{adv}} - C^{\mathrm{ret}}$*, i.e. as* Su*,* $u \in \mathcal{C}_0^\infty\left(M, \mathbb{C}^4\right)$*, and* Cv*,* $v \in \mathcal{C}_0^\infty\left(M, \left(\mathbb{C}^4\right)^*\right)$.

For each $\psi \in \mathcal{C}^\infty\left(M, \mathbb{C}^4\right)$ and $\varphi \in \mathcal{C}^\infty\left(M, \left(\mathbb{C}^4\right)^*\right)$, $\psi_t := \psi(t, \cdot)$ and $\varphi_t := \varphi(t, \cdot)$ are smooth $\mathbb{C}^4$-valued and $\left(\mathbb{C}^4\right)^*$-valued functions, and square-integrable on Σ with respect to vol_h for all $t \in (a, b)$, where we allow the possibilities $a = -\infty$ or $b = +\infty$. Hence, we have the L^2-expansions, valid in $L^2\left(\Sigma, \mathbb{C}^4; \mathrm{vol}_h\right)$ and $L^2\left(\Sigma, \left(\mathbb{C}^4\right)^*; \mathrm{vol}_h\right)$, respectively,

$$\psi_t = \sum_{z \in \mathbb{Z}'} \langle \chi_z \mid \psi_t \rangle_\Sigma \, \chi_z \qquad \text{and} \qquad \varphi_t = \sum_{z \in \mathbb{Z}'} \langle \zeta_z \mid \varphi_t \rangle_\Sigma \, \zeta_z, \tag{7.36}$$

where $\langle \chi_z \mid \psi_t \rangle_\Sigma$ and $\langle \zeta_z \mid \varphi_t \rangle_\Sigma$ are smooth functions in t with first derivatives[3] $\langle \chi_z \mid \frac{\partial \psi}{\partial t}(t, \cdot) \rangle_\Sigma$ and $\langle \zeta_z \mid \frac{\partial \varphi}{\partial t}(t, \cdot) \rangle_\Sigma$.

Now suppose $\psi \in \mathcal{C}^\infty\left(M, \mathbb{C}^4\right)$ and $\varphi \in \mathcal{C}^\infty\left(M, (\mathbb{C}^4)^*\right)$ are solutions of the inhomogeneous massive Dirac equations on $\mathbf{M}$,

$$\left(-\mathrm{i}\frac{\partial}{\partial t} \otimes \gamma^0 + \mathbb{1} \otimes \gamma^0 H^{\mathrm{sp}}\right)\psi = u \quad \text{and} \quad \left(\mathrm{i}\frac{\partial}{\partial t} \otimes \gamma^0 + \mathbb{1} \otimes H^{\mathrm{cosp}}(\cdot)\,\gamma^0\right)\varphi = v, \tag{7.37}$$

where $u \in \mathcal{C}_0^\infty\left(M, \mathbb{C}^4\right)$ and $v \in \mathcal{C}_0^\infty\left(M, \left(\mathbb{C}^4\right)^*\right)$; then for each $t \in (a, b)$,

$$\frac{\partial \psi}{\partial t}(t, \cdot) + \mathrm{i}\, H^{\mathrm{sp}} \psi_t = \mathrm{i}\, \gamma^0 u_t \qquad \text{and} \qquad \frac{\partial \varphi}{\partial t}(t, \cdot) - \mathrm{i}\, H^{\mathrm{cosp}} \varphi_t = -\mathrm{i}\, v_t \gamma^0. \tag{7.38}$$

Taking the L^2-inner products with χ_w and ζ_w, and using Corollary 7.2.3,

$$\frac{d}{dt}\langle \chi_w \mid \psi_t \rangle_\Sigma + \mathrm{i}\, \lambda_w \langle \chi_w \mid \psi_t \rangle_\Sigma = \langle \chi_w \mid \mathrm{i}\, \gamma^0 u_t \rangle_\Sigma \tag{7.39}$$

[3] If $\varphi \in \mathcal{C}^\infty\left(M, (\mathbb{C}^4)^*\right)$ is smooth and Σ is compact, $\lim_{h \to 0} \frac{\varphi_{t+h}(x) - \varphi_t(x)}{h} = \frac{\partial \varphi}{\partial t}(t, x)$ uniformly in $x \in \Sigma$, for each $t \in (a, b)$; due to the compactness of Σ and the uniform convergence, $\frac{\partial}{\partial t}\langle \zeta_z \mid \varphi_t \rangle_\Sigma = \langle \zeta_z \mid \frac{\partial \varphi}{\partial t}(t, \cdot) \rangle_\Sigma$. Iterating this argument and using the continuity of the L^2-inner product, we conclude smoothness. In the same way, one shows smoothness of $\langle \chi_z \mid \psi_t \rangle_\Sigma$ and $\frac{d}{dt}\langle \chi_z \mid \psi_t \rangle_\Sigma = \langle \chi_z \mid \frac{\partial \psi}{\partial t}(t, \cdot) \rangle_\Sigma$ for $\psi \in \mathcal{C}^\infty\left(M, \mathbb{C}^4\right)$.

and

$$\frac{d}{dt}\langle\zeta_w \mid \varphi_t\rangle_\Sigma - \mathrm{i}\,\lambda_w \langle\zeta_w \mid \varphi_t\rangle_\Sigma = \langle\zeta_w \mid -\mathrm{i}\, v_t\gamma^0\rangle_\Sigma \qquad \forall w \in \mathbb{Z}'. \tag{7.40}$$

These are ordinary and inhomogeneous first order differential equations with constant coefficients for the Fourier coefficients of ψ_t and φ_t. We find for the retarded and the advanced Green functions, and for the solutions of (7.39) and (7.40) defined by them [$z \in \mathbb{Z}'$, $u \in \mathcal{C}_0^\infty(M, \mathbb{C}^4)$, $v \in \mathcal{C}_0^\infty(M, (\mathbb{C}^4)^*)$ and $t \in (a,b)$]:

(7.41)

$$S_z^{\mathrm{ret}}(t,t') = \begin{cases} 0 & \text{if } a < t \le t' < b \\ e^{\mathrm{i}\lambda_z(t'-t)} & \text{if } a < t' \le t < b \end{cases}, \quad S_z^{\mathrm{adv}}(t,t') = \begin{cases} -e^{\mathrm{i}\lambda_z(t'-t)} & \text{if } a < t \le t' < b \\ 0 & \text{if } a < t' \le t < b \end{cases},$$

$$\left(S_z^{\mathrm{ret}}\langle\chi_z \mid \mathrm{i}\,\gamma^0 u_{t'}\rangle_\Sigma\right)(t) = \langle\chi_z \mid (S^{\mathrm{ret}}u)_t\rangle_\Sigma = \int_a^t e^{\mathrm{i}\lambda_z(t'-t)}\langle\chi_z \mid \mathrm{i}\,\gamma^0 u_{t'}\rangle_\Sigma\, dt', \tag{7.42}$$

$$\left(S_z^{\mathrm{adv}}\langle\chi_z \mid \mathrm{i}\,\gamma^0 u_{t'}\rangle_\Sigma\right)(t) = \langle\chi_z \mid (S^{\mathrm{adv}}u)_t\rangle_\Sigma = \int_t^b -e^{\mathrm{i}\lambda_z(t'-t)}\langle\chi_z \mid \mathrm{i}\,\gamma^0 u_{t'}\rangle_\Sigma\, dt', \tag{7.43}$$

$C_z^{\mathrm{ret/adv}} = S_{-z}^{\mathrm{ret/adv}}$ and

$$\left(C_z^{\mathrm{ret}}\langle\zeta_z \mid -\mathrm{i}\, v_{t'}\gamma^0\rangle_\Sigma\right)(t) = \langle\zeta_z \mid (C^{\mathrm{ret}}v)_t\rangle_\Sigma = \int_a^t e^{\mathrm{i}\lambda_z(t-t')}\langle\zeta_z \mid -\mathrm{i}\, v_{t'}\gamma^0\rangle_\Sigma\, dt', \tag{7.44}$$

$$\left(C_z^{\mathrm{adv}}\langle\zeta_z \mid -\mathrm{i}\, v_{t'}\gamma^0\rangle_\Sigma\right)(t) = \langle\zeta_z \mid (C^{\mathrm{adv}}v)_t\rangle_\Sigma = \int_t^b -e^{\mathrm{i}\lambda_z(t-t')}\langle\zeta_z \mid -\mathrm{i}\, v_{t'}\gamma^0\rangle_\Sigma\, dt'. \tag{7.45}$$

We conclude for $u \in \mathcal{C}_0^\infty(M, \mathbb{C}^4)$, $v \in \mathcal{C}_0^\infty(M, (\mathbb{C}^4)^*)$ and $t \in (a,b)$ that

$$(Su)_t = \sum_{z\in\mathbb{Z}'} \int_a^b -e^{\mathrm{i}\lambda_z t'}\langle\chi_z \mid \mathrm{i}\,\gamma^0 u_{t'}\rangle_\Sigma\, dt'\, e^{-\mathrm{i}\lambda_z t}\chi_z \tag{7.46}$$

and

$$(Cv)_t = \sum_{z\in\mathbb{Z}'} \int_a^b e^{-\mathrm{i}\lambda_z t'}\langle\zeta_z \mid \mathrm{i}\, v_{t'}\gamma^0\rangle_\Sigma\, dt'\, e^{\mathrm{i}\lambda_z t}\zeta_z, \tag{7.47}$$

which is to be understood in the L^2-sense. From this, one can also see that $(Su)^\dagger = Cu^\dagger$ and $(Cv)^\dagger = Sv^\dagger$ for all $u \in \mathcal{C}_0^\infty(M, \mathbb{C}^4)$ and for all $v \in \mathcal{C}_0^\infty(M, (\mathbb{C}^4)^*)$. In addition, it becomes apparent that

$$\|(Su)_t\|_\Sigma^2 = \sum_{z\in\mathbb{Z}'} \left| \int_a^b e^{\mathrm{i}\lambda_z t'} \langle \chi_z \mid \mathrm{i}\gamma^0 u_{t'} \rangle_\Sigma \, dt' \right|^2 \qquad \forall u \in \mathcal{C}_0^\infty(M,\mathbb{C}) \tag{7.48}$$

and

$$\|(Cv)_t\|_\Sigma^2 = \sum_{z\in\mathbb{Z}'} \left| \int_a^b e^{-\mathrm{i}\lambda_z t'} \langle \zeta_z \mid -\mathrm{i}\, v_{t'}\gamma^0 \rangle_\Sigma \, dt' \right|^2 \qquad \forall v \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*). \tag{7.49}$$

Both expressions are evidently constant in t.

The spinor field solution space $\mathcal{L}^{\mathrm{sp}} := S\mathcal{C}_0^\infty(M,\mathbb{C}^4)$ and the cospinor field solution space $\mathcal{L}^{\mathrm{cosp}} := C\mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*)$ become pre-Hilbert spaces with the inner products $\langle\cdot\mid\cdot\rangle_{\mathrm{sp}}$ and $\langle\cdot\mid\cdot\rangle_{\mathrm{cosp}}$ (cf. [San08, Lem.4.2.4]):

$$\langle Su \mid Su' \rangle_{\mathrm{sp}} = \mathrm{i}\int_M u^\dagger\, Su' \,\mathrm{vol}_{\mathbf{M}}, \qquad u,u' \in \mathcal{C}_0^\infty(M,\mathbb{C}^4), \tag{7.50}$$

and

$$\langle Cv \mid Cv' \rangle_{\mathrm{cosp}} = -\mathrm{i}\int_M Cv'\, v^\dagger \,\mathrm{vol}_{\mathbf{M}}, \qquad v,v' \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*). \tag{7.51}$$

The positive definiteness of these inner products is established by the identities (see e.g. [San08, Lem.4.2.4])

$$\langle Su \mid Su \rangle_{\mathrm{sp}} = \|(Su)_t\|_\Sigma^2 \qquad \text{and} \qquad \langle Cv \mid Cv \rangle_{\mathrm{cosp}} = \|(Cv)_t\|_\Sigma^2$$
$$\forall t \in (a,b),\ \forall u \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),\ \forall v \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*). \tag{7.52}$$

LEMMA 7.3.1. *$\{e^{-\mathrm{i}\lambda_z\cdot}\chi_z\}_{z\in\mathbb{Z}'}$ is an orthonormal basis for $(\mathcal{L}^{\mathrm{sp}},\langle\cdot\mid\cdot\rangle_{\mathrm{sp}})$ and an orthonormal basis for $(\mathcal{L}^{\mathrm{cosp}},\langle\cdot\mid\cdot\rangle_{\mathrm{cosp}})$ is given by $\{e^{\mathrm{i}\lambda_z\cdot}\zeta_z\}_{z\in\mathbb{Z}'}$.*

Proof: We prove this lemma for the cospinor solutions; indeed the arguments for the spinor solutions are the same. First of all, we show $e^{\mathrm{i}\lambda_z\cdot}\zeta_z \in \mathcal{L}^{\mathrm{cosp}}$. To this end, let $\sigma \in \mathcal{C}_0^\infty(a,b)$ have unit integral and let $w \in \mathbb{Z}'$. Then $-\mathrm{i}\sigma e^{\mathrm{i}\lambda_w\cdot}\zeta_w\gamma^0$ is of compact support and we find from (7.47) that

$$\left(C(-\mathrm{i}\sigma e^{\mathrm{i}\lambda_w\cdot}\zeta_w\gamma^0)\right)_t = \sum_{z\in\mathbb{Z}'} \int_a^b e^{-\mathrm{i}\lambda_z t'} \langle \zeta_z \mid \sigma(t')\, e^{\mathrm{i}\lambda_w t'}\zeta_w \rangle_\Sigma \, dt'\, e^{\mathrm{i}\lambda_z t}\zeta_z \tag{7.53}$$

$$= \int_a^b \sigma(t')\, dt'\, e^{\mathrm{i}\lambda_w t}\zeta_w \tag{7.54}$$

$$= e^{\mathrm{i}\,\lambda_w t}\zeta_w, \tag{7.55}$$

where the equation is to be understood in the L^2-sense. Because a smooth representative of an L^2-equivalence class is unique, we obtain the result $C\left(-\mathrm{i}\,\sigma e^{\mathrm{i}\,\lambda_w\,\cdot}\,\zeta_w\gamma^0\right) = e^{\mathrm{i}\,\lambda_w\,\cdot}\,\zeta_w$ and similarly, $S\left(\mathrm{i}\,\sigma e^{-\mathrm{i}\,\lambda_w\,\cdot}\,\gamma^0\chi_w\right) = e^{-\mathrm{i}\,\lambda_w\,\cdot}\,\chi_w$. With this result, it is not difficult to prove that $\{e^{\mathrm{i}\,\lambda_z\,\cdot}\,\zeta_z\}_{z\in\mathbb{Z}'}$ is an orthonormal system. We leave this to the reader and concentrate on completeness. Here, the simplest argument is to combine (7.52) with (7.49) to show that

$$\langle Cv \mid Cv\,\rangle_{\mathrm{cosp}} = \sum_{z\in\mathbb{Z}'}\left|\int_a^b e^{-\mathrm{i}\,\lambda_z t'}\langle\zeta_z \mid -\mathrm{i}\,v_{t'}\gamma^0\rangle_\Sigma\,dt'\right|^2 = \sum_{z\in\mathbb{Z}'}\left|\langle e^{\mathrm{i}\,\lambda_z\cdot}\zeta_z, Cv\rangle_{\mathrm{cosp}}\right|^2,$$

establishing completeness and concluding the proof. ❒

7.4 Quantisation of the free massive Dirac field and reference Hadamard state

In this section, we specify the quantum field theory of the free massive Dirac field in terms of the completion of the self-dual CAR-algebra. The self-dual CAR-algebra is thereby constructed from the solution pre-Hilbert spaces of the free massive Dirac equations (7.3) *and* (7.4)*,* $(\mathcal{L}^{\mathrm{sp}}, \langle\cdot\mid\cdot\rangle_{\mathrm{sp}})$ *and* $(\mathcal{L}^{\mathrm{cosp}}, \langle\cdot\mid\cdot\rangle_{\mathrm{cosp}})$*, which we have introduced in the last section. In view of applying the comparism test for Hadamard states, we also define our reference Hadamard state.*

We form the completions of $\mathcal{L}^{\mathrm{sp}}$ and $\mathcal{L}^{\mathrm{cosp}}$ with respect to $\langle\cdot\mid\cdot\rangle_{\mathrm{sp}}$ and $\langle\cdot\mid\cdot\rangle_{\mathrm{cosp}}$, which yields Hilbert spaces $\mathcal{H}^{\mathrm{sp}}$ and $\mathcal{H}^{\mathrm{cosp}}$. We continue denoting the inner products of $\mathcal{H}^{\mathrm{sp}}$ and $\mathcal{H}^{\mathrm{cosp}}$ by $\langle\cdot\mid\cdot\rangle_{\mathrm{sp}}$ and $\langle\cdot\mid\cdot\rangle_{\mathrm{cosp}}$. Owing to Lemma 7.3.1, $\{e^{-\mathrm{i}\,\lambda_z\,\cdot}\,\chi_z \mid z\in\mathbb{Z}'\}$ and $\{e^{\mathrm{i}\,\lambda_z\,\cdot}\,\zeta_z \mid z\in\mathbb{Z}'\}$ give rise to orthonormal bases in $\mathcal{H}^{\mathrm{sp}}$ and $\mathcal{H}^{\mathrm{cosp}}$, which will be denoted by $\{E^{-\mathrm{i}\,\lambda_z\,\cdot}\,X_z \mid z\in\mathbb{Z}'\}$ and $\{E^{\mathrm{i}\,\lambda_z\,\cdot}\,Z_z \mid z\in\mathbb{Z}'\}$.

The Dirac adjoint $\dagger : \mathcal{C}^\infty\left(M,\mathbb{C}^4\right) \longrightarrow \mathcal{C}^\infty\left(M,(\mathbb{C}^4)^*\right)$ and its inverse, which is also denoted by $\dagger$, descend to well-defined complex-conjugate linear bijections $\dagger : \mathcal{L}^{\mathrm{sp}} \longrightarrow \mathcal{L}^{\mathrm{cosp}}$ and $\dagger : \mathcal{L}^{\mathrm{cosp}} \longrightarrow \mathcal{L}^{\mathrm{sp}}$, which satisfy $\langle\psi^\dagger \mid \psi'^\dagger\rangle_{\mathrm{cosp}} = \langle\psi' \mid \psi\rangle_{\mathrm{sp}}$ for all $\psi,\psi'\in\mathcal{L}^{\mathrm{sp}}$ and $\langle\varphi^\dagger \mid \varphi'^\dagger\rangle_{\mathrm{sp}} = \langle\varphi' \mid \varphi\rangle_{\mathrm{cosp}}$ for all $\varphi,\varphi'\in\mathcal{L}^{\mathrm{cosp}}$. Hence, we obtain a complex-conjugate linear involution of $\dagger : \mathcal{L}^{\mathrm{sp}}\oplus\mathcal{L}^{\mathrm{cosp}} \longrightarrow \mathcal{L}^{\mathrm{sp}}\oplus\mathcal{L}^{\mathrm{cosp}}$ by defining $\left(\psi\oplus\varphi\right)^\dagger := \varphi^\dagger\oplus\psi^\dagger$ for all $\psi\in\mathcal{L}^{\mathrm{sp}}$ and for all $\varphi\in\mathcal{L}^{\mathrm{cosp}}$. Because of the involutive property, $\dagger$ is bounded with (operator) norm $\|\dagger\| = 1$ and extends thus continuously to $\mathcal{H} := \mathcal{H}^{\mathrm{sp}}\oplus\mathcal{H}^{\mathrm{cosp}} = \overline{\mathcal{L}^{\mathrm{sp}}\oplus\mathcal{L}^{\mathrm{cosp}}}$. We denote the inner product of $\mathcal{H}$ by $\langle\cdot\mid\cdot\rangle$, $\langle\cdot\mid\cdot\rangle := \langle\cdot\mid\cdot\rangle_{\mathrm{sp}}\oplus\langle\cdot\mid\cdot\rangle_{\mathrm{cosp}}$.

We may now form the self-dual CAR-algebra $\mathfrak{A} = \mathfrak{A}_{\mathrm{SDC}}\left(\mathcal{H},\langle\cdot\mid\cdot\rangle,\dagger\right)$, which is the unital *-algebra generated the abstract elements of the form $B\left(\Psi\oplus\Phi\right)$ and their conjugates $B(\Psi \oplus \Phi)^*$, $\Psi \oplus \Phi \in \mathcal{H}$, satisfying (see [Ara70, §2]):

(1) Linearity: for all $\lambda, \mu \in \mathbb{C}$, for all $\Psi \oplus \Phi, \Psi' \oplus \Phi' \in \mathcal{H}$,

$$B(\lambda \Psi \oplus \Phi + \mu \Psi' \oplus \Phi') = \lambda B(\Psi \oplus \Phi) + \mu B(\Psi' \oplus \Phi'). \tag{7.56}$$

(2) Canonical anticommutation relations (CARs): for all $\Psi \oplus \Phi, \Psi' \oplus \Phi' \in \mathcal{H}$,

$$B(\Psi \oplus \Phi) B(\Psi' \oplus \Phi')^* + B(\Psi' \oplus \Phi')^* B(\Psi \oplus \Phi) = \langle \Psi' \oplus \Phi' \mid \Psi \oplus \Phi \rangle \cdot 1_{\mathfrak{A}}. \tag{7.57}$$

(3) Hermiticity: for all $\Psi \oplus \Phi \in \mathcal{H}$,

$$B(\Psi \oplus \Phi)^* = B\big((\Psi \oplus \Phi)^\dagger\big). \tag{7.58}$$

$\mathfrak{A}$ has a unique C*-norm and we consider its completion $\overline{\mathfrak{A}}$ with respect to this norm.

