The Outer-Temperley-Lieb Algebra Structure and Representation Theory

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The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

We define a new algebra the Outer-Temperley-Lieb algebra, $OTL_n(\delta)$, as a fixed ring of the well known Temperley-Lieb algebra, with respect to an automorphism σ reflecting the known diagrammatic representations of the Temperley-Lieb elements in the vertical plane. We define the cell modules of the Outer-Temperley-Lieb algebra and determine that the algebra's semi-simplicity is dependant entirely on that of the Temperley-Lieb algebra. We are therefore able to give the complete representation theory of the Outer-Temperley-Lieb algebra when it is semi-simple. The induction and restriction of the standard modules to higher and lower rank $OTL_n(\delta)$ algebras is studied. We also begin a study of the representation theory of $OTL_n(\delta)$ when it is not semi-simple by describing a large family of homomorphisms between standard modules and conclude with a conjecture on the labelling set of the blocks of the Outer-Temperley-Lieb algebra in the non semi-simple cases.

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Introduction

Is this thesis we shall introduce a new family of algebras, the Outer-Temperley-Lieb algebras, derived as sub-algebras of the well known Temperley-Lieb algebra. The main objective will be to study aspects of the representation theory of the Outer-Temperley Lieb algebras. We believe these algebras to be new and as such can give no background specific to them. To begin however, we shall make brief review of studies of the parent algebra, which shall be key in the studying the Outer-Temperley-Lieb algebra.

The Temperley-Lieb algebras are a family of algebras originally introduced by Temperley and Lieb [30] when they came to light in their study of single bond transfer matrices for planer lattice models. The family is indexed by an integer n and a parameter δ, which is dependant on the model, with the algebra defined as a set of generators and relations satisfied by the transfer matrices. When abstracted over the complex field, the representation theory of these algebras has played a great part in further study of these models, such as the Potts model [19]. The algebras were also discovered independently by Jones [14] who went on to use them in the field of knot theory to define the *Jones polynomial*, by defining a trace on the algebras. It is well known that the Temperley-Lieb algebra can also be defined a a quotient of the type A *Hecke algebra*, which can be described as a quantum deformation of the group algebra of the symmetric group. As such, the representation theory of the type A Hecke algebra is known to be closely interconnected to the representation theory of the symmetric group, see for example the work by Mathas [24] or Murphy [26].

The representation theory of $TL_n(\delta)$ is a widely studied branch of algebra. Due to the applications of the Temperley-Lieb in physics it has been approached from the angle of both physics and pure mathematics. One very well known feature of the representation theory of $TL_n(\delta)$ is that its cell modules only fail to be irreducible for a finite collection of choices of the parameter δ. For δ not in this collection each algebra $TL_n(\delta)$ is known to be a semi-simple algebra. For δ in this collection, that is the *non-generic* δ, the study of the representation theory of $TL_n(\delta)$ is of great interest in both physics and mathematics, indeed probably of greater interest than the semi-simple case.

Martin [19] produced the first published study of the representation theory of $TL_n(\delta)$ for all possible values of δ over the complex field. In [19, Chapter 6] he defines and studies the Temperley-Lieb algebra at generic values of δ while in [19, Chapter 7] he considers what changes when δ is a non-generic value and concludes with the proof of his main theorem [19, Theorem 2, pg 171] which gives the complete representation theory of the Temperley-Lieb algebra, at non-generic δ, including a description of the blocks of the algebra. The book is written with the applications of $TL_n(\delta)$ to physics in mind, the preceding chapters give a background introduction to *statistical mechanics*, that is the branch of physics to which the Temperley-Lieb algebra is related, while later chapters apply the main results of the representation theory of $TL_n(\delta)$ to the physical models and to studying Hecke algebra.

In an independent study to Martin, Goodman and Wenzl [10] use the idempotents introduced by Wenzl [32] to construct the blocks of the Temperley-Lieb algebra when it fails to be semi-simple. Their approach uses the aforementioned relationship between the Temperley-Lieb and Hecke algebras. This leads to an explicit description of the blocks of $TL_n(\delta)$ and their radicals, although the more interesting case of $\delta = 0$ is not included in their work.

Westbury [33] studied the representation theory of the Temperley-Lieb algebras from a more algebraic point of view. Part of his work uses the idea that there exists a unique invariant bilinear form acting on each cell module of $TL_n(\delta)$, thus by the cellular structure of $TL_n(\delta)$ given by Graham and Leher [11], the question of when these cell modules are simple can be solved by determining the scenarios where this form is non-degenerate. Westbury does this by explicitly calculating the Gram matrices and their determinants for the cell modules with respect to these forms. While his work does present a useful introduction to the representation theory of the Temperley-Lieb algebra, it has been noticed that it contains a number of flaws, for example the recursion relation on Gram matrices given in [33, Chapter 5] is known to be incorrect and many of his statements are given without complete proofs or arguments.

Recently a new study of the complex representation theory of $TL_n(\delta)$ has been made in, at the time of writing, a pre-print by Ridout and Saint-Aubin [27] in which many of the results of Westbury [33] are corrected or proven in a slightly more efficient manner. Their approach gives a new proof of the theorem that the Temperley-Lieb algebra is semi-simple if and only if q is not a root of unity, where $\delta = q+q^{-1}$. One notable difference in this work to that of others is the introduction and use of a symmetric element F_n on which they rely heavily on for many proofs. It is used, for example, in their proof that the non-trivial radicals of the standard modules when q is a root of unity are irreducible, which leads to a full description of the set of homomorphisms between pairs of standard modules. They conclude by using F_n to construct the principal indecomposable modules of the Temperley-Lieb algebra. To date, this work appears to be the most straightforward and clearly explained algebraic exposition of the complex representation theory of $TL_n(\delta)$ and as such we shall make much reference to it throughout this thesis. Still it fails to address some points of interest such as providing an explanation of the minimal central idempotents and hence the blocks of $TL_n(\delta)$

The Temperley-Lieb algebra has a number of generalisations which have themselves and their representation theories been well studied. The two we shall make most use of are Martin and Saleur's *blob* algebra [22] and the *global* Temperley-Lieb algebra [5]. The global Temperley-Lieb algebra is now more commonly known as the *affine* Temperley-Lieb algebra. This is an infinite dimensional version of the Temperley-Lieb algebra which can also be represented by a diagram calculus as shown by Fan and Green [9]. As a generalisation of the Temperley-Lieb algebra, much of the study of the representation theory of the affine Temperley-Lieb algebra, see for example [12], can be specialised to give a new approach for deducing results for the representation theories of the Temperley-Lieb and blob algebras, see for example [5] for the latter. Since the Temperley-Lieb algebra can be defined as a quotient of the type A Hecke Algebra, it is often called itself the Temperley-Lieb algebra of type A. This has led to the consideration of the Temperley-Lieb algebras of type B and D by Green [13], in which it turns out the former is isomorphic to the blob algebra.

In the first chapter of this thesis we shall recall the preliminary results required to proceed with the chapters following. This will mainly be a review of the Temperley-Lieb algebra and the well known results regarding its representation theory, including the theorem which the values of the parameter δ for which $TL_n(\delta)$ is semi-simple and the construction of the labelling sets for blocks of $TL_n(\delta)$ when it is not semi-simple.

In chapter two we shall define our main object of study, the Outer-Temperley-Lieb algebra as a fixed ring, in the sense of Montgomery [25], of the Temperley-Lieb algebra. After giving an explicit formula for the dimension of the algebra we shall begin to explore its representation theory by defining a collection of modules, which we shall prove to be the full set of cyclic, indecomposable modules of the Outer-Temperley-Lieb algebra. We shall use these to determine that the Outer-Temperley-Lieb algebra is both cellular, in the sense of [11], and quasi-hereditary, in the sense of [4], and conclude the chapter by proving that, like the Temperley-Lieb algebra, the Outer-Temperley-Lieb algebra is semi-simple if and only if q (where $\delta = q + q^{-1}$) is not a root of unity.

The third chapter is dedicated to stating and proving restriction and induction rules for the standard modules of the Outer-Temperley-Lieb algebra. The rules for restriction are proven explicitly by splitting a standard module into parts depending on the shape of its boundary diagrams. We show these parts are sub-modules of the standard modules, isomorphic to standard modules of lower rank when restricting to a lower rank algebra. To complete the proof we use Ridout and Saint-Aubin's central elements F_n, [27] and a property of quasi-hereditary algebras demonstrated by Donkin [8]. The induction rules are proven by defining a pair of globalisation and localisation functors between the rank n and rank $n - 4$ Outer-Temperley-Lieb algebras. These are combined with a remark made in [6] which, for families of algebras such as ours, relates the induction of a module of an algebra to the globalisation of that module when it is restricted to a lower rank algebra.

In the final chapter we shall begin to study the representation theory of the Outer-Temperley-Lieb algebra when it is not semi-simple. Our main aim in this chapter will be to describe the blocks of the Outer-Temperley-Lieb algebra. We shall attempt to proceed by describing the space of homomorphisms between pairs of indecomposable modules, it will however become apparent that we are not able to give the complete description at this time. In section 4.2 we give a conjecture which if true would lead to a complete description when n is even, although we shall show that further work needs to be done when n is odd. In the final section we use the results of the previous sections to give two conjectures on the labelling set of the blocks of the Outer-Temperley-Lieb algebra in the case where n is even.

In appendix A we give some basic results on the box or q-number which are used in describing some of the indecomposable $OTL_n(\delta)$-modules homomorphisms in the non-semi-simple case. Appendix B recalls some results on bilinear forms and the Gram matrix of a module which we use for determining which parameter values, δ, make both the Temperley-Lieb and the Outer-Temperley-Lieb algebra semi-simple. Finally appendix C recalls some basic homological algebra used in the proof of the restriction theorem of the Outer-Temperley-Lieb standard modules.

Chapter 1

Preliminaries

This chapter is dedicated to reviewing known results which we shall require during the course of this thesis. We first define the Temperley-Lieb algebra, the parent algebra to the Outer-Temperley-Lieb algebra, in its original form given by [30]. We then develop the well known diagrammatic representation of this algebra which we will rely heavily on throughout this thesis. We also briefly define the well studied generalisation of the Temperley-Lieb algebra known as the *blob algebra*. Following this we recall the main results of the representation theory of the Temperley-Lieb algebra including a description of its cell modules and the well known theorem that determines the scenarios in which the Temperley-Lieb algebra is a semi-simple algebra. Following a review of the induction and restriction theorems for the Temperley-Lieb algebra we recall some of the theorems related to the study of the modules of the Temperley-Lieb algebra when it is not semi-simple and conclude the chapter by describing the labelling set of the blocks of the Temperley-Lieb algebra.

1.1 Temperley-Lieb algebra and boundary diagrams

In this section we define both the Temperley-Lieb algebra in the form of generators and relations and a type of diagram known as a boundary diagram. We then demonstrate how the Temperley-Lieb algebra can be represented using these boundary diagrams. Throughout the thesis we assume that k is a field, $\delta \in k$ and $n \in \mathbb{N}$, unless otherwise stated.

Definition 1.1.1. The Temperley-Lieb algebra, $TL_n(\delta)$, originally defined in [30], is the algebra defined by the generators $\{u_1, \ldots, u_{n-1}\}$ such that,

$$\begin{aligned} u_i^2 &= \delta u_i, && 1 \le i \le n-1 \ , \\ u_i u_j u_i &= u_i, && j = i \pm 1, \\ u_i u_j &= u_j u_i, && |i-j| > 1 \ , \end{aligned} \tag{1.1}$$

To give a clearer picture of how elements of $TL_n(\delta)$ interact, we can represent the algebra using a basis of cosets of *boundary diagrams*. We shall briefly review here how this is done, a more thorough description can be found in [19, Chapter 6]. We first describe what we mean by a diagram.

Let D_n be the set of all diagrams containing two parallel horizontal rows of n nodes connected in pairs by n non-crossing strings lying in the plane between the rows. If $d \in D_n$ is such a diagram, then the coset $c(d)$ is the subset of all diagrams of D_n with the same pairing of nodes as d. Note here that we choose to draw boundary lines in our diagrams along the top and bottom rows of nodes as in figure 1.1 and will often label our nodes starting with 1 on the left to n on the top edge and starting on the bottom edge with $1'$ on the left to n'.

We define a multiplication on cosets as follows. If $c(d_1), c(d_2)$ are two such cosets, join the strings at the top nodes of d_2 to the bottom nodes of d_1 and then remove the internal nodes and any closed loops to form a new diagram $d_1 d_2$ of D_n (see figure 1.2).

Figure 1.1: Two boundary diagrams of the same coset.

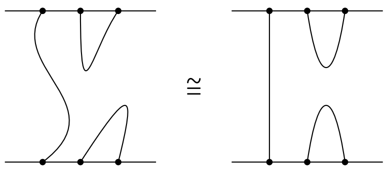
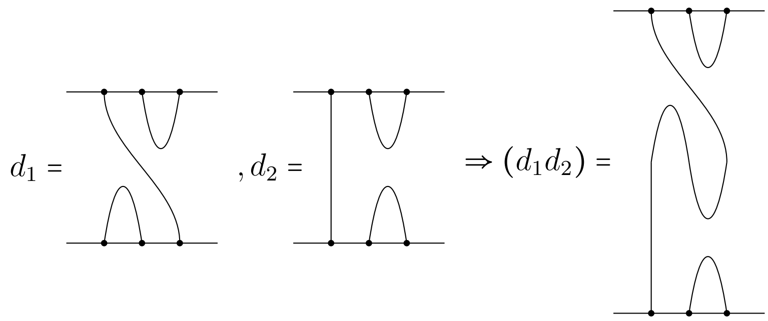


Figure 1.2: Example of two boundary diagrams forming the diagram (d_1d_2).



We can therefore form an algebra $A_n(\delta)$ over k with basis the set of all cosets of D_n and multiplication given by,

$$c(d_1)c(d_2) \mapsto \delta^l c(d_1d_2), \tag{1.2}$$

where $c(d_1d_2)$ is the coset of D_n containing d_1d_2 and l is the number of closed loops removed when forming d_1d_2.

We shall often refer to strings as lines. A line in a boundary diagram will be said to be propagating if its start and end nodes are on opposite edges of the diagram, otherwise it will be called an arc.

We define a map from $TL_n(\delta)$ to $A_n(\delta)$ by sending each generator u_i to the coset of diagram given in figure 1.3 and the identity I to the coset of the diagram in figure 1.4. It is shown in [19] that the multiplication of cosets of such diagrams, as given in (1.2), satisfies the identities in equation (1.1) and that this map is an isomorphism of algebras. From this point on we shall drop the notation $c(-)$. When we refer to the diagram d of an algebra we shall actually be referring to the coset represented by the boundary diagram d.

Figure 1.3: Boundary diagram representation of u_i in $TL_n(\delta)$.

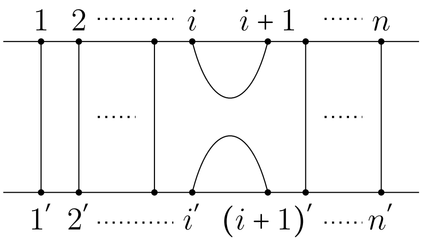
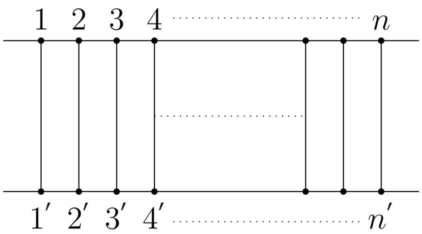


Figure 1.4: Boundary diagram representation of I in $TL_n(\delta)$.



Example 1.1.2. The algebra $TL_3(\delta)$ has two generators u_1, u_2. We can therefore write its basis as $\{I, u_1, u_2, u_1u_2, u_2u_1\}$ where I is the identity element. Figure 1.5 shows the boundary diagram representation of this basis.

With the above correspondence in mind we shall henceforth identify the elements u_i and the identity I with their corresponding diagrams.

We recall the formula for the dimension of both $A_n(\delta)$ and $TL_n(\delta)$.

Theorem 1.1.3. *The dimension of the algebra $TL_n(\delta)$ (and the diagram algebra $A_n(\delta)$) is given by the n^{th} Catalan number C_n where,*

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

Proof. See for example [27, Theorem 2.3] □

Figure 1.5: Boundary diagram representation of the basis of $TL_3(\delta)$.

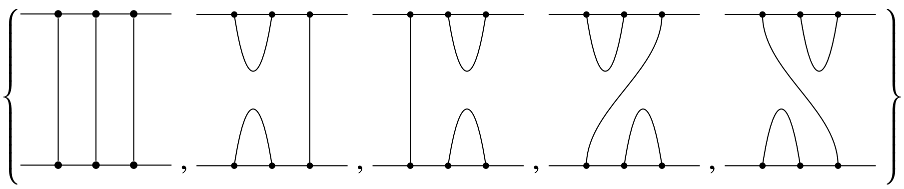
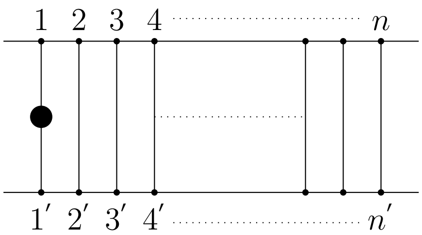


Figure 1.6: The generator e of the blob algebra.



1.2 The blob algebra

The *blob* algebra [22, 23], which we denote $b_n(\delta, \delta')$, $\delta, \delta' \in k$ is a generalisation of the Temperley-Lieb. It is generated by the set of generators $\{u_1, u_2, \ldots, u_n, e\}$ where the u_i are the generators of the Temperley-Lieb algebra and e, as shown in figure 1.6 is the identity diagram of the Temperley-Lieb algebra decorated with a 'blob' on its left-most propagating line. Diagrams of the blob algebra are multiplied in the same way as those of $TL_n(\delta)$ with some additional rules to handle the joining of lines with blobs. When concatenation of two blob diagrams creates a line with two blobs, these are replaced with a single blob and when it creates loops decorated by blobs, the resulting diagrams is multiplied by the parameter δ' to the power of the number of such loops. Diagrammatic versions of these rules can be found in [22, page 3] or [13, Figure 8]. In section 2.1 we shall recall a well known isomorphism between a sub-algebra of an even rank Temperley-Lieb algebra and a special case of the blob algebra. This isomorphism will be of use in calculating the dimension of the Outer-Temperley-Lieb algebra.

1.3 Representation theory of $TL_n(\delta)$

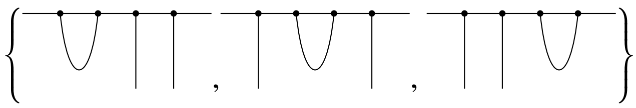
While it is not our aim to repeat the full description of the representation theory of the Temperley-Lieb algebra, the next four sections will be used to briefly describe some of the main results which will be of use to us in later chapters. We begin in this section by defining a collection of half diagrams and showing how they can be used to form a basis of the standard modules of the Temperley-Lieb algebra. We recall the formula for the dimensions of these modules and describe some of their properties that make them of interest to us, including their indecomposability.

Definition 1.3.1. Any $TL_n(\delta)$ boundary diagram d, as defined in section 1.1, can be cut such that all arcs on the top edge of d are above the cut, all arcs on the bottom edge of d are below the cut and each propagating line is only cut once. We shall call the diagrams created by such a cut *half n-diagrams*. If a half n-diagram's remaining lines extend below the boundary line we will call it a *top half n-diagram*, otherwise its lines will extend above the boundary line and will call it a *bottom half n-diagram*.

For any $n \in \mathbb{N}$ we can consider the set of all unique top half n-diagrams T_n of the boundary diagram basis of $TL_n(\delta)$. Note that we can multiply any diagram in this set by a $TL_n(\delta)$ boundary diagram by joining the strings at the bottom nodes of the $TL_n(\delta)$ diagram to those at the top nodes of the top half n-diagram then proceeding as we would with two $TL_n(\delta)$ boundary diagrams. If the final diagram in this process contains a string not connected to the top edge it is sent to zero. The result will therefore either be zero or another top half n-diagram multiplied by some factor of $\delta \in k$, thus kT_n, the k vector space with basis T_n, is a left $TL_n(\delta)$-module. In a similar way we can make kB_n, with basis B_n the set of all unique bottom half n-diagrams, a right $TL_n(\delta)$-module where multiplication joins the strings at the top nodes of the TL_n diagram to those at the bottom nodes of the bottom half diagram then proceeds as with $TL_n(\delta)$ multiplication.

We can also order the set of all top (or bottom) half n-diagrams by the number of

Figure 1.7: The diagram basis of $C_{4,2}$.



propagating lines which, in this sense, are the lines that begin but do not end on the top (or bottom) edge of the diagram.

Lemma 1.3.2. *Denote by $kT_{n,p}$ the vector subspace of kT_n, whose basis $T_{n,p}$ consists of all diagrams of T_n with p or fewer propagating lines. For each $0 \leq p \leq n$ with $n - p \in 2\mathbb{Z}$, $kT_{n,p}$ is a left $TL_n(\delta)$-sub-module of kT_n.*

Proof. As with multiplication of two $TL_n(\delta)$ diagrams, left multiplication of a $TL_n(\delta)$ diagram d with a diagram, say $m \in T_n$, cannot create any additional propagating lines. Therefore if $m \in kT_{n,p}$ the product dm can have no more than p propagating lines. We note that we must have $n - p \in 2\mathbb{Z}$ as the number of arcs in a valid half n-diagram is given by $\frac{n-p}{2}$ which cannot be valid with $n - p$ odd. □

Lemma 1.3.3. *Let $C_{n,p} := kT_{n,p}/kT_{n,p-1}$. Then for each $0 \leq p \leq n$ such that $n - p \in 2\mathbb{Z}$, $C_{n,p}$ has a diagram basis consisting of all diagrams of T_n with exactly p propagating lines. Furthermore each $C_{n,p}$ is a left $TL_n(\delta)$-module.*

Proof. The quotient $T_{n,p}/T_{n,p-1}$ sends any diagram with less than p propagating lines to zero, therefore left multiplication by any $d \in TL_n(\delta)$ with $m \in C_{n,p}$ will result in a diagram which is either the zero diagram or has exactly p propagating lines. □

Example 1.3.4. Figure 1.7 shows the diagram basis of the left $TL_n(\delta)$-module $C_{4,2}$.

Note that we shall often refer to an element of $C_{n,p}$ as a diagram, although we mean this to be taken as a top half n-diagram.

Lemma 1.3.5. *Let $a = \frac{n-p}{2}$, then for any $p \leq n$ with $n - p$ even the dimension of the $TL_n(\delta)$-module $C_{n,p}$ is given by the formula,*

$$c_{n,p} = \binom{n}{a} - \binom{n}{a-1}.$$

Proof. See for example the discussion leading up to [27, Equation 2.8] or the proposition at the end of [33, Section 2], noting that the statement in latter incorrectly uses $n - 1$ instead of n. □

Lemma 1.3.6. *For any $\delta \in k$ the left $TL_n(\delta)$-modules $C_{n,p}$ are cyclic, indecomposable modules. If we denote by $R_{n,p}$ the radical of the module $C_{n,p}$ then the quotient $C_{n,p}/R_{n,p}$ is irreducible.*

Proof. This is a well known result, see for example [27, Section 3]. □

We recall the original definition of a cellular algebra as given by Graham and Lehrer [11].

