Omega-categorical Simple Theories

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Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

The University of Leeds

School of Mathematics

November 2013

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Acknowledgements

I thank all the people and institutions who have been at one point or another involved in the development of this work. Special thanks go to Dugald, without whose generosity, gentle guidance, and expertise this would never have been completed; and to my parents, for all the obvious reasons.

I also thank all those who, although not directly involved in the results presented here, have contributed to my mental health and general enjoyment of life: David Bradley-Williams, Pedro Valencia, Ahmet Çevik, and Cemre Ceren Asarlı. And the staff at North Bar.

The work in this thesis was funded by EPSRC (grant EP/H00677X/1) and Conacyt (scholarship 309151).

A mis viejos

There is a time when a thing in the mind is a heavy thing to carry, and then it must be put down. But such is its nature that it cannot be set on a rock or shouldered off on to the fork of a tree, like a heavy pack. There is only one thing shaped to receive it, and that is another human mind. There is only one time when it can be done, and that is in a shared solitude.

(Theodore Sturgeon, *Scars* [41])

Abstract

This thesis touches on many different aspects of homogeneous relational structures. We start with an introductory chapter in which we present all the background from model theory and homogeneity necessary to understand the results in the main chapters.

The second chapter is a list of examples. We present examples of binary and ternary homogeneous relational stuctures, and prove the simplicity or non-simplicity of their theory. Many of these examples are well-known structures (the ordered rational numbers, random graphs and hypergraphs, the homogeneous K_n-free graphs), while others were constructed during the first stages of research. In the same chapter, we present some combinatorial results, including a proof of the TP2 in the Fraïssé limit of semifree amalgamation classes in the language of n-graphs, such that all the minimal forbidden configurations of the class of size at least 3 are all triangles.

The third chapter contains the main results of this thesis. We prove that supersimple finitely homogeneous binary relational structures cannot have infinite monomial SU-rank, show that primitive binary supersimple homogeneous structures of rank 1 are "random" in the sense that all their minimal forbidden configurations are of size at most 2, and partially classify the supersimple 3-graphs under the assumption of stable forking in the theories of finitely homogeneous structures with supersimple theory.

The fourth chapter is a proof of the directed-graph version of a well-known result by Erdős, Kleitman and Rothschild. Erdős *et al.* prove that almost all finite labelled triangle-free simple graphs are bipartite, and we prove that almost all finite labelled directed graphs in which any three distinct vertices span at least one directed arc consist of two disjoint tournaments, possibly with some directed arcs from one to the other.

Contents

List of Tables

§1. Introduction

> "Mike, applications of the compactness theorem are a dime a dozen. Go do something better."
> Saunders MacLane to Michael Morley [32].

1.1 Basic Model Theory

The work in this thesis is on homogeneous structures, an area where Group Theory, Graph Theory, Combinatorics, and Model Theory converge. Throughout, model-theoretic language and conventions are preferred, and so we will often talk of types, theories, ranks, forking, etc. From permutation group theory, we adopt the terms *transitive* and *primitive*. We say that the action of a group G on a structure M is transitive if for all $x, y \in M$ there exists $g \in G$ such that $x^g = y$; the group G acts primitively on M if the only equivalence relations left invariant by the action of G are the trivial equivalence relation (with equivalence classes of size 1) and the universal equivalence relation. In this section, we set up the basic language and present some general results that will be used in later chapters. This introductory chapter is largely based on [36], [34], [5], and [46].

Model Theory deals with the structures that satisfy a collection of sentences or axioms. It studies the semantics of the axiom system in a particular logic, most often first-order classical logic. An unintended consequence of this is a very relaxed attitude towards the distinctions between a formula $\varphi(x, \bar{a})$ and the set of its solutions in a particular model, $\{b \in M : M \models \varphi(b, \bar{a})\}$, and other related issues. This may be confusing, and we mention it here to alert the reader.

A language is a set of symbols, which come in three flavours: there are constant symbols, relational symbols, and function symbols. We will work exclusively in relational languages, that is, languages without function symbols (we do allow constant symbols, even though they are often identified with functions of arity 0). Formulas are well-formed strings of symbols from the language, and a sentence is a formula for which all variables are quantified.

A theory is simply a consistent set of sentences in a language L. We say that the theory is complete if it is maximal in the partial ordering of theories in the language by inclusion.

A structure M for the language L is a set M together with interpretations for each of the symbols of L. These interpretations are actual elements or tuples (for the constant symbols), subsets of various Cartesian products M^n (relations), and functions of the appropriate arity. A sentence σ is true in (or modeled by) a structure M if, after interpreting all the symbols from L present in σ in the structure M what we get is a true statement about M. This relation is expressed by $M \models \sigma$. Given a theory T in the language L, a model for T is a structure M for L such that for each sentence $\sigma \in T$ we have $M \models \sigma$. The work in this thesis fits very well in Fraïssé's Theory of Relations, and from that point of view, the concept of local isomorphism (bijections between finite subsets preserving all relations) is as important as that of a formula—in fact, it is possible to make a coherent exposition of Model Theory without refering to formulas, basing everything on local isomorphisms and the back-and-forth method, as Poizat did in his *Course* [36].

The most basic tool in first-order Model Theory is the Compactness Theorem; some even go as far as saying that the purpose of Model Theory is to make efficient use of it. There are two popular ways of proving this Theorem: as a corollary to Gödel's Completeness Theorem for first-order logic, and a more topological method using Łoś's ultraproduct construction. Of these two methods, the first has the advantage of giving a one-line proof of the Compactness Theorem, but sweeps a number of important facts under the carpet (it doesn't even illustrate where the name *compactness* comes from), and rests on a syntactic definition of proof. The ultrafilter proof requires only some basic knowledge of topology and gives some insights into the space of types of the theory.

Recall that a filter on an algebra of sets $\mathcal{A}$ is a set $U \subset \mathcal{P}(\mathcal{A})$ not containing $\varnothing$ such that if $u, u' \in U$ then $u \cap u' \in U$, and if $u \in U, v \in \mathcal{P}(\mathcal{A})$, and $u \subseteq v$, then $v \in U$. An ultrafilter is a maximal filter. It follows from Stone's Representation Theorem that any Boolean algebra is isomorphic to some algebra of sets, so we can transfer the concept of filter to Boolean algebras. In our case, the Boolean algebra to have in mind is the Tarski-Lindenbaum algebra of the language L. The ultrafilters of this algebra are the complete theories in the language L; if we add constant symbols $x, y, \ldots$ to the language (we think of these as variables—this is a technicality: the Tarski-Lindenbaum algebra consists only of *sentences* in the language; therefore, any free variables in a formula would automatically take it out of the algebra), the ultrafilters from the algebras associated with the new languages are what we call (complete) *types* in variables $x, y, \ldots$ In other words, a type $p(x)$ in an ambient theory T is a maximally consistent set of formulas modulo T-equivalence; equivalently, it is a completion of the theory T to the language $L \cup \{x\}$, or a consistent set of formulas with at most the variables displayed free, such that for all formulas $\varphi(x)$ in the language, either $\varphi(x)$ or $\neg\varphi(x)$ belong to it. For more on this, see [34] or [36].

We will assume that any filter is contained in a maximal filter (Tarski's ultrafilter axiom). Now we introduce ultraproducts, which are a way of creating new structures from existing ones using an ultrafilter for organising purposes. Let A be a nonempty set and U an ultrafilter on the power set of A, and for each $a \in A$ let S_a be a structure with nonempty universe M_a. We start by describing the universe of the ultraproduct of the S_a by the ultrafilter U. Consider the relation $\sim_U$ on the product $\prod S_a$ which holds for tuples $(\ldots, b_a, \ldots)$ and $(\ldots, c_a, \ldots)$ if the set $\{a : b_a = c_a\}$ is in U. It is routine to check that this is an equivalence relation; the universe of the ultraproduct is $S = \prod S_a / \sim_U$.

Now we decide how to interpret the symbols of the language. For any symbol $q \in L$, let q_a be its interpretation in the structure S_a:

1. If c is a constant symbol, its interpretation in the ultraproduct is the equivalence class of the tuple $(\ldots, c_a, \ldots)$.

2. If f is an n-ary function symbol, then, given an n-tuple $\alpha_1, \ldots, \alpha_n \in S$, then choose representatives $b_1, \ldots b_n$ of $\alpha_1, \ldots, \alpha_n$, $b_1 = (\ldots, b_{1,a}, \ldots), \ldots, b_n =$

$(\dots, b_{n,a}, \dots)$ and define the value of $f(\alpha_i, \dots, \alpha_n)$ to be the equivalence class of the tuple $(\dots, f(b_{1,a}, \dots, b_{n,a}), \dots)$.

3. If R is an n-ary relation symbol and $\beta_1, \dots, \beta_n \in S$, then choose representatives $b_1, \dots, b_n$ and say that $(\beta_1, \dots, \beta_n)$ satisfies R in the ultraproduct if the set of indices a such that $(b_{1,a}, \dots, b_{n,a})$ satisfies R in S_a belongs to U.

Theorem 1.1.1 (Łoś's Theorem) *Let U be an ultrafilter of subsets of I, and let the structures S_i be indexed by I, all for the same language L. Let $\varphi(\bar{x})$ be a formula in L, and $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a tuple from $\prod S_i/\sim_U$; let $a_1, \dots, a_n$ be representatives in $\prod S_i$ of $\alpha_1, \dots, \alpha_n$. Then $\prod S_i/\sim_U$ satisfies $\varphi(\alpha_1, \dots, \alpha_n)$ if and only if $\{i : S_i \models \varphi(a_{1,i}, \dots, a_{n,i})\}$ belongs to U.*

The proof of Łoś's Theorem is by induction on the complexity of formulas, and is mostly routine.

Fix a language L, and consider the set $\mathcal{T}$ of all complete theories in L. Given a sentence σ in L, let $\langle\sigma\rangle$ be the set of all theories $T \in \mathcal{T}$ containing σ. We claim that the collection $\mathcal{B}$ of all sets of the form $\langle\sigma\rangle$ is a base of open sets for a topology. To see this, notice that if $T \in \langle\sigma\rangle \cap \langle\tau\rangle$, then $T \models \sigma \wedge \tau$, and therefore $T \in \langle\sigma \wedge \tau\rangle$, and that complete theories are nonempty by definition.

The elements of $\mathcal{B}$ are basic open sets of some topology on $\mathcal{T}$. They are also closed sets, as the complement in $\mathcal{T}$ of $\langle\sigma\rangle$ is clearly $\langle\neg\sigma\rangle$. Therefore, $\mathcal{T}$ with the topology generated by $\mathcal{B}$ is a totally disconnected space. And if two complete theories $T, T' \in \mathcal{T}$ differ, then there exists some sentence σ belonging to T but not to T'. By maximality, this means that $\sigma \in T$ and $\neg\sigma \in T'$; therefore, $T \in \langle\sigma\rangle$ and $T' \in \langle\neg\sigma\rangle$, and $\mathcal{B}$ generates a Hausdorff topology on $\mathcal{T}$.

A filter in a topological space X is a filter in the power set of X. A point $x \in X$ is a limit of a filter F if every neighbourhood of x belongs to F, and x is a cluster point of F if x belongs to the closure of every member of F. In Hausdorff spaces, filters converge to at most one point. A topological space is compact if and only if every filter has a cluster point (Theorem 3.1.24 of [13]).

Notice that to prove compactness with this definition, it suffices to prove that every ultrafilter has a cluster point. Suppose that every ultrafilter has a cluster point, and let F be a filter on X. By Tarski's axiom, there exists an ultrafilter U extending F, and by our hypothesis U has a cluster point x. The point x belongs to the closure of every set in U, and in particular to the closure of every set in F. Therefore, x is a cluster point of F.

Theorem 1.1.2 (Compactness) *The space $\mathcal{T}$ of complete theories for a language L with the topology generated by $\mathcal{B} = \{\langle\sigma\rangle : \sigma \textit{ is a sentence in } L\}$ is compact and totally disconnected. Equivalently, a set of sentences Σ in L is consistent if and only if any finite subset of it is consistent.*

Proof
We have seen that $\mathcal{T}$ is Hausdorff and totally disconnected. Now we prove that every ultrafilter on $\mathcal{T}$ converges to some theory. For each theory $T \in \mathcal{T}$, let $M_T \models T$. Let U be an ultrafilter on $\mathcal{T}$. We claim that U converges to the theory θ of $\prod M_T/\sim_U$. Any neighbourhood A of θ contains some $\langle\sigma\rangle$ with $\theta \models \sigma$. By Łoś's Theorem, the set $\{T : M_T \models \sigma\} = \langle\sigma\rangle \in U$, and therefore $A \in U$.

For the equivalence, suppose first that $\mathcal{T}$ is compact. If any finite subset of Σ is consistent, then the sets $\langle\bigwedge_{i\in f}\sigma_i\rangle$ for $\sigma \in \Sigma$ and f a finite subset of a set indexing Σ form a net in $\mathcal{T}$, which by compactness converges to a theory T_0 satisfying all of Σ. If Σ is consistent, clearly all its finite subsets are consistent.

Now suppose that Σ is consistent whenever its finite subsets are consistent. This is equivalent to saying that every net in the totally disconnected space $\mathcal{T}$ converges. Therefore, $\mathcal{T}$ is compact. □

Now let us consider the space of completions of a theory T in the language L to the language $L \cup \{x_i : i \in I\}$. As we have remarked before, this is the space of types in the variables x_i. We can define a compact totally disconnected Hausdorff topology on this space just as we did for $\mathcal{T}$, using sets of the form $\langle\varphi(\bar{x})\rangle$ as basic open sets. This space can also be constructed as the space of ultrafilters on the Tarski-Lindenbaum algebra, a

Stone space denoted by $S_I(T)$ ($S_n(T)$ for finite indexing sets). So far, we have defined types without parameters; if we add constants to the language from a set A contained in some model of T and consider the completions of T in the language $L \cup \{c_a : a \in A\}$ then we get types over the set of parameters A.

In a topological space X, a point x_0 is isolated if the set $\{x_0\}$ is open. This condition translates in the case of type spaces to the existence of a formula $\varphi(\bar{x})$ consistent with T such that the type $p(\bar{x})$ consists of all the formulas with free variables from $\bar{x}$ implied by $T \cup \{\varphi(\bar{x})\}$, in symbols, $p(\bar{x}) = \{\psi(\bar{x}) : T \models \varphi(\bar{x}) \to \psi(\bar{x})\}$.

All isolated completions of T are realised in any model of a complete theory T: suppose that $p(\bar{x}) \in S(T)$ is isolated by $\varphi(\bar{x})$. Then the sentence $\exists \bar{x} \varphi(\bar{x})$ is in T and any model of T will have tuples realising φ, and therefore realising p. But non-isolated types can be omitted in models of T.

Theorem 1.1.3 (Omitting Types Theorem) *Let L be a countable language, T an L-theory and p a non-isolated n-type over $\varnothing$. Then there is a countable model of T omitting p.*

More generally, a "small" (meagre; compact Hausdorff spaces have the Baire property) set of non-isolated types from each $S_n(T)$ can be omitted.

Given a cardinal number κ, a complete L-theory T is said to be κ-categorical if all its models of size κ are isomorphic. All the theories in this thesis are ω-categorical. In an ω-categorical theory, the (unique up to isomorphism) countable model M embeds elementarily into every model of T (so M is a prime model of T) by the Löwenheim-Skolem Theorem. Suppose that T has a prime model M and $e : M \to N$ is an elementary embedding. If $\bar{a} \in M$ satisfies a type p, then $e(\bar{a})$ satisfies the same type p; if a type p realised in M were nonisolated, then we could find a model M' of the same cardinality $|M|$ into which M does not embed elementarily. Therefore, a prime model realises only the isolated types of the theory; if the language is countable, the converse can be proved by a back-and-forth argument. This gives us some more information about the topology of the type spaces of T:

Theorem 1.1.4 *A complete countable theory T has a prime model if and only if the set of isolated types in $S_n(T)$ is dense for each $n \in \omega$.*

Proof

Suppose that T has a prime model. We argued before that in this model the type of any $\bar{a} \in M$ is isolated. Let $\varphi(x)$ be a formula consistent with T and M a prime model for T. Then $\langle \varphi(x) \rangle$ is nonempty and $T \models \exists x \varphi(x)$. Find a tuple a in M satisfying φ. The type of a is isolated and is contained in $\langle \varphi(x) \rangle$. Therefore, the set of isolated types is dense.

If the isolated types form a dense set, then the non-isolated types form a closed set with empty interior, which is nowhere dense and therefore meagre. By Theorem 1.1.3, there is a countable model M omitting all the non-isolated types. This model M is prime. □

A more extreme case is when every type in $S_n(T)$ is isolated.

Theorem 1.1.5 (Ryll-Nardzewski) *A countable complete theory T is ω-categorical if and only if $S_n(T)$ is finite for all $n \in \omega$.*

Proof

If $S_n(T)$ is infinite for some n, then it cannot consist only of isolated types because $S_n(T)$ is compact. By the omitting types theorem, there exist countable models M, N such that M realises a non-isolated type $p \in S_n(T)$ and N omits p. These two countable models of T cannot be isomorphic.

And if all the $S_n(T)$ are finite, then all its elements are isolated. Therefore, in any model of T all tuples have an isolated type; moreover, if $\bar{a}$ and $\bar{b}$ have the same type in a model M of T and we extend the tuple $\bar{a}$ by adding an element a_0, then we can find a formula $\varphi(\bar{x}, y)$ isolating the type of $\bar{a}a_0$, so $\bar{a}$ satisfies $\exists \bar{x} \varphi(\bar{x}, y)$, as does $\bar{b}$. Therefore, we can find in M a b_0 for which $M \models \varphi(\bar{b}, b_0)$.

Under these conditions, any two models realise the same types over the empty set. These two remarks are enough to establish a back-and-forth system and find an isomorphism between any two countable models of T. □

A model M is said to be κ-saturated if for any $A \subset M$ of cardinality less than κ, M realises every type over A. If no κ is mentioned, a saturated model M is $|M|$-saturated. Under the GCH, one can prove the existence of saturated models of uncountable cardinality. Saturated models can be thought of as "universal domains" embedding other smaller infinite (non-saturated) models. A common formalism is to consider all the models that appear in a discussion to be elementary submodels of a fixed saturated model of strongly inaccessible cardinality, or of a cardinality at least as large as the successor of the supremum of all the cardinalities of models or sets involved. This model is called the monster model and denoted by $\bar{M}$ or $\mathfrak{C}$, and its elementary submodels are called small models, or simply models.

1.2 Simplicity

Following the great success of the 1980s in the study of stable theories, an effort was made to find similar results for the less restricted class of simple theories, originally defined by Shelah in [40] as theories without the tree property. The fundamental theorem in simplicity is the Independence Theorem, which states the conditions under which a common solution to two "sufficiently independent" types can be found.

A fundamental tool in stability theory and its variants is the use of indiscernible sequences.

An infinite sequence of tuples $(\bar{a}_i : i \in \omega)$ is said to be *indiscernible* over a set of parameters A if for all $i_1 < \ldots < i_n$, we have $\mathrm{tp}(\bar{a}_1, \ldots, a_n/A) = \mathrm{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}/A)$. Notice that by compactness, if we can find indiscernible sequences ordered by ω, then we can find indiscernible sequences ordered by any linear order. The existence of indiscernible sequences is proved using Ramsey's theorem or the stronger Erdős-Rado theorem. We do this next.

Theorem 1.2.1 (Ramsey's Theorem) *Let k, n be natural numbers and X be an infinite* *set. For each function $c : X^{[n]} \to k$ there exists an infinite $Y \subseteq X$ such that c is constant on $Y^{[n]}$.*

Proposition 1.2.2 *If M is $(|A|^+ + \kappa)$-saturated, then there is an infinite indiscernible sequence of length κ over A.*

Proof

Let P be the set of formulas on variables $(x_i : i < \kappa)$ stating that $x_i \neq x_j$ for all $i \neq j$ in κ, together with the formulas from

$$\{\varphi(x_{i_0}, \ldots, x_{i_n}) \leftrightarrow \varphi(x_0, \ldots, x_n) : n \in \omega, i_0 < \ldots < i_n < \kappa, \varphi \in L_A\}$$

Consider any finite subset s of P with k elements. Adding dummy variables if necessary, we can think of these formulas as being all on n free variables. Fix an enumeration $s = \{\varphi_0, \ldots, \varphi_{k-1}\}$ of s and define a function $c : M^n \to 2^k$ assigning to a tuple $\bar{m} = (m_1, \ldots, m_n)$ the subset of k corresponding to those formulas in s that $\bar{m}$ satisfies. By Ramsey's Theorem, there is an infinite monochromatic set satisfying s, and by compactness P is consistent; by saturation, there is an infinite sequence of length κ satisfying P in M, which is indiscernible over A by definition. □

Tuples belonging to an indiscernible sequence have strong invariance properties. Clearly, any two elements of an A-indiscernible sequence $(a_i : i \in I)$ of them have the same type over A, and therefore are conjugates in a saturated model. But there is more. It is clear that any two elements from an indiscernible sequence will have the same *strong* type, meaning that they will be in the same equivalence class of any equivalence relation definable over A with finitely many classes. Two elements in the same class of the transitive closure of the relation $a \sim_{ind_A} b$ that holds if a, b start an A-indiscernible sequence are said to have the same *Lascar strong* type over A; this relation holds if and only if a and b are conjugate under some automorphism fixing a model containing A.

In the main chapters of this thesis, we will establish some connections between the orbits under the natural action of the group of automorphisms of a structure and the solution sets of types. The following definitions will prove useful:

Definition 1.2.3 *Let A be a set of parameters and M a saturated model of T of cardinality greater than $|A|$.*

1. $\mathrm{Aut}(M/A)$ *is the group of automorphisms of M fixing A pointwise;* $\mathrm{Aut}_{\{A\}}(M)$*, is the group of automorphisms fixing A setwise.*
2. *An element is definable over A if it is fixed by all automorphisms of* $\mathrm{Aut}(M/A)$*. The set of all definable elements over A is the definable closure of A,* $\mathrm{dcl}(A)$.
3. *An element is algebraic over A if its orbit under* $\mathrm{Aut}(M/A)$ *is finite. The set of all elements which are algebraic over A is the algebraic closure of A,* $\mathrm{acl}(A)$.
4. *A relation R is A-invariant or invariant over A if $M \models R(\bar{c})$ implies $M \models R(\sigma(\bar{c}))$ for all $\sigma \in \mathrm{Aut}(M/A)$ and $\bar{c} \in M$.*

The contrast between the language in this section and the language in the section on basic model theory has to do with a change in direction in the discipline after Morley's proof of his famous categoricity theorem (originally conjectured by Jerzy Łoś) and with the historic East Coast/West Coast distinction of problems and methods in model theory. The modern focus is on more "geometric" properties, many of which are generalisations of situations arising in algebraic geometry, and the language of stability theory reflects this situation. A central feature in algebra is the concept of independence (linear independence in vector spaces, algebraic independence in fields, etc); when adapted to our level of generality, we come to the definition of *forking*.

Definition 1.2.4 *Let T be a complete L-theory, $\bar{a}$ a tuple in some Cartesian power of a small model M, and $\varphi(\bar{x}, \bar{y})$ an L-formula. We say that $\varphi(\bar{x}, \bar{a})$* k-divides *(over A) if there exists an infinite sequence $(\bar{a}_i : i \in \omega)$ of realisations of* $\mathrm{tp}(\bar{a})$ *(*$\mathrm{tp}(\bar{a}/A)$*) such that any k-element subset of $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is inconsistent. A partial (i.e., not necessarily complete) type $\pi(\bar{x})$ is said to* fork *over A if there are $n \in \omega$ and formulas $\varphi_0(\bar{x}), \dots, \varphi_{n-1}(\bar{x})$ such that $\pi(\bar{x})$ implies $\bigvee_{i<n} \varphi_i(\bar{x})$, and each φ_i k-divides over A.*

Using Ramsey's theorem, we can require that the sequences in the definition of forking/dividing be indiscernible over A. For (complete and partial) types $p(x)$, we say that p divides/forks over A if it implies a formula which divides/forks over A. As is often the case in model theory, the relation with all the desirable properties is *non*-forking. We use the symbols $A \underset{B}{\downarrow} C$ to mean "$\mathrm{tp}(a/BC)$ does not fork over B for any finite tuple a from A."

Dividing is the intuitively "correct" notion of dependence: if $\varphi(x, a)$ divides over A, then for some sequence $(a_i : i \in \omega)$ the set $\{\varphi(x, a_i) : i \in \omega\}$ is inconsistent, so any extension of $\mathrm{tp}(a/A)$ including or implying φ contains more information about the relations holding between a solution and the set Aa. But dividing has a technical disadvantage vis à vis forking: it is not always true that we can extend a partial type over B that does not divide over A to a complete type over B not dividing over A. This distinction turns out to be irrelevant in the case of simple theories, which we introduce next.

Definition 1.2.5 *A formula $\varphi(\bar{x}, \bar{a})$ consistent with a theory T is said to have the k-*tree property *if there is a tree of parameters $\{\bar{a}_s : s \in \omega^{<\omega}\}$ such that:*

1. *for all $f : \omega \to \omega$, the set $\{\varphi(\bar{x}, \bar{a}_{f\restriction n}) : n \in \omega\}$ is consistent with T, and*

2. *for all sequences $s \in \omega^{<\omega}$, the set $\{\varphi(\bar{x}, \bar{a}_{si} : i \in \omega\}$ is k-inconsistent with T.*

The theory T has the k-tree property if some formula consistent with it has the k-tree property, and is simple *if no formula has the k-tree property for any k.*

We will often prove that a theory is not simple by showing that it has the stronger tree property of the second kind, or TP2. Here's a definition for it:

Definition 1.2.6 *A theory T has the TP2 if there exists a formula $\varphi(\bar{x}, \bar{y})$ and an array of parameters $(\bar{a}^j_i : i, j \in \omega)$ such that:*

1. *for all functions $f : \omega \to \omega$, the set $\{\varphi(\bar{x}, \bar{a}^j_{f(j)}) : j \in \omega\}$ is consistent, and*

2. *each of the sets $\{a^j_i : i \in \omega\}$ is k-inconsistent.*

In a simple theory, a type $p \in S(B)$ divides over A if and only if it forks over A. Non-forking independence has other useful properties in simple theories. The following proposition is a synthesis of Propositions 5.3, 5.5, 5.6, 5.7, 5.18, and 5.20 in Casanovas' book [5].

Proposition 1.2.7 *The independence relation always has the following properties:*

Invariance *If* $f \in \mathrm{Aut}(\bar{M})$ *and* $A \underset{C}{\downarrow} B$, *then* $f(A) \underset{f(c)}{\downarrow} f(B)$.

Normality $A \underset{C}{\downarrow} B$ *if and only if* $A \underset{C}{\downarrow} CB$ *if and only if* $AC \underset{C}{\downarrow} B$.

Finite character *If* $a \underset{C}{\downarrow} b$ *for all finite tuples* $a \in A, b \in B$, *then* $A \underset{C}{\downarrow} B$.

Base monotonicity *If* $A \underset{C}{\downarrow} B$ *and* $B' \subseteq B$, *then* $A \underset{CB'}{\downarrow} B$.

Monotonicity *If* $A \underset{C}{\downarrow} B$, $A' \subseteq A$ *and* $B' \subseteq B$, *then* $A' \underset{C}{\downarrow} B'$.

Algebraic closure $\mathrm{acl}(A) \underset{A}{\downarrow} B$.

Closedness *The set of all complete types* $p(x) \in S(B)$ *which do not fork over* A *is closed in* $S(B)$.

If T *is simple, then the independence relation also satisfies:*

Local character *For any* B, C *there is some* $A \subseteq B$ *such that* $|A| \leq |T| + |C|$ *and* $C \underset{A}{\downarrow} B$.

Extension *Let* a *be a tuple (possibly infinite). For any* B, *there is some* $a' \equiv_A a$ *such that* $a' \underset{A}{\downarrow} B$.

Symmetry *For all* A, B, C, $A \underset{C}{\downarrow} B$ *if and only if* $B \underset{C}{\downarrow} A$.

Transitivity *Whenever* $B \subseteq C \subseteq D$, *if* $A \underset{B}{\downarrow} C$ *and* $A \underset{C}{\downarrow} D$, *then* $A \underset{B}{\downarrow} D$.

Reflexivity $B \underset{A}{\downfree} B$ *if and only if* $B \subseteq \mathrm{acl}(A)$.

Pairs Lemma $ab \underset{A}{\downfree} B$ *if and only if* $a \underset{A}{\downfree} B$ *and* $b \underset{Aa}{\downfree} B$.

Change of base *If* $ab \underset{A}{\downfree} B$*, then* $a \underset{A}{\downfree} b$ *if and only if* $a \underset{AB}{\downfree} b$.

- $A \underset{B}{\downfree} \mathrm{acl}(B)$
- $A \underset{B}{\downfree} C \leftrightarrow \mathrm{acl}(A) \underset{B}{\downfree} C \leftrightarrow A \underset{B}{\downfree} \mathrm{acl}(C) \leftrightarrow A \underset{\mathrm{acl}(B)}{\downfree} C$

Actually, simplicity is equivalent to nonforking independence satisfying any of local character, symmetry, or transitivity.

Let I be a linearly ordered set. An A-indiscernible sequence $(a_i : i \in I)$ is a *Morley sequence* if for every $i \in I$, $a_i \underset{A}{\downfree} \{a_j : j < i\}$. Most frequently, the existence of Morley sequences is proved using the Erdős-Rado theorem. The proof is not very illustrative, though. A more friendly way to find Morley sequences is using coheirs. Recall that a type $q \in S(B)$ is a coheir of $p \in S(M)$, $M \subset B$, if q is finitely satisfiable in M, meaning that any finite conjunction of formulas in q has a solution in M.

Given two models $M \prec N$ and a type $p \in S(M)$, we can find a coheir of p as follows. Consider the complete type p over M as an incomplete type π over N. Then π is finitely satisfiable in M, and so the family of clopen sets $P = \{\langle\varphi\rangle : \varphi \in \pi\}$ has the finite intersection property in $S(N)$: for any $\varphi_1, \ldots, \varphi_n$ in π, there is some $c \in M$ such that $tp(c) \in \langle\varphi_1\rangle \cap \ldots \cap \langle\varphi_n\rangle$. By compactness, the intersection of P is nonempty. We claim that any element q of $\bigcap P$ is a completion of π to N which is finitely satisfiable in M. The first assertion is clear; to prove the second, suppose for a contradiction that $q \in \bigcap P$ is not finitely satisfiable. This means that there exist $\psi_1, \ldots, \psi_n \in q$ such that $\psi_1 \wedge \ldots \wedge \psi_n$ have no solution in M. But in this situation $q \in \langle\psi_1 \wedge \ldots \psi_n\rangle \cap \bigcap P$, so there is a finite conjunction φ of formulas in π such that $\varphi \vdash \psi_i$ for each $i = 1, \ldots, n$, so any solution to φ is a solution to all the ψ_i. It follows from the fact that π is finitely satisfiable in M (as any type over M is) that φ has solutions in M, contradicting our assumption that q is not finitely satisfiable in M.

Proposition 1.2.8 *If $p \in S(M)$, $M \prec N \prec N'$, and $q \in S(N)$ is a coheir of p, then there is an extension of q to the model N' which is a coheir of p.*

Coheirs are nonforking extensions of the type they coinherit. A *coheir sequence* over A is a sequence $(a_i : i \in I)$ such that for some model $M \subset A$ and all $i < j \in I$, $\mathrm{tp}(a_i/A\{a_k : k < i\}) = \mathrm{tp}(a_j/A\{a_k : k < i\})$ and each $\mathrm{tp}(a_i/A\{a_k : k < i\})$ is finitely satifiable in M. We can find these sequences using 1.2.8: given any type $p \in S(A)$ finitely satisfiable in a model $M \subset A$, find an extension $p' \in S(\mathfrak{C})$ which is finitely satisfiable in M, and choose $a_i \models p' \restriction A\{a_j : j < i\}$.

Proposition 1.2.9 *A coheir sequence $(a_i : i \in I)$ over A is Morley over A*

Proof

It is clear that the sequence is indiscernible over A; and since coheirs are nonforking extensions, it is also an independent sequence. □

Using the Erdős-Rado Theorem, we can find Morley sequences even in models of a nonsimple theory. By the local character of forking, every type in a simple theory has a Morley sequence. The next proposition is an immediate consequence of Proposition 3.2.7 in Wagner's book [46], where it is phrased in terms of partial types and hyperimaginaries:

Proposition 1.2.10 *Let T be simple, a, b tuples and $\varphi(x, b)$ a formula over b. Then the following are equivalent:*

1. *$\varphi(x, b)$ does not fork over a.*
2. *$\varphi(x, b)$ does not divide over a.*
3. *$\{\varphi(x, b_i) : i \in \omega\}$ is consistent for all Morley sequences $(b_i : i \in \omega)$ in $\mathrm{tp}(b/a)$.*
4. *There is a Morley sequence $(b_i : i \in \omega)$ in $\mathrm{tp}(b/a)$ such that $\{\varphi(x, b_i) : i \in \omega\}$ is consistent.*

In Chapter 3, we will use the Lascar inequalities, which we include here for completeness.

Definition 1.2.11 *The* SU*-rank is the least function from the collection of all types over parameters in the monster model to* $\mathrm{On} \cup \{\infty\}$ *satisfying for each ordinal* α *that* $\mathrm{SU}(p) \geq \alpha + 1$ *if there is a forking extension* q *of* p *with* $\mathrm{SU}(q) \geq \alpha$ *.*

The SU-rank is invariant under definable bijections. Additionally, if q is a nonforking extension of p, then $\mathrm{SU}(q) = \mathrm{SU}(p)$. A theory T is supersimple if and only if $\mathrm{SU}(p) < \infty$ for all real types p. In the following theorem, we denote the Hessenberg sum of ordinals by $\oplus$.

Theorem 1.2.12 (Lascar inequalities) *The* SU*-rank satisfies the following inequalities:*

1. $\mathrm{SU}(a/bA) + \mathrm{SU}(b/A) \leq \mathrm{SU}(ab/A) \leq \mathrm{SU}(a/bA) \oplus \mathrm{SU}(b/A)$.
2. *Suppose* $\mathrm{SU}(a/Ab) < \infty$ *and* $\mathrm{SU}(a/A) \geq \mathrm{SU}(a/Ab) \oplus \alpha$*. Then* $\mathrm{SU}(b/A) \geq \mathrm{SU}(b/Aa) + \alpha$*.*
3. *Suppose* $\mathrm{SU}(a/Ab) < \infty$ *and* $\mathrm{SU}(a/A) \geq \mathrm{SU}(a/Ab) + \omega^\alpha n$*. Then* $\mathrm{SU}(b/A) \geq \mathrm{SU}(b/Aa) + \omega^\alpha n$*.*
4. *If* $a \underset{A}{\perp} b$*, then* $\mathrm{SU}(ab/A) = \mathrm{SU}(a/A) \oplus \mathrm{SU}(b/A)$*.*

1.3 Homogeneous Structures

Homogeneous structures appear in the work of Roland Fraïssé from the 1950s as a very special case of relational structures (see [18], [19]), but some trace the origins of the subject to Cantor's proof that any two countable dense linearly ordered sets without endpoints are isomorphic. That theorem is proved by a back-and-forth argument, which in model-theoretic terms says that the theory of $(\mathbb{Q}, <)$ eliminates quantifiers in the language $\{<\}$.

This subject is a meeting point for permutation group theory, model theory, and combinatorics. From the model-theoretic perspective, homogeneous structures have many desirable properties: they eliminate quantifiers, are prime, have few types, algebraic closure does not grow too quickly. All these properties made a full classification, at least for some restricted languages, accessible. There exist, for example, complete classifications of the finite and countably infinite homogeneous posets (Schmerl, [39]), graphs (Gardiner [21], Lachlan and Woodrow [31]), tournaments (Woodrow [47], Lachlan [27]), and digraphs (Cherlin [9]).