The smeared quantum Dirac spinor field is defined by

$$\Psi[v] := \overline{B}\,(0_{\mathcal{H}^{\mathrm{sp}}} \oplus CV), \qquad v \in \mathcal{C}_0^\infty(M, (\mathbb{C}^4)^*), \tag{7.59}$$

where CV is the element in $\mathcal{H}^{\mathrm{cosp}}$ defined by Cv, and the smeared quantum Dirac cospinor field is

$$\Psi^\dagger[u] := \overline{B}\,(SU \oplus 0_{\mathcal{H}^{\mathrm{cosp}}}), \qquad u \in \mathcal{C}_0^\infty(M, \mathbb{C}^4), \tag{7.60}$$

where SU is the element in $\mathcal{H}^{\mathrm{sp}}$ defined by Su.

It is possible to impose a $\mathrm{U}(1)$ global gauge group of unital *-automorphisms upon the unital C*-algebra $\overline{\mathfrak{A}}$; these unital *-automorphisms are determined by defining $g\,\overline{B}(\Psi \oplus \Phi) := \overline{B}((g\Psi) \oplus (\overline{g}\Phi))$ for all $\Psi \oplus \Phi \in \mathcal{H}$, where $g \in \mathbb{C}$, $|g| = 1$.

We will now introduce our reference Hadamard state $\omega_0 : \overline{\mathfrak{A}} \longrightarrow \mathbb{C}$. We define $Q^{\mathrm{sp}} : \mathcal{L}^{\mathrm{sp}} \longrightarrow \mathcal{L}^{\mathrm{sp}}$ to be the orthogonal projection onto the linear subspace of $\mathcal{L}^{\mathrm{sp}}$ which is spanned by the $e^{-\mathrm{i}\lambda_z \cdot}\chi_z$ with $z \in \mathbb{Z}^+ := \{z \in \mathbb{Z}' \mid z > 0\}$ (positive frequency spinor solutions), i.e.

$$Q^{\mathrm{sp}}\psi = Q^{\mathrm{sp}}\left(\sum_{z \in \mathbb{Z}'} \langle e^{-\mathrm{i}\lambda_z \cdot}\chi_z \mid \psi\rangle_{\mathrm{sp}}\, e^{-\mathrm{i}\lambda_z \cdot}\chi_z\right) = \sum_{z \in \mathbb{Z}^+} \langle e^{-\mathrm{i}\lambda_z \cdot}\chi_z \mid \psi\rangle_{\mathrm{sp}}\, e^{-\mathrm{i}\lambda_z \cdot}\chi_z \qquad \forall \psi \in \mathcal{L}^{\mathrm{sp}} \tag{7.61}$$

and extend continuously to $\mathcal{H}^{\mathrm{sp}}$. Similarly, we define $Q^{\mathrm{cosp}} : \mathcal{L}^{\mathrm{cosp}} \longrightarrow \mathcal{L}^{\mathrm{cosp}}$ to be the orthogonal projection onto the linear subspace of $\mathcal{L}^{\mathrm{cosp}}$ which is spanned by the $e^{\mathrm{i}\lambda_z \cdot}\zeta_z$

$z \in \mathbb{Z}^- := \{z \in \mathbb{Z}' \mid z < 0\}$ (positive frequency cospinor solutions), i.e.

$$Q^{\mathrm{cosp}}\varphi = Q^{\mathrm{cosp}}\left(\sum_{z\in\mathbb{Z}'}\langle e^{\mathrm{i}\lambda_z\cdot}\zeta_z \mid \varphi\rangle_{\mathrm{cosp}}\, e^{\mathrm{i}\lambda_z\cdot}\zeta_z\right) = \sum_{z\in\mathbb{Z}^-}\langle e^{\mathrm{i}\lambda_z\cdot}\zeta_z \mid \varphi\rangle_{\mathrm{cosp}}\, e^{\mathrm{i}\lambda_z\cdot}\zeta_z, \quad \forall\varphi \in \mathcal{L}^{\mathrm{cosp}} \tag{7.62}$$

and extend continuously to $\mathcal{H}^{\mathrm{cosp}}$. Observe the relations

$$Q^{\mathrm{sp}} = \mathrm{id}_{\mathcal{H}^{\mathrm{sp}}} - \dagger Q^{\mathrm{cosp}}\dagger \qquad \text{and} \qquad Q^{\mathrm{cosp}} = \mathrm{id}_{\mathcal{H}^{\mathrm{cosp}}} - \dagger Q^{\mathrm{sp}}\dagger. \tag{7.63}$$

Now, $P := Q^{\mathrm{sp}} \oplus Q^{\mathrm{cosp}}$ is a projection operator on $\mathcal{H}$ and so $0 \le P = P^* \le 1$. It is an easy exercise to verify that $P + \dagger P \dagger = \mathrm{id}_{\mathcal{H}}$. Thus, P meets [Ara70, (3.4) + (3.5)] and by [Ara70, Lem.3.3 + 4.3], P defines a gauge invariant [PS70], pure and quasifree state ω_0 on $\overline{\mathfrak{A}}$ which is uniquely determined by [Ara70, (3.3)], that is,

$$\omega_0\left(\overline{B}(\Psi\oplus\Phi)^*\,\overline{B}(\Psi'\oplus\Phi')\right) = \langle\Psi\oplus\Phi \mid P(\Psi'\oplus\Phi')\rangle \tag{7.64}$$

$$= \langle\Psi \mid Q^{\mathrm{sp}}\Psi'\rangle_{\mathrm{sp}} + \langle\Phi \mid Q^{\mathrm{cosp}}\Phi'\rangle_{\mathrm{cosp}} \quad \forall\Psi\oplus\Phi, \Psi'\oplus\Phi' \in \mathcal{H}. \tag{7.65}$$

Here, gauge invariance means that $\omega_0 \circ g = \omega_0$ for all $g \in \mathbb{C}$ with $|g| = 1$, and is manifest from the preceding expression. The state ω_0 is Hadamard [SV00; DH06] and the associated Wightman two-point distribution is

$$W_0^{(2)}\left[(u\oplus v)\otimes(u'\oplus v')\right] := \omega_0\left(\overline{B}(SU\oplus CV)\,\overline{B}(SU'\oplus CV')\right) \tag{7.66}$$

$$= \omega_0\left(\overline{B}\left(Sv^\dagger\oplus Cu^\dagger\right)^*\,\overline{B}(SU'\oplus CV')\right) \tag{7.67}$$

$$= \langle Sv^\dagger \mid Q^{\mathrm{sp}}Su'\rangle_{\mathrm{sp}} + \langle Cu^\dagger \mid Q^{\mathrm{cosp}}Cv'\rangle_{\mathrm{cosp}}, \quad u,u' \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),\ v,v' \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*). \tag{7.68}$$

In terms of the eigenfunctions χ_z and ζ_z,

$$\langle Sv^\dagger \mid Q^{\mathrm{sp}}Su'\rangle_{\mathrm{sp}}$$

$$= \sum_{w,z\in\mathbb{Z}'}\overline{\langle e^{-\mathrm{i}\lambda_w\cdot}\chi_w \mid Sv^\dagger\rangle}_{\mathrm{sp}}\,\langle e^{-\mathrm{i}\lambda_w\cdot}\chi_w \mid Q^{\mathrm{sp}}e^{-\mathrm{i}\lambda_z\cdot}\chi_z\rangle_{\mathrm{sp}}\,\langle e^{-\mathrm{i}\lambda_z\cdot}\chi_z \mid Su'\rangle_{\mathrm{sp}} \tag{7.69}$$

$$= \sum_{w,z\in\mathbb{Z}'}\overline{\langle e^{-\mathrm{i}\lambda_w\cdot}\chi_w \mid Sv^\dagger\rangle}_{\mathrm{sp}}\,\delta_{wz}\Theta(z)\,\langle e^{-\mathrm{i}\lambda_z\cdot}\chi_z \mid Su'\rangle_{\mathrm{sp}} \tag{7.70}$$

$$= \sum_{z\in\mathbb{Z}^+}\langle Sv^\dagger \mid e^{-\mathrm{i}\lambda_w\cdot}\chi_w\rangle_{\mathrm{sp}}\,\overline{\langle Su' \mid e^{-\mathrm{i}\lambda_z\cdot}\chi_z\rangle}_{\mathrm{sp}} \tag{7.71}$$

$$= \sum_{z\in\mathbb{Z}^+}\int_M e^{-\mathrm{i}\lambda_z t}\,v(t,x)\,\chi_z(x)\,\mathrm{vol}_{\mathbf{M}}\int_M e^{\mathrm{i}\lambda_z t'}\zeta_z(x')\,u'(t',x')\,\mathrm{vol}'_{\mathbf{M}} \tag{7.72}$$

$$= \sum_{z\in\mathbb{Z}^+} \int_M \int_M e^{-\mathrm{i}\lambda_z(t-t')} v(t,x)\,\chi_z(x)\,\zeta_z(x')\,u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}, \tag{7.73}$$

$$u' \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),\; v \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*),$$

where we have used Lemma 7.3.1, (7.61), (7.50) and (7.46), and equally

$$\langle Cu^\dagger \mid Q^{\mathrm{cosp}} Cv' \rangle_{\mathrm{cosp}}$$

$$= \sum_{w,z\in\mathbb{Z}'} \overline{\langle e^{\mathrm{i}\lambda_w \cdot}\zeta_w \mid Cu^\dagger\rangle}_{\mathrm{cosp}} \langle e^{\mathrm{i}\lambda_w \cdot}\zeta_w \mid Q^{\mathrm{cosp}} e^{\mathrm{i}\lambda_z \cdot}\zeta_z\rangle_{\mathrm{cosp}} \langle e^{\mathrm{i}\lambda_z \cdot}\zeta_z \mid Cv'\rangle_{\mathrm{cosp}} \tag{7.74}$$

$$= \sum_{w,z\in\mathbb{Z}'} \overline{\langle e^{\mathrm{i}\lambda_w \cdot}\eta_w \mid Cu^\dagger\rangle}_{\mathrm{cosp}} \delta_{wz}\Theta(-z) \langle e^{\mathrm{i}\lambda_z \cdot}\chi_z \mid Cv'\rangle_{\mathrm{cosp}} \tag{7.75}$$

$$= \sum_{z\in\mathbb{Z}^-} \langle Cu^\dagger \mid e^{\mathrm{i}\lambda_w \cdot}\zeta_w\rangle_{\mathrm{cosp}} \overline{\langle Cv' \mid e^{\mathrm{i}\lambda_z \cdot}\zeta_z\rangle}_{\mathrm{cosp}} \tag{7.76}$$

$$= \sum_{z\in\mathbb{Z}^-} \int_M e^{\mathrm{i}\lambda_z t}\zeta_z(x)\,u(t,x)\,\mathrm{vol}_{\mathbf{M}} \int_M e^{-\mathrm{i}\lambda_z t'} v'(t',x')\,\chi_z(x')\,\mathrm{vol}'_{\mathbf{M}} \tag{7.77}$$

$$= \sum_{z\in\mathbb{Z}^-} \int_M \int_M e^{\mathrm{i}\lambda_z(t-t')}\zeta_z(x)\,u(t,x)\,v'(t',x')\,\chi_z(x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}, \tag{7.78}$$

$$u \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),\; v' \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*),$$

where we have also used (7.62), (7.51) and (7.47). Putting everything together, the reference Wightman two-point distribution reads in terms of the eigenfunctions χ_z and ζ_z, $z \in \mathbb{Z}'$, of the spatial Dirac operators H^{sp} and H^{cosp}:

$$\begin{aligned} &W_0^{(2)}[(u\oplus v)\otimes(u'\oplus v')] \\ &= \sum_{z\in\mathbb{Z}^+} \int_M \int_M e^{-\mathrm{i}\lambda_z(t-t')} v(t,x)\,\chi_z(x)\,\zeta_z(x')\,u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}} \\ &\quad+ \sum_{z\in\mathbb{Z}^-} \int_M \int_M e^{\mathrm{i}\lambda_z(t-t')}\zeta_z(x)\,u(t,x)\,v'(t',x')\,\chi_z(x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\,, \end{aligned} \tag{7.79}$$

$$u,u' \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),\; v,v' \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*).$$

It will also be helpful to note that, using $Q^{\mathrm{cosp}} = \mathrm{id}_{\mathcal{H}^{\mathrm{cosp}}} - \dagger Q^{\mathrm{sp}} \dagger$, the reference Wightman two-point distribution can be written as

$$W_0^{(2)}[(u\oplus v)\otimes(u'\oplus v')] = \langle Sv^\dagger \mid Q^{\mathrm{sp}} Su'\rangle_{\mathrm{sp}} - \langle Sv'^\dagger \mid Q^{\mathrm{sp}} Su\rangle_{\mathrm{sp}} + \langle Cu^\dagger \mid Cv'\rangle_{\mathrm{cosp}} \tag{7.80}$$

$$u,u' \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),\; v,v' \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*).$$

7.5 FP-states for the quantised free massive Dirac field

Starting from the decription of the fermonic projector in [FR14a], *we present the construction of the FP-states in this section. Since it will not* *complicate our discussion and formulas at all, we deal with the whole family of the FP-states (unsoftened, softened and all the others) in one go.*

We now focus on 4-dimensional, oriented and globally hyperbolic ultrastatic slabs $\mathbf{M}$ with compact spatial section Σ and spin connections as constructed in Section 7.2; in particular, $a, b \in \mathbb{R}$ are now taken such that $-\infty < a < b < \infty$. Let $\mathbf{N}$ be the 4-dimensional, oriented and globally hyperbolic ultrastatic spacetime with exactly the same compact spatial section Σ and spin connections as $\mathbf{M}$ but with underlying smooth manifold $N = \mathbb{R} \times \Sigma$. By extension with zero, any $u \in \mathcal{C}_0^\infty(M, \mathbb{C}^4)$ can also be regarded as a compactly supported smooth $\mathbb{C}^4$-valued function on N. In this regard, $\tilde{\psi} = S_{\mathbf{N}} u \in \mathcal{L}_{\mathbf{N}}^{\mathrm{sp}}$ constitutes the unique solution of (7.3) on N which coincides with the solution $\psi = Su \in \mathcal{L}^{\mathrm{sp}}$ of (7.3) on M. Note, formulas or objects relating to $\mathbf{N}$ will be denoted using a subscript " $\mathbf{N}$ "; otherwise $\mathbf{M}$ is to be understood.

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be non-negative and integrable. Typically we will have in mind that f is either the characteristic function $\chi_{(a,b)}$ of $(a, b) \subseteq \mathbb{R}$, which will yield the unsoftened FP-state, or a compactly supported smooth function, which will yield a softened FP-state. Essential to our construction of FP-states is the non-degenerate Hermitean sesquilinear form

$$(7.81) \qquad <\cdot \,|\, \cdot>_{\mathrm{FP};f} : \mathcal{L}^{\mathrm{sp}} \times \mathcal{L}^{\mathrm{sp}} \longrightarrow \mathbb{C}, \qquad (\psi, \psi') \longmapsto \int_N f\, \tilde{\psi}^\dagger\, \tilde{\psi}'\, \mathrm{vol}_{\mathbf{N}},$$

which reduces to the one studied in [FR14a, (3.1)] in the case where $f = \chi_{(a,b)}$. Using the Cauchy-Schwarz inequality and (7.46),

$$(7.82) \quad |<\psi \,|\, \psi'>_{\mathrm{FP};f}| \le \int_{\mathbb{R}} f\, |\langle \tilde{\psi}_t \,|\, \gamma^0 \tilde{\psi}'_t \rangle_\Sigma|\, dt \le \int_{\mathbb{R}} f\, \|\tilde{\psi}_t\|_\Sigma\, \|\tilde{\psi}'_t\|_\Sigma\, dt = \int_{\mathbb{R}} f\, \|\tilde{\psi}\|_{\mathbf{N}}^{\mathrm{sp}}\, \|\tilde{\psi}'\|_{\mathbf{N}}^{\mathrm{sp}}\, dt$$

$$(7.83) \qquad \le \hat{f}(0)\, \|\psi\|_{\mathrm{sp}}\, \|\psi'\|_{\mathrm{sp}} \qquad \forall \psi, \psi' \in \mathcal{L}^{\mathrm{sp}},$$

where we have used (7.52) to give $\|\tilde{\psi}_t\|_2 = \|\tilde{\psi}\|_{\mathbf{N}}^{\mathrm{sp}}$ and $\hat{f}(0) := \int_{\mathbb{R}} f\, dt$. Thus, $<\cdot \,|\, \cdot>_{\mathrm{FP};f}$ is continuous and by [BB03, Thm.20.2.1], there is a unique self-adjoint bounded linear operator $A_f : \mathcal{H}^{\mathrm{sp}} \longrightarrow \mathcal{H}^{\mathrm{sp}}$ satisfying the identity $\langle \psi \mid A_f \psi' \rangle_{\mathrm{sp}} = <\psi \,|\, \psi'>_{\mathrm{FP};f}$ for all $\psi, \psi' \in \mathcal{L}^{\mathrm{sp}}$.

In order to proceed along the lines of [FR14a], which applies spectral theory and considers the bounded linear operators $\chi_{(-\infty,0)}(A_{\chi(a,b)})$ and $\chi_{(0,\infty)}(A_{\chi(a,b)})$, we spectrally decompose A_f. To this end and to get a good handling of explicit calculations, we compute the action of A_f on the elements of $\mathcal{H}^{\mathrm{sp}}$. On $\mathcal{L}^{\mathrm{sp}}$,

$$(7.84) \qquad \langle e^{-\mathrm{i}\lambda_w \cdot} \chi_w \mid A_f e^{-\mathrm{i}\lambda_z \cdot} \chi_z \rangle_{\mathrm{sp}} = <e^{-\mathrm{i}\lambda_w \cdot} \chi_w \,|\, e^{-\mathrm{i}\lambda_z \cdot} \chi_z>_{\mathrm{FP};f}$$

$$= \int_N f e^{\mathrm{i}\lambda_w t} e^{-\mathrm{i}\lambda_z t} \chi_w^\dagger \chi_z \,\mathrm{vol}_{\mathbf{N}} \tag{7.85}$$

$$= \int_{\mathbb{R}} f e^{\mathrm{i}(\lambda_w - \lambda_z)t} \langle \chi_w \mid \gamma^0 \chi_z \rangle_\Sigma \, dt \tag{7.86}$$

$$= \frac{\hat{f}(0)\,\mu}{\lambda_z} \delta_{wz} + \hat{f}(2\lambda_z) \sqrt{1 - \frac{\mu^2}{\lambda_z^2}}\, \delta_{-wz} \qquad \forall w, z \in \mathbb{Z}', \tag{7.87}$$

where $\hat{f}(2\lambda_z) := \int_{\mathbb{R}} f e^{\mathrm{i}2\lambda_z t}\, dt$, and thus

$$\langle e^{-\mathrm{i}\lambda_z \cdot} \chi_z \mid A_f \psi \rangle_{\mathrm{sp}} = \frac{\hat{f}(0)\,\mu}{\lambda_z} \langle e^{-\mathrm{i}\lambda_z \cdot} \chi_z \mid \psi \rangle_{\mathrm{sp}} + \hat{f}(2\lambda_z) \sqrt{1 - \frac{\mu^2}{\lambda_z^2}} \langle e^{-\mathrm{i}\lambda_{-z} \cdot} \chi_{-z} \mid \psi \rangle_{\mathrm{sp}}, \qquad \forall z \in \mathbb{Z}',\ \forall \psi \in \mathcal{L}^{\mathrm{sp}}. \tag{7.88}$$

Consequently, A_f acts on $\mathcal{H}^{\mathrm{sp}}$ by continuous extension of

$$A_f \psi = \sum_{z \in \mathbb{Z}'} \left(\frac{\hat{f}(0)\,\mu}{\lambda_z} \langle e^{-\mathrm{i}\lambda_z \cdot} \chi_z \mid \psi \rangle_{\mathrm{sp}} + \hat{f}(2\lambda_z) \sqrt{1 - \frac{\mu^2}{\lambda_z^2}} \langle e^{-\mathrm{i}\lambda_{-z} \cdot} \chi_{-z} \mid \psi \rangle_{\mathrm{sp}} \right) e^{-\mathrm{i}\lambda_z \cdot} \chi_z, \qquad \forall \psi \in \mathcal{L}^{\mathrm{sp}}. \tag{7.89}$$

LEMMA 7.5.1. *The spectrum of A_f is a pure point spectrum with non-zero eigenvalues*

$$\sigma_p(A_f) = \left\{ \pm \Xi_{f,z} := \pm \sqrt{ \left| \hat{f}(2\lambda_z) \right|^2 \left(1 - \frac{\mu^2}{\lambda_z^2} \right) + \frac{\hat{f}(0)^2 \mu^2}{\lambda_z^2} } \;\middle|\; z \in \mathbb{Z}' \right\}. \tag{7.90}$$