Definition 1.3.7. [11, Definition 1.1]. An associative algebra A is called a cellular algebra, over a commutative ring k, if there exists a *cell datum* $(\Delta, M_{\lambda\in\Delta}, \gamma, i)$, such that,

C1) For each $\lambda \in \Delta$, with Δ a poset, there is a finite set M_λ such that the map $\gamma : \coprod_{\lambda\in\Delta} M_\lambda \times M_\lambda \longrightarrow A$ is injective and its image a k-basis for A.

C2) For $\lambda \in \Delta$, if $m, n \in M_\lambda$ with $\gamma_\lambda(m, n) \in A$, then i is a k-linear anti-involution of A satisfying $i(\gamma_\lambda(m, n)) = \gamma_\lambda(m, n)$.

C3) For $\lambda \in \Delta$, $m, n \in M_\lambda$ and any $a \in A$ we can write

$$a\gamma_\lambda(m, n) = \sum_{p\in M_\lambda} r_a(p, m)\gamma_\lambda(p, n) + x, \tag{1.3}$$

where the $r_a(p, m)$ are coefficients which do not depend on n and x is some linear combination of basis elements with upper index $\mu < \lambda \in I$.

The modules $C_{n,p}$ are known to be the cell modules of $TL_n(\delta)$ and it is shown in of [11, Chapter 1] that $TL_n(\delta)$ is a *cellular algebra* with respect to the involution sending each diagram to its reflection in the horizontal plane and the ordered poset $\Delta = \{n, n-2, n-4, \ldots, x\}$, where $x = 1$ when n is odd and $x = 0$ when n is even.

We also recall the original definition of a *quasi-hereditary* algebra as can be found in [29, 4].

Definition 1.3.8. [29, 4]. Let J be an ideal of A, a finite dimensional k-algebra. If J is a projective left (or right) A-module, an idempotent, that is $J^2 = J$ and satisfies $J \operatorname{rad} AJ = 0$ then J is called a hereditary ideal. If there exists a finite chain

$$0 = J_0 \subset J_1 \subset \ldots \subset J_n = A \tag{1.4}$$

of ideals of A such that each J_i/J_{i-1} is a hereditary ideal of A/J_{i-1} for all i the A is called a *quasi-hereditary* algebra with equation (1.4) being a quasi-hereditary chain.

It is known, see for example [17, Proposition 4.1] that the Temperley-Lieb algebra is a quasi-hereditary algebra, unless $\delta = 0$ and n is even, and that the modules $C_{n,p}$ can be considered as standard modules in the quasi-hereditary sense.

1.4 Irreducibility of the cell modules

In this section we shall recall the result which determines the parameter choices for which the cell modules from the previous section are the complete set of irreducible modules of the Temperley-Lieb algebra and hence cause the Temperley-Lieb algebra to be a semi-simple algebra. The cell modules $C_{n,p}$ of $TL_n(\delta)$ are known to be irreducible except for certain choices of the parameter δ. We shall follow the usual convention of writing $\delta = q + q^{-1}$ with $q \in k$. The following result is well known for the Temperley-Lieb algebra, we shall state the version as given in [27].

Proposition 1.4.1. *[27, Proposition 4.5, Corollary 4.6]. When q is not a root of unity, the algebra $TL_n(\delta)$ is semi-simple and for all $0 \leq p \leq n$ such that $n-p \in 2\mathbb{Z}$ the modules $C_{n,p}$ are a complete set of unique irreducible modules of $TL_n(\delta)$.*

This result is often proven by determining the *Gram matrix*, as defined in definition B.2, of the cell modules, with respect to a contravariant bilinear form, see definition B.3. In appendix B we show that for A a finite dimensional algebra over a field and M an A-module, if the determinant of the Gram matrix of M with respect to this form is zero then the module cannot be irreducible. As $TL_n(\delta)$ is a cellular algebra, the proof of [11, Theorem 3.8] tells us that with respect to the contravariant bilinear form ϕ_p, $p \in \Delta$, as defined in [11, Definition 2.3], if the Gram matrix of M has a non-zero determinant then it is an irreducible module. It is shown in chapter 2 that ϕ_p is equivalent to the form $\langle -, - \rangle$ as defined in [27, Section 3].

The structure and determinants of the Gram matrices $G_{n,p}$ for the $TL_n(\delta)$ standard modules $C_{n,p}$ are well known, see for example [27, Section 4]. We note that a structure for these matrices is also given in [33] however, as mentioned in the introduction, the recursion used by the author to determine this structure is incorrect.

1.5 Induction and restriction of the standard modules

For any family of k-algebras A_n with an inclusion $A_{n-1} \subset A_n$, we may consider how the indecomposable modules of A_n behave when we restrict their action to A_{n-1}. If M is some left A-module we denote by $M \downarrow_{A_{n-1}}$ the restriction of the module M to the algebra A_{n-1} with respect to its inclusion in A. The Temperley-Lieb algebra has a well known inclusion of sub-algebras $TL_{n-1}(\delta) \hookrightarrow TL_n(\delta)$ sending each diagram d of an element of $TL_{n-1}(\delta)$ to the same boundary diagram with two additional nodes, one placed on the top and one placed on the bottom boundary line, to the left of all other nodes and a propagating line joining these two nodes. We can therefore consider how the standard modules $C_{n,p}$ of $TL_n(\delta)$ act when restricted to the algebra $TL_{n-1}(\delta)$.

The following is a well known result for $TL_n(\delta)$.

Proposition 1.5.1. *For each* $0 \leq p \leq n$ *such that* $n - p \in 2\mathbb{Z}$ *we have,*

$$C_{n,p} \downarrow_{TL_{n-1}(\delta)} \cong C_{n-1,p-1} \oplus C_{n-1,p+1}, \tag{1.5}$$

when $\delta = q + q^{-1}$ *is such that* $q^{p+1} \neq 1$.

Proof. Proof can be found in [27, Corollary 4.2]. □

We may also consider for such a family of algebras A_n how an A_{n-1} module N acts as a module for the algebra A_n. We construct from N the *induced* module $N \uparrow_{A_n} := A_n \otimes_{A_{n-1}} N$, where the action in A_n of N is given by $a(b \otimes n) = (ab) \otimes n$ for all $a, b \in A$, $n \in N$. In general the structure of the induced modules is harder to determine than that of the restricted modules, the following however has been proven for $TL_n(\delta)$.

Proposition 1.5.2. *For each* $0 \leq p \leq n$ *such that* $n - p \in 2\mathbb{Z}$ *we have,*

$$C_{n,p} \uparrow_{TL_{n+1}(\delta)} \cong C_{n+1,p-1} \oplus C_{n+1,p+1}, \tag{1.6}$$

when $\delta = q + q^{-1}$ *is such that* $q^{p+1} \neq 1$.

Proof. Chapter 5 of [27] is mostly dedicated to the proof of this result. □

1.6 Non-generic cases

When the Temperley-Lieb algebra is semi-simple we know by section 1.3 its full set of irreducible modules. For the values of δ where $TL_n(\delta)$ is not semi-simple more work is required to determine the simple modules and the composition factors of the standard modules. Many different studies of this have been made for various choices of k. We shall keep our review here to $k = \mathbb{C}$. Noting proposition 1.4.1, we shall assume here that q is a primitive l^{th} root of unity and denote by $S_{n,p}$ the irreducible quotients $C_{n,p}/R_{n,p}$. In the sense of [27, Corollary 4.2] we shall call the module $C_{n,p}$ *critical* if $q^{2(p+1)} = 1$. If we list the modules in order of the indexing set Δ then we can say that two modules C_{n,p_1}, C_{n,p_2} are symmetric if $p_2 < p_1 \leq n$, $0 < |p_1 - p_2| < l$ and that both modules lie symmetrically on either side of a critical module between them. It is shown in [27, Theorem 7.2] that the radicals of the standard modules are either trivial or irreducible themselves.

Proposition 1.6.1. *[27, Theorem 7.2]. The radical $R_{n,p}$ of the module $C_{n,p}$ is zero or irreducible. If C_{n,p_1} and C_{n,p_2} are two standard symmetric modules then $S_{n,p_1} \cong R_{n,p_2}$.*

Thus the standard modules of $TL_n(\delta)$ in the non-semi-simple cases will always have at most two irreducible composition factors, the radical $R_{n,p}$ and the quotient module $S_{n,p}$. This leads to a full description of the space of homomorphisms between indecomposable $TL_n(\delta)$-modules given by the following theorem.

Theorem 1.6.2. *[27, Theorem 7.3]. If C_{n,p_1}, C_{n,p_2} are two standard $TL_n(\delta)$-modules then when $p_1 = p_2$ or C_{n,p_1}, C_{n,p_2} are symmetric with $p_2 < p_1$ we have $\dim \operatorname{Hom}(C_{n,p_1}, C_{n,p_2}) = 1$. Otherwise $\dim \operatorname{Hom}(C_{n,p_1}, C_{n,p_2}) = 0$.*

1.7 The blocks of the Temperley-Lieb algebra

In this final section of this chapter we define the *blocks* of an algebra and recall the known labelling set of the blocks of the Temperley-Lieb algebra. We shall take our definition of blocks of an algebra from Benson [2] and some additional theory from Curtis and Reiner [7] although we note equivalent definitions can be found in most basic representation theory texts. Following this we shall give a brief description of the blocks of the Temperley-Lieb algebra.

For a finite dimensional algebra A, recall an idempotent of A is an element $e \in A$ such that $e^2 = e$. A pair of idempotents $e_i, e_j \in A$ are said to be orthogonal if $e_ie_j = e_je_i = 0$. Let $\{e_i \mid i \in \mathbb{N}\}$ be a set of primitive central idempotents of A, that is each e_i is orthogonal, in the centre of A and cannot be expressed as a sum of two orthogonal central idempotents of A. When A is considered as a two-sided ideal it is well known that there exists a one-one correspondence between a direct sum decomposition $A = B_1 \oplus B_2 \oplus \cdots \oplus B_n$ and the expression $1 = e_1 + e_2 + \cdots + e_n$ where each e_i is a primitive central idempotent and each B_i is a two sided ideal defined by $B_i = Ae_i$.

For an Artinian algebra A, that is A satisfies the property that every descending chain of left ideals stops, the decomposition,

$$A = B_1 \oplus B_2 \oplus \cdots \oplus B_n, \tag{1.7}$$

can be written such that each $B_i = Ae_i$ is an indecomposable two sided ideal of A. It is known that, up to reordering, such a decomposition is unique, see for example [2, Lemma 1.8.2].

Definition 1.7.1. [2, Definition 1.8.3]. The blocks of the algebra A are defined to be the indecomposable two sided ideals as in the decomposition given by equation (1.7).

One reason for studying the homomorphisms between standard modules of $TL_n(\delta)$ when it is not semi-simple is that they can help to determine to the blocks. We note part of a theorem and definition from [7].

Theorem 1.7.2. *[7, Theorem 55.2]. No two blocks of A have a composition factor in common when regarded as left A-module.*

Definition 1.7.3. [7, Definition 55.4]. If two irreducible left A-modules are composition factors of the same block then they are said to belong to the same block.

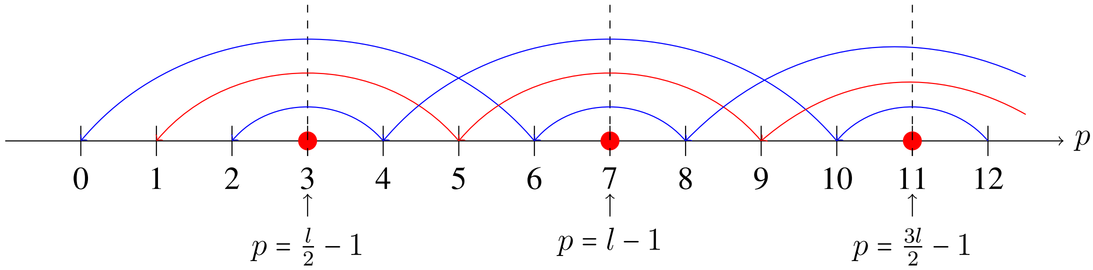
It follows from these that each irreducible A-module can only be a composition factor of exactly one block. Thus we may partially determine the labels of blocks of A by finding the A-module homomorphisms which map irreducible A-modules into indecomposable A-modules. The blocks of $TL_n(\delta)$ however can and have been determined by a more direct approach which we shall now briefly discuss.

Since the generators u_i of the Temperley-Lieb algebra satisfy the relation $u_i{}^2 = u_i$ we can use them to define a set of idempotents $e_i \in TL_n(\delta)$ by $e_i = \frac{1}{\delta}u_i$. Since they are each represented by single diagrams, these are primitive idempotents but they are not orthogonal nor central idempotents of $TL_n(\delta)$. A collection of central minimal idempotents for $TL_n(\delta)$ can be found in [10, Theorem 2.3], with the blocks of $TL_n(\delta)$ immediately following. Prior to this, Martin [19, Theorem 2, page 171] independently described the blocks of $TL_n(\delta)$ using isomorphism classes of quiver diagrams, however the notation used is somewhat difficult to incorporate with our current definitions.

In [10] the labels of the central minimal idempotents, and hence the blocks, are given by particular *Young diagrams*, λ, of size n and at most two parts, that is $\lambda = [\lambda_1, \lambda_2]$ and $\lambda_1 + \lambda_2 = n$. For such a diagram its *weight* is defined as the function $w(\lambda) := \lambda_1 - \lambda_2 + 1$. Such a diagram is said to be critical in [10] if l divides $w(\lambda)$, we believe however this should actually be when l divides $2w(\lambda)$, which is more consistent with [19, Theorem 2] and definitions given in other reviews such as [27].

The standard modules $C_{n,p}$ of $TL_n(\delta)$ have a correspondence with the Young diagrams of at most two parts. Each diagram $d \in C_{n,p}$ can be represented by the Young diagram whose first row contains the node i if the line leading from i is a propagating line or the leftmost part of an arc and whose second row contains each node j if the line at j is the rightmost part of an arc. A full explanation and example of this can be found in [33, Section 2]. We note that the second row of each such Young diagram will contain exactly one entry for each arc, therefore $\frac{n-p}{2}$ entries overall. We therefore correspond each $TL_n(\delta)$-module $C_{n,p}$ with the Young diagram $\lambda = [\frac{n+p}{2}, \frac{n-p}{2}]$. The weight $w(\lambda)$ of each such diagram is then $\frac{n+p-(n-p)}{2} + 1 = p + 1$ and the diagram, hence the module $C_{n,p}$ is critical if l divides $2(p + 1)$. Note that this is equivalent to our previous definition of a

Figure 1.8: Weight diagram representing the blocks of $TL_n(\sqrt{2})$.



critical module in section 1.6.

It is shown in [10] that the blocks of $TL_n(\delta)$ are labelled by the labels of the critical diagrams, that is the critical modules, and the orbits of the action of the affine reflection group $A_1^{(1)}$ on the non-critical diagrams, where $A_1^{(1)}$ acts on $\mathbb{Z}$ by reflection about $m\frac{l}{2} - 1$, $m \in \mathbb{Z}$. Let Δ denote the set of labels for the cell modules of $TL_n(\delta)$ as defined in 1.3, then we can say the blocks are labelled by the set,

$$\bigcup_{\substack{k \in \mathbb{Z}^+ \setminus \{0\} \\ \frac{k}{2}l - 1 \in \Delta}} \left\{ \frac{k}{2}l - 1 \right\}, \tag{1.8}$$

that is the set of labels of the critical modules of $TL_n(\delta)$, together with, for each $p \in \Delta$ such that $2(p+1) < l$, the sets,

$$\{p\} \cup \bigcup_{\substack{k \in \mathbb{Z}^+ \setminus \{0\} \\ kl - p - 2 \in \Delta}} \{kl - p - 2\} \cup \bigcup_{\substack{k \in \mathbb{Z}^+ \setminus \{0\} \\ kl + p \in \Delta}} \{kl + p\}. \tag{1.9}$$

We can show the blocks of $TL_n(\delta)$ in a weight diagram, see for example figure 1.8, where each label along the line $\mathbb{Z}$ corresponds to the indexes p of the standard modules $C_{n,p}$ and the red dots indicate the critical modules with the dashed lines extending from them being the critical walls. Note that the odd indices appear only when n is odd, and the even when n is even. The orbits for n odd are represented by the red lines, and those for n even by the blue lines.

Chapter 2

The Outer-Temperley-Lieb algebra

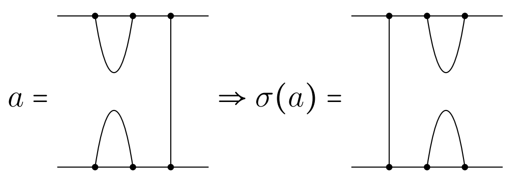
In this chapter we introduce the Outer-Temperley-Lieb Algebra, the main object of study of this thesis. This sub-algebra of the Temperley-Lieb algebra is believed to be a new algebra. In the first section we define an automorphism of the Temperley-Lieb algebra which will then be used to define the Outer-Temperley-Lieb algebra. We also use the automorphism to define what shall turn out to be a special case of the blob algebra with which we will use to determine the dimension of the Outer-Temperley-Lieb algebra. In the second section we begin to study some of the basic results of the representation theory of the Outer-Temperley-Lieb algebra. Using the bases of the standard modules of the Temperley-Lieb algebra we construct a set of modules for the Outer-Temperley-Lieb algebra and calculate their dimensions. This is followed by section 2.3 in which we define a pair of bilinear forms on these modules to determine that for all choices of δ these are cyclic indecomposable modules which have simple head. In section 2.4 we show that the Outer-Temperley-Lieb is a cellular algebra, in the sense of [11]. We then use a relation between a cellular and quasi-hereditary algebra as given in [17] to show in section 2.5 that the Outer-Temperley-Lieb algebra is also a quasi-hereditary algebra except in a few given cases. In the final section we use both fixed ring theory and the theory of Gram matrices with respect to contravariant bilinear forms to work towards proving the main theorem of this chapter which states that the Outer-Temperley-Lieb algebra is semi-simple if and only if the Temperley-Lieb algebra is semi-simple.

2.1 Definition and structure

In this section we define and give some properties of the Outer-Temperley-Lieb algebra. We begin by proving the existence of an automorphism σ of the Temperley-Lieb algebra which will then be used to define two algebras. The first, the Inner-Temperley-Lieb algebra is defined to be the algebra generated by the set of basis elements of the Temperley-Lieb algebra fixed by the automorphism σ while the second, the Outer--Temperley-Lieb algebra is defined as the fixed ring of the Temperley-Lieb algebra, in the sense of [25], with respect to the same automorphism. We recall how the Inner-Temperley-Lieb algebra can be related to the blob algebra introduced in 1.2 and use this relationship to determine the dimension of both the Inner and Outer Temperley-Lieb algebras for any rank n.

Definition 2.1.1. We define the map $\sigma : TL_n(\delta) \longrightarrow TL_n(\delta)$ as follows. If a is a diagram representative of a basis element of $TL_n(\delta)$, as introduced in section 1.1, then the diagram of $\sigma(a)$ is the reflection of the diagram of a in the vertical line (see for example figure 2.1). We extend this k-linearly to a vector space map.

Figure 2.1: The map σ acting on a diagram.



Proposition 2.1.2. *$\sigma : TL_n(\delta) \longrightarrow TL_n(\delta)$ is an automorphism, of the Temperley-Lieb algebra.*

Proof. We know that reflection in a fixed line is always a bijective map. Let $a_1, a_2 \in TL_n(\delta)$. Suppose there is a path in the diagram representation of a_1 from some node p to some node q then the diagram of $\sigma(a_1)$ will contain a path from nodes $n-p+1$ to $n-q+1$. If p and q both lie on the top edge of the diagram then the path from node p to q will also exists in the diagram of a_1a_2. Hence the diagram of $\sigma(a_1a_2)$ will contain a path from $n-p+1$ to $n-q+1$. However, the diagram of $\sigma(a_1)\sigma(a_2)$ will also contain a path from nodes $n-p+1$ to $n-q+1$ as they both lie on the top edge.

Similarly, we can show if there exists a path in the from some node r to some node s on the bottom edge of the diagram of a_2 then both $\sigma(a_1)\sigma(a_2)$ and $\sigma(a_1a_2)$ will contain the path $n-r+1$ to $n-s+1$.

Now suppose the diagram of a_1 contains a path from some node p on the top edge to some node q' on the bottom edge and that the diagram of a_2 contains a path from some node r on the bottom edge to the node q on the top edge. Then the diagram of a_1a_2, prior to removing the loops, will contain a path from p to r and hence the diagram of $\sigma(a_1a_2)$ will contain a path $n-p+1$ to $n-r+1$. Also the diagram of $\sigma(a_1)$ will contain a path from $n-p+1$ to $n-q+1$ and the diagram of $\sigma(a_2)$ will contain a path from $n-r+1$ to $n-q'+1$. Therefore the diagram of $\sigma(a_1)\sigma(a_2)$, prior to removing any loops, will contain a path from $n-p+1$ to $n-r+1$ as required.

We next suppose again that the diagram of a_1 contains a path from some node p on the top edge to some node q' on the bottom edge and an arc between some nodes t' and s' on the bottom edge. Then suppose the diagram a_2 contains a path from some node r on the bottom edge to the node s on the top edge and an arc between the nodes q and t on the top edge. Then the diagram of a_1a_2 prior to removing the loops will contain a path from p to r, via the now removed node t. Hence the diagram of $\sigma(a_1a_2)$ will contain a path $n-p+1$ to $n-r+1$. The diagram of $\sigma(a_1)$ will now contain a path from $n-p+1$ to $n-q'+1$ and an arc between $n-t'+1$ and $n-s'+1$. The diagram of $\sigma(a_2)$ will contain a path from $n-r+1$ to $n-s+1$ and an arc between $n-q+1$ and $n-t+1$. Therefore the diagram of

$\sigma(a_1)\sigma(a_2)$, prior to removing any loops, will contain a path from $n-p+1$ to $n-r+1$, via the removed $n-t+1$, as required. This case can be similarly extended to paths which pass through multiple arcs along the joining edges of the two diagrams.

The final case is if there exists a path from nodes p' to q' on the bottom edge of the diagram of a_1 and a path from nodes p to q on the top edge of the diagram of a_2. Then the diagram of a_1a_2, prior to removing the loops, will contain a loop through p and q. The diagram of $\sigma(a_1a_2)$, before removing loops, will therefore contain a loop through $n-p+1$ and $n-q+1$. Also the diagram of $\sigma(a_1)$ will contain a path from $n-p'+1$ to $n-q'+1$ on its bottom edge and the digram of $\sigma(a_2)$ will contain a path from $n-p+1$ to $n-q+1$ on its top edge. Thus the diagram of $\sigma(a_1)\sigma(a_2)$, before removing the loops, will contain a loop through $n-p+1$ and $n-q+1$. When we finally remove the loops in both $\sigma(a_1a_2)$ and $\sigma(a_1)\sigma(a_2)$ the loop through $n-p+1$ and $n-q+1$ in each diagram will cause each diagram to be multiplied by δ.

