Behavioural Biases and Evolutionary Dynamics in an Agent-Based Financial Market

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Submitted in accordance with
the requirements for the degree of

Doctor of Philosophy

University of Leeds

Leeds University Business School

Centre for Advanced Studies in Finance

March, 2014

Acknowledgements

First and foremost, I would like to take this opportunity to express my sincere appreciation to all who have helped me in any respect during the preparation of this thesis.

I owe my deepest gratitude to my lead supervisor, Prof. Klaus Reiner Schenk-Hoppé for encouraging me to pursue a doctoral degree when I was a MSc Student and for his invaluable help throughout my PhD study. This thesis could not be possible without his encouragement, guidance and support.

I would like to extend my deepest gratitude to my co-supervisor, Dr. Jan Palczewski whose enlightening lectures at my master level, constructive criticisms and extensive discussions around my PhD work have been of great value for me.

Special thanks go to my colleagues and administrative staff from the Centre for Advanced Studies in Finance at Leeds University Business School. I truly appreciate the continuous support and caring assistance you provided. I thank the participants at LUBS Annual Doctoral Conference 2011 for providing useful comments at an early stage of my PhD study.

I would like to gratefully acknowledge the valuable feedback received during my visit to the Hausdorff Research Institute for Mathematics at the University of Bonn in the framework of the Trimester Program Stochastic Dynamics in Economics and Finance. Many thanks to the program participants who discussed with me some possible improvements of this thesis. Particular appreciation goes to Prof. Rabah Amir, Prof. Igor Evstigneev and Prof. Klaus Reiner Schenk-Hoppé for inviting me to the Trimester Program.

I am sincerely grateful to Prof. Thorsten Hens for inviting me to the Department of Banking and Finance at the University of Zurich to do a research project which is closely related to my PhD study. His suggestions on my work and future research directions are gratefully acknowledged. I also thank Sabine Elmiger and Anastasiia Sokko for the time and effort they have devoted to working with me in Zurich.

Finally, I dedicate this thesis to my parents, Ming Zhang and Weixing Wang, and my wife, Xue Bai for their love, endless support and encouragement. Special thanks go to Xue for her patience, understanding, care and love.

Abstract

This research is devoted to the study of financial market dynamics in a framework which combines agent-based modelling and concepts from behavioural finance. The thesis explores, in an agent-based financial market model, the interlinkage between investor heterogeneity, bounded rationality, behavioural biases and the aggregate market dynamics.

We develop a dynamic equilibrium model of a financial market in the presence of heterogeneous, boundedly rational investors. The model combines a performance-driven strategy-switching mechanism of an adaptive belief system (Brock and Hommes, 1998) and an evolutionary finance model (Evstigneev, Hens and Schenk-Hoppé, 2011). A key feature of this new model is that it contains a combination of *passive* and *active* learning dynamics. Passive learning refers to the market force by which wealth accumulates on investment strategies which have done relatively well. Active learning refers to the switching behaviour by which investors actively move their wealth into strategies which have performed well in the recent or distant past. This thesis extends the literature by examining the joint effect of passive and active learning in relation to the evolutionary dynamics of financial markets.

By drawing in concepts from behavioural finance, we focus on the micro-level modelling of various heuristics and behavioural biases which may affect investors' active learning and financial forecasting, such as overconfidence, recency bias, sentiment, etc. We quantify the macro-level market impact of these behavioural elements and study the evolutionary prospects of market dynamics.

We show that the interaction between passive and active learning is crucial to understanding the market selection of dominant strategy or the survival of different strategies. Investors' bounded rationality and behavioural biases in active learning and financial forecasting play an important role in shaping the market dynamics. Our findings point to the causes of the persistence of market inefficiencies and a variety of stylised facts of financial market. The added value of drawing together agent-based modelling and behavioural finance on the study of financial markets dynamics is demonstrated.

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Chapter 1

Introduction

This thesis introduces a framework that combines agent-based modelling and concepts from behavioural finance to study the dynamics of financial markets. We develop an agent-based financial market model and use it to explore the interlinkage between investor heterogeneity, bounded rationality, behavioural biases and the aggregate market dynamics. The focus is on quantifying the effects of a number of important behavioural biases documented in behavioural finance. The goal of this research is two-fold. First, to contribute to the behavioural finance literature by providing insights for the macro-level market impact and evolutionary prospects of the presence of individual investors with various behavioural biases. Second, to contribute to explanatory power of agent-based modelling of financial markets by drawing in concepts from behavioural finance.

1.1 Background

The complexity of financial market presents a big challenge to the study of market behaviour and dynamics. Existing financial market theories and main approaches to coping with the analysis of financial markets diversify into several paradigms, by and large, from the traditional finance approach based on a representative, fully rational agent and market efficiency (e.g. Muth, 1961 and Fama, 1970, Lucas, 1972) to the behavioural finance approach based on the two pillars of *limits to arbitrage* and *investor psychology* (see Barberis and Thaler, 2003 and Shiller, 2003), and the agent-based approach based on financial market models with interacting groups of heterogeneous, boundedly rational agents (see LeBaron, 2006a and Hommes, 2006).

Traditional finance rests on normative theories and models of the behaviour of market participants and market dynamics. Traditional finance assumes that the mass of market participants can be modelled by a representative economic actor (agent) whose behaviour is narrowly defined as fully rational. Rational behaviour have two related but different aspects: *optimisation* and *rational expectations* (Sargent, 1993). Optimisation refers to the behaviour that individuals' financial decisions are derived from maximising their objectives (expected utility or profit), given his or her preference and constraints, adheres to the Subjective Expected Utility Theory (Savage, 1954). Rational expectations, as commented by Sargent (1993), assume not only individual rationality but also *consistent beliefs* according to which beliefs are perfectly consistent with realisations (i.e. a rational agent's expectation equals the true statistical expected value). It requires that the rational agent is able to collect all information including others' decisions, and process new information correctly as described by Bayes' law. Moreover, rational agent approach also involves the assumption of homogeneity of expectations, for instance, in the Capital Asset Pricing Model introduced in Sharpe (1964) and Lintner (1965).

Economic and finance studies have long been dominated by rational agent approaches despite the fact that the assumptions of homogeneity and rationality of investors have been heavily attacked due to lack of realism. Supporters of rational agent approaches often use the so-called "*as if*" argument by Friedman (1953) as a defense against this criticism. On the one hand, Milton Friedman argues that it is not the case that actual investors are fully rational. Instead, investors act "as if" rational. On the other hand, the market selection hypothesis popularised in economics and finance by Alchian (1950) and Friedman (1953) argues that only rational traders can survive in the process of market selection and determine asset prices. Markets punish irrational behaviour (e.g. traders who do not optimise) and eventually eliminate their impact on asset prices. The market selection process leads to the same outcome as the case where agents act as if they were rational. This point of view is one of the major arguments behind the rational agent approach and efficient market hypothesis. Furthermore, Friedman argues that whether a theory is realistic "enough" should be assessed by its predictions not assumptions.

However, in the past few decades, empirical and experimental studies have documented a number of striking discrepancies between real markets and traditional models and their theories in terms of both model predictions and assumptions. For example, the existence of the so-called market anomalies (e.g. Keim, 1988) and stylised facts (e.g. Cont, 2001) which characterise the dynamics of financial markets but are not reconcilable with the traditional finance paradigm. A detailed review of the rational agent approach and corresponding issues and debates will be provided in Chapter 2. Those issues and debates provided a major motivation for the emergence and development of other approaches to study financial markets.

In contrast to traditional finance, behavioural finance is built on descriptive theories of financial markets and their participants. The main difference between the two approaches is that the former focuses on the modelling of what investors should do, while the latter emphasises what investors really do. According to Barberis and Thaler (2003) and Shiller (2003), behavioural finance has two main building blocks: limits to arbitrage and investor psychology. On the one hand, limits to arbitrage argues that it can be difficult for rational investors to correct a mispricing caused by less rational investors through a process known as arbitrage. When a mispricing occurs, arbitrage can be too costly and risky, thereby allowing the mispricing to persist for a longer period.

On the other hand, the study of investor psychology uses laboratory experiments of human subject to discover and explain phenomena that are inconsistent with the narrowly defined rational behaviour in traditional finance paradigm. Over the past few decades, the behavioural finance literature has documented well a range of heuristics and behavioural biases which have substantial impact on investors' decision making but contradict to the assumption of rationality in traditional finance (see Shefrin (2000) and Barberis and Thaler (2003) for lists of known heuristics and biases that can arise in investors' decision making). The goal is not only to identify contradictive behaviour to economic rationality, but also to characterise investor's behaviour as accurately as possible.

The theory that investors' behaviour can be described by simple heuristics and biases which may potentially lead to market anomalies, has been proposed in behavioural finance. However, an important issue is that those heuristics and behavioural biases are found through laboratory experiments of individuals at the micro level. As commented by Rubinstein (2001) and LeBaron (2006b), how those behavioural quirks of economic actors affect the macro phenomena or whether they will appear at the macro are difficult questions and they are often ignored in the behavioural finance literature. Moreover, behavioural finance rarely addresses the evolution prospects of the presence of investors with heuristics and biases. Whether and how those commonly exhibited heuristics and biases impact the long-run market dynamics including investor adaptation and the outcome of market selection is still an open question in the behavioural finance literature. The answer for this question points to the central debate between traditional and behavioural finance approaches, and it is important for understanding the dynamics of financial markets.

Lo (2004) proposed an adaptive markets hypothesis arguing that market efficiency and behavioural alternatives may be reconcilable after applying the principles of evolution — competition, adaptation, and natural selection — to financial interactions. The author argued that most counterexamples to economic rationality, such as loss aversion, overconfidence, overreaction, mental accounting, and other behavioural biases, are consistent with an evolutionary model of individuals adapting to a changing environment via simple heuristics. Lo (2004, p.24) states that: *"Even at this early stage, though, it seems clear that an evolutionary framework is able to reconcile many of the apparent contradictions between efficient markets and behavioural exceptions. The former may be viewed as the steady-state limit of a population with constant environmental conditions, and the* *latter involves specific adaptations of certain groups that may or may not persist, depending on the particular evolutionary paths that the economy experiences."*

The framework of adaptive markets hypothesis is in its infancy, and it still lacks prediction power. However, this framework theoretically highlighted the importance of evolutionary dynamics on resolving conflict between traditional and behavioural finance, and on understanding the behaviour of financial markets and market participants.

During the past two decades, another influential approach emerged to tackle the complexity of financial markets: agent-based modelling of financial markets. This approach rests on the modelling of financial interactions of market participants with a view to studying the emergent properties generated from these interactions. In agent-based models, financial markets are viewed as complex evolving systems with interacting groups of learning, boundedly rational, heterogeneous agents using rule of thumb strategies (LeBaron, 2006a). Agents learn the mechanisms governing the economy over time rather than having acquired them fully before joining the market. Dynamics and emergent properties of the market are endogenously generated by the model itself as the result of interactions of market participants.

The literature on agent-based financial market models explores the link between agents' bounded rationality, heterogeneity, learning and market dynamics. Bounded rationality, which is originally proposed by Simon (1957), refers to the cognitive limitations of agents including limitations of both their knowledge of market environment and computational capacity. Herbert Simon argued that, because of these limitations, agents may seek a greater simplified but satisfactory investment strategy rather than performing optimisation in marking investment decisions.

In the literature on agent-based financial market models, as reviewed by

LeBaron (2006a) and Hommes (2006)[1], the heterogeneity of agents is commonly studied in combination with bounded rationality, for example, via modelling different rule of thumb investment styles (e.g. fundamental analysis and technical trading), different memory spans (observation horizons) in forming expectations, and different risk attitudes. Learning is studied by modelling the process that agents adaptively adjust their strategies or expectation rules. This usually involves the modelling of strategy-switching behaviour of agents and software agents using evolving strategies which are implemented by genetic algorithm or genetic programming. Agent-based financial market models have the goals to study the causes of instabilities of financial markets, market selection of investment strategies, and to reproduce and explain stylised facts of financial time series.

1.2 Motivation

A main feature of agent-based modelling of financial markets is that this approach offers the possibility to transparently model the link between the micro-level behaviour of investors and the aggregate market dynamics. In agent-based models, the aggregate market phenomena can be studied and explained through analysing agents' behaviour which are modelled at the micro level. Such an approach is especially suitable for studying and testing the macro-level impact of heuristics and behavioural biases documented in the behavioural finance literature. As mentioned by LeBaron (2006b), agent-based frameworks can be used as a testbed for drawing in behavioural results in both experimental and micro level financial markets. This helps to make sense of which aspects of individual psychology remain relevant at the macro level.

Moreover, agent-based modelling approach is also suitable to study the long-run prospects of the presence of individual investors who exhibit various heuristics and biases. An essential feature of agent-based modelling approach is that it addresses the dynamics of financial markets via an evolutionary perspective where

[1]LeBaron (2006a) reviews computational based financial market models, while Hommes (2006) surveys models which are constructed at least to be partially analytical tractable.

the terms of competition, learning, adaptation and market selection apply. This feature is consistent with the suggestion by Lo (2004) on using evolutionary dynamics to study financial markets.

According to LeBaron (2011), in the literature on agent-based financial market models, there are two commonly used principles of methodology on the study of evolutionary aspects of financial markets: the *passive* and *active* learning. Passive learning is similar to the Friedman's type of market selection, that is, evolutionary forces operating through wealth dynamics. Wealth accumulates on investment strategies which have done relatively well. The relative wealth of good investment strategies grows faster than those weaker strategies. Therefore, good strategies eventually dominate the market and wipe out weaker strategies. Active learning refers to the learning dynamics by which agents actively choose or switch among a set of fixed or evolving strategies, with some well defined objective functions in mind.

In the literature on agent-based financial market models, these are two strands of influential research, *evolutionary finance* and *adaptive belief systems*, which may represent well these two types of learning mechanisms. We include here a short introduction[2] to these two strands of research to illustrate their ideas on modelling financial markets and corresponding issues. These ideas and issues provide motivations and inspirations for this research. In addition, these two approaches form the basis of the financial market model presented in this thesis.

Evolutionary finance (see e.g. the survey by Evstigneev, Hens and Schenk-Hoppé, 2009) views financial markets as a heterogeneous population of frequently interacting portfolio strategies in competition for market capital. Market dynamics are driven by mutual feedback between the evolution of asset prices and the wealth managed by each investment strategy. Investment strategies which possess more relative wealth have greater impact on the determination of asset prices.

[2] A more detailed review of evolutionary finance and adaptive belief systems is presented in Chapter 2.

Financial markets select good investment strategies in the sense that their relative wealth grows faster than other competitive strategies. Good investment strategies eventually survive in market selection and weak strategies die out.

Evolutionary finance maintains a large degree of freedom on the modelling of investment strategies. The emphasis of this approach is on descriptive modelling of agents, which shuns any notion of utility maximisation. The main goal of evolutionary finance is to provide insights for the market selection of successful investment strategies, especially within a specific set of strategies. Its application aims to contribute to the portfolio choice of investors and to the valuation of financial assets. However, a limitation of evolutionary finance models is that they do not allow investors to actively switch among different investment strategies. The models exhibit pure passive learning dynamics.

Brock and Hommes (1997, 1998) proposed a notion of Adaptive Belief System (ABS, thereafter) to study the dynamics of financial markets. The main idea behind the ABS is that agents exhibit "*rational animal spirits*" which can be summarised as the following: i) agents can actively choose, at each date, from a finite set of different beliefs (expectation rule) of the future price of a risky asset; ii) belief selection is based on a performance measure such as past realised profits; iii) the selection is bounded rational in the sense that not all of the agents, but most of them, choose the belief which has the best past performance.

One key feature of ABS is that it captures adaptive-belief-updating of agents. The market behaviour including the process of market selection is driven by this adaptive-belief-updating based on the concept of rational animal spirits. However, a limitation of ABS is that it is unable to capture the mutual feedback between the price and wealth. Agents in ABS are assumed to have constant absolute risk aversion (CARA) utility which causes that the asset price evolves independently from agents' wealth. Therefore, ABS exhibits pure active learning dynamics.

LeBaron (2011) noted that, both passive and active learning have been extensively studied in agent-based literature in relation to agent adaptation and market selection (e.g. evolutionary finance and adaptive belief systems). However, studies of the combination of the two learning mechanisms have been sporadic. Since the combined learning mechanism may be more relevant to what we may observe in real markets, Blake LeBaron suggested that future research should take a combined learning mechanism into account in order to build a more realistic model, and more importantly, to study the interaction between the two different types of learning mechanisms.

Passive learning can be viewed as the "natural selection" of financial markets, while active learning involves the "subjective selection" of agents where heuristics and behavioural biases may become more relevant. For this reason, the subjective selection of agents may be consistent with or against to the natural selection of the market. Whether and how agents' subjective selection affects the natural selection of market is an important question since it points to the outcome of the survival of investment strategies and the long-run market dynamics, especially when heuristics and behavioural biases are involved. However, the answer for this question has rarely been explored in both agent-based and behavioural finance literature.

LeBaron (2006a, p.1128) states that: *"It is important to note that agent-based technologies are well suited for testing behavioural theories. They can answer two key questions that should be asked of any behavioural structure. First, how well do behavioural biases hold up under aggregation, and second which types of biases will survive in a coevolutionary struggle against others. Therefore, the connections between agent-based approaches and behavioural approaches will probably become more intertwined as both fields progress."* Although agent-based modelling can be used as a powerful tool to study concepts from behaviour finance as mentioned by LeBaron, existing contributions rarely addressed the confluence between heuristics-and-biases literature in behavioural finance and agent-based modelling.

So far, there are only very few previous contributions which explicitly studied the market impact of heuristics and behavioural biases. Examples of these studies are Lux and Marchesi (1999, 2000), Takahashi and Terano (2003) and Lovric et al. (2009). Lux and Marchesi (1999, 2000) studied social interaction between investors and herding. Takahashi and Terano (2003) focused on investors' overconfidence and loss aversion. Lovric et al. (2009) investigated the effect of investors' sentiment in expectation. However, none of these approaches used both passive and active learning to study the long-run prospects of these heuristics and biases.

1.3 Research Questions

Based on the issues and motivations mentioned above, this thesis aims to contribute to the confluence between agent-based modelling of financial markets and behavioural finance. We strive to develop an agent-based financial market model which combines both passive and active learning to study concepts from behavioural finance. This research has the objectives: i) to explore the interlinkage between investor heterogeneity, bounded rationality, behavioural biases and the aggregate market dynamics; ii) to provide insights for understanding the interaction of passive and active learning dynamics, especially when behavioural biases are relevant; iii) to study the macro-level market impact and the long-run prospects of some important heuristics and behavioural biases documented in behavioural finance.

In order to achieve these research objectives, we consider the following research questions.

Research Question 1

What behavioural aspects of investors may potentially and substantially impact their financial decisions and the dynamics of financial markets?

The complexity of financial markets stems from the mutual dependence between trading activities of market participants and market reactions (such as price dynamics) in response to market participants' trading behaviour. Understanding behavioural factors which affect market participants' trading activities is important for studying the dynamics of financial markets. The answer for this research question helps us to identify important behavioural factors of market participants, through which we can take these behavioural factors into account when constructing the financial market model.

In the past few decades, the behavioural finance literature has made a substantial progress on studying and characterising investors' behaviour. The literature on agent-based financial market models has also identified a number of important behavioural factors (regarding investor heterogeneity and bounded rationality) which may have remarkable impact on the market dynamics. In order to find the answer for research question 1, we conduct a literature survey (presented in Chapter 2) on behavioural finance and agent-based financial market models. We list a number of well-known heuristics and biases of investors documented in behavioural finance literature. We also review and discuss some important behavioural phenomena that have been studied in agent-based literature. The goal is to extract behavioural elements which have been identified to be important in shaping market dynamics.

Research Question 2

How to develop an agent-based financial market model which combines passive and active learning and maintains a large degree of freedom on modelling investors' behaviour?

As discussed previously, the combination of passive and active learning is important for studying investor adaptation, market selection of investment strategies, and the long-run market behaviour. Understanding the interaction between passive and active learning is crucial for the study of evolutionary prospects of the presence of investors with heuristics and biases. In order to study the effects of various heuristics and biases of investors, we need a framework which maintains a large degree of freedom on modelling investor behaviour so that different heuristics and biases can be added and implemented.

In the literature on agent-based financial market models, evolutionary finance models suit our needs for the degree of freedom on modelling of investors' behaviour. However, evolutionary finance models are based on pure passive learning dynamics. In contrast, ABS models have the advantage on characterising pure active learning dynamics. Based on these considerations, we develop an agent-based model which combines the strategy-switching mechanism of an ABS model (Brock and Hommes, 1997) and an evolutionary finance model (Evstigneev, Hens and Schenk-Hoppé, 2011).

This new model inherits the advantages of evolutionary finance approach: i) it maintains a large degree of freedoms on modelling of agents' behaviour — utility maximisation is not necessary; ii) multiple risky assets are allowed and increasing the number of risky assets is straightforward. The model also draws on the strength of ABS model by allowing investors to actively move their wealth among different investment strategies based on the past performance of each strategy. Moreover, our approach allows the coexistence of switching investors and non-switching investors. The main feature of this new model is that it captures the interaction between passive and active learning.

Research Question 3

How does investors' strategy-switching affect the price dynamics and the evolution of distribution of wealth managed by each investment strategy, especially when investor heterogeneity, bounded rationality and behaviour biases are associated with strategy-switching?

The research question points to the impact of the interaction between passive and active learning on the price dynamics of financial market. It addresses the role of investors' subjective selection of investment strategies in affecting the natural selection of by the market. Depending on behavioural aspects of investors, their subjective selections may not always be consistent with the natural selection of the market. This may affect the result of the survival of investment strategies leading to different market dynamics in the long-run.

To answer this research question, our approach allows investors to actively switch among different agent types such as the fundamentalist and trend follower. Investors bring (or take away) their wealth when they join (or leave) each agent type. We model a variety of behavioural factors regarding investor heterogeneity, bounded rationality, heuristics and behavioural biases which may impact investors' strategy-switching, such as differences of opinion of investors, better-than-average overconfidence, recency bias in performance evaluation, conservatism bias and herding. We conduct a series of simulation experiments to quantify the macro-level market impact of these behavioural phenomena associated with strategy-switching, and to study the long-run outcome for the survival of investment strategies and market dynamics.

Research Question 4

How do behavioural factors such as observation horizons, sentiment (optimism or pessimism), and recency bias affect investors' forecasting and the aggregate market dynamics?

This research question is related to the impact of investors' forecasting (expectations) about the future asset returns on the aggregate market dynamics. Previous contributions in the literature on agent-based financial market models (see heterogeneous agent models surveyed by Hommes, 2006) have shown that the mutual feedback between investors' expectations and the price dynamics is critical. Motivated by this finding, our research question 4 addresses the roles of some important behavioural factors such as observation horizons, sentiment (optimism or pessimism), and recency bias in affecting investors' expectations of asset returns and the price dynamics.

These behavioural factors are commonly studied in relation to the trend following behaviour, or equivalently, the positive feedback trading of investors. The trend following behaviour itself is often regarded as a manifestation of investor sentiment and recency bias in forecasting (see, e.g. De Long et al., 1990; Barberis, Shleifer, and Vishny, 1998). In both agent-based and behavioural finance literature, the presence of trend followers have been identified as an important source of instability and mispricing in financial markets. However, previous contributions which explicitly studied the impact of sentiment and recency bias on the trend following behaviour are rare.

In order to better understand the impact of observation horizons, sentiment and recency bias on the trend following behaviour, we focus on explicit micro-level modelling of these behavioural factors of trend followers and studying their macro-level impact on the price dynamics.

1.4 Outline of the Thesis

Chapter 2

The next chapter presents a literature survey which covers traditional finance with Market Selection Hypothesis and Efficient Market Hypothesis, behavioural finance with a list of heuristics and biases, and agent-based financial market models with an introduction to their market designs and a review of previous contributions. However, it is not our intention to provide a comprehensive review of each subject due to the vast body of the literature. The goal is to highlight important issues and central debates in economics and finance, and to give references to previous contributions which strive to tackle those issues and are closely related to our research by providing motivations, techniques, or inspirations.

Chapter 3

In Chapter 3, we develop a dynamic equilibrium model of a financial market in the presence of heterogeneous, boundedly rational agents. We explain how an evolutionary finance model and a strategy-switching mechanism in an ABS model are combined with focus on the modelling of various behavioural phenomena that are associated with investors' strategy-switching. We derive explicit solutions to the model in terms of asset prices and agents' wealth. A basic analysis of the existence and location(s) of steady state(s) is carried out. The aim is to investigate in our model the impact of some key parameters such as the risk-free rate of return and consumption rate on the aggregate economy. Since those parameters govern the expansion and shrinkage of the economy, they consequently affect the location(s) of the steady state(s) of the model. We present specifications for three agent types: fundamentalist, trend follower, and noise trader. Model dynamics will be explored by numerical simulations in the next two chapters.

Chapter 4

Based on the model presented in Chapter 3, this chapter explores numerically the impact of investors' strategy-switching behaviour and its related behavioural biases and on the aggregate market dynamics. A variety of behavioural phenomena in strategy-switching such as investor overconfidence, differences of opinion, recency bias in performance evaluation, conservatism bias and rational herding will be addressed. Our analysis focuses on investigating: i) the macro-level market impact and evolutionary prospects of these behavioural phenomena; ii) the interaction between passive and active learning and its outcome for the survival of investment strategies and long-run market dynamics. Our results point to new ideas on understanding the survival of investment strategies. Some persistent market phenomena at the macro level such as excess volatility, high trading volume and equity premium can be explained by investors' strategy-switching behaviour through tracing down to the micro-level foundation of investors' heuristic and behavioural biases.

Chapter 5

This chapter extends the model and analysis presented in previous two chapters to address the roles of observation horizons, sentiment (optimism and pessimism) and recency bias in affecting the trend followers' forecasting about future asset returns. Heterogeneous observations horizons of trend followers and switching among different observation horizons are allowed. We analyse numerically the impact of these behavioural elements on the trend followers' strategic behaviour as well as on the aggregate market dynamics. We offer a new idea on allowing behaviour aggregation via introducing and modelling a collective behaviour of the trend followers who have different observation horizons. This helps to understand the aggregate market dynamics such as the price dynamics, especially when the number of different observation horizons is large. The results documented in this chapter point to the causes of a number of stylised facts such as the absence of autocorrelation in asset returns, volatility clustering, negative skewness and excess kurtosis in return distribution.

Chapter 6

In the last chapter, we provide a summary and closing remarks of our approach on studying the dynamics of financial markets and our results. We introduce and discuss future research directions.

Chapter 2

Literature Review

2.1 Financial Market Theory

Dynamics of financial markets are characterised by a strong mutual dependence between the trading behaviours of market participants and the market environment in which trading behaviours are realised and evaluated. In order to understand the market dynamics, economic and finance theories face the central issue on how to model the behaviour of market participants.

The traditional approach in economics and finance rests on the assumption that the mass of investors can be considered as a representative, perfectly rational agent. Rational behaviour, by and large, can be summarised by two related but different concepts. The first addresses agents' decision making behaviour (e.g. the expected utility theory in Von Neumann and Morgenstern, 1944). Given an agent's risk preferences and constraints, decision rule is derived from maximising his or her expected utility, and adheres to the axioms of expected utility theory or at least Savage's notion of Subjective Expected Utility (Savage, 1954). The second concept is related to agents' expectations of future events (e.g. rational expectation hypothesis in Muth, 1961 and Lucas, 1972). Agents are assumed to be able to form *rational expectations* about future market outcomes, that is, as summarised by Hommes (2006): "*Beliefs are perfectly consistent with realisations and a rational agent does not make systematic forecasting errors. In a rational expectations equilibrium, forecasts of future variables coincide with the* *mathematical conditional expectations, given all relevant information.*" Based on the framework of a representative rational agent, seminal works including portfolio optimisation rules by Markowitz (1952) and Merton (1971), the static and intertemporal Capital Asset Pricing Model (CAPM) of Sharpe (1964), Lintner (1965), Mossin (1966) and Merton (1974), and the Efficient Market Hypotheses by Fama (1970) have long been a dominating paradigm in economics and finance.

Despite the dominance of this rational agent approach in the economic and finance literature, there had always been disagreement with the extreme assumption that economic agents behave rationally. Keynes (1936) argued that investors' sentiment and market psychology play an important role in financial markets. It is costly or impossible for investors to gather all relevant information to compute an objective measure of market fundamentals. Furthermore, Simon (1957) highlighted the fact of the limited capacity of human mind and computational abilities in gathering and analysing all aspects of a decision-marking problem in a complex environment. Due to these limitations, Simon argued that an "economic man" seeks a greater simplified but satisfactory solution rather than arriving at the optimal solution in decision-making process. Simon's view gave the birth to *bounded rationality*, and emphasised that modelling an economic man with bounded rationality rather than perfect rationality with optimal decision rules may be more accurate and realistic. This view was supported by studies of psychology laboratory experiments. For example, experimental evidence provided by Kahneman and Tversky (1973, 1979, 1986) had shown that human individuals often do not behave fully rational, possibly biased, in making investment decisions under uncertainty. Kahneman and Tversky documented that individual behaviour under uncertainty can best be described by simple heuristics and biases. Likewise, De Bondt (1998) pointed out that, over the past decades, psychologists and behavioural scientists have documented robust and systematic violations of principles of Expected Utility Theory, Bayesian Learning, and Rational Expectations - questioning their validity as a descriptive theory of decision making.

To clarify and and defend the assumption of agents' rational behaviour, Lucas (1978, p.1429) argued that "*As Muth (1961) made clear, this hypothesis (rational* *expectation) like utility maximisation is not 'behavioural': it does not describe the way of agents think about their environment, how they learn, process information, and so forth. It is rather a property likely to be (approximately) possessed by the* ***outcome*** *of this unspecified learning and adapting.*". Here, the "outcome" is related to the two important hypotheses in the finance literature, that is, the Market Selection Hypothesis of survival agent and the Efficient Market Hypothesis. These two hypotheses have long been the central subjects of debate in finance. Those ongoing debates played an important role in stimulating and motivating other approaches which are beyond the rational agent framework on studying financial markets, such as the behavioural finance and agent-based models of financial markets. Our research addresses the two cental subjects of debate by linking behavioural finance and agent-based financial market models.

2.1.1 Market Selection

The market selection hypothesis originated from the arguments documented by Alchian (1950) and Friedman (1953). These authors argued that, by and large, markets favour rational agents over irrational agents. Profit or utility maximising agents with correct beliefs will survive in the long run, while others who do not optimise will be driven out of the market. The evolutionary competitions would eventually lead to the same outcome as the case where agents act *as if* they were rational. The so-called "as if" argument in Friedman (1953), as being made clearly and prominently by Conlisk (1996) ("*The question is not whether people are unboundedly rational; of course they are not. The question is whether they act approximately as if unboundedly rational; they do.(p.683)*"), has long been used in economics and finance to defend the use of unrealistic assumptions of agents' rationality.

Alchian and Friedman's view of market selection highlighted the role of *optimisation* and *evolutionary forces* in wealth dynamics, and the consistency between the outcome of market selection and the assumption of rational agents, which is a central idea in support of the "as if" rationally. Based on this view, economists and financial experts can take a shortcut to the outcome by assuming rational agents from the beginning. Therefore, the question whether Friedman's hypothesis is valid becomes a crucial issue in justifying the rational agent approach, and more importantly in understanding the dynamics of financial markets.

Motivated by this issue, the process of market selection has been widely studied in the economics and finance literature, especially in models with heterogeneous agents. One direction of research on market selection in a heterogenous world is similar to Alchian and Friedman's idea, the process of market selection is studied based on evolutionary forces operating through wealth dynamics. Previous contributions, e.g. Sandroni (2000), Blume and Easley (1992, 2006) and the evolutionary finance literature surveyed in Evstigneev, Hens and Schenk-Hoppé (2009), are in line with this direction of research. These authors showed that the answer to Friedman's hypothesis is subject to conditions such as the cost of rational optimisation, completeness of the market, bounded or unbounded aggregate endowment, and so on. For example, in the presence of deliberation cost, survival logic may favour a cheap rule of thumb over a costly optimisation (Conlisk, 1996). Traders with correct beliefs survive in the long run, in a complete market model with bounded aggregate endowment (Blume and Easley, 2006).

In the finance literature, the market selection process and its effect have also been extensively studied with focuses on asset pricing and the outcome of agents' adaptive learning. Fama (1965) and Cootner (1964) argued that markets will select for rational investors, by which assets will eventually be priced efficiently (close to assets' intrinsic value). Similar arguments are agents can learn how to form rational expectations (e.g. Grossman and Stiglitz, 1976); forces of market selection can lead to the convergence to rational expectations equilibria (e.g. Bray, 1982 and Blume and Easley, 1998). These arguments point to another direction of research on the process of market selection in a heterogenous world. This direction focuses on the link between agents' adaptation, learning and the convergence to rational expectations equilibria.

The seminal work Brock and Hommes (1997) is a good example of this direction of research in the literature on agent-based financial market models. One novel idea of Brock and Hommes' approach is that the process of market selection is studied via agents' adaptive learning rather than evolutionary competition in wealth dynamics. Agents are assumed to have unbounded budgets. Agents' adaptation and learning are modelled via a belief updating process, that is, given a fixed set of beliefs (expectation rules), agents switch between different beliefs according to the past performance of each belief. The past performance of each belief is measured by the realised profit from the previous period of trade. The survival and extinction of each belief is measured by its population proportion. The authors modelled a scenario that a costly rational expectation rule versus a cheap naive expectation rule (which predicts tomorrow's price equals yesterday's price). They showed that when the intensity of switching is high, the cheap naive expectation rule will survive in market selection and chaotic price fluctuations will ensue.

Summarising, the market selection hypothesis played an important role in the emergence and development of heterogenous agent models. The two directions of research on the process of market selection revealed that the outcome of market selection is highly conditional.

2.1.2 Market Efficiency

The concept of market efficiency is another central issue of debate in the economics and finance literature. According to Fama (1970), an *efficient market*, known as *efficient market hypothesis*, is defined as a market in which prices fully reflect all available information. There are three versions of the hypothesis, and are labelled as "weak", "semi-strong" and "strong". The weak-form efficiency is concerned with the full reflection of all *past market information*. The semi-strong-form efficiency is concerned with the full reflection of *all public information*. The strong-form efficiency claims that price fully reflect *all information from public* *and private sources including hidden or inside information.*

The concept of rational expectations is the theoretical underpinning of the efficient market hypothesis. Since rational expectations require that agents exploit all relevant information to form their expectations about future market outcomes, efficient market hypothesis predicts that, prices are perfectly random and hence exhibit random walk behaviour. This view of market efficiency is interwind with random walk models (e.g. Malkiel, 1973). In the weak-form efficiency, future price changes cannot be predicted, and excess returns cannot be consistently archived from analysing the past information such as past prices. Similarly, the semi-strong-form efficiency implies that no one can consistently earn excess return based on public information. In the strong-form of efficiency, all prices in financial markets are correct and reflect market fundamentals. In this case, no one can consistently earn excess return.

However, a range of empirical studies led to the discussions of the so called anomalies and stylised facts characterising the dynamics of financial markets but are not reconcilable for the rational expectations and efficient market hypothesis paradigm. Shiller (1981) found that the movements in stock prices are much larger than movements in underlying economic fundamentals. Historical evidence stressed by Frankel and Froot (1986) and econometric test by West (1988) confirmed the existence of excess volatility in real financial market. Furthermore, other important stylised facts such as fat tails, excess kurtosis and negative/positive skewness in return distributions, volatility clustering, as well as some market anomalies including large price excursions from fundamentals, asset bubbles and equity premium are not well understood under the standard paradigm with rational agent and market efficiency. These issues appeal alternative approaches on studying financial markets.

2.1.3 Behavioural Finance

A possible reason behind the discrepancy between empirical findings and the predictions of traditional finance models may be attributed to the unrealistic assumption of rationality. Behavioural finance studies market anomalies by addressing the validity of the assumptions of rational decision-making and utility maximisation through laboratory experiments of individuals. As defined by Frankfurter and McGoun (2000, p.201): *"Behavioural finance, as a part of behavioural economics, is that branch of finance that, with the help of theories from other behavioural sciences, particularly psychology and sociology, tries to discover and explain phenomena inconsistent with the paradigm of the expected utility of wealth and narrowly defined rational behaviour. Behavioural economics is mostly experimental, using research methods that are rarely applied in the traditional, mainstream finance literature."*

According to Barberis and Thaler (2003), behavioural finance is built on two pillars: i) limits to arbitrage and ii) market psychology. In contrast to the efficient markets theory which believes that the mispricing of financial assets and its related arbitrage opportunities will be quickly exploited by rational traders leading to market efficiency, the concept of "limits to arbitrage" argues that arbitrage can be too costly, too risky, or simply impossible due to various constraints, so the market inefficiencies may persist for a longer period (Barberis and Thaler, 2003). Moreover, as reviewed by Rabin (1998) and Barberis and Thaler (2003), previous studies of market psychology in behavioural finance have documented extensive experimental evidence for departures from rational behaviour of investors. The aim of these studies is not only to find boundedly rational or irrational behaviour (of investors) which contradicts to the assumption of perfect rationality, but also to describe investors' behaviour as accurately as possible. The goal is to identify and characterise heuristics and biases which are commonly carried by investors in decision-making and study their effects.

In the past a few decades, the behavioural finance literature has documented a variety of psychological heuristics and behavioural biases which my have substantial impact on investors' decision-making and trading activities. Here, we list some important and well-known heuristics and biases. We describe and discuss briefly their definitions and effects. It is not our intention to give in this chapter a comprehensive review of all heuristics and biases studied in behavioural finance, but to explain and highlight those which are closely related to this research.

Loss Aversion

Loss aversion describes the phenomenon that individuals are physiologically more sensitive to potential losses than potential gains. Behavioural finance considers that investors are not always risk averse as suggested in the mainstream finance but loss averse. Tversky and Kahneman (1984) showed that individuals tend to strongly prefer avoiding losses to acquiring gains. Moreover, Tversky and Kahneman (1984, p.341) stated that: *"the psychophysics of value induce risk aversion in the domain of gains and risk seeking in the domain of losses."* Loss aversion is one of the central ideas behind the influential work of Prospect Theory proposed by Tversky and Kahneman (1979).

Representativeness and Availability

Kahneman and Tversky (1974) show that people rely on a heuristic of *representativeness* in forming subjective judgment. Representativeness refers to the phenomenon that: "*the subjective probability of an event, or a sample, is determined by the degree to which it: (i) is similar in essential characteristics to its parent population; and (ii) reflects the salient features of the process by which it is generated*" (Kahneman and Tversky, 1974, p.25). In other words, representativeness describes the tendency for people to categorise some events as typical or representative. This heuristic causes people to arrive at wrong judgements since the more representative events does not necessarily imply that they are more likely to happen. One example is that people judge the stock market changes as "bullish" or "bearish" market without paying attention to the likelihood that successive changes of price along one direction may rarely happen.

As shown in Kahneman and Tversky (1974), *availability* is another important heuristic according to which people assess the frequency of class or the probability of an event by how easy they can be brought to mind (e.g. how easy to be remembered or how easy it is to think of examples).

Overconfidence

In the literature of behavioural finance, there are two distinct manifestations of investor's overconfidence, *miscalibration* and *better-than-average effect*. Miscalibration refers to the tendency to overestimate the precision of one's information. For example, Shefrin (2000) states that when people are overconfident, they set overly narrow confidence bands in their predictions. Therefore, they get surprised more frequently than they anticipated. Similarly, Benos (1998), Caballé and Sákovics (2003), and Odean (1998) regard the phenomenon that investors underestimate the variance of a risky asset or overestimate its precision as a form of overconfidence. This type of overconfidence is known as miscalibration.

Different to miscalibration, a more general definition of overconfidence is that people overestimate their own capabilities, usually with respect to the average capability of others. For example, in the papers of Shiller (1999), Barberis and Thaler (2003), Hong and Stein (2003), and Glaser and Weber (2007), the phenomenon that people tend to judge themselves as better than others with respect to skills or information is regarded as a form of overconfidence. This is known as better-than-average overconfidence. Previous studies show that both types of overconfidence are able to cause high trading volume in financial markets. However, Glaser and Weber (2007) suggest that, even though widely used, miscalibration may not be the best proxy for overconfidence. Through an empirical study with combined psychometric measures of judgment biases (overconfidence scores) and field data (trading records), they could not relate measures of miscalibration to measures of trading volume, whereas they could do so with the better-than-average overconfidence.

Conservatism

Conservatism bias describes the phenomenon that people react conservatively to new information, and therefore they are too slow to change an established view. According to Ritter (2003, p.434): *"Conservatism suggests that when things* *change, people tend to be slow to pick up on the changes. In other words, they anchor on the ways things have normally been. When things change, people might underreact because of the conservatism bias. But if there is a long enough pattern, then they will adjust to it and possibly overreact, underweighting the long-term average.*"

Recency Bias

Recency bias refers to the tendency of investors to assign more importance to more recent observations compared to those farther in the past. Kahneman and Tversky (1973) find that people usually forecast future uncertain events by focusing on recent history and pay less attention to the possibility that such short history could be generated by chance.

These evidence of heuristics and biases which are obtained from laboratory experiments of individuals confirm the view of bounded rationality of Simon (1957). Simon's framework of bounded rationality and the heuristics and biases literature may be able to offer possible explanations for some empirical puzzles or anomalies which cannot be explained in traditional finance literature. However, as pointed out by Lo (2004), the evolutionary perspectives of financial markets in terms of competition, adaptation and natural selection are often ignored in Simon's framework and behavioural finance. The long-run prospects of the effects of investors' heuristics and biases have been rarely explored in the behavioural finance literature. Lo (2004) suggests that applying the principles of evolution to financial interaction may provide insights for understanding market dynamics as well as the effects of heuristics and biases.

Moreover, in the behavioural finance literature, heuristics and biases are found based on experiments of individuals at the micro level. How these behavioural quirks affect the dynamics of financial markets at the macro level or whether they will appear at the macro level are difficult questions. These important questions, as criticised by Rubinstein (2001) and LeBaron (2006b), are often ignored by many of behavioural finance literature. LeBaron (2006a, 2006b) pointed out that using agent-based financial market models to study concepts from behavioural finance may help to provide insights for understanding the macro-level market impact of heuristics and biases of individual investors.

2.2 Agent-Based Modelling of Financial Markets

The agent-based modelling approach serves itself as a bridge between disciplines of mathematics, economics, computer science, physics, psychology and many others (Axelrod, 1997). The field of agent-based modelling of financial markets brings multidisciplinary researchers to enrich our understandings of dynamics of financial markets. This branch of research is rooted in a vast body of literature with different taxonomies, for examples, agent-based computational economics (e.g. Tesfatsion, 2001, 2002, 2006) and its sub-discipline agent-based computational finance (e.g. LeBaron, 2006a), heterogenous agent models (e.g. Hommes, 2006), microscopic simulation (e.g. Levy, Levy, and Solomon, 2000), econophysics (e.g. Lux and Marchesi, 1999, 2000) and evolutionary finance (Evstigneev, Hens and Schenk-Hoppé, 2009). For convenience, we refer to models in this broad research area as agent-based financial market models.

The main idea behind this branch of research is that financial markets can be studied via models of artificial financial markets. In these models, financial markets are viewed as complex evolving systems with interacting groups of learning, boundedly rational, heterogeneous agents. Agents exhibit rule-governed behaviours by means they follow some simple "rule of thumb" that have been proven useful in the past to make their investment decisions (LeBaron, 2006a). In such an approach, the macro dynamics of the economy can be understood by analysing the micro interaction of agents, in which emergent properties arise endogenously rather than being imposed exogenously. This novel approach has also been known as "bottom-up" approach, and it has the intention to overcome the limitations of traditional approach which is restricted to the extreme assumption of rationality. As motivated in the previous section, issues of market selection of survival agents, efficient market and the interlinkage between agents' behaviour at the micro level and market phenomena at the macro level have been the central topics in the studies of agent-based models in economics and finance. Replicating and explaining a variety of stylised facts of financial markets is also an important objective for these agent-based models.

The main advantage of the agent-based approach is that it maintains a large degree of freedom on modelling financial markets and the behaviour of the market participants. As mentioned in Chan et al. (1999, p.6): "*Agent-based models can easily accommodate complex learning behaviour, asymmetric information, heterogeneous preferences, and ad hoc heuristics.*" However, the large degree of freedom of agent-based approach causes existing models to differ significantly in many aspects, such as the main tools which are used in the development and analysis of models, and market designs including the type and numbers of assets, agents' investment strategies and learning, time (discrete or continuous), price formation mechanisms, and so on. This in turn causes the existing contributions in the literature vary in market phenomena they explained, and the number of stylised facts they reproduced. In the remainder of this section, we sketch an overview of agent-based models in finance and economics with focus on introducing the main tools that are used to develop models, and different market designs.

2.2.1 Analytical and Computational Methods

In the vast body of contributions of the literature on agent-based financial market models, one can roughly recognise two groups of approaches. The first group consists of models which, at least to some extent, are analytical tractable. In this group, mathematics is the main tool in deriving models. Mathematics in combination of numerical tools are employed in analysing model dynamics. This group of models is extensively surveyed by Hommes (2006). In the second group, models are developed by using modern computer techniques. Analysis of the model dynamics relies on numerical simulations. See LeBaron (2006a) for extensive survey of those computational approaches.

Analytical models or partially analytical tractable models have the advantage that their dynamics can be at least partially studied by mathematics. Theoretical reasoning for the causes of the interesting model dynamics may be provided analytically. The basic idea of this type of modelling approach is that financial markets with interacting agents can be treated as a nonlinear dynamical system, in which dynamics can be studied by analysing the existence and location(s) of steady state(s) and investigating the stability conditions for the steady state(s). Bifurcation and chaos theories may be applied to help understand the resulting model dynamics. See examples of these approaches in Day and Huang (1990), Chiarella (1992), Lux (1995,1997,1998), Brock and Hommes (1997,1998), Chiarella and He (2001, 2002), Chiarella, Dieci and Gardini (2002, 2006), Gaunersdorfer and Hommes (2007), Evolutionary Finance models see survey in Evstigneev, Hens and Schenk-Hoppé (2009), and many others to quote only a few. However, a drawback of analytical models is that there are many limitations on what can be modelled, solved, or proven analytically.

Unlike analytical models, computational models offer a much higher degree of flexibility. The advantage of the computational models is attributed to the absence of restrictions on the side of assumptions. Computational models can be used to extend existing analytical models by inspecting the role of their assumptions, to develop new models that can capture more realistic elements. Dynamics can be characterised and understood by numerical simulations. This may overcome the limitations of analytical models which are restricted to mathematical tractability.

For example, agents' investment behaviour in analytical models is usually modelled by a simple investment strategy and it is assumed that the strategy stays fixed over time to ensure mathematical tractability. In contrast, computational models may use genetic algorithm and genetic programming to capture more complex behaviour of agents such as allowing investment strategies evolving over time. See seminal contributions e.g. Arthur, Holland, LeBaron, Palmer, and

Tayler (1997), LeBaron, Arthur, and Palmer (1999), Chen and Yeh (2001). However, as noted by LeBaron (2006a), the research questions in these computational approaches with genetic algorithms sometimes have more focus on developing and testing the genetic algorithms rather than focusing on explaining financial issues. Ghoulmie, Cont and Nadal (2005) also point out that although computational models have the strength in producing stylised facts in finance such as fat tail, excess volatility, and volatility clustering in return series, due to the complexity of these models it is not always clear which part of the model is responsible for generating those stylised facts.

2.2.2 Market Design

The design of the market is a central issue in agent-based modelling of financial markets since it determines the emergent properties that can be observed and studied. As surveyed by LeBaron (2006a), existing models in the agent-based literature differ in many aspects of market design. These design issues can be briefly categorised as assets, agents, learning, market mechanism, and time.

Asset

The assets which are traded by agents are one of the most important elements in building an artificial financial market. Models of agent-based financial markets have to deal with aspects of (i) type of assets, (ii) number of assets.

Regarding the type of assets, most of models assume that there are two types of assets that are available for trading, namely, a risk-free asset and a risky asset. The risk-free asset usually refers to cash in a bank account or a bond which pays a rate of interest. The risky asset sometimes refers to stock, currency or other forms of financial security. This kind of asset is risky in terms of their prices and payoffs. In the case of stock shares, the risky assets may pay dividends. The fundamental value of the risky asset can be modelled via an exogenous stochastic process (e.g. Farmer and Joshi, 2002) or through a discounted stream of dividends (e.g. Brock and Hommes, 1998).

Regarding the number of assets, the most commonly seen approach is that the market only consists of one risk-free and one risky asset. The single risky asset usually represent a stock index (like S&P 500 index). This market setting helps to simplify the market environment so that the aggregate dynamics such as dynamics of stock index can be studied conveniently. Moreover, the two-asset type of models are consistent with the so-called two-fund separation theorem (e.g. Tobin, 1958). Another approach is to allow multiple risky assets to be traded by agents. Studies of multi-asset models usually have the goal to explore agents' asset allocation problem with focus on the effect of the correlation between risky assets (Chiarella, Dieci and He, 2007), or the market selection of portfolio rules (Evstigneev, Hens and Schenk-Hoppé, 2009).

Agents and Learning

The design of agents plays an important role in the development of agent-based models in economics and finance since many important concepts such as heterogeneity, bounded rationality and learning are associated with the design of agents. According to the literature survey by LeBaron (2006a) and the more recent one Chen et al. (2012), the design of agents can be "labelled" as *N-type* and *autonomous*.

In the *N-type* agent design, a typical example is the so-called *fundamentalist-chartist* models, where agents are labelled as "fundamentalist" and "chartist". Fundamentalist refers to those investors whose trade are based on a perceived fundamental value of a risky asset. If the asset is overvalued (the price is above the fundamental value), a fundamentalist will take a short position. If the asset is undervalued (the price is below the fundamental value), a fundamentalist will take a long position. In contrast, the chartists are those investors whose trade are based on the "pattern" exhibited in past prices. One type of chartist is that investors try to extrapolate past movements of the price into the future, and they trade by following the trend identified in the past prices. This type of traders has also been known as trend followers. Another type of chartist is called contrarian traders who follow the opposite of the trend identified in the past prices. Besides the fundamentalist and the chartist, there is another commonly seen agent type which is called "noise trader" or "zero-intelligence trader". This type of agent basically trade randomly to provide liquidity and randomness to the markets.

The *N-type* agent design is usually concerned with static parameterised strategies. Agents' adaptive learning behaviour can be modelled at different levels. There is no adaptation in models with only zero-intelligence traders since they trade randomly, while a weak adaptation in models in which agents stick to those static parameterised strategies. Adaptive learning in the *N-type* agent design can be modelled by allowing agents to switch between a fixed set of static parameterised strategies. One good example along this line is the Adaptive Belief System which is proposed by Brock and Hommes (1997,1998).

The *autonomous* agent design is based on computer models. In contrast to the *N-type* agent design, the main idea of *autonomous* agent models is to allow agents discover investment strategies on their own rather than choosing from a fixed set of static strategies. Studies in this area use techniques of *genetic algorithms* and *genetic programming* to model the coevolution of investment strategies and market dynamics. These studies have an intention to model agents' behaviour more realistically. However, such an approach increases greatly the complexity of the modelling.

