Degrees of members of Π_1^0 classes

Ahmet Çevik

Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

The University of Leeds

School of Mathematics

August 2014

Acknowledgments

I would like to express my deepest gratitude to Andy Lewis for his great effort in supervising this thesis and for all his support, for having had patience with me and for his generosity through many years. I truly feel lucky to have a mentor like him as it has been a great experience to have been able to discuss mathematical issues with him and to learn so many things from his supervision. I hope we have a chance to continue our discussions in the future as well. I would like to thank Barry Cooper a lot for always giving me useful advice regarding academic issues whenever I needed help from a senior professional and for encouraging me to participate in conferences, workshops and so forth. I hope we stay in contact.

I would like to give a very special thank you to my external examiner Antonín Kučera for his detailed comments and carefully pointing out his suggestions which have been very useful for improving the quality of this thesis. I would also like to thank my internal examiner Peter Schuster for reading my correction report, for scheduling my viva and making me feel comfortable during that time. I should thank our postgraduate research secretary Jeanne Shuttleworth for helping us with our problems regarding formal procedures and for her patience.

I would like to thank my friends from Leeds including Andrés Aranda López, Joel Ronnie Nagloo, Michael Toppel, Pedro Francisco Valencia Vizcaíno, Rizos Sklinos, Serkan Aydın and many others for making Leeds a better place to live and for their company.

Last but not least, I would to thank my family for their continuous support, maybe not regarding mathematics but anything else.

This thesis is dedicated to my parents and my sister.

Abstract

In this thesis we study Turing degrees of members of Π^0_1 classes. We give two introductory chapters and then three main chapters which include new results.

In the first chapter we give some standard background for recursion theory, and we give an introduction to Π^0_1 classes in the second chapter.

The third chapter will be on the published work [1]. We show that for any degree $\mathbf{a} \geq \mathbf{0}'$, if a Π^0_1 class $\mathcal{P}$ contains members of every degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$, then $\mathcal{P}$ contains members of every degree. A local version of this result is also given. That is, when $\mathbf{a}$ is also Σ^0_2, it suffices in the hypothesis to have a member of every Δ^0_2 degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$. This result extends the Low Antibasis Theorem given in Kent and Lewis [2].

The fourth chapter has three subsections. The first subsection concerns an observation, which may be seen as a cupping non-basis analogue of Jockusch and Soare's capping basis theorem: We show that there exists a non-empty Π^0_1 class with no recursive member, such that no join of two sets in the class computes $\emptyset'$. The second one contains the principal result of the chapter, which concerns the relationship between the join property and the members of Π^0_1 classes. We show that there exists a non-empty Π^0_1 class with no recursive member, for which it also holds that no member satisfies the join property. Third subsection contains some future work where we give some open questions about the relation between minimal covers and Π^0_1 classes, and also about the relation between minimal covers and PA degrees.

In the fifth chapter we study the degree spectrum properties of a special kind of Π^0_1 classes that we introduce, so called Π^0_1 *choice* classes. A Π^0_1 choice class is a Π^0_1 class such that no two members have the same Turing degree.

Considering this restricted class leads us to some interesting antibasis theorems and technically innovative constructions.

Contents

Abbreviations

ANR	Array non-recursive
DNR	Diagonally non-recursive
dom$f$	Domain of the function f
FPF	Fixed point free
g.l.b.	Greatest lower bound
l.u.b.	Least upper bound
PA	Peano arithmetic
poset	Partially ordered set
r.e.	Recursively enumerable
ZFC	Zermelo-Fraenkel set theory with the Axiom of Choice

List of Figures

Chapter 1

Background on Recursion Theory

1.1 Introduction

The aim of this thesis is to add to our understanding of the Turing degrees of members of Π^0_1 classes. We shall be interested, in particular, in the degree spectra of Π^0_1 classes and their jump-inversion properties, in the relationship between the join property of members of Π^0_1 classes, minimal covers and degrees of members of Π^0_1 classes, and also in the degree spectra of a particular set of Π^0_1 classes which we shall refer to as choice classes.

The thesis contains five chapters in total. The first two chapters give the necessary background and motivation for the main chapters. In this chapter, without introducing Π^0_1 classes, we give our notation and some standard background for recursion theory. The material covered includes relative computability, properties of the Turing degrees, the arithmetical hierarchy, some examples of the construction methods which will be used in the later chapters, the Turing jump and so forth. The second chapter is devoted to the study of Π^0_1 classes in general. We introduce Π^0_1 classes, investigate their relationship with logic and describe some of the most important theorems from the literature. We also take a look at the relationship between Π^0_1 classes and PA degrees in that chapter.

In the third chapter, we give an exposition of the results appearing in [1]. We prove two *antibasis* theorems concerning Π^0_1 classes. These theorems extend the low antibasis theorem given in [2]. We show that for any degree $\mathbf{a} \geq \mathbf{0}'$, if a Π^0_1 class $\mathcal{P}$ contains members of every degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$, then $\mathcal{P}$ contains members of every degree. A local version of this result is also given. Namely that when $\mathbf{a}$ is also Σ^0_2, it suffices in the hypothesis to have a member of every Δ^0_2 degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$.

The fourth chapter has three subsections. The first subsection concerns a cupping non-basis analogue of Jockusch and Soare's capping basis theorem which appears in [3]. More specifically, we observe that there exists a non-empty Π^0_1 class with no recursive member such that no join of two sets in the class computes $\emptyset'$. The second subsection, which contains the main result for the fourth chapter, is about the relationship between the join property and the members of Π^0_1 classes. We show that there exists a non-empty Π^0_1 class with no recursive member such that no member satisfies the join property. In the final subsection we discuss some future work where we give some open questions about the relation between minimal covers and Π^0_1 classes, and also about the relation between minimal covers and PA degrees.

In the fifth chapter, we study the degree spectrum properties of a special kind of Π^0_1 class, so called Π^0_1 *choice* classes. A Π^0_1 choice class is a Π^0_1 class such that no two members have the same Turing degree. This property gives us some interesting results such as cardinality properties and proper antibasis theorems.

No background is assumed. All the necessary definitions and facts will be given here, and can also be found in any textbook on computability theory. For a more detailed account of computability theory, we refer the reader to [5], [6], [7], and [8].

1.2 Basics

Recursion theory is a branch of mathematical logic which originated from the study of recursive functions.[1] One of its main aims is to investigate the algorithmic relationship between non-computable sets, functions, and relations. The term computable refers to "algorithmically computable" mathematical objects.[2] We then must define what is meant by algorithmically computable. However, the notion of an algorithm or effective computation, carried out by the human mind, is not mathematically well defined. There have been different models of computation proposed which are believed to capture the class of intuitively computable functions. Kurt Gödel was the first logician who formally introduced general recursive functions in 1934 although he used primitive recursive functions in his well known paper on incompleteness [9]. On the other hand, Alonzo Church [10] introduced his *lambda calculus* as a model of computation. Alan Turing was perhaps the first to describe a really natural and universally accepted model of computation, the so called *Turing machine*, which is believed to capture the notion of algorithmic computability. Briefly, a *Turing machine* consists of an *infinite tape* divided into cells on which we write symbols from a finite set Σ of symbols, called the *alphabet*, a *tape head* which can read/write symbols and move left (L), right (R) or stay stationary (S), a finite set Q of *states*, and a set of *instructions* in the form of transition function $\delta : Q \times \Sigma \to Q \times \Sigma \times \{L, R, S\}$. The computation starts by reading the leftmost symbol of the input written on the tape of the machine. Then, depending on the state of the machine, if necessary we write a symbol on the tape cell, change our state, and move the tape head accordingly. The computation halts when the machine reaches a halting state $q_f \in Q$ and the output is whatever is written on the tape.

If we call the class of functions computable by Turing machines as *Turing computable functions*, it is a philosophical statement to claim that the class of Turing computable functions are exactly the class of algorithmically computable functions. This is called the *Church-Turing thesis.* The reason

[1]Recursion theory is contemporarily called *computability theory* by many mathematical logicians. We will adopt both and use them interchangeably.

[2]The term *algorithmically computable* is also known as *effectively computable.*

why this is not a mathematical statement but a philosophical statement is because there is no formal definition of algorithmic computation. On the other hand, a Turing machine is a well defined mathematical object. When studying computability theory, we generally work according to the assumption that the Church-Turing thesis is true.[3] Therefore, from now on, when we say a function is Turing computable (or just computable/recursive) we mean that it is actually algorithmically computable and vice versa.

Let ω denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. We let $\aleph_0$ denote the cardinality of ω. Lower case Latin letters such as $i, j, k, l, m, n, \ldots, x, y, z$ denote integers. We let $2^{<\omega}$ denote the set of all finite sequences of 0's and 1's. We denote sets of natural numbers with A, B, C, D and for a set A, $\overline{A}$ denotes the complement of A, i.e. $\omega - A$. We use standard set theoretic operators $\in, -, \cap, \cup, \subset, \supset$ for element of, difference, intersection, union, subset and superset (not necessarily proper), respectively. We form predicates with the usual notation of logic where $\wedge, \vee, \neg, \Longrightarrow, \Longleftrightarrow, \exists, \forall, \mu x$ denote respectively: and, or, not, implies, if and only if (iff), there exists, for all, and the least x. In addition, $\exists^\infty$ denotes that "there exists infinitely many x such that". We denote (possibly) partial functions on ω by lowercase Greek letters ϕ, φ, ψ and Turing functionals by uppercase Greek letters Φ, Θ, Ψ. We also use f, g, h for functions. For a function f, we let dom$f$ denote the domain of f. We use $\vec{x}$ to abbreviate $(x_1, x_2, \ldots, x_n)$. We use a similar abbreviation for quantifiers, i.e. $\exists\vec{x}$ means $\exists x_1 \exists x_2 \cdots \exists x_n$. For a binary relation $<$, we use $x, y, z < w$ to abbreviate $x < w$, $y < w$, and $z < w$.

We let $\langle ., . \rangle$ be a computable bijection $\omega \times \omega \to \omega$. Let ω^ω denote the set of all functions from ω to ω and let 2^ω be the power set of ω, i.e. the set of all subsets of ω. A *string* is a sequence of 0's and 1's. We denote strings $\in 2^{<\omega}$ by $\sigma, \tau, \rho, \upsilon, \eta$. We let $\emptyset$ denote the empty set and also the *empty string*, i.e. the string of length 0, depending on the context. We let $\sigma * \tau$ denote the concatenation of σ followed by τ. We let $\sigma \subset \tau$ denote that σ is an *initial segment* of τ. We say a string σ is *incompatible*[4] with τ if neither $\sigma \subset \tau$ nor

[3] In fact, by referring to the field as computability theory we are already believing that the Church-Turing thesis is true.

[4] We sometimes use the word *incomparable* for this.

$\tau \subset \sigma$. Otherwise we say that σ is *compatible* with τ. We say that σ *extends* τ if $\tau \subset \sigma$. If $\tau \subset \sigma$ we also say that σ is a *successor* of τ and τ is a *predecessor* of σ. If there is no σ' such that $\tau \subset \sigma' \subset \sigma$ then σ is an *immediate successor* of τ and τ is an *immediate predecessor* of σ. A set T of strings is *downward closed* if whenever $\sigma \in T$ and $\tau \subset \sigma$ then $\tau \in T$. For a set A, $A(i)$ denotes the $(i+1)$st bit of A and similarly for a string σ, $\sigma(i)$ denotes the $(i+1)$st bit of σ. Let $A \upharpoonright z$ denote the restriction of $A(x)$ to those $x < z$. We define the latter similarly for functions and strings. For any set A, let $|A|$ denote the cardinality of A. When we use this for strings $\sigma \in 2^{<\omega}$, let $|\sigma|$ denote the *length* of σ. Given a stage by stage enumeration of A, we let A_s denote the elements of A enumerated by the end of stage s.

Note that algorithms only yield partial functions, i.e. functions that may be undefined on some arguments, because we may not be able to give an output for an arbitrarily given argument. For example let $\psi(x) = \mu y\, [p(x, y) = 0]$, where $p(x, y)$ is some polynomial with integer coefficients and where $\mu x P(x)$ denotes "the least x such that $P(x)$". Then, ψ may be undefined for some values of x.

Definition 1. A function $f : \omega \to \omega$ is called *partial recursive* if it is effectively computable. If f is defined on every argument then f is *total*. In this case f is *total recursive* (or simply *recursive*).

Definition 2. Let S be any set. The *characteristic function* of S is given by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

We say that S is *recursive* if χ_S is recursive. Recursive relations are defined similarly.

Now every algorithm, hence Turing machine description, is of finite length. Therefore, we can enumerate partial recursive functions by Gödel numbering. We let the sequence $\{\varphi_i\}_{i\in\omega}$ be an effective enumeration of the partial recursive functions. We use the following notation:

$$\varphi_e \text{ is defined on } x \iff \varphi_e(x) \downarrow.$$

If φ_e is not defined on x, then we write $\varphi_e(x)\uparrow$. The following theorem, sometimes called the *fixed point theorem*, is one of the earliest and most important results in recursion theory and was proved by Kleene [12]. We simply state the theorem without proof here, but a proof can be found in any logic book covering the theory of recursive functions.

Theorem 1 (Recursion theorem)**.** For every recursive function f there exists some $n \in \omega$ (called a fixed point of f) such that $\varphi_n = \varphi_{f(n)}$.

We know that not every set is computable. Some sets which are not computable, however, may be *recursively enumerable*. For example, the set $\{i : i \text{ is a prime number}\}$ is computable because there is an algorithm for deciding whether or not a number is prime. However, the polynomial example that we gave earlier is not necessarily computable (depending on the particular polynomial).[5] In that example, we can only decide one way. That is, we can answer positively when there is a solution, but we may not be always able to answer negatively otherwise.

Definition 3. A set A is called *recursively enumerable* (r.e.) if there is an algorithm that enumerates the members of A. More precisely, A is r.e. if A is the domain of some partial recursive function. Let the e-th r.e. set be denoted by

$$W_e = \text{dom}\varphi_e = \{x : \varphi_e(x)\downarrow\}.$$

Now every recursive set is recursively enumerable since we can effectively enumerate the members of A by asking whether or not $n \in A$, for each $n \in \omega$ in turn. If $n \in A$ then we enumerate x into our enumeration set. The following theorem is a standard result about recursively enumerable sets saying that a set is recursive if and only if there is an enumeration for itself and for its complement.

Theorem 2 (Complementation Theorem)**.** A set A is recursive iff both A and $\overline{A}$ are recursively enumerable.

[5]Note that Hilbert's tenth problem is, given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients, devising a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers. This problem was shown to be unsolvable by a collection of works by Davis, Matiyasevich, Putnam and Robinson [11].

Proof. If A, hence $\overline{A}$, is recursive then both A and $\overline{A}$ are recursively enumerable. Now suppose that we have enumerations for A and $\overline{A}$. Then, for any given $n \in \omega$, n is going to appear in the enumeration list of $\overline{A}$ if it is not going to appear in the enumeration list of A. Similarly, if n is not going to appear in the enumeration list of $\overline{A}$ then it must appear in the enumeration list of A at some point. Hence, we can decide for any given $n \in \omega$ whether or not $n \in A$. □

We now describe the canonical example of a set which is recursively enumerable but not recursive. The corresponding decision problem is to decide whether or not a partial recursive function will ever be defined on a given argument. This is known as the *halting problem*, and its unsolvability may be seen as the main reason we have the Gödelian incompleteness.

Definition 4. Let $K = \{x : \varphi_x(x)\downarrow\}$ be the *halting set.*

Theorem 3. K is recursively enumerable.

Proof. K is the domain of the partial recursive function

$$\psi(x) = \begin{cases} x & \text{if } \varphi_x(x)\downarrow \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now ψ is partial recursive by Church-Turing thesis since $\psi(x)$ can be computed by applying the x-th partial recursive function to input x and giving output x only if $\varphi_x(x)$ converges. □

Theorem 4. K is not recursive.

Proof. If K had a recursive characteristic function χ_K, the following would be recursive.

$$f(x) = \begin{cases} \varphi_x(x) + 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K. \end{cases}$$

But f cannot be recursive since $f \neq \varphi_x$ for every x. □

Combining the fact that K is r.e. but not recursive with the complementation theorem, we have the following.

Corollary 1. $\overline{K}$ is not recursively enumerable.

1.3 Turing degrees

We now want to relativize the idea of computability. The basic idea is that while a set may not be computable, it may become computable if we work relative to another non-computable set, i.e. if we are given access to the characteristic function of another non-computable set. The intuition is to use information concerning the membership of one set to help compute another. Let A and B be two sets. We want B to be computable from A if we can answer "Is $n \in B$?" using an algorithm whose computation given input n uses finitely many pieces of information about membership in A. In this case, A is called the *oracle*. For this relativized form of computation, we use *oracle Turing machines*. An oracle Turing machine is like a standard Turing machine with an extra tape, called the *oracle tape*, on which the characteristic function of the oracle is written. Unlike the work tape of the machine, we do not write anything on the oracle tape, but only read from it. Then, we can define the transition function as $\delta : Q \times \Sigma_1 \times \Sigma_2 \to Q \times \Sigma_2 \times \{L, R, S\}^2$, where Σ_1 denotes the oracle tape alphabet and Σ_2 denotes the work tape alphabet. When computing, we read the characteristic function of A written on the oracle tape and we perform the given instructions in the usual manner. Since we use an oracle in our computation, whatever we compute is only computable relative to the oracle set.

We said that Turing machine procedures or descriptions, hence partial computable functions can be effectively listed. Recall that we denoted the e-th partial computable function by ψ_e. In that case, there was no use of an oracle. We now include oracles in the definition.

Definition 5. (i) A partial recursive function with an *oracle* for a set S is a function which is always able to answer whether $x \in S$ or not for any x.

(ii) We say that a function ψ is *recursive in A* (or *A-recursive*) if ψ is computable by a partial recursive function with oracle an A.

(iii) A set B is said to be *A-recursive*, written as $B \leq_T A$, if χ_B is A-recursive.

(iv) We let $\Psi_e(A)$ denote the e-th Turing functional with an oracle A.

(v) We write $W_e(A)$ to denote dom$\Psi_e(A)$. If $B = W_e(A)$ for some $e \in \omega$, then we say that B is *recursively enumerable in A*.

Now every set is identified by its characteristic function or its *characteristic sequence* of 0's and 1's. So we consider binary sequences of 0's and 1's to be initial segments of the characteristic sequences of sets or functions. It is worth noting that infinite binary strings can also code real numbers. So infinite strings can be considered as reals.

It makes sense then to say that $A \leq_T B$ if and only if there exists some $e \in \omega$ such that $A = \Psi_e(B)$.

Definition 6. Let $\Psi_e(A; x) \downarrow= y$ denote that $\Psi_e(A)$ on argument x is defined and equal to y. We let $\Psi_e(A; x)\,[s] \downarrow= y$ denote that $\Psi_e(A; x)$ converges in at most s stages and outputs the value y. For any $A \subset \omega$ and $n \in \omega$, $\Psi_e(A; n) \uparrow$ means it is not the case that $\Psi_e(A; n) \downarrow$. We also write $\Psi_e(x)$ in order to denote $\Psi_e(\emptyset; x)$.

Let $y + 1$ be the number of scanned non-empty cells in the oracle tape during the computation. In this case, y is the maximum number used in the membership test of A. Hence, this means we *used* the elements $z \leq y$ in our computation. We shall now define the use function more precisely.

Definition 7. For a given $e, x, s \in \omega$ and $A \subset \omega$, the *use function* $u(A; e, x, s)$ is 1+"the maximum number used in the computation" if $\Psi_e(A; x)\,[s] \downarrow$. Otherwise, $u(A; e, x, s) = 0$.

Theorem 5. (Use Principle)

(i) $\Psi_e(A; x) = y \Longrightarrow \exists s\ \exists \sigma \subset A\,[\Psi_e(\sigma; x)\,[s] = y]$,

(ii) $\Psi_e(\sigma; x)\,[s] = y \Longrightarrow \forall t \geq s\ \forall \tau \supset \sigma\,[\Psi_e(\tau; x)\,[t] = y]$,

(iii) $\Psi_e(\sigma; x) = y \Longrightarrow \forall A \supset \sigma\, [\Psi_e(A; x) = y]$.

This principle is important for later use. It implies that Ψ_e is continuous. The first item actually says that when a computation halts it does so in a finite number of stages and hence only a finite number of bits of the oracle tape can be scanned. The second item says that if a computation $\Psi_e(\sigma; x)$ is defined by the stage s, it will also be defined and give the same value for stages $t \geq s$ and for all extensions of σ. The third item says that if $\Psi_e(\sigma; x) \downarrow= y$ for some $\sigma \in 2^{<\omega}$ then the computation is also defined for all extensions of σ. For convenience, we assume that for any string $\sigma \in 2^{<\omega}$ and any $e, n \in \omega$, $\Psi_e(\sigma; n)$ is not defined when $|\sigma| < n$. Hence if this computation converges, it does so in at most $|\sigma|$ steps and $\Psi_e(\sigma; n')$ is defined for all $n' < n$.

It is also worth noting that there exists a universal Turing machine. That is, there exists some $i \in \omega$ such that for all A, j, n we have $\Psi_i(A; \langle j, n\rangle) = \Psi_j(A; n)$ if they are both defined, otherwise they are both undefined. The following is another known fact which easily follows from the relativization of previously given facts.

Theorem 6. The following are equivalent:

(i) B is r.e. in A.

(ii) $B = \emptyset$ or B is the range of some A-recursive total function.

Now we define the Turing degrees and the jump operator, both of which play a central role in computability theory.

Definition 8. (i) Let A and B be two sets. If $A \leq_T B$ and $B \leq_T A$, then we say that A and B are *Turing equivalent*, and this is denoted by $A \equiv_T B$.

(ii) We define the *Turing degree* (or degree of unsolvability) of a set $A \subset \omega$ to be

$$\mathbf{a} = \deg(A) = \{X \subset \omega : X \equiv_T A\}.$$

(iii) We write $\mathbf{D}$ for the collection of all such degrees, and define a partial ordering induced by $\leq_T$ on $\mathbf{D}$ by

$$\deg(B) \leq \deg(A) \Longleftrightarrow B \leq_T A.$$

We write $\deg(A) < \deg(B)$ if $A <_T B$, i.e. if $A \leq_T B$ and $B \not\leq_T A$.

(iv) We denote Turing degrees by lowercase boldface Latin letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

Definition 9. (i) A degree $\mathbf{a}$ is called *recursively enumerable* if it contains a recursively enumerable set. We let $\mathbf{R}$ denote the set of all recursively enumerable degrees with the same ordering as for $\mathbf{D}$.

(ii) We say that a degree $\mathbf{a}$ is *recursively enumerable in* $\mathbf{b}$ if $\mathbf{a}$ contains some set A r.e. in some set $B \in \mathbf{b}$.

Intuitively, if two sets are of the same degree then they can be thought of as equally difficult to compute. If $\mathbf{a} < \mathbf{b}$, this means that sets of degree $\mathbf{b}$ are more difficult to compute than those of degree $\mathbf{a}$.

Definition 10. We define the *join* $\mathbf{a} \cup \mathbf{b}$ of degrees $\mathbf{a} = \deg(A)$, $\mathbf{b} = \deg(B)$ by

$$\mathbf{a} \cup \mathbf{b} = \deg(A \oplus B) = \deg(\{2i : i \in A\} \cup \{2i + 1 : i \in B\}).$$

Definition 11. (i) A partially ordered set (poset) $\mathcal{L} = (L; \leq, \vee, \wedge)$ is called a *lattice* if any two elements have a least upper bound (also known as supremum, join, or union) and greatest lower bound (also known as infimum, meet, or intersection). If a and b are elements of $\mathcal{L}$, $a \vee b$ denotes the least upper bound (l.u.b.) of a and b, and $a \wedge b$ denotes the greatest lower bound (g.l.b.). If $\mathcal{L}$ contains a least element and greatest element, these are called the *zero* element 0 and *unit* element 1, respectively. In such a lattice, a is the *complement* of b if $a \vee b = 1$ and $a \wedge b = 0$.

(ii) A poset closed under union but not necessarily under intersection is called an *upper semi-lattice*. A poset closed under intersection but not necessarily under union is called a *lower semi-lattice*.

The basic properties of the structure $(\mathbf{D}, \leq)$ can be given as follows.

Theorem 7. (i) $\mathbf{D}$ has $2^{\aleph_0}$ elements.

(ii) There is a least degree $\mathbf{0}$ which is the set of all recursive sets.

(iii) Each degree $\mathbf{a}$ has $\aleph_0$ elements.

(iv) The set of degrees $\leq \mathbf{a}$, for a given degree $\mathbf{a}$, is countable, i.e. $|\{\mathbf{b} : \mathbf{b} \leq \mathbf{a}\}| \leq \aleph_0$.

(v) For any $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{D}$, the least upper bound is their join. Therefore, the degree structure forms an upper semi-lattice. However, the greatest lower bound may not always exist for $\mathbf{D}$ or $\mathbf{R}$. Hence, neither $\mathbf{D}$ nor $\mathbf{R}$ forms a lattice.

We can relativize the halting set to any set $A \in \omega$. This gives us the Turing jump and it gives us a chance to study the higher degrees in the Turing universe.

Definition 12. We define the *jump* A' of a set A to be

$$A' = K^A = \{x : \Psi_x(A; x) \downarrow\}.$$

The $(n+1)^{th}$ *jump* of A is defined to be $A^{(n+1)} = (A^{(n)})'$, where $A^{(1)} = A'$.

We can summarize some of the important properties of the jump operator as follows.

Theorem 8 (Jump Theorem)**.** Let $A, B \subset \omega$. Then,

(i) A' is recursively enumerable in A.

(ii) $A' \not\leq_T A$.

(iii) If $A \equiv_T B$ then $A' \equiv_T B'$.

(iv) If A is r.e. in B and $B \leq_T C$ then A is r.e. in C.

(v) A is r.e. in B iff A is r.e. in $\overline{B}$.

Let $\mathbf{a}' = \deg(A')$ for $A \in \mathbf{a}$. Note that $\mathbf{a}' > \mathbf{a}$ and $\mathbf{a}'$ is r.e. in $\mathbf{a}$. Let $\mathbf{0}^{(n)} = \deg(\emptyset^{(n)})$. Then, we have an infinite hierarchy of degrees

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(n)} < \cdots.$$

From the fact that the jump is strictly increasing, it follows that $\mathbf{D}$ has a least element but no maximal element. Note that $\mathbf{0}'$ is the degree of K which is Turing equivalent to $\emptyset'$.

1.4 Arithmetical hierarchy

In this section we describe another hierarchy, in which sets are classified according to the quantifier complexity of their definitions. We define the classes Σ_n^0, Π_n^0 (also denoted Σ_n, Π_n in the literature). The superscript denotes that we are working in first order logic. For the second order logic case, i.e. the analytical hierarchy, we refer the reader to [13]

Definition 13. (i) $\Sigma_0^0 = \Pi_0^0 = \Delta_0^0 =$ all recursive predicates.

For $n \geq 0$:

(ii) Σ_{n+1}^0 is the set of all relations of the form $(\exists \vec{y_l})R(\vec{x_k}, \vec{y_l})$, where $R \in \Pi_n^0$.

(iii) Π_{n+1}^0 is the set all relations of the form $(\forall \vec{y_l})R(\vec{x_k}, \vec{y_l})$, where $R \in \Sigma_n^0$.

(iv) $\Delta_{n+1}^0 = \Sigma_{n+1}^0 \cap \Pi_{n+1}^0$.
R is *arithmetical* if $R \in \bigcup_{n\in\omega}(\Sigma_n^0 \cup \Pi_n^0)$.

Let us give an example about determining the quantifier complexity of sets. For example, let Tot $= \{i : \varphi_i$ is a total function$\}$ be a set. We can argue that Tot is in Π_2^0 since

$$\begin{aligned} i \in \text{Tot} &\iff (\forall n)\varphi_i(n)\downarrow \\ &\iff (\forall n)(\exists s)\varphi_i(n)\,[s]\downarrow. \end{aligned}$$

It is easy to see that Tot is in fact a Δ_3^0 set. Since

$$\begin{aligned} i \in \text{Tot} &\iff (\exists m)(\forall n)\varphi_i(n)\downarrow \\ &\iff (\forall n)(\exists s)(\forall m)\varphi_i(n)\,[s]\downarrow, \end{aligned}$$

we have that Tot $\in \Sigma_3^0 \cap \Pi_3^0$. Since we can arbitrarily add dummy quantifiers such as $(\exists m)$ and $(\forall m)$, we have the relation that

$$\Sigma_n^0, \Pi_n^0 \subset \Delta_{n+1}^0 \subset \Sigma_{n+1}^0, \Pi_{n+1}^0 \cdots .$$

Note that $A \in \Sigma_n^0 \iff \overline{A} \in \Pi_n^0$, so Σ_n and Π_n are complementary. Before we give some important properties about the arithmetical hierarchy let us give the following definition first.