Proof: To show our claim, it is enough to consider A_f on $\mathcal{L}^{\mathrm{sp}}$; the rest follows by continuous extension. Let $\Xi \in \mathbb{C}$ and $\psi \in \mathcal{L}^{\mathrm{sp}}$, then $A_f \psi - \Xi \psi = 0$ if and only if

$$\left(\frac{\hat{f}(0)\,\mu}{\lambda_z} - \Xi \right) \langle e^{-\mathrm{i}\lambda_z \cdot} \chi_z \mid \psi \rangle_{\mathrm{sp}} + \hat{f}(2\lambda_z) \sqrt{1 - \frac{\mu^2}{\lambda_z^2}} \langle e^{-\mathrm{i}\lambda_{-z} \cdot} \chi_{-z} \mid \psi \rangle_{\mathrm{sp}} = 0 \qquad \forall z \in \mathbb{Z}', \tag{7.91}$$

which leads by recursion to

$$-\left(\frac{\hat{f}(0)^2\mu^2}{\lambda_z^2}-\Xi^2\right)=|\hat{f}(2\lambda_z)|^2\left(1-\frac{\mu^2}{\lambda_z^2}\right), \tag{7.92}$$

from which (7.90) follows. We find as normalised (with respect to $\langle\cdot\,|\,\cdot\rangle_{\mathrm{sp}}$) eigenvectors of A_f to the eigenvalues $\pm\Xi_{f,z}\neq 0$:

$$\kappa_{f,z}^{\pm}:=\sqrt{\frac{\Xi_{f,z}\pm\frac{\hat{f}(0)\mu}{\lambda_z}}{2\,\Xi_{f,z}}}\left(e^{-\mathrm{i}\lambda_z\cdot}\chi_z+\frac{\pm\Xi_{f,z}-\frac{\hat{f}(0)\mu}{\lambda_z}}{\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda_z^2}}}\,e^{-\mathrm{i}\lambda_{-z}\cdot}\chi_{-z}\right) \tag{7.93}$$

for $z\in\mathbb{Z}'$ such that $\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda^2}}\neq 0$; for $z\in\mathbb{Z}'$ such that $\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda^2}}=0$, we find as normalised eigenfunctions

$$e^{-\mathrm{i}\lambda_z\cdot}\chi_z\begin{cases}\text{with } z>0 \text{ for the case } +\Xi_{f,z}=+\hat{f}(0)\frac{\mu}{|\lambda_z|}\\ \text{with } z<0 \text{ for the case } -\Xi_{f,z}=-\hat{f}(0)\frac{\mu}{|\lambda_z|}\end{cases}. \tag{7.94}$$

We define the index sets $\mathbb{Z}^0:=\{z\in\mathbb{Z}'\mid \hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda^2}}=0\}$, $\mathbb{Z}^{+/0}:=\mathbb{Z}^+-\mathbb{Z}^0$ and $\mathbb{Z}^{-/0}:=\mathbb{Z}^--\mathbb{Z}^0$; then $\{\kappa_{f,z}^{\pm}\mid z\in\mathbb{Z}^{+/0}\}\cup\{e^{-\mathrm{i}\lambda_z\cdot}\chi_z\mid z\in\mathbb{Z}^0\}$ and $\{\kappa_{f,z}^{\pm}\mid z\in\mathbb{Z}^{-/0}\}\cup\{e^{-\mathrm{i}\lambda_z\cdot}\chi_z\mid z\in\mathbb{Z}^0\}$ are orthonormal bases for $\mathcal{L}^{\mathrm{sp}}$ each. Also, $\{\kappa_{f,z}^{+}\mid z\in\mathbb{Z}^{+/0}\}\cup\{\kappa_{f,z}^{-}\mid z\in\mathbb{Z}^{-/0}\}\cup\{e^{-\mathrm{i}\lambda_z\cdot}\chi_z\mid z\in\mathbb{Z}^0\}$ is an orthonormal basis for $\mathcal{L}^{\mathrm{sp}}$. ❐

With the results of Lemma 7.5.1, we obtain for the spectral decomposition of A_f

$$A_f=\sum_{z\in\mathbb{Z}^0}\frac{\hat{f}(0)\,\mu}{\lambda_z}\langle E^{-\mathrm{i}\lambda_z\cdot}X_z\,|\,\cdot\rangle_{\mathrm{sp}}\,E^{-\mathrm{i}\lambda_z\cdot}X_z+\sum_{z\in\mathbb{Z}^{+/0},s=\pm}s\,\Xi_{f,z}\langle Y_{f,z}^s\,|\,\cdot\rangle_{\mathrm{sp}}\,Y_{f,z}^s \tag{7.95}$$

or, equivalently,

$$A_f=\sum_{z\in\mathbb{Z}^0}\frac{\hat{f}(0)\,\mu}{\lambda_z}\langle E^{-\mathrm{i}\lambda_z\cdot}X_z\,|\,\cdot\rangle_{\mathrm{sp}}\,E^{-\mathrm{i}\lambda_z\cdot}X_z+\sum_{z\in\mathbb{Z}^{-/0},s=\pm}s\,\Xi_{f,z}\langle Y_{f,z}^s\,|\,\cdot\rangle_{\mathrm{sp}}\,Y_{f,z}^s, \tag{7.96}$$

where we have written $Y_{f,z}^s$ for the element in $\mathcal{H}^{\mathrm{sp}}$ which is defined by $\kappa_{f,z}^s$.

Proceeding by analogy with the fermionic projector description of [FR14a], we define $Q_f^{\mathrm{sp}}=\chi_{(0,\infty)}(A_f)$, which is the orthogonal projection onto the positive eigenspace of A_f, given by

$$Q_f^{\mathrm{sp}}=\sum_{z\in\mathbb{Z}^0\cap\mathbb{Z}^+}\langle E^{-\mathrm{i}\lambda_z\cdot}X_z\,|\,\cdot\rangle_{\mathrm{sp}}\,E^{-\mathrm{i}\lambda_z\cdot}X_z+\sum_{z\in\mathbb{Z}^{+/0}}\langle Y_{f,z}^+\,|\,\cdot\rangle_{\mathrm{sp}}\,Y_{f,z}^+. \tag{7.97}$$

We emphasise that Q_f^{sp} is not itself the fermionic projector, which is closely related to the complementary spectral projection $\chi_{(-\infty,0)}(A_f)$. Setting the factors $\frac{\Xi_{f,z}-\frac{\hat{f}(0)\mu}{\lambda_z}}{\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda_z^2}}}$ and $\frac{-\Xi_{f,z}-\frac{\hat{f}(0)\mu}{\lambda_z}}{\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda_z^2}}}$ to be zero by convention if $z\in\mathbb{Z}^0\cap\mathbb{Z}^+$ and $z\in\mathbb{Z}^0\cap\mathbb{Z}^-$, respectively, hence $\kappa^+_{f,z}=e^{-\mathrm{i}\lambda_z\cdot}\chi_z$ and $Y^+_{f,z}=E^{-\mathrm{i}\lambda_z\cdot}X_z$ in these cases, we can write Q^{sp}_f in closed form:

$$Q^{\mathrm{sp}}_f=\sum_{z\in\mathbb{Z}^+}\langle Y^+_{f,z}\mid\cdot\rangle_{\mathrm{sp}}\,Y^+_{f,z}. \tag{7.98}$$

In order to construct a gauge invariant, pure and quasifree state $\omega_{\mathrm{FP};f}$ on $\overline{\mathfrak{A}}$, we need to *double* Q^{sp}_f to a self-adjoint bounded linear operator $P_{\mathrm{FP};f}$ on $\mathcal{H}$ satisfying the two conditions [Ara70, (3.4) + (3.5)], that is, $0\le P^*_{\mathrm{FP};f}=P_{\mathrm{FP};f}\le 1$ and $P_{\mathrm{FP};f}+\dagger P_{\mathrm{FP};f}\dagger=\mathrm{id}_{\mathcal{H}}$. One possibility to obtain an orthogonal projection operator $Q^{\mathrm{cosp}}_f:\mathcal{H}^{\mathrm{cosp}}\longrightarrow\mathcal{H}^{\mathrm{cosp}}$ such that $P_{\mathrm{FP};f}:=Q^{\mathrm{sp}}_f\oplus Q^{\mathrm{cosp}}_f$ satisfies [Ara70, (3.4) + (3.5)] is to repeat the construction of Q^{sp}_f for cospinors. However, we take an equivalent "*quick and dirty approach*" and consider the orthogonal projection operator on $\mathcal{H}^{\mathrm{cosp}}$ defined by

$$Q^{\mathrm{cosp}}_f:=\mathrm{id}_{\mathcal{H}^{\mathrm{cosp}}}-\dagger Q^{\mathrm{sp}}_f\dagger \tag{7.99}$$

$$=\sum_{z\in\mathbb{Z}^0\cap\mathbb{Z}^-}\langle E^{\mathrm{i}\lambda_z\cdot}Z_z\mid\cdot\rangle_{\mathrm{cosp}}\,E^{\mathrm{i}\lambda_z\cdot}Z_z+\sum_{z\in\mathbb{Z}^{+/0}}\left\langle\left(Y^-_{f,z}\right)^\dagger\mid\cdot\right\rangle_{\mathrm{cosp}}\left(Y^-_{f,z}\right)^\dagger. \tag{7.100}$$

Then also $Q^{\mathrm{sp}}_f=\mathrm{id}_{\mathcal{H}^{\mathrm{sp}}}-\dagger Q^{\mathrm{cosp}}_f\dagger$, and $P_{\mathrm{FP}_f}=Q^{\mathrm{sp}}_f\oplus Q^{\mathrm{cosp}}_f:\mathcal{H}\longrightarrow\mathcal{H}$ has the required properties. Now, [Ara70, Lem.3.3 + 4.3] yield a gauge invariant, pure and quasifree state $\omega_{\mathrm{FP};f}:\overline{\mathfrak{A}}\longrightarrow\mathbb{C}$, *the FP-state associated with*[4] f, which is uniquely determined by

$$\omega_{\mathrm{FP};f}\left(\overline{B}(\Psi\oplus\Phi)^*\,\overline{B}(\Psi'\oplus\Phi')\right)=\langle\Psi\oplus\Phi\mid P_{\mathrm{FP};f}(\Psi'\oplus\Phi')\rangle \tag{7.101}$$

$$=\langle\Psi\mid Q^{\mathrm{sp}}_f\Psi'\rangle_{\mathrm{sp}}+\langle\Phi\mid Q^{\mathrm{cosp}}_f\Phi'\rangle_{\mathrm{cosp}} \tag{7.102}$$

$$\forall\Psi\oplus\Phi,\Psi'\oplus\Phi'\in\mathcal{H}.$$

If $f=\chi_{(a,b)}$, we call the FP-state $\omega_{\mathrm{FP};\chi_{(a,b)}}$ *unsoftened* and if $f\in\mathcal{C}^\infty_0\mathbb{R}$ is non-negative, we speak of a *softened* FP-state.

Notice, if $f=\chi_{(a,b)}$, then $Y^+_{f,z}$ converges (strongly, i.e. in norm) to $E^{-\mathrm{i}\lambda_z\cdot}X_z$ in $\mathcal{H}^{\mathrm{sp}}_{\mathbf{N}}$ in the limit $a\longrightarrow-\infty$ and $b\longrightarrow\infty$ for all $z\in\mathbb{Z}^+$; one may thus show that Q^{sp}_f converges strongly to Q^{sp} in this limit, recovering the orthogonal projection that defined the reference Hadamard state. Therefore, one may expect that for 4-dimensional, oriented and globally hyperbolic ultrastatic spacetimes with compact spatial section and spin connections as in Section 7.2, which are of "*infinite lifetime*", the unsoftened FP-state will coincide with the reference Hadamard state. As shown by [FR14b, Thm.5.1], this

[4] Recall that $f:\mathbb{R}\longrightarrow\mathbb{R}$ is non-negative and integrable.

is indeed the case[5].

The Wightman two-point distribution associated with an FP-state is

$$W^{(2)}_{\mathrm{FP};f}\left[(u\oplus v)\otimes(u'\oplus v')\right] := \omega_{\mathrm{FP};f}\left(\overline{B}\left(SU\oplus CV\right)\overline{B}\left(SU'\oplus CV'\right)\right) \tag{7.103}$$

$$= \omega_{\mathrm{FP};f}\left(\overline{B}\left(SV^\dagger\oplus CU^\dagger\right)^*\overline{B}\left(SU'\oplus CV'\right)\right) \tag{7.104}$$

$$= \langle Sv^\dagger \mid Q^{\mathrm{sp}}_f Su'\rangle_{\mathrm{sp}} + \langle Cu^\dagger \mid Q^{\mathrm{cosp}}_f Cv'\rangle_{\mathrm{cosp}} \tag{7.105}$$

$$u,u'\in C_0^\infty\left(M,\mathbb{C}^4\right),\ v,v'\in C_0^\infty\left(M,(\mathbb{C}^4)^*\right).$$

As in the case of the reference state, exploiting $Q^{\mathrm{cosp}}_f = \mathrm{id}_{\mathcal{H}^{\mathrm{cosp}}} - \dagger Q^{\mathrm{sp}}_f \dagger$ yields the useful expression

$$W^{(2)}_{\mathrm{FP};f}\left[(u\oplus v)\otimes(u'\oplus v')\right] = \langle Sv^\dagger \mid Q^{\mathrm{sp}}_f Su'\rangle_{\mathrm{sp}} - \langle Sv'^\dagger \mid Q^{\mathrm{sp}}_f Su\rangle_{\mathrm{sp}} + \langle Cu^\dagger \mid Cv'\rangle_{\mathrm{cosp}} \tag{7.106}$$

$$u,u'\in C_0^\infty\left(M,\mathbb{C}^4\right),\ v,v'\in C_0^\infty\left(M,(\mathbb{C}^4)^*\right).$$

For an expression of $W^{(2)}_{\mathrm{FP};f}$ in terms of the eigenfunctions χ_z and ζ_z, $z\in\mathbb{Z}'$, of the spatial Dirac operators H^{sp} and H^{cosp}, we refer the reader to (7.200) in the appendix to this chapter.

Equation (7.200) reveals that the unsoftened FP-state cannot possibly give rise to a natural state for the quantised free massive Dirac field, in the same way the SJ-state does not yield a natural state for the free and minimally coupled real scalar quantum field: consider $\mathbf{M}'\in\mathbf{Loc}$, where $M' := (a',b')\times\Sigma$ for some choice of $a',b'\in\mathbb{R}$ with $a<a'<b'<b$, $g' := g|_{M'}$, $[T'] := [T|_{M'}]$ and $[\Omega'] := [\Omega|_{M'}]$; then $W'^{(2)}_{\mathrm{FP};\chi_{(a',b')}}$ differs from $W^{(2)}_{\mathrm{FP};\chi_{(a,b)}}$ on functions in $C_0^\infty\left(M\times M,\left[\mathbb{C}^4\oplus(\mathbb{C}^4)^*\right]\otimes\left[\mathbb{C}^4\oplus(\mathbb{C}^4)^*\right]\right)$ whose compact support is entirely contained in M'. Indeed, take any $\sigma\in C_0^\infty\mathbb{R}$ with $\operatorname{supp}\sigma\subseteq(a',b')$ and for some $w\in\mathbb{Z}^+$, $u=0$, $u'=e^{-\mathrm{i}\lambda_w\cdot}\sigma\otimes\zeta_w^*$, $v=e^{\mathrm{i}\lambda_w\cdot}\sigma\otimes\chi_w^*$, $v'=0$ in (7.200). We find

$$W'^{(2)}_{\mathrm{FP};\chi_{(a',b')}}\left[\left(0\oplus e^{\mathrm{i}\lambda_w\cdot}\sigma\otimes\chi_w^*\right)\otimes\left(e^{-\mathrm{i}\lambda_w\cdot}\sigma\otimes\zeta_w^*\oplus 0\right)\right] = \left(\frac{1}{2}+\frac{\mu}{\frac{1-\cos(2\lambda_w(b'-a'))}{(b'-a')}\left(1-\frac{\mu^2}{\lambda_w^2}\right)+\mu^2}\right)\left(\int_{a'}^{b'}\sigma\,dt\right)^2, \tag{7.107}$$

$$W^{(2)}_{\mathrm{FP};\chi_{(a,b)}}\left[\left(0\oplus e^{\mathrm{i}\lambda_w\cdot}\sigma\otimes\chi_w^*\right)\otimes\left(e^{-\mathrm{i}\lambda_w\cdot}\sigma\otimes\zeta_w^*\oplus 0\right)\right] = \left(\frac{1}{2}+\frac{\mu}{\frac{1-\cos(2\lambda_w(b-a))}{(b-a)}\left(1-\frac{\mu^2}{\lambda_w^2}\right)+\mu^2}\right)\left(\int_{a}^{b}\sigma\,dt\right)^2, \tag{7.108}$$

which are easily seen to differ in general.

[5]We thank Simone Murro for sharing his insights on this matter and two anonymous referees for their helpful suggestions.

7.6 The unsoftened FP-state is not Hadamard in general

We can now establish that the unsoftened FP-state, $\omega_{FP;\chi(a,b)}$, cannot be a Hadamard state in general, using arguments analogue to [FV12c]*: when assuming that $\omega_{FP;\chi(a,b)}$ is Hadamard, we are almost always faced with a contradiction as follows. As mentioned at the beginning of this chapter, the comparism test yields that the difference $W^{(2)}_{FP;\chi(a,b)} - W^{(2)}_0$ is given by integration against a smooth function. This smooth function can be used to define a Hilbert-Schmidt operator, which is compact. However, by explicit calculation, we find a* (*non-constant and non-oscillating*) *sequence of eigenvalues of this compact operator which almost never converges to zero.*

At this point, it is requisite to specify what is to be understood exactly by "*integration against a smooth function*": to be precise, an FP-state $\omega_{\mathrm{FP};f}$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is non-negative and integrable, is Hadamard according to the comparism test if and only if there exists a smooth function $k \in \mathcal{C}^\infty\left(M \times M, [\mathbb{C}^4 \oplus (\mathbb{C}^4)^*] \otimes [\mathbb{C}^4 \oplus (\mathbb{C}^4)^*]\right)$ such that

$$\left(W^{(2)}_{\mathrm{FP};f} - W^{(2)}_0\right)[\sigma] = \int\limits_{M\times M} k^* \sigma \,\mathrm{vol}_{\mathbf{M}\times\mathbf{M}} \qquad \forall \sigma \in \mathcal{C}^\infty_0\left(M \times M, [\mathbb{C}^4 \oplus (\mathbb{C}^4)^*] \otimes [\mathbb{C}^4 \oplus (\mathbb{C}^4)^*]\right). \tag{7.109}$$

Since we can continuously identify $\mathcal{C}^\infty_0\left(M, \mathbb{C}^4 \oplus (\mathbb{C}^4)^*\right) \otimes \mathcal{C}^\infty_0\left(M, \mathbb{C}^4 \oplus (\mathbb{C}^4)^*\right)$ with a dense linear subspace of $\mathcal{C}^\infty_0\left(M \times M, [\mathbb{C}^4 \oplus (\mathbb{C}^4)^*] \otimes [\mathbb{C}^4 \oplus (\mathbb{C}^4)^*]\right)$, it suffices to establish (7.109) for σ of the form $\sigma = (u \oplus v) \otimes (u' \oplus v')$, where $u, u' \in \mathcal{C}^\infty_0\left(M, \mathbb{C}^4\right)$ and $v, v' \in \mathcal{C}^\infty_0\left(M, (\mathbb{C}^4)^*\right)$. The rest follows then by continuous extension. In fact, we need not consider the full difference $W^{(2)}_{\mathrm{FP};f} - W^{(2)}_0$ but only a *half* of it. Comparing (7.80) and (7.106), we see that

$$\begin{aligned}\left(W^{(2)}_{\mathrm{FP};f} - W^{(2)}_0\right)&[(u \oplus v) \otimes (u' \oplus v')] \\ &= \langle S v^\dagger \mid (Q^{\mathrm{sp}}_f - Q^{\mathrm{sp}}) S u' \rangle_{\mathrm{sp}} - \langle S v'^\dagger \mid (Q^{\mathrm{sp}}_f - Q^{\mathrm{sp}}) S u \rangle_{\mathrm{sp}} \\ &\qquad \forall u, u' \in \mathcal{C}^\infty_0\left(M, \mathbb{C}^4\right),\ \forall v, v' \in \mathcal{C}^\infty_0\left(M, (\mathbb{C}^4)^*\right)\end{aligned} \tag{7.110}$$

and as the two summands are of the same form, we conclude that $\omega_{\mathrm{FP};f}$ is Hadamard if and only if there exists $k \in \mathcal{C}^\infty\left(M \times M, \mathbb{C}^4 \otimes (\mathbb{C}^4)^*\right)$ such that

$$\langle S v^\dagger \mid (Q^{\mathrm{sp}}_f - Q^{\mathrm{sp}}) S u' \rangle_{\mathrm{sp}} = \int\limits_{M\times M} k^* \left(u' \otimes v\right) \mathrm{vol}_{\mathbf{M}\times\mathbf{M}} \qquad \forall u' \in \mathcal{C}^\infty_0\left(M, \mathbb{C}^4\right),\ \forall v \in \mathcal{C}^\infty_0\left(M, (\mathbb{C}^4)^*\right). \tag{7.111}$$

The left-hand side of (7.111) is computed in (7.201) in the appendix to this chapter.