It is natural that the complexity of models would increase along with the increase in number of different agents, or from *N-type* agents to *autonomous* agents. Chen et al. (2009) asked a sharp question on whether the increase in complexities of models would help to gain additional explanatory powers. LeBaron (2006a) suggested that the ability to reproduce stylised facts of finance could be potentially served as a benchmark in assessing the validity and explanatory powers of agent-based models in finance. For this reason, Chen et al. (2012) surveyed 50 agent-based financial market models on their ability in reproducing stylised facts. Among the 50 models, there are 38 *N-type* models and 12 *autonomous* agent models. The constituents of the *N-type* models are 18 (47%) 2-type models, 9 (24%)

3-type models, and 11 (29%) many-type models. The authors showed that, first, the increase in the number of different types of agents does not significantly increase both the number and types of different stylised facts that those models can reproduce. Second, *autonomous* agent models do not have stronger ability than *N-type* models in reproducing stylised facts in terms of the number of stylised facts and different types of stylised facts.

In relation to the learning dynamics of agent-based financial market model, LeBaron (2011) stressed two important but different types of learning: *passive* and *active* learning. Passive learning refers to the phenomenon that wealth accumulates on investment strategies which have done relatively well. The relative wealth of good investment strategies grows faster than those weaker strategies. Therefore, good strategies eventually dominate the market and wipe out weaker strategies. In the case of pure passive learning, investors do not change or switch among different investment strategies. Passive learning is similar to Friedman's idea on market selection. In agent-based literature, evolutionary finance models (Evstigneev, Hens and Schenk-Hoppé, 2009) serve a good example of pure passive learning.

Active learning refers to the phenomenon that there exists an active attempt for agents to switch among a set of fixed or evolving strategies, with some well defined objective functions in mind. In the case of pure active learning, investors have infinite budgets and the survival or distinction of investment strategies depends only on the population size of each strategy (i.e. the number of investors who choose each strategy). The model proposed by Brock and Hommes (1998) may represent well a pure active learning model.

LeBaron (2011) emphasised that both passive and active learning are important principles of methodology on modelling learning dynamics and they have been extensively studied in the past. However, studies of the combination of the two learning mechanisms have been rare. LeBaron suggested that it is important for future research to take into consideration the interaction between passive and active learning, since real markets may consist of both types of learning dynamics.

Market Mechanism

The market mechanism refers to the mechanism that how asset price is determined through demand and supply of market participants in agent-based models. In the literature on agent-based financial market models, there are three commonly used market mechanisms.

The first one is *Walrasian* market clearing mechanism. The market is viewed as finding an equilibrium price via a Walriasion auctioneer who equates the total demand of agents and the total supply of the underlying asset. In dynamic equilibrium model, such a market clearing mechanism often refers to Hicks's *Temporary Equilibrium* (Hicks, 1939) by which the price at each period of trade always clears the market. The market is always in equilibrium. Depending on the complexity of the model and model assumptions, the market clearing price can be computed either analytically (e.g. Brock and Hommes, 1998) or numerically (e.g. Levy, Levy and Solomon, 2000). This pricing mechanism can be computational costly, especially in those models which implicitly deal with the demand of each agent. A typical numerical procedure is that to compute the demand of each agent by testing different hypothetical prices until the market clearing price is found so that the aggregate demand of agents matches the total supply of the asset. A detailed discussion and illustration of this method is presented in Chapter 3.

The second commonly used market clearing mechanism is *market maker* clearing mechanism. In this mechanism, agents' aggregate demand is viewed as an excess demand which can be either positive (long position) or negative (short position); the market maker will take an offsetting position to cancel the excess demand and then announces a new price that moves the price in the direction of reducing the excess demand. For example, the price is computed by the following price impact equation:

$$S_{t+1} = S_t + \alpha \sum_i A_t^i(S_t), \tag{2.1}$$

where S_t and $A_t^i(S_t)$ denote the asset price and excess demand of agent i at time t. $\sum_i A_t^i(S_t)$ represents the aggregate excess demand of all agents except the market maker. The positive parameter α measures the speed that the market maker adjusts the price. The intuition behind this mechanism is that a positive excess demand raises the asset price, while a negative excess demand lowers the asset price. However, the market is never in equilibrium. An early example of this mechanism is Day and Huang (1990). Compared with Walrasian or temporary equilibrium, this market mechanism is more computationally efficient and it is easier to be tackled analytically.

The third commonly used market clearing mechanism is called *order book* or *double auction* pricing mechanism. The idea behind this mechanism is that the actual order book where agents post offers to buy and sell assets can be simulated. Orders of agents are matched using some well-defined procedure. As stressed by LeBaron (2006a), such a mechanism is more realistic since a detailed analysis of trading mechanisms is allowed. The model in Chiarella and Iori (2002) is a good example of this mechanism.

Time

Agent-based financial market models can be developed in either continuous- or discrete-time setting. The majority of agent-based models are in discrete-time with a fixed length of time step by which agents trade repeatedly at a conventional fixed time unit. This conventional time unit may represent a day, a month or a year. An important issue of discrete-time models is that comparison of models with different frequency of trade or planing horizon is difficult due to the lack of consistency in time.

2.3 Examples of Agent-Based Financial Market Models

In this section we review some important approaches in the literature on agent-based financial market models. However, it is not our intention to give a full review of existing contributions due to the vast body of this literature and the existence of good surveys such as LeBaron (2006a) for computational orientated models, Hommes (2006) for analytical models and Chen et al. (2012) from an econometrics viewpoint. We review those approaches which we believe are important, and are closely related to our research by providing the basic framework and/or motivations, and techniques for studying agent behaviour. We emphasise the market design issues and the behavioural aspects of these approaches, and summarise their important findings. By reviewing these approaches, we identify their advantages, potential weaknesses, and gaps on studying financial markets. We also explain why some particular issues are important to be addressed in this thesis.

2.3.1 Adaptive Belief Systems

Brock and Hommes (1997, 1998) introduced an *adaptive belief system* (ABS) to study the dynamics of financial markets in a heterogeneous, boundedly rational agent world. This approach offers an example on how the heterogeneity, bounded rationality, and evolution can be incorporated into a simple present discounted value asset pricing model (e.g. the model presented in Lucas, 1978). The main idea behind this approach is that financial markets can be regarded as a nonlinear dynamical system where the techniques of nonlinear dynamics, bifurcations, chaos theory, and complex system can be applied to study the market dynamics. During the past decade, the rapidly growing contributions in the Brock and Hommes type of framework played an important role in the development of the literature on agent-based financial market models. The existing ABS models provided insights for different perspectives of financial markets, and offered a deeper understanding of market behaviour. As a start, we use the model presented in

Brock and Hommes (1998) as an example to illustrate the market design and behavioural aspects of these ABS models.

Brock and Hommes (1998) consider a financial market which contains a risk-free asset paying fixed rate of return and a risky asset paying independently identically distributed (i.i.d.) dividend. The model is in discrete-time. Agents are assumed to have one-period myopic preferences of future wealth with Constant Absolute Risk Aversion (CARA) utility function. The demand of the risky asset by each agent is derived from maximising his or her utility myopically. It is assumed that agents have unbounded budgets (i.e. agents can borrow as much as they need) by which their optimal demand (derived from utility maximisation) can always be satisfied without a budget constraint. The net supply of the risky asset is assumed to be zero. The price of the risky asset is determined through Walrasian market clearing mechanism by equating the aggregate demand and supply.

The heterogeneity of agents is introduced via a finite set of different prediction strategies (expectation rules) of future price of the risky asset. The types of prediction strategies considered in Brock and Hommes (1998) are fundamentalist, trend follower, contrarian, and a rational expectation agent with perfect foresight (who not only knows the information of past prices and dividends, but also the market equilibrium equation and fractions of other types in the market). Agents are assumed to adapt their beliefs over time by choosing from those different strategies based upon their past performance as measured by the realised profits. This belief updating or type-switching behaviour is modelled via a so-called *discrete choice model* or *logit probabilities*.

A key parameter in this model is called the *intensity of choice* which measures how fast investors switch to the best performing strategy. If the intensity of choice is infinite, the entire mass of investors will immediately switch to the prediction strategy that has highest realised profit. If the intensity of choice is zero, the mass of investors distributes itself evenly across the set of available prediction strategies. Because of the type-switching behaviour, the resulting dynamical system is highly nonlinear. In order to maintain analytical tractability, those prediction strategies are specified in a simple and linear form. The authors assume that agents (except rational expectation agent) use past information such as past price and dividend in one period backward to formulate price forecasting.

Brock and Hommes emphasised that if agents are identical and they are all rational expectation agents, the model will essentially reduce to Lucas (1978) asset pricing model. In such a case, Brock and Hommes (1998, p.1237) pointed out: "*under homogeneous, rational expectations and the assumption that the dividend process of the risky asset is independently identically distributed (IID), the asset price dynamics is extremely simple: one constant price over time.*" The authors then highlight the importance of agent heterogeneity and adaptation by showing that the heterogeneity in terms of different expectation strategies, and agent adaptation in terms of performance-driven type-switching, are two crucial behavioural elements which lead to the emergence of rich and complicated dynamics of asset price, with bifurcation routes to strange attractors.

For example, the earlier work of Brock and Hommes (1997) showed that a high intensity of choice to switch between a costly rational and a "free ride" naive strategy, leads to market instability and chaotic price fluctuation. Brock and Hommes (1998) showed that even there are no information cost for rational expectation, the rational agents do not drive out strongly extrapolating trend followers. In addition, Brock and Hommes (1998) provided numerical evidence showing that, when the intensity of choice to switch between prediction strategies is high, the evolutionary dynamics of a heterogeneous agent financial market may lead to persistent deviations from the fundamental price, and highly irregular, chaotic asset price fluctuations.

The success of Brock and Hommes's approach is built on offering a novel idea of modelling agent heterogeneity and adaptation. A key aspect of the ABS model is that it exhibit expectation feedback and adaptation of agents. The importance of heterogeneous expectations and type-switching behaviour on generating and explaining various types of market dynamics has been illustrated and emphasised. Their approach has the advantage on providing a stylised model (i.e. close to a traditional asset pricing model) to study agent heterogeneity and bounded rationality. However, Brock and Hommes's approach also has several limitations. The most important one is caused by the assumption all agents have a CARA utility function. The characteristic of CARA type of utility functions leads to the phenomenon that agents' demand of the risky asset (which is derived from solving CARA utility maximisation) is independent of agents' wealth. This property causes that the price dynamics evolves independently of agents' wealth dynamics. The model lacks mutual feedback between wealth dynamics and price realisation.

However, the interdependence nature between price and wealth has been empirically and experimentally proven to be crucial in characterising the dynamics of financial markets, see, for example, Levy, Levy, and Solomon (2000) and Campbell and Viceira (2002). The survey presented in Levy, Levy, and Solomon (2000) showed that the hypothesis of CARA utility of agents is commonly rejected in both empirical and experimental studies. Instead, those evidence is in favour of the hypothesis that agents have Decreasing Absolute Risk Aversion (DARA) and Constant Relative Risk Aversion (CRRA) types of utility, by which the interdependence between price and wealth is critical.

Moreover, because of the assumption of CARA risk preference in Brock and Hommes's model, the growth in agents' wealth does not imply the increase in investments of the risky asset, while the wealth that agents allocate in the risk free asset will keep increasing due to the fixed risk-free rate of return. Therefore, the assumption CARA risk preference consequently leads to a limited contribution of the risky asset to agents' wealth. Furthermore, this property in conjunction with the assumption of unbounded budgets cause that passive learning through wealth dynamics cannot be captured by the model. The model only exhibits pure active learning dynamics. The process of market selection of investment strategies in such a model can only be studied by agents' type-switching behaviour. For example, the extinction of one prediction strategy is equivalent to the situation that all investors would never choose that strategy. This may rarely happen when agents are bounded rational.

During the last decade, based on the framework proposed by Brock and Hommes (1997,1998), many extensions of ABS have been made to study different perspectives of financial markets. Among those extensions, one goal is to reproduce and explain a variety of stylised facts of financial markets. Along this direction, Hommes (2002) and Gaunersdorfer and Hommes (2007) considered a scenario in which investors switch between two agent types: fundamentalist and trend follower. The authors assume that the type-switching is not only based on the past performance of the two types of strategies but also depends on the price deviations to the fundamental value. It is assumed that when the price deviation to fundamental value is high, the trend followers are more willing to switch to the strategy of fundamentalist. The authors showed that this "special" type-switching behaviour may be able to explain some stylised empirical findings such as the fat tail return distribution and volatility clustering. The models presented in Hommes (2002) and Gaunersdorfer and Hommes (2007) reproduced the stylised fact of volatility clustering remarkably well in the sense that the autocorrelation structures in the generated returns, absolute returns and squared returns series are very similar to those observed from S&P 500 data for the time period from 1960 to 2000.

In those models, price deviation to fundamental value is triggered and amplified by the trend followers. The trend followers become more and more profitable than the fundamentalist during the increase in the price deviation to fundamental value. If the type-switching is only driven by agents' past performances, it will eventually lead to the dominance of trend followers, hence, unbounded price dynamics (e.g. price goes to infinity). Therefore, technically speaking, their assumption that type-switching depends on the price deviation to fundamental value is to prevent the potential unbounded price dynamics. Such an assumption lacks support from reality, and their results appear to be "artificial". This drawback is also caused by the lack of budget constraint and mutual feedback between price and wealth.

In order to capture the interdependence between agents' demand, wealth and price, Chiarella and He (2001) extended the ABS proposed by Brock and Hommes (1998) to the case where agents have a CRRA type of utility function (logarithmic utility function). The authors considered the scenarios of two agent types, fundamentalists and trend followers, and proved analytically the existence of multiple equilibria and the convergence of the return and wealth proportions to the steady state. The model also reproduced the stylised facts of fat tail return distribution and volatility clustering. Nevertheless, the authors focused only on the case of fixed population fractions of agents. Recent contributions such as Anufriev, Bottazzi and Pancotto (2006) and Anufriev and Dindo (2010) are also in line with this direction of CRRA utility agents.

However, as mentioned by Brianzoni, Mammana and Michetti (2010), most heterogeneous agent models in the CRRA utility framework are based on the assumptions of fixed fractions of agents or non-switching agents. Moreover, models which allow agents to switch between different investment strategies, for instance Chiarella and He (2002), makes the following assumption: "*when agents switch from an old strategy to a new strategy, they agree to accept the average wealth level of agents using the new strategy. More precisely, the switching agent leaves his wealth to the group of origin.*"(Brianzoni, Mammana and Michetti, 2010, p.3). Under this assumption, the effect of wealth "redistribution" which is caused by the type-switching behaviour cannot be studied properly.

To overcome this issue, Brianzoni, Mammana and Michetti (2010) proposed a model under the assumption that agents who change their group will bring their wealth to the new group. As explained by the authors: " *As a consequence, the wealth of each group is updated from period t to $t+1$ not only as a consequence of portfolio growth of agents adopting the relative strategy, but also due to the flow of agents coming from the other group.*". However, the authors changed the Walrasian market clearing mechanism to a market maker clearing mechanism for the sake of analytical tractability. This is equivalent to assume that there exist another agent, the market maker, who uses his/her own inventory to clear the market. More importantly, Bottazzi, Dosi, and Rebesco (2005) and the recent contribution Anufriev and Panchenko (2009) showed that market clearing mechanism plays a larger role in shaping the time series properties than the behavioural aspects of the model. In models with the same behavioural aspects, the price dynamics which are generated from the Walrasian market clearing mechanism and the market maker clearing mechanism can be significantly different. For this reason, Brianzoni, Mammana and Michetti (2010) did not answer the question how the wealth redistribution among different agents affects the market dynamics in a Walrasian equilibrium. Motivated by such considerations, this question will be addressed in this thesis.

Within the Brock and Hommes's type of framework, the ABS model has been extended to many other directions. For example, Westerhoff (2004) extended the ABS model to a multi-asset market environment, and provided insights for the high degree of comovements in stock prices observed empirically. Chiarella, Dieci and He (2006) considered an ABS model of multi-asset market with market maker clearing mechanism. The authors paid particular attentions to the effect of correlation between the risky assets, and showed that investor's anticipated correlation and portfolio diversification do not always have a stabilising role, but rather may act as a further source of complexity in the financial market. He and Zheng (2010) extended the ABS model into continuous-time framework to investigate dynamics of moving average rules.

2.3.2 Econophysics Approaches

The model of Lux (1995, 1998) and Lux and Marchesi (1999, 2000) represents another important approach which is emerged from the field of the so-called Econophysics with focuses on studying the effect of contagion and herding behaviour of investors. Their approach draws inspirations from behavioural finance, whereas the design of the model is in spirit to models of multi-particle interaction in physics than to traditional asset-pricing models in finance. The authors provided another novel conceptual idea on modelling the investors' type-switching behaviour. Different to many agent-based models in finance which are mainly based on discrete-time framework, their model is developed in continuous time framework.

The model of Lux and Marchesi (1999, 2000) describes an asset market with a fixed number of individual traders. These traders are divided into two groups, named fundamentalists and chartists. The chartists are further divided into optimists (buyers) and pessimists (sellers). A chartist buys (sells) a fixed amount of the asset per period when he is optimistic (pessimistic). The demand of the asset by the fundamentalists is determined by the price deviation from the fundamental value and a constant parameter which measures the reaction speed of the fundamentalists to price deviations. The asset price is determined by the so-called market maker clearing mechanism.

In this model, the type-switching mechanism contains two elements: (i) the chartists switch between the optimistic and pessimistic beliefs; (ii) traders switch between a chartist and a fundamental trading strategy. The switching between two types of chartists is based on both the current price trend and the average opinion among chartists, where two constant parameters are introduced to measure the sensitivity of traders to price changes and their sensitivity to the average opinion. The average opinion of the chartists is measured by an opinion index which is determined by the difference between the proportions of optimists and pessimists. Such a type-switching models the contagion and herding behaviour, by which traders try to predict "what average opinion expects average opinion to be" through an opinion index.

Different to the switching between the two types of chartists, the switching between chartists and fundamentalists is based on the difference between each strategy's performance measures. The performance of the chartists is measured by the realised profit, while the performance of the fundamentalists is measure by expected arbitrage profit. The switching probabilities are formulated following the synergetics literature, originally developed in physics for interacting particle systems (e.g. Haken, 1983). Lux and Marchesi (1999, 2000) showed that the contagion and herding behaviour is important for generating unsystematic deviations of the market price from the fundamental price, heavy tails of return distributions, absence of autocorrelation in returns, and volatility clustering. The authors note that these results are fairly robust with respect to the choice of the parameters.

Lux and Marchesi's approach makes an important step towards bridging the field of Econophysics, agent-based modelling and behavioural finance. Such an approach offers behavioural explanations for some important market phenomena with focuses on market psychology. By comparing the switching mechanism proposed by Brock and Hommes with the one introduced by Lux and Marchesi, the latter brings new ideas on modelling the crowd effect of chartists by taking different types of chartists as a whole group and introducing an opinion index. This conceptual idea is motivated by the ant recruiting model of Kirman (1993), which has also been proposed as an analogy for herding behaviour of investors in financial markets. Lux and Marchesi's approach illustrates that not only the type-switching of agents between different investment styles, but also the "inner" switching within a same investment style between different opinions play an important role in determining the market dynamics.

Another novel idea of the Lux and Marchesi model is to assume that there is an asymmetry in the performance measure for chartists and fundamentalists. As pointed out by Goodhart (1988), this asymmetry may bias traders towards chartist strategies since chartists' switching is driven by realised profits, whereas fundamentalists' switching is driven by expected arbitrage profits which will not be realised until the price has reached the fundamental value. The asymmetry also reflects *limits-to-arbitrage* of fundamentalists, which is in line with the behavioural finance's point of view. In contrast, Brock and Hommes assume that the performance measure for every strategy is based on observable data such as the realised profits, by which investors tend to switch to strategies with higher past performance. This assumption coincides with some empirical evidence of the so-called *follow-the-leader-behaviour* in the mutual funds literature. For example, Friend et al. (1970) provided empirical evidence showing a significant tendency for groups of mutual funds to follow the prior investment choices of their more successful counterparts. They call this phenomenon follow-the-leader behaviour.

The model of Lux and Marchesi (1999, 2000) has been extended to many different contexts during the past decade. For example, Giardina and Bouchaud (2003) proposed a model with a larger set of strategies. Lux and Schornstein (2005) proposed a two-country general equilibrium model of the foreign exchange market with agents choosing consumption and interest strategies via genetic algorithms. Pape (2007) reformulate traders' behaviour as position-based trading, which is more realistic than the "fixed" demand of chartists in Lux and Marchesi (1999, 2000). The author also adds both a second risky asset and a risk-free bond into the market. Despite these models are in a very different framework, the authors find that the dynamics of returns seems to be governed by a similar mechanism like the one illustrated in Lux and Marchesi (2000). Their results indicate that the type-switching is still the key in generating return series similar to those observed in real markets. However, an important limitation of these models is that they cannot capture the interdependence between wealth and price which has been proven crucial in characterising the dynamics of financial markets.

2.3.3 Microscopic Simulation Models

The microscopic model presented in presented in Levy, Levy, and Solomon (1994, 1995, 2000) and Levy, Persky, and Solomon (1996) is another influential modelling approach of artificial financial market which is based on the Microscopic Simulation. Microscopic simulation emerged from physics and is a part of the Econophysics research. The idea behind microscopic simulation is that the underlying system can be modelled as a set of microscopic elements. By defining the microscopic interactions between those elements, this approach investigates how observed macroscopic features emerge from the interaction of those microscopic elements.

As a starting point, we use the model of Levy, Levy, and Solomon (1995) to illustrate their "architectures" of artificial financial markets and main contributions towards providing insights for the market dynamics. In Levy, Levy, and Solomon (1995), the financial market consists two investment options: a risk-free bond and a risky stock (or stock index). The model is in discrete-time. The risk-free bond pays a fixed interest per period and it has infinite supply. The risky stock pays a stochastic dividend with a constant average growth rate. The supply of the stock is finite and fixed to a positive number.

Similar to the Brock and Hommes's type of approaches, the model of Levy, Levy, and Solomon (1995) is also based on the expected utility framework. More realistically, the authors assume that agents have a CRRA type of utility by which the nature of interdependence between price and wealth can be captured. The optimal demand of the stock by each agent is derived numerically to maximise his or her expected utility. Short selling is not allowed. To avoid short selling, it is assumed that agents' demands which are expressed in terms of budget shares are bounded in an interval between 0 and 1 with superimposed boundaries 0.01 and 0.99. The stock price is numerically determined by the so-called temporary market clearing mechanism as introduced in previous section.

The heterogeneity of agents is introduced via the following aspects. First, the authors assume that agents can be heterogeneous with respect to an agent-specific noise. In their approach, a normally distributed noise term ϵ_i is added to the optimal demand X_i of an agent i. The optimal demand X_i for every agent is the same, but the actual demand $X_i^* = X_i + \epsilon_i$ is not since the noise of ϵ_i is drawn separately for each agent. To this extent, the noise is an agent-specific noise, which is the first factor introduced to induce agent heterogeneity. As explained by the authors, the agent-specific noise is to model the phenomenon that an utility maximiser may deviate from the optimal investment strategy for some reasons.

Second, the authors assume that agents can be heterogeneous in the expectation of the future distribution of the stock market return. In their model, it is assumed that agents keep track of the last L historical returns of the stock, and agents believe that each of the last L historical returns has an equal probability $1/L$ to reoccur in the next period. The parameter L measures the memory span of an agent, and it can be heterogeneous across the entire mass of agents. Third, it is assumed that agents can be heterogeneous with respect to the degree of risk aversion. The degree of risk aversion is measured by the risk aversion parameter in agents' utility functions. Levy, Levy, and Solomon (1995) did not consider the heterogeneous investment styles such as the fundamentalist and chartists.

The authors showed that, first, with homogeneous agents with no agent-specific noises, the stock price increases as a constant rate and is fully predictable. There is no trade taking place in the market (i.e. no shares exchanged between agents). When agent-specific noises are introduced, periodic (and predictable) booms and crashes in stock price are observed which contradict to market efficiency. The periodic booms and crashes are caused by the homogeneity of agents' memory span. Introduction of heterogeneous degrees of risk aversion does not affect the periodic booms and crashes. Only when agents have heterogeneous memory spans, those predictable booms and crashes disappear and more realistic price movements are observed.

Based on this approach, Levy, Persky, and Solomon (1996) investigated the price dynamics in relation to the wealth of those agents with different memory spans. Due to the interdependence between the price and agents' wealth, an agent with higher relative wealth has a greater impact on the movements of price. The authors show that the price dynamics are highly determined by the change of levels of agents' relative wealth. When the number of agents with different memory spans becomes large, agents' wealth dynamics become complex in terms of irregular shifts of levels of agents' relative wealth. The complexity of the price dynamics stems naturally due to the complex wealth dynamics of agents. The price dynamics can be understood by analysing the relative wealth of different agents together with their demands. Furthermore, Levy, Levy, and Solomon (2000) extended this approach to allow a full spectrum of agents with different memory spans. The authors assume that each agent j has a different memory span $L_j \in \mathbb{I}$ and L_j is distributed in the entire population according to a truncated ($L_j > 0$ since $L_j \leq 0$ are meaningless) normal distribution with average $\bar{L}$ and standard deviation σ_L. The authors show that under the presence of a full spectrum of agents the resulting price dynamics become more complex and are close to those observed in real markets.

Zschischang and Lux (2001) further investigated the role of agents' risk aversion in a Levy, Levy, Solomon model. The authors show that agents' degrees of risk aversion play an important role in the determination of a dominating agent, providing that agents' memory spans are not too short. They find that an agent with lower degree of risk aversion outperforms those with higher degrees of risk aversion. In addition, agents with constant investment proportions and low risk aversion are able to outperform other sophisticated agents. Recently, Anufriev and Dindo (2010) provided an analytical explanation for the observations of Zschischang and Lux (2001). The authors show that the market selection of dominating agents in Levy, Levy, Solomon model depends on the relation between the two parameters, the risk-free interest rate r, and the constant average growth rate of the dividend g. The relation between these two parameters determines the existence and the location(s) of the steady-state(s) of the underlying model, while the length of agents' memory span affects the stability of the steady state(s). Long memory spans of agents guarantee the stability of the steady state(s), whereas $g > r$ is an essential condition for the existence of a steady-state as the one observed by Zschischang and Lux (2001). In other words, Zschischang and Lux's results may not hold for the case where $g \leq r$. Anufriev and Dindo's explanation gives importance to the lengths of agents' memory spans and the two parameters r and g which govern the economy of the underlying model.

Summarising, previous contributions of the Levy, Levy, and Solomon type of models highlighted the role of agents' memory spans of past information (e.g. historical stock returns) in shaping the dynamics of financial markets. The heterogeneity with respect to agents' memory spans has been identified to be a crucial element in explaining the complexity of the market dynamics. However, this type of heterogeneity has been rarely addressed in the Brock and Hommes's type of ABS models. The reason is that most of ABS models are developed to be analytical tractable. Increasing agents' memory spans will increase the dimension of the underlying dynamical system and thus imposing difficulties for analytical treatment. Therefore, most ABS models assume that agents use simple price forecasting rules with short memory spans. In contrast to those ABS models, Levy, Levy and Solomon's approach emphasises that agent-based models should not rely on analytical tractability, the focus should be given on modelling agent in a more flexible and realistic way rather than being restricted to assumptions for the sake of tractability. We will incorporate this point of view into our approach in order to study more complex and realistic behaviour of agents.

Both the Levy, Levy, and Solomon type of models and the ABS type of models are based on the expected utility framework. The assumption that all agents are risk averse expected utility maximisers is in line with the main stream finance's point of view. Modelling financial market in this way helps reduce the huge degree of freedom caused by the nature of human's heterogeneity and bounded rationality, therefore a more stylised approach can be provided. These approaches are especially good for investigating the convergence of market dynamics to rational homogeneous expectation equilibrium. However, an important limitation is that modelling agents' portfolio rules as solutions to utility maximisation with risk aversion and heterogeneous expectations might be too restrictive, as they may ignore better performing portfolio rules and other forms of bounded rationality such as those behavioural heuristics and biases reviewed in previous section (section 2.1.3). Moreover, the experimental and empirical evidence of individuals' decision making presented in the behavioural finance literature are in favour of that individuals' decisions are driven by heuristics and biases rather than utility optimisations. Those evidence appeals a more open minded approach which is beyond the expected utility framework and allows descriptive modelling of various heuristics and behavioural biases.

2.3.4 Evolutionary Finance

Evolutionary finance models (see, e.g. the survey of Evstigneev, Hens and Schenk-Hoppé, 2009) focus on descriptive modelling of agents which allows heterogeneous decision rules on the formulation of agents' portfolios, e.g. decision rules which are driven by heuristics and behavioural biases, different portfolio optimisation methods, and/or other forms of bounded rationality. This approach aims to maintain the largest degree of freedom on the choice of portfolio rules without sacrificing the applicability of random dynamical systems as a modelling framework. The main goal of the evolutionary finance models is to provide insights for the market selection of successful investment strategies, especially within a specific set of strategies; its application aims to contribute to the portfolio choice of investors and to the valuation of financial assets.

The approach presented in Evstigneev, Hens and Schenk-Hoppé (2006, 2008, 2009) and Hens and Schenk-Hoppé (2005) considers a financial market where multiple risky assets paying stochastic dividends are traded by agents at discrete point in time. Agents' portfolios are represented by a set which contains investment proportions of agents' wealth. The price of a risky asset is endogenously determined through short-run temporary equilibrium by equating agents' aggregate demand and the net supply of the asset at each point in time. Price fluctuations and the dynamics of agents' wealth are driven by the interactions of investment strategies. The randomness stems from the stochastic dividend process. The main idea behind this approach is that financial markets can be understood as a heterogeneous population of frequently interacting portfolio strategies in competition for market capital.

In such a financial market model, the process of market selection and mutation can be studied via the evolutionary force operating through wealth dynamics. In evolutionary finance models, there exist a mutual feedback between price dynamics and evolution of wealth. This is consistent to the characterisation of CRRA type of utility, but the difference is that agents' demands of the risky assets are not necessarily derived from utility maximisation. The advantage of descriptive modelling of agents shuns any notion of utility maximisation. As pointed out by Evstigneev, Hens and Schenk-Hoppé (2009, p.510): "*This approach lets actions speak louder than intentions and money speak louder than happiness.*"

Following evolutionary finance approach, a number of important results are found including: the investment recommendation obtained is closely related to the Kelly rule, i.e. agent using Kelly rule will dominate the market asymptotically; the "irrational" trader who distributes his or her wealth equally across the assets performs much better than those sophisticated mean-variance optimisation strategies, which coincides with the empirical evidence documented in DeMiguel et al. (2009). Such findings again motivate that the dynamics of financial markets should be studied in a more broader framework with focuses on descriptive modelling of investors' behaviour to incorporate various forms of heterogeneity and bounded rationality rather than being restricted to expected utility world.

Recently, several extensions of the evolutionary finance model have been made. Palczewski and Schenk-Hoppé (2010a) generalised the approach presented Evstigneev, Hens and Schenk-Hoppé (2006, 2008, 2009) and extended the model to a continuous-time framework. Palczewski and Schenk-Hoppé (2010b) studied market selection in this continuous-time model and derived results on the asymptotic dynamics of the wealth distribution and asset prices for constant proportions investment strategies. Evstigneev, Hens and Schenk-Hoppé (2011) extended the model documented in Evstigneev, Hens and Schenk-Hoppé (2006) by adding a risk-free asset into the financial market model, which brings the model more closer to practical finance and to classical asset pricing models.

However, an important weakness of evolutionary finance models as well as the Levy, Levy, and Solomon type of models is that agents in those models only exhibit weak adaptation by means of sticking to static parameterised strategies. Those model only consist of pure passive learning dynamics. Introducing active learning, such as type-switching, into those models would be an important extension. This extension may help to provide insights for understanding the process of market selection of investment strategies under the interaction between passive and active learning.

Comparing those models reviewed previously in this section and the evolutionary finance models, the latter has the advantage of maintaining a large degree of freedom of modelling agent bounded rationality and heterogeneity. Evolutionary finance models are also suitable for simulation studies (see e.g. Hens et al., 2002 and Hens and Schenk-Hoppé, 2004). These desirable advantages make evolutionary finance models especially suitable for studies which have the goal to explore the dynamics of financial markets by linking agent-based models and behavioural finance. For this consideration, our study is based on an evolutionary finance model. We make extensions to incorporate those important behavioural elements regarding investor bounded rationality and heterogeneity, such as the type-switching and "inner" switching of agents (identified by, e.g. Brock and Hommes, 1997, 1998, and Lux and Marchesi, 1999, 2000) and heterogeneous memory spans (identified by Levy, Levy, and Solomon, 1994, 1995, 2000). Our research also addresses the macro-level market impact and evolutionary prospects of the presence of investors with various heuristics and behavioural biases.

2.4 Summary

This chapter surveys the literature on traditional finance, behavioural finance and agent-based modelling of financial markets. The central debates between traditional and behavioural finance on the study of dynamics of financial markets are discussed. We list a number of important and well-known heuristics and behavioural biases which may have substantial impact on investors' financial decisions. We also sketch an overview of agent-based financial market models by discussions of modelling aspects and by reviews of some previous contributions. A number of behavioural factors in relation to agent heterogeneity, bounded rationality and learning have been identified.

In the development of the literature on agent-based financial market models, agent heterogeneity, bounded rationality and learning attracted the most attentions of academics. The majority of studies in this field focuses on modelling different types of investment strategies, adaptation and evolution. Previous contributions which draw on behavioural finance to study the market impact of heuristics and behavioural biases have been rare. As emphasised by LeBaron

(2006a, 2006b), agent-based modelling of financial markets and behavioural finance can be complementary. Drawing together agent-based models and concepts from behavioural finance may improve our understanding of the dynamics of financial markets.

Chapter 3

An Evolutionary Finance Model with Strategy-Switching

3.1 Introduction

In this chapter, we construct a dynamic equilibrium model of a financial market in the presence of heterogeneous, boundedly rational agents. Our financial market model combines a performance-driven strategy-switching mechanism of adaptive belief systems (Brock and Hommes, 1998) and an evolutionary finance model (Evstigneev, Hens and Schenk-Hoppé, 2011). This new model inherits the advantages of the evolutionary finance approach but draws on the strengths of adaptive belief systems. The model has two main features. First, it captures the interaction between passive and active learning dynamics. Second, it addresses a variety of behavioural biases which may affect investors' strategy-switching behaviour.

As highlighted by LeBaron (2011), passive and active learning are two important principles of methodology in the design and construction of agent-based financial market models. Passive learning refers to the selection mechanism by which wealth accumulates on investment strategies with relative high profitability. Strategies with relative low profitability will eventually die out. In the case of pure passive learning, investors do not change or switch among different investment strategies. In contrast, active learning refers to the selection mechanism by which investors actively choose or switch among a set of strategies, with some well defined objective functions in mind. There is an active attempt by investors to switch to strategies with relative high profitability. In the case of pure active learning, investors have infinite budgets and the survival or extinction of investment strategies depends only on the population size of each strategy (i.e. the number of investors who choose each strategy). LeBaron (2011) pointed out that both forms of learning are important to financial market dynamics and have been extensively studied in the past, but the interaction between the two has been rarely explored.

As reviewed in Chapter 2, adaptive belief system may represent the strand of research on pure active learning dynamics, while evolutionary finance models may serve as a typical approach to pure passive learning dynamics. Combining the two allows the coexistence of active and passive learning in one model, the interaction between the two different types of learning dynamics can therefore be explored. In addition, active learning in our model is based on a realistic assumption by which switching investors will bring (take away) their wealth when they join (leave) investment strategies. This strategy-switching behaviour causes *flow of funds* among different investment strategies. Our combined model is able to characterise the coevolution between market prices and the redistribution of wealth when investors switch among a set of investment strategies.

Moreover, our model also makes a step towards bridging behavioural finance and agent-based modelling. We focus on the modelling of a variety of heuristics and behavioural biases documented in the behavioural finance literature, such as overconfidence, recency bias, conservatism, and herding. The goal is to study the macro-level impact and long-run prospects of these heuristics and behavioural biases using an agent-based model.

3.2 Model Description

Following Evstigneev, Hens and Schenk-Hoppé (2011), we consider a financial market in which a risk-free and $K \geq 1$ risky assets are traded at discrete points in time $t = 0, 1, ...$ The $k = 1, ..., K$ assets are risky in terms of their prices $S_{t,k}$ and their payoffs (dividends) $D_{t,k}$. The total supply of each risky asset is assumed to be fixed at 1. The risk-free asset ($k = 0$) is in perfect elastic supply and its price is exogenously given at 1. The price of the risk-free asset is used as a numeraire (cash) which expresses the market values of all the assets in the market. We will refer to holdings of the risk-free asset as balances in a bank account with a fixed net interest rate $r > 0$ per trading period.

The market participants are individual investors who are able to make a single choice from a finite set of $I > 1$ different investment strategies (prediction rules) available in the market. Each investment strategy $i = 1, ..., I$ at time t is characterised by a vector of investment proportions $\lambda_t^i = (\lambda_{t,0}^i, ...\lambda_{t,K}^i)$ satisfying $\lambda_{t,k}^i \geq 0$ and $\sum_{k=0}^{K} \lambda_{t,k}^i = 1$. A short position ($\lambda_{t,k}^i < 0$) is not allowed for all assets $k = 0, ..., K$. According to these investment proportions, each agent i distributes wealth across the $K + 1$ available assets. Here, the agent i refers to the group of individual investors who adopt investment strategy i. At the initial period, the wealth managed by each agent i equals $W_0^i > 0$. It is assumed that, each agent consumes (is taxed) a constant fraction, $c \in (0, 1)$, of his wealth in every period with the remainder being invested in assets. The constant c is the same for every agent and represents the intensity of consumption.

3.2.1 Strategy-Switching and Flow of Funds

We assume that, at each point $t > 0$ in time, every individual investor can update his or her choice from the fixed set of I investment strategies. Investors estimate the relative benefits of each choice and decide whether they should change investment strategy or not. In a dynamic context, the strategy updating behaviour may cause investors to switch among different investment strategies. Such an assumption of strategy-switching is common in agent-based modelling literature and its purpose is to model adaptation and active learning of investors.

Non-Switching and Switching Investors

It is assumed that all individual investors can be characterised by belonging to one of two possible types — *non-switching investors* and *switching investors*. Non-switching investors refer to investors who stay with their investment strategies between two successive points in time (e.g. the time interval $[t, t+1]$) with a probability of 1. The purpose of introducing such a type of investors is to capture investor overconfidence which will be explained in detail in next subsection.

In contrast, switching investors refer to investors who have positive probabilities to switch among investment strategies between two successive points in time. Before the trade starts at each point in time $t > 0$, switching investors will update their choices according to the past performance of each investment strategy $i \in \{1, ..., I\}$. Formally speaking, each switching investor chooses investment strategy i at time t with a probability $q_t^i > 0$ ($\sum_{i=1}^{I} q_t^i = 1$). The value of q_t^i is determined by the past performance of each investment strategy i.

Note that the terms of non-switching and switching are opposite properties of an individual investor. An investor *cannot* have both properties at the same time. Moreover, these two opposite properties are time-dependent. An individual investor can exhibit the property of non-switching *or* switching at different time. For example, a investor who belongs to non-switching investors at time t can become as a switching investor at time $t+1$. Such an assumption is to model the change in investor psychology with respect to time.

Wealth Reallocation and Flow of Funds

It is assumed that switching investors will bring (take away) their wealth when they join (leave) investment strategies. Under this assumption, the presence of switching investors causes wealth reallocation for different investment strategies. We call this wealth reallocation *flow of funds*.

In order to characterise the coexistence of non-switching investors and switching investors, we assume that, the wealth managed by switching investors at time t equals a $\beta \in [0,1]$ percent of the aggregate wealth in the market $\bar{W}_t$. The remainder $1-\beta$ percent of the aggregate wealth is managed by non-switching investors. Based on this assumption, at each point in time $t>0$, the wealth $\hat{W}^i$ reallocated to investment strategy i after strategy-switching is given by:

$$\hat{W}_t^i = (1-\beta)W_t^i + \beta q_t^i \bar{W}_t, \tag{3.2.1}$$

where W_t^i denotes the wealth managed by agent (investment strategy) i before the strategy-switching of investors at time t; $\bar{W}_t$ denotes the aggregate wealth in the market at time t; The constant parameter $\beta \in [0,1]$ governs the proportion between non-switching and switching investors. If $\beta = 0$, all individual investors are non-switching investors. None of the investors switches among investment strategies (i.e. $\hat{W}_t^i = W_t^i$). The model will then coincides with the original Evolutionary Finance model. If $\beta = 1$, every individual investor switches with a certain probability $q_t^i > 0$ to each strategy $i = 1, ..., I$. If $\beta \in (0,1)$, equation (3.2.1) describes the coexistence of non-switching and switching investors. Higher value of β represents higher proportion of switching investors (at wealth level) are present in the market vice and versa. Assuming that the number of individual investors in the market is sufficiently large, the term $\beta q_t^i \bar{W}_t$ in equation (3.2.1), therefore, represents the average amount (the expected value) of wealth which is allocated to agent i at time t by the switching investors. The term $(1-\beta)W_t^i$ measures the amount of wealth which has a probability of 1 to stay with agent i at time t. The total of the two terms is the wealth managed by agent i at time t after the strategy-switching of investors. It is also the available budget for agent i's investment in the period of [t,t+1).

Equation (3.2.1) embeds active learning into the evolutionary finance model. It allows us to explore in the model the impact of flow of funds on market dynamics. Moreover, the effect of the proportion between non-switching and switching investors can be investigated.

Switching Mechanism

Following Brock and Hommes (1998), the probability q_t^i for each switching investor to choose investment strategy i at time t is modelled by the *multinomial logit probabilities* of discrete choice:

$$q_t^i = \frac{\exp(\gamma f_{t-1}^i)}{\sum_{i=1}^{I} \exp(\gamma f_{t-1}^i)}, \tag{3.2.2}$$

where f_{t-1}^i denotes the performance measure of investment strategy i at time $t-1$. The value of q_t^i at time t depends on the past performance f_{t-1}^i (which is measured at time $t-1$). The intensity of choice parameter $\gamma \geq 0$ measures the sensitivity of switching investors to selecting more successful investment strategies. If $\gamma = \infty$, the entire mass of switching investors will choose to the investment strategy which has the highest performance. If $\gamma = 0$, switching investors will have a equal probability to choose each strategy i in the set of I strategies. The case of $\gamma \in [0, \infty)$ describes the failure of all switching investors to choose the best-performed investment strategy. Brock and Hommes (1997, 1998) attribute such a failure to investors' bounded rationality. Their studies focus only on the boundedly rational case ($\gamma \in [0, \infty)$).

The performance of each agent i is assessed based on realised simple returns[1]:

$$\phi_t^i = \frac{W_t^i - (1-c)\hat{W}_{t-1}^i}{(1-c)\hat{W}_{t-1}^i}, \tag{3.2.3}$$

where $(1-c)\hat{W}_{t-1}^i$ denotes the wealth managed by agent $i = 1, ..., I$ after strategy-switching and consumption at the point $t-1$ in time. It is the available budget for agent i at the time $t-1$. W_t^i denotes the payoff of agent i at time t.

[1]Because Brock and Hommes (1998) assume that agents have unlimited budgets, the performance measure in their model is computed based on realised profits. When agents have finite budgets, because of the existence of budget effect, using realised profits to measure performance can be biased. Therefore, we use realised returns to measure performance.

The performance measure for each agent i at the point $t \geq 1$ in time is given by the discounted realised returns:

$$f_t^i = \phi_t^i + \rho f_{t-1}^i, \tag{3.2.4}$$

where $\rho \in [0, 1]$ is a memory parameter or discounting factor measuring how strongly the past performances of each investment strategy are discounted for strategy selection. The case of $\rho = 1$ describes the phenomenon that the switching investors have infinite memory and they give equal weight to the realised returns at each point in time. In contrast, $\rho \in (0, 1)$ implies that the performance measure of each investment strategy is computed as a geometrically declining weighted average of past realised returns. In the case of $\rho = 0$, the performance of each strategy is measured by the most recent realised return only.

Remark

Since $\gamma \in [0, \infty)$ in (3.2.2) implies heterogeneous choices of investment strategies, an important question is that, for each strategy i, what is the condition which triggers the increase or decrease in the probability q^i between time t and $t + 1$? Such a question is important in understanding the model dynamics but ignored by Brock and Hommes (1997, 1998). We analyse here the condition which affects the increase or decrease in q^i between two successive points in time.

The increase or decrease in q^i between time t and $t+1$ can be represented by the sign of $q_{t+1}^i - q_t^i$ via a sign function $\text{sgn}(q_{t+1}^i - q_t^i)$. The positive or negative sign of $q_{t+1}^i - q_t^i$ refers to an increase or decrease.

Define the *improvement* of an investment strategy $i = 1, ..., I$ during a time interval $[t - 1, t]$ as:

$$\Delta_t^i = f_t^i - f_{t-1}^i,$$

and the *average improvement* of all investment strategies during the time interval $[t - 1, t]$ as:

$$\bar{\Delta}_t = \frac{\sum_{i=1}^{I} \Delta_t^i}{I}.$$

The condition which triggers the increase or decrease in q^i can be characterised by the following proposition.

Proposition 3.1. *During the time interval* $[t, t+1]$, *for each investment strategy* $i = 1, ..., I$, *the sign of* $q_{t+1}^i - q_t^i$ *is determined by the sign of* $\Delta_t^i - \bar{\Delta}_t$*:*

$$sgn(q_{t+1}^i - q_t^i) = sgn(\Delta_t^i - \bar{\Delta}_t).$$

Proof of Proposition 3.1. Rearranging equation (3.2.2) gives:

$$q_t^i = \frac{\exp(\gamma f_{t-1}^i)}{\sum\limits_{i=1}^{I} \exp(\gamma f_{t-1}^i)} = \frac{1}{1 + \sum_{j \neq i}^{I} \exp[\gamma(f_{t-1}^j - f_{t-1}^i)]}. \tag{3.2.5}$$

Inserting (3.2.5) into $\mathrm{sgn}(q_{t+1}^i - q_t^i)$ gives:

$$\begin{aligned}
\mathrm{sgn}(q_{t+1}^i - q_t^i) &= \mathrm{sgn}\left(\frac{1}{1 + \sum_{j \neq i}^{I} \exp[\gamma(f_t^j - f_t^i)]} - \frac{1}{1 + \sum_{j \neq i}^{I} \exp[\gamma(f_{t-1}^j - f_{t-1}^i)]}\right) \\
&= \mathrm{sgn}\Big(\sum_{j \neq i}^{I} \exp[\gamma(f_{t-1}^j - f_{t-1}^i)] - \sum_{j \neq i}^{I} \exp[\gamma(f_t^j - f_t^i)]\Big) \\
&= \mathrm{sgn}\Big(\sum_{j \neq i}^{I} [(f_t^i - f_{t-1}^i) - (f_t^j - f_{t-1}^j)]\Big) \\
&= \mathrm{sgn}\Big(\sum_{j \neq i}^{I} (\Delta_t^i - \Delta_t^j)\Big) \\
&= \mathrm{sgn}\Big(I\Delta_t^i - \sum_{j=1}^{I} \Delta_t^j\Big). \qquad (3.2.6)
\end{aligned}$$

Since dividing $I > 0$ on the right-hand side of (3.2.6) does not change its sign, therefore

$$\mathrm{sgn}(q_{t+1}^i - q_t^i) = \mathrm{sgn}\left(\Delta_t^i - \frac{\sum_{j=1}^{I} \Delta_t^j}{I}\right). \tag{3.2.7}$$

Using $\bar{\Delta}_t = \sum_{j=1}^{I} \Delta_t^j$ in (3.2.7) gives:

$$\mathrm{sgn}(q_{t+1}^i - q_t^i) = \mathrm{sgn}(\Delta_t^i - \bar{\Delta}_t).$$

□

Proposition 3.1 indicates that, based on the switching mechanism (3.2.2), the switching investors have a tendency to choose investment strategies which have higher improvements than the average level. Note that, during the time interval $[t-1, t]$, an investment strategy j which has the highest performance measure f_t^j at time t is not necessary to have higher improvement Δ_t^j than the average level. The "best performed" strategy j may lose investors at time $t+1$ if its improvement Δ_t^j falls below the average. Such a property reveals that some switching investors hold a different interpretation for the performance measure: the improvement of investment strategy. They are sensitive to the improvements and may overreact to the improvements when making choices of strategies.

3.2.2 Behavioural Aspects

Investor psychology may play an important role in the choice of investment strategies, especially when investors are boundedly rational. By linking to the behavioural finance literature, we now explain and discuss some psychological elements and behavioural phenomena which are covered by the strategy-switching behaviour modelled in the previous subsection.

Investor Overconfidence

Overconfidence is an important psychological element which affects individuals' decision making. As reviewed in Chapter 2, a key manifestation of overconfidence is better-than-average effect. It refers to the tendency that people overestimate their own capabilities, usually with respect to the average capability of others.

In the behavioural finance literature, see e.g. Shiller (1999), Barberis and Thaler (2003) and Glaser and Weber (2007), the phenomenon that investors have a tendency to rank themselves as above average with respect to investment skills or past portfolio performance is regarded as better-than-average overconfidence. Investors who exhibit better-than-average overconfidence may tend to focus on their own opinions in predictions and ignore those of other investors. As stated by Glaser and Weber (2007, p.6): *"an investor who regards himself as above average is more likely to maintain a specific opinion about the future performance of an asset even though he knows that other investors or the market hold a different opinion"*. Better-than-average overconfidence leads to *differences of opinion* among investors and reinforces these differences.

In the context of strategy-switching, better-than-average overconfident investors can be represented by those investors who are unwilling to change their strategies (prediction rules) between two successive trading periods. In contrast, investors who are not overconfident (or less confident) may intend to switch to investment strategies which they believe to be more profitable. In our model, overconfident investors are characterised the non-switching investors, while less confident investors are represented by the switching investors. Since our model allows the coexistence of the two types, the market impact of the proportion of overconfident investors can be explored by varying the value of the parameter $\beta \in [0, 1]$ in (3.2.1). Our approach to the modelling of investor overconfidence and the co-existence of non-switching and switching investors is similar to the one proposed by Dieci et al. (2006)[2].

Differences of Opinion

The foundation for the switching mechanism (3.2.2) is the randomized discrete choice framework of Manski and McFadden (1981), whereas Brock and Hommes (1997, 1998) utilized it in dynamic equilibrium models of financial markets to study the adaptation of investors. In discrete choice studies, not all agents necessarily choose the option (e.g. investment strategy) which is indicated (by the

[2]In Dieci et al. (2006), overconfidence and evolutionary adaption are characterised by fixed and adaptive switching fraction among two agent types (fundamentalist and trend followers), respectively. To our knowledge, this paper is the first one to explicitly model and examine the joint impact of the presence of non-switching and switching investors.

model) to have the highest performance measure. Such a phenomenon corresponds to the case of $\gamma \in [0, \infty)$ in (3.2.2). The finite value of γ implies that agents are heterogeneous in making choices of available options. The reason for such heterogeneity of agents is explained by McFadden (1981) as unmodelled idiosyncratic components in agents' utility function or randomness in agents' preferences, while Brock and Hommes (1997, 1998) attribute this heterogeneity to agents' bounded rationality. All these explanations point to the concept of differences of opinion.

The concept of differences of opinion has been widely discussed in economics and finance in relation to market anomalies. For example, as reviewed by Glaser and Weber (2007), the observed high levels of trading volume in financial markets can be explained by a strand of differences-of-opinion literature. Theoretical models of financial markets show that, differences of opinion among investors in terms of differences in prior beliefs (Varian, 1985, 1989) or differences in the way that investors interpret public information (Harris and Raviv, 1993; Kandel and Person, 1995) are able to cause high trading volume. These findings are confirmed by empirical studies such as Bamber, Barron, and Stober (1999), Antweiler and Frank (2004) and Glaser and Weber (2007). Moreover, Glaser and Weber (2007) argue that investors' better-than-average overconfidence is able to reinforce differences of opinion and lead to high trading volume.