Putting all the above scenarios together we deduce that for any $a_1, a_2 \in TL_n(\delta)$,

$$\sigma(a_1a_2) = \sigma(a_1)\sigma(a_2)$$

as required. □

Definition 2.1.3. For $\delta \in k$ we define $ITL_n(\delta)$ to be the sub-space of $TL_n(\delta)$ with basis consisting of the basis elements of $TL_n(\delta)$ that are fixed by σ.

Corollary 2.1.4. *The sub-space $ITL_n(\delta)$ is a sub-algebra of $TL_n(\delta)$.*

Proof. By definition, the identity in $TL_n(\delta)$ is fixed by σ therefore it is also in $ITL_n(\delta)$. Every element of $a \in ITL_n(\delta)$ will satisfy $a = \sigma(a)$, then if $a_1, a_2 \in ITL_n(\delta)$, we have $\sigma(a_1a_2) = \sigma(a_1)\sigma(a_2) = a_1a_2$ thus $a_1a_2 \in ITL_n(\delta)$. □

Let R be a ring and G a group of automorphisms of R. We shall denote by $|G|$ the order of G and by $g(x)$ the action of an element $g \in G$ on some $x \in R$.

Definition 2.1.5. We define, as in the beginning of [25, Chapter 0], the fixed sub-ring of G on R to be,

$$R^G := \{x \in R \mid g(x) = x, \text{ for all } g \in G\ \}. \tag{2.1}$$

Let G be the group generated by the reflection automorphism $\sigma : TL_n(\delta) \to TL_n(\delta)$ then G contains two elements, σ and the identity I.

Definition 2.1.6. For $\delta \in k$ we define $OTL_n(\delta)$ to be the the fixed ring of $TL_n(\delta)$ with respect to G, that is,

$$OTL_n(\delta) := TL_n(\delta)^G = \{d \in TL_n(\delta) \mid \sigma(d) = d\}. \tag{2.2}$$

Proposition 2.1.7. *The algebra $OTL_n(\delta)$ has a diagram basis O_n consisting of all diagrams in the basis of $TL_n(\delta)$ that are fixed by σ; and all the linear combinations $a+\sigma(a)$ formed from pairs of diagrams a, $\sigma(a)$ in the basis of $TL_n(\delta)$ which are not fixed by σ.*

Proof. Let $d \in OTL_n(\delta)$, then by definition $d = \sigma(d)$. Since d is also an element of $TL_n(\delta)$ we can write it as a sum,

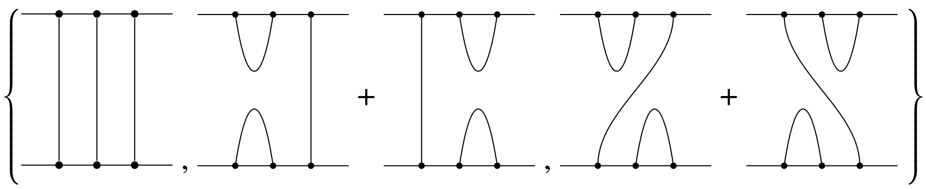
$$d = x_1a_1 + x_2a_2 + \cdots + x_ma_m, \;\; m \in \mathbb{N}, x_i \in k \tag{2.3}$$

where each a_i is unique in the basis of $TL_n(\delta)$. If any of the a_i are not such that $a_i = \sigma(a_i)$ then to satisfy $d = \sigma(d)$ one of the elements in the above sum must be the unique element a_j in the basis of $TL_n(\delta)$ satisfying $a_j = \sigma(a_i)$. We must also have $x_j = x_i$ to satisfy $d = \sigma(d)$, hence d is a linear combination of elements of O_n. □

Corollary 2.1.8. *For any $d \in OTL_n(\delta)$ we can write $d = a + \sigma(a)$ where $a \in TL_n(\delta)$.*

Proof. By above we can write any d in the form (2.3) where each a_i is in the basis of $TL_n(\delta)$. If the linear combination contains an element a_j in the basis of TL_n such that $a_j = \sigma(a_j)$ then we can replace it with $a_j = \frac{1}{2}a_j + \frac{1}{2}\sigma(a_j)$. □

Figure 2.2: Boundary diagram representation of the basis of $OTL_3(\delta)$



We shall refer to $ITL_n(\delta)$ as the Inner Temperley-Lieb algebra and $OTL_n(\delta)$ as the Outer-Temperley-Lieb algebra.

Lemma 2.1.9. *For each even $n \in \mathbb{N}$, $ITL_n(\delta)$ is isomorphic to the special case of the blob algebra, $b_{\frac{n}{2}}(\delta, 1)$, as defined in section 1.2.*

Proof. See for example, [20, Section 4.3] or [13, Lemma 5.7]. □

Corollary 2.1.10. *For each even $n \in \mathbb{N}$, the dimension of $ITL_n(\delta)$ is given by the formula,*

$$\dim ITL_n(\delta) = \binom{n}{n/2} = \frac{n!}{((n/2)!)^2}. \tag{2.4}$$

Proof. Apply the above lemma to the formula for the dimension of the blob algebra,

$$\dim b_n(\delta, \delta') = \binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$

as given in [13, Lemma 5.7]. □

Example 2.1.11. Let $n = 3$, in example 1.1.2 we show the basis diagram classes of $TL_3(\delta)$. Since the identity element I is the only basis element in $TL_3(\delta)$ fixed by σ we have $ITL_3 = k\{I\}$. However, when constructing a basis for $OTL_3(\delta)$ we also include any linear combinations of basis elements of $TL_3(\delta)$ which are fixed by σ. Hence as we have $u_1 = \sigma(u_2)$ in $TL_3(\delta)$ we get,

$$OTL_3(\delta) = k\{I, u_1 + u_2, u_1u_2 + u_2u_1\},$$

which can be represented as diagrams as shown in figure 2.2.

Lemma 2.1.12. *For any $n \in \mathbb{N}$;*

$$\begin{aligned}\dim OTL_n(\delta) &= \dim ITL_n(\delta) + \frac{1}{2}(\dim TL_n(\delta) - \dim ITL_n(\delta)) \\ &= \frac{1}{2}(\dim ITL_n(\delta) + \dim TL_n(\delta)).\end{aligned} \tag{2.5}$$

Proof. By definition, $OTL_n(\delta)$ contains any element of $TL_n(\delta)$ that is fixed by σ. Any element of $TL_n(\delta)$ is a linear combination of basis elements $s_1 d_1 + s_2 d_2 + \cdots + s_k d_k$ for some $k \in \mathbb{N}$, $s_i \in k$. For this element to be fixed by σ we must have,

$$s_1 d_1 + s_2 d_2 + \cdots + s_k d_k = s_1 \sigma(d_1) + s_2 \sigma(d_2) + \cdots + s_k \sigma(d_k), \tag{2.6}$$

that is for every $i \in [1, k]$ there exists $j \in [1, k]$ such that $d_i = \sigma(d_j)$. Thus for each basis element d_i in the combination (2.6) we either have $d_i = \sigma(d_i)$, that is $d_i \in ITL_n(\delta)$ or the basis element d_j such that $\sigma(d_i) = d_j$ is also in the combination. Thus the basis of $OTL_n(\delta)$ can be composed of the basis elements of $ITL_n(\delta)$, that is the basis elements of $TL_n(\delta)$ fixed by σ and for each basis element $a \in TL_n(\delta)$ not fixed by σ, the element $a + \sigma(a)$. This gives us $\dim ITL_n(\delta)$ basis elements from basis elements fixed by σ and $\frac{1}{2}(\dim TL_n(\delta) - \dim ITL_n(\delta))$ from elements not fixed by σ. The $\frac{1}{2}$ is due to the fact that if $a \in TL_n(\delta)$, then $\sigma(a) \in TL_n(\delta)$ so $a + \sigma(a)$ and $\sigma(a) + a \in OTL_n(\delta)$ but $a + \sigma(a) = \sigma(a) + a \in OTL_n(\delta)$ so we only need to include $a + \sigma(a)$. □

Proposition 2.1.13. *For any $n \in \mathbb{N}$, $\delta \in k$,*

$$\dim OTL_n(\delta) = \begin{cases} \frac{1}{2}\left[\frac{2n!}{(n+1)!n!} + \frac{(n-1)!}{(\frac{n+1}{2})!(\frac{n-1}{2})!}\right], & n \textit{ odd}, \\ \frac{1}{2}\left[\frac{2n!}{(n+1)!n!} + \frac{n!}{(\frac{n}{2}!)^2}\right], & n \textit{ even}. \end{cases} \tag{2.7}$$

Proof. For n odd each symmetric diagram of $TL_n(\delta)$ must have a propagating line from the node $\frac{n+1}{2}$ to the node $\frac{n+1}{2}'$. To the left of this line must then be a valid $TL_{\frac{n-1}{2}}(\delta)$ diagram say a, with the diagram $\sigma(a)$ on the right of the line to satisfy symmetry. Thus $\dim ITL_n(\delta) = \dim TL_{\frac{n-1}{2}}(\delta)$ and by applying both Theorem 1.1.3 and Lemma 2.1.12 we get,

$$\dim OTL_n(\delta) = \frac{1}{2}\left(\frac{(n-1)!}{(\frac{n+1}{2})!(\frac{n-1}{2})!} + \frac{2n!}{(n+1)!n!}\right).$$

For n even we can use the formula for $\dim ITL_n(\delta)$ given by corollary 2.1.10 to get the required result. □

Figure 2.3 gives, up to rank $n = 12$ the dimensions of the Inner, Outer and regular Temperley-Lieb algebras.

Figure 2.3: Table of dimensions of the Inner ($ITL_n(\delta)$), Outer ($OTL_n(\delta)$) and regular ($TL_n(\delta)$) Temperley-Lieb Algebras.

Rank n	$\dim TL_n(\delta)$	$\dim ITL_n(\delta)$	$\dim OTL_n(\delta)$
0	1	1	1
1	1	1	1
2	2	2	2
3	5	1	3
4	14	6	10
5	42	2	22
6	132	20	76
7	429	5	217
8	1430	70	750
9	4862	14	2438
10	16796	252	8524
11	58786	42	29414
12	208012	924	104468

2.2 Cell modules of $OTL_n(\delta)$

In the next few sections we shall begin studying the representation theory of the Outer-Temperley-Lieb algebra. We begin in this section by constructing the bases of a set of modules of the Outer-Temperley-Lieb algebra from the bases of modules of the Temperley-Lieb algebra. We then show how we can construct a formula for the dimensions of these modules.

Recall the definition of $TL_n(\delta)$ cell modules $C_{n,p}$ as defined in Lemma 1.3.3 for $0 \leq p \leq n$ with $n - p \in 2\mathbb{Z}$.

Definition 2.2.1. Let $T_{n,p}$ be the basis of $C_{n,p}$ given by the top-half diagrams as described in Lemma 1.3.3. We denote by $T^+_{n,p}$ the set,

$$T^+_{n,p} := \{d \mid \text{ for all } d \in T_{n,p}, d = \sigma(d)\} \cup \{d + \sigma(d) \mid \text{ for all } d \in T_{n,p}, d \neq \sigma(d)\}.$$

Then we define by $C^+_{n,p}$, the vector subspace of $C_{n,p}$ generated by $T^+_{n,p}$.

We define by $C^-_{n,p}$, the vector subspace of $C_{n,p}$ whose basis consists of the differences $d - \sigma(d)$ for every d in $T_{n,p}$ such that $d \neq \sigma(d)$, that is $C^-_{n,p}$ has basis $T^-_{n,p}$ given by,

$$T^-_{n,p} := \{d - \sigma(d) \mid \text{ for all } d \in T_{n,p}, d \neq \sigma(d)\}.$$

Lemma 2.2.2. *$C^+_{n,p}$ and $C^-_{n,p}$ are both left $OTL_n(\delta)$-modules.*

Proof. Let $b \in OTL_n(\delta)$ then we can write $b = a + \sigma(a)$ where $a \in TL_n(\delta)$.

First let $m \in C^+_{n,p}$. Note we can write $m = \alpha x + \alpha\sigma(x)$ with $x \in C_{n,p}$, $\alpha \in k$. This is clear if $m \neq \sigma(m)$, if $m = \sigma(m)$ we can write $m = \frac{1}{2}m + \frac{1}{2}\sigma(m)$. Therefore,

$$\begin{aligned} bm &= (a + \sigma(a))(x + \sigma(x)) = ax + a\sigma(x) + \sigma(a)x + \sigma(a)\sigma(x) \\ &= ax + \sigma(ax) + a\sigma(x) + \sigma(a\sigma(x)). \end{aligned}$$

But since $C_{n,p}$ is a module for $TL_n(\delta)$ we have $ax, \sigma(ax), a\sigma(x), \sigma(a\sigma(x)) \in C_{n,p}$ and thus $bm \in C^+_{n,p}$ as required.

Now let $m \in C^-_{n,p}$ then we can write $m = y - \sigma(y)$ for some $y \neq \sigma(y) \in C_{n,p}$. Then

$$\begin{aligned} bm &= (a + \sigma(a))(y - \sigma(y)) = ay - a\sigma(y) + \sigma(a)x - \sigma(a)\sigma(x)) \\ &= (ax - \sigma(ax)) + (a\sigma(x) - \sigma(a\sigma(x))). \end{aligned}$$

Figure 2.4: The diagram basis of $C^+_{4,2}$.

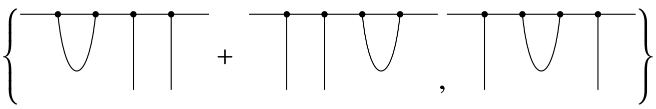
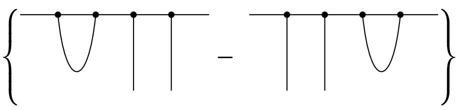


Figure 2.5: The diagram basis of $C^-_{4,2}$.



Again we know $ax, \sigma(ax), a\sigma(x), \sigma(a\sigma(x)) \in C_{n,p}$, thus $bm \in C^-_{n,p}$ as required. $\square$

Example 2.2.3. Figures 2.4 and 2.5 shows the diagram bases of the $OTL_4(\delta)$-modules $C^+_{4,2}$ and $C^-_{4,2}$ respectively.

In a similar way to calculating the dimension of $OTL_n(\delta)$ we can calculate the dimensions of these $OTL_n(\delta)$-modules as follows.

Lemma 2.2.4. *Let $p \leq n$ such that $n - p \in 2\mathbb{Z}$. Let $\#\mathrm{Sym}_{T_{n,p}}$ denote the number of elements $d \in T_{n,p}$ such that $d = \sigma(d)$. Then the dimensions of the $OTL_n(\delta)$-modules $C^+_{n,p}$ and $C^-_{n,p}$ are given by,*

$$\dim C^+_{n,p} = \frac{1}{2}(\dim C_{n,p} + \#\mathrm{Sym}_{T_{n,p}}), \tag{2.8}$$
$$\dim C^-_{n,p} = \frac{1}{2}(\dim C_{n,p} - \#\mathrm{Sym}_{T_{n,p}}). \tag{2.9}$$

Proof. The proof for $C^+_{n,p}$ is identical to the proof of Lemma 2.1.12 replacing $OTL_n(\delta)$ with $C^+_{n,p}$, $TL_n(\delta)$ with $C_{n,p}$ and $ITL_n(\delta)$ with $\#\mathrm{Sym}_{T_{n,p}}$. For $C^-_{n,p}$, by definition, its basis contains no basis element of $C_{n,p}$ such that $d = \sigma(d)$ hence we deduct the number of them from $\dim C_{n,p}$. The remaining basis elements $f \in T_{n,p}$ each appear in the basis of $C^-_{n,p}$ in the difference $f - \sigma(f)$, but since $f - \sigma(f) = -1(\sigma(f) - f)$, that is these elements are not linearly independent, each $(\sigma(f) - f)$ can be removed from the basis hence we multiply $\dim C_{n,p} - \#\mathrm{Sym}_{T_{n,p}}$ by $\frac{1}{2}$ to get the final result. $\square$

Note that it follows that $\dim C^+_{n,p} + \dim C^-_{n,p} = \dim C_{n,p}$. In the case where n is odd we can calculate the number of $d \in T_{n,p}$ such that $d = \sigma(d)$ since each such d must contain a propagating line at its $\frac{n+1}{2}^{th}$ node. The diagram to the left of this line is any valid element $f \in C_{\frac{n-1}{2},\frac{p-1}{2}}$, with $\sigma(f)$ appearing on the right of this line. Thus $\#\operatorname{Sym}_{T_{n,p}} = \dim C_{\frac{n-1}{2},\frac{p-1}{2}}$ and

$$\dim C^+_{n,p} = \frac{1}{2}(\dim C_{n,p} + \dim C_{\frac{n-1}{2},\frac{p-1}{2}}), \tag{2.10}$$
$$\dim C^-_{n,p} = \frac{1}{2}(\dim C_{n,p} - \dim C_{\frac{n-1}{2},\frac{p-1}{2}}), \tag{2.11}$$

for n odd.

In section 2.4 we show that the modules $C^+_{n,p}$, $C^-_{n,p}$ are the cell modules of $OTL_n(\delta)$ in the sense of [11] with respect to an involutive anti-automorphism which we shall define in the following section. We shall first begin to address the irreducibility of $C^+_{n,p}$, $C^-_{n,p}$.

2.3 Irreducibility of the $OTL_n(\delta)$ cell modules

In this section we show that the modules we defined in the previous section are indecomposable modules. Similar to [27] and [33] we define a bilinear contravariant (or invariant) form on our modules and show that it cannot be identically zero except in a few special cases which are handled at the end of the section. We can use this to show that these modules are the complete set of cyclic and indecomposable modules of the Outer-Temperley-Lieb algebra and that their quotient by their radicals, with respect to the defined forms, is simple.

Let $\downarrow$ be the map sending a $TL_n(\delta)$-diagram d to its reflection $d^\downarrow$ in the horizontal plane. Similar to proposition 2.1.2 (or see [27, Section 3]) we can show that $\downarrow$ is an anti-automorphism of $TL_n(\delta)$. Note that we can k-linearly extend this anti-isomorphism to $OTL_n(\delta)$, since reflections in the horizontal and vertical planes commute.

The $TL_n(\delta)$-modules $C_{n,p}$ have a well known bilinear form $\langle \cdot, \cdot \rangle : C_{n,p} \times C_{n,p} \to k$, see for example [27, Section 3]. This takes any two top half n-diagrams $x, y \in C_{n,p}$ and forms $\langle x, y \rangle \in k$ by reflecting x across a horizontal line and placing it atop y such that each node i in y is identified with the node i in x. If there is a propagating line in x not connected (whether directly or indirectly) to a propagating line in y then $\langle x, y \rangle = 0$, otherwise $\langle x, y \rangle = \delta^l$ where l is the number of loops formed in this composition. We can bilinearly extend $\langle \cdot, \cdot \rangle$ to all of $C_{n,p}$.

Lemma 2.3.1.

i) $\langle x, y \rangle = \langle y, x \rangle$ *for all* $x, y \in C_{n,p}$.

ii) $\langle x, dy \rangle = \langle d^{\downarrow} y, x \rangle$ *for all* $x, y \in C_{n,p}, d \in TL_n(\delta)$.

iii) $\langle x, x \rangle = \delta^{\frac{n-p}{2}}$ *for all* $x \in C_{n,p}$.

iv) $\langle x, y \rangle = \langle \sigma(x), \sigma(y) \rangle$.

Proof.

i) & ii) See [27, Chapter 3].

iii) The diagram $x \in C_{n,p}$ has exactly p propagating lines and therefore exactly $\frac{n-p}{2}$ arcs. When placing $x^{\downarrow}$ atop x, each single arc in x will be closed by the reflected arc in $x, x^{\downarrow}$, resulting in $\frac{n-p}{2}$ loops.

iv) Let $x^{\downarrow}$ be as above. Since $\sigma(x)^{\downarrow} = \sigma(x^{\downarrow})$ by properties of reflection, placing $\sigma(x)^{\downarrow}$ atop $\sigma(y)$ will result in the same number of loops as placing $\sigma(x^{\downarrow})$ atop $\sigma(y)$. But this is just the reflection in the vertical plane of $x^{\downarrow}$ placed atop y and will therefore have the same number of loops as $\langle x, y \rangle$. □

As described in section 1.3, any $TL_n(\delta)$ diagram d can be cut such that all arcs on the top edge of d are above the cut, all arcs on the bottom edge of d are below the cut and each propagating line is only cut once. We therefore have a well defined a map $|\cdot,\cdot| : C_{n,p} \times C_{n,p} \to TL_n(\delta)$ where for any $a, b \in C_{n,p}$, $|a, b|$ is the $TL_n(\delta)$ diagram which when cut as described above splits into the top half n-diagram a on the top edge and the bottom half n-diagram $b^{\downarrow}$ on the bottom edge.

Lemma 2.3.2. *If $a, b, d \in C_{n,p}$ then*

$$|a, b|d = \langle b, d\rangle a. \tag{2.12}$$

Proof. See [27, Lemma 3.2]. □

We note that, as described in sections 1.3, the involutive anti-automorphism $\downarrow$ is the automorphism associated with the cellular basis of $TL_n(\delta)$. As mentioned in section 1.4 it follows from the above that the form $\langle -, -\rangle$ is the unique form induced by the cellular structure, which can be used to determine the unique radical of the $TL_n(\delta)$ cell modules. Full details can be found in [11, Sections 1 and 6].

We can use the form $\langle\cdot,\cdot\rangle$ to define forms on the $OTL_n(\delta)$-modules $C^+_{n,p}$ and $C^-_{n,p}$.

If $X = x + \sigma(x), Y = y + \sigma(y) \in C^+_{n,p}$ where $x, y \in C_{n,p}$ then;

$$\begin{aligned}\langle X, Y\rangle &= \langle x + \sigma(x), y + \sigma(y)\rangle = \langle x, y\rangle + \langle x, \sigma(y)\rangle + \langle \sigma(x), y\rangle + \langle \sigma(x), \sigma(y)\rangle \\ &= 2\langle x, y\rangle + 2\langle x, \sigma(y)\rangle.\end{aligned}$$

We therefore define the form $[\cdot,\cdot]^+ : C^+_{n,p} \times C^+_{n,p} \to k$ such that,

$$[X, Y]^+ := \langle x, y\rangle + \langle x, \sigma(y)\rangle = \frac{1}{2}\langle X, Y\rangle. \tag{2.13}$$

If $X = x - \sigma(x), Y = y - \sigma(y) \in C^-_{n,p}$ where $x \neq \sigma(x), y \neq \sigma(y) \in C_{n,p}$ then;

$$\begin{aligned}\langle X, Y\rangle &= \langle x - \sigma(x), y - \sigma(y)\rangle = \langle x, y\rangle - \langle x, \sigma(y)\rangle - \langle \sigma(x), y\rangle + \langle \sigma(x), \sigma(y)\rangle \\ &= 2\langle x, y\rangle - 2\langle x, \sigma(y)\rangle.\end{aligned}$$

We therefore define the form $[\cdot,\cdot]^- : C^-_{n,p} \times C^-_{n,p} \to k$ such that,

$$[X,Y]^- := \langle x,y\rangle - \langle x,\sigma(y)\rangle = \frac{1}{2}\langle X,Y\rangle. \tag{2.14}$$

The following proposition follows immediately from the definitions of $[\cdot,\cdot]^+$ and $[\cdot,\cdot]^-$.

