Harmonic Vector Fields on Pseudo-Riemannian Manifolds

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Abstract

This thesis generalises the theory of harmonic vector fields to the non-compact pseudo-Riemannian case. After introducing the required background theory we consider the first variation of the local energies to find the Euler-Lagrange equations for this new case. We then introduce a natural closed conformal gradient field on pseudo-Riemannian warped products and find the Euler-Lagrange equations for harmonic closed conformal vector fields of this sort. We then give examples of such harmonic closed conformal fields, this leads to a harmonic vector fields on a 2-sphere with a rotationally symmetric singular metric. The harmonic conformal gradient fields on all hyperquadrics are then categorised up to congruence. The harmonic Killing fields on the 2-dimensional hyperquadrics are found, and shown to be unique up to congruence.

Contents

Dedicated to my wife, Philippa.

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Declaration

This thesis has not previously been accepted in substance for any degree and is not being concurrently submitted in candidature for any degree other than Doctor of Philosophy of the University of York. This thesis is the result of my own investigations, except where otherwise stated. Other sources are acknowledged by explicit references.

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Introduction

The progression of the theory of harmonic vector fields begins with harmonic maps. If vector fields are considered as maps between Riemannian manifolds, it is interesting to ask under what conditions these maps are harmonic. First a metric must be applied to the tangent bundle of the manifold, the natural choice being the Sasaki metric. This restricts harmonic vector fields to precisely those vector fields that are parallel, and thus of constant length. This further limits consideration to sphere bundles, of zero Euler characteristic. The introduction of a 2-parameter family of metrics (the generalised Cheeger-Gromoll metrics) on the tangent bundle removes this restriction. The generalised Cheeger-Gromoll metric on Riemannian vector bundles is semi-Riemannian for certain parameter values, meaning it is of variable signature, specifically it has index 1 outside a closed ball. The nature of this metric leads to our consideration of what happens if we consider harmonic vector fields on pseudo-Riemannian manifolds.

In Chapter 1 we cover background theory that will allow the thesis to proceed smoothly, especially the results of Chapter 2. Other theoretical underpinnings are introduced at the relevant juncture. While the majority of the results contained within Chapter 1 are well known in the Riemannian case, some additional work is required to ensure they apply in the pseudo-Riemannian. Thus this chapter is both the required background theory and the generalisation of said theory to this less familiar situation.

We assume the reader is familiar with Riemannian geometry and tensor calculus; see for instance [15, 20, 22] for excellent introductions to these. We define the basics of pseudo-Riemannian geometry from the view point of a Riemannian geometer, and note the similarities and differences this generalisation affords. For this O'Neill's book, [18], has been an invaluable resource; both as one of the few books on pseudo-Riemannian geometry and by tackling it in a clear and precise manner. We introduce vector bundles and principal bundles, primarily based on the work for the Riemannian case of [21]. These two types of bundles are combined in associated vector bundles. Ehresmann connections of principal bundles provide the motivation the connection map; the Ehresmann connection splits the tangent space into horizontal and vertical spaces. The link between principal bundles and vector bundles leads us to desire a similar splitting of the tangent space of the vector bundle. We find this from the connection map.

Key to the definition of the connection map is the concept of a mapping between manifolds. We therefore consider how principal, vector and associated vector bundles behave under the action of the pullback through a mapping. We do not define the connection map in the usual way; rather defining it in terms of the diagonal section of the square bundle. This gives the same characterising relation between the connection map and a linear connection on the vector bundle, which is vital for its use in Chapter 2. The advantage of our method is that it allows a more intuitive definition.

Chapter 2 begins with an introduction to the concept of a harmonic map, based on the summary of the compact Riemannian case in [11]. We generalise this to non-compact pseudo-Riemannian manifolds, finding the Euler-Lagrange equations. We then consider the conditions these impose on sections of a vector bundle, upon which we assign a metric, initially the natural Sasaki metric. Firstly we try to work with the conditions; restricting ourselves to sphere bundles, and then work around them; by altering the metric we impose on the vector bundle.

We assign the vector bundle the Sasaki metric and then examine the conditions on the section the Euler-Lagrange equations impose. These are very strong; namely the section is necessarily parallel, and hence of constant length. To work around this we then consider sections of the sphere bundle, and generalise the basic result of the Riemannian theory to pseudo-Riemannian sphere bundles. This only applies to vector bundles of zero Euler characteristic; so to move around these restrictions we next change the metric we impose on the vector bundle.

We introduce the 2-parameter family of metrics known as the generalised Cheeger-Gromoll metric. This family was first introduced for the Riemannian case in [3]. In the Riemannian case this metric is known to have interesting geometry, explored in [4]; most notably it has a variable signature. In the Riemannian case, for negative values of one parameter there is a Riemannian ball bundle, a degenerate sphere bundle and the remainder of the bundle that is of index $(n-1, 1)$. We find a similar result about the change of signature in the pseudo-Riemannian case.

For pseudo-Riemannian vector bundles the generalised Cheeger-Gromoll metric needs to be altered slightly to provide a consistent definition. This altered metric is then applied to harmonic sections by generalising the work of [3] to non-compact pseudo-Riemannian vector bundles using our generalised Cheeger-Gromoll metric. The altered metric has a degenerate measure zero subspace that we exclude in the definition of a (p, q)-harmonic sections. This results in an Euler-Lagrange equation valid on the entirety of the vector bundle. If we consider the Euler-Lagrange equations for a section that is entirely within the degenerate set then the section is trivially harmonic.

We next consider the first examples of pseudo-Riemannian (p, q)-harmonic sections. In Chapter 3 we examine the well-know class of closed conformal vector fields. There is no general classification of harmonic closed conformal vector fields, though a class specific reduction of the Euler-Lagrange equations can be found in terms of the conformal factor. We then move to the particular case of warped products, with a real interval as the warping factor. These have a natural closed conformal vector field, which reduce the Euler-Lagrange equations to an ordinary differential equation. This ODE is considered in general in Appendix A, while in Chapter 3 we give some examples of (p, q)-harmonic closed conformal vector fields. It is interesting to note that each of our examples is in fact a family of examples. With the harmonicity of the natural closed conformal vector field depending only on the warping function there is a free choice of any pseudo-Riemannian manifold for the warped factor. Particular choices of warped factor and warping function yield some of the hyperquadrics, shown in Chapter 4, and hence some of the pseudo-Riemannian space forms (including the Riemannian space forms). The same warping functions, however, can be applied to other warped factors. This is a generalisation of the results of [3], where conformal gradient fields were found on Riemannian space forms. We not only return the same results for a large number of pseudo-Riemannian space forms, but any such warped product. We use Jacobi elliptic warping functions to find an example of a harmonic vector field on the 2-sphere with a rotationally symmetric singular metric of constant sectional curvature.

In Chapter 4 we consider a particular class of manifolds, the hyperquadrics; in many cases these are isometric to the pseudo-Riemannian space forms. We begin with the general theory, and then look in particular at the dimension 2 space forms. We consider two classes of vector fields, conformal gradient fields and Killing fields. For conformal gradient fields we find a complete categorisation of the (p, q)-harmonic fields on each hyperquadric. In the case of Killing fields however we lack the normal form which would enable this. Instead we have a specific Euler-Lagrange equation that would lead to a categorisation when a normal form is found. In the 2-dimensional case we categorise all Killing fields on the six hyperquadrics.

The conformal gradient and Killing fields provide us with two classes of harmonic vector fields on the negative definite 2-sphere. This is an intriguing contrast to the positive definite 2-sphere on which no harmonic vector fields are known.

The classification of (p, q)-harmonic conformal gradient and Killing fields on the dimension 2 hyperquadrics leads us to consider if there is a family of (p, q)-harmonic conformal fields on these spaces, similar to that found on the hyperbolic space with a Kähler structure. In Section 4.5 we introduce para-Kähler geometry. This structure can only exist on neutral manifolds. On the neutral 2-dimensional hyperquadrics it allows us to map conformal gradient fields to Killing fields. Unfortunately it does not lead to a one parameter family of (p, q)-harmonic conformal vector fields as the Kähler structure does.

Chapter 1

Preliminaries

We begin with an introduction to the concepts that form the foundation of the content of this thesis. This covers the choice of conventions of pseudo-Riemannian geometry used throughout this thesis. We then introduce concepts of vector bundle theory that are key in defining the generalised Cheeger-Gromoll metric and performing calculus of variations to find the Euler-Lagrange equations for a harmonic section. Throughout this chapter M is a smooth orientable manifold of dimension n.

1.1 Pseudo-Riemannian geometry

There are various conventions in pseudo-Riemannian geometry that should be clarified. We begin with the basic definitions and then make our choice of conventions. All the concepts here are similar to those in Riemannian geometry, but have been extended from positive definite metrics to non-degenerate metrics of indefinite signature. The work here is based on [18]. While semi-Riemannian is often a synonym for pseudo-Riemannian it can also mean a metric of *variable* signature. For this reason we favour pseudo-Riemannian in our writing, and have altered work taken from [18] to match this.

1.1.1 Definition ([18, p. 54])**.** A *metric tensor* g on a smooth manifold M is a symmetric non-degenerate $(2, 0)$ tensor field on M of constant index. $\diamondsuit$

In Definition 1.1.1 non-degeneracy means that if, for all $x \in M$ and any tangent vector $X \in T_xM$, we have $g(X, Y) = 0$ for all tangent vectors $Y \in T_xM$ then $X = 0$. The definition of index will be clarified below (Definition 1.1.5).

1.1.2 Definition ([18, p. 54])**.** A *pseudo-Riemannian manifold* is a smooth manifold M furnished with a metric tensor g. $\diamondsuit$

One key difference between Riemannian and pseudo-Riemannian geometry is the notion of an orthonormal basis. Of necessity such a basis consists of mutually orthogonal vectors of both positive and negative unit"length", where by the "length" of a tangent vector X we understand $g(X, X)$. Henceforward we refer to this as the *pseudo-Riemannian length* of X.

1.1.3 Definition. An *orthonormal basis* of a pseudo-Riemannian manifold at a point $x \in M$

is a set of tangent vectors $\{E_i\}$ at x that span T_xM, such that

$$g_x(E_i, E_j) = \begin{cases} 0 & \text{if } i \neq j \\ \pm 1 & \text{if } i = j. \end{cases} \qquad \lozenge$$

Furthermore, vectors can be split into three distinct "causal types": space-like, light-like and time-like.
1.1.4 Definition ([18, p. 56]). A tangent vector X to M is:

- *space-like* if $g(X, X) > 0$ or $X = 0$,
- *null* or *light-like* if $g(X, X) = 0$ and $X \neq 0$,
- *time-like* if $g(X, X) < 0$. ◊

From these we can define the index of (M, g), which may be formulated in two different ways. It is a global property of the metric tensor and does not change as the point $x \in M$ varies.
1.1.5 Definition. The *index* of the metric tensor g is either the number of time-like basis vectors or a pair of numbers representing the number of space-like then time-like basis vectors. ◊

For example, a metric tensor with 3 space-like basis vectors and 4 time-like basis vectors has index 4 or index $(3, 4)$.

It will be useful to have shorthand notation for the causal type of the vectors in an orthonormal basis.
1.1.6 Definition. The *indicator* of an orthonormal basis vector E_i is

$$\epsilon_i = g(E_i, E_i) = \pm 1. \qquad \lozenge$$

Next we define a connection, then the Levi-Civita connection of a pseudo-Riemannian manifold.
1.1.7 Definition ([18, p. 58]). A *connection* D on a smooth manifold M is a function $D \colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that:

(D1) $D_V W$ is $C^\infty(M)$ linear in V,

(D2) $D_V W$ is $\mathbb{R}$-linear in W,

(D3) $D_V(fW) = (Vf)W + fD_V W$ for $f \in C^\infty(M)$,

where $\Gamma(TM)$ is the set of all smooth vector fields on M, and $C^\infty(M)$ is the set of smooth functions on M. Call $D_V W$ the *covariant derivative* of W with respect to V for the connection D ◊
1.1.8 Theorem ([18, p. 61]). *On a pseudo-Riemannian manifold (M, g) there is a unique connection D, called the* Levi-Civita connection, *such that:*

(D4) $[V, W] = D_V W - D_W V$, and

(D5) $Xg(V, W) = g(D_X V, W) + g(V, D_X W)$,

for all $X, V, W \in \Gamma(TM)$. The Levi-Civita connection is characterised by the Koszul formula:

$$2g(D_V W, X) = Vg(W, X) + Wg(X, V) - Xg(V, W) - g(V, [W, X]) + g(W, [X, V]) + g(X, [V, W]).$$

The curvature of a pseudo-Riemannian manifold is defined in the same way as that of a Riemannian manifold, and has similar properties.

1.1.9 Definition ([18, p. 74]). Let (M, g) be a pseudo-Riemannian manifold with Levi-Civita connection D. The function $R\colon \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ given by

$$R(X, Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z$$

is a $(3, 1)$ tensor field on M called the *Riemann tensor* of M. Note the reversal of sign convention from [18]. ◊

1.1.10 Proposition ([18, p. 75]). *If $X, Y, Z, V, W \in T_x(M)$ then:*

1. $R(X, Y) = -R(Y, X)$,
2. $g(R(X, Y)V, W) = -g(R(X, Y)W, V)$,
3. $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$,
4. $g(R(X, Y)V, W) = g(R(V, W)X, Y)$.

From this we define the sectional curvature of a tangent 2-plane.

1.1.11 Definition. Let $\Pi \subset T_xM$ be a non-degenerate 2-dimensional subspace, with basis $\{V, W\}$. The number

$$K(V, W) = g(R(V, W)V, W)/Q(V, W),$$

where

$$Q(V, W) = g(V, V)g(W, W) - g(V, W)^2,$$

is independent of choice of basis $\{V, W\}$, and is called the *sectional curvature* $K(\Pi)$ of Π. ◊

The non-degeneracy of Π in Definition 1.1.11 ensures the non-vanishing of $Q(V, W)$.

Orthonormal tangent bases may be extended to local frame fields, as in the Riemannian case.

1.1.12 Definition ([18, p. 84]). An orthonormal basis for a tangent space T_xM is called a *frame on M at x*, and a set $\{E_1, \ldots, E_n\}$ of n mutually orthogonal unit vector fields is called a *frame field* (since it assigns a frame at every point). In general there may not be a frame field on all of M, but local frame fields always exist. ◊

Vector fields can be expanded in terms of a local frame field. A common theme in the change from Riemannian to pseudo-Riemannian geometry is the incorporation of indicators into such expansions.

1.1.13 Proposition ([18, p. 84]). *Let $V \in \Gamma(TM)$ and let $\{E_i\}$ be a local frame field on M. Then on the domain of this frame field*

$$V = \sum_{i=1}^{n} \epsilon_i \, g(V, E_i)E_i,$$

and the metric tensor can therefore be expressed as

$$g(V, W) = \sum_i \epsilon_i \, g(V, E_i)g(W, E_i).$$

The pseudo-Riemannian versions of the gradient, divergence and Laplacian remain the same in principle as their Riemannian counterparts, with only the addition of indicator terms where appropriate required.

1.1.14 Definition ([18, p. 85]). The *gradient* $\operatorname{grad} f$ of a function $f \in C^\infty(M)$ is the vector field metrically dual to the differential $df \in \Gamma(T^*M)$, characterised by

$$g(\operatorname{grad} f, X) = df(X) = Xf \quad \text{for all } X \in \Gamma(M).$$

Thus

$$\operatorname{grad} f = \sum_i \epsilon_i \, df(E_i) E_i.$$

The notation ∇f is also used. ♢

1.1.15 Definition ([18, p. 86]). The *divergence* $\operatorname{div} V$ of a vector field $V \in \Gamma(TM)$ is the contraction with the covariant derivative D_V:

$$\operatorname{div} V = \sum_i \epsilon_i \, g(D_{E_i} V, E_i). \qquad ♢$$

1.1.16 Definition. The *Laplacian* Δf of a function $f \in C^\infty(M)$ is the divergence of its gradient:

$$\Delta f = -\operatorname{div} \operatorname{grad} f \in C^\infty(M). \qquad ♢$$

The Riemann tensor gives rise to two other curvatures, the Ricci curvature and the scalar curvature, defined as in the Riemannian case, with additional indicator terms.

1.1.17 Definition ([18, p. 87]). The *Ricci curvature tensor* Ric of (M, g) is the following contraction of the Riemann tensor:

$$\operatorname{Ric}(X, Y) = \sum_i \epsilon_i \, g(R(X, E_i)E_i, Y). \qquad ♢$$

1.1.18 Definition ([18, p. 88]). The *scalar curvature* S of (M, g) is the contraction of the Ricci tensor:

$$S = \sum_i \epsilon_i \operatorname{Ric}(E_i, E_i) = \sum_{i \neq j} K(E_i, E_j) = 2 \sum_{i<j} K(E_i, E_j). \qquad ♢$$

Finally we define some useful properties of vector fields. This will allow us to calculate local flows and apply isometries to find congruence classes.

1.1.19 Definition. Let X be a vector field over a manifold M. An integral curve X is a path $\alpha \colon I \to M$ such that $\alpha' = X(\alpha)$, i.e. $\alpha'(t) = X(\alpha(t))$ for all $t \in I$, where $I \subset \mathbb{R}$ is an open interval. ♢

1.1.20 Definition. Let $X \in \Gamma(TM)$. For every $x \in M$ let $\alpha_x \colon I_x \to M$ be the maximal integral curve of X such that $\alpha_x(0) = x$. Let $U_X \subset M \times \mathbb{R}$ be the open subset:

$$U_X = \{(x, t) : t \in I_x\}.$$

The *local flow* of the vector field X is the map

$$\psi \colon U_X \to M; \ \psi(x, t) = \alpha_x(t).$$

For each $t \in \mathbb{R}$ define $U_t \subset M$ to be the open set:

$$U_t = \{x \in M : (x, t) \in U_X\}.$$

The *stages* of the flow are the maps

$$\psi_t \colon U_t \to M; \ \psi_t(x) = \psi(x, t),$$

defined for all t such that $U_t \neq \emptyset$. The vector field is said to be *complete*, and the flow *global*, if the open set $U_X = M \times \mathbb{R}$ i.e. all maximal integral curves have domain $\mathbb{R}$. ♢

Proposition 1.1.21 allows stages of flow to be combined easily as well. It also allows the inverse of the flow to be found from the flow itself.

1.1.21 Proposition ([18, Lemma 1.54]). *Let ψ be the local flow of a vector field X. Then ψ is a local 1-parameter subgroup of diffeomorphisms of M:*

1. *$U_0 = M$ and $\psi_0(x) = x$ for all $x \in M$.*
2. *If $x \in U_t$ and $\psi_t(x) \in U_s$ then $x \in U_{s+t}$ and furthermore $\psi_s(\psi_t(x)) = \psi_{s+t}(x)$.*

It follows from Proposition 1.1.21 that if the vector field X is complete then each stage ψ_t is a diffeomorphism of M and the inverse map $(\psi_t)^{-1} = \psi_{-t}$.

Innocuous looking vector fields may not be complete. The following incomplete planar vector field is similar to the well known 1-dimensional example [21, Vol. I, §5].

1.1.22 Example. Let $M = \mathbb{R}^2$ and let X be the vector field

$$X(x, y) = (x^2 - y^2, 2xy);$$

thus when $\mathbb{R}^2 \cong \mathbb{C}$

$$X(z) = z^2.$$

Then it is easily checked that

$$\psi(z, t) = \frac{z}{1 - tz},$$

and the local flow domain is therefore

$$U_X = \{(z, t) \in \mathbb{C} \times \mathbb{R} : tz \neq 1\}.$$

Hence this vector field is not complete. The local 1-parameter subgroup property can be clearly seen:

$$\begin{aligned} \psi_s(\psi_t(z)) &= \psi_s(\frac{z}{1 - tz}) \\ &= \frac{z}{1 - tz} \frac{1 - tz}{(1 - tz) - sz} \\ &= \frac{z}{1 - (s + t)z} = \psi_{s+t}(z), \end{aligned}$$

for all $z \neq 1/t$ and $z \neq 1/(s + t)$.

In general, if the manifold M is compact or the vector field X is linear then X is complete.

1.2 Vector bundles and linear connections

In this section we explore the properties of vector bundles and connections, and their pullbacks. We begin with the basic definitions of vector bundle theory, before moving onto pullbacks and codifferentials. The main sources are [20, 22], each of which offers a fuller introduction to the concepts described. We begin with the property that defines a bundle, local triviality.

1.2.1 Definition ([22, §12.3]). A surjective smooth map $\pi \colon \mathcal{E} \to M$ of manifolds is said to be *locally trivial of rank r* if:

1. each fibre $\pi^{-1}(x)$ has the structure of a (real) vector space of dimension r and,
2. for each $x \in M$ there exists an open neighbourhood U of x and a diffeomorphism $\phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for all $y \in U$ the restriction $\phi|_{\pi^{-1}(y)}\colon \pi^{-1}(y) \to \{y\} \times \mathbb{R}^r$ is a vector space isomorphism.

Such an open set, U, is called a *trivialising open set* for $\mathcal{E}$, and ϕ is called a *trivialisation* of $\mathcal{E}$ over U ◊

Next, we define a vector bundle over a manifold. This is the basis for everything else we do throughout this thesis. It is analogous to the tangent bundle of a manifold, but rather than attaching copies of $\mathbb{R}^n$, we attach copies of a finite dimensional vector space.

1.2.2 Definition ([22, §12.3]). A C^∞ *vector bundle of rank* r is a triple $(\mathcal{E}, M, \pi)$ consisting of manifolds $\mathcal{E}$ and M and a surjective smooth map $\pi\colon \mathcal{E} \to M$ that is locally trivial of rank r. We call the manifold $\mathcal{E}$ the *total space* of the vector bundle and M the *base space*. With some abuse of language we say $\mathcal{E}$ is a *vector bundle* over M. The fibre $\pi^{-1}(x)$ is often denoted $\mathcal{E}_x$. ◊

There are a number of ways of constructing new vector bundles from old. For example, the *dual bundle* $\mathcal{E}^* \to M$ and, if $\mathcal{F} \to M$ is another vector bundle, the *direct sum* $\mathcal{E} \oplus \mathcal{F} \to M$, the *tensor product* $\mathcal{E} \otimes \mathcal{F} \to M$, the *exterior product* $\mathcal{E} \wedge \mathcal{F} \to M$ and the *symmetric product* $\mathcal{E} \odot \mathcal{F} \to M$.

1.2.3 Example. We outline the construction of the dual bundle $\xi\colon \mathcal{E}^* \to M$.

- The total space is $\mathcal{E}^* = \coprod_{x\in M}(\mathcal{E}_x)^*$, the disjoint union.
- If $\lambda \in (\mathcal{E}_x)^*$ then $\xi(\lambda) = x$.
- If $U \subset M$ is a trivialising open set for $\mathcal{E}$, with trivialising map ϕ, then U is also a trivialising open set for $\mathcal{E}^*$ with trivialising map:

  $$\psi\colon \xi^{-1}(U) \to U \times (\mathbb{R}^r)^*; \quad \lambda \mapsto (\xi(\lambda), \theta_\lambda),$$

  where

  $$\theta_\lambda(v) = \lambda(\phi^{-1}(\xi(\lambda), v)), \quad \text{for all } v \in \mathbb{R}^r.$$

- The topology of $\mathcal{E}^*$ is the pullback via ξ of the topology of M. The differential structure of $\mathcal{E}^*$ is that generated by the atlas:

  $$\{\xi^{-1}(U) : U \subset M \text{ is a trivialising open set for } \mathcal{E} \text{ and a chart domain for } M\}.$$

Analogous to vector fields on a manifold are sections of a vector bundle.

1.2.4 Definition ([11, §1.1]). A *section of a vector bundle* $\pi\colon \mathcal{E} \to M$ is a smooth map $\sigma\colon M \to \mathcal{E}$ such that $\pi \circ \sigma = I_M$, the identity map on M. We denote the vector space of all such sections by $\Gamma(\mathcal{E})$. ◊

We now define a fibre metric in a vector bundle in the context of pseudo-Riemannian geometry.

1.2.5 Definition. [11, §1.1] A *pseudo-Riemannian fibre metric* in a vector bundle $\pi\colon \mathcal{E} \to M$ is a section $g \in \Gamma(\mathcal{E}^* \odot \mathcal{E}^*)$ which induces on each fibre a non-degenerate inner product. For $e_1, e_2 \in \mathcal{E}_x$ we often use the alternative notation $g_x(e_1, e_2) = \langle e_1, e_2\rangle$. ◊

The following definition is analogous to that of a connection on a manifold (Definition 1.1.7).

1.2.6 Definition ([11, §1.1]). A *linear connection* in a vector bundle $\pi\colon \mathcal{E} \to M$ is a pairing

$$\nabla\colon \Gamma(TM) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E}),$$

written $\nabla\colon (X, \sigma) \mapsto \nabla_X \sigma$ such that:

1. The connection is additive in both factors:

$$\nabla_{X+Y}\sigma = \nabla_X\sigma + \nabla_Y\sigma, \quad \text{for all } X, Y \in \Gamma(TM),$$
$$\nabla_X(\sigma + \tau) = \nabla_X\sigma + \nabla_X\tau, \quad \text{for all } \sigma, \tau \in \Gamma(\mathcal{E}).$$

2. The connection is $C^\infty(M)$-bilinear in the first factor:

$$\nabla_{fX}\sigma = f\,\nabla_X\sigma, \quad \text{for all } f \in C^\infty(M).$$

3. The connection obeys a Leibniz type rule:

$$\nabla_X(f\sigma) = (Xf)\sigma + f\,\nabla_X\sigma, \quad \text{for all } f \in C^\infty(M).$$

It follows from (2) that the localisation $\nabla_X\sigma$ for $X \in TM$ is well-defined. This is called the *covariant derivative of* σ *in the direction* X. ♢

The previously described vector bundle constructions may be extended to linear connections. For example if $\mathcal{E}, \mathcal{F}$ are vector bundles over M with linear connections $\nabla^{\mathcal{E}}, \nabla^{\mathcal{F}}$ respectively then the linear connection ∇ in $\mathcal{E} \otimes \mathcal{F}$ may be defined:

$$\nabla_X(\sigma \otimes \tau) = (\nabla^{\mathcal{E}}_X\sigma) \otimes \tau + \sigma \otimes (\nabla^{\mathcal{F}}_X\tau),$$

for all $\sigma \in \Gamma(\mathcal{E})$ and $\tau \in \Gamma(\mathcal{F})$.

1.2.7 Definition. A *pseudo-Riemannian vector bundle* is a vector bundle $\pi\colon \mathcal{E} \to M$ equipped with a linear connection and *holonomy-invariant* fibre metric:

$$X\langle\sigma, \tau\rangle = \langle\nabla_X\sigma, \tau\rangle + \langle\sigma, \nabla_X\tau\rangle, \quad \text{for all } X \in TM \text{ and } \sigma, \tau \in \Gamma(\mathcal{E}).$$ ♢

The canonical example of a pseudo-Riemannian vector bundle is, of course, the tangent bundle of a pseudo-Riemannian manifold equipped with the Levi-Civita connection (Theorem 1.1.8).

Next we define the pullback of a vector bundle by a mapping of manifolds, followed by the pullback of a fibre metric and linear connection.

1.2.8 Definition. Let $\varphi\colon N \to M$ be a smooth map of manifolds and let $\pi\colon \mathcal{E} \to M$ be a vector bundle of rank r. The *pullback vector bundle* $\tilde{\pi}\colon \widetilde{\mathcal{E}} \to N$ has total space

$$\widetilde{\mathcal{E}} = \{(y, e) \in N \times \mathcal{E} \colon \varphi(y) = \pi(e)\},$$

equipped with subspace topology, and projection map

$$\tilde{\pi}(y, e) = y.$$

The alternative notations $\tilde{\mathcal{E}} = \varphi^*\mathcal{E} = \varphi^{-1}\mathcal{E}$ are also used. ♢

We claim that $\tilde{E}$ is indeed a vector bundle of rank r. The vector space structure on the fibres of $\tilde{\mathcal{E}}$ is clear. If $U \subset M$ is a trivialising open set for $\mathcal{E}$ with trivialising map ϕ then $\tilde{U} = \phi^{-1}(U) \subset N$ is a trivialising open set for $\tilde{\mathcal{E}}$ with trivialising map:

$$\tilde{\phi}\colon \tilde{\pi}^{-1}(\tilde{U}) \to \tilde{U} \times \mathbb{R}^r; \quad (y, e) \mapsto (y, \phi_2(e)),$$

where $\phi_2(e)$ is the $\mathbb{R}^r$-component of $\phi(e) \in U \times \mathbb{R}^r$.

Defining the map

$$\tilde{\varphi}\colon \tilde{\mathcal{E}} \to \mathcal{E}; \quad \tilde{\varphi}(y, e) = e$$

gives rise to the following commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{E}} & \xrightarrow{\tilde{\varphi}} & \mathcal{E} \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\varphi} & M \end{array}$$

This is a vector bundle morphism.

1.2.9 Definition. The *pullback* of a section $\sigma \in \Gamma(\mathcal{E})$ is the section $\tilde{\sigma} \in \Gamma(\widetilde{\mathcal{E}})$ defined:

$$\tilde{\sigma}(y) = (y, \sigma(\varphi(y)) \in \widetilde{\mathcal{E}}, \quad \text{for all } y \in N.$$

Note that $\varphi(y) = \pi(\sigma(\varphi(y)))$ since σ is a section of $\mathcal{E}$; thus $\tilde{\sigma}(y) \in \widetilde{\mathcal{E}}_y$. The alternative notations $\tilde{\sigma} = \varphi^* \sigma = \varphi^{-1}\sigma$ are also used. ◊

1.2.10 Definition. Let ∇ be a linear connection in $\mathcal{E}$. The *pullback connection* $\widetilde{\nabla}$ in $\widetilde{\mathcal{E}}$ is characterised on pullback sections by:

$$\widetilde{\nabla}_Y \tilde{\sigma} = (y, \nabla_{d\varphi(Y)}\sigma), \quad \text{for all } \sigma \in \Gamma(\mathcal{E}) \text{ and } Y \in T_yN.$$

Note that the pullback of a local frame of $\mathcal{E}$ is a local frame for $\widetilde{\mathcal{E}}$. The covariant derivative of an arbitrary section of $\widetilde{\mathcal{E}}$ can be formed by expanding it in terms of such local pullback frames, using the Leibniz property of ∇. ◊

In practice the following situation will be important. With a slight change of notation, let $\varphi\colon M \to N$. If $X \in \Gamma(TM)$ then a section $d\varphi(X)$ of $\varphi^{-1}(TN)$ may be defined as follows:

$$d\varphi(X)(x) = (x, d\varphi(X(x))), \quad \text{for all } x \in M.$$

Recall that a linear connection in TN is said to be symmetric if for all $A, B \in \Gamma(TN)$:

$$\nabla_A B - \nabla_B A = [A, B].$$

The following useful result is proved in [11, p. 5].

1.2.11 Proposition. *Let $\varphi : M \to N$ be a smooth mapping and let ∇ be a symmetric linear connection in TN with φ-pullback $\widetilde{\nabla}$. Then for all $X, Y \in \Gamma(TM)$:*

$$\widetilde{\nabla}_X(d\varphi(Y)) - \widetilde{\nabla}_Y(d\varphi(X)) = d\varphi[X, Y]. \tag{1.1}$$

1.2.12 Definition. Let $\varphi\colon (M,g) \to (N,h)$ be a smooth mapping of pseudo-Riemannian manifolds. Then the *second fundamental form of* φ is defined as follows:

$$(\nabla d\varphi)(X,Y) = (\nabla_X d\varphi)(Y) = D_X(d\varphi(Y)) - d\varphi(D_X Y), \quad \text{for all } X,Y \in \Gamma(TM),$$

where D_X denotes covariant differentiation along X with respect to the Levi-Civita connections of (M,g) and (N,h) where appropriate.
1.2.13 Corollary. *The second fundamental form is symmetric.*

The following is an application of this that will be used later.
1.2.14 Definition. A mapping $\varphi\colon (M,g) \to (N,h)$ of pseudo-Riemannian manifolds is an *isometry* if it is a diffeomorphism and

$$h(d\varphi(X), d\varphi(Y)) = g(X,Y), \quad \text{for all } X,Y \in \Gamma(TM).$$

1.2.15 Proposition. *If φ is an isometry then $\nabla d\varphi = 0$.*

Proof. Differentiating the definition of isometry leads to the equation

$$h((\nabla_X d\varphi)(Y), d\varphi(Z)) = -h(d\varphi(Y), (\nabla_X d\varphi)(Z)).$$

Then applying the symmetry of $\nabla d\varphi$:

$$\begin{aligned} h((\nabla_X d\varphi)(Y), d\varphi(Z)) &= -h(d\varphi(Y), (\nabla_Z d\varphi)(X)) \\ &= h((\nabla_Z d\varphi)(Y), d\varphi(X)) \\ &= h((\nabla_Y d\varphi)(Z), d\varphi(X)) \\ &= -h(d\varphi(Z), (\nabla_Y d\varphi)(X)) \\ &= -h(d\varphi(Z), (\nabla_X d\varphi)(Y)). \end{aligned}$$ □

We next define the covariant coderivative and establish its characterising property. Recall that if $\mathcal{E}$ is a vector bundle over M then sections of the vector bundle $T^*M \otimes \mathcal{E}$ are called $\mathcal{E}$-valued 1-forms on M. If (M,g) is a pseudo-Riemannian manifold then the Levi-Civita connection D induces a connection in T^*M and if $\mathcal{E}$ has a linear connection ∇ then taking the tensor product yields a connection in $T^*M \otimes \mathcal{E}$, which for notational simplicity we continue to denote by ∇.
1.2.16 Definition. Let ρ be an $\mathcal{E}$-valued 1-form on a pseudo-Riemannian manifold (M,g). Define the section $\nabla^*\rho$ of $\mathcal{E}$, the *covariant coderivative* of ρ, as follows:

$$\begin{aligned} \nabla^*\rho &= -\sum_i \epsilon_i \, (\nabla_{E_i}\rho)(E_i), \quad \text{where } \{E_i\} \text{ is a frame of } (M,g), \\ &= -\sum_i \epsilon_i \, \nabla\rho(E_i, E_i) \\ &= -\operatorname{trace} \nabla\rho. \end{aligned} \tag{1.2}$$

If $\rho = \nabla\sigma$ for $\sigma \in \Gamma(\mathcal{E})$ then $\nabla^*\nabla\sigma$ is also a section of $\mathcal{E}$, called the *rough Laplacian of* σ. Note that

$$\nabla^*\nabla\sigma = -\operatorname{trace} \nabla^2\sigma.$$ ◊

The following result generalises a well-known property of classical vector calculus.