Now, if such a smooth function k exists, it is clearly smooth and square-integrable on the smooth product manifold $M' \times M'$ with respect to the product measure $\mathrm{vol}_{\mathbf{M}'\times\mathbf{M}'}$, where $\mathbf{M}' = (M', g', [T'], [\Omega'])$; $M' := (a', b') \times \Sigma$ for some choice of $a', b' \in \mathbb{R}$ with $a < a' < b' < b$, $g' := g|_{M'}$, $[T'] := [T|_{M'}]$ and $[\Omega'] := [\Omega|_{M'}]$. Thus, k defines a Hilbert-Schmidt operator K on $L^2(M', \mathbb{C}^4; \mathrm{vol}_{\mathbf{M}'})$ via

$$\langle u \mid Ku' \rangle_{M'} := \int_{M'} u^*(t,x) \int_{M'} k^*(t,x,t',x')\, \gamma^0 u'(t',x')\, \mathrm{vol}'_{\mathbf{M}'}\, \mathrm{vol}_{\mathbf{M}'} \tag{7.112}$$

$$= \int_{M'\times M'} k^*(\gamma^0 u' \otimes u^\dagger \gamma^0)\, \mathrm{vol}_{\mathbf{M}'\times\mathbf{M}'} \qquad \forall u, u' \in \mathcal{C}_0^\infty(M', \mathbb{C}^4), \tag{7.113}$$

which is extended to all of $L^2(M', \mathbb{C}^4; \mathrm{vol}_{\mathbf{M}'})$ by continuity; $\langle \cdot \mid \cdot \rangle_{M'}$ denotes the inner product of $L^2(M', \mathbb{C}^4; \mathrm{vol}_{\mathbf{M}'})$. It is not difficult to see that the smooth functions $\frac{1}{\sqrt{b'-a'}}\, e^{-\mathrm{i}\lambda_z \cdot} \chi_z$, $z \in \mathbb{Z}'$, give rise to an orthonormal system in $L^2(M', \mathbb{C}^4; \mathrm{vol}_{\mathbf{M}'})$, which we denote by $\{\frac{1}{\sqrt{b'-a'}}\, E^{-\mathrm{i}\lambda_z \cdot} X_z \mid z \in \mathbb{Z}'\}$. Since K is Hilbert-Schmidt, it is compact, which we will show is contradictory in general. To this end we maintain (recall the abbreviations $c^\pm_{f,z} := \sqrt{\frac{\Xi_{f,z} \pm \frac{\hat{f}(0)\mu}{\lambda_z}}{2\Xi_{f,z}}}$ and $d^\pm_{f,z} := \frac{\pm\Xi_{f,z} - \frac{\hat{f}(0)\mu}{\lambda_z}}{\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda_z^2}}}$ from the appendix):

LEMMA 7.6.1. *For each $z \in \mathbb{Z}^+$,*

$$\nu_z := \frac{c^-_{f,z}}{b'-a'} = \frac{1}{\sqrt{2}\,(b'-a')} \sqrt{1 - \frac{\hat{f}(0)\,\mu}{\lambda_z \Xi_{f,z}}} \tag{7.114}$$

is an eigenvalue of K.

Proof: From the appendix, we gather (7.210):

$$\begin{aligned} &\left\langle E^{-\mathrm{i}\lambda_w \cdot} X_w \;\middle|\; K E^{-\mathrm{i}\lambda_z \cdot} X_z \right\rangle_{M'} \\ &\quad = (c^-_{f,w})^2 \left(|d^+_{f,-w}|^2\, \Theta(-w) - \Theta(w) \right) \delta_{wz} \\ &\quad + \left[(c^+_{f,w})^2\, \overline{d}^+_{f,w}\, \Theta(w) + (c^-_{f,w})^2 d^+_{f,-w}\, \Theta(-w) \right] \delta_{-wz} \end{aligned} \tag{7.115}$$

$$\forall w, z \in \mathbb{Z}'$$

and

$$\langle F \mid K E^{-\mathrm{i}\lambda_z \cdot} X_z \rangle_{M'} = 0 \qquad \forall z \in \mathbb{Z}' \tag{7.116}$$

whenever $\langle F \mid E^{-\mathrm{i}\lambda_z \cdot} X_z \rangle_{M'} = 0$ for all $z \in \mathbb{Z}'$. This implies

$$K E^{-\mathrm{i}\lambda_z \cdot} X_z = -\frac{(c^-_{f,z})^2}{b'-a'}\, E^{-\mathrm{i}\lambda_z \cdot} X_z + \frac{(c^+_{f,z})^2 d^+_{f,z}}{b'-a'}\, E^{-\mathrm{i}\lambda_{-z} \cdot} X_{-z} \tag{7.117}$$

and

$$KE^{-\mathrm{i}\lambda_{-z}\cdot}X_{-z} = \frac{(c^-_{f,z})^2}{b'-a'}\,E^{-\mathrm{i}\lambda_{-z}\cdot}X_{-z} + \frac{(c^+_{f,z})^2\,\overline{d}^+_{f,z}}{b'-a'}\,E^{-\mathrm{i}\lambda_z\cdot}X_z \qquad \forall z\in\mathbb{Z}^+. \tag{7.118}$$

We make again the important note that $\overline{d}^+_{f,z}$ denotes the complex conjugate of $d^+_{f,z}\in\mathbb{C}$ and does *not* stand for some kind of Dirac adjoint, which we denote by "$\dagger$". It is left as an exercise to the reader to check that if $z\in\mathbb{Z}^0$, then $E^{-\mathrm{i}\lambda_z\cdot}X_z$ and $E^{-\mathrm{i}\lambda_{-z}\cdot}X_{-z}$ are eigenvectors of K to the eigenvalue 0; recall that in this case $c^+_{f,z}=1$, $c^-_{f,z}=0$ and we have agreed to set $d^+_{f,z}=0$. We also leave it as an exercise to the reader to verify that

$$E^{-\mathrm{i}\lambda_z\cdot}X_z + \frac{c^-_{f,z}}{(1-c^-_{f,z})\,\overline{d}^+_{f,z}}\,E^{-\mathrm{i}\lambda_{-z}\cdot}X_{-z} \tag{7.119}$$

is an eigenvector of K to the eigenvalue $\frac{c^-_{f,z}}{b'-a'}$ for $z\in\mathbb{Z}^{+/0}$; recall $(c^+_{f,z})^2+(c^-_{f,z})^2=1$. ❒

LEMMA 7.6.2. *Let*[6] $a=-b$, *take* $f=\chi_{(-b,b)}$ *and consider the sequence* $\{\nu_z\}_{z\in\mathbb{Z}^+}$, *where* ν_z *is given by* (7.114). *The set of the* $b\in(0,\infty)$ *for which* $\lim_{z\to\infty}\nu_z=0$ *is of Lebesgue measure zero.*

Proof: Since $\Xi_{f,z}=\sqrt{|\hat f(2\lambda_z)|^2\,(1-\frac{\mu^2}{\lambda_z^2})+\frac{\hat f(0)^2\mu^2}{\lambda_z^2}}$ due to Lemma 7.5.1, we have $\lim_{z\to\infty}\nu_z=0$ if and only if $\lim_{z\to\infty}\lambda_z^2\,|\hat f(2\lambda_z)|^2\,(1-\frac{\mu^2}{\lambda_z^2})=0$. Since $\lim_{z\to\infty}|\lambda_z|=\infty$ by [LM89, Thm.III.5.8], it follows that $\lim_{z\to\infty}(1-\frac{\mu^2}{\lambda_z^2})=1$ and we conclude $\lim_{z\to\infty}\nu_z=0$ if and only if $\lim_{z\to\infty}\lambda_z^2\,|\hat f(2\lambda_z)|^2=0$. Now, because of $f=\chi_{(-b,b)}$, it holds $\hat f(2\lambda_z)=\frac{\sin(2b\lambda_z)}{\lambda_z}$ and we get the identity $\lambda_z^2\,|\hat f(2\lambda_z)|^2=\sin^2(2b\lambda_z)$. It is proven in [FV12c] (directly after Proposition 4.1) that the set $\{b\in(0,\infty)\mid \lim_{z\to\infty}\sin(2b\lambda_z)=0\}$ is of Lebesgue measure zero. ❒

THEOREM 7.6.3. *Let* $\mathbf{M}$ *be a 4-dimensional, oriented and globally hyperbolic ultrastatic slab with compact spatial section,* $a=-b$ *for* $b\in(0,\infty)$ *and spin connections as in* Section 7.2. *Then the unsoftened FP-state on the C*-completion of the self-dual CAR-algebra for the quantised free massive Dirac field fails to be Hadamard for all* $b\in(0,\infty)$ *outside a set of Lebesgue measure zero.*

Proof: If $b\in(0,\infty)$ is such that $\lim_{z\to+\infty}\nu_z\neq 0$, then K cannot be compact [Rud91, Thm.4.24(b)]. We conclude that the unsoftened FP-state cannot be a Hadamard state for such choices of b, and Lemma 7.6.2 completes the proof. ❒

Note that we have not determined what happens if $b\in(0,\infty)$ is taken from the aforesaid set of Lebesgue measure zero (if non-empty). Also note that the softened

[6] This assumption is made purely to simplify the formulas.

FP-states [$f \in \mathcal{C}_0^\infty((-b,b)\times\Sigma)$ non-negative] avoid the contradiction in the proof of the theorem due to [Hör90, (8.1.1)] (cf. [FJ98, Exercise 8.16]).

7.7 The softened FP-states are Hadamard

In this section, we show that the modifications of the unsoftened FP-state in the spirit of [BF14], *i.e. the softened FP-states $\omega_{FP;f}$, where $f \in \mathcal{C}_0^\infty\mathbb{R}$ is non-negative, are Hadamard states. Our strategy is, in principle, as follows. We first show that the difference $W^{(2)}_{FP;f} - W^{(2)}_0$ corresponds to integration against an L^2-function on M with respect to* $\mathrm{vol}_{\boldsymbol{M}}$*. This L^2-function can be pushed forward into an oriented compact Riemannian manifold along a smooth embedding. Using Sobolev spaces, Sobolev estimates and the Sobolev embedding theorem, we show that the resulting L^2-function (with respect to the structures of the oriented compact Riemannian manifold) features a smooth representative. Hence, by pulling back, $W^{(2)}_{FP;f} - W^{(2)}_0$ corresponds to integration against a smooth function and $\omega_{FP;f}$ is Hadamard by the comparism test.*

From now on let $f \in \mathcal{C}_0^\infty\mathbb{R}$ be non-negative and consider the resulting softened FP-state $\omega_{\mathrm{FP};f}$. As before in Section 7.6 to establish that $W^{(2)}_{\mathrm{FP};f} - W^{(2)}_0$ is smooth, i.e. is given by integration against a smooth function, we can equivalently show that

$$u' \otimes v \longmapsto \langle Sv^\dagger \mid (Q^{\mathrm{sp}}_f - Q^{\mathrm{sp}}) Su' \rangle_{\mathrm{sp}}, \qquad u' \in \mathcal{C}_0^\infty(M, \mathbb{C}^4),\ v \in \mathcal{C}_0^\infty(M, (\mathbb{C}^4)^*), \tag{7.120}$$

is given by integration against a smooth function [recall (7.110)]. We know that we can write (recall that " $*$" stands for the Hermitean conjugate, i.e. complex conjugation and transposition)

$$\langle Sv^\dagger \mid (Q^{\mathrm{sp}}_f - Q^{\mathrm{sp}}) Su' \rangle_{\mathrm{sp}} = \sum_{z \in \mathbb{Z}^+} \int_{M\times M} k_z^* (u' \otimes v)\, \mathrm{vol}_{\mathbf{M}\times\mathbf{M}} \tag{7.121}$$
$$u' \in \mathcal{C}_0^\infty(M, \mathbb{C}^4),\ v \in \mathcal{C}_0^\infty(M, (\mathbb{C}^4)^*),$$

where $k_z \in \mathcal{C}^\infty(M \times M, \mathbb{C}^4 \otimes (\mathbb{C}^4)^*)$ is read off from (7.201):

$$\begin{aligned} k_z = &-(c^-_{f,z})^2 \left[(e^{-\mathrm{i}\lambda_z \cdot}\gamma^0 \chi_z) \otimes (e^{\mathrm{i}\lambda_z \cdot}\zeta_z \gamma^0) \right] \\ &+ (c^+_{f,z})^2 d^+_{f,z} \left[(e^{\mathrm{i}\lambda_z \cdot}\gamma^0 \chi_{-z}) \otimes (e^{\mathrm{i}\lambda_z \cdot}\zeta_z \gamma^0) \right] \\ &+ (c^+_{f,z})^2\, \overline{d}^+_{f,z} \left[(e^{-\mathrm{i}\lambda_z \cdot}\gamma^0 \chi_z + d^+_{f,z} e^{\mathrm{i}\lambda_z \cdot}\gamma^0 \chi_{-z}) \otimes (e^{-\mathrm{i}\lambda_z \cdot}\zeta_{-z}\gamma^0) \right]. \end{aligned} \tag{7.122}$$

Recall again from the appendix that $c^\pm_{f,z} \coloneqq \sqrt{\frac{\Xi_{f,z} \pm \frac{\hat{f}(0)\mu}{\lambda_z}}{2\,\Xi_{f,z}}}$ and $d^\pm_{f,z} \coloneqq \frac{\pm\Xi_{f,z} - \frac{\hat{f}(0)\mu}{\lambda_z}}{\hat{f}(2\lambda_z)\sqrt{1 - \frac{\mu^2}{\lambda_z^2}}}$. Thus, to show that $\omega_{\mathrm{FP};f}$ is Hadamard, we can show that the series $\sum_{z\in\mathbb{Z}^+} k_z$ converges in $L^2\left(M\times M,\mathbb{C}^4\otimes\left(\mathbb{C}^4\right)^*;\mathrm{vol}_{\mathbf{M}\times\mathbf{M}}\right)$, where the standard L^2-inner product will be denoted by $\langle\cdot\,|\,\cdot\rangle_{M\times M}$, and has a smooth representative. Since the k_z are pairwise orthogonal for $z\in\mathbb{Z}^+$, $\sum_{z\in\mathbb{Z}^+} k_z$ converges in the L^2-sense if $\sum_{z\in\mathbb{Z}^+}\|k_z\|^2_{M\times M}<\infty$ (cf. [HS96, Lem.21.7]); in this case we also have the identity $\|\sum_{\mathbb{Z}^+} k_z\|_{M\times M}=\sum_{z\in\mathbb{Z}^+}\|k_z\|^2_{M\times M}$. Because of

$$\|k_z\|^2_{M\times M}=(b-a)^2\left[\left(c^-_{f,z}\right)^4+2\left(c^+_{f,z}\right)^4|d^+_{f,z}|^2+\left(c^+_{f,z}\right)^4|d^+_{f,z}|^4\right] \tag{7.123}$$

$$=(b-a)^2\left[2\left(c^-_{f,z}\right)^4+2\left(c^+_{f,z}\right)^4|d^+_{f,z}|^2\right] \tag{7.124}$$

$$=(b-a)^2\left[2\left(c^-_{f,z}\right)^4+2\left(c^-_{f,z}\right)^2\left(c^+_{f,z}\right)^2\right] \tag{7.125}$$

$$=(b-a)^2\left(2\left(c^-_{f,z}\right)^2\left[\left(c^-_{f,z}\right)^2+\left(c^+_{f,z}\right)^2\right]\right) \tag{7.126}$$

$$=(b-a)^2\left[2\left(c^-_{f,z}\right)^2\right] \tag{7.127}$$

$$=(b-a)^2\left(\frac{\Xi_{f,z}-\frac{\hat{f}(0)\mu}{\lambda_z}}{\Xi_{f,z}}\right) \tag{7.128}$$

$$=(b-a)^2\left(1-\frac{\hat{f}\left(0\right)\mu}{\lambda_z\Xi_{f,z}}\right)\qquad\forall\, z\in\mathbb{Z}^+, \tag{7.129}$$

the L^2-convergence follows from the $p=0$ case of the following lemma:

LEMMA 7.7.1. *For each $p=0,1,2,\ldots$, the sum*

$$\sum_{z\in\mathbb{Z}'}\left(1-\frac{\mathrm{sgn}\left(z\right)\hat{f}\left(0\right)\mu}{\lambda_z\Xi_{f,z}}\right)\lambda_z^p \tag{7.130}$$

converges absolutely.

Proof: Using the explicit form of $\Xi_{f,z}$ found in Lemma 7.5.1 and also the identity $\frac{\mathrm{sgn}(-z)\hat{f}(0)\mu}{\lambda_{-z}\Xi_{f,-z}}=\frac{\mathrm{sgn}(z)\hat{f}(0)\mu}{\lambda_z\Xi_{f,z}}$, we compute

$$\sum_{z\in\mathbb{Z}'}\left|\left(1-\frac{\mathrm{sgn}\left(z\right)\hat{f}\left(0\right)\mu}{\lambda_z\Xi_{f,z}}\right)\lambda_z^p\right|=\sum_{z\in\mathbb{Z}'}\left|1-\frac{\hat{f}\left(0\right)\mu}{|\lambda_z|\sqrt{|\hat{f}\left(2\lambda_z\right)|^2\left(1-\frac{\mu^2}{\lambda_z^2}\right)+\frac{\hat{f}(0)^2m^2}{\lambda_z^2}}}\right||\lambda_z|^p \tag{7.131}$$

$$=\sum_{z\in\mathbb{Z}'}\left|1-\frac{1}{\sqrt{|\hat{f}\left(2\lambda_z\right)|^2\frac{\lambda_z^2-\mu^2}{\hat{f}(0)^2m^2}+1}}\right||\lambda_z|^p \tag{7.132}$$

$$=\sum_{z\in\mathbb{Z}'}\left|1-\frac{1}{\sqrt{|\hat{g}\left(2\lambda_z\right)|^2\left(\lambda_z^2-\mu^2\right)+1}}\right||\lambda_z|^p \tag{7.133}$$

$$\le\sum_{z\in\mathbb{Z}'}\left|1-\frac{1}{|\hat{g}\left(2\lambda_z\right)|^2\lambda_z^2+1}\right||\lambda_z|^p \tag{7.134}$$

$$\le\sum_{z\in\mathbb{Z}'}\left|\frac{|\hat{g}\left(2\lambda_z\right)|^2\lambda_z^2}{|\hat{g}\left(2\lambda_z\right)|^2\lambda_z^2+1}\right||\lambda_z|^p \tag{7.135}$$

$$\leq \sum_{z\in\mathbb{Z}'} |\hat{g}(2\lambda_z)|^2 |\lambda_z|^{p+2}, \tag{7.136}$$

where $g := f/(\hat{f}(0)\mu) \in \mathcal{C}_0^\infty\mathbb{R}$; also, recall that $\lambda_z^2 - \mu^2 \geq 0$ due to the mass gap. By reason of [Hör90, (8.1.1)] (cf. [FJ98, Exercise 8.16]), there is a constant $C_N > 0$ for each $N \geq 0$ such that

$$|\hat{g}(2\lambda_z)| \leq \frac{C_N}{(1+2|\lambda_z|)^N}. \tag{7.137}$$

Hence for $N \geq 0$,

$$\sum_{z\in\mathbb{Z}'} \left| \left(1 - \frac{\operatorname{sgn}(z)\hat{f}(0)\mu}{\lambda_z \Xi_{f,z}}\right)\lambda_z^p \right| \leq \sum_{z\in\mathbb{Z}'} \frac{C_N^2}{(1+2|\lambda_z|)^{2N}} |\lambda_z|^{p+2} \leq \sum_{z\in\mathbb{Z}'} \frac{C_N^2}{2^{2N}} |\lambda_z|^{p+2-2N}. \tag{7.138}$$

According to [LM89, Chap.III, (5.6)], we know that there exists a constant $c > 0$ such that $d(\Lambda) \leq c\Lambda^{21/2}$ holds for all $\Lambda > 0$, where $d(\Lambda) = \dim(\bigoplus_{|\lambda|\leq\Lambda} E_\lambda)$ and E_λ is the eigenspace of the smooth $\mathbb{C}^4$- resp. $(\mathbb{C}^4)^*$-valued eigenfunctions of H^{sp} resp. H^{cosp} to the eigenvalue λ. Let $M(z) := \max\{w \in \mathbb{Z}' \mid |\lambda_w| = |\lambda_z|\}$, which exists by the finite multiplicity of the eigenvalues of H^{sp} and H^{cosp}; then by counting, we readily see $|z| \leq M(z) = d(|\lambda_z|) \leq c|\lambda_z|^{21/2}$ thanks to the way we have ordered the countably many eigenvalues of H^{sp} and H^{cosp} (see the end of Section 7.2). It follows $|\lambda_z| \geq k\sqrt[21]{z^2}$, where $k = \sqrt[21]{c^{-2}}$. Letting $N > \frac{p}{2} + 1$,

$$\sum_{z\in\mathbb{Z}'} \left| \left(1 - \frac{\operatorname{sgn}(z)\hat{f}(0)\mu}{\lambda_z \Xi_{f,z}}\right)\lambda_z^p \right| \leq \sum_{z\in\mathbb{Z}'} \frac{C_N^2}{2^{2N}|\lambda_z|^{2N-p-2}} \leq \sum_{z\in\mathbb{Z}'} \frac{C_N^2}{2^{2N} k^{2N-p-2} z^{2(2N-p-2)/21}}. \tag{7.139}$$

From this it follows that if we take $N > \frac{p+23}{2}$,

$$\sum_{z\in\mathbb{Z}'} \left| \left(1 - \frac{\operatorname{sgn}(z)\hat{f}(0)\mu}{\lambda_z \Xi_{f,z}}\right)\lambda_z^p \right| \leq \frac{C_N^2}{2^{2N} k^{2N-p-2}} \sum_{z\in\mathbb{Z}'} \frac{1}{z^2}, \tag{7.140}$$

which surely converges. ❒

We are thus capable of employing the sum $k = \sum_{z\in\mathbb{Z}^+} k_z$ as an L^2-function, with which we can continue to work. In particular, we can push k forward or pull k back. Our remaining task is to show that k features a smooth representative. This will be accomplished, as mentioned before, by smoothly embedding M into an oriented compact Riemannian manifold, pushing k forward and using Sobolev spaces, Sobolev estimates and the Sobolev embedding theorem.