Proposition 2.3.3. *The forms $[\cdot,\cdot]^+, [\cdot,\cdot]^-$ are symmetric bilinear forms on $C^+_{n,p}, C^-_{n,p}$ respectively, satisfying $[X,dY]^+ = [d^{\downarrow}X,Y]^+$, $[P,dQ]^- = [d^{\downarrow}P,Q]^-$ for all $X,Y \in C^+_{n,p}$, $P,Q \in C^-_{n,p}$, $d \in OTL_n(\delta)$.*

Denote by $R^+_{n,p}, R^-_{n,p}$ the radicals of the forms $C^+_{n,p}, C^+_{n,p}$ respectively, that is;

$$R^+_{n,p} = \{X \in C^+_{n,p} \mid [X,Y]^+ = 0 \text{ for all } Y \in C^+_{n,p}\},$$
$$R^-_{n,p} = \{X \in C^-_{n,p} \mid [X,Y]^- = 0 \text{ for all } Y \in C^-_{n,p}\}.$$

Lemma 2.3.4. *If $[\cdot,\cdot]^+$ is not identically zero on $C^+_{n,p}$ then $C^+_{n,p}$ is a cyclic, indecomposable $OTL_n(\delta)$-module and the quotient $C^+_{n,p}/R^+_{n,p}$ is irreducible.*

Proof. If $[\cdot,\cdot]^+$ is not identically zero then there exist $X,Y \in C^+_{n,p}$ such that $[Y,X]^+ = 1$. We can write $X = x+\sigma(x), Y = y+\sigma(y)$, then for every $Z = z+\sigma(z) \in C^+_{n,p}$ we can form the diagram $(|y,z| + |\sigma(y)\sigma(z)|) \in OTL_n(\delta)$. Then,

$$\begin{aligned}(|z,y| + |\sigma(z),\sigma(y)|)X &= |z,y|x + |z,y|\sigma(x) + |\sigma(z),\sigma(y)|x + |\sigma(z),\sigma(y)|\sigma(x)\\ &= \langle y,x\rangle z + \langle y,\sigma(x)\rangle z + \langle \sigma(y),x\rangle\sigma(z) + \langle\sigma(y),\sigma(x)\rangle\sigma(z)\\ &= \langle y,x\rangle(z+\sigma(z)) + \langle y,\sigma(x)\rangle(z+\sigma(z)) = [Y,X]^+Z = Z.\end{aligned}$$

Thus X generates $C^+_{n,p}$, that is $C^+_{n,p}$ is cyclic. Now if $X \notin R^+_{n,p}$ then there exists a Y such that $[Y,X]^+ = 1$, thus X generates $C^+_{n,p}/R^+_{n,p}$, that is every $X \neq 0$ in the quotient $C^+_{n,p}/R^+_{n,p}$ generates $C^+_{n,p}$ and hence $C^+_{n,p}/R^+_{n,p}$ is irreducible. Now suppose $C^+_{n,p} = A \oplus B$, if $A \nsubseteq R^+_{n,p}$ it must contain a generator, in which case $A = C^+_{n,p}$ and we are done. Otherwise $A \subseteq R^+_{n,p}$. Similarly, if $B \neq C^+_{n,p}$ then $B \subseteq R^+_{n,p}$. The remaining case is $A \subseteq R^+_{n,p}$ and

Figure 2.6: The diagrams $x, y \in C_{n,p}$, $p \geq 2$ such that $\langle x, y\rangle = 1$.

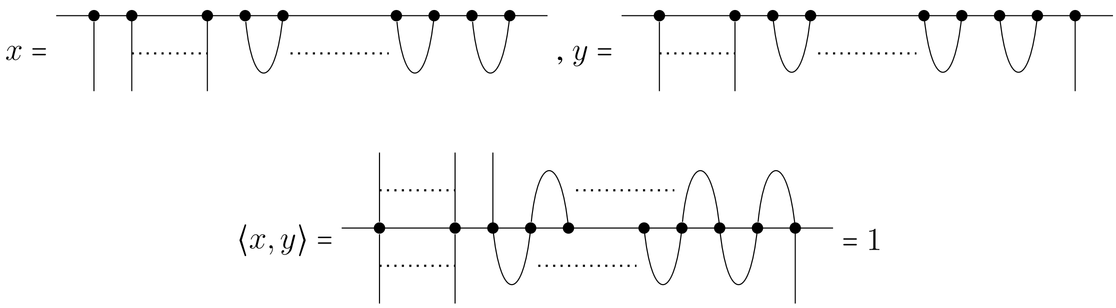
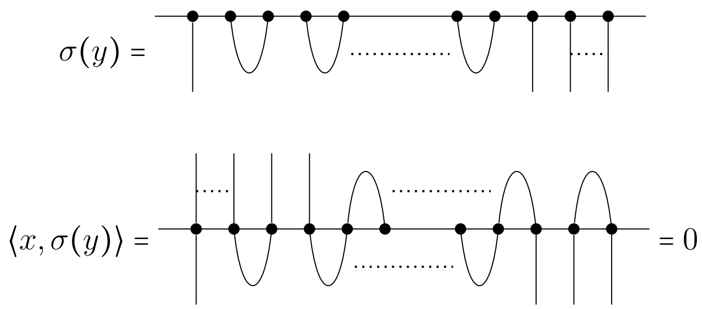


Figure 2.7: The diagram $\sigma(y)$ and its inner product with x from figure 2.6.



$B \subseteq R^{+}_{n,p}$, thus $A + B \subseteq R^{+}_{n,p}$. Therefore we have $R^{+}_{n,p} = C^{+}_{n,p}$ and hence $[\cdot,\cdot]^{+}$ is identically zero, a contradiction.

$\square$

We can similarly prove

Lemma 2.3.5. *If $[\cdot,\cdot]^{-}$ is not identically zero on $C^{-}_{n,p}$ then $C^{-}_{n,p}$ is a cyclic, indecomposable $OTL_n(\delta)$-module and the quotient $C^{-}_{n,p}/R^{-}_{n,p}$ is irreducible.*

Proposition 2.3.6. *Except for the cases $\delta = 0$ with $p = 0$ and $\delta = \pm 1$ with $p = 1$, $[\cdot,\cdot]^{+}, [\cdot,\cdot]^{-}$ are not identically zero on $C^{+}_{n,p}, C^{-}_{n,p}$ respectively.*

Proof. We will assume throughout this proof that $\delta \neq \pm 1$ when $p = 1$ and $\delta \neq 0$ when $p = 0$.

First suppose $\delta \neq 0$ and consider $p > 0$. If $p > 2$ then we may choose $X = x + \sigma(x), Y = y + \sigma(y)$ with $x, y \in C_{n,p}$ as in figure 2.6 and 2.7. Then;

$$[X, Y]^{+} = \langle x, y\rangle + \langle x, \sigma(y)\rangle = 1 + 0 \neq 0 \text{ for any } \delta.$$

If $p = 1$ however, there can be no $x, y \in C_{n,p}$ such that $\langle x, y\rangle = 0$. In this case n must be odd. If we choose the same x, y as above by simply removing all but the first propagating line in x and all but the last in y, we now get,

$$[X, Y]^{+} = \langle x, y\rangle + \langle x, \sigma(y)\rangle = 1 + \delta^{\frac{n-1}{2}}, \tag{2.15}$$

which is non-zero if and only if $\delta^{\frac{n-1}{2}} \neq -1$. If $\delta^{\frac{n-1}{2}} = -1$ we choose another pair $X' = x' + \sigma(x'), Y' = y' + \sigma(y')$ with x', y' as in figure 2.8. Now,

$$[X', Y']^{+} = \delta + \delta^{\frac{n-1}{2}-1}$$

and $\delta + \delta^{\frac{n-1}{2}-1} = 0$ only if $\delta^2 = -\delta^{\frac{n-1}{2}} = 1$, hence $\delta = \pm 1$ contradicting the hypothesis. Thus $[X', Y']^{+} \neq 0$.

For $p = 2$ the y chosen at the beginning of the proof, figure 2.6, becomes a symmetric element, that is $y = \sigma(y)$. Hence we can set $X = \frac{1}{2}y + \frac{1}{2}\sigma(y)$, then,

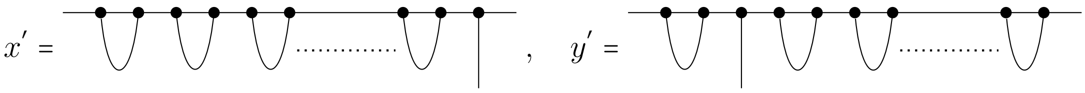
$$[X, X]^{+} = \langle \frac{1}{2}y, \frac{1}{2}y\rangle + \langle \frac{1}{2}y, \frac{1}{2}\sigma(y)\rangle = \frac{1}{2}\delta^{\frac{n-p}{2}} \neq 0 \text{ for } \delta \neq 0.$$

Similarly for $p = 0$, let $X = x \in C^{+}_{n,0}$ such that x is a single diagram, that is, not a linear combination of diagrams, satisfying $x = \sigma(x) \in C_{n,0}$. Then by writing $X = \frac{1}{2}x + \frac{1}{2}\sigma(x)$ we have;

$$[X, X]^{+} = \langle \frac{1}{2}x, \frac{1}{2}x\rangle + \langle \frac{1}{2}x, \frac{1}{2}\sigma(x)\rangle = \frac{1}{2}\delta^{\frac{n-p}{2}} \neq 0 \text{ for } \delta \neq 0.$$

When $\delta = 0$, $p \neq 0$ we can still argue as in the $\delta \neq 0$ case for $C^{+}_{n,p}$ unless $p = 2$. For $p = 2$ and $\delta = 0$ the x, y as in figure 2.6 satisfy $y = \sigma(y)$ and $\langle x, y\rangle = 1$. Thus setting $X = x + \sigma(x)$ and $Y = \frac{1}{2}y + \frac{1}{2}\sigma(y)$ we get $[X, Y] = 1 \neq 0$.

Figure 2.8: The diagrams $x', y' \in C_{n,1}$.



We now turn our attention to the modules $C^{-}_{n,p}$. If $X = x - \sigma(x) \in C^{-}_{n,p}$ with $x \neq \sigma(x) \in C_{n,p}$. We have;

$$[X, X]^{-} = \langle x, x\rangle - \langle x, \sigma(x)\rangle = \delta^{\frac{n-p}{2}} - \langle x, \sigma(x)\rangle.$$

Therefore, for $\delta \neq 0$, unless δ is such that there exists r with $\delta^r = 1$, we can only have $[X, X]^{-} = 0$ if $\langle x, \sigma(x)\rangle = \delta^{\frac{n-p}{2}}$.

Suppose $\langle x, \sigma(x)\rangle = \delta^{\frac{n-p}{2}}$, then each arc on the edge of $\sigma(x)$ must be closed into a loop by an arc on the edge of x, that is x must have an arc whenever $\sigma(x)$ has a arc. But this can only happen if $x = \sigma(x)$, a contradiction.

If δ is such that there exists r with $\delta^r = 1$, then arguing in the same way as with $[-,-]^{+}$, the product $[-,-]^{-}$ can be identically zero on $C^{-}_{n,p}$ only if $p = 1$ and $\delta^{\frac{n-1}{2}} = 1$. In this case we can again choose $X' = x' + \sigma(x)', Y' = y' + \sigma(y)'$ with x', y', as in figure 2.8. Then,

$$[X', Y']^{-} = \delta - \delta^{\frac{n-1}{2}-1},$$

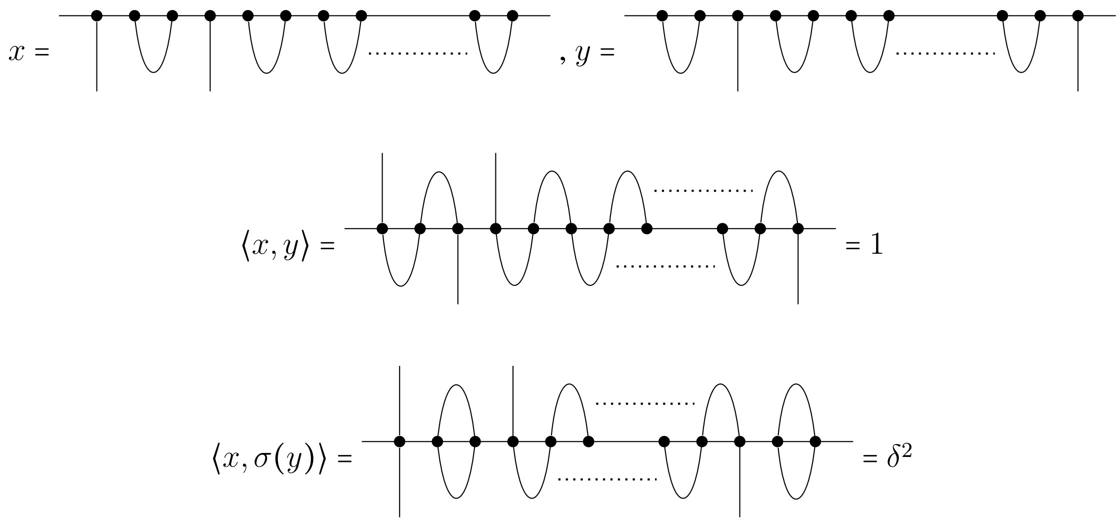
which is zero only if $\delta^2 = \delta^{\frac{n-1}{2}} = 1$, hence $\delta = \pm 1$ contradicting our hypothesis.

For $\delta = 0$, when $p \neq 2$ setting $X = x - \sigma(x)$, $Y = y - \sigma(y)$ for the x, y as in figure 2.6 we get,

$$[X, Y]^{-} = \langle x, y\rangle - \langle x, \sigma(y)\rangle = 1 + 0 \neq 0.$$

However, for $p = 2$ this will not work since y is such that $y = \sigma(y)$, hence $Y = y - \sigma(y) = 0$. Instead, for $n > 5$, as $C^{-}_{n,0}$ does not exist for $n \leq 5$, we can choose x, y to be the diagrams in $C_{n,2}$ such that x has propagating lines at nodes 1 and 3 and y has propagating lines at nodes 2 and n as shown in figure 2.9. Then $\langle x, y\rangle = 1$ and $\langle x, \sigma(y)\rangle = 0$ and hence $[X, Y]^{-} = 1 \neq 0$. $\square$

Figure 2.9: The diagrams $x, y \in C_{n,2}$ such that $[x+\sigma(x), y+\sigma(y)] = 1$.



We note that for $\delta = 0$, $[\cdot,\cdot]^-$ is identically zero on $C^-_{4,2}$, however as this module contains only one basis element it is already an irreducible module.

In the cases $\delta = 0$ with $p = 0$ and $\delta = \pm 1$ with $p = 1$ we know it is possible for the forms $[-,-]^+$, $[-,-]^-$ to be identically zero and therefore we cannot apply Lemmas 2.3.4, 2.3.5. To overcome this we shall attempt to renormalise our forms so that we can still show the modules $C^{\pm}_{n,p}$ are cyclic, indecomposable modules with irreducible quotients.

We first consider the case $\delta = 0, p = 0$. In [27, Section 3] this case is dealt with by renormalising the form $\langle\cdot,\cdot\rangle$ to define the new form,

$$\langle x, y\rangle' = \lim_{\delta\to 0} \frac{\langle x, y\rangle}{\delta}, \quad (x, y \in C_{n,0}). \tag{2.16}$$

Now for any two $x, y \in C_{n,0}$, the inner-product $\langle x, y\rangle'$ is 0 except when a single loop is formed, in which case it is $\lim_{\delta\to 0} \frac{\delta}{\delta} = 1$. The form inherits bi-linearity and invariance from $\langle\cdot,\cdot\rangle$. The following are also inherited from $\langle\cdot,\cdot\rangle$.

Lemma 2.3.7. *When $\delta = 0$, for all $x, y \in C_{n,0}$,*

i) $\langle x, y\rangle' = \langle y, x\rangle'$, *ii)* $\langle x, x\rangle' = \lim_{\delta\to 0}\delta^{\frac{n}{2}-1}$,

iii) $\langle x, y\rangle' = \langle\sigma(x), \sigma(y)\rangle'$.

Definition 2.3.8. When $\delta = 0$, we define the forms $[\cdot,\cdot]^{+'}$, $[\cdot,\cdot]^{-'}$ on $C^{+}_{n,0}$, $C^{-}_{n,0}$ respectively by;

$$[X, Y]^{+'} = \lim_{\delta\to 0}\frac{[X, Y]^{+}}{\delta}, \quad \text{for all } X, Y \in C^{+}_{n,0}. \tag{2.17}$$
$$[X, Y]^{-'} = \lim_{\delta\to 0}\frac{[X, Y]^{-}}{\delta}, \quad \text{for all } X, Y \in C^{-}_{n,0}. \tag{2.18}$$

The forms $[\cdot,\cdot]^{+'}$, $[\cdot,\cdot]^{-'}$ inherit their bilinearity, symmetry and invariance from $[\cdot,\cdot]^{+}$, $[\cdot,\cdot]^{-}$ respectively. Note that for $x, y \in C_{n,p}$,

$$[X, Y]^{+'} = \langle x, y\rangle' + \langle x, \sigma(y)\rangle', \quad X = x + \sigma(x), Y = y + \sigma(y),$$
$$[X, Y]^{-'} = \langle x, y\rangle' - \langle x, \sigma(y)\rangle', \quad X = x - \sigma(x), Y = y - \sigma(y), (x \neq \sigma(x), y \neq \sigma(y)).$$

Proposition 2.3.9. *For $\delta = 0$, $[\cdot,\cdot]^{+'}$, $[\cdot,\cdot]^{-'}$ are not identically zero on $C^{+}_{n,0}$, $C^{-}_{n,0}$ respectively.*

Proof. In $C_{n,p}$ we can choose x such that it is a single diagram with an arc from node 1 to node n and arcs from nodes i to $i+1$ where $i = 2, 4, \ldots, n-2$. We can also choose $y \in C_{n,0}$ such that it contains an arc from all nodes i to $i+1$ where $i = 1, 3, \ldots, n-1$. Then as shown in [27, Equation 3.14] $\langle x, y\rangle' = 1$. Since $x = \sigma(x)$ and $y = \sigma(y)$ as they are defined, set $X = x, Y = \frac{1}{2}y \in C^{+}_{n,0}$. Then by equations (2.13) and (2.17) we have $[X, Y]^{+'} = \frac{1}{2}(\langle x, y\rangle' + \langle x, y\rangle') = 1$.

For $n < 5$, $C^{-}_{n,0}$ does not exist by definition. For $C^{-}_{n,0}$, where $n \geq 5$, we set $X = x - \sigma(x), Y = y - \sigma(y)$ where $x, y \in C_{n,0}$ are as in figure 2.10. Then as shown in figure 2.11, $\langle x, y\rangle' = 1$ and $\langle x, y\rangle' = 0$ hence $[X, Y]^{-'} = 1$. □

We now turn our attention to the case of $\delta = \pm 1$, $p = 1$.

Figure 2.10: Diagrams $x, y \in C_{n,0}$ such that $[X, Y]^{-\prime} = 1$.

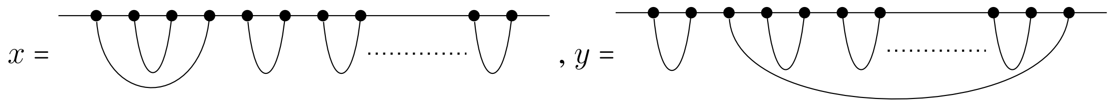
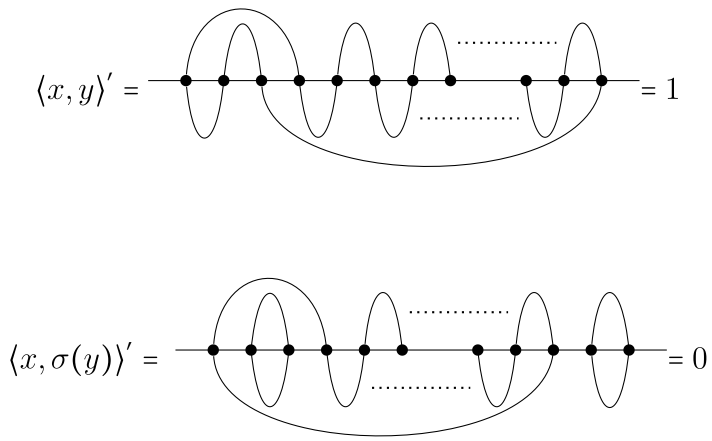


Figure 2.11: Inner products of the diagrams in figure 2.10.



Lemma 2.3.10. *When $\delta = -1$ and $\frac{n-1}{2}$ is even the form $[-,-]^{+}$ is not identically zero on $C^{+}_{n,1}$.*

Proof. When $\frac{n-1}{2}$ is even, $C^{+}_{n,1}$ contains an element x whose diagram has; a propagating line at the node $\frac{n+1}{2}$, for each node $i \in \{1, 3, 5, \ldots, \frac{n-3}{2}\}$ an arc from i to $i+1$ and for each node $i \in \{n, n-2, n-4, \ldots, \frac{n+5}{2}\}$ an arc from i to $i-1$. As this diagram contains chains of length one arcs on either side of the central propagating line, it satisfies $x = \sigma(x)$. Then $[x, x] = \delta^{\frac{n-1}{2}} \neq 0$ at $\delta = -1$ as required. $\square$

We shall handle the cases of $\delta = 1$, $p = 1$ and $\delta = -1$, $p = 1$, $\frac{n-1}{2}$ odd, similarly to that of $\delta = 0$, $p = 0$, by defining two more normalised forms.

Definition 2.3.11. When $\delta = -1$ with $\frac{n-1}{2}$ odd or $\delta = 1$ for any n we define the form

$[-,-]^{+''}$ on $C^+_{n,1}$ and the form $[-,-]^{-''}$ on $C^-_{n,1}$ as follows;

$$[X,Y]^{+''} = \lim_{\delta\to -1} \frac{[X,Y]^+}{\delta+1}, \quad \text{for all } X,Y \in C^+_{n,1}, \tag{2.19}$$

$$[X,Y]^{-''} = \lim_{\delta\to 1} \frac{[X,Y]^-}{\delta-1}, \quad \text{for all } X,Y \in C^-_{n,1}. \tag{2.20}$$

As with $[-,-]^{+'}, [-,-]^{-'}$, these new forms inherit bilinearity, symmetry and invariance from $[-,-]^+, [-,-]^-$ respectively. Note that for $\frac{n-1}{2}$ odd, $[-,-]^+$ is either zero or a contains a factor of $\delta+1$ on $C^+_{n,1}$ and hence is always zero when $\delta=-1$. Similarly $[-,-]^-$ always contains a factor of $\delta-1$ on $C^-_{n,1}$ and hence is always zero when $\delta=1$. Thus the normalised forms $[-,-]^{+''}$ and $[-,-]^{-''}$ are both well-defined. The use of the limit in these renormalised forms means we can show that they are not identically zero on $C^+_{n,1}$and $C^-_{n,1}$ respectively as we will now demonstrate.