Definition 14. (i) A set A is *many-one* reducible to B, written $A \leq_m B$, if there is a recursive function f such that $f(A) \subset B$ and $f(\overline{A}) \subset \overline{B}$, i.e. $x \in A$ iff $f(x) \in B$.

(ii) A set A is Σ_n^0-*complete* if $A \in \Sigma_n^0$ and $B \leq_m A$ for every $B \in \Sigma_n^0$. Π_n^0-complete and Δ_n^0-complete are defined similarly.

The following is known as Post's Theorem in the literature, and gives us some useful facts about the arithmetical hierarchy.

Theorem 9 (Post's Theorem)**.** Let $A \subset \omega$ and $n \geq 0$. Then:

(i) $\emptyset^{(n+1)}$ is Σ_{n+1}^0-complete

(ii) $A \in \Sigma_{n+1}^0 \iff A$ is r.e. in $\emptyset^{(n)}$

(iii) $A \in \Delta_{n+1}^0 \iff A \leq_T \emptyset^{(n)}$.

When $n = 1$, (iii) in Post's Theorem gives us the fact that $A \in \Delta_2^0 \iff A \leq_T \emptyset'$. Sets computable in $\emptyset'$ can also be characterized as approximating sequences:

Definition 15. We say that a recursive sequence $\{A_s\}_{s\in\omega}$ of finite sets is a Δ_2^0-*approximating sequence* for A if $A(x) = \lim_{s\to\infty} A_s(x)$ for all $x \in \omega$. We call A_s *the approximation to* A *at stage* s.

The following result, called limit lemma [14], is an important one and it will be used in the later sections.

Theorem 10 (Limit lemma, Shoenfield 1959)**.** $A \in \Delta_2^0 \iff$ there exists some recursive function g for which $\chi_A(x) = \lim_{s\to\infty} g(x,s)$.

Proof. Let g be given. Then,

$$\begin{aligned} x \in A &\iff (\forall s)(\exists t)(t \geq s \wedge g(x,t) = 1) \\ &\iff (\exists s)(\forall t)(t \geq s \rightarrow g(x,t) = 1). \end{aligned}$$

Therefore $A \in \Delta_2^0$. Now suppose that $A \in \Delta_2^0$. Then $A \leq_T K$. Let e be an index satisfying $\chi_A(x) = \Psi_e(K)$ and let g be such that $g(x,s) = \Psi_e(K_s; x)[s]$ if it is defined, otherwise $g(x,s) = 0$. Then $\chi_A(x)$ is the limit of the function g. □

1.5 Construction methods

In this section we give some known results, each using a different method of construction. We begin with showing that **D** is not linearly ordered. This was shown in [16].

Finite extension method

Definition 16. Two degrees $\mathbf{a}, \mathbf{b}$ are called *incomparable* if neither $\mathbf{a} \leq \mathbf{b}$ nor $\mathbf{b} \leq \mathbf{a}$.

We then have to show that there exist incomparable degrees. The main idea is that instead of considering a single complicated condition like $A \not\leq_T B$, we shall consider an infinite sequence $\{R_e\}_{e\in\omega}$ of simpler conditions. We call these conditions *requirements*. Here, each R_e will be $A \neq \Psi_e(B)$. At each stage of the construction we build more of the characteristic sequence of the sets we want to construct. We define strings σ_s and τ_s at stage s, so that ultimately we can define $A = \bigcup_{s\in\omega} \sigma_s$ and $B = \bigcup_{s\in\omega} \tau_s$. We use an oracle, specifically $\emptyset'$, at each stage of the construction when choosing σ_s and τ_s, and we choose these values so as to ensure that the next in our list of requirements is satisfied. We also ensure that $\sigma_s \subset \sigma_{s+1}$ for each $s \in \omega$. We call this method, ensuring that σ_{s+1} is a finite extension of σ_s for each s, the *finite extension method*.

Theorem 11 (Kleene and Post, 1954)**.** There exist incomparable degrees below $\emptyset'$.

Proof. We construct two sets A and B such that $A \not\leq_T B$ and $B \not\leq_T A$. We break these two conditions into infinite sequences of much simpler conditions and at each stage of the construction we aim to satisfy one. The requirements are as follows.

$$\begin{aligned} R_{2e} &\quad : \quad A \neq \Psi_e(B) \\ R_{2e+1} &\quad : \quad B \neq \Psi_e(A) \end{aligned}$$

We use the finite extension method to construct A and B. Let $A = \bigcup_{s\in\omega} \sigma_s$ and $B = \bigcup_{s\in\omega} \tau_s$. We satisfy a single requirement at each stage and once it is satisfied it will remain satisfied forever. Let $\sigma_0 = \tau_0 = \emptyset$. Suppose that σ_s and τ_s are given at stage $s + 1$.

If $s = 2e$, then we satisfy R_e. Let $x \in \omega$ be the first element such that $\sigma_s(x)$ is not defined yet. This means that we have not yet decided whether or not x should be in A. We decide this now and we use x to *witness* $A \neq \Psi_e(B)$. In other words, we satisfy $A(x) \neq \Psi_e(B; x)$. We should use diagonalization, i.e. make A on argument x different than $\Psi_e(B)$. Since we have not constructed B yet, we do not know if $\Psi_e(B; x)$ converges, i.e. is defined. However, we do know that if it converges then there exists some $\tau \subset B$ such that $\Psi_e(\tau; x)$ is defined as well, by the Use Principle. If there exists such τ we know that, by construction, since $\tau_s \subset B$, the string τ will be compatible with τ_s because B extends both. We may also suppose that τ is an extension of τ_s. In this case, we see if there exists a string $\tau \supset \tau_s$ such that $\Psi_e(\tau; x)$ converges.

If there is no such τ then $\Psi_e(B; x)$ will be undefined and since $A(x)$ will be defined because of being a total function, it does not matter what we do. In this case the requirement will be satisfied automatically and we let σ_{s+1} be the smallest extension of σ_s defined on x. Since nothing has to be done on B, we let $\tau_{s+1} = \tau_s$.

If such τ do exist, then $\Psi_e(B; x)$ will be defined. Then we must define τ_{s+1} in a way that B extends τ so that $\Psi_e(B; x) = \Psi_e(\tau; x)$. It suffices if we let $\tau_{s+1} = \tau$, where τ is the first such we found. However, there is one more point to be careful about A. Since $\Psi_e(B; x)$ is now defined, we need to make sure that it is different from $A(x)$. So, we let σ_{s+1} be the smallest extension of σ_s such that $\sigma_{s+1}(x) = 1 - \Psi_e(B; x)$.

If $s = 2e + 1$ then we just need to interchange the roles of A and B, the construction is the same.

Now A and B are computable in $\emptyset'$. The only non-recursive step in the construction is where we ask, given x and σ, if there exists some σ' extending σ such that $\Psi_e(\sigma'; x)$ is defined. It is easy to see that this is recursively enumerable. We consider all such σ' extending σ and we compute $\Psi_e(\sigma'; x)$ one step at a time in a dovetailing fashion. Hence, the construction is recursive in $\emptyset'$. □

Corollary 2. **D** is not linearly ordered.

Coinfinite extension method

We now introduce another method different than finite extension. We said that $(\mathbf{D}, \leq)$ is not a lattice. In order to show this we need to consider countable ideals of degrees.

Definition 17. Let $\mathcal{P}$ be an upper semi-lattice. Then $\mathcal{I} \subset \mathcal{P}$ is an *ideal* if

(i) Whenever $x, y \in \mathcal{I}$, $x \vee y$ is in $\mathcal{I}$;

(ii) Whenever $x \in \mathcal{I}$ and $y \leq x$, y is in $\mathcal{I}$.

Definition 18. A set $\mathbf{E}$ of Turing degrees is said to be *definable with parameters* if there is a finite set of degrees $\mathbf{a}_1, \ldots, \mathbf{a}_k$ and a formula in the language of partial orders $\mathcal{F}(x_0, \ldots, x_k)$ such that $\mathbf{a} \in \mathbf{E} \Longleftrightarrow \mathcal{F}(\mathbf{a}, \mathbf{a}_1, \ldots, \mathbf{a}_k)$ is true in the Turing degrees for all $\mathbf{a} \in \mathbf{E}$.

The next theorem, originally given in [17], shows that every countable ideal in the Turing degrees is definable with parameters. We need to give one more definition however.

Definition 19. Let $\mathcal{P}$ be a partial order and let $\mathcal{I} \subset \mathcal{P}$. We say that (x, y) is an *exact pair* for $\mathcal{I}$ if, for all $z \in \mathcal{P}$, $z \in \mathcal{I} \Longleftrightarrow (z \leq x$ and $x \leq y)$.

Theorem 12 (Spector, 1956)**.** Every countable ideal in the Turing degrees has an exact pair.

Corollary 3. The Turing degrees are not a lattice.

Proof. Let $\{\mathbf{x}_i\}_{i \in \omega}$ be a strictly increasing sequence of degrees (we can let $\mathbf{x}_{i+1} = \mathbf{x}_i'$ for example). Let $\mathcal{I}$ be the ideal generated by this sequence, i.e. $\mathbf{c} \in \mathcal{I} \Longleftrightarrow \exists \mathbf{x}_i \geq \mathbf{c}$. Let $(\mathbf{a}, \mathbf{b})$ be an exact pair for $\mathcal{I}$. If $\mathbf{c} \leq \mathbf{a}$ and $\mathbf{c} \leq \mathbf{b}$ then $\mathbf{c} \in \mathcal{I}$ and there exists $\mathbf{x}_i \geq \mathbf{c}$. Then, $\mathbf{x}_{i+1}$ is also below both $\mathbf{a}$ and $\mathbf{b}$ and is strictly above $\mathbf{c}$. Therefore, $\mathbf{a}$ and $\mathbf{b}$ have no greatest lower bound. □

We now prove the theorem.

Proof. Suppose that we are given an enumeration $\{X_s\}_{s \in \omega}$ of all sets which are of degree in $\mathcal{I}$. We construct A and B to satisfy the following requirements.

$\mathcal{P}_s : X_s \leq_T A$ and $X_s \leq_T B$;

$\mathcal{Q}_s :$ Let $s = \langle i, j \rangle$. $\Psi_i(A) = \Psi_j(B) = C \Longrightarrow (\exists k)(C = X_k)$.

The $\mathcal{P}_s$ requirements ensure that every degree is in the ideal is below both $\mathbf{a} = \deg(A)$ and $\mathbf{b} = \deg(B)$. Then the $\mathcal{Q}_s$ requirements ensure that anything computable in both A and B is of degree in the ideal.

A finite extension argument will not be sufficient here – we will need what is called a *coinfinite extension* argument. In a coinfinite extension argument, at each stage of the construction we define A and B on infinitely many arguments but at the end of each stage we also leave them undefined on an infinite number of arguments. The basic idea is to divide A and B into *columns* in order to ensure that each X_s is computable in these sets. The i-th column consists of all numbers of the form $\langle i, j\rangle$. If $A(\langle i, j\rangle) = X_i(j)$ for all but finitely many j, then this suffices to show that A will compute X_i. Similarly for B. Then, at each stage $s + 1$ we shall ensure that $X_s \leq_T A$ and $X_s \leq_T B$. To satisfy this we code X_s into the s-th columns of A and B in the following way:

We try to satisfy $\mathcal{Q}_s$ at stage $s + 1$.

Suppose that at the end of stage s we have already coded X_k into the k-th column of A and B for each $k < s$, and suppose that

(*) Outside the finite set of columns we have already used for coding we have decided only finitely many arguments of A and B.

Let α_s be the partial function specifying A on the arguments we have already decided. Let β_s be the similar partial function for B. Note that for all $s > 0$, α_s and β_s will be defined on infinitely many arguments and also undefined on infinitely many arguments. We see if there are any extensions $\alpha \supset \alpha_s$ and $\beta \supset \beta_s$ for which $\Psi_i(\alpha)$ and $\Psi_j(\beta)$ are incompatible, where $s = \langle i, j\rangle$ as we noted above.

If so, then there are finite extensions which satisfy this property. We can take these finite extensions, code X_s into what remains of the s-th columns of A and B, and maintain (*).

If there are no extensions which make $\Psi_i(\alpha)$ and $\Psi_j(\beta)$ incompatible, then we will be able to show that if $\Psi_i(A) = \Psi_j(B)$ and is total, then it is computable in the columns of A and B which we have already determined. This finite set of columns is basically the join of a finite number of sets of degree in the ideal, and so is of degree in the ideal. The following is the formal construction.

Stage 0. Let $\alpha_0 = \beta_0 = \emptyset$.

Stage $s+1$. Let $s = \langle i, j \rangle$. We see if there exist $\alpha \supset \alpha_s$ and $\beta \supset \beta_s$ such that $\Psi_i(\alpha)$ and $\Psi_j(\beta)$ are incompatible. If so, then let α and β be finite extensions of α_s and β_s respectively, satisfying that $\Psi_i(\alpha)$ and $\Psi_j(\beta)$ are incompatible. If not, then let $\alpha = \alpha_s$ and $\beta = \beta_s$.

Now let α_{s+1} be the least extension of α which is defined and equal to $X_s(j)$ on all arguments of the form $\langle s, j \rangle$ for which α is undefined. Let β_{s+1} be defined similarly in terms of β.

The construction satisfies the requirements. For verification, it suffices to consider what we do at stage $s+1$ when there are no α and β extending α_s and β_s satisfying that $\Psi_i(\alpha)$ and $\Psi_j(\beta)$ are incompatible. In this case we claim that if $\Psi_i(A)$ and $\Psi_j(B)$ are both total and equal, then this value is computable in $D = \oplus_{k=0}^{s-1} X_k$, and so is of degree in the ideal. Now D can decide which arguments α_s β_s are defined on, and compute their values on all such arguments. If $\Psi_i(A)$ and $\Psi_j(B)$ are both total and equal, then we use an oracle for D to compute the value of $\Psi_i(A; n)$ as follows. We find any extension α of α_s such that $\Psi_i(\alpha; n) \downarrow$. Now such an extension must exist since A extends α_s and $\Psi_i(A; n) \downarrow$. Then it must be the case that $\Psi_i(\alpha; n) = \Psi_i(A; n)$. In order to see this, suppose otherwise. Then let β be a finite extension of β_s which is compatible with B and such that $\Psi_j(\beta; n) \downarrow$. Since $\Psi_i(A) = \Psi_j(B)$ it must be the case that $\Psi_i(\alpha; n) \neq \Psi_j(\beta; n)$. This contradicts our hypothesis. □

Minimal degrees

A natural question is to ask whether or not $(\mathbf{D}, \leq)$ is dense, i.e. whether for any two distinct degrees there is another degree strictly between them. Spector [17] answers this question negatively in a stronger form. We now give a few definitions and the theorem. The simplified proof we follow is due to [15]. The non-density of $(\mathbf{D}, \leq)$ follows from the existence of minimal degrees:

Definition 20. A degree $\mathbf{a}$ is minimal if $\mathbf{a} > \mathbf{0}$ and there does not exist $\mathbf{b}$ such that $\mathbf{0} < \mathbf{b} < \mathbf{a}$, i.e., $(\forall \mathbf{c})(\mathbf{c} \leq \mathbf{a} \Rightarrow \mathbf{c} = \mathbf{0} \vee \mathbf{c} = \mathbf{a})$.

However, the existence of minimal degrees seem to require a bit more than finite extension where the idea there was to build an increasing sequence of strings σ_s, and then take their union $\bigcup_{s\in\omega}\sigma_s$. This can be thought of as building a decreasing sequence of open sets $T_s = \{X : X \supset \sigma_s\}$, and then taking their intersection $\bigcap_{s\in\omega} T_s$. This gives us a chance to work on more general sets like T_s.

Definition 21. A *tree* T is a function from $2^{<\omega}$ to $2^{<\omega}$ with the following properties:

(i) If $T(\sigma)$ is defined and $\tau \subset \sigma$, then $T(\tau)$ is defined and $T(\tau) \subset T(\sigma)$.

(ii) If one of $T(\sigma * 0)$ or $T(\sigma * 1)$ is defined, then both are defined and incompatible.

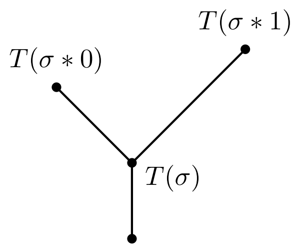

Figure 1.1: A segment of a tree.

The following terminology is standard.

Definition 22. Let T be a tree.

(i) A string σ is *in* T if it is in the range of T.

(ii) We say that a set $A \subset \omega$ *lies on* T if there exist infinitely many $\sigma \subset A$ in T.

(iii) A set A is a *branch* on T if A lies on T.

(iv) A *leaf* of T is a string $\sigma \in T$ such that $\tau \in T$ for no τ properly extending σ.

(v) T^* is called a *subtree* of T if every $\sigma \in T^*$ is also in T.

(vi) We say that T^* is the *full subtree* of T above σ if it consists of all strings on T extending σ.

We say that a tree T is *total* if it is total as a function from strings to strings. Otherwise we say that T is *partial*. We can think of subtrees for trees as extensions for finite strings. The *tree method* consists of building a decreasing sequence of $\{T_n\}_{n\in\omega}$ trees, where T_0 is the identity tree, i.e. a tree simply consisting of all strings, and such that T_{n+1} is a subtree of T_n. This method is more general because we are allowed to choose T_{n+1} to be any subtree of T_n. The following application of trees uses *recursive trees*, i.e. trees that are recursive as total recursive functions from strings to strings. We are interested in the range of a tree as in the standard terminology. We give two lemmas which are necessary for showing the existence of minimal degrees.

Lemma 1 (Diagonalization Lemma)**.** For any $e \in \omega$ and a recursive tree $T \subset 2^{<\omega}$, there is a recursive tree $Q \subset T$ such that for every A on Q, $A \neq \Psi_e$.

Proof. It is easy to see that since $T(0)$ and $T(1)$ are incompatible, at least one of them must disagree with $\Psi_e(x)$ for some $x \in \omega$. Let $T(i)$ be the one. Then we let Q be the full subtree of T above $T(i)$. □

If we want to construct a minimal degree we need to construct a set A such that

$$C \leq_T A \Rightarrow C \text{ is recursive or } A \leq_T C.$$

Definition 23. Let σ and τ be two strings. We say that σ and τ are *e-splitting* if, for some $x \in \omega$, $\Psi_e(\sigma; x)\downarrow \neq \Psi_e(\tau; x)\downarrow$. In this case, we say that σ and τ *e-split* on x.

Definition 24. A tree $T : 2^{<\omega} \to 2^{<\omega}$ is an *e-splitting tree* if any two strings in T which are incompatible are also e-splitting.

Definition 25. A tree $T : 2^{<\omega} \to 2^{<\omega}$ is an *e-nonsplitting tree* if no pair of strings in T are e-splitting.

Note that a tree which is not e-splitting does not necessarily have to be e-nonsplitting.

Lemma 2 (Spector, 1956)**.** For any $e \in \omega$, recursive tree T and A on T, if $\Psi_e(A)$ is total then:

(i) If T is an e-nonsplitting tree then $\Psi_e(A)$ is recursive.

(ii) If T is e-splitting then $A \leq_T \Psi_e(A)$.

Proof. We first prove (i). Suppose that A lies on T and that $\Psi_e(A)$ is total. Then for any given $x \in \omega$ we know that $\Psi_e(A;x)$ is defined and there must be some $\sigma \subset A$ such that $\Psi_e(\sigma;x)$ converges and gives the right value. We may suppose that σ is in T since A is on T. If there is T is e-nonsplitting then to compute $\Psi_e(A;x)$, it is enough to find any string $\tau \in T$ such that $\Psi_e(\tau;x)$ converges. Now $\Psi_e(\sigma;x)$ must be equal to $\Psi_e(\tau;x)$ since otherwise they would be an e-splitting. Hence their value must be equal to $\Psi_e(A;x)$.

For (ii), suppose that T is e-splitting. We show how to generate increasingly long segments of A recursively in $\Psi_e(A)$. Given $T(\sigma) \subset A$, since A lies on T either $T(\sigma * 0)$ or $T(\sigma * 1)$ will be included in A, and we have to decide which one of them is in A. Since T is e-splitting there exists some $x \in \omega$ such that $\Psi_e(\sigma * 0; x)$ and $\Psi_e(\sigma * 1; x)$ are defined and not equal. Then only one of them can be compatible with $\Psi_e(A;x)$, and this determines which of the two strings is included in A. □

Lemma 3 (Minimality Lemma)**.** (Spector, 1956) For any $e \in \omega$ and a recursive tree T, there is a recursive tree $Q \subset T$ such that one of the following holds:

(i) For every A on Q, if $\Psi_e(A)$ is total then $\Psi_e(A)$ is recursive.

(ii) For every A on Q, if $\Psi_e(A)$ is total then $A \leq_T \Psi_e(A)$.

Proof. We build Q with either no e-splitting on it, or as an e-splitting tree. If there is a string $\sigma \in T$ such that there is no e-splitting above σ, then Q has no e-splitting and Q is the full subtree of T above σ. If every string on T has two e-splitting extensions, then we can construct an e-splitting subtree Q of T by induction as follows:

Given $Q(\sigma)$, we let $Q(\sigma * 0)$ and $Q(\sigma * 1)$ be two e-splitting extensions of it for the first such e-splitting strings we found recursively. We will then be able to compute A from $\Psi_e(A)$ the same way as in the previous lemma. □

Now we can give the final theorem which follows from the given lemmas.

Theorem 13 (Spector, 1956)**.** There exists a minimal degree.

Proof. The requirements we need to satisfy are as follows.

$$\begin{array}{lcl} R_{2e} & : & A \neq \Psi_e \\ R_{2e+1} & : & C \leq_T A \Rightarrow C \text{ is recursive or } A \leq_T C \end{array}$$

We built a decreasing sequence of trees.

Stage 0. Let T_0 be the identity tree.

Stage $s = 2e + 1$. We let T_{2e+1} be the Q of the Diagonalization Lemma for $T = T_{2e}$.

Stage $s = 2e + 2$. We let T_{2e+2} be the Q of the Minimality Lemma for $T = T_{2e+1}$.

We then let $A = \bigcup_{s\in\omega} T_s(\emptyset)$ and so A will satisfy the requirements. □

If we analyze the proof we can see that it gives us the existence of a minimal degree below $\mathbf{0}''$. The reason is because of the question we ask in the Minimality Lemma, whether or not there is a string $\sigma \in T$ such that there is no e-splitting above σ. This uses a $\emptyset''$ oracle. Therefore, we can only assert the existence of a minimal degree below $\mathbf{0}''$. However, instead of using total recursive trees, if we are allowed to use partial trees, then we can prove that there is a minimal degree below $\mathbf{0}'$.

Theorem 14 (Sacks, 1961)**.** There exists a minimal degree below $\mathbf{0}'$.

The fact that there is minimal degree suffices to show that $(\mathbf{D}, \leq)$ is not dense. However, it is a remarkable result that this is not true for the recursively enumerable degrees.

Theorem 15 (Sacks, 1964)**.** Let $\mathbf{a}$ and $\mathbf{b}$ be two recursively enumerable degrees. Then there exists another recursively enumerable degree $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

This means that minimal degrees cannot be recursively enumerable.

1.6 Jump classes

We now know that there are degrees strictly between $\mathbf{0}$ and $\mathbf{0}'$, but how close are they to $\mathbf{0}$ or $\mathbf{0}'$? We give the jump hierarchy in this subsection. This will formalize the notion of sets being close to $\mathbf{0}$ or $\mathbf{0}'$. We know that the jump of $\mathbf{0}$ is $\mathbf{0}'$. Therefore, for degrees $\mathbf{a} \leq \mathbf{0}'$, $\mathbf{0}'$ is the least possible jump and $\mathbf{0}''$ is the greatest possible jump. We now show that, $\mathbf{0}$ is the not only degree whose jump is $\mathbf{0}'$.

Definition 26. A degree $\mathbf{a}$ is called *low* if $\mathbf{a}' = \mathbf{0}'$.

Spector [17] constructed a non-recursive Δ^0_2 low degree, hence gave the following result about the behavior of the jump operator.

Theorem 16 (Spector, 1956)**.** The jump operator is not one to one.

The next theorem, shown in [18], is known in the literature as jump inversion for Δ^0_2 degrees, and concerns the range of the Turing jump.

Theorem 17 (Friedberg, 1957)**.** If $\mathbf{b} \geq \mathbf{0}'$ then there exist a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{b}$.

A local version for this theorem for r.e. degrees is given by Shoenfield [14].

Theorem 18 (Shoenfield, 1959)**.** If $\mathbf{a} \in \Sigma^0_2$ and $\mathbf{a} \geq \mathbf{0}'$, then there exists a degree $\mathbf{b} < \mathbf{0}'$ such that $\mathbf{b}' = \mathbf{a}$.

We will see that the last two theorems are connected with the results in Chapter 3.

The question as to whether or not there exists a r.e. degree strictly between $\mathbf{0}$ and $\mathbf{0}'$ was asked by Emil Post and this was one of the oldest questions in recursion theory. This was answered positively in [19] and [20]. The fact that there is a Δ^0_2 degree strictly between $\mathbf{0}$ and $\mathbf{0}'$ does not answer the question. However, it can be shown that there are non-recursive low r.e. degrees. We give the proof because the method describes a *finite injury priority* construction.

Theorem 19 (Friedberg-Muchnik Theorem)**.** There exists a non-recursive r.e. low degree.

Proof. We construct a set A which is non-recursive r.e. and low. We define A by enumerating it as the construction progresses. This means we cannot use an oracle because the construction must be carried out effectively.

We first consider the requirements for making A non-recursive. In order to satisfy this, it is enough to ensure that $\overline{A}$ is infinite and that:

$$\mathcal{P}_e : |W_e| = \aleph_0 \Rightarrow W_e \cap A \neq \emptyset.$$

These requirements suffice because

$$A \text{ is recursive iff } A \text{ and } \overline{A} \text{ are r.e.}$$

Now we look to satisfy the lowness property. For this it is enough to consider the following requirements.

$$\mathcal{N}_e : (\exists^\infty s)\,[\Psi_e(A_s; e)\,[s]\downarrow] \Rightarrow \Psi_e(A; e)\downarrow.$$

Recall that $(\exists^\infty s)R(s)$ means "there are infinitely many s such that $R(s)$ holds". We let $A = \bigcup_{s\in\omega} A_s$, where A_s contains precisely those elements enumerated into A by the stage s. Satisfying $\mathcal{N}_e$ will give us a low degree. To see this, let g be a function which is defined in the following way: If $\Psi_e(A_s; e)\,[s]\downarrow$ then let $g(e, s) = 1$. Otherwise, let $g(e, s) = 0$. Let $g^*(e) = lim_{s\to\infty} g(e, s)$. Satisfaction of $\mathcal{N}_e$ means that this limit exists. Then g^* is the characteristic function of A'. Also note that since g is computable, g^* is computable when given an oracle for $\emptyset'$. In order to satisfy $\mathcal{N}_e$, for each e we define a *restraint function*

$$r_e(s) = u(A_s; e, e, s),$$

where $u(A_s; e, e, s)$ is the use function we defined earlier. We say that r_e gets *injured* at stage $s + 1$ if we enumerate $n < r_e(s)$ into A at this stage. One important property about this function is that if there exists a stage after which r_e is not injured, then $\mathcal{N}_e$ is satisfied and $\lim_{s\to\infty} r_e(s)$ is defined. To see this, suppose that r_e is not injured at any stage $\geq s_0$. If there is no stage $t \geq s_0$ such that $\Psi_e(A_t; e)\,[t]\downarrow$, then $\mathcal{N}_e$ will be satisfied and $\lim_{s\to\infty} r_e(s) = 0$. Otherwise, let t be the least such stage. Since we do not enumerate in a value less than $u(A_t; e, e, t)$ into A after stage t, this computation is preserved so that $\Psi_e(A;e)\downarrow$ and $r_e(s) = r_e(t)$ for all $s \geq t$. To satisfy all requirements, we give them priorities as $\mathcal{N}_0 > \mathcal{P}_0 > \mathcal{N}_1 > \mathcal{P}_1 > \cdots$, where $\mathcal{N}_0$ is the highest priority requirement. We agree that a requirement $\mathcal{P}_e$ is not allowed to injure any requirement $\mathcal{N}_i$ of a higher priority. Once $\mathcal{P}_e$ enumerates some value into A then this requirement will be satisfied so each $\mathcal{P}_e$ can enumerate only one element into A. For every $\mathcal{N}_i$ requirement, note that there are finitely many higher priority requirements $\mathcal{P}_e$. This means that after some stage $\mathcal{N}_i$ will not be injured. Therefore it will be satisfied and $\lim_{s\to\infty} r_i(s)$ will be defined. Then this means that we can satisfy each of the $\mathcal{P}_e$ requirements since W_e is infinite, then it will have a member greater than the limit values of all restraint functions of higher priority, and we can enumerate this number into A in order to satisfy the requirement. We define the construction as follows.