Let $\mathbf{M}'$ be the 4-dimensional, oriented and globally hyperbolic ultrastatic slab with exactly the same compact spatial section Σ and spin connections ∇^{sp} and ∇^{cosp} as $\mathbf{M}$

but with underlying manifold $M' = (a', b') \times \Sigma$ for $-\infty < a' < a < b < b' < \infty$. Let $\chi : (a', b') \longrightarrow \mathbb{R}$ be a smooth cut-off function with the properties $0 \leq \chi(t) \leq 1$ for all $t \in (a', b')$, $\chi(t) = 1$ for all $t \in [a, b]$ and $\operatorname{supp} \chi \subseteq (c, d)$ for some $a' < c < a < b < d < b'$; such a smooth function exists thanks to [Lee03, Prop.2.26]. As (7.122) exhibits, $(\chi \otimes \chi) k_z \in \mathcal{C}_0^\infty (M' \times M', \mathbb{C}^4 \otimes (\mathbb{C}^4)^*)$ for all $z \in \mathbb{Z}^+$ and the $(\chi \otimes \chi) k_z$ are pairwise orthogonal with respect to the standard L^2-inner product $\langle \cdot \mid \cdot \rangle_{M' \times M'}$. Clearly, if $(\chi \otimes \chi) k = (\chi \otimes \chi) \sum_{z \in \mathbb{Z}^+} k_z = \sum_{z \in \mathbb{Z}^+} (\chi \otimes \chi) k_z$, viewed in $L^2 (M' \times M', \mathbb{C}^4 \otimes (\mathbb{C}^4)^* ; \mathrm{vol}_{\mathbf{M'} \times \mathbf{M'}})$, has a smooth representative, then $k = \sum_{z \in \mathbb{Z}^+} k_z$ will have a smooth representative by restriction.

Reconsider now the 3-dimensional, oriented, connected and compact Riemannian manifold $(\Sigma, h, [\omega])$ with which $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{M'}$ are constructed. Also, consider the oriented and orthonormal smooth global framing (X_1, X_2, X_3) for the tangent bundle of Σ, τ_Σ. We also (uniquely) define a smooth vector field $Z \in \mathscr{X}(S^1)$ by $Z|_{S^1 \setminus \{(1,0)\}} := \frac{\partial}{\partial \varphi}$ and $Z|_{S^1 \setminus \{(-1,0)\}} := \frac{\partial}{\partial \psi}$, where we have used the two smooth charts of S^1, $\varphi^{-1} : (0, 2\pi) \longrightarrow S^1 \setminus \{(1,0)\}$, $t \longmapsto (\cos t, \sin t)$, and $\psi^{-1} : (-\pi, \pi) \longrightarrow S^1 \setminus \{(-1,0)\}$, $t \longmapsto (\cos t, \sin t)$; for the transitions of smooth charts, we have

$$\psi \circ \varphi^{-1} : (0, \pi) \cup (\pi, 2\pi) \longrightarrow (-\pi, 0) \cup (0, \pi), \quad t \longmapsto \begin{cases} t & \text{if } t \in (0, \pi) \\ t - 2\pi & \text{if } t \in (\pi, 2\pi) \end{cases}, \tag{7.141}$$

$$\varphi \circ \psi^{-1} : (-\pi, 0) \cup (0, \pi) \longrightarrow (0, \pi) \cup (\pi, 2\pi), \quad t \longmapsto \begin{cases} t + 2\pi & \text{if } t \in (-\pi, 0) \\ t & \text{if } t \in (0, \pi) \end{cases}. \tag{7.142}$$

We define X to be the smooth product manifold $S^1 \times \Sigma$ and equip it with the Riemannian metric $g_R := \mathrm{pr}_1^* g_{S^1} + \mathrm{pr}_2^* h$ and the orientation $[\Omega_R] := [\mathrm{pr}_1^* \omega_{S^1} \wedge \mathrm{pr}_2^* \omega]$, where g_{S^1} is the standard Riemannian metric and ω_{S^1} the standard orientation on S^1. The triple $\mathbf{X} = (X, g_R, [\Omega_R])$ constitutes an oriented compact Riemannian manifold, which is of dimension 4 and connected.

Consider now the smooth embedding

$$\jmath : (a', b') \longrightarrow S^1, \qquad t \longmapsto e^{2\pi \mathrm{i} (t - a')/(b' - a')}, \tag{7.143}$$

and the resulting smooth embedding

$$\iota : M' \longrightarrow X, \qquad \iota := \jmath \times \mathrm{id}_\Sigma . \tag{7.144}$$

We define for each $z \in \mathbb{Z}^+$, $\sigma_z \in \mathcal{C}^\infty (X \times X, \mathbb{C}^4 \otimes (\mathbb{C}^4)^*)$ by the pushforward

$$\sigma_z := (\iota \times \iota)_* (\chi \otimes \chi) k_z. \tag{7.145}$$

Note, the σ_z are pairwise orthogonal with respect to the standard L^2-inner product $\langle\cdot\mid\cdot\rangle_{X\times X}$ by construction. It is also easily seen that $(\chi\otimes\chi)\,k$ has a smooth representative in $L^2\left(M'\times M',\mathbb{C}^4\otimes(\mathbb{C}^4)^*;\mathrm{vol}_{\mathbf{M}\times\mathbf{M}}\right)$ if and only if $\sigma=\sum_{z\in\mathbb{Z}^+}\sigma_z$ exists in $L^2\left(X\times X,\mathbb{C}^4\otimes(\mathbb{C}^4)^*;\mathrm{vol}_{\mathbf{M}\times\mathbf{M}}\right)$ and has a smooth representative; indeed, the pullback of σ by $\iota\times\iota$ is a smooth representative for $(\chi\otimes\chi)\,k$. Establishing that $\sigma=\sum_{z\in\mathbb{Z}^+}\sigma_z$ has a smooth representative will now occupy us in the remainder of the section.

We will need to introduce various Sobolev spaces of $\mathbb{C}^4$- and $(\mathbb{C}^4)^*$-valued functions on X and $X\times X$, each of which can be defined as the completion of the space of smooth $\mathbb{C}^4$- and $(\mathbb{C}^4)^*$-valued functions in an appropriate Sobolev norm. In fact, there are many equivalent norms that can be used e.g. any linear connection or metric determines a corresponding basic Sobolev norm of order $s=0,1,2,\ldots$ [LM89, Chap.III, §2], and the norms induced by different linear connections and metrics are all equivalent. Furthermore, any elliptic linear differential operator of order s also induces an equivalent norm and therefore the same completion [LM89, Thm.III.5.2(iii)].

For our purposes, an especially convenient choice for elliptic linear differential operators $\mathfrak{D}^{\mathfrak{sp}}:\mathcal{C}^\infty(X,\mathbb{C}^4)\longrightarrow\mathcal{C}^\infty(X,\mathbb{C}^4)$ and $\mathfrak{D}^{\mathfrak{cosp}}:\mathcal{C}^\infty(X,(\mathbb{C}^4)^*)\longrightarrow\mathcal{C}^\infty(X,(\mathbb{C}^4)^*)$, which are of first order, is as follows because it allows us to recycle the smooth eigenfunctions χ_z and ζ_z, $z\in\mathbb{Z}'$, of the spatial Dirac operators H^{sp} and H^{cosp}:

$$\mathfrak{D}^{\mathfrak{sp}}\psi:=\gamma^0\left(-Z\otimes\mathbb{1}+\mathbb{1}\otimes H^{\mathrm{sp}}\right)\gamma^0\psi,\qquad \psi\in\mathcal{C}^\infty(X,\mathbb{C}^4),\tag{7.146}$$

and

$$\mathfrak{D}^{\mathfrak{cosp}}\varphi:=\left[\left(Z\otimes\mathbb{1}+\mathbb{1}\otimes H^{\mathrm{cosp}}\right)\varphi\gamma^0\right]\gamma^0,\qquad \varphi\in\mathcal{C}^\infty(X,(\mathbb{C}^4)^*).\tag{7.147}$$

"$\mathbb{1}$" denotes, depending on the context, the identity on $\mathcal{C}^\infty(\Sigma,\mathbb{C}^4)$, $\mathcal{C}^\infty(\Sigma,(\mathbb{C}^4)^*)$ or $\mathcal{C}^\infty(S^1,\mathbb{C})$. Also note that we have made use of the standard continuous identifications of $\mathcal{C}^\infty(S^1,\mathbb{C})\otimes\mathcal{C}^\infty(\Sigma,\mathbb{C}^4)$ with a dense linear subspace of $\mathcal{C}^\infty(S^1\times\Sigma,\mathbb{C}^4)$ and of $\mathcal{C}^\infty(S^1,\mathbb{C})\otimes\mathcal{C}^\infty(\Sigma,(\mathbb{C}^4)^*)$ with a dense linear subspace of $\mathcal{C}^\infty(S^1\times\Sigma,(\mathbb{C}^4)^*)$.

LEMMA 7.7.2. *The linear differential operators of first order $\mathfrak{D}^{\mathfrak{sp}}:\mathcal{C}^\infty(X,\mathbb{C}^4)\longrightarrow\mathcal{C}^\infty(X,\mathbb{C}^4)$ and $\mathfrak{D}^{\mathfrak{cosp}}:\mathcal{C}^\infty(X,(\mathbb{C}^4)^*)\longrightarrow\mathcal{C}^\infty(X,(\mathbb{C}^4)^*)$ are elliptic.*

Proof: We argue the claim for $\mathfrak{D}^{\mathfrak{cosp}}$; the proof for $\mathfrak{D}^{\mathfrak{sp}}$ is analogous. From (7.147) we obtain for the principal symbol of $\mathfrak{D}^{\mathfrak{cosp}}$ the expression $\sigma^{\mathfrak{cosp}}_{\mathfrak{D}}(\xi)=\xi_0+\mathrm{i}\,\xi_i\gamma^0\gamma^i$ for $\xi\in TX$ (cf. Lemma 7.2.1); hence $\det\left(\xi_0+\mathrm{i}\,\xi_i\gamma^0\gamma^i\right)=\left(\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2\right)^2$, which shows that $\sigma^{\mathfrak{cosp}}_{\mathfrak{D}}(\xi)$ is an isomorphism of complex vector spaces for all $\xi\in T^*X$ unless $\xi=0\in T^*X_{(t,x)}$ for $(t,x)\in X$. ❐

We may therefore introduce for $s=0,1,2,\ldots$ the Sobolev spaces specified as follows: $L^2_s\left(X,\mathbb{C}^4;\mathrm{vol}_{\mathbf{X}}\right)$ and $L^2_s\left(X,(\mathbb{C}^4)^*;\mathrm{vol}_{\mathbf{X}}\right)$ are defined to be the completions of $\mathcal{C}^\infty(X,\mathbb{C}^4)$

and $\mathcal{C}^\infty(X, (\mathbb{C}^4)^*)$ with respect to the norms defined by

$$\|\psi\|^2_{X,s} := \|\psi\|^2_{X,0} + \|(\mathfrak{D}^{\mathfrak{fp}})^s\psi\|^2_{X,0} \quad \text{and} \quad \|\varphi\|^2_{X,s} := \|\varphi\|^2_{X,0} + \|(\mathfrak{D}^{\mathfrak{cofp}})^s\varphi\|^2_{X,0} \quad \forall\psi \in \mathcal{C}^\infty(X, \mathbb{C}^4),\ \forall\varphi \in \mathcal{C}^\infty(X, (\mathbb{C}^4)^*), \tag{7.148}$$

where $\|\cdot\|_{X,0}$ denotes the ordinary L^2-norm. We do not distinguish notationally between these two norms, as it will always be clear which is intended. Note that the norms (7.148) are equivalent to the Sobolev norms $\|\cdot\|_{X,0} + \|(\mathfrak{D}^{\mathfrak{fp}/\mathfrak{cofp}})^s \cdot\|_{X,0}$ given by [LM89, Thm.III.5.2(iii)] thanks to the estimates

$$\sqrt{\|\cdot\|^2_{X,0} + \|(\mathfrak{D}^{\mathfrak{fp}/\mathfrak{cofp}})^s\cdot\|^2_{X,0}} \le \|\cdot\|_{X,0} + \|(\mathfrak{D}^{\mathfrak{fp}/\mathfrak{cofp}})^s\cdot\|_{X,0} \tag{7.149}$$

and

$$\|\cdot\|_{X,0} + \|(\mathfrak{D}^{\mathfrak{fp}/\mathfrak{cofp}})^s\cdot\|_{X,0} = \sqrt{\left(\|\cdot\|_{X,0} + \|(\mathfrak{D}^{\mathfrak{fp}/\mathfrak{cofp}})^s\cdot\|_{X,0}\right)^2} \tag{7.150}$$

$$\le \sqrt{2}\sqrt{\|\cdot\|^2_{X,0} + \|(\mathfrak{D}^{\mathfrak{fp}/\mathfrak{cofp}})^s\cdot\|^2_{X,0}}\,, \tag{7.151}$$

where we have used the binomial formulas and $(a+b)^2 \le 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$. In the same way, we define $L^2_s(X \times X, \mathbb{C}^4 \otimes (\mathbb{C}^4)^*; \mathrm{vol}_{\mathbf{X}\times\mathbf{X}})$ to be the completion of $\mathcal{C}^\infty(X \times X, \mathbb{C}^4 \otimes (\mathbb{C}^4)^*)$ with respect to the norm

$$\|\cdot\|^2_{X\times X,s} := \|\cdot\|^2_{X\times X,0} + \left\|\left((\mathfrak{D}^{\mathfrak{fp}})^s \otimes \mathbb{1} + \mathbb{1} \otimes (\mathfrak{D}^{\mathfrak{cofp}})^s\right)\cdot\right\|^2_{X\times X,0}, \tag{7.152}$$

where "$\mathbb{1}$" denotes the first time the identity on $\mathcal{C}^\infty(X, (\mathbb{C}^4)^*)$ and the second time the identity on $\mathcal{C}^\infty(X, \mathbb{C}^4)$. Each of these Sobolev spaces has a natural Hilbert space inner product compatible with the norms just given,

$$\langle\cdot\,|\,\cdot\rangle_{X,s} := \langle\cdot\,|\,\cdot\rangle_{X,0} + \langle(\mathfrak{D}^{\mathfrak{fp}})^s\cdot\,|\,(\mathfrak{D}^{\mathfrak{fp}})^s\cdot\rangle_{X,0}, \tag{7.153}$$

$$\langle\cdot\,|\,\cdot\rangle_{X,s} := \langle\cdot\,|\,\cdot\rangle_{X,0} + \langle(\mathfrak{D}^{\mathfrak{cofp}})^s\cdot\,|\,(\mathfrak{D}^{\mathfrak{cofp}})^s\cdot\rangle_{X,0}, \tag{7.154}$$

$$\langle\cdot\,|\,\cdot\rangle_{X\times X,s} := \langle\cdot\,|\,\cdot\rangle_{X\times X,0} + \langle(\mathfrak{D}^{\mathfrak{fp}})^s \otimes \mathbb{1} + \mathbb{1} \otimes (\mathfrak{D}^{\mathfrak{cofp}})^s\cdot\,|\,(\mathfrak{D}^{\mathfrak{fp}})^s \otimes \mathbb{1} + \mathbb{1} \otimes (\mathfrak{D}^{\mathfrak{cofp}})^s\cdot\rangle_{X\times X,0}. \tag{7.155}$$

Our choices are of course purpose-built so that the various norms interact well:

LEMMA 7.7.3. *For $s = 0, 1, 2, \ldots$, we have the estimate*

$$\|\psi \otimes \varphi\|^2_{X\times X,s} \le 2\,\|\psi\|^2_{X,s}\,\|\varphi\|^2_{X,s} \quad \forall\psi \in \mathcal{C}^\infty(X, \mathbb{C}^4),\ \forall\varphi \in \mathcal{C}^\infty(X, (\mathbb{C}^4)^*). \tag{7.156}$$

Proof: By the parallelogram law, we have in any Hilbert space H that $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in H$ (cf. [HS96, Lem.20.6]). With this in mind, we compute

$$\|\psi \otimes \varphi\|^2_{X\times X,s} = \|\psi \otimes \varphi\|^2_{X\times X,0} + \|(\mathfrak{D}^{\mathfrak{fp}})^s\psi \otimes \varphi + \psi \otimes (\mathfrak{D}^{\mathfrak{cofp}})^s\varphi\|^2_{X\times X,0} \tag{7.157}$$
$$\leq \|\psi\|^2_{X,0}\|\varphi\|^2_{X,0} + 2\|(\mathfrak{D}^{\mathfrak{fp}})^s\psi \otimes \varphi\|^2_{X\times X,0} + 2\|\psi \otimes (\mathfrak{D}^{\mathfrak{cofp}})^s\varphi\|^2_{X\times X,0} \tag{7.158}$$
$$\leq \|\psi\|^2_{X,0}\|\varphi\|^2_{X,0} + 2\|(\mathfrak{D}^{\mathfrak{fp}})^s\psi\|^2_{X,0}\|\varphi\|^2_{X,0} + 2\|\psi\|^2_{X,0}\|(\mathfrak{D}^{\mathfrak{cofp}})^s\varphi\|^2_{X,0} \tag{7.159}$$
$$\leq 2\|\psi\|^2_{X,s}\|\varphi\|^2_{X,s} \tag{7.160}$$
$$\forall\psi \in \mathcal{C}^\infty(X, \mathbb{C}^4),\ \forall\varphi \in \mathcal{C}^\infty(X, (\mathbb{C}^4)^*),$$

which shows our claim. ❐

We put now everything together to show that $\sigma = \sum_{z\in\mathbb{Z}^+}\sigma_z = (\iota\times\iota)_*(\chi\times\chi)\,k$ has a smooth representative and hence $k = \sum_{z\in\mathbb{Z}^+} k_z$. As a result, it follows that the softened FP-states are Hadamard states.