During the 1970s and 80s, stability theory was a rapidly growing subject. Abstractions from the dimension or rank concepts in "real life" theories were put to work, and whole families of theories were classified. Gardiner and Lachlan found that most finite homogeneous graphs and digraphs could be classified in a similar way: there was a partition of the set of structures into families parametrised by a few numbers. This parallel discovery led to Lachlan and Shelah's study of stable homogeneous structures ([9], [29]), and to Cherlin and Hrushovski's work on structures with few types in [10].

Definition 1.3.1 *A countable first-order structure M for the relational language $L = \{R_i : i \in I\}$ is* homogeneous *if any isomorphism between finite substructures extends to an automorphism of M.*

We will be dealing with finite languages practically all the time. It is essential to have a relational language; if the language has function symbols, we would have to change "finite substructures" to "finitely generated substructures" (functions can be iterated). Notice that this definition is stronger than the definition of homogeneity in model theory: the condition there is that partial elementary maps extend to automorphisms. This is one reason why our homogeneous structures are often called *ultra*homogeneous. Any partial elementary map is a local isomorphism, and so every ultrahomogeneous structure is homogeneous, but the converse is not true. The countability assumption is not necessary, but we will not consider homogeneous structures of any higher cardinality.

If M is any (not necessarily homogeneous) first-order relational structure, the set of all finite structures isomorphic to substructures of M is called the *age* of M, denoted by

$\mathrm{Age}(M)$. It is clear from the definition that the age of a homogeneous structure M is of particular importance if we wish to understand M.

Given countable relational structure M for a countable language, the following are true:

1. $\mathrm{Age}(M)$ has countably many members, since M itself is countable.

2. $\mathrm{Age}(M)$ is closed under isomorphism, by definition.

3. $\mathrm{Age}(M)$ is closed under forming substructures: given $A \in \mathrm{Age}(M)$, any substructure B of A will be finite, and a composition of the embeddings $B \to A$ and $A \to M$ proves that $B \in \mathrm{Age}(M)$.

4. $\mathrm{Age}(M)$ has the *Joint Embedding Property* or JEP: given two structures $A, B \in \mathrm{Age}(M)$, there exist embeddings $f : A \to C$ and $g : B \to C$ for some $C \in \mathrm{Age}(M)$.

The next theorem completes the picture:

Theorem 1.3.2 (Fraïssé) *Let L be a countable first-order relational language, and $\mathcal{C}$ a class of finite L-structures.*

1. *There exists a countable structure A whose age is equal to $\mathcal{C}$ if and only if $\mathcal{C}$ satisfies properties 1-4.*

2. *There exists a homogeneous structure A whose age is equal to $\mathcal{C}$ if and only if $\mathcal{C}$ satisfies 1-4 and the amalgamation property: given $A, B, C \in \mathcal{C}$ with embeddings $f_1 : A \to B$ and $f_2 : A \to C$, there exists $D \in \mathcal{C}$ and embeddings $g_1 : B \to D$ and $g_2 : C \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$. Furthermore, this structure is unique up to isomorphism.*

Consider a group of permutations G acting on a set X. Then G acts on each Cartesian power of X coordinatewise. Peter Cameron introduced the term *oligomorphic* action to describe the situation where G acts on a countably infinite set X and G has finitely many orbits on X^n for each natural number n. The following theorem is a more elaborate version of Theorem 1.1.5.

Theorem 1.3.3 *Let M be a countably infinite structure over a countable language and $T = \mathrm{Th}(M)$. The following are equivalent:*

1. *M is ω-categorical*
2. *Every type in $S_n(T)$ is isolated, for all $n \in \omega$*
3. *Each type space $S_n(T)$ is finite*
4. *$(M, \mathrm{Aut}(M))$ is oligomorphic*
5. *For each $n > 0$ there are only finitely many formulas $\varphi(x_1, \ldots, x_n)$ up to $\mathrm{Th}(M)$-equivalence.*

Proof

Condition 1 implies 2 by the omitting types theorem. The implication 2⇒3 is by compactness of the type spaces; 3 implies 4 by saturation (see Proposition 1.3.5). Condition 5 follows easily from 4. To prove that 5 implies 1, let M be a countable model of T. Notice that the type of any tuple in M^n is isolated by the conjunction of the finitely many formulas it satisfies together with the negations of the formulas it does not satisfy. From this it follows easily that M is a prime model of T. It is easy to prove that any two prime models of a complete theory in a countable language are isomorphic, but the argument is too long for the purposes of this introduction. See [34] for a detailed proof. □

Proposition 1.3.4 *Let M be a countably infinite structure homogeneous over a finite relational language. Then M is ω-categorical.*

Proof

The language is finite: there can be only finitely many isomorphism types of substructures of M of size n; by homogeneity, any two isomorphic finite substructures are in the same orbit, so by 1.3.3, M is ω-categorical. □

Proposition 1.3.5 *The unique model M of cardinality κ of a countable κ-categorical theory is saturated.*

Proof

By the Löwenheim-Skolem theorem and countability. □

Recall that a *small* substructure of a saturated model of cardinality κ is a substructure of any cardinality $\lambda < \kappa$. In homogeneous models, partial elementary maps extend to automorphisms. It is not hard to prove that saturated models are homogeneous; as a consequence,

Proposition 1.3.6 *In a saturated model M, two small substructures have the same type if and only if they belong to the same orbit under $\mathrm{Aut}(M)$.*

Proposition 1.3.7 *Let M be a countable ω-categorical structure and A a finite subset of M. A subset $X \subset M$ is definable over A if and only if X is a union of orbits of the set of automorphisms of M fixing A pointwise.*

Proof

This is a direct consequence of Proposition 1.3.3. □

Proposition 1.3.8 *Let M be a countable ω-categorical structure over a relational language L. Then M is homogeneous if and only if $\mathrm{Th}(M)$ eliminates quantifiers in the language L.*

Proof

If $\mathrm{Th}(M)$ eliminates quantifiers, homogeneity follows from saturation, by Proposition 1.3.5.

Given an n-tuple $\bar{a}$ in M, its isomorphism type in the language L can be expressed by a quantifier-free formula. And in a homogeneous structure, the isomorphism type of $\bar{a}$ determines its orbit under the action of $\mathrm{Aut}(M)$, and therefore its complete type. This is enough as we have shown that the quantifier-free type (i.e., the isomorphism type of the substructure induced on the tuple) determines the complete type of the tuple. □

Quantifier elimination is a matter of language; we can always force it on a structure by adding relation symbols to the language for each possible formula. If the structure we start with is ω-categorical, then we need only add finitely many predicates for each natural number n, corresponding to the finitely many elements of $S_n(T)$ or, equivalently by 1.3.3, to the orbits of $\mathrm{Aut}(M)$.

1.4 More specific context

The present work is an attempt to understand a restricted class of homogeneous structures with simple unstable theory, and as such, is a continuation of the work of Lachlan, Harrington, Cherlin and Shelah in the 1980's on stable homogeneous structures. They proved that all stable structures homogeneous in a finite relational language arise as a limit of finite homogeneous structures. The first stage was proving the result with the additional restriction of a binary language. All languages in this thesis are finte an relational.

During the first stages of our research, we worked with some non-binary structures. After some failed attempts to derive results even in very specific contexts (for example, we attempted to prove the existence of a 0-1 law for the universal homogeneous tetrahedron-free 3-hypergraph), it became clear that the combinatorics of non-binary structures can be intimidating or even intractable. One good reason for this is the following observation (Simon Thomas, [43]):

Observation 1.4.1 *Let M be a binary homogeneous structure, and for each $n \in \omega$, let t_n be the number of n-types realised in M. Let $A \in [M]^{<\omega}$ and let $M_1, \ldots, M_k$ be the descomposition of M into atoms over A. Then for each $1 \leq i \leq k$ and $n \in \omega$, the number of n-types over A realised in M_i is at most t_n.*

In Observation 1.4.1, the *atoms* (over A) are the solution sets of 1-types (over A) in the countable model M.

This observation fails for structures homogeneous in languages of higher arity. For example, in any homogeneous 3-hypergraph that is not complete, there is only one type of pairs (a, b) with $a \neq b$, but over any vertex $c \in M$ there are two types of pairs.

Simon Thomas proved in [43] that it is not possible to interpret a 'weak pseudoplane' in a homogeneous binary structure. In chapter 3 we prove some non-existence results for 3-graphs. A 3-graph is a complete graph with each edge coloured in one of three colours. We prove (in Theorems 3.5.37 and 3.6.2):

Theorem 1.4.2 *There are no primitive homogeneous 3-graphs with supersimple theory of* SU*-rank 2.*

Theorem 1.4.3 *Let M be a primitive homogeneous 3-graph, and suppose that if a, b are singletons and $a \not\mathop{\smile\hspace{-0.9em}|}_{\varnothing} b$, then the formula isolating $\mathrm{tp}(ab)$ is stable. Then the theory of M is of* SU*-rank 1.*

In the course of the proof we use "geometric" methods similar to those present in [43], defining an incidence structure on the 3-graphs. In Theorem 1.4.2 we have stable forking (the condition in Theorem 1.4.3 is satisfied over any set of parameters, not only $\varnothing$) by a result due to Assaf Peretz [35] stating that in supersimple ω-categorical theories, the elements of SU-rank 2 satisfy stable forking. Under stable forking, we can see Theorem 1.4.2 as the basis for an inductive argument for the non-existence of primitive homogeneous supersimple 3-graphs of rank higher than 1. It is our feeling that the hypothesis of stable forking should not be necessary to prove the conclusion of Theorem 1.4.3, and we are working towards eliminating stable forking from the statement of the theorem.

We worked with a number of examples of simple structures homogeneous in a finite relational language. At the time of this writing, all the examples of such structures we are aware of are actually supersimple, so we conjecture that all simple structures homogeneous in a finite relational language have supersimple theory (and by a result in this thesis, namely Theorem 3.2.7, those homogeneous in a binary language also have finite SU-rank). It is easy to prove that any *stable* finitely homogeneous (i.e., homogeneous in a finite relational language) structure is ω-stable, as any 1-type over a countable model M is determined by finitely many ϕ-types, which are definable.

Another observation deriving from the examples we know is that all the binary primitive (super)simple structures we know have SU-rank 1. Is it true that all primitive binary homogeneous supersimple structures have SU-rank 1? In Chapter 3, we observe that primitive simple homogeneous binary structures of rank 1 have trivial algebraic closure and are "random" in the sense that all the restrictions, or forbidden structures, of their age are of size 2 (Theorem 3.3.3). These rank 1 structures are the limit of a free amalgamation class $\mathcal{C}$, by which we mean, in the binary case, an amalgamation class in which there is a distinguished relation R that solves all the amalgamation problems $f : A \to C, g : A \to B$ where B, C are one-point extensions of A. By "solving" in this context we mean that if we consider B and C as extensions of a common substructure A, the L-structure defined on the union of B and C with $R(b, c)$ is an element of $\mathcal{C}$. (In the literature, the term "free amalgamation class" is sometimes used in the more restricted sense that the union of B and C, with no relations holding between elements from $B \setminus g(A)$ and $C \setminus f(A)$, is a solution to the amalgamation problem; we often assume that each 2-type of distinct elements is isolated by a relation in the language, so the definition we have given is more appropriate). This relates neatly with two of Cherlin's "outrageous conjectures" in [8] (Problem C2: Is every primitive infinite binary symmetric homogeneous structure generic for a free amalgamation class?, and Problem D: If Γ is infinite, primitive, binary, and finitely homogeneous, is $\mathrm{acl}(A) = A$ for all finite A?).

In the same line of thought, the simplicity of a binary structure seems to be very sensitive to the presence of large forbidden structures. In Chapter 2, we explore all of the examples in the Appendix to Cherlin's monograph [9], and prove that they have the TP2 (Corollary 2.2.3). These are all the known examples of primitive homogeneous structures in a binary language with up to four symmetric relations and non-free amalgamation, all of whose forbidden structures are of size greater than 2 are triangles. We go on to prove (Theorem 2.2.28) that any homogeneous n-graph in which all the minimal forbidden configurations are triangles and whose ages satisfy what Cherlin calls *semifree* amalgamation have the TP2 (all of Cherlin's examples from the monograph satisfy these hypotheses). Another family of examples is that of Urysohn spaces with finite diameter and integer distances. We prove that the only simple one is that with diameter 2, isomorphic to the Random Graph. All these examples are consistent with the idea that all primitive binary homogeneous structures with supersimple theory are random.

Question 1.4.4 *Is it true that the minimal forbidden configurations in any simple primitive binary homogeneous structure are of size at most 2?*

In the case of superstable ω-categorical theories, it is known that they are one-based and have finite Morley rank. The corresponding result for simple theories, namely that supersimple ω-categorical theories have finite SU-rank, has been open for a long time now. It is known, however, that supersimple ω-categorical one-based (and more generally, CM-trivial) theories have finite SU-rank (see [46], section 6.2.3). The following conjecture is a weakening of Problem 6.2.46 in Wagner's book.

Conjecture 1.4.5 *Supersimple finitely homogeneous relational structures are CM-trivial.*

In Chapter 3 (Theorem 3.2.7), we prove:

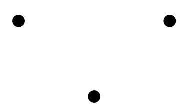
Theorem 1.4.6 *There are no binary finitely homogeneous structures with supersimple theory of infinite* SU*-rank of the form ω^α for any ordinal $\alpha \geq 1$.*

The theories we deal with are *low* (Proposition 3.1.1), a condition that allows us to use the amalgamation theorem very freely, as it implies in particular that to verify the equality of Lascar strong types of realisations of the types we wish to amalgamate, it suffices to verify the equality of their strong types. In most cases, the types we wish to amalgamate are 1-types over the empty set in a primitive structure. Under the condition of homogeneity, this means that any two realisations of the unique 1-type over $\varnothing$ will be of the same (Lascar) strong type over the empty set.

The structures we study in this thesis are purely combinatorial, in the sense that they do not interpret any algebraic structures which could give us information about them. In [1] and [44], Ben-Yaacov, Tomašić and Wagner prove an analogue of the group configuration theorem for simple theories. They find an almost hyperdefinable group of hyperimaginaries from a group configuration in a regular type. But the theories we deal with in this thesis do not fulfil their hypotheses. In the case of ω-categorical structures, this collapses to an interpretable group action, but we know the following fact from [33] (for the definition of homogenizable structure, see [11]; every homogeneous structure is homogenizable):

Theorem 1.4.7 *If M is a homogenizable relational structure, then it is not possible to interpret an infinite group in M.*

In Chapter 4 we look more closely into the combinatorics of a particular homogeneous binary structure, the universal homogeneous directed graph D not embedding a set of 3 vertices not spanning any directed arcs (what we call I_3-free digraphs). Namely, we investigate the almost sure theory of I_3-free digraphs.

$I_3 =$ 

The theory of D is nonsimple, but we are interested in it because in the cases we are aware of regarding simple binary relational structures (the random graph, random n-graphs, the random tournament), the almost sure theory coincides with the theory of the Fraïssé limit. On the other hand, in the case of triangle-free simple graphs the almost sure theory is, as a consequence of a result by Erdős, Kleitman and Rothschild, the theory the generic bipartite graph and so is supersimple of rank 1, but the theory of the universal homogeneous triangle-free graph is not simple. Our result is one more case where the theory of the Fraïssé limit of an amalgamation class and the almost sure theory of the structures in an age do not coincide. A *bitournament* is a digraph whose vertex set can be partitioned into two tournaments. Formally, what we prove is (Theorem 4.0.21):

Theorem 1.4.8 *Let $F(n)$ denote the set of labelled I_3-free digraphs on $\{0, \ldots, n-1\}$ and $T(n)$ denote the set of bitournaments on the same set. Then*

$$|F(n)| = |T(n)|(1 + o(1))$$

We conjecture that a similar result holds for other related digraphs, namely that almost all finite labelled I_m-free digraphs are m-multitournaments, and that the almost sure theory of the generic I_m-free digraph is supersimple of rank 1, and is that of the generic m-multitournament.

§2. Examples

In this chapter, we present a large number of examples of homogeneous relational structures. These examples have guided our thought and informed our conjectures. We start with the standard examples: the ordered rational numbers, the random graph, the random k-hypergraph, the universal homogenous K_n-free graphs, and a few ternary examples. In each case, we comment on the simplicity of the theory.

After that, we prove that all the examples of primitive binary structures with forbidden triangles presented by Cherlin in the appendix to his memoir [9] have nonsimple theory. We give a slight generalization to that fact and prove that any binary homogeneous complete edge-coloured graph with semifree amalgamation (see Definition 2.2.10) and all of whose minimal forbidden configurations are triangles, have the TP2. We also prove the TP2 for homogeneous integer-valued metric spaces of diameter greater than or equal to 2. We have two conjectures and a question related to this:

Conjecture 2.0.9 *Finitely homogeneous relational structures with simple theory have supersimple theory.*

Conjecture 2.0.10 *Binary finitely homogeneous relational structures with minimal forbidden configurations of size greater than 2 have nonsimple theory.*

Question 2.0.11 *If Conjecture 2.0.10 holds, is it true that all such structures have the TP2?*

All of the primitive structures with simple theory that we present in this chapter have supersimple theory of rank 1.

2.1 First examples

In this thesis we are interested in a number of properties that some homogeneous structures with simple theory have. Some of these are model-theoretical, some are combinatorial. We have formulated some conjectures which were motivated by various of examples, many of which have been studied before but not with an emphasis on these particular aspects. In this chapter, we will explore some of the examples and remark on their properties.

Our first example is $(\mathbb{Q}, <)$. This is the unique (up to isomorphism) countable model of the theory of dense linear orders without endpoints. The homogeneity of $(\mathbb{Q}, <)$ can be established by noticing that we can take any $a_1 < a_2 < \ldots < a_n$ to $b_1 < \ldots < b_n$ by a piecewise linear map, which is an automorphism of the structure. The theory of this structure is clearly unstable (a 1-type over $\mathbb{Q}$ corresponds to a Dedekind cut, and we know there are $2^{\aleph_0}$ of them) and not simple as it has the strict order property.

The universal homogeneous graph, also known as the random graph, is the archetypal example of a homogeneous simple binary structure. It is the Fraïssé limit of the amalgamation class of all finite graphs, and its theory is axiomatised by the set $\{\varphi_{n,m} : n, m \in \omega\}$, where $\varphi_{n,m}$ is $\forall v_1, \ldots, v_n \forall w_1, \ldots, w_m (D(v_1, \ldots, v_n, w_1, \ldots, w_m) \rightarrow \exists x(\bigwedge_{1 \leq i \leq n} R(x, v_i) \wedge \bigwedge_{1 \leq j \leq m} \neg R(x, w_j)))$. Here $D(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is the formula stating that the all the v_i and w_j are distinct. When phrased as "whenever V_1 and V_2 are finite disjoint sets of vertices in G, there exists a vertex v such that for all $v_1 \in V_1$ and $v_2 \in V_2$ the formula $R(v, v_1) \wedge \neg R(v, v_2)$ holds in G," the axiom schema $\phi_{n,m}$ is known as Alice's restaurant axiom. The theory of the random graph is supersimple unstable of SU-rank 1 and weakly eliminates imaginaries.

The universal homogeneous triangle-free graph is an interesting example. It is the Fraïssé limit of the family of all triangle-free graphs, and it fails to be simple (we prove the TP2 for this theory in Proposition 2.2.1). Its theory is axomatised by an axiom schema similar to Alice's restaurant axiom: given any two finite sets A, B such that there are no edges in A, there exists a vertex v such that v forms an edge with each element of A, and a nonedge with each element of B. The almost-sure theory of triangle-free graphs does not coincide with the theory of the Fraïssé limit by a result of Erdős, Kleitman, and Rothschild ([14]) saying that almost all triangle-free graphs are bipartite. It is an open question whether the universal homogeneous triangle-free graph is pseudofinite ([8]).

In languages of higher arity, we can mention the random k-hypergraph and the random structure. The random k-hypergraph is a higher-dimensional analogue of the random graph; its theory is unstable and is supersimple of S_1-rank 1 (as mentioned by Hrushovski in [23]). They are also interpretable in pseudofinite fields (see [2]). The random structure is pseudofinite by a result of Fagin [17].

If we construct the analogue for 3-hypergraphs of the homogeneous universal triangle-free graph, we get a universal homogeneous tetrahedron-free 3-hypergraph. Interestingly, though unstable, its theory is simple. This was one of the first signs of a difference between the binary and higher-arity cases that we noticed. If we go one step further, we find the "dunce-cap free" 3-hypergraph:

Proposition 2.1.1 *The family $\mathcal{C}$ of all finite 3-hypergraphs such that any four vertices span at most two edges is an amalgamation class.*

Proof
The family is clearly closed under isomorphism and substructure. The joint embedding property can be shown to hold by observing that the disjoint union of any two finite 3-hypergraphs from this family is still in the family. To prove the amalgamation property, suppose that we have $A, B, C \in \mathcal{C}$, such that A embeds into both B and C. Let D be $(B \times \{0\} \cup C \times \{1\})/ \sim_A$, where $(p_1, p_2) \sim_A (q_1, q_2)$ if p_1 and q_1 are images of the same element of A, let R be the ternary relation on D that holds on a triple of classes if R^C or R^B holds for some representatives of the classes. It is easy to verify that D is isomorphic to some structure in $\mathcal{C}$. □

Proposition 2.1.2 *The theory of the universal homogeneous dunce-cap free 3-hypergraph is not simple.*

Proof

We can interpret the universal homogeneous triangle-free graph in the duncecap-free hypergraph over one vertex. Take any $a \in M$ and define $x \sim y$ if $R(a, x, y)$ holds in M. Then $\Gamma = (M \setminus \{a\}, \sim)$ is isomorphic to the triangle-free graph. It is clear that Γ is triangle-free, and given two finite and disjoint sets of vertices $A, B \in M \setminus \{a\}$ with no edges in A, there exists a vertex $b \in M \setminus \{a\}$ such that for each of the vertices $v \in A$, we have $R(a, b, v)$ and forms no edges with the elements of B. In the interpreted graph, this b will be connected to all the elements of A and to none of the elements of B. $\square$

Not much is known about the tetrahedron-free and dunce-cap free hypergraphs. Indeed, some questions about them seem intractable. For example, establishing a 0-1 law for either of them would give much more detailed information about large hypergraphs in those classes than settling Turán's conjecture for tetrahedron-free 3-hypergraphs (for information on Turán problems, see [25]), or an analogous problem for the other family.

Lachlan and Tripp gave a classification of finite homogeneous 3-hypergraphs in [30], and found that they were related to projective planes and lines over finite fields. Lachlan and Tripp use the classification of finite 2-transitive groups in their proof.

2.2 Binary examples

In the binary case, things are, as far as we can see, less complicated. It seems to be the case that large (that is, of size larger than 2) forbidden configurations are an obstacle to simplicity in primitive structures; the easiest example of this is the homogeneous universal triangle-free graph. The argument we use to prove that it is not simple (a well-known fact) is also illustrative, and will occur again in a slightly more complicated form later.

Proposition 2.2.1 *The theory of the universal homogeneous triangle-free graph is not simple. In fact, it has the TP2.*

Proof

We will prove that the formula $\varphi(x, ab) : R(x, a) \wedge R(x, b)$ has the TP2. We claim that the set $\bigcup_{i\in\omega}\Sigma_i(\bar{x},\bar{y})$ is consistent with the theory of the universal homogeneous triangle-free graph, where $\bar{x}=(x^i_j : i,j\in\omega)$ and $\bar{y}=(y^i_j : i,j\in\omega)$ and each

$$\Sigma_i(\bar{x},\bar{y}) = \{R(x^i_j,y^i_k) : j<k<\omega\}\cup\{\neg R(x^i_j,y^i_j) : j\in\omega\}\cup \\ \cup\{\neg R(x^i_j,x^k_s)\wedge\neg R(x^i_j,y^k_s)\wedge\neg R(y^i_j,y^k_s) : i,j,k,s\in\omega, i\neq k\}$$

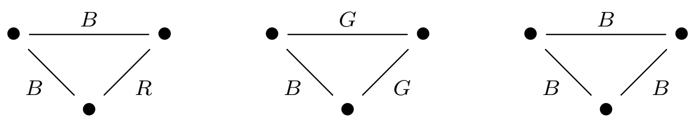
Each Σ_i says that the ith level of the array of parameters forms an infinite half-graph, and specifies that there are no edges towards any other level. As no triangles are implied, this countable array can be embedded into the universal homogeneous triangle-free graph. It is clear that it witnesses the TP2 for φ. □

2.2.1 Cherlin's primitive examples

Cherlin presented several examples of primitive homogeneous edge-coloured complete graphs in the appendix of his memoir [9], none of which has free amalgamation (though they are very close to having it, since in all of them any amalgamation problem can be solved using a relation from a distinguished proper subset of the language; Cherlin calls such amalgamation "almost free"). In this subsection, we show that their theories are not simple.

A connected graph is a metric space in the graph metric; if the associated metric space is homogeneous, then the graph is said to be *metrically homogeneous*. Cherlin has interpreted 20 of the examples in the appendix as metrically homogeneous graphs.

Cherlin uses notations of the type ABC to represent a triangle (three vertices in the complete edge-coloured graph) in which the sides are of type A,B,C. Therefore, his first example, listed as RBB, GGB, BBB (in the language $\mathcal{L}=\{R,G,B\}$) is a homogeneous primitive complete graph with edges coloured in R,G,B and omitting the triangles



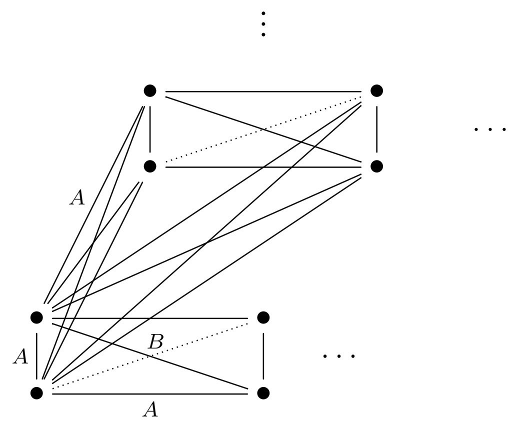
We will say that a theory T is non-simple by $Argument(P, Q; R)$ if we can apply the proof of Proposition 2.2.2 to T with the types P, Q, R instead of A, B, C. All of Cherlin's examples in the Appendix to [9] are non-simple.

Proposition 2.2.2 *Suppose that $\mathcal{M}$ transitive and homogeneous in a binary language, and that all its 2-types are symmetric. If its age is an amalgamation class all of whose minimal forbidden configurations are triangles, and there are three 2-types A, B, C for which*

1. *AAA, AAB, ABB are not forbidden*
2. *CCA is not forbidden, but CCB is*

Then $\mathrm{Th}(M)$ *has the 2-TP2.*

Proof
Let $\{\bar{c}^i_j\}_{i,j<\omega}$ be pairs $\bar{c}^i_j = \{a^i_j, b^i_j\}$ of type A, such that for all $i \in \omega$ the sequence $(\bar{a}^i_j)_{j\in\omega}$ is indiscernible, satisfying for all $s < k \in \omega$ $A(a^i_s, a^i_k), A(b^i_s, b^i_k), A(b^i_s, a^i_k)$ and $B(a^i_s, b^i_k)$. Such a sequence exists because it embeds no forbidden triangles by condition 1, and witnesses 2-dividing the formula $\varphi(x, a, b)$ of form $C(x, a) \wedge C(x, b)$ divides. Again by condition 1, we can connect these pairs as a monochromatic K_4 of colour A along all vertical lines: if $r < t$ then all pairs of different elements from $\{a^r_j, b^r_j, a^t_u, b^t_u\}$ are of type A. By conditions 1 and 2, the set $\{C(x, a^r_{f(r)}) \wedge C(x, b^r_{f(r)}) : r \in \omega\}$ is consistent for all $f : \omega \to \omega$, and the rows are 2-inconsistent (the following diagram shows the first few elements of this array in a simplified form).



□

Corollary 2.2.3 *All the examples in Table 2.1 have the TP2.*

Proof

By Proposition 2.2.2 used in each case as specified in Table 2.1 □

The table in page 34 summarises our application of Proposition 2.2.2 to Cherlin's examples. The first row is in the language $\{R, G, B\}$; all others have the language $\{R, G, A, X\}$ (we list the examples in each language separately; that is why #1 appears twice in the table).

2.2.2 Metric spaces

There is a clear connection with metric spaces: a finite integer-valued metric space can be thought of as a complete graph with edges coloured in finitely many colours omitting some triangles (corresponding to the triangle inequality). More formally, let $L = \{d_1, \dots, d_n\}$, where each d_i is a binary relation, and let $\mathcal{C}_n$ be the family of all finite L-structures A in which each of the d_i is symmetric and irreflexive, and for all pairs of distinct elements $a, b \in A$ exactly one of the d_i holds. Additionally, we impose the condition that there are no triangles $d_i(a, b) \wedge d_j(a, c) \wedge d_k(b, c)$ in which $i + j < k$ or $i + k < j$ or $j + k < i$.

Observation 2.2.4 *The family $\mathcal{C}_n$ of all finite metric spaces with integer distances and diameter at most n is a Fraïssé class for all $n \in \omega \setminus \{0\}$.*

Proof

The family $\mathcal{C}$ is clearly closed under isomorphism and substructure; it is also easy to see that there are only countably many different structures in $\mathcal{C}$ up to isomorphism. We proceed to prove the Joint Embedding Property and the Amalgamation Property.

Table 2.1: Non-simplicity of Cherlin's examples

Number	Forbidden triangles	Argument
#1	RBB, GGB, BBB	Argument(G,R;B)
#1	RXX, GAX, AXX	Argument(G,R;X)
#2	RXX, GAX, AXX, XXX	Argument(G,R;X)
#3	RXX, GAX, AXX, AAX	Argument(G,R;X)
#4	RXX, GAX, AXX, AAA	Argument(G,R;X)
#5	RXX, GAX, AXX, AAX, XXX	Argument(G,R;X)
#6	RXX, GAX, AXX, XXX, AAA	Argument(G,R;X)
#7	RXX, GAX, AAX, AXX, AAA	Argument(G,R;X)
#8	RXX, GAX, AAX, AXX, XXX, AAA	Argument(G,R;X)
#9	RXX, GAX, AAX, XXX	Argument(G,R;X)
#10	RXX, GAX, AAX, XXX, AAA	Argument(G,R;X)
#11	RXX, GGX, AXX, XXX	Argument(G,R;X)
#12	RXX, GGX, AAX, AXX, XXX	Argument(G,R;X)
#13	RXX, GGX, AXX, XXX, AAA	Argument(G,R;X)
#14	RXX, GGX, AAX, AXX, XXX, AAA	Argument(G,R;X)
#15	RXX, GAX, GGX, AXX, XXX	Argument(G,A;X)
#16	RXX, GAX, GGX, AAX, AXX, XXX	Argument(G,A;X)
#17	RXX, GAX, GGX, AXX, XXX, AAA	Argument(G,A;X)
#18	RXX, GAX, GGX, AAX, AXX, XXX, AAA	Argument(G,R;X)
#19	RXX, GAX, GGX, AAX, XXX	Argument(A,R;X)
#20	RXX, GAX, GGX, AAX, XXX, AAA	Argument(G,R;X)
#21	RAA, RXX, GAX, AAX, XXX	Argument(G,R;X)
#22	RAA, RXX, GAX, AAX, AXX	Argument(G,R;X)
#23	RAA, RXX, GAX, AAX, AXX, XXX	Argument(G,A;X)
#24	RAA, RXX, GAX, AXX, XXX, AAA	Argument(G,R;A)
#25	RAA, RXX, GAX, AAX, AXX, XXX, AAA	Argument(G,R;A)
#26	RRX, RAA, RXX, GAX, GXX, AAX, XXX	Argument(G,R;A)
#27	RRA, RRX, GAA, GAX, GXX, AAX, AXX, XXX, AAA	Argument(R,G;A)

To prove the AP, let $A \in \mathcal{C}$, and let $B = A \cup \{b\}, C = A \cup \{c\}$ be two one-point extensions of A. If we define the distance between b and c as $\delta = min\{min\{d(b,a) + d(c,a) : a \in A\}, n\}$, then for all $a \in A$ the inequality $d(b,c) \leq d(b,a) + d(c,a)$ holds, and since $\delta \leq n$, the structure thus defined on $B \cup C$ is an element of $\mathcal{C}$.

If A, B are nonempty elements of $\mathcal{C}$, then each of them embeds the one-point metric space and AP implies the JEP for this case. JEP follows trivially if one of them is empty. □

From the previous observation we get a universal countable integer-valued metric space U_n of diameter n for each $n \in \omega \setminus \{0\}$. By Fraïssé's Theorem, it is homogeneous, so whenever we have a partial self-isometry $f : \bar{a} \to \bar{a}'$ for finite subsets $\bar{a}, \bar{a}'$ of U_n, there is an isometry of U_n extending f.

If we allow only distances 0 and 1, then the triangle inequality does not impose any forbidden configurations, and M_2 is in fact the Random Graph. The situation is different if the diameter is an integer larger than two but we keep all other hypotheses. If U_n is a homogeneous metric space with diameter $n \geq 3$, then the triangle inequality does impose restrictions on the age. For example, a triangle with edges labelled 1,1,3 is not an element of $\mathcal{C}_n$. In fact, that triangle will be a forbidden configuration for all $n \geq 3$. From this it follows

Observation 2.2.5 *The theory of any homogeneous metric space with integer distances and finite diameter $n \geq 3$ has the TP2.*

Proof
This is a consequence of Proposition 2.2.2. In the language of Section 2.2.1, these theories have the TP2 by $Argument(d_2, d_3; d_1)$. □

If we drop the condition of having a finite diameter, but keep the integer distances, we get a family $\mathcal{C}_\infty$ consisting of all finite metric spaces with integer distances between every pair of elements. This time, the language is not finite.

Observation 2.2.6 *$\mathcal{C}_\infty$ is a Fraïssé class.*

Proof

Note that every finite metic space with integer distances can be thought of as a pair $(\{0, \ldots, n-1\}, (d_{00}, d_{01}, \ldots, d_{n-1,n-1})) \in \omega \times \omega^{<\omega}$ (the first element of the pair can be thought of as the set of points and the second as a distance matrix), and therefore there are only ω different such spaces, up to isomorphism.

To prove AP, let $A \in \mathcal{C}_\infty$, and let $B = A \cup \{b\}, C = A \cup \{c\}$ be two one-point extensions of A. Let $d(b, c) = min\{d(b, a) + d(a, c) : a \in A\}$. With this distance, all triangle inequalities hold and $B \cup C \in \mathcal{C}_\infty$. Again, the JEP follows from the AP. □

Consider an indiscernible sequence $(a_i : i \in \omega)$ in some model of $\mathrm{Th}(U_\infty)$, where U_∞ is the Fraïssé limit of $\mathcal{C}_\infty$. For any element b the set $\{d(b, a_i) : i \in \omega\}$ is finite, since $d(b, a_i)$ for $i > 0$ is bounded by $d(b, a_0) + d(a_0, a_1)$. Therefore, we have

Observation 2.2.7 *Given a sequence $(a_i : i \in \omega)$ indiscernible over the empty set and a finite set of parameters B, there is a subsequence $(a'_i : i \in \omega)$ that is indiscernible over B.*

Proof

Enumerate B as $b_0, \ldots, b_{k-1}$, and colour the sequence with $f : a_i \mapsto (d(a_i, b_0), d(a_i, b_1), \ldots, d(a_i, b_{k-1}))$. Since each of the coordinates can take only finitely many values, there is only a finite number of tuples in the range of f. So there is an infinite $A \subset \omega$ such that $|\{f(a_i) : i \in A\}| = 1$: all the a_i with $i \in A$ have the same type over B. Re-enumerate as $(a'_i : i \in \omega)$.