Our model characterises various types of differences of opinion among investors. First, the differences in prior beliefs is modelled by the set of different strategies. The presence of overconfident investors (non-switching investors) may reinforce this type of differences of opinion. Second, the switching mechanism (3.2.2) with $\gamma \in [0, \infty)$ is able to capture the differences of opinion in strategy selection and the phenomenon that investors hold different interpretations of the public performance measure (3.2.4) of each strategy. When $\gamma \in [0, \infty)$, each switching investor can be thought as having a private measurement or interpretation for the performance of each strategy. As revealed by Proposition 3.1, the improvement of strategy (i.e. $\Delta_t^i = f_t^i - f_{t-1}^i$) can be regarded as an example of this kind of private measurement or interpretation for the performance of each strategy. This may lead to the phenomenon that some switching investors choose strategies according to the value of public performance measure (i.e. the value of f_t^i), while some may make their choices based on the value of improvement of each strategy (i.e. the value of Δ_t^i).

In this switching mechanism, the distribution of investors among investment strategies becomes more (or less) diversified when the value of γ becomes low (or high). For this reason, the degree of differences of opinion among investors in strategy selection can be measured by the value of $\gamma \in [0, \infty)$. A lower (or higher) value of γ corresponds to a higher (or lower) degree of differences of opinion.

Conservatism Bias and Rational Herding

Different degrees of differences of opinion in strategy-switching may represent different behavioural phenomena, such as *conservatism* bias and *rational herding*. Edwards (1968) identified the phenomenon of conservatism which describes that people react conservatively to new information, and they are slow to change an established view. Such phenomenon can be captured when the value of γ is low (i.e. investors are less sensitive to the value of performance measure (3.2.4)).

Rational herding refers to the tendency they investors to react to information about the behaviour of other investors. According to Bruce (2010), rational herding happens because some investors believe others can perform better than themselves, therefore they follow or mimic others' behaviour. Such a phenomenon can be captured when the value of γ is high, since a high value of γ implies that investors are less conservative and a large proportion of investors will switch fast to the best performed investment strategy.

In our model, the market impact of the differences of opinion in strategy selection and its related behavioural phenomena can be studied by exploring how different values of γ in (3.2.2) affects the market dynamics.

Recency Bias

Recency bias is a cognitive bias which is related to the way that individuals order information. Recency bias refers to the tendency of individuals to assign more importance to more recent information compared to those farther in the past. In the behavioural finance literature, recency bias have been widely studied in relation to asset valuation and evaluation of funds' performance. For example, as stated in Pompian (2006, p.216): "*one of the most obvious and most pernicious manifestation of recency bias among investors pertains to their misuse of investment performance records for mutual funds and other types of funds. Investors track managers who produce temporary outsized returns during a one-,two-,or three-year period and make investment decisions based on such recent experience*".

In our model, recency bias in performance evaluation is modelled by the case of $\rho \in [0, 1)$ in the performance measure (3.2.4). The decrease in the value of $\rho \in [0, 1)$ represents the increase in the degree of recency bias. In the extreme case $\rho = 0$, the performance of each strategy is assessed by its most recent realised return. In contrast, the case of $\rho = 1$ describes that investors have infinite memory of past performances of each strategy and they are unbiased in performance evaluation. This setting allows us to explore how investors' recency bias in performance evaluation affects their selections of strategies as well as the resulting market dynamics.

3.2.3 Pricing Mechanism

In our model, the market clearing prices of all risky assets are determined by the equilibrium of asset supply (which is fixed to 1 for each risky asset) and demand. The price for each risky asset is thus given by:

$$S_{t,k} = (1 - c)\langle \lambda_{t,k}, \hat{W}_t \rangle, \quad k = 1, ..., K, \tag{3.2.8}$$

where $\langle \lambda_{t,k}, \hat{W}_t \rangle = \sum_{i=1}^{I} \lambda_{t,k}^i \hat{W}_t^i$ denotes the scalar product. The left-hand side of (3.2.8) represents the total value of the asset k. The right-hand side of (3.2.8) represents the total wealth invested in each risky asset k by all agents at the point in time t. Since the net supply of each risky asset is fixed at 1 for all t, equilibrium between the asset value and the amount invested implies the equality of (3.2.8).

It is assumed that the market is always in equilibrium. As reviewed in Chapter 2, such a market clearing mechanism, is called *temporary market equilibrium*. An important feature of (3.2.8) is that the market clearing price is affected by agents' wealth and the flow of funds.

The portfolio (in number of units) of each risky asset k held by an agent $i = 1, ..., I$ at time t is given by:

$$\hat{\theta}^i_{t,k} = \frac{(1-c)\lambda^i_{t,k}\hat{W}^i_t}{S_{t,k}}, \quad k = 1, ..., K. \tag{3.2.9}$$

Inserting equation (3.2.8) into (3.2.9) yields:

$$\hat{\theta}^i_{t,k} = \frac{\lambda^i_{t,k}\hat{W}^i_t}{\langle \lambda_{t,k}, \hat{W}_t \rangle}, \quad k = 1, ..., K. \tag{3.2.10}$$

The amount invested in risk-free asset by agent $i = 1, ..., I$ is given by:

$$\hat{\theta}^i_{t,0} = (1-c)\lambda^i_{t,0}\hat{W}^i_t. \tag{3.2.11}$$

3.2.4 Dividend Process

We assume an exogenous dividend process. Similar to Brock and Hommes (1998), the dividend payout (in cash) of each asset $k = 1, ..., K$ is assumed to be a sequence of i.i.d variables:

$$D_{t,k} = \bar{D}_k + \epsilon_{t,k}, \tag{3.2.12}$$

where $\bar{D}_k > 0$ is a constant, $\epsilon_{t,k} \sim \mathrm{N}(0, \sigma_k^2)$ is an i.i.d. variable with mean 0 and variance σ_k^2. The assumption of exogenous dividend process is common in agent-based literature, see e.g. Levy, Levy, and Solomon (1994, 1995, 2000), and Anufriev and Dindo (2010).

In the literature on agent-based financial market models, an alternative way of modelling dividend is to assume that dividend depends on an endogenous variable such as the asset price or the aggregate wealth. For instance, in the original evolutionary finance model presented in Evstigneev, Hens and Schenk-Hoppé (2011), the dividend process is assumed to depend on the aggregate wealth. Such an assumption leads to a case in which any change in the level of prices causes an immediate change in the level of dividends. However, as pointed out by Anufriev and Dindo (2010), the dividend policy of firms in reality is hardly responsive so fast to the performance of firms' assets. For this reason, the dividend process in our new model is assumed to be exogenous.

3.2.5 Dynamics of the Wealth Distribution

The wealth managed by each agent i evolves over time as:

$$W^i_{t+1} = \sum_{k=1}^{K} [S_{t+1,k} + D_{t+1,k}]\hat{\theta}^i_{t,k} + (1+r)\hat{\theta}^i_{t,0}, \tag{3.2.13}$$

where $\hat{\theta}^i_{t,k}$ and $\hat{\theta}^i_{t,0}$ are defined, respectively, by equation (3.2.10) and (3.2.11) representing agent i's portfolio holdings of each risky asset $k = 1, ..., K$ and the risk-free asset at time t. The first term on the right-hand side of equation (3.2.13) is the payoff of the portfolio of risky assets at time $t+1$, which contains the market value of the portfolio at time $t+1$ and dividends received by holding this portfolio between the time t and $t+1$. The second term is the agent's balance in the bank account after receiving the net interest at time $t+1$. The sum of the two terms gives the total wealth W^i_{t+1} of agent i at time $t+1$. Note that W^i_{t+1} is the wealth possessed by agent i immediately before the strategy-switching at time $t+1$.

Inserting equation (3.2.1), (3.2.8) and (3.2.11) into (3.2.13) yields:

$$\begin{aligned} W^i_{t+1} &= \sum_{k=1}^{K} \Big[(1-c)\big[\beta\langle\lambda_{t+1,k}, q^i_{t+1}\rangle\bar{W}_{t+1} + (1-\beta)\langle\lambda_{t+1,k}, W_{t+1}\rangle\big] + D_{t+1,k}\Big]\hat{\theta}^i_{t,k} \\ &\quad +(1+r)(1-c)\lambda^i_{t,0}\hat{W}^i_t, \end{aligned} \tag{3.2.14}$$

with $i = 1, ..., I$, or equivalently, in vector notation:

$$W_{t+1} = \hat{\Theta}_t\Big[(1-c)\big(\beta\Lambda_{t+1}q_{t+1}1^{\mathrm{T}}+(1-\beta)\Lambda_{t+1}\big)W_{t+1}+D_{t+1}\Big]+(1+r)(1-c)\Delta\lambda_{t,0}\hat{W}_t, \tag{3.2.15}$$

where (suppressing the time index) the matrix $\Lambda \in \mathbb{R}^{K\times I}$ is given by $\Lambda_{ki} = \lambda_k^i$ (agent's investment proportions for *risky* assets); $\hat{\Theta} \in \mathbb{R}^{I\times K}$ is given by $\hat{\Theta}_{ik} = \frac{\Lambda_{ki}\hat{W}^i}{(\Lambda\hat{W})_k}$ (agent's portfolio of risky assets); $1^{\mathrm{T}} = (1, ..., 1)$ is a row vector with I entries; $W \in \mathbb{R}^I$ and $\hat{W} \in \mathbb{R}^I$ are a column vectors of agents' wealth W^i (before strategy-switching) and $\hat{W}^i$ (after strategy-switching) respectively; $q \in \mathbb{R}^I$ is a column vector of the probabilities of switching investors to choose each investment strategy i; Id is a $\mathbb{R}^{I\times I}$ identity matrix; $\Delta\lambda_0$ is a $\mathbb{R}^{I\times I}$ matrix with entries λ_0^i (investment proportion for the risk-free asset) on its diagonal and zero otherwise.

Equation (3.2.15) is an implicit description of the wealth dynamics. Assuming that, at each point t in time, the investment proportion Λ_t does not depend on agents' contemporaneous wealth W_t. It is then straightforward to rewrite the wealth dynamics of (3.2.15) in semi-implicit form:

$$\begin{aligned}\Big[\mathrm{Id} - (1-c)\hat{\Theta}_t\Big(\beta\Lambda_{t+1}q_{t+1}1^T + (1-\beta)\Lambda_{t+1}\Big)\Big]W_{t+1}\\ = \hat{\Theta}_t D_{t+1} + (1+r)(1-c)\Delta\lambda_{t,0}\hat{W}_t,\end{aligned} \tag{3.2.16}$$

Following (3.2.16), an explicit representation of the wealth dynamics can be obtained only if the matrix on the far left of the equation is invertible.

Define the sets:

$$\mathcal{D} = \Big\{\hat{W} \in [0,\infty)^I : \sum_{i=1}^{I} \hat{W}^i > 0\Big\}$$

and

$$\mathcal{F} = \Big\{q \in [0,1]^I : \sum_{i=1}^{I} q^i = 1\Big\}$$

Theorem 3.1. *Assume that that all investment strategies are fully diversified, that is,* $\Lambda \in (0,1]^{K\times I}$ *and* $\lambda_0^i \in [0,1)$ *with* $\sum_{k=1}^{K}\Lambda_{ki} + \lambda_0^i = 1$ *for all* $i = 1, ..., I$*. Then the following matrix*

$$M = \left[\mathrm{Id} - \underbrace{(1-c)\hat{\Theta}}_{I\times K}\left(\underbrace{\beta\Lambda q 1^T}_{K\times I} + \underbrace{(1-\beta)\Lambda}_{K\times I}\right)\right] \in \mathbb{R}^{I\times I}$$

is inventible for every $\hat{W} \in \mathcal{D}$*,* $q \in \mathcal{F}$*,* $c \in (0,1)$ *and* $\beta \in [0,1]$*. Furthermore, the wealth dynamics* (3.2.16) *has an explicit form:*

$$W_{t+1} = \left[\mathrm{Id}-(1-c)\hat{\Theta}_t\Big(\beta\Lambda_{t+1}q_{t+1}1^T+(1-\beta)\Lambda_{t+1}\Big)\right]^{-1}\left[\hat{\Theta}_t D_{t+1}+(1+r)(1-c)\Delta\lambda_{t,0}\hat{W}_t\right], \tag{3.2.17}$$

and it is well-defined.

To prove the Theorem 3.1, an auxiliary Lemma which is similar to the one used in Palczewski and Schenk-Hoppé (2011) (see the lemma E.1(i), p.46) will be needed. Before we prove the Theorem 3.1, we first introduce the auxiliary Lemma and sketch its proof.

Lemma 3.1. *Suppose that* $A \in \mathbb{R}^{I\times K}$ *and* $B \in \mathbb{R}^{K\times I}$ *are matrices with non-negative entries, and* $I \geq 1$ *and* $K \geq 1$*. Assume that:*

(i) all column sums of A are strictly less than 1, i.e. $\sum_{i=1}^{I} A_{ik} < 1$ *for all* $k = 1, ..., K$*;*

(ii) all column sums of B are less than or equal to 1, i.e. $\sum_{k=1}^{K} B_{kj} \leq 1$ *for all* $j = 1, ..., I$*.*

Then the matrix $Id - AB$ *is invertible and its inverse maps the non-negative orthant into itself.*

Proof of Lemma 3.1. The matrix $C = Id - AB$ has a strict column-dominant diagonal. In the matrix C, the magnitude of the diagonal entry in a row is strictly larger than the sum of the magnitudes of all the other (non-diagonal) entries in that row:

$$C_{ii} > \sum_{j=1, j\neq i}^{I} |C_{ji}|, \text{ for all } i = 1, ..., I. \tag{3.2.18}$$

To see this, each entry in the matrix C is given by:

$$\mathbf{1}_{\{i=j\}} - \sum_{k=1}^{K} A_{ik} B_{kj}.$$

All non-diagonal entries are less than or equal to 0, while all diagonal entries are non-negative. Therefore, the condition (3.2.18) is equivalent to:

$$\sum_{i=1}^{I} \sum_{k=1}^{K} A_{ik} B_{kj} < 1, \text{ for all } j = 1, ..., I.$$

The following computation proves this strict inequality.

$$\sum_{i=1}^{I} \sum_{k=1}^{K} A_{ik} B_{kj} = \sum_{k=1}^{K} \underbrace{\Big(\sum_{i=1}^{I} A_{ik} \Big)}_{<1} B_{kj} < \sum_{k=1}^{K} B_{kj} \leq 1.$$

The property of strictly diagonal dominance (condition (3.2.18)) implies the matrix is inventible and its inverse maps the non-negative orthant into itself (see Corollary, p.22, and Theorem 23, p.24, in Murata, 1977).

□

Theorem 3.1 can be proved based on the above Lemma:

Proof of Thoerem 3.1. Let matrix $A = (1-c)\hat{\Theta} \in \mathbb{R}^{I\times K}$ and matrix $B = [\beta\Lambda q 1^T + (1-\beta)\Lambda] \in \mathbb{R}^{K\times I}$. Then the objective matrix $M = (\mathrm{Id} - AB) \in \mathbb{R}^{I\times I}$. Both A and B have non-negative entries. Since $c > 0$, the column sums of A are strictly less than one:

$$\sum_{i=1}^{I} A_{ik} = 1 - c < 1. \tag{3.2.19}$$

The column sums of B are less than or equal to 1:

$$\begin{aligned}\sum_{k=1}^{K} B_{ki} &= \beta \sum_{k=1}^{K}\sum_{i=1}^{I} \lambda_k^i q^i + (1-\beta)\sum_{k=1}^{K}\lambda_k^i \\ &= \underbrace{\beta \sum_{i=1}^{I}(1-\lambda_0^i) q^i}_{\leq \beta} + \underbrace{(1-\beta)\sum_{k=1}^{K}\lambda_k^i}_{\leq (1-\beta)} \\ &\leq \beta + 1 - \beta = 1. \end{aligned} \tag{3.2.20}$$

Lemma 3.1 implies that the matrix $M = \mathrm{Id} - AB$ is invertible. It follows immediately the result of the invertibility of the matrix M, the wealth dynamics (3.2.16) has the following explicit form:

$$W_{t+1} = \Big[\mathrm{Id} - (1-c)\hat{\Theta}_t\Big(\beta\Lambda_{t+1}q_{t+1}1^T + (1-\beta)\Lambda_{t+1}\Big)\Big]^{-1}\Big[\hat{\Theta}_t D_{t+1} + (1+r)(1-c)\Delta\lambda_{t,0}\hat{W}_t\Big].$$

Lemma 3.1 also implies that the inverse matrix in the above equation maps the non-negative orthant $[0,\infty)^I$ into itself. Since the term $\Big[\hat{\Theta}_t D_{t+1} + (1+r)(1-c)\Delta\lambda_{t,0}\hat{W}_t\Big]$ contains only non-negative coordinates, the wealth before strategy-switching W_{t+1} is non-negative, hence the wealth after strategy-switching $\hat{W}_{t+1}$ is also non-negative.

□

To summarise, we have derived an explicit description of the wealth dynamics. Theorem 3.1 ensures that the wealth dynamics of the model is well-defined, hence the price dynamics (3.2.8). The explicit formulation of wealth dynamics (3.2.17) will be used in building a C++ computer programme for numerical studies of the model dynamics.

The model derived in this section is in its *general form* which has features: i) it contains both cases in which the number of risky asset(s) $K = 1$ and $K > 1$; ii) it maintains a large degree of freedom on the specification of agents' behaviour by means of investment strategies. Since the two cases $K = 1$ and $K > 1$ require different agent behaviour, i.e. whether agents are required to deal with the asset allocation across several risky assets, we will treat these two cases separately. A detailed discussion of this point is presented in the next section.

3.3 The Two-Asset Model

The two-asset model refers to a special case which is obtained by setting the number of risky assets $K = 1$ in the general model presented in above section. In this case, the market contains only two types of asset: a risk-free asset (e.g. a bank account that pays a fixed rate of interest) and a risky asset which may serve as a proxy for the market portfolio (e.g. Standard & Poor's 500 index). This class of models is common in the economics and finance literature. The majority of agent-based financial market models are in line with this setup of market environment.

One of the main motivations of the two-asset model comes from the *mutual fund theorem* (also sometimes called the *two-fund separation theorem*). This theorem indicates that all investors should invest in two funds: the risk-free asset and a mutual fund (or a market portfolio) which contains a combination of risky assets available in the market (see, e.g. Tobin, 1958; Cass and Stiglitz, 1970). Modelling financial markets with two investment alternatives, a risk-free asset and a risky asset (which represents the market portfolio), is therefore consistent with mutual fund theorem.

In agent-based modelling literature, two-asset models have the advantage for studying the aggregate market behaviour because the single risky asset itself serves as a proxy for the market portfolio. Two-asset models are convenient for exploring the link between investors' behaviour at the micro-level and the aggregate market behaviour at the macro-level. In contrast, models with multiple risky assets are more suitable for studying the interaction between price dynamics of different risky assets (or different market indices) and/or the performance of various portfolio rules. Since a main goal of the thesis is to study the impact of investors' heuristics and behavioural biases on the aggregate market dynamics, our research in the remainder of the thesis will be based on the two-asset model.

The multiple risky assets model derived in the previous section serves as a general form of the model, which completes our evolutionary finance behavioural approach to asset pricing. The multiple risky asset model and its applications will be left for future researches. Our analysis of the single risky asset model presented in this thesis can be viewed as an important and crucial step towards the understanding of market behaviour and evolutionary dynamics of the multiple risky assets model.

3.4 Dynamical System and Steady State Equilibria

In the two-asset model, the expansion of the aggregate wealth and market value is governed by two important parameters: the risk-free rate of return r and the consumption rate c. Since the values of the two parameters are exogenously set, it is important to understand how different values of the two parameters affect the market dynamics. This section analyses in detail the macro-level impact of the risk-free rate of return and consumption rate.

The two-asset model can be treated as a discrete-time dynamical system. The effect of the parameter r and c can be studied by exploring how different values of the two parameters affect the existence and location(s) of the steady state(s) of the dynamical system. We first analyse the case when only one agent is present in the market. We then generalise the results to the case in which many agents are present in market.

3.4.1 Investment Function

In the two-asset model, the investment strategy of each agent $i = 1, ..., I$ corresponds to a vector of investment proportions $\lambda_t^i = (\lambda_{t,0}^i, \lambda_{t,1}^i)$ with $\lambda_{t,k}^i > 0$ and $\sum_{k=0}^{1} \lambda_{t,k}^i = 1$. Since the market impact of the parameter r and c does not depend on detailed specifications of investment strategies (i.e. the way that $\lambda_{t,k}^i$ is computed by each agent i), in this section, we still maintain a high generality of agent behaviour. We assume that, for each agent $i = 1, ..., I$, there exists an investment function g^i which maps the information set $\mathbb{I}_{t-1}$ into an agent's investment proportion for the risky asset $\lambda^i_{t,1}$:

$$\lambda^i_{t,1} = g^i(\mathbb{I}_{t-1}), \tag{3.4.1}$$

where the information set $\mathbb{I}_{t-1}$ contains all public available information such as historical prices, returns and dividend yields of the risky asset up to time $t-1$. The agent does not know or is able to compute the current price $S_{t,1}$ when he or she makes investment decision for $\lambda^i_{t,1}$. The current price $S_{t,1}$ and wealth of the agent W^i_t are simultaneously realised at time t.

Equation (3.4.1) defines a general behaviour by which agents transform past information into current investment decisions. Such a behaviour can be rational, (i.e. agent's investment function g is derived from utility or profit maximisation) or driven by heuristics and behavioural biases. Agents' behaviour (3.4.1) will be used in defining the model in terms of discrete-time dynamical system.

3.4.2 The Single Agent Case

Our analysis focuses on the *deterministic skeleton* of the two-asset model. Deterministic skeleton refers to the case where the dividend of the risky asset stays as a constant, i.e. $D_{t,1} = \bar{D}_1$ for all $t = 0, 1, ...$ in equation (3.2.12).

When there is only one agent in the market, the single agent determines the price of the risky asset which is proportional to the agent's wealth. The wealth of the single agent is identical to the aggregate wealth in the market. Since the expansion of the aggregate wealth is governed by the values of parameters r and c, the aggregate wealth hence the price of the risky asset in a steady state may consist of either increasing or decreasing sequences, or sequences of constants.

In order to characterise the behaviour of the asset price at a steady state, for convenience, we reformulate the model in terms of return of the risky asset $r^s_{t,1} = \frac{S_{t,1}-S_{t-1,1}}{S_{t-1,1}}$ and dividend yield $Y_{t,1} = \frac{D_{t,1}}{S_{t-1,1}}$. At a steady state, a positive or negative constant return corresponds to, respectively, an increasing or decreasing price series. Zero return corresponds to the case in which the asset price converges to a constant.

Dynamics of Asset Return and Dividend Yield

The dynamics of asset return and dividend yield can be described by the following proposition:

Proposition 3.2. *In the two-asset model with a single agent, i.e. $K = 1$ and $I = 1$, omitting agent-specific index, the return of the risky asset evolves as:*

$$r^s_{t+1,1} = r + \frac{(1+r)\big[(1-c)\lambda_{t+1,1} - \lambda_{t,1}\big] + (1-c)Y_{t+1,1}\lambda_{t+1,1}\lambda_{t,1}}{\lambda_{t,1} - (1-c)\lambda_{t+1,1}\lambda_{t,1}}, \tag{3.4.2}$$

and dividend yield evolves as:

$$Y_{t+1,1} = \frac{Y_{t,1}}{1 + r^s_{t,1}}, \tag{3.4.3}$$

provided that, $\lambda_{t,1} \in (0,1]$ and $\lambda_{t,0} \in [0,1)$ with $\lambda_{t,1} + \lambda_{t,0} = 1$ for all $t = 0, 1, ...,$ $D_{t,1} = \bar{D}_1$, and $c \in (0,1)$.

Proof of Proposition 3.2. Using notations of $r^s_{t,1} = \frac{S_{t,1} - S_{t-1,1}}{S_{t-1,1}}$ and $Y_{t,1} = \frac{D_{t,1}}{S_{t-1,1}}$, the wealth dynamics (3.2.13) under the conditions of $K = 1$ and $I = 1$ can be written as:

$$\begin{aligned} W_{t+1} &= (1-c)W_t[(1+r)\lambda_{t,0} + \lambda_{t,1}(r^s_{t,1} + Y_{t,1} + 1)] \\ &= (1-c)W_t[1 + r + \lambda_{t,1}(r^s_{t,1} + Y_{t,1} - r)]. \end{aligned} \tag{3.4.4}$$

The return of the risky asset can be rewritten as:

$$r^s_{t,1} = \frac{S_{t,1}}{S_{t-1,1}} - 1 = \frac{\lambda_{t+1,1}W_{t+1}}{\lambda_{t,1}W_t} - 1. \tag{3.4.5}$$

Inserting (3.4.4) into (3.4.5) yields:

$$r^s_{t,1} = \frac{\lambda_{t+1,1}}{\lambda_{t,1}}(1-c)[1 + r + \lambda_{t,1}(r^s_{t,1} + Y_{t,1} - r)] - 1. \tag{3.4.6}$$

Solving (3.4.6) for $r^s_{t,1}$ gives:

$$r^s_{t,1} = \frac{(1+r)(1-c)\lambda_{t+1,1} + (1-c)(Y_{t+1,1} - r)\lambda_{t+1,1}\lambda_{t,1} - \lambda_{t,1}}{\lambda_{t,1} - (1-c)\lambda_{t+1,1}\lambda_{t,1}}. \tag{3.4.7}$$

Here, the assumption $c \in (0, 1)$, $\lambda_{t,1} \in (0, 1]$ and $\lambda_{t,0} \in [0, 1)$ with $\lambda_{t,1} + \lambda_{t,0} = 1$ for all $t = 0, 1, ...$ ensures the above equation for returns is well-defined.

Adding both $\lambda_{t,1}r$ and $-\lambda_{t,1}r$ to the numerator of (3.4.7) and re-arranging the equation gives:

$$r^s_{t+1,1} = r + \frac{(1+r)\big[(1-c)\lambda_{t+1,1} - \lambda_{t,1}\big] + (1-c)Y_{t+1,1}\lambda_{t+1,1}\lambda_{t,1}}{\lambda_{t,1} - (1-c)\lambda_{t+1,1}\lambda_{t,1}}.$$

Under the assumption $D_{t,1} = \bar{D}_1$ for all $t = 0, 1, ...$, the dividend yield can be written as:

$$Y_{t+1,1} = \frac{D_{t+1,1}}{S_{t,1}} = \frac{\bar{D}_1}{S_{t,1}} \text{ and } Y_{t,1} = \frac{D_{t,1}}{S_{t-1,1}} = \frac{\bar{D}_1}{S_{t-1,1}},$$

which gives:

$$Y_{t+1,1} = Y_{t,1}\frac{S_{t-1,1}}{S_{t,1}} = \frac{Y_{t,1}}{1 + r^s_{t,1}}.$$

□

Dynamical System

Equations (3.4.1), (3.4.2) and(3.4.3) form a deterministic dynamical system:

$$\begin{cases} \lambda_{t+1,1} = g(\mathbb{I}_t), \\ r^s_{t+1} = r + \frac{(1+r)\big[(1-c)\lambda_{t+1,1} - \lambda_{t,1}\big] + (1-c)Y_{t+1}\lambda_{t+1,1}\lambda_{t,1}}{\lambda_{t,1} - (1-c)\lambda_{t+1,1}\lambda_{t,1}}, \\ Y_{t+1} = \frac{Y_t}{1+r^s_{t,1}}. \end{cases} \tag{3.4.8}$$

The dynamical system contains a vector of three time dependent variables (λ_1, r^s_1, Y_1). A steady state of the system (3.4.8) corresponds to constant vector $(\lambda_{*,1}, r^s_{*,1}, Y_{*,1})$ solving (3.4.8). The constant dividend yield i.e. $Y_{*,1} = \frac{Y_{*,1}}{1+r^s_{*,1}}$ implies that the yield is either positive or zero in steady state(s). Positive yield implies that the term of $r^s_{*,1}$ in the denominator has to be zero. Note that since the dividend is strictly positive ($\bar{D}_1 > 0$), the steady state with zero dividend yield can only be observed asymptotically, which corresponds to the case where the asset return $r^s_{*,1} > 0$ so that price eventually grows to infinity.

Different Types of Steady States

Our findings of the steady states of the dynamical system (3.4.8) are summarised by the following proposition:

Proposition 3.3. *(i) When* $c < \frac{r}{1+r}$*, the dynamics generated by the system (3.4.8) has two different types of steady states. The first type of steady state is characterised by:*

$$Y_{*,1} = 0, \quad r^s_{*,1} = r - \frac{c\cdot(1+r)}{1-(1-c)\lambda_{*,1}} \in (0,r), \quad \lambda_{*,1} \in \big(0, 1 - \frac{c}{r(1-c)}\big].$$

The second type of steady state is characterised by:

$$0 < Y_{*,1} < \frac{c}{1-c} < r, \quad r^s_{*,1} = 0, \quad \lambda_{*,1} = \frac{r - \frac{c}{1-c}}{r - Y_{*,1}} \in \big(1 - \frac{c}{r(1-c)}, 1\big].$$

(ii) When $c > \frac{r}{1+r}$*, the dynamical system (3.4.8) has only one type of steady state which is characterised by:*

$$Y_{*,1} \geq \frac{c}{1-c} > r, \quad r^s_{*,1} = 0, \quad \lambda_{*,1} = \frac{r - \frac{c}{1-c}}{r - Y_{*,1}} \in (0,1].$$

(iii) When $c = \frac{r}{1+r}$*, the dynamical system (3.4.8) has only one type of steady state which is characterised by:*

$$Y_{*,1} = r, \quad r^s_{*,1} = 0, \quad \lambda_{*,1} \in (0,1].$$

Proof of Proposition 3.3. The constant dividend yield $Y_{*,1} = \frac{Y_{*,1}}{1+r^s_{*,1}}$ implies that the yield is either positive or zero in steady state(s).

The case of zero dividend yield:

In a steady state, the asset return is given by:

$$r^s_{*,1} = r - \frac{c\cdot(1+r)}{1-(1-c)\lambda_{*,1}}. \tag{3.4.9}$$

Positive asset return $r^s_{*,1} > 0$ implies:

$$r > \frac{c\cdot(1+r)}{1-(1-c)\lambda_{*,1}}. \tag{3.4.10}$$

Since $\lambda_{*,1} \in (0,1]$ and $c \in (0,1)$, rearranging 3.4.10 gives:

$$0 < \lambda_{*,1} < 1 - \frac{c}{r(1-c)}. \tag{3.4.11}$$

In equation (3.4.11) $\lambda_{*,1}$ exists only if $1 - \frac{c}{r(1-c)} > 0$ which implies $c < \frac{r}{1+r}$. Therefore, in the case of zero dividend yield, the steady states only exist when both of the following conditions hold:

$$c < \frac{r}{1+r}, \text{ and } \lambda_{*,1} < 1 - \frac{c}{r(1-c)}, \tag{3.4.12}$$

and the steady states is characterised by:

$$Y_{*,1} = 0, \quad \lambda_{*,1} \in \big(0, 1 - \frac{c}{r(1-c)}\big), \quad r^s_{*,1} = r - \frac{c \cdot (1+r)}{1-(1-c)\lambda_{*,1}} \in (0, r).$$

The case of positive dividend yield:

If the constant dividend yield at steady-state is positive, the last equation of system (3.4.8) implies that the constant price return must be zero:

$$Y_{*,1} > 0 \text{ and } Y_{*,1} = \frac{Y_{*,1}}{r^s_{*,1}+1} \quad \Rightarrow \quad r^s_{*,1} + 1 = 1 \quad \Rightarrow r^s_{*,1} = 0,$$

which corresponds to the case that the price of the risky asset at a steady state is a constant. Positive dividend yield and zero price return implies that at a steady state:

$$\frac{(1-c)Y_{*,1}\lambda_{*,1} - c \cdot (1+r)}{1-(1-c)\lambda_{*,1}} = -r.$$

Rearranging the above equation gives:

$$(1-c)(r - Y_{*,1})\lambda_{*,1} = r(1-c) - c. \tag{3.4.13}$$

Equation (3.4.13) indicates that the value of $\lambda_{*,1}$ depends on the value of $r - Y_{*,1}$. Let us first consider the case when $Y_{*,1} \neq r$, this implies:

$$\lambda_{*,1} = \frac{r(1-c) - c}{(1-c)(r - Y_{*,1})} = \frac{r - \frac{c}{1-c}}{r - Y_{*,1}}. \tag{3.4.14}$$

If $Y_{*,1} < r$, the numerator on the right-hand side of 3.4.14 must be greater than zero to ensure positive $\lambda_{*,1}$, i.e. $r - \frac{c}{1-c} > 0$ which gives $c < \frac{r}{1+r}$. This requirement is the same as the first part of condition (3.4.11) in the zero dividend yield case. When $0 < \lambda_{*,1} \leq 1$, equation (3.4.14) implies the positive dividend yield satisfies $0 < Y_{*,1} \leq \frac{c}{1-c} < r$. In addition, $Y_{*,1} < r$ also implies $\frac{r-\frac{c}{1-c}}{r-Y_{*,1}} > 1 - \frac{c}{r(1-c)}$, therefore we can obtain that if:

$$c < \frac{r}{1+r} \text{ and } \lambda_{*,1} = \frac{r - \frac{c}{1-c}}{r - Y_{*,1}} \in \big(1 - \frac{c}{r(1-c)}, 1\big],$$

there exists another type of steady states which is characterised by:

$$0 < Y_{*,1} < \frac{c}{1-c} < r, \quad \lambda_{*,1} = \frac{r - \frac{c}{1-c}}{r - Y_{*,1}} \in \big(1 - \frac{c}{r(1-c)}, 1\big], \quad r^s_{*,1} = 0.$$

If $Y_{*,1} > r$, the numerator in the right-hand side of equation (3.4.14) must be smaller than zero to ensure positive $\lambda_{*,1}$, i.e. $r - \frac{c}{1-c} > 0$ which implies $c > \frac{r}{1+r}$. Furthermore, $0 < \lambda_{*,1} \leq 1$ in equation (3.4.14) implies that the positive dividend yield satisfies $Y_{*,1} \geq \frac{c}{1-c} > r$. Therefore, different to the case of $c < \frac{r}{1+r}$, when $c > \frac{r}{1+r}$ there is only one type of steady states which is characterised by:

$$Y_{*,1} \geq \frac{c}{1-c} > r, \quad \lambda_{*,1} = \frac{r - \frac{c}{1-c}}{r - Y_{*,1}} \in (0, 1], \quad r^s_{*,1} = 0.$$

Note that by equation (3.4.13) $Y_{*,1} = r \Leftrightarrow c = \frac{r}{1+r}$. When $c = \frac{r}{1+r}$, equation (3.4.11) in zero dividend yield case does not have solution satisfying the constraint $\lambda_{*,1} \in (0, 1]$. Therefore, when $c = \frac{r}{1+r}$, there exist one type of steady state which is characterised by:

$$Y_{*,1} = r, \quad \lambda_{*,1} \in (0, 1], \quad r^s_{*,1} = 0$$

□

Arbitrage-Free Equilibrium

In a steady state, the risky asset is no longer "risky" because it has constant price return and dividend yield. The asset can be treated like another "risk-free" asset which pays a constant rate of return $r^s_{*,1} + Y_{*,1}$. The arbitrage-free principle implies that the aggregate rate of return from the risky asset $r^s_{*,1} + Y_{*,1}$ should be equal to the risk-free rate of return. Otherwise for example if $r^s_{*,1} + Y_{*,1} < r$ the demand of the risky asset will decrease as it appears less attractive in comparison with the risk-free rate of return, which causes the decrease in price of the risky asset and therefore increase the dividend yield (as dividend is a constant). As a result of the arbitrage-free equilibrium, $r^s_{*,1}$ will eventually converge to zero and $Y_{*,1}$ converge to r. Similar situation holds if $r^s_{*,1} + Y_{*,1} > r$. Therefore, under the arbitrage-free equilibrium, we can conclude that $r^s_{*,1} + Y_{*,1} = r$ which happens either $r^s_{*,1} = r$ (price grows to infinity and dividend yield $Y_{*,1}$ is zero) or $r^s_{*,1} = 0$ and $Y_{*,1} = r$ (price converges to a constant and dividend yield equals risk-free rate of return). We cannot have both positive constant price return and dividend yield i.e. $r^s_{*,1} > 0$ and $Y_{*,1} > 0$ as the dividend is a constant.

The case of $r^s_{*,1} = 0$ and $Y_{*,1} = r$ corresponds to the steady state where $c = \frac{r}{1+r}$, the price at this arbitrage-free equilibrium steady state is given by $S_{*,1} = \frac{\bar{D}_1}{r}$. We will treat this price as a fundamental value of the risky asset. Note that the fundamental price derived in this way is identical to the one computed by using discounted dividend model from the finance literature i.e.:

$$S_{*,1} = \sum_{j=1}^{\infty} \frac{\bar{D}_1}{(1+r)^j} = \frac{\bar{D}_1}{r}$$

Our method in deriving the fundamental value of the risky asset is an alternative interpretation from the arbitrage-free equilibrium. Note that the dividend in the above derivation is assumed to be a constant, in the case of i.i.d. dividend process as in (3.2.12) with $\mathrm{E}_t[D_{t+j,1}] = \bar{D}_1$, $j = 1, 2, 3...,$, the above computation of fundamental price still holds, i.e. $S_{*,1} = \sum_{j=1}^{\infty} \frac{\mathrm{E}_t[D_{t+j,1}]}{(1+r)^j} = \frac{\bar{D}_1}{r}$. The arbitrage-free steady state in the i.i.d. dividend case refers to a stationary distribution of the return the risky asset, in which the asset price fluctuates closely around the constant fundamental price. The long run average total return of the risky asset (price return plus dividend yield) equals the constat risk-free rate of return.

To briefly summarise the effect of the consumption rate and risk-free rate of return: when $c < \frac{r}{1+r}$, two different types of steady states ($r^s_{*,1} > 0$ and $r^s_{*,1} = 0$) exist depending the value of $\lambda_{*,1}$, however, this case implies $r^s_{*,1} + Y_{*,1} < r$ which violates the arbitrage-free condition; when $c > \frac{r}{1+r}$, only one type of steady state ($r^s_{*,1} = 0$, $0 < Y_{*,1} < r$) exist, which also violates the arbitrage-free condition; when $c = \frac{r}{1+r}$, there exist an unique type of steady state where $r^s_{*,1} = 0$, $Y_{*,1} = r$

and $\lambda_{*,1} \in (0,1]$. This type of steady state satisfies the arbitrage-free condition.

3.4.3 The Multi-Agent Case

When there exists $I > 1$ different agents in the market, agents' wealth dynamics start to play an role in the dynamical system. The asset price and return are affected by agents' relative wealth and the flow of funds.

Let $\omega_t^i = \frac{W_t^i}{\bar{W}_t^i}$ and $\hat{\omega}_t^i = \frac{\hat{W}_t^i}{\bar{W}_t^i}$ denote, respectively, agent i's relative wealth at time t before strategy-switching and after strategy-switching. Since the wealth of agent i evolves as:

$$W_{t+1}^i = (1-c)\hat{W}_t^i[(1+r)\lambda_{t,0} + \lambda_{t,1}(r_{t,1}^s + Y_{t,1} + 1)]. \tag{3.4.15}$$

A straightforward computation shows that the agent's relative wealth before strategy-switching evolves as:

$$\omega_{t+1}^i = \hat{\omega}_t^i \frac{1 + r + (r_{t,1}^s + Y_{t,1} - r)\lambda_{t,1}}{1 + r + (r_{t,1}^s + Y_{t,1} - r)\sum_{i=1}^I \lambda_{t,1}\hat{\omega}_t^i}, \tag{3.4.16}$$

where $\hat{\omega}_t^i = (1-\beta)\omega_t^i + \beta q_t^i$ with $\beta \in [0,1]$.

Using the terms of relative wealth, the asset price at time $t+1$ in the presence of multiple agents can be reformulated as:

$$\begin{aligned} S_{t+1,1} &= (1-c)\sum_{i=1}^I \lambda_{t+1,1}[(1-\beta)W_{t+1}^i + \beta q_t^i \bar{W}_{t+1}] \\ &= (1-c)\bar{W}_{t+1}\sum_{i=1}^I \lambda_{t+1,1}[(1-\beta)\omega_{t+1}^i + \beta q_t^i] \\ &= (1-c)\bar{W}_{t+1}\sum_{i=1}^I \lambda_{t+1,1}\hat{\omega}_{t+1}^i. \end{aligned} \tag{3.4.17}$$

A steady state in the multi-agent case requires not only constant asset return $r^s_{*,1}$ and dividend yield $Y_{*,1}$ but also constant investment proportion $\lambda^i_{*,1}$, relative wealth ω^i_* and $\hat{\omega}^i_*$, and switching probability q^i_* for all $i = 1, ..., I$.

In a steady state, the price dynamics (3.4.17) can be reformulated using the constant investment proportion $\lambda^i_{*,1}$ and relative wealth $\hat{\omega}^i_*$ as:

$$S_{t+1,1} = (1-c)\bar{W}_{t+1} \sum_{i=1}^{I} \lambda^i_{*,1} \hat{\omega}^i_*. \tag{3.4.18}$$

According to the above equation, the market can be viewed as having one representative agent whose wealth equals the aggregate wealth $\bar{W}_{t+1}$ and investment proportion equals the constant $\bar{\lambda}_{*,1}$ with $\bar{\lambda}_{*,1} = \sum_{i=1}^{I} \lambda^i_{*,1} \hat{\omega}^i_*$. Therefore, the Proposition 3.3 in the single agent case holds for the multi-agent case in terms of $\bar{\lambda}_{*,1}$. This result indicate that the effect of the risk-free rate of return r and consumption rate c does not depend on the number of different agents as well as the flow of funds.

To summarise, we have shown in this section that different values of the parameter r and c affect the locations of steady states of the two-asset model. The effect of the parameter r and c is independent of the number of different agents presented in the market. An important finding is that the steady state under the condition $c = \frac{r}{1+r}$ corresponds to the arbitrage-free equilibrium. For this reason, we will set $c = \frac{r}{1+r}$ in our further analyses presented in this thesis.

3.5 Specification of Agent Behaviour

This section deals with the specification of agents' investment strategies in our two-asset model. In the finance literature, traditional finance approach assumes that the representative agent is risk averse and his or her demand is derived to maximise the expected utility of future wealth. Although agent-based modelling is able to maintain a large degree of freedom on specifying agents' behaviour, a commonly seen approach in agent-based models still replies on the assumption of risk averse utility maximising agents. We first review utility maximising approaches to specifying agent behaviour in agent-based literature. We discuss shortcomings and drawbacks of using utility maximising approaches in agent-based models. We then propose an alternative approach to specifying agents' decisions of investment proportions.

3.5.1 Review of Utility Maximising Approaches

In our model, agents' demands (number of shares) of the risky asset is characterised by their investment proportions $\lambda^i_{t,1}$ with $i = 1, ..., I$. The investment proportion $\lambda^i_{t,1}$ does not depend on the contemporaneous wealth of agent i. This implies that the demand (i.e. $\frac{\lambda^i_{t,1}\hat{w}^i_t}{S_{t,1}}$) of agent i for the risky asset is linearly increasing with the wealth $\hat{w}^i_t$ for a given S_t. Such property is consistent with the characterisation of constant relative risk aversion (CRRA).

We review, in this subsection, agent-based models with utility maximising agents which are consistent with the CRRA framework. We show that using utility maximising approach can be problematic in models with a Walrasian market clearing mechanism (or temporary market equilibrium clearing mechanism) and a fixed positive supply of the risky asset. Examples of those models are Chiarella and He (2001, 2002), Levy, Levy, and Solomon (1994, 1995, 2000) and Anufriev and Dindo (2010).

The Power Utility Case

One typical example of agents' utility function is the power (CRRA) utility of wealth in the form of:

$$U(W) = \begin{cases} \frac{1}{1-A}(W^{1-A} - 1), & \text{for} \quad A \neq 1 \\ \ln(W), & \text{for} \quad A = 1 \end{cases} \tag{3.5.1}$$

where the wealth $W > 0$ and the parameter A represents the relative risk aversion coefficient.

Each agent $i = 1, ..., I$ is assumed to seek an optimal value for the investment proportion $\lambda^i_{t,1}$ which maximises his or her expected utility of wealth at time $t+1$, as given by

$$\max_{\lambda^i_{t,1}} E^i_t[U(W^i_{t+1})]. \tag{3.5.2}$$

The two strands of researches, Chiarella and He (2001, 2002) and Levy, Levy, and Solomon (1994, 1995, 2000), represent two commonly used but different methods toward solving the CRRA utility maximisation problem (3.5.2) in agent-based models.

The model in the paper of Chiarella and He (2001) is based on a standard Walrasian market clearing mechanism. Under this market clearing mechanism, the asset price $S_{t,1}$ and the wealth W^i_t of agent i are simultaneously realised after the investment proportions $\lambda^i_{t,1}$ are computed by all agents $i = 1, ..., I$. In order to ensure the price $S_{t,1}$ can be solved explicitly, a commonly used and important assumption is that agents do not know or are not able to compute $S_{t,1}$ when computing $\lambda^i_{t,1}$ at time t. In other words, agents do not take into account the current price $S_{t,1}$ in their decisions of investment proportions at time t. Based on such an assumption, a closed-form approximation to the solution of the CRRA utility maximisation problem (3.5.2) is obtained by Chiarella and He (2001).

The authors show that, when the wealth W^i_t is given by the continuous time stochastic differential equation $dW^i = \mu(W^i)dt + \sigma(W^i)dz_t$ where dz_t is a Wiener process, the CRRA utility function in (3.5.1) can be rearranged as a stochastic differential utility. A discrete time approximation of the stochastic differential utility function can be obtained by Euler scheme, hence an approximate solution for the CRRA utility maximisation problem can be derived explicitly in the following form:

$$\lambda^i_{t,1} = \frac{E^i_t[R^s_{t+1,1} - R]}{A^i V^i_t[R^s_{t+1,1} - R]}, \tag{3.5.3}$$

where $R = 1 + r$ and $R^s_{t,1} = 1 + r^s_{t,1} + Y_{t,1}$ are the gross returns of the risk-free and the risky asset respectively; E_t and V_t denote, respectively, the operators of expectation and variance conditional on information available at time t.

The numerator of the optimal investment proportion (3.5.3) is the expected return of the risky asset in excess of the risk-free rate of return, which is forecasted by agent i at time t. The denominator of (3.5.3) represents the adjusted market risk which is estimated by agent i at time t. The intuition of (3.5.3) is straightforward, where the optimal investment proportion can be interpreted as a proportion of the expected excess return which is normalised by the risk undertaken.

However, using (3.5.3) in agent-based models with a Walrasian market clearing mechanism and a fixed positive supply of the risky asset may have several drawbacks. First, the optimal investment proportion (3.5.3) is obtained based on the assumption of a continuous time stochastic process of wealth which is an exogenous process. Such an assumption may not be true in agent-based models since the wealth dynamics in agent-based models are endogenously determined rather than exogenously imposed. Second, the value of the investment proportion in (3.5.3) can be outside the interval between 0 and 1. For instance, the value of (3.5.3) can be negative when agent i expects a negative excess return. A negative investment proportion represents short selling. Similarly, the value of (3.5.3) can exceed 1 which represent that agent i intends to borrow money to make the investment. As pointed out by Levy, Levy, and Solomon (2000) and and Anufriev and Dindo (2010), in agent-based models with Walrasian market clearing mechanism (or temporary market equilibrium clearing mechanism) and a fixed positive supply of the risky asset, allowing short selling and/or borrowing may cause negative wealth, bankruptcy, and negative prices.

To fix this problems, Chiarella and He assume that agents believe there is a reasonable large risk premium built into the risky asset so that agents' expected excess returns (the numerator of (3.5.3)) are always positive, and their investment proportions for the risky asset do not exceed 1. This is a highly hypothetical scenario of financial markets and the beliefs of their participants. If investors believe that the risky asset can always yield an positive excess return, there would be no incentive for investors to hold the risk-free asset which contradicts to the optimal solution in (3.5.3).

Levy, Levy, and Solomon (1994, 1995, 2000) represent another common approach to the CRRA utility maximising agents. It is based on the so-called temporary market equilibrium clearing mechanism. The main difference between the approaches of Chiarella and He (2001) and Levy, Levy, and Solomon (1994, 1995, 2000) is that the latter assume that agents take into account the current price $S_{t,1}$ when solving the CRRA utility maximisation problem for $\lambda_{t,1}^i$ at time t. Under this assumption, the optimal solution for $\lambda_{t,1}^i$ as well as the market clearing price $S_{t,1}$ have implicit solutions which may only be solved numerically. This is implemented by Levy, Levy, and Solomon via introducing a hypothetical price $S_{t,1}(h)$. Based on the hypothetical price $S_{t,1}(h)$, a hypothetical investment proportion $\lambda_{t,1}^i(h)$ for each agent $i = 1, ..., I$ is computed to solve the utility maximisation problem. The market clearing price $S_{t,1}$ is then determined through searching a $S_{t,1}(h)$ by which the hypothetical aggregate demand $N_t(h) = \frac{\sum_{i=1}^I \lambda_{t,1}^i(h)\hat{W}_t^i(h)}{S_{t,1}(h)}$ matches the fixed supply $N > 0$ of the risky asset. Agents' hypothetical investment proportions which solve the utility maximisation problem and the market clearing price are the optimal investment proportions. Such process is illustrated in Figure 3.1.

The market clearing mechanism used by Levy, Levy, and Solomon is equivalent to assume that every investor in the market knows other investors' strategies so that all investors are able to compute the current market clearing price. On the one hand, this assumption requires an extremely high degree of informational efficiency of the market so that every individual investors in the market is able to collect information about others' decision. On the other hand, this assumption requires that all investors have learned well the Nash equilibrium so their investment proportions depend on all other investors' strategies and are computed accordingly. Such an assumption seems unrealistic. Moreover, this numerical approach cannot guarantee the optimal investment portion lies in the interval between 0 and 1. This may cause negative wealth and price. In order to avoid such problems, an additional assumption is imposed by Levy, Levy and Solomon.

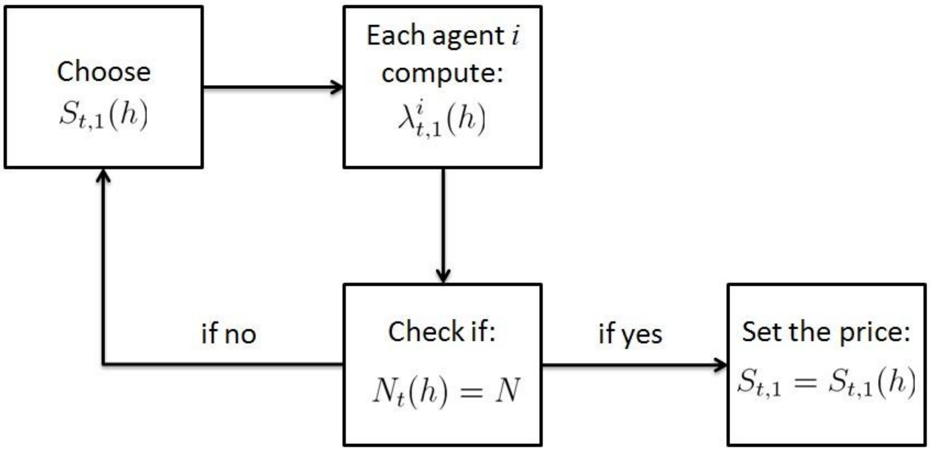

Figure 3.1: Flowchart of the market clearing mechanism in microscopic simulation models proposed by Levy, Levy, and Solomon (1994, 1995, 2000).