Proposition 2.3.12. *For $\delta=-1$, $\frac{n-1}{2}$ odd and $\delta=1$ with any n the forms $[-,-]^{+''}, [-,-]^{-''}$ are not identically zero on $C^+_{n,1}, C^-_{n,1}$ respectively. For $\delta=-1$, $\frac{n-1}{2}$ odd and $\delta=1$ with any n the forms $[-,-]^{+''}, [-,-]^{-''}$ are not identically zero on $C^+_{n,1}, C^-_{n,1}$ respectively.*

Proof. By choosing $X = x+\sigma(x) \in C^+_{n,1}$ with x as described above equation (2.15), we know that $[X,X]^+ = \delta^{\frac{n-1}{2}}+1$ which when $\delta=-1$ is zero for $\frac{n-1}{2}$ odd. However,

$$\begin{aligned}[X,X]^{+''} &= \lim_{\delta\to -1} \frac{\delta^{\frac{n-1}{2}}+1}{\delta+1} = \delta^{\frac{n-3}{2}} - \delta^{\frac{n-5}{2}} + \delta^{\frac{n-7}{2}} - \cdots + 1 \\ &= 1-(-1)+1-\cdots+1 \neq 0\end{aligned}$$

Thus for $\delta=-1$, $\frac{n-1}{2}$ odd, the form $[-,-]^{+''}$ is not identically zero on $C^+_{n,1}$.

Similarly we can choose $X = x-\sigma(x) \in C^-_{n,p}$ using the same x, then we have $[X,X]^- = \delta^{\frac{n-1}{2}}-1$. For $\delta=1$ this will always be zero, however,

$$\begin{aligned}[X,X]^{-''} &= \lim_{\delta\to 1} \frac{\delta^{\frac{n-1}{2}}-1}{\delta-1} = \delta^{\frac{n-3}{2}} + \delta^{\frac{n-5}{2}} + \delta^{\frac{n-7}{2}} + \cdots + 1 \\ &= 1+1+1+\cdots+1 \neq 0.\end{aligned}$$

Thus $[X,X]^{-''}$ is not identically zero on $C^-_{n,1}$. □

Denote by $R^{+'}_{n,0}$, $R^{-'}_{n,0}$,$R^{+''}_{n,1}$, $R^{-''}_{n,1}$ the radicals of the modules $C^{+}_{n,0}$, $C^{-}_{n,0}$, $C^{+}_{n,1}$, $C^{-}_{n,1}$ with respect to the forms $[\cdot,\cdot]^{+'}$, $[\cdot,\cdot]^{-'}$, $[\cdot,\cdot]^{+''}$, $[\cdot,\cdot]^{-''}$ respectively.

Lemma 2.3.13. *The forms $[\cdot,\cdot]^{+}$, $[\cdot,\cdot]^{-}$ are identically zero if and only if $\delta = 0$ and $p = 0$ or and $p = 1$ and either $\delta = -1$ with $\frac{n-1}{2}$ odd or $\delta = 1$ with any n. Then the modules $C^{+}_{n,0}, C^{-}_{n,0}, C^{+}_{n,1}, C^{-}_{n,1}$ are cyclic, indecomposable and have irreducible quotients $C^{+}_{n,0}/R^{+'}_{n,0}$, $C^{-}_{n,0}/R^{-'}_{n,0}$, $C^{+}_{n,1}/R^{+''}_{n,1}$, $C^{-}_{n,1}/R^{-''}_{n,1}$ respectively.*

Proof. Apply the same process in the proof of Lemma 2.3.4 using the forms $[\cdot,\cdot]^{+'}$, $[\cdot,\cdot]^{-'}$, $[\cdot,\cdot]^{+'}$, $[\cdot,\cdot]^{-'}$. □

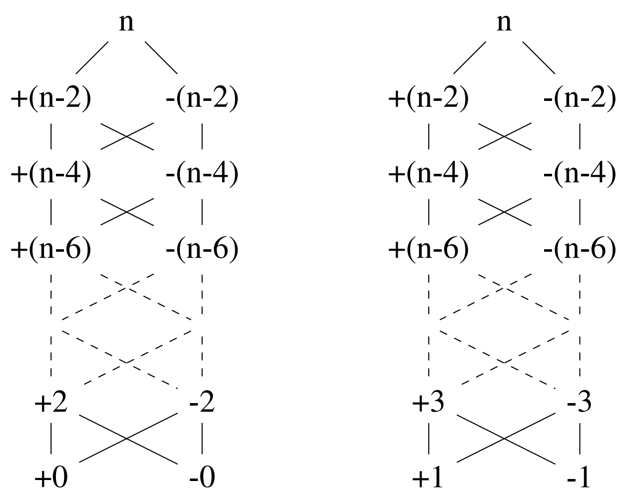
2.4 $OTL_n(\delta)$ as a cellular algebra

In this section we show that the Outer-Temperley-Lieb algebra satisfies the conditions to be a cellular algebra and that the modules defined in the previous section are the cell modules in the sense of [11]. Recall the original definition of a cellular algebra as given in definition 1.3.7 or [11, Definition 1.1]. As stated in section, 1.3 it is well known, and shown in [11, Chapter 1], that the Temperley Lieb algebra is a cellular algebra.

Since the definition of cellular requires a partially ordered set Δ, we choose the ordering given by the poset diagram in figure 2.12, where the index i appearing higher in the diagram than index j implies $i > j$. The following proposition is given without proof. The proof will follow from 2.4.4 as we will explain after the statement.

Proposition 2.4.1. *The Outer-Temperley-Lieb algebra is a cellular algebra with cell datum $(\Delta, C, |\cdot,\cdot|, \downarrow)$ such that $\Delta = \{+n, \pm n-2, \pm n-4 \cdots \pm 1(\ or\ 0)\}$, $C(\pm p) = C^{\pm}_{n,p}$, for any $\pm p \in \Delta$ and $\downarrow$ is the anti-involution, as described prior to claim 2.3.1,sending any diagram in $C^{\pm}_{n,p}$ to its image reflected in the horizontal plane.*

Figure 2.12: Poset diagrams for the indices $\lambda \in \Delta$. The diagram on the left is for when n even and on the right is when n is odd.



Instead of proving the result directly, we shall prove $OTL_n(\delta)$ satisfies an alternative definition of a cellular algebra. This definition is stated and proven to be equivalent to definition 1.3.7 in [16].

Definition 2.4.2. [16, Definition 3.2]. Let A be a k-algebra. Suppose i is an anti -automorphism on A such that $i^2 = \mathrm{id}$. A two-sided ideal of A, say J, is said to be a cell ideal if and only if $i(J) = J$ and there exists a left ideal I with $I \subset J$ such that I has finite k-dimension and there exists an A-bimodule isomorphism $\gamma : J \cong I \otimes_k i(I)$ allowing the following to commute.

$$\begin{tikzcd} J \arrow[r, "\gamma"] \arrow[d, "i"'] & I \otimes_k i(I) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\gamma"'] & I \otimes_k i(I) \end{tikzcd}$$

A is said to be cellular, with involution i, if and only if we can find a vector space decomposition $A = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$ (some $n \in \mathbb{N}$) such that $i(J'_j) = J'_j$ and by letting $J_j = \oplus_{l=1}^{j} J'_l$ we get a chain $0 \subset J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ of two sided ideals of A (each fixed by i) where for each j, $J'_j = J_j / J_{j-1}$ is a cell ideal of A/J_{j-1}.

The chain $0 \subset J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ is often called a cell chain. We shall show we can construct such a cell chain for $OTL_n(\delta)$, which will allow us to deduce that $OTL_n(\delta)$ is indeed a cellular algebra.

Let $\downarrow$ be the anti-automorphism on $OTL_n(\delta)$ as described prior to claim 2.3.1. As a reflection it is clear that $\downarrow^2$ is the identity map. For any $\pm p \in \Delta$, define,

$$J_{\pm p} = \{|X, Y| \mid \text{ for all } X, Y \in C^{\pm}_{n,p}\}.$$

Lemma 2.4.3. *Each $J_{\pm p}$, $\pm p \in \Delta$, is spanned by elements of $OTL_n(\delta)$ with exactly p propagating lines. Furthermore $J_{+p} + J_{-p}$ is the vector sub-space of $OTL_n(\delta)$ spanned by all elements with exactly p propagating lines.*

Proof. Given that the half diagrams of $C^{\pm}_{n,p}$ each have exactly p propagating lines, by the definition of $|-,-|$ any diagram in a linear combination of an element of $J_{\pm p}$ must also have p propagating lines. Let $x + \sigma(x), y + \sigma(y) \in C^{+}_{n,p}$ where $x, y \in C_{n,p}$ such that $x \neq \sigma(x), y \neq \sigma(y)$. Then,

$$|x + \sigma(x), y + \sigma(y)| = |x, y| + |\sigma(x), \sigma(y)| + |x, \sigma(y)| + |\sigma(x), y| \in J_{+p}.$$

Since $x \neq \sigma(x), y \neq \sigma(y)$ we must also have $x - \sigma(x), y - \sigma(y) \in C^{-}_{n,p}$. Then,

$$|x - \sigma(x), y - \sigma(y)| = |x, y| + |\sigma(x), \sigma(y)| - (|x, \sigma(y)| + |\sigma(x), y|) \in J_{-p}. \tag{2.21}$$

Therefore J_{+p} J_{-p} are sub-spaces of $OTL_n(\delta)$.

Now consider $J_{+p} + J_{-p}$ as a vector sub-space of $OTL_n(\delta)$. By the exchange lemma, see [15, Lemma 2.2.4], or by adding and subtracting the two equations above, we have $|x, y| + |\sigma(x), \sigma(y)| \in J_{+p} + J_{-p}$ and $|x, \sigma(y)| + |\sigma(x), y| \in J_{+p} + J_{-p}$.

Also if $z \in C_{n,p}$ with $z = \sigma(z)$ then $z \in C^+_{n,p}$ and

$$\begin{aligned} |x + \sigma(x), z| = |x, z| + |\sigma(x), z| \in J_{+p}, \\ |z, x + \sigma(x)| = |z, x| + |z, \sigma(x)| \in J_{+p}, \\ |z, z| \in J_{+p}. \end{aligned}$$

for any $x + \sigma x \in C^+_{n,p}$, $x \neq \sigma(x)$. Thus $|x, z| + |\sigma(x), z|, |z, x| + |z, \sigma(x)|, |z, z| \in J_{+p} + J_{-p}$.

Thus for every possible $x, y \in C_{n,p}$, the diagram $|x, y|$ has exactly p propagating lines and will appear in a linear combination in $J_{+p} + J_{-p}$, either on its own or as a sum with $\sigma(|x, y|) = |\sigma(x), \sigma(y)|$. However, if we denote by,

$$TL_n(\delta)^p = \{|x, y| \mid x, y \in C_{n,p}\},$$

then $TL_n(\delta)^p$ is the sub-space of *all* elements of $TL_n(\delta)$ containing exactly p propagating lines. Hence $J_{+p} + J_{-p}$ contains all elements of $TL_n(\delta)^p$ which are fixed by σ, but this is exactly the sub-space of $OTL_n(\delta)$ of all elements with exactly p propagating lines. □

Let,

$$\begin{aligned} W_i = \langle J_{+i} \rangle + \sum_{\substack{j<i \\ i-j \text{ even}}} \langle J_j + J_{-j} \rangle, \\ T_i = W_i + \langle J_{-i} \rangle, \end{aligned}$$

where $\langle X \rangle$ denotes the vector space spanned by the basis elements of X. Then with respect to the ordering given by figure 2.12 we have $W_i/T_{i-2} \cong J_{+i}$ and $T_i/W_i \cong J_{-i}$.

Proposition 2.4.4. *W_i and T_i are both ideals of $OTL_n(\delta)$ for all $i \leq n$. Furthermore, the chain,*

$$OTL_n(\delta) = T_n \supset W_n \supset T_{n-2} \supset W_{n-2} \supset T_{n-4} \supset W_{n-4} \cdots \supset T_{1/0} \supset W_{1/0} \supset 0, \qquad (2.22)$$

where the final index $(0/1)$ is 0 when n is even and 1 when n is odd, is a cell chain for $OTL_n(\delta)$ and hence $OTL_n(\delta)$ is cellular.

Proof. To show the chain is indeed a cell chain we require that for any $\pm p \in \lambda$, J_{-p} and J_{+p} are cell ideals of $OTL_n(\delta)/W_P$ and $OTL_n(\delta)/T_{p-2}$ respectively. Both quotients send all diagrams with less than p propagating lines to zero, and multiplication prevents us from forming diagrams with more than p propagating lines. It follows that J_{+p} must be an ideal of $OTL_n(\delta)/T_{p-2}$. For J_{-p} all elements are of the form in equation (2.21) where $x, y \in C_{n,p}$, $x \neq \sigma(x), y \neq \sigma(y)$have exactly p propagating lines. If we multiply by $A = a + \sigma(a) \in OTL_n(\delta)/W_p$ we get

$$\begin{aligned}(a\,|x,y| + \sigma(a)\,|\sigma(x),\sigma(y)|) + (a\,|\sigma(x),\sigma(y)| + \sigma(a)\,|x,y|)\\ - (a\,|x,\sigma(y)| + \sigma(a)\,|\sigma(x),y|) - (a\,|\sigma(x),y| + \sigma(a)\,|x,\sigma(y)|),\end{aligned} \tag{2.23}$$

where each element must have exactly p propagating lines. We first show that if one of

$$a\,|x,y| + \sigma(a)\,|\sigma(x),\sigma(y)| \quad \text{or} \quad a\,|x,\sigma(y)| + \sigma(a)\,|\sigma(x),y|,$$

is zero then the other term is zero and if one of

$$a\,|\sigma(x),\sigma(y)| + \sigma(a)\,|x,y| \quad \text{or} \quad a\,|\sigma(x),y| + \sigma(a)\,|x,\sigma(y)|$$

is zero then the other term is also zero. Assume $(a|x,y| + \sigma(a)|\sigma(x),\sigma(y)|) = 0$, then there exists a pair of propagating lines in $|x,y|$, say starting at points i, j on the top edge of x, which meet the endpoints of an arc in a causing them to become an arc in $a|x,y|$. However, by construction $|x,\sigma(y)|$ must also have a pair of propagating lines starting at points i, j on the top edge of x. These will also then be closed by the arc in a to form an arc in $a|x,\sigma(y)|$ hence it, along with $\sigma(a)|\sigma(x),y|$ will also be zero. A similar argument can be made for the other two terms.

We now claim that when $a|x,y| \neq 0$ then $a|x,y| = |ax,y|$. Since $a|x,y|$ is non zero, it has exactly p propagating lines, as do a and $|x,y|$. Thus since no new propagating lines can be formed in diagram multiplication, if the bottom half n-diagram part of $|x,y|$, that is $y^{\downarrow}$ contains any propagating lines, they must still be propagating lines in $a|x,y|$. Similarly since diagram multiplication cannot affect any arcs in $y^{\downarrow}$ the bottom half n-diagram of

$a|x,y|$ is also $y^{\downarrow}$. This means multiplication by a only affects the top half n-diagram part of $|x,y|$ and hence we may write $a|x,y| = |ax,y|$ as required. By making a similar argument for the other terms in equation (2.23) we may now write it as.

$$\begin{aligned}(|ax,y| + |\sigma(ax),\sigma(y)|) + (|a\sigma(x),\sigma(y)| + |\sigma(a)x,y|)\\ - (|ax,\sigma(y)| + |\sigma(ax),y|) - (|a\sigma(x),y| + |\sigma(a)x,\sigma(y)|)\end{aligned}$$

But ax and $a\sigma(x)$ are also elements of $C_{n,p}$, and thus this element is also a member of J_{-p}. Hence J_{-p} is an ideal as required.

By setting i in definition 2.4.2 to be the anti automorphism $\downarrow$ and the left ideal I of J_{-p} to be, for some fixed $Y^* \in C^-_{n,p}$, the span of $|X,Y^*|$ (in $OTL_n(\delta)/W_p$), $X \in C^-_{n,p}$. We can then define a map γ acting on each $d \in J_{-p}$ by sending each diagram in d_i in d to the tensor product of the diagram whose top half-diagram is d_i^t and the diagram whose bottom half n-diagram is d_i^b such that $|d_i^t, d_i^b| = d_i$. Then γ is a $OTL_n(\delta)/W_p$-bimodule isomorphism as the quotient allows us to only get zero or diagrams with exactly p propagating lines when multiplying diagrams. It can be seen by following the paths that the diagram in definition 2.4.2 does indeed commute hence J_{-p} is a cell ideal as required. J_{+p} can be show to be a cell ideal similarly by taking the left ideal I of J_{+p} to be $C^+_{n,p}$.

Thus the algebra $OTL_n(\delta)$ is cellular as claimed.

□

We note that the same proof applies if we reverse p and $-p$. Thus the set Δ is a poset as claimed.

2.5 $OTL_n(\delta)$ as a quasi-hereditary algebra

In this next section we shall determine which, if any, parameter choices allow $OTL_n(\delta)$ to be a quasi-hereditary algebra, see definition 1.3.8. As described in section 1.3, we know the Temperley-Lieb algebra is quasi-hereditary except for the particular cases where n is even and $\delta = 0$.

Since we have already shown that $OTL_n(\delta)$ is cellular, we make use of the main theorem of [17, Theorem 3.1] which determines the criterion for when a cellular algebra is also quasi-hereditary.

Theorem 2.5.1. *[17, Theorem 3.1]. Let A be a k-algebra. Suppose A is cellular with respect to an involution i, then the following are equivalent:*

a) Some cell chain of A, with respect to to some involution possibly other than i, is also a hereditary chain.

a') There is a cell chain, with respect to to some involution possibly other than i with length equal to the number of isomorphism classes of simple A modules.

b) A has finite global dimension.

c) The Cartan matrix of A has determinant one.

d) Any cell chain of A, with respect to any involution, is a hereditary chain.

Proposition 2.5.2. *The Outer-Temperley-Lieb algebra, $OTL_n(\delta)$, is a quasi-hereditary algebra except for the cases $\delta = 0$, n even and $\delta = \pm 1$, n odd.*

Proof. By 2.5.1 $a^{'}$) it is enough to show we can find a cell chain of $OTL_n(\delta)$ which has the same length as the number of simple modules. Consider the cell chain given in 2.4.4.

Since each cell ideal in this chain is formed out of a single standard module, it has length equal to the number of standard modules. The forms $[-,-]^+, [-,-]^-$ satisfy

$$|X,Y|D = 2[Y,D]^+X, \quad \text{any } X,Y,D \in C^+_{n,p},$$
$$|P,Q|E = 2[Q,E]^-P, \quad \text{any } P,Q,E \in C^-_{n,p}.$$

Thus $2[-,-]^+, 2[-,-]^-$ are the forms induced by the cellular structure of $OTL_n(\delta)$ in the sense of [11]. Thus by, Lemmas 2.3.4, 2.3.5 all the irreducible modules are quotients of standard modules whenever the products $2[-,-]^+$ and $2[-,-]^-$ are not identically zero. Furthermore, Lemma 2.3.13 shows that these products can only be identically zero on modules in the exceptional cases given in the hypothesis. □

2.6 Semi-simplicity of $OTL_n(\delta)$

It is already known when the Temperley-Lieb algebra is semi-simple, see proposition 1.4.1. In this section we shall work towards proving the final result of this chapter, theorem 2.6.7, which states that over a field k of characteristic zero, the Outer-Temperley-Lieb algebra is semi-simple if and only if the Temperley-Lieb algebra is semi-simple. We proceed first by considering the work on fixed rings found in [18, 25]. Since we have defined the Outer-Temperley-Lieb algebra as a fixed ring as shown in definition 2.1.6, we can use fixed ring theory to show that whenever the Temperley-Lieb algebra is semi-simple, the Outer-Temperley-Lieb algebra must be too. We then show the converse argument by considering how the Gram matrices of the standard $OTL_n(\delta)$-modules are related to those of the $TL_n(\delta)$-modules indexed by the same number of propagating lines.

The following theorem is originally due to Levitzki [18, Theorem 9].

Theorem 2.6.1. *[18, Theorem 9]. If R is a semi-simple Artinian ring, and G a finite group of automorphisms of R such that $|G|$ is invertible in R, then R^G, the fixed sub-ring of G on R, is also a semi-simple Artinian ring.*

Recall that in definition 2.1.6 we define $OTL_n(\delta)$, for any $n \in \mathbb{N}$, to be the fixed ring with respect to the group G generated by the automorphism $\sigma : TL_n(\delta) \longrightarrow TL_n(\delta)$.

Lemma 2.6.2. *Suppose k is a field not of characteristic two and $\delta = q + q^{-1} \in k$. Then when q is not a root of unity, $OTL_n(\delta)$ is semi-simple.*

Proof. By proposition 1.4.1 if q is not a root of unity, then $TL_n(\delta)$ is semi-simple. Since $|G| = 2$ is invertible in $TL_n(\delta)$ over k it follows from theorem 2.6.1 that $OTL_n(\delta) = TL_n(\delta)^G$ must also be semi-simple. □

This result does not yet give us the complete set of non semi-simple cases. Indeed, there may still be some $n \in \mathbb{N}$ and some $\delta \in k$ such that $TL_n(\delta)$ is not semi-simple but $OTL_n(\delta)$ is still semi-simple. We shall now use Gram matrices to show that we may extend the above lemma to include its converse.

Recall the definition of a *Gram matrix* as given in definition B.2. We shall denote by $G_{n,p}$ the Gram matrices of the standard $TL_n(\delta)$-modules $C_{n,p}$ with respect to the inner product $\langle -, - \rangle$ as described prior to Lemma 2.3.1. As mentioned in [27, Section 4] or section 1.4, we can use these matrices to determine the values of δ for which each $C_{n,p}$ is irreducible, by determining under what circumstances $\det G_{n,p} \neq 0$.

We shall similarly denote by $G^+_{n,p}, G^-_{n,p}$ the Gram matrices of the $OTL_n(\delta)$-modules $C^+_{n,p}$ with inner product $[-,-]^+$ and $C^-_{n,p}$, with inner product $[-,-]^-$ respectively. If the Gram matrix of a standard $OTL_n(\delta)$-module has zero determinant then by corollary B.5 the standard module is not irreducible. To enable our study of the standard $OTL_n(\delta)$-modules we shall present the $G_{n,p}$ in a less conventional way by using a change of basis. In what follows we shall assume $n \in \mathbb{N}$, $0 \leq p \leq n$ such that $n - p \in 2\mathbb{Z}$ and $\delta \in k$ such that $OTL_n(\delta)$ is a quasi-hereditary algebra unless otherwise stated.