The new sequence is still indiscernible over the empty set (so all pairs are at the same distance), and the types $\mathrm{tp}(x_0, \ldots, x_n/B)$ are isolated by $D_B(\bar{x}) = \bigwedge_{i=0}^{n} \bigwedge_{j=0}^{k} d(x_i, b_j) = c_{i,j}$. So $\mathrm{tp}(a_{i_0}, \ldots, a_{i_n}/B) = \mathrm{tp}(a_0, \ldots, a_n/B)$ and the sequence is indiscernible over B. □

Notice that the proof of Observation 2.2.7 depends only on the set $\{d(b, a_i) : i \in \omega\}$ being finite. In a finitely homogeneous structure, this condition is satisfied automatically for any formula $\varphi(x, a)$ in the place of $d(b, a)$; we will use this fact in Chapter 3 to prove that the theories of finitely homogeneous simple structures are low.

If we allow the metric to take all non-negative rational values, we obtain a universal homogeneous metric space, whose completion is the universal homogeneous Polish space:

Proposition 2.2.8 *There is a unique countable rational space U_0 which is homogeneous and embeds every finite rational space. Urysohn's space U is the completion of U_0.*

In his paper [45], A. Vershik proves that with probability one a random metric space is universal. More precisely, he proves that the random countable metric space is isometric to an everywhere dense subset of the Urysohn space. This is similar to the Erdős-Renyi construction of the Random Graph. These properties are related to Cameron's concepts of *ubiquity*.

Clearly, the argument from Proposition 2.2.5 proves the TP2 for U_0 and U, as the same array of parameters can be embedded into them.

2.2.3 Forbidden triangles

As we have seen, the universal homogeneous triangle-free graph is not simple. In the previous section, we proved that all but two of the universal homogeneous metric spaces with finite diameter and integer distances have the TP2, and that in those with a simple theory (diameter 1 and 2), the triangle inequality does not really impose any forbidden structures, as all triangles with sides of length 1 and 2 satisfy the triangle inequality. In all these cases, some triangles are forbidden. It is easy to prove that Henson's K_n-free graphs also have the TP2. All this seems to point towards the following conjecture.

Conjecture 2.2.9 *The minimal forbidden configurations of primitive binary homogeneous structures with simple theory are of size at most 2.*

We will require the following definitions:

Definition 2.2.10 *Let L be a relational language consisting exclusively of binary relations, $P, Q \in L$, and $\mathcal{C}$ an amalgamation class of finite L-structures. We assume that exactly one relation from the language holds for each pair of elements in the L-structures and that each relation is symmetric.*

1. *Let $\mathcal{C}$ be a family of isomorphism types of finite structures. We can define a partial order $\leq$ on $\mathcal{C}$ by $A \leq B$ if there is an embedding $A \to B$. In the case when $\mathcal{C} \subset \mathcal{D}$ are ages of relational structures and $B \in \mathcal{D}$, we say that B is a forbidden configuration of $\mathcal{C}$ if $B \in \mathcal{D} \setminus \mathcal{C}$; it is a* minimal forbidden configuration *if B is $\leq$-minimal in the set of all forbidden configurations of $\mathcal{C}$ with respect to $\mathcal{D}$. We will not make reference to $\mathcal{D}$ when the identity of $\mathcal{D}$ is clear from the context.*

2. *We say that $\mathcal{C}$ has PQ-*semifree *amalgamation if whenever $B = A \cup \{b\}$ and $C = A \cup \{c\}$ are one-point extensions in $\mathcal{C}$ of a common finite substructure $A \in \mathcal{C}$, then at least one of the structures defined on the union of B and C with $P(b, c)$ or $Q(b, c)$ belongs to $\mathcal{C}$. The predicates P and Q are assumed to be distinct.*

3. *We say that $\mathcal{C}$ has P-*free *amalgamation if whenever $B = A \cup \{b\}$ and $C = A \cup \{c\}$ are one-point extensions in $\mathcal{C}$ of a common finite substructure $A \in C$, then the structure defined on the union of B and C with $P(b, c)$ belongs to $\mathcal{C}$.*

4. *We denote the set of (isomorphism types) of minimal forbidden configurations of $\mathcal{C}$ (with respect to the age of the random L-structure) of size n by $\mathcal{F}orb^n(\mathcal{C})$; the set of all minimal forbidden configurations of $\mathcal{C}$ is $\mathcal{F}orb(\mathcal{C})$.*

5. *A* triangle *over $X \subseteq L$ is the isomorphism type of an X-structure on 3 vertices. We denote a triangle as the sequence RST of predicates that hold in the unordered pairs of vertices.*

6. *Let P_i be a set of amalgamation problems of one-point extensions of structures in $\mathcal{C}$. We say that the P_i have a* common solution in *$X \subset L$ if there exists $R \in X$ such that R is a solution to each of the P_i.*

In this subsection, we prove a weak version of this conjecture (Theorem 2.2.28), namely:

Theorem 2.2.11 *If $\mathcal{C}$ is a PQ-semifree amalgamation class $\mathcal{C}$ of edge-coloured graphs restricted by triangles with primitive Fraïssé limit Γ_L, then either $\mathcal{F}orb^3(\mathcal{C}) = \varnothing$ or the theory of Γ_L has the TP2.*

In our argument, we strengthen Cherlin's hypothesis of "almost free" amalgamation to say that any amalgamation problem of one-point extensions can be solved using one of two predicates. As we will see (Proposition 2.2.19), it follows that the subages consisting of finite structures realising only those two types have free amalgamation, and therefore we can embed in $\mathcal{M}$ any countable structure realising only those types. This is very useful when building indiscernible sequences.

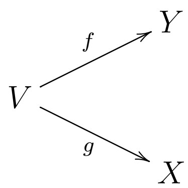
Our conjecture for primitive binary homogeneous simple structures is that they are "random" in the sense that all of their minimal forbidden configurations are of size 2. The examples we have presented so far certainly point in that direction, but all of our proofs are ad hoc and depend on detailed information about the set of forbidden configurations. At the moment, we are not aware of any method suitable to prove our conjecture. The reason is that we have no way, other than the amalgamation property and the Independence Theorem, of establishing relations between the minimal forbidden configurations. In other words, when we are trying to build an array of parameters witnessing a tree property, we need some information about the forbidden configurations to ensure that the array will be embeddable into the structure under scrutiny, and so far we have not found an effective way of obtaining this kind of information from the Amalgamation Property and the Independence Theorem.

Remark 2.2.12 An amalgamation class with PQ-semifree amalgamation has in particular the *disjoint amalgamation property*: the embeddings $f : A \to B$ and $g : A \to C$ can be chosen to be inclusions, and if $(B \setminus A) \cap (C \setminus A) = \varnothing$, then a solution to the problem is a structure D on $B \cup C$ which is in the amalgamation class.

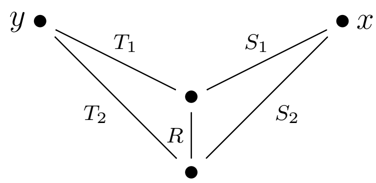
We aim to show that if M is a primitive homogeneous L-structure in which all relations are symmetric, with a nonempty set of minimal forbidden configurations of size greater than 2, all of which are triangles, and whose age has PQ-semifree amalgamation, then the theory of M has the TP2. The proof of this fact is not hard, but is somewhat laborious and involves a good deal of fiddling with amalgamation problems.

Throughout this section, we assume that the isomorphism types of a loop $R(x,x)$ and a directed edge $R(x,y) \wedge \neg R(y,x)$ are in $\mathcal{F}orb^2(\mathcal{C})$ for all $R \in L$, and that all unordered pairs are coloured by exactly one relation in the language (i.e., the isomorphism type of a solution to $\bigwedge_{R \in L} \neg R(x,y)$ is in $\mathcal{F}orb^2(\mathcal{C})$). Additionally, we assume that $\mathcal{F}orb^n(\mathcal{C}) = \varnothing$ for all $n > 3$. We will summarize these conditions by saying "$\mathcal{C}$ is a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles."

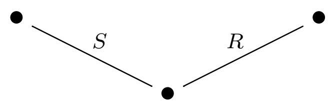
Most proofs in this section involve using the Amalgamation Property to show that we can embed a particular array of parameters in M. In practice, we use only amalgamation problems of one-point extensions of one- and two-element structures. Suppose that $V \in \mathcal{C}$ is a two-element structure on $\{v_1, v_2\}$ and that $R(v_1, v_2)$ holds in V. If



is an amalgamation problem in $\mathcal{C}$, and X, Y are one-point extensions of V, we will assume that f, g are inclusions and if $X = V \cup \{x\}, Y = V \cup \{y\}$, with relations $S_1(x, v_1), S_2(x, v_2), T_1(y, v_1), T_2(y, v_2)$, we will write the amalgamation problem as



If the amalgamation problem of one-point extensions we are considering is over a one-point structure, then we are looking for a predicate to complete a triangle. We will refer to the problem



as $RS\_$.

Definition 2.2.13 *Let $\mathcal{C}$ be a PQ-semifree amalgamation class of L-structures, where L is a language consisting exclusively of binary relations, and let $L' \subset L$. We use $\mathcal{C} \restriction_{L'}$ to denote the set of all $L' \cup \{P, Q\}$-structures in $\mathcal{C}$.*

Observation 2.2.14 *Let $\mathcal{C}$ be a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles. For all $L' \subset L$, $\mathcal{C} \restriction_{L'}$ is an amalgamation class. We denote its Fraïssé limit by $\Gamma_{L'}$.*

Proof

This follows immediately from the definiton of PQ-semifree amalgamation. □

In particular, $\mathcal{C} \restriction_{\{P,Q\}}$ is an amalgamation class, and its Fraïssé limit $\Gamma_{\{P,Q\}}$ is a homogeneous graph restricted by triangles.

Remark 2.2.15 If $\mathcal{C}$ is a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles, then by Ramsey's theorem, at least one of PPP, QQQ is in $\mathcal{C}$. We will assume $PPP \in \mathcal{C}$.

Remark 2.2.16 By the Lachlan-Woodrow Theorem 3.4.11, $\Gamma_{\{P,Q\}}$ or its complement is isomorphic to one of the following:

1. The random graph if $\mathcal{F}orb^3(\mathcal{C} \restriction_{\{P,Q\}}) = \varnothing$
2. $K^P_\omega[K^Q_n]$ or $K^Q_n[K^P_\omega]$ if $QQP \in \mathcal{F}orb^3(\mathcal{C} \restriction_{\{P,Q\}})$ or $PPQ \in \mathcal{F}orb^3(\mathcal{C} \restriction_{\{P,Q\}})$, respectively.
3. The homogeneous universal triangle-free graph if $\mathcal{F}orb^3(\mathcal{C} \restriction_{\{P,Q\}})$ is $\{QQQ\}$.

In the last of these cases, the theory of the Fraïssé limit cannot be simple. Our first goal is to prove that we need not consider the second case.

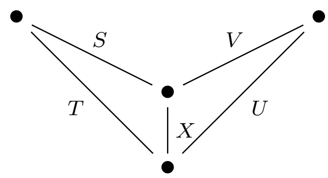
Proposition 2.2.17 *Let $\mathcal{C}$ be a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles. If $n \geq 2$ amalgamation problems of the form $R_iS_i\_$, $i \in n$, have a common solution in L, then they have a common solution in $\{P, Q\}$.*

Proof

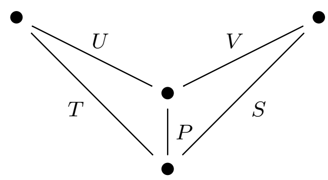
We proceed by induction on n.

For $n = 2$, if two problems $ST\_$ and $UV\_$ have a common solution X, but no common solution in $\{P, Q\}$, then either $STP \in \mathcal{C}$ and $UVP \in \mathcal{F}orb(\mathcal{C})$, or $STQ \in \mathcal{C}$ and $UVQ \in \mathcal{F}orb(\mathcal{C})$.

If $STP \in \mathcal{C}$ and $UVP \in \mathcal{F}orb(\mathcal{C})$, then $UVQ \in \mathcal{C}$ by semifree amalgamation. The amalgamation problem

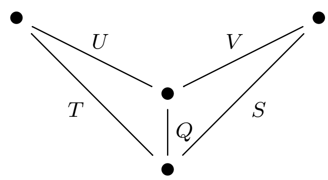


has a solution in $\{P, Q\}$. If P is a solution to this problem, then $VSP, TUP \in \mathcal{C}$, and the problem



has solution Q as $UVP \in \mathcal{F}orb(\mathcal{C})$. This implies that Q is a common solution to $ST\_$ and $UV\_$, contradiction.

And if $STQ \in \mathcal{C}$ and $UVQ \in \mathcal{F}orb(\mathcal{C})$, then the problem



shows that P is a common solution to $UV\_$ and $ST\_$. This completes the proof for $n = 2$.

Now suppose that any $k \geq 2$ problems $R_1S_{1\_}, \ldots, R_kS_{k\_}$ with a common solution have a common solution in $\{P, Q\}$. Consider $k + 1$ problems $R_1S_{1\_}, \ldots, R_{k+1}S_{k+1\_}$ with a common solution X. If P is not a solution to $R_{k+1}S_{k+1\_}$, then the common solution to

$R_2S_2\_, \ldots, R_{k+1}S_{k+1}\_$ is Q; similarly, the system $R_1S_1\_, R_3S_3\_, \ldots, R_{k+1}S_{k+1}\_$ has solution Q. Therefore, Q is a common solution to the $k+1$ problems. □

Proposition 2.2.18 *If $\mathcal{C}$ is a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles, and there is a predicate $R \in L$ such that $PPP, PPR, RRP \in \mathcal{C}$ and $RRR \in \mathcal{F}orb(\mathcal{C})$, then the theory of Γ_L has the TP2.*

Proof

We will present an array of parameters $(\bar{a}^i_j)$, where each $\bar{a}^i_j$ is an edge of type $P(b^i_j, c^i_j)$, testifying the TP2 for the formula $R(x,b) \wedge R(x,c)$.

Let $\Sigma(x^i_j, y^n_m)_{i,j,n,m\in\omega}$ be the set containing formulas saying that the x^i_j and y^n_m are all distinct, $P(x^i_j, y^n_m)$ for all combinations of i,j,n,m such that one of $i = n \wedge j = m$, $i \neq n$, $P(x^i_j, x^n_m)$ and $P(y^i_j, y^n_m)$ whenever $(i,j) \neq (n,m)$, and $R(x^i_j, y^n_m)$ for $i = n \wedge j \neq m$. We claim that Σ is consistent with the theory of Γ_L. This is clear as the only triangles implied by a solution to a finite subset of Γ_L are PPR, PPP, and PRR. Let $(\bar{a}^i_j)_{i,j\in\omega} = (b^i_j, c^i_j)_{i,j\in\omega}$ be a solution of Σ in Γ_L.

Now notice that for all $i \in \omega$, the set $\{R(x, b^i_j) \wedge R(x, c^i_j) : j \in \omega\}$ is 2-inconsistent because the triangle RRR is forbidden, and for any $f : \omega \to \omega$, $\{R(x, \bar{a}^i_{f(i)}) : i \in \omega\}$ is consistent, as only triangles of type PRR and PPP are implied by such a set. Therefore, $R(x,b) \wedge R(x,c)$ has the TP2. □

Proposition 2.2.19 *Let $\mathcal{C}$ be a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles such that the theory of the Fraïssé limit does not have the TP2. If $\mathcal{C} \restriction_{\{P,Q\}}$ is the age of an imprimitive homogeneous graph, then Γ_L is imprimitive as well.*

Proof

The structure $\Gamma_{\{P,Q\}}$ is homogeneous in the language $\{P, Q\}$ by Observation 2.2.14, so one of P, Q defines an equivalence relation. Suppose without loss that P is an equivalence relation in $\Gamma_{\{P,Q\}}$, so we have $PPQ \in \mathcal{F}orb(\mathcal{C} \restriction_{\{P,Q\}})$. Note that, by semifree amalgamation, this implies $PQQ \in \mathcal{C}$, since the problem $PQ\_$ has a solution in $\{P, Q\}$; by the same reason, $PPP \in \mathcal{C}$.

Suppose for a contradiction that Γ_L is primitive. We will prove by induction that a disjunction of predicates defines a proper equivalence relation on $\Gamma_{\{R_1,\ldots,R_k\}}$ for each $k \leq n$, so at some point we exhaust the (finite) language and reach a contradiction.

Since Γ_L is primitive, P does not define an equivalence relation on Γ_L. Therefore, there exists a predicate $R \in L$ such that $PPR \in \mathcal{C}$. We will prove that each of the problems $PP\_, RR\_, PR\_$ can be solved with any predicate from $\{P, R\}$, and that PPQ, RRQ, PRQ are forbidden; from this it will follow that $P \vee R$ defines a proper equivalence relation in $\Gamma_{\{P,Q,R\}}$.

First, suppose that $PQR \in \mathcal{C}$. Then $PP\_$ and $PQ\_$ have R as a common solution, so by Proposition 2.2.17 they have a common solution in $\{P, Q\}$. But this is impossible since $PPQ \in \mathcal{F}orb(\mathcal{C})$. Therefore, $PQR \in \mathcal{F}orb(\mathcal{C}), PPR \in \mathcal{C}$.

To prove $RRQ \in \mathcal{F}orb(\mathcal{C})$, suppose for a contradiction $RRQ \in \mathcal{C}$. Then $QR\_$ and $PP\_$ have R as a common solution, so they have a common solution in $\{P, Q\}$. Again this is impossible as $PPQ, PQR \in \mathcal{F}orb(\mathcal{C})$. Therefore, $RRP \in \mathcal{C}$. From this it follows that $RRR \in \mathcal{C}$, as otherwise Proposition 2.2.18 would imply that Γ_L has the TP2. Therefore, all triangles over $\{P, R\}$ are in $\mathcal{C}$ and PPQ, RRQ, PRQ are forbidden. It follows that $P \vee R$ defines an equivalence relation on $\Gamma_{\{P,Q,R\}}$. This constitutes our basis for induction.

For the inductive step, suppose that $P \vee R_1 \vee \ldots \vee R_k$ defines an equivalence relation on $\Gamma_{\{P,Q,R_1,\ldots,R_k\}}$ and that every triangle over $\{P, R_1, \ldots, R_k\}$ is in $\mathcal{C}$. We aim to show that there exists a relation R_{k+1} such that $P \vee R_1 \vee \ldots \vee R_k \vee R_{k+1}$ defines an equivalence relation on $\Gamma_{\{P,Q,R_1,\ldots,R_k,R_{k+1}\}}$.

By primitivity of Γ_L there exists R_{k+1} such that some triangle XYR_{k+1} is in $\mathcal{C}$, for some $X, Y \in \{P, R_1, \ldots, R_k\}$.

Claim 2.2.20 *If XYR_{k+1} is in $\mathcal{C}$ for some $X, Y \in \{P, R_1, \ldots, R_k\}$, then $PPR_{k+1} \in \mathcal{C}$.*

Proof

By the induction hypothesis, $XYP \in \mathcal{C}$, so $R_{k+1}X\_$ and $PX\_$ have Y as common solution. By Proposition 2.2.17 and the induction hypothesis ($PXQ \in \mathcal{F}orb(\mathcal{C})$), the triangles PPX, PXR_{k+1} are in $\mathcal{C}$. Now $PP\_, PR_{k+1}\_$ have X as a common solution, so these two problems have a common solution in $\{P, Q\}$. Since $PPQ \in \mathcal{F}orb(\mathcal{C})$, we get $PPR_{k+1} \in \mathcal{C}$. □

Our next goal is to prove that all the triangles XYQ with $X, Y \in \{P, R_1, \ldots, R_{k+1}\}$ are forbidden. We know by the induction hypothesis that all those triangles in which $X, Y \in \{P, R_1, \ldots, R_k\}$ are forbidden, so we need only prove that $XR_{k+1}Q$ is forbidden for all $X \in \{P, R_1, \ldots, R_{k+1}\}$.

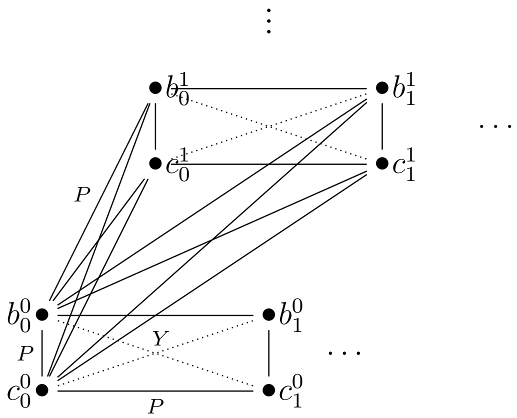
First, $PR_{k+1}Q \in \mathcal{F}orb(\mathcal{C})$, as otherwise we would have R_{k+1} as a common solution to $PP\_$ and $PQ\_$, so by Proposition 2.2.17 they would have a common solution in $\{P, Q\}$, which is impossible, since $PPQ \in \mathcal{F}orb(\mathcal{C})$.

Now suppose for a contradiction that $XR_{k+1}Q \in \mathcal{C}$ for some $X \in \{R_1, \ldots, R_k\}$. Then $XQ\_$ and $PP\_$ have R_{k+1} as a common solution, but there is no common solution to these problems in $\{P, Q\}$, contradicting Proposition 2.2.17. Therefore, $XR_{k+1}Q \in \mathcal{F}orb(\mathcal{C})$ for all $X \in \{P, R_1, \ldots, R_k\}$. Finally $QR_{k+1}R_{k+1} \in \mathcal{F}orb(\mathcal{C})$, since $PP\_$ and $QR_{k+1}\_$ do not have a common solution in $\{P, Q\}$. This shows that $XYQ \in \mathcal{F}orb(\mathcal{C})$ for all $X, Y \in \{P, R_1, \ldots, R_{k+1}\}$. By semifree amalgamation, $XYP \in \mathcal{C}$ for all $X, Y \in \{P, R_1, \ldots, R_{k+1}\}$.

The last step in the induction is to prove that all triangles over $\{P, R_1, \ldots, R_{k+1}\}$ are in $\mathcal{C}$. We already know that every triangle XYZ over $\{P, R_1, \ldots, R_k\}$ is in $\mathcal{C}$ and that for all $X, Y \in \{P, R_1, \ldots, R_{k+1}\}$ the triangle XYP is in $\mathcal{C}$, so it suffices to prove that all the triangles XYR_{k+1} with $X, Y \in \{R_1, \ldots, R_{k+1}\}$ are in $\mathcal{C}$.

The triangle $R_{k+1}R_{k+1}R_{k+1}$ is in $\mathcal{C}$ by Proposition 2.2.18, since we have $PPP, PPR_{k+1}, PR_{k+1}R_{k+1} \in \mathcal{C}$. All triangles $R_{k+1}R_{k+1}X$ with $X \in \{R_1, \ldots, R_k\}$ are in $\mathcal{C}$ by Proposition 2.2.2 because we have $PPP, PPX, PXX, PR_{k+1}R_{k+1} \in \mathcal{C}$. The same argument proves that $XXR_{k+1} \in \mathcal{C}$ for all $X \in \{R_1, \ldots, R_k\}$.

So we need only prove $XYR_{k+1} \in \mathcal{C}$ for distinct $X, Y \in \{R_1, \ldots, R_k\}$. We have $YYY, YYP, PPY, PPP \in \mathcal{C}$ by the induction hypothesis, and since $\mathcal{C}$ is restricted by triangles, any finite P, Y-structure is in $\mathcal{C}$. The array of parameters $\bar{a}^i_j = (b^i_j, c^i_j)$ with $i, j \in \omega$ in which $Y(b^i_j, c^i_s)$ holds for all natural numbers i and all $s \neq j$, and in which all other edges are of type P (see illustration below), witnesses the TP2 for the formula $R_{k+1}(x, b) \wedge X(x, c)$ if $XYR_{k+1} \in \mathcal{F}orb(\mathcal{C})$, so we must have $XYR_{k+1} \in \mathcal{C}$.



We conclude that the disjunction of all the predicates in the language, except Q, defines an equivalence relation in Γ_L, contradicting our hypothesis of primitivity. Therefore, if $\Gamma_{\{P,Q\}}$ is imprimitive, then so is Γ_L. □

It follows from Proposition 2.2.19 and the Lachlan-Woodrow Theorem 3.4.11 that we may assume $\mathcal{F}orb^3(\mathcal{C} \upharpoonright_{\{P,Q\}}) = \varnothing$. Now we proceed to prove that in this situation all the semifree amalgamation classes restricted by triangles with primitive limit have nonsimple theory.

Observation 2.2.21 *Let $\mathcal{C}$ be a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles, and suppose $\mathcal{F}orb^3(\mathcal{C} \upharpoonright_{\{P,Q\}}) = \varnothing$. If for some $R \in L \setminus \{P, Q\}$, we have $RRQ \in \mathcal{F}orb(\mathcal{C})$ (similarly, $RRP \in \mathcal{F}orb(\mathcal{C})$), then $R(x, a)$ has the TP2 in the theory of Γ_L.*

Proof

As there are no forbidden PQ-triangles, any countable PQ-structure can be embedded in Γ_L. In particular, the array of parameters $(a^i_j)_{i,j\in\omega}$, where for each $i \in \omega$ the set $\{a^i_j : j \in \omega\}$ is a Q-clique and all other edges are of type P, can be embedded in Γ_L.

By PQ-semifree amalgamation, $RRP \in \mathcal{C}$, and therefore any vertical path $\{R(x, a^i_{f(i)}) : i \in \omega\}$ is consistent, but for any i the set $\{R(x, a^i_j) : j \in \omega\}$ is 2-inconsistent, as $RRQ \in \mathcal{F}orb(\mathcal{C})$. □

Definition 2.2.22 *Let RST be a triangle over L, where $|\{R, S, T\}| \geq 2$. If $RST \in \mathcal{F}orb(\mathcal{C})$, then we say that RST is a* special forbidden triangle *every triangle over any nonempty $\ell \subsetneq \{R, S, T\}$ is in $\mathcal{C}$. If $|\{R, S, T\}| = 2$, we call RST a* bicoloured *triangle; if $|\{R, S, T\}| = 3$, RST is a* tricoloured *triangle.*

Proposition 2.2.23 *Suppose that $\mathcal{C}$ is a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles with a primitive Fraïssé limit M, and $\mathcal{F}orb^3(\mathcal{C}\restriction_{\{P,Q\}}) = \varnothing$. Assume that RST over $L \setminus \{P, Q\}$ is a special forbidden triangle. Then the M has the TP2.*

Proof

Suppose for a contradiction that the Fraïssé limit is NTP2. There are two cases to consider:

I If RST is a tricoloured special forbidden triangle, then $\mathcal{C}\restriction_{\{S,T\}}$ has $S-$ and $T-$ free amalgamation. It is easy to show that $R(x, a) \wedge S(x, b)$ has the TP2; the argument is the same as in Observation 2.2.18 with T in the position of R and S in the position of P in the array of parameters.

II Suppose RST is a bicoloured special forbidden triangle, say $RST = RSS$. The problems $SS\_$, $RS\_$ have solutions in $\{P, Q\}$. By Observation 2.2.21, $SSP, SSQ, RRP, RRQ \in \mathcal{C}$. We will prove tha these conditions are, under NTP2, inconsistent with PQ-semifree amalgamation.

Claim 2.2.24 $PPR \in \mathcal{F}orb(\mathcal{C})$.

Proof

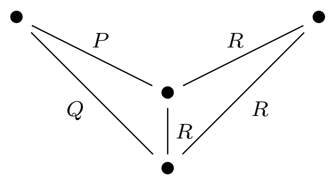
If $PPR \in \mathcal{C}$, then we have all PR-triangles in $\mathcal{C}$, and therefore all countable PR-graphs can be embedded in M (as the class is restricted by triangles). In particular, we can find an array (a^i_j) of vertices in which each level (fix i) is an infinite R-clique and all other edges are of type P. This array witnesses the TP2 for the formula $S(x, a)$. □

Claim 2.2.25 $QQR \in \mathcal{F}orb(\mathcal{C})$.

Proof

By the same argument as in the preceding claim. □

Now we can see that the problem



has no solution in $\{P, Q\}$, contradicting PQ-semifree amalgamation. Therefore, there are no special forbidden bicoloured triangles in $\mathcal{F}orb(\mathcal{C})$.

This concludes our proof. □

It follows from Proposition 2.2.23 that in any PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles with a primitive limit, if $\mathcal{F}orb^3(\mathcal{C} \restriction_{\{P,Q\}}) = \varnothing$, then all forbidden configurations are either monochromatic or have at least one edge in $\{P, Q\}$. Now we eliminate the former of these two possibilities:

Proposition 2.2.26 *Let $\mathcal{C}$ be a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles with a primitive limit, and assume $\mathcal{F}orb^3(\mathcal{C} \restriction_{\{P,Q\}}) = \varnothing$. If for some $R \in L \setminus \{P, Q\}$ we have $RRR \in \mathcal{F}orb(\mathcal{C})$, then the theory of Γ_L has the TP2.*

Proof

It follows from Observation 2.2.21 that $RRP, RRQ \in \mathcal{C}$. Therefore, $RQ\_, PR\_$ have, by Proposition 2.2.17 a common solution in $\{P, Q\}$.

If P is a common solution to $RQ\_, PR\_$, then $RQP, PPR \in \mathcal{C}$ and the set of PR-structures embeddable in Γ_L is the age of all PR-graphs which are RRR-free. It follows that the theory of Γ_L has the TP2.

If Q is a common solution to $RQ\_, PR\_$, then $RQP, QQR \in \mathcal{C}$, and the same argument (with Q replacing P) shows that the theory of Γ_L has the TP2. □

We have proved so far that in any PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles $\mathcal{C}$, if $\mathcal{C} \upharpoonright_{\{P,Q\}}$ is the age of an imprimitive homogeneous graph, then the limit of $\mathcal{C}$ is imprimitive as well, so we need only concern ourselves with those $\mathcal{C}$ in which $\mathcal{F}orb(\mathcal{C}) \upharpoonright_{\{P,Q\}} = \varnothing$, and in this case there are no special or monochromatic forbidden triangles.

Observation 2.2.27 *If $\mathcal{C}$ is a PQ-semifree amalgamation class of edge-coloured graphs restricted by triangles with primitive NTP2 limit and $\mathcal{F}orb(\mathcal{C}) \upharpoonright_{\{P,Q\}} = \varnothing$, then all the elements of $\mathcal{F}orb^3(\mathcal{C})$ have at least one edge in $\{P, Q\}$.*

Proof

Consider a forbidden triangle RST. By Proposition 2.2.23, RST is not special.

If RST is a bicoloured triangle, then, because it is not special, there is a monochromatic forbidden triangle, and by Proposition 2.2.26, the theory of the limit is not simple.

If RST is a tricoloured triangle, then by Proposition 2.2.23, at least one of $RSS, RRS, RRT, RTT, STT, SST, RRR, SSS, TTT$ is a minimal forbidden configuration. By Proposition 2.2.26, $RRR, SSS, TTT \in \mathcal{C}$, and therefore the forbidden bicoloured triangle over $\{R, S, T\}$ is special, and by Proposition 2.2.23, the theory of the limit cannot be NTP2. We conclude that in all the cases we are interested in, all the forbidden triangles have at least one edge in $\{P, Q\}$. □

We are ready to prove:

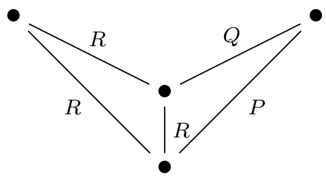
Theorem 2.2.28 *If $\mathcal{C}$ is a PQ-semifree amalgamation class $\mathcal{C}$ of edge-coloured graphs restricted by triangles with primitive Fraïssé limit Γ_L, then either $\mathcal{F}orb^3(\mathcal{C}) = \varnothing$ or the theory of Γ_L has the TP2.*

Proof

It follows from Proposition 2.2.19 and Remark 2.2.16 that we may assume that $\mathcal{F}orb^3(\mathcal{C} \restriction_{\{P,Q\}}) = \varnothing$. By Observation 2.2.27, all the forbidden triangles have at least one edge in $\{P, Q\}$. This leaves us with four cases to consider:

1. For some $R \in L \setminus \{P, Q\}$, $RRP \in \mathcal{F}orb(\mathcal{C})$. In this case, it follows from Observation 2.2.21 that the theory of Γ_L cannot be simple.

2. For some $R \in L \setminus \{P, Q\}$, $RPP \in \mathcal{F}orb(\mathcal{C})$. The triangle PQR is forced to be in $\mathcal{C}$ by semifree amalgamation, and the amalgamation problem



implies that $QQR \in \mathcal{C}$. Therefore, we have $QQR, QRR, RRR, PPQ \in \mathcal{C}$ and $RPP \in \mathcal{F}orb(\mathcal{C})$, and by Proposition 2.2.2 ($Argument(Q, R; P)$) there is a formula with the TP2 in the theory of Γ_L.

3. For some distinct $R, S \in L \setminus \{P, Q\}$, $RSP \in \mathcal{F}orb(\mathcal{C})$. Then semifree amalgamation forces $RSQ \in \mathcal{C}$. By Case 2, all of SSQ, SSP, RRP, RRQ are in $\mathcal{C}$. The array of pairs $(\bar{a}^i_j)_{i,j\in\omega} = (b^i_j, c^i_j)_{i,j\in\omega}$, where $P(b^i_j, c^n_m)$ holds if $i = n, j \neq m$, and $Q(b^i_j, c^n_m)$ holds for all other i, j, n, m, and all the edges between b^i_j, b^n_m and c^i_j, c^n_m are of type Q, can be embedded into Γ_L because all the finite PQ-structures can be embedded into Γ_L, and witnesses the TP2 for the formula $R(x, b) \wedge S(x, c)$.

4. For some $R \in L \setminus \{P, Q\}$, $RPQ \in \mathcal{F}orb(\mathcal{C})$. By semifree amalgamation, $RPP, RQQ \in \mathcal{C}$, and by Observation 2.2.21, $RRP, RRQ \in \mathcal{C}$. The array of

pairs $(\bar{a}^i_j)_{i,j\in\omega} = (b^i_j, c^i_j)_{i,j\in\omega}$, where $P(b^i_j, c^i_j)$ holds for all $i, j \in \omega$, $Q(b^i_j, c^i_k)$ holds for all other $j \neq k$, and all other edges are of type P, witnesses the TP2 for $R(x, b) \wedge P(x, c)$.

□

§3. Supersimple Homogeneous Binary Structures

This chapter contains an analysis of binary homogeneous structures with supersimple theory. The opening section contains general results on binary structures, the first of which states that if T is the theory of a finitely homogeneous structure, and we know additionally that T is simple, then T is *low*. The proof is very easy in the homogeneous case. A related result by Casanovas and Wagner [6] is that every ω-categorical supersimple theory is low. Our result is used in arguments that involve the Independence Theorem, mainly to verify easily the condition of equality of Lascar strong types in that Theorem.

In Section 3.2, we prove a general result saying that finitely homogeneous binary relational structures with supersimple theory cannot have monomial infinite SU-rank. This can be thought of as a first approximation to proving that all the structures we are interested in have finite rank.

The third section in this chapter is concerned with structures of SU-rank 1. We prove that all primitive supersimple unstable binary homogeneous structures satisfy extension axioms which, in the case of graphs, translate to Alice's restaurant axiom.

Then we move on to structures of SU-rank 2. Things are more complicated in this case, and we focus on those unstable structures with three binary symmetric irreflexive predicates R, S, T, which are assumed to be disjoint (as subsets of M^2) and such that every pair of distinct elements from the structure satisfy exactly one of them. We call these structures *3-graphs*.

We use a result due to Assaf Peretz [35] saying that supersimple theories of rank 2 have stable forking, and therefore assume throughout Section 3.5 that the forking relations in a 3-graph of SU-rank 2 are stable. It follows from this that we have only one forking relation (assumed to be R), as more of them would imply that all the relations are stable, and therefore the structure is stable. The stable 3-graphs were classified by Lachlan in [28], and we use his classification in some of our work.

The classification of homogeneous 3-graphs looks very hard. There exist uncountably many of them, by a simple variation on Henson's proof of the existence of uncountably many homogeneous directed graphs [22], so additional conditions like stability and simplicity are probably necessary to achieve partial classifications (though in the case of digraphs, Cherlin obtained a complete classification, presented in [9]).