Stage $s = 0$. Let $A_0 = \emptyset$.

Stage $s+1$. Given A_s, we see if there exists least $i \leq s$ such that $W_{i,s} \cap A_s = \emptyset$ and

$$(*)\ \exists s\,[x \in W_{i,s} \wedge x > 2i \wedge (\forall e \leq i)\,[r_e(s) < x]],$$

where, $W_{i,s}$ is the domain of $\Psi_i\,[s]$. If there is such i, then we enumerate the least x satisfying $(*)$ into A, i.e. $A_{s+1} = A_s \cup \{x\}$. If there is no such i we let $A_{s+1} = A_s$. This ends the construction. For verification, the fact that $\overline{A}$ is infinite follows from the fact that each requirement $\mathcal{P}_e$ enumerates at most one element into A, and if it does enumerate an x into A then $x > 2e$. The fact that every requirement is satisfied follows by induction on the priority ranking of the requirements. □

Definition 27. A degree $\mathbf{a}$ is called *high* if $\mathbf{a}' = \mathbf{0}''$.

The following is due to [21].

Theorem 20 (Sacks, 1963)**.** There exists a high degree $\mathbf{a} < \mathbf{0}'$.

Definition 28. We say that a function f *dominates* a function g if $f(n) \geq g(n)$ for all but finitely many n.

A nice characterization of high degrees is given in terms of domination properties, by Martin [23].

Theorem 21 (Martin, 1966)**.** Let $\mathbf{a}$ be a degree. Then, $\mathbf{a}' \geq \mathbf{0}''$ iff there is a function recursive in $\mathbf{a}$ which dominates every recursive function.

The low and high jump classes can then be extended to give a richer hierarchy, as follows.

Definition 29. For any $n \geq 0$, we define the following:

$$\text{Low}_n = \{\mathbf{a} : \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\}.$$
$$\text{High}_n = \{\mathbf{a} : \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\}.$$

Clearly, $\text{Low}_n \subset \text{Low}_{n+1}$ and $\text{High}_n \subset \text{High}_{n+1}$. In fact, Sacks [22] proved the following.

Theorem 22 (Sacks, 1963)**.** For every n, $\text{Low}_{n+1} - \text{Low}_n \neq \emptyset$. Similarly for High_n degrees.

Chapter 2

Degrees of Peano Arithmetic and Π^0_1 Classes

A Π^0_1 class is basically an effectively closed subset in Cantor space 2^ω.[1] The study of determining the complexity of the members of Π^0_1 classes continues to be of strong interest after many years of analysis and investigation. It is also an important topic since Π^0_1 classes are closely related to recursively axiomatizable theories such as Zermelo-Fraenkel Set Theory (ZFC) or Peano Arithmetic (PA). They are also related to topology, algebra, and combinatorics such as graph theory. In this chapter we introduce Π^0_1 classes which are essential for our study. We start with giving standard definitions including an alternative characterization of Π^0_1 classes in terms of complete and consistent extensions of axiomatizable theories. We then give some basis and non-basis results that are known in the literature such as low basis theorem, hyperimmune-free basis theorem and so forth. We also introduce the degrees of (models of) Peano Arithmetic and give some relevant results related to that. We finally mention variants of Π^0_1 classes and their basic properties.

[1] Occasionally we use subsets of Baire space ω^ω but unless we explicitly state that, we will be working in Cantor space.

2.1 Cantor Space and Topology

We start by defining the Cantor space and its topology before we define Π^0_1 classes. For a detailed account of general topology, we refer the reader to [24], [25]. We now work with second order objects, i.e. sets of subsets of ω rather than just subsets of ω.

Definition 30. *Cantor space* is 2^ω with the following topology. For every $\sigma \in 2^{<\omega}$, we define the *basic open set*

$$[\![\sigma]\!] = \{A : A \in 2^\omega \ \& \ A \supset \sigma\}.$$

The *open sets* of Cantor space are unions of basic open sets. A set $A \subset 2^{<\omega}$ is an *open representation* of the open set

$$[\![A]\!] = \bigcup_{\sigma \in A} [\![\sigma]\!].$$

Definition 31. (i) We say that $\mathcal{A} \subset 2^\omega$ is *effectively open* if $\mathcal{A} = [\![A]\!]$ for an r.e. set $A \subset 2^{<\omega}$.

(ii) We let $\mathcal{A} \subset 2^\omega$ be a Σ^0_1 *class* if there exists a recursive predicate $\varphi(n, X)$ s.t. $X \in \mathcal{A} \iff \exists n \varphi(n, X)$ where n ranges over ω and X ranges over 2^ω.

We may say that a class $\mathcal{A}$ being effectively open is logically equivalent to $\mathcal{A}$ being Σ^0_1. We shall now define closed classes. It is known from general topology that a set is *closed* if its complement is open. However, we shall give the effective analogue of this using trees which will lead us defining Π^0_1 classes. In fact, Π^0_1 classes can be defined in several different ways. Hence we should not give only one.

From now on we simply define *tree* T as a set of finite strings which is downward closed, i.e. if $\sigma \in T$ and $\tau \subset \sigma$ then $\tau \in T$. Note that this definition is different than the one given in Chapter 1.

Definition 32. We say that a tree T is *recursive* if for any string σ, we can decide whether or not $\sigma \in T$.

Definition 33. (i) Let $T \subset 2^{<\omega}$ be a tree. The set of infinite paths through T is

$$[T] = \{A : \forall n(A \upharpoonright n \in T)\}.$$

(ii) A class $\mathcal{A} \subset 2^\omega$ is Π^0_1 if there exists a recursive predicate $\varphi(n, X)$ s.t. $X \in \mathcal{A} \Longleftrightarrow \forall n\varphi(n, X)$ where n ranges over ω and X ranges over 2^ω.

(iii) A class $\mathcal{A} \subset 2^\omega$ is *effectively closed* if its complement is effectively open.

Definition 34. We let $\{\Lambda_i\}_{i\in\omega}$ be an effective listing of downward closed recursive sets of strings such that for any Π^0_1 class $\mathcal{P}$ there exists i such that $\mathcal{P}$ is the set of all infinite paths through Λ_i.

Now from the definitions, it is easy to see the following.

Theorem 23. Let $\mathcal{A} \subset 2^\omega$ be a class. The following are equivalent.

(i) $\mathcal{A} = [T]$ for some recursive tree T.

(ii) $\mathcal{A}$ is effectively closed.

(iii) $\mathcal{A}$ is a Π^0_1 class.

Using trees is a convenient way of representing open and closed sets. We know that if $\mathcal{A}$ is a closed set there is a tree T such that $\mathcal{A} = [T]$. Let $A = 2^{<\omega} - T$. Now T is downward closed and we may assume that A is upward closed. Moreover, A defines the open set $[\![A]\!] = 2^\omega - [T]$ which is equal to $\overline{\mathcal{A}}$. Notice that A and T are complementary in $2^{<\omega}$. So $[\![A]\!]$ which is an open set and $[T]$, a closed set, are complementary in 2^ω.

Since we are working in Cantor space, we shall mention the compactness property of it. We are particularly interested in *König's Lemma* but compactness can also be provided by the following.

Theorem 24. (i) Let $\{T_n\}_{n\in\omega}$ be a decreasing sequence of trees such that $[T_n] \neq \emptyset$ for every $n \in \omega$, and that $T_\omega = \bigcap_{n\in\omega} T_n$. Then $[T_\omega]$ is non-empty.

(ii) Let $\{\mathcal{A}_i\}_{i\in\omega}$ be a countable sequence of closed sets such that $\bigcap_{i\in F} \mathcal{A}_i \neq \emptyset$ for every finite set $F \subset \omega$. Then, $\bigcap_{i\in\omega} \mathcal{A}_i$ is non-empty as well.

(iii) Any open cover $[\![A]\!] = 2^\omega$ has a finite open subcover $F \subset A$ such that $[\![F]\!] = 2^\omega$.

We now give König's Lemma [26].

Lemma 4 (König's Lemma)**.** If T is a finitely branching infinite tree, then T has an infinite path.

Proof. Let T be a finitely branching infinite tree. We define a set $A = \bigcup_{s\in\omega} \sigma_s$ on T by induction. We let $\sigma_0 = \emptyset$, i.e. the root of T. Given σ_s in T such that there are infinitely many extensions in T, let σ_{s+1} be an immediate successor of σ_s in T such that σ_s has infinitely many extensions in T. Now it exists because σ_s has infinitely many extensions in T, but only finitely many immediate successors since T is finitely branching. Therefore, at least one of the immediate successors must have infinitely many extensions in T. □

We shall now give some notation for trees.

Definition 35. Let $T \subset 2^{<\omega}$ be a tree.

(i) For any given $\sigma \in T$, we let T_σ be the *subtree of nodes compatible with* σ and be defined as

$$T_\sigma = \{\tau \in T : \sigma \text{ is compatible with } \tau\}.$$

(ii) A path $A \in [T]$ is said to be *isolated* if there exists a string σ such that $[T_\sigma] = \{A\}$. Otherwise A is called a *limit point*.

Note that when σ isolates A we have $[\![\sigma]\!] \cap [T] = \{A\}$, and there are no incompatible infinite extensions of σ in T.

Definition 36. We say that $\sigma \in T$ is *infinitely extendible* in T if there exists some $A \supset \sigma$ such that $A \in [T]$.

The next theorem is the effective analogue of compactness property.

Theorem 25. Let $T \subset 2^{<\omega}$ be a recursive tree.

(i) If $[T]$ is non-empty then there exists a set $A \in [T]$ such that $A \leq_T \emptyset'$.[2]

(ii) Whenever $[T]$ is non-empty, the leftmost branch of $[T]$, i.e. lexicographically least member, is of r.e. degree.[3]

[2]This is also known as Kreisel's basis theorem which will be extended by the low basis theorem.

[3]Lexicographical order means the dictionary order.

Proof. (i) We use an oracle for $\emptyset'$ to choose $A \in [T]$ such that $A = \bigcup_{n\in\omega} \sigma_n$ is defined as follows.

Let $\sigma_0 = \emptyset$. Given σ_n such that σ_n is infinitely extendible, we let σ_{n+1} be $\sigma_n * 0$ if $\sigma_n * 0$ is infinitely extendible and we let σ_{n+1} be $\sigma_n * 1$ otherwise.

(ii) Let A be the leftmost branch on T, i.e. lexicographically least member of $[T]$. Then A is Turing equivalent to B which is the set of finite strings strictly to the left of A, and which is an r.e. set. □

The following theorem gets important for further analysis when we introduce Π^0_1 classes that are countable.

Theorem 26. Let T be a recursive tree. If $A \in [T]$ is isolated, then A is recursive.

Proof. Let T be a recursive tree and let A be a path on T. Suppose that A is isolated. Then there exists a string $\sigma \subset A$ such that no path on T except A extends σ. Then, by König's Lemma, for any $n > |\sigma|$, there is a unique $\tau \supset \sigma$ such that the subtree of T above τ is infinite. So in order to compute $A \restriction n$ for $n > |\sigma|$, we find $m \geq n$ such that exactly one $\tau \supset \sigma$ of length n has an extension of length m in T. Then $A \restriction n = \tau$. □

Definition 37. A Π^0_1 class is called *special* if it does not contain any recursive member.

The next corollary is particularly an important consequence as it will be used later on. It says that Π^0_1 classes with no recursive member are quite "dense".

Corollary 4. If $\mathcal{P}$ is a special Π^0_1 class, then $\mathcal{P}$ has cardinality $2^{\aleph_0}$.

Proof. By the previous theorem, since there is no recursive member in $\mathcal{P}$, every branch splits. Therefore the number of infinite branches is $2^{\aleph_0}$. □

Corollary 5. Let $\mathcal{P}$ be a finite Π^0_1 class. Then every member of $\mathcal{P}$ is recursive.

Proof. Let T be a recursive tree such that $\mathcal{P} = [T]$ and that T has only finitely many paths. Therefore, every member of T is isolated. Hence, they are all recursive. □

2.2 Axiomatizable theories

In this section we give the link between axiomatizable theories and Π^0_1 classes. This will show the strong connection between logical theories and Π^0_1 classes.

Definition 38. (i) A *theory* is a set of sentences in the formal language of first order arithmetic closed under logical deduction.

(ii) Let T be a theory. Then T is called *consistent* if no contradiction can be derived from T, i.e. for any statement S in the language of T, $S \wedge \neg S$ is not provable from T.

(iii) Let T be a theory. Then T is called *complete* if either S or $\neg S$ is provable from T for any given sentence S in the language of T.

Definition 39. Let T be a theory and let R be a set of sentences in the language of T.

(i) We say that R is an *extension* of T if $T \subset R$.

(ii) R is a *complete (consistent)* extension if R is complete (consistent).

(iii) T is *recursively axiomatizable* if it has a recursive set of axioms.

(iv) T is *decidable* if it is recursive. Otherwise T is called *undecidable.*

A classical result [28] is that any consistent theory has a complete and consistent extension which follows from Zorn's Lemma.[4] It is also known that r.e. sets are associated with recursively axiomatizable theories. That is, every r.e. degree contains a recursively axiomatizable theory and vice versa [29]. Π^0_1 classes can be viewed as complete and consistent extensions of recursively axiomatizable theories. The following theorem is due to Shoenfield [30].

Theorem 27 (Shoenfield, 1960)**.** The set of complete and consistent extensions of a recursively axiomatizable theory is a Π^0_1 class.

Proof. Let T be a theory. By Gödel numbering, we can enumerate the set of all sentences, say $S_0, S_1, \ldots$, in the formal language of T. So any sentence

[4]Zorn's Lemma: If a partially ordered set P has the property that every totally ordered subset has an upper bound in P, then P contains a maximal element.

can be identified with its index $i \in \omega$. Then T can be represented by the set $\{i : S_i \in T\}$ and a class of sets of sentences can be represented by a class in 2^ω. We let $T \vdash_s S_i$ mean that S_i is provable from T in s steps. When we say s steps, we mean the number of derivations or the length of the proof. Note that this type of provability is a recursive relation since it is bounded. Let $\Gamma(T)$ be the class of complete and consistent extensions of T. Then $\Gamma(T)$ can be represented by the set of infinite branches through a recursive tree $Q \subset 2^{<\omega}$ which is defined in a way that σ is in Q iff the following conditions hold.

(i) For any $i < n$, if $T \vdash_s S_i$ then $\sigma(i) = 1$.

(ii) For any $i, j < n$, if $T \vdash_s S_i \Rightarrow S_j$ and $\sigma(i) = 1$, then $\sigma(j) = 1$.

(iii) For any $i, j, k < n$, if $S_k = (S_i \wedge S_j)$, $\sigma(i) = 1$ and $\sigma(j) = 1$, then $\sigma(k) = 1$.

(iv) For any $i, j < n$, if $\sigma(i) = 1$ and $S_j = \neg S_i$, then $\sigma(j) = 0$.

(v) For any $i, j < n$, if $S_j = \neg S_i$, then either $\sigma(i) = 1$ or $\sigma(j) = 1$.

Let f be an infinite branch on Q and let $\Delta = \{S_i : f(i) = 1\}$. Now the conditions (i), (ii), and (iii) ensure that Δ is a theory and the first item ensures that Δ extends T. The fourth condition ensures that Δ is consistent. The last condition ensures that Δ is a complete theory. □

The converse of this theorem is provided by Ehrenfeucht [31].

Theorem 28 (Ehrenfeucht, 1961)**.** Any Π^0_1 class can be represented as the set of complete and consistent extensions of a recursively axiomatizable theory.

Proof. We give the proof which appears in [32]. We let the language $\mathcal{L}$ consists of a countable sequence $A_0, A_1, \ldots$ of propositional variables. For any $S \in 2^\omega$, we can define a complete and consistent theory $\Delta(S)$ in the language $\mathcal{L}$ to be the set of consequences of $\{C_i\}$ such that $C_i = A_i$ if $S(i) = 1$ and $C_i = \neg A_i$ otherwise. So for any Π^0_1 class $\mathcal{P}$ we construct a theory Γ such that $\Delta(\mathcal{P}) = \{\Delta(S) : S \in \mathcal{P}\}$ is the set of complete and consistent extensions of Γ. For each string σ such that $|\sigma| = n$, we let $P_\sigma = \wedge_{i=0}^{n-1} C_i$, where $C_i = A_i$ if $\sigma(i) = 1$, and $C_i = \neg A_i$ otherwise. Let T be a given recursive tree such that $\mathcal{P} = [T]$ and we define the theory $\Gamma(T)$ consists of all sentences $P_\sigma \Rightarrow A_n$ such that $\sigma \in T$

but $\sigma * 0 \notin T$, and similarly of all sentences $P_\sigma \Rightarrow \neg A_n$ such that $\sigma \in T$ but $\sigma * 1 \notin T$, where σ is of length n.

It suffices if we show that $\Delta(\mathcal{P})$ equals the set of complete and consistent extensions of $\Gamma(T)$. Assume that S is in $\mathcal{P}$ and let $\Delta(S)$ be the set of consequences of $\{C_i\}$ for $i \in \omega$. Note that any sentence $\gamma \in \Gamma(T)$ is either of the form $P_\sigma \Rightarrow A_n$ or $P_\sigma \Rightarrow \neg A_n$ for some $\sigma \in T$ of length say n. We need to look at different cases. Suppose that $\sigma = \sigma \restriction n$ such that $\sigma \subset S$. If $\sigma * 0 \notin T$ then we know that $S(n) = 1$ so that $C_n = A_n$ is in $\Delta(S)$, hence $\Delta(S)$ proves $P_\sigma \Rightarrow A_n$. Otherwise, i.e. if $\sigma * 1 \notin T$, it must be the case that $\Delta(S)$ proves $P_\sigma \Rightarrow \neg A_n$. Therefore $\Delta(S)$ is a complete and consistent extension of $\Gamma(T)$. Now let Δ be a complete and consistent extension of $\Gamma(T)$. Then, by the definition of completeness, Δ either proves A_i or its negation for each $i \in \omega$. We let $C_i = A_i$ if Δ proves A_i and let $C_i = \neg A_i$ otherwise. Then define $S \in 2^\omega$ such that $S \in \mathcal{P}$ iff Δ proves A_i. Hence it is easy to see that Δ is equal to $\Delta(S)$. We still need to show that $S \in \mathcal{P}$. Suppose that $S \notin \mathcal{P}$. Then there is some $n \in \omega$ such that $S \restriction n \in T$ but $S \restriction n + 1 \notin T$. Then $P_\sigma = \wedge_{i=0}^{n-1} C_i$ so that Δ proves P_σ and that $P_\sigma \Rightarrow \neg C_i$ is a sentence in the theory $\Gamma(T)$ so Δ contradicts $\Gamma(T)$. □

The last theorem was modified by Jockusch and Soare [3] and they showed that the theory could be taken to be propositional.

Now from these theorems, Π^0_1 classes can be viewed as the set of complete and consistent extensions of an axiomatizable theory.

Separating Sets

It is worth giving another natural example of Π^0_1 classes in another form. The class of so called separating sets of a pair of disjoint r.e. sets is also a natural example of a Π^0_1 class.

Definition 40. Let A and B be disjoint r.e. sets. Then C is a *separating set* for A and B if $A \subset C$ and $B \cap C = \emptyset$. Let $S(A, B)$ denote the class of separating sets for A and B. If A and B have no separating set, then they are called *recursively inseparable.*

The notion of recursively inseparable sets originally was introduced by Kleene in [33]. Shoenfield proved in [34] that every nonzero r.e. degree contains a pair of recursively inseparable sets. Shoenfield again in [30] showed that for any pair of r.e. sets A and B, $S(A, B)$ is a Π^0_1 class. Notice that $S(A, \emptyset)$ is the class of supersets of A and $S(\emptyset, B)$ is the class of subsets of the Π^0_1 set $\{0,1\}^\omega - B$. It is easy to observe that $S(A, B)$ is finite iff $A \cup B$ is cofinite. So A, B, and the separating set C are all recursive in this case. Otherwise the class of separating sets for A and B is a perfect set in a sense that there are no isolated points. So the class has the cardinality $2^{\aleph_0}$. But then countably infinite Π^0_1 classes cannot be represented by any class of separating sets. So the class of separating sets of a pair of disjoint r.e. sets can be seen as a Π^0_1 class but the other way around is not true.

2.3 Basis theorems

The main motivation of this thesis is the investigation of degrees of members of Π^0_1 classes. This is usually provided by theorems which tell what kind of members are contained or not contained in Π^0_1 classes. The investigation of the degrees of members of Π^0_1 classes has been studied by many researchers but two of the most well known papers, and earliest, in this field are by Jockusch and Soare [3], [4]. In this section we give some important properties about the members of Π^0_1 classes. We start with so called *basis* theorems.

Definition 41. (i) A class $\mathcal{A}$ of sets is a *basis* for Π^0_1 classes if every non-empty Π^0_1 class has a member in $\mathcal{A}$. A set of degrees is a basis for Π^0_1 classes if the union of the set of degrees is.

(ii) Anything which is not a basis is called *non-basis*.

So a basis theorem is a theorem which asserts that every non-empty Π^0_1 class contains a member of a particular kind. A well known result is the low basis theorem, shown in [3]. This theorem extends Kreisel's basis theorem which says that every Π^0_1 class contains a Δ^0_2 set. Recall that a degree $\mathbf{a}$ is low if $\mathbf{a}' = \mathbf{0}'$.

Theorem 29. Every non-empty Π^0_1 class contains a member of low degree.

Proof. Let $\mathcal{P}$ be a non-empty Π^0_1 class such that $\mathcal{P} = [T]$ for a recursive tree T. We build a set A on T such that $A' \leq_T \emptyset'$.

We let $T_0 = T$. Given T_e, in order to decide the membership of e in A' we consider the following:

$$U_e = \{\sigma : \sigma \in T_e \text{ and } \Psi_e(\sigma; e) \uparrow\}.$$

Now U_e is a downward closed set of strings and can be finite or infinite.

If it is infinite, we let $T_{e+1} = U_e$. In this case $e \notin A'$ for any A on T_{e+1}.

If it is finite, we let $T_{e+1} = T_e$. Now in this case we have that $e \in A'$ because $\Psi_e(\sigma; e) \uparrow$ for only finitely many strings on T_{e+1}. Then the computation must converge for sufficiently large strings.

Now A is in T because $T_0 = T$, and whenever $A \in \bigcap_{e \in \omega} T_e$ we have that $A' \leq_T \emptyset'$. The reason that this is so is because we can decide on the case distinction by König's lemma and using an oracle for $\emptyset'$. □

Corollary 6. There exists a complete and consistent extension of PA which is of low degree.

Proof. The corollary follows from the fact that any axiomatizable theory, particularly PA, can be viewed as a Π^0_1 class together with the low basis theorem. □

We give another basis theorem but it is necessary to give some definitions first.

Definition 42. A set A is of *hyperimmune-free* degree if for every function f such that $f \leq_T A$, there exists a computable function g which majorizes f, i.e. $g(n) \geq f(n)$ for all $n \in \omega$.

Let us discuss what this intuitively means. If A is of hyperimmune-free degree then A has no ability to compute fast growing functions. This means that for every $f \leq_T A$ there is a recursive function g which grows at least as quickly as f. It is clear that $\mathbf{0}$ is a hyperimmune-free degree. However, the minimal degree construction given in the previous chapter can be modified to get a minimal degree which is hyperimmune-free. A way to construct hyperimmune-free degrees would be as follows. Suppose that $f = \Psi_i(A)$ for some $i \in \omega$, and if A is on a total recursive i-splitting tree then we have that $\Psi_i(\sigma; n) \downarrow$, for every σ of level $n+1$ in the tree by induction on n. Since the tree is total as a function we can compute all computations of the form $\Psi_i(\sigma; n)$ for any σ of level $n+1$. Then we can define $g(n)$ to be larger than what $\Psi_i(\sigma; n)$ gives as an output for all σ of level $n+1$. The idea of the proof of the next theorem is similar to this argument and that of low basis theorem.

Theorem 30. Every non-empty Π^0_1 class contains a member of hyperimmune-free degree.

Proof. Let $\mathcal{P}$ be a non-empty Π^0_1 class such that $\mathcal{P} = [T]$ for some recursive tree T. We construct a set A of hyperimmune-free degree such that $A \in [T]$ and that whenever $\Psi_e(A)$ is total, it is majorized by a recursive function.

We let $T_0 = T$. Given T_e, we consider the following set.

$$U_{\langle e,n\rangle} = \{\sigma : \sigma \in T_e \text{ and } \Psi_e(\sigma; n) \uparrow\}.$$

As in the previous theorem, there are two cases we need to look at. Now $U_{\langle e,n\rangle}$ is again a downward closed set of strings and can be finite or infinite.

If it is infinite for some $n \in \omega$, we let $T_{e+1} = U_{\langle e,n\rangle}$ for that n. Now in this case for any A on T_{e+1}, $\Psi_e(A)$ will be partial so the requirement is automatically satisfied.

If however $U_{\langle e,n\rangle}$ is finite for all n, then we let $T_{e+1} = T_e$. Then in this case, for any A on T_{e+1} we find a recursive function that majorizes $\Psi_e(A)$. For this, we look for a level m in a computable fashion such that $\Psi_e(\sigma; n)$ is defined for all σ of level m. We finally let $g(n)$ be greater than the value of $\Psi_e(\sigma; n)$ for all σ of level m. □

Corollary 7. There exists a complete and consistent extension of PA which of hyperimmune-free degree.

In [35], Diamondstone, Dzhafarov and Soare give a result which is a strong version of the low basis theorem.

Definition 43. A set A is *truth-table* reducible to a set B, denoted $A \leq_{tt} B$, if there is a total Turing functional Ψ_e such that $A = \Psi_e(B)$.

Definition 44. A set A is called *superlow* if $A' \leq_{tt} \emptyset'$.

The following is the *superlow basis theorem*, given in [35]. We omit the proof.

Theorem 31. If $\mathcal{P}$ is a non-empty Π^0_1 class then it contains a member of superlow degree.

In the same paper, the following result is given.

Theorem 32. Every special Π^0_1 class has a member of degree that is Low_{n+1} but not Low_n.

We said earlier that any class of degrees which does not form a basis is a non-basis. An example for a non-basis theorem, given in [4], would be the fact that the class of r.e. degrees strictly below $\mathbf{0}'$ does not form a basis. We earlier showed that every Π^0_1 class contains a member of r.e. degree. However, it does not necessarily have to contain a member of degree strictly below $\mathbf{0}'$.

Another non-basis theorem would be again by Jockusch and Soare [4] that the class of recursive sets does not form a basis since there exists a special Π^0_1 class, i.e. all members are non-recursive. It is worth giving the construction of such classes since we will use special Π^0_1 classes in the later chapters.

Theorem 33. There exists a Π^0_1 class which does not contain a recursive member.