Lemma 2.6.3. *Let $C_{n,p}$ be a standard $TL_n(\delta)$-module with ordered basis $B_{n,p}$. For the same n, p we can construct a basis $B^+_{n,p}$ of the $OTL_n(\delta)$-module $C^+_{n,p}$ and a basis $B^-_{n,p}$ of the $OTL_n(\delta)$-module $C^-_{n,p}$ from $B_{n,p}$ such that $B^+_{n,p} \uplus B^-_{n,p}$ is also a basis for $C_{n,p}$.*

Proof. Let $B_{n,p} = \{b_1, b_2, b_3, \dots, b_m, c_1, c_2, \dots, c_l\}$, m even, where each $b_i \in C_{n,p}$ is such that $\sigma(b_i) = b_{m-i+1}$ and each $c_j \in C_{n,p}$ is such that $\sigma(c_j) = c_j$. Using the exchange lemma, see [15, Lemma 2.2.4], we may rewrite this basis as;

$$\{b_1 + b_m, b_2 + b_{m-1}, \dots, b_{\frac{m}{2}} + b_{\frac{m}{2}+1}, c_1, c_2, \dots, c_l, b_1 - b_m, b_2 - b_{m-1}, \dots, b_{\frac{m}{2}} - b_{\frac{m}{2}+1}\}. \tag{2.24}$$

However, by definition $\{b_1 + b_m, b_2 + b_{m-1}, \dots, b_{\frac{m}{2}} + b_{\frac{m}{2}+1}, c_1, c_2, \dots, c_l\}$ is a basis for $C^+_{n,p}$ and $\{b_1 - b_m, b_2 - b_{m-1}, \dots, b_{\frac{m}{2}} - b_{\frac{m}{2}+1}\}$ is a basis for $C^-_{n,p}$. Thus let $B^+_{n,p} = \{b_1 + b_m, b_2 + b_{m-1}, \dots, b_{\frac{m}{2}} + b_{\frac{m}{2}+1}, c_1, c_2, \dots, c_l\}$ and $B^-_{n,p} = \{b_1 - b_m, b_2 - b_{m-1}, \dots, b_{\frac{m}{2}} - b_{\frac{m}{2}+1}\}$. □

Lemma 2.6.4. *For any $a + \sigma(a) \in C^+_{n,p}$, $b - \sigma(b) \in C^-_{n,p}$ we have $\langle a + \sigma(a), b - \sigma(b)\rangle = 0$.*

Proof. We have,

$$\langle a + \sigma(a), b - \sigma(b)\rangle = \langle a, b\rangle - \langle a, \sigma(b)\rangle + \langle \sigma(a), b\rangle - \langle \sigma(a), \sigma(b)\rangle.$$

However, by claim 2.3.1, $\langle \sigma(a), \sigma(b)\rangle = \langle a, b\rangle$ and $\langle a, \sigma(b)\rangle = \langle \sigma(a), b\rangle$. □

Corollary 2.6.5. *The Gram matrix $G_{n,p}$ of the $TL_n(\delta)$ standard module $C_{n,p}$ can be written in the form;*

$$G_{n,p} = \begin{pmatrix} 2G^+_{n,p} & 0 \\ 0 & 2G^-_{n,p} \end{pmatrix}.$$

Proof. We choose the basis of $C_{n,p}$ so that it is of the same form as equation (2.24). Let $a + \sigma(a), b + \sigma(b) \in B^+_{n,p}$, then,

$$\begin{aligned} \langle a + \sigma(a), b + \sigma(b)\rangle &= \langle a, b\rangle + \langle a, \sigma(b)\rangle + \langle \sigma(a), b\rangle + \langle \sigma(a), \sigma(b)\rangle \\ &= 2\langle a, b\rangle + 2\langle a, \sigma(b)\rangle = 2[a + \sigma(a), b + \sigma(b)]^+. \end{aligned}$$

Similarly, for $c - \sigma(c), d - \sigma(d) \in B^-_{n,p}$, we have,

$$\begin{aligned} \langle c - \sigma(c), d - \sigma(d)\rangle &= \langle c, d\rangle - \langle c, \sigma(d)\rangle - \langle \sigma(c), d\rangle + \langle \sigma(c), \sigma(d)\rangle \\ &= 2\langle c, d\rangle - 2\langle c, \sigma(d)\rangle = 2[c - \sigma(d), d - \sigma(d)]^-. \end{aligned}$$

The zeros in the matrix follow from Lemma 2.6.4. □

Theorem 2.6.6. *Suppose $\delta = q + q^{-1}$ is such that $TL_n(\delta)$ is not semi-simple, that is q is a root of unity. Then $OTL_n(\delta)$ is also not semi-simple.*

Proof. When q is a root of unity there exists a pair (n, p) where $0 \leq p \leq n$ with $n - p \in 2\mathbb{Z}$ such that the module $C_{n,p}$ is not irreducible and hence $\det G_{n,p} = 0$. However, by corollary 2.6.5 we have $\det G_{n,p} = (\det 2G^{+}_{n,p})(\det 2G^{-}_{n,p})$ so for $\det G_{n,p} = 0$ at least one of $\det 2G^{+}_{n,p}$, $\det 2G^{-}_{n,p}$ must be zero. This implies that at least one of the $OTL_n(\delta)$ standard modules $C^{+}_{n,p}$, $C^{-}_{n,p}$ is not irreducible for this choice of n, p and δ, therefore $OTL_n(\delta)$ cannot be semi-simple. □

The combination of this and corollary 2.6.5 therefore implies the following.

Theorem 2.6.7. *When k is a field not of characteristic two, the Outer-Temperley-Lieb algebra is semi-simple if and only if the Temperley-Lieb algebra is semi-simple.*

We note that there are cases where our choices of δ cause the Gram matrices of $C^{+}_{n,p}$ and $C^{-}_{n,p}$ with respect to the inner products $[-,-]^{+}, [-,-]^{-}$ to be identical to the zero matrix, namely the cases $\delta = 0$, $p = 0$ and $\delta = \pm 1$, $p = 1$. In these cases a zero Gram matrix determinant cannot be used to determine irreducibility. We instead apply the above using the renormalised forms $[-,-]^{+'}, [-,-]^{-'}$ for $\delta = 0$ $[-,-]^{+''}$ for $\delta = -1$, $\frac{n-1}{2}$ odd and $[-,-]^{-''}$ for $\delta = 1$ as defined in definitions 2.3.8 and 2.3.11 respectively.

We shall finish by using the cellular structure of the Outer-Temperley-Lieb algebra as given in section 2.4 to describe the structure of the standard $OTL_n(\delta)$-modules irrespective of algebra being semi-simple.

Proposition 2.6.8. *Assume δ is such that the forms $[-,-]^{+}, [-,-]^{-}$ are non-zero, that is $\delta \neq 0$ with $p = 0$ and $\delta \neq \pm 1$ with $p = 1$. When the determinants of the Gram matrix of the cell modules $C^{+}_{n,p}$, $C^{-}_{n,p}$ with respect to the forms $[-,-]^{+}, [-,-]^{-}$, are non-zero, the modules $C^{+}_{n,p}$, $C^{-}_{n,p}$ are irreducible $OTL_n(\delta)$-modules.*

Proof. Note that as argued in the proof of proposition 2.5.2 the forms $2[-,-]^{+}, 2[-,-]^{-}$ are the forms induced by the cellular structure of $OTL_n(\delta)$ in the sense of [11]. Since $[-,-]^{+}, [-,-]^{-}$ are non zero if and only if $2[-,-]^{+}, 2[-,-]^{-}$ are non zero, respectively, the proof follows from corollary B.11 and Lemma B.12. $\square$

Chapter 3

Restriction and Induction of the $OTL_n(\delta)$ standard modules

Recall from section 1.5 the descriptions of restriction and inductions for a family of k-algebras A_n such that $A_{n-1} \subset A_n$. As mentioned in section 1.5 it is known that the Temperley-Lieb algebra is such a family of algebras and as we shall show in section 3.1 the Outer-Temperley-Lieb algebra is also. Therefore in this chapter, we consider the action on the standard (in the quasi-heriditary sense), modules for $OTL_n(\delta)$, defined in the previous chapter, for higher and lower rank Outer-Temperley-Lieb algebras. In section 3.1 we shall work towards proving theorem 3.1.15 which shows how the standard modules of $OTL_n(\delta)$ decompose when restricted to the algebra of rank $n-2$. The completion of the proof will require some knowledge of homological algebra; we give the required basics in appendix C. In the second section we construct an idempotent E_n of the Outer-Temperley-Lieb algebra and use this to construct an isomorphism between the rank $n-4$ Outer-Temperley-Lieb algebra and the algebra $E_n OTL_n E_n$. This in turn is used to define globalisation and localisation functors on the categories of $OTL_n(\delta)$-modules and prove a property of the $OTL_n(\delta)$-module $OTL_n E_n$ when considered as a certain bimodule. These ideas will be used in the third and final section of this chapter to prove theorem

3.3.3 which shows how the standard modules of $OTL_n(\delta)$ decompose when we induce them so as to consider them as modules for $n+2$ rank Outer-Temperley-Lieb algebra. In what follow we will mostly omit the parameter δ from our notation, writing TL_n for $TL_n(\delta)$ and OTL_n for $OTL_n(\delta)$.

3.1 Restriction of the standard modules of $OTL_n(\delta)$

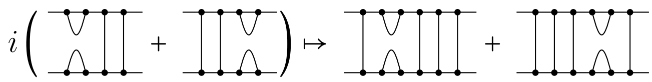
In this section we work towards proving theorem 3.1.15 which describes how the standard modules of OTL_n decompose when restricted to the Outer-Temperley-Lieb algebra of rank $n-2$. We begin by defining a inclusion map from OTL_{n-2} into OTL_n derived from an inclusion on boundary diagrams. We similarly define an inclusion map on the standard modules $C^{\pm}_{n-2,p-2} \hookrightarrow C^{\pm}_{n,p}$ by defining a map of the top half n-diagrams which allows us to consider $C^{\pm}_{n-2,p-2}$ as a sub-module of $C^{\pm}_{n,p}$ when restricted to OTL_{n-2}. When we quotient by this sub-module we can quotient again by the space generated by all basis diagrams in the first quotient that have propagating lines at both outer edges. We show that this space contains $C^{-}_{n-2,p}$ as a sub-module and $C^{+}_{n-2,p}$ as quotient module while the second quotient module is isomorphic to a copy of $C^{+}_{n-2,p+2}$. At the end of the section we use the central element defined in [27, Appendix A] and a well known property of quasi-hereditary algebras to show that the sequences formed by these sub-modules and quotient modules split to give a direct sum decomposition of $C^{\pm}_{n,p}$ when restricted to OTL_{n-2}.

Proposition 3.1.1. *For all $n \in \mathbb{N}$, OTL_{n-2} is a sub-algebra of OTL_n.*

Proof. We define an inclusion map $i : OTL_{n-2} \hookrightarrow OTL_n$ by sending each linear combination of diagrams $d \in OTL_{n-2}$ to the linear combination containing each diagram of d with the following changes.

- The label of each existing node is increased by one.

Figure 3.1: The diagrammatic action of the inclusion $i : OTL_{n-2} \hookrightarrow OTL_n$.



- Two new nodes are added to the left of the diagram, one on the top and one on the bottom edge, these are labelled 1 and $1'$ respectively and joined by a propagating line.

- Two new nodes are added to the right of the diagram with one on the top and one on the bottom edge, these are labelled n and n' respectively and are also joined by a propagating line.

Figure 3.1 shows the inclusion sending a diagram of OTL_4 to one of OTL_6. Note that this inclusion sends the identity in OTL_{n-2} to that in OTL_n. Since each individual diagram in the linear combination d is sent to one with a propagating line on both the left and right edges, any symmetric pair of diagrams in d will remain a symmetric pair under this inclusion and any individual diagram in d which was symmetric will remain symmetric under the inclusion. □

Proposition 3.1.2. *Given the above inclusion, denote by $C^+_{n,p} \downarrow_{OTL_{n-2}}, C^-_{n,p} \downarrow_{OTL_{n-2}}$ the corresponding restriction of $C^+_{n,p}, C^-_{n,p}$ respectively to OTL_{n-2}. Then for $n > 0$, $C^+_{n-2,p-2}$ is a sub-module of $C^+_{n,p} \downarrow_{OTL_{n-2}}$ and $C^-_{n-2,p-2}$ a sub-module of $C^-_{n,p} \downarrow_{OTL_{n-2}}$.*

Proof. As with the inclusion of OTL_{n-2} in OTL_n, we can define an inclusion $C^+_{n-2,p-2} \hookrightarrow C^+_{n,p}$ (and similarly $C^-_{n-2,p-2} \hookrightarrow C^-_{n,p}$) by sending each linear combination of diagrams $d \in C^+_{n-2,p-2}$ ($C^-_{n-2,p-2}$) to the linear combination containing all diagrams of d with the following modifications,

- The label of each existing node is increased by one.

- A new node labelled 1 is introduced to the left of the diagram with a propagating line attached.

- A new node labelled n is introduced to the right of the diagram with a propagating line attached.

Since the inclusion defined on OTL_{n-2} will preserve these two new propagating lines, this inclusion is an injective homomorphism of OTL_{n-2}-modules. $\square$

We therefore have two short exact sequences,

$$0 \longrightarrow C^{+}_{n-2,p-2} \longrightarrow C^{+}_{n,p}\downarrow_{OTL_{n-2}} \longrightarrow Q^{+}_{n,p} := C^{+}_{n,p}\downarrow_{n-2} / C^{+}_{n-2,p-2} \longrightarrow 0, \tag{3.1}$$

$$0 \longrightarrow C^{-}_{n-2,p-2} \longrightarrow C^{-}_{n,p}\downarrow_{OTL_{n-2}} \longrightarrow Q^{-}_{n,p} := C^{-}_{n,p}\downarrow_{n-2} / C^{-}_{n-2,p-2} \longrightarrow 0. \tag{3.2}$$

The modules $Q^{+}_{n,p}$ and $Q^{-}_{n,p}$ are quotient modules of OTL_{n-2} with bases of cosets represented by basis elements of $C^{+}_{n,p}$ and $C^{-}_{n,p}$, respectively, which do not have any diagrams in their linear combination containing propagating lines at both the left-most and right-most nodes.

For $p > 0$ denote by S_l the subset of the basis $T_{n,p}$, as defined in section 1.3, of the TL_n standard module $C_{n,p}$ such that any diagram $a \in S_l$ has an arc extending from the left-most node and a propagating line at the right-most node. Similarly let $S_r \subset T_{n,p}$ be such that any diagram $b \in S_r$ has a propagating line at the left-most node and an arc extending from the right-most node. Denote by $X_{n,p}$ the subset of basis elements of $Q^{+}_{n,p}$ which contain linear combinations of diagrams that are in S_l and S_r only and let $S_{n,p} = \langle X_{n,p} \rangle$.

Proposition 3.1.3. *$S_{n,p}$ is a OTL_{n-2}-sub-module of $Q^{+}_{n,p}$.*

Proof. By definition, none of the diagrams in the linear combination of some basis element $d \in S_{n,p}$ can be symmetric, therefore if $a \in C_{n,p}$ is in the linear combination of any such d, $\sigma(a)$ must also be in the linear combination. We may therefore write any

d in the basis of $S_{n,p}$ as $d = a + \sigma(a)$ where a is a basis element in $S_l \subset C_{n,p}$. Let $x + \sigma(x) \in OTL_{n-2}$. Then

$$xd + \sigma(x)d = xa + \sigma(xa) + x\sigma(a) + \sigma(x\sigma(a)).$$

Now the diagram xa will contain a propagating line on the right-most node by the inclusion of OTL_{n-2} in OTL_n. If x causes the arc leading from the left-most node in a to become a propagating line in xa then $xa = 0$ and hence $\sigma(xa) = 0$. Otherwise this arc must be closed by x to form an arc leading from the left-most node in xa. Similarly the diagram $x\sigma(a)$ will contain a propagating line on the left-most node by inclusion. If x causes the arc leading from the right-most node in $\sigma(a)$ to become a propagating line in $x\sigma(a)$ then $x\sigma(a) = 0$ and hence $\sigma(x\sigma(a)) = 0$. Otherwise this arc must be closed by x to form an arc leading from the right-most node in $x\sigma(a)$. Thus xa is either zero or an element of S_l and $x\sigma(a)$ is either zero or an element of S_r. Since xd contains both xa, $x\sigma(a)$ and their reflections we have $xd \in S_{n,p}$. □

Corollary 3.1.4. *There exists a short exact sequence,*

$$0 \longrightarrow S_{n,p} \longrightarrow Q^+_{n,p} \xrightarrow{\gamma^+} C^+_{n-2,p+2} \longrightarrow 0\,. \tag{3.3}$$

That is $S_{n,p}$ is a OTL_{n-2}-sub-module of $Q^+_{n,p}$ and the map $\gamma^+ : Q^+_{n,p}/S_{n,p} \to C^+_{n-2,p+2}$ sends the linear combinations $d \in Q^+_{n,p}/S_{n,p}$ to the linear combinations containing all diagrams of d after their left-most and right-most arcs have been cut to form four new propagating lines, with the left-most and right-most propagating lines and nodes then removed, see figure 3.2.

Proof. By proposition 3.1.3 $Q^+_{n,p}/S_{n,p}$ is a OTL_{n-2}-quotient-module with basis of cosets of all basis elements of $C^+_{n,p}$ containing linear combination of only the diagrams which have arcs leading from both the left-most and right-most nodes. We note that if $p \geq n-2$ then the quotient $Q^+_{n,p}/S_{n,p}$ is empty as there can exist no diagram with more than one are in these cases. Henceforth we assume $p < n-2$

Figure 3.2: The map γ^+ acting on a general element of $Q^+_{n,p}/S_{n,p}$.

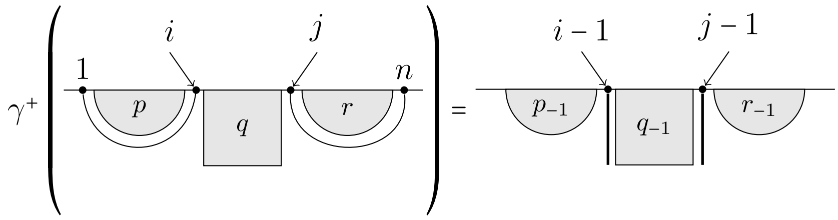
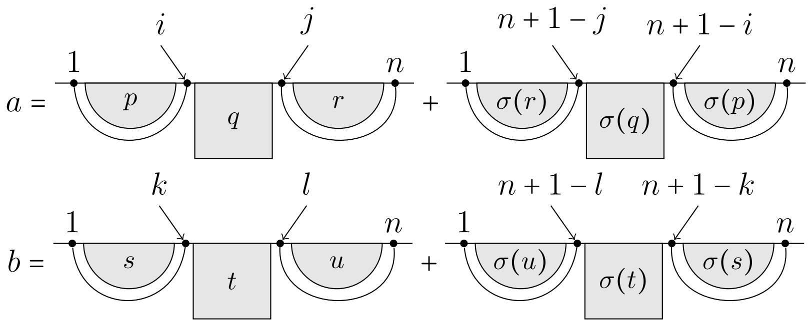


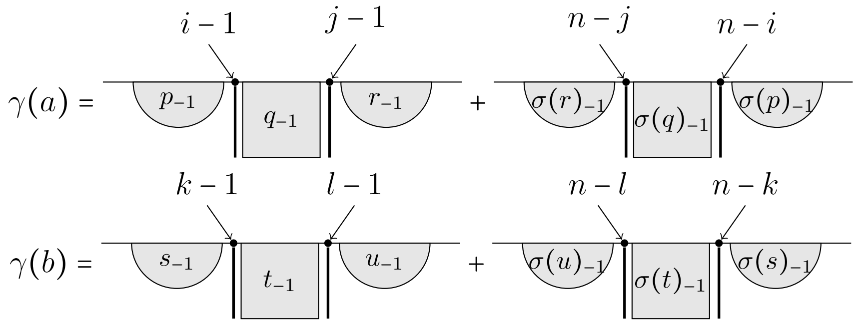
Figure 3.3: Two coset representative basis elements a, b of $Q^+_{n,p}/S_{n,p}$.



We note that the map γ^+ is injective for if we take two basis elements $a, b \in Q^+_{n,p}/S_{n,p}$ and suppose $\gamma^+(a) = \gamma^+(b)$, we can write a, b as in figure 3.3 where the p, r, s, u denote arrangements of non crossing arcs and q, t denote arrangements of non crossing arcs and propagating lines. Applying γ to a and b we get the diagrams as shown in figure 3.4 where the notation x_{-1} denotes the arrangement x with one subtracted from the label of each node. As $\gamma^+(a) = \gamma^+(b)$ and $p, \sigma(r), s, \sigma(u)$ can only contain arrangements of arcs, in each diagram $i-1, n-j, k-1, n-l$ must be the nodes containing the left-most propagating lines respectively. Similarly, as $r, \sigma(p), u, \sigma(s)$ can only contain arrangements of arcs, $j-1, n-i, l-1, n-k$ must be the nodes containing the right-most propagating line in each respective diagram. It follows that either,

$$i-1 = k-1, j-1 = l-1 \text{ and } s_{-1} = p_{-1}, t_{-1} = q_{-1}, u_{-1} = r_{-1}, \quad \text{or}$$
$$n-j = k-1, n-i = l-1 \text{ and } s_{-1} = \sigma(r)_{-1}, t_{-1} = \sigma(q)_{-1}, u_{-1} = \sigma(p)_{-1}.$$

Figure 3.4: Application of gamma to the coset representative basis elements a, b of $Q^+_{n,p}/S_{n,p}$.

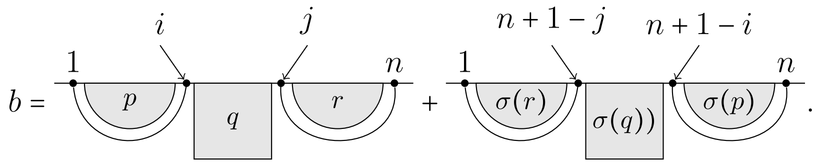


Hence either,

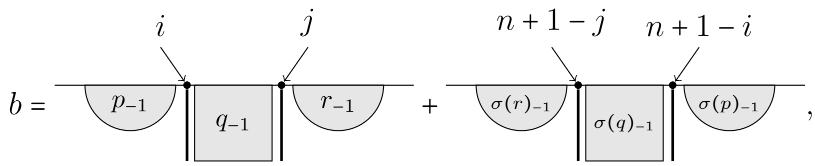
$$i = k, j = l \text{ and } s = p, t = q, u = r, \quad \text{or}$$

$$n - j + 1 = k, n - i + 1 = l \text{ and } s = \sigma(r), t = \sigma(q), u = \sigma(p).$$

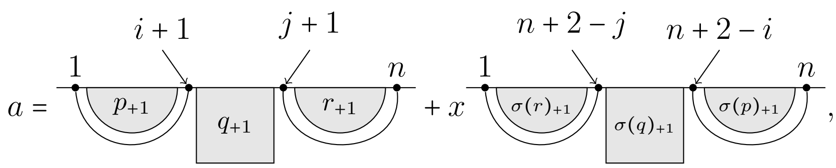
In either case we get,



The map γ^+ is also surjective. Suppose $b \in C^+_{n-2,p+2}$, then b must contain at least two propagating lines. Suppose the left-most propagating line in b sits at node i and the right-most propagating line in b sits at node j, then we may write,



where p, r are arrangements of non crossing arcs and q is an arrangement of non crossing arcs and propagating lines. Now let,



where the notation x_{+1} denotes the arrangement x with one added from the label of each node. Then $a \in Q^+_{n,p}/S_{n,p}$ since all its diagrams have arcs extending from the left-most and right-most nodes and $\gamma^+(a) = b$.