1.2.17 Proposition. *Let ρ be an $\mathcal{E}$-valued 1-form on M. Then for all smooth functions $f\colon M \to \mathbb{R}$ we have:*

$$\nabla^*(f\rho) = f\,\nabla^*\rho - \rho(\nabla f).$$

Proof. From Definition 1.2.16:

$$\begin{aligned}
\nabla^*(f\rho) &= -\sum_i \epsilon_i\,(\nabla_{E_i}(f\rho))(E_i) \\
&= -\sum_i \epsilon_i\,(E_i(f)\rho + f\nabla_{E_i}\rho)(E_i) \\
&= -\sum_i \epsilon_i\,E_i(f)\rho(E_i) - f\sum_i \epsilon_i\,(\nabla_{E_i}\rho)(E_i) \\
&= -\sum_i \epsilon_i\,\rho(E_i(f)E_i) + f\,\nabla^*\rho \\
&= -\rho(\nabla f) + f\,\nabla^*\rho, \quad \text{by Definition 1.1.14.}
\end{aligned}$$

□

It is useful to note the following well known differential topological result. Recall that if α is a $\mathcal{E}$-valued p-form on M then the *support* of α is

$$\operatorname{supp}(\alpha) = \operatorname{cl}\{x \in M : \alpha(x) \neq 0\},$$

where cl denotes the topological closure.
1.2.18 Theorem (Stokes' Theorem, [21, Vol. I, Theorem 8.4])**.** *Let M be an orientable n-dimensional manifold-with-boundary, and let ρ be a $(n-1)$-form on M with compact support. Then*

$$\int_{\partial M} \rho = \int_M d\rho.$$

The following consequence of Stokes' Theorem is particularly useful. We recall that if (M, g) is an orientable pseudo-Riemannian manifold the volume element $\operatorname{vol}(g)$ is the unique n-form which takes value 1 on all positively oriented frames.
1.2.19 Theorem (Divergence Theorem)**.** *Let Y be a compactly supported vector field on an orientable pseudo-Riemannian manifold (M, g). Then*

$$\int_M \operatorname{div}(Y)\operatorname{vol}(g) = 0.$$

Proof. Let η be the 1-form on M metrically dual to Y:

$$\eta(X) = g(Y, X), \quad \text{for all } X \in \Gamma(TM).$$

Then the divergence of Y may be characterised as

$$\operatorname{div}(Y) = -\delta\eta,$$

where δ is the exterior coderivative of (M, g). Recall that the space of p forms is denoted $\Omega^p(M)$. In general, $\delta\colon \Omega^p(M) \to \Omega^{p-1}(M)$ may be expressed:

$$\delta\alpha = -s(-1)^{n(p+1)} * d * \alpha,$$

where d is the exterior derivative and $*$ is the Hodge star operator (see for instance [7]). The number $s = \epsilon_1 \cdots \epsilon_n$, where the ϵ_i are the indicator symbols for any frame of M. (In the familiar Riemannian case $s = 1$.) The Hodge star operator is the unique linear isomorphism $*\colon \Omega^p(M) \to \Omega^{n-p}(M)$ characterised by

$$\alpha \wedge *\beta = \tilde{g}(\alpha, \beta)\,\mathrm{vol}(g), \quad \text{for all } \alpha, \beta \in \Omega^p(M),$$

where $\tilde{g}$ is the metric induced on $\Omega^p(M)$. In particular, if f is a function (0-form) on M then:

$$*f = f\,\mathrm{vol}(g).$$

It follows from this, and the involution formula:

$$**\,\alpha = s(-1)^{p(n-p)}\alpha,$$

that:

$$\begin{aligned}(\operatorname{div} Y)\,\mathrm{vol}(g) &= *(\operatorname{div} Y) = -*\delta\eta \\ &= s**d*\eta \\ &= s^2(-1)^{n-1}d(*\eta) \\ &= (-1)^{n-1}d(*\eta).\end{aligned}$$

If Y is compactly supported then so is η and hence $*\eta$, so by Stokes' Theorem (1.2.18):

$$\int_M (\operatorname{div} Y)\,\mathrm{vol}(g) = \int_M (-1)^{n-1}d(*\eta) = (-1)^{n-1}\int_{\partial M} *\eta = 0,$$

since $\partial M = \emptyset$. □

1.2.20 Proposition (Integration by Parts). *Let σ be a section of a pseudo-Riemannian vector bundle $\mathcal{E}$ over an orientable pseudo-Riemannian manifold (M, g) and let ρ be a $\mathcal{E}$-valued 1-form on M, with either σ or ρ compactly supported. Then*

$$\int_M \langle \rho, \nabla\sigma\rangle\,\mathrm{vol}(g) = \int_M \langle \nabla^*\rho, \sigma\rangle\,\mathrm{vol}(g).$$

Proof. First note, for all $X, Y \in TM$:

$$(\nabla_X\rho)(Y) = \nabla_X(\rho(Y)) - \rho(D_XY), \tag{1.3}$$

where, on the right hand side, Y has been extended to a local vector field. Then by definition, for a frame $\{E_i\}$ of M:

$$\begin{aligned}\langle \rho, \nabla\sigma\rangle &= \sum_i \epsilon_i\,\langle \rho(E_i), \nabla_{E_i}\sigma\rangle \\ &= \sum_i \epsilon_i\,[E_i\langle\rho(E_i),\sigma\rangle - \langle\nabla_{E_i}(\rho(E_i)),\sigma\rangle], \quad \text{by holonomy invariance} \\ &= \sum_i \epsilon_i\,[E_i\langle\rho(E_i),\sigma\rangle - \langle\nabla_{E_i}\rho(E_i) + \rho(D_{E_i}E_i),\sigma\rangle], \quad \text{by (1.3)} \\ &= \langle -\sum_i \epsilon_i\,\nabla_{E_i}\rho(E_i),\sigma\rangle + \sum_i \epsilon_i\,[E_i\langle\rho(E_i),\sigma\rangle + \langle\rho(D_{E_i}E_i),\sigma\rangle] \\ &= \langle\nabla^*\rho,\sigma\rangle + \sum_i \epsilon_i\,[E_i(\eta(E_i)) + \eta(D_{E_i}E_i)] \\ &= \langle\nabla^*\rho,\sigma\rangle + \sum_i \epsilon_i\,\nabla_{E_i}\eta(E_i),\end{aligned}$$

where η is the 1-form on M defined:

$$\eta(X) = \langle \rho(X), \sigma \rangle.$$

If Y is the vector field on M metrically dual to η:

$$\eta(X) = g(Y, X),$$

then:

$$(\nabla_Z \eta)(X) = g(\nabla_Z Y, X).$$

Therefore:

$$\begin{aligned} \langle \rho, \nabla \sigma \rangle &= \langle \nabla^* \rho, \sigma \rangle + \sum_i \epsilon_i \, g(\nabla_{E_i} Y, E_i) \\ &= \langle \nabla^* \rho, \sigma \rangle + \operatorname{div}(Y). \end{aligned}$$

Since either ρ or σ is compactly supported, η and hence Y is compactly supported. The result follows on integrating over M, using the Divergence Theorem (1.2.19). □

The result of Proposition 1.2.20 generalises to $\mathcal{E}$-valued p-forms, see [7, Theorem 0.2.18], but we will have no need for $p > 1$.

1.3 Principal fibre bundles and Ehresmann connections

We next review the basic definitions and theory of principal bundles. This motivates the definition of the connection map.

1.3.1 Definition. [21, Vol. II, §8]. Let M be a C^∞ manifold and G a Lie group. A *principal G-bundle over M* is triple $(Q, \xi, \cdot)$ where:

1. Q is a C^∞ manifold (the *total space* of the principal bundle).
2. $\xi\colon Q \to M$ is a C^∞ submersion (the *projection map* of the bundle) onto M (the *base space* of the principal bundle), satisfying

$$\xi(q \cdot g) = \xi(q), \quad \text{for all } q \in Q \text{ and } g \in G.$$

3. The map $\cdot$ (the *action* of G) is a C^∞ map $(q, g) \mapsto q \cdot g$ from $Q \times G$ to Q such that

$$q \cdot (gh) = (q \cdot g) \cdot h, \quad \text{for all } q \in Q \text{ and } g, h \in G.$$

4. The following version of the local triviality condition is satisfied. For each $x \in M$ there is a neighbourhood U of x and a diffeomorphism $\psi\colon \xi^{-1}(U) \to U \times G$ of the form $\psi(q) = (\xi(q), \psi_2(q))$ where ψ_2 is G-equivariant:

$$\psi_2(q \cdot g) = \psi_2(q) g, \quad \text{for all } q \in Q \text{ and } g \in G. \qquad \Diamond$$

1.3.2 Definition. Since ξ is a submersion its fibres are smooth submanifolds. Let $i\colon \xi^{-1}(x) \to Q$ be the inclusion map. The *vertical subspace at* $q \in \xi^{-1}(x)$ is the linear subspace

$$V_q = di(T_q\xi^{-1}(x)).$$

Equivalently:

$$V_q = \ker d\xi(q).$$

Tangent vectors in V_q are called *vertical tangent vectors at* q. $\diamondsuit$

When a Lie group G acts on a manifold P there is a "infinitesimal action" of its Lie algebra $\mathfrak{g}$ on the vector fields $\Gamma(TP)$, as follows:

- For every $a \in \mathfrak{g}$ we have a curve $t \mapsto \exp ta$ in G.
- For every $p \in P$ this gives a path in P, $c_p(t) = p \cdot (\exp ta)$.
- Define $\alpha(a)(p) = c_p'(0)$. Then $\alpha(a)$ is a vector field on P, and $\alpha\colon \mathfrak{g} \to \Gamma(TP)$ is a Lie algebra homomorphism.

1.3.3 Definition. The vector field $\alpha(a)$ is called the *fundamental vector field generated by* a. $\diamondsuit$

We now define a connection in a principal bundle.

1.3.4 Definition ([21, Vol. II, §8-16])**.** An *Ehresmann* connection in a principal G-bundle $\xi\colon Q \to M$ is a C^∞ $\mathfrak{g}$-valued 1-form ω on Q such that:

1. fundamental vector fields map to their generating element

$$\omega(\alpha(a)) = a, \quad \text{for all } a \in \mathfrak{g},$$

2. ω is equivariant in the following sense:

$$\omega(X \cdot g) = Ad(g^{-1})\omega(X), \quad \text{for all } g \in G \text{ and } X \in TQ,$$

where Ad is the adjoint action of the group G on its Lie algebra $\mathfrak{g}$ and $X \cdot g = dR_g(X)$ where $R_g\colon Q \to Q$, $R_g(q) = q \cdot g$. $\diamondsuit$

The connection provides a preferred complement to the vertical distribution on Q.

1.3.5 Definition. The *horizontal subspace* H_q of T_qQ is

$$H_q = \ker \omega(q).$$

This is of the same dimension as M because $\omega(q)\colon T_qQ \to \mathfrak{g}$ is surjective. $\diamondsuit$

1.3.6 Proposition. *[21, Vol. II, Proposition 8.4] The horizontal distribution H on Q defined by an Ehresmann connection ω is C^∞ and has the following properties:*

1. *The distribution splits the tangent space at each point $q \in Q$,*

$$T_qQ = V_q \oplus H_q.$$

2. *The distribution is invariant under the group action on Q,*

$$H_{q\cdot g} = H_q \cdot g, \quad \textit{for all } g \in G.$$

We now combine the theories of principal bundles and vector bundles.

Let $\xi\colon Q \to M$ be a principal G-bundle and let V be a r-dimensional real vector space with a representation of G; i.e. a left group action of G on V by linear isomorphisms. Then G acts freely on elements of $Q \times V$ on the right by

$$(q, v) \cdot g = (q \cdot g, g^{-1} \cdot v),$$

and the quotient map $\mu\colon Q \times V \to \mathcal{E}$ to the orbit space $\mathcal{E}$ is a principal G-bundle. For convenience we abbreviate $\mu(q, v) = q \cdot v$.

Define a map $\pi\colon \mathcal{E} \to M$ by $\pi(q \cdot v) = \xi(q)$. This map is well-defined. For, if $q' \cdot v' = q \cdot v$ then there exists $g \in G$ such that $q' = q \cdot g$, hence $\xi(q') = \xi(q)$.

Let $U \subset M$ be a trivialising open set for Q, with trivialising map ψ and define a mapping $\phi : \pi^{-1}(U) \to U \times V$ by:

$$\phi(q \cdot v) = (\xi(q), \psi_2(q) \cdot v).$$

Then ϕ is well-defined. For, if $q' \cdot v' = q \cdot v$ then there exists $g \in G$ such that $v' = g^{-1} \cdot v$, hence

$$\psi_2(q') \cdot v' = \psi_2(q \cdot g) \cdot (g^{-1} \cdot v) = (\psi_2(q) \cdot g) \cdot (g^{-1} \cdot v) = \psi_2(q) \cdot v.$$

The restrictions $\phi|_{\pi^{-1}x}$ are invertible (because G acts on V by isomorphisms) and may be used to transfer the vector space structure of V to the fibres of $\mathcal{E}$ with respect to which $\phi|_{\pi^{-1}x}$ is a linear isomorphism. This linear structure is also invariant under change of trivialisation. We conclude that π is a locally trivial map of rank r (see Definition 1.2.1) and hence that $\mathcal{E}$ is a vector bundle of rank r over M.

1.3.7 Definition. The vector bundle $\pi\colon \mathcal{E} \to M$ constructed above is called the *associated vector bundle*, and denoted by $Q \times_G V$. ♢

If $\pi_Q\colon Q \times V \to Q$ is the projection map then there is the following commutative diagram of bundles:

$$\begin{array}{ccc} Q & \xleftarrow[\pi_Q]{} & Q \times V \\ \Big\downarrow \xi & & \Big\downarrow \mu \\ M & \xleftarrow[\pi]{} & \mathcal{E} \end{array}$$

The bundles ξ and μ are principal bundles whereas the bundles π and π_Q are vector bundles.

We shall find it useful to lift tangent vectors to M, and sections of $\mathcal{E}$, to Q as follows.

1.3.8 Definition. Let $X \in T_xM$ then a *lift of* X to Q is $A \in T_qQ$, where $\xi(q) = x$ and $d\xi(A) = X$. ♢

1.3.9 Definition. Let $\sigma \in \Gamma(\mathcal{E})$. The *equivariant lift of* σ is the unique equivariant map $s\colon Q \to V$ such that

$$q \cdot s(q) = \sigma(\xi(q)), \quad \text{for all } q \in Q.$$

In this case *equivariance* means

$$s(q \cdot g) = g^{-1}s(q), \quad \text{for all } g \in G.$$ ♢

We now relate the ideas of an Ehresmann connection in a principal bundle and a linear connection in a vector bundle.

1.3.10 Definition. Let $\mathcal{E} = Q \times_G V$ and let $\sigma \in \Gamma(\mathcal{E})$ with equivariant lift s. Suppose ω is an Ehresmann connection in Q. Then the *associated linear connection* ∇ in $\mathcal{E}$ is defined by

$$\nabla^{\omega}_{X}\sigma = q \cdot D^{\omega}s(A), \quad \text{for all } X \in T_xM,$$

where $\xi(q) = x$, A is a lift of X to T_qQ and $D^{\omega}s(A) = ds(A^h)$, where A^h is the horizontal component of A. ♢

In Definition 1.3.10 A^h is called the *horizontal lift* of X to T_qQ and is uniquely determined. It is easily checked that the definition of ∇^{ω} is independent of the choice of $q \in \xi^{-1}(x)$.

The definition of a pullback of a principal G-bundle $\xi : Q \to M$ by a mapping $\varphi : N \to M$ is analogous to that of a pullback vector bundle. The total space is

$$\widetilde{Q} = \{(y, q) \in N \times Q : \varphi(y) = \xi(q)\},$$

the G-action is $(y, q)\cdot g = (y, q\cdot g)$ for all $g \in G$. The bundle projection is therefore $\tilde{\xi}(y, q) = y$. It is easily checked that $\widetilde{Q}$ is locally trivialisable. If the mapping $\varphi\colon \widetilde{Q} \to Q$ is defined

$$\tilde{\varphi}(y, q) = (\varphi(y), q), \quad \text{for all } (y, q) \in \widetilde{Q},$$

then the following commutative diagram:

$$\begin{array}{ccc} \widetilde{Q} & \xrightarrow{\tilde{\varphi}} & Q \\ \Big\downarrow{\scriptstyle \tilde{\xi}} & & \Big\downarrow{\scriptstyle \xi} \\ N & \xrightarrow{\varphi} & M \end{array}$$

is a morphism of principal G-bundles.

1.3.11 Definition. Let ω be an Ehresmann connection for Q. Then the *pullback Ehresmann connection* $\tilde{\omega}$ in $\widetilde{Q}$ is defined to be

$$\tilde{\omega}(\tilde{A}) = \omega(d\tilde{\phi}(\tilde{A})), \quad \text{for all } \tilde{A} \in T\widetilde{Q}.$$

Notice that $\tilde{\omega} = \tilde{\phi}^*(\omega)$, the pullback form. ♢

It is easily checked that $\tilde{\omega}$ defined in Definition 1.3.11 satisfies the properties of an Ehresmann connection.

The above described constructions of associated vector bundles and their linear connections "commute" with the operation of pullback.

1.3.12 Proposition. *Let $\xi\colon Q \to M$ be a principal G-bundle with connection ω and let $\mathcal{E} = Q \times_G V$ be an associated vector bundle. Suppose $\varphi\colon N \to M$ is a smooth mapping.*

(1) *The pullback vector bundle $\widetilde{\mathcal{E}}$ is naturally isomorphic to the associated bundle $\widetilde{Q} \times_G V$.*
(2) *Under this isomorphism the pullback of the linear connection in $\mathcal{E}$ associated to ω is associated to the pullback Ehreshmann connection $\tilde{\omega}$.*

Proof. In summary, the isomorphism $\widetilde{\mathcal{E}} \to \widetilde{Q} \times_G V$ is

$$(y, q \cdot v) \mapsto (y, q) \cdot v,$$

for all pairs $(y, q) \in N \times Q$ such that $\varphi(y) = \xi(q)$. If s is the equivariant lift of a section $\sigma \in \Gamma(\mathcal{E})$ then the equivariant lift of the pullback section $\tilde{\sigma} \in \Gamma(\tilde{\mathcal{E}})$ is $s \circ \tilde{\varphi}$. Therefore if $Y \in T_y N$ and $B \in T_{(y,q)}\widetilde{Q}$ is a lift of Y then

$$D^{\tilde{\omega}} \tilde{s}(B) = d\tilde{s}(B^h) = d(s \circ \tilde{\varphi})(B^h) = ds((d\tilde{\varphi}(B))^h),$$

since $d\tilde{\varphi}(\widetilde{H}) = H$. The linear connection associated to $\tilde{\omega}$ is (Definition 1.3.10):

$$\nabla^{\tilde{\omega}}_Y \tilde{\sigma} = (y, q) \cdot ds((d\tilde{\varphi}(B))^h).$$

On the other hand, the pullback of the linear connection $\nabla = \nabla^{\omega}$ to $\widetilde{\mathcal{E}}$ is (Definition 1.2.10):

$$\widetilde{\nabla}_Y \tilde{\sigma} = (y, \nabla_{d\varphi(Y)} \sigma) = (y, q \cdot D^{\omega} s(d\tilde{\varphi}(B))) = (y, q \cdot ds((d\tilde{\varphi}(B))^h).$$

Thus $\nabla^{\tilde{\omega}} = \widetilde{\nabla}$ under the natural vector bundle isomorphism. □

1.4 The connection map

The connection map for a linear connection in a vector bundle is analogous to the connection form for an Ehresmann connection in a principal bundle. We outline the construction, which is based on [23], in a way which we consider to be slightly more insightful than the standard approach, for example that in [14]. The key idea of our approach is the square of a vector bundle, which we now define.

1.4.1 Definition. Let $\pi : \mathcal{E} \to M$ be a vector bundle. The *square of* $\mathcal{E}$ is the pullback vector bundle $\pi_2 : \pi^*\mathcal{E} \to \mathcal{E}$ of $\mathcal{E}$ over itself:

$$\pi^*\mathcal{E} = \{(e, f) \in \mathcal{E} \times \mathcal{E} \colon \pi(e) = \pi(f)\},$$

where $\pi_2(e, f) = e$. If we define $\xi \colon \pi^*\mathcal{E} \to \mathcal{E}$ by $\xi(e, f) = f$ then there is the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E} & \xleftarrow[\xi]{} & \pi^*\mathcal{E} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi_2} \\ M & \xleftarrow[\pi]{} & \mathcal{E} \end{array}$$

◊

The square has a natural section.

1.4.2 Definition. The *diagonal section* $\chi \in \Gamma(\pi^*\mathcal{E})$ is defined as follows:

$$\chi \colon \mathcal{E} \to \pi^*\mathcal{E}; \quad \chi(e) = (e, e), \quad \text{for all } e \in \mathcal{E}.$$ ◊

We now assume that $\mathcal{E}$ has linear connection ∇. This may be pulled back to a linear connection $\widetilde{\nabla}$ in $\pi^*\mathcal{E}$ (Definition 1.2.10).

1.4.3 Definition. We define a $\pi^*\mathcal{E}$-valued one-form κ on $\mathcal{E}$ as follows:

$$\kappa(A) = \widetilde{\nabla}_A \chi, \quad \text{for all } A \in T\mathcal{E}.$$

The *connection map* $K\colon T\mathcal{E} \to \mathcal{E}$ is then defined

$$K(A) = \xi \circ \kappa(A)). \qquad \lozenge$$

The connection map defined in Definition 1.4.3 is characterised by the following property.

1.4.4 Theorem. *Let $\mathcal{E}$ be a vector bundle over M with linear connection ∇ and connection map K. Then*

$$\nabla_X \sigma = K(d\sigma(X)), \tag{1.4}$$

for all $X \in TM$ and $\sigma \in \Gamma(\mathcal{E})$.

Proof. Suppose $X \in T_xM$ and $X = x'(0)$ for some smooth path $x(t) \in M$ with $x(0) = x$. Then the covariant derivative of σ may be expressed in terms of parallel translation $\rho_t\colon \mathcal{E}_x \to \mathcal{E}_{x(t)}$ along the path $x(t)$ as follows:

$$\nabla_X \sigma = \frac{d}{dt}\Big|_{t=0} (\rho_t)^{-1}(\sigma(x(t))).$$

Define the path $e(t)$ in $\mathcal{E}$ by $e(t) = \sigma(x(t))$; then $e = e(0) = \sigma(x)$ and $d\sigma(X) = e'(0)$. The $\widetilde{\nabla}$-parallel translation in $\widetilde{\mathcal{E}} = \pi^*\mathcal{E}$ along $e(t)$ is:

$$\tilde{\rho}_t\colon \widetilde{\mathcal{E}}_e \to \widetilde{\mathcal{E}}_{e(t)}; \quad \tilde{\rho}_t(e, f) = (e(t), \rho_t(f)).$$

Therefore

$$\begin{aligned}
\widetilde{\nabla}_{d\sigma(X)} \chi &= \frac{d}{dt}\Big|_{t=0} (\tilde{\rho}_t)^{-1}(\chi(e(t))) \\
&= \frac{d}{dt}\Big|_{t=0} (\tilde{\rho}_t)^{-1}(e(t), e(t)) \\
&= \frac{d}{dt}\Big|_{t=0} (\tilde{\rho}_t)^{-1}(e(t), \sigma(x(t))) \\
&= \frac{d}{dt}\Big|_{t=0} (\tilde{\rho}_t)^{-1}(e(t), \sigma \circ \pi(e(t))) \\
&= \frac{d}{dt}\Big|_{t=0} (\tilde{\rho}_t)^{-1}(\tilde{\sigma}(e(t))) \\
&= \widetilde{\nabla}_{d\sigma(X)} \tilde{\sigma}
\end{aligned}$$

which by Definition 1.2.10 of the pullback linear connection

$$\begin{aligned}
&= (e, \nabla_{d\pi(d\sigma(X))} \sigma) \\
&= (e, \nabla_X \sigma).
\end{aligned}$$

Therefore

$$K(d\sigma(X)) = \xi(\widetilde{\nabla}_{d\sigma(X)} \chi) = \nabla_X \sigma.$$

$\square$

The connection map plays a similar role to the Ehresmann connection form in a principal bundle, by splitting the tangent bundle $T\mathcal{E}$ into vertical and horizontal subbundles. In particular this will enable us to define various pseudo-Riemannian metrics on $\mathcal{E}$.

1.4.5 Definition. If $\pi\colon \mathcal{E} \to M$ is any vector bundle then the *vertical subbundle* $V \subset T\mathcal{E}$ is defined as:

$$V_e \subset T_e\mathcal{E};\ V_e = \ker d\pi(e).$$

If $\mathcal{E}$ has a linear connection with connection map K then the *horizontal subbundle* $H \subset T\mathcal{E}$ is defined as:

$$H_e \subset T_e\mathcal{E};\ H_e = \ker K(e).$$

1.4.6 Proposition. *Each tangent space $T_e\mathcal{E}$ splits as follows:*

$$T_e\mathcal{E} = H_e \oplus V_e.$$

Finally we note that if $\varphi\colon N \to M$ is a smooth map then the connection map $\widetilde{K}\colon T\widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}$ for the pullback connection $\widetilde{\nabla}$ in $\widetilde{\mathcal{E}} = \varphi^*\mathcal{E}$ satisfies the relation

$$\tilde{\varphi} \circ \widetilde{K} = K \circ d\tilde{\varphi}, \tag{1.5}$$

where

$$\tilde{\varphi}\colon \widetilde{\mathcal{E}} \to \mathcal{E};\quad \tilde{\varphi}(y,e) = e.$$

To see this let $Y \in T_yN$ and denote $d\varphi(Y) = X \in T_{\varphi(y)}M$, and let $\tilde{\sigma}(y) = (y, \sigma\circ\varphi(y)) \in \Gamma(\widetilde{\mathcal{E}})$ for $\sigma \in \Gamma(\mathcal{E})$. Note

$$T_{(y,e)}\widetilde{\mathcal{E}} = \{(Y,A) : Y \in T_yN,\ A \in T_e\mathcal{E},\ d\varphi(Y) = d\pi(A)\}.$$

Then

$$d\tilde{\varphi}\colon T\widetilde{\mathcal{E}} \to T\mathcal{E};\quad d\tilde{\varphi}(Y,A) = A,$$

and

$$d\tilde{\sigma}(Y) = (Y, d\sigma(X)),$$

hence

$$d\tilde{\varphi}(d\tilde{\sigma}(Y)) = d\tilde{\varphi}(Y, d\sigma(X)) = d\sigma(X). \tag{1.6}$$

Then

$$\begin{aligned}
\tilde{\varphi} \circ \widetilde{K}(d\tilde{\sigma}(Y)) &= \tilde{\varphi} \circ \widetilde{\nabla}_Y\tilde{\sigma}, \quad \text{by Theorem 1.4.4}\\
&= \tilde{\varphi} \circ (y, \nabla_{d\varphi(Y)}\sigma), \quad \text{by Definition 1.2.10}\\
&= \nabla_X\sigma\\
&= K(d\sigma(X)), \quad \text{by Theorem 1.4.4}\\
&= K(d\tilde{\varphi}(d\tilde{\sigma}(Y))), \quad \text{by (1.6)}\\
&= K \circ d\tilde{\varphi}(d\tilde{\sigma}(Y)).
\end{aligned}$$

1.5 Harmonic maps

Harmonic maps were first introduced in 1964 [12] and enjoy a richly developed theory. With a view to Chapter 2, where it is important to work on non-compact spaces, we will consider the derivation of the Euler-Lagrange equations arising from local energy and compactly supported variations. Let (M, g) and (N, h) be pseudo-Riemannian manifolds and let $\varphi\colon (M, g) \to (N, h)$ be a smooth mapping. The energy density of φ is defined as follows:

$$\mathrm{e}(\varphi) = \frac{1}{2} h(d\varphi, d\varphi) = \frac{1}{2} \sum \epsilon_i \, h(d\varphi(E_i), d\varphi(E_i)),$$

where $\{E_i\}$ is a frame on M.

1.5.1 Definition. An open set $U \subset M$ is *relatively compact* if $\mathrm{cl}(U)$ is compact. ♢

1.5.2 Definition. The *local energies* of $\varphi\colon M \to N$ are

$$\mathrm{E}(\varphi; U) = \frac{1}{2} \int_U \mathrm{e}(\varphi) \, \mathrm{vol}(g),$$

for any relatively compact $U \subset M$. If M itself is compact then we simply write $\mathrm{E}(\varphi; M) = \mathrm{E}(\varphi)$. ♢

The following definitions are required when applying calculus of variations to these local functionals.

1.5.3 Definition. Let $I \subset \mathbb{R}$ be an open interval containing 0. Let $\Phi\colon M \times I \to N$ be a smooth map and for each $t \in I$ define the smooth map $\varphi_t\colon M \to N$ by $\varphi_t(x) = \Phi(x, t)$. If $\varphi_0 = \varphi$ then Φ is said to be a 1-*parameter variation* of $\varphi\colon M \to N$. The variation Φ is *supported in* $U \subset M$ if $\varphi_t(x) = \varphi(x)$ for all $x \in M \setminus U$ and $t \in I$. ♢

1.5.4 Definition. The *variation field* of the 1-parameter variation φ_t of φ is, for all $x \in M$,

$$v(x) = \frac{d}{dt}\Big|_{t=0} \varphi_t(x) \in T_{\varphi(x)} N. \qquad ♢$$

Note that v is a section of the pullback bundle $\varphi^{-1}(TN) \to M$. Also, if the variation is compactly supported then the variation field v will also be compactly supported.

A map is harmonic if it is a critical point of the local energies in the space of smooth maps $M \to N$.

1.5.5 Definition. A smooth map of manifolds $\varphi\colon M \to N$ is *harmonic* if

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}(\varphi_t; U) = 0,$$

for all relatively compact open $U \subset M$ and all variations φ_t supported in U. ♢

We now compute the Euler-Lagrange equations for this variational problem, for which the following definition is important.

1.5.6 Definition. The *tension field* of the map $\varphi\colon M \to N$ is

$$\tau(\varphi) = \mathrm{trace} \, \nabla d\varphi = \sum_i \epsilon_i \, (\nabla_{E_i} d\varphi)(E_i),$$

where $\{E_i\}$ is a frame field of M and ∇ is the connection in the bundle $T^*M \otimes \varphi^{-1}(TN)$ obtained by tensoring the Levi-Civita connection of (M, g) with the pullback of the Levi-Civita connection of (N, h). Note that

$$\tau(\varphi) = -\nabla^* d\varphi,$$

when $d\varphi$ is regarded as a $\varphi^{-1}(TN)$-valued 1-form on M. $\Diamond$

The following technical lemma is a vital tool for the analysis of mappings between non-compact pseudo-Riemannian manifolds.

1.5.7 Definition. Let $U \subset M$ be a neighbourhood of the point $x \in M$. Then a *bump function about* x is a smooth function $\Lambda\colon M \to [0,1] \subset \mathbb{R}$ such that $\Lambda(x) = 1$ and $\Lambda(y) = 0$ for all $y \in M \setminus U$. $\Diamond$

1.5.8 Lemma. *Let σ be a section of a pseudo-Riemannian vector bundle $\pi\colon \mathcal{E} \to M$ of rank r over a pseudo-Riemannian manifold (M, g). Suppose that*

$$\int_M \langle \sigma, \zeta \rangle \operatorname{vol}(g) = 0,$$

for all $\zeta \in \Gamma(\mathcal{E})$ compactly supported. Then $\sigma = 0$.

Proof. Let $x \in M$ be an arbitrary point and let $\{e_i\}$ a local frame of $\mathcal{E}$ defined over a relatively compact open set U. Let Λ be a bump function about x vanishing outside U. For each $k \leq r$ define the section $\varsigma_k = \Lambda\langle \sigma, e_k\rangle e_k \in \Gamma(\mathcal{E})$. This section is compactly supported. Let $\zeta = \varsigma_k$. Then

$$\begin{aligned} 0 &= \int_M \langle \sigma, \varsigma_k \rangle \operatorname{vol}(g) \\ &= \int_M \langle \sigma, \Lambda\langle \sigma, e_k\rangle e_k \rangle \operatorname{vol}(g) \\ &= \int_M \Lambda\langle \sigma, e_k\rangle^2 \operatorname{vol}(g) \end{aligned}$$

The integrand is positive and the bump function Λ is non-zero, thus $\langle \sigma, e_k\rangle^2 = 0$. Repeating for each k shows σ vanishes on the neighbourhood U of x. Therefore as x is arbitrary $\sigma = 0$. $\square$

1.5.9 Theorem. *A smooth mapping φ of pseudo-Riemannian manifolds is harmonic if and only if $\tau(\varphi) = 0$.*

Proof. The full technical details of the Riemannian version of the following computation are given in [11]; we simply note the modifications that are required for the pseudo-Riemannian case.

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathrm{e}(\varphi_t) &= \frac{d}{dt}\Big|_{t=0} \frac{1}{2} h(d\varphi_t, d\varphi_t) \\ &= \frac{1}{2} \sum_i \epsilon_i \frac{d}{dt}\Big|_{t=0} h(d\varphi_t(E_i), d\varphi_t(E_i)) \\ &= \sum_i \epsilon_i\, h(d\varphi(E_i), \nabla_{E_i} v) \\ &= h(d\varphi, \nabla v). \end{aligned}$$

The first variation of the local energy is therefore:

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}(\varphi_t, U) = \int_U h(d\varphi, \nabla v) \operatorname{vol}(g) = \int_M h(d\varphi, \nabla v) \operatorname{vol}(g),$$

since v hence ∇v is compactly supported. Then Proposition 1.2.20 (integration by parts) yields

$$\int_M h(d\varphi, \nabla v)\,\mathrm{vol}(g) = \int_M h(\nabla^* d\varphi, v)\,\mathrm{vol}(g) = -\int_M h(\tau(\varphi), v)\,\mathrm{vol}(g).$$

It follows from Lemma 1.5.8 that φ is harmonic if and only if $\tau(\varphi) = 0$. □

Chapter 2

Harmonic Sections of Pseudo-Riemannian Vector Bundles

2.1 Harmonic sections of vector bundles and sphere bundles

The standard way to define a pseudo-Riemannian metric on the total space of a vector bundle is the following construction of Sasaki, [19].

2.1.1 Definition. Let $\pi\colon \mathcal{E} \to M$ be a pseudo-Riemannian vector bundle over a pseudo-Riemannian manifold (M, g). The *Sasaki metric* h on $\mathcal{E}$ is defined as follows:

$$h(A, B) = g(d\pi(A), d\pi(B)) + \langle K(A), K(B)\rangle, \quad \text{for all } A, B \in \Gamma(T\mathcal{E}),$$

where $K\colon T\mathcal{E} \to \mathcal{E}$ is the connection map for the linear connection in $\mathcal{E}$. Then h is a pseudo-Riemannian metric on $\mathcal{E}$ whose signature is the sum of the signatures of g and $\langle\, ,\rangle$. $\Diamond$

Now let σ be a section of a vector bundle $\pi\colon \mathcal{E} \to M$.