Proof. The construction is simply by diagonalization. We recall that the notation $\Psi_e\,[s]$ denotes the computation of Ψ_e after s steps. We define T such that σ of length n is in T iff $\Psi_e(e)\,[n] \neq \sigma(e)$ for each $e < n$. Note that $[T]$ is non-empty since $A \in [T]$ for A defined as follows:

$$A(e) = \begin{cases} 1 - \Psi_e(e) & \text{if } \Psi_e(e)\downarrow \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $B \in [T]$ and any $e \in \omega$ such that Ψ_e is total, it is easy to see that $B \neq \Psi_e$. □

2.4 PA Degrees and Π^0_1 Classes

Peano arithmetic (PA), established by Giuseppe Peano [36], is a formal axiomatic system containing a set of axioms for natural number arithmetic. The theory of Peano arithmetic is of course known to be undecidable by the incompleteness theorem.

Definition 45. A degree is called *PA* if it contains a set which (computably) codes a complete and consistent extension of Peano arithmetic.

First observation is that PA degrees cannot be recursive since the theory is undecidable. We cover the properties of PA degrees in the next few chapters but now we shall give the relationship between Π^0_1 classes and PA degrees. We start with Scott Basis Theorem [37].

Theorem 34. If $\mathbf{a}$ is a PA degree then $\mathbf{D}(\leq \mathbf{a})$ forms a basis for Π^0_1 classes.

Proof. Let $\mathcal{P}$ be a Π^0_1 class such that $\mathcal{P} = [T]$ for some recursive tree T and let A codes a complete and consistent extension of PA. We compute a path $B = \bigcup_{s\in\omega} \sigma_s$ on T recursively in A. Let $\sigma_0 = \emptyset$. Given σ_s of length s, we consider all such $\tau \supset \sigma_s$ in T of length $s+1$, say τ_0 and τ_1. We then let σ_{s+1} be τ_0 if

(*) there exists some m such that τ_0 has an extension in T of length m but τ_1 does not.

Otherwise we let σ_{s+1} be τ_1.

Note that (*) is expressible in PA since the statement is Σ^0_1 and since that Peano arithmetic is Σ^0_1-complete. Now if (*) is true then is it provable in A. Let us call this true statement ψ_0. Similarly if we exchange τ_0 and τ_1 in (*), and let us call it ψ_1, assuming that ψ_1 is true, it is also provable. But since A is consistent, $\psi_0 \wedge \psi_1$ cannot be provable. So we simply choose τ_i if ψ_i is provable in A, for $i = \{0,1\}$. Note that any string which does not infinitely extend σ_s will be eliminated and the remaining path which we choose will be the one which infinitely extends σ_s. □

The converse of the previous theorem is provided by Solovay (unpublished).

Theorem 35 (Solovay, unpublished)**.** If $\mathbf{D}(\leq \mathbf{a})$ is a basis for Π^0_1 classes then $\mathbf{a}$ is a PA degree.

Proof. We give the proof due to [7] and [27]. The claim follows from the upward closure of PA degrees which we now show here. We let A be a complete and consistent extension of PA recursive in a set C. It suffices to build a tree recursively in A containing sets which code complete and consistent extensions of PA. Then the path which is determined by C has the same Turing degree of C. Let $\{\phi_n\}_{n\in\omega}$ be an effective enumeration of sentences in the language of PA. We let $A_\emptyset$ be the theory of PA. Given A_σ, we shall follow two steps.

For completeness, suppose that we are given ϕ_n such that $n = |\sigma|$. Our aim is to add either ϕ_n or its negation to A_σ preserving consistency. As in the previous proof we let ψ_0 be true if and only if there is some $m \in \omega$ which codes a proof of ϕ_n in A_σ but $\neg\phi_n$ cannot be proved by the proof coded by any $n < m$. Similarly we let ψ_1 be true if and only if there is some $m \in \omega$ which codes a proof of $\neg\phi_n$ in A_σ but ϕ_n cannot be proved by the proof coded by any $n < m$. Now since ψ_0 and ψ_1 are Σ^0_1 statements and since A is Σ^0_1-complete, ψ_i is provable in A iff ψ_i is true, for $i = \{0, 1\}$. By the consistency of A, $\psi_0 \wedge \psi_1$ cannot be provable at the same time. We recursively decide in A which one of them is suitable to be provable.

If ψ_1 is provable in A then it must be the case that ψ_0 is not provable in A_σ. Therefore $\neg\phi_n$ is consistent with it. Then we can let $A'_\sigma = A_\sigma \cup \{\neg\phi_n\}$. Otherwise we let $A'_\sigma = A_\sigma \cup \{\phi_n\}$.

Now it is a known fact by [38] and [39] that the sets of provable and refutable sentences of PA are a recursively inseparable pair of r.e. sets.[5] This also applies to A'_σ since it extends PA. So we can recursively find a sentence ψ which is neither provable nor refutable in A'_σ. Let $A_{\sigma*0} = A'_\sigma \cup \{\psi\}$ and let $A_{\sigma*1} = A'_\sigma \cup \{\neg\psi\}$. Now let $S = \bigcup_{\sigma\subset C} A_\sigma$. If $A \leq_T C$ then S is a complete extension of PA. Given S we can compute C as follows. Suppose that we have computed $C \restriction n$ and $A_{C\restriction n}$. We know whether ϕ_n or $\neg\phi_n$ is in S. Since we can know $A'_{C\restriction n}$ we can find a sentence ψ and decide which of ψ or $\neg\psi$ is in S. Then we can know $C \restriction n+1$ and $A'_{C\restriction n+1}$. Hence S and C are Turing equivalent. □

Corollary 8. PA degrees are upward closed, i.e. if $\mathbf{a}$ is a PA degree and $\mathbf{b} \geq \mathbf{a}$

[5] In fact they are effectively inseparable pair of r.e. sets. That is, there is an effective method which, given a potential index for a recursive separator S, finds a counter example to that S is a separating set, i.e. a number which demonstrates that S does not separate the two r.e. sets.

then **b** is a PA degree as well.

Proof. Follows immediately from the theorem. □

Corollary 9. Every PA degree computes a member in every Π^0_1 class.

The following result, given in [3], is sufficient to show that PA degrees cannot be minimal.

Theorem 36. If **a** is a PA degree then any countable poset is embeddable in the ordering of degrees $\leq \mathbf{a}$.

Then, so far we can say that PA degrees cannot be minimal, recursive, or incomplete r.e. We will give more about PA degrees in the following chapters.

2.5 Variants of Π^0_1 classes

Since the fifth chapter of this thesis concerns a variant of Π^0_1 classes, it it useful to give other variants of Π^0_1 classes considered so far by some researchers. Some variants include countable Π^0_1 classes, minimal and thin Π^0_1 classes.

Countable Π^0_1 classes

Countable Π^0_1 class is a Π^0_1 class whose cardinality is countable. Perhaps the most well known papers in this area are [40], [41], [42]. We showed earlier that any countable Π^0_1 class contains an isolated member and any isolated member is recursive. So every countable Π^0_1 class contains a recursive member. We also showed earlier that if $\mathcal{P}$ is a finite Π^0_1 class then every member of $\mathcal{P}$ is recursive. The study of countable Π^0_1 classes are investigated through the generalization of the notion of isolated points.

Definition 46. A set $\mathcal{A} \subset 2^\omega$ is called *perfect* if there is no $f \in \mathcal{A}$ and an open set $\mathcal{O}$ such that $\mathcal{O} \cap \mathcal{A} = \{f\}$, i.e. it has no isolated points.

The well known Cantor-Bendixson Theorem states the following.

Theorem 37. Any closed set $\mathcal{P} \subset 2^\omega$ is the union of a perfect set $\mathcal{K}$ and a countable set S.

The following definition is not the original version but it is provably equivalent.

Definition 47. A set is called *ranked* if it is a member of a countable Π^0_1 class.

For example, it was shown in [40] that for any computable ordinal α, each $\emptyset^{(\alpha)}$ is Turing equivalent to a ranked set. Cenzer and Smith [41] showed that $\emptyset'$ is not a ranked set. In the same paper it was shown that every r.e. set is Turing equivalent to an r.e. ranked set. Some negative results by the same authors were shown including that there is an r.e. set which is not ranked. Jockusch and Shore [43] showed that there exists a Σ^0_2 set which is not Turing equivalent to any ranked set. In an unpublished work of Soare, mentioned in [40], it was shown that any Δ^0_2 set is Turing equivalent to a ranked set. These results are interesting in their own right and are related to our study.

Minimal and Thin Π^0_1 classes

Another variant is minimal and thin Π^0_1 classes which were introduced in [42].

Definition 48. A Π^0_1 class $\mathcal{P} \subset \{0,1\}^\omega$ is *thin* if every Π^0_1 subclass $\mathcal{Q}$ of $\mathcal{P}$ is the intersection of $\mathcal{P}$ with some clopen set, i.e. a set which is both open and closed.

Definition 49. An infinite Π^0_1 class $\mathcal{P}$ is called *minimal* if every Π^0_1 class $\mathcal{Q} \subset \mathcal{P}$ is either finite or cofinite in $\mathcal{P}$.

A thin Π^0_1 class can be thought of as the analogue of maximality in a lattice of r.e. sets under inclusion. So thin Π^0_1 classes are strongly connected with the lattice of r.e. sets and also with recursive combinatorics. This type of classes are useful for such analysis. We shall mention some of the degree theoretic properties of members of such classes rather than their relationship with the lattice of r.e. sets. For a more lattice theoretic treatment, we refer the reader to [44]. Now any isolated member of a Π^0_1 class is recursive. A result in [42] says that any recursive member in a thin Π^0_1 class is isolated. The same authors also showed that for any ordinal $\alpha > 1$ no set of degree $\mathbf{0}^{(\alpha)}$ can be a member of a thin Π_1^0 class. Moreover, there exists an r.e. degree $\mathbf{a}$ such that no set of degree $\mathbf{a}$ is a member of a thin Π_1^0 class. The connection between minimal and thin Π_1^0 classes is also given in the same work. In [42], the authors show that if $\mathcal{P}$ is a thin Π_1^0 class and the set of isolated points is a singleton then $\mathcal{P}$ is minimal. Another result, provided in the same work, is that if $\mathcal{P}$ is minimal and contains a non-recursive member then $\mathcal{P}$ is thin.

Another variant of Π_1^0 classes, introduced by Binns [45], is small Π_1^0 classes. These classes have been investigated with respect to Medvedev and Muchnik degrees.

Chapter 3

Antibasis theorems and jump inversion

3.1 Introduction

This chapter establishes a connection between degrees of members of Π^0_1 classes and the Turing jump. The chapter contains so called *antibasis* theorems for Π^0_1 classes. We prove two antibasis theorems concerning Π^0_1 classes. The first theorem concerns the global structure of the Turing degrees, and the second concerns the degrees below $\mathbf{0}'$. We show that for any degree $\mathbf{a} \geq \mathbf{0}'$, if a Π^0_1 class $\mathcal{P}$ contains members of every degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$, then $\mathcal{P}$ contains members of every degree. A local version of this result is also given. Namely that when $\mathbf{a}$ is Σ^0_2 and $\mathbf{a} \geq \mathbf{0}'$, it suffices in the hypothesis to have a member of every Δ^0_2 degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{a}$. These theorems extend the low antibasis theorem given in [2] which is the main motivation of this chapter.

In the previous chapter we gave some basis theorems including the low basis theorem and the hyperimmune-free basis theorem. A basis theorem tells us that every non-empty Π^0_1 class contains a member of a particular kind. For example, the low basis theorem says that every non-empty Π^0_1 class contains a member of low degree $\mathbf{a}$, i.e. $\mathbf{a}' = \mathbf{0}'$. Similarly, the hyperimmune-free basis theorem

This chapter is based on my published work [1].

says that every non-empty Π^0_1 class contains a member of hyperimmune-free degree. This type of theorem is often proved by the method of forcing with Π^0_1 classes (also known as Jockusch-Soare forcing). The idea behind forcing with Π^0_1 classes is similar to forcing in set theory [46] but it is in fact simpler. In forcing with Π^0_1 classes, we successively move from a set to one of its subsets in order to *force* satisfaction of a given requirement. This is a very general technique and can be used to obtain many useful results about the members of Π^0_1 classes. A non-basis theorem gives a set of degrees which does not constitute a basis for Π^0_1 classes. For example, not every Π^0_1 class contains a recursive set, i.e. there exists a Π^0_1 class such that all members are non-recursive.

An *antibasis* theorem, on the other hand, tells us that a Π^0_1 class cannot have all/any members of a particular kind without having a member of every degree. Kent and Lewis [2] proved the low antibasis theorem which says that if a Π^0_1 class contains a member of every low degree then it contains a member of every degree. Recall that a set $\mathcal{A} \subset 2^\omega$ is *perfect* if there is no $f \in \mathcal{A}$ and an open set $\mathcal{O}$ such that $\mathcal{O} \cap \mathcal{A} = \{f\}$, or basically if $\mathcal{A}$ has no isolated points.

Definition 50. A tree is *perfect* if every infinitely extendible string in the tree has at least two incompatible extensions.

So if a tree T is perfect then $[T]$ must be uncountable and it has no isolated points. But this does not mean that it does not contain a computable member. It is worth noting here that if a Π^0_1 class $\mathcal{P}$ contains all paths through a perfect computable tree T, then it has a member of every degree. To see this, suppose that $\mathcal{P}$ contains all paths through T of this kind. Given any set B, we can then define a set $C \in [T]$ such that $C = \bigcup_{s \in \omega} \sigma_s$ which is of the same degree as of B. We define σ_0 to be the string at level 0 in T. Given σ_s, we let σ_{s+1} to be the leftmost successor of σ_s in T if $B(s) = 0$. Otherwise, define σ_{s+1} to be the rightmost successor of σ_s in T. Since there exists a Π^0_1 class which contains a member of every degree, any antibasis theorem for such classes is not expected to be "proper" in a sense that there will always be a Π^0_1 class which actually contains a member belonging to the relevant set. This is same as the problem that basis results are not proper too since a Π^0_1 class can be taken to be the empty set. However, antibasis results will get more concrete in Chapter 5 when we introduce Π^0_1 choice classes.

We begin with some definitions. The motivation comes from the notion of *invisible* degrees.

Definition 51. A degree $\mathbf{a}$ is called *invisible* if any Π^0_1 class which contains a member of degree $\mathbf{a}$ contains a member of every degree.

Definition 52. A set $T \subset 2^{<\omega}$ is said to be *dense* if for every τ there is some $\sigma \supset \tau$ in T.

Definition 53. A set A is *weakly 2-generic* if for every dense set of strings T such that T is Σ^0_2, there exists $\sigma \subset A$ such that $\sigma \in T$. A degree is weakly 2-generic if it contains a weakly 2-generic set.

Definition 54. For any $\mathcal{P} \subset 2^\omega$, define $S(\mathcal{P})$, the *degree spectrum* of $\mathcal{P}$, to be the set of all Turing degrees $\mathbf{a}$ such that there exists $A \in \mathcal{P}$ of degree $\mathbf{a}$.

Jockusch, Kent and Lewis [2] showed that if $\mathbf{a}$ is weakly 2-generic then it is invisible. We give the proof in that paper.

Theorem 38 (Jockusch, Kent and Lewis, 2010)**.** Every weakly 2-generic degree is invisible.

Proof. We try to define a string $\sigma(i, j, \tau)$ for every i, j, τ such that $\Psi_i(\sigma(i, j, \tau))$ is in Λ_j if and only if Λ_j contains a member of every degree, i.e. if $S([\Lambda_j]) = \mathbf{D}$. We start by letting T be an i-splitting set of strings enumerated in a computable fashion such that τ is the root of T, i.e. the only element of level 0, and that whenever σ is not a leaf of T, σ has exactly two successors and for any leaf σ of T there does not exist an i-splitting set of strings above σ. We suppose that we enumerate strings in T which properly extend leaves of T that are already enumerated into T. We also suppose that the strings in T are ordered first according to their level and then from left to right.

Now if there is a least string $\sigma \in T$ such that either σ is a leaf of T or else $\Psi_i(\sigma) \notin \Lambda_j$ then define $\sigma(i, j, \tau)$ to be that string. Otherwise, $\sigma(i, j, \tau)$ remains undefined.

Suppose that $[\Lambda_k]$ does not contain a member of every degree. Then for every $i \in \omega$, the set

$$T_i = \{\sigma(i, j, \tau) : \tau \in 2^{<\omega}\}$$

is dense and Σ^0_2. If A is weakly 2-generic then for every $i \in \omega$ there is some $\sigma \subset A$ such that $\sigma \in T_i$. So either $\Psi_i(A)$ is partial or computable, or $\Psi_i(A) \notin [\Lambda_j]$. Since weakly 2-generic sets are not computable the result follows immediately. □

Definition 55. Let $\mathbf{E}$ be a class of Turing degrees. We say that $\mathbf{E}$ is an *antibasis* for Π^0_1 classes if whenever a Π^0_1 class contains a member of every degree $\mathbf{a} \in \mathbf{E}$, it contains a member of every degree.

Note that every singleton containing an invisible degree is an antibasis for Π^0_1 classes. So since weakly 2-generic degrees are invisible, any singleton containing a weakly 2-generic set is an antibasis. Kent and Lewis [2] gives the following definition for the initial motivation to low antibasis theorem.

Definition 56. A set of degrees α is called a *sufficiency set* for a degree $\mathbf{a}$ if every Π^0_1 class that contains a member of every degree in α also contains a member of degree $\mathbf{a}$.

In [2], the authors argue that for every countably infinite sufficiency set α for $\mathbf{a}$ there is a proper subset β of α such that β is a sufficiency set for $\mathbf{a}$. The following theorem, so called the low antibasis theorem, suffices to show that there is some $\mathbf{a}$ and a countable set α which is a sufficiency set for $\mathbf{a}$ such that no finite subset of α is a sufficiency set for $\mathbf{a}$.

Theorem 39 (Low Antibasis Theorem)**.** If a Π^0_1 class contains a member of every low degree, then it contains a member of every degree.

Another antibasis theorem given in the same paper considers generalized low degrees. We shall give the definition for generalized low degrees first and then state the theorem.

Definition 57. For $n \geq 1$, a degree $\mathbf{a}$ is called *generalized low*$_n$ (GL_n) if $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n-1)}$. A degree $\mathbf{a}$ is called *generalized high*$_n$ (GH_n) if $\mathbf{a}^{(n)} = (\mathbf{a} \vee \mathbf{0}')^{(n)}$.

Theorem 40 (Kent and Lewis, 2010)**.** If $\mathbf{b}$ is non-GL_2 and $\mathcal{P}$ is a Π^0_1 class which does not contain a member of every degree then there exists some nonzero $\mathbf{a} \leq \mathbf{b}$ such that $\mathcal{P}$ does not contain any member of nonzero degree below $\mathbf{a}$.

We extend the low antibasis theorem to every jump level below $\mathbf{0}'$. This gives us an idea about the relationship between the degrees of members of Π^0_1 classes and the Turing jump. However, first we need to modify the definition of $\sigma(i, j, \tau)$ that we gave earlier.

3.2 Modifying $\sigma(i, j, \tau)$

The definition of $\sigma(i, j, \tau)$ given earlier was for finite strings. We need to modify this definition in order to make it work for strings with infinite domain as well since this is necessary for the second theorem. For τ which is partial computable with computable domain and for every i, j, we define $\sigma(i, j, \tau)$ as follows: We let T be an i-splitting set of strings, which is recursively enumerable (in some generic fashion) such that:

(i) all strings in T are compatible with τ;

(ii) each element which is not a leaf has precisely two immediate successors;

(iii) for any σ' which is a leaf of T there does not exist an i-splitting set of strings above σ' compatible with τ;

(iv) at each stage of the enumeration of T we only enumerate strings which properly extend leaves of the set of strings previously enumerated into T.

So roughly speaking, when τ is a finite string, T is the recursively enumerable i-splitting tree above τ. When τ has infinite domain, T is a recursively enumerable i-splitting tree in which all strings are compatible with τ.

Note. Of course the notion of string extension for infinite strings is different than that for finite strings. If θ is an infinite string with partial domain then θ' extends θ when some of the undefined bits in θ get defined in θ' and the defined bits of θ are just kept compatible with θ'.

Let the strings in T be ordered first according to their level and then from left to right. If there exists a string σ' in T such that either σ' is a leaf of T, or else $\Psi_i(\sigma') \notin \Lambda_j$ then define $\sigma(i, j, \tau)$ to be the least such string, where Λ_j is as defined earlier in the introduction part of the previous chapter. If there exists no such string then $\sigma(i, j, \tau)$ is undefined.

Definition 58. For any degree $\mathbf{a} \geq \mathbf{0}'$, let $\text{Jump}^{-1}(\mathbf{a}) = \{\mathbf{b} : \mathbf{b}' = \mathbf{a}\}$. Similarly, for any degree $\mathbf{a} \geq \mathbf{0}'$ such that $\mathbf{a}$ is recursively enumerable in $\mathbf{0}'$, let $\text{Jump}^{-1}_{\leq \mathbf{0}'}(\mathbf{a}) = \{\mathbf{b} : \mathbf{b} \leq \mathbf{0}' \text{ and } \mathbf{b}' = \mathbf{a}\}$.

So $\text{Jump}^{-1}(\mathbf{a})$ is basically the jump inversion set of $\mathbf{a}$, i.e. the set of all degrees whose jump is $\mathbf{a}$. Similarly, $\text{Jump}^{-1}_{\leq \mathbf{0}'}(\mathbf{a})$ gives those of below $\mathbf{0}'$ when $\mathbf{a}$ is recursively enumerable in and above $\mathbf{0}'$.

3.3 First theorem

The following theorem concerns the global structure of the Turing degrees.

Theorem 41. For any $\mathbf{a} \geq \mathbf{0}'$ and any Π^0_1 class $\mathcal{P}$, if $\text{Jump}^{-1}(\mathbf{a}) \subset S(\mathcal{P})$ then $\mathcal{P}$ contains a member of every degree.

Proof. Recall that, as shown earlier, if a Π^0_1 class contains all paths through a perfect computable tree, then it has a member of every degree. Given a set $A \geq_T \emptyset'$, let j be such that $[\Lambda_j] = \mathcal{P}$ does not contain a member of every degree. Let $\sigma(i,j,\tau)$ be defined as modified, for any given i, τ. Note that, since $\mathcal{P}$ does not have a member of every degree, $\sigma(i,j,\tau)$ is defined for all i, τ, since otherwise Λ_j is a superset of the perfect tree which is the set of all strings $\Psi_i(\tau')$ for $\tau' \in T$, with T as specified in the definition of $\sigma(i,j,\tau)$.

We will define $B = \bigcup_{i \in \omega} \sigma_i$ such that each σ_i is finite, which is non-recursive such that $B' \equiv_T A$ and such that if $\Psi_i(B)$ is total and non-recursive then it is not an element of $[\Lambda_j]$. Now here we do not have to consider the case that τ has infinite domain in the definition of $\sigma(i,j,\tau)$. The formal construction is as follows.

At stage $s = 0$, define $\sigma_0 = \emptyset$.

If $s = 4i+1$, define $\sigma_{4i+1} = \sigma(i,j,\sigma_{4i})$.

If $s = 4i+2$, then we see if there exists some $\sigma \supset \sigma_{4i+1}$ such that $\Psi_i(\sigma; i) \downarrow$. If so, we let $\sigma_{4i+2} = \sigma$ for smallest such σ. Otherwise we just let σ_{4i+2} be some $\sigma \supset \sigma_{4i+1}$.

If $s = 4i+3$, find the smallest $\sigma \supset \sigma_{4i+2}$ such that σ is not an initial segment of $\Psi_i(\emptyset)$. Then we let $\sigma_{4i+3} = \sigma$.

If $s = 4i + 4$, we code the i-th element of A into B simply by $\sigma_{4i+4} = \sigma_{4i+3} * \langle A(i) \rangle$.

3.3.1 Verification

Note that the first three steps are recursive in $\emptyset'$ which is recursive in A by hypothesis. The fourth step is recursive in A since we use it directly. Hence the construction is recursive in A. Since $i \in B' \iff \Psi_i(\sigma_{4i+2}; i) \downarrow$ we have $B' \leq_T A$. The construction is also recursive in $\emptyset' \oplus B$ since the action at stage $4i + 4$ simply adds one bit which can be determined by B. Then $i \in A$ if and only if $B(|\sigma_{4i+4}|) = 1$, so $A \leq_T \emptyset' \oplus B$. Since $B \oplus \emptyset' \leq_T B'$ we have $A \leq_T B'$. Also note that if $\Psi_e(B)$ is total and non-recursive then it is not an element of $[\Lambda_j]$. This is satisfied at stage $4i + 1$. □

So the first theorem basically says that for any degree $\mathbf{a} \geq \mathbf{0}'$, if a Π^0_1 class contains members of every degree whose jump is $\mathbf{a}$ then it contains members of every degree. We now prove the next theorem which concerns the degrees below $\mathbf{0}'$.

3.4 Second theorem

Now we know from the first theorem that if $\mathbf{a} \geq \mathbf{0}'$ is the jump of $\mathbf{b}$ and if a Π^0_1 class $\mathcal{P}$ contains a member of every such $\mathbf{b}$, then $\mathcal{P}$ contains a member of every degree. But the theorem does not quite say where $\mathbf{b}$ can be placed in the jump hierarchy. The next theorem considers degrees below $\mathbf{0}'$. Hence, it gives us more precisely what is sufficient for a Π^0_1 class to contain a member of every degree. We cannot however use strings with finite extensions in the second theorem since we need to work with strings having infinite domain.

Definition 59. A *coinfinite condition* is a partial function $\theta : \omega \to \omega$ with coinfinite recursive domain. A coinfinite condition is *recursive* if it is recursive as a partial function.

The proof of the following theorem uses coinfinite conditions and jump inversion theorem for r.e. degrees.

Theorem 42. For any $\mathbf{c} \geq \mathbf{0}'$ which is recursively enumerable in $\mathbf{0}'$ and any Π^0_1 class $\mathcal{P}$, if $\mathrm{Jump}^{-1}_{\leq \mathbf{0}'}(\mathbf{c}) \subset S(\mathcal{P})$ then $\mathcal{P}$ contains a member of every degree.

Proof. Given a degree $\mathbf{c} \geq \mathbf{0}'$ which is r.e. in $\mathbf{0}'$, let j be such that $[\Lambda_j] = \mathcal{P}$ does not contain a member of every degree. We aim to construct a set $A = \bigcup_{s \in \omega} \sigma_s$ by coinfinite extension such that $A \leq_T \emptyset'$ and $A' \equiv_T C$ for $C \in \mathbf{c}$ and such that $\Psi_i(A) \notin [\Lambda_j]$ for any i, if $\Psi_i(A)$ is total and non-recursive.

Let $C \in \mathbf{c}$ be r.e. in $\emptyset'$ such that $\emptyset' \leq_T C$. To satisfy $C \leq_T A'$ we want to make sure that $x \in C \iff \lim_{s \to \infty} A(\langle x, s \rangle) = 1$, so that $C \leq_T A'$ by the relativized limit lemma (see Theorem 10). Choose a one-one enumeration f of C recursive in $\emptyset'$. When a new element x appears in f, we put the x-th column of C in A with finitely many exceptions. To make sure that $A' \leq_T C$ we will prove the existence of some function g which is recursive in C such that $\Psi_e(A; e) \downarrow$ if and only if $\Psi_e(\sigma_{g(e)}; e) \downarrow$. Now we begin with the formal construction.

At stage $s = 0$ we let $\sigma_0 = \emptyset$. At each next stage,

If $s = 3i + 1$ then $\sigma_{3i+1} = \sigma(i, j, \sigma_{3i})$. Note that we can compute this value using an oracle for $\emptyset'$ since σ_{3i} is partial computable with computable domain.

If $s = 3i + 2$ then, given σ_{3i+1}, choose some $n \in \omega$ such that $\sigma_{3i+1}(n) \uparrow$. Then define

$$\sigma_{3i+2}(n) = \begin{cases} 1 - \Psi_i(\emptyset; n) & \text{if } \Psi_i(\emptyset; n) \downarrow \\ 0 & \text{otherwise.} \end{cases}$$

If $s = 3i + 3$, given σ_{3i+2}, we look for the least $e \leq 3i + 2$ such that $\Psi_e(\sigma_{3i+2}; e) \uparrow$ and such that there exists a string σ compatible with σ_{3i+2} such that $\Psi_e(\sigma; e) \downarrow$ and giving only value 0 to elements of the columns with index smaller than e, when σ_{3i+2} is not already defined on them. If e exists, then let σ be the smallest string compatible with σ_{3i+2} and then define σ_{3i+3} as follows.

$$\sigma_{3i+3}(x) = \begin{cases} \sigma_{3i+2}(x) & \text{if } \sigma_{3i+2}(x) \downarrow \\ \sigma(x) & \text{if } \sigma(x) \downarrow \\ 1 & \text{if } x = \langle f(i), z \rangle, \text{ otherwise} \\ 0 & \text{if } x = \langle n, z \rangle \wedge n \neq f(i) \wedge n, z \leq 3i + 2 \end{cases}$$

In this case we also say that g *receives attention* with respect to argument e at stage s.