THEOREM 7.7.4. *Let* $\mathbf{M}$ *be a* 4*-dimensional, oriented and globally hyperbolic ultrastatic slab with compact spatial section and spin connections as in* Section 7.2*. Then the softened FP-states on the* C^**-completion of the self-dual CAR-algebra for the quantised free massive Dirac field are Hadamard states.*

Proof: As we have argued so far in this section, we are left to show that $\sum_{z\in\mathbb{Z}^+}\sigma_z \in L^2(X\times X, \mathbb{C}^4\otimes(\mathbb{C}^4)^*; \mathrm{vol}_{\mathbf{X}\times\mathbf{X}})$ has a smooth representative. We will first argue that the σ_z are pairwise orthogonal with respect to $\langle\cdot\mid\cdot\rangle_{X\times X,s}$ for $s = 0,1,2,\ldots$, which will allow us to apply [HS96, Lem.21.7], that is, to check if $\sum_{z\in\mathbb{Z}^+}\|\sigma_z\|_{X\times X,s} < \infty$ for all $s = 0,1,2,\ldots$. For each $z\in\mathbb{Z}^+$, we define two functions $S^1\smallsetminus\{(1,0)\} \longrightarrow (a',b')$, $\alpha_z := (\chi\circ\jmath^{-1})\,e^{-\mathrm{i}\lambda_z\jmath^{-1}}$ and $\beta_z := (\chi\circ\jmath^{-1})\,e^{\mathrm{i}\lambda_z\jmath^{-1}}$; then

$$\mathfrak{D}^{\mathfrak{fp}}(\alpha_z\gamma^0\chi_z) = (-Z(\alpha_z) + \lambda_z\alpha_z)\,\gamma^0\chi_z, \tag{7.161}$$
$$\mathfrak{D}^{\mathfrak{cofp}}(\beta_z\zeta_z\gamma^0) = (Z(\beta_z) + \lambda_z\beta_z)\,\zeta_z\gamma^0 \qquad \forall z\in\mathbb{Z}^+, \tag{7.162}$$

and we obtain by induction for $s = 0,1,2,\ldots$

$$(\mathfrak{D}^{\mathfrak{fp}})^s(\alpha_z\gamma^0\chi_z) = \left(\sum_{k=0}^{s}\binom{s}{k}(-1)^{s-k}Z^{s-k}(\alpha_z)\,\lambda_z^k\right)\gamma^0\chi_z \tag{7.163}$$

and

$$(\mathfrak{D}^{\mathfrak{cofp}})^s(\beta_z\zeta_z\gamma^0) = \left(\sum_{k=0}^{s}\binom{s}{k}Z^{s-k}(\beta_z)\,\lambda_z^k\right)\zeta_z\gamma^0 \qquad \forall z\in\mathbb{Z}^+. \tag{7.164}$$

Note that for $z \in \mathbb{Z}^+$,

$$p_{s,z} := \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} Z^{s-k}(\alpha_z)\, \lambda_z^k \quad \text{and} \quad q_{s,z} := \sum_{k=0}^{s} \binom{s}{k} Z^{s-k}(\beta_z)\, \lambda_z^k \tag{7.165}$$

are compactly supported smooth $\mathbb{C}$-valued functions on S^1, which are polynomials in λ_z; the only other dependency on z is of the form $e^{\mathrm{i}\lambda_z J^{-1}}$. The formulas (7.163) and (7.164) entail that the σ_z are pairwise orthogonal with respect to $\langle \cdot \mid \cdot \rangle_{X\times X,s}$ for all $s = 0, 1, 2, \ldots$ and we may thus consider $\sum_{z\in\mathbb{Z}^+} \|\sigma_z\|_{X\times X,s}$ for $s = 0, 1, 2, \ldots$. To estimate this sum, we compute for $s = 0, 1, 2, \ldots$

$$\|(\mathfrak{D}^{\mathfrak{fp}})^s(\alpha_z\gamma^0\chi_z)\|_{X,0}^2 = \int_{S^1} |p_{z,s}|^2 \operatorname{vol}_{S^1} \leq 2\pi P_{s,z} \tag{7.166}$$

and

$$\|(\mathfrak{D}^{\mathfrak{cofp}})^s(\beta_z\zeta_z\gamma^0)\|_{X,0}^2 = \int_{S^1} |q_{z,s}|^2 \operatorname{vol}_{S^1} \leq 2\pi Q_{s,z} \qquad \forall z \in \mathbb{Z}^+, \tag{7.167}$$

where vol_{S^1} is the standard volume form on S^1 and

$$P_{s,z} := \max\left\{ \left(\sum_{k=0}^{s} \binom{s}{k} \left| Z^{s-k}(t;\alpha_z) \right| |\lambda_z^k| \right)^2 \,\middle|\, t \in S^1 \right\}, \tag{7.168}$$

$$Q_{s,z} := \max\left\{ \left(\sum_{k=0}^{s} \binom{s}{k} \left| Z^{s-k}(t;\alpha_z) \right| |\lambda_z^k| \right)^2 \,\middle|\, t \in S^1 \right\}; \tag{7.169}$$

$P_{s,z}$ and $Q_{s,z}$ are polynomials in $|\lambda_z|$ of degree $2s$ with non-negative real coefficients and have no other dependencies on z. With this and Lemma 7.7.3, we compute for $s = 0, 1, 2, \ldots$

$$\|\sigma_z\|_{X\times X,s}^2 = \|\,(\iota\times\iota)_*\,(\chi\otimes\chi)k_z\|_{X\times X,s}^2 \tag{7.170}$$

$$\begin{aligned}
&\leq (c_{f,z}^-)^4 \left\|\alpha_z\gamma^0\chi_z \otimes \beta_z\zeta_z\gamma^0\right\|_{X\times X,s}^2 \\
&\quad + (c_{f,z}^+)^4 |d_{f,z}^+|^2 \left\|\alpha_{-z}\gamma^0\chi_{-z} \otimes \beta_z\zeta_z\gamma^0\right\|_{X\times X,s}^2 \\
&\quad + (c_{f,z}^+)^4 |d_{f,z}^+|^2 \left\|\alpha_z\gamma^0\chi_z \otimes \beta_{-z}\zeta_{-z}\gamma^0\right\|_{X\times X,s}^2 \\
&\quad + (c_{f,z}^+)^4 |d_{f,z}^+|^4 \left\|\alpha_{-z}\gamma^0\chi_{-z} \otimes \beta_{-z}\zeta_{-z}\gamma^0\right\|_{X\times X,s}^2
\end{aligned} \tag{7.171}$$

$$\begin{aligned}
&\leq 2\,(c_{f,z}^-)^4 \left\|\alpha_z\gamma^0\chi_z\right\|_{X,s}^2 \left\|\beta_z\zeta_z\gamma^0\right\|_{X,s}^2 \\
&\quad + 2\,(c_{f,z}^+)^2 (c_{f,z}^-)^2 \left\|\alpha_{-z}\gamma^0\chi_{-z}\right\|_{X,s}^2 \left\|\beta_z\zeta_z\gamma^0\right\|_{X,s}^2 \\
&\quad + 2\,(c_{f,z}^+)^2 (c_{f,z}^-)^2 \left\|\alpha_z\gamma^0\chi_z\right\|_{X,s}^2 \left\|\beta_{-z}\zeta_{-z}\gamma^0\right\|_{X,s}^2 \\
&\quad + 2\,(c_{f,z}^-)^4 \left\|\alpha_{-z}\gamma^0\chi_{-z}\right\|_{X,s}^2 \left\|\beta_{-z}\zeta_{-z}\gamma^0\right\|_{X,s}^2
\end{aligned} \tag{7.172}$$

$$\leq 8\pi\left(c^{-}_{f,z}\right)^{2}\left(P_{0,z}+P_{s,z}\right)\left(Q_{0,z}+Q_{s,z}\right) \qquad \forall z\in\mathbb{Z}^{+}. \tag{7.173}$$

Hence,

$$\|\sigma_z\|^2_{X\times X,s}=\left(1-\frac{\hat{f}(0)\,\mu}{\lambda_z\Xi_{f,z}}\right)R_{s,z}, \tag{7.174}$$

where $R_{s,z}$ is a polynomial in $|\lambda_z|$ of degree $4s$, which is the only dependence of $R_{s,z}$ on z. Due to Lemma 7.7.1, it follows that $\sum_{z\in\mathbb{Z}^+}\|\sigma_z\|^2_{X\times X,s}<\infty$ for each $s=0,1,2,\ldots$ and thus $\sigma=\sum_{z\in\mathbb{Z}^+}\sigma_z$ exists in $L^2_s\left(X\times X,\mathbb{C}^4\otimes\left(\mathbb{C}^4\right)^*;\mathrm{vol}_{\mathbf{X}\times\mathbf{X}}\right)$ for each $s=0,1,2,\ldots$. As a result of the Sobolev embedding theorem [LM89, Thm.III.2.15(1)], σ features a smooth representative. In conclusion, $(\chi\otimes\chi)\,k=(\iota\times\iota)^*\,\sigma$ has a smooth representative in $L^2\left(M'\times M',\mathbb{C}^4\otimes\left(\mathbb{C}^4\right)^*;\mathrm{vol}_{\mathbf{M}'\times\mathbf{M}'}\right)$ and by restriction, $k=\sum_{z\in\mathbb{Z}^+}k_z$ has a smooth representative in $L^2\left(M\times M,\mathbb{C}^4\otimes\left(\mathbb{C}^4\right)^*;\mathrm{vol}_{\mathbf{M}\times\mathbf{M}}\right)$. All in all, this implies that the softened FP-states are Hadamard. ❐

7.8 Quantum fluctuations in the unsoftened FP-state

On the next few pages, we want to discuss further unphysical properties of the unsoftened FP-state. To be concrete, we argue that the normal ordered energy density of the free massive Dirac quantum field has almost always infinite quantum fluctuations in the unsoftened FP-state. Our treatment is motivated by [FV13]*, where the existence of physically interesting normal ordered quantities with infinite quantum fluctuations in the SJ-state was established.*

Let $\mathbf{M}$ again be a 4-dimensional, oriented and globally hyperbolic ultrastatic slab with compact spatial section Σ and spin connections as in Section 7.2. In a Fock space representation, where the Fock vacuum Ω_f represents the FP-state $\omega_{\mathrm{FP};f}$ with $f:\mathbb{R}\longrightarrow\mathbb{R}$ non-negative and integrable, the quantum Dirac spinor field and the quantum Dirac cospinor field take the form

$$\begin{aligned}\Psi_f[v]=&\sum_{z\in\mathbb{Z}^+}c^{+}_{f,z}\int_M v(t,x)\left[e^{-\mathrm{i}\lambda_z t}\chi_z(x)+d^{+}_{f,z}\,e^{\mathrm{i}\lambda_z t}\chi_{-z}\right]\mathrm{vol}_{\mathbf{M}}\;b_z\\&+\sum_{z\in\mathbb{Z}^-}c^{-}_{f,z}\int_M v(t,x)\left[e^{-\mathrm{i}\lambda_z t}\chi_z(x)+d^{-}_{f,z}\,e^{\mathrm{i}\lambda_z t}\chi_{-z}\right]\mathrm{vol}_{\mathbf{M}}\;d^{\dagger}_z,\end{aligned} \tag{7.175}$$

$$v\in C^\infty_0\left(M,(\mathbb{C}^4)^*\right),$$

and

(7.176)
$$\Psi_f^\dagger[u] = \sum_{z\in\mathbb{Z}^+} c_{f,z}^+ \int_M \left[e^{\mathrm{i}\lambda_z t}\zeta_z(x) + \overline{d}_{f,z}^+ e^{-\mathrm{i}\lambda_z t}\zeta_{-z}\right] u(t,x)\,\mathrm{vol}_{\mathbf{M}}\, b_z^\dagger + \sum_{z\in\mathbb{Z}^-} c_{f,z}^- \int_M \left[e^{\mathrm{i}\lambda_z t}\zeta_z(x) + \overline{d}_{f,z}^- e^{-\mathrm{i}\lambda_z t}\zeta_{-z}\right] u(t,x)\,\mathrm{vol}_{\mathbf{M}}\, d_z,$$
$$u \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),$$

where $c_{f,z}^\pm := \sqrt{\frac{\Xi_{f,z} \pm \frac{\hat{f}(0)\mu}{\lambda_z}}{2\,\Xi_{f,z}}}$, $d_{f,z}^\pm := \frac{\pm\Xi_{f,z} - \frac{\hat{f}(0)\mu}{\lambda_z}}{\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda_z^2}}}$ as in the appendix to this chapter and $b_z^\dagger, d_z^\dagger, b_z$ and d_z are the creation and the annihilation operators, respectively. As a consistency check, one easily verifies

(7.177)
$$\langle\Omega_f \mid \left(\Psi_f[v]\,\Psi_f^\dagger[u'] + \Psi_f^\dagger[u]\,\Psi_f[v']\right)\Omega_f\rangle = W_{\mathrm{FP};f}^{(2)}\left[(u\oplus v)\otimes(u'\oplus v')\right]$$
$$\forall u,u' \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),\ \forall v,v' \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*).$$

For comparism, in a Fock space representation where the Fock vacuum Ω_0 represents our reference Hadamard state ω_0, we find

(7.178)
$$\Psi_0[v] = \sum_{z\in\mathbb{Z}^+} \int_M v(t,x)\,e^{-\mathrm{i}\lambda_z t}\chi_z(x)\,\mathrm{vol}_{\mathbf{M}}\, b_z + \sum_{z\in\mathbb{Z}^-} \int_M v(t,x)\,e^{-\mathrm{i}\lambda_z t}\chi_z(x)\,\mathrm{vol}_{\mathbf{M}}\, d_z^\dagger,$$
$$v \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*),$$

and

(7.179)
$$\Psi_0^\dagger[u] = \sum_{z\in\mathbb{Z}^+} \int_M e^{\mathrm{i}\lambda_z t}\zeta_z(x)\,u(t,x)\,\mathrm{vol}_{\mathbf{M}}\, b_z^\dagger + \sum_{z\in\mathbb{Z}^-} \int_M e^{\mathrm{i}\lambda_z t}\zeta_z(x)\,u(t,x)\,\mathrm{vol}_{\mathbf{M}}\, d_z,$$
$$u \in \mathcal{C}_0^\infty(M,\mathbb{C}^4),$$

for the quantum Dirac spinor field and the quantum Dirac cospinor field. They satisfy the identity

(7.180)
$$\langle\Omega_0 \mid \left(\Psi_0[v]\,\Psi_0^\dagger[u'] + \Psi_0^\dagger[u]\,\Psi_0[v']\right)\Omega_0\rangle = W_0^{(2)}\left[(u\oplus v)\otimes(u'\oplus v')\right]$$
$$\forall u,u' \in \mathcal{C}_0^\infty(X,\mathbb{C}^4),\ \forall v,v' \in \mathcal{C}_0^\infty(M,(\mathbb{C}^4)^*).$$

With normal ordering defined in the usual way, we look into the fluctuations of the quantity $:\!\Psi_f^\dagger\gamma^0\dot{\Psi}_f - \dot{\Psi}_f^\dagger\gamma^0\Psi_f\!:$ in the Fock vacuum Ω_f, which is $(-2\,\mathrm{i}\,\mu)$-times the normal ordered energy density $:\rho_f:$ (recall that $\mu := \frac{mc}{\hbar}$ is the reduced mass). We first compute

$$
\begin{aligned}
:\Psi_f^\dagger\gamma^0\dot{\Psi}_f:(h\otimes g)\,\Omega_f = \sum_{\substack{w\in\mathbb{Z}^+\\ z\in\mathbb{Z}^-}} \int_M & c_{f,w}^+ \left[e^{\mathrm{i}\lambda_w t}\,\overline{\chi}_w^\top(x) + \overline{d}_{f,w}^+\, e^{-\mathrm{i}\lambda_w t}\,\overline{\chi}_{-w}^\top(x)\right] \\
& \times c_{f,z}^-\,\mathrm{i}\lambda_z\left[-e^{-\mathrm{i}\lambda_z t}\chi_z(x) + d_{f,z}^-\,e^{\mathrm{i}\lambda_z t}\chi_{-z}(x)\right] \\
& \times h(t)\,g(x)\,\mathrm{vol}_{\mathbf{M}}\; b_w^\dagger d_z^\dagger\Omega_f, \\
& \qquad h\in\mathcal{C}_0^\infty\mathbb{R},\; g\in\mathcal{C}^\infty(\Sigma,\mathbb{C}).
\end{aligned} \tag{7.181}
$$

Taking g to be the constant $1_{\mathbb{C}}$-function on Σ, we can perform the integration over Σ and make use of the orthonormality relations of the χ_z. We obtain for (7.181):

$$
:\Psi_f^\dagger\gamma^0\dot{\Psi}_f:(h\otimes 1_{\mathbb{C}})\,\Omega_f = \sum_{w\in\mathbb{Z}^+}\int_{-b}^{b} c_{f,w}^+\,c_{f,w}^-\;\mathrm{i}\lambda_{-w}\left[d_{-w}^- - \overline{d}_{f,w}^+\right]h(t)\,dt\,b_w^\dagger d_{-w}^\dagger\Omega_f \tag{7.182}
$$

$$
= 2\,\mathrm{i}\,\hat{h}(0)\sum_{w\in\mathbb{Z}^+}(c_{f,w}^+)^2\,\overline{d}_{f,w}^+\,\lambda_w\,b_w^\dagger d_{-w}^\dagger\Omega_f \qquad \forall h\in\mathcal{C}_0^\infty\mathbb{R}. \tag{7.183}
$$

In the same way, we compute

$$
:\dot{\Psi}_f^\dagger\gamma^0\Psi_f:(h\otimes 1_{\mathbb{C}})\,\Omega_f = -2\,\mathrm{i}\,\hat{h}(0)\sum_{w\in\mathbb{Z}^+}(c_{f,w}^+)^2\,\overline{d}_{f,w}^+\,\lambda_w\,b_w^\dagger d_{-w}^\dagger\Omega_f \qquad \forall h\in\mathcal{C}_0^\infty\mathbb{R} \tag{7.184}
$$

and hence

$$
:\Psi_f^\dagger\gamma^0\dot{\Psi}_f - \dot{\Psi}_f^\dagger\gamma^0\Psi_f:(h\otimes 1_{\mathbb{C}})\,\Omega_f = 4\,\mathrm{i}\,\hat{h}(0)\sum_{w\in\mathbb{Z}^+}(c_{f,w}^+)^2\,\overline{d}_{f,w}^+\,\lambda_w\,b_w^\dagger d_{-w}^\dagger\Omega_f \qquad \forall h\in\mathcal{C}_0^\infty\mathbb{R}. \tag{7.185}
$$

Since $:\Psi_f^\dagger\gamma^0\dot{\Psi}_f - \dot{\Psi}_f^\dagger\gamma^0\Psi_f:(h\otimes 1_{\mathbb{C}})$ has vanishing expectation value in the Fock vacuum Ω_f for all $h\in\mathcal{C}_0^\infty\mathbb{R}$, its fluctuation is simply given by $\|:\Psi_f^\dagger\gamma^0\dot{\Psi}_f - \dot{\Psi}_f^\dagger\gamma^0\Psi_f:(h\otimes 1_{\mathbb{C}})\,\Omega_f\|$, where

$$
\|:\Psi_f^\dagger\gamma^0\dot{\Psi}_f - \dot{\Psi}_f^\dagger\gamma^0\Psi_f:(h\otimes 1_{\mathbb{C}})\,\Omega_f\|^2 = 16\,\hat{h}(0)^2\sum_{w\in\mathbb{Z}^+}(c_{f,w}^+)^4\lambda_w^2\,|d_{f,w}^+| \tag{7.186}
$$

$$
= 16\,\hat{h}(0)^2\sum_{w\in\mathbb{Z}^+}(c_{f,w}^+)^2\,(c_{f,w}^-)^2\lambda_w^2 \tag{7.187}
$$

$$
= 4\,\hat{h}(0)^2\sum_{w\in\mathbb{Z}^+}\left(1-\frac{\hat{f}(0)^2\,\mu^2}{\lambda_w^2\,\Xi_{f,w}^2}\right)\lambda_w^2 \qquad \forall h\in\mathcal{C}_0^\infty\mathbb{R}. \tag{7.188}
$$

<u>THEOREM</u> 7.8.1. *Let* $\mathbf{M}$ *be a* 4*-dimnsional, oriented and globally hyperbolic ultra-static slab with compact spatial section,* $a = -b$ *for* $b\in(0,\infty)$ *and spin connections as* *in* Section 7.2. *We consider the unsoftened FP-state, i.e.* $f = \chi_{(-b,b)}$. *Then the set of the* $b \in (0,\infty)$ *for which the normal ordered energy density* $:\rho_{\chi_{(-b,b)}}:$ *has finite quantum fluctuations in the Fock vacuum* $\Omega_{\chi_{(-b,b)}}$ *corresponding to the unsoftened FP-state* $\omega_{\chi_{(-b,b)}}$ *is of Lebesgue measure zero.*

Proof: For $f = \chi_{(-b,b)}$, (7.188) further computes to

$$4\,\hat{h}(0)^2 \sum_{w\in\mathbb{Z}^+} \left(1 - \frac{1}{\frac{\lambda_w^2 \hat{\chi}_{(-b,b)}(2\lambda_w)^2}{\hat{\chi}_{(-b,b)}(0)^2 \mu^2}\left(1 - \frac{\mu^2}{\lambda_w^2}\right) + 1}\right)\lambda_w^2 \qquad \forall h \in \mathcal{C}_0^\infty \mathbb{R}, \tag{7.189}$$

which can only converge if the term in the round brackets goes to zero as w goes to infinity (and stronger than λ_w^2 goes to infinity). Evidently, this can only happen if it holds that $\lim_{w\to\infty} \lambda_w^2\, \hat{\chi}_{(-b,b)}\,(2\lambda_w)^2 = \lim_{w\to\infty} \sin^2(2\lambda_w b) = 0$. Recalling the results of [FV12c], directly after Proposition 4.1, the $b \in (0,\infty)$ for which this can happen form a set of Lebesgue measure zero (cf. the proof of Lemma 7.6.2). ❐

We make no statement for $b \in (0,\infty)$ in the aforementioned set of Lebesgue measure zero (if non-empty). Also observe that the softened FP-states elude our argument for the divergence of quantum fluctuations by [Hör90, (8.1.1)] (cf. [FJ98, Exercise 8.16]).