2.1.2 Definition. The section σ is *parallel* with respect to a linear connection ∇ in $\mathcal{E}$ if

$$\nabla_X \sigma = 0,$$

for all $X \in TM$. $\Diamond$

If σ is a parallel section of a pseudo-Riemannian vector bundle then

$$X\langle\sigma, \sigma\rangle = 2\langle\nabla_X\sigma, \sigma\rangle = 0, \quad \text{for all } X \in TM.$$

Hence σ has constant pseudo-Riemannian length.

If the vector bundle $\mathcal{E}$ is pseudo-Riemannian, and the base is a pseudo-Riemannian manifold (M, g), then the possibility of σ being a harmonic map with respect to the Sasaki metric may be considered. However the following weaker condition is more natural.

2.1.3 Definition. The section σ is said to be a *harmonic section* of $\mathcal{E}$ if

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}(\sigma_t; U) = 0,$$

for all relatively compact open $U \subset M$ and all variations σ_t supported in U, where each σ_t is a section of $\mathcal{E}$.

The energy density of σ with respect to the Sasaki metric is easy to calculate:

$$\begin{aligned} 2\,\mathrm{e}(\sigma) &= h(d\sigma, d\sigma) \\ &= \sum_i \epsilon_i\, h(d\sigma(E_i), d\sigma(E_i)) \\ &= \sum_i \epsilon_i\, [g(E_i, E_i) + \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle], \quad \text{By Definition 2.1.1 and Theorem 1.4.4} \\ &= n + \langle \nabla\sigma, \nabla\sigma\rangle. \end{aligned} \tag{2.1}$$

Using the splitting $T\mathcal{E} = V \oplus H$ (Proposition 1.4.6), the differential may be decomposed

$$d\sigma = d^v\sigma + d^h\sigma,$$

where $d^v\sigma(X)$ is the V-component of $d\sigma(X)$ etc. This allows us to define the *vertical energy density* of σ to be

$$\mathrm{e}^v(\sigma) = \frac{1}{2} h(d^v(\sigma), d^v(\sigma)),$$

and similarly the horizontal energy density $\mathrm{e}^h(\sigma)$. Since V and H are orthogonal with respect to the Sasaki metric:

$$\mathrm{e}(\sigma) = \mathrm{e}^v(\sigma) + \mathrm{e}^h(\sigma).$$

Inspection of (2.1) shows that

$$\mathrm{e}^v(\sigma) = \frac{1}{2}\langle \nabla\sigma, \nabla\sigma\rangle; \quad \mathrm{e}^h(\sigma) = \frac{n}{2}.$$

It follows that σ is a harmonic section if and only if it is a critical point with respect to variations through sections of the *vertical energy functional:*

$$\mathrm{E}^v(\sigma; U) = \int_U \mathrm{e}^v(\sigma)\,\mathrm{vol}(g) = \frac{1}{2}\int_U \langle \nabla\sigma, \nabla\sigma\rangle\,\mathrm{vol}(g). \tag{2.2}$$

When the bundle is Riemannian (2.2) shows that parallel sections are harmonic, and the following result is a partial converse.

2.1.4 Theorem ([16,17]). *A section of a Riemannian vector bundle over a compact pseudo-Riemannian base manifold, where the vector bundle is equipped with the Sasaki metric, is a harmonic section if and only if it is parallel.*

Proof. Consider the variation $\sigma_t = \sigma + t\sigma = (t+1)\sigma$. Then by (2.2):

$$2\,\mathrm{E}^v(\sigma_t) = (t+1)^2 \int_M \langle \nabla\sigma, \nabla\sigma\rangle\,\mathrm{vol}(g).$$

Therefore

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}^v(\sigma_t) = \int_M \langle \nabla\sigma, \nabla\sigma\rangle\,\mathrm{vol}(g).$$

Hence if σ is a harmonic map and $\langle\,,\rangle$ is positive definite then $\nabla\sigma = 0$. □

2.1.5 Corollary. *If $\mathcal{E}$ is a Riemannian vector bundle over a compact pseudo-Riemannian base manifold, and the Euler class $\chi(\mathcal{E}) \neq 0$, then the only harmonic section of $\mathcal{E}$ is the zero section.*

It is not self-evident from (2.2) that parallel sections of a pseudo-Riemannian vector bundle are harmonic. To see this we need to compute the first variation of the vertical energy.

2.1.6 Theorem. *Let σ be a section of a pseudo-Riemannian vector bundle over a pseudo-Riemannian base manifold. Then σ is a harmonic section if and only if $\nabla^*\nabla\sigma = 0$.*

Proof. Let $U \subset M$ be a relatively compact open set and let σ_t be a variation of the section σ through sections with support in U. The variation field may be viewed as a (compactly supported) section $\alpha \in \Gamma(\mathcal{E})$ as follows:

$$\alpha(x) = \frac{d}{dt}\Big|_{t=0} \sigma_t(x),$$

bearing in mind that $t \mapsto \sigma_t(x)$ is a curve in the fibre $\mathcal{E}_x$. Then

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mathrm{e}^v(\sigma_t) &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\langle\nabla\sigma_t, \nabla\sigma_t\rangle \\ &= \frac{1}{2}\sum_i \epsilon_i \frac{d}{dt}\Big|_{t=0}\langle\nabla_{E_i}\sigma_t, \nabla_{E_i}\sigma_t\rangle \\ &= \sum_i \epsilon_i \langle\nabla_{E_i}\alpha, \nabla_{E_i}\sigma\rangle \\ &= \langle\nabla\alpha, \nabla\sigma\rangle.\end{aligned}$$

Therefore

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}^v(\sigma_t; U) = \int_M \langle\nabla\alpha, \nabla\sigma\rangle \operatorname{vol}(g) = \int_M \langle\alpha, \nabla^*\nabla\sigma\rangle \operatorname{vol}(g),$$

by Proposition 1.2.20. The result now follows from Lemma 1.5.8. □

The restrictions imposed by Theorem 2.1.4 and Corollary 2.1.5 may be overcome to a certain extent by confining attention to sphere bundles.

2.1.7 Definition. Let $\pi\colon \mathcal{E} \to M$ be a pseudo-Riemannian vector bundle. Then for any $k \neq 0$ the *sphere bundle* $\pi\colon S\mathcal{E}(k) \to M$ is the subbundle of $\mathcal{E}$ with total space

$$S\mathcal{E}(k) = \{e \in \mathcal{E} : \langle e, e\rangle = k\}.$$

Note that if the fibre metric is positive (resp. negative) definite then it is implicit that $k > 0$ (resp. $k < 0$). Furthermore this is no longer a vector bundle, but a fibre bundle. ◊

A section σ of $\mathcal{E}$ with constant pseudo-Riemannian length k may be regarded as a section of $S\mathcal{E}(k)$. The manifold $S\mathcal{E}(k)$ may be given a pseudo-Riemannian metric by restricting the Sasaki metric of $\mathcal{E}$. Definition 2.1.3 may then be adapted by requiring σ to be a critical point of vertical energy with respect to variations through sections of pseudo-Riemannian length k, in which case σ is said to be a harmonic section of $S\mathcal{E}(k)$. The following result has been established in the Riemannian case, see for instance [24].

2.1.8 Theorem. *Let $\mathcal{E}$ be a pseudo-Riemannian vector bundle over a pseudo-Riemannian manifold, and let $\sigma \in \Gamma(\mathcal{E})$ be a section of constant pseudo-Riemannian length k. Then σ is a harmonic section of $S\mathcal{E}(k)$ if and only if it satisfies*

$$\nabla^*\nabla\sigma = \frac{1}{k}\langle\nabla\sigma, \nabla\sigma\rangle\sigma.$$

Proof. Let σ_t be a compactly supported variation of σ through sections of pseudo-Riemannian length k:

$$\langle\sigma_t, \sigma_t\rangle = k.$$

Therefore:

$$0 = \frac{d}{dt}\Big|_{t=0} \langle \sigma_t, \sigma_t \rangle = \langle \alpha, \sigma \rangle,$$

where α is the variation field. By the calculation of Theorem 2.1.6

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}(\sigma_t; U) = \int_M \langle \alpha, \nabla^* \nabla \sigma \rangle \,\mathrm{vol}(g).$$

Define $\hat{\sigma} = \sigma/\sqrt{|k|}$. Then $\hat{\sigma}$ is a section of pseudo-Riemannian length $\epsilon = k/|k|$, and the component of $\nabla^*\nabla\sigma$ in the direction of $\hat{\sigma}$ is $\epsilon\langle \nabla^*\nabla\sigma, \hat{\sigma}\rangle\hat{\sigma}$. Therefore

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}(\sigma_t; U) = \int_M \langle \alpha, \nabla^*\nabla\sigma - \epsilon\langle \nabla^*\nabla\sigma, \hat{\sigma}\rangle\hat{\sigma} \rangle \,\mathrm{vol}(g).$$

The integrand is now the inner product of two sections of the vector subbundle $\sigma^\perp \subset \mathcal{E}$. Applying Lemma 1.5.8 to this subbundle, it follows that σ is a harmonic section of $S\mathcal{E}(k)$ if and only if

$$\nabla^*\nabla\sigma = \epsilon\langle \nabla^*\nabla\sigma, \hat{\sigma}\rangle\hat{\sigma} = \frac{\epsilon}{|k|}\langle \nabla^*\nabla\sigma, \sigma\rangle\sigma = \frac{1}{k}\langle \nabla^*\nabla\sigma, \sigma\rangle\sigma.$$

Finally we note

$$\begin{aligned}
-\langle \nabla^*\nabla\sigma, \sigma\rangle &= \sum_i \epsilon_i \,\langle \nabla^2_{E_i,E_i}\sigma, \sigma\rangle \\
&= \sum_i \epsilon_i \,[\langle \nabla_{E_i}\nabla_{E_i}\sigma - \nabla_{\nabla_{E_i}E_i}\sigma, \sigma\rangle] \\
&= \sum_i \epsilon_i \,[E_i\langle \nabla_{E_i}\sigma, \sigma\rangle - \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle - \tfrac{1}{2}\nabla_{E_i}E_i\langle \sigma, \sigma\rangle] \\
&= \sum_i \epsilon_i \,[\tfrac{1}{2}E_i(E_i\langle \sigma, \sigma\rangle) - \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle - \tfrac{1}{2}\nabla_{E_i}E_i\langle \sigma, \sigma\rangle] \\
&= -\sum_i \epsilon_i \,\langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle, \quad \text{since } \langle \sigma, \sigma\rangle = k \\
&= -\langle \nabla\sigma, \nabla\sigma\rangle.
\end{aligned}$$

□

The case $\mathcal{E} = TM$ where M is a Riemannian manifold and the fibre metric of $\mathcal{E}$ is the Riemannian metric of M has been extensively studied (see for instance [13]). Harmonic sections in this case, with $k = 1$, are referred to as *harmonic unit vector fields.* Of course this theory is only applicable to manifolds with zero Euler characteristic.

2.2 The generalised Cheeger-Gromoll metric on vector bundles over pseudo-Riemannian manifolds

From now on we assume that $\pi\colon \mathcal{E} \to (M, g)$ is a pseudo-Riemannian vector bundle over a pseudo-Riemannian base space (M, g).

The definition of harmonic sections of such a vector bundle $\mathcal{E} \to M$ (Definition 2.1.3) assumes the Sasaki metric on the total space $\mathcal{E}$. A recent approach to removing the restrictions imposed by Theorems 2.1.4 and 2.1.8 and Corollary 2.1.5, proposed in [3], was to replace this metric by a member of the family of *generalised Cheeger-Gromoll metrics*. This is a 2-parameter family of semi-Riemannian metrics $h_{p,q}$ on $\mathcal{E}$, for all $p, q \in \mathbb{R}$, which contains amongst others the Sasaki metric ($h_{0,0}$), Cheeger-Gromoll metric ($h_{1,1}$) and the stereographic metric ($h_{2,0}$).

In this and subsequent sections, we generalise the work of Benyounes et al. from Riemannian vector bundles over Riemannian manifolds to pseudo-Riemannian vector bundles over pseudo-Riemannian manifolds. This requires careful modification of the generalised Cheeger-Gromoll metrics to ensure they remain meaningful in the pseudo-Riemannian case. The calculation of the Euler-Lagrange equations is also carefully reconsidered. This necessitates the consideration of non-compact base manifolds.

2.2.1 Definition. Let $\pi\colon \mathcal{E} \to (M, g)$ be a pseudo-Riemannian vector bundle over a pseudo-Riemannian manifold (M, g). Suppose the fibre metric is $\langle\, ,\rangle$ and the connection map is $K\colon T\mathcal{E} \to \mathcal{E}$. Define

$$\mathcal{E}' = \{e \in \mathcal{E} : \langle e, e\rangle \neq -1\}.$$

Then $\mathcal{E}' \subset \mathcal{E}$ is a dense open subset. For each $(p, q) \in \mathbb{R}^2$ define a symmetric $(2, 0)$-tensor $h_{p,q}$ on $\mathcal{E}'$ as follows:

$$h_{p,q}(A, B) = g(d\pi(A), d\pi(B)) + \omega^p(e)[\langle K(A), K(B)\rangle + q\langle K(A), e\rangle\langle e, K(B)\rangle],$$

for all $A, B \in T_e\mathcal{E}'$, where $\omega(e) = 1/|1 + \langle e, e\rangle|$. If $q = 0$ then $h_{p,q}$ is a pseudo-Riemannian metric on $\mathcal{E}'$ with the same signature as the Sasaki metric. Otherwise $h_{p,q}$ is of variable signature and therefore defines a semi-Riemannian metric on $\mathcal{E}'$. More precisely, if $q < 0$ (resp. $q > 0$) then $h_{p,q}$ has the same signature as the Sasaki metric in the region where $\langle e, e\rangle < -1/q$ (resp. $> -1/q$). Furthermore $h_{p,q}$ degenerates on the sphere bundle $S\mathcal{E}(-1/q)$ and:

- If $q < 0$ then the index of $h_{p,q}$ increases by 1 in the space-like region where $\langle e, e\rangle > -1/q$.
- If $q > 0$ then the index of $h_{p,q}$ decreases by 1 in the time-like region where $\langle e, e\rangle < -1/q$.

Proof. First note that the horizontal part of this is always the same signature as the base manifold and the $\omega^p(e)$ term is a scaling and does not effect the signature. Denote the relevant part:

$$\beta_q(A, B) = \langle K(A), K(B)\rangle + q\langle K(A), e\rangle\langle e, K(B)\rangle.$$

Let $q > 0$. Consider some $e \in \mathcal{E}$ with $\langle e, e\rangle < -1/q$. Let $\{e_i\}$ be a frame of $\mathcal{E}$ with $e = ae_j$ for some e_j time-like. Then $-a^2 < -1/q$. Now consider the uniquely determined frame of $T\mathcal{E}$, $\{A_i\}$, given by $K(A_i) = e_i$. The indicator terms κ_i of this frame are:

$$\begin{aligned}\kappa_i = \beta_q(A_i, A_i) &= \langle e_i, e_i\rangle + q\langle e_i, e\rangle\langle e, e_i\rangle \\ &= \begin{cases}\epsilon_i, & \text{if } i \neq j \\ \epsilon_j + qa^2, & \text{if } i = j.\end{cases}\end{aligned}$$

In the first case the indicator terms are the same as the fibre metric of the vector bundle. In the second case note that $\epsilon_j = -1$ and that $qa^2 > 1$. Hence $\kappa_j > 0$ and hence $K(A_j)$ is space-like. This means the total index of the metric has decreased by one from the Sasaki metric.

Similarly in the case $q < 0$ the index increases by one from the Sasaki metric for $\langle e, e\rangle > -1/q$. □

The parameters (p, q) are known as the *metric parameters*. ♢

2.2.2 Definition. The (p, q)*-energy density* of section $\sigma \in \Gamma(\mathcal{E})$ is

$$\mathrm{e}_{p,q}(\sigma) = \tfrac{1}{2} h_{p,q}(d\sigma, d\sigma).$$

The (p, q)*-energy* of σ over relatively compact open set $U \subset M$ is

$$\mathrm{E}_{p,q}(\sigma; U) = \int_U \mathrm{e}_{p,q}(\sigma)\,\mathrm{vol}(g). \qquad ♢$$

2.2.3 Definition. The section σ is said to be a (p, q)*-harmonic section* of $\mathcal{E}$ if

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}_{p,q}(\sigma_t; U) = 0,$$

for all relatively compact open $U \subset \sigma^{-1}\mathcal{E}'$ and all variations σ_t supported in U, where each σ_t is a section of $\mathcal{E}$. Note that $\sigma_t(U) \subset \mathcal{E}'$ for all sufficiently small t.

The (p, q)-energy density of σ may be calculated as follows, using Definition 2.2.1 and Theorem 1.4.4:

$$\begin{aligned} 2\,\mathrm{e}_{p,q}(\sigma) &= h_{p,q}(d\sigma, d\sigma) \\ &= \sum_i \epsilon_i\, h_{p,q}(d\sigma(E_i), d\sigma(E_i)) \\ &= \sum_i \epsilon_i\, [g(E_i, E_i) + \omega^p(\sigma)(\langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle + q\langle \nabla_{E_i}\sigma, \sigma\rangle^2)] \\ &= n + \omega^p(\sigma)(\langle \nabla\sigma, \nabla\sigma\rangle + qg(\nabla F, \nabla F)), \end{aligned} \tag{2.3}$$

where $F = \frac{1}{2}\langle\sigma, \sigma\rangle$. The last step to (2.3) goes as follows:

$$\begin{aligned} g(\nabla F, \nabla F) &= \sum_i \epsilon_i\, g(\nabla F, E_i)^2), \quad \text{by Proposition 1.1.13} \\ &= \sum_i \epsilon_i\, (E_i(F))^2, \quad \text{by Definition 1.1.14} \\ &= \tfrac{1}{4}\sum_i \epsilon_i\, (E_i\langle\sigma, \sigma\rangle)^2 \\ &= \sum_i \epsilon_i\, \langle \nabla_{E_i}\sigma, \sigma\rangle^2. \end{aligned} \tag{2.4}$$

Note that this requires the vector bundle to be pseudo-Riemannian. Again, using the splitting $T\mathcal{E} = V \oplus H$ (Proposition 1.4.6), the differential may be decomposed

$$d\sigma = d^v\sigma + d^h\sigma,$$

where $d^v\sigma(X)$ is the V-component of $d\sigma(X)$ etc. This allows us to define the *vertical* (p, q)*-energy density* of σ to be

$$\mathrm{e}^v_{p,q}(\sigma) = \tfrac{1}{2} h_{p,q}(d^v(\sigma), d^v(\sigma)),$$

and similarly the horizontal (p,q)-energy density $\mathrm{e}^h_{p,q}(\sigma)$. Since V and H are orthogonal with respect to the metric $h_{p,q}$:

$$\mathrm{e}_{p,q}(\sigma) = \mathrm{e}^v_{p,q}(\sigma) + \mathrm{e}^h_{p,q}(\sigma).$$

Inspection of (2.3) shows that

$$\mathrm{e}^v_{p,q}(\sigma) = \tfrac{1}{2}\omega^p(\sigma)(\langle\nabla\sigma,\nabla\sigma\rangle + qg(\nabla F,\nabla F)); \quad \mathrm{e}^h_{p,q}(\sigma) = \frac{n}{2}.$$

Notice once again the horizontal energy density is constant. It follows that σ is a (p,q)-harmonic section if and only if it is a critical point with respect to variations through sections of the *vertical* (p,q)*-energy functional:*

$$\mathrm{E}^v_{p,q}(\sigma;U) = \int_U \mathrm{e}^v_{p,q}(\sigma)\,\mathrm{vol}(g) = \frac{1}{2}\int_U \omega^p(\sigma)(\langle\nabla\sigma,\nabla\sigma\rangle + qg(\nabla F,\nabla F))\,\mathrm{vol}(g). \tag{2.5}$$

2.3 Euler-Lagrange equations

From this point we are roughly following the calculation of the Euler-Lagrange equations in the Riemannian case [3]. Some care is required to take into account both the move to pseudo-Riemannian geometry and the alterations to the generalised Cheeger-Gromoll metric.

Let σ be a section of $\mathcal{E}$ and let σ_t be a variation of σ through sections that is supported in a relatively compact open set $U \subset \sigma^{-1}\mathcal{E}'$. Define:

$$\Sigma\colon M\times\mathbb{R}\to\mathcal{E}; \quad \Sigma(x,t) = \sigma_t(x).$$

Let $\pi_1\colon M\times\mathbb{R}\to M$ be projection onto the first factor: $(x,t)\mapsto x$. Then Σ may be viewed as a section of the pullback bundle $\pi_1^{-1}\mathcal{E}\to M\times\mathbb{R}$.
2.3.1 Definition. Let $v(t)$ denote the variation field:

$$v_t = v(t) = \frac{d}{dt}\sigma_t.$$

Since σ_t is a variation through sections $v(t)$ is vertical. Define a family of sections ρ_t of $\pi\colon\mathcal{E}\to M$ as follows:

$$K\circ v_t = \rho_t. \qquad \Diamond$$

We note the following well known facts in two Lemmas.
2.3.2 Lemma. *Let ∂_t be the unit vector field on $M\times\mathbb{R}$ in the positive $\mathbb{R}$-direction. Then the covariant derivative of Σ with respect to the π_1-pullback connection is the π_1-pullback of ρ_t. That is:*

$$\nabla_{\partial_t}\Sigma = \pi_1^{-1}\rho_t.$$

Proof. Recall Theorem 1.4.4, and note that:

$$v_t\circ\pi_1 = d\tilde{\pi}_1\circ d\Sigma(\partial_t).$$

Then, if $\widetilde{K}$ is the connection map for the pullback connection:

$$\begin{aligned}\tilde{\pi}_1 \circ \nabla_{\partial_t}\Sigma &= \tilde{\pi}_1 \circ \widetilde{K}(d\Sigma(\partial_t)) \\ &= K \circ d\tilde{\pi}_1(d\Sigma(\partial_t)), \quad \text{by (1.5)} \\ &= K(v_t \circ \pi_1) \\ &= \rho_t \circ \pi_1.\end{aligned}$$

□

2.3.3 Lemma. *The curvature of the pullback connection satisfies:* $R(TM, T\mathbb{R}) = 0$.

For the purposes of our main calculation it is convenient to split the first variation into two pieces V_1 and V_2 as follows,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \mathrm{E}^v_{p,q}(\sigma_t; U) &= \frac{1}{2}\int_U \frac{d}{dt}\Big|_{t=0} \omega^p(\sigma_t)(\langle \nabla\sigma, \nabla\sigma\rangle + qg(\nabla F, \nabla F)) \operatorname{vol}(g) \\ &+ \frac{1}{2}\int_U \omega^p(\sigma) \frac{d}{dt}\Big|_{t=0} (\langle \nabla\sigma_t, \nabla\sigma_t\rangle + qg(\nabla F_t, \nabla F_t)) \operatorname{vol}(g) \\ &= V_1 + V_2.\end{aligned}$$

We consider each of the components of the calculation in turn, introducing $\alpha = dF \otimes \sigma$, an $\mathcal{E}$-valued 1-form on M, and denoting $\rho = \rho_0$.

2.3.4 Lemma.

1. $\frac{d}{dt}\Big|_{t=0} \omega^p(\sigma_t) = -2p\epsilon\omega^{p+1}(\sigma)\langle \sigma, \rho\rangle$,
   where $\epsilon = \frac{1+2F}{|1+2F|} = \pm 1$ *is the sign of* $1 + \langle \sigma, \sigma\rangle$.

2. $\frac{d}{dt}\Big|_{t=0} \langle \nabla\sigma_t, \nabla\sigma_t\rangle = 2\langle \nabla\rho, \nabla\sigma\rangle$.

3. $\frac{d}{dt}\Big|_{t=0} g(\nabla F_t, \nabla F_t) = 2\langle \alpha, \nabla\rho\rangle + 2\langle \nabla_{\nabla F}\sigma, \rho\rangle$.

Proof. 1. By elementary calculus:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} \omega^p(\sigma_t) &= \frac{d}{dt}\Big|_{t=0} |1 + \langle \sigma_t, \sigma_t\rangle|^{-p} \\ &= -p\epsilon|1 + \langle \sigma_t, \sigma_t\rangle|^{-p-1} \frac{d}{dt}\Big|_{t=0} \langle \sigma_t, \sigma_t\rangle \\ &= -2p\epsilon|1 + \langle \sigma, \sigma\rangle|^{-p-1}\langle \rho, \sigma\rangle \\ &= -2p\epsilon\omega^{p+1}(\sigma)\langle \sigma, \rho\rangle.\end{aligned}$$

2. Summing over i:

$$\frac{d}{dt}\Big|_{t=0} \langle \nabla\sigma_t, \nabla\sigma_t\rangle = 2\epsilon_i \left\langle \nabla_{\partial_t}\nabla_{E_i}\Sigma, \nabla_{E_i}\sigma_t\right\rangle\Big|_{t=0}$$

swapping the order of covariant differentiation, using Lemma 2.3.3:

$$= 2\epsilon_i \left\langle \nabla_{E_i}\nabla_{\partial_t}\Sigma, \nabla_{E_i}\sigma_t\right\rangle\Big|_{t=0}$$

and using Lemma 2.3.2 to evaluate the covariant derivative:

$$\begin{aligned}&= 2\epsilon_i \left\langle \nabla_{E_i}\rho_t, \nabla_{E_i}\sigma_t \right\rangle\Big|_{t=0}\\ &= 2\langle \nabla\rho, \nabla\sigma\rangle.\end{aligned}$$

Note that we make no notational distinction between the connection ∇ in $\mathcal{E}$ and its π_1-pullback.

3. Summing over i:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|_{t=0} g(\nabla F_t, \nabla F_t) &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \epsilon_i \left\langle \nabla_{E_i}\sigma_t, \sigma_t\right\rangle^2, \quad \text{by (2.4)}\\ &= \epsilon_i \left\langle \nabla_{E_i}\sigma, \sigma\right\rangle \frac{d}{dt}\Big|_{t=0} \langle \nabla_{E_i}\sigma_t, \sigma_t\rangle\\ &= \epsilon_i\, E_i(F)(\langle \nabla_{\partial_t}\nabla_{E_i}\Sigma, \sigma\rangle\Big|_{t=0} + \langle \nabla_{E_i}\sigma, \rho\rangle)\end{aligned}$$

using Lemma 2.3.3 to swap the order of covariant differentiation and Lemma 2.3.2 to evaluate the covariant derivatives:

$$= \epsilon_i\, g(\nabla F, E_i)(\langle \nabla_{E_i}\rho, \sigma\rangle + \langle \nabla_{E_i}\sigma, \rho\rangle)$$

and using the definition $\alpha = dF \otimes \sigma$:

$$\begin{aligned}&= \epsilon_i \left\langle \nabla_{E_i}\rho, \alpha(E_i)\right\rangle + \langle \nabla_{\nabla F}\sigma, \rho\rangle\\ &= \langle \alpha, \nabla\rho\rangle + \langle \nabla_{\nabla F}\sigma, \rho\rangle.\end{aligned}$$

□

2.3.5 Corollary. *The pieces of the first variation are:*

$$\begin{aligned}V_1 &= -p\epsilon \int_U \omega^{p+1}(\sigma)\langle\sigma,\rho\rangle[\langle\nabla\sigma,\nabla\sigma\rangle + qg(\nabla F, \nabla F)]\,\mathrm{vol}(g),\\ V_2 &= \int_U \omega^p(\sigma)(\langle\nabla\sigma + q\alpha, \nabla\rho\rangle + q\langle\nabla_{\nabla F}\sigma, \rho\rangle)\,\mathrm{vol}(g).\end{aligned}$$

We recall Proposition 1.2.17, for a $\mathcal{E}$-valued 1-form β and smooth function $f\colon M \to \mathbb{R}$:

$$\nabla^*(f\beta) = f\nabla^*(\beta) - \beta(\nabla f). \tag{2.6}$$

Let $f = \omega^p(\sigma)$. Then

$$\begin{aligned}\nabla f &= p\omega^{p-1}(\sigma)\nabla(\omega(\sigma))\\ &= -2p\omega^{p-1}(\sigma)\epsilon|1+2F|^{-2}\nabla F\\ &= -2p\omega^{p+1}(\sigma)\nabla F. \end{aligned}\tag{2.7}$$

2.3.6 Proposition. *The codifferential of* $\gamma = \omega^p(\sigma)(\nabla\sigma + q\alpha)$ *is*

$$\nabla^*\gamma = \omega^p(\sigma)(\nabla^*\nabla\sigma + q\nabla^*\alpha) + 2p\epsilon\omega^{p+1}(\sigma)(\nabla_{\nabla F}\sigma + qg(\nabla F, \nabla F)\sigma).$$

Proof. Let $\beta = \nabla\sigma + q\alpha$ and $f = \omega^p(\sigma)$. Then $\gamma = f\beta$ and by (2.7) and (2.6):

$$\begin{aligned}\nabla^*\gamma = \nabla^*(f\beta) &= \omega^p(\sigma)(\nabla^*\nabla\sigma + q\nabla^*\alpha) - (\nabla\sigma + q\alpha)(-2p\epsilon\omega^{p+1}(\sigma)\nabla F) \\ &= \omega^p(\sigma)(\nabla^*\nabla\sigma + q\nabla^*\alpha) + 2p\epsilon\omega^{p+1}(\sigma)(\nabla_{\nabla F}\sigma + qg(\nabla F, \nabla F)\sigma).\end{aligned}$$ □

2.3.7 Lemma. *The codifferential of* $\alpha = dF \otimes \sigma$ *is:*

$$\nabla^*\alpha = (\Delta F)\sigma - \nabla_{\nabla F}\sigma.$$

Proof. Taking the covariant derivative of the tensor product:

$$\nabla\alpha = (\nabla dF) \otimes \sigma + dF \otimes \nabla\sigma,$$

and applying the definition of the coderivative (Definition 1.2.16):

$$\begin{aligned}\nabla^*\alpha &= (\nabla^* dF)\sigma - \sum_i \epsilon_i\, dF(E_i)\nabla_{E_i}\sigma \\ &= (\Delta F)\sigma - \sum_i \epsilon_i\, \nabla_{dF(E_i)E_i}\sigma \\ &= (\Delta F)\sigma - \nabla_{\nabla F}\sigma, \quad \text{by Definition 1.1.14.}\end{aligned}$$ □

Applying Lemma 2.3.7 to Proposition 2.3.6 yields:
2.3.8 Proposition. *The codifferential of Proposition 2.3.6 is:*

$$\nabla^*\gamma = \omega^p(\sigma)(\nabla^*\nabla\sigma + q((\Delta F)\sigma - \nabla_{\nabla F}\sigma)) + 2p\epsilon\omega^{p+1}(\sigma)(\nabla_{\nabla F}\sigma + qg(\nabla F, \nabla F)\sigma).$$

We are now in a position to calulate the Euler-Lagrange equations for (p,q)-harmonic sections.
2.3.9 Theorem. *The section* $\sigma \in \Gamma(\mathcal{E})$ *is* (p,q)*-harmonic if and only if*

$$\tau_{p,q}(\sigma) = 0, \tag{2.8}$$

where

$$\tau_{p,q}(\sigma) = T_p(\sigma) - \phi_{p,q}(\sigma)\sigma,$$

with

$$\begin{aligned}T_p(\sigma) &= (1+2F)\nabla^*\nabla\sigma + 2p\,\nabla_{\nabla F}\sigma, \\ \phi_{p,q}(\sigma) &= p\langle\nabla\sigma, \nabla\sigma\rangle - pq\, g(\nabla F, \nabla F) - q(1+2F)\Delta F.\end{aligned}$$

Proof. The first variation of the local vertical (p,q)-energy is

$$\frac{d}{dt}\Big|_{t=0} \mathrm{E}^v_{p,q}(\sigma_t; U) = V_1 + V_2,$$

where by Corollary 2.3.5:

$$\begin{aligned}V_1 &= -p\int_M \epsilon\omega^{p+1}(\sigma)\langle[\langle\nabla\sigma, \nabla\sigma\rangle + qg(\nabla F, \nabla F)]\sigma, \rho\rangle\,\mathrm{vol}(g), \\ V_2 &= \int_M \langle\nabla^*\gamma + q\omega^p(\sigma)\nabla_{\nabla F}\sigma, \rho\rangle\,\mathrm{vol}(g).\end{aligned}$$

Note that we have used integration by parts (Proposition 1.2.20) to rewrite V_2 in divergence form. Now by Proposition 2.3.8:

$$V_2 = \int_M \langle \omega^p(\sigma)(\nabla^*\nabla\sigma + q(\Delta F)\sigma) + 2p\epsilon\omega^{p+1}(\sigma)(\nabla_{\nabla F}\sigma + qg(\nabla F, \nabla F)\sigma), \rho\rangle \operatorname{vol}(g),$$

by a cancellation of terms

$$= \int_M \epsilon\omega^{p+1}(\sigma)\langle \epsilon|1+2F|[\nabla^*\nabla\sigma + q(\Delta F)\sigma] + 2p(\nabla_{\nabla F}\sigma + qg(\nabla F, \nabla F)\sigma), \rho\rangle \operatorname{vol}(g).$$

Therefore:

$$V_1 + V_2 = \int_M \epsilon\omega^{p+1}(\sigma)\langle \tau_{p,q}(\sigma), \rho\rangle \operatorname{vol}(g),$$

noting that

$$\epsilon|1+2F| = 1+2F.$$

The result now follows from Lemma 1.5.8. □

2.3.10 Remarks.

1. The Euler-Lagrange equations (2.8) make sense on all of M not just on $\sigma^{-1}(\mathcal{E}')$. We thus redefine "$(p,q)$-harmonic section" to mean a section of $\mathcal{E}$ that satisfies Equation (2.8).

2. If the section σ has $\langle\sigma,\sigma\rangle \equiv -1$, i.e. $M \setminus \sigma^{-1}(\mathcal{E}') = \emptyset$, then σ is $(0,q)$-harmonic for all $q \in \mathbb{R}$.

3. If $\langle\sigma,\sigma\rangle = k \neq -1$ then the Euler-Lagrange equations are:

   $$(1+k)\nabla^*\nabla\sigma = p\langle\nabla\sigma, \nabla\sigma\rangle\sigma. \tag{2.9}$$

   Therefore by Theorem 2.1.8 σ is a harmonic section of the sphere bundle $S\mathcal{E}(k)$ if and only if σ is (p,q)-harmonic for $p = \frac{1+k}{k}$ and all q.

   If $\langle\sigma,\sigma\rangle = k = -1$ then the sections are $(0,q)$-harmonic for all q, as noted in Remark 2 and there is no link between the (p,q)-harmonic sections and the harmonic sections $S\mathcal{E}(-1)$.