If e does not exist we define σ_{3i+3} as above but we take $\sigma = \emptyset$. That is we define σ_{3i+3} in this case as

$$\sigma_{3i+3}(x) = \begin{cases} \sigma_{3i+2}(x) & \text{if } \sigma_{3i+2}(x)\downarrow \\ 1 & \text{if } x = \langle f(i), z\rangle, \text{ otherwise} \\ 0 & \text{if } x = \langle n, z\rangle \wedge n \neq f(i) \wedge n, z \leq 3i+2 \end{cases}$$

We then let $A = \bigcup_{s\in\omega} \sigma_s$. Since the construction of A is recursive in $\emptyset'$, $A \leq_T \emptyset'$ is satisfied.

3.4.1 Verification

Lemma 5. $C \leq_T A'$.

Proof. Since the columns that correspond to the elements of $\overline{C}$ are only finitely affected by the construction, the last clause in the definition of σ_{3i+3} ensures that A is total. We have that $A \leq_T \emptyset'$ by construction and $x \in C \iff \lim_{s\to\infty} A(\langle x, s\rangle) = 1$. So $C \leq_T A'$ is satisfied by the relativized limit lemma.

Lemma 6. $A' \leq_T C$.

Proof. We show how to construct the function g such that $\Psi_e(A; e)\downarrow$ if and only if $\Psi_e(\sigma_{g(e)}; e)\downarrow$. Choose s' large enough so that the elements smaller than e which are in C have been generated before stage s'. We can find such s' recursively in C. Then let $s'' \geq s' + 4e$ be congruent to 3 mod 4, and define $g(e) = s''$. Now we have $\Psi_e(A; e)\downarrow \iff \Psi_e(\sigma_{s''}; e)\downarrow$ since if $\Psi_e(\sigma_{s''}; e)\uparrow$ and $\Psi_e(\sigma; e)\downarrow$ for some extension σ of $\sigma_{s''}$ which is correctly defined on higher priority columns, then g would receive attention with respect to argument e at stage s''. □

A natural consequence of the theorem is the *high antibasis theorem* of course.

Corollary 10 (High Antibasis Theorem)**.** The class of high degrees is an antibasis for Π^0_1 classes.

A more general corollary can be given as follows.

Corollary 11. If a Π^0_1 class contains members of every degree of any non-recursive jump level below $\mathbf{0}'$, then it contains members of every degree.

Chapter 4

Join Property and effectively closed sets

In this chapter we give two new results for Π^0_1 classes. Recall that a Π^0_1 class is called *special* if it does not contain a recursive member. We first give the cupping non-basis theorem which says that there exists a special Π^0_1 class such that no join of two members computes $\emptyset'$. This gives the non-basis cupping analogue of Jockusch and Soare's capping basis theorem for Π^0_1 classes which says that every non-empty Π^0_1 class has members whose degrees form a minimal pair. The second, and the primary result for this chapter, is about the relation between the join property and members of Π^0_1 classes. We show that there exists a non-empty special Π^0_1 class such that no member satisfies the join property. As a future work, we end the chapter by giving some open questions, one on the relation between minimal covers and PA degrees, and the other on the relation between minimal covers and degrees of members of Π^0_1 classes, at the end of the chapter.

4.1 Related work

We start by giving a couple of results which can be found in [35]. First we shall give a useful module for upper cone avoidance.

Lemma 7 (Upper cone avoidance)**.** If C is a non-recursive set and T is an infinite recursive tree and $i \in \omega$, then there exists an infinite recursive subtree $T_0 \subset T$ such that $C \neq \Psi_i(A)$ for any $A \in [T_0]$. Index for T_0 can be found recursively in $\emptyset' \oplus C$ from i and an index for T.

Proof. For every $e \in \omega$, define the following set.

$$U_e = \{\sigma \in T : \Psi_i(\sigma; e) \uparrow \text{ or } \Psi_i(\sigma; e) \downarrow \neq C(e)\},$$

where the computation is bounded by the length of σ here. Now for every $e \in \omega$, U_e is a recursive tree and its index can be found recursively in C from i and from an index for T. We shall show that there exists some e such that U_e is infinite. If not, then for any $e \in \omega$ it could be possible to find a level $m \in \omega$ and some $k \in \omega$ such that $\Psi_i(\sigma; e) \downarrow = k$ for all σ of length m. But then we would have that $C(e) = k$, and C would be recursive which contradicts the hypothesis that C is non-recursive. We then find the least e such that U_e is infinite and we let T_0 be that U_e. So we have that $\Psi_i(A) \neq C$ as required for all $A \in [T_0]$. □

Corollary 12. If C is a non-recursive set, then every non-empty Π^0_1 class has a member which does not compute C.

Lemma 8 (Lower cone avoidance)**.** If C is a non-recursive set and T is an infinite recursive tree with no recursive paths and $i \in \omega$, then there exists an infinite recursive subtree $T_0 \subset T$ such that $A \neq \Psi_i(C)$ for any $A \in [T_0]$. Index for T_0 can be found recursively in C' from i and an index for T.

Proof. Now since T has no recursive paths, T must be perfect. That is every infinitely extendible string in T must have two incompatible extensions, say σ and τ in T. Let n be the smallest value smaller than the length of σ and τ such that $\sigma(n) \neq \tau(n)$. We ask recursively in C' if $\Psi_i(C; n)$ is defined. If so, then at least one of the strings, say σ, must disagree with $\Psi_i(C; n)$. We then let T_0 be all strings in T which are compatible with σ. Then we have that $A(n) = \sigma(n)$ for all $A \in [T_0]$. Hence, $\Psi_i(C) \neq A$ for all $A \in [T_0]$. Otherwise, we let $T_0 = T$ and the result follows automatically in this case. □

Corollary 13. If C is a non-recursive set, then every special non-empty Π^0_1 class has a member which is not recursive in C.

It is possible to combine upper and lower cone avoidance modules and as well as the low basis theorem to get a single basis result. The following was originally proved in [3]. We give the proof due to [35].

Theorem 43. Let $C_0, C_1, \ldots$ be a sequence of non-recursive sets, and let $D = \oplus_{j\in\omega} C_j'$. Then every special non-empty Π^0_1 class contains a member A which is Turing incomparable with each C_i and satisfies $A' \leq_T D$.

Proof. Let $\mathcal{P}$ be a non-empty special Π^0_1 class and let T be a recursive tree such that $\mathcal{P} = [T]$. We construct a sequence of recursive trees $T = T_0 \supset T_1 \supset \cdots$. We let $T = T_0$. Given T_s,

If $s = 3e$ for some $e \in \omega$, we apply the low basis theorem on T_s and let T_{s+1} be the resulting tree from there.

If $s = 3\langle i, j\rangle + 1$ for some $i, j \in \omega$, we apply Lemma 7 on T_s and on C_j. Then we let T_{s+1} be the resulting tree T_0 from the lemma.

If $s = 3\langle i, j\rangle + 2$, then we similarly apply Lemma 8.

Now since $\emptyset' \leq_T D$ and $C_j' \leq_T D$ for all j, in either case D is sufficient to find an index for T_{s+1} from an index for T_s. We finally let $A \in \bigcap_{s\in\omega} [T_s]$. Note that the first modulo stage ensures the lowness. The second modulo stage ensures that $C \not\leq_T A$, and the third one ensures that $A \not\leq_T C$. □

Corollary 14. Let $C_0, C_1, \ldots$ be a sequence of non-recursive low sets. Then every non-empty special Π^0_1 class contains a member that is low and Turing incomparable with each C_i.

Definition 60. Let **a** and **b** be two Turing degrees. Then we say that **a** and **b** form a *minimal pair* if they are non-recursive and their greatest lower bound is **0**, i.e. $\forall \mathbf{c}(\mathbf{c} \leq \mathbf{a} \wedge \mathbf{c} \leq \mathbf{b} \Rightarrow \mathbf{c} = \mathbf{0})$.

Lemma 9 (Minimal pair basis)**.** If C is a set and T is an infinite recursive tree, and $i, j \in \omega$, then there exists an infinite recursive subtree $T_0 \subset T$ such that if $\Psi_i(A) = \Psi_j(C) = B$ for some $A \in [T_0]$ and some set B then B is recursive.

Proof. Given T, we ask if there exist strings $\sigma, \tau \in T$ which are infinitely extendible and an $x \in \omega$ such that $\Psi_i(\sigma; x) \downarrow \neq \Psi_i(\tau; x) \downarrow$. We can do this using an oracle for $\emptyset''$ because it is easy to see the question that if a given string in a recursive tree is infinitely extendible can be expressed by a Π^0_1 statement.

If there are no such strings, then whenever $\Psi_i(A)$ is total for some $A \in [T]$ it must be recursive. To compute the value of $\Psi_i(A; x)$ we proceed as follows. We first see if $\Psi_i(A)$ is partial on a non-empty subclass of $[T]$. Note that this is a $\emptyset''$-question. If there is such subclass then the theorem automatically holds. Otherwise, $\Psi_i(A)$ is total on $[T]$ and there are two subcases:

i) There is an i-splitting on $[T]$.

ii) There is no such splitting.

In case (i), we choose the string σ such that $\Psi_i(\sigma; x)$ is defined and agrees with $\Psi_i(A; x)$. In case (ii), we find a level on T such that all strings $\sigma \in T$ at this level yield the same output value, on argument x, which gives the value of $\Psi_i(A; x)$. We then we let T_0 be the full subtree above σ.

Now suppose that such σ and τ do exist. We fix the least such and we use an oracle for C' to see whether or not $\Psi_j(C; x)$ converges. If not, the lemma holds automatically and we can let $T_0 = T$. If it converges, then one of the output of two computations, i.e. $\Psi_i(\sigma; x)$ and $\Psi_i(\tau; x)$, must be different from that of $\Psi_j(C; x)$. If $\Psi_i(\sigma; x)$ is the different one, we let T_0 be the set of all strings in T compatible with σ. Otherwise we let T_0 be the set of all strings in T compatible with τ. Hence, we have $\Psi_i(A) \neq \Psi_j(C)$ for any $A \in [T_0]$. □

Theorem 44. For any non-recursive set C of degree $\mathbf{c}$, any non-empty special Π^0_1 class contains a member B of degree $\mathbf{b}$ such that $\mathbf{b}$ and $\mathbf{c}$ form a minimal pair, i.e. $\mathbf{b} \wedge \mathbf{c} = \mathbf{0}$.

Proof. Let $\mathcal{P}$ be a non-empty special Π^0_1 class such that $\mathcal{P} = [T]$ for some recursive tree T. We construct a sequence of recursive trees $T = T_0 \supset T_1 \supset \cdots$. We first let $T = T_0$. Suppose that we are given T_s, and let $s = \langle i, j \rangle$ for some $i, j \in \omega$. We apply the previous lemma on T_s and we let T_{s+1} be the resulting tree obtained from there. Then we let $B \in \bigcap_{s \in \omega} [T_s]$. Clearly, $A = \Psi_i(B) = \Psi_j(C)$ for some A if A is recursive. Hence, $\mathbf{b}$ and $\mathbf{c}$ form a minimal pair. □

An interesting result given in [47] which we omit the proof is the following.

Theorem 45 (Jockusch and Soare, 1971)**.** There exist special Π^0_1 classes $\mathcal{P}$ and $\mathcal{Q}$ such that for any $A \in \mathcal{P}$ of degree $\mathbf{a}$ and $B \in \mathcal{Q}$ of degree $\mathbf{b}$, $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

Another result related with the connection between minimal degrees and Π_1^0 classes is given by Groszek and Slaman [48].

Theorem 46 (Groszek and Slaman, 1997)**.** There exists a non-empty Π_1^0 class such that every member computes a minimal degree.

4.2 Cupping Non-basis theorem

Jockusch and Soare's capping basis theorem [3] which we gave earlier says that any non-empty special Π_1^0 class contains members of degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$. We now give the proof of the cupping non-basis analogue of this theorem. In fact, we show something more stronger.

Theorem 47. There exists a non-empty special Π_1^0 class $\mathcal{P}$ such that $\emptyset' \not\leq_T A \oplus B$ for any $A \in \mathcal{P}$, $B \in \mathcal{P}$.

Proof. We aim to satisfy $\emptyset' \not\leq_T A \oplus B$ by constructing a set T such that $\mathcal{P} = [T]$ and an r.e. set D such that $D \not\leq_T A \oplus B$.

The requirements are:

R_{2e+1} : If $S \in \mathcal{P}$ then $S \neq \Psi_e(\emptyset)$
R_{2e+2} : If $A \in \mathcal{P}$ and $B \in \mathcal{P}$ then $\Psi_e(A \oplus B) \neq D$.

At stage $s = 0$, enumerate $\emptyset$ into T.
At stage $s > 0$,

(i) Find the least string $\tau \in T$ such that τ is of level $2e+1$ and $\tau \subset \Psi_e(\emptyset)\,[s]$. Let $\tau_0 \in T$ be the immediate predecessor of τ and let τ_1 be a leaf of T extending τ_0 and incompatible with τ. Stop enumerating any strings extending τ_0 in T, then enumerate two incompatible extensions of τ_1 into T.

(ii) If the enumerated strings are at even level, we consider all pairs $\{\sigma_0, \tau_1\}$ of strings of that level in T such that one of σ_0 or τ_1 is in $\{\sigma_0, \tau_1\}$. For each such pair fix some value $n \in \omega$ not yet enumerated in D. Find the least $\tau \in T$, $\sigma \in T$ of level $2e+2$ such that there exists $\sigma' \supset \sigma$ and $\tau' \supset \tau$ such that $\Psi_e(\sigma' \oplus \tau'; n) \downarrow = 0$ and $D(n) = 0$, for that fixed n. If they exist, enumerate n into D. Then stop enumerating any extensions of σ and τ in T, and then enumerate σ' and τ', or some extensions of them, into T.

After these instructions, choose two incompatible strings σ, τ extending each leaf of T, and enumerate these strings into T. This ends the construction.

4.2.1 Verification

We shall first show that $[T]$ is a Π^0_1 class. For this we need to show explicitly that there exists a downward closed computable set of strings Λ such that $[T] = [\Lambda]$. We let Λ be the set of all strings which are initial segments of strings in T at any stage. We next show that Λ is downward closed, computable and $[\Lambda] = [T]$. Now Λ is computable since we enumerate in strings that only extend strings in Λ of the previous stage. Clearly, every infinitely extendible string in T is also in Λ by the defintion of Λ. The opposite direction is also true. By contrapositive, suppose that σ is not infinitely extendible in Λ. Then σ must be a leaf of T in which case σ is not infinitely extendible in T since otherwise σ would be infinitely extendible in Λ.

Lemma 10. R_{2e+1} is satisfied.

Proof. Suppose that $S \in [T]$ and $S = \Psi_e(\emptyset)$ for some e. Then for all $\sigma \subset S$, where $\sigma \subset \Psi_e(\emptyset)$, we have $\sigma \in T$. Let σ_0 be the immediate predecessor of σ. Then any extensions of σ_0, compatible with σ, are not enumerated into T. But then, $\sigma \subset \Psi_e(\emptyset)$ is in T for finitely many σ's. A contradiction.

Lemma 11. R_{2e+2} is satisfied.

Proof. Suppose the contrary that there exist $A \in [T]$ and $B \in [T]$ such that $\Psi_e(A \oplus B) = D$ for some e. Then there are $\sigma \subset A$ and $\tau \subset B$ in T, and there exist $\sigma' \supset \sigma$, $\tau' \supset \tau$ such that $\Psi_e(\sigma' \oplus \tau'; n) = D(n)$ which, according to our construction, is a contradiction. This proves the theorem.

□

Corollary 15. There exists a special non-empty Π^0_1 class $\mathcal{P}$ such that $\mathbf{a} \vee \mathbf{b} \neq \mathbf{0}'$ for any two members of $\mathcal{P}$ of degrees $\mathbf{a}$ and $\mathbf{b}$.

Corollary 16. There exists a special non-empty Π^0_1 class $\mathcal{P}$ in which there exist members of degrees $\mathbf{a}$ and $\mathbf{b}$ such that $(\mathbf{a}, \mathbf{b})$ forms a minimal pair and that $\mathbf{a} \vee \mathbf{b} \neq \mathbf{0}'$.

Proof. Follows from the cupping non-basis theorem and Theorem 44. □

4.3 Join property and Π^0_1 classes

In this section we establish a connection between the join property and members of Π^0_1 classes. This will be the primary result for this chapter.

Definition 61. A degree $\mathbf{a}$ satisfies the *join property* if for all non-zero $\mathbf{b} < \mathbf{a}$ there exists $\mathbf{c} < \mathbf{a}$ such that $\mathbf{b} \vee \mathbf{c} = \mathbf{a}$.

Theorem 48. There exists a non-empty special Π^0_1 class such that no member satisfies the join property.

Proof. We construct a non-empty special Π^0_1 class $\mathcal{P}$ and we define a functional φ such that for any $A \in \mathcal{P}$ we satisfy the following requirements.

$P_i : \varphi(A) \neq \Psi_i(\emptyset)$

$R_{\langle i,j \rangle} : \Psi_j(\Psi_i(A) \oplus \varphi(A)) = A \Longrightarrow \Theta_{i,j}(\Psi_i(A)) = A$

for some functional $\Theta_{i,j}$ we aim to construct for each given $i, j \in \omega$.

Before writing the construction let us give the idea of the proof. The construction has stages at which we act to satisfy a requirement and each stage is aimed to satisfy one or more desired properties. We will place modules on strings. At any stage of the construction, we refer to σ as a *leaf* if σ has a module placed on it, and no proper extension of σ has modules placed on it.

The Π^0_1 class $\mathcal{P}$ is defined via its complement, i.e. $A \notin \mathcal{P}$ iff there exists some stage at which A does not extend a leaf. So we shall start with a single module placed on $\emptyset$. During the construction, modules placed on leaves will place further modules.

Now consider the module α placed on σ. We have to decide which requirements are *active* at α at any given point of the construction.

For the P_i requirements this is easy. The *level* of α is the number of proper initial segments of σ on which modules are placed. At any given point, the P_i requirement active at α is P_n, where n is the level of σ.

For the $R_{\langle i,j\rangle}$ requirements a little more work is needed. Consider the module α placed on σ, and let σ_0, σ_1 be the 'successors' of σ on which modules are placed such that σ_0 and σ_1 are incompatible. Roughly speaking if $R_{\langle i,j\rangle}$ is active at α then the module α will search for both

(i) an extension $\sigma_1^* \supset \sigma_1$ on which a module is placed such that

$$\sigma_1 \subset \Psi_j(\Psi_i(\sigma_1^*) \oplus \varphi(\sigma_1^*))$$

(ii) an extension $\sigma_0^* \supset \sigma_0$ on which a module is placed such that

$$\sigma_0 \subset \Psi_j(\Psi_i(\sigma_0^*) \oplus \varphi(\sigma_0^*)).$$

When (i) occurs we shall say that α is *complete* for all triples (i, j, σ') such that $\sigma' \supset \sigma_1$. Similarly, when (ii) occurs we shall say that α is complete for all triples (i, j, σ') such that $\sigma' \supset \sigma_0$. If a module is complete then we will be putting a φ splitting there which is what the P_i requirements are looking for. Until then $R_{\langle i,j\rangle}$ requirements will be active. This pattern will continue in the general picture.

We decide whether or not the pair (i, j) requires attention at α, placed on σ, as follows. The pair (i, j) *requires attention* at α unless there exists β placed on a proper initial segment σ' of σ such that $R_{\langle i,j\rangle}$ is active at β but β is not complete for (i, j, σ).

Now we can specify which $R_{\langle i,j\rangle}$ requirements are active at α. We will denote the set of $R_{\langle i,j\rangle}$ requirements active at α placed on a string σ by Π_α. It will be determined with their indices. For example, if $\Pi_\alpha = \{(m, n)\}$ then only $R_{\langle m,n\rangle}$ active at α.

If $\sigma = \emptyset$ then $\Pi_\alpha = \{R_{\langle 0,0\rangle}\}$.

Suppose that $\sigma \neq \emptyset$ and is of level $n > 0$. Let σ' be the initial segment of σ on which a module β is placed of level $n - 1$. If β is complete for all (i, j, σ) such that $(i, j) \in \Pi_\beta$, then

$$\Pi_\alpha = \Pi_\beta \cup \{(i, j)\}$$

where (i, j) is the least pair not in Π_β which requires attention at α. Otherwise Π_α is the set of all $(i, j) \in \Pi_\beta$ which require attention at α.

For a given module, there will be a finite set of requirements which are active at any given point. At each stage, the module performs the instructions for all of these (in order of priority).

We have to be careful about one case. Suppose that for some distinct $A, B \in \mathcal{P}$, we have $\Psi_j(\Psi_i(A) \oplus \varphi(A)) = A$ and $\Psi_j(\Psi_i(B) \oplus \varphi(B)) = B$. Then, if $\Psi_i(A) = \Psi_i(B)$ we have a problem since $\Theta_{i,j}$ cannot be asked to map compatible strings to incompatible values. In order to avoid this problematic situation, we shall proceed roughly as follows. Suppose the $R_{\langle i,j\rangle}$ requirement is working above σ, i.e. a module is placed on σ at which $R_{\langle i,j\rangle}$ is active. Let σ_0 and σ_1 be two incompatible successors of σ on which modules are placed. We shall ensure at all later stages, that for each $\sigma_1^* \supset \sigma_1$ there exists $\sigma_0^* \supset \sigma_0$ such that $\varphi(\sigma_0^*)$ and $\varphi(\sigma_1^*)$ are compatible. Similarly, for each $\sigma_0^* \supset \sigma_0$ we shall ensure that there exists $\sigma_1^* \supset \sigma_1$ such that $\varphi(\sigma_1^*)$ and $\varphi(\sigma_0^*)$ are compatible. Now, if we find $\sigma_1^* \supset \sigma_1$, for example, with $\Psi_j(\Psi_i(\sigma_1^*) \oplus \varphi(\sigma_1^*)) \supset \sigma_1$ we can take $\sigma_0^* \supset \sigma_0$ such that $\varphi(\sigma_0^*)$ and $\varphi(\sigma_1^*)$ are compatible. We can then enumerate axiom such that $\varphi(\sigma_0^*) = \varphi(\sigma_1^*)$ and remove all extensions of σ_0 and σ_1 except σ_0^* and σ_1^*. Now we can then enumerate the axiom $\Theta_{i,j}(\Psi_i(\sigma_1^*)) \supset \sigma_1$. Since $\varphi(\sigma_0^*) = \varphi(\sigma_1^*)$ and since σ_0 and σ_1 are incompatible, if we subsequently find some $\sigma_0^+ \supset \sigma_0^*$ such that $\Psi_j(\Psi_i(\sigma_0^+) \oplus \varphi(\sigma_0^+)) \supset \sigma_0$ then $\Psi_i(\sigma_0^+)$ and $\Psi_i(\sigma_1^*)$ must be incompatible. So we can now enumerate the axiom $\Theta_{i,j}(\Psi_i(\sigma_0^+)) \supset \sigma_0$.

We maintain a downward closed set of strings Φ which contains possible φ values. We define the φ values of the form $\varphi(\sigma) \supset \tau$. At any given stage, $\varphi(\sigma)$ is the longest τ for which we have enumerated some axiom $\varphi(\sigma') \supset \tau$ with $\sigma' \subset \sigma$ (or $\varphi(\sigma) = \emptyset$ if we have enumerated no such axioms). If want to satisfy the $R_{\langle i,j\rangle}$ requirements, as long as $\Psi_i(\sigma) \oplus \varphi(\sigma)$ seems to be computing σ, via Ψ_j, we also have enumerate axioms for $\Theta_{i,j}$. When we enumerate an axiom of the form $\Theta_{i,j}(\sigma) = \tau$, we ensure the following:

(i) $\Psi_j(\Psi_i(\sigma) \oplus \varphi(\sigma))$ already maps to an initial segment of $A \supset \sigma$ of length longer than n, where n is the least value such that $\Theta_{i,j}(\sigma; n) \uparrow$.

(ii) Internal consistency of the axioms. That is, we do not want to enumerate an axiom where $\Theta_{i,j}(\sigma; n) = k$ holds when there is already some

τ compatible with σ for which we have enumerated an axiom where $\Theta_{i,j}(\tau; n) = 1 - k$ holds.

Instructions for P_i requirements at α, placed on σ:

If $\Pi_\alpha = \emptyset$ and α does not have successor modules then we follow the instructions in (1). Otherwise we follow the instructions in (2). We initially let $\Phi = \emptyset$.

(1) We check to see whether there exist two incompatible strings $\tau_0, \tau_1 \in \Phi$ extending $\varphi(\sigma)$.

If so, choose such τ_0, τ_1 of shortest possible length, choose σ_0, σ_1 extending σ such that σ_0 and σ_1 are incompatible, place modules on σ_0 and σ_1 and then enumerate the axioms

$$\varphi(\sigma_0) \supset \tau_0,\ \varphi(\sigma_1) \supset \tau_1.$$

If not, let $\tau' \supset \varphi(\sigma)$ be the longest extension of $\varphi(\sigma)$ in Φ. Choose two incompatible strings σ_0 and σ_1 extending σ, place modules on σ_0 and σ_1, and enumerate the axioms

$$\varphi(\sigma_0) \supset \tau' * 0,\ \varphi(\sigma_1) \supset \tau' * 1.$$

Also, enumerate $\tau' * 0$ and $\tau' * 1$ into Φ.

(2) Unless already declared successful at α, the strategy searches for strings σ_0, σ_1 extending σ on which modules are placed such that $\varphi(\sigma_0)$ and $\varphi(\sigma_1)$ are incompatible and either

(a) $\varphi(\sigma_0) \subset \Psi_i(\emptyset)$ or

(b) $\varphi(\sigma_1) \subset \Psi_i(\emptyset)$

If the module, for instance, finds that (a) occurs (we follow similar instructions with roles exchanged when (b) occurs) then it

(i) declares itself successful at α,

(ii) chooses $\sigma_1' \supset \sigma_1$ which is a leaf,

(iii) removes all strings from Φ which extend $\varphi(\sigma)$ and are incompatible with $\varphi(\sigma_1')$,

(iv) runs the 'Φ-adjustment' procedure with argument (σ, σ_1').

Instructions for $R_{\langle i,j\rangle}$ requirements at α, placed on σ.

The module is initially in *state 0.*

If α does not have successors, then choose two incompatible strings σ_0 and σ_1 extending σ, and place modules on σ_0 and σ_1.

Let the successors of σ be σ_0, σ_1 such that they are incompatible. While in state 0 the strategy searches for either

(a) $\sigma_0^* \supset \sigma_0$ such that

$$\Psi_j(\Psi_i(\sigma_0^*) \oplus \varphi(\sigma_0^*)) \supset \sigma_0$$

or

(b) $\sigma_1^* \supset \sigma_1$ such that

$$\Psi_j(\Psi_i(\sigma_1^*) \oplus \varphi(\sigma_1^*)) \supset \sigma_1.$$

If it finds that (a) holds then we perform the following instructions (the instructions for (b) are similar).

(i) Let $\sigma_1^+ \supset \sigma_1^*$ and $\sigma_0^+ \supset \sigma_0^*$ be leaves such that $\varphi(\sigma_1^+)$ is compatible with $\varphi(\sigma_0^+)$. Note that we shall prove in the verification that the Φ-adjustment procedure guarantees such strings indeed exist.

(ii) Enumerate axioms so that $\varphi(\sigma_1^+) = \varphi(\sigma_0^+)$.

(iii) Remove all modules from proper extensions of σ.

(iv) Place modules on σ_0^+ and σ_1^+.

(v) If α is complete for all $(i, j) \in \Pi_\alpha$ then

(v-a) We check to see whether there exist incompatible extensions of $\varphi(\sigma_0^+)$ in Φ. If so, let τ_0 and τ_1 be shortest such strings. Otherwise let τ be the longest extension of $\varphi(\sigma_0^+)$ in Φ. Then define $\tau_0 := \tau * 0$, $\tau_1 := \tau * 1$, and enumerate these strings into Φ.