Thus we have shown that γ^+ is an isomorphism of vector spaces. Now let $x + \sigma(x) \in OTL_{n-2}$ and d be in the basis of $Q^+_{n,p}/S_{n,p}$. Then we can write d as $d = a + \sigma(a)$ where $a \in C_{n,p}$ has an arcs leading from both the left-most and right-most nodes. Let $(x + \sigma(x))d = xd + \sigma(x)d$, then as in the proof of proposition 3.1.3 we have,

$$xd + \sigma(x)d = xa + \sigma(xa) + x\sigma(a) + \sigma(x\sigma(a)).$$

If at least one of the left-most or right-most arcs in a is turned into propagating lines by x then the diagram $xa = 0$ by the quotient and $\sigma(xa) = 0$. Similarly if at least one of the left-most or right-most arcs in $\sigma(a)$ is also turned into propagating lines by x then $x\sigma(a) = 0$ by the quotient and $\sigma(x\sigma(a))$, therefore $\gamma^+(xd + \sigma(x)d) = \gamma^+(0) = 0$.

Suppose, without loss of generality, that the left-most arc in a is turned into a propagating line by x. Then there exists a path starting at the endpoint of this arc, i say, travelling back and forth through any nodes, which do not already have a path through them, in a and x and ending by joining a propagating line in a, say at node k. If we similarly suppose, without loss of generality, that the left-most arc in $\sigma(a)$ is turned into a propagating line by x. Then there exists a path starting at the endpoint of this arc, j say, travelling back and forth through any nodes, which do not already have a path through them, in $\sigma(a)$ and x and ending by joining a propagating line in $\sigma(a)$, say at node l. Now,

$$(x + \sigma(x))\gamma^+(d) = x\gamma^+(d) + \sigma(x)\gamma^+(d) = x\gamma^+(a + \sigma(a)) + \sigma(x)\gamma^+(a + \sigma(a)).$$

Denote by $a', \sigma(a)'$ the diagrams formed from $a, \sigma(a)$ respectively in $\gamma^+(d)$, that is $\gamma^+(d) = a' + \sigma(a)'$. By definition, a' will have propagating lines at node $i - 1$ and $k - 1$ and $\sigma(a)'$ will have propagating line at node $j - 1$ and $l - 1$. But by the definitions of inclusion and γ^+ there will exist paths in x such that there is a path in xa' joining $i - 1$

with $k-1$ and such that there is a path in $x\sigma(a)'$ joining $j-1$ with $l-1$. Since these paths reduce the number of propagating lines in both diagrams we have $xa' = \sigma(xa') = 0$ and $x\sigma(a)' = \sigma(x)a' = 0$, that is $(x+\sigma(x))\gamma^+(d) = 0$.

If neither of the left-most and right-most arcs in both a and $\sigma(a)$ are turned into propagating lines by x then there must exist propagating lines in the inclusion of x in OTL_n, say at nodes t, u and paths in xa such that each of these arcs is joined to one of these propagating lines. Similarly there must exist propagating lines in the inclusion of x in OTL_n, say at nodes v, w and paths in $x\sigma(a)$ such that each of these arcs is joined to one of these propagating lines. By definition, the diagrams formed from xa and $x\sigma(a)$ in $\gamma^+(xd+\sigma(x)d)$ will have propagating lines at $t-1, u-1, v-1, w-1$. However, x must also have propagating lines at $t-1, u-1, v-1, w-1$ and there will exist paths in xa' joining the propagating lines formed by the left-most and right-most arcs in a to $t-1$ and $u-1$. Similarly there will exist paths in $x\sigma(a)'$ joining the propagating lines formed by the left-most and right-most arcs in $\sigma(a)$ to $v-1$ and $w-1$. By arguing the same for the reflected diagrams $\sigma(xa)$ and $\sigma(x\sigma(a))$ and noting that all other nodes not affected by the above construction have labels reduced by one, we can deduce that $\gamma^+((x+\sigma(x))d) = (x+\sigma(x))\gamma^+(d)$.

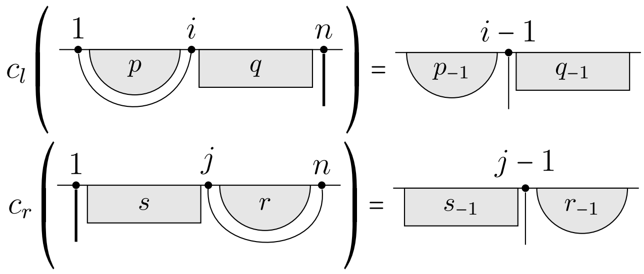
The case that only one of the left-most or right-most arcs in a is turned into a propagating line by x but neither of the left-most or right-most arcs in $\sigma(a)$ is turned into a propagating line by x can be handled similarly by combining the argument in our first case for a with the argument in the previous case or $\sigma(a)$. □

Using the methods in proposition 3.1.3 and corollary 3.1.4 we can similarly prove the following.

Corollary 3.1.5. *There exists a short exact sequence,*

$$0 \longrightarrow S_{n,p} \longrightarrow Q^-_{n,p} \xrightarrow{\gamma^-} C^-_{n-2,p+2} \longrightarrow 0\,. \tag{3.4}$$

Figure 3.5: Action of c_l and c_r on elements of S_l and S_r respectively.



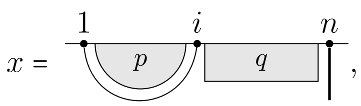
That is $S_{n,p}$ is a sub-module of $Q^-_{n,p}$ under restriction to OTL_{n-2} and the map γ^- : $Q^-_{n,p}/S_{n,p} \to C^-_{n-2,p+2}$ sends the linear combination $d \in Q^-_{n,p}/S_{n,p}$ to the linear combination containing all diagrams of d after their left-most and right-most arcs have been cut to form four new propagating lines, with the left-most and right-most propagating lines and nodes then removed.

It remains to determine if $S_{n,p}$ contains any sub-modules of OTL_{n-2}.

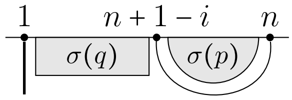
Definition 3.1.6. Define a set map $c_l : S_l \longrightarrow C_{n-2,p}$ sending $a \in S_l$ to the same diagram but with the arc from the left-most node split into two propagating lines. This is followed by removing the left-most and right-most nodes, along with the propagating lines extending from them and reducing the label of all other nodes by 1 (see figure 3.5). Similarly define a set map $c_r : S_r \to C_{n-2,p}$ sending $b \in S_r$ to the same diagram but with the arc from the right-most split into two propagating lines. This is followed by removing the left-most and right-most nodes, along with the propagating lines extending from them and reducing the label of all other nodes by one (see figure 3.5).

Lemma 3.1.7. *For any $x \in S_l$, $\sigma(c_l(x)) = c_r(\sigma(x))$.*

Proof. Let,

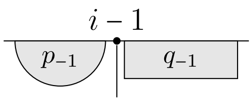


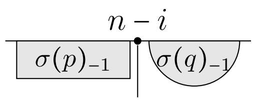
where p is an arrangement of arcs and q is an arrangement of arcs and propagating lines. Then

$\sigma(x) =$ 

,

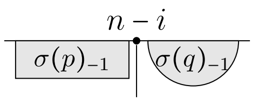
We also have,

$c_l(x) =$ 

$\Rightarrow \sigma(c_l(x)) =$ 

,

and

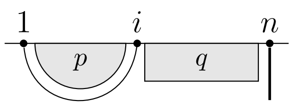
$c_r(\sigma(x)) =$ 

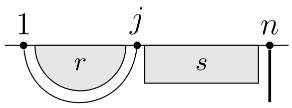
.

Hence $\sigma(c_l(x)) = c_r(\sigma(x))$ as required. □

Lemma 3.1.8. *The maps c_l, c_r are bijections.*

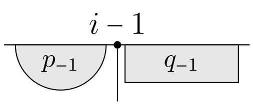
Proof. Let

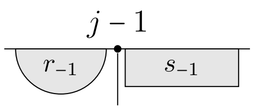
$a =$ 

and $b =$ 

,

and suppose $c_l(a) = c_l(b)$. Then we must have

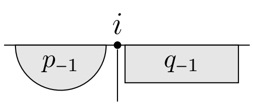


$=$ 

.

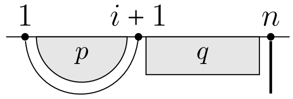
However, by definition, both p_{-1} and r_{-1} only contain arrangements of arcs, therefore $i-1$ and $j-1$, must be the left-most propagating lines in $c_l(a)$ and $c_l(b)$ respectively, that is $i = j$. It follows that we must have $p_{-1} = r_{-1}$ and $q_{-1} = s_{-1} \Rightarrow p = r$ and $q = s$. Therefore $a = b$ as required.

Let $a \in C_{n-2,p}$, then since $p > 0$, a must have at least one propagating line. Suppose node i contains the left-most propagating line in a. Then,

$a =$ 

,

where p_{-1} is an arrangement of arcs and q_{-1} an arrangement of arcs and propagating lines. Now let b be the diagram,

$b =$ 

.

Then since b has an arc leading from node one and a propagating line at node n, $b \in S_l$ and $c_l(b) = a$.

For c_r apply the result of Lemma 3.1.7 to the above. □

Proposition 3.1.9. *For any $x \in TL_n$, $xc_l(a) = c_l(xa)$ and $xc_r(a) = c_r(xa)$.*

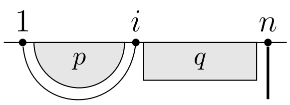
Proof. For c_r we refer to the second paragraph of the proof of [27, Proposition 4.1], the result for c_l can then be deduced from Lemma 3.1.7. □

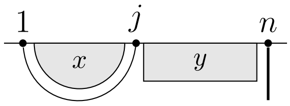
We can extend c_l, c_r k-linearly to become vector space isomorphisms from the vector spaces generated by the sets S_l, S_r respectively to $C_{n,p}$.

We now construct a set $S^-_{n,p}$ from S_l, where for each $a \in S_l$ which does not already appear in a combination in $S^-_{n,p}$, if there exists $b \neq a \in S_l$ such that $c_l(a) = c_r(\sigma(b))$ then include $a + \sigma(a) - (b + \sigma(b))$ in $S^-_{n,p}$.

Lemma 3.1.10. *Where such an element b exists it is unique.*

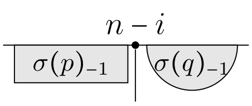
Proof. Let $c \neq b \in S_l$ be such that $c_l(a) = c_r(\sigma(c))$, then $c_r(\sigma(c)) = c_r(\sigma(b))$. Then we have,

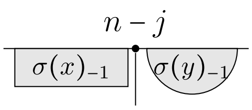
$b =$ 

and $c =$ 

,

where p, x are arrangements of non-crossing arcs and q, y are arrangements of non crossing arcs and propagating lines. Then,

$c_r(\sigma(b)) =$ 

and $c_r(\sigma(c)) =$ 

.

Since both these diagrams are equal and $\sigma(p)_{-1}, \sigma(x)_{-1}$ both contain only arrangements of arcs, $n-i$ and $n-j$ are the right-most propagating lines in $c_r(\sigma(b))$ and $c_r(\sigma(c))$ respectively. Therefore we must have $n-i = n-j \Rightarrow i = j$ and it follows that $\sigma(p)_{-1} = \sigma(x)_{-1}$ and $\sigma(q)_{-1} = \sigma(y)_{-1} \Rightarrow x = p$ and $q = y$. □

Proposition 3.1.11. *$S^-_{n,p}$ forms the basis of a sub-module of $S_{n,p}$ under restriction to $OTL_{n-2}(\delta)$.*

Proof. Any element in $d \in kS^-_{n,p}$ has the form $a + \sigma(a) - (b + \sigma(b))$, where $a, b \in S_l$. By the definition of S_l we must have $a + \sigma(a), b + \sigma(b) \in S_{n,p}$, therefore $d \in S_{n,p}$ also. Let $x + \sigma(x) \in OTL_{n-2}(\delta)$, then,

$$\begin{aligned}(x + \sigma(x))d &= xa + x\sigma(a) - (xb + x\sigma(b)) + \sigma(x)a + \sigma(x)\sigma(a) - (\sigma(x)b + \sigma(x)\sigma(b)) \\ &= [xa + \sigma(x)a + \sigma(xa + \sigma(x)a)] - [xb + \sigma(x)b + \sigma(xb + \sigma(x)b)].\end{aligned}$$

Using the results from proposition 3.1.9 we have,

$$\begin{aligned}c_r(\sigma(xb + \sigma(x)b)) &= c_r(\sigma(xb)) + c_r(\sigma(\sigma(x)b)) = c_r(\sigma(x)\sigma(b)) + c_r(x\sigma(b)) \\ &= \sigma(x)c_r(\sigma(b)) + xc_r(\sigma(b)) = \sigma(x)c_l(a) + xc_l(a) \\ &= c_l(xa + \sigma(x)a).\end{aligned}$$

Thus $(x + \sigma(x))d \in kS^-_{n,p}$ also and hence $kS^-_{n,p}$ is a sub-module of $S_{n,p}$. □

Proposition 3.1.12. *There exists the following short exact sequence,*

$$0 \longrightarrow C^-_{n-2,p} \longrightarrow S_{n,p} \longrightarrow C^+_{n-2,p} \longrightarrow 0\,. \tag{3.5}$$

That is $C^-_{n-2,p}$ is a sub-module of $S_{n,p}$ and $S_{n,p}/C^-_{n-2,p} \cong C^+_{n-2,p}$.

Proof. We define a map $\theta^- : S^-_{n,p} \to C^-_{n-2,p}$ such that for any $d \in S^-_{n,p}$ with $d = a + \sigma(a) - (b + \sigma(b))$, $a, b \in S_l$,

$$\theta^-(d) = c_l(a) - c_l(b).$$

The map θ^- inherits the k-linear and bijective properties from c_l. Now let $x + \sigma(x) \in OTL_{n-2}$, then,

$$\begin{aligned}
\theta^-((x+\sigma(x))d) &= \theta^-(xa+\sigma(x)a+\sigma(xa+\sigma(x)a)-(xb+\sigma(x)b+\sigma(xb+\sigma(x)b)) \\
&= c_l(xa+\sigma(x)a)-c_l(xb+\sigma(x)b) \\
&= c_l(xa)+c_l(\sigma(x)a)-c_l(xb)-c_l(\sigma(x)b) \\
&= x(c_l(a)-c_l(b))+\sigma(x)(c_l(a)-c_l(b)) = (x+\sigma(x))(c_l(a)-c_l(b)) \\
&= (x+\sigma(x))\theta^-(d).
\end{aligned}$$

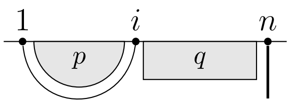
Hence θ^- is a module isomorphism.

The quotient $S_{n,p}/C^-_{n-2,p}$ has a basis of cosets of elements of the form $a + \sigma(a) + (b + \sigma(b))$ where there is a $b \neq a$, $b \in S_l$ satisfying $c_l(a) = c_r(\sigma(b))$ and elements of the form $a + \sigma(a)$ where there is no such b. We define a map $\theta^+ : S_{n,p}/C^-_{n-2,p} \longrightarrow C^+_{n-2,p}$ such that,

$$\theta^+(d) = \begin{cases} c_l(a)+c_l(b), & \text{if } d = a+\sigma(a)+(b+\sigma(b)), a,b \in S_l, c_l(a) = c_r(\sigma(b)), \\ c_l(a), & \text{otherwise.} \end{cases} \tag{3.6}$$

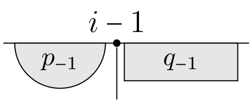
As with θ^-, θ^+ inherits k-linear and bijective properties from c_l. Suppose $x + \sigma(x) \in OTL_{n-2}(\delta)$, then if $d \in S_{n,p}/C^-_{n-2,p}$ has the form $d = a + \sigma(a) + (b + \sigma(b))$ with $c_l(a) = c_r(\sigma(b))$ we can use the same argument as for θ^+, replacing the minus signs with plus signs, to show $\theta^+((x + \sigma(x))d) = (x + \sigma(x))\theta^+(d)$.

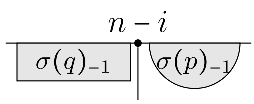
If no such b exists then we claim $c_l(a) = \sigma(c_l(a))$. Let,

$a =$ 

,

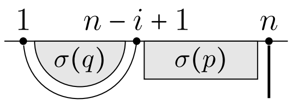
where p contains an arrangement of non-crossing arcs and q contains an arrangement of non-crossing arcs and propagating lines. We have two cases, If q contains no propagating lines then,

$c_l(a) =$ 

and $\sigma(c_l(a)) =$ 

,

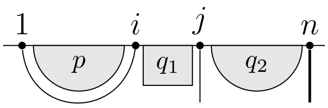
where $q_{-1}, \sigma(p)_{-1}$ and $\sigma(q)_{-1}$ all contain only arcs. Let,

$b =$ 

,

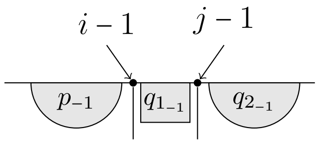
then $c_l(b) = \sigma(c_l(a)) \Rightarrow c_r(\sigma(b)) = c_l(a)$ by Lemma 3.1.7. However, our assumption is that no such $b \neq a$ exists, therefore we must have $b = a$ and thus $i = n - i + 1 \Rightarrow i - 1 = n - i$ and $q = \sigma(p)$. Thus $c_l(a) = \sigma(c_l(a))$ as required.

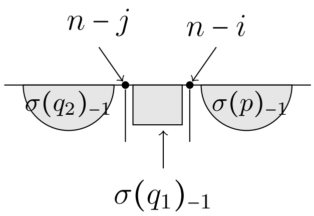
If q contains at least one propagating line, let j be the maximum node in q which has a propagating line extending from it. Then,

$a =$ 

,

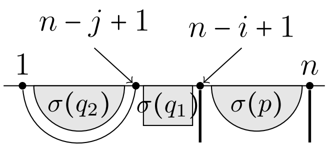
where q_1 is an arrangement of non-crossing arcs and propagating lines and q_2 is an arrangement of non-crossing arcs. Now,

$c_l(a) =$ 

and $\sigma(c_l(a)) =$ 

.

We therefore define

$b =$ 

.

Then $c_l(b) = \sigma(c_l(a)) \Rightarrow \sigma(c_l(b)) = c_l(a) \Rightarrow c_r(\sigma(b)) = c_l(a)$. However, we have assumed that no such $b \neq a$ exists, therefore we must have $b = a$ and thus $q_2 = \sigma(p), q_1 = \sigma(q_1)$ and $j = n - i + 1, i = n - j + 1 \Rightarrow i - 1 = n - j$ and $j - 1 = n - i$. Therefore $c_l(a) = \sigma(c_l(a))$ as required.

We now let $x+\sigma(x) \in OTL_{n-2}(\delta)$. Then $(x+\sigma(x))(a+\sigma(a)) = xa+\sigma(xa)+\sigma(x)a+\sigma(x\sigma(a))$. Now $c_r(\sigma(\sigma(x)a)) = c_r(x\sigma(a)) = xc_r(\sigma(a)) = x\sigma(c_l(a))$. But by the above $\sigma(c_l(a)) = c_l(a)$, hence $c_r(\sigma(\sigma(x)a)) = xc_l(a) = c_l(xa)$. Therefore we have,

$$\begin{aligned}\theta^+((x+\sigma(x))d) &= \theta^+(xa+\sigma(x)a+\sigma(xa+\sigma(x)a)) \\ &= c_l(xa)+c_l(\sigma(x)a) \\ &= xc_l(a)+\sigma(x)c_l(a) = (x+\sigma(x))c_l(a) = (x+\sigma(x))\theta^+(d),\end{aligned}$$

as required. □

For the cases $p=0$, $p=1$, we first note that the modules $C^+_{n-2,p-2}$ and $C^-_{n-2,p-2}$ do not exist hence $Q^+_{n,p} \cong C^+_{n,p} \downarrow_{OTL_{n-2}}$ and $Q^-_{n,p} \cong C^-_{n,p} \downarrow_{OTL_{n-2}}$. However, unlike when $p=0$, the sub-module $S_{n,p}$ of $Q^+_{n,p}, Q^-_{n,p}$ still exists when $p=1$ and so we can still derive the sequences (3.1), (3.3), (3.5). When $p=0$ we may instead define a second inclusion map $C^+_{n-2,0} \hookrightarrow C^+_{n,0}$ by sending each linear combination of diagrams $d \in C^+_{n-2,0}$ to the linear combination containing all diagrams of d modified such that,

- The label of each existing node is increased by one.
- A new node is introduced to the left of the diagram and another new node introduced to the right of the diagram.
- The two new nodes are connected by an arc.

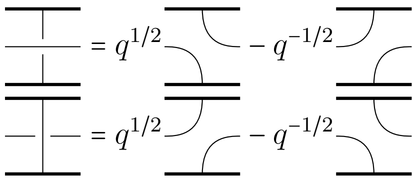
An inclusion $C^-_{n-2,0} \hookrightarrow C^-_{n,0}$ can be defined similarly. The inclusion sending OTL_{n-2} to OTL_n will preserve the arcs joining the new nodes in each diagram thus these two inclusions are injective OTL_{n-2}-module homomorphisms. The quotients $Q^{+'}_{n,0} := C^+_{n,0} \downarrow_{OTL_{n-2}} / C^+_{n-2,0}$ and $Q^{-'}_{n,p} := C^-_{n,p} \downarrow_{OTL_{n-2}} / C^-_{n-2,0}$ will thus have a basis of all elements of $C^+_{n,0}, C^-_{n,0}$ respectively, which do not have an arc connecting the left-most and right-most nodes. On these quotients we may therefore apply the maps γ^+, γ^- as defined in corollaries 3.1.4 and 3.1.5 respectively. Hence, there exist the following exact sequences,

$$0 \longrightarrow C^{+}_{n-2,0} \longrightarrow C^{+}_{n,p} \downarrow_{OTL_{n-2}} \xrightarrow{\gamma^{+}} C^{+}_{n-2,2} \longrightarrow 0, \tag{3.7}$$

$$0 \longrightarrow C^{-}_{n-2,0} \longrightarrow C^{-}_{n,p} \downarrow_{OTL_{n-2}} \xrightarrow{\gamma^{-}} C^{-}_{n-2,2} \longrightarrow 0. \tag{3.8}$$

To give a complete restriction rule it remains to check if the sequences we have derived throughout this section split. In order to do this we shall make use of a central element $F_n \in TL_n$ defined in [27, Appendix A].