The main result of Section 3.5 is that there do not exist primitive homogeneous 3-graphs with supersimple theory of SU-rank 2. We use this result in Section 3.6, to prove that if ω-categorical supersimple structures of finite SU-rank have stable forking, then there do not exist supersimple primitive homogeneous 3-graphs of any finite rank higher than 1.

It is open whether we can weaken the hypothesis of supersimplicity to simplicity, as is whether we can omit the hypothesis of stable forking. More specifically, are all simple finitely homogeneous structures supersimple? In the stable case, we know that superstable ω-categorical theories are 1-based [7]; is it true that supersimple finitely homogeneous structures are 1-based? If the answer to the latter question is *yes*, and binary homogeneous supersimple structures have finite rank, then we should be able to use Theorem 4.2 of [3] (1-based theories with finite SU-rank and weak elimination of imaginaries have stable forking) to eliminate the stable forking hypothesis.

3.1 General results on binary supersimple structures

We collect in this section a number of results that will be used later. We start with an easy but extremely useful proposition saying that all the theories we are interested in are *low*. The relevance of this is that in arguments using the Independence Theorem, lowness allows us to perform the amalgamation of the nonforking extensions $\mathrm{tp}(b/AB)$ and $\mathrm{tp}(c/AC)$ if $\mathrm{stp}(b/A) = \mathrm{stp}(c/A)$. This condition is generally easier to verify than the standard $\mathrm{Lstp}(b/A) = \mathrm{Lstp}(c/A)$, and in many of the cases that we will encounter, satisfied automatically.

Recall that a simple theory is low if for every formula $\varphi(\bar{x}, \bar{a})$ there exists a natural number n_φ such that given any indiscernible sequence $(\bar{a}_i : i \in \omega)$, if the set $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$ is inconsistent, then it is n_φ-inconsistent.

Proposition 3.1.1 *Let T be an ω-categorical simple theory eliminating quantifiers in a finite relational language. Then T is low.*

Proof
Let $\varphi(x, a)$ be a formula in L. Denote by m the highest arity for a relation in L, and let $\ell(a)$ be the length of the tuple a. Given any indiscernible sequence $(a_i : i \in \omega)$, the first m tuples of the sequence determine the type over $\varnothing$ of $a_{i_0}...a_{i_k}$ for any $i_0 < \ldots < i_k$ and any $k < \omega$.

By the Ryll-Nardzewski theorem, there are ony finitely many types of $(l(a)\times m)$-tuples, so there are only finitely many kinds of indiscernible sequences over $\varnothing$. We claim that, given an A-indiscernible sequence $(d_i : i \in \omega)$, the set $D = \{\varphi(x, d_i) : i \in \omega\}$ is consistent if and only if for any $\varnothing$-indiscernible sequence $(c_i : i \in \omega)$ such that $\mathrm{tp}(d_0 \ldots d_{m-1}) = \mathrm{tp}(c_0 \ldots c_{m-1})$, the set $C = \{\varphi(x, c_i) : i \in \omega\}$ is consistent. If D is consistent, then viewing $(d_i : i \in \omega)$ as indiscernible over $\varnothing$ shows one direction.

For the other direction, suppose that C is consistent but D is k-inconsistent for some $k \in \omega$. Let u satisfy C. In particular, u satisfies $\varphi(x, c_0) \wedge \ldots \wedge \varphi(x, c_{k-1})$. Using homogeneity, there is an automorphism σ of M taking $c_0...c_{k-1}$ to $d_0...d_{k-1}$, so $\sigma(u)$ contradicts the k-inconsistency of D.

Let $\Phi_j(x) = \{\varphi(\bar{x}, i) : i \in I_j\}$. If $\Phi_j(x)$ is inconsistent, then by indiscernibility it is n_j-inconsistent for some minimal $n_j \in \omega$. If we define $n_\varphi := \max_{j\in\{1,\ldots,k\}} n_j$, then it is clear that for any indiscernible sequence I of $\ell(\bar{a})$-tuples, if $\{\varphi(x, i) : i \in I\}$ is inconsistent, then it is n_φ-inconsistent. □

The next theorem appears as Theorem 6.4.6 in Wagner's book [46].

Theorem 3.1.2 *Let T be a low theory. Then Lascar strong type is the same as strong type, over any set A.*

The immediate corollary is:

Corollary 3.1.3 *Let T be an ω-categorical simple theory eliminating quantifiers in a finite relational language. Then the Lascar strong type of any tuple is the same as its strong type, over any set A.* □

Recall that an equivalence relation with finitely many classes is referred to as a *finite equivalence relation*. The classes of an A-definable finite equivalence relation correspond to strong types over A in a saturated model.

Proposition 3.1.4 *If M is a binary homogeneous simple structure in which there are no $\varnothing$-definable finite equivalence relations on M, then for each $n \in \omega$ greater than 1, whenever $a_1, \dots, a_n$ are pairwise independent elements of M, we have for each $1 \leq i \leq n$ that $a_i \mathop{\smile\!\!\!\!\!\!|}\, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$.*

Proof

We proceed by induction on n. The proposition is trivial for $n = 2$; suppose that it holds for all $n \leq n_0$ and $a_1, \dots, a_{n_0+1}$ are pairwise independent but such that $\mathrm{tp}(a_1/a_2, \dots, a_{n_0+1})$ divides over $\varnothing$. By the induction hypothesis, $a_1 \mathop{\smile\!\!\!\!\!\!|}\, a_2, \dots, a_{n_0}$ and $a_1 \mathop{\smile\!\!\!\!\!\!|}\, a_{n_0+1}$, so those two types are nonforking extensions of $\mathrm{tp}(a_1)$. We also have $a_{n_0+1} \mathop{\smile\!\!\!\!\!\!|}\, a_2, \dots, a_{n_0}$ by induction. Let $b \models \mathrm{tp}(a_1/a_{n_0+1})$ and $b' \models \mathrm{tp}(a_1/a_2, \dots, a_{n_0})$; this also ensures that $\mathrm{stp}(b) = \mathrm{stp}(b')$, and because $\mathrm{Th}(M)$ is low by 3.1.1, they are of the same Lascar strong type. Therefore, $\mathrm{Lstp}(b/\varnothing) = \mathrm{Lstp}(b'/\varnothing)$. By the Independence Theorem, $\mathrm{Lstp}(b) \cup \mathrm{tp}(a_1/a_{n_0+1}) \cup \mathrm{tp}(a_1/a_2, \dots, a_{n_0})$ is a consistent set of formulas and is realised by some $a' \mathop{\smile\!\!\!\!\!\!\mid}\, a_2, \ldots, a_{n_0+1}$. But in this case, because the language is binary, $\mathrm{tp}(a_1/a_2, \ldots, a_{n_0+1}) = \mathrm{tp}(a'/a_2, \ldots, a_{n_0+1})$, a contradiction. □

By Proposition 3.1.1, we can carry out the argument in Proposition 3.1.4 over any set of parameters, as in any low theory $a \equiv_A^{\mathrm{stp}} b$ if and only if $a \equiv_A^{\mathrm{Lstp}} b$.

Reformulating 3.1.4 for sequences:

Observation 3.1.5 *In a binary homogeneous primitive simple structure, if $(a_i : i \in \omega)$ is an $\varnothing$-indiscernible sequence of singletons such that $a_0 \mathop{\smile\!\!\!\!\!\!\mid}\, a_1$, then $(a_i : i \in \omega)$ is a Morley sequence over $\varnothing$.* □

Remark 3.1.6 The argument in Proposition 3.1.4 can be carried out in finitely homogeneous binary simple structures even over sets of parameters as long as we guarantee that the realisations of the types we want to amalgamate have the same *strong* type over the set of parameters, by Proposition 3.1.1.

Definition 3.1.7 *Let L be a finite relational language in which each relation is binary. We will say that a family $\mathcal{B}$ of finite L-structures is the* age of a random L-structure *if B is an amalgamation class and all the minimal forbidden structures of B (cf. Definition 2.2.10) are of size at most 2.*

Proposition 3.1.8 *Let M be a binary homogeneous simple structure in which there are no $\varnothing$-definable finite equivalence relations on M. Suppose that all the relations in $L = \{R_1, \ldots, R_m\}$ are realised in M, and $R_1, \ldots, R_k$ are the only forking relations. Then the subfamily of $\mathrm{Age}(M)$ consisting of all finite $\{R_1, \ldots, R_k\}$-free substructures of M is the age of a random $L \setminus \{R_1, \ldots, R_k\}$-structure.*

Proof

We aim to show that any finite structure not realising any of $R_1, \ldots, R_k$ embeds in M. All the $\{R_1, \ldots, R_k\}$-free structures of size 2 are realised in M because the R_i isolate 2-types. Consider an $\{R_1, \ldots, R_k\}$-free structure B on $n + 1$ points. We wish to show that this structure can be embedded into M, or, equivalently, that its isomorphism type belongs to $\mathrm{Age(M)}$.

Let $A = \{a_1, \ldots, a_n\}$ realise the substructure of B on the first n points, embedded in M, so $a_1 \mathop{\smile\!\!\!\!\!\!\,|} a_2, \ldots, a_n$. By the induction hypothesis, the type p_1 of a_{n+1} in B over a_1, and p_2, the type of a_{n+1} over $a_2, \ldots, a_n$ are nonforking extensions of the unique strong type over the empty set, which by lowness (Proposition 3.1.1) is Lascar strong, and therefore by the Independence Theorem there is a single element b of M simultaneously satisfying both types, so using that B is a binary structure, we get $\mathrm{tp}(b/a_1) \cup \mathrm{tp}(b/a_2, \ldots, a_n) \vdash \mathrm{tp}(b/a_1, \ldots, a_n)$, and conclude that B can be embedded into M. □

By the same argument:

Observation 3.1.9 *Let M be a homogeneous 3-graph of* SU*-rank 2 with no definable finite equivalence relations on M, and suppose S, T are nonforking relations. Then all finite S, T structures can be embedded into the* SU*-rank 2 homogeneous 3-graphs $S(a)$ and $T(a)$ for any vertex a.*

Proof
This is a direct consequence of Proposition 3.1.8. □

Observation 3.1.10 *In a transitive supersimple ω-categorical structure of finite* SU*-rank, for all $a, b \in M$ we have $\mathrm{SU}(a/b) = \mathrm{SU}(b/a)$.*

Proof
By the Lascar inequalities (Theorem 1.2.12),

$$\mathrm{SU}(a/b) + \mathrm{SU}(b) = \mathrm{SU}(ab) = \mathrm{SU}(b/a) + \mathrm{SU}(a)$$

By transitivity, there is a unique 1-type over $\varnothing$ in M, and hence $\mathrm{SU}(a) = \mathrm{SU}(b)$ for all $a, b \in M$. The result follows. □

The following observation is folklore, but we include a proof for completeness.

Observation 3.1.11 *In a primitive ω-categorical structure,* $\mathrm{acl}(a) = \{a\}$.

Proof

The relation $x \sim y$ that holds if $\mathrm{acl}(x) = \mathrm{acl}(y)$ is an equivalence relation. It is clearly reflexive and transitive, and it is symmetric because if $y \in \mathrm{acl}(x)$, then $\mathrm{acl}(y) \subseteq \mathrm{acl}(x)$ and $|\mathrm{acl}(y)| = |\mathrm{acl}(x)|$, so the algebraic closures of x and y are equal as, by ω-categoricity, they are finite sets. Hence $\sim$ is a symmetric relation, and clearly invariant. By primitivity, the $\sim$-classes are finite, and this relation is trivial.

Definition 3.1.12 *A* (complete) n-edge-coloured *graph is a structure* $(M, R_1, \ldots, R_n)$ *in which each R_i is binary, irreflexive and symmetric; also, for all distinct $x, y \in M$ exactly one of the R_i holds and $n \geq 2$. Sometimes we refer to these structures as n-graphs or simply graphs. We assume that all the relations in the language are realised in a homogeneous n-graph. If $R_i(x, y)$ holds, we often say that there is an edge of colour i or R_i between x and y, or that (x, y) is an edge of colour i (R_i).*

For any relation P in the language of an n-graph M and any tuple $\bar{a}$, $P(\bar{a})$ denotes the set $\{\bar{x} \in M : P(\bar{a}, \bar{x})\}$.

□

Given a natural number m and an irreflexive symmetric relation R, we denote the structure on m vertices $v_0, \ldots, v_{m-1}$ in which for all distinct v_i, v_j the formula $R(v_i, v_j)$ holds by K_m^R. In the following observation, a *minimal* finite equivalence relation is a proper finite equivalence relation with minimal number of classes.

Observation 3.1.13 *If $(M; R_0, \ldots, R_k)$ is a simple homogeneous transitive $k+1$-graph in which R_0 is a minimal finite equivalence relation with m classes, and R_1 is a nonforking relation realised between any two R_0-classes, then M embeds $K_m^{R_1}$.*

Proof

First note that we can embed the triangle $R_1R_1R_1$ across any three R_0-classes. To see this, consider a, b with $R_1(a, b)$. By transitivity, a and b are of the same type over the empty set. The relation R_1 is realised between any two classes; consider a', b' in the same R_0-class such that $R_1(a, a')$ and $R_1(b, b')$. Then a' and b' have the same (Lascar) strong type over $\varnothing$ and $\mathrm{tp}(a'/a), \mathrm{tp}(b'/b)$ are nonforking extensions of the unique 1-type over the empty set; we can apply the Independence Theorem to find an element c in the same R_0 class such that abc is a $K_3^{R_1}$.

The result follows by iterating the same argument, amalgamating nonforking (R_1) extensions of smaller complete graphs over the empty set. We can only iterate as many times as the number of R_0-classes. □

Observation 3.1.14 *Let M be a simple homogeneous 3-graph in which R defines an equivalence relation. If for any pair of distinct R-classes C, C' only one of S, T is realised transversally to C, C', then the S, T-graph induced on a set X containing exactly one element from each R-class is homogeneous.*

Proof

Consider the graph defined on M/R with predicates $\hat{S}, \hat{T}$ which hold of two distinct classes $a/R, b/R$ if for some/any $\alpha \in a/R, \beta \in b/R$ we have $S(\alpha, \beta)$ (respectively, $T(\alpha, \beta)$). This graph is clearly isomorphic to the graph induced on X.

Claim 3.1.15 *The graph interpreted in M/R as described in the preceding paragraph is homogeneous in the language $\{\hat{S}, \hat{T}\}$.*

Proof

Let π denote the quotient map $M \to M/R$. Given two isomorphic finite substructures A, A' of M/R, then any transversals to $\pi^{-1}(A)$ and $\pi^{-1}(A')$ are isomorphic, so by the homogeneity of M there exists an automorphism σ taking $\pi^{-1}(A)$ to $\pi^{-1}(A')$. The map $\pi\sigma\pi^{-1}$ is an automorphism of M/R taking A to A'. □

And the result follows. □

3.2 A result on the rank of supersimple binary structures

We prove in this section that supersimple structures homogeneous over a finite binary relational language cannot have infinite monomial SU-rank, that is, with an SU-rank of the form ω^α for some ordinal $\alpha > 0$.

Throughout this section, whenever we speak of a transitive binary homogeneous supersimple structure we will assume, in addition to the stated hypotheses, that M homogeneous in the finite binary language $L = \{R_1, \ldots, R_n\}$, where each R_i isolates a 2-type over the empty set, and that $R_1, \ldots, R_k$ are forking relations: $R_i(x, y)$ implies that $\mathrm{tp}(x/y)$ divides over $\varnothing$ for $i \in \{1, \ldots, k\}$, while all other relations in L are assumed to be nonforking. Finally, we assume that there are no $\varnothing$-definable equivalence relations with finitely many classes. This last assumption is innocuous because if E is such a relation, then we could carry out the argument in some infinite class.

Proposition 3.2.1 *Let M be a transitive binary homogeneous supersimple structure of* SU-*rank ω^α, for some $\alpha \geq 1$. Then M is imprimitive: the relation F given by $F(a, b)$ if* $\mathrm{tp}(a/b)$ *divides over $\varnothing$ is an equivalence relation.*

Proof
Define F by $F(x, y) \leftrightarrow \bigvee_{i=1}^{k} R_i(x, y) \vee x = y$. This relation is clearly reflexive, and by simplicity (symmetry of forking) it is symmetric; now suppose that $M \models \exists z(F(a, z) \wedge F(z, y))$. This means that for some $i, j \leq k$ and $b, c \in M$ such that $F(a, c) \wedge F(c, b)$, $R_i(a, c) \wedge R_j(c, b)$ holds (the R_i are mutually exclusive because they isolate distinct 2-types), or $a = c \vee b = c$, in which case transitivity holds trivially.

In the case $\alpha = 1$, forking implies that $\mathrm{SU}(c/a)$ and $\mathrm{SU}(b/ca)$ are both finite, so the Lascar inequalities yield $\mathrm{SU}(bc/a) \leq \mathrm{SU}(c/a) + \mathrm{SU}(b/ca) < \omega$, and since $\mathrm{SU}(b/a) \leq \mathrm{SU}(bc/a)$, we get $\mathrm{SU}(b/a) < \omega$. By transitivity of M, $\mathrm{SU}(b) = \omega$, and therefore $\mathrm{tp}(b/a)$ divides over $\varnothing$ and F is transitive.

And if $\alpha > 1$, then write $\mathrm{SU}(c/a) = \omega^{\beta_1} c_1 + \ldots + \omega^{\beta_k} c_k$ and $\mathrm{SU}(b/ac) = \omega^{\gamma_1} d_1 + \ldots + \omega^{\gamma_s} d_s$. Since $\mathrm{tp}(c/a)$ and $\mathrm{tp}(b/ac)$ are forking extensions of the unique 1-typeover $\varnothing$, we know that $\beta_1, \gamma_1 < \alpha$. By the Lascar inequalities, $\mathrm{SU}(bc/a) \leq \mathrm{SU}(c/a) \oplus \mathrm{SU}(b/ca)$, and the leading term of the ordinal on the right-hand side of this inequality has leading term $\lambda = \omega^{\max\{\beta_1,\gamma_1\}}e$, where e is the coefficient corresponding to $\max\{\omega^{\beta_1}, \omega^{\gamma_1}\}$. The properties of the Cantor normal form tell us that $\lambda < \omega^\alpha$, so we get as before $\mathrm{SU}(b/a) < \omega^\alpha$ and $\mathrm{tp}(b/a)$ divides over $\varnothing$. In any case, F is transitive. □

Proposition 3.2.2 *Let M be a transitive binary homogeneous supersimple structure of* SU-*rank ω^α for some $\alpha \geq 1$, and let $S_2^{*M}(A)$ denote the set of 2-types of distinct elements over A in M that are nonforking over $\varnothing$. There exists a partition $\{R_{k+1} \dots R_n\} = P_1 \cup \dots \cup P_m$ of $S_2^{*M}(\varnothing)$ and a bijection $f : \{1, \dots, m\} \to S_2^{*M/F}$ such that $(\alpha, \beta) \in (M/F)^2$ realises $f(i)$ if and only if for all $a \in \alpha$ and $b \in \beta$, $\mathrm{tp}(a, b) \in P_i$.*

Proof

By transitivity of M and invariance of F, all classes are isomorphic. By the definition of F, only relations in $\{R_{k+1} \dots R_n\}$ hold between elements of different classes, and by homogeneity, if $R_t(a, b)$ holds for some $a \in \alpha$ and $b \in \beta$, then it holds for some elements of any pair of equivalence classes of the same 2-type as α, β. The conclusion follows. □

Observation 3.2.3 *Let M be a transitive binary homogeneous supersimple structure of* SU-*rank ω^α. Consider M/F as a structure in the language $\{P_1, \dots, P_m\}$ from Proposition 3.2.2. Then we may assume that M/F is primitive.*

Proof

If M/F is imprimitive, then by ω-categoricity there are only finitely many $\varnothing$-definable equivalence relations. If one of them, E, has infinitely many classes, then the formula $E(x, b)$ divides over $\varnothing$ in M and therefore E equals F. In the case where E has finitely many classes, then at least one of them will contain a cofinal set of F-classes and will therefore be a rank ω^α structure with the same language as M/F and fewer $\varnothing$-definable equivalence relations. □

Proposition 3.2.4 *Let M be a transitive binary homogeneous supersimple structure of* SU*-rank ω^α for some $\alpha \geq 1$. If $a_1, \ldots, a_n \in M$ belong to different F-classes, then $a_1 \mathop{\smile\!\!\!\!\!\!\;\,|}\, a_2, \ldots a_n$.*

Proof

The proposition clearly holds for $n = 2$. Now suppose that it holds up to $n = k$ and we are given $a_1, \ldots, a_{k+1}$ in different F-classes and $\mathrm{tp}(a_1/a_2 \ldots a_{k+1})$ divides over $\varnothing$. Then by the induction hypothesis, we have $a_1 \mathop{\smile\!\!\!\!\!\!\;\,|}\, a_2 \ldots a_k$ and $a_1 \mathop{\smile\!\!\!\!\!\!\;\,|}\, a_{k+1}$. Let $b \models \mathrm{tp}(a_1/a_{k+1})$ and $b' \models \mathrm{tp}(a_1/a_2 \ldots a_k)$; again by the induction hypothesis, $a_{k+1} \mathop{\smile\!\!\!\!\!\!\;\,|}\, a_2 \ldots a_k$; $\mathrm{Lstp}(b) = \mathrm{Lstp}(b')$ because of the way we chose them and lowness of the theory (cf. Proposition 3.1.4). We can apply the Independence Theorem to obtain $a \models \mathrm{Lstp}(b) \cup \mathrm{tp}(b/a_{k+1}) \cup \mathrm{tp}(b'/a_2 \ldots a_k)$. Because the language is binary, this implies that $\mathrm{tp}(a/a_2 \ldots a_{k+1}) = \mathrm{tp}(a_1/a_2 \ldots a_{k+1})$, a contradiction since one of them divides over $\varnothing$ and the other does not. $\square$

In the next proposition, we use $\ulcorner a \urcorner$ to denote the imaginary element corresponding to the F-class of a.

Proposition 3.2.5 *Let M be a transitive binary homogeneous supersimple structure of* SU*-rank ω^α for some $\alpha \geq 1$. In M/F, $\mathrm{tp}(\ulcorner a \urcorner / A)$ divides over $\varnothing$ iff $\ulcorner a \urcorner \in A$.*

Proof

The "if" part is clear. For the "only if" part, we proceed as follows:

It is clear by Proposition 3.2.4 that the proposition holds if $|A| = 1$. More generally, suppose that $\ulcorner a \urcorner \not\mathop{\smile\!\!\!\!\!\!\;\,|}\, \ulcorner a_1 \urcorner, \ldots, \ulcorner a_m \urcorner$ for some $m > 1$. Then we can find $b_i \in \ulcorner a_i \urcorner$ such that $\ulcorner a \urcorner \not\mathop{\smile\!\!\!\!\!\!\;\,|}\, b_1 \ldots b_m$, but in this case, for any $b \in \ulcorner a \urcorner$, since $\mathrm{tp}(b/b_1 \ldots b_m) \vdash \mathrm{tp}(\ulcorner b \urcorner / b_1 \ldots b_m)$ because $\ulcorner b \urcorner \in \mathrm{dcl}(b)$, we have $b \not\mathop{\smile\!\!\!\!\!\!\;\,|}\, b_1 \ldots b_m$, contradicting Proposition 3.2.4. $\square$

Proposition 3.2.6 *Let M be a transitive binary homogeneous supersimple structure of* SU*-rank ω^α for some $\alpha \geq 1$. Then* $\mathrm{SU}(M/F) = 1$

Proof
By primitivity, there is a unique 1-type p over $\varnothing$. Suppose $q \in S(A)$ is a forking extension of p. Then by Proposition 3.2.5, q includes the formula $x = a$ for some $a \in A$, so q itself has no forking extensions and therefore has rank 0. It follows that $\mathrm{SU}(p) = 1$. □

Theorem 3.2.7 *There are no binary finitely homogeneous structures with supersimple theory of infinite* SU*-rank of the form ω^α, for any ordinal $\alpha > 0$.*

Proof
As we have seen, in this case forking would be a definable equivalence relation with classes of lower rank. But in this case, M/F would be of infinite rank, contradicting Proposition 3.2.6. □

3.3 Binary homogeneous structures of SU-rank 1

In this section we investigate supersimple binary homogeneous structures of SU-rank 1. Under this assumption, $\mathrm{tp}(a/B)$ forks over $A \subset B$ iff $a \in \mathrm{acl}(B) \setminus \mathrm{acl}(A)$, and algebraic closure on an SU-rank 1 structure induces a pregeometry.

3.3.1 The primitive case

Proposition 3.3.1 *Let M be a binary homogeneous structure with supersimple theory of* SU*-rank 1 such that* $\mathrm{Aut}(M)$ *acts primitively on M. Then* $\mathrm{acl}(a, b) = \{a, b\}$.

Proof
Suppose not. Then there is $c \in \mathrm{acl}(ab) \setminus (\mathrm{acl}(a) \cup \mathrm{acl}(b))$ and $a \underset{\varnothing}{\perp\!\!\!\perp} b$, since by primitivity

(Observation 3.1.11) $\mathrm{acl}(a) = a$. By primitivity, there is only one strong type of elements over $\varnothing$, and since the rank is finite, this implies that all elements are of the same Lascar strong type. So we have $\mathrm{Lstp}(a) = \mathrm{Lstp}(b)$. Take two elements c', c'' realising $\mathrm{tp}(c/a)$ and $\mathrm{tp}(c/b)$ respectively. Note that $c' \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} a$ and $c'' \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} b$.

Therefore we can apply the Independence Theorem to produce $d \models \mathrm{Lstp}(a) \cup \mathrm{tp}(c/a) \cup \mathrm{tp}(c/b)$ with $d \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} ab$. Since the language is binary, $\mathrm{tp}(d/ab) = \mathrm{tp}(c/ab)$ (an algebraic type), so $d \in \mathrm{acl}(ab)$ which contradicts $d \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} ab$. □

A stronger statement is:

Proposition 3.3.2 *Under the hypotheses of 3.3.1,* $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a) = A$.

Proof

We prove this by induction on $|A|$. The case $|A| = 1$ is true by primitivity and $|A| = 2$ is Proposition 3.3.1. Now suppose that the result holds for sets of cardinality k, and let $A = \{a_1, \ldots a_{k+1}\}$.

Suppose that the equality does not hold, and take $b \in \mathrm{acl}(A) \setminus \bigcup_{a \in A} \mathrm{acl}(a)$. By the rank 1 assumption, $a_{k+1} \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} A_0$, where $A_0 = A \setminus \{a_{k+1}\}$. Now take b_0 realizing $\mathrm{tp}(b/A_0)$ and b_1 realizing $\mathrm{tp}(b/a_{k+1})$. By the induction hypothesis, $b_0 \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} A_0$ and $b_1 \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} a_{k+1}$. By primitivity, $\mathrm{Lstp}(b_0) = \mathrm{Lstp}(b_1)$ (over the empty set), so we can apply the Independence Theorem to get a $\beta \models \mathrm{tp}(b_0/A_0) \cup \mathrm{tp}(b_1/a)$ with $\beta \mathop{\smile\!\!\!\!\!\!\mid}_{\varnothing} A$. By rank 1, β is not algebraic over A.

But $\mathrm{tp}(\beta/A) = \mathrm{tp}(b/A)$; indeed, $\mathrm{tp}(\beta/A_0) = \mathrm{tp}(b/A_0)$, which implies that $\mathrm{tp}(\beta/\alpha) = \mathrm{tp}(b/\alpha)$ for all $\alpha \in A_0$, and also $\mathrm{tp}(\beta/a) = \mathrm{tp}(b/a)$. Since the language is binary, this implies that $\mathrm{tp}(\beta/A) = \mathrm{tp}(b/A)$. This is a contradiction (because b is algebraic over A.) □

Let $D(\bar{x})$ denote the formula expressing that the elements of the tuple $\bar{x}$ are all different. Recall that the theory of the Random Graph is axiomatised by the set of sentences $\{\phi_{n,m} : n, m \in \omega\}$, where $\phi_{n,m}$ is $\forall v_1, \ldots, v_n \forall w_1, \ldots, w_m (D(v_1, \ldots, v_n, w_1, \ldots, w_m) \rightarrow$

$\exists x(\bigwedge_{1\leq i\leq n} R(x,v_i) \wedge \bigwedge_{1\leq j\leq m} \neg R(x,w_j))$. When phrased as "whenever V_1 and V_2 are finite disjoint sets of vertices in G, there exists a vertex v such that for all $v_1 \in V_1$ and $v_2 \in V_2$ the formula $R(v,v_1) \wedge \neg R(v,v_2)$ holds in G," the axiom schema $\phi_{n,m}$ is known as *Alice's restaurant axiom.*

We will assume for the rest of this section that M is a binary relational structure, homogeneous in a language $L = \{R_1,\ldots,R_n\}$, and that each 2-type over $\varnothing$ of distinct elements is isolated by one of the relations in the language. Our aim is to show that supersimple primitive binary homogeneous structures are very similar to the random graph, in the sense that we can prove analogues of Alice's restaurant axioms in them. As in other proofs in this chapter, at the core of the argument is the Independence Theorem.

Proposition 3.3.3 *Let M be a countable relational structure homogeneous in the binary language $L = \{R_1,\ldots,R_n\}$, and assume that each complete 2-type over $\varnothing$ is isolated by one of the R_i. Suppose that $R_1,\ldots,R_m$ are symmetric relations and $R_{m+1},\ldots,R_n$ are antisymmetric. If M is primitive and $Th(M)$ is supersimple of* SU*-rank 1, then for any collection $\{A_1,\ldots,A_m,A_{m+1},A'_{m+1},\ldots,A_n,A'_n\}$ of pairwise disjoint finite sets of elements from M there exists $v \in M$ such that*

$$M \models \bigwedge_{i\in\{1,\ldots m\}} (\bigwedge_{v_i\in A_i} R_i(v,v_i)) \wedge \bigwedge_{i\in\{m+1,\ldots,n\}} (\bigwedge_{v_i\in A_i} R_i(v,v_i) \wedge \bigwedge_{w_i\in A'_i} R_i(w_i,v))$$

Proof

To prove this, we use Proposition 3.3.2 and the Independence Theorem. We may assume that all the A_i, A'_i are of the same size, and will prove this proposition for $|A_i| = 1$ (it will be clear that the same argument can be iterated for larger sets). By Proposition 3.3.2, $a_1 \mathop{\smile\!\!\!\!\!\mid} a_2$ if $a_1 \neq a_2$, and for any A,B,C, $A \mathop{\smile\!\!\!\!\!\mid}_C B$ if $(A\setminus C)\cap(B\setminus C) = \varnothing$. Let $A_i = \{a_i\}$ and $A'_j = \{a'_j\}$ for $m+1 \leq j \leq n$, and assume all the a_i are different and therefore pairwise independent. Then by homogeneity, there exist b_i with $R_i(a_i,b_i)$, and $\mathrm{tp}(b_i/a_i)$ does not fork over $\varnothing$. By primitivity, $\mathrm{Lstp}(b_1) = \mathrm{Lstp}(b_2)$, so we can apply the Independence Theorem and find $b_{12} \models \mathrm{Lstp}(b_1)\cup\mathrm{tp}(b_1/a_1)\cup\mathrm{tp}(b_2/a_2)$ satisfying $b_{12} \mathop{\smile\!\!\!\!\!\mid} a_1a_2$.

Now we have $b_{12} \mathop{\smile\!\!\!\!\!\!\downarrow} a_1a_2$ and we know $a_1a_2 \mathop{\smile\!\!\!\!\!\!\downarrow} a_3$ and $\mathrm{tp}(b_3/a_3)$ does not fork over $\varnothing$. Also, by primitivity $\mathrm{Lstp}(b_3) = \mathrm{Lstp}(b_{12})$ and we can apply the independence theorem again. Iterating this process, we find $\alpha \models \mathrm{Lstp}(b_1) \cup \mathrm{tp}(b_1/a_1) \cup \ldots \cup \mathrm{tp}(b_n/a_n)$ independent from $a_1, \ldots, a_m, a_{m+1}, a'_{m+1}, \ldots, a_n, a'_n$. $\square$

3.3.2 Finite equivalence relations

If M is a transitive, imprimitive rank 1 structure in which all the definable equivalence relations have infinite classes, then it follows from the rank hypothesis that each of the equivalence relations has finitely many classes. From homogeneity and transitivity it follows that if E is a definable equivalence relation on M and $\neg E(a,b)$, then a/E and b/E are homogeneous structures with the same age, and each has fewer definable equivalence relations than M. By ω-categoricity, there are only finitely many definable equivalence relations, so that M is in fact the union of finitely many primitive homogeneous structures (which are the equivalence classes of the finest definable equivalence relation on M with infinite classes) in which all invariant equivalence relations have finite classes. Our next goal is to describe how two classes of a finite equivalence relation in a rank 1 binary homogeneous structure can relate to each other.

The archetypal example of an imprimitive simple unstable binary homogeneous structure with a finite equivalence relation is the Random Bipartite Graph. It is the Fraïssé limit of the family of all bipartite graphs with a specified partition or equivalence relation; it is not homogeneous as a graph, but is homogeneous in the language $\{R, E\}$, where E is interpreted as an equivalence relation. To axiomatise this theory, it suffices to express that E is an equivalence relation with exactly two infinite classes, R is a graph relation, and that for any finite disjoint subsets A_1, A_2 of the same E-class there exists a vertex v in the opposite class such that $R(v,a)$ holds for all $a \in A_1$ and $\neg R(v,a')$ holds for all $a' \in A_2$.

If A, B are different classes of the finest definable finite equivalence relation E on M, we will say that a relation R holds *transversally* or *across* A, B if there exist $a \in A$ and $b \in B$ such that $R(a,b) \vee R(b,a)$. Relations which hold transversally for some pair of E-classes are refered to as *transversal relations*. Notice that by homogeneity any relation holding across E-classes does not hold within a class, and vice-versa. By quantifier elimination and our assumption on the disjointness of the binary relations, E is defined by a disjunction of atomic formulas $\bigvee_{i\in I} R_i(x,y)$ for some $I \subset \{1,\ldots n\}$. Therefore, the transversal relations are those in $L \setminus \{R_i : i \in I\}$. We assume that each 2-type of distinct elements is isolated by a relation in the language; therefore, each relation is either symmetric or antisymmetric.

Given two E-classes A, B, if only one symmetric relation R holds across A, B then we say that R is *complete bipartite* in A, B, for the reason that if we forget the structure within the classes, what we obtain is a complete bipartite graph. All other relations are *null* across A, B in this case, i.e., not realised across these classes.

If D is an antisymmetric relation realised across A, B, we say that the ordered pair of classes (A, B) is *directed for* D if all the D-edges present in $A \cup B$ go in the same direction, that is, if either $\forall(c, c' \in A \cup B)(D(c, c') \rightarrow c \in A \wedge c' \in B)$ or $\forall(c, c' \in A \cup B)(D(c, c') \rightarrow c \in B \wedge c' \in A)$. A dramatic example of a D-directed pair of E-classes is when $\forall a \in A \forall b \in B(D(a, b))$. We adopt the convention that if (A, B) is directed for D, then the D-edges go from A to B. If (A, B) is not directed for any D, then we say that (A, B) is an *undirected* pair of E-classes.