(v-b) Let τ_0 and τ_1 be as above. Enumerate axioms $\varphi(\sigma_1^+) \supset \tau_1$, $\varphi(\sigma_0^+) \supset \tau_0$.

(vi) Enumerate the axiom $\Theta_{i,j}(\Psi_i(\sigma_0^+)) \supset \sigma_0$.

(vii) Run the 'Φ-adjustment' procedure with argument (σ, σ_0^+).

(viii) Declare the requirement complete with respect to (i, j) and complete with respect to all triples (i, j, σ') such that $\sigma' \supset \sigma_0$.

(ix) Declare the module to be in *state 1*.

While in state 1, let $k \in \{0, 1\}$ be such that the requirement is not complete with respect to (i, j, σ_k). We search for $\sigma_k^* \supset \sigma_k$ such that $\Psi_j(\Psi_i(\sigma_k^*) \oplus \varphi(\sigma_k^*)) \supset \sigma_k$.

If σ_k^* is found then choose $\sigma_k^+ \supset \sigma_k^*$ which is a leaf, remove all modules from extensions of σ_k, place a module on σ_k^+. Enumerate the axiom $\Theta_{i,j}(\Psi_i(\sigma_k^+)) = \sigma_k$. Then we run the '$\Phi$-adjustment' procedure with argument (σ, σ_k^+). Declare *state 2*.

In state 2, we do nothing.

Subroutine for Φ-adjustment procedure with argument (σ, σ'):

Remove all strings from Φ which are compatible with $\varphi(\sigma)$ and incompatible with $\varphi(\sigma')$. We also remove any module α placed on a string τ such that $\varphi(\tau)$ has just been removed from Φ.

If α placed on σ'' now has precisely one successor σ''' then let $\sigma^{(iv)} \supset \sigma'''$ be a leaf. Remove all modules from proper extensions of σ'', place modules on $\sigma^{(iv)} * 0$ and $\sigma^{(iv)} * 1$.

If Π_α (α at σ) is empty or if α is complete for all $(i, j) \in \Pi_\alpha$ then we check to see if there exist incompatible extensions of $\varphi(\sigma^{(iv)})$ in Φ. If so, let τ_0 and τ_1 be shortest such strings. Otherwise, let τ be the longest extension of $\varphi(\sigma^{(iv)})$ in Φ, and define $\tau_0 := \tau * 0$, $\tau_1 := \tau * 1$, and then enumerate τ_0 and τ_1 into Φ. Also enumerate axioms

$$\varphi(\sigma^{(iv)} * 0) = \tau_0,$$
$$\varphi(\sigma^{(iv)} * 1) = \tau_1.$$

4.3.1 Verification

First we shall explain why Φ-adjustment procedure avoids us having $\Psi_i(A) = \Psi_i(B)$ when $\varphi(A) = \varphi(B)$ for $i \in \omega$, and $A, B \in \mathcal{P}$ such that $A \neq B$. Suppose that a module is placed on some σ at which $R_{\langle i,j \rangle}$ is active. Let $\sigma_0 \subset A$ and $\sigma_1 \subset B$ be two incompatible successors of σ on which modules are placed. Suppose that the module for $R_{\langle i,j \rangle}$ requirements finds, say $\sigma_0^* \supset \sigma_0$. We choose $\sigma_1^+ \supset \sigma_1^*$ and $\sigma_0^+ \supset \sigma_0^*$ so that $\varphi(\sigma_1^+)$ and $\varphi(\sigma_0^+)$ are the same. Suppose that we later find, for example, $\tau \supset \sigma_1^+$ for which $\Psi_j(\Psi_i(\tau) \oplus \varphi(\tau)) \supset \sigma_1$. Now the reason $\Psi_i(\tau)$ must be incompatible with $\Psi_i(\sigma_0^*)$ is because σ_0 is incompatible with σ_1, and since we have $\Psi_j(\Psi_i(\sigma_0^*) \oplus \varphi(\sigma_0^*)) \supset \sigma_0$ and that $\varphi(\tau)$ is compatible with $\varphi(\sigma_0^*)$ by the Φ-adjustment procedure, if the φ parts are compatible, then the Ψ_i parts cannot be compatible since this would contradict the monotonicty of Turing functionals that they cannot map compatible strings to incompatible values. We shall argue more about why Φ-adjustment procedure ensures that σ_1^+ and σ_0^+ do exist with compatible φ values. This is given by the following lemma. We also argue that φ is total.

Lemma 12. The Φ-adjustment procedure ensures the existence of strings σ_0^+ and σ_1^+ with compatible φ values. Moreover, φ is a total functional.

Proof. For the totality of φ, note that if σ is a leaf, then the Φ-adjustment procedure will always define the φ values of some two incompatible extensions of σ. Then we are able to find incompatible strings extending each leaf where their φ values are defined. This is ensured by the last paragraph of the subroutine.

Next we show the existence of σ_0^+ and σ_1^+ with compatible φ values. We prove by induction. In the base case, we take the empty string and we define the φ values of two incompatible extensions of the empty string to be equal to each other. Now suppose that σ is a string on which a module is placed and let σ_0 and σ_1 be two incompatible extensions of σ such that for any $\sigma_0^+ \supset \sigma_0$ there exists some $\sigma_1^+ \supset \sigma_1$, and vice versa, such that $\varphi(\sigma_0^+)$ and $\varphi(\sigma_1^+)$ are compatible. It is ensured then by the first sentence of the subroutine followed by the last part where we define the φ values that when Φ is non-empty and whenever we leave a string, it will be an extension of σ_1^+ for which there exists some extension of σ_0^+, and vice versa, such that we define compatible φ values of these strings in Φ. We also want to make sure that every module has at least two successors. This is guaranteed by the second paragraph of the subroutine when we define $\sigma^{(iv)}$. □

Lemma 13. Axioms enumerated for $\Theta_{i,j}$ are consistent for any $i, j \in \omega$.

Proof. Suppose that we enumerated an axiom $\Theta_{i,j}(\Psi_i(\sigma)) \supset \tau$ such that $\sigma \supset \tau$. First note that, for monotonicity, if $\sigma' \supset \sigma$ then $\Theta_{i,j}(\Psi_i(\sigma'))$ will extend $\Theta_{i,j}(\Psi_i(\sigma))$ since $\Psi_i(\sigma') \supset \Psi_i(\sigma)$. For consistency we first need to show that $\Psi_j(\Psi_i(\sigma) \oplus \varphi(\sigma))$ already maps to an initial segment of $A \supset \sigma$ of length longer than n, where n is the least value such that $\Theta_{i,j}(\Psi_i(\sigma); n) \uparrow$. This is satisfied by (a) and (b) in the instructions for the $R_{\langle i,j \rangle}$ requirement. Next we need to ensure that if σ and σ' are two incompatible strings and if $\Psi_i(\sigma)$ is compatible with $\Psi_i(\sigma')$, then we make sure not to enumerate $\Theta_{i,j}(\Psi_i(\sigma)) = \tau$ and $\Theta_{i,j}(\Psi_i(\sigma')) = \tau'$ such that τ and τ' are incompatible. This is guaranteed by having $\varphi(\sigma) = \varphi(\sigma')$. Since we make sure that φ parts are compatible, the same argument given in the beginning of verification suffices to show that we do not map compatible strings to incompatible values. Hence, internal consistency is preserved. □

Lemma 14. For every $i, j \in \omega$, P_i and $R_{\langle i,j \rangle}$ requirements are satisfied.

Proof. For P_i requirements, this is ensured by steps (ii) and (iii) in the instructions for P_i. For $R_{\langle i,j \rangle}$ requirements, this follows from the lemmas. □

4.4 Future work

We finish this chapter by giving some open questions. We first consider the connection between PA degrees and minimal covers. First we shall give some important properties of PA degrees. The following theorem, for which we omit the proof, gives us a nice relation between Π^0_1 classes and PA degrees [35].

Theorem 49. There exists a Π^0_1 class such that every member is of PA degree.

This theorem has some consequences. By the low basis theorem, we now can say that there exists a PA degree which is low. Similarly, by the hyperimmune-free basis theorem, there exists a PA degree which is hyperimmune-free. It is also known that each PA degree strictly bounds another PA degree and that PA degrees are upward closed. So clearly every degree above $\mathbf{0}'$ is PA.

Definition 62. A degree $\mathbf{a}$ satisfies the *cupping property* if for any degree $\mathbf{b} > \mathbf{a}$ there exists some $\mathbf{c} < \mathbf{b}$ such that $\mathbf{a} \vee \mathbf{c} = \mathbf{b}$.

The following theorem was proven in [50]

Theorem 50 (Kučera, 1985)**.** Every PA degree satisfies the cupping property.

A different classification of PA degrees is provided by $\{0, 1\}$-valued diagonally non-recursive functions. We shall give the definition and some properties.

Definition 63. A function $f : \omega \to \omega$ is *fixed point free (FPF)* if $\Psi_e \neq \Psi_{f(e)}$ for all $e \in \omega$. A degree is FPF if it contains an FPF function.

Definition 64. A function $f : \omega \to \omega$ is called *diagonally non-recursive (DNR)* if $f(e) \neq \Psi_e(e) \downarrow$ for every $e \in \omega$. A degree is DNR if it contains a DNR function.

Definition 65. A function f is said to be n-valued if $f(e) < n$ for all $e \in \omega$.

It is known that a degree is DNR iff it is FPF [49]. We are mainly interested in $\{0, 1\}$-valued DNR functions. The following result is a known fact and we give the proof which appears in [51].

Theorem 51. A degree is PA if and only if it contains a $\{0, 1\}$-valued DNR function.

Proof. The fact that PA degrees compute $\{0, 1\}$-valued DNR functions follows from Solovay's unpublished result (Theorem 35) and the fact that the set of all $\{0, 1\}$-valued DNR functions is a Π^0_1 class. It is also clear to see that the degrees containing $\{0, 1\}$-valued DNR functions are upward closed. Now suppose that a degree $\mathbf{a}$ contains a $\{0, 1\}$-valued DNR function f. Then f can compute a path on any Π^0_1 class as follows. We let $\mathcal{P}$ be the set of all infinite paths of Λ for some downward closed computable set of strings Λ. We define $\sigma_0 = \emptyset$. Suppose that we are given σ_s which is infinitely extendible in $[\Lambda]$. We then look for the least $j \leq 1$ and l such that $\sigma_s * j$ has no extension in Λ of length l. Let i be such that $\Psi_i(i)$ is defined and $j = \Psi_i(i)$ if such j exists. Then $\sigma_s * f(i)$ is infinitely extendible in $[\Lambda]$, so we define σ_{s+1} to be $\sigma_s * f(i)$. □

Another classification of PA degrees was given by Lewis [52] as follows.

Definition 66. A tree T is $\mathbf{a}$-*incapable* if no path on T is of degree $\geq \mathbf{a}$.

Theorem 52 (Lewis, 2007)**.** If $\mathbf{a}$ is PA then it computes a perfect $\mathbf{a}$-incapable tree.

We know that $\{0, 1\}$-valued DNR degrees cannot be minimal. However, the following was given in [53].

Theorem 53 (Kumabe and Lewis, 2009)**.** There exists a minimal degree which is FPF.

We want to know the relationship between minimal covers and PA degrees. We therefore ask the following question.

Open question. Does there exist a PA degree which is a minimal cover for a non-PA degree?

Some related results are known in the literature. Recall that the modulus function of K, $m_K(n)$ is defined as the least s such that $\Psi_m(m)\,[s]\downarrow$ for every $m \leq n$, where $m \in K$.

Definition 67. A degree $\mathbf{a}$ is called *array non-recursive (ANR)* if there is a function $f \leq_T A$ for $A \in \mathbf{a}$ which is not dominated by the modulus function of K

ANR degrees and PA degrees share some properties. For example, it was shown by Downey, Jockusch, Stob in [54] that no ANR degree is minimal.

Definition 68. A degree is *2-minimal* if it is a minimal cover for a minimal degree. More generally, a degree is $n+1$-*minimal* if it is a minimal cover for some n-minimal degree.

An interesting result, given by Cai [55], which might be related to the open question is the following.

Theorem 54 (Cai, 2010)**.** There exists a 2-minimal ANR degree.

In fact, Cai in [56] asked a stronger form of the open question we gave above, asking whether or not there exists a 2-minimal PA degree. Certainly, this has a negative answer below $\mathbf{0}'$ by the following two results by Cai [57] and Kučera [58].

Theorem 55 (Cai, 2014)**.** If $\mathbf{a}$ is n-minimal for some $n \in \omega$ then it cannot compute any non-recursive r.e. degree.

Theorem 56 (Kučera, 1986)**.** Every fixed point free (in particular PA) degree $\leq \mathbf{0}'$ bounds a non-recursive r.e. degree.

So to answer our question positively, we could aim to look for a PA degree which is a minimal cover for an incomplete r.e. degree perhaps. Kučera in [59] shows that there exists an incomplete high PA degree which computes a high incomplete r.e. degree. This theorem might be helpful for the investigation of finding an answer to the open question we proposed.

We can extend the notion of minimal degree to minimal upper bounds.

Definition 69. A degree $\mathbf{b}$ is a *minimal cover* for a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and there does not exist a degree $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

Note that every degree has a minimal cover since the minimal degree construction can be relativized.

Now one could also investigate the relation between minimal covers and degrees of members of Π^0_1 classes. We know, by Grozsek and Slaman [48], that there exists a non-empty Π^0_1 class such that every member computes a minimal degree. We give the following open question that whether every Π^0_1 class contains a minimal upper bound.

Open question. Does there exist a non-empty special Π^0_1 class such that no member is a minimal cover?

It can be seen this is related to the join property result. So it is likely that the question has a positive answer.

Chapter 5

Choice Classes

5.1 Introduction

This chapter is devoted to the study of so called Π_1^0 *choice classes*. A Π_1^0 choice class is another variant of Π_1^0 classes with a restriction on its elements. The work in this chapter can be considered as a Π_1^0 choice class analogue of the work by Kent and Lewis [2]. We first give some properties about the structure of the degree spectra of Π_1^0 choice classes. We show that the existential theory of this structure is decidable. We then prove the existence of Turing degrees which are not contained in any degree spectrum of a Π_1^0 choice class but can be contained in the degree spectrum of some Π_1^0 class which is not necessarily choice. We define Π_1^0 choice classes as follows.

Definition 70. A Π_1^0 class is called a *choice* class if no two members have the same Turing degree.

We study the basic properties of Π_1^0 choice classes. Define $\mathfrak{P}_c = \{S(\mathcal{P}) : \mathcal{P}$ is a Π_1^0 choice class$\}$, where $S(\mathcal{P})$ is the degree spectrum of $\mathcal{P}$, i.e. the set of all degrees $\mathbf{a}$ such that there exists $A \in \mathcal{P}$ of degree $\mathbf{a}$. We denote the elements of $\mathfrak{P}_c$ by α, β, γ. We define $\mathfrak{P}$ in the same manner for Π_1^0 classes which are not necessarily choice. We study the structure $(\mathfrak{P}_c, <)$ where the elements are ordered by inclusion. We also investigate degrees which are called *choice invisible* degrees that are not contained in any of the degree spectra of Π_1^0

choice classes. This gives us proper antibasis results. Note that if we consider Π^0_1 classes, since they can contain members of every degree, an antibasis result for such classes makes sense for those that do not contain members of every degree. However, as we will see, one does not need to worry about this case for Π^0_1 choice classes.

5.2 Properties of $(\mathfrak{P}_c, <)$

It is known that Π^0_1 choice classes do exist. An example of a Π^0_1 choice class would be a Π^0_1 class such that each member is incomparable with each other. The existence proof of such class is given in Theorem 4.7 of [3]. The first observation is that if $\mathcal{P}$ is a Π^0_1 choice class then $S(\mathcal{P}) \neq \mathbf{D}$. This is true, as proved in [2], because a Π^0_1 class $\mathcal{P}$ contains all paths through a perfect computable tree T iff it has members of every degree. To see why this is enough to ensure that $\mathcal{P}$ is not a Π^0_1 choice class, suppose that $\mathcal{P}$ contains all paths through T of this kind. Given any set B, we can then define a set $C_B \in [T]$ such that $C_B = \bigcup_{s\in\omega} \sigma_s$ which is of the same degree as of B. We define σ_0 to be the string at level 0 in T. Given σ_s, we let σ_{s+1} to be the leftmost successor of σ_s in T if $B(s) = 0$. Otherwise, define σ_{s+1} to be the rightmost successor of σ_s in T. Then for $B' \neq B$ but of the same degree as B, $C_{B'} \neq C_B$ but $C_{B'} \in [T]$ and $C_B \in [T]$. Note that the same argument suffices to show the other direction.

Since no Π^0_1 choice class contains a member of every degree, in particular there exists a Π^0_1 class $\mathcal{P}$ such that $S(\mathcal{P}) \neq S(\mathcal{Q})$ for any Π^0_1 choice class $\mathcal{Q}$. Another interesting observation is that Π^0_1 choice classes appear to have cardinality restrictions. First of all, a Π^0_1 choice class $\mathcal{P}$ cannot be finite unless it has a single element, because the members of finite classes are all recursive. In fact, we will show something stronger than this.

Let us recall that a subset A of a topological space is *dense in itself* if A contains no isolated points. Consider the Cantor topology on 2^ω. The *Cantor-Bendixson derivative* of $\mathcal{P}$ is the set of non-isolated points of $\mathcal{P}$ according to the Cantor topology and is denoted by $D(\mathcal{P})$. The iterated derivative $D^\alpha(\mathcal{P})$ is defined for all ordinals α by transfinite recursion:

(i) $D^0(\mathcal{P}) = \mathcal{P}$;

(ii) $D^{\alpha+1}(\mathcal{P}) = D(D^\alpha(\mathcal{P}))$;

(iii) $D^\beta(\mathcal{P}) = \bigcap_{\alpha<\beta} D^\alpha(\mathcal{P})$ for any limit ordinal β.

The following theorem is a sufficient condition for the statement that every non-empty element except $\{\mathbf{0}\}$ of $\mathfrak{P}_c$ is uncountable.

Theorem 57. Any countably infinite Π^0_1 class has members of the same degree.

Proof. Let $\mathcal{P}$ be a countably infinite Π^0_1 class. We show there are at least two recursive members in $\mathcal{P}$. For this it suffices to show that in fact there are at least two isolated points. Suppose that, for the sake of contradiction, $\mathcal{P}$ is countable and has only one isolated point, say A. Let $\mathcal{Q} = \mathcal{P} - \{A\}$. Now $\mathcal{Q}$ is still a closed set because A is an isolated point. So $\mathcal{Q}$ is a Π^0_1 class and contains no isolated point, hence $D(\mathcal{Q}) = D$. Then, $\mathcal{Q}$ is dense in itself and it is perfect. Therefore, $\mathcal{Q}$ is uncountable. But then, $\mathcal{P}$ is uncountable since $\mathcal{Q} \subset \mathcal{P}$. A contradiction. □

Corollary 17. Every non-empty Π^0_1 choice class is uncountable unless it has a single element.

Since there does not exist a Π^0_1 choice class which contains a member of every degree, it is natural to ask first if there exists a maximal element of $(\mathfrak{P}_c, <)$. It is known that there is no maximal element of $\mathfrak{P}$ for special Π^0_1 classes. This is provided by Jockusch and Soare [4]. The theorem says that if $\mathcal{P}$ is a special Π^0_1 class then there exists a nonzero r.e. degree $\mathbf{a} \notin S(\mathcal{P})$. On the other hand, for every degree $\mathbf{a}$ with $\mathbf{0} < \mathbf{a} \leq \mathbf{0}'$ there exists a special Π^0_1 class $\mathcal{P}'$ with $\mathbf{a} \in S(\mathcal{P}')$. Then $\mathcal{P}' \cup \mathcal{P}$ is a special Π^0_1 class and it properly includes $\mathcal{P}$.

We know that for $(\mathfrak{P}, <)$, the greatest element is $\mathbf{D}$, but since $\mathbf{D} \notin \mathfrak{P}_c$, we first ask if there exists a maximal element in the case for Π^0_1 choice classes. We now show that there is no maximal element of $(\mathfrak{P}_c, <)$ and we do not need to worry this time about the cases where the given class contains a member of every degree since the set of all Turing degrees is not a degree spectrum of a Π^0_1 choice class.

To prove this we are given a Π^0_1 choice class $\mathcal{P}$ such that $S(\mathcal{P}) = \alpha$, where $\alpha \neq \mathbf{D}$ of course, and we construct a Π^0_1 choice class $\mathcal{Q} \supset \mathcal{P}$ with $S(\mathcal{Q}) = \beta$ and $\alpha < \beta$. A way to construct $\mathcal{Q}$ is to add reals in $\mathcal{P}$ to extend it to a larger class $\mathcal{Q}$. We may call $\mathcal{Q}$ the *choice extension* of $\mathcal{P}$. Note that we do not need $\mathcal{Q} - \mathcal{P}$ to be infinite since it would be sufficient to add a single element whose degree is not the same as the degree of any member in $\mathcal{P}$. Other kinds of extensions are possible as well. A few notions which we are not going to discuss here were introduced by Cenzer [60] on the minimal extensions of Π^0_1 classes, and by Lawton [61] on minor superclasses of Π^0_1 classes.

Before we show how to construct $\mathcal{Q}$, it will be useful first to prove the following theorem, which holds for Π^0_1 classes, and is well known. Now it is easy to observe that any countable set of Turing degrees is not the degree spectrum of a Π^0_1 choice class unless it is $\{\mathbf{0}\}$. However, we have the following result.

Lemma 15. For any nonzero recursively enumerable degree $\mathbf{a}$, $\{\mathbf{0}, \mathbf{a}\}$ is the degree spectrum of a Π^0_1 class.

Proof. Let $A \subset \omega$ be an r.e. set of degree $\mathbf{a}$. Suppose that we are given an enumeration of A,

$$f(n) = \mu s(A_s \restriction n = A \restriction n).$$

So $f(n)$ shows how long we have to wait until the enumeration of A is correct up to the initial segment of length n. The idea behind the proof is to code the enumeration function on a path of the Π^0_1 class that we are constructing and let all other paths be recursive. We also construct a set, we call Λ^*, which will be used in the construction. The role of Λ^* is to put delimiters in a way so that we can fill 0's above some strings in the class to get a some kind of enumeration distance between the enumerated elements in A, i.e. number of stages required for the enumeration of the next element. We keep adding zeros above some strings in case we change our mind about the enumeration function of A, say $f(n)$, and need to come back, for some n, and increase the coded value. So once a string is an initial segment in Λ^*, we will always have zeros added on each stage of the construction. We put 1 after a sequence of zeros when the distance between the 1's gets sufficient enough to code the enumeration function up to stage s.

We define a Π^0_1 class $\mathcal{P}$ which has one element B such that

$$B = 0^{f(0)+1}10^{f(1)+1}10^{f(2)+1}1\cdots$$

and such that all other elements end with an infinite sequence of 0's. We define $\mathcal{P}$ to be the set of all infinite paths through $\Lambda = \bigcup_{s\in\omega}\Lambda_s$ which we define in the construction. Let $f_s(n) = \mu s' \leq s$ such that $A_{s'} \upharpoonright n = A_s \upharpoonright n$. So $f_s(n)$ is a function that shows how f would look if the enumeration of A, by stage s, were correct up to the initial segment of length n. We then let $\tau_s = 0^{f_s(0)+1}10^{f_s(1)+1}10^{f_s(2)+1}1\cdots$ be the approximation of B at stage s. The construction is as follows:

Stage 0. Enumerate $\emptyset$ into Λ_0, and let $\Lambda^* = \emptyset$.

Stage $s+1$. Given Λ_s, for each leaf $\tau \in \Lambda_s$,

(i) Enumerate $\tau * d$ into Λ_{s+1} (for $d = \{0,1\}$) if $\tau * d \subset \tau_s$ for some value of d.

If $d = 1$, then enumerate $\tau * d$ also into the set Λ^*.

(ii) If τ has an initial segment in Λ^*, then enumerate $\tau * 0$ into Λ_{s+1}.

Now since f and A are both computable in each other, f is non-recursive by hypothesis. However, note that f_s is recursive. As s increases, $f_s(n)$ can only get larger for a given argument $n \in \omega$. If we ever want to change our guess about $f_s(n)$, we come back and increase our guess. It is easy to see that every $f_s(n)$ gets changed finitely many times. Also note that Λ^* contains strings that end with a 1, so step (ii) at stage $s+1$ guarantees that there is an extension succeeded with all zeros. Hence, this step provides us that every string which does not become an initial segment of τ_s anymore is chosen to become an infinite computable path in Λ since it ends with infinite zeros. $\square$

In fact, we can modify the previous proof to get something stronger. We now want to prove the following.

Lemma 16. If α is the degree spectrum of a Π^0_1 class $\mathcal{P}$ then for any recursively enumerable degree $\mathbf{a} \notin \alpha$, $\alpha \cup \{\mathbf{a}\}$ is the degree spectrum of a Π^0_1 class.

Proof. We aim to add above some strings a "copy" of a given Π^0_1 class which has the degree spectrum α instead of just adding 0's as in the previous proof.

We again use the functions f and f_s in the same way we used in the previous lemma. If we never have to come back and increase our guess about $f_s(n)$, for a given argument n, then we are fine since we will be leaving a copy of $\mathcal{P}$ above some string. If we come back to increase our guess, we kill all but one branch and we increase our guess about $f_s(n)$ by raising the delimiter symbol for coding the enumeration distance. However, since there will be zeros and ones in the copy of $\mathcal{P}$, particularly on the branch we leave, we have to use another delimiter to code the enumeration distance in stages. For this purpose we build our new Π^0_1 class with a degree spectrum $\alpha \cup \{\mathbf{a}\}$ as a subset of $\{0,1,2\}^\omega$ and use the distance between 2's instead to code the enumeration function f.

We should first define what we mean by a copy of $\mathcal{P}$. Here a copy of $\mathcal{P} = [\Lambda]$ for some downward closed recursive set of strings Λ, is just defined by the set $\{\tau * \sigma : \sigma \in \Lambda\}$ for any τ.

Suppose that $\mathcal{P} = [\Lambda]$ is a Π^0_1 class with a degree spectrum α and suppose that we are given a recursively enumerable degree $\mathbf{a} \notin \alpha$. We build a downward closed set of strings $\Upsilon = \bigcup_{s\in\omega} \Upsilon_s$ as a subset of $\{0,1,2\}^{<\omega}$ such that $\mathcal{Q} = [\Upsilon]$ is a Π^0_1 class with a degree spectrum $\alpha \cup \{\mathbf{a}\}$. So we consider Υ like a ternary tree containing many copies of Λ. When building $\mathcal{Q}$, we begin to place a copy of $\mathcal{P}$ above strings that end with a 2 in Υ_s in the form of a set of strings in Λ up to a certain length at each stage of the construction. When putting the bits of Λ into Υ_s, we put Λ up to strings of length $f_s(0)+1,\ f_s(0)+1,\ f_s(2)+1$, and so on. We consider a set Π_s of strings of the form

$$\{0,1\}^{f_s(0)+1}2\{0,1\}^{f_s(1)+1}2\{0,1\}^{f_s(2)+1}2\cdots$$

Since we are enumerating the branches of Λ between 2's, if we let $\Lambda \upharpoonright n$ denote the set of strings in Λ of length n then we can put the strings in Π_s of the form

$$(\Lambda \upharpoonright f(0)+1)2(\Lambda \upharpoonright f(1)+1)2(\Lambda \upharpoonright f(2)+1)2\cdots$$

The construction is similar to the one in the previous lemma with a few modifications. Now we do not have τ_s but Π_s.

At stage 0, we enumerate $\emptyset$ into Υ_0 and let $\Upsilon^* = \emptyset$.

At stage $s+1$. Given Π_s and Υ_s, let σ be a leaf of Υ_s.

(i) We enumerate $\sigma * d$ into Υ_{s+1} for $d = \{0, 1, 2\}$ if there exists a string $\tau \in \Pi_s$ such that $\sigma * d \subset \tau$.

If $d = 2$, then we enumerate $\sigma * d$ also into Υ^*.

(ii) To put a copy of $\mathcal{P}$, we see if σ has an initial segment in Υ^*. If so then enumerate each $\sigma * (\Lambda \upharpoonright f_s(n) + 1)$, for a given $n \in \omega$, into Υ_{s+1}.