The authors assume that agents' investment proportions are constrained to lie in an interval $[a, b] \subset (0, 1)$. The parameters a and b represent the minimum and maximum level of the investment proportion respectively. Under this assumption, if investors' optimal solution for investment proportions exceed one of the two boundaries, their investment proportion will be set to the nearest boundary.

The Mean-Variance Optimization Case

Another commonly used approach on specifying agents' investment proportions is to assume that agents are myopic mean-variance maximisers. Anufriev and Dindo (2010) may serve as an example for this approach. In the paper of Anufriev and Dindo (2010), each agent $i = 1, ..., I$ is assumed to have a mean-variance utility of the next period total return:

$$U^i = E^i_t\Big[(1-\lambda^i_{t,1})R + \lambda^i_{t,1}R^s_{t+1,1}\Big] - \frac{A^i}{2}V^i_t\Big[(1-\lambda^i_{t,1})R + \lambda^i_{t,1}R^s_{t+1,1}\Big]. \tag{3.5.4}$$

The first order condition of the maximisation problem (i.e. $\max_{\lambda^i_{t,1}} U^i$) leads to a solution which is exactly the same as the one in (3.5.3). This mean-variance utility maximising approach provides an alternative way to justify the optimal investment proportion in the form of (3.5.3). In addition, Anufriev and Dindo (2010) assume that agents are homogeneous with respect to the conditional variance, and the conditional variance stays as a constant σ^2, i.e. $V_t^i = \sigma^2$ for all $i = 1, ..., I$. This is another commonly seen assumption which has been widely used in agent-based models, for example, Brock and Hommes (1997, 1998), Hommes (2002), Chiarella and He (2002) and Gaunersdorfer and Hommes (2007). Under such an assumption, the optimal solution in (3.5.3) can be simplified to the following form:

$$\lambda_{t,1}^i = \frac{E_t^i[R_{t+1,1}^s - R]}{A^i \sigma^2}. \tag{3.5.5}$$

As discussed above, the solution in (3.5.3) as well as the simplified one in (3.5.5) may cause negative wealth and price. To solve this issue, similar to Levy, Levy, and Solomon (1994,1996,2000), Anufriev and Dindo assume that agents' investment proportions in the form of (3.5.5) are bounded in the interval of $[a, b] \subset (0, 1)$ so that the investment proportion $\lambda_{t,1}^i$ of an agent i is given by:

$$\lambda_{t,1}^i = \min\Big\{b, \max\Big\{a, \frac{E_t^i[R_{t+1,1}^s - R]}{A^i \sigma^2}\Big\}\Big\}. \tag{3.5.6}$$

An important drawback of the investment function (3.5.6) is that agents react very differently over gains (positive expected excess returns) and losses (negative expected excess return). As illustrated in Figure 3.2, when the expected excess return is positive, the investment proportion (for the risky asset) increases linearly along with the increase in the expected excess return. However, when the expected excess return is negative, even for a very small negative value, agents will always hold the minimum level of investment proportion which is close to zero. In other words, agents are indifferent with respect to different degrees of negative expected returns. Investment function (3.5.6) implies that agents' investment proportions are very sensitive to the sign of the expected excess return. When agents' expected excess returns shift between the positive and negative values, the asymmetry in agents' reactions to expected gains and losses may cause unrealistic dynamics of sudden booms and crashes in agents' investment proportions as well as in the price. This is exactly what happens in the simulations of Anufriev and Dindo (2010).

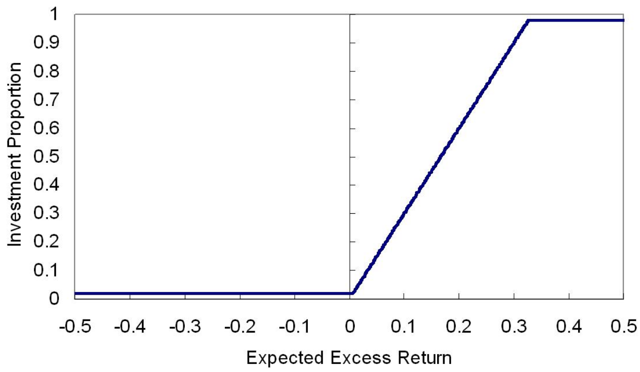

Figure 3.2: An example of the investment function used by Anufriev and Dindo (2010).

To summarise, the commonly used investment function (3.5.3) which is derived from expected utility maximisation may cause the value of the investment proportion $\lambda^i_{t,1}$ to lie outside of the interval between 0 and 1. This may lead to negative wealth and price in models with a Walrasian market clearing mechanism and a fixed positive supply of the risky asset. In order to fix this problem, additional assumptions are required, such as introducing a reasonable large risk premium to rule out negative expected excess returns of agents (as used in Chiarella and He, 2002) or imposing a constrained investment proportion $\lambda^i_{t,1} \in [a, b] \subset (0, 1)$ (as adopted by Anufriev and Dindo, 2010). The former assumption seems highly hypothetical which lacks empirical evidence, while the latter one may case unrealistic price dynamics. Moreover, the assumption of constrained investment proportions causes that all agents will hold a minimum investment proportion when negative expected excess returns are encountered. The homogeneity in agents' behaviour in reaction to different degrees of negative expected excess returns seems unrealistic.

3.5.2 Endogenous Risk

We discuss here another shortcoming of using utility maximising approach in agent-based models: the *endogenous risk* is often ignored by utility maximising agents. The concept of endogenous risk, as defined by Danielsson and Shin (2003), refers to the risk from shocks that are generated and amplified within the system. It stands in contrast to exogenous risk, which refers to shocks that arrive from outside the system. According to Danielsson and Shin (2003), endogenous risk appears in a financial market if individuals react together to the market dynamics or the individual actions affect the market dynamics. The authors pointed out that financial markets in reality do exhibit endogenous risk [3].

In agent-based models, individual investors are commonly modelled by different agent types such as fundamentalist and chartist which characterise some typical investment styles in real markets. Asset prices in agent-based models are endogenously determined by agents' demands and the supplies of assets. During such a price discovery process, agents' actions do affect the asset prices. Both conditions mentioned by Danielsson and Shin (2003) for the appearance of endogenous risk are main characteristics of agent-based models. The endogenous risk is therefore an important sources of risk in agent-based models of financial markets.

Moreover, the impact of agents' actions on price dynamics may increase if i) the model contains only a small number of different agent types, such as the commonly discussed fundamentalist-chartist models; ii) the proportion (in terms of population size and/or wealth) of a particular agent type is (or becomes) large. On the one hand, a small number of different agent types implies that each agent type in the model may play an important role in affecting price dynamics. On the other hand, a substantial increase or decrease in a large agent's demand of an asset may trigger a significant increase or decrease in the asset price. For this

[3] Danielsson and Shin (2003) use examples of the 1987 stock market crash, the 1998 Long Term Capital Management crisis, and the collapse of dollar/yen in October 1998 to illustrate the existence and effect of endogenous risk in financial markets.

reason, the endogenous risk turns out to be a crucial risk in agent-based model with only a few agent types. Agents in those "few-type" models have to deal with the endogenous price risk which is generated by their own activities.

It is well known that an important assumption behind the utility maximising approach in the modern portfolio theory is that all investors are price takers and their actions do not influence the price. Based on this assumption, the price dynamics in traditional finance models with utility maximising agents are often modelled by a stochastic process which is exogenously given. Investors in those models only face to external (e.g. fundamental) sources of risk of asset prices. This approach contradicts to the nature of agent-based models. Therefore, modelling agents as utility maximisers who are not aware of the endogenous price risk in agent-based models is inappropriate, and may cause unrealistic price dynamics (such as the "booms and crashes" type of dynamics) and poor performance of utility maximising agents. Such a problem is especially prominent in models with a standard Walrasian equilibrium and a fixed positive supply of the risky asset. The reason is discussed as follows.

In agent-based models with a standard Walrasian market clearing mechanism, a widely used assumption is that agents do not know or are not able to compute the current market clearing price $S_{t,1}$ when making decisions for investment proportions λ_t^i at time t. Based on this assumption, for example, the current price $S_{t,1}$ is excluded by utility maximising agents in the computation of the conditional mean $E_t^i[R_{t+1,1}^s - R]$ (the expected excess return) and conditional variance $V_t^i[R_{t+1,1}^s - R]$ (the risk estimated) in the investment function (3.5.3). Moreover, it is common to assume that the conditional variance stays as a constant (as mentioned above) or depends on past information such as asset prices up to time $t-1$ (see, e.g. Chiarella and He, 2001). Therefore, the endogenous price risk of $S_{t,1}$ at time t is ignored by utility maximisation agents whose investment proportions at the same time are computed to maximise their expected utility at time $t+1$. Such a behaviour may lead to an unexpected loss at time t for agents before their expected utilities can be maximized at time $t+1$. This phenomena is illustrated by the following example.

Suppose that agent i represents a large group of investors whose investment proportion $\lambda^i_{t-1,1} \in [a, b]$ with $[a, b] \subset (0, 1)$ at time $t-1$ is large, say 70% of wealth. The large investment proportion corresponds to a relatively high demand of the risky asset of agent i at time $t-1$. If this agent expects that the excess return at time $t+1$ will be negative, the investment function (3.5.6) implies that this agent will hold the minimum level of investment proportion $\lambda^i_{t,1} = a$ at time t, say $a = 1\%$, to minimise his or her expected loss for time $t+1$. The minimum level of investment proportion corresponds to an extremely low demand of agent i for the risky asset at time t. Such a low demand as well as the sharp decrease in demand between time $t-1$ and time t may have cause a significant decrease (e.g. a crash) in price $S_{t,1}$ at time t (if the supply of the risky asset is fixed and positive). The significant decrease in $S_{t,1}$ will lead to an unexpected loss for agent i at time t since agent i did not take $S_{t,1}$ or his or her own impact on $S_{t,1}$ (the endogenous price risk) into consideration when making decision for $\lambda^i_{t,1}$. This behaviour is equivalent to sell the risky asset at a low price at time t.

In contrast, if the agent i in the above example invested a moderate level of investment proportion at time t (e.g. 40% of wealth or higher) rather than holding the minimum level at 1%, such a behaviour may have a chance to mitigate the potential loss caused by his or her own activity at time t, or it may even lead to a potential gain at time t if the price $S_{t,1}$ is increased due to the value of $\lambda^i_{t,1}$. For this reason, when a negative expected excess return is encountered by agents at time t, if agents were aware of the endogenous price risk caused by their own activities, they have to seek a tradeoff between the potential losses (or gains) caused by their own activities at time t and their expected losses for time $t+1$ (i.e. the negative expected excess return). Such a tradeoff can be possibly realised by holding a moderate (or high) level of investment proportion at time t. The value of the investment proportion should somehow depend on the degree of the negative expected excess return. For example, a large (or small) absolute value of the negative expected excess return should correspond to a lower (or higher) investment proportions at time t.

Holding a positive and moderate level of investment proportion when the expected excess return is negative represents risk seeking behaviour of agents. Such a behaviour contradicts to the risk averse utility maximisation approach but it is consistent with the concept of *loss aversion* (as documented in Tversky and Kahneman, 1984) especially the *myopic loss aversion* (as shown in Thaler et al., 1997). According to Thaler et al. (1997), investors who exhibit both myopia and loss aversion are more willing to accept risks in order to mitigate losses.

In the case where a positive expected excess return is encountered by agents, the awareness of endogenous price risk may imply that agents would not invest as much as possible but a reasonable value for the investment proportion at time t. The value of the investment proportion should also be positively correlated to the value of the expected excess return. This is because that a very large investment proportion may significantly increase the price $S_{t,1}$, which is equivalent to buy the risky asset at a relatively high price at time t. This may leave agents in a position with a high risk of having losses at time $t + 1$ if the price drops at time $t + 1$. Agents' behaviour for gains under the awareness of endogenous price risk can be therefore characterised by risk aversion.

To summarise, in agent-based models with a standard Walrasian equilibrium and a fixed positive supply of the risky asset, since short-selling and borrowing are not allowed, the awareness of endogenous risk may imply that agents tend to be risk seeking for losses but risk averse for gains. This phenomenon is consistent with the prospect theory in behavioural finance. Moreover, for each agent, the endogenous risk may increase along with the increase in the absolute value of agent's expected excess return. This property should be reflected by agents' investment functions of expected excess returns in order to prevent overreaction and losses which are caused by their own activities.

In contrast, the utility maximisation approach which ignores the endogenous risk may put agents into a situation with unexpected losses caused by their own activities. For this reason, the utility maximisation agents in models with a Walrasian equilibrium and a fixed positive supply of the risky asset usually perform poorly. As shown in Zschischang and Lux (2001) and Anufriev and Dindo (2010), even agents with constant investment proportions can possibly outperform utility maximising agents. Similar results are also obtained in Evstigneev, Hens and Schenk-Hoppé (2009) with a model of multiple risky assets. The poor performance of utility maximisation agents cast doubt on the validity of using utility maximisation (without considering the endogenous price risk) to characterise agents' behaviour. For this consideration, it would be more reasonable to assume that agents are aware of the endogenous price risk caused by their own activities.

3.5.3 A Sigmoid Investment Function

In order to properly specify agents' demand for the risky asset, we make the following two crucial assumptions for all agents $i, = 1, ..., I$:

A1: Agent i does not know or be able to compute the current market clearing price $S_{t,1}$ when making decision for the investment proportion $\lambda^i_{t,1}$ at time t;

A2: Agent i is informed about the *endogenous price risk* in the sense that agent i is aware of his or her investment proportion $\lambda^i_{t,1}$ may be positively correlated to the market clearing price $S_{t,1}$ at time t.

As discussed above, based on the assumption A1 and A2, when a negative expected excess return is encountered by agents at time t, they may not hold the minimum investment proportion (for the risky asset) as suggested by (3.5.6) but a positive investment proportion with its value depends on the degree of the negative expected return at time $t + 1$. Similarly, when a positive expected excess return is encountered by agents at time t, the risk aversion as well as the presence of endogenous price risk imply that the values of agents' investment proportions at time t also depends on the degree of the positive expected excess returns. In order to capture these properties, we assume that the investment proportion $\lambda^i_{t,1}$ for all agents $i =, 1, ..., I$ is determined through a bounded investment function of expected excess returns $g\big(E^i_t[R^s_{t+1,1} - R]\big) \in (0, 1)$ such that $\lambda^i_{t,1} = g\big(E^i_t[R^s_{t+1,1} - R]\big)$ increases monotonically along with the increase in the expected excess return $E^i_t[R^s_{t+1,1} - R] \in \mathbb{R}$. The investment function is a concave for the positive expected excess returns, and a convex for negative expected excess returns. The concavity and convexity represent the behaviour that agents expect endogenous risk of the market price increases as the increase in the absolute value of expected excess return.

A natural candidate for such a investment function is a "S-shaped" sigmoid function which is bounded in the interval $(0, 1)$. Since the point of $E^i_t[R^s_{t+1,1} - R] = 0$ in the investment function is a special point which connects the domains of negative and positive expected excess returns as well as agents' risk preferences between risk seeking and risk averse, we assume that agents are risky neutral when the expected excess return is zero. In this case, agents are indifferent between the risk-free asset and the risky asset, so they equally distribute their wealth across the two assets, i.e. $\lambda^i_{t,1} = \lambda^i_{t,0} = 0.5$. We use the following sigmoid function to characterise agents' behaviour:

$$\lambda^i_{t,1} = \frac{1}{\pi}\arctan\big(\alpha^i E^i_t[R^s_{t+1,1} - R]\big) + \frac{1}{2}, \tag{3.5.7}$$

where α^i is a scaling parameter for each agent i.

The qualitative behaviour [4] of the investment function (3.5.7) is illustrated in Figure 3.3. Compared with the investment function (3.5.6) which is derived from utility maximisation, the revised one in (3.5.7) characterises agents' risk preferences as risk seeking for losses and risk averse for gains. This property of agents'

[4]We choose the S shaped arctangent function, re-scaled to (0,1), to represent agents' investment functions. Other re-scaled sigmoid functions with different curvatures are also consistent with our assumptions of the awareness of the endogenous price risk, therefore do not affect the generality of the simulation results (presented in Chapter 4 and 5.). Similar results can be obtained by using other sigmoid functions e.g. $\frac{1}{2}(\tanh(x) + 1)$.

risk preferences coincides with the experimental evidence documented in the behavioural finance literature. The concept of the awareness of the endogenous price risk was used to explain in an agent-based model: 1)the risk seeking behaviour of agents for losses; 2) the concavity/convexity of agents' investment functions for positive/negarive expected excess returns. Similar sigmoid investment functions were used in Chiarella, Dieci and Gardini (2002, 2006)[5] but in models with a zero net supply [6] of the risk asset and/or a different market clearing mechanism.

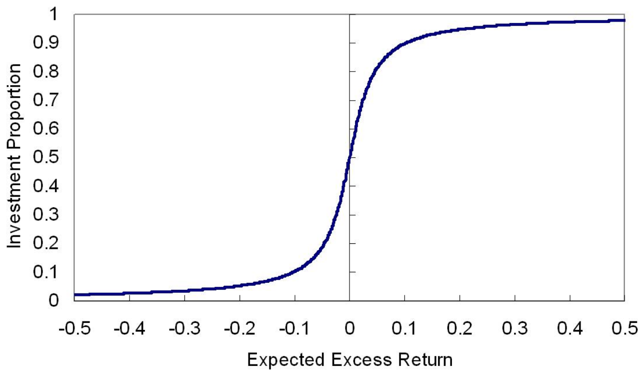

Figure 3.3: An example of the investment function defined in equation (3.5.7).

3.5.4 Different Types of Agents

It is assumed that agents are heterogeneous about the conditional expectation of the excess return. Following the classic approach in the literature on agent-based financial market models, we consider three types of agents: fundamentalists, trend followers and noise traders.

[5]These papers are based on utility maximisation approaches with a zero net supply of the risky asset. Sigmoid investment functions are introduced to model chartists' state-dependent beliefs about risk.

[6]Short selling is allowed in models with a zero net supply. Therefore, these models with utility maximising agents do not have the shortcomings and drawbacks discussed in Section 3.5.1 of this thesis.

Fundamentalists:
At time t, the fundamentalists believe that the price of the risky asset at time $t+1$ will move towards the fundamental value $S_1^* = \frac{\bar{D}_1}{r}$. The fundamentalist's forecasting rule is given by

$$E_t^F[R_{t+1}^s - R] = \frac{S_1^* + \bar{D}_1}{S_{t-1,1}} - R, \tag{3.5.8}$$

where $E_t^F[R_{t+1,1}^s - R]$ denotes the fundamentalists' expectation of the excess return at time $t+1$.

The fundamentalists' investment proportion for the risky asset is given by:

$$\lambda_{t,1}^F = \frac{1}{\pi}\arctan\big(\alpha^F E_t^F[R_{t+1,1}^s - R]\big) + \frac{1}{2}, \quad \alpha^F \geq 0, \tag{3.5.9}$$

where the scaling parameter α^F measures the degree of reaction of the fundamentalists towards a speculative opportunity. The investment in the risk-free asset is proportional to $1 - \lambda_{t,1}^F$.

Equations (3.5.8) and (3.5.9) describe that the fundamentalists use the fundamental value of the risky asset as a benchmark in their investment decisions. For example, if the risky asset is fair valued at time t (i.e. $S_{t,1} = S_1^*$), the fundamentalists believe that there is no arbitrage opportunity in the market. The risky asset yields the same rate of return as the risk-free asset. Risk neutral at this special point implies that the fundamentalists are indifferent between the risky asset and the risk-free asset. Therefore, they equally distribute their wealth between the risky asset and the risk-free asset. If the risky asset is undervalued/overvalued at time t (i.e. $S_{t,1} < S_1^*$ or $S_{t,1} > S_1^*$), the fundamentalists will invest more/less than one half of their wealth into the risky asset. The fundamentalists stabilise the market by pushing the price of the risky asset to the fundamental value, i.e. their investment proportion for the risky asset decreases/increases from 0.5 when the risky asset is overvalued/undervalued.

Trend Followers:
It is assumed that the trend followers use a rolling window which contains $L \geq 1$

past returns of the risky asset $\{R^s_{t-L,1}, R^s_{t-L+1,1}, ..., R^s_{t-1,1}\}$ to estimate the future return R^s_{t+1} of the risky asset. The parameter L measures the observation horizon (or the memory span) of the trend followers. It is assumed that, at time t, the trend followers believe that each past return in the rolling window will have an equal probability to repeat at the next point in time $t+1$, that is, $\mathbb{P}(R^s_{t+1,1} = R^s_{t-j,1}) = \frac{1}{L}$ for $j = 1, ..., L$. The trend followers' expectation of next period's excess return is therefore given by:

$$E^T_t[R^s_{t+1,1} - R] = \frac{1}{L}\sum_{j=1}^{L} R^s_{t-j,1} - R. \tag{3.5.10}$$

This way of modelling the trend followers' return forecasting is in line with the probabilistic decision theory, i.e. trend followers' forecasting is implemented by using a probability weighting function of a set of possible outcomes (past returns observed). Based on such a specification of trend followers, behavioural biases can be easily modelled through the way they assign probabilities to each past return in the rolling window when computing the expected excess return. Here, equation (3.5.10) represents only the simplest or unbiased case in which the trend followers assign identical probability mass to each return observed. Extensions of a more complicated case e.g. trend followers with sentiment (pessimistic or optimistic), and/or recency bias will be discussed and studied in Chapter 5.

The trend followers' investment proportion for the risky asset is given by:

$$\lambda^T_{t,1} = \frac{1}{\pi}\arctan\big(\alpha^T E^T_t[R^s_{t+1,1} - R]\big) + \frac{1}{2}, \quad \alpha^T = L, \tag{3.5.11}$$

where the scaling parameter $\alpha^T = L$ is to ensure that trend follower's investment proportion for the risky asset is properly scaled with respect to their observation horizon L.

Noise Traders:
In our model, the noise trading strategy represents a group of individual investors whose investment behaviour deviate from both the fundamentalists and the trend

followers. The noise traders' expectation of excess return is computed randomly, and it follows a wide sense stationary AR(1) process:

$$B_t = bB_{t-1} + \sigma_\epsilon \epsilon_t,\ \ \epsilon_t \sim N(0,1), \tag{3.5.12}$$

where $b \in (0,1)$ and σ_ϵ are constant.

The noise traders' investment proportion for the risky asset is given by

$$\lambda_{t,1}^N = \frac{1}{\pi}\arctan\big(B_t\big) + \frac{1}{2}. \tag{3.5.13}$$

The random investment proportion computed by (3.5.12) and (3.5.13) represent an "aggregate behaviour" of many individual investors whose investment behaviour are heterogenous and random. For this reason, we assume that the probability $q_t^N \in [0,1]$ of a switching investor who becomes a noise trade is fixed as a constant rather than depending on the past performance of the noise trading strategy, i.e. $q_t^N = q^N \in [0,1]$. The noise trading strategy does not participate the performance-driven strategy-switching. Instead, there is a constant fraction $q^N\beta$ of the aggregate wealth which will be allocated into the noise traders' strategy at each point $t > 0$ in time, and the $(1-q^N)\beta$ fraction of the aggregate wealth will flow between the fundamentalists' strategy and the trend followers' strategy depending on their past performances.

3.5.5 Illustration of Agents' Behaviour

This subsection uses numerical simulations to illustrate the investment behaviour of those different types of agents modelled above and the resulting price dynamics. As shown in Figure 3.4, the fundamentalists invest more (less) than 50% of wealth into the risky asset (than into the risk-free asset) when the risky asset is undervalued (overvalued). This behaviour of the fundamentalists helps to stabilise the market. In contrast, the trend followers' investment proportion for the risky asset follows the general trend of the price series.

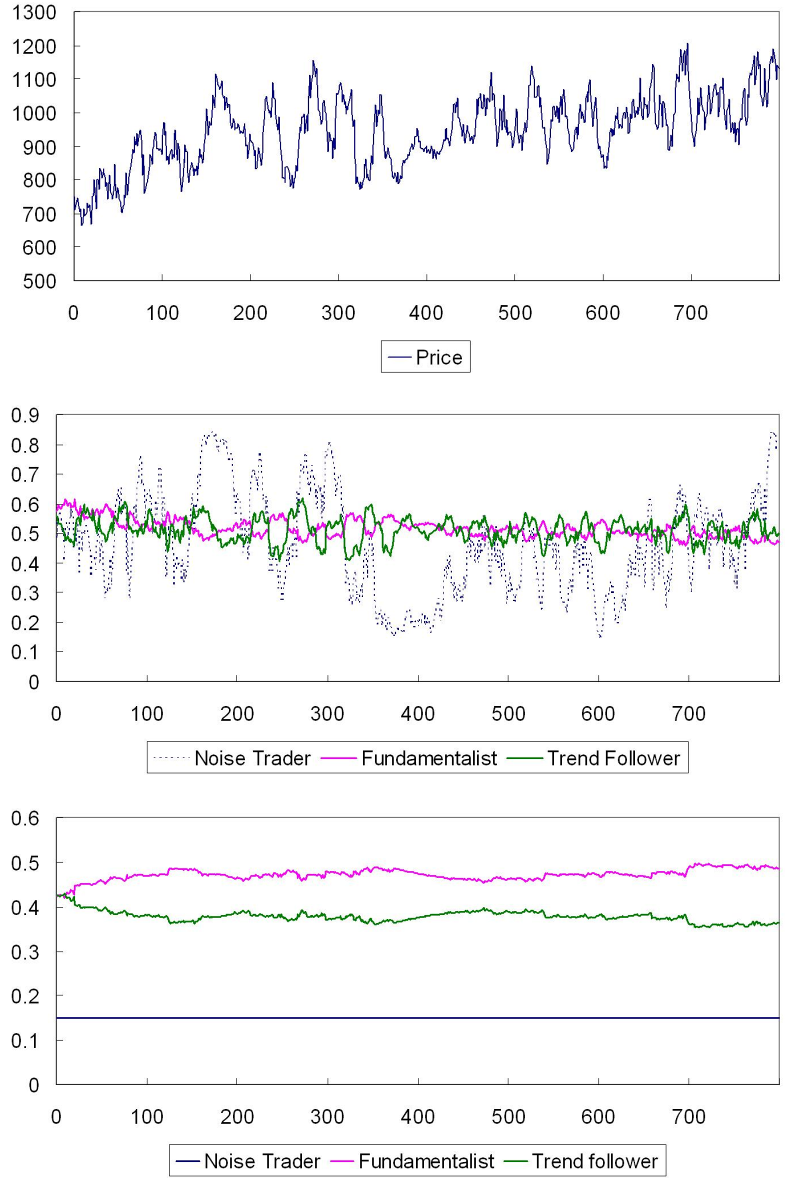

Figure 3.4: Illustration of agent behaviour and market dynamics: price dynamics (top), agents' investment proportions for the risky asset (middle), probabilities for an individual investor to choose each agent type (bottom). Parameters: $r = 0.01$, $c = \frac{r}{1+r}$, $\bar{D}_1 = 10$, $\beta = 1$, $\gamma = 5$, $\rho = 0.99$, $\alpha^F = 0.75$, $\alpha^T = L = 20$, $b = 0.97$, $\sigma_\epsilon = 0.2$. The fundamental value of the risky asset is $S_1^* = \frac{\bar{D}_1}{r} = 1000$.

The trend in the price maybe triggered by the noise traders but amplified by the trend followers. The noise traders' investment proportion for the risky asset fluctuates wildly and randomly. The resulting price dynamic reflects the interaction among the three different types of agents. These results indicate that our investment function (3.5.7) in conjunction with the specifications of agents' expectations (equations (3.5.8), (3.5.10) and (3.5.12)) are able to capture properly the nature of the three typical types of traders.

In Figure 3.4, the probability of a switching investor who becomes a noise trader is fixed at 0.15, while the probability of a switching investor who becomes a fundamentalist or a trend follower varies over time depending on the past performance of those two investment strategies. This strategy-switching behaviour causes flow of funds and affects agents' wealth dynamics. The market price is determined by agents' investment behaviour and wealth shares. The realised market price evaluates agents' performance in terms of profit or loss which affects the strategy-switching as well as agents' investment behaviour in a new round.

The key features of our model is that it captures: i) the mutual feedback between wealth and price dynamics; ii) the interaction between active and passive learning dynamics; iii) a variate of behavioural biases which are associated with active learning of agents. A detailed analysis of the impact of investors' strategy-switching behaviour and its related behavioural biases on the aggregate market dynamics will be presented in the next chapter.

3.6 Summary

In this chapter, we presented a model which combines the strategy-switching mechanism introduced by Brock and Hommes (1997) and the evolutionary finance model documented in Evstigneev, Hens and Schenk-Hoppé (2011). This new model is able to capture the interaction of active and passive learning dynamics and a number of behavioural phenomena which are associated with investors' strategy-switching. The general form of the model allows a risk-free asset and multiple risky assets are traded in a financial market. The special case in which the market consists of a risk-free asset and a single risky asset is studied separately. The study of the two-asset model has the goal to explore the impact of various behavioural biases on the aggregate market dynamics.

We analysed the effect of the risk-free rate of return r and the consumption rate c in the two-asset model. We showed that different values of these two important parameter can affect the locations of steady states of the two asset model. A proper relation between the parameters has been identified (i.e. $c = \frac{r}{1+r}$) by which the unique type of steady state of the two-asset model corresponds to an arbitrage-free equilibrium. Such a condition ensures that the equity premium (the return of the risky asset in excess of the risk-free rate) in the unique type of steady state is not created exogenously by the values of those two parameters.

We discussed the shortcomings and drawbacks of using utility maximisation agents in models with a Walrasian market clearing mechanism and a fixed positive supply of the risky asset. We then proposed a sigmoid investment function to characterise agents' behaviour as risk averse for gains and risk seeking for losses. This behaviour is consistent with the prospect theory in behavioural finance. Three different types of agents named fundamentalists, trend followers and noise traders have been specified to capture some typical investment styles in real markets.

The two-asset model with heterogenous agents and various heuristics and behavioural biases is no longer analytical tractable. We will use numerical tools to study the model dynamics. The model is programmed in C++ programming language for simulation studies[7]. In the next two chapters, we will explore numerically the impact of various behaviour of investors on the aggregate market dynamics.

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[7]For readers' interest, the C++ source code is available upon request. The two-asset model is programmed with compatibility to the multi-asset framework. Increasing the number of risky assets in the computer programme is straightforward.

Chapter 4

Behavioural Biases in Strategy-Switching

4.1 Introduction

Based on the model proposed in Chapter 3, this chapter explores numerically the impact of investors' strategy-switching behaviour and its related behavioural biases on the aggregate market dynamics. A variety of behavioural phenomena including investor overconfidence, differences of opinion, recency bias in performance evaluation, conservatism bias and rational herding will be addressed. Moreover, the interaction between passive and active learning dynamics will be investigated. Our analysis focuses on both the short-term dynamics and long-run prospects of the model. The findings are expected to contribute to new ideas and concepts on understanding the process of market selection and the causes of some persistent market phenomena such as asset bubbles, high trading volume, equity premium and excess volatility.

In the behavioural finance literature, a key manifestation of investor overconfidence refers to the tendency that investors judge themselves as above average in terms of investment skills or past performances. This type of overconfidence is known as better-than-average overconfidence. For instance, Shiller (1999), Barberis and Thaler (2003), Hong and Stein (2003) and Glaser and Weber (2007) regard investors' psychology of "better than others" as a form of overconfidence.

These authors point out that investors who exhibit better-than-average overconfidence are more likely to maintain their own opinions in predictions even though they know that other investors may hold a different opinion. These papers show that better-than-average overconfidence may lead to differences of opinion among investors and high levels of trading volume in financial markets.

The concept of differences of opinion has long been proposed as an explanation for the high trading volume in financial markets. The strand of differences-of-opinion literature is motivated by Varian (1985, 1989) due to the mere plausibility that differences of opinion are present in every day life. Varian (1989) finds that, in a theoretical model of financial markets, the trading volume is entirely driven by differences of opinion in terms of differences in prior beliefs among investors. Moreover, Harris and Raviv (1993) and Kandel and Person (1995) show that differences of opinion in terms of differences in the way investors interpret public information may also cause high trading volume. These findings are empirically confirmed by Bamber, Barron, and Stober (1999), Antweiler and Frank (2004) and Glaser and Weber (2007). Furthermore, among others, Glaser and Weber (2007) highlight that better-than-average overconfidence psychologically reinforces theses differences of opinion and thereby leads to high trading volume.

Although the market impact of better-than-average overconfidence and differences of opinion has been extensively studied, previous contributions in these areas rarely address the evolutionary perspectives of financial markets such as competition, adaptation and market selection. The long-term prospects of the presences of overconfident investors and differences of opinion in an evolutionary context have not been well explored. Theoretical models and empirical studies mentioned above are usually silent about the role of adaptation and market selection in affecting the presences of overconfidence and differences of opinion. Our analysis aims to provide insights for this point. We address investors' better-than-average overconfidence and differences of opinion through an evolutionary approach based on an agent-based model. The long-run market impact of these behavioural phenomena is explored by simulation experiments.

As explained in Chapter 3, we model investors' better-than-average overconfidence through their propensity to switch among different investment strategies. Investors who are more (less) confident about their skills or information are less (more) likely to switch to or mimic other investors' strategies. Overconfident investors in our model are represented by the non-switching investors who stay with their investment strategies between two successive points in time with a probability of 1. The less confident investors are represented by the switching investors who have positive probabilities to switch among investment strategies. The proportion between overconfident and switching investors is governed by a parameter $\beta \in [0, 1]$. This parameter measures the percentage of the aggregate wealth that is managed by the switching investors at each point in time. The remainder $1 - \beta$ percentage of the aggregate wealth belongs to the overconfident investors. If $\beta = 0$, all investors are overconfident investors who consistently stay with their strategies. In this case, the model exhibits pure passive learning dynamics. In contrast, the case $\beta \in (0, 1]$ describes the presence of switching investors. The model then exhibits both passive and active learning dynamics. The market impact of the presence of overconfidence investors as well as the interaction between passive and active learning will be explored through simulation experiments of different values of the parameter $\beta \in [0, 1]$.

Moreover, our model characterises various types of differences of opinion. First, differences in prior beliefs are modelled by a set of different and fixed investment strategies. The presence of overconfident investors maintains this type of differences of opinion. Second, our model addresses differences of opinion in strategy selection. A key feature of our strategy-switching mechanism (as the one used by Brock and Hommes, 1997, 1998) is that, if the value of the intensity of choice parameter γ is finite, not all investors necessarily choose the strategy which is indicated (by the model) to have the highest performance measure. Investors may hold different opinions in selecting investment strategies. Furthermore, investors under this switching mechanism may have different interpretations of the public performance measure of each strategy (see Proposition 3.1 in Chapter 3). The degree of differences of opinion in strategy-switching is measured by the value of parameter γ. As discussed in Chapter 3, investors' conservatism bias and herding type of behaviour are associated with the degree of differences of opinion in strategy-switching. Investors' conservative or herding type of behaviour can be observed when the value of γ is relatively low or high. This chapter studies the effect of different types of differences of opinion in relation to market selection and adaptation as well as how different values of γ impact the market dynamics.

Our analysis also addresses investors' recency bias in performance evaluation which is regarded by pernicious Pompian (2006) as one of the most obvious and most pernicious manifestation of recency bias among investors. In our model, recency bias in performance evaluation is modelled by setting the value of the discounting (or memory) parameter ρ in performance measure (equation 3.2.4) to an interval of $[0, 1)$. In this case, investors assess the performance of each strategy by assigning more weight to more recent realised returns of each strategy. In the extreme case $\rho = 0$, the performance measure of each strategy equals its most recent realised return. In contrast, the case $\rho = 1$ describes the phenomenon that investors have infinite memory and are unbiased in performance evaluation. We analyse the market impact of investors' recency bias in performance evaluation by varying the value of parameter $\rho \in [0, 1]$.

This chapter is organised as follows. In section 4.2, we analyse the classic fundamentalist-trend follower scenario with focus on the case where the risky asset pays a constant dividend. In this case, the model generates deterministic dynamics. We then, in section 4.3, include noise traders to study how the additional source of (exogenous) randomness affects the market dynamics. Under the presence of noise traders, the model generate stochastic dynamics. Section 4.4 summaries and remarks our findings.

4.2 Analysis of Deterministic Dynamics

We analyse the classic scenario in which only two agent types, fundamentalists and trend followers, are present in the market. In this two-type model, the unique source of randomness is the dividend process. We refer to the two-type model with a constant dividend process (i.e. $D_{t,1} = \bar{D}_1$ for all $t = 0, 1, 2, ...$) as the *deterministic skeleton*. Our analysis presented in this section focuses on the deterministic dynamics of the model. We will show that the impact of an i.i.d. dividend process on market dynamics is limited and even negligible if time represented by each period of the model is small (e.g. a week or a day).

In our simulations, each time period represents one week. Those behavioural phenomena mentioned in introduction section will be analysed through experiments of different values of parameters $\beta \in [0, 1]$, $\gamma \in [0, \infty)$ and $\rho \in [0, 1]$. Other model parameter values are summarised in Table 4.1. At the initial time $t = 0$, the wealth of the two agent types is set equally to 1000; the value of performance measure of each agent type is set to 0. This ensures that the competition of the two agent types starts with the same advantages. Unless stated otherwise the initial conditions and parameter values in Table 4.1 are fixed in all numerical analyses presented in this section.

Table 4.1: Parameters and their Default Values for the Two-Type Model

Parameters	Value	Explanation
r	0.001	Interest rate per week (yielding 5.3% per annual)
c	0.000999	Consumption rate (its value is given by $c = \frac{r}{1+r}$)
$\bar{D}_1$	1	Mean of the i.i.d. dividend process
σ_1	0.2	Standard deviation of the i.i.d. dividend process
α^F	0.25	Scaling parameter for fundamentalists
L	30	Observation horizon of trend followers

4.2.1 The Non-Switching Case and Overconfidence

We start with the case where all individual investors are overconfident (i.e. $\beta = 0$ in equation (3.2.1)). In this case, all investors stick to their own investment strategies for all $t \geq 0$. The model exhibits pure passive learning dynamics, i.e. wealth accumulates on investment strategies with relative high profitability and none of investors switches among different investment strategies.

According to the specification for fundamentalists (equation (3.5.8)), the fundamentalists in our model can be regarded as "informed" investors who are endowed with superior information of the fundamentals (dividend process and intrinsic value) of the risky asset. They believe that the price of the risky asset is determined by its fundamentals. In contrast, the trend followers (equation (3.5.10)) extrapolate the "trend" of the price series from realised prices in the past and form their expectation accordingly. Similar specifications for fundamentalists and trend followers are used by Levy, Levy, and Solomon (2000) who regard these two agent types as rational informed investors and efficient market believers. In our model, the rationality of both agent types is bounded as they do not take other's investment behaviour into account when forming their expectations or investment strategies.

The dynamics of the deterministic skeleton of the two-type model when $\beta = 0$ are illustrated in Figure 4.1. The price dynamics and agents' investment proportions for the risky asset show that the trend followers destabilise the market by pushing the price away from the fundamental value ($S_{*,1} = \frac{\bar{D}_1}{r} = 1000$), while the fundamentalists stabilise the market by trading oppositely to the trend followers. The scaling parameter for the fundamentalists is set to $\alpha^F = 0.25$ which describes the phenomenon that the fundamentalists do not react strongly when the price deviates from the fundamental value. The purpose of modelling such a phenomenon is to capture the concept of limits to arbitrage. In the behavioural finance literature, limits to arbitrage argues that the arbitrage can be too costly or too risky, or simply impossible due to various constraints, so the market inefficiencies may persist for a longer period (Barberis and Thaler, 2003). As shown in Figure 4.1, the interaction between the destabilising and the stabilising forces causes the oscillatory price dynamics in excess to the dividend (constant).

In Figure 4.1, the initial wealth of both agent types is set to 1000. The dynamics of agents' wealth shares show that the fundamentalists outperform the trend followers. The long-term behaviour of the price indicates that the level of

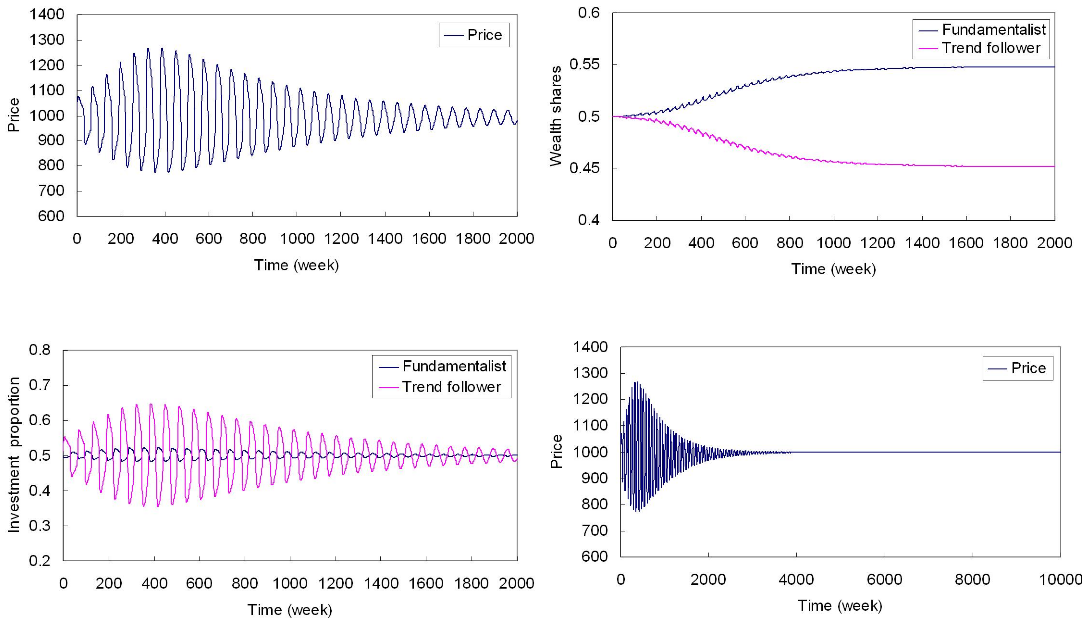


Figure 4.1: Market dynamics generated under a constant dividend $D_t = \bar{D} = 1$ with $\beta = 0$ (the non-switching case): time series of price (top left), wealth shares (top right), agents' investment proportions for the risky asset (bottom left), and the long-term price behaviour (bottom right).

price oscillation, in general, diminishes along with the increase in the fundamentalists' wealth share. The price eventually converges to the fundamental value. The survival of the trend followers is due to that the convergence of the price happened before the extinction of the trend followers. When the price converged to the fundamental value, the risky asset yields the same rate of return as the risk-free asset. Both agent types hold the same and a constant investment proportion (50% of their wealth) for the risky asset. For this reason, agents' wealth shares stay unchanged. However, as long as there exist persistent and significant shocks to the price, the trend followers will adapt themselves to the price changes quickly and vary their investment proportions. In such a case, the trend followers may keep losing money to the fundamentalists. This phenomenon will be illustrated in the in the next section under the presence of noise traders.

Our simulation results of the non-switching case ($\beta = 0$) indicate that, if all investors are overconfident (non-switching) investors, the process of market selection will play a main role in affecting the long-term market dynamics. The informed investors (the fundamentalist) outperform other investors (the trend followers) and tend to dominate the market. This observation is consistent with the famous Market Selection Hypothesis (MSH) in Alchian (1950) and Friedman (1953). Moreover, the price swing in excess to the fundamental (dividend), in the non-switching case, is only a temporary market phenomenon which happens when the trend followers have sufficiently large wealth share to destabilise the price. Along with the process of market selection, the Efficient Market Hypothesis (EMH) holds asymptotically in this non-switching case with pure passive learning dynamics. These results also reveal that, in the context of pure passive learning, the market impact of differences of opinion in terms of different strategies (or prior beliefs) may be eliminated by the market selection force. Although investors' better-than-average overconfidence in terms of non-switching helps to maintain differences of opinion, it cannot prevent the process of market selection.

Figure 4.2 illustrates the long-term price dynamics under an i.i.d. dividend process. It shows that the stochastic dividend process does not change the long-term qualitative behaviour of the price. The price fluctuates closely around the fundamental value in the long run. Those small changes of the price around the fundamental value in the long run (e.g. periods 4,000 - 10,000) are caused by the i.i.d. dividend process. Since the time period in our simulations represents one week, the dividend to price ratio is very small (0.001 on average), the effect of randomness contributed by the i.i.d. dividend process is therefore very weak and even negligible (in the sense of affecting agents' forecasts and the price dynamics). For this reason, the random dividend does not affect the qualitative behaviour of the model dynamics. Our analysis of the fundamentalist-trend follower scenario will then focus on the deterministic skeleton only. The analysis of the deterministic skeleton is able to provide basic intuitions which help to understand the more complex scenario, for instance, the scenario with the presence of the noise traders.

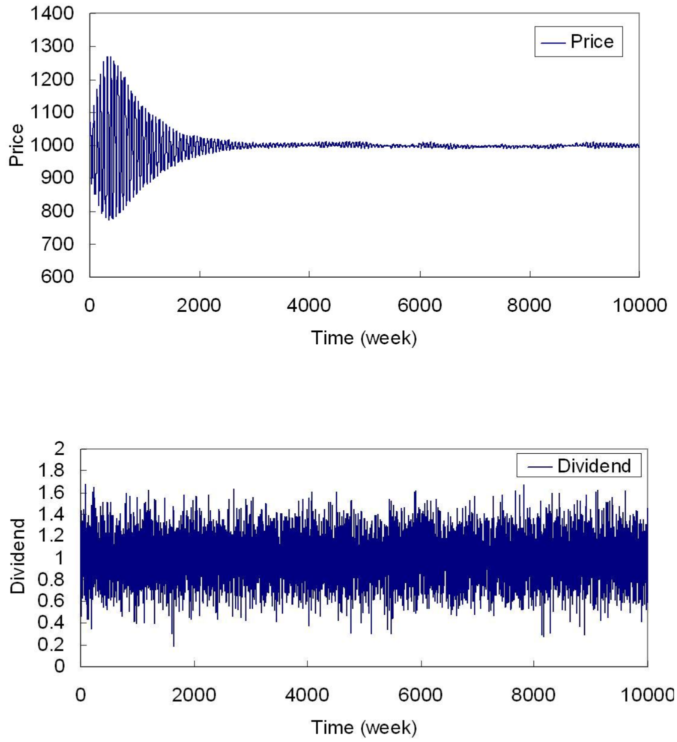


Figure 4.2: The long term price behaviour under i.i.d. dividend process with $\beta = 0$ (the non-switching case): price (top), i.i.d. dividend process (bottom) with mean $\bar{D}_1 = 1$ and standard deviation $\sigma_1 = 0.2$.

4.2.2 Strategy-Switching and Behavioural Biases

One interesting phenomenon in the non-switching case (Figure 4.1) is that the fundamentalists's investment strategy is not consistently more profitable than that of the trend followers. The increase in the fundamentalists' wealth share is not monotonic, the trend followers can gain higher profit at some periods than the fundamentalists. These varying performances of investment strategies may provide incentives to individual investors who are less confident about their own investment strategies to switch to or mimic others' investment strategies which they believe to be more profitable at some moments.

In order to explore the market impact of investors' strategy-switching behaviour and its related behavioural biases, we first analyse the case in which all investors are switching investors by setting the value of parameter β to 1 (in equation (3.2.1)). We then analyse, in the next subsection, the case of coexistence of non-switching and switching investors by varying the value of the parameter β in the interval $(0, 1)$.

In the case of $\beta = 1$, the presence of the switching investors causes that the model exhibits both passive and active learning dynamics. Wealth still accumulates on investment strategies which have relative high profitability, but there is an active attempt by investors to move their wealth into strategies which they believe to be more profitable. We analyse the impact of the interaction between passive and active learning on the aggregate market dynamics.

Unbiased Performance Evaluation

We start with the case in which the switching investors are unbiased in the evaluation of strategies' performances. The term of unbiased refers to the phenomenon in which investors have infinite memory of past returns of each strategy and weigh equally these past returns when evaluating the performance of each strategy. This phenomenon corresponds to the case when the discounting parameter $\rho = 1$ in the performance measure (3.2.4).

Figure 4.3 depicts the dynamics of asset price and agents' wealth shares for the case of $\rho = 1$ under different values of the parameter γ (the intensity of choice). Since the strategy-switching is involved, agents' wealth shares at each point in time in Figure 4.3 are computed as the proportions of the aggregate wealth which are allocated to each agent type after strategy-switching (i.e. $\frac{\hat{W}_t^i}{W_t}$). As shown in Figure 4.3, if investors are unbiased in performance evaluation, the fundamentalists will dominate the market and the price will converge to the fundamental value. The convergence of the market price and dominance of the fundamentalists are consistent with the result of the non-switching case in which the model exhibits pure passive learning dynamics. Compared with pure passive learning

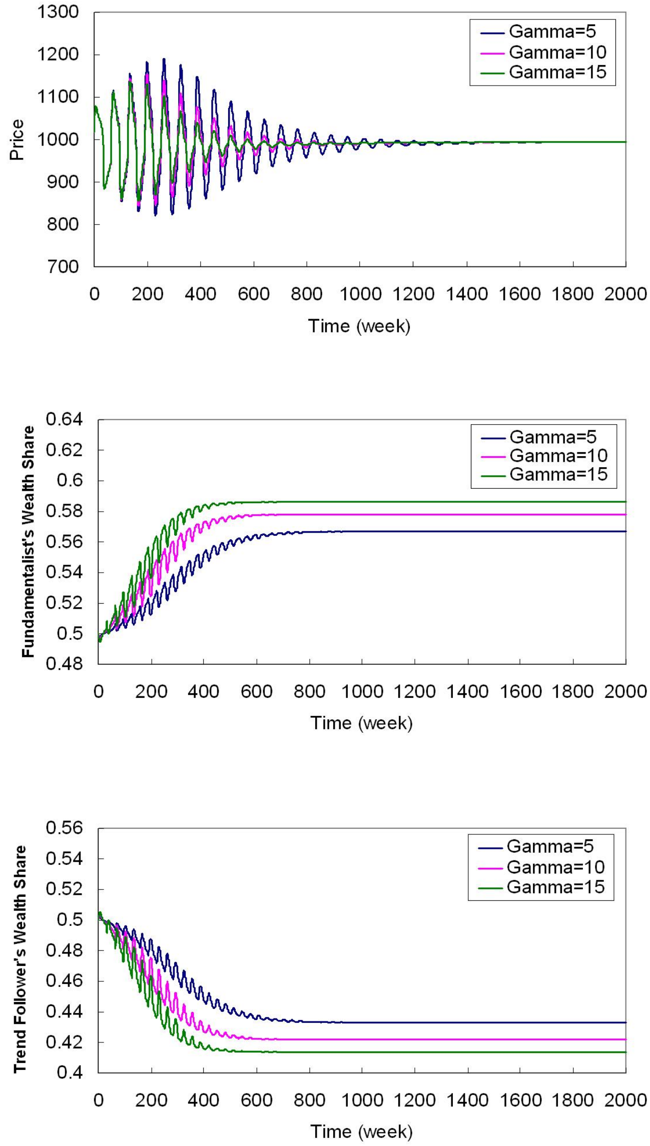


Figure 4.3: Dynamics of asset price and agents' wealth shares (after strategy-switching) under different values of the parameter γ with $\beta = 1$ and $\rho = 1$. Price dynamics (top), wealth shares of the fundamentalists $\frac{\hat{W}_t^F}{W_t}$(bottom left), wealth shares of the trend followers $\frac{\hat{W}_t^T}{W_t}$(bottom right).

dynamics in Figure 4.1, the results in Figure 4.3 shows that active learning may change the speed of convergence of the asset price. The speed of convergence under active learning is affected by the value of the intensity of choice γ. Increasing the value of γ will increase the speed of convergence.