4. If σ is parallel then σ is (p,q)-harmonic for all (p,q).

2.4 Preharmonic sections

A useful concept it that of a preharmonic section. This is a both relatively quick way to determine if a section is likely to be harmonic, and a way in which to simplify the Euler-Lagrange equations.

2.4.1 Definition. A section σ of a pseudo-Riemannian vector bundle $\pi\colon \mathcal{E} \to M$ is *p-preharmonic* if $T_p(\sigma)$ is pointwise collinear with σ, and *preharmonic* if it is p-preharmonic for all p. Preharmonicity means:

1. There exists a smooth function $\nu\colon M \to \mathbb{R}$ such that $\nabla^*\nabla\sigma = \nu\sigma$, for example if σ is an eigenfunction of the rough Laplacian.

2. There exists a smooth function $\zeta\colon M \to \mathbb{R}$ such that $\nabla_{\nabla F}\sigma = \zeta\sigma$.

The function ζ introduced in the Riemannian case in [5], is called the *spinnaker* of σ. $\Diamond$

The following Weitzenböck formula generalises the calculation used at the end of the proof of Theorem 2.1.8.

2.4.2 Lemma (Weitzenböck identity). *Let σ be a section of a pseudo-Riemannian vector bundle $\pi\colon \mathcal{E} \to M$. Then*

$$\langle \nabla^*\nabla\sigma, \sigma\rangle = \langle \nabla\sigma, \nabla\sigma\rangle + \Delta F.$$

Proof. Let $\{E_i\}$ be a frame of M. Then summing over i:

$$\begin{aligned}
-\langle \nabla^*\nabla\sigma, \sigma\rangle &= \langle \epsilon_i\, \nabla^2_{E_i,E_i}\sigma, \sigma\rangle \\
&= \epsilon_i\, \langle \nabla_{E_i}\nabla_{E_i}\sigma - \nabla_{D_{E_i}E_i}\sigma, \sigma\rangle \\
&= \epsilon_i\, [E_i\langle \nabla_{E_i}\sigma, \sigma\rangle - \langle \nabla_{E_i}\sigma, \nabla_{E_i}\sigma\rangle] - \epsilon_i\, \tfrac{1}{2} D_{E_i}E_i\langle \sigma,\sigma\rangle \\
&= \epsilon_i\, [E_i(E_iF) - (D_{E_i}E_i)F] - \langle \nabla\sigma, \nabla\sigma\rangle \\
&= -\nabla^* dF - \langle \nabla\sigma, \nabla\sigma\rangle, \quad \text{by Definition 1.2.16,} \\
&= -\Delta F - \langle \nabla\sigma, \nabla\sigma\rangle, \quad \text{by Definition 1.1.16.}
\end{aligned}$$

$\square$

2.4.3 Theorem. *Let σ be a preharmonic section of the vector bundle $\pi\colon \mathcal{E} \to M$. Then σ is (p,q)-harmonic if and only if*

$$(p + q + 2qF)\Delta F + 2p(1 + qF)\zeta + (1 + 2(1-p)F)\nu = 0.$$

Proof. Let $\nabla^*\nabla\sigma = \nu\sigma$ and $\nabla_{\nabla F}\sigma = \zeta\sigma$. Then

$$T_p(\sigma) = ((1 + 2F)\nu + 2p\zeta)\sigma. \tag{2.10}$$

Now

$$\begin{aligned}
\langle \nabla F, \nabla F\rangle &= \nabla F(F) \\
&= \tfrac{1}{2}\nabla F\langle \sigma, \sigma\rangle \\
&= \langle \nabla_{\nabla F}\sigma, \sigma\rangle \\
&= \langle \zeta\sigma, \sigma\rangle \\
&= 2F\zeta,
\end{aligned}$$

and use of the Weitzenböck identity (Lemma 2.4.2) yields

$$\langle \nabla\sigma, \nabla\sigma\rangle = \nu\langle \sigma, \sigma\rangle - \Delta F.$$

Therefore

$$\begin{aligned}
\phi_{p,q}(\sigma) &= p(2F\nu - \Delta F) - pq(2F\zeta) - q(1 + 2F)\Delta F \\
&= 2p(\nu - q\zeta)F - (p + q + 2qF)\Delta F.
\end{aligned} \tag{2.11}$$

The result follows by applying Theorem 2.3.9 to (2.10) and(2.11). $\square$

2.5 Harmonic vector fields

We now consider the situation where $\mathcal{E} = TM$, and M is a pseudo-Riemannian manifold. The Levi-Civita connection of M will now be denoted by ∇ and the metric g on M will often be denoted $\langle\,,\rangle$.

2.5.1 Definition. A vector field σ on M is said to be a *harmonic vector field* if σ is a (p,q)-harmonic section of the vector bundle $TM \to M$ for some (p,q). In this case (p,q) are said to be *metric parameters for* σ.

The metric parameters for a harmonic vector field need not be uniquely determined. This was already known in the Riemannian case from [5] and we will give non-Riemannian examples, such as Theorem 4.3.8.

There is a natural action of the isometry group of (M,g) on vector fields.

2.5.2 Definition. Let $\varphi\colon (M,g) \to (M,g)$ be an isometry. Then for any vector field σ on M define a vector field $\varphi.\sigma$ by:

$$(\varphi.\sigma)(x) = d\varphi(\sigma(\varphi^{-1}(x))).$$

The vector fields σ and $\varphi.\sigma$ are said to be *congruent*.

Although the equations for a vector field to be harmonic are somewhat complicated, they are in fact invariant under this action.

2.5.3 Theorem. *Let σ be a harmonic vector field on M and let φ be an isometry of M. Then $\varphi.\sigma$ is also harmonic, with the same metric parameters.*

Proof. It follows from the fact that $\nabla d\varphi = 0$ (Proposition 1.2.15) that

$$\nabla_X(\varphi.\sigma) = \varphi.\nabla_X\sigma, \quad \text{for all } X \in \Gamma(TM).$$

Therefore

$$\nabla^*\nabla(\varphi.\sigma) = \varphi.\nabla^*\nabla\sigma.$$

Since $\langle\varphi.\sigma, \varphi.\sigma\rangle = \langle\sigma,\sigma\rangle$ we deduce that

$$T_p(\varphi.\sigma) = \varphi.T_p(\sigma).$$

Furthermore

$$\langle\nabla(\varphi.\sigma), \nabla(\varphi.\sigma)\rangle = \langle\varphi.\nabla\sigma, \varphi.\nabla\sigma\rangle = \langle\nabla\sigma,\nabla\sigma\rangle,$$

and the remaining terms of $\phi_{p,q}(\varphi.\sigma)$ are the same as those of $\phi_{p,q}(\sigma)$. Therefore

$$\phi_{p,q}(\varphi.\sigma) = \phi_{p,q}(\sigma).$$

It follows that

$$\tau_{p,q}(\varphi.\sigma) = \varphi.\tau_{p,q}(\sigma).$$

□

2.5.4 Remark. Although harmonic vector fields are invariant under isometry it is perhaps surprising to learn that they are not invariant under simple rescaling. Examples of this were already noted in [3]. As a consequence when solving the Euler-Lagrange equations scale factors cannot be neglected.

Chapter 3

Harmonic Closed Conformal Vector Fields

From now on we will deal with tangent bundles and vector fields, rather than vector bundles and sections. We consider the general class of vector fields known as closed conformal fields. This generalises the class of conformal gradient fields, all of which are preharmonic. To find examples we consider warped products; specifically where the warping factor is an interval of the real line. These warped products have a natural closed conformal field that allows us to reduce the Euler-Lagrange equations for harmonic vector fields to an ordinary differential equation of the warping function. An analysis of this equation can be found in Appendix A.

Every pseudo-Riemannian space form may be regarded as a warped product (see for instance [8]). For example, the Riemannian n-sphere is the warped product of the Riemannian $(n-1)$-sphere with an interval, warped by the cosine function. This will allow us to generate examples of harmonic vector fields on most pseudo-Riemannian space forms, but notably the construction breaks down on the Riemannian 2-sphere.

3.1 Geometry of closed conformal vector fields

We begin by introducing conformal vector fields and considering the geometric objects that are involved in the Euler-Lagrange equations.

3.1.1 Definition. A *conformal vector field* σ on a pseudo-Riemannian manifold (M, g) is characterised by

$$\mathrm{L}_\sigma \, g = 2\psi_\sigma g,$$

for a smooth function $\psi_\sigma \in C^\infty(M)$ (see below). $\Diamond$

3.1.2 Definition. A *closed* vector field $\sigma \in \Gamma(TM)$ is one whose metrically dual 1-form ω is closed. That is

$$d\omega = 0.$$

$\Diamond$

For computational purposes the following characterisations are useful.

3.1.3 Lemma. *A vector field $\sigma \in \Gamma(TM)$ is conformal is and only if*

$$g(\nabla_X \sigma, Y) + g(X, \nabla_Y \sigma) = 2\psi_\sigma g(X, Y), \quad \textit{for all } X, Y \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection.

Proof. We note that

$$\begin{aligned}(\mathrm{L}_\sigma\, g)(X, Y) &= \sigma\langle X, Y\rangle - \langle \mathrm{L}_\sigma\, X, Y\rangle - \langle X, \mathrm{L}_\sigma\, Y\rangle \\ &= \sigma\langle X, Y\rangle - \langle [\sigma, X], Y\rangle - \langle X, [\sigma, Y]\rangle \\ &= \langle \nabla_\sigma X - [\sigma, X], Y\rangle + \langle X, \nabla_\sigma Y - [Y, \sigma]\rangle \\ &= \langle \nabla_X \sigma, Y\rangle + \langle X, \nabla_Y \sigma\rangle.\end{aligned}$$

Notice that both properties of the Levi-Civita connection have been used. □

It follows from the lemma that the *conformal factor* $\psi_\sigma = \frac{1}{n} \operatorname{div} \sigma$. For:

$$\begin{aligned}2n\psi_\sigma = \sum_i \epsilon_i\, 2\psi_\sigma g(E_i, E_i) &= \sum_i \epsilon_i \left[g(\nabla_{E_i}\sigma, E_i) + g(E_i, \nabla_{E_i}\sigma)\right] \\ &= 2\sum_i \epsilon_i\, g(\nabla_{E_i}\sigma, E_i) \\ &= 2\operatorname{div}(\sigma), \quad \text{by Definition 1.1.15.}\end{aligned}$$

3.1.4 Lemma. *A vector field $\sigma \in \Gamma(TM)$ is closed if and only if*

$$g(\nabla_X \sigma, Y) - g(X, \nabla_Y \sigma) = 0, \quad \textit{for all } X, Y \in \Gamma(TM).$$

Proof. The exterior derivative of a 1-form ω may be written as follows:

$$d\omega(X, Y) = X\omega(Y) - Y\omega(Y) - \omega([X, Y]).$$

When $\omega(X) = g(X, \sigma)$ we have:

$$\begin{aligned}d\omega(X, Y) &= X\langle Y, \sigma\rangle - Y\langle X, \sigma\rangle - \langle [X, Y], \sigma\rangle \\ &= \langle \nabla_X Y, \sigma\rangle + \langle Y, \nabla_X \sigma\rangle - \langle \nabla_Y X, \sigma\rangle - \langle X, \nabla_Y \sigma\rangle - \langle [X, Y], \sigma\rangle \\ &= \langle Y, \nabla_X \sigma\rangle - \langle X, \nabla_Y \sigma\rangle + \langle \nabla_X Y - \nabla_Y X - [X, Y], \sigma\rangle \\ &= \langle Y, \nabla_X \sigma\rangle - \langle X, \nabla_Y \sigma\rangle.\end{aligned}$$

Note again, both properties of the Levi-Civita connection are used. □

Lemmas 3.1.3 and 3.1.4 can be combined to give the covariant derivative of a closed conformal vector field.
3.1.5 Theorem. *Let $\sigma \in \Gamma(TM)$ be a closed conformal vector field. Then:*

$$\nabla_X \sigma = \psi_\sigma X, \quad \textit{for all } X \in TM,$$

where $\psi_\sigma = \frac{1}{n} \operatorname{div} \sigma$.

Proof. Adding Lemmas 3.1.3 and 3.1.4 yields:

$$g(\nabla_X \sigma, Y) = \psi_\sigma \, g(X, Y).$$

Then for a frame of $\{E_i\}$ of M:

$$\begin{aligned}\nabla_X \sigma &= \sum_i \epsilon_i \, g(\nabla_X \sigma, E_i) E_i \\ &= \sum_i \epsilon_i \, \psi_\sigma \, g(X, E_i) E_i \\ &= \psi_\sigma \sum_i \epsilon_i \, g(X, E_i) E_i \\ &= \psi_\sigma X.\end{aligned}$$

□

The converse is also true.

3.1.6 Proposition. *Let $\sigma \in \Gamma(TM)$ be a vector field with covariant derivative:*

$$\nabla_X \sigma = \psi_\sigma X, \quad \textit{for all } X \in TM.$$

Then σ is closed and conformal.

Proof. This follows directly by substituting the covariant derivative into Lemmas 3.1.3 and 3.1.4. □

We now calculate the various components of the Euler-Lagrange equation (2.8), to enable us to see when a closed conformal vector field is harmonic.

3.1.7 Proposition. *Let $\sigma \in \Gamma(TM)$ be a closed conformal vector field with conformal factor ψ_σ. Then*

$$\langle \nabla \sigma, \nabla \sigma \rangle = n \psi_\sigma^2.$$

Proof. It follows from the result of Theorem 3.1.5

$$\nabla_X \sigma = \psi_\sigma X,$$

that

$$\begin{aligned}\langle \nabla \sigma, \nabla \sigma \rangle &= \sum_i \epsilon_i \, \langle \nabla_{E_i} \sigma, \nabla_{E_i} \sigma \rangle \\ &= \sum_i \epsilon_i \, \langle \psi_\sigma E_i, \psi_\sigma E_i \rangle \\ &= \sum_i \epsilon_i \, \psi_\sigma^2 \, \langle E_i, E_i \rangle \\ &= n \psi_\sigma^2.\end{aligned}$$

□

3.1.8 Proposition. *Let $\sigma \in \Gamma(TM)$ be a closed conformal vector field with conformal factor ψ_σ. If $F = \frac{1}{2} \langle \sigma, \sigma \rangle$ then*

$$\nabla F = \mathrm{grad}(F) = \psi_\sigma \, \sigma,$$

hence

$$\langle \nabla F, \nabla F \rangle = 2 \psi_\sigma^2 F.$$

Proof. Decompose the gradient into its components:

$$\begin{aligned}\operatorname{grad}(F) &= \sum_i \epsilon_i \langle \nabla F, E_i \rangle E_i \\ &= \sum_i \epsilon_i \, E_i(F) E_i \\ &= \sum_i \epsilon_i \langle \nabla_{E_i} \sigma, \sigma \rangle E_i \\ &= \sum_i \epsilon_i \langle \psi_\sigma E_i, \sigma > E_i \\ &= \psi_\sigma \sum_i \epsilon_i \langle E_i, \sigma \rangle E_i \\ &= \psi_\sigma \, \sigma.\end{aligned}$$

Then the second part follows from

$$\langle \nabla F, \nabla F \rangle = \langle \psi_\sigma \, \sigma, \psi_\sigma \, \sigma \rangle = 2\psi_\sigma^2 F.$$ □

3.1.9 Proposition. *Let $\sigma \in \Gamma(TM)$ be a closed conformal vector field with conformal factor ψ_σ. The covariant derivative in the direction of the gradient of F is*

$$\nabla_{\nabla F} \sigma = \psi_\sigma^2 \sigma.$$

Proof. The result follow directly from Theorem 3.1.5 and Proposition 3.1.8. □

3.1.10 Proposition. *Let $\sigma \in \Gamma(TM)$ be a closed conformal vector field with conformal factor ψ_σ. Then the rough Laplacian of σ is*

$$\nabla^* \nabla \sigma = -\nabla \psi_\sigma = -\operatorname{grad}(\psi_\sigma).$$

Proof. First consider the second covariant derivative

$$\begin{aligned}\nabla^2_{X,Y} \sigma &= \nabla_X (\nabla_Y \sigma) - \nabla_{\nabla_X Y} \sigma \\ &= \nabla_X (\psi_\sigma Y) - \psi_\sigma \nabla_X Y \\ &= (X\psi_\sigma) Y + \psi_\sigma \nabla_X Y - \psi_\sigma \nabla_X Y \\ &= (X\psi_\sigma) Y,\end{aligned}$$

then note that the rough Laplacian is

$$\begin{aligned}\nabla^* \nabla \sigma &= -\operatorname{trace} \nabla^2 \sigma \\ &= -\sum_i \epsilon_i \, \nabla^2_{E_i, E_i} \sigma \\ &= -\sum_i \epsilon_i \, (E_i \psi_\sigma) E_i \\ &= -\operatorname{grad} \psi_\sigma.\end{aligned}$$ □

3.1.11 Proposition. *Let $\sigma \in \Gamma(M)$ be a closed conformal vector field with conformal factor ψ_σ. The Laplacian of the function F is*

$$\Delta F = -n\psi_\sigma^2 - \langle \sigma, \nabla \psi_\sigma \rangle.$$

Proof. Consider the Laplacian as $\Delta F = -\operatorname{div}(\operatorname{grad}(F))$. Then

$$\begin{aligned} -\Delta F &= \sum_i \epsilon_i \langle \nabla_{E_i}(\psi_\sigma \sigma), E_i \rangle, \quad \text{by Proposition 3.1.8} \\ &= \sum_i \epsilon_i \langle (E_i \psi_\sigma)\sigma, E_i \rangle + \sum_i \epsilon_i \langle \psi_\sigma \nabla_{E_i}\sigma, E_i \rangle \\ &= \sum_i \langle \sigma, \epsilon_i (E_i \psi_\sigma) E_i \rangle + \sum_i \epsilon_i \langle \psi_\sigma^2 E_i, E_i \rangle \\ &= \langle \sigma, \nabla \psi_\sigma \rangle + n\psi_\sigma^2. \end{aligned}$$

□

Combining Propositions 3.1.7 to 3.1.11 yields the following result.

3.1.12 Theorem. *Let $\sigma \in \Gamma(TM)$ be a closed conformal vector field with conformal factor ψ_σ. Then by* (2.8) *of Theorem 2.3.9:*

$$\begin{aligned} T_p(\sigma) &= -(1+2F)\nabla \psi_\sigma + 2p\psi_\sigma^2 \sigma, \\ \phi_{p,q}(\sigma) &= pn\psi_\sigma^2 - 2pq\psi_\sigma^2 F + q(1+2F)(n\psi_\sigma^2 + \langle \sigma, \nabla \psi_\sigma \rangle). \end{aligned}$$

In order to make further progress further assumptions on σ are necessary.

3.2 Warped products

We introduce warped products, both in the case of a general warping factor and the case where the warping factor is a real interval. It is the latter case that provides us with our first examples of harmonic vector fields on pseudo-Riemannian manifolds. In fact warped products provide us with a plethora of examples. This is because the solutions to the Euler-Lagrange equations do not depend on the geometry of the warped factor, so in fact we can choose any manifold we like and find a harmonic closed conformal vector field on its warped product with suitable warping function and real interval.

We begin with an introduction to warped products, following [18] which itself summarises the constructions first introduced in [6]. This is then simplified to the case where the warping factor is a space-like real interval. The spur to consider this type of manifold was the paper [8], in which this type of closed conformal field is studied and the link between these warped products and space forms noted.

3.2.1 Definition. Let $(B, \langle\,,\rangle_B), (F, \langle\,,\rangle_F)$ be pseudo-Riemannian manifolds, let $f\colon B \to \mathbb{R}^+$ be a positive function on B. The *warped product* $M = B \times_f F$ is the product manifold $B \times F$ with the warped metric g:

$$g_x(X,Y) = \langle d\pi_B(X), d\pi_B(Y) \rangle_B + (f \circ \pi_B(x))^2 \langle d\pi_F(X), d\pi_F(Y) \rangle_F,$$

for all $X, Y \in T_xM$, where $\pi_B\colon M \to B$ and $\pi_F\colon M \to F$ are the projection maps. Call B the *warping factor*, F the *warped factor* and f the *warping function.* Note that the index of the warped metric is the sum of the indices of the factors. ◊

We now consider the Levi-Civita connection of this warped product, followed by its curvatures. If $X \in \Gamma(TB)$ then the *lift* of X to M is the vector field $(X, 0) \in \Gamma(TM)$, and similarly if $V \in \Gamma(TF)$ then its lift to M is the vector field $(0, V) \in \Gamma(TM)$. It is convenient to make no notational distinction between a vector field and its lift.

3.2.2 Proposition ([18, Prop. 7.35]). *Let $M = B \times_f F$ be a warped product, with vector fields $X, Y \in \Gamma(TB)$ and $V, W \in \Gamma(TF)$. Then the connection on M is characterised by:*

1. $\nabla_X Y \in TB$ *is the lift of* $\nabla^B_X Y$ *on* B*,*
2. $\nabla_X V = \nabla_V X = (X(f)/f)V$*,*
3. $\mathrm{nor}\, \nabla_V W = -f\, \langle V, W\rangle_F\, \nabla f$*,*
4. $\tan \nabla_V W \in TF$ *is the lift of* $\nabla^F_V W$ *on* F*,*

where "$\tan$*" denotes the component tangent to the fibres (the tangent spaces of the warped factor, TF) and "*nor*" denotes the component tangent to the base (the tangent spaces of the warping factor, TB).*

Proof. We clarify a detail in the proof of [18] arising in (3).

$$\begin{aligned} g(\nabla_V W, X) &= -g(W, \nabla_V X) \\ &= -(Xf/f)g(W, V), \quad \text{by (2)} \\ &= -(Xf)f\langle W, V\rangle_F, \quad \text{by Definition 3.2.1} \\ &= -\langle \nabla f, X\rangle_B\, f\, \langle W, V\rangle_F \\ &= -g(f\, \langle W, V\rangle_F\, \nabla f, X). \end{aligned}$$

□

3.2.3 Proposition ([18, Prop. 7.42]). *Under the same hypotheses as Proposition 3.2.2, the Riemann tensor of M is characterised by:*

1. $R(X, Y)Z \in TB$ *is the lift of* $R^B(X, Y)Z$ *on* B*,*
2. $R(V, X)Y = ((\nabla df)(X, Y)/f)V$*,*
3. $R(X, Y)V = R(V, W)X = 0$*,*
4. $R(X, V)W = f\, \langle V, W\rangle_F\, \nabla_X(\nabla f)$*,*
5. $R(V, W)U = R^F(V, W)U - \langle \nabla f, \nabla f\rangle_B[\langle V, U\rangle_F W - \langle W, U\rangle_F V]$.

3.2.4 Proposition ([18, Prop. 7.43]). *Under the same hypotheses as Proposition 3.2.2, with d the dimension of F, the Ricci curvature tensor of M is characterised by*

1. $\mathrm{Ric}(X, Y) = \mathrm{Ric}^B(X, Y) - (d/f)(\nabla df)(X, Y)$*,*
2. $\mathrm{Ric}(X, V) = 0$*,*
3. $\mathrm{Ric}(V, W) = \mathrm{Ric}^F(V, W) - f^2\langle V, W\rangle_F f^\sharp$*, where*

$$f^\sharp = \frac{\Delta f}{f} + (d-1)\frac{\langle \nabla f, \nabla f\rangle}{f^2},$$

where Δf is the Laplacian on B.

Note that it follows from Corollary 1.2.13 that ∇df is symmetric.

We now apply this to the special case where the warping factor is a space-like real interval, and so the warping function is a strictly positive function of one variable.

3.2.5 Proposition. *Let $M = I \times_f F$, where $I \subset \mathbb{R}$ is an interval. Let $\partial_t \in \Gamma(TM)$ be the lift of the standard tangent vector d/dt on I, and let $V, W \in \Gamma(TF)$. Then the connection of M is characterised by:*

1. $\nabla_{\partial_t}\partial_t = 0$,
2. $\nabla_{\partial_t}V = \nabla_V\partial_t = \frac{f'}{f}V$,
3. $\operatorname{nor}\nabla_V W = -\langle V, W\rangle_F\, ff'\partial_t$,
4. $\tan\nabla_V W \in TF$ *is the lift of* $\nabla^F_V W$ *on* F.

Proof. Note that I is flat and $\partial_t(f\circ\pi_B) = f'\circ\pi_B$. Then all follow from Proposition 3.2.2. □

In a similar vein the curvatures can also be simplified.

3.2.6 Proposition. *Under the same hypotheses as Proposition 3.2.5, the Riemann tensor of M is characterised by:*

1. $R(V,\partial_t)\partial_t = \frac{f''}{f}V$,
2. $R(\partial_t,\partial_t)V = R(V,W)\partial_t = 0$,
3. $R(\partial_t, V)W = -\langle V, W\rangle_F\, ff''\partial_t$,
4. $R(V,W)U = R^F(V,W)U - (f')^2[\langle V,U\rangle_F W - \langle W,U\rangle_F V]$.

Proof. For $\{E_i\}$ a frame of M, with $E_1 = \partial_t$:

$$\begin{aligned}
\nabla f &= \sum_i \epsilon_i\,\langle\nabla f, E_i\rangle E_i\\
&= \langle\nabla f,\partial_t\rangle\partial_t + \sum_{i>1}\epsilon_i\,\langle\nabla f, E_i\rangle E_i\\
&= (\partial_t f)\partial_t + \sum_{i>1}\epsilon_i\,(E_i f)E_i\\
&= f'\partial_t, \quad \text{as } f \text{ is a function of } I \text{ only},
\end{aligned}$$

and

$$\begin{aligned}
\nabla df(\partial_t,\partial_t) &= \partial_t(df(\partial_t)) - df(\nabla_{\partial_t}\partial_t), \quad \text{by Definition 1.2.12}\\
&= \partial_t(\partial_t f)\\
&= \partial_t(f')\\
&= f''.
\end{aligned}$$

The result then follows from Proposition 3.2.3. □

3.2.7 Proposition. *The sectional curvatures of the warped product $M = I\times_f F$ are:*

$$\begin{aligned}
K(\partial_t, V) &= -\frac{f''}{f},\\
K(V,W) &= \frac{1}{f^2}[K^F(V,W) - (f')^2],
\end{aligned}$$

where $V, W \in TF$ are orthonormal in F.

Proof. We have:

$$\begin{aligned}
K(\partial_t, V) &= \frac{g(R(\partial_t, V)\partial_t, V)}{g(\partial_t, \partial_t)g(V, V) - g(\partial_t, V)^2} \\
&= -\frac{f''}{f}, \quad \text{by Proposition 3.2.6,} \\
K(V, W) &= \frac{g(R(V, W)V, W)}{g(V, V)g(W, W) - g(V, W)^2} \\
&= \frac{1}{f^4}[g(R^F(V, W)V, W) - (f')^2 g([\langle V, V\rangle_F W - \langle W, V\rangle_F V], W)], \quad \text{by Prop. 3.2.6} \\
&= \frac{1}{f^2}[K^F(V, W) - (f')^2].
\end{aligned}$$

□

3.2.8 Proposition. *Under the same hypotheses as Proposition 3.2.5, where F has dimension $n-1$, the Ricci curvature tensor of M is characterised by:*

1. $\mathrm{Ric}(\partial_t, \partial_t) = (1-n)\frac{f''}{f}$,
2. $\mathrm{Ric}(\partial_t, V) = 0$,
3. $\mathrm{Ric}(V, W) = \mathrm{Ric}^F(V, W) - f^2\langle V, W\rangle_F\, f^\sharp$, *where*

$$f^\sharp = -\frac{f''}{f} - \frac{(f')^2}{f^2}.$$

Proof. Note that $\nabla f = f'\partial_t$ and $(\nabla df)(\partial_t, \partial_t) = f''$. Let $\{E_i\}$ be a frame of M with $E_1 = \partial_t$. Then:

$$\begin{aligned}
\Delta f &= -\sum_i \epsilon_i\, g(\nabla_{E_i}(\nabla f), E_i) \\
&= -g(\nabla_{\partial_t}(f'\partial_t), \partial_t) - \sum_{i>1} \epsilon_i\, g(\nabla_{E_i}(f'\partial_t), E_i) \\
&= -g((\partial_t f')\partial_t, \partial_t) - g(f'\nabla_{\partial_t}\partial_t, \partial_t) - \sum_{i>1} \epsilon_i\, g(f'\nabla_{E_i}\partial_t, E_i) \\
&= -g(f''\partial_t, \partial_t) - \sum_{i>1} \epsilon_i\, g(f'(\frac{f'}{f}E_i), E_i), \quad \text{by Proposition 3.2.5} \\
&= -f'' - \frac{(f')^2}{f}\sum_{i>1} \epsilon_i\, g(E_i, E_i) \\
&= -f'' + (1-n)\frac{(f')^2}{f}.
\end{aligned}$$

Then the result follows from Proposition 3.2.4. □

In the extra special case where the warped factor is also one dimensional we can say even more about the curvature of the warped product, and in fact find the Gauss curvature of the surface.

3.2.9 Proposition. *Let $M = I \times_f F$, where $I \subset \mathbb{R}$ is an interval and F is a 1-dimensional Riemannian manifold. Let V be the lift of the standard unit vector field on F, and define the unit vector field $U = V/f$. Then the connection of M is characterised by:*

1. $\nabla_{\partial_t}\partial_t = 0$,
2. $\nabla_{\partial_t}U = (\partial_t(f)/f)U = \frac{f'}{f}U$,
3. $\operatorname{nor}\nabla_U U = -\frac{f'}{f}\partial_t$,
4. $\tan\nabla_U U = 0$.

3.2.10 Proposition. *Under the same hypotheses as Proposition 3.2.9, the Riemann tensor of M is characterised by*

1. $R(U,\partial_t)\partial_t = \frac{f''}{f}U$,
2. $R(\partial_t,U)U = -\frac{f''}{f}\partial_t$.

3.2.11 Proposition. *Under the same hypotheses as Proposition 3.2.9, the Gauss curvature of M is*

$$\kappa(t,s) = -\frac{f''(t)}{f(t)},$$

for all $(t,s) \in N$.

Proof. Note that the Gauss curvature is the same as the sectional curvature (Definition 1.1.11) of the 2-plane spanned by ∂_t and U:

$$\begin{aligned}\kappa(t,s) = K(\partial_t,U) &= \frac{\langle R(\partial_t,U)U,\partial_t\rangle}{\langle U,U\rangle} \\ &= \langle -\frac{f''}{f}\partial_t,\partial_t\rangle \\ &= -\frac{f''}{f}.\end{aligned}$$

□

3.3 Closed conformal vector fields on warped products

We now consider only the case of a space-like real interval as the warping factor. This gives rise to a natural closed conformal vector field on the warped product that is tangent to the warping factor. First we prove that such a vector field is in fact closed and conformal. Note the change of notation from the previous section.

3.3.1 Lemma. *Let $N = I \times_f M$ be a warped product of the open interval $I \subset \mathbb{R}$ and a pseudo-Riemannian manifold M by the warping function $f\colon I \to \mathbb{R}^+$ with metric*

$$g(\,,\,) = \langle\,,\,\rangle_{\mathbb{R}} + (f\circ\pi_I)^2\langle\,,\,\rangle_M.$$

Then there exists a closed conformal vector field

$$\sigma = a(f\circ\pi_I)\partial_t,$$

where $\pi_I\colon N \to I$ is projection onto the first factor and $a \in \mathbb{R}$ is constant. This has conformal factor

$$\psi_\sigma = a(f'\circ\pi_I).$$

Proof. Consider in turn the covariant derivatives with respect to $\partial_t \in TI$ and $V \in TM$, and for convenience write $f = f \circ \pi_I$ etc. It follows from Proposition 3.2.5 that:

$$\begin{aligned}\nabla_{\partial_t}\sigma &= \nabla_{\partial_t}(af\partial_t)\\ &= \partial_t(af)\partial_t + af\,\nabla_{\partial_t}\partial_t\\ &= af'\partial_t,\end{aligned}$$

and

$$\begin{aligned}\nabla_V\sigma &= \nabla_V(af\partial_t)\\ &= V(af)\partial_t + af\,\nabla_V\partial_t\\ &= 0 + af(\frac{f'}{f}V)\\ &= af'V.\end{aligned}$$

The result then follows from Proposition 3.1.6. □

Applying Theorem 3.1.12 leads to the following result.

3.3.2 Theorem. *Let σ be the closed conformal vector field on the warped product $N = I \times_f M$ defined as in Lemma 3.3.1. Then σ is (p,q)-harmonic if the warping function satisfies the second order non-linear ordinary differential equation:*

$$(g')^2[p(n-2) + nq + q(n-p)g^2]g + g''(1+g^2)(1+qg^2) = 0, \tag{3.1}$$

where $g = af$.

Proof. For a frame $\{E_i\}$ of N with $E_1 = \partial_t$:

$$\begin{aligned}\nabla\psi_\sigma &= \nabla(af')\\ &= \sum_i \epsilon_i E_i(af')E_i\\ &= \partial_t(af')\partial_t + \sum_{i>1}\epsilon_i E_i(af')E_i\\ &= (af'')\partial_t = g''\partial_t.\end{aligned}$$

Note that:

$$\begin{aligned}2F = \langle\sigma,\sigma\rangle &= \langle g\partial_t, g\partial_t\rangle = g^2,\\ \langle\sigma,\nabla\psi_\sigma\rangle &= \langle g\partial_t, g''\partial_t\rangle = gg''.\end{aligned}$$

Substituting these into Theorem 3.1.12 yields:

$$\begin{aligned}T_p(\sigma) &= -(1+g^2)g''\partial_t + 2p(g')^2g\partial_t,\\ \phi_{p,q}(\sigma)\sigma &= [pn(g')^2 - pq(g')^2g^2 + q(1+g^2)(n(g')^2 + gg'')]g\partial_t.\end{aligned}$$

Comparison of coefficients of ∂_t and rearrangement yields the stated equation. □

The ODE (3.1) is analysed in Appendix A.

3.4 Jacobi elliptic functions

We now consider one particular class of functions that give interesting results in the warped products defined in Section 3.2. These are the Jacobi elliptic functions. More particularly we consider them only for real parameters, and hence when they are real valued. A good introduction is provided in [2]. We use their work as a basis for the introduction below.

3.4.1 Definition ([2, §1.7])**.** Consider first the integral

$$x = \int_0^{\psi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}.$$

Then the *Jacobi elliptic functions* are

$$\begin{aligned} \operatorname{sn}(x,k) &= \sin(\psi), \\ \operatorname{cn}(x,k) &= \cos(\psi), \\ \operatorname{dn}(x,k) &= \sqrt{1 - k^2 \operatorname{sn}(x)}. \end{aligned}$$

Note that the abbreviations $\operatorname{sn}(x,k) = \operatorname{sn}(x)$ etc. are often used. ♢

3.4.2 Proposition. *Let $k = 0, 1$. Then the Jacobi elliptic functions simplify to the trigonometric and hyperbolic functions respectively.*

Proof. Substitute $k = 0, 1$ into Definition 3.4.1 and solve. □

Note that although $k = 1$ gives the hyperbolic functions, they are not the expected ones; for example $\operatorname{sn}(t, 1) = \tanh(t)$.