□

We now want to show that the last lemma holds for Π^0_1 choice classes.

Theorem 58. If α is the degree spectrum of a Π^0_1 choice class $\mathcal{P}$ then for any recursively enumerable degree $\mathbf{a} \notin \alpha$, $\alpha \cup \{\mathbf{a}\}$ is the degree spectrum of a Π^0_1 choice class.

Proof. Now if we want the last lemma to work for Π^0_1 choice classes we have to make some modifications because we do not want to have multiple copies of the given class $\mathcal{P} = [\Lambda]$, for some downward closed computable set of strings Λ, in the class $\mathcal{Q} = [\Upsilon]$ that we construct. One idea is to copy mutually disjoint parts of the given class $\mathcal{P}$ into different parts of $\mathcal{Q}$. However, there are some technical difficulties. If we are given a Π^0_1 choice class $\mathcal{P}$ such that $\mathcal{P} = [\Lambda]$ for some downward closed computable set of strings Λ, with a degree spectrum α and if $\mathbf{a} \notin \alpha$ is an r.e. degree, then we construct our new Π^0_1 choice class $\mathcal{Q}$ having degree spectrum $\alpha \cup \{\mathbf{a}\}$ in the following way.

Since we want to enumerate mutually disjoint subclasses of $\mathcal{P}$, above various strings in Υ, we have to decide which parts of $\mathcal{P}$ we should take. For this we approximate a sequence of pairwise mutually incompatible strings $\{\sigma_s\}_{s\in\omega}$ in Λ. For $i \in \omega$, let $[\sigma_i]$ denote the set of infinite branches of $\{\tau \in \Lambda : \tau \text{ is compatible with } \sigma_i\}$. Each σ_s will satisfy $\mathcal{P} \cap [\sigma_s] \neq \emptyset$ and if A is the leftmost path in $[\Lambda]$ then we should have that $\mathcal{P} = \{A\} \cup \bigcup_{i\in\omega}([\sigma_i] \cap \mathcal{P})$. For any $s \in \omega$, let us denote the class $[\sigma_s]$ by $\mathcal{P}^s$. Now for any $s \in \omega$, $\mathcal{P}^s$ is a Π^0_1 choice class since $\mathcal{P}^s \subset \mathcal{P}$ and also for any $s, t \in \omega$, $\mathcal{P}^s \cup \mathcal{P}^t$ is a choice class because of the fact that they are mutually disjoint subclasses of $\mathcal{P}$. Instead of adding the entire class as in the previous lemma, we keep on adding the mutually disjoint subclass of $\mathcal{P}$ above different strings in Υ. Namely, we add

$\mathcal{P}^s$ for every $s \in \omega$. As we keep on enumerating strings into $\mathcal{Q}$, one of the following problems might occur in $\mathcal{P}^s$.

(i) We find out that the set of infinite paths in $[\Lambda]$ above our present approximation to some σ_s is empty.

(ii) We eventually find out that the set of infinite paths above some σ_s turns out to be the whole class $\mathcal{P}$.

These cause problems because we have to code the enumeration function of the given set of r.e. degree **a** on an infinite path of $\mathcal{Q}$ and we might need to change our guess about the sequence of mutually incomparable strings. So we have to change our mind about the values σ_s, and so about the various $\mathcal{P}^s$ of which we are placing copies in $\mathcal{Q}$. Note that it is also a problem that even if we add copies of all $\mathcal{P}^s$ into $\mathcal{Q}$, we will still miss the leftmost branch $A \in \mathcal{P}$ because for any $i, j \in \omega$, σ_i is incompatible with σ_j and if one looks at Figure 5.2, in any kind of mutually incompatible sequence of strings for forming a sequence of mutually disjoint subclasses of $\mathcal{P}$, the leftmost path will not be covered by the mutually incompatible sequence of strings. This leftmost path, however, is of r.e. degree, just like **a**. Then, instead of enumerating a single r.e. set into $\mathcal{Q}$, we also have to enumerate the leftmost branch of $\mathcal{P}$ that we miss. But then we have to be careful about not duplicating the branches of $\mathcal{P}$ when we put copies. We can solve this by enumerating the bits of $\mathcal{P}^s$ on two r.e. branches in an alternating fashion. That is, since we enumerate in two r.e. branches, we put the bits of $\mathcal{P}^s$ into the first r.e. branch then enumerate $\mathcal{P}^{s+1}$ into the second, $\mathcal{P}^{s+2}$ into the first again and so on.

When we approximate the sequence $\{\sigma_s\}_{s\in\omega}$ problem (i) or (ii) may occur. To overcome these problems, it suffices to ensure that for each $i \in \omega$ the class $\mathcal{P} \cap [\sigma_i]$ is non-empty and that each branch on Λ, except the leftmost branch, extends some σ_i.

Regarding problem (ii), if there exists a string $\sigma \in \Lambda$ such that the set of infinite paths above σ is actually the entire class, then the set of infinite paths above any string $\tau \in \Lambda$ which is incompatible with σ must be empty. However, we may still have finite branches above σ. If this is the case then we have to work on the subtree above σ. If we denote the subtree of Λ above σ by Λ' and if we let $\mathcal{P}' = [\Lambda']$ be a Π^0_1 class, clearly $\mathcal{P}'$ is a Π^0_1 choice class since

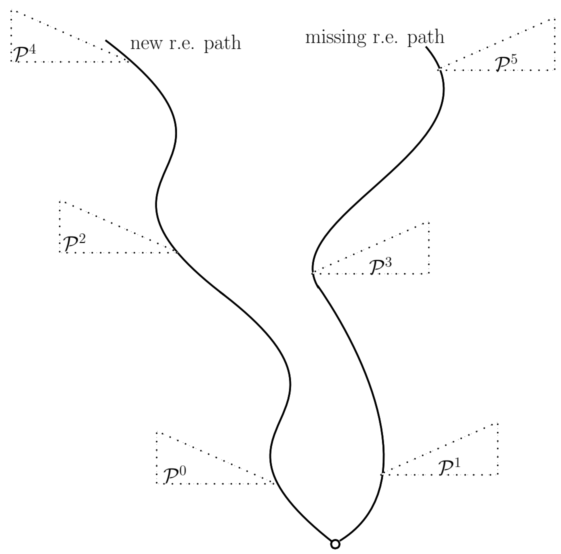

Figure 5.1: Two r.e. paths on $\mathcal{Q}$.

$\Lambda' \subset \Lambda$. Moreover, $S(\mathcal{P}') = S(\mathcal{P})$ since $\mathcal{P}' - \mathcal{P}$ has no infinite branch, and in fact $\mathcal{P} = \mathcal{P}'$.

We shall now give the construction of the sequence of mutually incompatible strings.

Now let $\Lambda \upharpoonright n$ denote the elements of Λ of length n. We assume further that $\mathcal{P}$ has no isolated members, i.e. there does not exist any finite σ such that $\mathcal{P}$ has precisely one element extending σ. We can assume this because we can separately enumerate in any isolated path to our new class at the very end of its construction.

Let A be the leftmost element of $\mathcal{P}$. The following construction produces an approximation to a sequence $\{\sigma_i\}_{i\in\omega}$ such that the members of this sequence are pairwise incompatible and satisfy:

a) For each $i \in \omega$, $\mathcal{P} \cap [\sigma_i]$ is non-empty.

b) For all $B \in \mathcal{P}$ except A, there exists $i \in \omega$ with $\sigma_i \subset B$.

We define values $\sigma_i[s]$ for a finite number of i at each stage s of the construction. So $\sigma_i[s]$ shows our guess for σ_i at stage s. For each i we shall ensure $\sigma_i[s]$ is defined and takes the same value for all sufficiently large s.

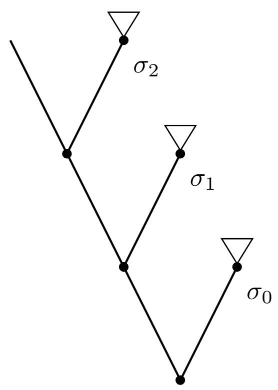

Figure 5.2: A simple example of how $\{\sigma_i\}_{i\in\omega}$ could be formed.

At stage $s = 0$, we let $\tau = \emptyset$.

At stage $s > 0$, let τ be the leftmost element of Λ of length s. Perform the following iteration until instructed to stop:

Step i. Let ρ_i be the rightmost element of $\Lambda \upharpoonright s$ which does not extend any $\sigma_j[s]$ with $0 \leq j < i$. If $\rho_i = \tau$ then terminate the iteration, and proceed to the next stage of the construction. Otherwise, let υ_i be the longest string which is an initial segment of both τ and ρ_i. Define $\sigma_i[s] = \upsilon_i * 1$, and proceed to step $i + 1$ of the iteration.

Now we verify that the sequence $\{\sigma_i[s]\}_{s\in\omega}$ converges for every $i \in \omega$ and satisfies the desired properties in (a) and (b) written above. Recall that A is the leftmost member of $\mathcal{P}$. Let B_0 be the rightmost element of $\mathcal{P}$, and let υ_0 be the longest string which is an initial segment of both A and B_0. Given B_i and υ_i, let B_{i+1} be the rightmost element of $\mathcal{P}$ extending $\upsilon_i * 0$, and let υ_{i+1} be the longest string which is an initial segment of both A and B_{i+1}.

For each i we wish to show:

(a) For all sufficiently large s we have:

$\sigma_i[s] \downarrow= \upsilon_i * 1$.

(b) All elements of $\mathcal{P}$ extending σ_i or to the right of σ_i, extend some σ_j for $j \leq i$.

Suppose (a) and (b) are true for all $j < i$, and let s be large enough that, for all $s' \geq s$ and all $j < i$, $\sigma_j[s'] \downarrow = \upsilon_j * 1$.

Now let $s' > s$ be sufficiently large that there do not exist any elements of $\Lambda \upharpoonright s'$ strictly to the right of υ_i, other than those which extend some σ_j for $j < i$ (the fact that such an s' exists follows from the compactness of Cantor space, i.e. König's Lemma).

Then at all stages $s'' \geq s'$ we have $\sigma_i[s''] \downarrow = \upsilon_i * 1$, and (b) also clearly holds as required.

Now that we have $\{\sigma_i\}_{i\in\omega}$, we describe how to construct $\mathcal{Q}$. The construction of $\mathcal{Q}$ uses the previous lemma but modified as described here. For the construction we shall have a supermodule μ which handles two submodules; one for the new r.e. branch and one for the missing r.e. branch. Let us call them κ and λ, respectively. Now κ will use Lemma 16 but instead we put $\mathcal{P}_i$ such that $i = 2j$ for every $j \in \omega$ above the j-th enumeration point. Then, in the limit, we obtain on this side a Π^0_1 choice class with an an r.e. branch of degree $\mathbf{a}$ with single copy of each $\mathcal{P}_i$ such that $i = 2j$ for every $j \in \omega$. Module λ is defined similarly for the missing leftmost path of r.e. degree and subclasses $\mathcal{P}_i$ such that $i = 2j + 1$ for every $j \in \omega$. Again, we eventually obtain a Π^0_1 choice class containing a member of degree $\deg(A)$ with single copy of each $\mathcal{P}_i$ such that $i = 2j + 1$ for every $j \in \omega$. The supermodule μ passes the control to κ at even stages and passes to λ at odd stages to fully obtain $\mathcal{Q}$. Then $\mathcal{Q}$ is clearly a Π^0_1 choice class such that $S(\mathcal{Q}) = \alpha \cup \{\mathbf{a}\}$. □

The idea can be easily modified to get the same result for Δ^0_2 degrees. Instead of coding the modulus function for r.e. sets, we code the modulus function for Δ^0_2 sets and the construction becomes similar. Then, since a Π^0_1 choice class cannot contain members of every Δ^0_2 degree, this makes sure that the following corollaries hold.

Corollary 18. $(\mathfrak{P}_c, <)$ has no maximal element.

Definition 71. We say that β is a *minimal cover* for α if there is no $\gamma \in \mathfrak{P}_c$ strictly between α and β.

Corollary 19. For every $\alpha \in \mathfrak{P}_c$, there exists a minimal cover for α in $\mathfrak{P}_c$.

Definition 72. We say that a poset P has the *meet property* if for any a there exists some b such that $a \wedge b$ gives the least element of P.

We now want to show that $(\mathfrak{P}_c, <)$ has the meet property. This almost follows from a theorem due to Cole and Simpson [62]. However, to get the desired result we need to modify it for Π^0_1 choice classes. The original theorem is as follows and the proof is given in [2].

Theorem 59 (Simpson and Cole, 2007)**.** For any special Π^0_1 class $\mathcal{P}_0$ there exists a special Π^0_1 class $\mathcal{P}_1$ such that no member of $\mathcal{P}_1$ computes any member of $\mathcal{P}_0$.

We modify this theorem for Π^0_1 choice classes.

Theorem 60. For any special Π^0_1 class $\mathcal{P}_0$ there exists a special Π^0_1 choice class $\mathcal{P}_1$ such that no member of $\mathcal{P}_1$ computes any member of $\mathcal{P}_0$.

Proof. Let $\mathcal{P}_0$ be given such that $\mathcal{P}_0 = [\Lambda]$ for some downward closed computable set of strings Λ. We define an approximation to a set of strings T such that $\mathcal{P}_1 = [T]$ is a Π^0_1 choice class which satisfies the statement of the theorem. For each level of T, we aim to satisfy a single requirement for those strings at that level. Specifically, all those strings at level $2i+1$ will be defined so as to satisfy

Ξ_i : If $A \in \mathcal{P}_1$ and $\Psi_i(A)$ is total then $\Psi_i(A) \notin \mathcal{P}_0$.

For those strings at level $2i+2$, we should aim to satisfy the choiceness property (in fact we satisfy something stronger in the construction). That is,

Θ_i : If $A \in \mathcal{P}_1$ and $C \in \mathcal{P}_1$ then $A \neq \Psi_i(C)$ or $C \neq \Psi_i(A)$.

At stage $s = 0$, enumerate $\emptyset$ into T.

At stage $s > 0$,

(i) Find the least string $\tau \in T$ such that τ is of level $2i+1$, $\Psi_i(\tau)[s]$ is compatible with some string in Λ of length s and there is some leaf τ' of T extending τ such that $\Psi_i(\tau')[s]$ properly extends $\Psi_i(\tau)[s]$. If this is the case then we remove all strings extending τ from T except τ'.

(ii) We find the least string $\tau \in T$ such that $\tau \subset \Psi_i(\sigma)[s]$ for some $\sigma \in T$ of level $2i+2$ which is incompatible with τ. If such τ exists, we remove all strings extending τ from T and enumerate two incompatible extensions of σ into T.

After these instructions, choose two incompatible strings extending each leaf of T, and enumerate these strings into T.

We claim that $\mathcal{P}_1$ is a Π^0_1 class. The argument is standard. For this we let Υ be the set of all strings which are initial segments of strings in T at any stage. We show that Υ is downward closed, computable and $[\Upsilon] = [T]$. Now Υ is computable since we enumerate in strings that only extend strings in Υ of the previous stage. Clearly, every infinitely extendible string in T is also in Υ by the definition of Υ. The opposite direction is also true. By contrapositive, suppose that σ is not infinitely extendible in Υ. Then σ must be a leaf of T in which case σ is not infinitely extendible in T since otherwise σ would be infinitely extendible in Υ. Approximation to T converges, i.e. requirements are satisfied. Now it is easy to see that step (ii) simply ensures that no branch of $\mathcal{P}_1$ computes another. For the Ξ_i requirements, suppose that for some least i there is a sequence $\{\tau_j\}_{j\in\omega}$ of strings such that each τ_j is a string of level $2i+1$ in T at some stage of the construction and $\tau_j \subset \tau_{j+1}$ for all j. Let $A = \bigcup_{j\in\omega} \tau_j$. Then $\Psi_i(A)$ is computable and is in $\mathcal{P}_0$. A contradiction. □

Corollary 20. $(\mathfrak{P}_c, <)$ has the meet property.

The following theorem is another observation about the structure of the degree spectra of Π^0_1 choice classes.

Theorem 61. (i) $(\mathfrak{P}_c, <)$ has a least element and it is defined as $\mathbf{0}_{\mathfrak{P}_c} = \mathbf{0}_{\mathfrak{P}} = \emptyset$.

(ii) We say that $\alpha > \mathbf{0}_{\mathfrak{P}_c}$ in $\mathfrak{P}_c$ is *minimal* if there does not exist $\beta \in \mathfrak{P}_c$ with $\mathbf{0}_{\mathfrak{P}_c} < \beta < \alpha$. Then, $(\mathfrak{P}_c, <)$ has only one minimal element, i.e. $\{\mathbf{0}\}$.

Proof. There is nothing to prove for (i).

We prove (ii). Obviously $\{\mathbf{0}\}$ is minimal. Suppose that there is another minimal element of $\mathfrak{P}_c$, say α. Then there would be a Π^0_1 choice class $\mathcal{P}$ such that $S(\mathcal{P}) = \alpha$ and $\mathcal{P} = [\Lambda]$ for some downward closed computable set of strings Λ. Note that $S(\mathcal{P})$ must be uncountable. Take two immediate incompatible extensions, σ and τ, of any element of Λ. Remove every extension of τ and let $\mathcal{R}$ be the resulting class with the degree spectrum β. Now, $\mathcal{R}$ is a Π^0_1 choice class such that $\mathcal{R} \subset \mathcal{P}$ and hence $\beta < \alpha$. A contradiction. □

We state the following conjecture for which we shall give a proof for a special case and then discuss about possible solutions to prove the general case. The reader can skip to Section 5.3 without loss of continuity.

Conjecture. $(\mathfrak{P}_c, <)$ is an upper semilattice.

The requirement here is that given two Π^0_1 choice classes $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P} = [\Lambda]$ and $\mathcal{Q} = [\Upsilon]$ for some downward closed computable sets of strings Λ and Υ, to get a Π^0_1 choice class (in which the elements are not Turing equivalent to another via *any* pair of Turing functionals) with a degree spectrum $S(\mathcal{P}) \cup S(\mathcal{Q})$, we enumerate the elements of $\mathcal{Q}$ into the copy of $\mathcal{P}$. We only give an informal proof here for a *fixed* pair of Turing functionals. Hence, note that $\mathcal{Q}$ will be a Π^0_1 "choice" class in a sense that with respect to the given fixed pair of Turing functionals. So we will enumerate those elements which are not Turing equivalent to any of the members of $\mathcal{P}$ with respect to a given pair of Turing functionals. We do this by avoiding exception points which is defined as follows.

Definition 73. Let $\mathcal{P}$ and $\mathcal{Q}$ be two Π^0_1 classes. An *exception point* for $\mathcal{Q}$ is a path $A \in \mathcal{Q}$ such that $\Psi_i(A) = B$ and $\Psi_j(B) = A$ for a given $i, j \in \omega$ and some $B \in \mathcal{P}$.

We take a sequence $\{\sigma_k\}_{k\in\omega}$ of mutually incompatible strings for $\mathcal{Q}$ as we constructed in Theorem 58. The idea is roughly that at some point we try to add in everything in $\mathcal{Q}$ above some σ_k, but that later, we may decide, actually, for $\eta \supset \sigma_k$, that we do not want to add in everything above η. Then later, for another $\eta' \supset \sigma_k$ which is incompatible with η, we might decide we do not want to add everything above there either, and so on. Then later for some $\tau \supset \eta$ we might decide that we do want to add in the strings above τ and etc. Ultimately we do not want to add the exception points into the Π^0_1 choice class we wish to construct. Let $\{\tau_k\}_{k\in\omega}$ be an effective enumeration of the terminal strings of Λ. We take a copy of Λ and we start adding the strings in Υ above σ_k into Λ above the k-th terminal string of Λ, assuming that the strings in Λ are ordered first by length and then from left to right. We also fix some $n_k \in \omega$ for each τ_k such that $n_{k+1} > n_k$. We stop adding the extensions of $\eta \in \Upsilon$ whenever we find such $\eta \supset \sigma_k$ which computes some $\tau \in \Lambda$ via Ψ_i up to the initial segment of length n_k and vice versa via Ψ_j, for a fixed pair of indices $i, j \in \omega$. In this case we say that η *carries risk* up to n_k. When a string carries risk up to some n_k, this does not completely mean that there exists $A \supset \eta$ such that $\Psi_i(A) = B$ and $\Psi_j(B) = A$ for some $B \in \mathcal{P}$. Therefore, we need to check if there exist infinitely many extensions of η which carry risk up to all sufficiently large n_k's. One thing we will be sure is that if $\Psi_i(A) \neq B$ or $\Psi_j(B) \neq A$, there will be some $k \in \omega$ such that $A \in \mathcal{Q}$ and $B \in \mathcal{P}$ do not compute each other up to the initial segment of length n_k. So if this is the case A will eventually be added into the copy of $\mathcal{P}$, particularly it will be added above some $\tau_k \in \Lambda$. To see if there exist infinitely many extensions of η which carry risk up to all sufficiently large n_k's, we start putting η again and all its initial segments above the next terminal string of Λ for which we take the next sufficiently large n_{k+1} for the enumeration of the subtree of Υ above σ_{k+1}. If we ever find out that some $\eta' \supset \eta$ carries risk up to n_{k+1} we stop enumerating the strings above η' and continue enumerating it above another terminal and so on.

For a fixed $t = \langle i, j \rangle$, we give the construction of Λ_t as follows. We define Λ_t as a subset of $\{0, 1, 2\}^{<\omega}$ as in Theorem 58.

We fix some sufficiently large n_k for each τ_k such that $n_{k+1} > n_k$. We take a copy of $\mathcal{P}$ in the form of downward closed computable sets of strings Λ_t such that $\mathcal{P}_t = [\Lambda_t]$, where $t = \langle i, j \rangle$. We shall add strings of elements of Υ into Λ_t.

At stage 0, we define $\Lambda_t[0] = \Lambda$ (where, for any $n \in \omega$, $\Lambda_t[n]$ denotes Λ_t defined at stage n).

Whenever we decide on the new value of $\sigma_k[s]$ (as in Theorem 58) we perform the following instructions.

At stage $s > 0$, we assume that we are given $\Lambda_t[s-1]$.

For each $k < s$, suppose that $\sigma_k[s]$ is given. Consider the set T of strings in Υ above $\sigma_k[s]$ up to length s (relative to $\sigma_k[s]$). We enumerate those strings $\eta \in T$ into $\Lambda_t[s-1]$ above τ_k such that there is no $\tau \in \Lambda$ of length $\leq |\eta|$ satisfying that $\Psi_i(\eta) = \tau$ and $\Psi_j(\tau) = \eta$ up to the initial segment of length n_k (We assume that $\tau_k * 2$ has already been enumerated before we start to put strings in, indicating the starting point of the information content of $\mathcal{Q}$).

If there is such $\tau \in \Lambda$ satisfying that $\Psi_i(\eta) = \tau$ and $\Psi_j(\tau) = \eta$ up to the initial segment of length n_k, we stop enumerating any string extending η into

$\Lambda_t[s-1]$ above the point τ_k. To keep checking (for later stages) if the extensions of η carry risk up to larger values of n, i.e. n_l for $l > k$, we add the strings in $\{\eta^* \in \Upsilon : \eta^* \text{ is compatible with } \eta\}$ to the set of strings above $\sigma_{k+1}[s]$, hence this way we will be able to continue to enumerate η and its extensions into Λ_t above some other terminal string where we take n_{k+1} for that. We define $\Lambda_t[s]$ to be the set of strings we enumerate by the end of this stage union $\Lambda_t[s-1]$.

Now this construction gives us $\mathcal{P}_t$. However, again note that we add in $\mathcal{P}_t$ the elements of $\mathcal{Q}$ which are not Turing equivalent to any of the elements of $\mathcal{P}$ only via a fixed pair of Turing functionals (Ψ_i, Ψ_j), where $t = \langle i, j \rangle$. Let S be the leftmost member of $\mathcal{Q}$. Then, using Theorem 58, we let $\mathcal{R} = \mathcal{P}_t \cup \{S\}$. Then we have that $S(\mathcal{R}) = S(\mathcal{P}) \cup S(\mathcal{Q})$ with respect to the fixed pair of Turing functionals with indices (i, j).

Now we shall give the verification.

Let $t = \langle i, j \rangle$ be fixed. Let $S(\mathcal{P}) = \alpha$ and $S(\mathcal{Q}) = \beta$. We shall argue that $\mathcal{R}$ is a Π^0_1 choice class, with respect to t, and has the degree spectrum $\alpha \cup \beta$. It is clear that $\mathcal{P}_t$ is a Π^0_1 class and that $S(\mathcal{P}_t) \supset \alpha$ since $\Lambda_t[0] = \Lambda$ for every $t \in \omega$ and that for any given $\Lambda_t[n]$ we recursively construct $\Lambda_t[n+1]$. If $A \in \mathcal{P}$ and $B \in \mathcal{Q}$ such that $\Psi_i(A) = B$ and $\Psi_j(B) = A$ for $i, j \in \omega$ then $B \notin \mathcal{R}$ since otherwise there would exist some $k \in \omega$ and $\eta \subset B$ such that η carries no risk up to $n_{k'}$ for all $k' > k$. Therefore it must be that either $\Psi_i(A) \neq B$ or $\Psi_j(B) \neq A$ for $A \in \mathcal{P}$. Also, $S(\mathcal{R}) \supset \beta$ since every infinite branch except the leftmost one is extended by some σ_k by Theorem 58 and since we can enumerate the missing r.e. path by the same result. This completes the argument.

Now the argument gives us a degree spectrum of the class, for a fixed pair indices $i, j \in \omega$,

$$\mathcal{P} \cup \{A \in \mathcal{Q} : \text{there exists no } B \in \mathcal{P} \text{ such that } \Psi_i(A) = B \text{ and } \Psi_j(B) = A\}.$$

Of course this class does not necessarily have to be a real Π^0_1 choice class since the enumerated element might be Turing equivalent to some element in $\mathcal{P}$ via some other pair of functionals. We now want to give an idea about how one might prove the conjecture. However, it is important to note that we do not give an actual proof here. If we want to prove the conjecture we need to look at all pairs of Turing functionals. To work with all pairs of Turing functionals, one thing we could do is to work simultaneously on infinitely many copies of $\mathcal{P}$, say $\{\mathcal{P}_t\}_{t\in\omega}$, where $t = \langle i, j\rangle$ according to some fixed computable bijection $\omega \times \omega \to \omega$, and work with (Ψ_i, Ψ_j) for that $\mathcal{P}_t = [\Lambda_t]$. We add the elements of $\mathcal{Q}$ into $\mathcal{P}_t$ which are not Turing equivalent to any of the members in $\mathcal{P}_t$ via (Ψ_i, Ψ_j). Now for each t we get the elements of $\mathcal{P}$ together with the elements of $\mathcal{Q}$ which are not Turing equivalent to any member of $\mathcal{P}$ via only (Ψ_i, Ψ_j). We would like to *intersect* each $\mathcal{P}_t$ to get such elements of $\mathcal{Q}$, hence obtain those which are not Turing equivalent to any member of $\mathcal{P}$ (via any pair of Turing functionals). However, taking simply $\bigcap_{t\in\omega} \mathcal{P}_t$ does not work here, because we have to be careful about the possibility that a member in $\mathcal{Q}$ might get enumerated above different terminal strings in different copies of $\mathcal{P}$. So when we take the intersection of all $\mathcal{P}_t$'s it might not give us the desired elements of $\mathcal{Q}$ since the sets we want to obtain might have different initial segments in each $\mathcal{P}_t$ up to the point where we start to enumerate in. This is why we want to construct the class as a subset of $\{0, 1, 2\}$ as in Theorem 58. So recall that we enumerated strings from Υ above terminal strings in Λ. Let us call them *enumeration points*. We can certainly have a recursive enumeration for enumeration points for a given Π^0_1 class since we can enumerate its terminal strings. Let $[\Lambda^e_t]$ denote the set of all infinite branches above the e-th enumeration point of the t-th copy of Λ. Now let $\mathcal{P}^+ = \bigcap_t \bigcup_e [\Lambda^e_t] \cup S$, where S is the leftmost branch of Λ. The problem here is that we need to show, for each t, that $\bigcup_e [\Lambda^e_t]$ is actually a Π^0_1 choice class. That is, we need to show there exists a downward closed computable set of strings Λ^*_t such that $[\Lambda^*_t] = \bigcup_e [\Lambda^e_t]$. If one could show this then it would be possible to prove the conjecture. One would also show $(\mathfrak{P}_c, <)$ also forms a lower semilattice by modifying the proof of Theorem 8.1 in [2], hence show that the structure is a lattice. We end the discussion here. Anything stated after the argument for the conjecture remains as a future study.