The value of γ can be interpreted by the degree of differences of opinion among investors on selecting strategies. Higher (lower) value of γ represents lower (higher) degree of differences of opinion. As discussed in Chapter 3, investors' conservatism bias or rational herding type of behaviour can be observed when the degree of differences of opinion is high or low.

The negative correlation between the degree of differences of opinion and the speed of convergence in Figure 4.3 indicates that conservatism bias may delay the process of market selection and the convergence of the price (market efficiency), while rational herding has the opposite market impact. The intuition is as follows. A lower value of γ implies a higher degree of differences of opinion among investors in selecting investment strategies. In this case, investors exhibit conservatism bias and therefore are slower in picking the strategy which is indicated by the model to have the highest performance measure. In contrast, investors' rational herding type of behaviour when the value of γ is high implies that they are more quickly to move to the best strategy.

However, provided that investors are unbiased in performance evaluation, conservatism bias and rational herding affect neither the long-run market efficiency nor the outcome of market selection. The market behaviour in the case of strategy-switching with unbiased performance evaluation is consistent with the one observed from the non-switching case. For this reason, one can easily check that the case of coexistence of non-switching and switching investors under $\rho = 1$ will lead to similar results as those shown in Figure 4.3.

Recency Bias in Performance Evaluation

We investigate the case when the switching investors exhibit recency bias in performance evaluation (i.e. $\rho < 1$ in (3.2.4)). Surprisingly, our results indicate that, if all investors are switching investors and exhibit recency bias in performance evaluation ($\beta = 1$, and $\rho < 1$), the price may not converge to the fundamental value, as illustrated in Figure 4.4.

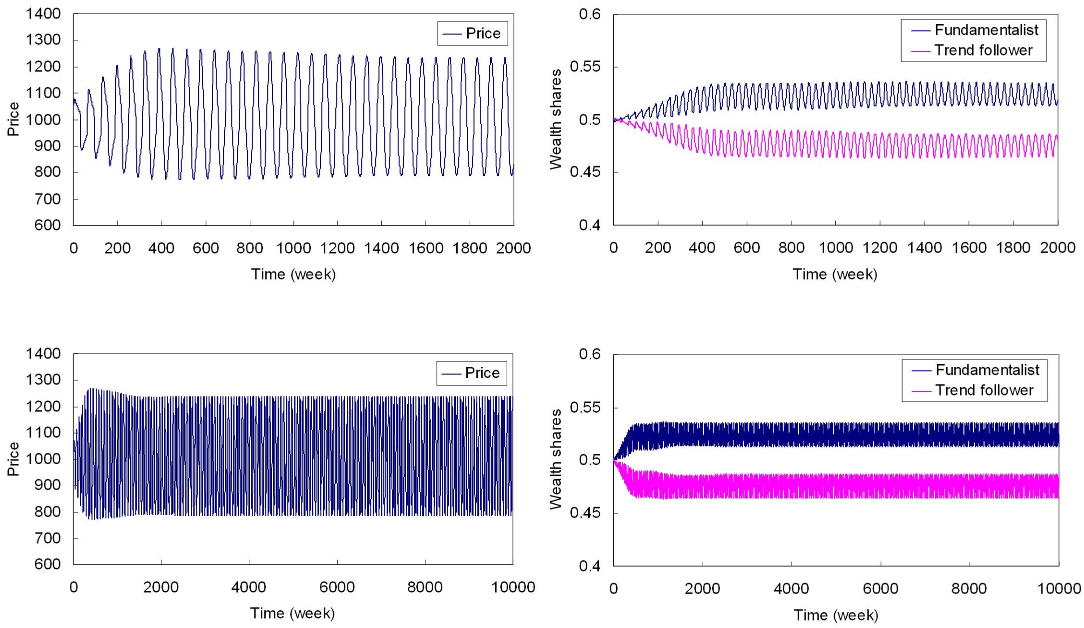


Figure 4.4: Dynamics of asset price and agents' wealth shares (after strategy-switching) when $\rho = 0.99$, $\beta = 1$ and $\gamma = 5$: short-term price (top left) and agents' wealth shares $\frac{\hat{W}_t^i}{\hat{W}_t}$ (top right); the long-term price behaviour (bottom left) and agents' wealth shares (bottom right).

In Figure 4.4, the value of parameter ρ is set to 0.99, which leads to the case that investors assign slightly more weight to more recent observed realised returns of investment strategies when evaluating the performances of these investment strategies. The performance measure of each investment strategy is computed as a geometrically declining weighted average of realised returns. This setting describes the phenomenon that the old performance of each investment strategy vanishes gradually (the weight approaches to 0) in the investors' memory.

Figure 4.4 shows that when investors exhibit a weak degree of recency bias ($\rho = 0.99$), the fundamental steady state in which the price converges to the fundamental value becomes unstable. The price converges to a stable limit cycle rather than a fixed point at the fundamental value. The excess price swing and excess volatility becomes a persistent market phenomenon as a result of the strategy-switching behaviour with recency bias. As shown in Figure 4.4, the recency bias in performance evaluation causes that agents' wealth shares exhibit oscillatory dynamics and also converge to stable limit cycles. Both agent types survive in the long run. On average, the fundamentalists possess more wealth than the trend followers.

Figure 4.5 plots the bifurcation diagram of the asset price with respect to different values of $\rho \in [0, 1]$. This bifurcation diagram indicates that the case of $\rho \in [0, 1)$ (recency bias) destabilises the fundamental steady state and leads to complicated price behaviour rather than the convergence.

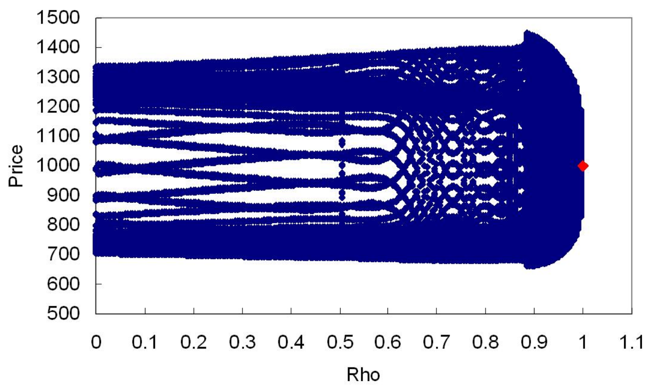


Figure 4.5: The bifurcations of the price with respect to ρ. Other parameters: $\beta = 1$ and $\gamma = 5$.

To analyse the market impact of differences of opinion when the switching investors exhibit recency bias in performance evaluation, Figure 4.6 plots the bifurcations of the price with respect to different values of γ between 0 and 10.

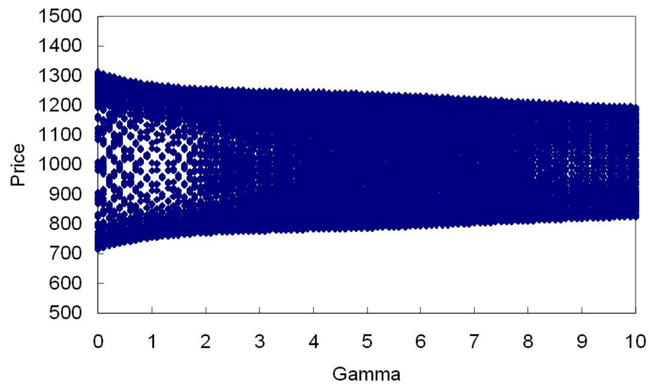


Figure 4.6: The bifurcations of the price with respect to γ between 0 and 10. Other parameters: $\beta = 1$ and $\rho = 0.99$.

As shown in Figure 4.6, the range of price variations decreases along with the increase in the value of parameter γ. Since a lower value of γ corresponds to a higher degree of differences of opinion, the result in Figure 4.6 indicates that the range of price variations is positively correlated with the degree of differences of opinion. A larger degree of differences of opinion among the switching investors is able to cause a larger range of price variations. Different values of $\gamma \in [0, 10]$ cannot lead to the convergence of the asset price.

To further illustrate the impact of the parameter γ on the market dynamics, Figure 4.7 depicts the long-term price dynamics and agents' wealth shares in the cases of $\gamma = 10$, 15 and 20 respectively. The result in Figure 4.7 indicates that recency bias in conjunction with rational herding type of behaviour (high value of γ, e.g. $\gamma = 20$) may destabilise the limit cycles of the asset price. As illustrated in the case of $\gamma = 20$ in Figure 4.7, the strategy-switching between the fundamentalists and the trend followers becomes very strong which causes large booms and crashes in the asset price.

Compared with the case of unbiased performance evaluation, investors' recency bias in the performance-driven strategy-switching turns out to be a crucial behavioural element which causes the persistence of market inefficiency. The

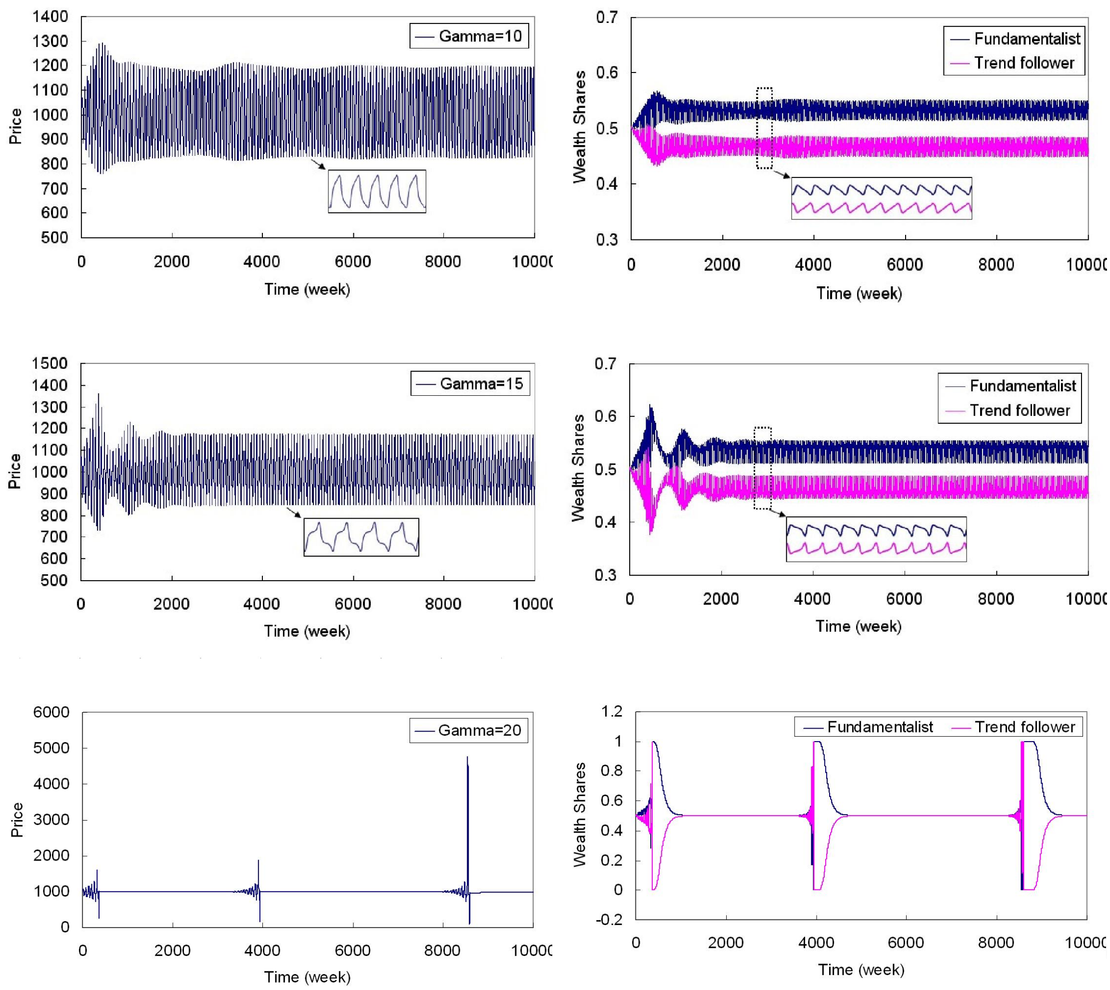


Figure 4.7: Long-term dynamics of asset price and agents' wealth shares (after strategy-switching) under different values of the parameter γ with $\beta = 1$ and $\rho = 0.99$: $\gamma = 10$ (top), $\gamma = 15$ (middle), $\gamma = 20$ (bottom).

recency bias in performance evaluation helps to maintain and reinforces the differences of opinion in strategy-switching. This result also reveals that a sufficient condition for the EMH and MSH to be valid in the model with strategy-switching is that investors have infinite memory of past performances of each strategy and are unbiased in performance evaluation ($\rho = 1$ or very near 1)[1].

[1]Due to the complexity of our model, analytical proof of this matter is difficult. Our

This finding shows that the result of active learning may not always agree with the one of pure passive learning but subject to conditions such as the unbiased performance evaluation. Active learning is sensitive to behavioural biases. The interaction between passive and active learning under the presence of recency bias in performance evaluation may change the outcome of market selection and cause the long-term market inefficiency.

4.2.3 Coexistence of Non-Switching and Switching Investors

We now study the case in which non-switching and switching investors coexist in the market. We analyse how different proportions between the overconfident investors and switching investors (the value of parameter β) affect the market dynamics. Because different values of $\beta \in (0, 1)$ do not change the convergence of the market price if the value parameter ρ equals 1, our analysis focuses on the case in which investors exhibit recency bias in performance evaluation ($\rho < 1$). We set the parameter $\rho = 0.99$ and $\gamma = 5$ in following numerical analyses.

Figure 4.8 compares the price dynamics when $\beta = 1$, $\beta = 0.1$ and $\beta = 0.001$. It shows that larger proportion of overconfident investors (at wealth level) decreases the level of price oscillation in the long-run. The intuition is straightforward. The price oscillation is caused by the interaction between different investment strategies. As explained above, it is not the overconfident investors but the switching investors with recency bias who help to maintain and reinforce the wealth of different investment strategies. When the proportion of the overconfident investors increases, the amount of money (β percent of the aggregate wealth) that flows among different investment strategies becomes small. The market impact of the presence of switching investors with recency bias becomes weak, which leads to a lower wealth level of the trend followers hence a lower level of price oscillation. This result casts a doubt on what will happen to the long-run price behaviour if

numerical analysis shows that the price will not converge to the fundamental value in long run (100,000 periods) if $\rho \in [0, 0.99995]$. However, the convergence may happen if $\rho \geq 0.99996$.

the proportion of the overconfident investors is very large. We are particularly interested in the question whether the price will converge to the fundamental value if there is only a very small fraction of the aggregate wealth flowing among investment strategies at each period of time.

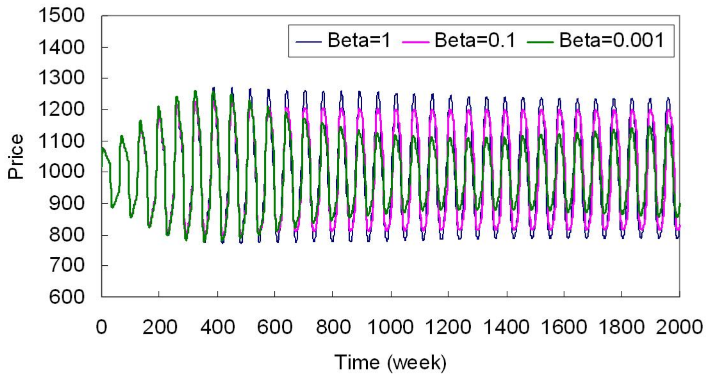


Figure 4.8: Comparison of the price behaviour under different values of the parameter β with $\gamma = 5$ and $\rho = 0.99$.

Figure 4.9 illustrates the long-run dynamics of the deterministic model when $\beta = 0.0001$. Surprisingly, even though there is only 0.01% of the aggregate wealth flowing among investment strategies at each period of time, the price does not converge to the fundamental value in long-run. The price exhibits clustered oscillatory dynamics in the first 10,000 weeks and it then converges to a limit cycle. This clustered price oscillation can be understood by analysing the probabilities of investors to choose each investment strategy and agents' wealth shares. Strong oscillation in the price is triggered when trend followers have larger wealth shares, but trend followers perform poorly during the phase of strong price oscillation. Investors therefore tend to switch to the fundamentalists during the periods of strong price oscillation. This capital outflow accelerates the decrease in trend followers' wealth share, which in turn decreases the level of price oscillation leading to a phase of weak price oscillation.

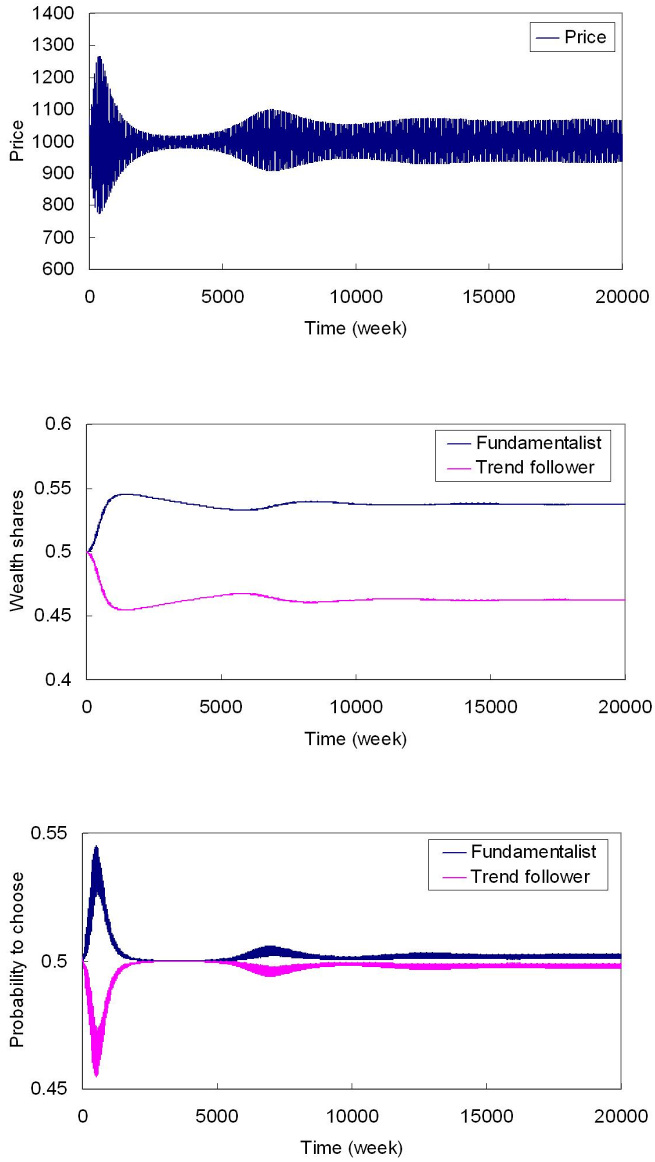


Figure 4.9: Long-term market dynamics when $\beta = 0.0001$, $\gamma = 5$ and $\rho = 0.99$: price (top), agents' wealth shares (after strategy-switching) $\frac{\hat{W}_t^i}{W_t}$ (middle), probabilities for individual investors to choose each investment strategy (bottom).

In the phase of weak price oscillation, trend followers' performance is improved. Due to investors' recency bias, more and more investors start to switch to trend followers. The capital inflow increases trend followers' wealth share which eventually triggers another phase of strong price oscillations. The excess market price swings and excess volatility under the flow of funds with recency bias becomes a persistent market phenomenon. The result of the case with $\beta = 0.0001$ and $\rho = 0.99$ reveals that even a very small proportion of the mobile capital with recency bias is able to cause the persistence of market inefficiency.

We emphasise that the i.i.d. dividend process does not change the general qualitative behaviour of the model dynamics. Under i.i.d. dividend process, the parameter $\beta = 0$ implies that the price converges to a stationary distribution in which the price fluctuates closely to the fundamental value. In contrast, if $\beta > 0$ and $\rho < 1$, the price exhibits predictable cycles and these price cycles are persistent in the long run[2].

From a modeler's perspective, introducing randomness *only* from the dividend process to the discrete time model has important limitations. As each time period in discrete time model can only represent a fixed length of time which can be a day, a week or a year etc., but the dividend to price ratio naturally decreases from longer time period (e.g. a year) to shorter time period (e.g. a day). The effect of the randomness contributed by the dividend becomes weaker when the length of time represented by each time period is shorter. In order to investigate how stronger randomness affects the interaction between agents as well as the resulting model dynamics, other sources of randomness are needed. For example, the randomness contributed by the noise traders.

4.3 Analysis of Stochastic Dynamics

In order to study the market impact of investors' overconfidence and strategy-switching behaviour when the dynamical system is affected by exogenous ran-

[2]Given the complexity of our model, using analytical method to analyse the asymptotic behaviour of the model is difficult. The long-term price dynamics is characterised by numerical simulations, the time simulated (20,000 weeks) is sufficient long for a real market.

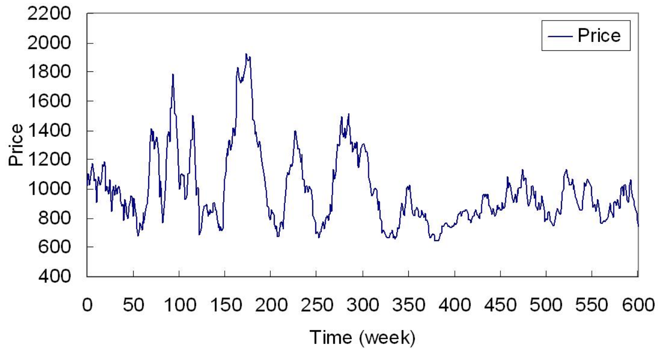
domness, we add noise traders into our simulations. Parameter values for the noise traders are specified as: $b = 0.97$ and $\sigma_\epsilon = 0.2$. Other model parameter values are the same as those presented in Table 4.1. The initial wealth for the fundamentalists, trend followers and noise traders are equally set. We first analyse the non-switching case when every individual investor is overconfident ($\beta = 0$). We then investigate the case of $\beta \in (0, 1]$ and each switching investor has a fixed probability $q^N > 0$ to become a noise trader.

4.3.1 Noise Traders in the Non-Switching Case

Figure 4.10 illustrates the short-term dynamics of asset price and agents' investment proportions generated by the three-type model when all investors exhibit better-than-average overconfidence. As shown in Figure 4.10, the price dynamics under the presence of noise traders become more complicated. The frequency and the size of the price cycles (price bubbles) are less predictable than those in the fundamentalist-trend follower scenario. The trend followers' investment proportion for the risky asset is strongly affected by the presence of noise traders resulting that the trend followers tend to chase and amplify the "trend" caused by the noise traders.

The long-term dynamics of asset price and agents' wealth shares of this non-switching case are illustrated in Figure 4.11. The long-term price dynamics indicate that the random fluctuations of the price in the non-switching case are only a temporary market phenomenon. The price tends to converge to the fundamental value in the long run, which is consistent with the prediction of EMH for real markets. The long-term wealth shares of agents in Figure 4.11 reveal that the noise traders will be driven out of the market, while the fundamentalists tend to dominate the market.

Similar to the fundamentalist-trend follower scenario, the survival of the trend follower is due to that the price converged before the trend followers vanish from the market. In our model, the noise traders represent the aggregate behaviour



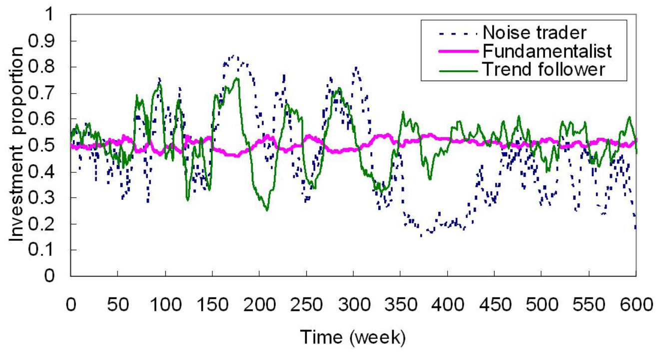


Figure 4.10: Short-term dynamics of asset price (top) and agents' investment proportions for the risky asset (bottom). $\beta = 0$.

of a group of "irrational" individual investors whose investment proportions for the risky asset are computed randomly. Our results in the non-switching case indicate that irrational traders (noise traders) will be driven out of the market, which agrees with the MSH. We can now summarise that, the EMH and MSH hold asymptotically in our model of overconfident investors and pure passive learning dynamics. The excess price fluctuations which are caused by investor overconfidence (in terms of non-switching) in conjunction with differences of opinion (in terms of different strategies) under the pure passive learning are only a temporary market phenomenon. Although investor overconfidence may psychologically reinforce differences of opinion through investors' minds (as argued by Glaser and Weber, 2007), the effect of differences of opinion and the resulting excess fluctu-

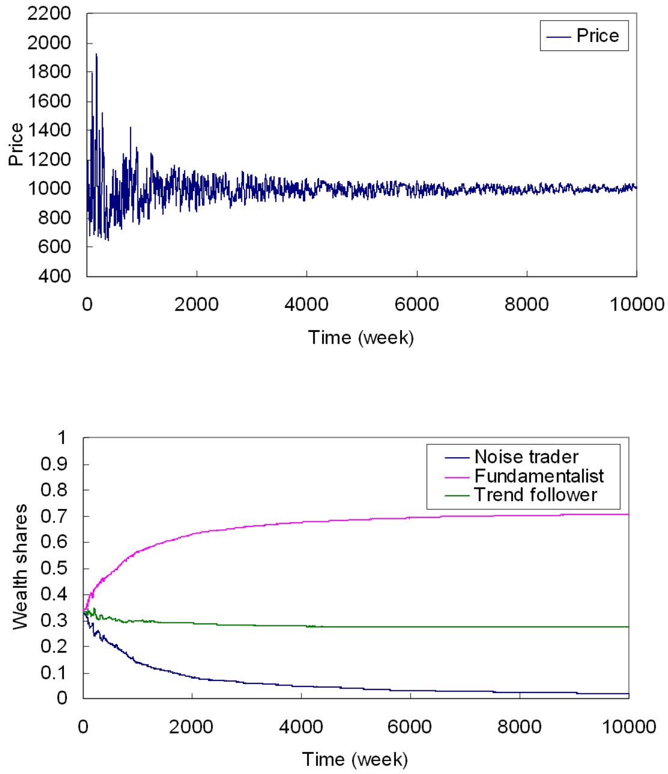


Figure 4.11: Long-term dynamics of asset price (top) and agents' wealth shares (bottom). $\beta = 0$.

ations of the price will be eventually destroyed by the process of market selection.

4.3.2 Market Dynamics under Strategy-Switching

Behavioural finance argues that humans often depart from rationality in a consistent manner. Individuals' investment decisions are easily influenced by individuals' prejudices and perceptions which do not meet the criteria of rationality. In order to better model this behaviour, we assume that each switching investor will have a fixed probability q^N to become a noise trader at each time $t > 0$. This assumption implies that, in the case of $\beta \in (0, 1]$, the wealth possessed by the noise traders at time $t > 0$ equals βq^N percent of the aggregate wealth at that time.

Unbiased Performance Evaluation

We analyse here the case where individual investors are unbiased in performance evaluation ($\rho = 1$). In this case, investors give weight equally to all realised returns of each investment strategy. The performance measure for each strategy (equation (3.2.4)) is given by the accumulated return without discounting. In order to better understand the market impact of investors' strategy-switching behaviour, we focus on the case in which all investors are switching investors ($\beta = 1$). This is to ensure that the effect of the presence of the overconfident investors is eliminated.

Figure 4.12 illustrates the case when all investors are switching investors. It shows that if there is a fixed fraction of the noise traders ($q^N = 0.15$) and the value of the parameter ρ is set to 1, the trend followers will be driven out of the market. The size of price bubbles decreases along with the decrease in the trend followers' wealth share. This result reveals that the survival of the trend followers in the non-switching case (as illustrated in Figure 4.1 and 4.11) is *only* caused by the convergence of the price to the fundamental value. As soon as there exist significant and persistent shocks to the price, for instance the random shocks imposed by the noise traders (as illustrated in Figure 4.12), the trend followers will be driven out of the market. For this reason, we can summarise that in our model the MSH holds for the both cases, one with $\beta = 0$ (the non-switching case) and the other one with $\beta = 1$ and $\rho = 1$ (the case of strategy-switching with unbiased performance evaluation). The excess price fluctuations after the extinction of the trend followers are caused by the interaction between the fixed fraction of noise traders and the fundamentalists.

The bottom panel of Figure 4.12 shows that the value of the intensity of choice γ is positively correlated with the speed of the market selection. A lower (higher) value of γ is able to delay (accelerate) the process of market selection. This observation confirms our finding in the fundamentalist-trend follower scenario.

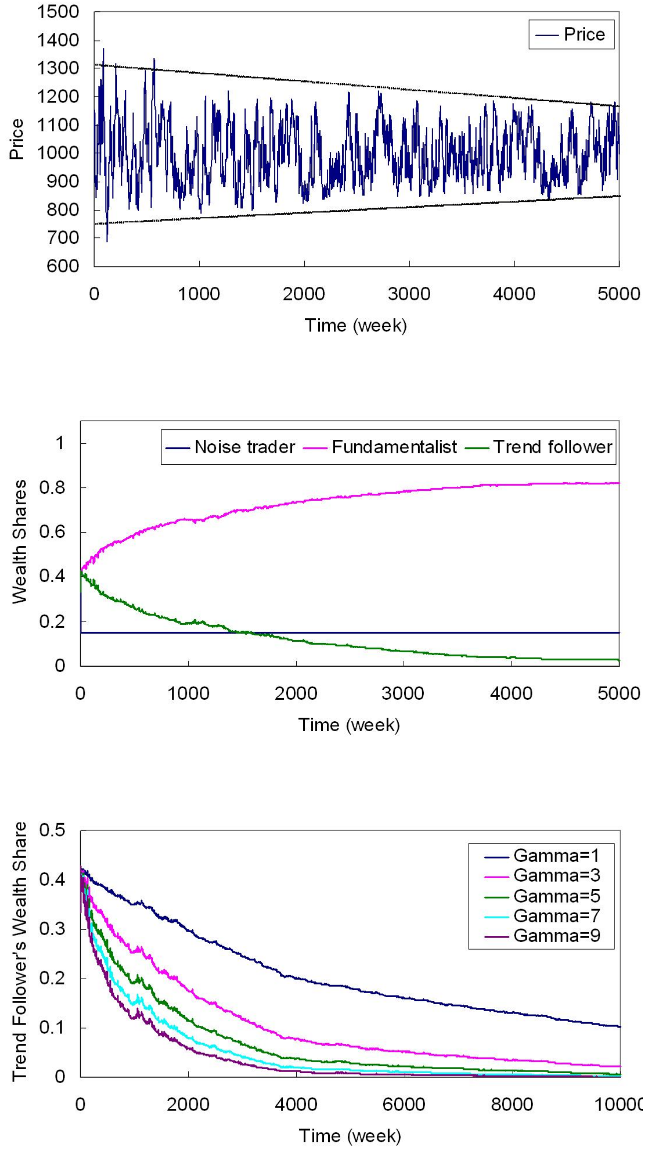


Figure 4.12: Price dynamics (top), agents' wealth shares after strategy-switching (middle), comparison of the trend followers' wealth shares (after strategy-switching) under different values of γ (bottom). Parameters: $\beta = 1$, $\rho = 1$ and $\gamma = 5$ (for top and middle panels).

Switching investors' conservative or herding type of behaviour when the value of γ is low or high may change the speed of the process of market selection. Provided that investors are unbiased in performance evaluation ($\rho = 1$), all investors will eventually move to the best strategy (except the noise traders). Investors' differences of opinion in strategy-switching (i.e. $\gamma \in [0, \infty)$) will be eventually destroyed by market selection in conjunction with adaptation. We can summarise that, in our model, the simulation results in the non-switching case and the case of strategy-switching with unbiased performance evaluation agree with the prediction of MSH. Moreover, the differences of opinion in terms of differences in investment strategies (or prior beliefs) or differences in strategy selections cannot lead to the long-term excess fluctuations of the price in the case of pure passive learning or active learning with unbiased performance evaluation.

Recency Bias in Performance Evaluation

If investors exhibit recency bias (e.g. $\rho = 0.99$) in performance evaluation, as illustrated in Figure 4.13, the MSH is no longer valid. The three agent types coexist in the long-run. The survival of the trend followers causes large fluctuations of the price. The price dynamics are more complex than in the case of unbiased performance evaluation illustrated above.

Compared with the fundamentalist-trend follower scenario, the presence of noise traders causes irregular shifts in agents' wealth shares rather than those stable and predictable cycles as illustrated in Figure 4.4. In the case without noise traders, Figure 4.4 shows that the trend followers' wealth shares exhibit stable cycles but are consistently lower than the wealth shares of the fundamentalists. In contrast, the result in Figure 4.13 indicates that the presence of noise traders may enhance the performance of the trend followers. Trend followers can have higher wealth shares than the fundamentalists at some moments especially when large price bubbles occur (e.g. the large bubbles between periods 10,000 and 15,000 in Figure 4.13).

However, compared with the results in the non-switching case and the unbiased performance evaluation case, the new results in Figure 4.13 indicate that

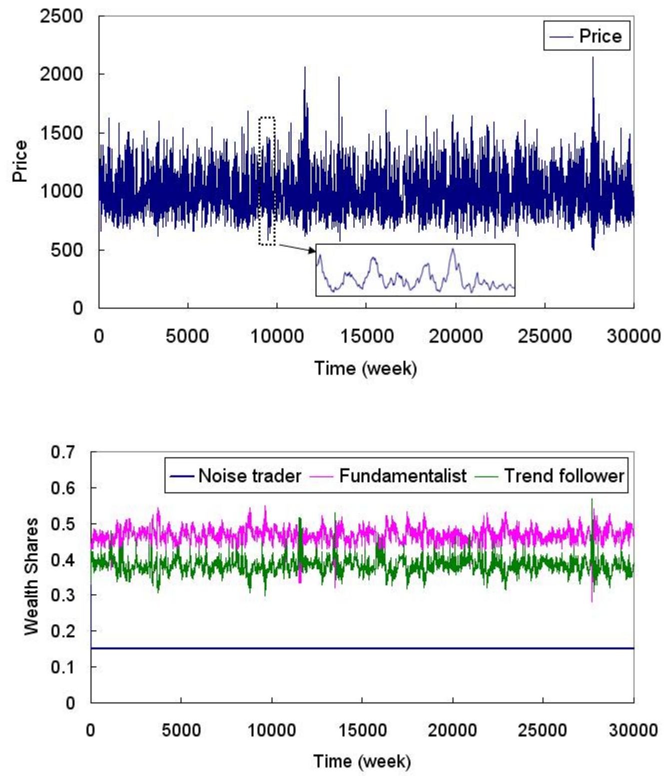


Figure 4.13: Long-term price dynamics (top) and agents' wealth shares after strategy-switching (bottom) when $\beta = 1$, $\gamma = 5$ and $\rho = 0.99$.

it is not the presence of additional exogenous randomness (the randomness contributed by the noise traders) but investors' recency bias in performance evaluation which changes the outcome of adaptation and market selection. This observation confirms our previous finding that recency bias in performance evaluation plays an important role in affecting the long-run result of adaptation and market selection. This finding again reveals that the outcome of active learning is not necessarily consistent with the one of passive learning due to the sensitivity of active learning to behavioural biases. Active learning in conjunction with recency bias in performance evaluation may cause persistence of market inefficiency.

To analyse the role of the intensity of choice parameter γ in affecting the market dynamics, Table 4.2 shows, under different values of the parameter γ, the long-run average values of agents' wealth shares (after strategy-switching), price volatility, trading volume and return of the risky asset (including dividend yield, i.e. $\frac{S_{t,1}+D_{t,1}}{S_{t-1,1}} - 1$). Based on this table, the market impact of the parameter γ is illustrated in Figure 4.14. These results are obtained through 100 independent runs of 30,000-period numerical simulations with the same parameters but different (random) paths of the noise traders' investment proportions. The parameter β is set to 1 (all investors are switching investors), the discounting factor in performance measure ρ is set to 0.99 (recency bias). We compute the average values of periods 25,001-30,000 over the 100 simulations. The price volatility is measured by the standard deviation of the price time series. The trading volume is the average trading volume per period in percentage of outstanding shares. The trading volume at each period of time is computed as the absolute value of the number of shares that change hands during each period:

$$\mathrm{Vol}_t = \frac{\sum_{i=1}^{I} |\hat{\theta}_t^i - \hat{\theta}_{t-1}^i|}{2},$$

where $\hat{\theta}_t^i$ is given by equation (3.2.9) measuring the number units of the risky asset held by each agent $i = 1, ..., I$ at time t.

Table 4.2: Long-Term Average Values of Wealth Shares, Price Volatility, Trading Volume and Return of the Risky Asset under Different Values of γ

Value of γ	Agents' Wealth Shares			Price Volatility	Trading volume	Return
	Noise Trader	Fundamentalist	Trend Follower			
$\gamma = 1$	15%	43.6477%	41.3523%	193.2590	1.3582%	0.1822%
$\gamma = 2$	15%	44.6561%	40.3439%	188.2102	1.3382%	0.1776%
$\gamma = 3$	15%	45.5267%	39.4733%	182.1131	1.3268%	0.1750%
$\gamma = 4$	15%	46.3862%	38.6138%	174.3130	1.3215%	0.1737%
$\gamma = 5$	15%	47.1491%	37.8509%	173.1808	1.3197%	0.1711%
$\gamma = 6$	15%	47.8385%	37.1615%	170.1561	1.3152%	0.1689%
$\gamma = 7$	15%	48.4815%	36.5185%	166.4110	1.3221%	0.1669%
$\gamma = 8$	15%	49.2264%	35.7736%	167.8360	1.3382%	0.1677%
$\gamma = 9$	15%	49.8564%	35.1436%	171.0581	1.3453%	0.1676%
$\gamma = 10$	15%	50.5956%	34.4044%	174.8700	1.3577%	0.1696%

The results in Table 4.2 and Figure 4.14 show that the value of the parameter γ is positively correlated with the average wealth share of the fundamentalists

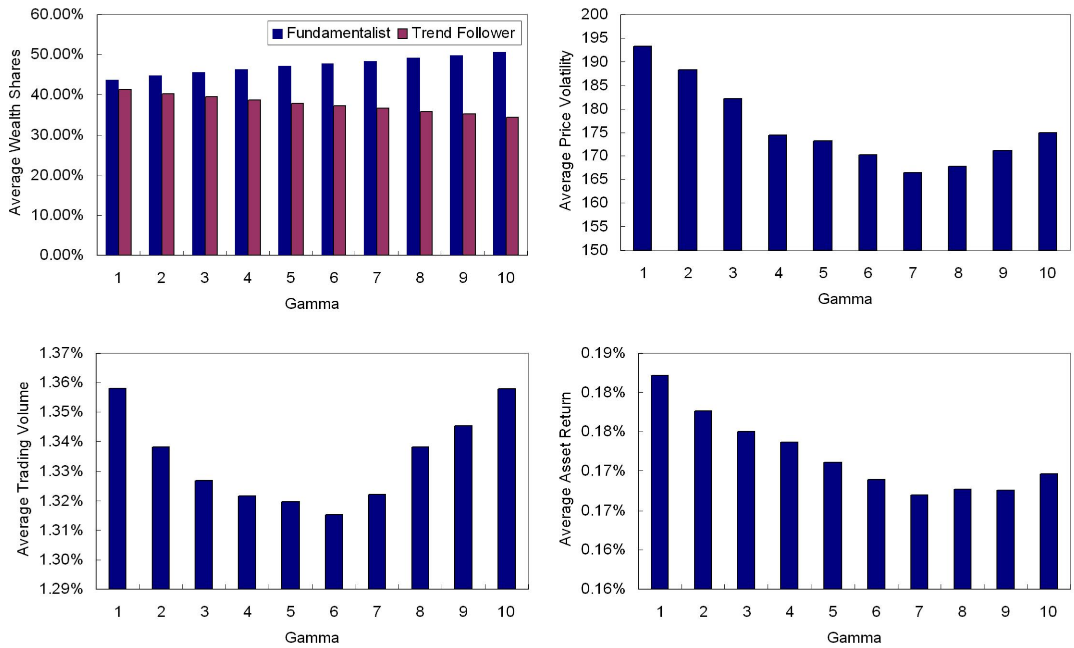


Figure 4.14: Illustration of the impact of different values of γ on: average wealth shares of agents (top left), average price volatility (top right), average trading volume (bottom left) and average asset return (bottom right). The results are based on Table 4.2.

but negatively correlated with the one of the trend followers. The average price volatility, trading volume and asset return exhibit a "U" shape with respect to different values of γ in the interval $[1, 10]$. Both a lower and higher values of γ can lead to higher levels of price volatility and trading volume. However, the causes for these observed high price volatility and trading volume are quite different.

When the value of γ is relatively low (e.g. $\gamma = 1$), investors are conservative in strategy-switching leading to a more diversified wealth distribution among investment strategies, or equivalently, a higher degree of differences of opinion. As illustrated by Figure 4.15, when $\gamma = 1$, the wealth shares (after strategy-switching) for both fundamentalists and trend followers are stabilized roughly at 42.5% indicating a high degree of differences of opinion among the entire population of individual investors. In this case, the high price volatility and trading volume arise due to the presence of a higher degree of differences of opinion. This finding is in general consistent with the view of the differences-of-opinion literature (see, e.g. Varian, 1985, 1989; Harris and Raviv, 1993; Kandel and Person, 1995; Antweiler and Frank , 2004; Glaser and Weber, 2007) that higher degree of differences of opinion leads to a higher degree of trading volume.

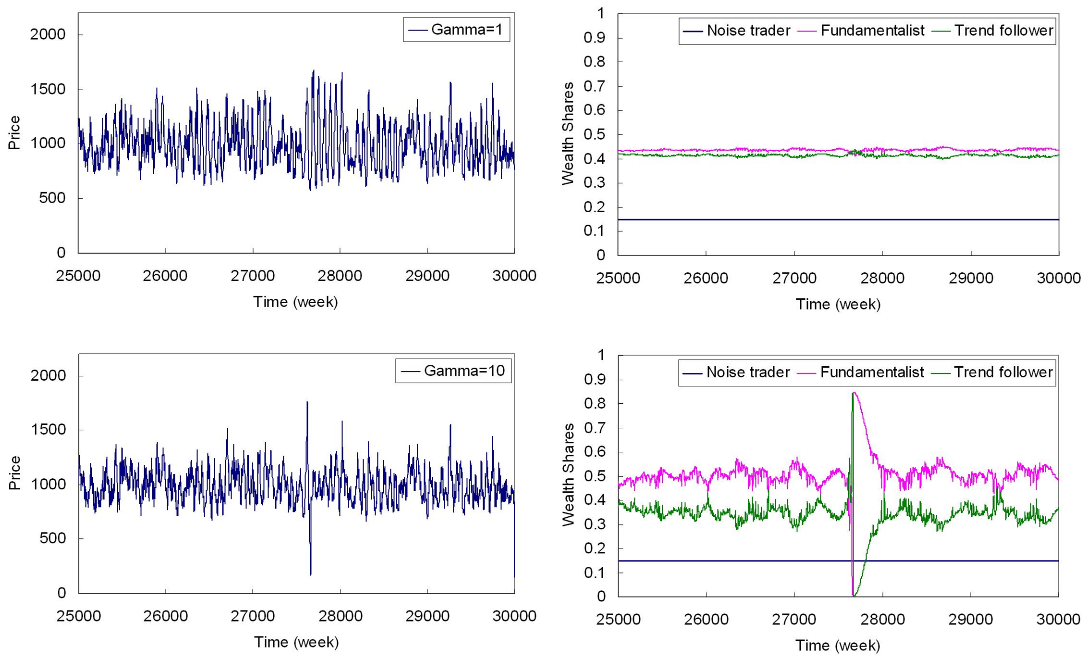


Figure 4.15: Illustration of the dynamics of asset price and agents' wealth shares (after strategy-switching) for periods 25,000-30,000 under different values of γ: $\gamma = 1$ (top), $\gamma = 10$ (bottom). Other parameters: $\beta = 1$ and $\rho = 0.99$.

When the value of γ is relatively high (e.g. $\gamma = 10$), as illustrated by Figure 4.15, investors' herding type of behaviour leads to a less diversified wealth distribution among investment strategies or a lower degree of differences of opinion. Investors' herding behaviour in strategy-switching causes wild fluctuations of agents' wealth shares and thereby booms and crashes of the price leading to high price volatility and trading volume. These findings reveal that not only a higher degree of differences of opinion but also the herding type of behaviour with a lower degree of differences of opinion can cause high price volatility and trading volume. This finding points to an alternative explanation to the differences-of-opinion literature on explaining the high trading volume observed in real markets.

Moreover, the differences-of-opinion literature usually ignores the evolutionary perspectives of financial markets such as adaptation and market selection. We have shown that, in a evolutionary context, excess fluctuations of the price which are caused solely by differences of opinion (in terms of different investment strategies or prior beliefs, different views in strategy selections and different interpretations of the public performance measure) are only a *temporary* market phenomenon. These differences of opinion are not sufficient to explain the *persistence* of high trading volume, whereas additional insights can be obtained by analysing investors' heuristics and biases in strategy-switching. Our simulation results show that the high trading volume is triggered by differences of opinion and amplified by conservatism bias or herding behaviour, while it is investors' recency bias in performance evaluation which maintains the persistence of differences of opinion and high trading volume.

4.3.3 Market Impact of Flow of Funds

We analyse the case in which both the overconfident investors and the switching investors are present in the market by varying the value of the parameter β between 0 and 1. In this case, only β percent of the aggregate wealth flows among different investment strategies at each point in time. We are particularly interested in how the proportion of the mobile capital affects the market dynamics. We set the parameter $\rho = 0.99$ and $\gamma = 5$ in following numerical analyses.

The top panel in Figure 4.16 illustrates the price dynamics under different values of β, the probability that a switching investor becomes a noise trader is set to 0.15. Similar to the fundamentalist-trend follower scenario, the result indicates that larger proportion of the overconfident investors decreases the level

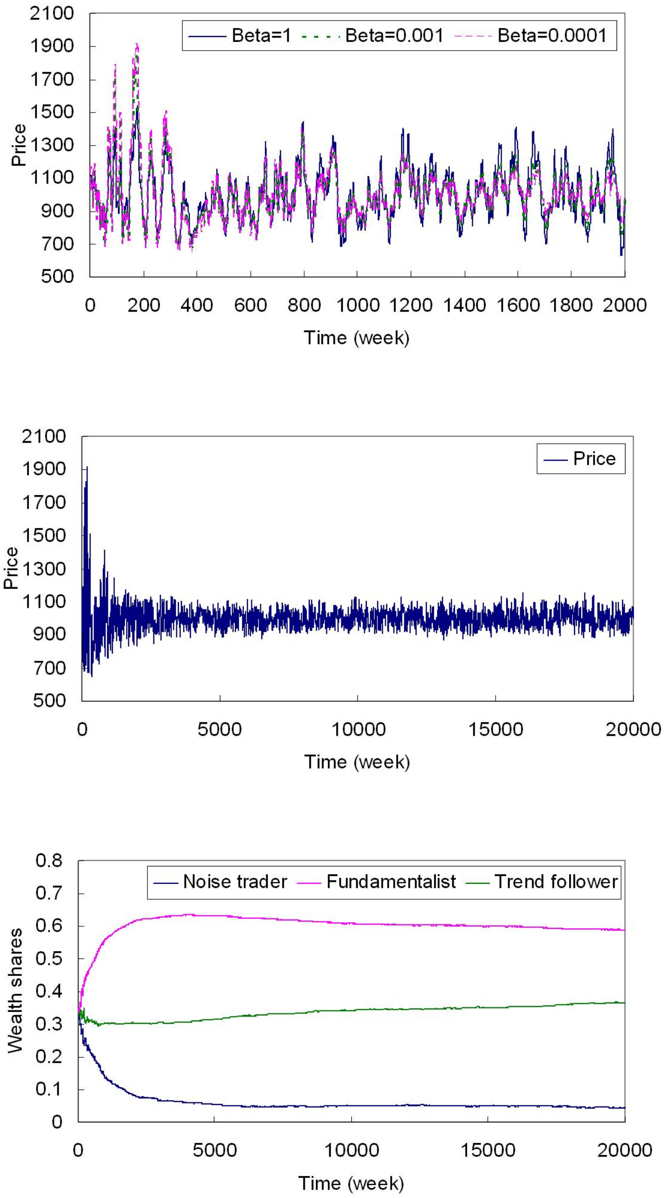


Figure 4.16: Comparison of short-term price dynamics when $\beta = 1$, $\beta = 0.001$, $\beta = 0.0001$ (top), long-term price behaviour when $\beta = 0.0001$ (middle), long-term dynamics of agents' wealth shares after strategy-switching (bottom).

of price oscillations. For $\beta = 0.0001$, the long-term price behaviour and agents' wealth shares in Figure 4.16 show that even though there is only 0.01% of the aggregate wealth flowing among investment strategies, the trend followers survive in long run. The price does not converge to the fundamental value. The excess price fluctuations to the fundamental become a persistent phenomenon rather than a temporary phenomenon as the one in the non-switching case. The case of $\beta = 0.0001$ confirms that the flow of funds under recency bias can cause the survival of trend followers and noise traders and the persistence of market inefficiency.

Table 4.3 illustrates the impact of different proportions of the mobile capital on agents' wealth shares, price volatility, trading volume and return of the risky asset (including dividend yield). These results are the average values which are computed over periods 25,001-30,000 based on 100 independent runs. The case $\beta = 0$ in Table 4.3 is considered as a benchmark because the price asymptotically equals the fundamental value. The EMH and MSH hold in this case.

Table 4.3: Long-Term Average Values of Wealth Shares, Price Volatility, Trading Volume and Return of the Risky Asset under Different Values of β

Value of β	Agents' Wealth Shares			Price Volatility	Trading volume	Return
	Noise Trader	Fundamentalist	Trend Follower			
$\beta = 0$	0.7458%	72.0832%	27.1710%	4.8756	0.0564%	0.1002%
$\beta = 0.0001$	4.9212%	59.1249%	35.9539%	50.0466	0.4571%	0.1070%
$\beta = 0.001$	10.7858%	52.1970%	37.0172%	117.0211	0.9643%	0.1392%
$\beta = 0.01$	14.3314%	48.0738%	37.5948%	160.9642	1.2715%	0.1660%
$\beta = 1$	15.0000%	47.1502%	37.8498%	172.9907	1.3182%	0.1710%

Table 4.3 shows that, for $\beta = 0$, the fundamentalists tend to dominate the market, while the noise traders are near extinction (average wealth share of 0.75%). The long-term average price volatility and trading volume are very small, which means the price fluctuates closely to the fundamental value. There is almost no trade between agents. The return of the risky asset (0.1002%) is almost identical to the risk-free rate of return (0.1%). In contrast, for $\beta = 0.0001$, the dynamics change dramatically. The average wealth shares of both the noise traders

and the trend followers increase significantly. The average wealth shares of the noise traders and the trend followers stabilise at 4.92% and 35.95% respectively. The time series of the wealth shares of the two types of agents under different β are illustrated in Figure 4.17, which shows that their wealth shares when $\beta > 0$ stabilise in long run.