3.4.3 Theorem. *[2, Theorem 3.4] The Jacobi elliptic functions* $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$ *have periods* $4K, 4K, 2K$ *respectively, where*

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

Proof. Since $\sin$ and $\cos$ have period 2π, and $\sin^2$ and $\cos^2$ have period π:

$$\begin{aligned} x + 2K &= \int_0^{\psi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \\ &= \Big(\int_0^{\psi} + \int_{\psi}^{\psi + \pi} \Big) \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \\ &= \int_0^{\psi + \pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \end{aligned}$$

Noting that $\sin(\psi + \pi) = -\sin \psi$:

$$\operatorname{sn}(x + 2K) = \sin(\psi + \pi) = -\sin \psi = -\operatorname{sn}(x).$$

Similarly

$$\operatorname{cn}(x + 2K) = -\operatorname{cn}(x).$$

However:

$$\operatorname{dn}(x + 2K) = \sqrt{1 - k^2 \operatorname{sn}^2(x + 2K)} = \sqrt{1 - k^2 \operatorname{sn}^2(x)} = \operatorname{dn}(x).$$ □

3.4.4 Proposition. *The Jacobi elliptic functions satisfy the square identities:*

$$\operatorname{sn}^2(x) + \operatorname{cn}^2(x) = 1,$$
$$k^2 \operatorname{sn}^2(x) + \operatorname{dn}^2(x) = 1.$$

3.4.5 Proposition ([2, p. 17])**.** *The derivatives of the Jacobi elliptic functions are:*

$$\frac{d}{dx}\operatorname{sn}(x) = \operatorname{cn}(x)\operatorname{dn}(x),$$
$$\frac{d}{dx}\operatorname{cn}(x) = -\operatorname{sn}(x)\operatorname{dn}(x),$$
$$\frac{d}{dx}\operatorname{dn}(x) = -k^2\operatorname{sn}(x)\operatorname{cn}(x).$$

It is also useful to record the following addition formulæ.
3.4.6 Proposition ([2, p. 32])**.** *The addition formulæ for the Jacobi elliptic functions are:*

$$\operatorname{sn}(x+y) = \frac{\operatorname{sn}(x)\operatorname{cn}(y)\operatorname{dn}(y) + \operatorname{sn}(y)\operatorname{cn}(x)\operatorname{dn}(x)}{\Delta(x,y)},$$
$$\operatorname{cn}(x+y) = \frac{\operatorname{cn}(x)\operatorname{cn}(y) - \operatorname{sn}(x)\operatorname{sn}(y)\operatorname{dn}(x)\operatorname{dn}(y)}{\Delta(x,y)},$$
$$\operatorname{dn}(x+y) = \frac{\operatorname{dn}(x)\operatorname{dn}(y) - k^2\operatorname{sn}(x)\operatorname{sn}(y)\operatorname{cn}(x)\operatorname{cn}(y)}{\Delta(x,y)},$$

where $\Delta(x,y) = 1 - k^2\operatorname{sn}^2(x)\operatorname{sn}^2(y)$.

The following integrals of the Jacobi elliptic functions will be useful.
3.4.7 Theorem ([10, §5, p. 46])**.** *The Jacobi elliptic functions have the following integrals:*

$$\int \operatorname{sn}(x)dx = \frac{1}{k}\ln\frac{1 + k\operatorname{sn}^2(x/2)}{1 - k\operatorname{sn}^2(x/2)},$$
$$\int \operatorname{cn}(x)dx = \frac{\operatorname{sn}(x)\arccos(\operatorname{dn}(x))}{\sqrt{1-\operatorname{dn}^2(x)}},$$
$$\int \operatorname{cn}^3(x)dx = \frac{1}{2k^2}\operatorname{sn}(x)\Big[\operatorname{dn}(x) + \frac{(2k^2-1)\arccos(\operatorname{dn}(x))}{\sqrt{1-\operatorname{dn}^2(x)}}\Big].$$

Proof. Dixon, in [10], has a constructive method for integrating sn using the double angle formula (Proposition 3.4.6). First make the substitution $x = 2u$:

$$\int \operatorname{sn}(x)dx = 2\int \operatorname{sn}(2u)du,$$

then apply the double angle formula:

$$\int \operatorname{sn}(x)dx = \int \frac{4\operatorname{sn}(u)\operatorname{cn}(u)\operatorname{dn}(u)}{1 - k^2\operatorname{sn}^4(u)}du.$$

Next use the substitution $z = \mathrm{sn}^2(u)$:

$$\begin{aligned}\int \mathrm{sn}(x)dx &= 2\int \frac{dz}{1-k^2z^2}\\ &= \frac{1}{k}\ln\frac{1+kz}{1-kz}\\ &= \frac{1}{k}\ln\frac{1+k\,\mathrm{sn}^2(x/2)}{1-k\,\mathrm{sn}^2(x/2)}.\end{aligned}$$

The remainder may be verified by differentiation of the right hand side. □

3.5 Examples

In this section we make use of the results of Section 3.3 to provide our first examples of harmonic closed conformal vector fields. We first consider space forms as warped products, using trigonometric and hyperbolic warping functions. We then modify this construction using Jacobi elliptic warping functions. This ultimately leads us to an interesting example (Section 3.6): a smooth vector field on the 2-sphere which is harmonic with respect to a rotationally symmetric *singular* Riemannian metric of finite total curvature.

3.5.1 Theorem. *Let N be the warped product*

$$N = I \times_f M,$$

where $f = b\sin(t/b)$, $b \in \mathbb{R}^+$, $t \in I = (0, b\pi)$, and M is a pseudo-Riemannian manifold. Then $\sigma = a(f \circ \pi_I)\partial_t$ is (p,q)-harmonic if and only if:

$$\begin{aligned}n &> 2,\\ p &= n+1,\\ q &= 2-n,\\ ab &= \pm\sqrt{\frac{-1}{q}}.\end{aligned}$$

Proof. First note that

$$f = b\sin(t/b), \quad f' = \cos(t/b), \quad f'' = \frac{-\sin(t/b)}{b},$$

and substitute these into (3.1), multiplying through by b^2 and substituting $c = ab$ to give:

$$\begin{aligned}&c^2\cos^2(t/b)[p(n-2)+nq+q(n-p)c^2\sin^2(t/b)]c\sin(t/b)\\ &-c\sin(t/b)(1+c^2\sin^2(t/b))(1+qc^2\sin^2(t/b)) = 0.\end{aligned}$$

Noting $c^2 \neq 0$ and $\sin \neq 0$ on I this may be expressed as the following polynomial in $\cos^2(t/b)$:

$$\begin{aligned}&c^2\cos^2(t/b))[p(n-2)+nq+q(n-p)c^2(1-\cos^2(t/b))]\\ &-(1+c^2(1-\cos^2(t/b)))(1+qc^2(1-\cos^2(t/b))) = 0.\end{aligned}$$

Inspection of the coefficients yields:

$$\begin{aligned}
\cos^4: &\quad 0 = -c^4 q(n-p) - qc^4 \\
\cos^2: &\quad 0 = c^2(p(n-2)+nq) + c^4 q(n-p) + (1+c^2)qc^2 + (1+qc^2)c^2 \\
\cos^0: &\quad 0 = (1+c^2)(1+qc^2).
\end{aligned}$$

The leading coefficient gives one of three conditions:

$$c^4 = 0, \quad q = 0, \quad p = n+1,$$

the first of which can only happen if σ is trivial. If $q = 0$ then it follows from the constant term that $c^2 = -1$. We deduce that $p = n+1$. From the constant term we see $c^2 = -1/q$. Substituting these into the linear term yields:

$$\begin{aligned}
0 &= -(p(n-2)+nq) + (n-p) - (q-1) + q(0) \\
&= -(n+1)(n-2) - nq + -1 - q + 1 \\
&= -(n+1)(n-2) - q(n+1),
\end{aligned}$$

which gives $q = 2-n$. □

Note that Theorem 3.5.1 yields a preferred scale for the harmonic vector field, and the metric parameters are uniquely determined. Furthermore the metric parameter $q < 0$ which means that even if M is Riemannian the generalised Cheeger-Gromoll metric $h_{p,q}$ is necessarily semi-Riemannian. If the warped factor is the round unit $(n-1)$-sphere then the warped product is the doubly punctured round n-sphere (of radius b) and the result coincides with that of [3]. It is interesting to observe that when generalised to warped products the dimensional restriction $n > 2$ still arises.

Applying a phase shift to Theorem 3.5.1 yields the following.

3.5.2 Corollary. *Let N be the warped product*

$$N = I \times_f M,$$

where $f = b\cos(t/b)$, $b \in \mathbb{R}^+$, $t \in I = (-b\pi/2, b\pi/2)$, and M is a pseudo-Riemannian manifold. Then $\sigma = a(f \circ \pi_I)\partial_t$ is (p,q)-harmonic if and only if:

$$\begin{aligned}
n &> 2, \\
p &= n+1, \\
q &= 2-n, \\
ab &= \pm\sqrt{\frac{-1}{q}}.
\end{aligned}$$

We now consider hyperbolic warping functions.

3.5.3 Theorem. *Let N be the warped product*

$$N = \mathbb{R}^+ \times_f M,$$

where $f = b\sinh(t/b)$, $b \in \mathbb{R}^+$, $t \in \mathbb{R}^+$, and M is a pseudo-Riemannian manifold. Then $\sigma = a(f \circ \pi_I)\partial_t$ is (p,q)-harmonic if and only if:

$$n > 2, \quad p = \frac{1}{2-n}, \quad q = 0, \quad ab = \pm 1,$$

or:

$$n \geq 2, \quad p = n + 1, \quad q = \frac{1 + n - n^2}{n}, \quad ab = \pm 1.$$

Proof. The first part is similar to that of Theorem 3.5.1. First note that :

$$f = b \sinh(t/b), \quad f' = \cosh(t/b), \quad f'' = \frac{\sinh(t/b)}{b},$$

then substitute these into (3.1), multiply through by b^2 and substitute $c = ba$ to obtain:

$$\begin{aligned} &c^2 \cosh^2(t/b)[p(n-2) + nq + q(n-p)(cf)^2] c \sinh(t/b) \\ &\quad + c \sinh(t/b)(1 + c^2 \sinh^2(t/b))(1 + qc^2 \sinh^2(t/b)) = 0. \end{aligned}$$

Noting $c^2 \neq 0$ and $\sinh \neq 0$ on I yields the following polynomial in $\cosh^2(t/b)$:

$$\begin{aligned} &c^2 \cosh^2(t/b)[p(n-2) + nq + q(n-p)c^2(\cosh^2(t/b) - 1)] \\ &\quad + (1 + c^2(\cosh^2(t/b) - 1))(1 + qc^2(\cosh^2(t/b) - 1)) = 0. \end{aligned}$$

Inspection of the coefficients yields:

$$\begin{aligned} \cosh^4 &: \quad 0 = c^4 q(n-p) + qc^4, \\ \cosh^2 &: \quad 0 = c^2(p(n-2) + nq) - q(n-p)c^4 + qc^2(1 - c^2) + c^2(1 - qc^2), \\ \cosh^0 &: \quad 0 = (1 - c^2)(1 - qc^2). \end{aligned}$$

The leading coefficient leaves three possibilities:

$$c^4 = 0, \quad q = 0, \quad p = n + 1,$$

the first of which can only happen if σ is trivial.

If $q = 0$ then if follows from the constant term that $c^2 = 1$. Then the linear term yields:

$$0 = p(n-2) + 1,$$

which gives the first result.

Suppose $p = n + 1$. Then the constant term yields two possibilities:

$$c^2 = 1, \quad qc^2 = 1.$$

If $qc^2 = 1$ then the linear term yields:

$$\begin{aligned} 0 &= (p(n-2) + nq) - (n-p) + (q-1) \\ &= (n+1)(n-2) + (n+1)q. \end{aligned}$$

This gives $q = 2 - n$ hence the contradiciton $c^2 = 1/(2-n)$. If $c^2 = 1$ then the linear term yields:

$$\begin{aligned} 0 &= ((n+1)(n-2) + nq) - q(n - n - 1) + (1 - q) \\ &= (n+1)(n-2) + qn + 1, \end{aligned}$$

and this gives the final result. □

In contrast to Theorem 3.5.1, Theorem 3.5.3 asserts the existence of a harmonic vector fields for dimension three and higher which has two pairs of metric parameters. Furthermore there is a unique harmonic vector field in dimension two which is metrically unique with parameters $(3, -1/2)$. This generalises the situation for conformal gradient fields on the real hyperbolic space H^n [5], which may be obtained by setting $M = H^{n-1}$. For the metric parameter $q = 0$ the generalised Cheeger-Gromoll metric $h_{p,q}$ is a warped version of the Sasaki metric.

3.5.4 Theorem. *Let N be the warped product*

$$N = \mathbb{R} \times_f M,$$

where $f = b\cosh(t/b)$, $b \in \mathbb{R}^+$, $t \in \mathbb{R}$, and M is a pseudo-Riemannian manifold. Then $\sigma = a(f \circ \pi_I)\partial_t$ is (p,q)-harmonic if and only if:

$$\begin{aligned} n &> 2, \\ p &= n+1, \\ q &= 2-n, \\ ab &= \pm\frac{1}{\sqrt{n-2}}. \end{aligned}$$

Proof. Using the same method of proof as Theorem 3.5.3 yields the following polynomial in $\sinh^2(t/b)$:

$$\begin{aligned} &c^2\sinh^2(t/b))[p(n-2) + nq + q(n-p)c^2(1+\sinh^2(t/b))] \\ &\quad + (1 + c^2(1+\sinh^2(t/b)))(1 + qc^2(1+\sinh^2(t/b))) = 0, \end{aligned}$$

where $c = ab$. Inspection of the coefficients yields:

$$\begin{aligned} \sinh^4:&\quad 0 = c^4 q(n-p) + qc^4, \\ \sinh^2:&\quad 0 = c^2[p(n-2) + nq + qc^2(n-p)] + qc^2(1+c^2) + (1+qc^2), \\ \sinh^0:&\quad 0 = (1+c^2)(1+qc^2). \end{aligned}$$

These are the same equations as those obtained for the warping function $t \mapsto b\sin(t/b)$ in Theorem 3.5.1. □

We now turn our attention to the Jacobi elliptic functions.

3.5.5 Theorem. *Let N be the warped product*

$$N = I \times_f M,$$

where $f(t) = b\,\mathrm{sn}(t/b)$, $b \in \mathbb{R}^+$, $t \in I = (0, 2bK)$, and M is a pseudo-Riemannian manifold. Then $\sigma = a(f \circ \pi_I)\partial_t$. Then σ is (p,q)-harmonic if and only if $n > 2$ and:

$$k = 0, \quad p = n+1, \quad q = 2-n, \quad ab = \frac{1}{\sqrt{n-2}},$$

or

$$k = 1, \quad p = n+2, \quad q = 2-n, \quad ab = \frac{1}{\sqrt{n-2}}.$$

Proof. By Proposition 3.4.5:

$$f = b\,\mathrm{sn}(t/b), \quad f' = \mathrm{cn}(t/b)\,\mathrm{dn}(t/b), \quad f'' = \frac{-1}{b}\,\mathrm{sn}(t/b)(\mathrm{dn}^2(t/b) + k^2\,\mathrm{cn}^2(t/b)).$$

Substituting these into (3.1) yields:

$$\begin{aligned} 0 &= c^2\,\mathrm{cn}^2(t/b)\,\mathrm{dn}^2(t/b)[p(n-2) + nq + q(n-p)c^2\,\mathrm{sn}^2(t/b)]c\,\mathrm{sn}(t/b) \\ &\quad - c\,\mathrm{sn}(t/b)(\mathrm{dn}^2(t/b) + k^2\,\mathrm{cn}^2(t/b))(1 + c^2\,\mathrm{sn}^2(t/b))(1 + qc^2\,\mathrm{sn}^2(t/b)). \end{aligned}$$

Use of the identities $\mathrm{cn}^2 + \mathrm{sn}^2 = 1$ and $k^2\,\mathrm{sn}^2 + \mathrm{dn}^2 = 1$, and noting that $c^2 \neq 0$ and $\mathrm{sn} \neq 0$ on I yields:

$$\begin{aligned} 0 &= c^2\,\mathrm{cn}^2(t/b)(1 - k^2 + k^2 cn^2(t/b))[p(n-2) + nq + q(n-p)c^2(1 - cn^2(t/b))] \\ &\quad - ((1 - k^2 + k^2 cn^2(t/b)) + k^2\,\mathrm{cn}^2(t/b))(1 + c^2(1 - cn^2(t/b)))(1 + qc^2(1 - 1 - cn^2(t/b))) \\ &= c^2\,\mathrm{cn}^2(t/b)(1 - k^2 + k^2 cn^2(t/b))[p(n-2) + nq + q(n-p)c^2 - q(n-p)c^2 cn^2(t/b))] \\ &\quad - ((1 - k^2 + 2k^2 cn^2(t/b))(1 + c^2 - c^2 cn^2(t/b)))(1 + qc^2 - qc^2 cn^2(t/b))), \end{aligned}$$

which is a cubic polynomial in $\mathrm{cn}^2(t/b)$. Inspection of the coefficients yields the following four equations:

$$\begin{aligned} \mathrm{cn}^6: &\quad 0 = -c^4 k^2 q(n-p) - 2k^2 qc^4, \\ \mathrm{cn}^4: &\quad 0 = -c^4(1-k^2)q(n-p) + c^2k^2(p(n-2) + nq + q(n-p)c^2) \\ &\qquad - (1-k^2)qc^4 + 2k^2c^2(1+qc^2) + 2k^2qc^2(1+c^2), \\ \mathrm{cn}^2: &\quad 0 = c^2(1-k^2)(p(n-2) + nq + q(n-p)c^2) \\ &\qquad + (1-k^2)c^2(1+qc^2) + (1-k^2)qc^2(1+c^2) - 2k^2(1+c^2)(1+qc^2), \\ \mathrm{cn}^0: &\quad 0 = (1-k^2)(1+c^2)(1+qc^2). \end{aligned}$$

The leading coefficient gives the following four possibilities:

$$c = 0, \quad k = 0, \quad q = 0, \quad p = n + 2,$$

the first of which can only happen if σ is trivial. If $k = 0$ then the remaining equations reduce to those in the proof of Theorem 3.5.1.

If $q = 0$ then the remaining equations become:

$$\begin{aligned} \mathrm{cn}^4: &\quad 0 = c^2k^2p(n-2) + 2k^2c^2, \\ \mathrm{cn}^2: &\quad 0 = c^2(1-k^2)p(n-2) - (1-k^2)c^2 - 2k^2(1+c^2), \\ \mathrm{cn}^0: &\quad 0 = (1-k^2)(1+c^2). \end{aligned}$$

The constant term gives $k = 1$, and the second equation leads to the contradiction $c^2 = -1$.

Finally if $p = n + 2$ then the remaining equations become:

$$\begin{aligned} \mathrm{cn}^4: &\quad 0 = 2c^4(1-k^2)q + c^2k^2((n+2)(n-2) + nq - 2qc^2) \\ &\qquad - (1-k^2)qc^4 + 2k^2c^2(1+qc^2) + 2k^2qc^2(1+c^2), \\ \mathrm{cn}^2: &\quad 0 = c^2(1-k^2)((n+2)(n-2) + nq - 2qc^2) \\ &\qquad + (1-k^2)c^2(1+qc^2) + (1-k^2)qc^2(1+c^2) - 2k^2(1+c^2)(1+qc^2), \\ \mathrm{cn}^0: &\quad 0 = (1-k^2)(1+c^2)(1+qc^2). \end{aligned}$$

The constant term implies either of two possibilities:

$$k = 1, \quad qc^2 = -1.$$

If $k = 1$ the remaining equations become:

$$\begin{aligned} \mathrm{cn}^4: &\quad 0 = ((n+2)(n-2) + nq - 2qc^2) + 2(1+qc^2) + 2q(1+c^2), \\ \mathrm{cn}^2: &\quad 0 = 2(1+c^2)(1+qc^2). \end{aligned}$$

The cn^2 coefficient gives $qc^2 = -1$ and then:

$$\begin{aligned} 0 &= ((n+2)(n-2) + nq + 2) + 2(q-1) \\ &= (n+2)(n-2+q), \end{aligned}$$

which gives the second result.

If $p = n+2$ and $qc^2 = -1$ then the remaining equations reduce to:

$$\begin{aligned} \mathrm{cn}^4: &\quad 0 = 1 - k^2((n+2)(n-2) + nq + 2 + 2(q-1) + 1), \\ \mathrm{cn}^2: &\quad 0 = (1-k^2)((n+2)(n-2) + nq + 2 + (q-1)). \end{aligned}$$

The cn^2 coefficient gives two possible conditions:

$$k = 1, \quad q = \frac{3-n^2}{n+1}.$$

If $k = 1$ we recover the previous case. However if $q = (3-n^2)/(n+1)$ then:

$$\begin{aligned} k^2 &= \frac{1}{n^2 - 3 + (n+2)q} \\ &= \frac{1}{3-n^2}, \end{aligned}$$

which is a contradiction. □

If $k = 0$ then the warping function in Theorem 3.5.5 is that of Theorem 3.5.1 and the two results agree. However if $k = 1$ then $f(t) = b\tanh(t/b)$. In both cases the metric parameters are uniquely determined and $q < 0$, and the dimensional restriction $n > 2$ applies.

3.5.6 Theorem. *Let N be the warped product*

$$N = I \times_f M$$

where $f(t) = b\,\mathrm{cn}(t/b)$, $b \in \mathbb{R}^+$, $t \in I = (-bK, bK)$, and M is a pseudo-Riemannian manifold. Let $\sigma = a(f \circ \pi_I)\partial_t$. Then σ is (p,q)-harmonic if and only if:

$$n > 2, \quad k = 0, \quad p = n+1, \quad q = 2-n, \quad ab = \frac{1}{\sqrt{n-2}},$$

or:

$$n \geq 2, \quad k^2 = \frac{n+1}{(n+2)(n-1)}, \quad p = n+2, \quad q = \frac{3-n^2}{n+1}, \quad ab = \sqrt{\frac{n+1}{n^2-3}}.$$

Proof. By Proposition 3.4.5:

$$f = b\,\mathrm{cn}(t/b), \quad f' = -\,\mathrm{sn}(t/b)\,\mathrm{dn}(t/b), \quad f'' = \frac{1}{b}\,\mathrm{cn}(t/b)(k^2\,\mathrm{sn}^2(t/b) - \mathrm{dn}^2(t/b)).$$

Substituting these into (3.1) yields:

$$\begin{aligned} 0 = {} & c^2\,\mathrm{sn}^2(t/b)\,\mathrm{dn}^2(t/b)[p(n-2) + nq + q(n-p)c^2\,\mathrm{cn}^2(t/b)]c\,\mathrm{cn}(t/b) + \\ & + (c\,\mathrm{cn}(t/b)(k^2\,\mathrm{sn}^2(t/b) - \mathrm{dn}^2(t/b)))(1 + c^2\,\mathrm{cn}^2(t/b))(1 + qc^2\,\mathrm{cn}^2(t/b)). \end{aligned}$$

Using the identities $\mathrm{cn}^2 + \mathrm{sn}^2 = 1$ and $k^2\,\mathrm{sn}^2 + \mathrm{dn}^2 = 1$, and noting $c^2 \neq 0$ and $\mathrm{cn} \neq 0$ on I yields:

$$\begin{aligned} 0 &= c^2(\mathrm{sn}^2(t/b) - k^2\,\mathrm{sn}^4(t/b))[p(n-2) + nq + q(n-p)c^2(1 - \mathrm{sn}^2(t/b))] + \\ &\quad + (2k^2\,\mathrm{sn}^2(t/b) - 1)(1 + c^2(1 - \mathrm{sn}^2(t/b)))(1 + qc(1 - \mathrm{sn}^2(t/b))) \\ &= c^2(\mathrm{sn}^2(t/b) - k^2\,\mathrm{sn}^4(t/b))[p(n-2) + nq + q(n-p)c^2 - q(n-p)c^2\,\mathrm{sn}^2(t/b)] + \\ &\quad + (2k^2\,\mathrm{sn}^2(t/b) - 1)(1 + c^2 - c^2\,\mathrm{sn}^2(t/b))(1 + qc^2 - qc^2\,\mathrm{sn}^2(t/b)), \end{aligned}$$

which is a cubic polynomial in $\mathrm{sn}^2(t/b)$. Inspection of the coefficients yields the following four equations, for $c \neq 0$:

$$\begin{aligned} \mathrm{sn}^6 : \quad & 0 = k^2(q(n-p) + 2q), \\ \mathrm{sn}^4 : \quad & 0 = k^2[p(n-2) + nq + qc^2(n-p)] + c^2q(n-p) + qc^2 + 2k^2[(1 + qc^2) + q(1 + c^2)], \\ \mathrm{sn}^2 : \quad & 0 = c^2[p(n-2) + nq + qc^2(n-p)] + c^2(1 + qc^2) + qc^2(1 + c^2) + 2k^2(1 + c^2)(1 + qc^2), \\ \mathrm{sn}^0 : \quad & 0 = (1 + c^2)(1 + qc^2). \end{aligned}$$

The leading coefficient gives three possibilities:

$$k = 0, \quad q = 0, \quad p = n + 2,$$

If $k = 0$ then the remaining equations reduce to those in the proof of Theorem 3.5.1.

If $q = 0$ then the remaining equations become:

$$\begin{aligned} \mathrm{sn}^4 : \quad & 0 = k^2[p(n-2) + 2], \\ \mathrm{sn}^2 : \quad & 0 = c^2[p(n-2) + nq + qc^2(n-p)] + c^2 + 2k^2(1 + c^2), \\ \mathrm{sn}^0 : \quad & 0 = 1 + c^2, \end{aligned}$$

the constant term provides a contradiction.

Finally if $p = n + 2$ then the remaining equations become:

$$\begin{aligned} \mathrm{sn}^4 : \quad & 0 = k^2[p(n-2) + nq + qc^2(n-p)] + c^2q(n-p) + qc^2 + 2k^2[(1 + qc^2) + q(1 + c^2)], \\ \mathrm{sn}^2 : \quad & 0 = c^2[p(n-2) + nq + qc^2(n-p)] + c^2(1 + qc^2) + qc^2(1 + c^2) + 2k^2(1 + c^2)(1 + qc^2), \\ \mathrm{sn}^0 : \quad & 0 = (1 + c^2)(1 + qc^2). \end{aligned}$$

The constant term implies $qc^2 = -1$, whereupon the remaining equations become:

$$\begin{aligned} \mathrm{sn}^4 : \quad & 0 = k^2[(n+2)(n-2+q)] + 1, \\ \mathrm{sn}^2 : \quad & 0 = (n+2)(n-2) + (n+1)q + 1. \end{aligned}$$

The sn^2 coefficient gives $q = (3 - n^2)/(n + 1)$, and the remaining equation is then:

$$\begin{aligned} 0 &= k^2\Big[\frac{n+2}{n+1}((n+1)(n-2) + 3 - n^2)\Big] + 1 \\ &= k^2\Big[\frac{n+2}{n+1}(1-n)\Big] + 1. \end{aligned}$$

Rearranging for k^2 gives the result. □

If $k = 0$ then the warping function in Theorem 3.5.6 is that of Corollary 3.5.2 and the two results agree. However if $k = (n + 1)/(n + 2)(n - 1)$ then $f(t)$ is a genuine Jacobi elliptic function, and in this case there is no restriction on dimension. In both cases the metric parameters are uniquely determined and $q < 0$.

3.5.7 Theorem. *Let N be the warped product*

$$N = \mathbb{R} \times_f M$$

where $f(t) = b\,\mathrm{dn}(t/b)$, $b \in \mathbb{R}^+$, $t \in \mathbb{R}$, and M is a pseudo-Riemannian manifold. Let $\sigma = a(f \circ \pi_I)\partial_t$. Then σ is (p, q)-harmonic if and only if:

$$k^2 = \frac{(n+2)(n-1)}{n+1}, \quad p = n+2, \quad q = \frac{3-n^2}{n+1}, \quad ab = \sqrt{\frac{n+1}{n^2-3}}.$$

Proof. By Proposition 3.4.5:

$$f = b\,\mathrm{dn}(t/b), \quad f' = -k^2\,\mathrm{sn}(t/b)\,\mathrm{cn}(t/b), \quad f'' = \frac{k^2}{b}\,\mathrm{dn}(t/b)(\mathrm{sn}^2(t/b) - \mathrm{cn}^2(t/b)).$$

Substituting these into (3.1) yields:

$$\begin{aligned} &c^2k^4\,\mathrm{sn}^2(t/b)\,\mathrm{cn}^2(t/b)[p(n-2) + nq + q(n-p)c^2\,\mathrm{dn}^2(t/b)]c\,\mathrm{dn}(t/b) \\ &\quad + k^2c\,\mathrm{dn}(t/b)(\mathrm{sn}^2(t/b) - \mathrm{cn}^2(t/b)))(1 + c^2\,\mathrm{dn}^2(t/b))(1 + qc\,\mathrm{dn}^2(t/b)) = 0. \end{aligned}$$

Using the identities $\mathrm{cn}^2 + \mathrm{sn}^2 = 1$, $k^2\,\mathrm{sn}^2 + \mathrm{dn}^2 = 1$, and noting $c^2 \neq 0$ and $\mathrm{dn} \neq 0$ on $\mathbb{R}$ yields:

$$\begin{aligned} 0 &= c^2k^4(\mathrm{sn}^2(t/b) - \mathrm{sn}^4(t/b))[p(n-2) + nq + q(n-p)c^2(1 - k^2\,\mathrm{sn}^2(t/b))] \\ &\quad + k^2(2\,\mathrm{sn}^2(t/b) - 1)(1 + c^2(1 - k^2\,\mathrm{sn}^2(t/b)))(1 + qc^2(1 - k^2\,\mathrm{sn}^2(t/b))), \\ &= c^2k^4(\mathrm{sn}^2(t/b) - \mathrm{sn}^4(t/b))[p(n-2) + nq + q(n-p)c^2 - k^2q(n-p)c^2\,\mathrm{sn}^2(t/b)] \\ &\quad + k^2(2\,\mathrm{sn}^2(t/b) - 1)(1 + c^2 - k^2c^2\,\mathrm{sn}^2(t/b))(1 + qc^2 - k^2qc^2\,\mathrm{sn}^2(t/b)), \end{aligned}$$

which is a cubic polynomial in $\mathrm{sn}^2(t/b)$. Note that if $k = 0$ then by definition $dn^2 \equiv 1$ and σ is the parallel vector field $c\partial_t$ on the unwarped product. Inspection of coefficients yields the following four equations, for $c \neq 0$ and $k \neq 0$:

$$\begin{aligned} \mathrm{sn}^6 &: \quad 0 = q(n-p+2), \\ \mathrm{sn}^4 &: \quad 0 = [p(n-2) + nq + c^2q(n-p)] + c^2k^2q(n-p) + 2[(1+qc^2) + q(1+c^2)] + qc^2k^2, \\ \mathrm{sn}^2 &: \quad 0 = c^2k^2[p(n-2) + nq + c^2q(n-p)] + 2(1+c^2)(1+qc^2) \\ &\qquad\quad + k^2([c^2(1+qc^2) + qc^2(1+c^2)] + 2(1+c^2)(1+qc^2), \\ \mathrm{sn}^0 &: \quad 0 = (1+c^2)(1+qc^2). \end{aligned}$$

The leading coefficient gives two possibilities:

$$q = 0, \quad p = n + 2,$$

If $q = 0$ then the remaining equations become:

$$\begin{aligned}
\mathrm{sn}^4 &: \quad 0 = p(n-2) - 2,\\
\mathrm{sn}^2 &: \quad 0 = c^2 p(n-2) + 2k^2(1+c^2) + k^2c^2 + 2(1+c^2),\\
\mathrm{sn}^0 &: \quad 0 = 1 + c^2,
\end{aligned}$$

and the constant term provides a contradiction.

If $p = n + 2$ then the remaining equations become:

$$\begin{aligned}
\mathrm{sn}^4 &: \quad 0 = [(n+2)(n-2) + nq - 2c^2q] - 2c^2k^2q + 2[(1+qc^2) + q(1+a^2)] + qc^2k^2,\\
\mathrm{sn}^2 &: \quad 0 = c^2k^2[(n+2)(n-2) + nq - 2c^2q] + 2(1+c^2)(1+qc^2)\\
&\qquad + k^2([c^2(1+qc^2) + qc^2(1+c^2)] + 2(1+c^2)(1+qc^2),\\
\mathrm{sn}^0 &: \quad 0 = (1+c^2)(1+qc^2).
\end{aligned}$$

The constant term implies $qc^2 = -1$, in which case the remaining equations are:

$$\begin{aligned}
\mathrm{sn}^4 &: \quad 0 = (n+2)(n-2+q) + k^2,\\
\mathrm{sn}^2 &: \quad 0 = (n+2)(n-2) + (n+1)q + 1.
\end{aligned}$$

The sn^2 coefficient then gives $q = (3-n^2)/(n+1)$, and the remaining equation is:

$$\begin{aligned}
0 &= \frac{n+2}{n+1}((n-2)(n+1) + 3 - n^2) + k^2\\
&= \frac{n+2}{n+1}(1-n) + k^2.
\end{aligned}$$

Rearranging for k^2 gives the result. □

3.5.8 Theorem. *Let N be the warped product*

$$N = \mathbb{R}^+ \times_f M$$

where $f(t) = bt$, $b \in \mathbb{R}^+$, $t \in \mathbb{R}^+$, and M is a pseudo-Riemannian manifold. Let $\sigma = a(f \circ \pi_I)\partial_t$. Then σ is (p,q)-harmonic if and only if:

$$p = 0, \quad q = 0,$$

or:

$$n = 2, \quad q = 0,$$

or:

$$p = n, \quad q = 2 - n.$$

Proof. First note:

$$f = bt, \quad f' = b, \quad f'' = 0.$$

Substituting these into (3.1) yields:

$$c^3t[p(n-2)+nq+q(n-p)c^2t^2]=0,$$

which is a polynomial in t. Noting that $c \neq 0$ and inspecting coefficients yields:

$$\begin{aligned} t^3: &\quad 0 = p(n-2)+nq, \\ t: &\quad 0 = q(n-p). \end{aligned}$$

The t coefficient gives two possibilities:

$$q=0, \quad p=n.$$

If $q=0$ then the t^3 coefficient yields two possibilities:

$$p=0, \quad n=2,$$

which are the first two results.