Appendix: some computations and formulas

We provide an expression for the Wightman two-point distribution $W^{(2)}_{FP;f}$ associated with an FP-state $\omega_{FP;f}$, $f:\mathbb{R}\longrightarrow\mathbb{R}$ non-negative and integrable, in terms of the eigenfunctions χ_z and ζ_z of the spatial Dirac operators H^{sp} and H^{cosp}, respectively. We also compute the left-hand side of (7.111) *and prove our claims made in* Lemma 7.6.1 *regarding the matrix elements of the Hilbert-Schmidt operator K. Notice that we adopt the notation from* Section 7.5 *and* Section 7.6 *throughout.*

For our first item, we compute

$$\langle Sv^\dagger \mid Q^{\mathrm{sp}}_f Su'\rangle_{\mathrm{sp}} = \sum_{z\in\mathbb{Z}^+} \langle Sv^\dagger \mid \kappa^f_{z+}\rangle_{\mathrm{sp}} \langle \kappa^f_{z+} \mid Su'\rangle_{\mathrm{sp}} \tag{7.190}$$

$$= \sum_{z\in\mathbb{Z}^+} \langle Sv^\dagger \mid c^+_{f,z} e^{-\mathrm{i}\lambda_z \cdot}\chi_z + c^+_{f,z} d^+_{f,z} e^{\mathrm{i}\lambda_z \cdot}\chi_{-z}\rangle_{\mathrm{sp}} \langle c^+_{f,z} e^{-\mathrm{i}\lambda_z \cdot}\chi_z + c^+_{f,z} d^+_{f,z} e^{\mathrm{i}\lambda_z \cdot}\chi_{-z} \mid Su'\rangle_{\mathrm{sp}} \tag{7.191}$$

$$= \sum_{z\in\mathbb{Z}^+} (c^+_{f,z})^2 \int_M v(t,x)\left[e^{-\mathrm{i}\lambda_z t}\chi_z(x) + d^+_{f,z} e^{\mathrm{i}\lambda_z t}\chi_{-z}(x)\right]\mathrm{vol}_{\mathbf{M}} \int_M \left[e^{\mathrm{i}\lambda_z t'}\zeta_z(x') + \overline{d}^+_{f,z} e^{-\mathrm{i}\lambda_z t'}\zeta_{-z}(x')\right] u'(t',x')\,\mathrm{vol}'_{\mathbf{M}} \tag{7.192}$$

$$\begin{aligned} &= \sum_{z\in\mathbb{Z}^+} (c^+_{f,z})^2 \int_M\int_M e^{-\mathrm{i}\lambda_z(t-t')} v(t,x)\,\chi_z(x)\,\zeta_z(x')\,u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}} \\ &\quad + \sum_{z\in\mathbb{Z}^+} (c^+_{f,z})^2\,\overline{d}^+_{f,z} \int_M\int_M e^{-\mathrm{i}\lambda_z(t+t')} v(t,x)\,\chi_z(x)\,\zeta_{-z}(x')\,u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}} \\ &\quad + \sum_{z\in\mathbb{Z}^+} (c^+_{f,z})^2 d^+_{f,z} \int_M\int_M e^{\mathrm{i}\lambda_z t} v(t,x)\,\chi_{-z}(x)\left[e^{\mathrm{i}\lambda_z t'}\zeta_z(x') + \overline{d}^+_{f,z} e^{-\mathrm{i}\lambda_z t'}\zeta_{-z}(x')\right] u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}} \end{aligned} \tag{7.193}$$

$$\forall u' \in \mathcal{C}^\infty_0(M,\mathbb{C}^4),\ \forall v \in \mathcal{C}^\infty_0(M,(\mathbb{C}^4)^*),$$

where we have defined $c^\pm_{f,z} := \sqrt{\frac{\Xi_{f,z} \pm \frac{\hat{f}(0)\mu}{\lambda_z}}{2\,\Xi_{f,z}}}$ and $d^\pm_{f,z} := \frac{\pm\Xi_{f,z} - \frac{\hat{f}(0)\mu}{\lambda_z}}{\hat{f}(2\lambda_z)\sqrt{1-\frac{\mu^2}{\lambda_z^2}}}$ for the sake of clarity and economy; observe the important identities $c^+_{f,-z} = c^-_{f,z}$, $(c^+_{f,z})^2 + (c^-_{f,z})^2 = 1$, $d^+_{f,-z} = -\overline{d}^-_{f,z}$, $(c^+_{f,z})^2\,|d^+_{f,z}|^2 = (c^-_{f,z})^2$ and $(c^-_{f,z})^2\,|d^-_{f,z}|^2 = (c^+_{f,z})^2$. In addition, we also compute

$$\langle Cu^\dagger \mid Cv' \rangle_{\text{cosp}} = -\mathrm{i} \int_M Cv'\, u \,\mathrm{vol}_{\mathbf{M}} \tag{7.194}$$

$$= -\mathrm{i} \int_a^b \langle u^\dagger \gamma^0 \mid (Cv')_t \rangle_\Sigma \, dt \tag{7.195}$$

$$= -\mathrm{i} \int_a^b \left\langle u^\dagger \gamma^0 \,\middle|\, \sum_{z \in \mathbb{Z}'} \int_a^b e^{-\mathrm{i}\lambda_z t'} \langle \zeta_z \mid \mathrm{i}\, v'_{t'} \gamma^0 \rangle_\Sigma \, dt' \, e^{\mathrm{i}\lambda_z t} \zeta_z \right\rangle_\Sigma dt \tag{7.196}$$

$$= \sum_{z \in \mathbb{Z}'} \int_a^b e^{\mathrm{i}\lambda_z t} \langle u^\dagger \gamma^0 \mid \zeta_z \rangle_\Sigma \, dt \int_a^b e^{-\mathrm{i}\lambda_z t'} \langle \zeta_z \mid v'_{t'} \gamma^0 \rangle_\Sigma \, dt' \tag{7.197}$$

$$= \sum_{z \in \mathbb{Z}'} \int_M e^{\mathrm{i}\lambda_z t} \zeta_z(x)\, u(t,x) \,\mathrm{vol}_{\mathbf{M}} \int_M e^{-\mathrm{i}\lambda_z t'} v'(t',x')\, \chi_z(x') \,\mathrm{vol}'_{\mathbf{M}} \tag{7.198}$$

$$= \sum_{z \in \mathbb{Z}'} \int_M \int_M e^{\mathrm{i}\lambda_z (t-t')} \zeta_z(x)\, u(t,x)\, v'(t',x')\, \chi_z(x') \,\mathrm{vol}_{\mathbf{M}} \,\mathrm{vol}'_{\mathbf{M}} \tag{7.199}$$

$$\forall u \in \mathcal{C}_0^\infty(M, \mathbb{C}^4),\ \forall v' \in \mathcal{C}_0^\infty(M, (\mathbb{C}^4)^*),$$

where we have used (7.47). Plugging (7.193) and (7.199) into (7.106), we finally obtain

$$\begin{aligned} W^{(2)}_{\mathrm{FP};f}\left[(u \oplus v) \otimes (u' \oplus v')\right] = & \sum_{z \in \mathbb{Z}^+} (c^+_{f,z})^2 \int_M \int_M e^{-\mathrm{i}\lambda_z (t-t')} v(t,x)\, \chi_z(x)\, \zeta_z(x')\, u'(t',x') \,\mathrm{vol}_{\mathbf{M}} \,\mathrm{vol}'_{\mathbf{M}} \\ & + \sum_{z \in \mathbb{Z}^+} (c^+_{f,z})^2\, \overline{d}^+_{f,z} \int_M \int_M e^{-\mathrm{i}\lambda_z (t+t')} v(t,x)\, \chi_z(x)\, \zeta_{-z}(x')\, u'(t',x') \,\mathrm{vol}_{\mathbf{M}} \,\mathrm{vol}'_{\mathbf{M}} \\ & + \sum_{z \in \mathbb{Z}^+} (c^+_{f,z})^2\, d^+_{f,z} \int_M \int_M e^{\mathrm{i}\lambda_z t} v(t,x)\, \chi_{-z}(x) \left[e^{\mathrm{i}\lambda_z t'} \zeta_z(x') + \overline{d}^+_{f,z} e^{-\mathrm{i}\lambda_z t'} \zeta_{-z}(x') \right] u'(t',x') \,\mathrm{vol}_{\mathbf{M}} \,\mathrm{vol}'_{\mathbf{M}} \end{aligned} \tag{7.200}$$

$$
\begin{aligned}
&+\sum_{z\in\mathbb{Z}^-}(c^-_{f,z})^2\int_M\int_M e^{\mathrm{i}\lambda_z(t-t')}\zeta_z(x)\,u(t,x)\,v'(t',x')\,\chi_z(x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\\
&+\sum_{z\in\mathbb{Z}^-}(c^-_{f,z})^2\,d^-_{f,z}\int_M\int_M e^{\mathrm{i}\lambda_z(t+t')}\zeta_z(x)\,u(t,x)\,v'(t',x')\,\chi_{-z}(x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\\
&+\sum_{z\in\mathbb{Z}^-}(c^-_{f,z})^2\,\overline{d}^-_{f,z}\int_M\int_M e^{-\mathrm{i}\lambda_z t}\zeta_{-z}(x)\,u(t,x)\,v'(t',x')\left[e^{-\mathrm{i}\lambda_z t'}\chi_z(x')+d^-_{f,z}e^{\mathrm{i}\lambda_z t'}\chi_{-z}(x')\right]\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}},\\
&\qquad\qquad u,u'\in\mathcal{C}^\infty_0(M,\mathbb{C}^4),\ v,v'\in\mathcal{C}^\infty_0(M,(\mathbb{C}^4)^*).
\end{aligned}
$$

We turn to the second item of this appendix. From (7.73), we see that the left-hand side of (7.111) is given by

$$
\begin{aligned}
\langle Sv^\dagger \mid (Q^{\mathrm{sp}}_f-Q^{\mathrm{sp}})Su'\rangle_{\mathrm{sp}} = &-\sum_{z\in\mathbb{Z}^+}(c^-_{f,z})^2\int_M\int_M e^{-\mathrm{i}\lambda_z(t-t')}v(t,x)\,\chi_z(x)\,\zeta_z(x')\,u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\\
&+\sum_{z\in\mathbb{Z}^+}(c^+_{f,z})^2\,\overline{d}^+_{f,z}\int_M\int_M e^{-\mathrm{i}\lambda_z(t+t')}v(t,x)\,\chi_z(x)\,\zeta_{-z}(x')\,u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\\
&+\sum_{z\in\mathbb{Z}^+}(c^+_{f,z})^2\,d^+_{f,z}\int_M\int_M e^{\mathrm{i}\lambda_z t}v(t,x)\,\chi_{-z}(x)\left[e^{\mathrm{i}\lambda_z t'}\zeta_z(x')+\overline{d}^+_{f,z}e^{-\mathrm{i}\lambda_z t'}\zeta_{-z}(x')\right]u'(t',x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\\
&\qquad\qquad \forall u'\in\mathcal{C}^\infty_0(M,\mathbb{C}^4),\ \forall v\in\mathcal{C}^\infty_0(M,(\mathbb{C}^4)^*).
\end{aligned}
\tag{7.201}
$$

For our third and final item of this appendix, we let $n\geq 2$, consider $\sigma_n\in\mathcal{C}^\infty((a,b),\mathbb{R})$ such that $0\leq\sigma_n(t)\leq 1$ for all $t\in(a,b)$, $\sigma_n(t)=1$ on $[a'+\frac{\varepsilon}{n},b'-\frac{\varepsilon}{n}]$ and $\operatorname{supp}\sigma_n\subseteq(a'+\frac{\varepsilon}{2n},b'-\frac{\varepsilon}{2n})$ for $\varepsilon>0$ small enough. Such σ_n exist thanks to [Lee03, Prop.2.26]. Then surely we have $\lim_{n\to\infty}\sigma_n e^{-\mathrm{i}\lambda_z\cdot}\chi_z=e^{-\mathrm{i}\lambda_z\cdot}\chi_z$ and $\lim_{m\to\infty}\sigma_n e^{\mathrm{i}\lambda_z\cdot}\zeta_z=e^{\mathrm{i}\lambda_z\cdot}\zeta_z$ on M' pointwise. For each $m,n\geq 2$, $k^*\left[(\sigma_n e^{-\mathrm{i}\lambda_z\cdot}\chi_z)\otimes(\sigma_m e^{\mathrm{i}\lambda_w\cdot}\zeta_w)\right]$ is a smooth function and thus bounded from above on $M'\times M'$ by

$$
\max\left\{\left|k^*(t,x,t',x')\left[\left(\sigma_n(t')\,e^{-\mathrm{i}\lambda_z t'}\chi_z(x')\right)\otimes\left(\sigma_m(t)\,e^{\mathrm{i}\lambda_w t}\zeta_w(x)\right)\right]\right|\ \middle|\ t,t'\in[a',b'],\ x,x'\in\Sigma\right\}.
\tag{7.202}
$$

Hence, we can apply Lebesgue's dominated convergence theorem in the following computation of matrix elements for K:

$$\langle E^{-\mathrm{i}\lambda_w\cdot} X_w \mid K E^{-\mathrm{i}\lambda_z\cdot} X_z\rangle_{M'} = \int\limits_{M'\times M'} k^*\left[(e^{-\mathrm{i}\lambda_z\cdot}\gamma^0\chi_z)\otimes(e^{\mathrm{i}\lambda_w\cdot}\zeta_w\gamma^0)\right]\mathrm{vol}_{\mathbf{M}'\times\mathbf{M}'} \tag{7.203}$$

$$= \int\limits_{M'\times M'} \lim_{m,n\to\infty} k^*\left[(\sigma_n\, e^{-\mathrm{i}\lambda_z\cdot}\gamma^0\chi_z)\otimes(\sigma_m\, e^{\mathrm{i}\lambda_w\cdot}\zeta_w\gamma^0)\right]\mathrm{vol}_{\mathbf{M}'\times\mathbf{M}'} \tag{7.204}$$

$$= \lim_{m,n\to\infty}\int\limits_{M'\times M'} k^*\left[(\sigma_n\, e^{-\mathrm{i}\lambda_z\cdot}\gamma^0\chi_z)\otimes(\sigma_m\, e^{\mathrm{i}\lambda_w\cdot}\zeta_w\gamma^0)\right]\mathrm{vol}_{\mathbf{M}'\times\mathbf{M}'} \tag{7.205}$$

$$= \lim_{m,n\to\infty}\int\limits_{M\times M} k^*\left[(\sigma_n\, e^{-\mathrm{i}\lambda_z\cdot}\gamma^0\chi_z)\otimes(\sigma_m\, e^{\mathrm{i}\lambda_w\cdot}\zeta_w\gamma^0)\right]\mathrm{vol}_{\mathbf{M}\times\mathbf{M}} \tag{7.206}$$

$$\begin{aligned} &= -\lim_{m,n\to\infty}\sum_{k\in\mathbb{Z}^+}(c^-_{f,k})^2\int\limits_M\int\limits_M e^{-\mathrm{i}\lambda_k(t-t')}\sigma_m(t)\,e^{\mathrm{i}\lambda_w t}\zeta_w(x)\,\gamma^0\chi_k(x)\,\zeta_k(x')\,\sigma_n(t')\,e^{-\mathrm{i}\lambda_z t'}\gamma^0\chi_z(x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\\ &\quad+\lim_{m,n\to\infty}\sum_{k\in\mathbb{Z}^+}(c^+_{f,k})^2\,\overline{d}^+_{f,k}\int\limits_M\int\limits_M e^{-\mathrm{i}\lambda_k(t+t')}\sigma_m(t)\,e^{\mathrm{i}\lambda_w t}\zeta_w(x)\,\gamma^0\chi_k(x)\,\zeta_{-k}(x')\,\sigma_n(t')\,e^{-\mathrm{i}\lambda_z t'}\gamma^0\chi_z(x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}}\\ &\quad+\lim_{m,n\to\infty}\sum_{k\in\mathbb{Z}^+}(c^+_{f,k})^2\,d^+_{f,k}\int\limits_M\int\limits_M e^{\mathrm{i}\lambda_k t}\sigma_m(t)\,e^{\mathrm{i}\lambda_w t}\zeta_w(x)\,\gamma^0\chi_{-k}(x)\left[e^{\mathrm{i}\lambda_k t'}\zeta_k(x')+\overline{d}^+_{f,k}e^{-\mathrm{i}\lambda_k t'}\zeta_{-k}(x')\right]\sigma_n(t')\,e^{-\mathrm{i}\lambda_z t'}\gamma^0\chi_z(x')\,\mathrm{vol}_{\mathbf{M}}\,\mathrm{vol}'_{\mathbf{M}} \end{aligned} \tag{7.207}$$

$$\begin{aligned} &= -\lim_{m,n\to\infty}\sum_{k\in\mathbb{Z}^+}(c^-_{f,k})^2\int\limits_a^b\int\limits_a^b e^{-\mathrm{i}\lambda_k(t-t')}\sigma_m(t)\,e^{\mathrm{i}\lambda_w t}\,\delta_{wk}\delta_{kz}\,\sigma_n(t')\,e^{-\mathrm{i}\lambda_z t'}dt\,dt'\\ &\quad+\lim_{m,n\to\infty}\sum_{k\in\mathbb{Z}^+}(c^+_{f,k})^2\,\overline{d}^+_{f,k}\int\limits_a^b\int\limits_a^b e^{-\mathrm{i}\lambda_k(t+t')}\sigma_m(t)\,e^{\mathrm{i}\lambda_w t}\,\delta_{wk}\delta_{-kz}\,\sigma_n(t')\,e^{-\mathrm{i}\lambda_z t'}dt\,dt'\\ &\quad+\lim_{m,n\to\infty}\sum_{k\in\mathbb{Z}^+}(c^+_{f,k})^2\,d^+_{f,k}\int\limits_a^b\int\limits_a^b e^{\mathrm{i}\lambda_k t}\sigma_m(t)\,e^{\mathrm{i}\lambda_w t}\,\delta_{w-k}\left(e^{\mathrm{i}\lambda_k t'}\delta_{kz}+\overline{d}^+_{f,k}e^{-\mathrm{i}\lambda_k t'}\delta_{-kz}\right)\sigma_n(t')\,e^{-\mathrm{i}\lambda_z t'}dt\,dt' \end{aligned} \tag{7.208}$$

$$= -(c^-_{f,w})^2\,\Theta(w)\,\delta_{wz}+(c^+_{f,w})^2\,\overline{d}^+_{f,w}\,\Theta(w)\,\delta_{-wz}+(c^+_{f,-w})^2\,d^+_{f,-w}\,\Theta(-w)\left(\delta_{-wz}+\overline{d}^+_{f,-w}\delta_{wz}\right) \tag{7.209}$$

$$= (c^-_{f,w})^2 \left[|d^+_{f,-w}|^2 \, \Theta(-w) - \Theta(w) \right] \delta_{wz} + \left[(c^+_{f,w})^2 \, \overline{d}^+_{f,w} \, \Theta(w) + (c^-_{f,w})^2 \, d^+_{f,-w} \, \Theta(-w) \right] \delta_{-wz} \qquad \forall w, z \in \mathbb{Z}'. \tag{7.210}$$

Now suppose that $F \in L^2(M', \mathbb{C}^4; \mathrm{vol}_{\mathbf{M}'})$ is orthogonal to $E^{-\mathrm{i}\lambda_z \cdot} X_z$ for all $z \in \mathbb{Z}'$, i.e. $\langle F \mid E^{-\mathrm{i}\lambda_z \cdot} X_z \rangle_{M'} = 0$ for all $z \in \mathbb{Z}'$; then, from the previous calculation, we can also read off that $\langle F \mid K E^{-\mathrm{i}\lambda_z \cdot} X_z \rangle_{M'} = 0$ for all $z \in \mathbb{Z}'$.

Summary

In this thesis, we have studied the appearance and the application of category theory in algebraic and locally covariant quantum field theory. Notably, we have devoted much study to the notions of colimits and left Kan extensions, trying to clarify K. Fredenhagen's universal algebra from the point of view of category theory, and applying them to the quantum field theory of the free Maxwell field in curved spacetimes. Categorical concepts also played the key role for the general scheme to understand C.J. Isham's twisted quantum fields from the point of view of algebraic and locally covariant quantum field theory. Additionally, we have constructed new Hadamard states for the quantised free massive Dirac field. Here, we give a summary of our achievements and point out some missed opportunities.

K. Fredenhagen's universal algebra

We have taken up the position to view the universal algebra firstly introduced in[Fre90] as the universal object $\varinjlim F$ of the colimit for a net $\mathbf{B} \longmapsto \mathfrak{A}(\mathbf{B})$ of the local (unital) $(C)^*$-algebras viewed as a functor $F : \mathscr{J} \longrightarrow \mathbf{(C)^*Alg_1^m}$. Here, $\mathscr{J}$ could in principle be any small category, however, we usually thought of $\mathscr{J}$ as the poset (viewed as a category) which is defined by a choice of spacetime regions for a **Loc**-object **M** and their inclusions into each other. Also, we allowed the local (unital) $(C)^*$-algebras Fi, $i \in \mathscr{J}$, to be general (unital) $(C)^*$-algebras and not just (unital) $(C)^*$-subalgebras of one (unital) $(C)^*$-algebra or of each other. To the same effect, the (unital) *-monomorphisms $F\mu_{ij}$, $\mu_{ij} \in \mathscr{J}(i,j)$, $i, j \in \mathscr{J}$, were not restricted to be inclusion maps. As we established by examples, $\mathbf{{}^*Alg^m}$, $\mathbf{{}^*Alg_1^m}$, $\mathbf{C^*Alg^m}$ and $\mathbf{C^*Alg_1^m}$ are not cocomplete categories but dropping any requirement of injectivity and allowing instead for general (unital) *-homomorphisms, we obtained the cocomplete categories $\mathbf{{}^*Alg}$, $\mathbf{{}^*Alg_1}$, $\mathbf{C^*Alg}$ and $\mathbf{C^*Alg_1}$. This is by no means a new result, see [Ped99; KT02]; nevertheless, this insight has taught us to view F as a functor $F : \mathscr{J} \longrightarrow \mathbf{(C)^*Alg_1}$, i.e. to drop the restriction to unital *-monomorphisms and to allow for general unital *-homomorphisms, if we want the colimit and hence the universal algebra always to exist. However, as we have discussed in addressing some of the criticism leveled against the universal algebra by [RV12] and R. Brunetti[7], the existence of the colimit does not guarantee the non-triviality of the universal algebra, i.e. the universal algebra may turn out to be the zero algebra.