Observation 3.3.4 *Let M be a binary homogeneous imprimitive transitive relational structure in which there are proper nontrivial invariant equivalence relations with infinite classes. Let E be the finest such equivalence relation in M. If (A, B) is a directed pair of equivalence classes for some $D \in L$, then no symmetric relations are realised across A, B and for all antisymmetric relations D' in the language realised across A, B, either (A, B) or (B, A) is directed for D'.*

Proof

The first assertion follows from the fact that if $R(a, b)$ for some symmetric relation R, where $a \in A$ and $b \in B$, then by homogeneity there would exist an automorphism taking $a \rightarrow b$ and $b \rightarrow a$, which is impossible by invariance of E and the fact that (A, B) is directed for D. Similarly, if for some directed relation D' we had $a, a' \in A$ and $b \in B$

with $D'(a,b) \wedge D'(b,a')$ then by homogeneity there would exist an automorphism of M taking ab to ba', again impossible since (A,B) is directed for D. $\square$

Observation 3.3.5 *Let M be a binary homogeneous imprimitive transitive relational structure with supersimple theory of* SU*-rank 1 in which there are proper nontrivial invariant equivalence relations with infinite classes. Let E be the finest such equivalence relation in M, and assume that* $\mathrm{Aut}(M)$ *acts primitively on each E-class. If $a_1,\ldots,a_n$, $n \geq 2$, are distinct E-equivalent elements of M, then $a_1 \mathop{\smile\!\!\!\!\!\mid}\, a_2,\ldots,a_n$.*

Proof

We proceed by induction on n. For the case $n=2$, let a_1,a_2 be distinct elements of M, $E(a_1,a_2)$. In the situation described, each of the relations that imply E is non-algebraic, since otherwise the action of $\mathrm{Aut}(M)$ on a_1/E would not be primitive. It follows that the relation isolating $\mathrm{tp}(a_1a_2)$ is nonforking, so $a_1 \mathop{\smile\!\!\!\!\!\mid}\, a_2$.

Now suppose that any k distinct E-equivalent elements of M are independent. Suppose for a contradiction that $a_1,\ldots,a_{k+1}$ are pairwise independent E-equivalent elements of M, and $a_{k+1} \not\mathop{\smile\!\!\!\!\!\mid}\, a_1,\ldots a_k$. By the induction hypothesis, $a_1 \mathop{\smile\!\!\!\!\!\mid}\, a_2,\ldots,a_k$, $a_{k+1} \mathop{\smile\!\!\!\!\!\mid}\, a_1$ and $a_{k+1} \mathop{\smile\!\!\!\!\!\mid}\, a_2,\ldots,a_k$. Let $b_1 \models \mathrm{tp}(a_{k+1}/a_1)$ and $b_2 \models \mathrm{tp}(a_{k+1}/a_2,\ldots,a_k)$; these are nonforking extensions of the unique 1-type over $\varnothing$ to a_1 and $a_2,\ldots,a_k$, and are of the same strong type. Therefore, by the Independence Theorem, there exists c satisfying $\mathrm{tp}(a_{k+1}/a_1) \cup \mathrm{tp}(a_{k+1}/a_2,\ldots,a_k)$ in the same class as a_{k+1}, independent (i.e., non-algebraic) from $a_1,\ldots,a_k$. But then $\mathrm{tp}(c/a_1,\ldots,a_k) = \mathrm{tp}(a_{k+1}/a_1,\ldots,a_k)$ because the language is binary, which is impossible as the type on the left-hand side of the equality is non-algebraic, while the other one is algebraic. $\square$

Given a pair of E-classes A,B, denote the set of nonforking transversal relations realised in $A \cup B$ by $\mathcal{I}(A,B)$. If (A,B) is a directed pair of classes, then $\mathcal{I}^*(A,B)$ is the set of nonforking relations D realised in $A \cup B$ such that $D(a,b)$ for some $a \in A, b \in B$. Note that for directed pairs, $\mathcal{I}(A,B) = \mathcal{I}^*(A,B) \cup \mathcal{I}^*(B,A)$.

Proposition 3.3.6 *Let M be a binary homogeneous imprimitive transitive relational structure with supersimple theory of* SU*-rank 1 in which there are proper nontrivial invariant equivalence relations with infinite classes. Let E be the finest such equivalence relation in M, and assume that $\mathrm{Aut}(M)$ acts primitively on each E-class. Suppose that (A, B) is a D_1-directed pair of E-classes. Enumerate $\mathcal{I}^*(A, B) = \{D_1, \ldots, D_n\}$ and $\mathcal{I}^*(B, A) = \{Q_1, \ldots, Q_m\}$. Then for all finite disjoint $V_1, \ldots, V_n \subset B$ and $W_1, \ldots, W_m \subset A$ there exist $c \in A$ and $d \in B$ such that $D_i(c, v)$ holds for all $v \in V_i$ ($1 \leq i \leq n$) and $Q_j(w, d)$ holds for all $w \in W_j$ ($1 \leq j \leq m$).*

Proof

We will prove only that for all finite disjoint $V_1, \ldots, V_n \subset B$ there exists $c \in B$ such that $D_i(c, v)$ holds for all $v \in V_i$; the same argument produces the d from the statement.

We proceed by induction on $k = |V_1| + \ldots + |V_n|$, with an inner induction argument. If $k = n$, so $V_i = \{b_i\}$ then by Observation 3.3.5 we have $b_1 \mathop{\smile\!\!\!\!\!\!\;|}\, b_2$. There exist $a, a' \in A$ such that $D_1(a, b_1) \wedge D_2(a', b_2)$; since D_1 and D_2 are nonforking relations, $a \mathop{\smile\!\!\!\!\!\!\;|}\, b_1$ and $a' \mathop{\smile\!\!\!\!\!\!\;|}\, b_2$, and since a, a' are E-equivalent, they have the same strong type. By the Independence Theorem, there exists $c_{12} \in A$ such that $c_{12} \mathop{\smile\!\!\!\!\!\!\;|}\, b_1 b_2$ and $D_1(c_{12}, b_1) \wedge D_2(c_{12}, b_2)$. Now suppose that for $t \leq n-1$, we can find $c_{1\ldots t} \mathop{\smile\!\!\!\!\!\!\;|}\, a_1, \ldots, a_t$ such that $D_1(c_{1\ldots t}, b_1) \wedge \ldots \wedge D_t(c_{1\ldots t}, b_t)$. Given distinct $b_1, \ldots, b_{t+1}$ with $t + 1 \leq n$, it follows from Observation 3.3.5 that $b_{t+1} \mathop{\smile\!\!\!\!\!\!\;|}\, b_1, \ldots, b_t$. By the induction hypothesis, there exists $c_{1\ldots t} \mathop{\smile\!\!\!\!\!\!\;|}\, b_1, \ldots, b_t$ satisfying $\bigwedge_{i=1}^{t} D_i(c_{1\ldots t}, b_i)$; and we know that there exists $c_{t+1} \in A$ such that $D_{t+1}(c_{t+1}, b_{t+1})$. Since D_{t+1} is nonforking, $c_{t+1} \mathop{\smile\!\!\!\!\!\!\;|}\, b_{t+1}$, and by the Independence Theorem, there exists $c_{1\ldots t+1} \mathop{\smile\!\!\!\!\!\!\;|}\, b_1, \ldots, b_{t+1}$ such that $\bigwedge_{i=1}^{t+1} D_i(c_{1\ldots t+1}, b_i)$. This concludes, by induction, the case $k = n$. The same argument proves the inductive step on k. □

By the same argument, we can prove:

Proposition 3.3.7 *Let M be a binary homogeneous imprimitive transitive relational structure with supersimple theory of* SU*-rank 1 in which there are proper nontrivial* *invariant equivalence relations with infinite classes. Let E be the finest such equivalence relation in M, and assume that* $\mathrm{Aut}(M)$ *acts primitively on each E-class. Suppose that (A, B) is an undirected pair of E-classes, $\mathcal{I}(A, B) = \{R_1, \ldots, R_k\} \cup \{D_1, \ldots, D_s\}$, where each R_i is symmetric and each D_j is antisymmetric. Then for all finite disjoint subsets $V_1, \ldots, V_k, W_1, \ldots, W_s, W_1', \ldots, W_s' \subset B$ there exists $c \in A$ such that $R_i(c, v)$ for all $v \in V_i$, $D_j(c, w)$ for all $w \in W_j$, and $D_j(w, c)$ for all $w \in W_j'$.*

We remark here that if all the relations are symmetric, Proposition 3.3.7 says that a nonforking transversal relation R occurs across a pair of E-classes A, B in one of three ways, namely:

1. Complete, that is, only one relation is realised across A, B,
2. Null, so R is not realised in $A \cup B$
3. Random bipartite: it satisfies that given two disjoint nonempty finite subsets V, V' of A (B), there is a vertex v in B (A) that is R-related to all vertices from V and to none from V'

The results in this section tell us exactly what to expect from binary supersimple homogeneous structures of SU-rank 1. Even though we did not phrase it as a list of structures, Proposition 3.3.7 is essentially a classification result for imprimitive binary homogeneous structures of SU-rank 1 in which one of the relations defines an equivalence relation with infinite classes. Our next proposition is, in the same sense, a classification of unstable imprimitive simple 3-graphs (language $\{R, S, T\}$, all relations symmetric and irreflexive, each pair of distinct vertices realises exactly one of them) in which one of the predicates defines a finite equivalence relation. This result is of interest in the final sections of this chapter; we make implicit use Proposition 3.1.1:

Proposition 3.3.8 *Let M be a transitive simple unstable homogeneous 3-graph in which R defines an equivalence relation with $m < \omega$ classes. Then M has supersimple theory of* SU*-rank 1, the structure induced on each pair of classes is isomorphic to the Random Bipartite Graph, and for all $k \leq m$ and all k-sets of R-classes X, any S, T-graph of size k is realised as a transversal to X.*

Proof

The first assertion follows easily from transitivity (only one 1-type q_0 over $\varnothing$) and the fact that if $\varphi(x, \bar{a})$ is a formula not implying $x = a_i$ for some $a_i \in \bar{a}$, then φ does not divide over $\varnothing$, so the only forking extensions to the unique 1-type over $\varnothing$ are algebraic. To see this, consider any such $\varphi(x, \bar{a})$. We may assume that φ is not algebraic, as in that case we would already know that any extension of q_0 implying φ is algebraic and so of SU-rank 0. Let c realise this formula, $c \notin \bar{a}$. We wish to prove that $c \mathop{\smile\!\!\!\!\!\!|}\, \bar{a}$; by simplicity, this is equivalent to proving $\bar{a} \mathop{\smile\!\!\!\!\!\!|}\, c$.

Let $\varphi'(\bar{x}, c)$ be the formula isolating $\mathrm{tp}(\bar{a}/c)$. Consider any $\varnothing$-indiscernible sequence $I = (c_i : i \in \omega)$ such that $c \in I$. This is an infinite sequence contained in the R-class of c. Colour the elements of I according to the types they realise over $\bar{a}$. Since $\bar{a}$ is finite, there are only finitely many colours, and by the pigeonhole principle there is an infinite monochromatic subset I' of I. Then we have $I' \equiv_c I$ and I' is indiscernible over $\bar{a}$, so $\varphi(x, \bar{a})$ does not divide over $\varnothing$ and the SU-rank of q_0 (and therefore M) is 1.

The relation R is clearly stable in M, so S and T must be unstable. By instability, there are parameters a_i, b_i ($i \in \omega$) such that $S(a_i, b_j)$ holds iff $i \leq j$. Since R is stable, we have $T(a_i, b_j)$ for all $j < i$ in this sequence of parameters. If we consider the $a_i b_i$ as pairs of type S and colour the pairs of distinct pairs in the sequence by the type they satisfy over $\varnothing$, then using Ramsey's theorem we can extract an infinite $\varnothing$-indiscernible sequence of pairs, which we also call a_i, b_i. By indiscernibility, the new a_i and b_i form monochromatic cliques, which are of colour R because there are no other infinite monochromatic cliques in M. This proves that S and T are realised as transversals to any pair of R-classes. By homogeneity, all pairs of classes are isomorphic.

The relation R is clearly nonforking in M. By instability, both S and T are non-algebraic, so for any $a \in M$ the sets $S(a)$ and $T(a)$ contain infinite R-cliques. It follows that S and T are nonforking transversal relations, so by Proposition 3.3.7 the structure on any pair of R-classes is isomorphic to the Random Bipartite Graph. Using the Independence Theorem, we can embed any S, T-graph of size k as a transversal to a union of k R-classes, for any $k \leq m$. □

3.4 Primitive homogeneous 3-graphs of SU-rank 2

Let M be a simple homogeneous 3-graph of SU-rank 2 (the language is $\{R, S, T\}$). Of the three relations R, S, T (all of which are realised in M), we assume that R is stable and forking, and S, T are nonforking. This assumption (not needed in the proof of Theorem 3.4.2 below) is justified by a suitable version of the Stable Forking Conjecture in graphs of rank greater than or equal to 3. Given any $a \in M$, consider $R(a)$. This is a definable set of rank at most 1 by our assumptions on R and the rank of M. What is the structure of $R(a)$? The main theorem of this section is:

Theorem 3.4.1 *Suppose that supersimple binary homogeneous structures have stable forking, and let M be a primitive supersimple homogeneous 3-graph. Then the theory of M is of* SU*-rank 1.*

To prove Theorem 3.4.1, we prove first

Theorem 3.4.2 *There are no supersimple primitive homogeneous 3-graphs of* SU*-rank 2.*

Theorem 3.4.2 is proved by arguing first that R defines an equivalence relation on $R(a)$ with finitely many classes. We use the imprimitivity blocks of the R-neighbourhoods to define an incidence structure. This incidence structure is a semilinear space. The analysis divides into two main cases, depending on the R-diameter of the 3-graph; most of the work goes into proving the non-existence of primitive homogeneous supersimple 3-graphs of SU-rank 2 and R-diameter 2. The case with $\mathrm{diam}_R(M) = 3$ is considerably easier.

The proof of Theorem 3.4.1 rests on the possibility of defining the semilinear space. We use this observation to start an inductive argument on the rank of the structure. Theorem 3.4.2 is the basis for induction in that proof.

3.4.1 Preliminary notes, notation and assumptions

Our objective in this section is to study the structure of some countable homogeneous n-graphs (definition below) with supersimple theory of SU-rank 2. Most of our results are for 3-graphs, but some hold in more general contexts.

The symbol M denotes a countable homogeneous structure throughout this section. The language may vary, though. Most of the time we refer to the relational language $L = \{R, S, T\}$, where each relation is assumed binary, symmetric and irreflexive. For most of the section, we will assume that the SU-rank of $\mathrm{Th}(M)$ is 2; by a result of Assaf Peretz [35], the rank 2 elements in a supersimple ω-categorical theory have stable forking: the statement "$\mathrm{tp}(a/B)$ divides over $A \subseteq B$" is witnessed by a stable formula. In statements where the language is $\{R, S, T\}$, we assume that R is a forking relation ($R(a, b)$ implies $\mathrm{tp}(a/b)$ divides over $\varnothing$), and therefore stable. In view of Lachlan's classification of stable homogeneous 3-graphs (see Theorem 3.5.4), we may suppose that $\mathrm{Th}(M)$ is unstable. Since any Boolean combination of stable formulas is stable, it follows that both S and T are unstable, therefore nonforking. Statements for the language $\{R_1, \ldots, R_n\}$ may be more general and refer to ω-categorical homogeneous n-graphs. Note that if all relations are nonforking then a primitive structure M is random in the sense that all its minimal forbidden structures are of size 2 (examples: the Random Graph, Random n-edge-coloured graphs), by the Independence Theorem argument used in the proof of Theorem 3.3.3.

Recall that for any relation P and tuple $\bar{a}$, $P(\bar{a}) = \{\bar{x} \in M | P(\bar{a}, \bar{x})\}$. We sometimes refer to this set as the P-neighbourhood of $\bar{a}$. In Definition 3.1.12, we defined an n-graph to be a structure $(M, R_1, \ldots, R_n)$ in which each R_i is binary, irreflexive and symmetric; also, we assume that for all distinct $x, y \in M$ exactly one of the R_i holds and $n \geq 2$. Finally, if M is a homogeneous n-graph, we assume that for each $i \in \{1, \ldots, n\}$ there exist $a_i, b_i \in M$ such that $R_i(a_i, b_i)$ holds in M.

Some definitions:

Definition 3.4.3

1. *A* path *of colour i and length n between x and y is a sequence of distinct vertices $x_0, x_1, \ldots, x_n$ such that $x_0 = x$, $x_n = y$ and for $0 \leq j \leq n-1$ the edge (x_j, x_{j+1}) is of colour i.*

2. *Two vertices x, y in an edge-coloured graph $(M, R_1, \ldots, R_n)$ are* R_i-connected *if there exists a path of colour i between them; a subset A of M is R_i-connected if any $a, a' \in A$ are R_i-connected by a path in A. A maximal R_i-connected subset of M is an* R_i-connected component.

3. *The* R_i-distance *between two vertices x, y in an edge-coloured graph, denoted by $d_i(x, y)$, is the length of a minimal R_i-path between x and y (∞ if no such path exists). The* R_i-diameter *of an R_i-connected graph A is defined as the supremum of* $\{d_i(x, y) | x, y \in A\}$.

4. *An n-graph is* R-multipartite *with k ($k > 1$ possibly infinite) parts if there exists a (not necessarily definable) partition $P_1, \ldots, P_k$ of its vertex set into nonempty subsets such that if two vertices x, y are R-adjacent then they do not belong to the same P_i. We will say that G is* R-complete-multipartite *if G is R-multipartite with at least two parts and for all pairs a, b from distinct classes, $R(a, b)$ holds.*

5. *For any relation R, $n \in \omega$, and a, $R^n(a)$ is the set of vertices at R-distance n from a.*

6. *A* half-graph *for colour R with m pairs in an n-coloured graph M is a set of vertices $\{a_i : i \in m\} \cup \{b_i : i \in m\} \subset M$ such that $R(a_i, b_j)$ holds iff $i < j$.*

We often divide binary relations in two groups: forking and nonforking. We mean:

Definition 3.4.4 *Let $L = \{R_1, \ldots, R_n\}$ be a binary relational language. We say that R_i is a* forking *relation if $R(a, b)$ implies that $tp(a/b)$ forks over $\varnothing$. Otherwise, R_i is* nonforking.

By simplicity, forking and dividing coincide, so in our statements and arguments we usually prove or use dividing instead of forking. We assume that all relations in the language are realised in M.

3.4.2 More facts about homogeneous n-graphs

This is a short section with a few useful observations about homogeneous edge-coloured graphs.

Observation 3.4.5 *In any homogeneous transitive n-graph $(M, R_1, \ldots, R_n)$, if $R_i(a)$ is an R_i-complete graph, then for any $b \in R_i(a)$ we have $\{a\} \cup R_i(a) = \{b\} \cup R_i(b)$*

Proof
If $c \in R_i(b) \setminus R_i(a)$, then both a and c are in $R_i(b)$, which is R_i-complete by transitivity, and therefore $R_i(a, c)$ holds, contradiction. □

Observation 3.4.6 *If $(M, R_1, \ldots, R_n)$ is an ω-categorical n-graph, then each connected component of (M, R_i) has finite diameter.*

Proof
Each of the R_i-distances is preserved by automorphisms. If one of the connected components of (M, R_i) has infinite diameter, then there are infinitely many 2-types, contradicting ω-categoricity. □

As a consequence of this observation, in ω-categorical edge-coloured graphs the relation $E_i(x, y)$ which holds if there is a path of colour i between x and y is definable. Also, in primitive n-coloured graphs, each (M, R_i) is connected, since the equivalence relation $x \sim_{R_i} y$ that holds if x and y are R_i-connected is invariant under $\mathrm{Aut}(M)$.

Observation 3.4.7 *If $(M, R_1, \ldots, R_n)$, where $n > 1$, is a primitive homogeneous n-graph, then for all i with $1 \leq i \leq n$, the structure $R_i(a)$ is not R_i-complete.*

Proof
Suppose not. Then, using Observation 3.4.5 and homogeneity, there is i with $1 \leq i \leq n$

such that for all a, b with $R_i(a, b)$ we have $\{a\} \cup R_i(a) = \{b\} \cup R_i(b)$. Hence, $\{a\} \cup R_i(a)$ is an R_i-connected component. This contradicts primitivity, since $|R_i(a)| > 0$ and as $n > 1$, $\{a\} \cup R_i(a) \neq M$. □

Observation 3.4.8 *If $(M, R_1, \ldots, R_n)$ is a homogeneous n-graph, then the diameter of each connected component of (M, R_i) is at most n.*

Proof

Suppose there are $a, b \in M$ at R_i-distance $n + 1$, so there are distinct $a = x_0, x_1, \ldots, x_{n+1} = b$ such that $R_i(x_j, x_{j+1})$ for $0 \leq j \leq n$ and R_i does not hold in any other pair from $\{x_0, \ldots, x_{n+1}\}$. Then the n pairs (a, x_j) $(2 \leq j \leq n + 1)$ are coloured in $n - 1$ colours, so at least two of them have the same colour. Using homogeneity, there is an automorphism of M taking the pair with the smaller index in the second coordinate to the other pair, and therefore we can find a shorter path from a to b. □

Observation 3.4.9 *If $(M; R, S, T)$ is a countable transitive 3-graph of R-diameter 2 where all three predicates are realised, then S and T are realised in $R(a)$ for any $a \in M$.*

Proof

For any a, there exist $c_1, c_2 \in M$ such that $S(a, c_1)$ and $T(a, c_2)$. Since the R-diameter of the graph is 2, there exist $b_1, b_2 \in R(a)$ such that $R(b_1, c_1)$ and $R(b_2, c_2)$. Therefore, the triangles RRS and RRT are in $\mathrm{Age}(M)$. The conclusion follows by transitivity. □

Proposition 3.4.10 *Let $(M; R_1, \ldots, R_n)$ be an R_i-connected transitive homogeneous n-graph. If for some $a \in M$ the set $R_i(a)$ is R_i-complete-multipartite, then M is R_i-complete-multipartite (and in particular is not primitive).*

Proof

For simplicity, we will write R and not R_i. Note first that the partition of $R(a)$ is invariant over a, defined by $R(a,x) \wedge R(a,y) \wedge \neg R(x,y) =: E_a(x,y)$. Take any $b \in R(a)$. By homogeneity, $R(b)$ consists of a/E_b together with $R(a) \setminus (b/E_a)$. We claim that this is all there is in M. First note that there are no more classes in $R(b) \setminus R(a)$: if we had $c \in R(b) \setminus R(a)$ not E_b-equivalent to a, then by homogeneity we would have $R(a,c)$, contradicting $c \notin R(a)$. Therefore, $a/E_b \cup R(a)$ is an R-connected component of M; by connectedness, it is all of M, $\mathrm{diam}_R(M) = 2$, and $\neg R(x,y)$ is an equivalence relation. □

The following theorem was proved by Lachlan and Woodrow in [31]:

Theorem 3.4.11 (Lachlan-Woodrow 1980) *Let G be an infinite homogeneous graph. Then either G or G^c is of one of the following forms:*

1. *$I_m[K_n]$, where at least one of m, n is infinite,*
2. *Generic omitting K_{n+1},*
3. *Generic (the Random Graph)*

Remark 3.4.12 From this list, graphs in the first category are ω-stable of SU-rank 1 if only one of m, n is infinite; the graph $I_\omega[K_\omega]$ is of rank 2. The random graph is supersimple unstable of SU-rank 1, and the homogeneous K_n-free graphs are not simple.

Observation 3.4.13 *If $(M; R, S, T)$ is a homogeneous primitive simple 3-coloured graph in which R is a forking relation and S, T are nonforking, then there are no infinite S- or T-cliques in $R(a)$.*

Proof

By 3.1.5, an infinite S- or T-clique is a Morley sequence over $\varnothing$. By Proposition 1.2.10, since $R(x,a)$ divides over $\varnothing$, for any Morley sequence $(a_i : i \in \omega)$ the set $\{R(x,a_i) : i \in \omega\}$ is inconsistent. □

A related result in a richer language:

Proposition 3.4.14 *If* $(M; R_1, \ldots, R_n)$ *is a simple primitive homogeneous* n*-graph in which* $R_1, \ldots, R_m$ *are forking non-algebraic relations and* $R_{m+1}, \ldots, R_n$ *are nonforking, then for each* $1 \leq i \leq m$ *there exists a* j *in the same range, such that for any* $a \in M$*, the set* $R_i(a)$ *embeds infinite* R_j*-cliques.*

Proof
Consider $R_i(a)$ for any $a \in M$, with $1 \leq i \leq m$. As M is primitive, $R_i(x, a)$ is infinite (Observation 3.1.11). By simplicity, for any Morley sequence $(a_j : j \in \omega)$, the set $\{R_i(x, a_j) : j \in \omega\}$ is inconsistent. In a primitive homogeneous simple n-graph, any infinite clique of a nonforking colour is a Morley sequence over $\varnothing$ in any enumeration (by primitivity and the Independence Theorem). Therefore, in $R_i(a)$ there are no infinite cliques of any nonforking colour. By Ramsey's theorem, there are infinite monochromatic sets of pairs, which are necessarily of some forking colour. □

And a result we will later quote:

Proposition 3.4.15 *In a supersimple unstable primitive rank 1 homogeneous* n*-graph* $(M; R_1, \ldots, R_n)$*,* $n > 1$*, each of the* R_i *is unstable.*

Proof
In SU-rank 1 structures, forking is algebraic, so $\mathrm{tp}(a/b)$ forks iff over $\varnothing$ iff $a \in \mathrm{acl}(b) \setminus \mathrm{acl}(\varnothing)$. Therefore, each relation is non-algebraic, by primitivity, and so each relation is nonforking. Using the Independence Theorem to amalgamate partial structures over the empty set (cf. 3.1.4, 3.1.8), we can embed infinite half-graphs for each of the R_i into M, witnessing instability. See also Theorem 3.3.3. □

From these observations two possible pictures emerge for the structure of 3-graphs of rank 2 with relations R, S, T: either (M, R) has diameter 2, or it has diameter 3. In the latter case, since $\mathrm{Aut}(M)$ preserves the R-distance, for any $a \in M$ the sets $S(a)$ and $T(a)$ correspond to R-distance 2 and 3 from a, so $\mathrm{Aut}(M, R) = \mathrm{Aut}(M, R, S, T)$.

3.5 Semilinear 3-graphs of SU-rank 2

In this section we present the proof of Theorem 3.4.2. Assaf Peretz proved in [35] that the elements of SU-rank 2 in an ω-categorical supersimple structure have stable forking. Therefore we may assume that the forking relation R is stable. We cannot have more than one forking relation because we assume that each relation in the language isolates a 2-type, so by Peretz's theorem, if we had two forking relations then both would be stable, which would imply that the third (which is equivalent to the negation of the other two) is also a stable relation, and the theory of the homogeneous 3-graph would be stable; and by Theorem 3.5.4 (due to Lachlan), there are no primitive stable 3-graphs. Here we start a case-by-case analysis of these graphs.

Observation 3.5.1 *If M is a primitive supersimple ω-categorical relational structure of* SU*-rank 2, and R is a forking relation, then $R(a)$ is a set of rank 1.*

Proof
Given any $a \in M$, $R(a)$ is a set of rank at most 1. If it were of rank 0, then the set of solutions of $R(x, a)$ would be finite, and therefore any element satisfying it would be in the algebraic closure of a, impossible by Observation 3.1.11. Therefore, the rank of $R(a)$ is 1. □

Proposition 3.5.2 *Suppose that M is a simple primitive homogeneous R, S, T-graph, the formula $R(x, a)$ forks, and S, T are unstable, nonforking relations. Then M embeds K_n^R for all $n \in \omega$.*

Proof
Being K_n^R-free would either force $R(a)$ to be algebraic, contradicting primitivity by Observation 3.1.11, or contradict, by Ramsey's Theorem, Observation 3.4.13. □

Observation 3.5.3 *Equivalence relations definable in an* SU*-rank 1 structure cannot have infinitely many infinite classes.*

Proof

Let p be a type in $S_1(\varnothing)$, φ a formula defining an equivalence relation E_φ in M, and $a \in M$. If E_φ has infinitely many infinite classes, then the extension p_1 of p to $\{a\}$ which includes $\varphi(x, a)$ divides over $\varnothing$. Since each class is infinite, $\mathrm{SU}(p_1) \geq 1$, and therefore $\mathrm{SU}(p) \geq 2$, a contradiction. □

Note that if M is a simple 3-graph in which R is stable, then R is still a stable relation in the (homogeneous, simple) structure $R(a)$, since any model of the theory of $R(a)$ can be defined in a model of the original theory, and therefore witnesses for instability in $R(a)$ theory would also witness instability in the original theory.

What can we say about $R(a)$? We will show in the next section that the action of $\mathrm{Aut}(M/a)$ is imprimitive on $R(a)$, and that the vertices together with the imprimitivity blocks of their neighbourhoods form a semilinear space. In our argument, we will use Lachlan's classification of stable homogeneous 3-graphs:

Theorem 3.5.4 (Lachlan 1986, [28]) *Every stable homogeneous 3-graph is isomorphic to one of the following:*

1. P_{**}
2. Z
3. Z'
4. Q^i_*
5. P^i_*
6. $P^i[K^i_m]$
7. $K^i_m[Q^i]$
8. $Q^i[K^i_m]$
9. $K^i_m[P^i]$
10. $K^i_m \times K^j_n$
11. $K^i_m[K^j_n[K^k_p]]$

where $\{i, j, k\} = \{R, S, T\}$ *and* $1 \leq m, n, p \leq \omega$.

Items 1 to 5 are finite 3-graphs; for 6-11, if at least one of m, n, p is infinite, the 3-graph is infinite. We will not explain what Z, the asterisks, and primes mean, since we are concerned only with infinite graphs. In the j, k-graph P^i there are five vertices, and both the j-edges and the k-edges form a pentagon. The j, k-graph Q^i is defined on 9 vertices; the j- and k-edges form a copy of $K_3 \times K_3$.

For $1 \leq m, n \leq \omega$, $K^i_m \times K^j_n$ is the graph with vertex set $m \times n$ and relations

$$((a_1, b_1), (a_2, b_2)) \in \begin{cases} i & \text{if } a_1 \neq a_2 \wedge b_1 = b_2 \\ j & \text{if } a_1 = a_2 \wedge b_1 \neq b_2 \\ k & \text{if } a_1 \neq a_2 \wedge b_1 \neq b_2 \end{cases}$$

where we again assume $\{i, j, k\} = \{R, S, T\}$.

And if G, H are 3-graphs, then $G[H]$ is the 3-graph with vertex set $V(G) \times V(H)$ and in which the 3-graph induced on $\{(a, v) : v \in V(H)\}$ is isomorphic to H for each $a \in V(G)$, and for any function $f : V(G) \to V(H)$, the 3-graph induced on $\{(a, f(a)) : a \in V(G)\}$ is isomorphic to G. More formally, $P((a, b), (c, d))$ holds in $G[H]$ if $a = c$ and $H \models P(b, d)$, or if $G \models P(a, c)$, where $P \in \{R, S, T\}$.

We summarise some properties of some of these infinite stable homogeneous 3-graphs in the table in page 83. We present only those structures that may appear as $R(a)$ in a primitive homogeneous 3-graph.

3.5.1 Lines

In this subsection we define the main tool that we will use to eliminate candidates to be primitive homogeneous 3-graphs of SU-rank 2, a family of sets we call *lines*. Thus we interpret an incidence structure in M in which lines are infinite and each point belongs to a finite number of lines. It is tempting to try to see this structure as a pseudoplane and use a general result of Simon Thomas on the nonexistence of binary omega-categorical pseudoplanes (see [43]), but our incidence structure falls short of being a pseudoplane or even a weak pseudoplane, which is what Thomas uses in his proof. It is a semilinear space (see Definition 3.5.6), which under some conditions also qualifies as a generalised

Table 3.2: Some stable homogeneous 3-graphs

Structure	Equivalence relations	U-rank
$P^R[K^R_\omega]$	R	1
$K^R_\omega[Q^R]$	$S \vee T$	1
$Q^R[K^R_\omega]$	R	1
$K^R_\omega[P^R]$	$S \vee T$	1
$K^R_\omega \times K^S_n$	R, S	1
$K^R_\omega \times K^T_n$	R, T	1
$K^R_\omega[K^S_n[K^T_p]]$	$S \vee T, T$	1
$K^R_\omega[K^T_n[K^S_p]]$	$S \vee T, S$	1
$K^S_m[K^R_\omega[K^T_p]]$	$T \vee R, T$	1
$K^T_m[K^R_\omega[K^S_p]]$	$S \vee R, S$	1
$K^S_m[K^T_n[K^R_\omega]]$	$R \vee T, R$	1
$K^T_m[K^S_n[K^R_\omega]]$	$R \vee S, R$	1

quadrangle (cf. Observation 3.5.18, see the paragraph preceding it for the definition of generalised quadrangle).

Remark 3.5.5 By Observation 3.4.8, the R-diameter of a homogeneous 3-graph is at most 3. We are interested in "proper" 3-graphs, that is, structures in which all three colours are realised; therefore, we may assume that the R-diameter of M is either 2 or 3. If the R-diameter of M is 3, then, as the automorphism group of M preserves R-distance, we may adopt the convention that $S(a)$ and $T(a)$ correspond to $R^2(a)$ and $R^3(a)$ (cf. the paragraph after Proposition 3.4.15). Note that in R-diameter 3, the triangle RRT is forbidden, and therefore the R-neighbourhood of any vertex a is an R, S-graph, stable by the stability of R.

Definition 3.5.6 *A semilinear space S is a nonempty set of elements called* points *provided with a collection of subsets called* lines *such that any pair of distinct points is contained in at most one line and every line contains at least three points.*

Remark 3.5.7 As we have mentioned before, these structures are related to weak pseudoplanes. Given a structure M and a definable family $\mathcal{B}$ of infinite subsets of M, the incidence structure $P = (M, \mathcal{B})$ is a weak pseudoplane if for any distinct $X, Y \in \mathcal{B}$ we have $|X \cap Y| < \omega$ and each $p \in M$ lies in infinitely many elements of $\mathcal{B}$. The connection between our semilinear spaces and weak pseudoplanes is, then, that a semilinear space interpreted (i.e., the lines form a definable family of subsets of M) in a homogeneous structure in which each line is infinite and each point lies in infinitely many lines is a weak pseudoplane. In all the semilinear spaces that we will encounter in this chapter, lines are infinite and each point belongs to finitely many lines.

The rest of this chapter consists of a study of the properties of a semilinear space definable in homogeneous primitive 3-graphs of SU-rank greater than or equal to 2.

Proposition 3.5.8 *Let M be an infinite 3-graph such that $Aut(M)$ acts transitively on M, $R(a)$ is infinite for any a, and R is a transitive relation on $R(a)$ such that the reflexive closure of R on $R(a)$ has finitely many equivalence classes. Denote by $\ell(a, b)$ the maximal*

R-clique in M containing the R-edge ab. Then $(M, \mathcal{L})$, where $\mathcal{L} = \{\ell(a,b) : M \models R(a,b)\}$, is a semilinear space.

Proof

We start by justifying our use of *the* when we said that $\ell(a,b)$ is "the maximal R-clique in M containing the R-edge ab." Since we chave $R(a,b)$, we know that $b \in R(a)$, so it is an element of one of the finitely many classes of R in $R(a)$. Let b/R^a denote the R-equivalence class of b in $R(a)$; then $\{a\} \cup b/R^a$ is an infinite clique containing a, b. We claim that any clique containing a, b is a subset of $\{a\} \cup b/R^a$. To see this, let K be an R-clique containing a, b, and let $x \neq a, b \in K$. Such an x exists because R partitions an infinite set into finitely many subsets. Since K is a clique, we have that $x \in R(a)$, and as R defines an equivalence relation on $R(a)$ and $R(x,b)$ holds, we have that $x \in b/R^a$. Therefore, $x \in \{a\} \cup b/R(a)$ and $\ell(a,b)$ denotes this set.

So we have that two distinct points (vertices) belong to at most one element of $\mathcal{L}$. Any line contains at least three points, by transitivity of M and the fact that R forms infinite cliques within $R(a)$. □

Definition 3.5.9 *A 3-graph is* semilinear *if it satisfies the hypotheses of Proposition 3.5.8. In particular, whenever we refer to a semilinear 3-graph in this chapter we assume that points are incident with only finitely many lines.*

Definition 3.5.10 *If M is a semilinear 3-graph and $R(a,b)$ holds in M, then $\ell(a,b)$ is the imprimitivity block in $R(a)$ to which b belongs, together with the vertex a. Equivalently, it is the largest R-clique in M containing a and b. We refer to these sets as* lines.