If $p=n$ then the t^3 coefficient yields $q=2-n$, the final result. □

Note that the vector fields in Theorem 3.5.8 do not have a preferred scale. If we exclude the $(0,0)$-harmonic case then for $n>2$ they are metrically unique. If $n=2$ then there is a continuum of metric parameters, the first such example for harmonic vector fields of non-constant pseudo-Riemannian length. In the special case when $M=S^{n-1}$ we obtain the harmonic conformal gradient fields on $\mathbb{R}^n$, thus completing the classification on Riemannian space forms, the non-flat cases of which were given in [5].

3.6 The 2-sphere

In this section we consider the warped product of an interval with a circle, with warping function the second Jacobi elliptic function cn. This warped product can be embedded into a topological 2-sphere but the metric does not extend smoothly. The harmonic vector field on the warped product given by Theorem 3.5.6 may be realised as a smooth vector field on the 2-sphere but the failure of the metric to extend means that this realisation cannot be regarded as a harmonic vector field.

3.6.1 Theorem. *Let N be the warped product*

$$N = I \times_f S^1$$

where $f(t)=b\,\mathrm{cn}(t/b)$, $b\in\mathbb{R}^+$, $t\in I=(-bK,bK)$. Then:

$$\int_N \kappa dA = 4\pi\sqrt{1-k^2}.$$

Proof. Proposition 3.2.11 states the Gauss curvature of a two-dimensional warped product at a point (t,s) is:

$$\kappa(t,s) = -\frac{f''(t)}{f(t)}.$$

By Proposition 3.4.5:

$$f = b\,\mathrm{cn}(t/b), \quad f' = -\,\mathrm{sn}(t/b)\,\mathrm{dn}(t/b), \quad f'' = \frac{1}{b}\,\mathrm{cn}(t/b)(k^2\,\mathrm{sn}^2(t/b) - \mathrm{dn}^2(t/b)).$$

Therefore:

$$\kappa(t,s) = \frac{\mathrm{cn}(t/b)(\mathrm{dn}^2(t/b) - k^2\,\mathrm{sn}^2(t/b))}{b^2\,\mathrm{cn}(t/b)}.$$

The surface area element is:

$$\begin{aligned} dA &= |g|dt \wedge ds \\ &= b\,\mathrm{cn}(t/b)dt \wedge ds. \end{aligned}$$

Hence the total curvature integral is:

$$\begin{aligned} \int_N \kappa dA &= \int_N -\frac{f''}{f} b\,\mathrm{cn}(t/b)dt \wedge ds \\ &= \int_{-bK}^{bK} \frac{\mathrm{cn}(t/b)(\mathrm{dn}^2(t/b)) - k^2\,\mathrm{sn}^2(t/b)}{b^2\,\mathrm{cn}(t/b)} b\,\mathrm{cn}(t/b)dt \int_{S^1} ds \\ &= 2\pi \int_{-bK}^{bK} \frac{1}{b}\,\mathrm{cn}(t/b)(\mathrm{dn}^2(t/b)) - k^2\,\mathrm{sn}^2(t/b))dt. \end{aligned}$$

Using $\mathrm{sn}^2 + \mathrm{cn}^2 = 1$ and $\mathrm{dn}^2 + k^2\,\mathrm{sn}^2 = 1$ yields:

$$\int_N \kappa dA = 2\pi \int_{-bK}^{bK} \frac{1}{b}\,\mathrm{cn}(t/b)(1 - 2k^2 + 2k^2\,\mathrm{cn}^2(t/b))dt.$$

Substituting $bu = t$ yields:

$$\int_N \kappa dA = 2\pi \int_{-K}^{K} [(1-2k^2)\,\mathrm{cn}(u) + 2k^2\,\mathrm{cn}^3(u)]du.$$

Using Theorem 3.4.7 yields:

$$\begin{aligned} \frac{1}{2\pi}\int_N \kappa dA &= \Big[(1-2k^2)\Big(\frac{\mathrm{sn}(u)\arccos(\mathrm{dn}(u))}{\sqrt{1-\mathrm{dn}^2(u)}}\Big) \\ &\quad + 2k^2\Big(\frac{1}{2k^2}\,\mathrm{sn}(u)[\mathrm{dn}(u) + \frac{(2k^2-1)\arccos(\mathrm{dn}(u))}{\sqrt{1-\mathrm{dn}^2(u)}}]\Big)\Big]_{-K}^{K} \\ &= \Big[(1-2k^2)\Big(\frac{\mathrm{sn}(u)\arccos(\mathrm{dn}(u))}{\sqrt{1-\mathrm{dn}^2\,ut)}}\Big) + \mathrm{sn}(u)\Big(\mathrm{dn}(u) - \frac{(1-2k^2)\arccos(\mathrm{dn}(u))}{\sqrt{1-\mathrm{dn}^2(u)}}\Big)\Big]_{-K}^{K} \\ &= \Big[(1-2k^2)\Big(\frac{\mathrm{sn}(u)\arccos(\mathrm{dn}(u))}{\sqrt{1-\mathrm{dn}^2(u)}}\Big) + \mathrm{sn}(u)\,\mathrm{dn}(u) + \mathrm{sn}(u)\frac{(2k^2-1)\arccos(\mathrm{dn}(u))}{\sqrt{1-\mathrm{dn}^2(u)}}\Big)\Big]_{-K}^{K} \\ &= [\mathrm{sn}(u)\,\mathrm{dn}(u)]_{-K}^{K}. \end{aligned}$$

Finally use:

$$\mathrm{sn}(K) = -\,\mathrm{sn}(-K) = 1, \quad \mathrm{dn}(K) = \mathrm{dn}(-K) = \sqrt{1-k^2},$$

to give:

$$\int_N \kappa dA = 4\pi\sqrt{1-k^2}.$$

□

3.6.2 Corollary. *For $k \in (0, 1]$ the warped product of Theorem 3.6.1 cannot be isometrically embedded into any Riemannian 2-sphere.*

Proof. By the Gauss-Bonnet theorem the existence of such an isometric embedding necessitates the total Gauss curvature is 4π. □

Since $k = \sqrt{3}/2$ when $n = 2$ in Theorem 3.5.6, it follows from Corollary 3.6.2 that the harmonic vector field constructed on the warped product cannot be regarded as a harmonic vector field on the 2-sphere. This leaves open the question of whether there is in fact any harmonic vector field on the 2-sphere, round or otherwise. No such vector fields are currently known.

Chapter 4

Harmonic Vector Fields on Constant Curvature Spaces

In this chapter we focus on pseudo-Riemannian spaces of constant curvature and generalise results of [5] on harmonic conformal gradient and harmonic Killing fields on the Riemannian space forms. This gives rise to some interesting new examples. For instance, in contrast to the Riemannian 2-sphere the pseudo-Riemannian space form anti-isometric to it (a pseudo-Riemannian sphere of index 2) is shown to have harmonic vector fields. Our main goal is to classify up to congruence the harmonic Killing fields on 2-dimensional spaces of constant curvature.

4.1 Hyperquadrics and space forms

Our definition of space form is:
4.1.1 Definition ([18, Definition 8.22]). A *space form* is a simply connected complete pseudo-Riemannian manifold of constant sectional curvature. ♢

The classification of space forms depends on the following result.
4.1.2 Proposition. *[18, Proposition 8.23] Two space forms are isometric if and only if they have the same dimension, index and sectional curvature.*

The isometry class of the space form of dimension n, index v, and sectional curvature C is denoted by $M(n, v, C)$. In the following result we show how an appropriate warping of a positively curved space form yields a simply connected manifold of arbitrary constant curvature.
4.1.3 Theorem. *Each of the following is an n-dimensional pseudo-Riemannian manifold of index $v < n$ and constant curvature C: If $v < n$ then:*

1. $\mathbb{R}^+ \times_f M(n-1, v, 1)$ *where* $f(t) = t$, $C = 0$,
2. $I \times_f M(n-1, v, 1)$ *where* $f(t) = b\sin(t/b)$ *and* $I = (0, b\pi)$, $C = 1/b^2$,
3. $\mathbb{R}^+ \times_f M(n-1, v, 1)$ *where* $f(t) = b\sinh(t/b)$, $C = -1/b^2$,

Proof. Note that each of these warped products have dimension n and index v. As products of simply connected manifolds they are also simply connected. It remains to show that they have the stated constant sectional curvatures.

If $N = \mathbb{R}^+ \times_t M(n-1, v, 1)$ then:

$$f = t, \quad f' = 1, \quad f'' = 0.$$

Proposition 3.2.7 then yields the sectional curvatures:

$$\begin{aligned} K(\partial_t, V) &= -\frac{f''}{f} = 0, \\ K(V, W) &= \frac{1}{f^2}[K^M(V, W) - (f')^2] = \frac{1}{t}[1 - 1] = 0. \end{aligned}$$

If $N = I \times_f M(n-1, v, 1)$ then:

$$f = b\sin(t/b), \quad f' = \cos(t/b), \quad f'' = -\frac{1}{b}\sin(t/b).$$

Proposition 3.2.7 then yields the following sectional curvatures:

$$\begin{aligned} K(\partial_t, V) &= -\frac{f''}{f} = \frac{1}{b^2}, \\ K(V, W) &= \frac{1}{f^2}[K^M(V, W) - (f')^2] \\ &= \frac{1}{b^2\sin^2(t/b)}[1 - \cos^2(t/b)] = \frac{1}{b^2}. \end{aligned}$$

If $N = \mathbb{R}^+ \times_f M(n-1, v, 1)$ then:

$$f = b\sinh(t/b), \quad f' = \cosh(t/b), \quad f'' = \frac{1}{b}\sinh(t/b).$$

Proposition 3.2.7 then yields the following sectional curvatures:

$$\begin{aligned} K(\partial_t, V) &= -\frac{f''}{f} = -\frac{1}{b^2}, \\ K(V, W) &= \frac{1}{f^2}[K^M(V, W) - (f')^2] \\ &= \frac{1}{b^2\sinh^2(t/b)}[1 - \cosh^2(t/b)] = -\frac{1}{b^2}. \end{aligned}$$ □

For practical purposes, and to explore the geometry of space forms with a view to calculations in later sections, we work with a non-homogeneous model of space forms as hyperquadrics: those hypersurfaces of constant pseudo-Riemannian distance from the origin in pseudo-Euclidean space. Although metrically complete, in some cases these hyperquadrics are only locally isometric to the corresponding space form.

4.1.4 Definition. The *pseudo-Euclidean space* $\mathbb{R}^{n+1}_v$ of index v is the vector space $\mathbb{R}^{n+1}$ equipped with the metric:

$$\langle x, y\rangle = x_1y_1 + \cdots + x_{n+1-v}y_{n+1-v} - \cdots - x_{n+1}y_{n+1}.$$

Define the quadratic form Q by:

$$Q\colon \mathbb{R}^{n+1}_v \to \mathbb{R}; \quad Q(x) = \langle x, x\rangle. \qquad \Diamond$$

The hyperquadrics can be divided into two classes: the pseudo-spheres, of positive sectional curvature and the pseudo-hyperbolic spaces, of negative sectional curvature. The non-flat Riemannian space forms are (connected components of) the index zero case of each of these classes.

4.1.5 Definition. The *pseudo-sphere* $S^n_v(r)$ of dimension n, index v and radius r is the hyperquadric:

$$S^n_v(r) = \{x \in \mathbb{R}^{n+1}_v : Q(x) = r^2\}.$$

The *pseudo-hyperbolic space* $H^n_v(r)$ of dimension n, index v and radius r is the hyperquadric:

$$H^n_v(r) = \{x \in \mathbb{R}^{n+1}_{v+1} : Q(x) = -r^2\}. \qquad \Diamond$$

The relationship between hyperquadrics and space forms is described in the following two propositions.

4.1.6 Proposition ([18, Corollary 8.24])**.** *For $n \geq 2$:*

$$M(n, v, C) = \begin{cases} S^n_v(r), & \text{if } C = 1/r^2 \text{ and } 0 \leq v \leq n-2 \\ \mathbb{R}^n_v, & \text{if } C = 0 \\ H^n_v(r), & \text{if } C = -1/r^2 \text{ and } 2 \leq v \leq n. \end{cases}$$

4.1.7 Remark. Since $S^n_v(r)$ and $H^n_{n-v}(r)$ are homeomorphic to $S^{n-v}\times\mathbb{R}^v$, these hyperquadrics are all simply connected and hence space forms.

4.1.8 Proposition ([18, Corollary 8.26])**.** *For $n \geq 2$:*

- *$M(n, n, C)$ is one component, $S^n_n(r)^0$, of $S^n_n(r)$ for $C = 1/r^2$.*
- *$M(n, 0, C)$ is hyperbolic n-space: one component, $H^n(r)$, of $H^n_0(r)$ for $C = -1/r^2$.*
- *$M(n, n-1, C) = \widetilde{S}^n_{n-1}(r)$, the universal pseudo-Riemannian covering manifold of $S^n_{n-1}(r)$ for $C = 1/r^2$.*
- *$M(n, 1, C) = \widetilde{H}^n_1(r)$, the universal pseudo-Riemannian covering manifold of $H^n_1(r)$ for $C = -1/r^2$.*

The necessity for universal covers in Proposition 4.1.8 is illustrated by $S^2_1(r)$ which is a hyperboloid of one sheet in 3-space and hence topologically a cylinder.

We recall from Chapter 3 harmonic vector fields were constructed on warped products; however many of these manifolds are incomplete. We now show how some of these constructions may be completed.

4.1.9 Theorem. *The warped products given in Theorem 4.1.3 can be isometrically embedded into the space-like region Σ of $\mathbb{R}^n_v$, or suitable hyperquadrics, for $v < n-1$.*

Proof. Consider the space form $M(n-1,v,1)$ as the pseudo-sphere embedded in $\mathbb{R}^n_v$. We first consider $N=\mathbb{R}^+\times_t M(n-1,v,1)$. Define:

$$\varphi\colon N\to\Sigma\setminus\{0\};\quad (t,x)\mapsto tx=y.$$

Then $dy=tdx+dt$ and the line element is $dy^2=dt^2+t^2dx^2$ as expected.

Now consider $N=I\times_f M(n-1,v,1)$ where $f=b\sin(t/b)$ and $I=(0,2b\pi)$. If $p=(b,0)\in\mathbb{R}^n_v$ then define:

$$\varphi\colon N\to S^n_v(b)\setminus\{\pm p\};\quad (t,x)\mapsto(b\cos(t/b),b\sin(t/b)x)=y.$$

Then

$$d(b\cos(t/b))=-\sin(t/b)dt,\quad d(b\sin(t/b)x)=b\sin(t/b)dx+\cos(t/b)xdt$$

and the line element is:

$$dy^2=(Q(x)\cos^2(t/b)+\sin^2(t/b))dt^2+b^2\sin^2(t/b)dx^2=dt^2+b^2\sin^2(t/b)dx^2,$$

as expected.

Now consider $N=\mathbb{R}^+\times_f M(n-1,v,1)$ where $f=b\sinh(t/b)$. If $p=(0,b)\in H^n_v(b)$ then define:

$$\varphi\colon N\to H^n_v(b)\setminus\{p\};\quad (t,x)\mapsto(b\sinh(t/b)x,b\cosh(t/b))=y.$$

Then

$$d(b\cosh(t/b))=\sinh(t/b)dt,\quad d(b\sinh(t/b)x)=b\sinh(t/b)dx+\cosh(t/b)xdt$$

and the line element is:

$$dy^2=(Q(x)\cosh^2(t/b)-\sinh^2(t/b))dt^2+b^2\sinh^2(t/b)dx^2=dt^2+b^2\sinh^2(t/b)dx^2,$$

as expected. □

It follows from Theorem 4.1.9 that if $v<n-1$ the harmonic vector fields σ constructed in Theorems 3.5.1, 3.5.3 and 3.5.8 may be transferred to harmonic vector fields on

$$\Sigma\setminus\{0\},\quad S^n_v(b)\setminus\{\pm(b,0)\},\quad H^n_v(b)\setminus\{(0,b)\},$$

respectively, with $n>2$ for the pseudo-spheres and $n\geq 2$ otherwise. In the first case if $v=0$ then σ extends to a smooth harmonic vector field on $\Sigma=\mathbb{R}^n$ which vanishes at the origin, namely a rotationally symmetric radial field. In the latter two cases σ may then be extended to smooth harmonic vector fields on $S^n_v(b),H^n_v(b)$ which vanish at the puncture points. The restriction on the index is certainly satisfied in the Riemannian case, thus recovering the results from [5] and adding to them the flat case.

4.1.1 Isometries and anti-isometries

In pseudo-Riemannian geometry the concept of an isometry is complemented by that of an anti-isometry.

4.1.10 Definition. A mapping $\varphi\colon (M,g) \to (N,h)$ of pseudo-Riemannian manifolds is an *anti-isometry* if

$$h(d\varphi(X), d\varphi(Y)) = -g(X,Y), \quad \text{for all } X, Y \in \Gamma(TM).$$

Note that for two pseudo-Riemannian n-manifolds to be anti-isometric the sum of their indices must equal n.

There is an anti-isometry that pairs up the pseudo-spheres and pseudo-hyperbolic spaces.
4.1.11 Proposition. *The anti-isometry*

$$\varphi\colon \mathbb{R}^{n+1}_v \to \mathbb{R}^{n+1}_{n+1-v}; \quad \varphi(x_1,\dots,x_{n+1}) = (x_{n+1-v},\dots,x_{n+1},x_1,\dots,x_{n-v}),$$

carries $S^n_v(r)$ anti-isometrically onto $H^n_{n-v}(r)$ and vice-versa.

In pseudo-Riemannian geometry anti-isometric spaces are sometimes considered to be identical; for example:
4.1.12 Proposition. *If φ is an anti-isometry then $\nabla d\varphi = 0$.*

Proof. The proof of Proposition 1.2.15 works just as well for anti-isometries. □

However from the point of view of harmonic vector fields anti-isometries are not so natural.
4.1.13 Definition. Let $\varphi\colon (M,g) \to (N,h)$ be an anti-isometry and let $\sigma \in \Gamma(TM)$. The *push-forward* of σ is the vector field $\varphi.\sigma \in \Gamma(TN)$ defined:

$$(\varphi.\sigma)(y) = d\varphi(\sigma(\varphi^{-1}(y))), \quad \text{for all } y \in N. \qquad \diamondsuit$$

Remark. Push-forward is defined for any diffeomorphism between manifolds.

The Euler-Lagrange equations for harmonic vector fields are *not* invariant under anti-isometry. That is, if σ is a harmonic vector field in (M,g) then its push-forward $\varphi.\sigma$ by an anti-isometry $\varphi\colon (M,g) \to (N,h)$ need not be a harmonic vector field on (N,h). This is because:

$$h(\varphi.\sigma, \varphi.\sigma) \circ \varphi = -g(\sigma,\sigma).$$

Hence the term $1 + 2F$ in the Euler-Lagrange equations does not transform correctly. A concrete example is given in Section 4.3 (see Remark 4.3.11).

We now introduce the pseudo-orthogonal groups (called the semi-orthogonal groups by O'Neill), which are the symmetry groups of most of the space forms.
4.1.14 Definition ([18, p. 234]). The *pseudo-orthoganal group $O_v(n)$* is the subgroup of all matrices $A \in GL(n,\mathbb{R})$ that preserve the scalar product of $\mathbb{R}^n_v$:

$$\langle Ax, Ay\rangle = \langle x, y\rangle, \quad \text{for all } x, y \in \mathbb{R}^n_v.$$

This is a closed subgroup of $GL(n,\mathbb{R})$ and is thus a Lie group. ♢

4.1.15 Proposition ([18, p. 237]). *Elements of the pseudo-orthoganal group $O_v(n)$ can be block decomposed as follows:*

$$A = \begin{pmatrix} A_S & B \\ B^t & A_T \end{pmatrix}$$

where $A_S \in O(n-v)$ and $A_T \in O(v)$ are orthogonal matrices, and B is an arbitrary $(n-v)\times v$ matrix. Thus $O_v(n)$ decomposes into four disjoint sets according to the signs of $\det A_S$ *and* $\det A_T$*:*

$$O_v^{++}(n), \quad O_v^{+-}(n), \quad O_v^{-+}(n), \quad O_v^{--}(n).$$

4.1.16 Proposition ([18, Prop. 9.8, 9.9]). *The isometry groups of the hyperquadrics (or their components) are as follows:*

$$\begin{aligned} I(S_v^n) &= O_v(n+1) \quad \text{if } v<n,\\ I(H_v^n) &= O_{v+1}(n+1) \quad \text{if } v>0,\\ I(H^n) &= O_1^{++}(n+1) \cup O_1^{-+}(n+1),\\ I(S_n^n(r)^0) &= O_n^{++}(n+1) \cup O_n^{+-}(n+1). \end{aligned}$$

4.2 The 2-dimensional case

We consider more carefully the two dimensional hyperquadrics, and alternative projective models for those that are non-compact.

We recall that there are six two dimensional hyperquadrics, namely:

- The Riemannian 2-sphere and its anti-isometric counterpart, which are spheres in $\mathbb{R}_0^3$ and $\mathbb{R}_3^3$ respectively.
- The hyperbolic plane and its anti-isometric counterpart, which are connected components of hyperboloids of two sheets in $\mathbb{R}_1^3$ and $\mathbb{R}_2^3$ respectively.
- The neutral hyperquadrics, S_1^2 and H_1^2, which are hyperboloids of one sheet in $\mathbb{R}_1^3$ and $\mathbb{R}_2^3$ respectively.

Note that the index 0 and 2 hyperquadrics are in fact space forms, whereas the neutral hyperquadrics are not.

We recall the following result of standard vector calculus.

4.2.1 Lemma. *The differential of a map $F\colon \mathbb{R}_v^3 \to \mathbb{R}_u^3$ at a point $x=(x_1,x_2,x_3)\in\mathbb{R}_v^3$ acting on the vector $y=(y_1,y_2,y_3)\in T\mathbb{R}_v^3$ is*

$$\begin{aligned} dF_x :& T_x\mathbb{R}_v^3 \to T_{F(x)}\mathbb{R}_u^3,\\ dF_{(x)}(y) =& \Big(\sum_i \frac{\partial F^1}{\partial x_i} y_i, \sum_i \frac{\partial F^2}{\partial x_i} y_i, \sum_i \frac{\partial F^3}{\partial x_i} y_i\Big). \end{aligned}$$

To examine the behaviour at infinity of vector fields on the non-compact hyperquadrics, it is helpful to project them onto a bounded model space. The Beltrami disc can be used for the hyperbolic plane and its anti-isometric counterpart. For the neutral hyperquadrics we consider an analogous model, the *cylinder model*. This is obtained by projecting points on the hyperboloid along rays through the origin, onto an enclosed cylinder.

Let $C^2(b)$ be the set:

$$C^2(b)\{(x,y,z)\in\mathbb{R}_2^3 : -b<x<b, y^2+z^2=b^2\}$$

Let $H^2_1 \subset \mathbb{R}^3_2$ be parametrised as

$$H^2_1(b) = \{(b\sinh(u), b\cosh(u)\cos(v), b\cosh(u)\sin(v)) : u \in \mathbb{R}, v \in (0, 2\pi)\}$$

The projection is the point on the line through $(x, y, z) \in H^2_1(b)$ and $(0, 0, 0)$ intersecting the cylinder. The line is

$$L = (1 - t)(0, 0, 0) + t(x, y, z) = t(x, y, z)$$

Substitute in the parametrisation of H^2_1 and solve for points on the cylinder. Note that $y^2 + z^2 = b^2 + x^2$, from the definition of the pseudo-hyperbolic space. Therefore $t = b/\sqrt{b^2 + x^2}$. This gives the projection

$$F\colon H^2_1(b) \to C^2; \quad F\colon (x, y, z) \mapsto (\frac{bx}{\sqrt{b^2 + x^2}}, \frac{by}{\sqrt{b^2 + x^2}}, \frac{bz}{\sqrt{b^2 + x^2}}).$$

Note also that the inverse can be calculated, and is

$$F^{-1}\colon C^2 \to H^2_1; \quad F^{-1}\colon (x, y, z) \mapsto (\frac{x}{r}, \frac{y}{r}, \frac{z}{r}),$$

where $r^2 = -x^2 + y^2 + z^2$. Also of note is the differential

$$dF\colon TH^2_1 \to TC; \quad dF_{(x,y,z)}(u, v, w) = b(\frac{u}{(b^2 + x^2)^{3/2}}, \frac{-xyu + v(b^2 + x^2)}{(b^2 + x^2)^{3/2}}, \frac{-xzu - w(b^2 + x^2)}{(b^2 + x^2)^{3/2}}).$$

The pseudo-sphere $S^2_1(b) \subset \mathbb{R}^3_1$ also has a projective cylinder model $C^2(b)$. This can be constructed in a similar way to that of $H^2_1(b)$.

4.3 Harmonic conformal gradient fields

We now consider conformal gradient fields on the hyperquadrics. This is an adaptation of work done in [5] on Riemannian space forms. The methods used are extrinsic by nature, making full use of the ambient pseudo-Euclidean space.

4.3.1 Definition. Let $M = S^n_v(1)$ or $M = H^n_v(1)$ in the appropriate pseudo-Euclidean space $\mathbb{V}$ with metric $\langle\,,\rangle$, where $\epsilon = \pm 1$ is the curvature of the hyperquadric. Note that the equation of the hyperquadric is $\langle x, x\rangle = \epsilon$. Let $a \in \mathbb{V}$ and $\mu = \langle a, a\rangle$. Let $\alpha\colon M \to \mathbb{R}$ be the restriction of the covector field metrically dual to a:

$$\alpha(x) = \langle x, a\rangle, \quad \text{for all } x \in M.$$

Then the *conformal gradient field* σ on M with pole a is:

$$\sigma = \operatorname{grad}\alpha = \nabla\alpha.$$

The gradient is, of course, that intrinsic to the hyperquadric given by $\langle\nabla\alpha, X\rangle = d\alpha(X)$ for all $X \in TM$.

4.3.2 Proposition. *For all $X \in TM$:*

$$\langle\sigma, X\rangle = \langle a, X\rangle.$$

Proof. By Definition 4.3.1:

$$\begin{aligned}\langle \sigma, X\rangle &= \langle \nabla\alpha, X\rangle \\ &= d\alpha(X),\end{aligned}$$

and the linearity of α gives:

$$d\alpha(X) = \alpha(X) = \langle a, X\rangle.$$

□

4.3.3 Proposition. *The conformal gradient field is:*

$$\sigma(x) = a - \epsilon\alpha(x)x.$$

Proof. Let $\{E_i\}$ be frame of M at x; then $\{x, E_i\}$ is a frame of $\mathbb{V}$. Decompose σ:

$$\sigma(x) = \sum_i \epsilon_i \langle \sigma(x), E_i\rangle E_i = \sum_i \epsilon_i \langle a, E_i\rangle E_i,$$

by Proposition 4.3.2. Next note that:

$$a = \sum_i \epsilon_i \langle a, E_i\rangle E_i + \langle x, x\rangle\langle a, x\rangle x.$$

Thus:

$$\sigma(x) = a - \epsilon\langle a, x\rangle x.$$ □

The pseudo-Riemannian length, $2F$, is a key measurement.

4.3.4 Proposition. *If σ is a conformal gradient field with pole a then:*

$$2F = \langle \sigma, \sigma\rangle = \mu - \epsilon\alpha^2.$$

Proof. From Proposition 4.3.3:

$$\begin{aligned}2F &= \langle a - \epsilon\langle a, x\rangle x, a - \epsilon\langle a, x\rangle x\rangle \\ &= \langle a, a\rangle - 2\langle a, \epsilon\langle a, x\rangle x\rangle + \langle \epsilon\langle a, x\rangle x, \epsilon\langle a, x\rangle x\rangle \\ &= \mu - 2\epsilon\alpha^2(x) + \epsilon^3\alpha^2(x).\end{aligned}$$ □

It follows from Proposition 4.3.4 that $F(x) = 0$ if and only if $\mu\epsilon > 0$ and $x = \pm a/\sqrt{|\mu|}$. However in this case $\sigma(x) = 0$, therefore σ is either space-like or time-like, although it is not possible to discern which from the signs of μ and ϵ. If $\mu\epsilon < 0$ then σ has no zeros.

4.3.5 Proposition. *If σ is a conformal gradient field then:*

$$\nabla_X \sigma = -\epsilon\alpha\, X, \quad \textit{for all } X \in TM.$$

Proof. Note the Gauss formula for all vector fields $Y \in \Gamma(TM)$ and $X \in T_xM$ (see for instance [18]):

$$\nabla_X Y = D_X Y + \epsilon\langle X, Y\rangle x,$$

where D is directional differentiation in $\mathbb{V}$ and x is interpreted as a unit normal vector. Therefore:

$$\begin{aligned}\nabla_X\sigma &= D_X\sigma + \epsilon\langle X,\sigma\rangle x\\ &= D_X(a-\epsilon\alpha(x)x) + \epsilon\alpha(X)x, \quad \text{by Propositions 4.3.2 and 4.3.3}\\ &= -\epsilon\alpha(X)x - \epsilon\alpha(x)X + \epsilon\alpha(X)x\\ &= -\epsilon\alpha(x)X.\end{aligned}$$

□

4.3.6 Proposition. *If σ is a conformal gradient field then:*

$$\nabla^2_{X,Y}\sigma = -\epsilon\langle\sigma, X\rangle Y.$$

Proof. The second covariant derivative is:

$$\nabla^2_{X,Y}\sigma = \nabla_X(\nabla_Y\sigma) - \nabla_{\nabla_X Y}\sigma.$$

By Proposition 4.3.5:

$$\begin{aligned}\nabla^2_{X,Y}\sigma &= \nabla_X(-\epsilon\alpha Y) + \epsilon\alpha\,\nabla_X Y\\ &= -\epsilon\alpha(X)Y - \epsilon\alpha\,\nabla_X Y + \epsilon\alpha\nabla_X Y,\end{aligned}$$

and Proposition 4.3.2 gives the result. □

We consider next the terms of $T_p(\sigma)$ in (2.8) and conclude σ is preharmonic.
4.3.7 Lemma. *If σ is a conformal gradient field then σ preharmonic, with:*

$$\nu = \epsilon, \quad \zeta = \epsilon(\mu - 2F).$$

Proof. By calculation:

$$\begin{aligned}\nabla^*\nabla\sigma &= -\operatorname{trace}\nabla^2\sigma\\ &= -\sum_i \epsilon_i\,\nabla^2_{E_i,E_i}\sigma\\ &= \sum_i \epsilon_i\,\epsilon\langle\sigma, E_i\rangle E_i, \quad \text{by Proposition 4.3.6}\\ &= \epsilon\sigma,\end{aligned}$$

hence $\nu = \epsilon$. Furthermore:

$$\begin{aligned}\nabla F &= -\epsilon\alpha(x)\sum_i \epsilon_i\,\langle\sigma,(E_i)\rangle E_i\\ &= -\epsilon\alpha\sigma. \end{aligned}\tag{4.1}$$

Therefore:

$$\begin{aligned}\nabla_{\nabla F}\sigma &= -\epsilon\alpha(x)(\nabla F)\\ &= -\epsilon\alpha(x)(-\epsilon\alpha\sigma)\\ &= \alpha^2\sigma\\ &= \epsilon(\mu-2F)\sigma, \quad \text{by Proposition 4.3.4,}\end{aligned}$$

hence $\zeta = \epsilon(\mu - 2F)$. □

4.3.8 Theorem. *Let σ be a conformal gradient field with pole a on a hyperquadric $M \subset \mathbb{V}$. If $\mu \geq 0$ then σ is (p,q)-harmonic if and only if:*

$$n > 2, \quad \mu = 1/(n-2), \quad p = n+1, \quad q = 2-n.$$

If $\mu < 0$ then σ is (p,q)-harmonic if and only if $\mu = -1$ and:

$$p = n+1, \quad q = \frac{1+n-n^2}{n}.$$

or:

$$n > 2, \quad p = 1/(2-n), \quad q = 0.$$

Proof. By Lemma 4.3.7 σ is preharmonic. Hence the harmonic equations simplify to:

$$(p+q+2qF)\Delta F + 2p(1+qF)\zeta + (1+2(1-p)F)\nu = 0,$$

where $\nu = \epsilon$ and $\zeta = \epsilon(\mu - 2F)$. The Laplacian of F is:

$$\begin{aligned}
\Delta F &= -\operatorname{div}\operatorname{grad} F \\
&= -\sum_i \epsilon_i \langle \nabla_{E_i}(\nabla F), E_i \rangle \\
&= -\sum_i \epsilon_i \langle \nabla_{E_i}(-\epsilon\alpha\sigma), E_i \rangle, \quad \text{by (4.1)} \\
&= \sum_i \epsilon_i \langle \epsilon\alpha(E_i)\sigma + \epsilon\alpha\nabla_{E_i}\sigma, E_i \rangle, \quad \text{by Proposition 4.3.5} \\
&= \epsilon\langle \sigma, \sum_i \epsilon_i \langle \sigma, E_i \rangle E_i \rangle + \sum_i \epsilon_i \langle -\epsilon^2\alpha^2 E_i, E_i \rangle, \quad \text{by Proposition 4.3.5 again} \\
&= \epsilon\langle \sigma, \sigma \rangle - n\alpha^2 \\
&= \epsilon 2F(1+n) - \epsilon n\mu, \quad \text{by Proposition 4.3.4.}
\end{aligned}$$

Therefore the harmonic equations reduce to the following polynomial in F:

$$0 = (p+q+2qF)(2F(1+n) - n\mu) + 2p(1+qF)((\mu - 2F)) + (1+2(1-p)F).$$

Note the cancellation of ϵ and hence the independence of the result from the sign of the curvature of the hyperquadric. Inspection of coefficients yields:

$$\begin{aligned}
F^2 : &\quad 0 = q(n+1-p), \\
F : &\quad 0 = 1 + p(n-2) + q(n+1) + q\mu(p-n), \\
F^0 : &\quad 0 = 1 + \mu(p(2-n) - nq).
\end{aligned}$$

The leading term gives two possibilities:

$$q = 0, \quad p = n+1.$$

If $p = n+1$ the remaining equations become:

$$\begin{aligned}
F : &\quad 0 = 1 + (n+1)(n-2+q) + q\mu, \\
F^0 : &\quad 0 = 1 + \mu((n+1)(2-n) - nq),
\end{aligned}$$

which imply:

$$(n-2+q)(1+\mu)=0.$$

Thus either $q = 2-n$ or $\mu = -1$. If $q = 2-n$ then $\mu = 1/(n-2)$, the first result. If $\mu = -1$ then:

$$nq = 1+n-n^2,$$

which is the second result.