In this general context of the existence and non-triviality of colimits in specific categories, we have unfortunately missed out on investigating the existence and non-triviality of colimits in categories of von Neumann or W^*-algebras with unital *-homo/monomorphisms or in categories of topological (unital) *-algebras with continuous (unital) *-homo/monomorphisms, which are also important to quantum field theory and mathematical physics in general. Though we do not expect any surprises or huge differences to the analysis and results of this thesis, establishing cocompleteness or the failure of cocompleteness for such categories is worthwhile and not just for the sake of completeness. It would also be insightful to compute the universal algebra in more examples of quantum field theories in curved spacetimes. In particular, as suggested by K. Fredenhagen, it would be interesting to obtain the universal algebra for quantum field theories in non-globally hyperbolic spacetimes, taking only account of the quantum field theory on the globally hyperbolic open subsets. As far as we know, this has only been done for the free massive scalar field in Minkowski hal$F$-space by [Som06].

Our main technical result concerned the universal algebra of some specific nets $F : \mathcal{J} \longrightarrow \mathbf{{}^*Alg_1^m}$ of local unital *-algebras, which were derived from a functor $G : \mathcal{J} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}^{\mathbf{m}}$ via a functorial quantisation prescription $Q : \mathbf{pSympl}_{\mathbb{K}}^{\mathbf{m}} \longrightarrow \mathbf{{}^*Alg_1^m}$ as $F = Q \circ G$. The physical interpretation was thereby that the local (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic spaces[8] represent the classical field theory in terms of very basic linear observables whose Poisson brackets are given by the (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic form. The quantisation functor Q promotes the basic linear observables of the classical field theory to smearings of the quantum field such that their Poisson brackets relate to the commutation relations according to the correspondence principle. In our main technical result, we have established a relation between the colimit for F, viewed as a functor $F : \mathcal{J} \longrightarrow \mathbf{{}^*Alg_1}$, and the colimit for G, viewed as a functor $G : \mathcal{J} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, which asserts that under the right circumstances, the quantisation functor $Q : \mathbf{pSympl}_{\mathbb{K}} \longrightarrow \mathbf{{}^*Alg_1}$ preserves the colimit, i.e. $\operatorname{colim} F \cong Q(\operatorname{colim} G)$. The "*right circumstances*" refer mainly to the commutation relations in the universal algebra of F, $\varinjlim F$, which we know to exist by the cocompleteness of $\mathbf{{}^*Alg_1}$, and require certain commutators in the universal algebra to be multiples of the identity. This is remarkable insofar that we have obtained a criterion for the existence of the colimit for G (the categories $\mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{pSympl}_{\mathbb{K}}^{\mathbf{m}}$ and $\mathbf{Sympl}_{\mathbb{K}}$ are not cocomplete) by giving

[7] Private communication.

[8] As before in this thesis, by "*space*" we mean a *linear* space, i.e. a vector space and not a manifold.

the (complexified if $\mathbb{K} = \mathbb{C}$) pre-symplectic form of $\varinjlim G$ in terms of the commutation relations in the universal algebra $\varinjlim F$ such that $Q(\varinjlim G) \cong \varinjlim F$. Hence, the (universal) classical field theory is obtained with significant guidance of the (universal) quantum field theory such that its quantisation yields the (universal) quantum field theory. This seems to indicate the involvement of a classical limit procedure and it would be interesting to see if this can be made more precise e.g. in terms of a classical limit functor. Further investigation is needed to shed more light on this issue however. Now, on the other hand, the result is important because it allows us to characterise the universal algebra $\varinjlim F$ more concretely as $Q(\varinjlim G)$, opposed to an abstract characterisation by generators and relations. It becomes thus also a helpful tool to establish the non-triviality of the universal algebra, and we have applied the main technical theorem many times in this thesis to concretely compute universal algebras.

To be more explicit, we have applied colimit constructions and left Kan extensions in three examples: the free and minimally coupled real scalar field, the free Maxwell field and $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields. We treated the free and minimally coupled real scalar field in more general terms of smooth differential p-forms and determined that by considering colimits, hence the universal algebra, and left Kan extensions, the standard (complexified) symplectic spaces, the standard simple unital *-algebras and the standard locally covariant quantum field theory are recovered. This result was independent of any topological restrictions which we have imposed on the starting situation. The free Maxwell field and $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields will be dicussed in detail further below.

Provided that there are enough interesting examples and applications to justify the effort, one could also look into analogue statements for other quantisation prescriptions such as the Weyl quantisation prescription or quantisation prescriptions for fermionic field theories. We have some results regarding Weyl quantisation: it is well-known that the Weyl quantisation prescription for symplectic spaces gives rise to a functor $W : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{C^*Alg_1^m}$ ([BGP07, Sec.4.2], [BF09, Sec.1.6]). Using the methods of [BHR04], this functor can be extended to a functor $W : \mathbf{pSympl}_{\mathbb{R}} \longrightarrow \mathbf{C^*Alg_1}$. Alternatively, one can construct $W : \mathbf{pSympl}_{\mathbb{R}} \longrightarrow \mathbf{C^*Alg_1}$ as the pointwise left Kan extension of $W : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{C^*Alg_1^m}$, viewed as a functor $W : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{C^*Alg_1}$, along the inclusion functor $K : \mathbf{Sympl}_{\mathbb{R}} \longrightarrow \mathbf{pSympl}_{\mathbb{R}}$.

PROPOSITION 8.9.2. *Let $\mathcal{J}$ be a small category, $F : \mathcal{J} \longrightarrow \mathbf{pSympl}_{\mathbb{R}}$ a functor and write $Fi = (V_i, \omega_i)$ for all $i \in \mathcal{J}$. Let $(V, v : F_\omega \circ F \dot{\longrightarrow} \Delta V)$ be the colimit for $F_\omega \circ F : \mathcal{J} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$, where $F_\omega : \mathbf{pSympl}_{\mathbb{R}} \longrightarrow \mathbf{Vec}_{\mathbb{R}}$ is the forgetful functor that forgets about the pre-symplectic form, and $(A, \alpha : W \circ F \dot{\longrightarrow} \Delta A)$ the colimit for $W \circ F : \mathcal{J} \longrightarrow \mathbf{C^*Alg_1}$. Both colimits exist thanks to Theorem 2.2.10. With the notation introduced, the following statements are equivalent:*

(a) $\operatorname{colim} F$ *exists and* $W(\operatorname{colim} F) = \operatorname{colim}(W \circ F)$.

(b) *There exists a pre-symplectic form* ω *on* V *turning* v *into a cocone* $v : F \dot{\longrightarrow} \Delta(V,\omega)$ *such that*

$$\alpha_i(W_i(x_i))\,\alpha_j(W_j(y_j)) = e^{\mathrm{i}\,\omega(v_j(y_j),v_i(x_i))}\,\alpha_j(W_j(y_j))\,\alpha_i(W_i(x_i)) \qquad \forall x_i \in V_i,\ \forall y_j \in V_j,\ \forall i,j \in \mathcal{J}. \tag{8.211}$$

Dynamical locality of the free Maxwell field

We also applied colimit constructions and left Kan extensions to the example of the free Maxwell field in terms of the field strength tensor, leading to the classical and the quantised F-theory ("F" for field strength tensor) of the free Maxwell field. There, we took a modest conservative approach and started with considering only those $\mathbf{M} \in \mathbf{Loc}$ for which the F-description was given by (complexified) symplectic spaces and simple unital *-algebras in the standard way, and coincided with the A-description ("A" for vector potential) of the free Maxwell field, which was the case for $H^1_{\mathrm{dR}}(M;\mathbb{K}) = H^2_{\mathrm{dR}}(M;\mathbb{K}) = H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$. On this basis, we obtained functors $\mathcal{F}$, $\mathfrak{F} : \mathbf{Loc}_{\{2,m-2\}} \longrightarrow \mathbf{Sympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1^m}$ and showed that for $\mathbf{M} \in \mathbf{Loc}$, the same colimit and in particular the same universal algebra for the restrictions of $\mathcal{F}$ and $\mathfrak{F}$ to $\mathrm{loc}^q_{-\mathbf{M}}$ are acquired for any choice of $q \subseteq \mathbb{N} \setminus \{0\}$ with $2, m-2 \in q$ or $q = ©$. Similarly, regardless of $q \subseteq \mathbb{N} \setminus \{0\}$ with $2, m-2 \in q$ or $q = ©$, we obtained the same left Kan extensions along the inclusion functor $K_q : \mathbf{Loc}_q \longrightarrow \mathbf{Loc}$ for the restrictions of $\mathcal{F}$ and $\mathfrak{F}$ to $\mathbf{Loc}_q$.

Studying the properties of these left Kan extensions, which we termed the classical and the quantised universal F-theory of the free Maxwell field, we asserted that they both fail the principles of local covariance, which was already known from [DL12], and dynamical locality, even in a weakened sense. Getting to the bottom of this failure, namely non-trivial radicals in the classical field theory and non-trivial centres in the quantised field theory for $\mathbf{M} \in \mathbf{Loc}$ with $H^2_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$ or $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$, we were led to considering a reduced classical and quantised F-theory for the free Maxwell field, which was shown to be locally covariant and dynamically local. However, this came at the price of sacrificing the sensitivity of the universal F-theory to the topology of curved spacetimes.

We also briefly looked into the A-description of the free Maxwell field and considered functors $\mathcal{A}$, $\mathfrak{A} : \mathbf{Loc} \longrightarrow \mathbf{pSympl}_{\mathbb{K}}$, $\mathbf{{}^*Alg_1}$, which are not locally covariant theories but were regarded in the recent publication [SDH14]. To our business of computig colimits and left Kan extensions, $\mathcal{A}$ and $\mathfrak{A}$ were valuable reference functors. As was to be expected, the A-description turned out to coincide with the F-description by the means of a natural isomorphism under certain topological restrictions, namely for $\mathbf{M} \in \mathbf{Loc}$

such that $H^1_{\mathrm{dR}}(M;\mathbb{K}) = H^2_{\mathrm{dR}}(M;\mathbb{K}) = H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) = 0$. As a bit of a surprise, we found however that computing colimits proved to be a lot trickier than in the F-decription. On abstract categorical grounds, the colimits for the restrictions of $\mathcal{A}$ and $\mathfrak{A}$ to $\mathrm{loc}^q_{\mathbf{M}}$ turned out to be the same as the colimits for the restrictions of $\mathcal{F}$ and $\mathfrak{F}$ to $\mathrm{loc}^q_{\mathbf{M}}$ whenever $q \subseteq \mathbb{N} \setminus \{0\}$ with $1, 2, m-2 \in q$ or $q =$ ©. Thus, the universal objects of these colimits differed significantly from $\mathcal{A}\mathbf{M}$ and $\mathfrak{A}\mathbf{M}$ whenever $H^1_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$ or $H^2_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$ or $H^{m-2}_{\mathrm{dR}}(M;\mathbb{K}) \neq 0$, and could not be expressed in a closed form by just considering coclosed smooth differential 1-forms. This was due to the fact that contributions from the V-description of the free Maxwell field are picked up if colimit constructions are performed in the A-description. The V-description is essentially a description of the free Maxwell field in terms of a smooth differential 3-form potential $V \in \Omega^3(M;\mathbb{K})$ such that $\delta V = F$, opposed to the vector potential $A \in \Omega^1(M;\mathbb{K})$ which satisfies $dA = F$. The V-description is related to electromagnetic duality and basically arises from interchanging the role of the electric field (resp. electric charges) with the magnetic field (resp. magnetic charges); both A- and V-description are dealt with on an equal footing in the F-description. For $\mathbf{M} \in \mathbf{Loc}$ and $q \subseteq \mathbb{N} \setminus \{0\}$ with $1 \notin q$, $2 \notin q$ or $m-2 \notin q$, we did not succeed in obtaining closed expressions for the colimits of $\mathcal{A}$ and $\mathfrak{A}$ restricted to $\mathrm{loc}^q_{\mathbf{M}}$. We suspect that there is none besides in terms of generators and relations.

C. J. Isham's twisted quantum fields

In order to understand twisted quantum fields [Ish78b; AI79b] from the perspective of algebraic and locally covariant quantum field theory, we introduced an abstract categorical framework, which allowed us, more generally, to consider twisted variants of generic locally covariant theories, not necessarily referring to traditional twisted (quantum) fields. We argued that C.J. Fewster's ideas [Few13] on the automorphisms of a locally covariant theory and their interpretation as the global gauge group of the theory play the key role in this general formalism, in the same way Lie groups play the key role for smooth principal bundles.

Adopting the general scheme to locally covariant theories $F : \mathbf{Loc} \longrightarrow \mathbf{Phys}$, we infered that we can only sensibly talk about twisted variants for F on single $\mathbf{Loc}$-objects. Hence, we needed to restrict our attention to restrictions of F to single $\mathbf{Loc}$-objects $\mathbf{M}$, which resulted in the functors $U : (K_s \downarrow \mathbf{M}) \xrightarrow{P_{\mathbf{M}}} \mathbf{Loc}_s \xrightarrow{K_s} \mathbf{Loc} \xrightarrow{F} \mathbf{Phys}$, $\mathbf{M} \in \mathbf{Loc}$. To make sure that we were really considering only elements of the global gauge group of the theory, which are the automorphisms of F, $\mathrm{Aut}\,F$, or a suitable subcollection thereof, we regarded for our general scheme only automorphisms $\varepsilon : U \xrightarrow{\dot\sim} U$ of the form $\varepsilon_{\mathbf{A} \xrightarrow{f} \mathbf{M}} = \eta_{\mathbf{A}}$ for all $\mathbf{A} \xrightarrow{f} \mathbf{M} \in (K_s \downarrow \mathbf{M})$, where $\eta : F \xrightarrow{\dot\sim} F$. We showed that this naturally leads to the classification of twisted variants by the isomorphism classes of *flat* smooth principal bundles and to a relatively simple construction theorem. This was different to C.J. Isham's classification of twisted quantum fields by the isomorphism classes of smooth principal bundles, resulting, in principle, in more and even entirely new twisted quantum field theories which have been overlooked before.

We gave some examples for the classification of twisted variants, which were motivated from free scalar field theory, the theory of the free Maxwell field and the theory of the free Dirac field. To be more specific, we took the global gauge group to be $G = \mathbb{R}^n$, $n = 1, 2, \ldots$, ($\Longrightarrow$ shift-twisted free and minimally coupled real scalar fields), $G = \mathrm{SL}(2;\mathbb{C})$ ($\Longrightarrow$ twisted free Dirac field) and $G = \mathrm{U}(1)$ ($\Longrightarrow$ twisted free Maxwell field). As far as we know, none of the resulting twisted (quantum) field theories and their properties have been investigated yet. In particular, a twisted free Dirac field is entirely new, due to our classification of twisted variants by the isomorphism classes of flat smooth principal bundles. According to C.J. Isham's classification by the isomorphism classes of smooth principal bundles, there were no twisted free Dirac fields, meaning that all possible twists had been credited to inequivalent spin-frame projections previously [Ish78a; DHI79] and not to non-trivial spinor and cospinor bundles [in the sense of flat smooth vector bundles with structure group $\mathrm{SL}(2;\mathbb{C})$].

As a concrete example for a twisted quantum field theory, we constructed $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields, that is, twisted variants of multiple free and minimally coupled real scalar fields of the same mass, in the spirit of [Ish78b; AI79b], and demonstrated that this example fits nicely into the general scheme of twisted variants for locally covariant theories. In doing so, we observed that the universal algebra of the local unital *-algebras of the smeared quantum field, which are used in the descripton of twisted quantum fields according to the general scheme, is precisely the global unital *-algebra of the smeared quantum field used in the description of twisted quantum fields according to C.J. Isham. Using left Kan extensions, we also studied some further properties of $\mathrm{O}(n)$-twisted free and minimally coupled real scalar fields. Namely, we showed that the time-slice axiom is obeyed and on this basis, we computed the relative Cauchy evolution and the classical stress-energy-momentum tensor. We even began to compute the dynamical net, which we expect to coincide with the kinematical net. For mass = 0, this is an entirely new aspect since the free and minimally coupled massless real scalar field is known to fail dynamical locality [FV12b]. This failure is due to the constant solutions of the homogeneous massless Klein-Gordon equation, which may be avoided by the twisted variants (a smooth vector bundle is trivial if and only if it allows for a non-zero constant cross-section). A future aim is therefore, of course, to complete the computation of the dynamical net and to establish dynamical locality.

At this point, it should be stressed that we have only dealt with algebraic aspects of twisted (quantum) field theories and, more generally, twisted variants of generic locally covariant theories. We thus do not claim that the general scheme has the final word to say in the matter of twisted (quantum) fields and yields a complete picture. For example, it is still conceivable that a quantum field theory is algebraically, i.e. according to our general scheme, a twisted variant of another quantum field theory but as soon as states are included they become equivalent, i.e. the same quantum field theory according to our general scheme. It is also possible that due to taking states into consideration, the gauge group might need to be enlarged or reduced. Accordingly, it is very much conceivable that our general scheme will be subjected to modifications once more examples of twisted quantum field theories and discussions of their states are available in the future.

An opportunity, we have missed out on, was suggested to us by K. Fredenhagen and C.J. Fewster many times. Basically, since the twisted and the untwisted quantum field theory share the same observables, the observables should "*know*" about all possible twisted variants and it should be possible to reconstruct the twisted quantum field theory from the observables analogue to [Fre94, Sec.II.1]. In the light of the Doplicher-Haag-Roberts analysis of superselection sectors, which achieved a reconstruction of the field algebra from the observable algebra and some of whose concepts we borrowed for our general scheme, this seems more than plausible.

Another important task for the future is to make contact, if at all possible, to the topological superselection sectors of [BR09], which had been C.J. Fewster's original motivation to look into twisted quantum fields, and to understand them from the point of view of algebraic and locally covariant quantum field theory. Indeed, ideas stemming from the Doplicher-Haag-Roberts analysis of superselection sectors had been prominent in the development of our general scheme for twisted variants of locally covarint theories and there appear to be some structural similarities with [BR09], in particular with the structures elaborated in [RRV09], namely locally constant bundles. Hence, it is conceivable that a connection exists and that the general scheme needs to be altered to establish this connection.

FP-states for the quantised free massive Dirac field

In the last part of the thesis, we took a detour from the general scheme and without any reference to category theory or locally covariant quantum field theory at all, we constructed a family of new Hadamard states for the quantised free massive Dirac field on 4-dimensional, oriented and globally hyperbolic ultrastatic slabs with compact spatial section. Utilising a recent description [FR14a] of F. Finster's fermionic projector

[Fin98; Fin06] and the methods supplied by [Ara70], we defined a gauge invariant, pure and quasifree state, the unsoftened FP-state, on the C^*-completion of the self-dual CAR-algebra. Following the analysis of [FV12c; FV13] of the SJ-state for the quantised free and minimally coupled real scalar field, we compared the unsoftened FP-state with a reference Hadamard state and asserted that it can almost always be ruled out that the unsoftened FP-state is Hadamard. We also determined that the unsoftened FP-state features infinite quantum fluctuations, e.g. the quantum fluctuations of the normal ordered energy density are infinite. Inspired by the Brum-Fredenhagen modification of the the SJ-state [BF14], we modified the construction of the unsoftened FP-state by the means of non-negative integrable functions on the real line, thus yielding a whole family of FP-states, where the unsoftened FP-state is obtained by taking the characteristic function of the time interval of the slab. By taking compactly supported non-negative smooth functions, we obtained the softened FP-states, which turned out to be always Hadamard, applying again the comparism test with a reference Hadamard state.

We would like to emphasise on the close resemblance with the analysis of the SJ-state for the quantised free and minimally coupled real scalar field in [FV12c; FV13; BF14]. Of course, these references have served as a model for our analysis of the FP-states. Comparing our argument for the failure of the unsoftened FP-state to be Hadamard in general, which was carried over from [FV12c], with the failure of the SJ-state to be Hadamard in general, the unsoftened FP-state is as "*badly*" behaved regarding the Hadamard property as the SJ-state. However, in comparism with the Brum-Fredenhagen modification of the SJ-state [BF14], it can be said that the unsoftened FP-state is "*better*" behaved than the SJ-state because in its modification yielding Hadamard state, a compactly supported non-negative smooth function is deployed only once in our construction while [BF14] employs a compactly supported smooth function at two places in an essential way (compare (7.81) with [BF14, (19)]).

Like [BF14], we have not investigated any of the concrete physical properties of the softened FP-states other than the Hadamard condition, which we must leave as a task for the future. It would also be important to extend our constructions and results, if possible, to other globally hyperbolic spacetimes of "*finite lifetime*" which are not ultrastatic slabs or possess non-compact Cauchy surfaces. Likewise, extracting states from the description of the fermionic projector for globally hyperbolic spacetimes of "*infinite lifetime*" given in [FR14b] and discussing their properties, is a task worth picking up in the future. In the case of ultrastatic spacetimes, it is already known that the FP-state will be the standard vacuum state, hence Hadamard, due to [FR14b, Thm.5.1].

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