We have introduced semilinear 3-graphs because a good deal of the analysis of homogeneous primitive 3-graphs of SU-rank 2 depends more on this combinatorial property than on any simplicity or rank assumptions. The next two results establish that anything we prove about semilinear 3-graphs is also true of homogeneous primitive 3-graphs of SU-rank 2.

Observation 3.5.11 *Suppose M is a primitive homogeneous supersimple 3-graph of* SU-*rank 2, where R is a forking relation, S, T are nonforking, and $a \in M$. Then $R(a)$ is imprimitive.*

Proof

If the R-diameter is 2, then all three predicates are realised in $R(a)$ (see Observation 3.4.9). By Proposition 3.5.1, $R(a)$ is a 3-graph of rank 1, so it cannot be primitive and unstable by Proposition 3.4.15, as it would embed infinite S-cliques, contradicting Observation 3.4.13. And by Lachlan's Theorem 3.5.4, $R(a)$ cannot be primitive and stable (see the table of 3-graphs without infinite S- or T-cliques in page 83).

If the R-diameter is 3, then $R(a)$ is a homogeneous RS-graph. It follows from the Lachlan-Woodrow Theorem 3.4.11 and simplicity that $R(a)$ is isomorphic to $I_m[K_\omega]$ or to $I_\omega[K_n]$ $(m, n \in \omega)$. □

Proposition 3.5.12 *If M is a homogeneous supersimple primitive 3-graph of* SU-*rank 2, then R defines an equivalence relation on $R(a)$ with finitely many infinite classes.*

Proof

We know from Observation 3.5.11 that $R(a)$ is imprimitive. By quantifier elimination and our assumption that exactly one of R, S, T holds for any pair of vertices in M, to show that R defines an equivalence relation on $R(a)$, an invariant equivalence relation on $R(a)$ is defined by a disjunction of at most two predicates from L. Our two main cases depend on the R-diameter of M.

I. If $\mathrm{diam}_R(M) = 3$, then $R(a)$ is a homogeneous R, S-graph, which must be stable since R is stable and in which both R and S are realised, by Observation 3.4.7. The formula $S(x, y)$ does not define an equivalence relation on $R(a)$ by Proposition 3.4.10. Therefore, R is an equivalence relation on $R(a)$ and by Observation 3.4.13, this equivalence relation has finitely many classes, each of which is infinite by homogeneity and the fact that $R(a)$ is an infinite set.

II. If $\mathrm{diam}_R(M) = 2$, then all predicates are realised in $R(a)$.

By Proposition 3.4.10 the relation $S \vee T$ does not define an equivalence relation.

If $R \vee S$ defines an equivalence relation on $R(a)$, then it must have finitely many classes as any transversal to $R \vee S$ is a T-clique and T does not form infinite cliques in $R(a)$. Each $R \vee S$-class in $R(a)$ is a homogeneous graph, so by the Lachlan-Woodrow Theorem 3.4.11 it must be of the form $K_n^S[K_\omega^R]$, since $K_n^R[K_\omega^S]$ is impossible because S forms infinite cliques in it. It follows that $R(a)$ is isomorphic to $K_m^T[K_n^S[K_\omega^R]]$, and R defines an equivalence relation on $R(a)$ with $m \times n$ infinite classes (see the table on page 83). The same argument shows that if $R \vee T$ defines an equivalence relation on $R(a)$, then R is also an equivalence relation there, with finitely many infinite classes.

If S defines an equivalence relation on $R(a)$, then it is a stable relation on $R(a)$, its classes are finite, and $R(a)$ is a stable 3-graph of one of the forms 6-11 from Lachlan's Theorem 3.5.4. We can eliminate all those stable graphs in which $S \vee T$, $R \vee S$, or $R \vee T$ defines an equivalence relation, since we have already dealt with those cases. In all other cases (see the table on page 83), R defines an equivalence relation with finitely many infinite classes.

□

Observation 3.5.11 and Proposition 3.5.12 tell us that in supersimple homogeneous primitive 3-graphs of SU-rank 2 the forking predicate R defines an equivalence relation on $R(a)$ with finitely many infinite classes. We summarise this in a lemma for easier reference:

Lemma 3.5.13 *Primitive homogeneous supersimple 3-graphs of* SU-*rank 2 are semilinear. The lines of the semilinear space are infinite and each point is incident with finitely many lines.*

Proof

By primitivity, none of the relations R, S, T is algebraic (cf. Observation 3.1.11), so $R(a)$

is infinite. The transitivity of the 3-graph follows trivially from primitivity. Observation 3.5.11 and Proposition 3.5.12 prove that (the reflexive closure of) R is an equivalence relation on $R(a)$ with finitely many infinite classes. □

We have defined a semilinear space over a homogeneous structure, but there is no reason for it to be homogeneous as a semilinear space. This observation differentiates our work from Alice Devillers' study of homogeneous semilinear spaces (see [12]).

In Devillers' formulation, a semilinear space is a two-sorted structure with one sort for points and another for lines; it is homogeneous if the usual condition on the extensibility of local isomorphisms between finite configurations of points and lines is satisfied.

Our semilinear space is defined in a primitive homogeneous supersimple 3-coloured graph. It is clear that we have two types of non-collinear points, corresponding to S- and T-edges in the coloured graph. If the diameter of the graph is 2, then we will see that $n = |R(c) \cap R(a)|$ and $m = |R(d) \cap R(a)|$ are not necessarily equal for $c \in S(a)$ and $d \in T(a)$, even though ac and ad are isomorphic as incidence structures. Any automorphism of the semilinear space extending the isomorphism $a \mapsto a, c \mapsto d$ would necessarily take $R(c) \cap R(a)$ to $R(d) \cap R(a)$, impossible. Thus, we cannot expect our linear spaces to be homogeneous in the sense of Devillers.

We will use the semilinear space to analyse the structure of SU-rank 2 graphs. Any two distinct vertices belong to at most one line and two distinct lines intersect in at most one vertex. Any given vertex belongs only to a finite number of lines, each of which is infinite. As a consequence:

Observation 3.5.14 *Suppose that M is a semilinear 3-graph and $a \in M$. Then for all $d \in R^2(a)$ and ℓ a line through a, $|R(d) \cap \ell| < 2$.*

Proof

If we had two different points b_1, b_2 on $\ell \cap R(d)$, then as we have $R(b_1, b_2)$ we get that b_1, b_2 belong to the same line through d. But then $b_1, b_2 \in \ell(a, b_1) \cap \ell(d, b_1)$, contradicting the fact, obvious from Definition 3.5.6 that the intersection of two distinct lines in a semilinear space is either empty or a singleton. □

The situation in primitive semilinear 3-graphs is essentially different from that in primitive structures of SU-rank 1. Compare our next observation with Proposition 3.3.1.

Observation 3.5.15 *Let M be a primitive homogeneous semilinear 3-graph. If the R-distance between a and b is 2, then* $\mathrm{acl}(a, b) \neq \{a, b\}$.

Proof

The vertices a and b belong to a finite number of lines. Since the R-distance from a to b is 2, there exists at least one element $c \in R(a)$ such that $R(c, b)$ holds. There is at most one such c in any line through a. These points are algebraic over a, b and distinct from them. □

Observation 3.5.14 implies that the lines of the semilinear space interpreted in a semilinear 3-graph do not form triangles.

The sets $R(a), S(a), T(a)$ are homogeneous in the language L, so having the same type over a is equivalent to being in the same orbit under $\mathrm{Aut}(M/a)$. Therefore, we cannot have more than 2 nested a-invariant/definable proper nontrivial equivalence relations in any of them, as we would need more than 3 types of edges to distinguish them. For the same reason, the number of lines through a that $R(c)$ meets for $c \in R^2(a)$ is invariant under a-automorphisms (which fix the set of lines through a) as c varies in an a-orbit.

3.5.2 The nonexistence of primitive homogeneous 3-graphs of R-diameter 2 and SU-rank 2

We know by Lemma 3.5.13 that finitely many lines are incident with any vertex $a \in M$ in a primitive supersimple homogeneous 3-graph of SU-rank 2. Recall from subsection 3.5.1 that that two lines intersect in at most one point (by Observation 3.5.14 or by the definition of a semilinear space). The main question to ask is: if a and b are not R-related, how many lines containing a can the R-neighbourhood of b meet?

Proposition 3.5.16 *Let M be a homogeneous primitive semilinear 3-graph with simple theory in which S and T are nonforking predicates, and suppose that* $\mathrm{diam}_R(M) = 2$. *If for every $b \in R(a)$ and each line ℓ through b other than $\ell(a,b)$ we have that $\ell \cap S(a)$ and $\ell \cap T(a)$ are both nonempty, then either $\ell \cap S(a)$ and $\ell \cap T(a)$ are both infinite, or one of them is of size 1 and the other is infinite.*

Proof

Clearly, at least one of $\ell \cap S(a)$ and $\ell \cap T(a)$ is infinite. Suppose for a contradiction that $1 < |\ell \cap S(a)| < \omega$. The formula $S(x,a)$ does not divide over $\varnothing$; therefore, for any indiscernible sequence $(a_i)_{i\in\omega}$ the set $\{S(x,a_i) : i \in \omega\}$ is consistent by simplicity. In particular when $R(a_0,a_1)$ holds. Therefore, $S(a)$ embeds infinite R-cliques and by homogeneity every R-related pair in $S(a)$ is in one such clique. Take any $c, c' \in \ell \cap S(a)$, and let X be an infinite R-clique in $S(a)$ containing them. Consider $d \in X \setminus \ell$; $X \subset \ell(c,d)$ and $b \notin \ell(c,d)$, and the same is true of c'. But both belong to $\ell(b,c)$. Therefore, there are two points which lie on two different lines, contradiction. □

Our next observation is crucial to proving that there are no homogeneous 3-graphs of rank 2 and diameter 2. We mentioned before that the incidence structure interpreted in M by the lines and vertices is close to being a generalised quadrangle. Recall that a generalised quadrangle (see [42]) is an incidence structure of points and lines with possibly infinite parameters s and t satisfying:

1. any two points lie on at most one line,
2. any line is incident with exactly $s+1$ points, and any point with exactly $t+1$ lines, and
3. if x is a point not incident with a line L, then there is a unique point incident with L and collinear with x.

In [33], Macpherson proves:

Theorem 3.5.17 *Let M be a homogenizable structure. Then it is not possible to interpret in M any of the following:*

(i) an infinite group,

(ii) an infinite projective plane,

(iii) an infinite generalised quadrangle, or

(iv) an infinite Boolean algebra.

Observation 3.5.18 *If M is homogeneous primitive semilinear 3-graph and $\mathrm{diam}_R(M) = 2$, then it is not the case that for all $b \in R^2(a)$ the set $R(b)$ intersects all lines containing a.*

Proof
In this case, the incidence structure interpreted in M with lines of the form $\ell(x, y)$ and vertices as points is a generalised quadrangle with infinite lines and as many lines through a point as R-classes in $R(a)$, contradicting Theorem 3.5.17. □

The following observation will help us find different points c, c' in $S(a)$ or $T(a)$ such that $R(c)$ and $R(c')$ meet the same lines through a. Recall that given a subset B of M, the group of all automorphisms of M fixing B setwise is denoted by $\mathrm{Aut}(M)_{\{B\}}$.

Observation 3.5.19 *Let M be a primitive homogeneous semilinear 3-graph of R-diameter 2. Let X be a set of lines incident with a. Then $\mathrm{Aut}(M/a)_{\{\bigcup X\}}$ acts transitively on $\ell \setminus \{a\}$ for all $\ell \in X$.*

Proof
Note that at least one of RSS, RTT is realised in $R(a)$. Assume without loss of generality that RSS is realised in $R(a)$. Let b, b' be elements of $R(a)$ satisfying $R(b, b')$. Enumerate the lines in X as $\ell_1, \ldots, \ell_k$, and assume $b, b' \in \ell_k \setminus \{a\}$. We can find elements $d_1 \in \ell_1, \ldots, d_{k-1} \in \ell_{k-1}$ such that $S(b, d_i) \wedge S(b', d_i)$ for $i \in \{1, \ldots, k-1\}$, so $\mathrm{tp}(b/d_1, \ldots, d_{k-1}) = \mathrm{tp}(b'/d_1, \ldots, d_{k-1})$. By homogeneity, there is an automorphism of M fixing $a, d_1, \ldots, d_{k-1}$ (and therefore fixing $\bigcup X$ setwise) taking b to b'. □

If only one of S, T is realised in the union of two lines through a, then each pair of R-classes in $R(a)$ is isomorphic to a complete bipartite graph (the parts of the partition are R-cliques and the edges are of colour S or T), so we have two orbits of pairs of lines through a. This is not the case if all relations are realised in the union of two lines through a.

Observation 3.5.20 *Let M be a primitive homogeneous simple semilinear 3-graph of R-diameter 2. If in $R(a)$ all relations are realised in the structure induced on a pair of lines through a, and there are m lines through a, then there is only one orbit of k-sets of lines over a, for all $k \leq m$.*

Proof

There are two cases, depending on whether we can find witnesses to the instability of S, T within $R(a)$.

If $R(a)$ is a stable structure, then it is isomorphic to $K_m^S \times K_\omega^R$ or to $K_m^T \times K_\omega^R]$, by Lachlan's Theorem 3.5.4, Observation 3.4.13, and the hypothesis that all relations are realised in the structure induced on a pair of incident lines. In any of these structures there are monochromatic transversal cliques of size m, so the observation follows by invariance and homogeneity.

If we can find witnesses to the instability of S, T within $R(a)$, then $R(a)$ is isomorphic to a simple unstable homogeneous 3-graph in which R defines a finite equivalence relation. By Proposition 3.3.8, we can embed transversal monochromatic cliques, and again the observation follows by invariance and homogeneity. □

Notice that if for some element $b \in R(a)$ and some line ℓ through b different from $\ell(a, b)$ the sets $\ell \cap S(a)$ and $\ell \cap T(a)$ are both nonempty, then by homogeneity we can transitively permute the lines through b whilst fixing ab, and therefore all lines through any $b \in R(a)$, except $\ell(a, b)$, meet both orbits over a in $R^2(a)$. Furthermore, the size of the intersections does not change and is either 1 or infinite, with at least one of them infinite. To put it differently, if one line through b (not $\ell(a, b)$) is almost entirely contained (the point b is assumed to be in $R(a)$) in $S(a)$, then each line is almost entirely contained in one orbit. Now we prove that all lines that meet $R^2(a)$ meet both $S(a)$ and $T(a)$.

We will need the following well-known fact from permutation group theory (see, for example, 2.16 in [4]) to strengthen Observation 3.5.19 :

Theorem 3.5.21 *Let G be a permutation group on a countable set Ω, and let A, B be finite subsets of Ω. If G has no finite orbits on Ω, then there exists $g \in G$ with $Ag \cap B = \varnothing$.*

Proposition 3.5.22 *Suppose that M is a primitive simple homogeneous semilinear 3-graph with m lines through each point, and let $X = \{\ell_i : 1 \leq i \leq k\}$ be a set of lines through $a \in M$. Then for any transversal A to the k lines in X, there exists a transversal B to X such that $B \cong A$ and $B \cap A = \varnothing$.*

Proof

This is a direct consecuence of Theorem 3.5.21 and Observation 3.5.19. □

Proposition 3.5.23 *Let M be a simple homogeneous primitive semilinear 3-graph of R-diameter 2, in which all predicates are realised in the structure induced on a pair of incident lines, and $a \in M$. Then for all $b \in R(a)$, each line $\ell \neq \ell(a, b)$ through b meets both $S(a)$ and $T(a)$.*

Proof

First note that it is not possible to have $R(b) \cap S(a) = \varnothing$ or $R(b) \cap T(a) = \varnothing$. To see this, suppose for a contradiction that $R(b) \cap S(a) = \varnothing$; moving b by homogeneity within $R(a)$, it follows that $R(b') \cap S(a) = \varnothing$ for all $b' \in R(a)$, contradicting the assumption that vertices in $S(a)$ are at R-distance 2 from a. Similarly, $R(b) \cap T(a) \neq \varnothing$. Therefore, this proposition can only fail if we have at least 3 lines through a.

Suppose for a contradiction that there are $m \geq 3$ lines incident with a and for all $b \in R(a)$ and $\ell \neq \ell(a, b)$ through b, $\ell \setminus \{b\} \subset S(a)$ or $\ell \setminus \{b\} \subset T(a)$. By Observation 3.5.18, we may assume that $k = |R(c) \cap R(a)| < m$ for all $c \in S(a)$. We define two binary relations on $S(a)$: for $c, c' \in S(a)$, $E(c, c')$ holds if $R(c)$ and $R(c')$ meet the same lines through a, and $C(c, c')$ holds if there exists $b \in R(a)$ such that b, c, c' are collinear. Given two elements $x, y \in S(a)$, let $\#(x, y)$ denote the number of R-classes in $R(a)$ that $R(x)$ and $R(y)$ meet in common, that is $\#(x, y) = |\{z \in R(x) \cap R(a) : \exists w(w \in R(y) \cap R(a) \wedge (R(w, z) \vee w = z))\}|$.

I. If $k = 1$, then there are at least four types of unordered pairs of vertices in $S(a)$. We prove this assertion as follows: let b_c denote the unique element in $R(c) \cap R(a)$ for $c \in S(a)$. The relations $\hat{P}(c, c')$ that hold if $P(b_c, b_{c'})$ is true ($P \in \{R, S, T\}$) are invariant and imply that b_c, c, c' are not collinear. It follows from the assumption that for all $\ell \neq \ell(a, b)$ through $b \in R(a)$ the set $\ell \setminus \{b\}$ is contained in $S(a)$ or in $T(a)$ and the first paragraph of this proof that C is also realised in $S(a)$. That gives us too many types of distinct unordered pairs of elements in $S(a)$.

II. If $2 \leq k < m$, then we have two subcases:

(i) If $m - k \geq 2$, then we can find at least five types of unordered pairs of elements in $S(a)$. The proof is as follows: Observation 3.5.20 implies that E is a nontrivial proper equivalence relation on $R(a)$ and that it has $\binom{m}{k}$ classes. Now we claim that there are at least four types of E-inequivalent elements in $S(a)$. By Observation 3.5.20 there exist pairs of elements $c, c' \in S(a)$ with $\neg E(c, c') \wedge \#(c, c') = k-1$. Using Proposition 3.5.22, we can find pairs which additionally satisfy $R(c) \cap R(c') \cap R(a) \neq \varnothing$ and $R(c) \cap R(c') \cap R(a) = \varnothing$. We can follow the same argument in the case $\#(c, c') = k - 2$ to find two more types of unordered pairs of distinct elements from $S(a)$, giving a total of at least five.

(ii) Suppose then that $m - k = 1$, so E has m equivalence classes. There is at least one line through b almost entirely contained in $T(a)$, so we are left with at most $m - 2$ lines through b distinct from $\ell(a, b)$ which may meet $S(a)$.

Claim 3.5.24 *$R(b)$ meets $m - 1$ E-classes in $S(a)$.*

Proof

We know that $R(b) \cap S(a) \neq \varnothing$. Let $c \in R(b) \cap S(a)$. By hypothesis, $|R(c) \cap R(a)| = m - 1$. Let X denote $R(c) \cap R(a)$. By Observation 3.5.20, we can find a-translates of X_i, $i \leq m-1$, to any of the $m-1$ sets of $m-1$ lines through a that include the R-class to which b belongs. And by the transitivity of $\mathrm{Aut}(M/a)$ on $R(a)$ we can find translates Y_i in those sets of lines such that $b \in Y_i$. By homogeneity, each of the automorphisms taking X to Y_i moves c to a new E-class.

Clearly, $R(b)$ does not meet the E-class of elements whose R-neighbourhoods meet all the lines in $R(a)$ except $\ell(a, b)$. □

As the lines are infinite and E has only finitely many classes, for each line ℓ through b that meets $S(a)$ there is at least one E-class that contains infinitely many elements of ℓ. Since we have at most $m - 2$ lines through b that meet $S(a)$ and $R(b)$ meets $m - 1$ E-classes, there is at least one line that meets more than one E-class. Note that if a line ℓ meets more than one E-class, then the intersection of ℓ with each of the E-classes it meets is infinite, by homogeneity as elements in each class have the same type over ab and at least one of the intersections of ℓ with an E-class is infinite.

Again by homogeneity (we can permute the lines over b that meet $S(a)$ whilst fixing ab), each line through b that meets $S(a)$ meets more than one class.

As $k \geq 2$, there exist $b_1, b_2 \in R(a)$ such that for some $c \in S(a)$ we have $\ell(b_i, c) \setminus \{b_i\} \subset S(a)$ $(i = 1, 2)$. Take $c' \in \ell(b_1, c) \cap S(a)$ and $c'' \in \ell(b_2, c) \cap S(a)$, both distinct from c and E-equivalent to c. Such elements exist because the intersections of lines through b_i with the E-class of c are infinite, by homogeneity and the fact that at least one of the intersections is infinite, so we have $E(c, c') \wedge R(c, c')$. Also, c' and c'' are not R-related (because the lines of the semilinear space do not form triangles, cf. Observation 3.5.14), but are E-equivalent since we have $E(c, c')$ and $E(c, c'')$, so at least one of $E(c', c'') \wedge S(c'c,'')$ and $E(c', c'') \wedge T(c', c'')$ is realised. This gives us at least two types of E-equivalent pairs.

Now we will show that there are at least two types of E-inequivalent pairs. By Proposition 3.5.22, we can find pairs of E-inequivalent elements with no common R-neighbours in $R(a)$ and also pairs of E-inequivalent elements with common R-neighbours in $R(a)$. Again, we get at least four types of unordered pairs of distinct elements from $S(a)$.

□

Proposition 3.5.25 *Let M be a primitive homogeneous semilinear 3-graph with supersimple theory and R-diameter 2, and assume that R is a forking relation and S, T are nonforking. Then each element is incident with at least three lines.*

Proof

Note first that if in any pair of lines through a only two of the predicates in the language are realised, then we get the result automatically because R, S, T are realised in $R(a)$ by the diameter 2 hypothesis. So we may assume that in the structure induced by M on a pair of lines through a all predicates are realised.

By Observation 3.4.7, each vertex belongs to at least two lines.

If $R(a)$ has exactly two imprimitivity blocks, then by homogeneity for any $b \in R(a)$ the set $R(b)$ consists of two infinite R-cliques as well, one of which is $\ell(a,b) \setminus \{b\}$. Therefore, $R(b) \cap R^2(a)$ is an infinite R-clique, and by Proposition 3.5.23, $R(b) \cap R^2(a)$ meets both $S(a)$ and $T(a)$, as by the diameter 2 hypothesis both S and T are realised in $R(a)$.

Claim 3.5.26 *For all $b \in R(a)$, $\ell^b \cap S(a)$ and $\ell^b \cap T(a)$ are infinite.*

Proof

Suppose that each vertex is incident with two lines. Then for all $b \in R(a)$, there is a unique line through b that meets $R^2(a)$; let ℓ^b denote that line, for each $b \in R(a)$.

Proposition 3.5.16 tells us that either $\ell^b \cap S(a)$ and $\ell^b \cap T(a)$ are both infinite, or one of them is of size 1 and the other is infinite. As this line is uniquely determined for each

$b \in R(a)$, if we had, say $|\ell^b \cap S(a)| = 1$ and $|R(c) \cap R(a)| = 1$ for all $c \in S(a)$, then this would establish a definable bijection between $R(a)$ and $S(a)$. This is impossible as the rank of $R(a)$ is lower than that of $S(a)$.

Therefore, in orded to establish the claim, we need to eliminate the case where $|\ell^b \cap S(a)| = 1$ and $|R(c) \cap R(a)| = 2$.

By Observation 3.5.18, if these conditions are satisfied then $|R(d) \cap R(a)| = 1$ for all $d \in T(a)$. Given any $c \in S(a)$, the set $R(c)$ consists of two infinite R-cliques by homogeneity; from these two cliques, two vertices belong to $R(a)$.

Therefore, for any $c \in S(a)$, all relations in the language are realised in $R(c) \cap T(a)$. Define $Q(d, d')$ on $T(a)$ to hold if there exists $c \in S(a)$ such that $R(d, c) \wedge R(d', c)$.

We claim that $Q \wedge R$, $Q \wedge S$, $Q \wedge T$ are realised in $T(a)$. The reason is that both lines through c are almost entirely contained in $T(a)$: c and the two vertices in $R(c) \cap R(a)$ are the only elements of $R(c)$ not in $T(a)$, since any other element of $R(c) \cap S(a)$ would be forced to be an element of $R(b_1)$ or of $R(b_2)$, contradicting $R(b) \cap S(a) = 1$ for all $b \in R(a)$. Our claim follows from the transitivity of $\mathrm{Aut}(M/c)$ on $R(c)$ and Theorem 3.5.21.

Now, since S does not divide over $\varnothing$ in M we must have the triangle SSR in $\mathrm{Age}(M)$ (otherwise, S would divide, as witnessed by an $\varnothing$-indiscernible sequence $(e_i)_{i \in \omega}$ with $R(e_0, e_1)$). Notice that we have an additional a-definable equivalence relation F on $T(a)$ with two classes, $F(d, d')$ holds if $R(d)$ and $R(d')$ meet the same line through a. If Q and F were satisfied simultaneously by a pair from $T(a)$ then $F \wedge R$, $F \wedge S$, $F \wedge T$ (realised because R, S, T are realised in the union of any two incident lines), and $\neg F$ already give us too many relations on $T(a)$. And if they are not simultaneously realised by any pair, then any F-equivalent pair is Q-inequivalent, so this together with the three relations from the preceding paragraph give us four types of unordered pairs of distinct elements from $T(a)$. $\square$

By Observation 3.5.18, we may also assume that $|R(c) \cap R(a)| = 1$ for all $c \in T(a)$.

Consider the relation $W(x, y)$ on $T(a)$ that holds if there exists a $b \in R(a)$ such that $R(b, x) \wedge R(y, b)$. This is clearly a symmetric and reflexive relation, and if $W(x, y)$ and $W(y, z)$, then there exist $b, b' \in R(a)$ such that $R(x, b) \wedge R(y, b)$ and $R(y, b') \wedge R(z, b')$. The hypothesis that $|R(c) \cap R(a)| = 1$ for all $c \in R^2(a)$ implies $b = b'$, as they are both R-related to y and in $R(a)$. Therefore, x, y, z are all collinear with b and $W(x, z)$. Given a vertex $c \in T(a)$ denote by b_c the unique element of $R(c) \cap R(a)$, and define $\hat{P}(c, c')$ on $T(a)$ if $P(b_c, b_{c'})$ holds for $P \in \{R, S, T\}$. This gives us at least four types of unordered pairs of distinct elements in $T(a)$: W-equivalent and three types of W-inequivalent pairs (corresponding to $\hat{R}, \hat{S}, \hat{T}$). □

Observation 3.5.27 *If M is a primitive homogeneous simple semilinear 3-graph with* $\mathrm{diam}_R(M) = 2$ *in which any point a is incident with at least three lines, then it is not possible for all $c \in R^2(a)$ to satisfy $|R(c) \cap R(a)| = 1$.*

Proof

In this case, the sets $R(b) \cap R^2(a)$, $b \in R(a)$, partition the set of maximal rank $R^2(a)$ into infinitely many infinite parts, each consisting of at least 2 infinite R-cliques. By Proposition 3.5.16, at least one of $S(a)$ and $T(a)$ is partitioned into infinitely many infinite R-cliques by the family of sets $\ell \setminus \{b\}$, where $b \in R(a)$ and ℓ is a line through b not containing a. We may assume it is $S(a)$.

Define the relation $Q(c, c')$ on $S(a)$ to hold if there exists $b \in R(a)$ such that $R(b, c) \wedge R(b, c')$ holds. We claim that Q is an equivalence relation. It is clearly symmetric and reflexive. Now suppose $Q(x, y) \wedge Q(y, z)$. Then there exist $b, b' \in R(a)$ such that $R(b, x) \wedge R(b, y)$ and $R(b', y) \wedge R(b', z)$, so $b = b'$ since $|R(y) \cap R(a)| = 1$. Therefore, $Q(x, z)$ holds.

A Q-equivalence class consists of a finite number $m > 1$ of R-cliques, because we assume that at least three lines are incident with a. Define the binary relations $\hat{P}(c, c')$ to hold if $\neg Q(c, c')$ and $P(b, b')$, where $\{b\} = R(c) \cap R(a)$, $\{b'\} = R(c') \cap R(a)$ and $P \in \{R, S, T\}$. This gives 3 types of Q-inequivalent pairs, plus at least two more types of Q-equivalent pairs (collinear and not collinear), so we have too many types of unordered pairs of elements from $S(a)$. □

By Observation 3.4.13, not all R-free structures can be embedded into $R(a)$. If $R(a)$ is a stable 3-graph, then it must be of one of the forms 6-11 in 3.5.4, as all the others are finite. Observation 3.4.13 implies that only one of m, n, p is ω (and the corresponding superindex is R).

The sets of relations realised with endpoints in different classes of an equivalence relation partition the set of types of pairs of classes in a homogeneous binary structure. In our case, there can be no more than 2 types of pairs of R-classes in $R(a)$. This is implicitly used in the proof of our next result:

Proposition 3.5.28 *There are no primitive supersimple homogeneous 3-graphs of R-diameter 2 such that all relations are realised in the union of any two maximal R-cliques in $R(a)$.*

Proof

By Propositions 3.5.18 and 3.5.27, we have two cases to analyse:

I. For some $c \in R^2(a)$, $|R(c) \cap R(a)| = 1$. By homogeneity, this is true for all the elements of the orbit of c under the action or $\mathrm{Aut}(M/a)$. Without loss of generality, assume $S(a, c)$. We can define $E(x, y)$ on $S(a)$ if $R(x)$ and $R(y)$ meet the same line through a, and refine this equivalence relation with $E'(x, y)$ if they meet the same line at the same point. These two are equivalence relations, and $E'(x, y) \to E(x, y)$. For E-inequivalent pairs, since both S and T are realised in $R(a)$, we can define $\hat{S}(x, y)$ and $\hat{T}(x, y)$ if S (respectively, T) holds between the elements of the intersections $R(x) \cap R(a)$ and $R(y) \cap R(a)$. Notice that both $\hat{S}$ and $\hat{T}$ are realised, as any element in $R(a)$ has a neighbour in $S(a)$. We have too many 2-types of distinct elements over a, since $E'(x, y) \wedge x \neq y, E(x, y) \wedge \neg E'(x, y), \hat{S}(x, y) \wedge \neg E(x, y), \hat{T}(x, y) \wedge \neg E(x, y)$ are all realised.

II. For no element b of $R^2(a)$ does $|R(b) \cap R(a)| = 1$ hold. Then, without loss of generality, the elements of $S(a)$ satisfy $|R(b) \cap R(a)| = k$, where $1 < k < m$.

Define $E(c, c')$ on $S(a)$ if $R(c)$ and $R(c')$ meet the same lines through a. There are two subcases to analyse:

(i) If $m - k \geq 3$, then define $P_i(c, c')$ on $S(a)$ for $0 \leq i \leq \min\{k, m-k\}$ to hold if $R(c) \cup R(c')$ meet a total of $k + i$ lines through a. The P_i are invariant under $\mathrm{Aut}(M/a)$ and mutually exclusive; therefore all cases with $\min\{k, m-k\} \geq 3$ are impossible, as we would get at least four types of pairs of distinct elements in $S(a)$. This leaves us with only one more possible case, namely $m - k \geq 3, k = 2$, since the case $m - k \geq 3, k = 1$ is covered in Case I.

Suppose then that $m - k \geq 3$ and $k = 2$. We claim that there are two types of pairs satisfying P_1. Let $\{b, b'\} = R(c) \cap R(a)$ for some $c \in S(a)$, and take any line ℓ through a not including b or b'. By homogeneity, there exists a $b'' \in \ell$ satisfying the same relation with b' as b. Therefore, there exists $c' \in S(a)$ satisfying $P_1(c, c')$ and the relation $Q(c, c')$ defined by $\exists x(R(a, x) \wedge R(c, x) \wedge R(c', x))$. Using Proposition 3.5.23, we can find pairs d, d' in $S(a)$ satisfying $P_1(d, d')$ and $R(d) \cap R(d') \cap R(a) = \varnothing$. Therefore, we have at least four types of pairs of distinct elements from $S(a)$, as the relations E, $P_1 \wedge Q$, $P_1 \wedge \neg Q$, P_2 are all realised.

(ii) Suppose $m - k = 1$. By Proposition 3.5.19, there exist unordered pairs of distinct elements satisfying E in $S(a)$, and P_1 (defined as in Case II(i)) is realised by homogeneity and Observation 3.5.20.

Notice that there are two types of pairs satisfying $P_1(c, c')$, namely those with $R(c) \cap R(c') \cap R(a) = \varnothing$, and those with $R(c) \cap R(c') \cap R(a) \neq \varnothing$. Both are realised by Proposition 3.5.22.

This leaves us with two possibilities: for distinct $c, c' \in S(a)$, either $E(c, c')$ implies $R(c) \cap R(c') \cap R(a) = \varnothing$ (this can happen if the structure on any pair of lines through a is that of a perfect matching and $R(c)$ picks a transversal clique of the matching colour), or we can have $E(c, c') \wedge R(c) \cap R(c') \cap R(a) \neq \varnothing$. In the latter case, we have found four types of pairs of unordered distinct elements from $S(a)$.

Therefore, assume that $E(c, c')$ implies $R(c) \cap R(c') \cap R(a) = \varnothing$ for all $c \neq c'$ in $S(a)$. We claim that this can only happen in the situation described before, namely if the structure on two lines is that of a matching and for all pairs $b, b' \in R(c) \cap R(a)$, the edge bb' is of the colour of the matching predicate, say T. This claim follows from the argument of Proposition 3.5.19: if for some edge bb' in $R(c) \cap R(a)$ we were able to find some b'' collinear with b' such that bb'' and bb' are of colour T, then by homogeneity we could find a c' E-equivalent to c with $b \in R(c) \cap R(c') \cap R(a)$.

It follows that in the situation we are considering T is an algebraic predicate in $R(a)$ and the set of K^T_{m-1} in $R(a)$ is in definable bijection with $S(a)$ by the function taking a T-clique $\bar{c}$ to the unique element of $\bigcap\{R(c) : c \in \bar{c}\} \cap S(a)$. This is impossible, since the rank of $S(a)$ is greater than that of the set of T-cliques in $R(a)$, as T is algebraic.

(iii) If $m - k = 2$, then the relations E, P_1, P_2 defined in Case II(i) are realised in $S(a)$. As in case (i), there are two types of pairs c, c' satisfying P_1: some with $R(c) \cap R(c') \cap R(a) \neq \varnothing$ and some with $R(c) \cap R(c') \cap R(a) = \varnothing$, by the same argument as in Case II(i).

□

Proposition 3.5.28 eliminates all cases where $R(a)$ is unstable, as in this case for some infinite R-cliques A, B in $R(a)$ the induced structure is isomorphic to the Random Bipartite Graph. But Proposition 3.5.28 also covers some stable cases (for example, if S or T is a perfect matching on the union of the two R-cliques). The only cases that remain are those in which $R(a)$ is stable and the induced structure on any pair of R-cliques in $R(a)$ is isomorphic to a complete bipartite graph, that is, those cases in which for all pairs of lines ℓ_1, ℓ_2 through a and all $(b_1, b_2), (c_1, c_2) \in \ell_1 \times \ell_2$, $\mathrm{tp}(b_1 b_2) = \mathrm{tp}(c_1 c_2)$. In all of these cases, $R(a)$ is stable.

Proposition 3.5.29 *Let M be a homogeneous primitive semilinear 3-graph of R-diameter 2 with finitely many lines through each point. If all types of pairs are realised in*

$R(a)$, but not in any pair of lines through a, then it is not possible for any $c \in R^2(a)$ to satisfy $|R(c) \cap R(a)| > 3$.