If $q = 0$ the remaining equations become:

$$\begin{aligned} F: &\quad 0 = 1 + p(n-2), \\ F^0: &\quad 0 = 1 + \mu p(2-n). \end{aligned}$$

The F-coefficient implies $p = 1/(2-n)$ and the constant coefficient $\mu = (n-2)p = -1$, the last result. □

It is interesting to note that this result does not depend on the curvature of the hyperquadric. It does however depend on the index of the ambient space: if the ambient space is strictly positive (resp. negative) definite then necessarily $\mu > 0$ (resp. $\mu < 0$). It should also be noted that these harmonic conformal gradient fields are not necessarily metrically unique; if $n > 2$ and $\mu < 0$ there are two sets of metric parameters.

4.3.9 Example. Let $M = S^2_1$, the 2-dimensional de Sitter space. Then the conformal gradient field with pole $(0,0,1)$ is $(3,-1/2)$-harmonic. This vector field has no fixed points, and up to congruence it is the unique conformal gradient field that is harmonic (Theorem 4.3.12).

4.3.10 Example. Let $M = H^2_2$, the negative definite 2-sphere. Then the conformal gradient field with pole $(0,0,1)$ is $(3,-1/2)$-harmonic. This vector field has two fixed points at $\pm(0,0,1)$, and up to congruence it is the unique conformal gradient field that is harmonic (Theorem 4.3.12). This is a contrast to the positive definite 2-sphere, which has no harmonic conformal gradient fields.

4.3.11 Remark. Example 4.3.10 is the first example of a harmonic vector field on a smooth 2-sphere. That being said the sphere in question is not *Riemannian.* Although H^2_2 and S^2_0 are anti-isometric, and are often therefore thought of of being equivalent, the anti-isometry does not transport the harmonic vector field on H^2_2 to one on S^2_0. The anti-isometry φ from H^2_2 to S^2_0 is in fact the identify map, hence $\varphi.\sigma = \sigma$. This is also a conformal gradient field on S^2_0, the difference being $\langle\varphi.\sigma, \varphi.\sigma\rangle = -\langle\sigma,\sigma\rangle$. That σ is harmonic but $\varphi.\sigma$ is not shows that the Euler-Lagrange equations for harmonic vector fields are not invariant under anti-isometry.

4.3.1 Congruence of conformal gradient fields on hyperquadrics

Whilst there is a lot of choice in the pole vector used to generate the harmonic conformal gradient fields, they do have specified length. We shall show that any two conformal gradient field generated by vectors of the same length are congruent, and hence these should be considered to be *one* harmonic vector field.

4.3.12 Theorem. *The congruence classes of conformal gradient fields on the hyperquadrics are defined by $\mu = \langle a, a\rangle$, where a is the pole vector of the field.*

Proof. Let M be a hyperquadric and let $\sigma, \tilde{\sigma} \in \Gamma(TM)$ be conformal gradient fields with poles $a, \tilde{a}$ respectively, such that $\mu = \tilde{\mu}$. We aim to find a congruence between σ and $\tilde{\sigma}$.

Consider the isometry groups of the hyperquadrics, given in Proposition 4.1.16. Then there is some $F \in O^{++}(n+1, u)$, where u is the index of the ambient pseudo-Euclidean space, such that $\tilde{a} = F(a)$, and thus $\mu = \tilde{\mu}$. The function $\tilde{\alpha}(x)$ is:

$$\tilde{\alpha}(x) = \langle \tilde{a}, x \rangle = \langle F(a), x \rangle = \langle a, F^{-1}(x) \rangle.$$

Thus:

$$\tilde{\alpha} = \alpha \circ F^{-1}.$$

For all $X \in T_xM$:

$$\begin{aligned}
\langle \nabla \tilde{\alpha}, X \rangle &= d\tilde{\alpha}(X) \\
&= d\alpha(dF^{-1}(X)) \\
&= \langle \nabla \alpha, dF^{-1}(X) \rangle \\
&= \langle \nabla \alpha, F^{-1}(X) \rangle \\
&= \langle F(\nabla \alpha), X \rangle \\
&= \langle dF(\nabla \alpha), X \rangle,
\end{aligned}$$

where $\nabla \alpha$ is evaluated at $F^{-1}(x)$. Therefore

$$\tilde{\sigma}(x) = \nabla \tilde{\alpha}(x) = dF(\nabla \alpha(F^{-1}(x))) = dF \circ \sigma(F^{-1}(x)),$$

which shows that $\tilde{\sigma}$ is congruent to σ. $\square$

4.4 Harmonic Killing fields

We now consider Killing fields on the hyperquadrics. This is an adaptation of work done in [5] on Riemannian space forms. The methods used are extrinsic by nature, making full use of the ambient pseudo-Euclidean space. First we recall the definition, and some useful properties of, Killing fields.

4.4.1 Definition. A *Killing field* σ of pseudo-Riemannian manifold (M, g) is characterised by

$$\mathrm{L}_\sigma g = 0. \qquad \Diamond$$

We state two well known properties of Killing fields.

4.4.2 Proposition ([18, Prop. 9.23]). *A vector field σ is Killing if and only if the stages ϕ_t of its (local) flow are isometries.*

4.4.3 Lemma ([25]). *If σ is a Killing field on a pseudo-Riemannian manifold then:*

$$\nabla^* \nabla \sigma = \mathrm{Ric}(\sigma).$$

The Lie algebra of $O(n+1, u)$ is the space of linear transformations of $\mathbb{R}^{n+1}_u$ that are skew-symmetric with respect to $\langle\, , \rangle$ in the following sense:

$$\langle A(x), y \rangle + \langle x, A(y) \rangle = 0, \quad \text{for all } x, y \in \mathbb{R}^{n+1}_u.$$

The isometry group of the hyperquadric is $O(n+1,u)$, given in Proposition 4.1.16, hence the Killing fields on the hyperquadric are the restrictions of skew-symmetric transformations to the hyperquadric.

4.4.4 Proposition. *Let $\mathbb{R}^{n+1}_u$ be the pseudo-Euclidean space of dimension $n+1$, index u and let A be a skew-symmetric linear transformation. Then the components of the matrix $A = (a_{ij})$ with respect to a frame $\{e_i\}$ satisfy:*

$$a_{ij} = -\epsilon_i\,\epsilon_j\,a_{ji},$$

where $\epsilon_i = \langle e_i, e_i\rangle$.

Proof. Let $x, y \in \mathbb{R}^{n+1}_u$ and decompose:

$$x = \sum_i \epsilon_i\,x^i e_i, \quad y = \sum_i \epsilon_i\,y^i e_i.$$

Then:

$$\begin{aligned}\langle A(x), y\rangle &= \langle \epsilon_i\,a_{ij}\,x^j e_i, y\rangle = \epsilon_i\,a_{ij}\,x^j y^i\\ \langle x, A(y)\rangle &= \langle \epsilon_i\,x, a_{ij}\,y^j e_i\rangle = \epsilon_i\,a_{ij}\,y^j x^i.\end{aligned}$$

Therefore by skew-symmetry:

$$\begin{aligned}0 &= \epsilon_i\,a_{ij}\,x^j y^i + \epsilon_j\,a_{ji}\,y^i x^j\\ &= (a_{ij} + \epsilon_i\,\epsilon_j\,a_{ji})y^i x^j.\end{aligned}$$

□

We now find the covariant derivative of the Killing field.

4.4.5 Proposition. *Let σ be a Killing field on hyperquadric $M \subset \mathbb{V}$ with curvature ϵ represented by skew-symmetric matrix A. Then, for all $X \in T_xM$:*

$$\nabla_X\sigma(x) = A(X) - \epsilon\langle A(X), x\rangle x.$$

Proof. The covariant derivative is characterised by the Gauss formula (see for instance [18]):

$$\nabla_X Y = D_X Y + \epsilon\langle X, Y\rangle x.$$

Applying this to σ:

$$\nabla_X\sigma(x) = D_X A(x) + \epsilon\langle X, A(x)\rangle x,$$

and since A is linear and skew symmetric:

$$\nabla_X\sigma(x) = A(X) - \epsilon\langle A(X), x\rangle x.$$

□

Next we consider the rough Laplacian and investigate the preharmonicicty of a Killing field. The calculation given in the proof of the following result provides a cross-check of the expression for the covariant derivative obtained in Proposition 4.4.5.

4.4.6 Proposition. *If σ is a Killing field on a hyperquadric M of curvature ϵ then:*

$$\nabla^*\nabla\sigma = \epsilon(n-1)\sigma.$$

Proof. Consider the second covariant derivative, given by

$$\nabla^2_{X,Y}\sigma = \nabla_X \nabla_Y \sigma - \nabla_{\nabla_X Y}\sigma.$$

Using the result of Proposition 4.4.5:

$$\nabla^2_{X,Y}\sigma = \nabla_X(A(Y) - \epsilon\langle A(Y), x\rangle x) - A(\nabla_X Y) + \epsilon\langle A(\nabla_X Y), x\rangle x.$$

We now use the Gauss formula once again:

$$\begin{aligned}\nabla^2_{X,Y}\sigma &= D_X(A(Y) + \epsilon\langle A(Y), x\rangle x) + \epsilon\langle X, A(Y) + \epsilon\langle A(Y), x\rangle x\rangle x \\ &\quad - A(D_X Y - \epsilon\langle X, Y\rangle x) - \epsilon\langle A(D_X Y - \epsilon\langle X, Y\rangle x), x\rangle x \\ &= 2\epsilon\langle X, A(Y)\rangle x - \epsilon\langle X, Y\rangle A(x) - \epsilon\langle A(Y), x\rangle X,\end{aligned}$$

noting that $\langle X, x\rangle = 0$, $\langle A(x), x\rangle = 0$ and $D_X x = X$. The rough Laplacian is:

$$\begin{aligned}\nabla^*\nabla\sigma &= -\operatorname{trace}\nabla^2\sigma \\ &= \sum_i \epsilon_i\,[-2\epsilon\langle E_i, A(E_i)\rangle x + \epsilon\langle E_i, E_i\rangle A(x) + \epsilon\langle A(E_i), x\rangle E_i], \\ &= \sum_i \epsilon_i\,[\epsilon\langle E_i, E_i\rangle A(x) - \epsilon\langle E_i, A(x)\rangle E_i], \quad \text{by the skew-symmetry of } A \\ &= \epsilon\sigma\sum_i \epsilon_i - \epsilon\sum_i \langle E_i, A(x)\rangle E_i \\ &= \epsilon(n-1)\sigma.\end{aligned}$$

□

4.4.7 Proposition. *If σ is a Killing field on a hyperquadric M of curvature ϵ then:*

$$\nabla_{\nabla F}\sigma(x) = -A^3(x) - 2\epsilon F\sigma(x).$$

Remark. Since A is skew-symmetric so is A^3 and therefore defines a vector field on M.

Proof. First consider ∇F, where $2F = \langle\sigma, \sigma\rangle$

$$\begin{aligned}\nabla F &= \sum_i \epsilon_i dF(E_i)E_i \\ &= \sum_i \epsilon_i\langle\nabla_{E_i}\sigma, \sigma\rangle E_i \\ &= \sum_i \epsilon_i\langle A(E_i) - \epsilon\langle A(E_i), x\rangle x, \sigma\rangle E_i \\ &= \sum_i \epsilon_i\langle A(E_i), A(x)\rangle E_i \\ &= -\sum_i \epsilon_i\langle E_i, A^2(x)\rangle E_i \\ &= -A^2(x) + \epsilon\langle A^2(x), x\rangle x \\ &= -A^2(x) - \epsilon\langle\sigma, \sigma\rangle x.\end{aligned}$$

Consider the covariant derivative of σ in the direction of ∇F, noting that the skew-symmetry of $A^3(x)$ implies $\langle A^3(x), x\rangle = 0$:

$$\begin{aligned}\nabla_{\nabla F}\sigma &= A(\nabla F) - \epsilon\langle A(\nabla F), x\rangle x \\ &= A(-A^2(x) - \epsilon\langle\sigma,\sigma\rangle x) - \epsilon\langle A(-A^2(x) - \epsilon\langle\sigma,\sigma\rangle x), x\rangle x \\ &= -A^3(x) - \epsilon\langle\sigma,\sigma\rangle A(x) \\ &= -A^3(x) - 2\epsilon F\sigma.\end{aligned}$$

□

4.4.8 Proposition. *A Killing field on a hyperquadric is preharmonic if and only if the corresponding matrix A satisfies:*

$$A^3(x) = \lambda A(x),$$

where $\lambda \in \mathbb{R}$, in which case the spinnaker is:

$$\zeta = -(\lambda + 2\epsilon F).$$

Proof. From Proposition 4.4.7 we know that σ is preharmonic if and only if

$$A^3(x) - \lambda(x)A(x) = 0,$$

for some smooth function $\lambda\colon M \to \mathbb{R}$. We differentiate this equation:

$$A^3(X) - \lambda(x)A(X) = d\lambda(X)A(x), \quad \text{for all } X \in T_xM.$$

It follows from these two equations that for each $x \in M$ the linear map $A^3 - \lambda(x)A$ has rank at most one. However non-trivial skew-symmetric transformations of pseudo-Euclidean space have rank at least two by Proposition 4.4.4. Therefore $A^3 - \lambda(x)A = 0$, hence $d\lambda(X)A(x) = 0$. Since $\ker(A)$ is a subspace of codimension at least 2, whose intersection with M is a submanifold of dimension at most $n-1$, it follows that $d\lambda = 0$ on an open dense subset and hence by continuity everywhere. Since M is connected it follows that λ is constant. The expression for ζ then follows from Proposition 4.4.7. □

Finally, we calculate the Laplacian of the pseudo-Riemannian length of the Killing field.

4.4.9 Proposition. *If σ is a Killing field on a hyperquadric M of curvature ϵ then the Laplacian of $F = \frac{1}{2}\langle\sigma,\sigma\rangle$ is:*

$$\Delta F = 2(n+1)\epsilon F - \langle A, A\rangle,$$

where $\langle A, A\rangle$ is the norm squared of A in the ambient pseudo-Euclidean space.

Proof. Recall $\Delta F = -\operatorname{div}\nabla F$ and from Proposition 4.4.7 $\nabla F = -A^2(x) - \epsilon\langle\sigma,\sigma\rangle x$. We use the Gauss formula to calculate covariant derivative of ∇F:

$$\begin{aligned}\nabla_X(\nabla F) &= D_X(-A^2(x) - \epsilon\langle\sigma,\sigma\rangle x) + \epsilon\langle X, -A^2(x) - \epsilon\langle\sigma,\sigma\rangle x\rangle x \\ &= -A^2(X) - \epsilon\langle\sigma,\sigma\rangle X + \epsilon\langle X, -A^2(x) - \epsilon\langle\sigma,\sigma\rangle x\rangle x \\ &= -A^2(X) - \epsilon\langle\sigma,\sigma\rangle X - \epsilon\langle A(X), A(x)\rangle x\end{aligned}$$

Therefore:

$$\begin{aligned}
\operatorname{div}(\nabla F) &= \sum_i \epsilon_i \langle \nabla_{E_i}(\nabla F), E_i\rangle \\
&= \sum_i \epsilon_i \langle -A^2(E_i) - \epsilon\langle\sigma,\sigma\rangle E_i - \epsilon\langle A(E_i), A(x)\rangle x, E_i\rangle \\
&= \sum_i \epsilon_i \langle -A^2(E_i) - \epsilon\langle\sigma,\sigma\rangle E_i, E_i\rangle \\
&= \sum_i \epsilon_i \langle -A^2(E_i), E_i\rangle - \sum_i \epsilon_i\langle \epsilon\langle\sigma,\sigma\rangle E_i, E_i\rangle \\
&= -n\epsilon\langle\sigma,\sigma\rangle + \sum_i \epsilon_i \langle A(E_i), A(E_i)\rangle \\
&= -n\epsilon\langle\sigma,\sigma\rangle + \langle A, A\rangle - \epsilon\langle A(x), A(x)\rangle \\
&= \langle A, A\rangle - \epsilon\langle\sigma,\sigma\rangle - n\epsilon\langle\sigma,\sigma\rangle, \text{since } A(x) = \sigma(x).
\end{aligned}$$

□

4.4.10 Proposition. *Let $\mathbb{R}^n_v$ be the pseudo-Euclidean space of dimension n, index v and let A be a skew-symmetric linear transformation with matrix (a_{ij}) with respect to a frame. Then:*

$$\langle A, A\rangle = 2\sum_{i<j} \epsilon_i\, \epsilon_j\, a_{ij}^2.$$

Proof. If $\{e_i\}$ is a frame of $\mathbb{R}^n_v$ then:

$$\begin{aligned}
\langle A, A\rangle &= \sum_i \epsilon_i \langle A(e_i), A(e_i)\rangle \\
&= \sum_i \epsilon_i \langle \sum_j a_{ij} e_j, \sum_j a_{ij} e_j\rangle \\
&= \sum_{i,j} \epsilon_i\, \epsilon_j a_{ij}^2.
\end{aligned}$$

Note from Proposition 4.4.4 that $a_{ji}^2 = a_{ij}^2$. □

4.4.11 Remark. From Propositions 4.4.6, 4.4.7 and 4.4.9 and Theorem 2.4.3 the Euler-Lagrange equation for a Killing field on a hyperquadric to be (p, q)-harmonic is:

$$\begin{aligned}
0 &= (p + q + q2F)(\epsilon(n+1)2F - \langle A, A\rangle) - p(2 + q2F)(\lambda + \epsilon 2F) \\
&\quad + (1 + (1-p)2F)\epsilon(n-1) \\
&= \epsilon(q(n+1) - pq)(2F)^2 \\
&\quad + (\epsilon(p+q)(n+1) - q\langle A, A\rangle - pq\lambda - 2p\epsilon + \epsilon(1-p)(n-1))(2F) \\
&\quad - ((p+q)\langle A, A\rangle + 2p - \epsilon(n-1)).
\end{aligned} \tag{4.2}$$

Since $A^3 - \lambda A = A(A - \sqrt{\lambda} I)(A + \sqrt{\lambda} I)$ standard matrix theory ensures A has eigenvalues $0, \pm\sqrt{\lambda}$ over $\mathbb{C}$, and when $\lambda \neq 0$ A must be diagonalisable over $\mathbb{C}$ since its minimal polynomial must have linear factors. The implications of this for $\langle A, A\rangle$, λ and F, and the analysis of (4.2), is however unclear.

4.4.1 The 2-dimensional case

We consider now the Killing fields on the 2-dimensional hyperquadrics. Firstly we find a general relationship between $\langle A, A\rangle$, the pseudo-Riemannian length of A, and the eigenvalue λ (Proposition 4.4.8) and then solve equation (4.2) with this relation.

4.4.12 Proposition. *Let σ be a Killing field on a 2-dimensional hyperquadric of curvature ϵ, whose representing matrix A with respect to a frame of $\mathbb{R}^3_u$ is:*

$$A = \begin{pmatrix} 0 & a & b \\ -\epsilon_1\,\epsilon_2\,a & 0 & c \\ -\epsilon_1\,\epsilon_3\,b & -\epsilon_2\,\epsilon_3\,c & 0 \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. Then σ is preharmonic, and:

$$\lambda = -\epsilon_1\,\epsilon_2\,a^2 - \epsilon_1\,\epsilon_3\,b^2 - \epsilon_2\,\epsilon_3\,c^2.$$

Proof. We calculate A^3 and compare with A:

$$\begin{aligned} A^3 &= \begin{pmatrix} 0 & a & b \\ -\epsilon_1\,\epsilon_2\,a & 0 & c \\ -\epsilon_1\,\epsilon_3\,b & -\epsilon_2\,\epsilon_3\,c & 0 \end{pmatrix}^3 \\ &= \begin{pmatrix} 0 & a & b \\ -\epsilon_1\,\epsilon_2\,a & 0 & c \\ -\epsilon_1\,\epsilon_3\,b & -\epsilon_2\,\epsilon_3\,c & 0 \end{pmatrix} \begin{pmatrix} -\epsilon_1\,\epsilon_2\,a^2 - \epsilon_1\,\epsilon_3\,b^2 & -\epsilon_2\,\epsilon_3\,ac & ac \\ -\epsilon_1\,\epsilon_3\,bc & -\epsilon_1\,\epsilon_2\,a^2 - \epsilon_2\,\epsilon_3\,c^2 & -\epsilon_1\,\epsilon_2\,ab \\ \epsilon_1\,\epsilon_3\,ac & -\epsilon_1\,\epsilon_3\,ab & -\epsilon_1\,\epsilon_3\,b^2 - \epsilon_2\,\epsilon_3\,c^2 \end{pmatrix} \\ &= (-\epsilon_1\,\epsilon_2\,a^2 - \epsilon_1\,\epsilon_3\,b^2 - \epsilon_2\,\epsilon_3\,c^2) \begin{pmatrix} 0 & a & b \\ -\epsilon_1\,\epsilon_2\,a & 0 & c \\ -\epsilon_1\,\epsilon_3\,b & -\epsilon_2\,\epsilon_3\,c & 0 \end{pmatrix}. \end{aligned}$$

It follows from Proposition 4.4.8 that σ is preharmonic. □

4.4.13 Corollary. *Under the same hypotheses as Proposition 4.4.12 the Laplacian of $F = \frac{1}{2}\langle\sigma, \sigma\rangle$ is:*

$$\Delta F = 6\epsilon F + 2\lambda,$$

and the rough laplacian is:

$$\nabla^*\nabla\sigma = \epsilon\sigma.$$

Proof. It follows from Propositions 4.4.10 and 4.4.12 that $\langle A, A\rangle = -2\lambda$, and the expression for ΔF follows from Proposition 4.4.9. The rough Laplacian follows from Proposition 4.4.6 with $n = 2$. □

With this information we can now solve the Euler-Lagrange equations for harmonic Killing fields on dimension two hyperquadrics.

4.4.14 Theorem. *Let σ be a Killing field of non-constant pseudo-Riemannian length on a 2-dimensional hyperquadric of curvature ϵ. Then σ is (p, q)-harmonic if and only if:*

$$p = 3, \quad q = -1/2, \quad \lambda = \epsilon.$$

Proof. The Killing field σ is the restriction of the action of the skew-symmetric matrix A to M. Now σ is preharmonic (Proposition 4.4.12) with $\nu = \epsilon$ (Corollary 4.4.13) and $\zeta = \epsilon(2F - \lambda)$ (Proposition 4.4.8) where λ follows from Proposition 4.4.12.Therefore by Theorem 2.4.3 σ is (p, q)-harmonic if and only if:

$$0 = 2(p + q + 2qF)(3\epsilon F + \lambda) + 2p(1 + qF)(-\lambda - 2\epsilon F) + \epsilon(1 + 2(1 - p)2F).$$

This is a polynomial in F, giving the equations:

$$\begin{aligned} F^2: &\quad 0 = (3 - p)q, \\ F^1: &\quad 0 = (p + q)3\epsilon + 2q\lambda - 2p\epsilon + pq(-\lambda) + \epsilon(1 - p), \\ F^0: &\quad 0 = 2\lambda(p + q) + 2p(-\lambda) + \epsilon. \end{aligned}$$

The leading coefficient implies two possibilities:

$$q = 0, \quad p = 3.$$

If $q = 0$ then the F coefficient leads to a contradiction. If $p = 3$ then the remaining equations are:

$$\begin{aligned} F^1: &\quad 0 = \epsilon + 3\epsilon q - q\lambda, \\ F^0: &\quad 0 = \epsilon + 2q\lambda. \end{aligned}$$

Adding these equations implies $q = -1/2$. Then the constant term gives $\lambda = \epsilon$. □

4.4.15 Remark. In the case where σ is of constant pseudo-Riemannian length then we refer to Remarks 2.3.10. If $\langle\sigma\,\sigma\rangle = -1$ then σ is $(0, q)$-harmonic for all q. If $\langle\sigma\,\sigma\rangle = k \neq -1$ then, noting $\Delta F = 0$, by Lemma 2.4.2:

$$\langle\nabla\sigma, \nabla\sigma\rangle = \epsilon k,$$

and the Euler-Lagrange equations simplify to that of (2.9), in particular:

$$(1 + k)\epsilon\sigma = p\epsilon k\sigma,$$

and thus σ is (p, q)-harmonic for all q when $p = \frac{1+k}{k}$, and hence σ is also a harmonic section of the sphere bundle $S(TM)(k)$.

Note that once again the Riemannian 2-sphere is excluded from having harmonic vector fields.

4.4.16 Theorem. *There are no harmonic Killing fields on the Riemannian 2-sphere.*

Proof. In the Riemannian case $\lambda < 0$ (Proposition 4.4.12), but $\epsilon > 0$. □

We can deduce a number of additional results from Theorem 4.4.14. For example, we can pick any suitable skew-symmetric matrix and use it to generate a harmonic Killing field.

4.4.17 Example. Let $M = H^2_1$, the neutral pseudo-hyperbolic space in $\mathbb{R}^3_2$, otherwise known as anti-de Sitter space. Then the Killing field generated by:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is $(3, -1/2)$-harmonic. For, in this case $\epsilon = -1$ and $\lambda = -1$, and the result follows from Theorem 4.4.14.

4.4.18 Example. Let $M = H^2_2$, the index 2 pseudo-hyperbolic space in $\mathbb{R}^3_3$ anti-isometric to the Riemannian 2-sphere. Then the Killing field generated by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is $(3, -1/2)$-harmonic. For, in this case $\epsilon = -1$ and $\lambda = -1$, thus the result follows fromTheorem 4.4.14.

4.4.2 Congruences of Killing fields on 2-dimensional hyperquadrics

Now we shall show that although there may seem to be many harmonic Killing fields on a given hyperquadric (apart from the Riemannian 2-sphere), modulo the action of the isometry group the harmonic Killing field is unique. Note that the Lie algebra of the isometry group is three dimensional.

First we shall note that congruence classes for Killing fields on the index two, dimension two pseudo-hyperbolic space can be found by comparison with those of the 2-sphere.

4.4.19 Theorem. *All Killing fields on H^2_2 have two fixed points, and are determined up to congruence by the value of λ.*

Proof. Let σ and $\bar{\sigma}$ be two Killing fields on H^2_2. We first show that the anti-isometry φ of Proposition 4.1.11 between H^2_2 and S^2_0 preserves Killing fields. For $X, Y \in \Gamma(TS^2_0)$ we have:

$$\langle \nabla_X(\varphi.\sigma), Y\rangle + \langle X, \nabla_Y(\varphi.\sigma)\rangle = \langle \varphi.(\nabla_{\varphi^{-1}.X})\sigma, Y\rangle + \langle X, \varphi.(\nabla_{\varphi^{-1}.Y})\sigma\rangle,$$

because φ is totally geodesic (Proposition 4.1.12)

$$= -\langle \nabla_{\varphi^{-1}.X}\sigma, \varphi^{-1}.Y\rangle - \langle \varphi^{-1}.X, \nabla_{\varphi^{-1}.Y}\sigma\rangle,$$

because φ is an anti-isometry

$$= 0, \quad \text{because } \sigma \text{ is Killing.}$$

Since Killing fields on S^2_0 have two fixed points it follows that the same will be true for Killing fields on H^2_2. Furthermore if σ and $\bar{\sigma}$ are congruent with a congruence T, then $\varphi.\bar{\sigma}$ and $\varphi.\sigma$ are congruent with congruence $\varphi \circ T \circ \varphi^{-1}$. For if $\bar{\sigma} = T.\sigma$:

$$\begin{aligned} \varphi.\bar{\sigma} &= \varphi.T.\sigma \\ &= \varphi.T.\varphi^{-1}.(\varphi.\sigma) \\ &= (\varphi \circ T \circ \varphi^{-1}).(\varphi.\sigma), \end{aligned}$$

and $\varphi \circ T \circ \varphi^{-1}$ is an isometry of S^2_0. The congruence classes of Killing fields on S^2_0 are determined by the value of λ [5], which is invariant under the anti-isometry. □

To enable us to find the congruence classes of the Killing fields on other hyperquadrics we first consider a familiar case, the hyperbolic plane H^2 (note that this is the upper sheet of the hyperquadric H^2_0). We start by considering the fixed points of the Killing fields using the well-known Beltrami disc model.

4.4.20 Theorem. *The fixed points of Killing fields on H^2 are categorised by $\lambda = c^2 - a^2 - b^2$. The Killing fields have:*

1. *one fixed point if $\lambda < 0$,*
2. *one fixed point on the boundary at infinity if $\lambda = 0$,*
3. *two fixed points on the boundary at infinity if $\lambda > 0$.*

Proof. Project H^2 onto the Beltrami disc:

$$B^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$

with the map

$$F\colon H^2 \to B^2; \quad F\colon (x, y, z) \mapsto (x/z, y/z).$$

The inverse map is

$$F^{-1}\colon B^2 \to H^2; \quad F^{-1}\colon (x, y) \mapsto (\frac{x}{\sqrt{1-r^2}}, \frac{y}{\sqrt{1-r^2}}, \frac{1}{\sqrt{1-r^2}}),$$

where $r^2 = x^2 + y^2$.

The differential of the map F is

$$dF_{(x,y,z)}(u, v, w) = (\frac{zu - xw}{z^2}, \frac{zv - yw}{z^2}).$$

Then σ projects to $\tilde{\sigma}$ on B^2, where

$$\tilde{\sigma}(x, y) = (F.\sigma)(x, y) = dF(\sigma(F^{-1}(x, y))) = (b + ay - bx^2 - cxy, c - ax - bxy - cy^2).$$

Note that $\tilde{\sigma}$ extends smoothly across ∂B^2. Then $\tilde{\sigma}(x, y) = (0, 0)$ for $(x, y) \in B^2 \cup \partial B^2$ if and only if

$$\begin{aligned} b(1 - x^2) + y(a - cx) &= 0, \\ c(1 - y^2) - x(a + by) &= 0. \end{aligned}$$

If $\lambda < 0$ then the unique solution is $(\bar{x}, \bar{y}) = (c/a, -b/a)$, which corresponds to the point $\frac{1}{\sqrt{\lambda}}(c, -b, a) \in H^2$. If $\lambda = 0$ then the unique solution is $(\bar{x}, \bar{y}) = (c/a, -b/a)$, which has $\bar{x}^2 + \bar{y}^2 = 1$ and is hence on ∂B^2 and therefore corresponds to a point at infinity of H^2. Finally if $\lambda > 0$ then there are two solutions

$$\begin{aligned} x_\infty &= \frac{ac \pm b\sqrt{\lambda}}{b^2 + c^2}, \\ y_\infty &= -\frac{ab \pm c\sqrt{\lambda}}{b^2 + c^2}, \end{aligned}$$

both of which lie in ∂B^2 and hence correspond to points at infinity of H^2. □

We now exhibit a matrix normal form for Killing fields on H^2_0 in each of the three classes described by Theorem 4.4.20. Since H^2_0 has two sheets the Killing fields have twice the number of fixed points as those on H^2.

4.4.21 Lemma. *Let $\sigma \in \Gamma(TH_0^2)$ be a Killing field on H_0^2 with generating skew-symmetric matrix $A = (a_{ij})$. Suppose $\lambda < 0$, where $\lambda = -\sum_{i<j} \epsilon_i \epsilon_j a_{ij}^2$. Then σ is congruent to $\tilde{\sigma}$, the Killing field with generating skew-symmetric matrix:*

$$\tilde{A} = \begin{pmatrix} 0 & a_0 & 0 \\ -a_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $a_0^2 = -\lambda$.

Proof. Consider the infinitesimal isometry of H_0^2 represented by skew-symmetric matrix of $\mathbb{R}_1^3$:

$$\begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ u & v & 0 \end{pmatrix}.$$

When considered as a linear vector field on $\mathbb{R}_1^3$ this has flow given by the following 1-parameter family of matrices:

$$\Phi_t = \begin{pmatrix} 1 + u^2(\cosh(t) - 1) & uv(\cosh(t) - 1) & u \sinh(t) \\ uv(\cosh(t) - 1) & 1 + v^2(\cosh(t) - 1) & v \sinh(t) \\ u \sinh(t) & v \sinh(t) & \cosh(t) \end{pmatrix}.$$

This is an elementary calculation involving a linear system of first order ODEs.

Choose u and v to be:

$$u = \frac{-c}{\sqrt{b^2 + c^2}}, \quad v = \frac{b}{\sqrt{b^2 + c^2}},$$

where $a = a_{12},\ b = a_{13},\ c = a_{23}$. Let t_0 is the unique parameter such that $\Phi_{t_o}(\rho) = \tilde{\rho}$, where ρ is the zero of σ and $\tilde{\rho}$ is the zero of $\tilde{\sigma}$. Then using $\rho = \frac{1}{\sqrt{\lambda}}(c, -b, a)$ from the proof of Theorem 4.4.20 it follows that

$$\cosh(t_0) = a/a_0, \quad \sinh(t_0) = \frac{\sqrt{b^2 + c^2}}{a_0}.$$

We claim that the desired congruence is Φ_{t_0}. Writing $d = (a_0 - a)/\sqrt{b^2 + c^2}$ we compute:

$$\begin{aligned} & a_0^2 \Phi_{-t_0} A \Phi_{t_0} \\ &= \begin{pmatrix} -a_0 + c^2 d & -bcd & c \\ -bcd & -a_0 + b^2 d & -b \\ c & -b & -a \end{pmatrix} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{pmatrix} \begin{pmatrix} -a_0 + c^2 d & -bcd & -c \\ -bcd & -a_0 + b^2 d & b \\ -c & b & -a \end{pmatrix} \\ &= \begin{pmatrix} 0 & a_0^3 & 0 \\ -a_0^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore $\Phi_{-t_0} A \Phi_{t_0} = \tilde{A}$ and hence σ and $\tilde{\sigma}$ are congruent. □

4.4.22 Lemma. *Suppose σ and A are as defined in Lemma 4.4.21, and $\lambda = 0$. Then σ is congruent to $\tilde{\sigma}$, the Killing field with generating skew-symmetric matrix:*

$$\tilde{A} = \begin{pmatrix} 0 & a_0 & 0 \\ -a_0 & 0 & c_0 \\ 0 & c_0 & 0 \end{pmatrix},$$

where $a_0^2 = b^2 - a^2$ and $c_0 = -c$.