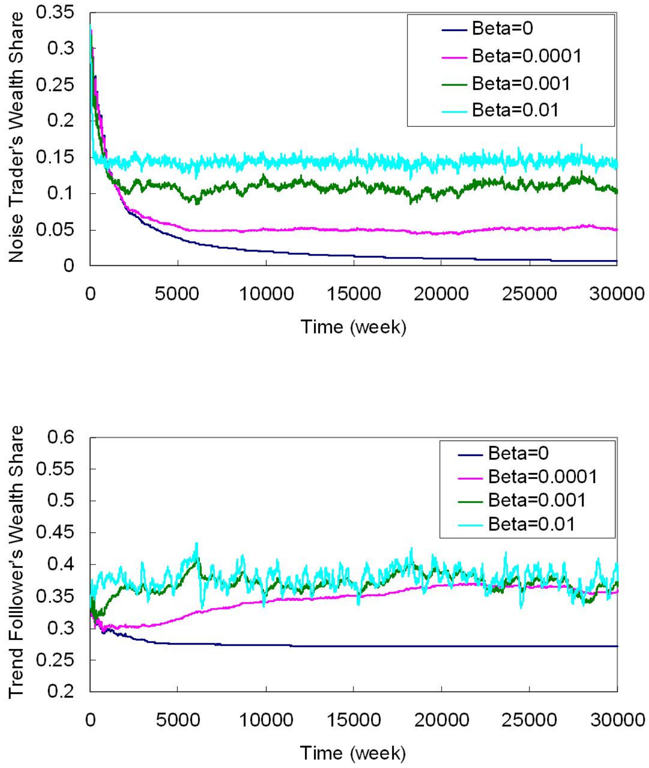


Figure 4.17: Dynamics of the wealth shares (after strategy-switching) of the noise traders (top) and the trend followers (bottom) under different values of β.

Since the probability q^N of a switching investor who becomes a noise trader in our simulation is set to 0.15, for $\beta = 0.0001$ the percentage of the aggregate wealth flowing to the noise trading strategy per period is $\beta q^N = 0.000015$. This small amount of mobile capital has a dramatic impact on noise traders' wealth share. Because of the small amount of flowing capital, noise traders' average wealth share increased from 0.75% to 4.92%. Moreover, the trend followers' wealth share increased from 27.1710% to 35.95%. Along with the increase in the wealth shares of the noise traders and the trend followers, the long-run average price volatility and trading volume are also significantly increased. For $\beta = 1$, the price volatility and trading volume become much larger than those in the $\beta = 0$ case. The average return of the risky asset (0.171%) in the $\beta = 1$ case yields 0.071% excess return per week.

The results in Table 4.3 indicate that the increase in the parameter β increases the (average) proportion of wealth managed by the noise traders and the trend followers, but it decreases the proportion of wealth managed by the fundamentalists. The increase in β also significantly increases price volatility, trading volume and excess return. This effect is highly nonlinear. The wealth shares of the noise traders and trend followers, the price volatility, trading volume and excess return all increase rapidly with the increase in β when β is small. The price volatility, trading volume and return of the risky asset are negatively correlated with the average wealth share of the fundamentalists but positively correlated with the wealth shares of the noise traders and the trend followers. This result indicates that the fundamentalists stabilise, while noise traders and trend followers destabilise the market.

The excess volatilities, high trading volume and the excess returns in Table 4.3 can be explained by analysing agents' wealth shares. Comparing the results between the cases $\beta = 0.0001$ and $\beta = 0.001$, the noise traders' wealth share increases by 5.86%, while the trend followers' wealth share only increases by 1.06%. However, the price volatility, trading volume and return of the risky asset all increase dramatically.

This result indicates that the noise traders might be blamed rather than the trend followers for the excess volatility, high trading volume and excess return of the risky asset. In order to investigate how the wealth shares of the noise traders and the trend followers can affect the market dynamics, the following experiment is carried out. For $\beta = 1$, the proportion of the aggregate wealth which is allocated to the noise traders at each period equals the q^N. By varying the value of q^N in the case of $\beta = 1$, we replicate the average wealth shares of the noise traders in the cases of $\beta = 0.0001$ and $\beta = 0.001$. We compute, for $\beta = 1$, the price volatility, trading volume, asset return and agents' average wealth shares by setting $q^N = 4.9212\%$ and $q^N = 10.7858\%$ respectively. We then compare the results with those documented in Table 4.3. The comparison of the results is summarised in Table 4.4.

Table 4.4: The Impact of Agents' Wealth Shares on the Price Volatility, Trading Volume and Return

Cases	Agents' Wealth Shares			Price volatility	Trading volume	Return
	Noise Trader	Fundamentalist	Trend Follower			
A1: $\beta = 0.0001$, $q^N = 15\%$	4.9212%	59.1249%	35.9539%	50.0466	0.4571%	0.1070%
A2: $\beta = 1$, $q^N = 5.2130\%$	4.9212%	49.5489%	45.5299%	121.3550	0.6632%	0.1206%
B1: $\beta = 0.001$, $q^N = 15\%$	10.7858%	52.1970%	37.0172%	117.0211	0.9643%	0.1392%
B2: $\beta = 1$, $q^N = 10.6172\%$	10.7858%	48.1162%	41.098%	145.3331	1.0605%	0.1457%

As can be seen from Table 4.4, the cases A2 and B2 replicate the noise traders' average wealth shares in the cases A1 and B1 respectively. Comparing the results between the cases A1 and A2 shows that the trend followers' wealth shares increases from 35.9539% to 45.5299%. Along with the increase in the trend followers' wealth share, the price volatility, trading volume and return of the risky asset also increase dramatically. A similar situation also happens in the cases B1 and B2. This result indicates that not only the noise traders, but also the trend followers play an important role in producing excess volatilities, high trading volume and excess returns of the risky asset. The underlying intuition is that the impact of the noise traders on the price dynamics can be captured and amplified by the trend followers due to their trend-chasing behaviour.

Summarising, the presences of noise traders and trend followers, or equivalently, the differences of opinion in terms of different investment strategies (or prior beliefs) are able to explain the observed high price volatility, trading volume and excess return. These differences of opinion are maintained and reinforced by investors' strategy-switching behaviour with recency bias in performance evaluation rather than investors' better-than-average overconfidence.

This finding points to a conjecture of Glaser and Weber (2007) which states that better-than-average overconfidence psychologically causes and maintains differences of opinion among investors leading to high trading volume. Based on evolutionary perspectives of financial markets, we agree with that overconfidence reinforces differences of opinion through investors' minds. However, we argue that the market selection force in wealth dynamics and/or investors' adaptive behaviour in terms of performance-driven strategy selection may weaken or even eliminate the effect of better-than-average overconfidence in maintaining and reinforcing the differences of opinion. The high trading volume originated from better-than-average overconfidence is only a temporary rather than a persistent phenomenon of financial markets. Instead, the flow of funds among different investment strategies in conjunction with recency bias in performance evaluation may explain various persistent phenomena of financial markets such as the persistence of asset bubbles, excess price volatility, high trading volume and excess return of the risky asset.

4.3.4 Agents' Performance under Flow of Funds

We now analyse the performance of each agent type in the presence of flow of funds with recency bias. Table 4.5 shows the long-term average values of simple returns (as defined in equation 3.2.3) of different agent types corresponding to experiments documented in Table 4.3.

As shown in Table 4.3, for $\beta = 0$, all three agents have similar long-term average returns which are close to the risk-free rate of return (0.1%). This is due to the convergence of the price at the fundamental value. The risky asset then yields the same rate of return as the risk-free asset. For $\beta > 0$, the fundamentalists always have the highest long-term average return than the other two agent types. The noise traders' long-term average returns are less than the risk-free

Table 4.5: Average Returns of Different Agents

Case	Noise Trader	Fundamentalist	Trend Follower
$\beta = 0$	0.0966%	0.1002%	0.1001%
$\beta = 0.0001$	0.0815%	0.1043%	0.1001%
$\beta = 0.001$	0.0722%	0.1229%	0.1003%
$\beta = 0.01$	0.0693%	0.1438%	0.1013%
$\beta = 1$	0.0703%	0.1456%	0.1015%

rate indicating that they make losses. Along with the increase in β, the fundamentalists' long-term average return increases, while the noise traders' long-term average return decreases. The change in the trend follower's long-term average return is not significant with respect to the change in the value of parameter β.

Comparing the results in Table 4.5 and 4.3, we observe that the noise traders' long-term average return is negatively correlated with the price volatility, which indicates that noise traders do not make profit from the volatility they create. The more volatility the noise traders create the less average return they can get. In contrast, the fundamentalists' long-term average return is positively correlated with the price volatility. This indicates that their investment strategy is successful (in terms of long-term average return) to against high volatility and the flow of funds. The long-term average return of the trend followers is almost independent of the price volatility.

To further investigate the profitability of these agent types, we compute M-periods simple moving average returns of each agent type i:

$$\mathrm{SMA}_{\mathrm{t}}^{\mathrm{i}} = \frac{1}{\mathrm{M}} \sum_{\mathrm{j}=0}^{\mathrm{M}-1} \phi_{\mathrm{t}-\mathrm{j}}^{\mathrm{i}},$$

where ϕ_t^i (given by equation (3.2.3)) denotes the simple return of agent i at each time period t.

Figure 4.18 illustrates the 10-, 30-, and 90-week moving average returns of different agent types for the case of recency bias in performance evaluation ($\beta = 1$, $\gamma = 5$ and $\rho = 0.99$). The results are based on periods 29,800-30,000. Figure 4.18 shows that in terms of 10- and 30-week moving average returns, both the noise traders and the trend followers are able to outperform the fundamentalists at some periods especially when price bubbles happen. In the short term, noise traders and trend followers can generate quick and large profits during price bubbles, but they also suffer large losses immediately after the burst of a price bubble.

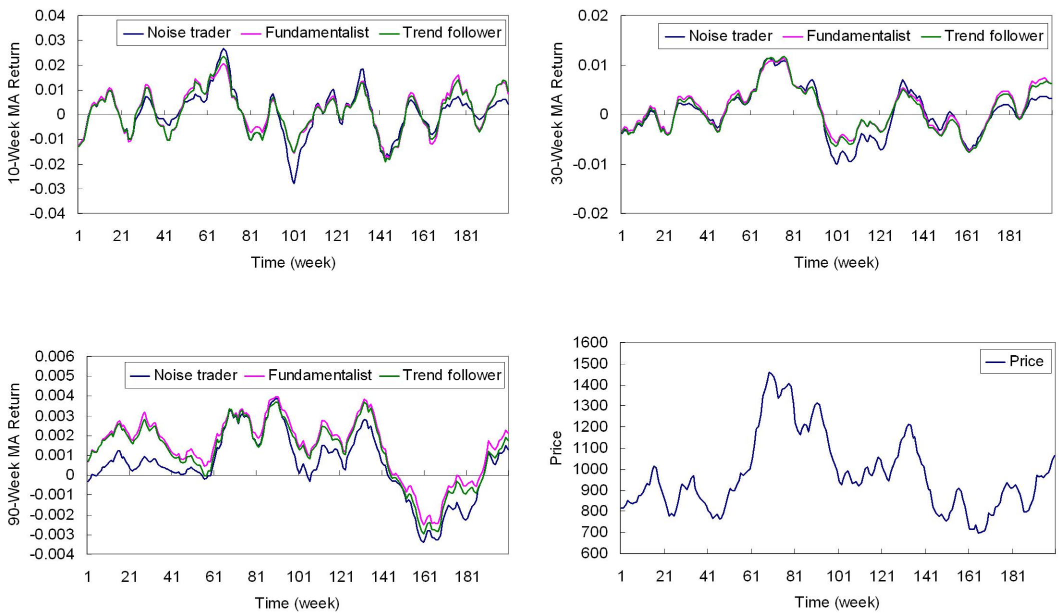


Figure 4.18: Time series of 10-week (top left), 30-week (top right) and 90-week (bottom right) moving average returns of different agents in the case with $\beta = 1$, $\gamma = 5$ and $\rho = 0.99$. The underlying price dynamics (bottom right).

In contrast, the fundamentalists make profits or losses smoothly and steadily along with fluctuations of the price. Therefore, the advantage of the fundamentalists' strategy lies in its long-term profitability. As shown in Figure 4.18, the fundamentalists almost always outperform the other two agents in 90-week moving average return. This findings also suggest that if an investor who intend to make profits using the fundamentalists' strategy, he or she must stay with the strategy for a reasonable long period. The fundamentalists' strategy may not be suitable for short-term speculation, but it has advantage in the long-term on against high price volatility and the effect of flow of funds.

4.4 Concluding Remarks

This chapter studied numerically the dynamics of the behavioural model proposed in Chapter 3. Our analysis addressed the market impact of a variety of behavioural phenomena including investor overconfidence, differences of opinion, recency bias in performance evaluation, conservatism bias and rational herding. The focus was given to evolutionary perspectives of financial markets where adaptation, market selection and both passive and active learning dynamics apply. Our results point to the famous MSH and EMH as well as the causes for some persistent market phenomena such as asset bubbles, excess volatility, high trading volume and equity premium.

We studied the case where all investors are better-than-average overconfident investors and none of them switches among investment strategies. In this case, the model exhibits pure passive learning dynamics. Our simulation results of this non-switching case indicate that the process of market selection plays an important role in shaping the long-term market dynamics. The fundamentalists (informed trader) dominate the market in long run, which is consistent with MSH. The price converges to the fundamental value, which agrees the EMH. The excess volatility or high trading volume originated from better-than-average overconfidence together with differences of opinion in terms of different prior beliefs is only a temporary market phenomenon. The market selection force will eventually eliminate the market impact of overconfidence and differences of opinion.

We also analysed the case where all investors are switching investors. In this case, the model exhibits both passive and active learning dynamics. We showed that, if investors have infinite memory of past performances and are unbiased in performance evaluation, the long-run market behaviour is consistent with the one observed in the non-switching case. Investors' conservative or herding type of behaviour in strategy-switching is able to delay or accelerate the process of market selection and the convergence of market price. The differences of opinion in terms of differences in strategy selection and different interpretations for the public performance measure will be destroyed by investors' adaptive behaviour in conjunction with the market selection force. All switching investors will eventually move to the best performed strategy leading to "homogeneous behaviour" of investors.

In contrast, if the switching investors exhibited recency bias in performance evaluation, we showed that both the fundamentalists and trend followers will survive in long run. The survival of trend followers causes the persistence of market inefficiencies such as asset bubbles and excess volatility. Moreover, our results indicate that investors' conservatism bias in strategy-switching is able to cause a more diversified wealth distribution among different strategies therefore reinforce the differences of opinion leading to high trading volume. Furthermore, we showed that investors' herding type of behaviour in strategy-switching may also cause high trading volume even booms and crashes in the price. This observation provides an alternative explanation for the high trading volume in addition to the concept of differences of opinion.

Such a finding is not entirely new. A similar result was obtained by Brock and Hommes (1998) showing that larger values of the intensity of choice parameter destabilise the market. However, in their model, the wealth dynamics do not affect the price dynamics. The model therefore exhibits only pure active learning dynamics. We provided supportive evidence in a model which captures both passive and active learning. More importantly, we showed that the destabilising effect of the intensity of choice may highly depend on how the performance of each strategy is evaluated by investors as well as the interaction between passive and active learning dynamics. If investors have infinite memory of past performances are unbiased in performance evaluation, the presence of active learning does not affect the outcome of pure passive learning. The price converges to the fundamental value asymptotically and the destabilising effect of the intensity of choice does not exist. If investors exhibit recency bias in performance evaluation, the outcome of active learning disagrees the one of pure passive learning leading to the persistence of market inefficiency. The destabilising effect of the intensity of choice arises in this case. This result indicates that active learning is sensitive to behavioural biases. Investors' behavioural biases such as the recency bias in performance evaluation may play an important role in affecting the process of market selection of survival strategies as well as the long-term market dynamistic.

Previous studies on the process of market selection are mainly based on the approach in which evolutionary forces operate through wealth dynamics. Financial market models which are in line with this approach often exhibit pure passive learning dynamics only, see for example, Sandroni (2000), Blume and Easley (2006) and the evolutionary finance literature surveyed in Evstigneev, Hens and Schenk-Hoppé (2009). In contrast with this approach, our analysis highlighted the importance of using models with both passive and active learning to study the process of market selection. We have shown that investors' heuristics and biases such as conservatism bias, herding and recency bias in performance evaluation may become more relevant under the presence of active learning. We have also demonstrated that analysing the effect of these heuristics and biases and the interaction between passive and active learning dynamics is able to provide additional insights for understanding the process of market selection as well as the long-term market behaviour. Moreover, according to LeBaron (2011), one may expect both passive and active learning are present in real markets.

To check the robustness of our results, we carried out experiments in which both overconfident and switching investors are present in the market. The results in these experiments were obtained based on 100 independent runs. We showed that even a very small amount mobile capital (e.g. 0.01% of the aggregate wealth) with recency bias has a substantial impact on the market dynamics leading to the survival of different agent types and the persistence of market inefficiencies. Moreover, the presence of overconfident investors cannot explain the persistence of excess volatility and high trading volume, whereas the flow of funds with recency bias can provide insights for these observed market phenomena.

Although the concepts of better-than-average overconfidence and differences of opinion have long been proposed as explanations for the observed high trading volume in financial markets (see the review by Glaser and Weber, 2007), previous studies in these areas are usually silent about the long-run prospects and evolutionary perspectives of financial markets. Based on our simulation results, none of these two concepts is sufficient to explain the persistence of high trading volume in a evolutionary context. We agree with the view of Glaser and Weber (2007) that better-than-average overconfidence psychologically maintains differences of opinion among investors. However, we argue that the effect of the presence of overconfident investors as well as differences of opinion may be eliminated by evolutionary forces such as adaptive behaviour of investors and/or the process of market selection.

Summarising, our results presented in this chapter highlighted the role of evolutionary forces in affecting the market dynamics. We demonstrated the added values for using an agent-based model with both passive and active learning to study concepts from behavioural finance and the process of market selection. Our findings may contribute to new ideas on understanding the market selection of survival strategies and the causes of the persistence of asset bubbles, excess volatility, high trading volume and equity premium.

Chapter 5

Observation Horizons and Behavioural Biases in Forecasting

5.1 Introduction

This chapter is devoted to studying behavioural elements which may impact trend followers' forecasting about future returns of the risky asset. We extend the model and analysis presented in previous two chapters to address the roles of observation horizon, sentiment (optimism and pessimism) and recency bias in affecting the trend followers' forecasting. Heterogeneous observation horizons of trend followers and switching among different observation horizons are allowed. We analyse numerically the impact of these behavioural elements on the trend followers' strategic behaviour as well as on the aggregate market dynamics. Our findings point to the causes of a variety of stylised facts such as the absence of autocorrelation in asset returns, volatility clustering, negative skewness and excess kurtosis in asset return distribution.

During the past a few decades, investors' trend following behaviour, or equivalently, the positive feedback trading, has been extensively studied. In the behavioural finance literature, the trend following behaviour is commonly studied in relation to the concept of limits to arbitrage. The existence of trend followers and various limits of arbitrage have been proposed to explain the occurrences of mispricing in financial markets (see overviews by Barberis and Thaler, 2003).

Similarly, in the literature on agent-based financial market models, the presence of trend followers has been identified as an important source of instability in financial markets (see LeBaron, 2006a and Hommes, 2006). However, in behavioural finance studies and agent-based models, investor psychology and biases such as sentiment and recency bias are usually left implicitly in the trend following behaviour itself. This causes difficulties in distinguishing and quantifying the roles of different behavioural biases in affecting the trend following behaviour as well as the market dynamics. Previous contributions which explicitly studied different behavioural biases of trend followers are rare. Our research on trend followers' sentiment and recency bias presented in this chapter aims to contribute to this area.

In the finance literature, investor sentiment have been widely discussed in relation to limits of arbitrage and mispricing (De Long et al., 1990), underreaction and overreaction of stock prices (Barberis, Shleifer, and Vishny, 1998), market crashes (Hong and Stein, 2003), explanation and predication of stock returns (Baker and Wurgler, 2007) and many other aspects. However, a commonly seen issue is that studies on investor sentiment differ from one another in terms of the definition and measurement for sentiment, and no consensus has been reached. As stated by Baker and Wurgler (2007, p.130): *"Now, the question is no longer, as it was a few decades ago, whether investor sentiment affects stock prices, but rather how to measure investor sentiment and quantify its effects."*

Our study addresses investor sentiment through its phycology underpinnings. Sentiment is defined in terms of optimism and pessimism. We refer optimism (or pessimism) to the tendency that investors give importance to positive (or negative) events in their information set when forming return forecasts. In our model, trend followers' forecasting is formulated by a probability weighting function of historical returns of the risky asset, where the observation horizon governs the sample size. We regard optimism (or pessimism) as the phenomenon that trend followers assign larger decision weights to larger positive (or negative) historical returns. This approach to sentiment has a key feature that investors' decision weights for each possible outcome are not necessary equal to its probability but subject to heuristics and behavioural biases. Modelling sentiment in this way is in line with the probabilistic decision theory, such as the Prospect Theory (Kahneman and Tversky, 1979).

As reviewed in Chapter 2, Kahneman and Tversky (1974) have identified two important heuristics: representativeness and availability. The representativeness heuristic refers to the tendency for people to categorise some events as typical or representative and ignore the laws of probability. The availability heuristic describes the phenomenon that people assess the frequency of class or the probability of an event by how easy they can be brought to mind. Consistent with these two heuristics, our model extension of optimism (or pessimism) characterises the phenomena: i) optimistic (or pessimistic) trend followers categorise large positive (or negative) historical returns as typical performances of the risky asset; ii) they assigned larger decision weights to these typical performances due to these typical performances are more easy to be recalled.

This conceptual model of sentiment mentioned above is implemented via an optimism-pessimism index which is originally proposed by Kaymak and van Nauta Lemke (1998) in the fuzzy decision making literature. Using this index to study investor sentiment in the literature on agent-based financial market models was firstly advocated by Lovric et al. (2009). One important advantage of this approach is that the degree of investors' optimism or pessimism can be intuitively measured by a single parameter: the optimism-pessimism index. Based on this approach, we will explore the market impact of different degrees of optimism and pessimism by varying the value of the optimism-pessimism index.

In addition to sentiment, we also study the impact of recency bias on trend followers' forecasting. Recency bias is modelled by the phenomenon that trend followers assign larger decision weights to more recent observations of historical returns of the risky asset. We expect that recency bias in forecasting may reinforce the trend following behaviour and affect sentiment leading to frequent shifts between optimism or pessimism. As pointed out by Offerman and Sonnemans (2004), when traders exhibit recency bias in asset valuation, traders who are not sure of the intrinsic value of a stock will be too optimistic about its value when the firm is winning and too pessimistic when it is losing. We analyse how the combined effect of sentiment and recency bias impact the aggregate market dynamics.

Our analysis will start from a simple case where the trend followers have homogeneous observation horizons. The purpose is to provide a basic understanding of the roles of trend followers' observation horizon, sentiment and recency bias in affecting the aggregate market dynamics. We then propose another extension of the model to study a more complicated but realistic case in which the trend followers have heterogeneous observation horizons. In this case, the whole population of trend followers are divided into subgroups which are characterised by different observation horizons.

In the literature on agent-based financial market models, the heterogeneity with respect to investors' observation horizons has been previously studied by Levy, Persky, Solomon (1996) and Levy, Levy, and Solomon (2000). Compared with their approaches, our approach to heterogenous observation horizons has features: i) switching among different observation horizons is allowed; ii) the collective behaviour of the trend followers who have different observation horizons can be modelled and observed. This helps to understand the resulting price dynamics when the number of different subgroups of trend followers is large. Our analysis aims to explore the market impact of the heterogeneity in trend followers' observation horizons, especially when behavioural biases such as sentiment and recency bias are relevant.

This chapter is organised as follows. We employ a so-called incremental approach according to which extensions of new behaviour are introduced gradually to the original model. The market impact of newly added behaviour is explored by comparing the results of model extensions with the results of the original model. In Section 5.2, we introduce and analyse model extensions of sentiment and recency bias in a homogeneous observation horizons case. Section 5.3 extends the model and analysis to a heterogeneous observation horizons case. Section 5.4

summarises our findings.

5.2 Homogenous Observation Horizons

This section focuses on the case where the trend followers have homogeneous observation horizons $L \geq 1$. The model with original specification for the trend followers' forecasting rule (equation (3.5.10) in Chapter 3) is referred to as a benchmark model. We first analyse the benchmark model with focus on the role of the trend followers' observation horizon in affecting the market dynamics. We then modify the benchmark model gradually to incorporate sentiment and recency bias in the trend followers' forecasting. The market impact of sentiment and recency bias will be explored by comparing the results of modified model with those of the benchmark model.

Consistent with the previous chapter, each time period in our simulation experiments represents one week. Model parameters values are the same as those documented in Table 4.1. Previous analysis has identified that, in our model, the existence of switching investors (i.e. $\beta \in (0, 1]$) and recency bias in performance evaluation (i.e. $\rho \in [0, 1)$) are prior conditions for the survival of the trend followers in the long run. To ensure the survival of the trend followers and the persistence of their market impact, our analysis presented in this chapter is based on the assumption that every individual investors are switching investor and they exhibit recency bias in strategy-switching. The parameter values of β and ρ are set to 1 and 0.99 respectively. The value of the intensity of choice parameter γ is set to 5. The set of model parameter values is fixed for all numerical analyses presented in this Chapter. Moreover, all market phenomena (including short-term dynamics) illustrated in this chapter are persistent phenomena due to the survival of the trend followers in the long run, unless stated otherwise.

5.2.1 The Basic Forecasting Rule of Trend Followers

In our model, it is assumed that the group of trend followers uses a rolling window which contains $L \geq 1$ historical returns of the risky asset $\{R^s_{t-L}, R^s_{t-L+1}, ..., R^s_{t-1}\}$ to forecast the future return R^s_{t+1} of the risky asset. The value of L measures the length of the observation horizon of the trend followers.

We refer to the original specification (3.5.10) as the *basic forecasting rule* of the trend followers. The basic forecasting rule characterises the phenomenon that the group of trend followers follows a simple heuristic according to which they assign identical probability mass to each observed historical return i.e. $\mathbb{P}(R^s_{t+1,1} = R^s_{t-j,1}) = \frac{1}{L}$ for $j = 1, ..., L$. The trend followers' expectation (forecasting rule) of the future return at time t is given by $E^T_t[R^s_{t+1,1}] = \frac{1}{L}\sum_{j=1}^{L} R^s_{t-j,1}$. We explore here the role of trend followers' observation horizon L in affecting the market dynamics.

We first analyse the deterministic model in which the fundamentalists and trend followers are present in the market and the dividend is assumed to be a constant. Figure 5.1 illustrates market dynamics generated by the deterministic model with different values of observation horizon L. These results are sampled in the long run when the price converges to a stable limit cycle (as illustrated by Figure 4.4 in the previous chapter). As shown in Figure 5.1, the price exhibits predictable cycles due to the interaction between the fundamentalists and the trend followers who use basic forecasting rule. The size and duration of each price cycle (bubble) are positively correlated with the length of trend followers' observation horizon. Longer observation horizons are able to cause the larger and longer price bubbles.

We now analyse the case in which the fundamentalists, the trend followers and the noise traders are present in the market. Due to the presence of the noise traders, the model generates stochastic dynamics. Figure 5.2 illustrates the market dynamics generated by the stochastic model for the cases $L = 10$ and $L = 20$ respectively. In this figure, the proportion of noise traders is set to 15% (i.e.

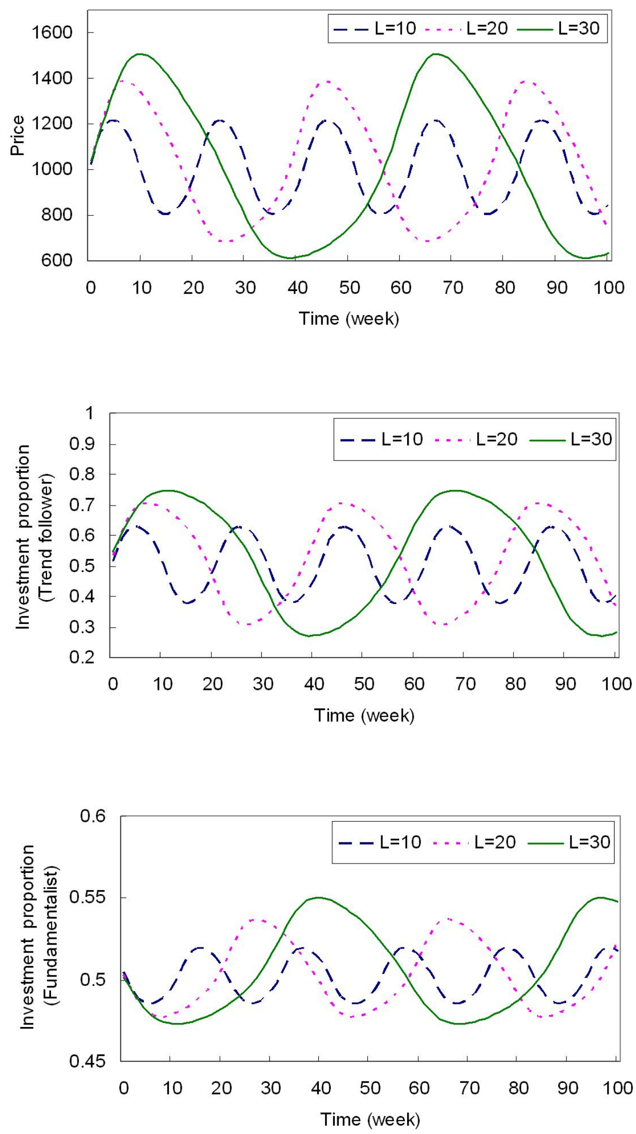


Figure 5.1: Comparison of short-term dynamics under different values of the parameter L when the fundamentalists and the trend followers are present in the market: price dynamics (top), the trend followers' investment proportion for the risky asset (middle), the fundamentalists' investment proportion for the risky asset (bottom).

$q^N = 0.15$). In order to compare agents' behaviour between the case $L = 10$ and $L = 20$, the path of the noise traders' random investment proportion is fixed in the two independent simulations (i.e. the value of the seed of the random number generator is fixed).

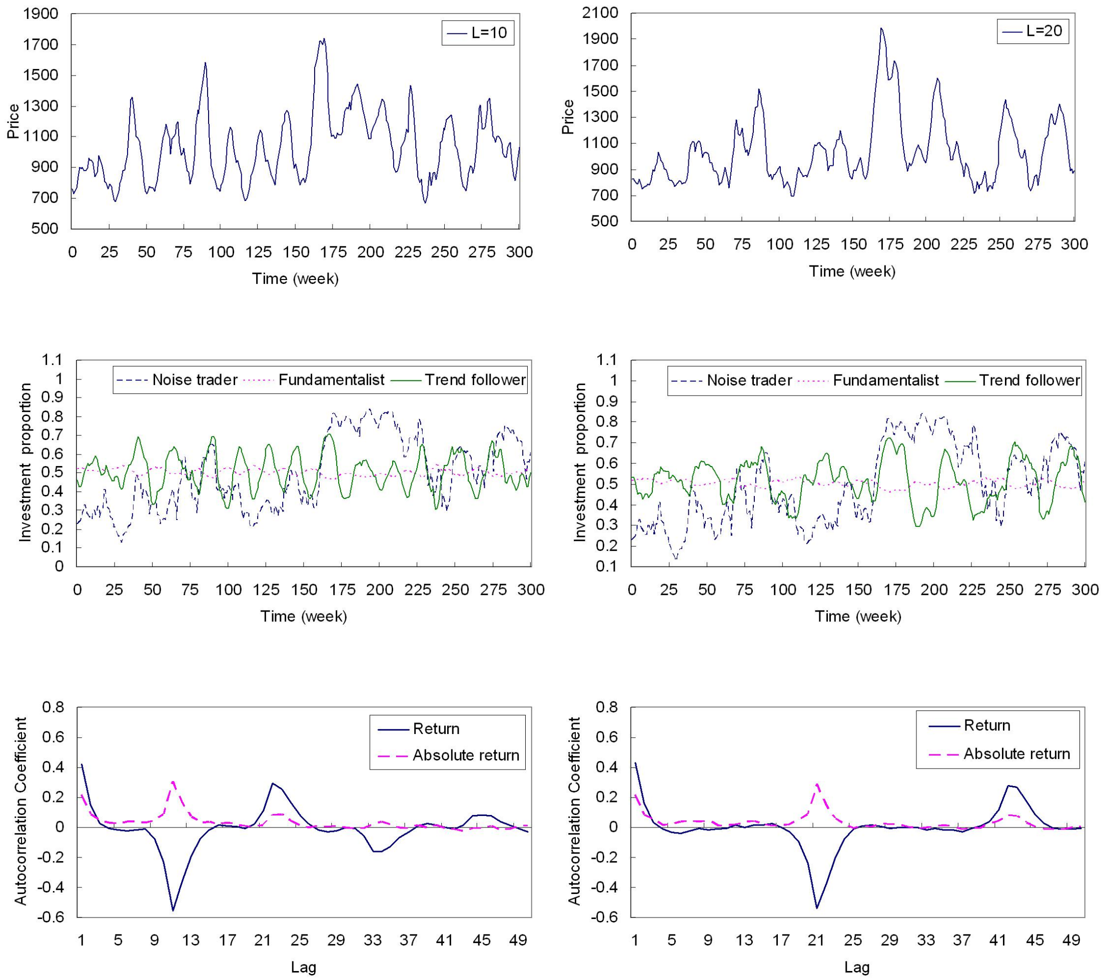


Figure 5.2: Illustration of price dynamics and agents' investment proportions for the risky asset when $L = 10$ (left column) and $L = 20$ (right column). The bottom panels are autocorrelation coefficients of log returns and absolute returns.

As shown in Figure 5.2, under the presence of the noise traders, the size and duration of the price bubbles are less predictable than those in the scenario without the noise traders. In the case where the trend followers have a shorter observation horizon ($L = 10$), their investment proportions are strongly affected by the noise traders' behaviour resulting that the trend followers tend to chase the trend caused by the noise traders. In the case where the trend followers have a longer observation horizon ($L = 20$), their investment proportions are less sensitive to the noise traders' behaviour resulting smoother price dynamics than in the shorter observation horizon case.

The autocorrelation coefficients of log returns (bottom panels of Figure 5.2) reveal that the return dynamics is characterised by the length of trend followers' observation horizon: in the case $L = 10$, large autocorrelation coefficients (either in positive or negative) appear in every 10 lags; in the case $L = 20$, large autocorrelation coefficients appear in every 20 lags. Here, an important issue is that these autocorrelation coefficients of weekly returns are too large to be realistic.

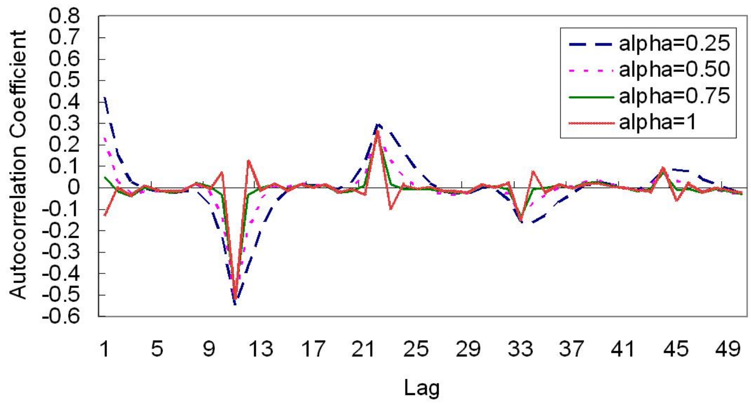


Figure 5.3: Autocorrelation coefficients of returns when the fundamentalists exhibit different degrees of reaction $\alpha^F = 0.25$, $\alpha^F = 0.5$ and $\alpha^F = 0.75$. The proportion of the noise traders: $q^N = 0.15$.

To understand the impact of the fundamentalists on return dynamics, the top panel of Figure 5.3 compares the autocorrelation coefficients of returns generated in four different cases where the fundamentalists have different degrees of reaction (when price deviates from the fundamental value): $\alpha^F = 0.25$, $\alpha^F = 0.5$, $\alpha^F = 0.75$ and $\alpha^F = 1$. The result shows that the stabilising force of the fundamentalists has a significant impact on the first two lags of the autocorrelation coefficients of returns. The presence of the fundamentalists with stronger reaction implies that returns are less autocorrelated in the first two lags. However, the impact on the remainder lags are insignificant indicating that the fundamentalists with different degrees of reaction do not affect the property that the return dynamics is characterised by the length of observation horizon L of the trend followers.

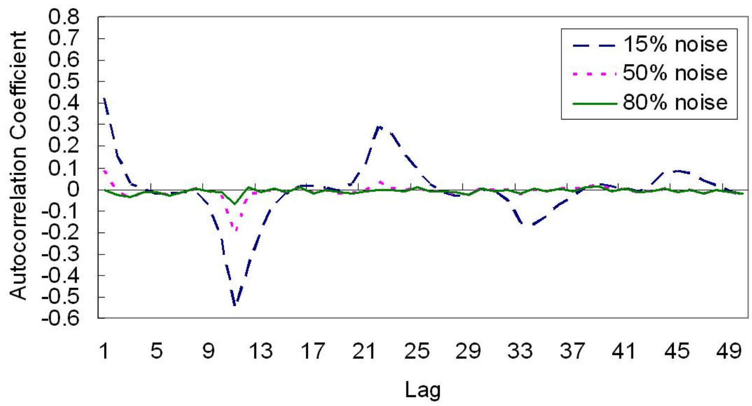


Figure 5.4: Autocorrelation coefficients of returns generated under different proportions of the noise traders. Degree of reaction of the fundamentalist: $\alpha^F = 0.25$.

Figure 5.4 depicts the impact of different proportions of the noise traders on return dynamics. The autocorrelation coefficients show that the return dynamics is still characterised by the length of trend followers' observation horizon unless the group of noise traders owns a very large proportion of the aggregate wealth (e.g. above 80%). These results reveal that the length of the trend followers' observation horizon plays an important role in shaping the price and return dynamics.

The predictable and unrealistic autocorrelation structure (which is caused by the presence of trend followers with homogeneous observation horizons) cannot be removed by the fundamentalists with different degrees of reaction or a reasonable proportion of the noise traders.

5.2.2 Forecasting with Sentiment: Optimism and Pessimism

In this subsection, we extend the basic forecasting rule of the trend followers to incorporate sentiment. Sentiment is defined and modelled in terms of optimism and pessimism via an optimism-pessimism index which is originally proposed by Kaymak and van Nauta Lemke (1998) in the fuzzy decision making literature. This approach is utilised by Lovric et al. (2009) in agent-based modelling literature to study the market impact of different degrees of optimism or pessimism. The trend followers' forecasting rule of excess return with sentiment is given by:

$$E_t^T[R_{t+1}^s - R] = \Big(\frac{1}{L}\sum_{j=1}^{L}(R_{t-j,1}^s)^{\tau}\Big)^{\frac{1}{\tau}} - R, \tag{5.2.1}$$

where $\tau \in \mathbb{R}$ is the optimism-pessimism index measuring how close the value of the forecasted excess return $E_t^T[R_{t+1}^s - R]$ to the maximum or minimum value of historical return in sample. The higher (lower) the value of the parameter τ is, the closer the forecasted excess return is to the maximum (or minimum) value of historical return in sample. This is equivalent to assign larger probability masses to larger positive (or negative) historical returns in sample. In the extreme case, $\tau \to \infty$ (or $\tau \to -\infty$) implies that the forecasted return equals the maximum (or minimum) value in sample.

The value of the parameter τ can be used to describe the trend followers' degree of optimism and pessimism. If $\tau = 1$, the forecasting rule is identical to the basic forecasting rule, i.e. the trend followers assign identical probability mass to each observation of excess return in the rolling window (the forecasted excess return is the arithmetic mean). In this case, trend followers exhibit neither optimism nor pessimism. If $\tau > 1$, the forecasting rule corresponds to optimistic trend follower, because trend followers assign larger probabilities to observations of larger excess returns in the rolling window. In contrast, if $\tau < -1$, the forecasting rule corresponds to pessimistic trend follower, because trend followers assign larger probabilities to observations of smaller excess returns in the rolling window. As mentioned in introduction, this approach to sentiment is consistent with the heuristics of representativeness and availability documented by Kahneman and Tversky (1974). Based on this approach, we explore how different values of the optimism-pessimism index τ affects market dynamics.

We first analyse the deterministic case in which the noise traders are absent and the dividend is a constant. Figure 5.5 compares the market dynamics generated under different values of τ: $\tau = 1$ (without sentiment), $\tau = 30$ (optimism) and $\tau = -30$ (pessimism). The trend followers' observation horizon in these simulations is set to 10 weeks ($L = 10$). Simulation results are sampled in the long run when the price exhibits stable cycles.

As shown in Figure 5.5, the price dynamics under $\tau = 1$ is similar to those under $\tau = 30$ or $\tau = -30$. This observation indicates that, in the deterministic model, the price dynamics is not very sensitive to the trend followers' sentiment. The trend followers' investment proportions (Figure 5.5, middle) show that, on average, the optimistic (or pessimistic) trend followers tend to invest larger (or smaller) fraction of wealth into the risky asset than the trend followers without sentiment. The optimistic trend followers cause the price to be clustered around the "top" (maximum value) of the price cycle, while the pessimistic trend followers cause the price to be clustered at the "bottom" (minimum value) of the price cycle. The trend followers' wealth shares (Figure 5.5, bottom) show that the trend followers have a slightly better performance in the case without sentiment. The variations of wealth shares in the three different cases are not strong.

We now add the noise traders into our simulation experiments. The goal is to explore how different values of τ affect the price dynamics under the presence of the noise traders. Figure 5.6 compares the market dynamics generated under

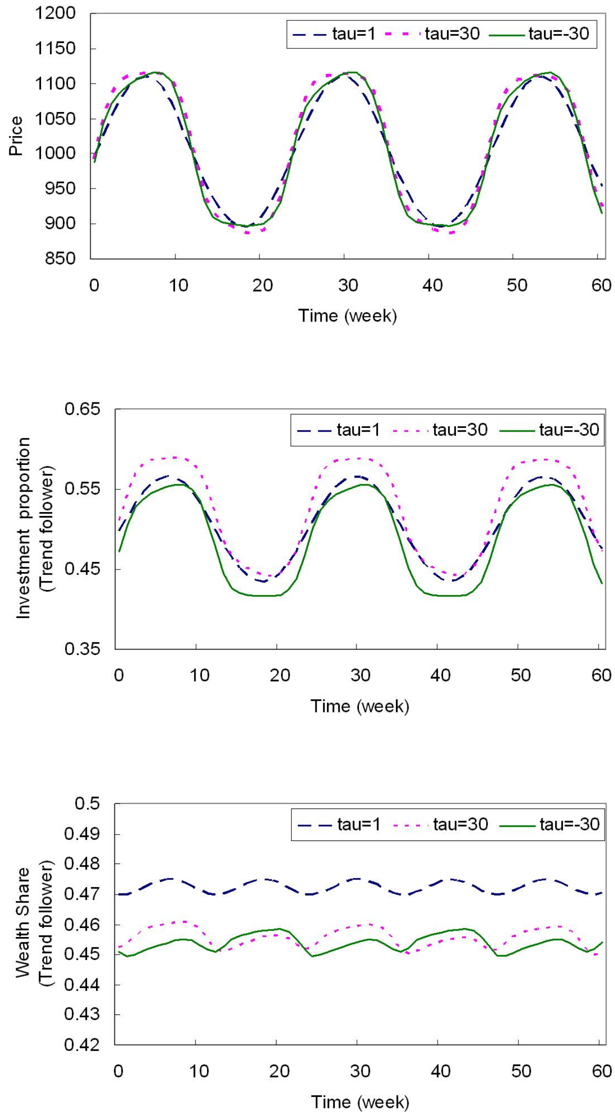


Figure 5.5: Illustration of the impact of the trend followers' sentiment when the fundamentalists and the trend followers are present in the market: price dynamics (top), the trend followers' investment proportion for the risky asset (middle), the trend followers' wealth shares after strategy-switching (bottom). $L = 10$.

different degrees of optimism of the trend followers. The results are sampled after 29,000 periods in order to illustrate the long-run dynamics. Figure 5.6 shows

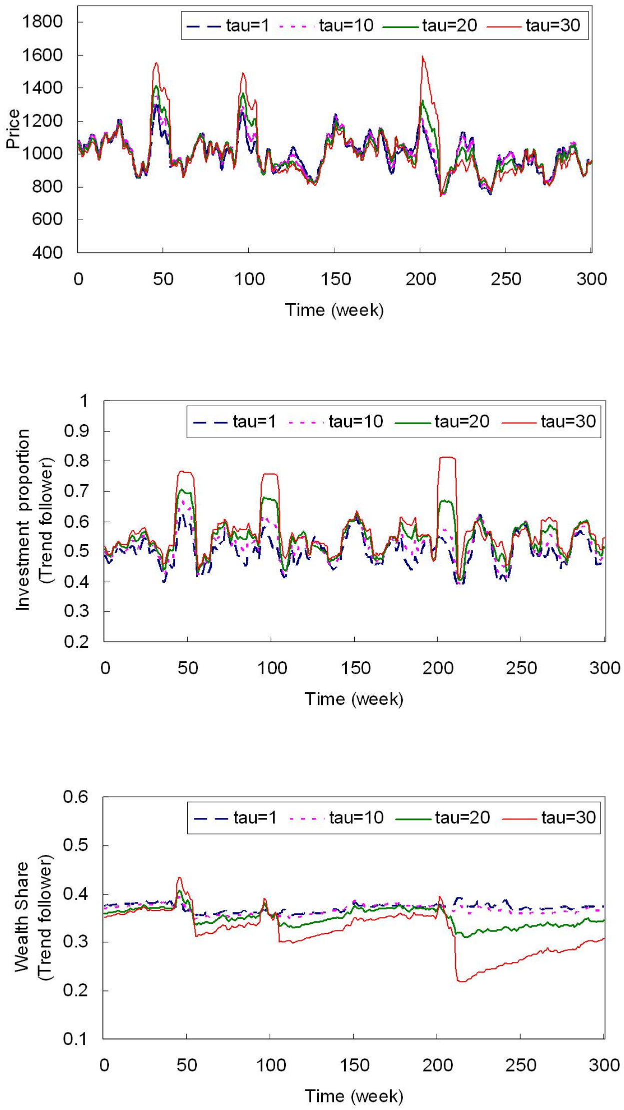


Figure 5.6: Illustration of the impact of the trend followers' optimism when the fundamentalists, trend followers and noise traders are present in the market: price dynamics (top), the trend followers' investment proportion for the risky asset (middle), the trend followers' wealth shares after strategy-switching (bottom). Parameters: $L = 10$, $q^N = 0.15$.

that, under the presence of the noise traders, the price becomes sensitive to the trend followers' sentiment. The size of booms and crashes in the price is positively correlated with the degree of the trend followers' optimism.

The reason for this phenomenon is as follows. The presence of the noise traders may randomly trigger large returns in the price. When these large returns are observed by the optimistic trend followers, according to the degree of optimism, the trend followers may sharply increase their investment proportions for the risky asset leading to a market boom. The size of this market boom is linked to the degree of the trend followers' optimism. Those large returns generated in the boom will also be observed by the optimistic trend followers. Therefore, the optimistic trend followers will continue to invest large proportions of wealth into the risky asset, which helps to maintain the price bubble for some periods (as illustrated in periods 45-55 in Figure 5.6).

However, these large returns in the boom will be gradually pushed out of the trend followers' observation horizon with length $L = 10$. For this reason, 10 periods after a market boom, the optimistic trend followers face a set of smaller returns. This may cause a sharp decrease in the trend followers' investment proportion for the risky asset leading to a dramatic market crash. The size of a market crash after a boom is also linked to the degree of the trend followers' optimism. Compared with the case $\tau = 1$, the trend followers in the optimism cases (i.e. $\tau > 1$) are able to gain higher wealth shares when market booms appear. However, the optimistic trend followers have lower wealth shares when market crashes happen. On average, the trend followers in the case $\tau = 1$ have higher wealth shares than in the optimism cases.

Figure 5.7 illustrates the market impact of the trend followers' pessimism under the presence of the noise traders. In contrast with the optimism case, the pessimistic trend followers may sharply decrease their investment proportion for the risky asset when large negative historical returns are observed. This behaviour causes crashes in the price. A market crash is usually followed by a market boom after L (the length of observation horizon) periods. The degree of pessimism is positively correlated with the size of market crashes and booms. Compared with the case $\tau = 1$, the trend followers in the pessimism cases (i.e. $\tau < 1$) have lower wealth shares on average. However, those pessimistic trend followers are able to gain higher wealth shares when market crashes happen.

The simulation results illustrated in Figure 5.6 and Figure 5.7 indicate that the trend followers' sentiment in forecasting is able to significantly affect the size of price bubbles (i.e. the range of the price deviation from the fundamental value). However, sentiment has a minor impact on the duration (with respect to time) of the price bubbles. The same as the results of the case without sentiment, the duration of price bubbles in the optimism or pessimism cases is mainly determined by the length of the trend followers' observation horizon.

Figure 5.8 depicts the long-term return dynamics generated in three different cases: $\tau = 1$, $\tau = 30$ and $\tau = -30$. It shows that log returns of the risky asset in the case $\tau = 1$ are mainly clustered around their mean value. Large movements in the price rarely happen. However, in the cases $\tau = 30$ and $\tau = -30$, large returns happened frequently indicating that sentiment (optimism and pessimism) destabilises the market and cause large movements in the price.

The autocorrelation coefficients of log returns (Figure 5.8, bottom right) reveal that the trend followers' sentiment causes the returns to be slightly less autocorrelated than in the case where trend followers use basic forecasting rule without sentiment. However, optimism or pessimism cannot destroy the unrealistic autocorrelation structure which is imposed by the homogeneous observation horizons of the trend followers. The length of trend followers' observation horizon still plays an important role in shaping the price and return dynamics.

Figure 5.9 shows the histograms and statistics corresponding to the log returns illustrated in Figure 5.8 under different values of τ. When the trend followers use basic forecasting rule without sentiment ($\tau = 1$), log returns exhibit a bell shaped distribution which is similar to the normal distribution in its shape. However, the Jarque−Bera test shows that the skewness and kurtosis do not match the normal

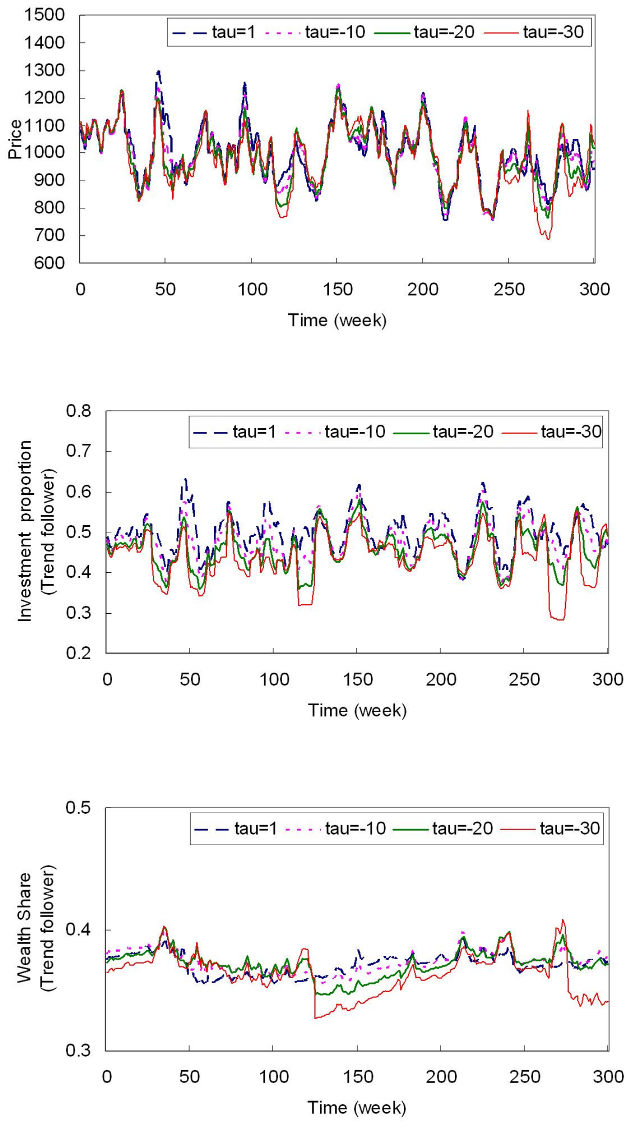


Figure 5.7: Illustration of the impact of the trend followers' pessimism when the fundamentalists, trend followers and noise traders are present in the market: price dynamics (top), the trend followers' investment proportion for the risky asset (middle), the trend followers' wealth shares after strategy-switching (bottom). Parameters: $L = 10$, $q^N = 0.15$.