5.3 Decidability of the ∃-theory of $(\mathfrak{P}_c, <)$

Next, we consider the existential (∃) theory of $(\mathfrak{P}_c, <)$ and observe that it is decidable indeed. By the ∃-theory of $(\mathfrak{P}_c, <)$, we mean the set of sentences in the first order language of partial orders that are true about the degree spectra of Π^0_1 choice classes, and that are of the form $\exists x_1 \exists x_2 \cdots \exists x_k R(x_1, \ldots, x_k)$ for some $k \in \omega$, where $R(x_1, \ldots, x_k)$ is a quantifier free expression with free variables $x_1, \ldots, x_k$.

Theorem 62. The $\exists$-theory of $(\mathfrak{P}_c, <)$ is decidable.

Proof. We define a countable infinite *independent* sequence $\{\mathcal{P}_n\}_{n\in\omega}$ of Π^0_1 choice classes with degree spectra $\{\alpha_n\}_{n\in\omega}$, i.e. a sequence satisfying that $\alpha_k \not\subset \alpha_{k_1} \cup \cdots \cup \alpha_{k_n}$ with $k \neq k_i$ for any of the k_i's.

We begin with a Π^0_1 choice class $\mathcal{P} = [\Lambda]$ for some downward closed recursive set of strings Λ such that all members in $\mathcal{P}$ are Turing incomparable. Let $\{\sigma_i\}_{i\in\omega}$ be a sequence of mutually pairwise incomparable set of finite strings in Λ the same manner in Theorem 58. Given any $n \in \omega$, we let $\mathcal{P}_n$ to be the Π^0_1 choice class above σ_n, i.e. the set of all infinite strings in $\mathcal{P}$ extending σ_n. Note that this is a Π^0_1 choice class because all members are still Turing incomparable since $\mathcal{P}_n \subset \mathcal{P}$. If we take any finite set $J \subset \omega$ and take $\mathcal{P}' = \bigcup_{n\in J} \mathcal{P}_n$, which is a Π^0_1 choice class since $\mathcal{P}' \subset \mathcal{P}$ and $\mathcal{P}$ contains members that are Turing incomparable, then it is easy to see that $\alpha_m \not\subset S(\mathcal{P}')$ for $m \notin J$. We still have to show that there exists an embedding from any finite partially ordered set into the structure of the degree spectra of Π^0_1 choice classes. We assert this in the next lemma.

Lemma 17. Any finite partially ordered set is embeddable in $(\mathfrak{P}_c, <)$.

Proof. Let $\mathcal{M} = \langle M, \leq \rangle$ be a finite partially ordered set and let $M = \{x_i : i < n\}$. We define an order preserving bijection from $\mathcal{M}$ into $\mathfrak{P}_c$. Let $\{\alpha_i\}_{i\in\omega}$ be an independent sequence of degree spectra for Π^0_1 choice classes and let $\mathcal{P}_i$ has the degree spectrum α_i. For each $k < n$, let $F(k)$ be the set of all i such that $x_i \leq x_k$. Put $\mathcal{Q}_k = \bigcup_{i\in F(k)} \mathcal{P}_i$ and define β_k to be the degree spectrum of $\mathcal{Q}_k$. We define an embedding as follows: $g(x_i) = \beta_i$ for every $i < n$. In order to verify that this is indeed an embedding we must show that for all $i, j < n$, $x_i \leq x_j \Leftrightarrow \mathcal{Q}_i \subset \mathcal{Q}_j$. Suppose first that $x_i \leq x_j$. Then, $F(i) \subset F(j)$ so the result follows immediately. Next, suppose that $\mathcal{Q}_i \subset \mathcal{Q}_j$ and $x_i \not\leq x_j$ in order to derive a contradiction. Then $\mathcal{P}_i \subset \mathcal{Q}_i \subset \mathcal{Q}_j$, so $\mathcal{P}_i \subset \bigcup_{k\in F(j)} \mathcal{P}_k$ and $i \notin F(j)$, which contradicts the fact that $\{\mathcal{P}_i\}_{i\in\omega}$ is an independent sequence of Π^0_1 choice classes. Now, the reason this works is because an existential statement of the theory of $(\mathfrak{P}_c, <)$ asserts the existence of finitely many degree spectra $\alpha_1, \ldots, \alpha_k$ and for i, j it asserts that $\alpha_i < \alpha_j$, while for other pairs of i, j it asserts that $\alpha_i \not\leq \alpha_j$. Since we have just showed existence of an independence sequence, it only remains to check whether or not the statement is satisfiable by running through finite number of possibilities, which is a decidable process so this completes the proof of the theorem. □

5.4 Choice invisible degrees

Next, we want to show that there exists a degree such that no Π^0_1 choice class contains a member of that degree but can be contained in a Π^0_1 class which does not contain a member of every degree. These kinds of results are often associated with antibasis theorems. Examples of antibasis theorems can be seen in [2] and [1]. When proving antibasis theorems for Π^0_1 classes, we usually exclude the case that the given class might contain a member of every degree. Then for Π^0_1 choice classes, it is more concrete to have an antibasis result since there is no such Π^0_1 choice class at all which contains a member of every degree. This way we avoid the exception of having a Π^0_1 class containing a member of every degree.

Definition 74. A degree is called *invisible* if no Π^0_1 class contains a member of that degree unless it contains a member of every degree. A degree is *choice invisible* if no Π^0_1 choice class contains a member of that degree.

Let $\mathbf{I}$ denote the set of all invisible degrees for Π^0_1 classes and let $\mathbf{CI}$ denote the set of all choice invisible degrees. Every invisible degree is choice invisible. But we ask if the relation $\mathbf{I} \subset \mathbf{CI}$ is strict and we will show that $\mathbf{CI} - \mathbf{I}$ is indeed non-empty.

Recall that a degree is PA if it contains a set which codes a complete and consistent extension of Peano Artihmetic according to some computable bijection between sentences of first order language of arithmetic and the natural numbers. Although we give a more precise definition later, let us call for now a degree Martin-Löf random (1-random) if it contains a random set. It is worth noting that every degree $\mathbf{a} \geq \mathbf{0}'$ is 1-random. They are also PA since $\mathbf{0}'$ is a PA degree and PA degrees are upward closed. Moreover, if $\mathbf{a}$ is PA and 1-random, then $\mathbf{0}' \leq \mathbf{a}$. For a detailed account of the theory of algorithmic randomness we refer the reader to [27] and [64]. We first consider hyperimmune-free PA degrees for our purpose and then we consider 1-random sets.

Definition 75. (Kent and Lewis, 2010) We say that $\alpha \neq \mathbf{0}_{\mathfrak{P}}$ is *subclass invariant* if for any Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \alpha$ and any non-empty Π^0_1 class $\mathcal{P}' \subset \mathcal{P}$, $S(\mathcal{P}') = \alpha$. We say that $\alpha \neq \mathbf{0}_{\mathfrak{P}}$ is *weakly subclass invariant* if there exists a Π^0_1 class $\mathcal{P}$ with $S(\mathcal{P}) = \alpha$ and for any non-empty Π^0_1 class $\mathcal{P}' \subset \mathcal{P}$, $S(\mathcal{P}') = \alpha$.

Now, any α which is minimal must be subclass invariant. If α is subclass invariant, suppose that $\mathcal{P}$ be a Π^0_1 class such that $S(\mathcal{P}) = \alpha$ and suppose that $\mathcal{P}'$ is a non-empty Π^0_1 class with $S(\mathcal{P}') \subset \alpha$. Then let $\mathcal{Q} = \{0 * A : A \in \mathcal{P}\} \cup \{1 * A : A \in \mathcal{P}'\}$ be a Π^0_1 class. Note that $S(\mathcal{Q}) = \alpha$, so $\mathcal{Q}$ witnesses the fact that α is not subclass invariant which is a contradiction. So subclass invariancy is equivalent to minimality.

Theorem 63. (Kent and Lewis, 2010) Suppose that α is weakly subclass invariant. If a Π^0_1 class contains any member of any hyperimmune-free degree in α then it contains a member of every degree in α.

Then, by hyperimmune-free basis theorem, any non-empty Π^0_1 class which contains only members of degree in α contains a member of hyperimmune-free degree in α. Hence, by the theorem above, we have the fact that α is minimal if and only if it is weakly subclass invariant.

Recall that a degree is PA if and only if it contains a $\{0,1\}$-valued DNR function. Let $\boldsymbol{r}$ be the set of all 1-random degrees and let $\boldsymbol{p}$ be the set of all PA degrees. Kent and Lewis [2] showed that both $\boldsymbol{r}$ and $\boldsymbol{p}$ are minimal in $(\mathfrak{P}, <)$. This is not the case for Π^0_1 choice classes. In fact, we show that $\boldsymbol{r}$ and $\boldsymbol{p}$ are not in $\mathfrak{P}_c$. The reason is that if a Π^0_1 class contains a member of hyperimmune-free PA degree, then it contains a member of every PA degree. This is basically followed by the hyperimmune-free basis theorem and by the fact that any non-empty Π^0_1 class containing only $\{0,1\}$-valued DNR functions contains a member of every PA degree. The proof of the latter fact, originally proved in [65], appears in [2]. We modify that proof to get the desired result. But first we need to give a lemma which is necessary for our claim.

Lemma 18. If there exists a Π^0_1 choice class which contains a member of hyperimmune-free PA degree, then there exists a non-empty Π^0_1 choice class which contains only $\{0,1\}$-valued DNR functions.

Proof. Let $\mathcal{P}$ be a Π^0_1 choice class containing a hyperimmune-free PA member A. Then there exists a set B which is $\{0,1\}$-valued DNR such that $A \equiv_{tt} B$. This means there are total Turing functionals Ψ_m and Ψ_n such that $\Psi_m(A) = B$ and $\Psi_n(B) = A$. We then let $\mathcal{Q}$ contain all sets C such that $\Psi_m(C) = D$ and $\Psi_n(D) = C$, where D is a member of $\mathcal{P}$. We then let $\mathcal{Q}'$ be the elements of $\mathcal{Q}$ which are $\{0,1\}$-valued DNR. Now we need to argue that $\mathcal{Q}'$ is a non-empty Π^0_1 choice class. Now an infinite string is $\{0,1\}$-valued DNR if and only if there is no finite stage at which we see that some initial segment of it is not $\{0,1\}$-valued DNR. So then, we take a downward closed and computable set of strings Λ such that $\mathcal{Q} = [\Lambda]$. To form Λ' such that $\mathcal{Q}'$ is the set of infinite paths on Λ', we enumerate Λ but whenever we see that any finite string σ is not $\{0,1\}$-valued DNR, we stop enumerating in any extensions of σ. Then let $\mathcal{Q}'$ be the set of infinite paths through Λ'. Clearly, $\mathcal{Q}'$ is a non-empty Π^0_1 choice class containing only $\{0,1\}$-valued DNR functions. □

Theorem 64. Any non-empty Π^0_1 class $\mathcal{P}$ containing only $\{0,1\}$-valued DNR functions contains a member of every PA degree. Moreover, $\mathcal{P}$ contains members of the same degree.

Proof. The proof uses forcing with Π^0_1 classes. If Λ is computable and downward closed then consider $\Psi_i(\emptyset)$ such that $\Psi_i(\emptyset; i) \downarrow= n$ if and only if there exists some $l > i$ such that $\tau(i) = n$ for all $\tau \in \Lambda$ of length l. By the uniformity of the recursion theorem (Theorem 1), there exists a computable function f such that, whenever $[\Lambda_j]$ is non-empty and contains only $\{0,1\}$-valued DNR functions, there exist sets $A, B \in [\Lambda_j]$ with $A(f(j)) = 0$ and $B(f(j)) = 1$. Here one can also use Lemma 2.6 in [63].

Assume that we are given j_0 such that $[\Lambda_{j_0}] = \mathcal{P}$ is non-empty and contains only $\{0,1\}$-valued DNR functions. Let A be a $\{0,1\}$-valued DNR function. We construct $B = \bigcup_{s\in\omega} \sigma_s$ which is in $\mathcal{P}$ and is of the same degree as A. We define an infinite descending sequence $[\Lambda_{j_0}] \supset [\Lambda_{j_1}] \supset [\Lambda_{j_2}] \supset \cdots$ for approximating B in $\mathcal{P}$.

Suppose that we are given j_0 such that $[\Lambda_{j_0}]$ is non-empty.

At stage 0: Define $\sigma_0 = \emptyset$.

At stage $s > 0$: Suppose that we have already decided j_{s-1} and σ_{s-1}. Suppose also that there exists $C \in \left[\Lambda_{j_{s-1}}\right]$ with $C(f(j_{s-1})) = A(s-1)$.

Using an oracle for A, we can therefore compute σ of length $f(j_{s-1})+1$ such that $\sigma(f(j_{s-1})) = A(s-1)$ which is an initial segment of some $C \in \left[\Lambda_{j_{s-1}}\right]$. This follows from the fact that any $\{0,1\}$-valued DNR function computes a member of any non-empty Π^0_1 class such that every member is $\{0,1\}$-valued DNR.

We then define $\sigma_s = \sigma$. Then define j_s so that $[\Lambda_{j_s}]$ is the set of all $C \in \left[\Lambda_{j_{s-1}}\right]$ which extends σ.

The fact that B computes A follows from the fact that an oracle for B allows us to retrace every step of the construction defining B.

This proves the first part. Now to show that there are two members of the same degree, suppose that $\mathcal{P} = [\Lambda]$ is a Π^0_1 class, for some downward closed computable set of strings Λ, such that $\mathcal{P}$ contains only $\{0,1\}$-valued DNR functions. We take two incompatible strings σ_0 and σ_1 in Λ. Now since every member of the set of all infinite branches above σ_0 and σ_1 is $\{0,1\}$-valued DNR, they both contain a member of every PA degree by the previous part. Hence, they contain members of the same degree and therefore so does $\mathcal{P}$. □

Corollary 21. $\mathbf{CI} - \mathbf{I}$ is non-empty. Moreover, $\boldsymbol{p}$ is not a subset of the degree spectrum of any Π^0_1 choice class.

Proof. It follows from Lemma 18 and Theorem 64 that hyperimmune-free PA degrees are choice invisible but not invisible.

5.4.1 Random sets and Π^0_1 choice classes

We first review *Lebesgue measure* for Cantor space. Intuitively, a set $\mathcal{A}$ of binary reals is measured by estimating how much of the interval $[0,1] = N_\emptyset$ it covers. This is done by covering $\mathcal{A}$ with sets that can be measured, i.e. sets that can be expressed by countable unions of open intervals N_σ, and taking the infimum of the measure of such covers.

Definition 76. (i) Let $\mathcal{A} \subset 2^\omega$ be a set. We say $\{N_\sigma\}_{\sigma\in\Gamma}$ is a *covering* of $\mathcal{A}$ if $\mathcal{A} \subset \bigcup_{\sigma\in\Gamma} N_\sigma$.

(ii) The Lebesgue *outer measure* μ^* is given by:

$$\mu^*(\mathcal{A}) = Inf\{\textstyle\sum_{\sigma\in\Gamma} 2^{-|\sigma|} : \{N_\sigma\}_{\sigma\in\Gamma} \text{ is a covering of } \mathcal{A}\}.$$

(iii) $\mathcal{A}$ is Lebesgue *measurable* if for each $\mathcal{X} \subset 2^\omega$ we have

$$\mu^*(\mathcal{X}) = \mu^*(\mathcal{X} \cap \mathcal{A}) + \mu^*(\mathcal{A} \cap \overline{\mathcal{X}}).$$

If A is measurable, the Lebesgue *measure* of A is $\mu(A) = \mu^*(A)$.

Now we shall give the definition for 1-random sets more precisely as follows.

Definition 77. A class $\mathcal{P} \subset 2^\omega$ is of Σ^0_1*-measure zero* if there is a recursively enumerable sequence of Σ^0_1 classes $\mathcal{B}_0, \mathcal{B}_1, \ldots$ such that $\forall n(\mu(\mathcal{B}_n) < 2^{-n})$ and $\mathcal{P} \subset \bigcap_{n\in\omega} \mathcal{B}_n$. A set $B \subset \omega$ is called *1-random (Martin-Löf random)* if the class $\{B\}$ is not of Σ^0_1-measure zero.

Although Π^0_1 choice classes can contain a member of PA degree, we now shall argue that 1-random sets are too "computationally related" to be a member of a Π^0_1 choice class. The following result can be found in [66].

Theorem 65 (Kautz, 1991)**.** If a Π^0_1 class contains a 1-random set, then it is of positive measure.

The next theorem was shown by Kučera [50].

Theorem 66 (Kučera, 1985)**.** If a Π^0_1 class is of positive measure then it contains a member of every 1-random degree.

The following result shows that Π^0_1 choice classes do not contain random sets.

Theorem 67. No Π^0_1 choice class contains a 1-random set.

Proof. Suppose that a Π^0_1 class $\mathcal{P} = [\Lambda]$, for some downward closed computable set of strings Λ, contains a 1-random set. Then it is of positive measure. Hence, it must contain at least two 1-random sets, say A and B, since the class of sets which are not 1-random is of measure 0 and any class of positive measure must contain positive measure of sets which are 1-random. Similar to Theorem 64, let $\sigma_0 \subset A$ and $\sigma_1 \subset B$ be two incompatible strings in Λ such that they are infinitely extendible. Then the set of all infinite branches above each σ_i, for $i = \{0, 1\}$, is of positive measure. Hence, they both contain members of every 1-random degree. Therefore, $\mathcal{P}$ must contain members of the same degree. This contradicts the definition of Π^0_1 choice classes. $\square$

Bibliography

[1] A. Çevik: Antibasis theorems for Π^0_1 classes and the jump hierarchy. Archive for Mathematical Logic, Vol. 52, Issue 1-2, pp. 137-142 (2013).

[2] T. Kent and A. E. M. Lewis: On the degree spectrum of a Π^0_1 class. Trans. Amer. Math. Soc. Vol. 362, pp. 5283-5319 (2010).

[3] C. Jockusch and R. I. Soare: Π^0_1 classes and degrees of theories. Trans. Amer. Math. Soc. Vol. 173, pp. 33-56 (1972).

[4] C. Jockusch and R. I. Soare: Degrees of members of Π^0_1 classes. Pacific J. Math. Vol. 40, pp. 605-616 (1972).

[5] R. I. Soare: Recursively Enumerable Sets and Degrees, Perspectives in Mathematical Logic. Springer-Verlag, Berlin (1987).

[6] S. B. Cooper: Computability theory, Chapman & Hall, CRC Press, Boca Raton, FL, New York, London (2004).

[7] P. Odifreddi: Classical Recursion Theory Vol.I & Vol.II., North-Holland, Studies in Logic and the Foundations of Mathematics (1999).

[8] M. Lerman: The Degrees of Unsolvability, Springer (1983).

[9] K. Gödel: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, Monatsch Math. und Phys. Vol. 38, pp. 173-198 (1931).

[10] A. Church: An unsolvable problem of elementary number theory. Amer. J. Math, Vol. 58, pp. 345-363 (1936).

[11] Y. Matiyasevich: Hilbert's Tenth Problem, MIT Press, Cambridge, Mass., London (1993).

[12] S. C. Kleene: General recursive functions of natural numbers. Math. Ann., Vol. 112, pp. 727-742 (1936).

[13] H. Rogers: Theory of recursive functions and effective computability. McGraw Hill (1967).

[14] J. R. Shoenfield: On degrees of unsolvability. Ann. Math., Vol. 60, pp. 644-653 (1959).

[15] J. R. Shoenfield: A theorem on minimal degrees. Journal of Symbolic Logic, Vol. 31, pp. 539-544 (1966).

[16] S. C. Kleene and E. L. Post: The uppersemilattice of degrees of recursive unsolvability. Ann. Math., Vol. 59, pp. 379-407 (1954).

[17] C. Spector: On degrees of recursive unsolvability. Ann. Math., Vol. 64, pp. 581-592 (1956).

[18] R. M. Friedberg: A criterion for completeness of degrees of unsolvability. Journal of Symbolic Logic, Vol. 22, pp. 159-160 (1957).

[19] R. M. Friedberg: Two recursively enumerable sets of incomparable degrees of unsolvability. Proc. Nat. Acd. Sci., Vol. 43, pp. 236-238 (1957).

[20] A. A. Muchnik: Negative answer to the problem of reducibility in the theory of algorithms. Dokl. Acad. Nauk. Vol. 108, pp. 194-197 (1956).

[21] G. E. Sacks: On the degrees less than $\mathbf{0}'$. Ann. Math., Vol. 77, pp. 211-231 (1963).

[22] G. E. Sacks: Recursive enumerability and the jump operator. Trans. Amer. Math. Soc., Vol. 108, pp. 223-239 (1963).

[23] D. A. Martin: Classes of recursively enumerable sets and degrees of unsolvability. Zeit. Math. Log. Grund. Math., Vol. 12, pp. 295-310 (1966).

[24] J. Munkres: Topology (Second edition), Prentice Hall, Incorporated (2000).

[25] B. Mendelson: Introduction to Topology (Third Edition), Dover Books, Boston (1990).

[26] D. König: Sur les correspondances multivoques des ensembles. Fund. Math., Vol. 8, pp. 114-134 (1926).

[27] R. Downey and D. Hirshfeldt: Algorithmic Randomness and Complexity, Springer-Verlag (2010).

[28] L. Henkin: The completeness of first-order functional calculus. Journal of Symbolic Logic, Vol. 14, pp. 159-166 (1949).

[29] S. Feferman: Degrees of unsolvability associated with classes of formalized theories. Journal of Symbolic Logic, Vol. 22, pp. 161-175 (1957).

[30] J. R. Shoenfield: Degrees of Models. Journal of Symbolic Logic, Vol. 25, pp. 233-237 (1960).

[31] A. Ehrenfeucht: Separable theories. Bull. Acad. Polon. Sci. Sr. Sci. Math. Astronom. Phys., Vol. 9, pp. 17-19 (1961).

[32] D. Cenzer: Classes in Recursion Theory. Handbook of Computability Theory, Vol. 140, Studies in Logic and the Foundations of Mathematics, North-Holland, pp. 37-39 (1999).

[33] S. C. Kleene: A symmetric form of Gödel's theorem. Neder. Akad., Wetenshappen Proc. Ser. A, Vol. 53, pp. 800-802 (1950).

[34] J. R. Shoenfield: Degrees of formal systems. Journal of Symbolic Logic, Vol. 23, pp. 389-392 (1958).

[35] D. E. Diamondstone, D. D. Dzhafarov, and R. I. Soare: Π^0_1 classes, Peano arithmetic, randomness, and computable domination. Notre Dame J. Formal Logic, Vol. 51(1), pp. 127-159 (2010).

[36] G. Peano: Arithmetices principia, nova methodo exposita. Torino, trans. in Van Heijenoort (1967), pp. 85-97 (1889).

[37] D. Scott: Algebras of sets binumerable in complete extensions of arithmetic. Proc. Symp. Pure Appl. Math., Vol. 5, pp. 117-121 (1962).

[38] B. J. Rosser: Extensions of some theorems of Gödel and Church. Journal of Symbolic Logic, Vol. 1, pp. 87-91 (1936).

[39] H. Putnam and R. M. Smullyan: Exact separation of recursively enumerable sets within theories. Proc. Amer. Math. Soc., Vol. 11, pp. 574-577 (1960).

[40] D. Cenzer, P. Clote, R. L. Smith, R. I. Soare, and S. S. Wainer: Members of countable Π^0_1 classes. Annals of Pure and Applied Logic, Vol. 31, pp. 145-63 (1986).

[41] D. Cenzer and R. L. Smith: On the ranked points of a Π^0_1 set. Journal of Symbolic Logic, Vol. 54, pp. 975-991 (1989).

[42] D. Cenzer, R. Downey, C. Jockusch. R. A. Shore: Countable thin Π^0_1 classes. Annals of Pure and Applied Logic, Vol. 59, pp. 79-139 (1993).

[43] C. Jockusch and R. A. Shore: Pseudo-jump operators II transfinite iterations, hierarchies and minimal covers. Journal of Symbolic Logic, Vol. 49, pp. 1206-1236 (1984).

[44] P. Cholak, R. Coles, R. Downey, and E. Herrmann: Automorphisms of the lattice of Π^0_1 classes; Perfect thin classes and ANC degrees. Trans. Amer. Math. Soc., Vol. 353, pp. 4899-4924 (2001).

[45] S. Binns: Small Π^0_1 classes. Archive for Mathematical Logic, Vol. 45, pp. 393-410 (2006).

[46] P. J. Cohen: The independence of the continuum hypothesis. Proc. Nat. Acd. Sci., Vol. 50, pp. 1143-1148 (1963).

[47] C. Jockusch and R. I. Soare: A minimal pair of Π^0_1 classes. Journal of Symbolic Logic, Vol. 36, pp. 66-78 (1971).

[48] M. J. Groszek and T. A. Slaman: Π^0_1 classes and minimal degrees. Annals of Pure and Applied Logic, Vol. 87, pp. 117-144 (1997).

[49] C. Jockusch: Degrees of functions with no fixed points. Logic, Methodology and Philosophy of Science VIII. Proceedings of the Eighth International Congress of Logic, Methodology and Philosophy of Science, Vol. 126, pp. 191-201 (1989).

[50] A. Kučera: Measure, Π^0_1 classes and complete extensions of PA. Springer Lecture Notes in Mathematics, Vol. 1141, pp. 245-259, Karger (1985).

[51] A. E. M. Lewis: The search for natural definability in the Turing degrees. Unpublished lecture notes (2012).

[52] A. E. M. Lewis: Π^0_1 classes, strong minimal covers and hyperimmune-free degrees. Bulletin of the London Mathematical Society, Vol. 39, pp. 892-910 (2007).

[53] M. Kumabe and A. E. M. Lewis: A fixed point free minimal degree. Journal of the London Mathematical Society, Vol. 80 pp. 785-797 (2007).

[54] R. Downey, C. Jockusch, and M. Stob: Array non-recursive degrees and genericity, in Computability, Enumerability, Unsolvability (Cooper, Slaman, Wainer, eds.), London Mathematical Society Lecture Notes Series Vol. 224, pp. 93-105, Cambridge University Press (1996).

[55] M. Cai: A hyperimmune minimal degree and an ANR 2-minimal degree. Notre Dame Journal of Formal Logic, Vol. 51, pp. 443-455 (2010).

[56] M. Cai: Minimality and Jump Classes, CiE 2011 conference presentation notes, http://cie2011.fmi.uni-sofia.bg/files/slides/Cai.pdf (last accessed: 13.09.2013).

[57] M. Cai: 2-minimality, jump classes and a note on natural definability. Annals of Pure and Applied Logic, Vol. 165, pp. 724-741 (2014).

[58] A. Kučera: An alternative, priority-free solution to Post's problem. Springer Lecture Notes in Computer Science, Springer-Verlag, Vol. 233, pp. 493-500 (1986).

[59] A. Kučera: On the use of diagonally non-recursive functions. Studies in Logic and the Foundations of Mathematics, Vol. 129 (Logic Colloquium'87), pp. 219-239, Elsevier (1989).

[60] D. Cenzer and F. Riazati: Minimal extensions of Π^0_1 classes. Mathematical Logic Quarterly, Vol. 51, pp. 206-216 (2005).

[61] L. Lawton: Almost complemented Π^0_1 classes. Archive for Mathematical Logic, Vol. 45(5), pp. 555-568 (2006).

[62] J. A. Cole and S. G. Simpson: Mass problems and hyperarithmeticity. J. Math. Log., Vol. 7, pp. 125-143 (2007).

[63] A. Kučera, T. A. Slaman: Low upper bounds of ideals. Journal of Symbolic Logic, Vol. 74(2), pp. 517-534 (2009).

[64] A. Nies: Computability and Randomness. Oxford University Press, New York (2012).

[65] S. G. Simpson: Mass problems and randomness. Bull. Symbolic Logic, Vol. 11, pp. 1-27 (2005).

[66] S. M. Kautz: Degrees of Random Sets, Cornell University, PhD Thesis (1991).