Proof

Claim 3.5.30 *If* $\operatorname{tp}(ac) = \operatorname{tp}(ac')$*, then* $\operatorname{tp}(R(c) \cap R(a)) = \operatorname{tp}(R(c') \cap R(a))$*.*

Proof

By homogeneity, there exists an automorphism $\sigma \in \operatorname{Aut}(M/a)$ taking $c \mapsto c'$; this automorphism takes $R(c) \cap R(a)$ to $R(c') \cap R(a)$. □

Claim 3.5.31 *Under the hypotheses of Proposition 3.5.29, the isomorphism type of $R(c) \cap R(a)$ for any $c \in R^2(a)$ depends only on the set of lines through a that $R(c)$ meets.*

Proof

By Observation 3.5.14, the set $R(c) \cap R(a)$ is transversal to a set of k lines through a, and by the hypotheses of Proposition 3.5.29, all transversals to the same set of k lines are isomorphic. □

Now suppose that for some $c \in S(a)$ we have $|R(c) \cap R(a)| > 3$. By Claim 3.5.30, the intersections of the R-neighbourhood of any two elements of $S(a)$ with $R(a)$ are isomorphic; let E be the (not necessarily proper) equivalence relation on $S(a)$ that holds for elements that meet the same set of lines through a. Claim 3.5.31 says that if $A = R(c) \cap R(a)$ for some $c \in S(a)$ and we take any other set B transversal to the same set of $k > 3$ lines then there exists an automorphism taking A to B over a that moves c to an E-equivalent element of $S(a)$. Therefore, the a-invariant relations $P_i(c, c')$ holding if $E(c, c') \wedge |R(c) \cap R(a)| = i$ for $i \in \{0, \ldots, k-1\}$ are all realised. As $k \geq 4$, this gives us too many invariant relations on pairs over a. This completes the proof of Proposition 3.5.29. □

Proposition 3.5.32 *There are no homogeneous primitive 3-graphs of* SU*-rank 2 and* R*-diameter 2.*

Proof

We know by Proposition 3.5.25 that the number m of lines through a is greater than or equal to 3, and that all types of pairs are realised in $R(a)$, but not in any pair of lines through a (Proposition 3.5.28). By Proposition 3.5.29, for all $c \in R^2(a)$ we have $|R(c) \cap R(a)| \leq 3$. Assume that $k = \max\{|R(c) \cap R(a)|, |R(d) \cap R(a)|\}$, where $c \in S(a)$ and $d \in T(a)$.

I. First we prove that $k = 3$ is impossible. Let $E(c, c')$ be the equivalence relation on $S(a)$ that holds if $R(c)$ and $R(c')$ meet the same lines through a. The key observation in this case is that the graph induced on $R(c) \cap R(a)$ is a finite homogeneous graph of size 3, so it must be a monochromatic triangle (see also Gardiner's classification [21] of finite homogeneous graphs).

   We start by arguing that E is always a proper equivalence relation on $S(a)$ if $k = 3$. By the preceding paragraph, $R(c) \cap R(a)$ is a complete graph in S or T. If E were universal in $S(a)$, then it follows either that there are only three lines through a (impossible as in that case one of the predicates would not be realised in $R(a)$), or, assuming without loss that $R(c) \cap R(a)$ is isomorphic to K_3^S, that $R(a)$ is isomorphic to $K_m^T[K_n^S[K_\omega^R]]$. In the latter case, we must have $n = 3$ because otherwise we could move by homogeneity the K_3^S corresponding to $R(c) \cap R(a)$ to another set of 3 lines in the same $R \vee S$-class and find E-inequivalent elements. Finally, if $m > 1$ then again we have that E is a proper equivalence relation, depending on which $R \vee S$-class in $R(a)$ the set $R(c)$ meets. We reach a contradiction in any case; E is a proper equivalence relation on $R(a)$.

   Suppose for a contradiction that for $c \in S(a)$ we have $|R(c) \cap R(a)| = 3$. Since E is a proper equivalence relation, we have at least 4 invariant and exclusive relations on $S(a)$: E-inequivalent and three ways to be E-equivalent, as we can define $I_i(c, c')$ on $S(a)$ to hold if $E(c, c')$ and $|R(c) \cap R(c') \cap R(a)| = i$ for $i \in \{0, 1, 2\}$ (these relations are realised because the intersection of the R-neighbourhoods of c and a

is a complete monochromatic graph, so any two transversals to the lines that $R(c)$ meets are isomorphic); this already gives us too many invariant relations on pairs from $S(a)$.

II. Assume $\max\{|R(c) \cap R(a)|, |R(d) \cap R(a)|\} \leq 2$ ($c \in S(a), d \in T(a)$). By Observation 3.5.27 and Proposition 3.5.25, it must be equal to 2. Suppose that the maximum is reached in $S(a)$. The equivalence relation $E(c, c')$ that holds on $S(a)$ if $R(c)$ and $R(c')$ meet the same lines through a is proper: since $m \geq 3$ and $k = 2$, we can use homogeneity to move an element of $R(c) \cap R(a)$ to any line not containing any elements of $R(c) \cap R(a)$; this automorphism moves c to an element of $S(a)$ that is not E-equivalent with c. Therefore we have at least four types of pairs on $S(a)$: two satisfying $E(c, c')$ (one with $R(c) \cap R(c') \cap R(a)$ empty, the other with $R(c) \cap R(c') \cap R(a)$ nonempty), and, similarly, two with $\neg E(c, c')$.

We have exhausted the list of possible cases. The conclusion follows. □

3.5.3 The nonexistence of primitive homogeneous 3-graphs of R-diameter 3 and SU-rank 2

By homogeneity, if the R-diameter of the graph is 3, then, since R-distance is preserved under automorphisms, if there are a, b, c such that $S(a, c) \wedge R(a, b) \wedge R(b, c)$, then all pairs c, c' with $S(c, c')$ consist of vertices at R-distance 2; and similarly $T(a)$ would be the set of vertices at R-distance 3 from a. From this point on, we will follow the conventions $S(a) = R^2(a)$ and $T(a) = R^3(a)$.

The situation in diameter 3 is considerably simpler than in diameter 2, as the sets $S(a)$ and $T(a)$ are more clearly separated. The first thing to notice is that if the R-diameter of M is 3, then RRT is a forbidden triangle, as T corresponds to R-distance 3.

Proposition 3.5.33 *Suppose that M is a semilinear homogeneous primitive 3-graph of R-diameter 3 and that each point a is incident with $m < \omega$ lines. Then it is not possible for any $b \in S(a)$ to be collinear with m elements from $R(a)$.*

Proof

The R-neighbourhood of b has m R-connected components by transitivity. But by homogeneity and diameter 3, b is adjacent to some element of $R^3(a)$. Therefore, if $R(b)$ meets each line through a, then $R(b)$ has at least $m+1$ R-connected components, contradicting homogeneity. $\square$

Proposition 3.5.34 *Let M be a semilinear homogeneous primitive 3-graph with $\mathrm{diam}_R(M) = 3$ and $m < \omega$ lines through each point, and let k denote $|R(b) \cap R(a)|$ for any $b \in S(a)$. Then $k = 1$.*

Proof

By Proposition 3.5.33, $k < m$. The main point here is that we get the conclusion of Observation 3.5.20 for free in this situation, as the intersection of any pair of lines through a with $R(a)$ is isomorphic to a complete bipartite graph (edges given by S, non-edges given by R). We can define an equivalence relation E on $S(a)$ holding for c, c' if $R(c)$ and $R(c')$ meet the same lines through a. By Proposition 3.5.33 and homogeneity, E is a nontrivial proper equivalence relation on $S(a)$ with $\binom{m}{k}$ classes. Notice that for any E-equivalent c, c', the isomorphism types of $R(c) \cap R(a)$ and $R(c') \cap R(a)$ are the same over a, and in fact are the same as the isomorphism type of any set transversal to k lines. Therefore, we can define $P_i(c, c')$ for $0 \leq i < k$ if $E(c, c') \wedge |R(c) \cap R(c') \cap R(a)| = i$. All of these relations are realised by homogeneity, and invariant over a. This implies $k \leq 2$.

Now we eliminate the case $k = 2$. If $|R(c) \cap R(a)| = 2$ for $c \in S(a)$, then by Proposition 3.5.33 we have at least 3 lines through a, and the relation E defined in the preceding paragraph is a proper nontrivial equivalence relation. By the same argument, there are at least two types of E-equivalent pairs, plus at least two types of E-inequivalent pairs, depending on whether the intersections of their R-neighbourhoods meet $R(a)$ or not. The conclusion follows. $\square$

The situation is similar to what we had in diameter 2 after Observation 3.5.18, but we have the additional information $|R(b) \cap R(a)| = 1$ for $b \in S(a)$.

Proposition 3.5.35 *Let M be a semilinear primitive homogeneous 3-graph of R-diameter 3 and $m < \omega$ lines through each point. Then $m = 2$.*

Proof

By Proposition 3.5.34, for any $b \in S(a)$ we have $|R(b) \cap R(a)| = 1$. Let m denote the number of lines through a. We know by Observations 3.4.7 and 3.5.11 that $m \geq 2$. Now suppose for a contradiction that $m \geq 3$. Define E_1, E_2 on $S(a)$ by

$$E_1(c, c') \leftrightarrow R(c) \cap R(a) = R(c') \cap R(a)$$

$$E_2(c, c') \leftrightarrow R(b, b') \vee b = b'$$

where $\{b\} = R(c) \cap R(a)$ and $\{b'\} = R(c') \cap R(a)$. The relation E_2 holds iff $R(c)$ and $R(c')$ intersect the same line through a; E_1 holds iff they meet $R(a)$ at the same point. There are m E_2-classes and each of them contains infinitely many E_1-classes. Since $m \geq 3$ and the R-diameter of M is 3, each E_1-class contains at least two infinite disjoint cliques, corresponding to the lines through a particular $b \in R(a)$. Therefore, we can define an invariant $F(c, c')$ if $E_1(c, c') \wedge R(c, c')$, breaking each E_1-class into finitely many R-cliques.

We have only three 2-types over a in $S(a)$, corresponding to R, S, T, but we need at least four invariant relations for these three nested equivalence relations. □

Proposition 3.5.36 *There are no primitive homogeneous 3-graphs of* SU*-rank 2 and R-diameter 3.*

Proof

We know by Propositions 3.5.34 and 3.5.35 that under the hypotheses of this proposition we have $|R(c) \cap R(a)| = 1$ for all $c \in S(a)$ and there are exactly two lines through each point in M. So far, the main characters in our analysis have been $R(a)$ and $R^2(a)$. Now the structure on $R^3(a)$ will also come into play. The structure of $S(a)$ in diameter 3 and a single element in $|R(a) \cap R(c)|$ consists, by Proposition 3.5.35, of two E_2-classes, each divided into infinitely many E_1-classes (R-cliques), where E_1, E_2 are as in the proof of Proposition 3.5.35. We have two subcases:

I. Suppose that S holds between E_1-classes contained in the same E_2-class. Take $d \in T(a)$. The set $R(d) \cap R^2(a)$ meets each E_1-class in at most one vertex and one E_2 class (T holds across E_2-classes; if $R(d) \cap R^2(a)$ met both E_2-classes, then the triangle RRT would be realised, contradicting our assumption that $T(a) = R^3(a)$). Therefore, we can define an equivalence relation on $T(a)$ with two classes: define

$$F(d, d') \leftrightarrow \exists(c, c' \in S(a))(c \in R(d) \cap S(a) \wedge c' \in R(d') \cap S(a) \wedge E_2(c, c'))$$

So $F(d, d')$ holds iff $R(d)$ and $R(d')$ meet the same E_2-class in $R^2(a)$. We have a further subdivision into cases, depending on how many E_1-classes $R(d)$ meets:

(i) If $|R(d) \cap R^2(a)| = 1$, then we can define on $T(a)$ two more equivalence relations:

$$F'(e, e') \leftrightarrow E_1(c, c')$$

$$F''(e, e') \leftrightarrow R(e) \cap S(a) = R(e') \cap S(a)$$

where $\{c\} = R(e) \cap S(a)$ and $\{c'\} = R(e') \cap S(a)$. The condition $|R(d) \cap R^2(a)| = 1$ ensures that these relations are transitive. Clearly, $F'' \to F' \to F$; and as there are two lines through any vertex, F is a proper nontrivial equivalence relation. To prove that F' and F'' are both realised and different, take any $c \in S(a)$. There are two lines incident with it, one of which is its E_1-class, together with some point from $R(a)$; the other line, ℓ, through c is almost entirely contained in $T(a)$. Two points on $\ell \cap T(a)$ satisfy F'', and F-equivalent points in $T(a)$ on lines through different elements from $S(a)$ satisfy $F' \wedge \neg F''$ if the elements from $S(a)$ belong to the same E_1-class, and they satisfy $F \wedge \neg F'$ if the elements from $S(a)$ are E_2-equivalent and S-related.This gives us three nested invariant equivalence relations in $T(a)$. This rules out the possibility of $|R(d) \cap R^2(a)| = 1$ in the situation of Case I(i).

(ii) If $R(d)$ meets more than one E_1-class, then by homogeneity, since any vertex lies on two lines, it has to intersect exactly two of E_1-classes. Note that $R(d) \cap S(a)$ is contained in a single E_2-class, because the triangle RRT is forbidden. Again, we find too many types realised on $T(a)$. For any pair $d, d' \in T(a)$, the number of E_1-classes that $R(d) \cup R(d')$ meets is invariant under a-automorphisms. Notice that it is not possible for $|(R'(d) \cup R(d')) \cap R^2(a)|$ to be 2, as in that case d and d' would belong to two different lines: by homogeneity, each element $c \in S(a)$ lies on two lines, one of which is its E_1-class; therefore, if $d, d' \in T(a)$ are such that $R(d,c) \cap R(d',c) \neq \varnothing$, then c, d, d' must be collinear. Define $F_1(d,d')$ on $T(a)$ if $R(d)$ and $R(d')$ meet the same two E_1-classes, and $P(d,d')$ if $R(d) \cap R(d') \cap S(a) \neq \varnothing$. There are pairs satisfying all of $F_1 \wedge P, F_1 \wedge \neg P, \neg F_1 \wedge P, \neg F_1 \wedge \neg P$, giving us four invariant relations on pairs from $T(a)$.

II. If T holds between E_1-classes contained in the same E_2-class, then S holds between E_2-classes (as each E_1-class is an R-clique). Again, we have two subcases, depending on $|R(d) \cap S(a)|$ for $d \in T(a)$:

(i) If $|R(d) \cap S(a)| = 1$ for $d \in T(a)$, then we can define an equivalence relation $E'(e,e')$ on $T(a)$ holding if $R(e)$ and $R(e')$ meet the same E_2-class in $S(a)$. We will show that we already have three invariant and mutually exclusive relations on unordered pairs in each of the E' classes. Define $\hat{R}, \hat{T}$ on $T(a)$ by $\hat{P}(e,e')$ iff P holds for the points in the intersection of $R(e)$ and $R(e')$ with $S(a)$ ($P \in \{R, T\}$), and $C(e,e')$ if e, e' are collinear with some $c \in S(a)$, which happens if $R(e) \cap R(e') \cap S(a) \neq \varnothing$. We would need at least one more predicate to separate the E'-classes.

(ii) And if $|R(d) \cap S(a)| = 2$ for $d \in T(a)$, then the intersection with each E_2-class is of size one, as otherwise the triangle RRT would be realised. Then we can count the total number of E_1-classes that $R(e)$ and $R(e')$ meet, which can be 4, 3, or 2. And in the cases where this number is 3 or 2, we have another two relations, depending on whether $R(e) \cap R(e') \cap S(a)$ is empty or not. Again, we find too many invariant and mutually exclusive relations on unordered pairs of distinct elements from $T(a)$.

□

We can now prove that no primitive homogeneous supersimple 3-graphs have SU-rank 2.

Theorem 3.5.37 *There are no homogeneous primitive simple 3-graphs of* SU*-rank 2.*

Proof

By Observation 3.4.8, the diameter of a primitive homogeneous simple 3-graph of SU-rank 2 is either 2 or 3. Propositions 3.5.32 and 3.5.36 say that both situations are impossible. □

3.6 Higher rank

We have now proved that there are no homogeneous primitive supersimple 3-graphs of SU-rank 2. In this section, we see that result as the basis for an inductive argument on the rank of the theory, under the assumption of stable forking. We remark that in the course of the proof of nonexistence of supersimple 3-graphs of rank 2, we only use the rank 2 hypothesis to prove that we can define in M a semilinear space with finitely many lines through each point. Also, for most of the analysis simplicity suffices, and we require supersimplicity only in Propositions 3.5.25 and 3.5.28 (and, indirectly, Proposition 3.5.32 because the proof uses 3.5.25 and 3.5.28); in these results we use the fact that the theory is ranked by SU, but the specific value of its rank is irrelevant.

Therefore, if we prove that supersimple homogeneous 3-graphs of rank 3 or greater are semilinear with finitely many lines through each point, then the rest of the argument from Section 3.5 is valid in higher rank.

Proposition 3.6.1 *Suppose that supersimple binary finitely homogeneous structures satisfy stable forking. Let M be a homogeneous primitive supersimple 3-graph of* SU*-rank $k \geq 2$. Then M is semilinear.*

Proof

Independently of the rank, if $\mathrm{diam}_R(M) = 3$, then $R(a)$ is a stable RS-graph. It cannot be primitive by Theorem 3.4.11. And S is not an equivalence relation by Proposition

3.4.10; therefore, R is an equivalence relation on $R(a)$ with finitely many infinite classes (by Proposition 3.4.13).

So we need only worry about those cases with $\mathrm{diam}_R(M) = 2$. We proceed by transfinite induction on k. The case $k = 2$ corresponds to Lemma 3.5.13. Suppose that up to $k \geq 3$, we know that there are no primitive homogeneous supersimple 3-graphs of SU-rank $k - 1$ (for $k = 3$, this is the content of Theorem 3.5.37).

If we are given a homogeneous primitive supersimple 3-graph of SU-rank $k + 1$ and R-diameter 2, then we may assume by stable forking that S and T are nonforking, so we know that $R(a)$ is a supersimple homogeneous 3-graph of rank at most k. It follows that either $R(a)$ is imprimitive or it is of rank 1 as a structure in its own right (it could have a higher rank as a subset of M due to external parameters). If $R(a)$ is imprimitive, the same arguments as in Proposition 3.5.12 show that R is an equivalence relation; by Observation 3.4.13, it has finitely many classes.

Now we argue that $R(a)$ is not primitive. By the induction hypothesis, if $R(a)$ were primitive, then its rank would be 1.

The structure on $R(a)$ cannot be stable, as in that case it would be one of Lachlan's infinite stable 3-graphs from Theorem 3.5.4, all of which are imprimitive.

And $R(a)$ cannot be isomorphic to a primitive unstable 3-graph of rank 1, as by Proposition 3.4.15 primitivity contradicts the stability of R. Therefore, $R(a)$ is imprimitive and R defines an equivalence relation on $R(a)$ with finitely many classes, by Observation 3.4.13. This proves the proposition for all successor ordinals $k \geq 3$.

If the SU-rank of the structure is a limit ordinal λ, and we know that there are no homogeneous supersimple primitive 3-graphs of rank γ for all ordinals γ satisfying $1 < \gamma < \lambda$, then we again have that $R(a)$ is, as a structure in its own right, a 3-graph of rank $\delta < \gamma$. It follows that it is either imprimitive or $\delta = 1$. The same arguments as in the proof for successor ordinals prove that R defines an equivalence relation with finitely many classes on $R(a)$. □

Proposition 3.6.1 tells us that we can define a semilinear space on M just as we did in subsection 3.5.1. The analysis from subsections 3.5.2 and 3.5.3 translates verbatim to this more general setting, as the rank hypothesis was used there only to ensure that M interprets a semilinear space. As a consequence,

Theorem 3.6.2 *Suppose that supersimple binary finitely homogeneous structures satisfy stable forking, and let M be a primitive supersimple homogeneous 3-graph. Then the theory of M is of* SU*-rank 1.*

□

Under stable forking, all the homogeneous supersimple unstable primitive 3-graphs of finite SU-rank have rank 1. We know from Section 3.3 that those are random, and that in the case of imprimitive structures with finitely many classes, the transversal relations in a pair of classes are null, complete or random. This gives us a reasonably clear image of what a classification should look like, but more work is needed to prove it, particularly in the class of imprimitive structures with infinite classes.

Our next conjecture is a tentative classification of supersimple homogeneous 3-graphs assuming stable forking. We do not mention stable forking as a hypothesis because we have reason to believe that homogeneous simple 3-graphs satisfy stable forking.

Conjecture 3.6.3 *The following is a list of all supersimple infinite transitive homogeneous n-graphs with $n \in \{2, 3\}$:*

1. *Stable structures:*
   (a) $I_\omega[K_n]$ *or its complement* $K_\omega[I_n]$ *for some* $n \in \omega + 1$
   (b) $P^i[K^i_m]$
   (c) $K^i_m[Q^i]$
   (d) $Q^i[K^i_m]$
   (e) $K^i_m[P^i]$

   (f) $K^i_m \times K^j_n$

   (g) $K^i_m[K^j_n[K^k_p]]$

2. *Unstable structures:*

   (a) *Primitive structures:*

      i. *The random graph* $\Gamma^{S,T}$

      ii. *The random 3-graph* $\Gamma^{R,S,T}$

   (b) *Imprimitive structures with infinite classes:*

      i. $K^R_m[\Gamma^{S,T}]$, $m \in \omega + 1$

      ii. $\Gamma^{S,T}[K^R_\omega]$

      iii. $B^{S,T}_n * K^R_\omega$, $n \in \omega + 1$, $n \geq 2$

   (c) *Imprimitive structures in which the equivalence relation has finite classes:*

      i. *Structures in which both unstable predicates are realised across any two equivalence classes:* $C(\Gamma)$

      ii. *Structures in which only one of the unstable predicates is realised across any two equivalence classes:* $\Gamma^{S,T}[K^R_n]$, $n \in \omega$.

Here $B^{S,T}_n * K^R_\omega$ is the 3-graph consisting of n copies of K^R_ω in which the structure on the union of any two maximal infinite R-cliques is isomorphic to the random bipartite graph, and all S, T-structures of size $k \leq n$ are realised transversally in the union of any k maximal infinite R-cliques. The meaning of $C(\Gamma)$ is explained below.

All the stable graphs and 3-graphs are in our list by Theorems 3.4.11 and 3.5.4, and Remark 3.4.12.

Given an supersimple unstable homogeneous graph or 3-graph Δ, if it is primitive then it has to be isomorphic to the Random Graph $\Gamma^{S,T}$ or the Random 3-graph $\Gamma^{R,S,T}$, by Proposition 3.4.15 assuming stable forking. If Δ is imprimitive and the equivalence classes are finite, it follows by instability and the stable forking hypothesis that the equivalence relation is defined by the stable relation R. This case is not very well understood, but the examples of 3-graphs of this form that we are aware of are finite covers of a reduct of some homogeneous graph. To give an example, enumerate the random graph as $\{w_i : i \in \omega\}$, and define a 3-graph $C(\Gamma)$ on countably many vertices $\{v_i : i \in \omega\}$ where R holds for pairs of vertices of the form $v_{2n}v_{2n+1}$,

$$S(v_i, v_j) \text{ if } \begin{cases} i \neq j, i = 2m, j = 2n, E(w_m, w_n) \\ i \neq j, i = 2m+1, j = 2n+1, E(w_m, w_n) \\ i \neq j, i = 2m, j = 2n+1, \neg E(w_m, w_n) \\ i \neq j, i = 2m+1, j = 2n, \neg E(w_m, w_n) \end{cases}$$

and all other pairs of distinct vertices satisfy T (E denotes the edge relation in the random graph). This structure is a finite cover in the sense of Evans (see [15], [16]) of a reduct of the random graph. Its theory is supersimple of rank 1, as it can be interpreted in $\Gamma \times \{0, 1\}$. The conjecture here is that given a finitely homogeneous binary structure G in which there is a proper nontrivial equivalence relation with finite classes, we can find a binary homogeneous structure H without any equivalence relations with finite classes such that G is a finite cover of a reduct of H. In the more restricted case of 3-graphs, we conjecture that $C(\Gamma)$ is the only homogeneous 3-graph with an equivalence relation with finite classes in which both S and T are realised in the structure induced on the union of two R-classes.

Continuing with our Δ, if the R-classes are finitely many and infinite, then, as the structure is unstable and homogeneous, it follows that the other two predicates are realised across any two R-classes. This case is almost completely covered by Proposition 3.3.8: Δ should be $B_n^{S,T} * K_\omega^R$ for some $n \geq 2$ in ω, though we still need to prove that these are the only homogeneous simple 3-graphs satisfying the conditions on transversals mentioned in Proposition 3.3.8. It seems likely that a little group theory and a study of homogeneous multipartite graphs (including the related but not identical structures in [24]) will settle the issue. The case with infinitely many infinite classes probably requires a different approach.

Finally, if Δ is imprimitive and the equivalence relation is defined as a disjunction of two predicates S, T in the language, then the other predicate R is stable and each class is a primitive and (because S, T are unstable) unstable graph. Therefore, each $S \vee T$-class is isomorphic to the random graph and Δ is isomorphic to $K_m^R[\Gamma^{S,T}]$ for some $m \in \omega + 1$.

§4. An asymptotic result

The work on this chapter is only tangentially related to the rest of the thesis. The main result can be stated informally as saying that almost all finite directed graphs in which any three vertices span at least one directed edge consist of two tournaments with some directed edges between them. This is a directed-graphs version of the following theorem by Erdős, Kleitman, and Rothschild (Theorem 2 in [14]):

Theorem 4.0.4 *Let T_n be the number of labelled triangle-free graphs on a set of n vertices, and S_n be the number of labelled bipartite graphs on n vertices. Then*

$$T_n = S_n(1 + o(\frac{1}{n})).$$

So the proportion of triangle-free graphs on n vertices that are not bipartite is negligible for large n.

Now we explain the link connecting this work to the rest of the thesis. Recall that a sentence σ is *almost surely true* (respectively, *almost surely false*) if the fraction $\mu_n(\sigma)$ of structures with universe $\{0, \ldots, n-1\}$ satisfying σ converges to 1 (0) as n approaches infinity. Fagin [17] proved:

Theorem 4.0.5 *Fix a relational language L. For every first-order sentence σ over L, $\mu_n(\sigma)$ converges to 0 or to 1.*

Given an L-sentence τ with $\mu_n(\tau) > 0$ for all n, denote by $\mu_n(\sigma|\tau)$ the conditional probability $\mu_n(\sigma|\tau) = \mu_n(\sigma \wedge \tau)/\mu_n(\tau)$. These conditional probabilities need not converge, but for some special cases they do converge. Given a relational language L and appropriate τ, let $T_{as}(L;\tau)$ be the set of L-sentences σ with $\lim_{n\to\infty}\mu_n(\sigma|\tau) = 1$. We call this the *almost sure theory of* L. It follows from Gaifman's [20] and Fagin's work that T is consistent and complete when τ is $\forall x(x = x)$; Fagin proved in [17] that T is also consistent and complete in the cases where L is the language $\{R\}$ and τ expresses one of the following:

1. R is a graph relation,
2. R is a tournament predicate symbol.

We can think of Fraïssé's construction as a way to associate a complete first-order theory with infinite models (the theory of the Fraïssé limit) to a countable hereditary family of finite structures with the JEP and AP; Fagin's theorem provides us with an alternative way of associating a first-order theory with a family of finite structures, namely the almost sure first-order theory of the language in question (possibly with some restrictions, represented by the sentences τ).

In the studied cases of simple binary relational structures (the random graph, random n-graphs, the random tournament), the almost sure theory coincides with the theory of the Fraïssé limit. On the other hand, in the known cases where τ is such that the conditional probabilities $\mu_n(\sigma|\tau)$ converge, and the class of finite structures satisfying τ is the age of a non-simple homogeneous structure, the almost sure theory is simple (in fact, supersimple of SU-rank 1). For example, it is known that the almost sure theory of triangle-free graphs is the theory of the Random Bipartite Graph (the proof has two stages, the first of which is Theorem 4.0.4; the second step is proving that almost all bipartite graphs satisfy the appropriate extension axioms); and whilst the generic triangle-free graph is not simple, the generic bipartite graph has supersimple theory of SU-rank 1. Similarly, the almost-sure theory of partial orders is, by a result due to Kleitman and Rothschild [26], the theory of the generic 3-level partial order in which every element of the bottom level is less than every element of the top level; this theory is supersimple of SU-rank 1.

Definition 4.0.6

1. *A* digraph *is a pair* (G, E) *where* G *is a set and* E *is a subset of* $G \times G$ *such that for all* $g \in G$ $(g, g) \notin E$ *and* $(g, g') \in E$ *implies* $(g', g) \notin E$*. We will often denote a digraph* (G, E) *by* G *and write* $g \to g'$ *if* $(g, g') \in E$*.*

2. *A digraph* G *is* I_3*-free if every subset of three distinct vertices spans at least one arrow.*

3. *A* tournament *is a digraph* G *in which for all distinct* x, y*, either* $x \to y$ *or* $y \to x$ *holds. A* bitournament *is a digraph whose vertex set can be partitioned into two tournaments* T_1, T_2 *(we allow arrows from one tournament to the other).*

4. *Given two vertices* x, y *in a digraph* G*, we write* $x \not\sim y$ *if* $x \not\to y$*,* $y \not\to x$*, and* $x \neq y$*. If* v *is a vertex in a digraph* G*, then* $\Delta(v) = \{x \in G : x \not\sim v\}$*. If* $Q \subset G$*, then* $\Delta(Q) = (\bigcup_{v \in Q} \Delta(v)) \setminus Q$*.*

5. *We denote the set of* I_3*-free digraphs on* $\{0, \ldots, n-1\}$ *by* $F(n)$*, and the set of bitournaments on the same set by* $T(n)$*.*

The following is the main theorem of this chapter.

Theorem 4.0.7 $|F(n)| = |T(n)|(1 + o(1))$

Remark 4.0.8 *Given an* I_3*-free digraph* (D, E)*, the graph* $(D, Q(D))$*, where* Q *is the set of pairs* $(d, d') \in D^2$ *such that* $(d, d'), (d', d) \notin E$ *and* $d' \neq d$*) is a triangle-free graph. Conversely, if we start with a triangle-free graph* G*, any orientation of the complement of* G *is an* I_3*-free digraph.*

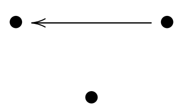
Proposition 4.0.9 *There exists a universal homogeneous* I_3*-free digraph* $\mathcal{D}$ *and it is a primitive structure.*

Proof

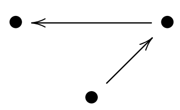
We will show that the family $\mathcal{C}$ of all finite I_3-free digraph satisfies Fraïssé's conditions. It is clear that $\mathcal{C}$ is countable (up to isomorphism) and closed under induced substructures.

Given two structures $A, B \in \mathcal{C}$, we can embed both A and B in the structure defined on $A \times \{0\} \cup B \times \{1\}$ where for all $b \in B$ and all $a \in A$ we have $R((a, 0), (b, 1))$. The amalgamation property follows from the fact that given an amalgamation problem $f_1 : A \to B$ and $g_1 : A \to C$, let D be $(B \times \{0\} \cup C \times \{1\})/ \sim$, where $(b, 0) \sim (c, 1)$ if there exists $a \in A$ such that $f_1(a) = b$ and $g_1(a) = c$, and define a digraph relation on D by $((p, i)/ \sim) \to ((q, j)/ \sim)$ if there exist representatives of the classes that are related in B or C, or if that condition fails and $i < j$.

If $\mathcal{D}$ were imprimitive, then the reflexive closures of $\not\sim$ or the relation $x \to y \vee y \to x$ would define an equivalence relation on $\mathcal{D}$, by quantifier elimination. But these relations are not transitive as $\mathcal{D}$ embeds the triangles



and



□

Proposition 4.0.10 *The theory of the universal homogeneous I_3-free digraph is not simple.*

Proof

We will prove that the formula $\psi(x, a, b) = x \not\sim a \wedge x \not\sim b$ has the TP2. Let $\{(a^i_j, b^i_j) : i, j \in \omega\}$ be an array of parameters such that $c^i_s \to c^i_t$ for $s < t$ and $c \in \{a, b\}$, $a^i_s \to b^i_t$ if $s \leq t$ and $a^i_s \not\sim a^j_t$ for $t < s$, and there are no other pairs satisfying $c^i_s \not\sim d^j_t$ $(c, d \in \{a, b\})$. Any such array of parameters can be embedded into the universal homogeneous I_3-free digraph as elements from different levels $L_i = \{(a^i_j, b^i_j) : j \in \omega\}$ are in a directed edge, and therefore no I_3 embeds into any level. Each level L_i witnesses 2-dividing for ψ, and each branch is a tournament. Therefore, ψ has the TP2. □

Remark 4.0.11 It is tempting to argue that given an I_3-free digraph, the associated graph obtained as in Remark 4.0.8 is almost always a bipartite graph, and so an orientation of its complement will be a bitournament. But formalising this argument is not as straightforward as it seems.

The general strategy we will follow consists of breaking up the set $F(n)$ into four parts: the bitournaments and three classes $A(n), B(n), C(n)$. We prove that as n tends to infinity the proportion of I_3-free digraphs in $A(n) \cup B(n) \cup C(n)$ becomes negligible. In this chapter, we use logarithms base 2, and when making assertions of the type $n = \log m$, where n is an integer, by $\log m$ we mean the integral part of $\log m$.

Definition 4.0.12

1. $A(n) = \{\Gamma \in F(n) : \exists v \in \Gamma(|\Delta(v)| \leq \log(n))\}$
2. $B(n) = \{\Gamma \in F(n) \setminus A(n) : \exists v \in \Gamma \exists Q \subset \Delta(v)(|Q| = \log(n) \wedge |\Delta(Q)| \leq (1/2 - 1/10^6)n)\}$
3. $C(n) = \{\Gamma \in F(n) \setminus (A(n) \cup B(n)) : \exists x, y \in \Gamma(x \not\rightarrow y \wedge \exists Q_x \subseteq \Delta(x), Q_y \subseteq \Delta(y)(|Q_x| = |Q_y| = \log(n) \wedge |\Delta(Q_x) \cap \Delta(Q_y)| \geq n/100))\}$

We follow the techniques and ideas from [38] and [37].

Observation 4.0.13 *Let G be an I_3-free digraph and $v \in G$. Then $\Delta(v)$ is a tournament, $v \in \Delta(\Delta(v))$, and $\Delta(v) \cap \Delta(\Delta(v)) = \varnothing$.*

Proof
There can be no undirected arcs between any elements of $\Delta(v)$ as any such pair would form an I_3 with v. Take any $y \in \Delta(v)$. Then $y \not\rightarrow v$, so $v \in \Delta(y) \subset \Delta(\Delta(v))$. And if $x \in \Delta(v) \cap \Delta(\Delta(v))$, then $x \not\rightarrow v$ and $x \not\rightarrow y$ for all $y \in \Delta(v)$, so xyv forms an I_3. □

Definition 4.0.14 *A* pinwheel *on n vertices $v_0, \ldots, v_{n-1}$ is a digraph in which $v_i \not\rightarrow v_{i+1}$ (addition is modulo n) for each $i \in n$. Equivalently, it is an orientation of the complement of a Hamiltonian graph on n vertices. We will abuse notation and denote a pinwheel by C_n even though there are several isomorphism types of pinwheels of the same size.*

Lemma 4.0.15 *If n is sufficiently large, then $F(n) \subseteq T(n) \cup A(n) \cup B(n) \cup C(n)$*

Proof

Suppose for a contradiction that for all n, $F(n)$ is a proper superset of $T(n) \cup A(n) \cup B(n) \cup C(n)$, and let $\Gamma \in F(n) \setminus (T(n) \cup A(n) \cup B(n) \cup C(n))$. This means that every vertex v in Γ, $|\Delta(v)| > \log(n)$ and all nonempty subsets Q of $\Delta(v)$ of size $\log n$ satisfy $|\Delta(Q)| > (1/2 - 1/10^6)n$. As $\Gamma \notin C(n)$, if $x \not\sim y$ and $x \neq y$, then $|\Delta(Q_x) \cap \Delta(Q_y)| < n/100$, where Q_x and Q_y are any subsets of $\Delta(x), \Delta(y)$ of size $\log(n)$.