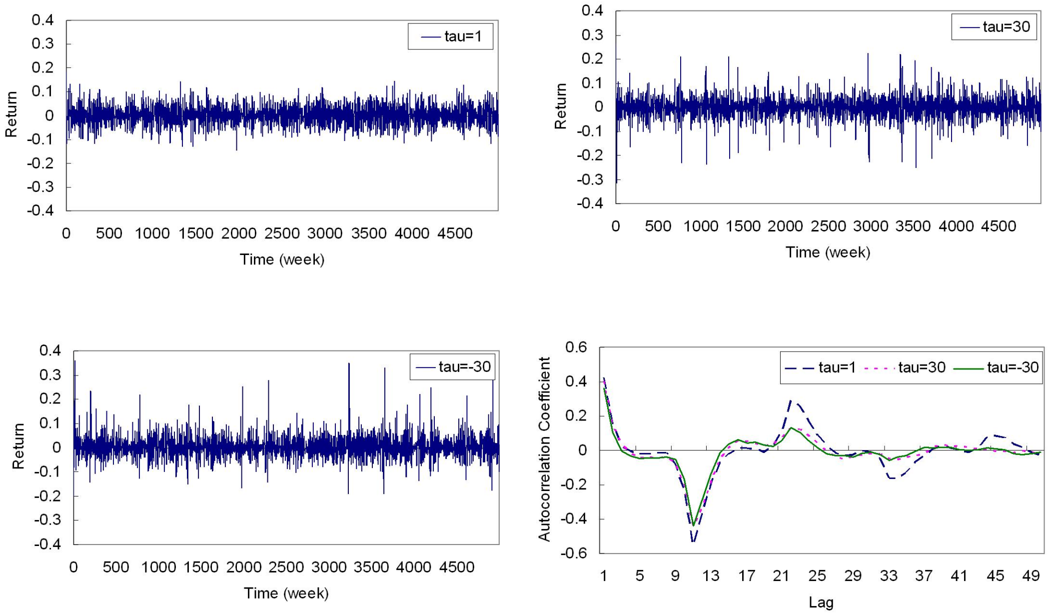


Figure 5.8: Illustration of return dynamics under different values of τ: $\tau = 1$ (top left), $\tau = 30$ (top right), $\tau = -30$ (bottom left); autocorrelation coefficients of log returns (bottom right). Parameters: $L = 10$, $q^N = 0.15$.

distribution. When the trend followers are optimistic ($\tau = 30$), the distribution of log returns exhibits large excess kurtosis, fat tails and negative skewness. When the trend followers are pessimistic ($\tau = -30$), the distribution of log returns also exhibit excess kurtosis, fat tails but positive skewness.

In both optimism and pessimism cases, the volatilities (standard deviations) of log returns are larger than in the case without sentiment. Moreover, the excess kurtosis and fat tails in return distributions are qualitatively consistent with the stylised facts of real-world financial time series as those illustrated by Cont (2001). These results indicate that the price and return dynamics generated under the presence of sentiment are closer to those in real markets. However, the autocorrelation coefficients of returns are still too large to be realistic.

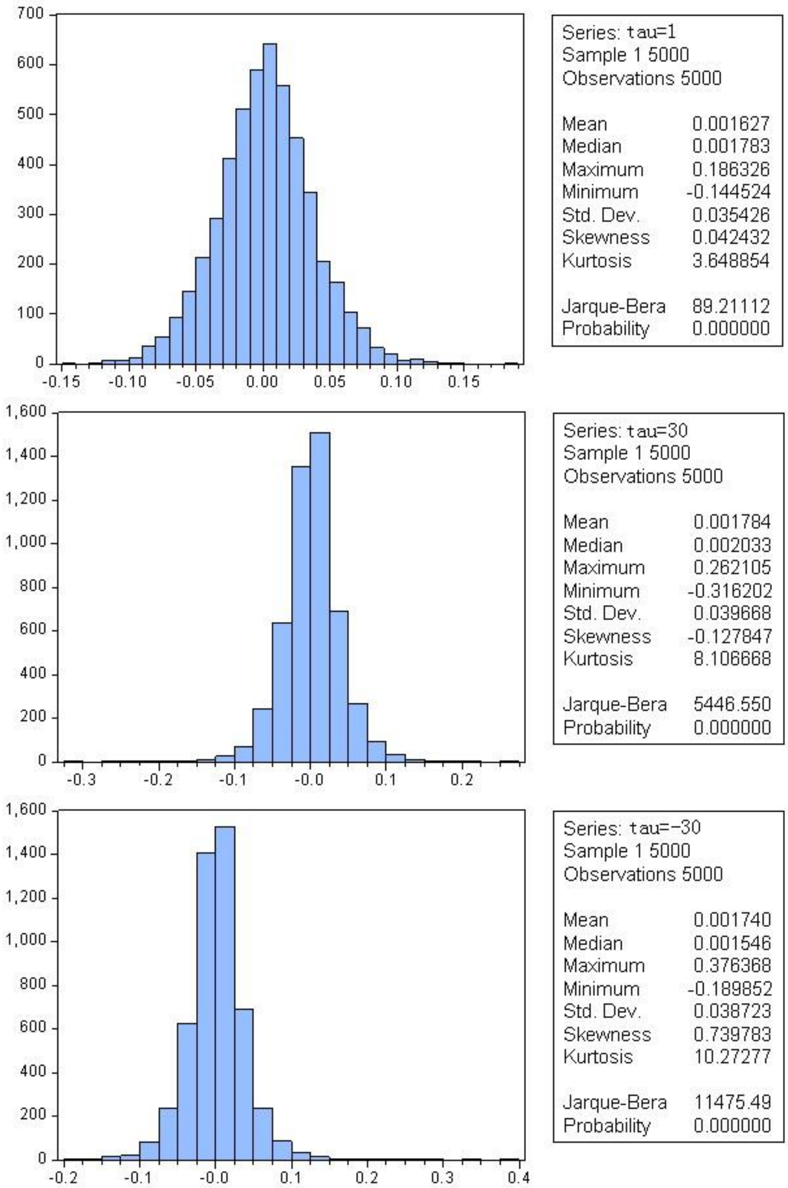


Figure 5.9: Histogram and statistics of log returns under different values of τ: $\tau = 1$ (top), $\tau = 30$ (middle), $\tau = -30$ (bottom). Parameters: $L = 10$, $q^N = 0.15$.

Compared with the results of Lovric et al. (2009) who modelled sentiment in a similar fashion, our results in general agree with their finding that investor optimism is related to market booms and crashes. However, Lovric et al. (2009)

showed that investor pessimism has little impact on the price, which is in contrast with our results. The reason is that, in their model, the trend followers (called efficient market believers in their paper) who exhibit pessimism tend to be driven out of the market in the competition with a group of rational informed investors (i.e. the fundamentalists who are informed about the dividend process). In the optimism case in their paper, the trend followers who exhibit optimism may outperform the rational informed investors and dominate the market. The long-run average wealth share of the optimistic trend followers is positively correlated with the degree of optimism.

The paper of Lovric et al. (2009) is based on an microscopic simulation model proposed by Levy, Levy and Solomon (2000). As reviewed in Chapter 2, this model does not allow investors to switching between different agent types. The results on the survival of different agent types in the Levy, Levy and Solomon model (LLS model) have been critically evaluated by Zschischang and Lux (2001) and Anufriev and Dindo (2010). These authors revealed that an important property of the LLS model is that the rational informed investors do not necessarily dominate the market. Investors who are more aggressive (i.e. less risk averse) to invest in the risky asset may dominate the market. This special property of the LLS model explains the survival of the optimistic investors and the extinction of the pessimistic investors documented in Lovric et al. (2009).

According to Anufriev and Dindo (2010), the property of the LLS model regarding the survival of aggressive investors depends on the relation between the values of the interest rate r of the risk-free asset and the growing rate g of the dividend of the risky asset. The property comes from the assumption of $r < g$. Under this assumption, at a steady state where returns of the risk-free asset and the risky asset are constant, the risky asset yields a higher return than the risk-free asset. However, this is not in case in our model. As explained in Chapter 3, such a steady state in our model corresponds to an arbitrage-free equilibrium in which the risky asset pays the same return as the risk-free asset. The survival of the trend follower type in our model is due to the strategy-switching of investors.

5.2.3 Forecasting with Recency Bias and Sentiment

We now model the phenomenon that the trend followers exhibit recency bias in forecasting. Recency bias is modelled through a set of exponentially decaying probability masses, by which the trend followers assign larger probability mass to the more recent observations of returns in the rolling window of size L. At time $t > 0$, the most recent return observation $R^s_{t-1,1}$ is initially assigned a decision weight of μ:

$$\omega(R^s_{t-1,1}) = \mu, \quad \mu \in \mathbb{R} \text{ and } \quad \mu \in [0,1], \tag{5.2.2}$$

the decision weights of older observations are iteratively reduced by a factor $1-\mu$,

$$\omega(R^s_{t-j,1}) = \omega(R^s_{t-j+1,1})(1-\mu), \quad j = 2, ..., L, \tag{5.2.3}$$

these weights are then normalised to give the probability mass function:

$$\text{pmf}(R^s_{t-j,1}) = \mathbb{P}(R^s_{t+1,1} = R^s_{t-j,1}) = \frac{\omega(R^s_{t-j,1})}{\sum_{j=1}^{L} \omega(R^s_{t-j,1})}. \tag{5.2.4}$$

Therefore, trend followers' forecast rule with recency bias is given by:

$$E^T_t[R^s_{t+1} - R] = \sum_{j=1}^{L} \text{pmf}(R^s_{t-j,1})(R^s_{t-j,1}) - R. \tag{5.2.5}$$

When recency bias is coupled with sentiment, trend followers' forecast rule becomes:

$$E^T_t[R^s_{t+1} - R] = \Big(\sum_{j=1}^{L} \text{pmf}(R^s_{t-j,1})(R^s_{t-j,1})^{\tau}\Big)^{\frac{1}{\tau}} - R. \tag{5.2.6}$$

In this specification of recency bias, the discounting factor $1-\mu$ measures the speed of the exponential decay of the probability mass. Therefore, the initial weight μ governs the degree of the recency bias of the trend followers. To illustrate how it works, Figure 5.10 illustrates the probability mass functions under different initial weights, $\mu = 0.1$ and $\mu = 0.3$, respectively. The observation horizon is set to 10. The higher the parameter μ is, the higher degree of recency bias the trend followers exhibit in their return forecast. In the case $\mu = 0$, the forecasting rule is identical to the basic forecast rule (i.e. $\mathbb{P}(R^s_{t+1,1} = R^s_{t-j,1}) = \frac{1}{L}$) in which the trend followers do not exhibit recency bias.

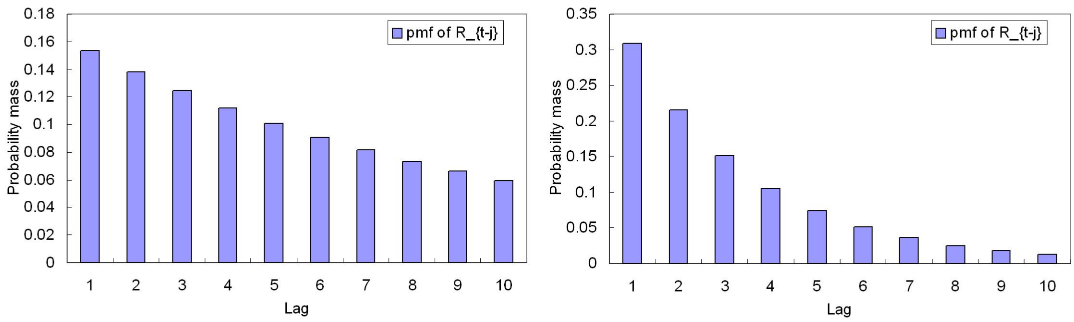


Figure 5.10: Illustration of probability mass function when $L = 10$. The initial weight: $\mu = 0.1$ (left), $\mu = 0.3$ (right).

To explore the market impact of recency bias in trend followers' forecasting, Figure 5.11 compares the price dynamics generated in different cases where the trend followers exhibit different degrees of recency bias in forecasting: $\mu = 0.1$ and $\mu = 0.3$. The top panel shows the price dynamics generated in the deterministic case with absence of the noise traders. The middle panel depicts the price dynamics generated in the stochastic case in which the noise traders possess 15% aggregate wealth. The bottom panel shows the autocorrelation coefficients of 5,000-period log returns in the stochastic case. The price and return dynamics are sampled in the long run (i.e. after 29,000 periods).

As can be seen in Figure 5.11, unlike the effect of sentiment which mainly affects the size but not the duration of price bubbles, the presence of recency bias in the trend followers' forecasting is able to affect both the size and duration of price bubbles. The most significant impact of the recency bias is that it makes the duration of bubbles shorter and therefore increases the frequency of bubbles. This property can be found in both the deterministic case and the stochastic case. These observations indicate that the recency bias in trend followers' forecasting is able to reinforce the trend following behaviour causing the trend followers to

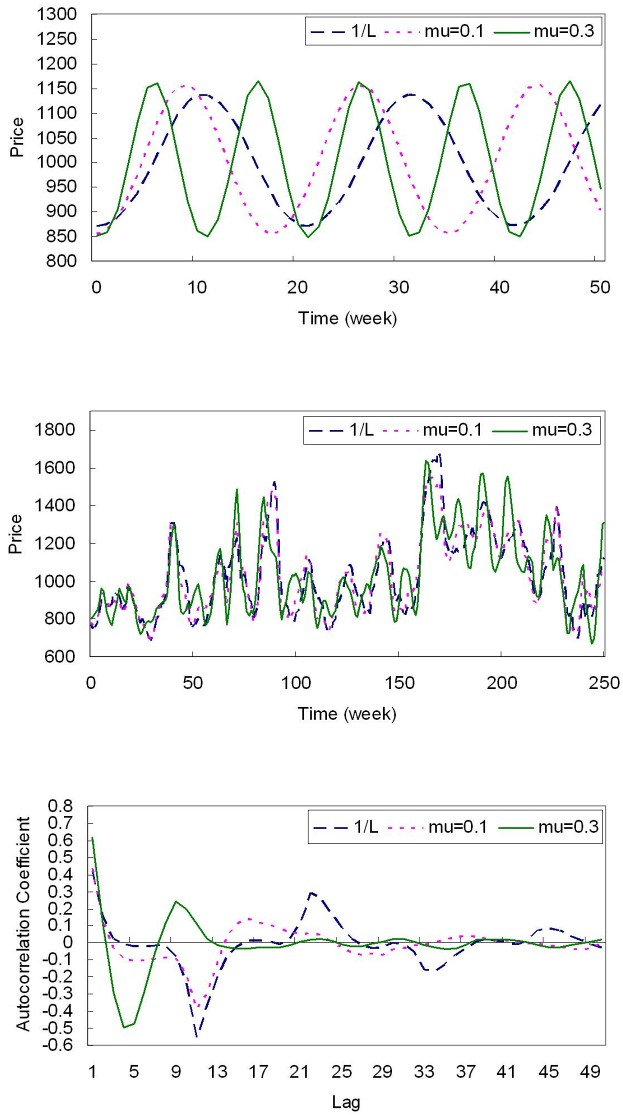


Figure 5.11: Comparison of price dynamics generated under different degrees of recency bias of the trend followers: price dynamics in the deterministic case (top), price dynamics under the presence of the noise traders (middle), autocorrelation coefficients of log returns under the presence of the noise traders (bottom). Parameters: $L = 10$, $q^N = 0.15$.

become optimistic (or pessimistic) when the price is increasing (or decreasing).

Figure 5.12 compares the price dynamics generated when the trend followers exhibit recency bias ($\mu = 0.3$) and different sentiment: $\tau = 1$ (no sentiment), $\tau = 30$ (optimism) and $\tau = -30$ (pessimism). The results show that when the recency bias is coupled with sentiment, these two behavioural biases are able to significantly affect both the size and duration of price bubbles leading to more complicated price dynamics with large booms and crashes. The autocorrelation coefficients of returns reveal that, when trend followers exhibit both the recency bias and sentiment in forecasting, returns are less autocorrelated. However, these autocorrelation coefficients are still too large compared with those observed in real markets under weekly frequency.

Figure 5.13 shows the histograms and statistics of returns corresponding to the case illustrated in Figure 5.12. The results indicate that, without sentiment, the recency bias itself cannot cause high excess kurtosis in return distribution. Recency bias with optimism (or pessimism) leads to high excess kurtosis and positive (or negative) skewness in return distribution. These observations are opposite to those when the trend followers only exhibit sentiment. In the case where the trend followers only exhibit sentiment but not recency bias, optimism (or pessimism) may imply negative (or positive) skewness.

To summarise, our analysis presented in this section showed that the trend followers' observation horizon, sentiment and recency bias in forecasting are important behavioural factors which may strongly affect the price and return dynamics on the macro level. The case where the trend followers use basic forecasting rule without sentiment and recency bias showed that the size and duration of the price bubbles as well as the autocorrelation structure of returns are characterised by the length trend followers' homogeneous observation horizons. This observation is consistent with those obtained by Levy and Levy (1996) and Levy, Levy, and Solomon (2000) in models which have a similar specification for the forecasting rule of the trend followers.

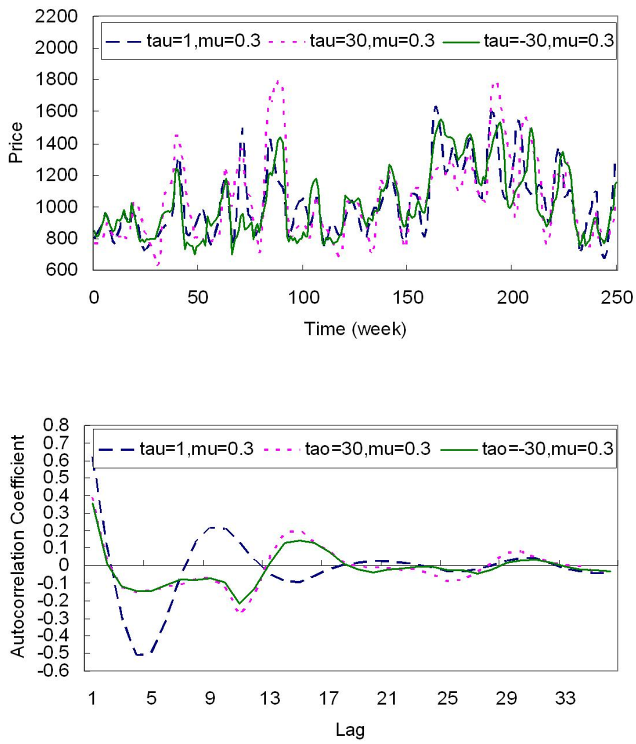


Figure 5.12: Comparison of price dynamics (top) and autocorrelation structures (bottom) generated under trend followers with recency bias and different sentiments. Three types of agents, the fundamentalist, trend follower and noise trader (q^N=0.10) are present in the market. $L = 10$.

After adding sentiment into the trend followers' forecasting, our results showed that sentiment in terms of optimism or pessimism is able to affect the size of price bubbles. Moreover, sentiment plays a main role in producing large volatility in the price and return dynamics and the excess kurtosis in return distributions. However, sentiment has little impact on the duration of the price bubbles. In contrast with the effect of sentiment, recency bias mainly affects the duration of the price bubbles. The presence of recency bias in forecasting reinforces the trend following behaviour, which increases the frequency of the price bubbles.

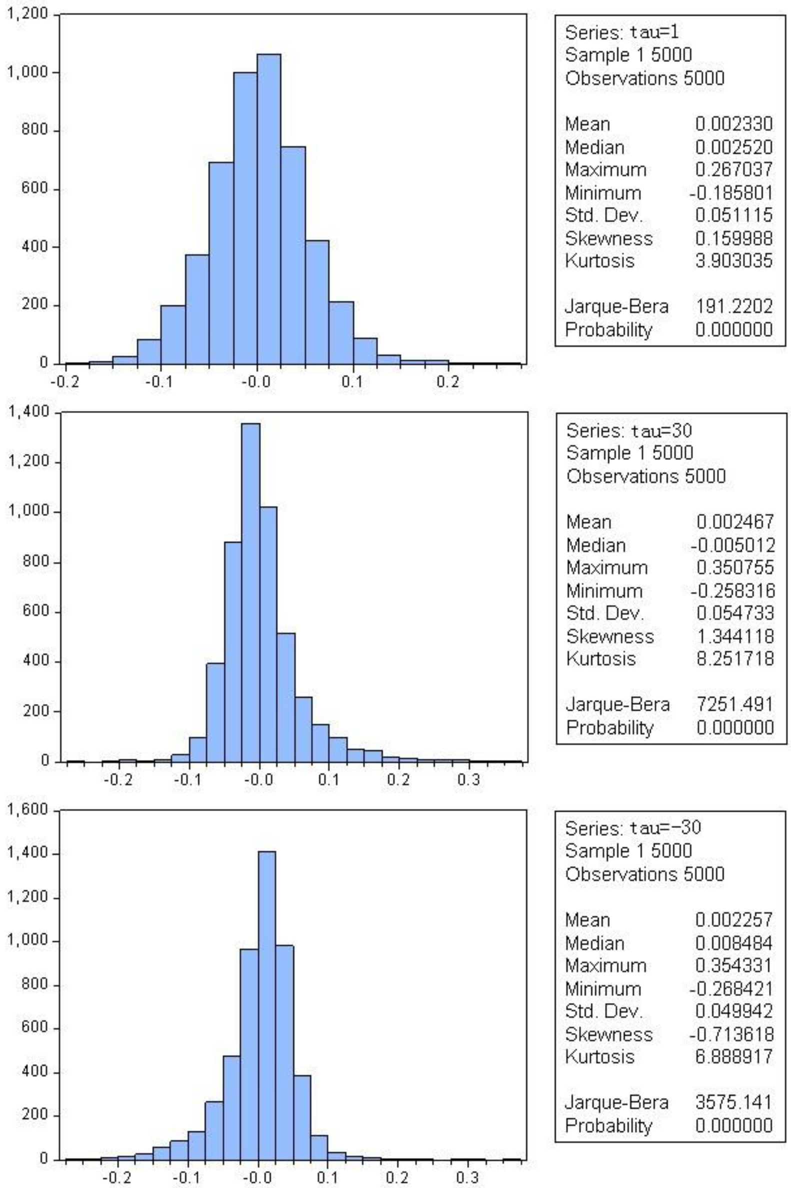


Figure 5.13: Histogram and statistics of asset returns when $\mu = 0.3$ and different values of the parameter τ: $\tau = 1$ (top), $\tau = 30$ (middle), $\tau = -30$ (bottom). $L = 10$.

When sentiment is coupled with recency bias, both the size and duration of the price bubbles are strongly affected resulting less predictable price and return dynamics than in the case without sentiment and recency bias. Statistical features of the return distribution in the case with sentiment and recency bias are more closer to those observed from real markets. These results indicate that behavioural biases such as sentiment and recency bias in forecasting help to make the price and return dynamics become more realistic.

However, our model cannot reproduce the stylised fact of absence of autocorrelations in returns. The reason is that the homogeneity with respect to the observation horizon of the trend followers imposed too much regularities in the price and return dynamics. We showed that these strong regularities cannot be destroyed by the fundamentalists with different degrees of reaction, a moderate proportion of the noise traders, or the trend followers' sentiment and recency bias in forecasting.

Levy, Levy, and Solomon (2000) pointed out that the assumption of homogeneous observation horizons of investors may share the same spirit as the representative agent approach in economic modelling. As mentioned by many previous contributions in the economics and finance literature, Levy, Levy, and Solomon (2000, p.167) state that: *"One justification for using a representative agent in economic modelling is that although investors are heterogeneous in reality, one can model their collective behaviour with one representative or "average" investor."*

Following this justification, the homogeneous observation horizons of investors can be regarded as the collective behaviour of investors who have heterogeneous observation horizons. Levy, Levy, and Solomon (2000) carried a simulation experiment to test the validity of the assumption of homogeneous observation horizons in representing the collective behaviour of heterogeneous observation horizons. In their experiment, the authors simulated two subpopulations of efficient market believers (EMBs, i.e. trend followers in our model) with different sizes of observation horizons $L = 5$ and $L = 15$. They showed that the resulting market dynamics is completely different to the dynamics generated under one "average" investor who has observation horizon $L = 10$. This phenomenon was explained by the authors as "the nonlinear interaction between the different subpopulation" and "it can be partly understood by looking at the wealth of each subpopulation". Such a finding made the authors question the validity of using representative agent in economic modelling.

Indeed, modelling the collective behaviour of the two subpopulation of EMBs one with $L = 5$ and the other one $L = 15$ as one representative investor with $L = 10$ is not appropriate, especially in models with features of nonlinear interaction between different investor types and mutual dependence between wealth and price. The nonlinearity which is originated from investors' strategies and wealth dynamics makes the modelling of the collective behaviour of heterogeneous investors become difficult. The assumption of homogeneous observation horizons can help to simplify the model, thereby an initial and basic understanding of the model dynamics can be obtained. However, such an assumption is not appropriate for modelling a representative agent or the collective behaviour of agents who have different observation horizons.

In order to tackle this issue, the following section will address the heterogeneity with respect to the trend followers' observation horizons. Our approach to heterogeneous observation horizons provides an example on how the aggregation of heterogeneous behaviour or a representative agent can be achieved in an agent-based model.

5.3 Heterogeneous Observation Horizons

This section extends the model and analysis to study the case where the trend followers have heterogeneous observation horizons. Before we introduce our model extension regarding the heterogeneous observation horizons, we first briefly review an existing approach by Levy, Levy, and Solomon (2000) to the heterogeneous observation horizons of investors. The purpose is to illustrate some important issues on modelling heterogeneous observation horizons.

To relax the assumption of homogeneous observation horizons, Levy, Levy, and Solomon (2000) assume that each investor j in the EMBs type has a different observation horizon $L_j > 0$ and L_j is distributed in the population according to a truncated normal distribution with average $\bar{L}$ and standard deviation σ_L (since $L_j \leq 0$ is meaningless). The population of the EMBs type is defined in terms of the number of investors. Their model does not allow investors to change their observation horizons or switch between different observation horizons.

An advantage of this approach is that it allows a full spectrum of EMBs with different observation horizons. The price and return dynamics generated become more realistic than in the homogeneous observation horizons case. However, a side effect is that, as noted by Levy, Levy, and Solomon (2000), the price and return dynamics in this heterogeneous observation horizons case also become difficult to be understood. In order to understand the price dynamics, because of the mutual dependence between wealth and price in their model, one has to analyse the wealth dynamics of each EMB investor who has a different observation horizon L_j together with his or her demand of the asset. A large number of EMB investors with different observation horizons makes such analysis very difficult. Here, the distribution of L_j does not help to understand the price dynamics. This is because that the distribution of L_j does not necessarily match the distribution of wealth fractions of each different L_j.

Moreover, to understand the long-term price behaviour, one has to take into consideration the evolution of wealth fraction of each EMB investor who has a different observation horizon. It is important to examine whether some EMB investors will be driven out of the market in the long run. If only one EMB investor who has a particular length of observation horizon survives, the market impact of heterogeneous observation horizons will disappear in the long run. However, the long-term wealth dynamics of each EMB investors was not reported by Levy, Levy, and Solomon (2000). Therefore, whether their findings in the heterogeneous observation horizons case are only temporary phenomena or persistent phenomena is an open question.

Different to the approach proposed by Levy, Levy, and Solomon (2000), our approach focuses on measuring and monitoring the wealth dynamics of each subgroup of the trend followers who use a different observation horizon. The population size of each subgroup of the trend followers is measured in terms of wealth proportions rather than the number of investors. This is because that it is not the number of investors but investors' wealth which affects the price. Switching among different observation horizons is allowed. In addition, the collective behaviour of the trend followers who have different observation horizons is properly modelled. This helps to understand the aggregate market dynamics.

5.3.1 Modelling Heterogeneous Observation Horizons

Our model extension regarding heterogeneous observation horizons of the trend followers is based on the following three main assumptions:

Assumption 1: Heterogenous Observation Horizons.
The trend followers are divided into a finite number $J > 1$ of subgroups, each subgroup $j = 1, ..., J$ uses a different size of rolling window $L_j > 0$ to forecast future returns.

Assumption 2: Inner-Switching of the Trend Followers.
Individual investors firstly choose their investment styles between the two general styles: fundamentalist and trend follower. Then the individual investors who have chosen the trend follower style continue to choose their forecasting rules which are characterised by L_j. In another words, the strategy-switching firstly happens between the two general styles of fundamentalist and trend follower. After that, the strategy-switching happens between the subgroups of trend followers.

This assumption is similar to the one used by Lux and Marchesi (1999,2000). The authors assume that strategy-switching happens not only between fundamentalist and chartist, but also between two subgroups of chartist namely bullish investors and bearish investors.

Assumption 3: A Performance-Driven Mechanism for Inner-Switching. The same as the mechanism for the switching between fundamentalist and trend follower, the group of trend followers switch among subgroups with different observation horizons according to the past performance of each subgroup. At time t, the population $n_t^j \geq 0$ of each subgroup $j = 1, ..., J$ of the trend followers is measured by the fraction of the total wealth of the trend followers, satisfying $\sum_{j=1}^J n_t^j = 1$. The population n_t^j from time t to $t+1$ is updated according to the past performance of each subgroup:

$$n_t^j = \frac{\exp(\gamma^T f_{t-1}^{T_j})}{\sum_{j=1}^J \exp(\gamma^T f_{t-1}^{T_j})}, \tag{5.3.1}$$

where γ^T is the intensity of choice of the trend followers; $f_{t-1}^{T_j}$ is the performance measure of each subgroup j of the trend followers at time $t-1$. The performance measure is given by the discounted realised returns:

$$f_t^{T_j} = r + \lambda_{t-1}^{T_j}(r_{t,1}^s - r) + \rho^T f_{t-1}^{T_j}, \tag{5.3.2}$$

where $\lambda_{t-1}^{T_j}$ is the investment proportion for the risky asset computed by the trend follower subgroup j who use observation horizon L_j; ρ^T is a discounting parameter which measures how fast the past performance is discounted. When $\rho^T = 0$, the performance measure $f_t^{T_j}$ is the portfolio return of each subgroup j at time t.

Consensus Investment Proportions

Based on the above three assumptions, it is possible to define the *consensus investment proportions* for the entire population of trend followers. At each point in time t, the amount of money invested into the risky asset by the entire population of trend followers is:

$$\sum_{j=1}^{J}(1-c)\hat{W}_t^T n_t^j \lambda_{t,1}^{T_j} = (1-c)\hat{W}_t^T \sum_{j=1}^{J} n_t^j \lambda_{t,1}^{T_j}, \tag{5.3.3}$$

where $\hat{W}_t^T = q_t^T \bar{W}_t$ is total wealth managed by the entire population of trend followers after the strategy-switching between fundamentalist and trend follower. The term $(1-c)\hat{W}_t^T n_t^j$ describes the available budget of each subgroup of the trend followers at time t after the inner-switching.

Following equation (5.3.3), one can define the consensus investment proportions across all the subgroups of trend followers: $\bar{\lambda}^T_{t,1} = \sum_{j=1}^{J} n^j_t \lambda^{T_j}_{t,1}$ and $\bar{\lambda}^T_{t,0} = 1 - \bar{\lambda}^T_{t,1}$, so that

$$(1-c)\hat{W}^T_t \sum_{j=1}^{J} n^j_t \lambda^{T_j}_{t,1} = (1-c)\hat{W}^T_t \bar{\lambda}^T_{t,1}. \tag{5.3.4}$$

Equation (5.3.4) implies that these subgroups of trend followers can be treated as one representative agent with investment proportions $\bar{\lambda}^T_{t,0}$ and $\bar{\lambda}^T_{t,1}$. Individual investors can thereby update their choices between the two general investment styles, fundamentalist and trend follower, according the past performance of the strategies of the fundamentalists and the "representative" trend follower, as defined before by equations (3.2.2) to (3.2.4) in Chapter 3.

Introducing the consensus investment proportions for the entire population of the trend followers helps us to understand the resulting dynamics through monitoring the aggregate behaviour of the representative trend follower. The aggregate behaviour is able to correctly and accurately reflect the result of the interaction of the heterogeneous trend followers. Our model provides an example on how the aggregation and a "representative agent" can be implemented in an agent-based model, through which our model contributes to the literature on enhancing the explanatory power of the agent-based modelling approach.

In the remainder of this section, we use numerical simulations to explore the impact of trend followers' heterogeneous observation horizons and the inner-switching on the aggregate market dynamics. In Section 5.3.2, we start with the case where the trend followers with heterogeneous observation horizons use the basic forecasting rule without sentiment and recency bias. We then study, in Section 5.3.3 and 5.3.4, the effects of sentiment and recency bias in forecasting.

5.3.2 Numerical Results under the Basic Forecasting Rule

This section studies the effect of heterogeneous observation horizons of the trend followers in the case where the trend followers use the basic forecasting rule without sentiment and recency bias. we first examine the role of the market selection in the inner-switching of the trend followers. We investigate whether the process of market selection reduces or even destroy the effect of the heterogeneity with respect to the observation horizons of the trend followers.

Our analyses presented in Chapter 4 have shown that, when strategy-switching is allowed, the market selection hypothesis holds in our model if investors had infinite memory and they did not exhibit recency bias in performance evaluation ($\rho = 1$). In the case with inner-switching, it is equivalent to set the value of the parameter ρ^T to 1. Figure 5.14 illustrates the long-term dynamics when the fundamentalists, three subgroups of trend followers with $L = 5$, $L = 25$ and $L = 45$ and the noise traders are present in the market. The population dynamics (wealth fractions) of the three subgroups with $\rho^T = 1$ (Figure 5.14, top left) reveals that those subgroups with longer observation horizons ($L = 25$ and $L = 45$) tend to be driven out of the market. Only the subgroup $L = 5$ survives in the long run. The autocorrelation coefficients of returns of the risky asset (Figure 5.14, bottom left) show that the return dynamics is characterised by the shortest observation horizon $L = 5$. The longer observation horizons, $L = 25$ and $L = 45$, have little impact on the return dynamics.

In contrast, when trend followers exhibit weak recency bias ($\rho^T = 0.99$) in performance evaluation, the population dynamics (Figure 5.14, top right) show that all the three subgroups survive. This result indicates that the inner-switching in conjunction with the recency bias in performance evaluation is a key behavioural element which helps to maintain the heterogeneity in the trend followers' observation horizons. The autocorrelation coefficients of returns (Figure 5.14, bottom right) reveal that the return dynamics is characterised by the lengthes of all the three different observation horizons of the trend followers. Large negative autocorrelation coefficients appear at 6th, 26th and 46th lag. However, these

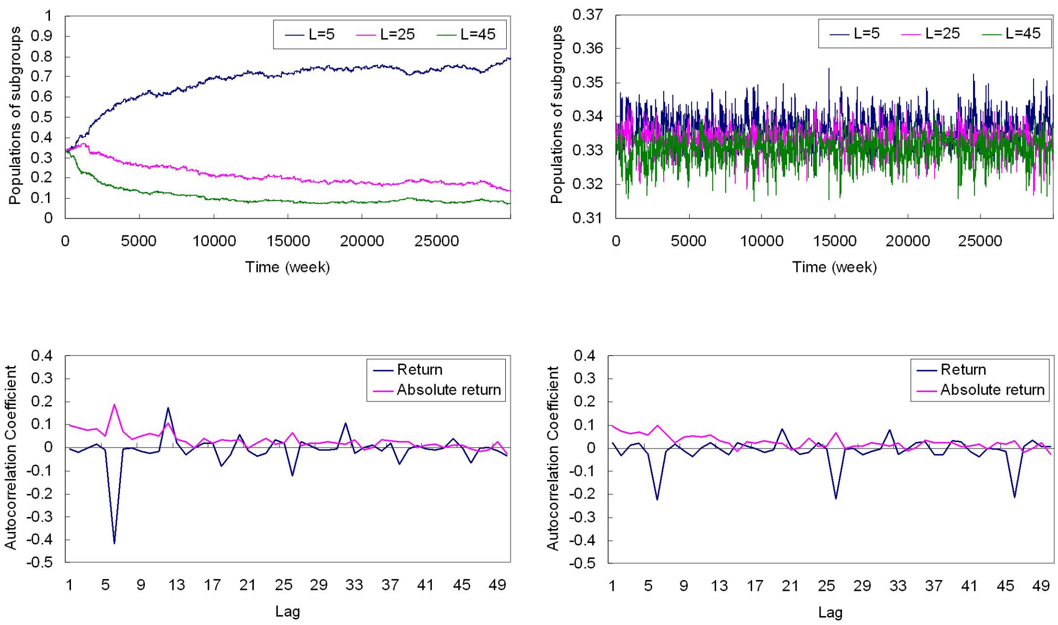


Figure 5.14: Long-term dynamics when the fundamentalists, three subgroups of trend followers with $L = 5$, $L = 25$ and $L = 45$ and the noise traders are present in the market. Population (wealth fraction) of each subgroup of the trend followers: $\rho^T = 1$ (top left), $\rho^T = 0.99$ (top right). Autocorrelation coefficients of returns and absolute return: $\rho^T = 1$ (bottom left), $\rho^T = 0.99$ (bottom right). Other parameter: $\gamma^T = 5$, $\alpha^F = 0.90$, $q^N = 0.1$.

autocorrelation coefficients are too large to be realistic.

Figure 5.15 depicts the short-term dynamics corresponding to the scenario with $\rho^T = 0.99$ in Figure 5.14. The price dynamics (Figure 5.15, top left) can be understood by looking at the consensus investment proportion of the trend followers together with investment proportions of other agent types (Figure 5.15, top right). The consensus investment proportion reflects the aggregate behaviour of the three subgroups of trend followers, which can be further analysed by looking at the population fraction (Figure 5.15, bottom left) and the investment proportion (Figure 5.15, bottom right) of each subgroup of trend followers.

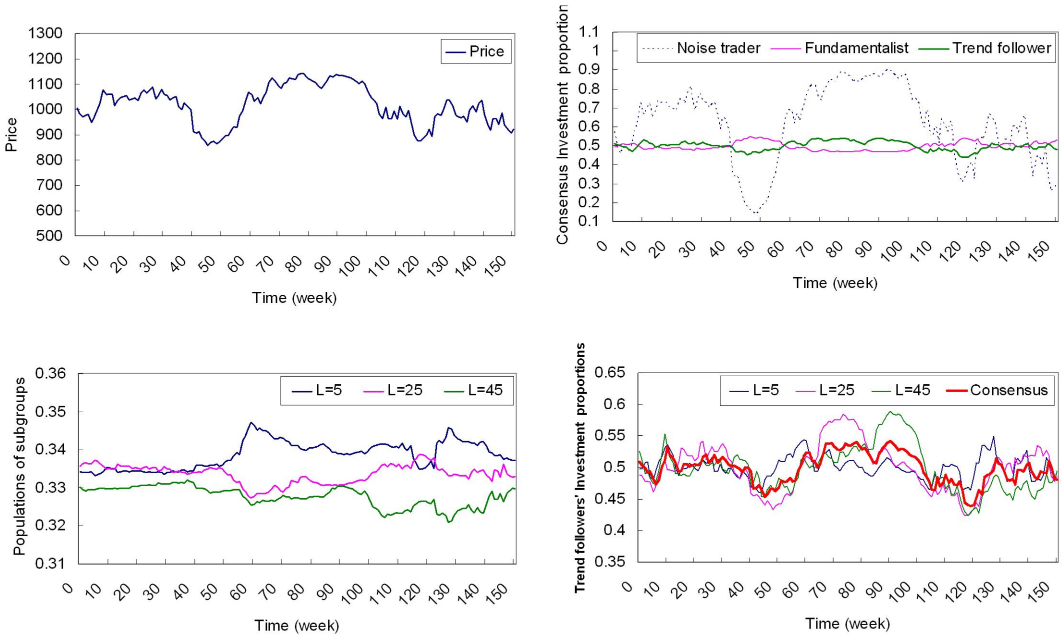


Figure 5.15: Short-term dynamics when the fundamentalists, three subgroups of trend followers with $L = 5$, $L = 25$ and $L = 45$ and the noise traders are present in the market, $\gamma^T = 5$ and $\rho^T = 0.99$: price dynamics (top left); agents' investment proportions for the risky asset (top right); population (wealth fraction) of each subgroup of the trend followers (bottom left); each subgroup's investment proportion for the risky asset (bottom right).

The bottom right panel of Figure 5.15 reveals that the trend followers with different observation horizons may hold different views of the short-term price trend. For example, between periods 50 and 60, the subgroup of trend followers with $L = 45$ believe that the price is decreasing and they therefore decrease their ivestment proportions for the risky asset, while the other two subgroups believe that the price is increasing and they increase their investment proportions for the risky asset. When the behaviour of these subgroups of trend followers is aggregated, those subgroups with larger wealth fractions have stronger impact on the consensus investment proportion, and there exist a "cancellation" effect among the different behaviour of these subgroups of trend followers. Due to this cancellation effect, the consensus investment proportion, on average, appears less volatile than those of each subgroup. An obvious market impact of this cancellation effect is that it helps to reduce the volatility of the price of the risky asset.

To examine how the number of different subgroups of trend followers can affect the market dynamics, we conduct an experiment with 50 subgroups of trend followers whose observation horizons start from $L_1 = 1$, $L_2 = 2$ to $L_{50} = 50$. The parameter ρ^T is set to 0.99. Figure 5.16 depicts the dynamics of the 50 subgroups case. In this case, the large number of subgroups of trend followers would make the model become complicated in understanding the aggregate dynamics such as the price dynamics. In order to illustrate the long-run market behaviour, results illustrated in Figure 5.16 are sampled after 25,000 periods.

As shown in the top right panel of Figure 5.16, investment proportions of each subgroup of trend followers may differ from one another significantly. It is difficult to analyse the price dynamics by looking at the investment proportions and wealth fractions of all the 50 subgroups of trend followers at one time. In this case, our model starts to exhibit its advantage on allowing the collective behaviour (consensus investment proportions) of the trend followers to be observed. The price dynamics can be understood by looking at the consensus investment proportions of each agent type (Figure 5.16, middle left).

Our results show that the return dynamics (Figure 5.16, middle right) exhibits volatility clustering which is a well-known stylised fact of real-world financial time series. A quantitative manifestation of this fact is that, as documented by Cont (2001), while returns themselves are uncorrelated, absolute returns display a positive, significant and slowly decaying autocorrelation coefficient function ranging from a few minutes to a several weeks. This property is illustrated in the bottom row of Figure 5.16. The bottom row compares the autocorrelation structures of returns from our simulation and those from the S&P 500 index (in the period of 1950 to 2010 at weekly frequency). Compared with the three subgroups case, the result of the 50 subgroups case turns out that *"more is different"*. When there

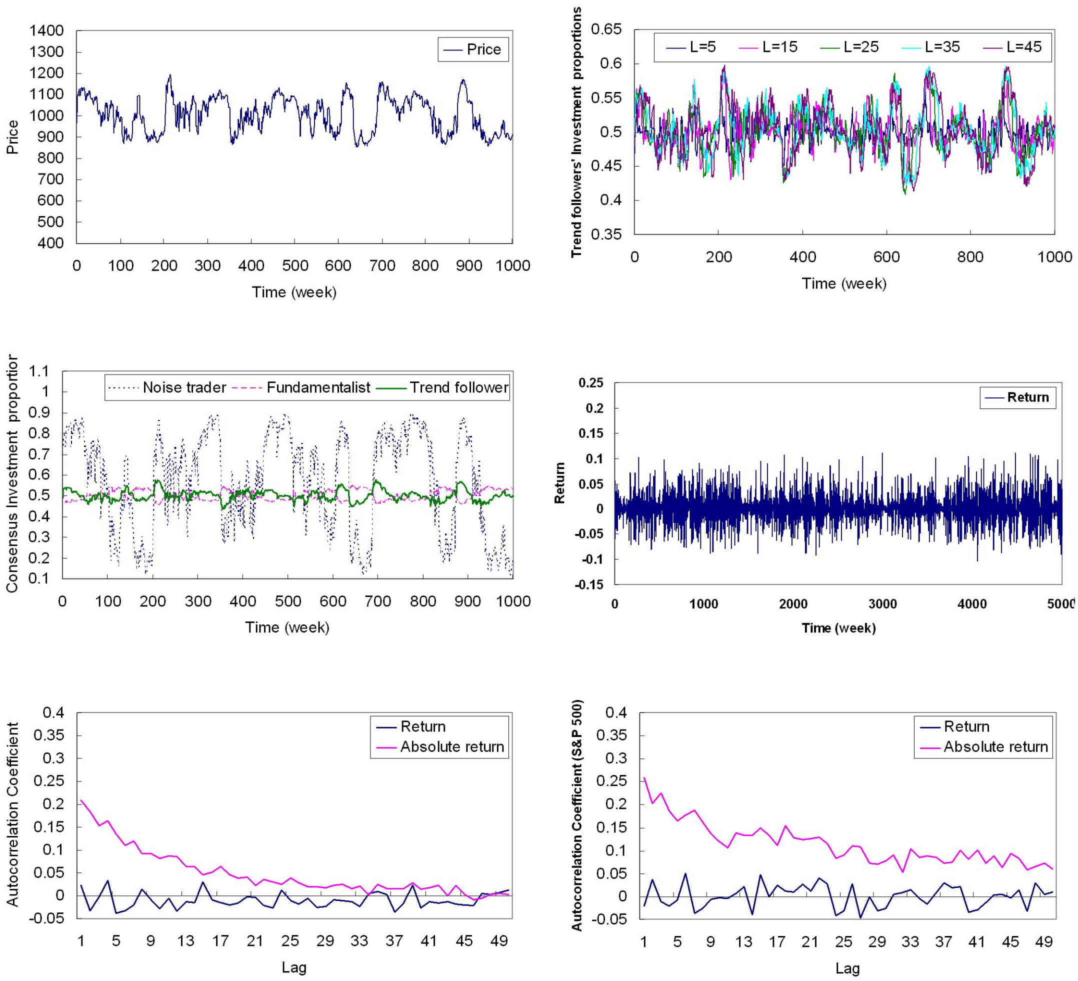


Figure 5.16: The fundamentalists, 50 subgroups of trend followers with $L_1 = 1$ to $L_{50} = 50$ and the noise trader are present in the market: price dynamics (top left); investment proportions of selected subgroups of trend followers (top right); consensus investment proportions for the risky asset (middle left); long-term dynamics of the returns of the risky asset (middle right); autocorrelation coefficients of returns and absolute returns (bottom left); autocorrelation coefficients of S&P 500 index returns and absolute returns (bottom right). Parameter: $\gamma^T = 5$, $\rho^T = 0.99$, $\alpha^F = 0.90$, $q^N = 0.1$.

are 50 subgroups of trend followers, the return dynamics become more realistic. The autocorrelation coefficients of returns are insignificant at 5% level for almost all of the lags, which is close to those from the S&P 500 index.

In order to explore the effect the parameters γ^T (intensity of choice) and ρ^T (discounting parameter in performance measure) in the process of inner-switching, we conduct experiments with different values of these two parameters. Three different scenarios denoted by S1, S2 and S3 are considered. In scenario S1, the values of the parameters γ^T and ρ^T are set to 5 and 0.99 respectively. This scenario describes the phenomenon that the trend followers exhibit a lower intensity of choice and weak recency bias in the inner-switching. Scenario S2 corresponds to the case where parameter values are: $\gamma^T = 50$ and $\rho^T = 0.99$. It represents the phenomenon that the trend followers exhibit weak recency bias but a high intensity of choice in the inner-switching. Scenario S3 describes the phenomenon that the trend followers exhibit a high intensity of choice $\gamma^T = 50$ and a high degree of recency bias $\rho^T = 0$ in the inner-switching.

Figure 5.17 illustrates the population dynamics of some selected subgroups (out of the total 50 subgroups) and the trend followers' consensus investment proportions in three different scenarios. The results show that, the population dynamics of the selected subgroups in the three different scenarios exhibit significant differences. However, the impact on the trend followers' consensus investment proportion is minor. The trend followers' consensus investment proportions in the three different scenarios do not exhibit significant differences. This observation indicates that, the aggregate dynamics such as the consensus investment proportions and the price dynamics is not sensitive to the trend followers' intensity of choice(γ^T) and different degrees of recency bias ($\rho^T \in [0, 1)$) in the inner-switching.

The most significant impact of the inner-switching of the trend followers is that, as illustrated in Figure 5.14, it leads to the survival of subgroups with different observation horizons. Without inner-switching or without recency bias in performance evaluation, the market impact of the heterogeneous observation

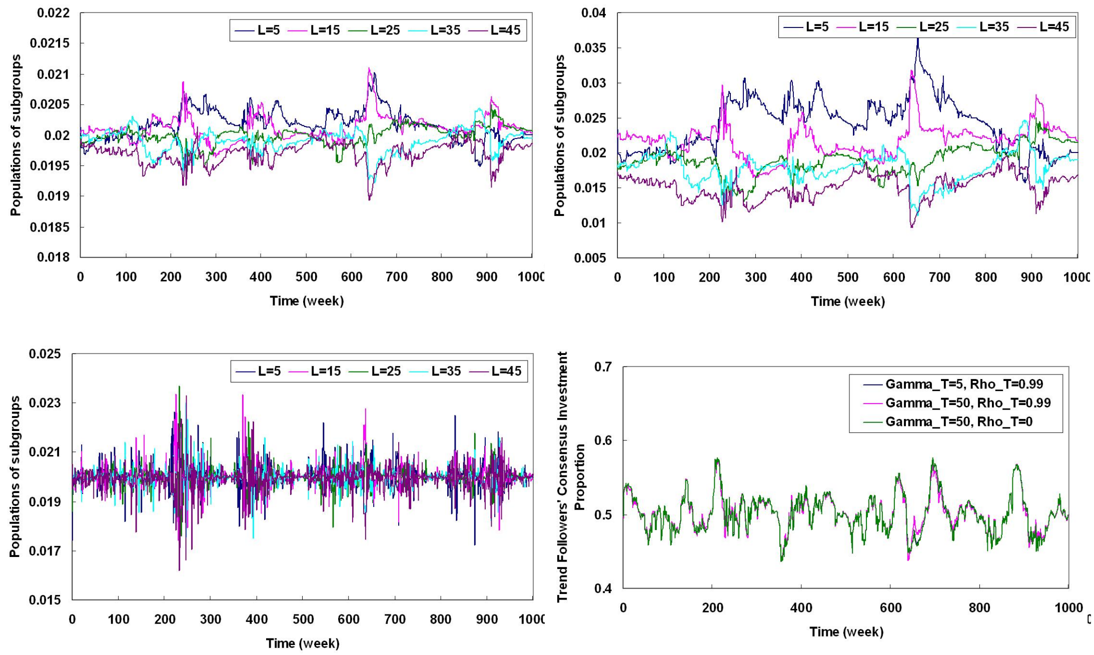


Figure 5.17: Population (wealth fractions) dynamics of selected subgroups of trend followers in three different scenarios. S1: $\gamma^T = 5$ and $\rho^T = 0.99$ (top left). S2: $\gamma^T = 50$ and $\rho^T = 0.99$ (top right). S3: $\gamma^T = 50$ and $\rho^T = 0$ (bottom left). Comparison of trend followers' consensus investment proportions (bottom right).

horizons of the trend followers may be eliminated by the process of market selection. The inner-switching with recency bias helps to maintain the heterogeneity of the trend followers with respect to the observation horizon. The observed statistical properties of return series such as the absence of autocorrelation in returns and volatility clustering are caused by the heterogeneity which stems from a large number of subgroups of trend followers with different observation horizons.