Proof. Consider the infinitesimal isometry of H_0^2 represented by skew-symmetric matrix of $\mathbb{R}_1^3$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

When considered as a linear vector field on $\mathbb{R}_1^3$ this has flow given by the following 1-parameter family of matrices:

$$\Phi_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix}.$$

The desired congruence is then Φ_{t_0} where

$$\cosh(t_0) = -a/a_0, \quad \sinh(t_0) = b/a_0.$$

For:

$$\begin{aligned}
\Phi_{-t_0} A \Phi_{t_0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t_0) & -\sinh(t_0) \\ 0 & -\sinh(t_0) & \cosh(t_0) \end{pmatrix} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh(t_0) & \sinh(t_0) \\ 0 & \sinh(t_0) & \cosh(t_0) \end{pmatrix} \\
&= \frac{1}{a_0^2} \begin{pmatrix} a_0 & 0 & 0 \\ 0 & -a & -b \\ 0 & -b & -a \end{pmatrix} \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{pmatrix} \begin{pmatrix} a_0 & 0 & 0 \\ 0 & -a & b \\ 0 & b & -a \end{pmatrix} \\
&= \frac{1}{a_0^2} \begin{pmatrix} 0 & aa_0 & ba_0 \\ a^2 - b^2 & -bc & -ac \\ 0 & -ac & -bc \end{pmatrix} \begin{pmatrix} a_0 & 0 & 0 \\ 0 & a & -b \\ 0 & -b & a \end{pmatrix} \\
&= \frac{1}{a_0^2} \begin{pmatrix} 0 & a^2 a_0 - b^2 a_0 & -aba_0 + aba_0 \\ (-a^2 + b^2) a_0 & -abc + abc & b^2 c - a^2 c \\ 0 & -a^2 c + b^2 c & abc - abc \end{pmatrix} \\
&= \begin{pmatrix} 0 & a_0 & 0 \\ -a_0 & 0 & c_0 \\ 0 & c_0 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore $\Phi_{-t_0} A \Phi_{t_0} = \tilde{A}$ and hence σ and $\tilde{\sigma}$ are congruent. □

4.4.23 Lemma. *Suppose σ and A are as defined in Lemma 4.4.21, and $\lambda > 0$. Then σ is congruent to $\tilde{\sigma}$, the Killing field with generating skew-symmetric matrix:*

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_0 \\ 0 & c_0 & 0 \end{pmatrix},$$

where $c_0^2 = \lambda$.

Proof. This time consider the infinitesimal isometry of H_0^2 represented by the following skew-symmetric matrix of $\mathbb{R}_1^3$:

$$\begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & \gamma \\ 0 & \gamma & 0 \end{pmatrix},$$

with
$$\alpha = \frac{c - c_0}{\sqrt{(c-c_0)^2 - a^2}}, \quad \gamma = \frac{a}{\sqrt{(c-c_0)^2 - a^2}}.$$
Then $\gamma^2 - \alpha^2 = -1$. This has flow matrix:
$$\Phi_t = \begin{pmatrix} -\gamma^2 + \alpha\cos(t) & \alpha\sin(t) & \alpha\gamma(1-\cos(t)) \\ -\alpha\sin(t) & \cos(t) & \gamma\sin(t) \\ \alpha\gamma(\cos(t)-1) & \gamma\sin(t) & \alpha^2 - \gamma\cos(t) \end{pmatrix}.$$
Let $t_0 \in \mathbb{R}$ satisfy:
$$\sin(t_0) = \frac{b\sqrt{(c-c_0)^2 - a^2}}{c_0(c-c_0)}.$$
Then:
$$\begin{aligned} \Phi_{t_0}\tilde{A}\Phi_{-t_0} &= \begin{pmatrix} -\gamma^2 + \alpha\cos(t_0) & \alpha\sin(t_0) & \alpha\gamma(1-\cos(t_0)) \\ -\alpha\sin(t_0) & \cos(t_0) & \gamma\sin(t_0) \\ \alpha\gamma(\cos(t_0)-1) & \gamma\sin(t_0) & \alpha^2 - \gamma\cos(t_0) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_0 \\ 0 & c_0 & 0 \end{pmatrix} \cdot \\ &\quad \cdot \begin{pmatrix} -\gamma^2 + \alpha\cos(t_0) & -\alpha\sin(t_0) & \alpha\gamma(1-\cos(t_0)) \\ \alpha\sin(t_0) & \cos(t_0) & -\gamma\sin(t_0) \\ \alpha\gamma(\cos(t_0)-1) & -\gamma\sin(t_0) & \alpha^2 - \gamma\cos(t_0) \end{pmatrix} \\ &= c_0 \begin{pmatrix} 0 & \alpha\gamma(1-\cos(t_0)) & \alpha\sin(t_0) \\ -\alpha\gamma(1-\cos(t_0)) & 0 & \alpha^2\cos(t_0) - \gamma^2 \\ \alpha\sin(t_0) & \alpha^2\cos(t_0) - \gamma^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ b & c & 0 \end{pmatrix}, \end{aligned}$$
using the values of $\sin(t_0), \alpha, \gamma$. Hence σ and $\tilde{\sigma}$ are congruent. □

Finally, we summarise these results in Theorem 4.4.24. The H_0^2 is the already known Riemannian case of [5].

4.4.24 Theorem. *The congruence classes of Killing fields on H_0^2 and S_2^2 are determined by the value of λ. In particular there is a unique, up to congruence, harmonic Killing field on each hyperquadric.*

Proof. Lemmas 4.4.21 to 4.4.23 show that all Killing fields with the same value of λ are congruent. Recall that σ is harmonic if and only if $\lambda = \epsilon$. Thus if $M = H_0^2$ then σ has a unique fixed point whereas if $M = S_2^2$ then σ has two fixed points at infinity and in both cases σ is uniquely determined up to congruence. □

We now conduct an analogous analysis of congruence classes of Killing fields on the the neutral hyperquadrics. Since these are anti-isometric it suffices to consider H_1^2. We study fixed points using the cylinder model of Section 4.2.

4.4.25 Theorem. *The fixed points of Killing fields on H_1^2 are categorised by $\lambda = a^2 + b^2 - c^2$. The Killing fields have:*

1. *no fixed points if $\lambda < 0$,*
2. *two ideal fixed points, one on each component of the boundary at infinity, if $\lambda = 0$,*
3. *two fixed points if $\lambda > 0$.*

Proof. We first map the Killing field to the cylinder model of H_1^2 given in Section 4.2 and then consider the fixed points there. The components of $\sigma(x)$ are

$$u = ay + bz, v = ax + cz, w = bx - cy.$$

The differential of the map $F\colon H_1^2 \to C^2$ is:

$$dF_{(x,y,z)}(u,v,w) = \left(\frac{u}{(1+x^2)^{3/2}}, \frac{-xyu + v(1+x^2)}{(1+x^2)^{3/2}}, \frac{-xzu - w(1+x^2)}{(1+x^2)^{3/2}} \right).$$

The inverse map is:

$$(x,y,z) = F^{-1}(\bar{x},\bar{y},\bar{z}) = \left(\frac{\bar{x}}{\sqrt{1-\bar{x}^2}}, \frac{\bar{y}}{\sqrt{1-\bar{x}^2}}, \frac{\bar{z}}{\sqrt{1-\bar{x}^2}} \right).$$

The projection $\bar{\sigma}$ of σ to the cylinder is therefore:

$$\begin{aligned}\bar{\sigma} &= dF_{(x,y,z)}(u,v,w)\\ &= \left(\frac{ay+bz}{(1+x^2)^{3/2}}, \frac{-xy(ay+bz) + (ax+cz)(1+x^2)}{(1+x^2)^{3/2}}, \frac{-xz(ay+bz) - (bx-cy)(1+x^2)}{(1+x^2)^{3/2}} \right)\\ &= ((a\bar{y}+b\bar{z})(1-\bar{x}^2), -\bar{x}\bar{y}(a\bar{y}+b\bar{z}) + (a\bar{x}+c\bar{z}), -\bar{x}\bar{z}(a\bar{y}+b\bar{z}) - (b\bar{x}-c\bar{y})).\end{aligned}$$

Note that $\bar{\sigma}$ extends smoothly across ∂C^2; i.e. when $\bar{x} = \pm 1$. Then $\bar{\sigma}(\bar{x},\bar{y},\bar{z}) = 0$ for $(\bar{x},\bar{y},\bar{z}) \in C^2 \cup \partial C^2$ if and only if the following simultaneous equations are satisfied:

$$\begin{aligned}0 &= (a\bar{y}+b\bar{z})(1-\bar{x}^2),\\ 0 &= -\bar{x}\bar{y}(a\bar{y}+b\bar{z}) + (a\bar{x}+c\bar{z}),\\ 0 &= -\bar{x}\bar{z}(a\bar{y}+b\bar{z}) - (b\bar{x}-c\bar{y}).\end{aligned}$$

The first of these gives two possibilities:

$$a\bar{y}+b\bar{z} = 0, \quad \bar{x} = \pm 1.$$

If $a\bar{y}+b\bar{z} = 0$ then, noting that on C we have $\bar{y}^2+\bar{z}^2 = 1$:

$$\bar{y} = \pm\frac{b}{\sqrt{a^2+b^2}}, \quad \bar{z} = \mp\frac{a}{\sqrt{a^2+b^2}}.$$

The remaining simultaneous equations then give

$$\bar{x} = \pm\frac{c}{\sqrt{a^2+b^2}}.$$

If $\lambda = 0$ then $a^2+b^2 = c^2$ and there are two fixed points, namely $\pm(1, b/c, -a/c) \in \partial C$, one on each component of the boundary at infinity.

If $\lambda > 0$ then $a^2+b^2 > c^2$ and so $-1 < c/\sqrt{a^2+b^2} < 1$ and there are two fixed points at

$$\pm\left(\frac{c}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, \frac{-a}{\sqrt{a^2+b^2}}\right) \in C.$$

These correspond to:

$$\pm\left(\frac{c}{\sqrt{a^2+b^2-c^2}}, \frac{b}{\sqrt{a^2+b^2-c^2}}, \frac{-a}{\sqrt{a^2+b^2-c^2}}\right) \in H^2_1.$$

Finally if $\lambda < 0$ then $a^2 + b^2 < c^2$, hence $c/\sqrt{a^2+b^2} > 1$, therefore there are no fixed points in $C \cup \partial C$ and hence none in H^2_1.

If $\bar{x} = \pm 1$ then we recover fixed points at $\pm(1, b/c, -a/c) \in \partial C$ as before. □

We can then use Theorem 4.4.25 to find a normal form for each type of Killing field and show that any other Killing field of the same type is congruent to it.

4.4.26 Lemma. *Let $\sigma \in \Gamma(TH^2_1)$ be a Killing field on H^2_1 with generating skew-symmetric matrix A. Suppose $\lambda < 0$, where $\lambda = a^2 + b^2 - c^2$. Then σ is congruent to $\tilde{\sigma}$, the Killing field with generating skew-symmetric matrix:*

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_0 \\ 0 & -c_0 & 0 \end{pmatrix},$$

where $c_0^2 = -\lambda$.

Proof. Consider infinitesimal isometry:

$$\begin{pmatrix} 0 & \alpha & \gamma \\ \alpha & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}$$

where $\alpha = b/\sqrt{a^2+b^2}$ and $\gamma = -a/\sqrt{a^2+b^2}$. Then $\alpha^2 + \gamma^2 = 1$. This has flows matrix:

$$\Phi_t = \begin{pmatrix} \cosh(t) & \gamma\sinh(t) & \alpha\sinh(t) \\ \gamma\sinh(t) & \alpha^2 - \gamma^2\cosh(t) & -\alpha\gamma(1-\cosh(t)) \\ \alpha\sinh(t) & -\alpha\gamma(1-\cosh(t)) & \gamma^2 + \alpha^2\cosh(t) \end{pmatrix}.$$

The desired congruence is then Φ_{t_0} for $\cosh(t_0) = -c/c_0$. For:

$$\begin{aligned} c_0^2\Phi_{-t_0}A\Phi_{t_0} &= \begin{pmatrix} -c & a & -b \\ a & \frac{b^2c_0-a^2c}{a^2+b^2} & \frac{ab(c_0-c)}{a^2+b^2} \\ -b & \frac{ab(c_0-c)}{a^2+b^2} & \frac{a^2c_0+b^2c}{a^2+b^2} \end{pmatrix} \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} \begin{pmatrix} -c & -a & b \\ -a & \frac{b^2c_0-a^2c}{a^2+b^2} & \frac{ab(c_0-c)}{a^2+b^2} \\ b & \frac{ab(c_0-c)}{a^2+b^2} & \frac{a^2c_0+b^2c}{a^2+b^2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_0^3 \\ 0 & -c_0^3 & 0 \end{pmatrix}. \end{aligned}$$

□

4.4.27 Lemma. *Suppose σ and A are as defined in Lemma 4.4.26, and $\lambda = 0$. Then σ is congruent to $\tilde{\sigma}$, the Killing field with generating skew-symmetric matrix:*

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Proof. Consider the infinitesimal isometry:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix}.$$

This has flow matrix:

$$\Phi_t = \gamma \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix}.$$

The desired congruence is then Φ_{t_0} for $\gamma^2 = 1/c, \cos(t_0) = \frac{ac}{a^2+b^2}, \sin(t_0) = \frac{-bc}{a^2+b^2}$. For:

$$\begin{aligned}
\Phi_{-t_0} A \Phi_{t_0} &= \gamma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t_0) & -\sin(t_0) \\ 0 & \sin(t_0) & \cos(t_0) \end{pmatrix} \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t_0) & \sin(t_0) \\ 0 & -\sin(t_0) & \cos(t_0) \end{pmatrix} \\
&= \frac{1}{c(a^2+b^2)} \begin{pmatrix} a^2+b^2 & 0 & 0 \\ 0 & ac & bc \\ 0 & -bc & ac \end{pmatrix} \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} \begin{pmatrix} a^2+b^2 & 0 & 0 \\ 0 & ac & -bc \\ 0 & bc & ac \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\end{aligned}$$

□

4.4.28 Lemma. *Suppose σ and A are as defined in Lemma 4.4.26, and $\lambda > 0$. Then σ is congruent to $\tilde{\sigma}$, the Killing field with generating skew-symmetric matrix:*

$$\tilde{A} = \begin{pmatrix} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $a_0^2 = \lambda$.

Proof. Consider

$$\begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & -\gamma \\ 0 & -\gamma & 0 \end{pmatrix},$$

where

$$\alpha = \frac{c}{\sqrt{(a-a_0)^2 - c^2}}, \quad \gamma = \frac{(a-a_0)}{\sqrt{(a-a_0)^2 - c^2}}.$$

Then $\alpha^2 - \gamma^2 = -1$. This has flow:

$$\Phi_t = \begin{pmatrix} \mu^2 - \alpha^2\cos(t) & \alpha\sin(t) & \alpha\mu(1-\cos(t)) \\ \alpha\sin(t) & \cos(t) & \mu\sin(t) \\ \alpha\mu(\cos(t)-1) & -\mu\sin(t) & \mu^2\cos(t) - \alpha^2 \end{pmatrix}.$$

Then the desired congruence is Φ_{t_0}, where t_0 is the unique parameter such that $\phi_{t_0}(\rho^\pm) = \tilde{\rho}^\pm$, for $\rho^\pm$ the zeros of σ and $\tilde{\rho}^\pm$ the zeros of $\tilde{\sigma}$:

$$\sin(t_0) = \frac{b\sqrt{(a-a_0)^2 - c^2}}{a_0(a-a_0)}.$$

Then:

$$\begin{aligned}
\Phi_{t_0} A \Phi_{-t_0} &= \begin{pmatrix} \mu^2 - \alpha^2\cos(t_0) & \alpha\sin(t_0) & \alpha\mu(1-\cos(t_0)) \\ \alpha\sin(t_0) & \cos(t_0) & \mu\sin(t_0) \\ \alpha\mu(\cos(t_0)-1) & -\mu\sin(t_0) & \mu^2\cos(t_0) - \alpha^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & a_0 & 0 \\ a_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \\
&\cdot \begin{pmatrix} \mu^2 - \alpha^2\cos(t_0) & -\alpha\sin(t_0) & \alpha\mu(1-\cos(t_0)) \\ -\alpha\sin(t_0) & \cos(t_0) & -\mu\sin(t_0) \\ \alpha\mu(\cos(t_0)-1) & \mu\sin(t_0) & \mu^2\cos(t_0) - \alpha^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -a_0\alpha^2 + a_0\mu^2\cos(t_0) & -\mu a_0\sin(t_0) \\ -a_0\alpha^2 + a_o\mu^2\cos(t) & 0 & -\alpha\mu a_0 + \alpha\mu a_0\cos(t_0) \\ -\mu a_0\sin(t_0) & \alpha\mu a_0 - \alpha\mu a_0\cos(t_0) & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix},
\end{aligned}$$

using the values of $\sin(t_0), \alpha, \gamma$. Hence σ and $\tilde{\sigma}$ are congruent. □

These can be combined into the following result:
4.4.29 Theorem. *The congruence classes of Killing field on H^2_1 and S^2_1 are determine by λ. Therefore there is a unique, up to congruence, harmonic Killing field on each hyperquadric.*

Proof. Lemmas 4.4.26 to 4.4.28 show that all Killing fields with the same value of λ are congruent. Since the harmonic Killing fields are determined by $\lambda = \epsilon$ they are unique up to congruence. □

4.5 Para-Kähler geometry

We now consider a generalisation of Kähler geometry to pseudo-Riemannian manifolds. The introduction to this theory is based on the survey article [9], which also covers other cases of para-complex geometry. We go on to use the theory to provide a link between conformal and Killing fields, and then harmonic vector fields on neutral hyperquadrics. This is similar to the work in [5] linking conformal gradient fields to Killing fields through the Kähler structure of the hyperbolic plane.

We begin by formally defining a term that has already appeared in our discussion of 2-dimensional hyperquadrics.
4.5.1 Definition. A *neutral pseudo-Riemannian manifold* is one with dimension $2n$ and index n. ◊

Next we define the class of para-complex structures that are particularly relevant to pseudo-Riemannian geometry.

4.5.2 Definition. An *almost para-Hermitian (APH) structure* J on a pseudo-Riemannian manifold (M, g) is a $(1, 1)$ tensor field on M satisfying:

- APH1 $J^2(X) = X$ for all $X \in TM$,
- APH2 $g(JX, Y) = -g(X, JY)$ for all $X, Y \in T_xM$, all $x \in M$.

◊

4.5.3 Lemma. *Let (M, g) be a pseudo-Riemannian manifold with APH structure J. Then J interchanges time-like and space-like vectors.*

Proof. If $X \in TM$ then using (APH2):

$$g(JX, JX) = -g(X, J^2(X)),$$

and using (APH1):

$$-g(X, J^2(X)) = -g(X, X),$$

□

4.5.4 Corollary. *A pseudo-Riemannian manifold with almost para-Hermitian structure is neutral.*

This limits the number of possible para-Hermitian manifolds considerably, especially amongst the hyperquadrics. In each even dimension it leaves us with one anti-isometric pair.

4.5.5 Proposition. *Let (M, g) be a pseudo-Riemannian manifold with APH structure J. Then J satisfies*

$$\nabla_X J(JY) = -J\nabla_X J(Y),$$

and

$$g(\nabla_X J(Y), Z) = -g(Y, \nabla_X J(Z)),$$

for all $X, Y, Z \in T_xM$

Proof. First consider the derivative of the identity $J^2 = 1$:

$$0 = (\nabla_X J) \circ J + J \circ \nabla_X J,$$

which is the first relation. Expand $g(\nabla_X J(Y), Z)$ using the skew-symmetry of J and the metric property of covariant derivation:

$$\begin{aligned} g(\nabla_X J(Y), Z) &= g(\nabla_X (JY) - J(\nabla_X Y), Z) \\ &= X.g(JY, Z) - g(JY, \nabla_X Z) + g(\nabla_X Y, JZ) \\ &= -X.g(Y, JZ) + g(Y, J(\nabla_X Z)) + g(\nabla_X Y, JZ) \\ &= -g(\nabla_X Y, JZ) - g(Y, \nabla_X JZ) + g(Y, J(\nabla_X Z)) + g(\nabla_X Y, JZ) \\ &= -g(Y, \nabla_X JZ - J(\nabla_X Z)) \\ &= -g(Y, \nabla_X J(Z)). \end{aligned}$$

□

We now define a para-Kähler structure, a sub class of almost para-Hermitian structures.

4.5.6 Definition. A *para-Kähler* structure J on pseudo-Riemannian manifold (M, g) is an almost para-Hermitian structure on M satisfying

$$\nabla J = 0.$$

◊

We prove that for a neutral 2-dimensional pseudo-Riemannian manifold such a structure can always be defined.

4.5.7 Definition. Let (M, g) be a 2-dimensional neutral orientable pseudo-Riemannian manifold. Then there exists *globally* defined line subbundles $L_1, L_2 \subset TM$ with each L_i light-like and $L_1 \cap L_2 = 0$. Label these such that if $\{A, B\}$ is a positively oriented basis of T_xM with $A \in L_1, B \in L_2$ then $A + B$ is space-like and $A - B$ is time-like. Then define J as

$$JA = A, \quad JB = -B. \qquad \diamondsuit$$

4.5.8 Proposition. *Let (M, g) be a dimension 2 neutral orientable pseudo-Riemannian manifold. Then the structure J defined in Definition 4.5.7 is a para-Kähler structure on M.*

Proof. The structure J satisfies (APH1). To see that it satisfies (APH2):

$$\begin{aligned}
&g(JA, A) = g(A, A) = 0,\\
&g(JA, B) = g(A, B) = -g(A, -B) = -g(A, JB),\\
&g(JB, B) = -g(B, B) = 0.
\end{aligned}$$

Therefore J is almost para-Hermitian. Let $\{S, T\}$ be an orthonormal basis, where S is space-like and T is time-like. Because J interchanges space-like and time-like vectors (Lemma 4.5.3) we may assume $T = JS$. It is sufficient to consider the components of ∇J with respect to such a basis. Using the relations of Proposition 4.5.5, we first consider $\nabla_S J(S)$:

$$\begin{aligned}
g(\nabla_S J(S), S) &= -g(S, \nabla_S J(S)) = 0,\\
g(\nabla_S J(S), T) &= -g(S, \nabla_S J(JS))\\
&= g(S, J\nabla_S J(S))\\
&= -g(T, \nabla_S J(S)).
\end{aligned}$$

So $\nabla_S J(S) = 0$. Next consider $\nabla_S J(T)$:

$$\begin{aligned}
g(\nabla_S J(T), S) &= -g(T, \nabla_S J(S)) = 0,\\
g(\nabla_S J(T), T) &= -g(T, \nabla_S J(T)) = 0.
\end{aligned}$$

Therefore $\nabla_S J = 0$. Similarly:

$$\begin{aligned}
&g(\nabla_T J(S), S) = -g(S, \nabla_T J(S)) = 0,\\
&g(\nabla_T J(S), T) = -g(S, \nabla_T J(JS)) = g(S, J\nabla_T(JS)) = -g(T, \nabla_T(JS)) = 0,
\end{aligned}$$

so $\nabla_T J(S) = 0$. Finally consider $\nabla_T J(T)$:

$$\begin{aligned}
g(\nabla_T J(T), S) &= -g(T, \nabla_T J(S)) = 0,\\
g(\nabla_T J(T), T) &= -g(T, \nabla_T J(T)) = 0.
\end{aligned}$$

Therefore $\nabla_T J = 0$ and so $\nabla J = 0$. $\square$

4.5.1 Application to harmonic fields

We now apply the theory of para-Kähler geometry to harmonic vector fields. In particular we consider how the para-Kähler structure acts on closed conformal fields and Killing fields. We then produce a map under which the Euler-Lagrange equations of a harmonic vector field are invariant, by combining the para-Kähler structure and the anti-isometry of hyperquadrics.

4.5.9 Lemma. *Let (M, g) be a neutral pseudo-Riemannian manifold with para-Kähler structure J. Let σ be a closed conformal field on M. Then $J\sigma$ is a Killing field.*

Proof. The vector field σ is closed conformal, so has the property, for all $X \in TM$:

$$\nabla_X \sigma = \psi_\sigma X,$$

by Theorem 3.1.5. A Killing field τ satisfies:

$$g(\nabla_X \tau, Y) + g(X, \nabla_Y \tau) = 0,$$

for all $X, Y \in TM$. Consider:

$$g(\nabla_X(J\sigma), Y) + g(X, \nabla_Y(J\sigma)) = g(J\nabla_X \sigma, Y) + g(X, J\nabla_Y \sigma),$$

since J is para-Kähler

$$\begin{aligned} &= g(J(\psi_\sigma X), Y) + g(X, J(\psi_\sigma Y)) \\ &= -\psi_\sigma g(X, JY) + \psi_\sigma g(X, JY) = 0, \end{aligned}$$

hence $J\sigma$ is Killing. □

4.5.10 Remark. In general the para-Kähler structure does not map Killing fields to closed conformal fields. This can be seen by comparing the dimension of the Lie algebra of Killing with the vector space of closed conformal fields.

For example on space forms the Lie algebra of Killing fields has dimension $n(n+1)/2$, given in [18]. However since the pole vector of a conformal gradient field may be arbitrarily chosen the space of conformal gradient fields on space forms has dimension $n + 1$. These two dimensions coincide only when $n = 2$.

In fact in two dimensions the para-Kähler structure acting on a Killing field *does* produce a closed conformal field.

4.5.11 Lemma. *Let (M, g, J) be a dimension 2 neutral para-Kähler manifold. Let τ be a Killing field on M. Then $J\tau$ is a closed conformal field.*

Proof. It suffices to show that J maps closed conformal fields injectively into Killing fields. If σ is a closed conformal field such that $J\sigma = 0$ then, since σ is either time-like or space-like by Proposition 4.3.4 and J swaps time-like and space-like vectors, it follows that $\sigma = 0$. □

The culmination of this is the invariance of the Euler-Lagrange equations under the combination of para-Kähler structure and anti-isometry.

4.5.12 Lemma. *Let (M, g) be a neutral pseudo-Riemannian manifold with para-Kähler structure J and let $\varphi\colon (M, g) \to (N, h)$ be an anti-isometry. Then the Euler-Lagrange equations for harmonic vector fields,* (2.8)*, are invariant under the map $\sigma \mapsto \varphi.(J\sigma)$.*

Proof. Since J is para-Kähler:

$$\nabla_X(J\sigma) = J\nabla_X\sigma, \quad g(J\sigma, J\sigma) = -g(\sigma, \sigma).$$

Let $\tilde{\sigma} = \varphi.(J\sigma)$. Note that:

$$\begin{aligned} h(\tilde{\sigma}, \tilde{\sigma}) &= h(\varphi.(J\sigma), \varphi.(J\sigma)) \\ &= -g(J\sigma, J\sigma) \\ &= g(\sigma, \sigma). \end{aligned}$$

Now by Proposition 4.1.12:

$$\begin{aligned} \nabla^*\nabla\tilde{\sigma} &= \varphi.(J\nabla^*\nabla\sigma), \\ \nabla_{\nabla\tilde{F}}\tilde{\sigma} &= \varphi.(J\nabla_{\nabla F}\sigma), \\ \langle\nabla\tilde{F}, \nabla\tilde{F}\rangle &= \langle\varphi.(J\nabla F), \varphi.(J\nabla F)\rangle = \langle\nabla F, \nabla F\rangle, \\ \langle\nabla\tilde{\sigma}, \nabla\tilde{\sigma}\rangle &= \langle\varphi.(J\nabla\sigma), \varphi.(J\nabla\sigma)\rangle = \langle\nabla\sigma, \nabla\sigma\rangle, \\ \Delta\tilde{F} &= \Delta F \\ 1 + \langle\tilde{\sigma}, \tilde{\sigma}\rangle &= 1 + \langle\sigma, \sigma\rangle. \end{aligned}$$

Thus the components of the harmonic equations under this transformation are:

$$\begin{aligned} T_p(\tilde{\sigma}) &= \varphi.(J(T_p(\sigma)), \\ \phi_{p,q}(\tilde{\sigma}) &= \phi_{p,q}(\sigma), \end{aligned}$$

and thus

$$T_p(\tilde{\sigma}) - \phi_{p,q}(\tilde{\sigma})\tilde{\sigma} = \varphi.(J(T_p(\sigma) - \phi_{p,q}(\sigma)\sigma)).$$

Then $\tilde{\sigma}$ is (p, q)-harmonic if and only if σ is (p, q)-harmonic. □

From this we can conclude that there are (p, q)-harmonic Killing fields on any even dimensional neutral hyperquadric on which there exists a para-Kähler structure; namely the images of (p, q)-harmonic closed conformal fields under the aforementioned correspondence.

In particular, the harmonic closed conformal and Killing fields found on H_1^2 and S_1^2 are related in this way.

4.5.13 Remark. There is no way to recreate the generation of a family of harmonic conformal vector fields using para-Kähler geometry on neutral space forms, as [3] did with Kähler geometry on the hyperbolic plane. This is due to the difficulty in combining space-like and time-like vector fields into a family, while maintaining the Euler-Lagrange equations. The problem, once again, is the metric term, $1 + \langle\sigma, \sigma\rangle$. To preserve this term implies that all members of the family have the same pseudo-Riemannian length, which cannot be the case.

Further Work

There have been a number of questions left unanswered as a result of the work in this thesis. The area of (p, q)-harmonic sections is a very new one, but the generalisations to pseudo-Riemannian spaces we have given open up even more avenues of exploration.

The first notable open conjecture is that there are no (p, q)-harmonic vector fields on the Riemannian 2-sphere. This could perhaps be approached analogously to the Hopf differential of minimal surface theory to translate the problem into one of holomorphic geometry.

The geometry of the pseudo-Riemannian generalised Cheeger-Gromoll metric should be explored, in the vein of [4]. Whilst we have described its signature in Chapter 2 we have not explored the geometry any further. This may provide insight into the differences arising in the pseudo-Riemannian case.

The second variation of local energy for (p, q)-harmonic sections could be looked at. As yet there is no result even for the Riemannian case, so both could be considered in one. This is an exercise in calculus of variations that will tax the resolve of the most ardent computational differential geometer.

The generalisation of the results on warped products to time-like intervals as well as space-like intervals is a small amendment to the work in this thesis. With an additional term to indicate the casual type of the interval this generalisation would modify the Euler-Lagrange equations found for the natural closed conformal vector field on a warped product. This would provide additional solutions for a given warping function, when the interval is time-like. It would also allow more manifolds to be represented as warped products. For example all hyperquadrics can be represent as warped products, if we allow a time-like interval, and the well-known Robertson-Walker space-time is such an example.

Appendix A

General Analysis of the Warping Function Ordinary Differential Equation

The ODE given by Theorem 3.3.2 is:

$$(g')^2[p(n-2)+nq+q(n-p)g^2]g+g''(1+g^2)(1+qg^2)=0. \tag{A.1}$$

Although this is non-linear second order it can be solved using standard techniques. Define $v(g)=g'$, and rewrite (A.1) as the following system of first order equations:

$$\begin{aligned} v(g) &= g' \\ \frac{dv}{dg} &= vR(g), \end{aligned} \tag{A.2}$$

where $R(g)$ is the rational function:

$$R(g)=-\frac{[p(n-2)+nq+q(n-p)g^2]g}{(1+g^2)(1+qg^2)}.$$

Then (A.2) has solution:

$$\left|\frac{dg}{dt}\right|=\exp\int R(g)dg. \tag{A.3}$$

We evaluate the integral using partial fractions [1].

If $q=0$ then:

$$R(g)=\frac{p(2-n)g}{1+g^2},$$

and

$$\int R(g)dg=\frac{p(2-n)}{2}\ln|1+g^2|+C.$$

If $q=1$ then:

$$R(g)=-\frac{[p(n-2)+n+(n-p)g^2]g}{(1+g^2)^2}=-\frac{(n-p)g}{1+g^2}-\frac{p(n-1)g}{(1+g^2)^2},$$

and

$$\int R(g)dg = \frac{p-n}{2}\ln|1+g^2| + \frac{p(n-1)}{2}\frac{1}{1+g^2} + C.$$

If $q > 0$ and $q \neq 1$ then:

$$\begin{aligned} R(g) &= -\frac{[p(n-2)+nq+q(n-p)g^2]g}{(1+g^2)(1+qg^2)} \\ &= \frac{p(n-2+q)}{q-1}\frac{g}{1+g^2} + \frac{q(n+p-np-nq)}{q-1}\frac{g}{1+qg^2}, \end{aligned}$$

and

$$\int R(g)dg = \frac{p(n-2+q)}{2(q-1)}\ln|1+g^2| + \frac{n+p-np-nq}{2(q-1)}\ln|1+qg^2| + C.$$

If $q < 0$ then:

$$\begin{aligned} R(g) &= -\frac{[p(n-2)+nq+q(n-p)g^2]g}{(1+g^2)(1+\sqrt{|q|}g)(1-\sqrt{|q|}g)} \\ &= \frac{p(n-2+q)}{q-1}\frac{g}{1+g^2} + \frac{q(n+p-np-nq)}{2(q-1)}\left[\frac{g}{1+\sqrt{|q|}g} + \frac{g}{1-\sqrt{|q|}g}\right], \end{aligned}$$

and

$$\begin{aligned} \int R(g)dg &= \frac{p(n-2+q)}{2(q-1)}\ln|1+g^2| \\ &\quad + \frac{q(n+p-np-nq)}{2|q|(1-q)}[\ln|1+\sqrt{|q|}g| + \ln|1-\sqrt{|q|}g|] + C \\ &= \frac{p(n-2+q)}{2(q-1)}\ln|1+g^2| + \frac{n+p-np-nq}{2(q-1)}\ln|1+qg^2| + C. \end{aligned}$$

The general form of differential equation (A.3), for suitable α and β, is then:

$$\left|\frac{dg}{dt}\right| = C|1+g^2|^\alpha|1+qg^2|^\beta, \tag{A.4}$$

if $q \neq 1$, or

$$\left|\frac{dg}{dt}\right| = C|1+g^2|^\alpha \exp\left(\frac{\beta}{1+g^2}\right), \tag{A.5}$$

if $q = 1$.

For any triple p, q, n local solutions exist; however these may not extend to a global solution on the interval.

We briefly indicate how the solutions of Section 3.5 may be obtained.

The solution is linear if $\alpha = \beta = 0$, in which case:

$$\left|\frac{dg}{dt}\right| = C.$$

If $q = 0$ then:

$$\alpha = \frac{p(2-n)}{2}, \quad \beta = 0,$$

hence either $p = 0$ or $n = 2$ (c.f. Theorem 3.5.8). Similarly if $q \neq 0, 1$ then:

$$\alpha = \frac{p(n-2+q)}{2(q-1)}, \quad \beta = \frac{n+p-np-nq}{2(q-1)}, \tag{A.6}$$

hence $p = n, q = 2-n$ (c.f. Theorem 3.5.8). If $q = 1$ then $\beta = 0$ only if $n = 1$, which is impossible.

In general if $q \neq 0, 1$ then (A.4) may be written:

$$\left|\frac{dg}{dt}\right| = C|1+g^2|^\alpha |1+qg^2|^\beta,$$

where α, β are given by (A.6). It follows that $\alpha = \beta = 1/2$ if and only if

$$p = n+2, \quad q = \frac{3-n^2}{n+1}, \tag{A.7}$$

in which case g is the inverse of an elliptic integral. This explains the appearance of the Jacobi elliptic functions as solutions in Section 3.5 when the parameters are given by (A.7).

Similar analyses are possible for trigonometric and hyperbolic functions by taking one of α, β to be zero.

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