Claim 4.0.16 Γ *contains no pinwheels C_5, C_7 or C_9.*

Proof

The idea of the proof is the same in all cases: if we had a pinwheel on $\{v_0, \ldots, v_{2m}\}$ for $m = 2, 3, 4$, then as Γ is not in $B(n) \cap A(n)$ we know that there is a subset Q_{v_i} of $\Delta(v_i)$ of size $\log n$ such that $R_{v_i} := \Delta(Q_{v_i})$ contains approximately half the vertices of the digraph. This implies that the R_{v_i} have large intersection for i even (odd), so the only way to satisfy that condition is if R_{v_0} contains almost all the vertices of the digraph, but then there are not enough vertices left for R_{v_1}. We present the formal proofs next.

Suppose that there is a C_5 on a set of vertices $\{v_0, \ldots, v_4\}$. Denote by R_{v_i} the set $\Delta(Q_{v_i})$, where $Q_{v_i} \subseteq \Delta(v_i)$ is of size $\log(n)$. For any distinct x, y with $x \not\sim y$,

$$|R_x \cup R_y| = |R_x| + |R_y| - |R_x \cap R_y| \geq n(1 - 2/10^6 - 1/100)$$

and

$$|\bar{R}_x \cap \bar{R}_y| = n - |R_x \cup R_y| \leq (2/10^6 + 1/100)n,$$

where $\bar{R}_x$ stands for the complement of R_x in the vertex set of Γ. Notice that as $|R_{v_1} \cap R_{v_2}| < n/100$ and $|\bar{R}_{v_0} \cap \bar{R}_{v_1}| \leq n(2/10^6 + 1/100)$, then

$$|R_{v_0} \cap R_{v_2}| \geq |R_{v_2}| - |R_{v_2} \cap R_{v_1}| - |\bar{R}_{v_0} \cap \bar{R}_{v_1}| \geq n(\frac{1}{2} - \frac{3}{10^6} - \frac{2}{100}).$$

Similarly, $|R_{v_0} \cap R_{v_3}| \geq n(\frac{1}{2} - \frac{3}{10^6} - \frac{2}{100})$. This gives us

$$\begin{aligned}|R_{v_0}| &= |R_{v_0} \cap R_{v_2}| + |R_{v_0} \cap R_{v_3}| - |R_{v_0} \cap R_{v_2} \cap R_{v_3}| + |R_{v_0} \cap \bar{R}_{v_2} \cap \bar{R}_{v_3}| \geq \\ &\geq n(\frac{1}{2} - \frac{3}{10^6} - \frac{2}{100}) + n(\frac{1}{2} - \frac{3}{10^6} - \frac{2}{100}) - \frac{n}{100} = \\ &= n(1 - \frac{6}{10^6} - \frac{5}{100})\end{aligned}$$

So $R(v_0)$ is almost all the digraph. Now,

$$\begin{aligned}|R_{v_1}| &= |R_{v_1} \cap \bar{R}_{v_0}| + |R_{v_1} \cap R_{v_0}| \leq \\ &\leq n(\frac{6}{10^6} + \frac{5}{100}) + \frac{n}{100} = \\ &= n(\frac{6}{10^6} + \frac{6}{100}) < n(\frac{1}{2} - \frac{1}{10^6}),\end{aligned}$$

which contradicts $\Gamma \notin B(n)$.

Suppose now that we have a pinwheel on 7 vertices $v_0, \ldots, v_6$. Our estimate for $|R_{v_0} \cap R_{v_2}|$ is still valid, and by the same argument we know $|R_{v_0} \cap R_{v_5}| > n(1/2 - 3/10^6 - 2/100)$. Now we estimate $|R_{v_0} \setminus R_{v_3}|$ (the calculations hold for $|R_{v_0} \setminus R_{v_4}|$ as well).

$$\begin{aligned}|R_{v_0} \cap R_{v_3}| &= |R_{v_0} \cap R_{v_2} \cap R_{v_3}| + |R_{v_0} \cap R_{v_3} \cap \bar{R}_{v_2}| \leq \\ &\leq |R_{v_3} \cap R_{v_2}| + |R_{v_0} \setminus R_{v_2}| < \\ &< \frac{n}{100} + |R_{v_0}| - |R_{v_0} \cap R_{v_2}| < \\ &< \frac{n}{100} - n(\frac{1}{2} - \frac{3}{10^6} - \frac{2}{100}) + |R_{v_0}|\end{aligned}$$

Therefore, $|R_{v_0} \setminus R_{v_3}| > n(1/2 - 3/10^6 - 3/100)$. Similarly, $|R_{v_0} \setminus R_{v_4}| > n(1/2 - 3/10^6 - 3/100)$. Now we use this information to get a new estimate of $|R_{v_0}|$.

$$\begin{aligned}|R_{v_0}| &= |R_{v_0} \setminus R_{v_3}| + |R_{v_0} \setminus R_{v_4}| - |R_{v_0} \cap \bar{R}_{v_3} \cap \bar{R}_{v_4}| + |R_{v_0} \cap R_{v_3} \cap R_{v_4}| \\ &> 2n(\frac{1}{2} - \frac{3}{10^6} - \frac{3}{100}) - n(\frac{2}{10^6} + \frac{1}{100}) \geq \\ &\geq n(1 - \frac{8}{10^6} - \frac{7}{100})\end{aligned}$$

Again, R_{v_0} contains almost all the vertices in Γ. As before, this contradicts $\Gamma \notin B(n)$:

$$|R_{v_1}| = |R_{v_1} \cap \bar{R}_{v_0}| + |R_{v_1} \cap R_{v_0}| < n(\frac{8}{10^6} + \frac{8}{100}) < n(\frac{1}{2} - \frac{1}{10^6})$$

Finally, suppose that there is a pinwheel on nine vertices in Γ. We know that $|R_{v_0} \cap \bar{R}_{v_3}| \geq n(1/2 - 3/10^6 - 3/100)$ and $|R_{v_0} \cap \bar{R}_{v_3} \cap \bar{R}_{v_4}| \leq n(2/10^6 + 1/100)$. From this, we derive

$$\begin{aligned}|R_{v_0} \cap \bar{R}_{v_3} \cap R_{v_4}| &= |R_{v_0} \cap \bar{R}_{v_3}| - |R_{v_0} \cap \bar{R}_{v_3} \cap \bar{R}_{v_4}| > \\ &> n(\frac{1}{2} - \frac{3}{10^6} - \frac{3}{100}) - n(\frac{2}{10^6} + \frac{1}{100}) = \\ &= n(\frac{1}{2} - \frac{5}{10^6} - \frac{4}{100})\end{aligned}$$

It follows that $|R_{v_0} \cap R_{v_4}| > n(\frac{1}{2} - \frac{5}{10^6} - \frac{4}{100})$, and by the same argument (going down the other side of the pinwheel), $|R_{v_0} \cap R_{v_5}| > n(\frac{1}{2} - \frac{5}{10^6} - \frac{4}{100})$. It follows that

$$|R_{v_0}| = |R_{v_0} \cap R_{v_4}| + |R_{v_0} \cap R_{v_5}| - |R_{v_0} \cap R_{v_4} \cap R_{v_5}| + |R_{v_0} \cap \bar{R}_{v_4} \cap \bar{R}_{v_5}| \geq$$
$$\geq 2n(\frac{1}{2} - \frac{5}{10^6} - \frac{4}{100}) - \frac{n}{100} = n(1 - \frac{1}{10^5} - \frac{9}{100})$$

As a consequence, $|\bar{R}_{v_0}| < n(\frac{1}{10^5} + \frac{9}{100})$. Therefore,

$$|R_{v_1}| = |R_{v_1} \cap \bar{R}_{v_0}| + |R_{v_1} \cap R_{v_0}| < n(\frac{1}{10^5} + \frac{9}{100}) + \frac{n}{100} =$$
$$= n(\frac{1}{10^5} + \frac{1}{10}) < n(\frac{1}{2} - \frac{1}{10^6}),$$

contradicting $\Gamma \notin B(n)$. □

Now we describe how to find a partition of Γ into two tournaments. For readability, we will use U_v to denote $\Delta(\Delta(v))$. Take an arbitrary non-arc $x \not\sim y$; then as Γ is I_3-free, $\Delta(x) \cap \Delta(y) = \varnothing$ and $U_x \cap U_y = \varnothing$ because any $z \in U_x \cap U_y$ would form a C_5 with x, y, x', y', for some $x' \in \Delta(x)$ and $y' \in \Delta(y)$. For the same reason (no C_5), U_x and U_y are tournaments. Let $W = V(\Gamma) \setminus (\Delta(x) \cup U_x \cup \Delta(y) \cup U_y)$, and $W_x = \{v \in W : R_v \cap R_x \neq \varnothing\}$, $W_y = \{v \in W : R_v \cap R_y \neq \varnothing\}$; again $W_x \cap W_y = \varnothing$ because there are no C_9s in Γ.

We know that $|R_x \cup R_y| \geq n(1 - 2/10^6 - 1/100)$, and since $\Gamma \notin B(n)$, for all $v \in W$ we have $|R_v| \geq n(1/2 - 1/10^6)$; therefore, $R_v \cap (R_x \cup R_y) \neq \varnothing$ and every vertex in W is in W_x or W_y. Our partition consists of $W_x \cup U_x \cup \Delta(y)$ and $W_y \cup U_y \cup \Delta(x)$.

We claim that $W_x \cup U_x \cup \Delta(y)$ is a tournament. By Observation 4.0.13, $\Delta(x)$ is a tournament.

Now consider $w \in \Delta(x)$ and $w' \in U_y$. We argued before that $U_x \cap U_y = \varnothing$, so $U_y \subseteq V(\Gamma) \setminus U_x$, and therefore $U_y \subseteq \bigcap_{v \in \Delta(x)}(V(\Gamma) \setminus \Delta(v))$, so there is a directed edge between w and w'. Thus, $U_y \cup \Delta(x)$ is a tournament.

If w and w' are in W_x, then there exist $p \in \Delta(v), p' \in \Delta(v'), q, q' \in R_x, r, r' \in \Delta(x)$. From all these vertices, $p \neq p'$ because the digraph is I_3-free. So we have $w \not\sim p \not\sim q$ and $w' \not\sim p' \not\sim q'$. If $q = q'$, then a directed edge is forced between w and w' because Γ is C_5-free. Similarly, an edge is forced if $r \neq r'$ because Γ is C_7-free. Finally, even if all the vertices are distinct, an edge is forced because $r, r' \in \Delta(x)$, so a C_9 would be formed if $w \not\sim w'$.

Finally, suppose for a contradiction that $w \in W_x, w' \in U_x$ and $w \not\sim u$. Then there exist $q_w \in Q_w$ and $r_w \in R_w \cap R_x$ such that $w \not\sim q_w \not\sim r_w$. We also have either a $\not\sim$-path of length 2 $r_w \not\sim v \not\sim u$ with $v \in \Delta(x)$ or a $\not\sim$-path of length 4 $r_w \not\sim d \not\sim x \not\sim d' \not\sim u$ with $d, d' \in \Delta(x)$; in the first case we get a C_5 and in the second, a C_7, contradicting in any case Claim 4.0.16. Therefore, Γ is a bitournament, contradiction. □

Lemma 4.0.17 $|T(n+1)| \geq 6^{n/2}|T(n)|$

Proof

There are $|T(n)|$ bitournaments on $[n]$. From a bitournament T on $[n]$, we can build a bitournament on $[n+1]$ by adding the vertex $n+1$ to the smaller of the tournaments in a given partition of T into two tournaments, which is of size at most $n/2$. Now we connect the vertex to the rest of the digraph: we need to make at least $3^{n/2}$ choices to connect it to the other tournament and at most $2^{n/2}$ choices to connect it to the smaller tournament. In total, at least $6^{n/2}$ choices for each tournament in $T(n)$, and the result follows. □

We wish to prove that the sets $A(n)$, $B(n)$, and $C(n)$ are negligible in size when compared to $T(n)$. The next step is to find bounds for their sizes relative to that of $F(n)$.

Lemma 4.0.18 *For sufficiently large n,* $\log(\frac{|A(n)|}{|F(n-1)|}) \leq n + \log^2 n + \log n - 1$

Proof

To construct a digraph in $A(n)$, we need to

1. Select a vertex v that will satisfy the condition in the definition of $A(n)$: n possible choices;
2. Select the neighbourhood $\Delta(v)$ of size at most $\log n$: $\sum_{i=0}^{\log n} \binom{n-1}{i}$ choices;
3. Choose a digraph structure on $[n] \setminus \{v\}$: $|F(n-1)|$ choices;

4. Connect v to $[n] \setminus \Delta(v)$: at most 2^{n-1} choices;

In total, this gives the following estimates:

$$|A(n)| \leq n(\sum_{i=0}^{\log n} \binom{n-1}{i})2^{n-1}|F(n-1)|$$
$$\leq nn^{\log n}2^{n-1}|F(n-1)|$$

So

$$\log(\frac{|A(n)|}{|F(n-1)|}) \leq \log n + \log^2 n + n - 1,$$

as desired. □

Lemma 4.0.19 *For sufficiently large n, $\log(\frac{|B(n)|}{|F(n-\log n)|}) \leq \beta n \log n + n + \frac{3}{2}\log^2 n - \frac{1}{2}\log n$, where $\beta = \frac{1+\alpha}{2} + \frac{1-\alpha}{10^6}$, and $\alpha = \log 3$.*

Proof
All the digraphs in $B(n)$ can be constructed as follows:

1. Choose a set Q of size $\log n$: $\binom{n}{\log n}$ choices;
2. Choose a tournament structure on Q: $2^{\binom{\log n}{2}}$ choices;
3. Choose a digraph structure on $[n] \setminus Q$: $|F(n - \log n)|$ choices;
4. Choose $R = \Delta(Q)$: at most 2^n choices;
5. Connect Q to R: $3^{(\log n)|R|}$ choices;
6. Connect Q to $[n] \setminus R$: $2^{(\log n)|[n]\setminus R|}$ choices

So we have

$$|B(n)| \leq \binom{n}{\log n} 2^{\binom{\log n}{2}} |F(n - \log n)| 2^n 3^{\log n|R|} 2^{\log n|[n]\setminus R|} \tag{4.1}$$

From this expression, the factor $3^{\log n|R|}2^{\log n|[n]\setminus R|}$ depends on the size of R. We claim that $3^{\log n|R|}2^{\log n|[n]\setminus R|}$, and therefore the expression 4.1, is maximised when $|R|$ is maximal, i.e., $|R| = n(1/2 - 1/10^6)$.

$$\log(3^{\log n|R|}2^{\log n|[n]\setminus R|}) = \alpha \log n|R| + \log n(n - |R|) = \\ = n \log n + (\alpha - 1)|R| \log n$$

This expression is, as a function of $|R|$, a linear polynomial with positive slope $(\alpha - 1)$. Therefore, the value of the expression in equation 4.1 is maximal when $|R| = n(1/2 - 1/10^6)$ is maximal, as claimed. Let us continue with the calculations:

$$|B(n)| \leq \binom{n}{\log n} 2^{\binom{\log n}{2}} |F(n - \log n)| 2^n 3^{\log n|R|} 2^{\log n|[n]\setminus R|} \leq \\ \leq \binom{n}{\log n} 2^{\binom{\log n}{2}} |F(n - \log n)| 2^n 3^{n \log n(\frac{1}{2} - \frac{1}{10^6})} 2^{n \log n(\frac{1}{2} + \frac{1}{10^6})} = \\ = \binom{n}{\log n} 2^{\binom{\log n}{2} + n + n \log n(\frac{1}{2} + \frac{1}{10^6})} |F(n - \log n)| 3^{n \log n(\frac{1}{2} - \frac{1}{10^6})}$$

Therefore,

$$\log(\frac{|B(n)|}{|F(n - \log n)|}) \leq \log \binom{n}{\log n} + \binom{\log n}{2} + n + n \log n(\frac{1}{2} + \frac{1}{10^6}) + \\ + \alpha n \log n(\frac{1}{2} - \frac{1}{10^6}) \leq \\ \leq \log^2 n + \frac{\log^2 n - \log n}{2} + n + n \log n(\frac{1}{2}(\alpha + 1) + \frac{1}{10^6}(1 - \alpha)) = \\ = \beta n \log n + n + \frac{3}{2} \log^2 n - \frac{1}{2} \log n.$$

□

Lemma 4.0.20 *For large enough n, $\log(\frac{|C(n)|}{|F(n-2)|}) \leq \gamma n + 2 \log^2 n + 2 \log n$, where $\gamma = 1 + \frac{4}{10^6} + \frac{3}{100} + \alpha(1 - \frac{2}{10^6} - \frac{2}{100})$*

Proof

Counting the elements in $C(n)$ is harder than counting $B(n)$ or $A(n)$, so we will give a rougher bound. All the elements in $C(n)$ can be found in the following way:

1. Choose two elements x, y, which will satisfy $x \not\sim y$: $n \times (n-1) < n^2$ choices.

2. Choose an I_3-free structure for $[n] \setminus x, y$: $|F(n-2)|$ options

3. Choose neighbourhoods Q_x, Q_y in $\Delta(x), \Delta(y)$ of size $\log n$. The $\not\sim$-neighbourhoods of x and y are disjoint because the digraph is I_3-free. So we have $\binom{n-1}{\log n}\binom{n-2-\log n}{\log n} \leq \binom{n-2}{\log n}^2 \leq n^{2\log n}$ choices. Notice that at this point the neighbourhoods R_x, R_y of Q_x and Q_y are determined by the I_3-free structure for $[n] \setminus \{x, y\}$, but we will only count those cases in which $|R_x \cap R_y| \geq \frac{n}{100}$.

4. Connect x, y to $[n]\setminus\{x, y\}$: We have already decided how to connect x, y to Q_x, Q_y, so we need to decide:

   (a) If $u \in R_x \cap R_y$, then there are only 4 possible ways to connect x, y to u.

   (b) If $u \in R_x \setminus R_y$ or $u \in R_y \setminus R_x$, then there are 6 possible ways to connect x, y to u.

   (c) If $u \in$ the complement of $R_x \cup R_y$, there are 8 ways to connect x, y to u.

   Therefore, we have

   $$4^{|R_x\cap R_y|}6^{|R_x\setminus R_y|+|R_y\setminus R_x|}8^{n-|R_x\cup R_y|} \tag{4.2}$$

   choices to make at this point. We claim that the expression 4.2 is maximised when $|R_x \cap R_y|$ and $|R_x \cup R_y|$ are minimised.

   $$\begin{aligned}&\log(4^{|R_x\cap R_y|}6^{|R_x|+|R_y|-2|R_x\cap R_y|}8^{n-|R_x|-|R_y|+|R_x\cap R_y|}) =\\ &2|R_x \cap R_y| + (1+\alpha)(|R_x| + |R_y| - 2|R_x \cap R_y|) + 3(n - |R_x| - |R_y| + |R_x \cap R_y|)\\ &= 3n + (3-2\alpha)|R_x \cap R_y| + (\alpha - 2)(|R_x| + |R_y|).\end{aligned}$$

   This is a linear polynomial in variables $|R_x \cap R_y|$, $|R_x|$,$|R_y|$, and is clearly maximal (in $(0, n]^3$) when the variables are minimised, as $3-2\alpha$ and $\alpha-2$ are both negative. By hypothesis, this happens when $|R_x \cap R_y| = \frac{1}{100}n$ and $|R_x| = |R_y| = n(1/2 - 1/10^6)$. Therefore, we have at most

$$
\begin{aligned}
&4^{|R_x\cap R_y|}6^{|R_x\setminus R_y|+|R_y\setminus R_x|}8^{n-|R_x\cup R_y|} = \\
&= 4^{|R_x\cap R_y|}6^{|R_x|+|R_y|-2|R_x\cap R_y|}8^{n-(|R_x|+|R_y|-|R_x\cap R_y|)} \leq \\
&\leq 4^{\frac{1}{100}n}6^{n(1-\frac{2}{10^6}-\frac{2}{100})}8^{n(\frac{2}{10^6}+\frac{1}{100})} = \\
&= 2^{\frac{2}{100}n}2^{n\log 6(1-\frac{2}{10^6}-\frac{2}{100})}2^{3n(\frac{2}{10^6}+\frac{1}{100})} = \\
&= 2^{n\log 6(1-\frac{2}{10^6}-\frac{2}{100})+\frac{2}{100}n+3n(\frac{2}{10^6}+\frac{1}{100})} = \\
&= 2^{n(1+\alpha)(1-\frac{2}{10^6}-\frac{2}{100})+\frac{2}{100}n+3n(\frac{2}{10^6}+\frac{1}{100})} = \\
&= 2^{n(1+\frac{4}{10^6}+\frac{3}{100}+\alpha(1-\frac{2}{10^6}-\frac{2}{100}))} = \\
&= 2^{\gamma n}
\end{aligned}
$$

ways to connect x, y to the rest of the digraph.

In total, this gives us

$$\frac{|C(n)|}{|F(n-2)|} \leq n^2 n^{2\log n} 2^{\gamma n}$$

So

$$\log(\frac{|C(n)|}{|F(n-2)|}) \leq 2\log n + 2\log^2 n + \gamma n$$

$\square$

Theorem 4.0.21 $|F(n)| = |T(n)|(1 + o(1))$.

Proof

Set $\eta = 2^{\frac{1}{3000}}$. We will prove that there exists a constant $c \geq 1$ such that for all n,

$$|F(n)| \leq (1 + c\eta^{-n})|T(n)| \tag{4.3}$$

holds. Let n_0 be a natural number large enough for all our estimates from Lemmas 4.0.18 to 4.0.20 to hold, and choose a $c \geq 1$ such that $|F(n)| \leq (1 + c\eta^{-n})|T(n)|$ for all $n \leq n_0$. We use this as a basis for induction on n.

Suppose that for all $n' < n$ equation 4.3 holds. From Lemma 4.0.15, we have

$$|F(n)| \leq |T(n)| + |A(n)| + |B(n)| + |C(n)|$$

If we show that the ratio $\frac{|X(n)|}{|T(n)|}$, where X is any of A, B, C, is at most $\frac{c}{3}\eta^{-n}$, the result will follow. We will use Lemmas 4.0.18 to 4.0.20 and induction to prove these bounds.

1.

$$\begin{aligned}\frac{|A(n)|}{|T(n)|} &= \frac{|A(n)|}{|F(n-1)|}\frac{|F(n-1)|}{|T(n-1)|}\frac{|T(n-1)|}{|T(n)|} \leq \\ &\leq 2^{n+\log^2 n+\log n-1}(1+c\eta^{-(n-1)})6^{-\frac{1}{2}(n-1)} \leq \\ &\leq 2c2^{n+\log^2 n+\log n-1}2^{-\frac{\log 6}{2}(n-1)} = \\ &= c2^{n+\log^2 n+\log n-\frac{1+\alpha}{2}(n-1)} = \\ &= c2^{n(\frac{1-\alpha}{2})+\log^2 n+\log n+\frac{\alpha+1}{2}}\end{aligned}$$

The leading term in the exponent of 2 is $n(\frac{1-\alpha}{2})$. Notice that $1-\alpha<0$, so as n_0 is assumed to be a very large number,

$$c2^{n(\frac{1-\alpha}{2})+\log^2 n+\log n+\frac{\alpha+1}{2}} \leq \frac{c}{3}\eta^{-n}$$

2.

$$\begin{aligned}\frac{|B(n)|}{|T(n)|} &= \frac{|B(n)|}{|F(n-\log n)|}\frac{|F(n-\log n)|}{|T(n-\log n)|}\prod_{i=1}^{\log n}\frac{|T(n-i)|}{|T(n-i+1)|} \leq \\ &\leq 2^{\beta n\log n+n+\frac{3}{2}\log^2 n-\frac{1}{2}\log n}(1+c\eta^{-(n-\log n)})\prod_{i=1}^{\log n}6^{-\frac{1}{2}(n-i)} \leq \\ &= 2^{\beta n\log n+n+\frac{3}{2}\log^2 n-\frac{1}{2}\log n}(1+c\eta^{-(n-\log n)})6^{-\frac{1}{2}(\sum_{i=1}^{\log n}(n-i))} \leq \\ &\leq 2^{\beta n\log n+n+\frac{3}{2}\log^2 n-\frac{1}{2}\log n}(1+c\eta^{-(n-\log n)})6^{-\frac{1}{2}(\log n(n-\frac{1}{2}\log n+1))} \leq \\ &\leq c2^{\beta n\log n+n+\frac{3}{2}\log^2 n-\frac{1}{2}\log n+1}6^{-\frac{1}{2}(\log n(n-\frac{1}{2}\log n+1))} = \\ &= c2^{\beta n\log n+n+\frac{3}{2}\log^2 n-\frac{1}{2}\log n+1+(1+\alpha)(-\frac{1}{2}(\log n(n-\frac{1}{2}\log n+1)))}\end{aligned}$$

For readability, we will continue our calculations on the exponent of 2 until we reach a more manageable expression:

$$\begin{aligned}
&\beta n\log n+n+\frac{3}{2}\log^2 n-\frac{1}{2}\log n+1+(1+\alpha)(-\frac{1}{2}(\log n(n-\frac{1}{2}\log n+1)))= \\
&\quad=\frac{3}{2}\log^2 n-\frac{1}{2}\log n+n+\beta n\log n+1-\frac{1}{2}n\log n-\frac{1}{4}\log^2 n- \\
&\quad\quad-\frac{1}{2}\log n-\frac{\alpha}{2}n\log n+\frac{\alpha}{2}\log^2 n-\frac{\alpha}{2}\log n= \\
&\quad=(\frac{3+\alpha}{2}-\frac{1}{4})\log^2 n+n\log n(\beta-\frac{1}{2}-\frac{\alpha}{2})- \\
&\quad\quad-\log n(1+\frac{\alpha}{2})+n+1= \\
&\quad=n\log n(\beta-\frac{1}{2}-\frac{\alpha}{2})+n+\frac{5+2\alpha}{4}\log^2 n-\log n(\frac{2+\alpha}{2})+1= \\
&\quad=\frac{1-\alpha}{10^6}n\log n+n+\frac{5+2\alpha}{4}\log^2 n-\log n(\frac{2+\alpha}{2})+1
\end{aligned}$$

Therefore,

$$\frac{|B(n)|}{|T(n)|}\le c2^{\frac{1-\alpha}{10^6}n\log n+n+\frac{5+2\alpha}{4}\log^2 n-\log n(\frac{2+\alpha}{2})+1}$$

The leading term in the exponent is $\frac{1-\alpha}{10^6}n\log n$, and $1-\alpha<0$. For sufficiently large n,

$$c2^{\frac{1-\alpha}{10^6}n\log n+n+\frac{5+2\alpha}{4}\log^2 n-\log n(\frac{2+\alpha}{2})+1}<\frac{c}{3}\eta^{-n}$$

3.

$$\begin{aligned}
\frac{|C(n)|}{|T(n)|}&=\frac{|C(n)|}{|F(n-2)|}\frac{|F(n-2)|}{|T(n-2)|}\frac{|T(n-2)|}{|T(n-1)|}\frac{|T(n-1)|}{|T(n)|}\le \\
&\le 2^{\gamma n+2\log n+2\log^2 n}(1+c\eta^{-(n-2)})6^{-\frac{1}{2}(n-2)}6^{-\frac{1}{2}(n-1)}\le \\
&\le 2^{\gamma n+2\log n+2\log^2 n}2c6^{-\frac{1}{2}(n-2)}6^{-\frac{1}{2}(n-1)}= \\
&=2^{\gamma n+2\log n+2\log^2 n}2c6^{-\frac{1}{2}(2n-3)}= \\
&=c2^{\gamma n+2\log n+2\log^2 n+1-\frac{\log 6}{2}(2n-3)}= \\
&=c2^{(\gamma-\log 6)n+2\log n+2\log^2 n+1+\frac{3}{2}\log 6}
\end{aligned}$$

Now, $\gamma-\log 6=1+\frac{4}{10^6}+\frac{3}{100}+\alpha(1-\frac{2}{10^6}-\frac{2}{100})-(1+\alpha)=\frac{4}{10^6}+\frac{3}{100}-\frac{2\alpha}{10^6}-\frac{2\alpha}{100}<0$, so $\frac{|C(n)|}{|T(n)|}<\frac{c}{3}\eta^{-n}$. Therefore,

$$\frac{|F(n)|}{|T(n)|}\le 1+c\eta^{-n}$$

and we conclude that the proportion of I_3-free digraphs on n vertices which are not bitournaments becomes negligible as n tends to infinity. □

Bibliography

[1] BEN-YAACOV, I., TOMAŠIĆ, I., AND WAGNER, F. O. The group configuration in simple theories and its applications. *Bulletin of Symbolic logic 8*, 2 (2002), 283–298.

[2] BEYARSLAN, O. Random hypergraphs in pseudofinite fields. *J. Inst. Math. Jussieu 9* (2010), 29–47.

[3] BROWER, D., AND HILL, C. D. On weak elimination of hyperimaginaries and its consequences. *arXiv preprint arXiv:1210.7883* (2012).

[4] CAMERON, P. J. *Oligomorphic permutation groups*, vol. 152 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1990.

[5] CASANOVAS, E. *Simple theories and hyperimaginaries*, vol. 39 of *Lecture Notes in Logic*. Cambridge University Press, 2011.

[6] CASANOVAS, E., AND WAGNER, F. O. Local supersimplicity and related concepts. *Journal of Symbolic Logic 67*, 2 (2002), 744–758.

[7] CHERLIN, G., HARRINGTON, L., AND LACHLAN, A. H. $\aleph_0$-categorical, $\aleph_0$-stable structures. *Annals of Pure and Applied Logic 28*, 2 (1985), 103–135.

[8] CHERLIN, G. L. Combinatorial problems connected with finite homogeneity. *Contemporary Mathematics 131* (1993), 3–3.

[9] CHERLIN, G. L. *The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous N-tournamentrs*, vol. 621 of *Memoirs of the American Mathematical Society*. American Mathematical Soc., 1998.

[10] CHERLIN, G. L., AND HRUSHOVSKI, E. *Finite structures with few types*, vol. 152 of *Annals of Mathematics Studies*. Princeton University Press, 2003.

[11] COVINGTON, J. Homogenizable relational structures. *Illinois Journal of Mathematics 34*, 4 (1990), 731–743.

[12] DEVILLERS, A. Ultrahomogeneous semilinear spaces. *Proc. London Math. Soc 3* (2000), 35–58.

[13] ENGELKING, R. *General topology*. Heldermann, 1989.

[14] ERDŐS, P., KLEITMAN, D. J., AND ROTHSCHILD, B. L. Asymptotic enumeration of K_n-free graphs. In *International Colloquium on Combinatorial Theory*, vol. 2.

[15] EVANS, D. M. Splitting of finite covers of $\aleph_0$-categorical structures. *Journal of the London Mathematical Society 54*, 2 (1996), 210–226.

[16] EVANS, D. M., IVANOV, A. A., AND MACPHERSON, H. D. Finite covers. In *Model Theory of Groups and Automorphism Groups*, vol. 244 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1995.

[17] FAGIN, R. Probabilities on finite models. *The Journal of Symbolic Logic 41*, 1 (1976), 50–58.

[18] FRAÏSSÉ, R. Sur l'extension aux relations de quelques propriétés des ordres. In *Annales Scientifiques de l'École Normale Supérieure* (1954), vol. 71, Société mathématique de France, pp. 363–388.

[19] FRAÏSSÉ, R. Theory of relations. *Studies in Logic and the Foundations of Mathematics 145* (1986).

[20] GAIFMAN, H. Concerning measures in first order calculi. *Israel journal of mathematics 2*, 1 (1964), 1–18.

[21] GARDINER, A. Homogeneous graphs. *Journal of Combinatorial Theory, Series B 20*, 1 (1976), 94–102.

[22] HENSON, C. W. Countable homogeneous relational structures and $\aleph_0$-categorical theories. *Journal of Symbolic Logic* (1972), 494–500.

[23] HRUSHOVSKI, E. Pseudo-finite fields and related structures. *Model theory and applications 11* (1991), 151–212.

[24] JENKINSON, T., TRUSS, J., AND SEIDEL, D. Countable homogeneous multipartite graphs. *European Journal of Combinatorics 33*, 1 (2012), 82–109.

[25] KEEVASH, P. Hypergraph turán problems. *Surveys in Combinatorics* (2011), 83–140.

[26] KLEITMAN, D. J., AND ROTHSCHILD, B. L. Asymptotic enumeration of partial orders on a finite set. *Transactions of the American Mathematical Society 205* (1975), 205–220.

[27] LACHLAN, A. H. Countable homogeneous tournaments. *Transactions of the American Mathematical Society 284*, 2 (1984), 431–461.

[28] LACHLAN, A. H. Binary homogeneous structures II. *Proceedings of the London Mathematical Society 3*, 3 (1986), 412–426.

[29] LACHLAN, A. H., AND SHELAH, S. Stable structures homogeneous for a finite binary language. *Israel Journal of Mathematics 49*, 1-3 (1984), 155–180.

[30] LACHLAN, A. H., AND TRIPP, A. Finite homogeneous 3-graphs. *Mathematical Logic Quarterly 41*, 3 (1995), 287–306.

[31] LACHLAN, A. H., AND WOODROW, R. E. Countable ultrahomogeneous undirected graphs. *Transactions of the American Mathematical Society* (1980), 51–94.

[32] MAC LANE, S. The education of Ph.D.s in mathematics. In *A century of mathematics in America*, P. L. Duren, R. Askey, and U. C. Merzbach, Eds. American Mathematical Soc., 1989.

[33] MACPHERSON, H. D. Interpreting groups in ω-categorical structures. *Journal of Symbolic Logic* (1991), 1317–1324.

[34] MARKER, D. *Model theory: an introduction*. Springer, 2002.

[35] PERETZ, A. Geometry of forking in simple theories. *Journal of Symbolic Logic* (2006), 347–359.

[36] POIZAT, B. *A course in model theory: an introduction to contemporary mathematical logic*. Springer Verlag, 2000.

[37] PRÖMEL, H. J., SCHICKINGER, T., AND STEGER, A. A note on triangle-free and bipartite graphs. *Discrete mathematics 257*, 2 (2002), 531–540.

[38] PRÖMEL, H. J., STEGER, A., AND TARAZ, A. Asymptotic enumeration, global structure, and constrained evolution. *Discrete Mathematics 229*, 1 (2001), 213–233.

[39] SCHMERL, J. H. Countable homogeneous partially ordered sets. *Algebra Universalis 9*, 1 (1979), 317–321.

[40] SHELAH, S. Simple unstable theories. *Annals of Mathematical Logic 19*, 3 (1980), 177–203.

[41] STURGEON, THEODORE (EDITED BY PAUL WILLIAMS) *The Complete Stories of Theodore Sturgeon, volume V: The Perfect Host*. North Atlantic Books, 1998.

[42] THAS, K. *Symmetry in finite generalized quadrangles,* Frontiers in Mathematics *vol. 1*.

[43] THOMAS, S. The nonexistence of a binary homogeneous pseudoplane. *Mathematical Logic Quarterly 44*, 1 (1998), 135–137.

[44] TOMAŠIĆ, I., AND WAGNER, F. O. Applications of the group configuration theorem in simple theories. *Journal of Mathematical Logic 3*, 02 (2003), 239–255.

[45] VERSHIK, A. M. Random metric spaces and universality. *Russian Mathematical Surveys 59*, 2 (2004), 259.

[46] WAGNER, F. O. *Simple theories,* volume 503 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 2000.

[47] WOODROW, R. E. *Theories with a finite number of countable models and a small language*. Ph.D. thesis, Simon Fraser University, Burnaby, British Columbia, Canada, 1976.