5.3.3 Numerical Results under the Forecasting Rule with Sentiment

Our previous analysis of the trend followers' sentiments (optimism and pessimism) focused only on the case in which the observation horizons of the trend followers are homogeneous (Section 5.2.2). When the trend followers have heterogenous observation horizons, sentiment may have different impact on these heterogenous trend followers. One may expect that sentiment may have stronger impact on those trend followers with longer observation horizons because they use larger sample sizes in forecasting. This subsection addresses the effect of sentiment when the trend followers have heterogeneous observation horizons.

The forecasting rule of each subgroup of trend followers is defined exactly as in equation (5.2.1), but the homogeneous observation horizon L in (5.2.1) is replaced by L^j (the observation horizon of subgroup j). The values of the optimism-pessimism index τ^j for each subgroup j of trend followers is identically set, i.e. $\tau^j = \tau$ for $j = 1, ..., J$. This value of τ represents the general sentiment of the whole population of trend followers.

Figure 5.18 illustrates the case where the fundamentalists, the noise traders and 50 subgroups of optimistic ($\tau = 25$) trend followers are present in the market. Trend followers' observation horizons are uniformly distributed in $[1, 50]$ (e.g. $L_1 = 1$, $L_2 = 2$,..., $L_{50} = 50$). The price dynamics (Figure 5.18, top left) shows that optimism is able to affect the size of the price bubbles, but it has a minor impact on the duration of the price bubbles. The trend followers' investment proportions (Figure 5.18, top right) indicate that the sentiment has stronger impact on those trend followers with longer observation horizons. The trend followers with longer observation horizons are more optimistic than those with shorter observation horizons.

As shown in Figure 5.18, when a price bubble starts to emerge, the trend followers' optimism implies sharp increases in the price. However, when a price bubble starts to burst, the downswing in the price is not as sharp as the upswing

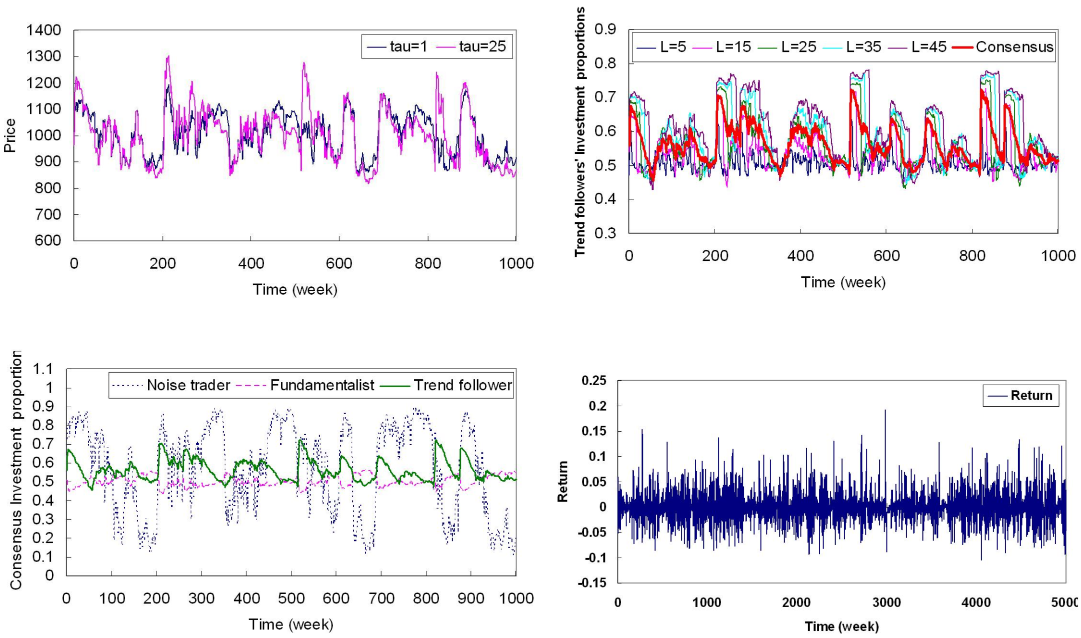


Figure 5.18: The case where the fundamentalists, 50 subgroups of optimistic ($\tau = 25$) trend followers with $L_1 = 1$ to $L_{50} = 50$ and the noise trader are present in the market: price dynamics (top left); investment proportions of selected subgroups of trend followers (top right); consensus investment proportions for the risky asset (bottom left); long-term returns of the risky asset (bottom right). Parameters: $\gamma^T = 5$, $\rho^T = 0.99$, $\alpha^F = 0.90$, $q^N = 0.1$.

during the phase of the growth of the price bubble. The consensus investment proportions of the trend followers reveal that, during the phase of the burst of price bubbles, there exists a significant cancellation between the investment proportions of each subgroup which leads to slower and smaller successive decreases in the price. This phenomenon can also be observed from the long-term asset returns (Figure 5.18, bottom right). The magnitudes of positive returns are on average larger than the magnitudes of negative returns. In addition, large positive returns happened more frequently than large negative returns.

This observation is in contrast to the those in the homogeneous observation horizons case where optimism of the trend followers also causes large negative returns frequently, usually a few periods after large positive returns. This property can also be observed from the investment proportions of each selected subgroup, where a large boom is usually followed by a large crash after a "cooling off" phase. The length of the cooling off phase for each subgroup of trend followers is characterised by the length of their observation horizon. Because these subgroups of trend followers have different lengths of cooling off phase, when their investment proportions are aggregated, the cancellation effect among investment proportions of these subgroups causes that the decrease part in the consensus investment proportion is in general not as sharp as the increase part. This property leads to asymmetric volatility in the price.

Figure 5.19 depicts the case with 50 subgroups of pessimistic ($\tau = -25$) trend followers. Similar to the optimistic case, the trend followers with longer observation horizons are more pessimistic than those with shorter observation horizons. We find that the trend followers' pessimism is able to cause sharp downswings of the price. In contrast with the optimistic case, the cancellation effect in the investment proportions of the trend followers exhibits in the upswing parts of the price. The cancellation effect causes that the upswings in the price are in general less steep than the downswings in the price. The return dynamics in Figure 5.19 reveals that the magnitudes of large negative returns are larger than those of large positive returns. Moreover, large negative returns happen more frequently than large positive returns.

Figure 5.20 shows the autocorrelation coefficients of returns and absolute returns for three different cases: $\tau = 1$ (without sentiment), $\tau = 25$ (optimism) and $\tau = -25$ (pessimism). We find that the trend followers' sentiment, in general, does not affect the statistical property of absence of autocorrelation in returns. Compared with the $\tau = 1$ case, optimism or pessimism causes only a slightly higher autocorrelation coefficient for the first lag of the returns. The remainder lags are insignificant at 5% level. Absolute returns in both the optimism and pessimism cases exhibit a positive, significant and slowly decaying autocorrelation

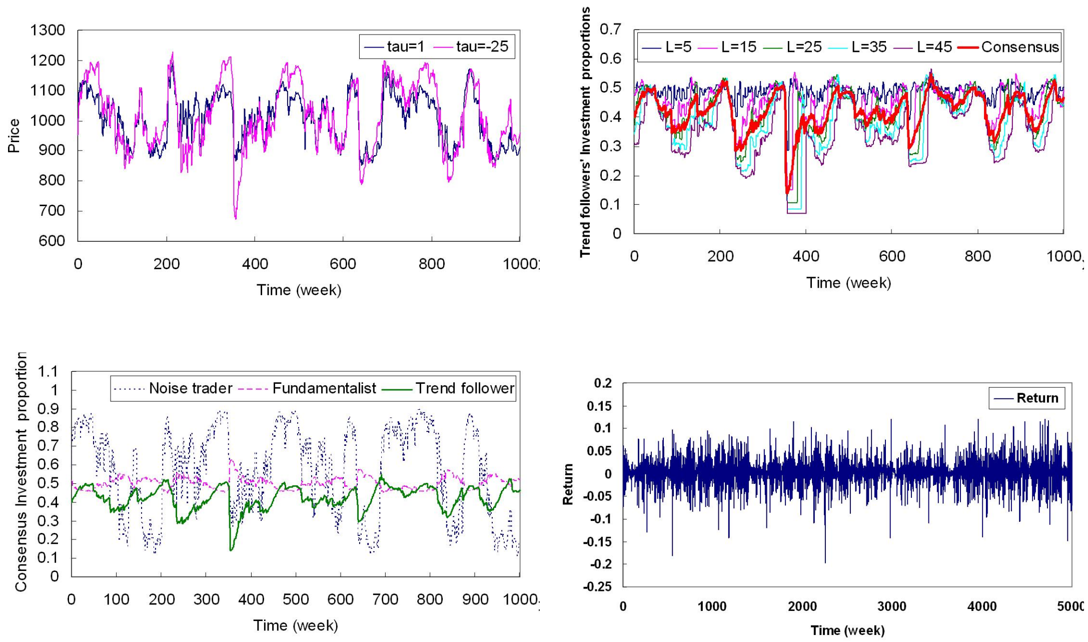


Figure 5.19: The case where the fundamentalists, 50 subgroups of pessimistic trend followers ($\tau = -25$) with $L_1 = 1$ to $L_{50} = 50$ and the noise trader are present in the market: price dynamics (top left); investment proportions of selected subgroups of trend followers (top right); consensus investment proportions for the risky asset (bottom left); long-term returns of the risky asset (bottom right). Parameters: $\gamma^T = 5$, $\rho^T = 0.99$, $\alpha^F = 0.90$, $q^N = 0.1$.

coefficient function which is similar to the one in the $\tau = 1$ case. This result indicates that the trend followers' sentiment does not affect the property of volatility clustering in return series.

To examine the impact of the trend followers' sentiments on the distribution of the aggregate market returns, Table 5.1 compares the summary statistics of weekly returns generated in three different cases ($\tau = 1$, $\tau = 25$ and $\tau = -25$) with those of the S&P 500 index at weekly frequency. The generated returns correspond to those illustrated in Figures 5.16, 5.18 and 5.19, which are sampled from the periods 25,001−30,000 in each independent run. The S&P 500 index

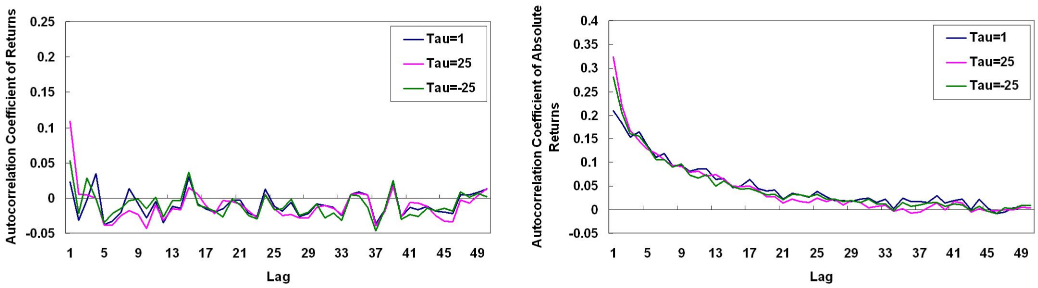


Figure 5.20: Autocorrelation coefficients of returns (left) and absolute returns (right) for the cases with $\tau = 1$ (unbiased), $\tau = 25$ (optimism) and $\tau = -25$ (pessimism); the fundamentalists, 50 subgroups of pessimistic trend followers ($\tau = -25$) with $L_1 = 1$ to $L_{50} = 50$ and the noise trader are present in the market. Parameters: $\gamma^T = 5$, $\rho^T = 0.99$, $\alpha^F = 0.90$, $q^N = 0.1$.

data are sampled from the period of 1950 to 2010 at weekly frequency.

It is well known that the aggregate stock market returns are not normally distributed at relatively high frequencies (less than one month). The finance literature (e.g. Fama, 1965; French et al., 1987; Cont, 2001; Hong and Stein, 2003) has documented that the distribution of aggregate stock returns exhibits excess kurtosis, negative skewness or a closely related property asymmetric volatility, as illustrated by the S&P 500 index data in Table 5.1.

Table 5.1: Summary Statistics of Returns - The Effect of Sentiment

Case	Mean	Maximum	Minimum	Std	Skewness	Kurtosis
Unbiased: $\tau = 1$	0.1302%	11.0732%	-10.0727%	1.9589%	0.1319	3.8616
Optimism: $\tau = 25$	0.1487%	19.2997%	-10.6868%	2.2856%	0.3565	6.1442
Pessimism: $\tau = -25$	0.1417%	11.0479%	-19.6947%	2.2352%	-0.2907	6.2576
S&P 500 (1950-2010)	0.1547%	14.1161%	-18.1955%	2.0689%	-0.3537	8.2353

We find that the trend followers' sentiment is able to cause excess kurtosis in return distribution. As shown in Table 5.1, the kurtosis value 3.86 in the case without sentiment is close to the one of the normal distribution, while the kurtosis values in the optimism and pessimism cases are closer to the one obtained from S&P 500 index data. Moreover, in the pessimism case, the return distribution exhibits negative skewness which is consistent with the stylised fact. The negative skewness of the generated returns stems from the trend followers' pessimism in conjunction with the cancellation effect which was found in the investment proportions of the trend followers.

The intuition is as follows. Pessimism in forecasting implies that the trend followers tend to decrease sharply their investment proportions for the risky asset when large negative returns are observed. Such behaviour causes that the asset price decreases significantly. When the price starts to increase, the consensus investment proportions of the trend followers reveals that there exists a cancellation between the investment proportions of each subgroup of trend followers due to they have different lengths of cooling off phase. This cancellation effect causes that the magnitudes of the increases of the price are in general smaller than the magnitudes of the decreases of the price. Pessimism in conjunction with the cancellation effect may serve as a new idea or mechanism which explains the asymmetric volatility.

5.3.4 Numerical Results under the Forecasting Rule with Recency Bias and Sentiment

In this subsection, we analyse the effect of recency bias in the trend followers' forecasting. The forecasting rule of each subgroup j of the trend followers is defined by equation 5.2.5 with the homogeneous L replaced by L^j. We assume that the value of parameter μ^j (which measures the degree of recency bias) for each subgroup j are the same, i.e. $\mu^j = \mu$ for $j = 1, ..., J$.

Figure 5.21 illustrates the market dynamics when the trend followers exhibit recency bias in forecasting. The market participants are the fundamentalists, 50 subgroups of the trend followers whose observation horizons start from $L_1 = 1$ to $L_{50} = 50$, and the noise traders. The top left panel of Figure 5.21 compares the price dynamics in the case with $\mu = 0.02$ and those in the case with unbiased trend followers ($\mu = 0$). The result indicates that the trend followers' recency bias is able to affect both the size and duration of the price bubbles. Compared with the unbiased case, the price in the case with $\mu = 0.02$ increases (or decreases) more sharply when the bubble grows (or bursts).

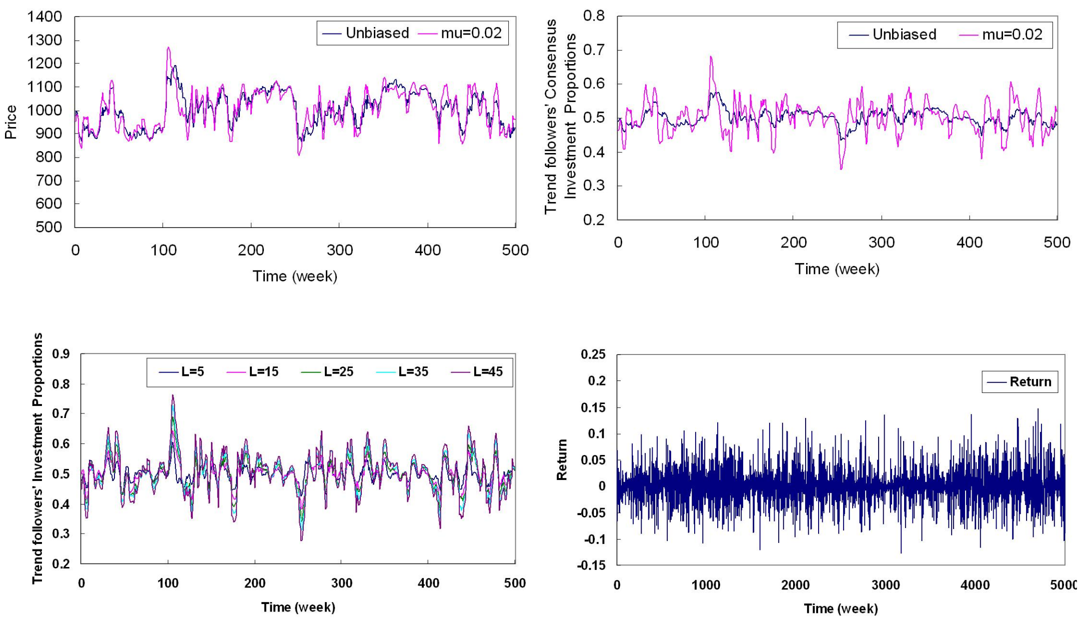


Figure 5.21: Market dynamics when the trend followers exhibit recency bias in forecasting. The fundamentalists, 50 subgroups of trend followers with $L_1 = 1$ to $L_{50} = 50$ and the noise trader are present in the market: comparison of the price dynamics (top left); comparison of the consensus investment proportions of the trend followers (top right); investment proportions of selected subgroups of trend followers (bottom left); long-term return dynamics (bottom right). Parameters: $\mu = 0.02$, $\gamma^T = 5$, $\rho^T = 0.99$, $\alpha^F = 0.90$, $q^N = 0.1$.

This phenomenon can be explained by analysing the trend followers' consensus investment proportions (Figure 5.21, top right). The trend followers' consensus investment proportions reveal that recency bias causes the trend followers become optimistic (or pessimistic) when the bubble grows (or bursts), which accelerates the increase (or decrease) in the price. The long-term return dynamics (Figure 5.21, bottom right) shows that the trend followers' recency bias is able to cause larger returns (the maximum return is 14.81% and the minimum return is −12.70%) than those from the unbiased case (illustrated in Figure 5.16, the maximum return is 11.07% and the minimum return is −10.072%).

Figure 5.22 depicts the market dynamics when the trend followers exhibit both recency bias and sentiment in forecasting. The left (right) column of Figure 5.22 shows the case where the trend followers exhibit recency bias and optimism (pessimism). We find that recency bias in conjunction with the optimism cause the price increases dramatically when the price bubble grows (Figure 5.22, top left). When the price bubble bursts, due to the cancellation effect in the investment proportions of each subgroup of the trend followers (Figure 5.22, middle left), the decreases in the price are smaller (in magnitudes) than those in the periods when the bubble grows.

Compared with the unbiased case, the price in the case with $\mu = 0.02$ and $\tau = 25$ exhibits larger crashes. These larger crashes are partly caused by recency bias rather than optimism itself. This property is also observable in the long-term return dynamics (Figure 5.22, bottom left). When recency bias is coupled to optimism, the trend followers are able to cause not only large positive returns but also large negative returns (the minimum return is −14.60%). This observation of large negative returns is in contrast with the one in the case where the trend followers exhibit only optimism. Without recency bias in forecasting, the optimistic trend followers cannot cause large negative returns (as shown in Table 5.1, the minimum return is −10.69% which is close to the one in the unbiased case).

When the trend followers exhibit recency bias and pessimism in forecasting, Figure 5.22 shows that the trend followers are able to cause dramatic decreases in the price when the price bubble bursts. When the price bubble grows, the increases in the price (in magnitudes) are smaller than those decreases in the

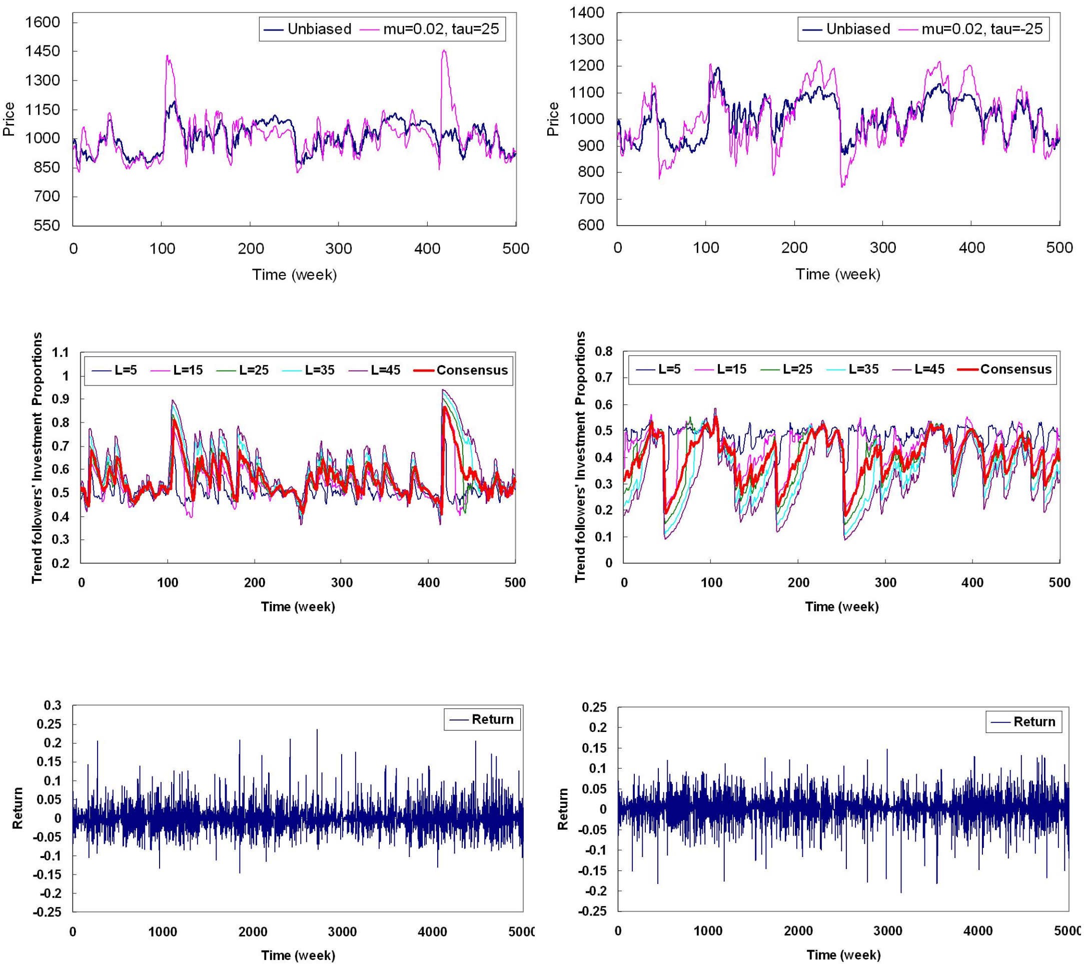


Figure 5.22: Market dynamics when the trend followers exhibit recency bias and different sentiments in forecasting. The fundamentalists, 50 subgroups trend followers with $L_1 = 1$ to $L_{50} = 50$ and the noise trader are present in the market. Optimistic trend follower case (left column); pessimistic trend follower case (right column). Comparison of the price dynamics (top row); investment proportions of selected subgroups of trend followers (middle row); long-term return dynamics (bottom row). Parameters: $\mu = 0.02$, $\gamma^T = 5$, $\rho^T = 0.99$, $\alpha^F = 0.90$, $q^N = 0.1$.

periods of the burst of the bubble. In addition, the recency bias helps to generate large positive returns which are missing from the case where the pessimistic trend followers do not exhibit recency bias in forecasting. These results indicate that recency bias in conjunction with sentiment causes the return dynamics become more realistic.

Figure 5.23 shows the autocorrelation coefficients of returns and absolute returns corresponding to those illustrated in Figure 5.21 and Figure 5.22. The autocorrelation coefficients of returns show that the recency bias causes returns in the first lag are slightly positive correlated, while the autocorrelation at the remainder lags are insignificant at 5% level. When recency bias is coupled to sentiment, the presence of sentiment does not impose extra autocorrelations in returns. The autocorrelation coefficient function of absolute returns indicate that recency bias and sentiment in forecasting do not affect the property of volatility clustering in return series.

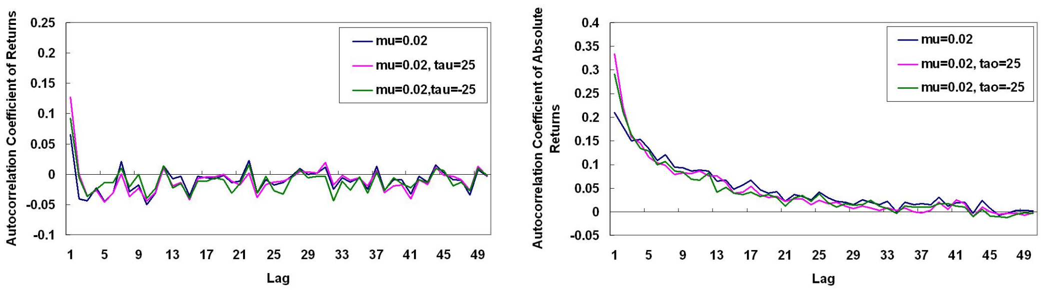


Figure 5.23: Autocorrelation coefficients of returns (left) and absolute returns (right) for the cases with $\mu = 0.02$ (recency bias only), $\mu = 0.02$ and $\tau = 25$ (recency bias with optimism) and $\mu = 0.02$ and $\tau = -25$ (recency bias with pessimism). Parameters: $\mu = 0.02$, $\gamma^T = 5$, $\rho^T = 0.99$, $\alpha^F = 0.90$, $q^N = 0.1$.

Table 5.2 compares the summary statistics of weekly returns generated in three different cases (recency bias only, recency bias with optimism and recency bias with pessimism) with those of the S&P 500 index at weekly frequency. The generated weekly returns correspond to those in Figure 5.21 and 5.22, which are sampled from periods 25,001−30,000.

Table 5.2: Summary Statistics of Returns - The Effect of Recency Bias with Sentiment

Case	Mean	Maximum	Minimum	Std	Skewness	Kurtosis
Recency Bias Only: $\mu = 0.02$, $\tau = 1$	0.1398%	14.8091%	-12.7021%	2.3054%	0.1581	4.2574
Recency Bias with Optimism: $\mu = 0.02$, $\tau = 25$	0.1531%	23.8011%	-14.6001%	2.5082%	0.5623	7.9593
Recency Bias with Pessimism: $\mu = 0.02$, $\tau = -25$	0.1528%	14.7202%	-20.4694%	2.4107%	-0.3852	8.0785
S&P 500 (1950-2010)	0.1547%	14.1161%	-18.1955%	2.0689%	-0.3537	8.2353

The results in Table 5.2 indicate that the trend followers' recency bias in forecasting is also able to cause excess kurtosis. However, the excess kurtosis caused by recency bias is too small compared with the one of S&P 500 index. When recency bias is coupled to sentiment, optimism implies high excess kurtosis and positive skewness, while the pessimism causes also high excess kurtosis but negative skewness. The results in the case of recency bias with pessimism are in general closer to the those from S&P 500 data. These results indicate that recency bias and pessimism in the trend followers' forecasting may be a potential candidate which is able to explain the stylised facts of negative skewness and excess kurtosis in return distributions.

5.4 Conclusion

The previous chapter has shown that, in our model, the presence of strategy-switching and recency bias in performance evaluation may lead to the survival of the trend followers' strategy. Based on this result, this chapter explored the market impact of the survival of the trend followers with focus on the interlinkage between the trend followers' forecasting and the aggregate market dynamics. The original model presented in Chapter 3 has been extended to study the roles of observation horizons, sentiment and recency bias in affecting the trend followers' forecasting. A series of numerical experiments has been conducted to quantify the market impact of these behavioural elements which are associated with the trend followers' forecasting.

Our analysis started from the case where the trend followers have homogeneous observation horizons. We showed that the homogeneity of the trend followers with respect to the observation horizon leads to unrealistic and predictable bubble dynamics in the price of the risky asset. In the case where the trend followers use basic forecasting rule without sentiment and recency bias, the size and duration of price bubbles as well as the autocorrelation structure of return series are characterised by the length of the homogeneous observation horizons of the trend followers.

In the homogeneous observation horizons case, we studied the market impact of the trend followers' sentiment and recency bias in forecasting. Different to the commonly used approach which regards the trend following behaviour itself as a manifestation of investor sentiment and recency bias in forecasting (see, e.g. De Long et al., 1990; Barberis, Shleifer, and Vishny, 1998), our approach helps to distinguish the effects of sentiment and recency bias in the trend following behaviour. We showed that, sentiment in the trend followers' forecasting has a primary impact on changing the size of price bubbles but not the duration. The degree of optimism or pessimism of the trend followers is positively correlated with the size of market booms or crashes. The presence of optimism or pessimism in the trend followers' forecasting is able to cause excess kurtosis in return distribution.

Different to the effect of sentiment, recency bias in the trend followers' forecasting mainly affects the duration of price bubbles. Recency bias in forecasting is able to reinforce the trend following behaviour causing optimism (or pessimism) when the price is increasing (or decreasing). For this reason, recency bias in forecasting also has impact on the size of price bubbles. When the trend followers exhibit sentiment and recency bias at the same time, both the size and duration of price bubbles can be strongly affected leading to less predictable price dynamics. Moreover, we showed that optimism (or pessimism) in conjunction with recency bias leads to excess kurtosis and positive (or negative) skewness in return distribution. These statistical properties of generated returns are qualitatively close to those from real-world financial time series.

However, our results of the homogeneous observation horizons case indicate that the homogeneity in the trend followers' observation horizons imposed too much regularities in the price and return dynamics. These regularities cannot be removed by the presence of the fundamentalists with different degrees of reaction, a reasonable proportion of the noise traders, or behavioural biases such as sentiment and recency bias in the trend followers' forecasting. Without adding a very large proportion of the noise traders (e.g. assuming that the noise traders possess above 80% aggregate wealth at each period), our model with homogeneous observation horizons cannot reproduce the stylised fact: the absence of autocorrelations in returns.

Previous contributions with the assumption of homogeneous observation horizons often rely on exogenous noises (usually very strong) to destroy the strong regularities imposed by the homogeneity. We illustrated that this issue can be solved naturally once investors have heterogenous observation horizons. To model heterogeneous observation horizons of the trend followers, the whole group of trend followers are divided into subgroups which are characterised by different observation horizons. In order to capture active learning, the inner-switching of the trend followers among different subgroups are allowed. Moreover, a consensus investment proportion is defined to represent the aggregate behaviour of the trend followers with different observation horizons. The entire population of trend followers can be regarded as one representative agent who makes investment according to the consensus investment proportion. This setting helps us to understand the generated price dynamics, especially when the number of different subgroups of the trend followers is large.

Our analysis showed that the heterogeneity of the trend followers with respect to the observation horizon plays an important role in shaping the aggregate market dynamics. A large number of subgroups of the trend followers with different observation horizons helps to reproduce the stylised facts of absence of autocorrelations and volatility clustering in return series. However, our results also highlighted the role of market selection in models with heterogeneous observation horizons. We showed that those subgroups with longer observation horizons may be driven out of the market if the trend followers were unbiased in performance evaluation in the inner-switching (i.e. $\rho^T = 1$). In this case, the market impact of heterogeneous observation horizons may be eliminated by the force of market selection. In contrast, recency bias (i.e. $\rho^T < 1$) in the inner-switching leads to the survival of subgroups with different observation horizons.

In the heterogeneous observation horizons case, our analysis of the trend followers' sentiment revealed that sentiment have stronger impact on those trend followers who have longer observation horizons. This property leads to a significant difference among investment proportions of different subgroups of the trend followers. When these investment proportions are aggregated, there exist a significant cancellation effect among the investment proportions of different subgroups. This cancellation effect in conjunction with pessimism cause that the price movements in market downswings are larger than in market upswings. This observation is consistent with the so-called asymmetric volatility which is observed in real markets. When recency bias in forecasting is taken into account, the trend followers' pessimism in conjunction with recency bias lead to excess kurtosis and negative skewness in return distribution, which are similar to those from the S&P 500 index.

Summarising, the model and analyses presented in this chapter helped to quantify the effects of some important behavioural elements such as the observation horizons, sentiment and recency bias which may affect the trend following behaviour of investors. We showed that these micro-level behavioural elements play an important role in shaping the market dynamics at the macro-level. Our results demonstrated the ability of our model on reproducing a number of important stylised facts. Those behavioural elements considered in this chapter helped to explain the causes of these stylised facts.

Chapter 6

Conclusion

6.1 Summary of the Research

This PhD thesis introduces a framework that combines agent-based modelling and concepts from behavioural finance to study the dynamics of financial markets. A number of important heuristics and behavioural biases documented in the behavioural finance literature are studied in an agent-based financial market model which consists of a combination of passive and active learning dynamics. The focus is on micro-level modelling of investors' heterogeneity, bounded rationality, heuristics and biases which may affect investors' active learning and forecasting with a view to assessing their impact on the aggregate market dynamics and the survival of investment strategies. The goal is to explore the roles of these behavioural factors in affecting the interaction between passive and active learning; to contribute insights for the macro-level impact and evolutionary prospects of the presence of boundedly rational, heterogeneous investors with various heuristics and biases.

In Chapter 2 of this thesis, we conduct a literature survey which covers areas of traditional finance, behavioural finance, and agent-based models of financial markets. It is not our intention to provide a comprehensive review of each subject due to the vast body of the literature. Our review of the traditional and behaviour finance literature aims to highlight major issues and debates between the two different approaches on studying the behaviour of market participants and dynamics of financial markets. These issues and debates point to the motivation of our research on using agent-based modelling together with concepts from behavioural finance to study the dynamics of financial markets.

By reviewing the behavioural finance literature, we list a number of important heuristics and behavioural biases which may impact investors' financial decisions at the micro level. These behavioural elements are identified from laboratory experiments based on real-world individuals. Incorporating these behavioural elements into agent-based financial market model helps to build realistically the micro-level modelling of investors' behaviour. Our review of the agent-based literature aims to provide an overview of agent-based financial market models with focus on their market designs and behavioural aspects. The purpose is to identify issues, gaps, and lessons learned in this rapidly growing body of literature. Overall, the survey presented in Chapter 2 provides motivations, techniques and inspirations for our research. It also answers our research question 1 by identifying important behavioural aspects of investors.

Chapter 3 addresses our research question 2 by developing an agent-based financial market model which combines a performance-driven strategy-switching mechanism of adaptive belief systems (Brock and Hommes, 1998) and an evolutionary finance model (Evstigneev, Hens and Schenk-Hoppé, 2011). This new model inherits the advantages of the evolutionary finance approach on maintaining a large degree of freedom on modelling investors' behaviour. It also draws on the strengths of adaptive belief systems on allowing active learning of investors.

The financial market model contains three commonly studied agent types: fundamentalist, trend follower, and noise trader. It is assumed that investors can switch between fundamentalist and trend follower according to the past performances of the two agent types. The coexistence of switching investors and non-switching investors are allowed. Switching investors will bring (or take away) their wealth when they join (or leave) each investment strategies. A main property of the model is that investors' performance-driven strategy-switching behaviour causes wealth reallocation among different investment strategies (agent types).

We call this property "flow of funds". Since the wealth managed by each agent type affects the price dynamics which in turn affects agents' performance and the flow of funds, there exist a feedback loop between the wealth dynamics of each agent type, the price dynamics and the flow of funds. The model characterises the coevolution of asset prices and the redistribution of wealth when investors switch among different investment strategies. The survival of investment strategies and long-run market dynamics are determined by both passive and active learning dynamics.

By incorporating concepts from behavioural finance, we focus on the modelling of behavioural fundamentals which underpins and affects investors' strategy-switching behaviour, such as better-than-average overconfidence, differences of opinion, recency bias in performance evaluation, conservatism bias and rational herding. The whole financial market model therefore has key features: i) it captures the interaction between passive and active learning dynamics. ii), it addresses a variety of behavioural biases which may affect investors strategy-switching behaviour.

An explicit solution to the wealth dynamics is derived in Chapter 3. This explicit formulation of wealth dynamics is for building a computer programme of the model and increasing its computational efficiency. A basic analytical study of the existence and location(s) of steady state(s) of the model is carried out. This analysis reveals the impact of the values of the risk-free rate of return r and consumption rate c on the aggregate economy of the model. A proper relation between the values of the two parameters has been identified (i.e. $c = \frac{r}{1+r}$) by which the unique type of steady state of the model corresponds to an arbitrage-free equilibrium.

Based on the model developed in Chapter 3, Chapter 4 uses numerical simulations to explore the interaction between passive and active learning dynamics, and the macro-level market impact and long-run prospects of behavioural biases associated with strategy-switching, such as better-than-average overconfidence, differences of opinion, recency bias in performance evaluation, conservatism bias and rational herding. The results documented in this chapter answers our research question 3. We summarise and list here the main findings in relation to answers of research question 3.

- We analysed the pure passive learning case where all investors are better-than-average overconfident investors and none of them switches among investment strategies. Our results show that the process of market selection plays an important role in shaping the long-term market dynamics. Only the fundamentalists survive in the long-run. The asset price converges to the fundamental value. These results agree with the prediction of Market Selection Hypothesis (MSH) and Efficient Market Hypothesis (EMH) for real markets. Moreover, we found that the excess volatility and high trading volume originated from investors' better-than-average overconfidence together with differences of opinion (in terms of different prior beliefs) are only temporary market phenomena under pure passive learning. The evolutionary forces operating through wealth dynamics will eventually eliminate the market impact of overconfidence and differences of opinion.

- We analysed the case where all investors are switching investors. In this case the model exhibits both passive and active learning dynamics. We found that, if the parameter $\rho = 1$ meaning that investors have infinite memory of past performances and are unbiased in performance evaluation, the long-run outcome of active learning agrees with the passive learning: all investors will eventually move to the fundamentalists and the price converges to the fundamental value. This results is consistent with the one obtained in the pure passive learning case. In addition, we found that under $\rho = 1$ the process of market selection and the convergence of market price can be delayed (or accelerated) by investors' conservative (or herding) type of behaviour in strategy-switching.

- If the switching investors exhibit recency bias in performance evaluation (i.e. $\rho \in [0, 1)$), our results show that both the fundamentalists and trend followers survive in long run. The survival of trend followers leads to the persistence of market inefficiencies such as asset bubbles and excess volatility. Investors' conservatism bias in strategy-switching reinforces the effect of difference of opinions leading to diversified wealth distribution among different agent types and high trading volume. Moreover, investors' herding type of behaviour when the intensity of choice is high may also cause high trading volume even large booms and crashes in the price. These results turn out that investor active learning is sensitive to heuristics and behavioural biases. Under the presence of various heuristics and biases, the outcome of active learning may not be consistent with the outcome of passive learning leading to different behaviour of the market in the long-run.

- We studied the case where better-than-average investors (i.e. non-switching investors) and switching investors coexist in the market. In this case, the proportion of these two types of investors is governed by a parameter $\beta \in [0, 1]$ which measures the percentage of aggregate wealth managed by the switching investors at each period. We conducted simulation experiments to explore how the values of β impact the market dynamics. We showed that, if $\rho \in [0, 1)$, even a very small amount mobile capital (e.g. 0.01% of the aggregate wealth) with recency bias has a substantial impact on the market dynamics leading to the survival of different agent types and the persistence of market inefficiencies. The persistence of high trading volume and excess volatility is caused by the flow of funds with recency bias in performance evaluation rather than the presence of overconfidence investors. The robustness of these results is checked based on 100 independent simulations with different seeds in random number generators.

- According to the literature review by Glaser and Weber (2007), the concepts of better-than-average overconfidence and differences of opinion have long been proposed as explanations for the observed high trading volume in financial markets. However, previous contributions in these areas usually ignore the long-run prospects of the effect of better-than-average overconfidence and differences of opinion. Our approach addressed these two concepts in an agent-based model with focus on the evolutionary perspective of the financial market. We showed these two concepts may be related to the short-term high trading volume. However, none of them is sufficient to explain the persistence of high trading volume in a evolutionary context. Based on our findings, we agree with the view of Glaser and Weber (2007) that better-than-average overconfidence psychologically maintains differences of opinion among investors. Nevertheless, we argue that the market impact of better-than-average overconfidence and differences of opinion may be eventually eliminated by evolutionary forces operating through both passive and active learning. In contrast, the flow of funds and recency bias in performance evaluation maybe potential candidates on explaining the survival of different investment strategies and the persistence of market inefficiencies.

The Chapter 5 of this thesis addresses the answer for our last research question. Based on a previous finding that the presence of strategy-switching and recency bias in performance evaluation may lead to the survival of the trend followers' strategy, Chapter 5 extends the model and analysis presented in previous two chapters to study the market impact of the survival of the trend followers. The focus is on exploring some important behavioural factors such as observation horizons, sentiment (optimism and pessimism) and recency bias on affecting the trend followers' forecasting about future asset returns.

In the trend followers' forecasting, the length of observation horizon affects their sample size. Optimism (pessimism) is modelled as the tendency that trend followers give more importance to the positive (negative) events (e.g. realised past returns of the risky asset within their observation horizons) when forming return forecasting. Recency bias is modelled by the tendency that the trend followers assign more importance to more recent observations of asset returns in forecasting. Heterogeneous observation horizons of the trend followers and the inner-switching among different observation horizons are allowed.

Moreover, our approach allows the behaviour of the trend followers with different observation horizons to be aggregated via introducing and modelling a consensus investment proportion of the trend followers. This tool helps us analyse and understand the complex market dynamics emerged in our model, especially when the number of subgroups of the trend followers with different observation horizons is large. We conducted a series of numerical experiments to quantify the macro-level market impact of observation horizons, sentiment and recency bias on the trend following behaviour and the aggregate market dynamics. We summarise here our main findings related to these extensions.

- We studied the case where the trend followers have homogeneous observation horizons. Our analysis showed that, if the trend followers adopted the basic forecasting rule without sentiment and recency bias, the size and duration of asset bubbles as well as the autocorrelation structure of return series are characterised by the length of the trend followers' homogeneous observation horizons. The homogeneous observation horizons lead to unrealistic and predictable bubble dynamics in the price. These findings agree with those documented by Levy and Levy (1996) and Levy, Levy, and Solomon (2000) in models which have a similar specification for the forecasting rule of the trend followers.

- In the homogeneous observation horizons case, we explored the effect of the trend followers' sentiment and recency bias in forecasting. Our results show that sentiment in the trend followers' forecasting has a primary impact on changing the size of price bubbles but not the duration. The degree of optimism of the trend followers is positively correlated with the size of market booms or crashes, which supports the observations documented in Lovric et al. (2009). We found that the degree of pessimism of the trend followers is also positively correlated to the size of booms and crashes, which is in contrast to the finding of Lovric et al. (2009). The reason for this contradiction has been discussed and explained. Moreover, our results indicate that both optimism and pessimism may lead to excess kurtosis in return distributions.

- Different to the effect of sentiment, we found that recency bias in the trend followers' forecasting mainly affects the duration of price bubbles. Our results show that recency bias in forecasting is able to reinforce the trend following behaviour causing optimism (or pessimism) when the price is increasing (or decreasing). Therefore, recency bias in forecasting may also impact on the size of price bubbles. In addition, we showed that if the trend followers exhibited sentiment and recency bias at the same time, both the size and duration of price bubbles can be significantly affected resulting less predictable dynamics of the price and return. However, our analysis showed that the homogeneous observation horizons of the trend followers imposed strong regularities (e.g. autocorrelations) in the price and return dynamics. These strong regularities cannot be removed by the presence of the fundamentalists with different degrees of reaction (i.e. different values of α^F), a reasonable proportion of the noise traders (e.g. less than 80% aggregate wealth is managed by the noise traders), or behavioural biases such as sentiment and recency bias in the trend followers' forecasting. The model with homogeneous observation horizons failed to reproduce the stylised fact: the absence of autocorrelations in return series.

- We studied the case where the trend followers have heterogenous observation horizons and they switch among subgroups with different observation horizons according the past performance of each subgroup. Our results revealed that, the heterogeneity with respect to observation horizons and the inner-switching play an important role in shaping the price and return dynamics. The inner-switching with recency bias in performance evaluation leads to the survival of subgroups with different observation horizons. The heterogenous observations in conjunction with the inner-switching help to reproduce the stylised facts of absence of autocorrelations in returns and volatility clustering. Compared with an existing approach to heterogeneous observation horizons by Levy, Levy, and Solomon (2000) which does not allow investors to change observation horizons, our approach and results highlighted the importance of the switching among different observation horizons. We argue that without considering investors' active learning (e.g.

the inner-switching), evolutionary force and the process of market selection may reduce or eventually eliminate the market impact of the heterogeneous observation horizons.

- Levy, Levy, and Solomon (2000) pointed out that allowing investors to have heterogeneous observation horizons present difficulties on understanding the resulting dynamics due to the complexity imposed by the large number of investors with heterogeneous observation horizons. Our approach tackles this issue by modelling a consensus behaviour of the trend followers with different observation horizons. This tool helps us to analyse the aggregate dynamics through monitoring the aggregate behaviour of the entire population of the trend followers. By using this tool, we found that, in the case where the trend followers exhibit sentiment, when the investment proportions of each subgroup of the trend followers are aggregated, there exist a significant cancellation effect among the investment proportions of different subgroups. This cancellation effect in conjunction with pessimism cause that the price movements in market downswings are larger than in market upswings, which is consistent with the so-called asymmetric volatility observed in real markets. Moreover, our results indicate that the trend followers' pessimism in conjunction with recency bias may lead to excess kurtosis and negative skewness in return distribution, which are consistent with the statistical properties of real-world financial time series such as those of S&P 500 index.

6.2 Closing Remarks and Outlook

This thesis has the intention to bring closer agent-based modelling of financial markets and behavioural finance. One of the main contributions of this research is that it explores the added value of drawing together agent-based modelling and concepts from behavioural finance on the study the dynamics of financial markets.

Compared with those *autonomous* agent models which are based on genetic algorithm or genetic programming, our financial market model is relatively simple in the sense that agents use static parameterised investment strategies. However, it is our intention to keep agent behaviour as simple as possible at this stage. The main motivation comes from the common criticism that in agent-based models with complex behaviour of agents it is not always clear which aspect of agents' behaviour is responsible for the generation of some certain market phenomena such as the stylised facts. Moreover, the survey of Chen et al. (2012) shows that autonomous agent models do not have stronger ability than N-type (static agent) models in reproducing and explaining stylised facts (in terms of the number and different types of stylised facts reproduced and explained).

The goal of using autonomous agent with genetic algorithm or genetic programming in agent-based models is to model agents' behaviour more realistically. For this purpose, instead of using autonomous agent, we draw on the strength of behavioural finance to incorporate investor psychology into the micro-level modelling of agents. We strive to make the interlinkages between individual investors' psychology and the aggregate market dynamics as transparent as possible. Such an approach helps to explain those endogenous market phenomena through tracking down to individual investors' psychological elements which underpin, affect and govern their trading activities.

Our simulation results documented in the Chapter 4 and 5 have demonstrated the added value of bringing together agent-based models and behavioural finance. On the one hand, we showed that agent-based modelling can be used to provide insights for the macro-level market impacts and evolutionary prospects of a number of important heuristics and behavioural biases documented in the behavioural finance literature. On the other hand, we found that many important endogenous market phenomena such as market inefficiencies and stylised facts can be explained not only by investors' behaviour (e.g. trend following, strategy-switching etc.) but also their psychological elements and cognitive biases such as overconfidence, recency bias, conservatism, sentiment and so on. This finding illustrated that drawing in concepts from the behavioural finance literature helps to increase the explanatory power of agent-based modelling of financial markets.

Another important contribution of this research is that it explores the interaction between passive and active learning, especially when heuristics and behavioural biases are involved in active learning. Our approach brings together two research areas: adaptive belief systems and evolutionary finance. By combining the strategy-switching mechanism of adaptive belief system and an evolutionary finance model, we propose a new concept: the flow of funds. In the combined financial market model, the feedback loop between wealth dynamics of each agent type, the flow of funds, and the price dynamics links together the passive and active learning. The flow of funds represents investors' active learning where heuristics and biases may apply. Therefore, the effect of investors heuristics and biases may enter into the feedback loop and affect the interaction between passive and active learning as well as the market dynamics.

Our results presented in this thesis indicate that both passive and active learning play important roles in affecting the survival of investment strategies and in shaping the long-run market dynamics. We showed that, in economies with properties of self-financing and bounded endowments, the passive learning which represents the market selection force in wealth dynamics may have the power to reduce or eliminate the market impact of the heterogeneity of investment strategies leading to a dominant behaviour. In the case of pure passive learning, because of the process of market selection, as predicted by MSH and EMH, investor heterogeneity, bounded rationality, heuristics and behavioural biases may not impact the long-run outcome of the financial markets.

In contrast to passive learning, active learning which represents investors' adaptive belief updating process is sensitive to investor bounded rationality, heuristics and behavioural biases. We showed that, because of investors' bounded rationality and various behavioural biases, active learning investors may not always choose the best profitable investment strategy. The interaction between passive and active learning with bounded rationality and behavioural biases may lead to the survival of different investment strategies and the persistence of market inefficiencies. Investors' recency bias in performance evaluation has been identified to be a crucial behavioural element which may impact the outcome of active learning. These results highlighted the importance of passive learning and behavioural aspects of active learning on understanding the dynamics of financial markets. Our findings therefore suggest that future research on agent-based financial market models should take both passive and active learning into account. The interaction between passive and active learning is critical on studying the dynamics of financial markets.

We believe that our research should be extended and developed to several directions in the future, by and large, can be summarised by the studies of agent behaviour and extensions of market mechanism. Our research in this thesis only makes a first step towards bridging agent-based models and behavioural finance. Future research of agent-based models should work closely with behaviour finance. One way of doing this is to incorporate more behavioural elements into the modelling of agent behaviour. The goal is to enrich the explanatory power of the modelling approach and to offer more insights for the impact of micro-level behavioural elements on the macro-level market dynamics. The other way is to equip laboratory experiments of human subjects to test and validate the parametric values used in the modelling of agent behaviour and the hypothesis and conjectures obtained from studying the model. Hommes (2011) serves a good example of this direction of research.

Another direction of future research is to move onto the study of the multi-asset model documented in the Chapter 3 which is proposed as a general form of the evolutionary finance model with flow of funds. Our analysis presented in this thesis primarily focused on the two-asset model. The aim is to explore the link between agents' behavioural elements at the micro-level and the aggregate market behaviour. Compared with the study of the two-asset model, the study of the multi-asset model involves a different research objective which addresses the combined effect of the interaction of heterogeneous agents and diversification among multiple risky assets. This direction of research will have focuses on, for example, the spillover of instabilities among the risky assets through the wealth effect and the flow of funds, and the impact of the flow of funds on the performance of portfolio rules with asset allocations. Our study of the two-asset model in this thesis is able to provide crucial understandings of the origin of the instabilities for the future research of the multi-asset model.

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