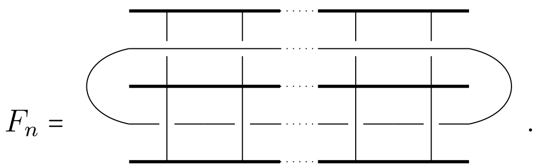
Definition 3.1.13. [27, Appendix A] Let



(3.9)

(3.10)

Then F_n is defined to be the diagram,

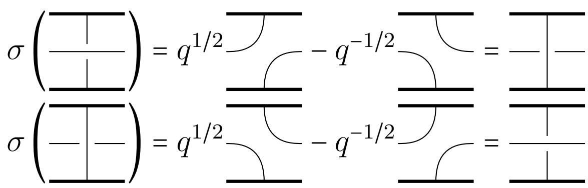


(3.11)

It is proven in [27, Appendix A] that F_n is a central element of TL_n and that F_n acts on any TL_n standard module $C_{n,p}$ as the identity times $f_{n,p} = q^{p+1} + q^{-(p+1)} \in k$.

Lemma 3.1.14. *F_n is a central element of OTL_n. Furthermore F_n acts on the standard OTL_n-modules $C^{+}_{n,p}, C^{-}_{n,p}$ as the identity times $f_{n,p}$.*

Proof. Applying σ to the shorthand diagrams in (3.9), (3.10) we get,



(3.12)

(3.13)

Therefore since σ is linear we can deduce $\sigma(F_n) = F_n$, that is $F_n \in OTL_n$.

Now since F_n is a central $TL_n(\delta)$ element, if we apply it to any $a + \sigma(a) \in OTL_n$, such that $a \in TL_n$, we get $F_n a + F_n\sigma(a) = aF_n + \sigma(a)F_n$. Thus F_n is central in OTL_n.

Finally for any $a+\sigma(a) \in C^+_{n,p}$ we have $F_n a+F_n\sigma(a) = f_{n,p}a+f_{n,p}\sigma(a) = f_{n,p}(a+\sigma(a))$ since both $a, \sigma(a) \in C_{n,p}$. We similarly show $F_n(a - \sigma(a)) = f_{n,p}(a - \sigma(a))$ for any element $a - \sigma(a) \in C^-_{n,p}$. □

Theorem 3.1.15. *Recall we can write $\delta = q + q^{-1}$, $q \in k$. For any $n \in \mathbb{N}$ and $0 \le p \le n$ such that $n - p \in 2\mathbb{Z}$, when $q^{2(p+1)} \ne \pm 1$ and $q^{2(p+2)} \ne 1$ the restriction of the standard $OTL_n(\delta)$-modules, $C^\pm_{n,p}$, to $OTL_{n-2}(\delta)$ causes them to decompose as follows;*

$$C^+_{n,p} \downarrow_{OTL_{n-2}} \cong \begin{cases} C^+_{n-2,0} \oplus C^+_{n-2,2}\,, & p = 0, \\ C^+_{n-2,p-2} \oplus C^+_{n-2,p} \oplus C^-_{n-2,p} \oplus C^+_{n-2,p+2}\,, & p > 0. \end{cases} \tag{3.14}$$

$$C^-_{n,p} \downarrow_{OTL_{n-2}} \cong \begin{cases} C^-_{n-2,0} \oplus C^-_{n-2,2}\,, & p = 0, \\ C^-_{n-2,p-2} \oplus C^+_{n-2,p} \oplus C^-_{n-2,p} \oplus C^-_{n-2,p+2}\,, & p > 0. \end{cases}\,, \tag{3.15}$$

where $C^+_{n-2,n} = C^-_{n-2,n} = 0$.

Proof. Since F_{n-2} is central on $OTL_{n-2}(\delta)$, left-multiplication by $F_{n-2} - \lambda I$ is a homomorphism from any module M to itself where $\lambda \in \mathbb{C}$. Therefore the eigenspaces of F_{n-2} on M are sub-modules. For $p > 0$, when we apply this to $C^+_{n,p} \downarrow_{OTL_{n-2}}$, F_{n-2} has at most four eigenvalues since the modules $C^+_{n-2,p}$, $C^+_{n-2,p-2}$, $C^+_{n-2,p+2}$, $C^-_{n-2,p}$ are all indecomposable. Three of these eigenvalues are $f_{n-2,p-2}$ with eigenspace $C^+_{n-2,p-2}$, $f_{n-2,p+2}$ with eigenspace $C^+_{n-2,p+2}$ and $f_{n-2,p}$ which is the eigenvalue for both eigenspaces $C^+_{n-2,p}$, $C^-_{n-2,p}$. If these eigenvalues are distinct then the sequences (3.1), (3.3) will split. We have,

$$\begin{aligned} f_{n-2,p+2} - f_{n-2,p} &= (q - q^{-1})(q^{p+2} - q^{-(p+2)}), \\ f_{n-2,p} - f_{n-2,p-2} &= (q - q^{-1})(q^{p} - q^{-p}), \\ f_{n-2,p+2} - f_{n-2,p-2} &= (q^2 - q^{-2})(q^{p+1} - q^{-(p+1)}). \end{aligned}$$

If $(q - q^{-1}) = 0$ or $(q^{p+1} - q^{-(p+1)}) = 0$ then we would have $q^{2(p+1)} = 1$ contradicting the hypothesis. If $(q^{p+2} - q^{-(p+2)}) = 0$, $q^{2(p+2)} = 1$ and if $(q^2 - q^{-2}) = 0$ then $q^{2(p+1)} = \pm 1$ both contradicting the hypothesis. Thus the eigenvalues $f_{n-2,p+2}$, $f_{n-2,p}$, $f_{n-2,p-2}$ are distinct as required.

The modules $C^+_{n-2,p}$, $C^-_{n-2,p}$ both have the same eigenvalue with respect to F_n so we cannot use the above to prove the sequence (3.5) splits. We shall defer completion of the proof of the $p > 0$ case to the end of this section.

For $p = 0$, F_{n-2} has at most two eigenvalues when applying the homomorphism $F_{n-2} - \lambda I$ to $C^+_{n,0} \downarrow_{OTL_n(\delta)}$, since both $C^+_{n,0}$ and $C^+_{n,2}$ are indecomposable for our choice of q. Splitting of the sequence (3.7) therefore happens when the eigenvalues $f_{n-2,0}$, $f_{n-2,2}$ are distinct which by the above only fails when $(q - q^{-1})(q^2 - q^{-2}) = 0$, that is when either $q^1 = \pm 1$ or $q^2 = \pm 1$ contradicting the hypothesis.

Since F_n also acts on any standard $OTL_n(\delta)$-module $C^-_{n,p}$ as the identity times $f_{n,p}$ the proof of the decomposition of $C^-_{n,p} \downarrow_{OTL_{n-2}}$ for any $0 \leq p \leq n$, $n - p \in 2\mathbb{Z}$, can be derived in the same way as the above. $\square$

To complete the above proof we shall use the some properties of quasi-hereditary algebras.

Let A be a finite dimensional algebra with $\mathrm{mod}-A$ the category of finite dimensional left A-modules. We denote by Ext^m_A the m^{th} extension bifunctor as defined in definition C.9.

The following proposition is given as by Donkin as part of [8, Proposition A2.2] as a property of $\mathrm{mod}-A$ when it can be considered as a *high weight category* as defined in [8, Definition A2.1]. However the author shows in [8, Proposition A3.7] that if $\mathrm{mod}-A$ is a high weight category then this is equivalent to A being a quasi-hereditary algebra. We can therefore give the proposition as follows.

Proposition 3.1.16. *[8, Proposition A2.2]. For A a quasi-hereditary algebra let $\{M(\lambda) \mid \lambda \in \Delta\}$ be the set of standard A-modules in the quasi-hereditary sense. Then for any $\lambda \in \Delta$ if $\mathrm{Ext}^1_S(M(\lambda), M(\mu)) \neq 0$, for some $\mu \in \Delta$ then $\mu > \lambda$.*

Proof of Theorem 3.1.15 continued. We have shown in proposition 2.5.2 that $OTL_n(\delta)$ is a quasi-hereditary algebra except when $\delta = 0$, n even and $\delta = \pm 1$, n odd. Since the exceptional cases are excluded by our hypothesis we can assume for the purpose of this theorem that $OTL_n(\delta)$ is quasi-hereditary. Now by proposition 3.1.16 if $\mathrm{Ext}^1_S(C^{\pm}_{n,p}, C^{\pm}_{n,p}) \neq 0$ for any two standard $OTL_n(\delta)$-modules $C^{\pm}_{n,p}, C^{\pm}_{n,p}$ then we must have $\pm p < \pm s$ in the sense of the ordering given by the poset diagram in figure 2.12. However, $+p$ and $-p$ are the incomparable in the sense of this ordering, so we must have $\mathrm{Ext}^1_S(C^{+}_{n,p}, C^{-}_{n,p}) = 0$ for any $p \in \Delta$, and hence $\mathrm{Ext}^1_S(C^{+}_{n-2,p}, C^{-}_{n-2,p}) = 0$. Thus by theorem C.11 the sequence (3.5) splits completing the proof. □

3.2 Globalisation and localisation

In this section we construct an idempotent E_n of the Outer-Temperley-Lieb algebra using known idempotents of the Temperley-Lieb algebra. We use this idempotent to define an isomorphism between the algebras OTL_{n-4} and $E_n OTL_n E_n$. Using this isomorphism we can define the pair of well known functors, the globalisation and localisation functors, between the categories of modules of the algebras OTL_{n-4} and OTL_n. We conclude this section by considering the OTL_n module $OTL_n E_n$ and show in proposition 3.2.5 that it is isomorphic to OTL_{n-2} when considered as a (OTL_{n-2}, OTL_{n-4})-bimodule. This along with the globalisation and localisation functions will be used in the final section of this chapter to prove an induction theorem, theorem 3.3.3, for the standard modules of the Outer-Temperley-Lieb algebra. We first recall the definition of the globalisation and localisation functors.

For a finite-dimensional k-algebra A let e be an idempotent in A. Then the category of modules $eAe - \mathrm{mod}$ is embedded within the category $A - \mathrm{mod}$ and we can define the localisation functor between these two categories by

$$F : \mathrm{mod} - A \longrightarrow \mathrm{mod} - eAe,$$
$$M \mapsto eM,$$

and the globalisation functor by

$$G : \mathrm{mod} - eAe \longrightarrow \mathrm{mod} - A,$$
$$N \mapsto Ae \otimes_{eAe} N.$$

We note it is well known that the functor F sends simple A-modules to either simple eAe-modules or zero and that for any eAe-module M, we have $FG(M) = M$.

As described in section 1.7, for $\delta \neq 0$ the Temperley-Lieb algebra has a collection of well known idempotents, namely $e_i = \frac{1}{\delta} u_i$ where the u_i are the generators of TL_n as given in definition 1.1.1.

Proposition 3.2.1. *For $\delta \neq 0$, $n \geq 4$ the element $E_n = e_1 e_{n-1} \in TL_n$ is an idempotent in OTL_n.*

Proof. $e_1 = \sigma(e_{n-1}) \in TL_n$ and by definition $e_1 e_{n-1} = e_{n-1} e_1$. Therefore $\sigma(e_1 e_{n-1}) = \sigma(e_1)\sigma(e_{n-1}) = e_{n-1} e_1 = e_1 e_{n-1} \in OTL_n$. Now ,

$$(e_1 e_{n-1})^2 = e_1 e_{n-1} e_1 e_{n-1} = e_1 e_{n-1} e_{n-1} e_1 = e_1 e_1 e_{n-1} e_{n-1} = e_1 e_{n-1},$$

as e_1, e_{n-1} are idempotents in TL_n. □

Proposition 3.2.2. *There exists an isomorphism $\Theta : OTL_{n-4} \longrightarrow E_n OTL_n E_n$.*

Proof. Any diagram in an element of $E_n OTL_n E_n$ has the form as shown in figure 3.6 where the grey box can contain any valid OTL_{n-4} diagram. Multiplication of any two

Figure 3.6: A typical diagram in an element of $E_nOTL_nE_n$.

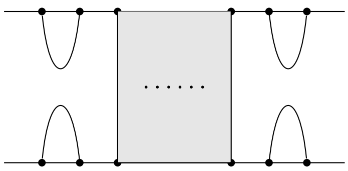
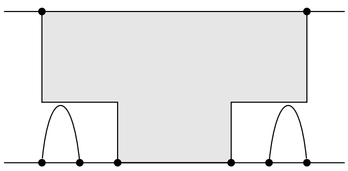


Figure 3.7: A typical basis diagram of TL_nE_n.



diagrams of this form will always result in a diagram of the same form. We therefore define Θ to be the k-linear extension of the map sending each diagram a in a basis element OTL_n to the diagram of the form in figure 3.6 where the grey box is replaced by the diagram a. $\square$

Definition 3.2.3. Given the above isomorphism Θ we can define globalisation, G, and localisation, F, functors on the module categories of OTL_n and OTL_{n-4} as follows

$$\begin{aligned} F : \operatorname{mod}-OTL_n &\longrightarrow \operatorname{mod}-OTL_{n-4}, \\ M &\mapsto E_nM, \\ G : \operatorname{mod}-OTL_{n-4} &\longrightarrow \operatorname{mod}-OTL_n, \\ N &\mapsto OTL_nE_n \otimes_{OTL_{n-4}} N. \end{aligned}$$

We now consider the OTL_n-module OTL_nE_n. Note that OTL_nE_n has a basis of diagrams consisting of all basis elements of TL_nE_n which are fixed by σ and for each diagram d in the basis of TL_nE_n which is not fixed by σ, the combination $d + \sigma(d)$, where each diagram in TL_nE_n has form as in figure 3.7.

Definition 3.2.4. We define an action ψ on elements of TL_nE_n as follows. On both the top and bottom edges we cut the lines emitting from the first and n^{th} nodes; we then remove the first and n^{th} nodes and the lines attached to them. This will leave either two or four nodes whose lines do not have endpoints. If there are two we join the lines emitting from them to form a new line and if there are four we form two new lines by joining the two nodes which were previously connected to 1 and $1'$ and the two nodes which were previously connected to n and n'. We finally reduce the label of each remaining node by one.

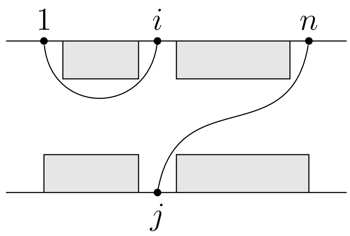
Proposition 3.2.5. *Define $\Psi : OTL_nE_n \to OTL_{n-2}$ to be the map sending each element $d \in OTL_nE_n$, such that $d = a_1 + a_2 + \cdots + a_j$, for some j with each $a_j \in TL_nE_n$, to the element $\psi(a_1) + \psi(a_2) + \cdots + \psi(a_j)$. Then Ψ is an isomorphism of left OTL_{n-2}-modules.*

Proof. We first need to check ψ does not form any diagrams with crossing lines and that it commutes with σ. For this, and the majority of the proof, we shall consider the action on the left hand side of the diagram only, noting that the same may applied to the right hand side unless we state otherwise.

Suppose $d \in TL_nE_n$ is such that $\psi(d)$ contains a pair of crossing lines. Assume this pair appears on the left hand side of the diagram, then since d did not contain any crossing lines one of the pair must be the new line $\{1', i\}$ from the first node on the bottom edge attached to the node i on one of the edges. Therefore in d there must be a line $\{1, i+1\}$ from the first node on the top edge to $i+1$. Suppose i is on the top edge of $\psi(d)$ corresponding to $i+1$ on the top edge of d. Then the line crossed by $\{1', i\}$ is some $\{j, k\} \in \psi(d)$ with j on the top edge and k on either edge such that $j < i$. However, this means the line $\{j+1, k+1\}$ exists in d with $j+1 < i+1$ on the top edge. Thus $\{1, i+1\}$ will cross $\{j+1, k+1\}$ in d, a contradiction. If i is on the bottom edge of $\psi(d)$ then the line that crosses $\{1', i\}$ will be some $\{p, q\}$ with $p < i$ on the bottom edge and q on either edge. However, this means $\{p+1, q+1\}$ will exist in d with $p+1 < i+1$ on the bottom edge. Thus $\{1, i+1\}$ will cross $\{p+1, q+1\}$ in d, again a contradiction.

We next check that ψ commutes with σ. Consider the diagram $d \in TL_nE_n$ which

Figure 3.8: The diagram $d \in TL_nE_n$ which contains an arc $\{1, i\}$ on its top edge for some i and a propagating line $\{n, j\}$ for some j on the bottom edge.



contains an arc $\{1, i\}$ on its top edge for some i and a propagating line $\{n, j\}$ for some j on the bottom edge (see figure 3.8). The diagram $\sigma(d)$ will therefore contain an arc $\{n - i + 1, n\}$ on its top edge and a propagating line $\{1, n - j + 1\}$. Hence $\psi(\sigma(d))$ will contain an arc $\{1', n - j\}$ on its bottom edge and a propagating line $\{n - i, n'\}$. Since all other lines are preserved by ψ we only need show that these two lines also appear in $\sigma(\psi(d))$. But by definition, $\psi(d)$ will contain a propagating line $\{i - 1, 1'\}$ and an arc $\{j - 1, n'\}$ on its bottom edge, therefore $\sigma(\psi(d))$ will contain an arc $\{1,', n - j\}$ on its bottom edge and a propagating line $\{n - i, n'\}$ as required.

Since we have now shown ψ sends diagrams of TL_nE_n to TL_{n-2} diagrams and that it commutes with σ, we can deduce by k-linear extension that Ψ is a valid morphism from OTL_nE_n to OTL_{n-4}. Now consider OTL_nE_n, OTL_{n-4} as OTL_{n-4} modules. To show the module action is respected by Ψ we only need to consider how it acts diagrammatically on the nodes whose lines are cut since all other lines are preserved. Let $d \in OTL_nE_n$, $a \in OTL_{n-2}$ and consider the left hand side of the product ad, with respect to the inclusion i. First suppose that there is a line from node 1 on the top edge of ad to some j on the bottom edge of ad. Then by definition, $\psi(ad)$ will contain an arc from $1'$ on the bottom edge to $j - 1$. For such a line to exist in ad there must be a line in d, $\{i, j\}$ with $i \geq 1$ on the top edge of d and a collection of arcs in d and a which join to form a line $\{1, i\}$ in ad. If $i = 1$ then the collection of arcs is empty and so by definition $\psi(d)$ will contain an arc $1'$ to $j - 1$. This arc will therefore appear in $a\psi(d)$ since it is unaffected by action from the top. If $i > 1$ then there is a k on the top edge of d such that $\{1, k\}$ is a arc in d. Therefore in $\psi(d)$ we have a propagating line $\{k-1, 1'\}$. Now when applying a on top the collection of arcs which joined 1 to i through k in ad will join $k-1$ to $i-1$. But since $k-1$ is joined to $1'$ and $i-1$ is joined to $j-1$ we must get an arc $\{1', j\}$.

Now suppose j lies on the top edge of ad, that is $\{1, j\}$ is an arc on the top edge of ad. Then in $\psi(ad)$ there will be a propagating line $\{j-1, 1'\}$. For such a line to exist in ad there must be a propagating line $\{j-1, i\}$ in a and a collection of arcs in a and d which form a path from 1 to i in ad. Let the arc extending from 1 in d end at point k on the top edge of d. Then in $\psi(d)$ we get a propagating line $\{k-1, 1'\}$. When applying a to $\psi(d)$ the collection of arcs which joined 1 to i through k in ad will join $k-1$ to $i-1$. But since $k-1$ is joined to $1'$ and $i-1$ is joined to $j-1$ we get a propagating line $\{j-1, 1'\}$.

To show Ψ is an isomorphism of OTL_n-modules, we check that ψ is a bijective map from the diagram basis of TL_nE_n to that of TL_{n-2}, the result will then follow by k-linearly extending to all of TL_nE_n then restricting elements to OTL_nE_n. Let $b \in TL_{n-2}$ and suppose b contains the two lines $\{1', i\}$, $\{j, (n-2)'\}$ for some i, j. We construct a diagram a from b as follows. First increase the label of all nodes of b by one, then remove the lines $\{2', i+1\}$ and $\{j+1, (n-1)'\}$. Introduce two new nodes on the top edge of b, one labelled 1 to the left of 2 and one labelled n to the right of $n-1$. Similarly, introduce two new nodes on the bottom edge of b, one labelled $1'$ to the left of $2'$ and one labelled n' to the right of $(n-1)'$. Finally create four new lines $\{1', 2'\}$, $\{(n-1)', n'\}$, $\{1, i+1\}$, $\{j+1, n\}$. Now $a \in TL_nE_n$ since its edges contain n nodes and the new lines $\{1', 2'\}, \{(n-1)', n'\}$ are arcs on the bottom edge of a which make it of the required form. Furthermore, by definition $\psi(a)$ will contain a line $\{1', i\}$ and a line $\{j, n'\}$ and since both ψ and the construction of a preserves all other lines in b we deduce that $\psi(a) = b$.

Now suppose there exist $d_1, d_2 \in TL_nE_n$ such that $\psi(d_1) = \psi(d_2) = a$. Let $\{1', i\}$ be a line in a for some i. If i sits on the top edge of a then both d_1 and d_2 must contain an arc $\{1, i\}$. Similarly if i sits on the bottom edge of a then both d_1 and d_2 must contain a propagating line $\{1, i\}$. However, in either case ψ preserves all other lines in both d_1 and d_2 on the left hand side, thus for $\psi(d_1) = \psi(d_2)$ all other lines on the left hand side in d_1 must be the same as those in d_2. Applying the same arguments to the right hand side we must have $d_1 = d_2$. $\square$

Given that Ψ sends basis elements of OTL_nE_n to those of OTL_{n-2} it is also an isomorphism of right OTL_{n-4} modules. We therefore have $OTL_nE_n \cong OTL_{n-2}$ as an (OTL_{n-2}, OTL_{n-4})-bimodule.

3.3 Induction of the standard modules of $OTL_n(\delta)$

In this final section we use the results from the previous section to show how the standard modules of OTL_n decompose when induced to the Outer-Temperley-Lieb algebra of rank $n+2$. Following [6, axiom $A4$] a remark is made about the relationship between induction, restriction and globalisation of modules satisfying certain criteria. So that we may make full use of it we shall state this remark as a Lemma. The decomposition of the induced standard modules will then follow directly from this. At the end of the chapter we encode the restriction and induction theorems we have given in the form of a *Bratelli diagram*.

Lemma 3.3.1. *[6]. Let A_n be a family of quasi-hereditary algebras with idempotents $e_n \in A_n$ such that $e_nA_ne_n \cong A_{n-2}$. Denote by M_n a standard A_n-module. If for all $n \geq 1$ $A_ne_n \cong A_{n-1}$ as a left A_{n-1}, right A_{n-2}-bimodule then $G_n(M_n)\downarrow_{A_{n+1}} \cong M_n \uparrow_{A_{n+1}}$.*

Thus by considering the odd and even rank Outer-Temperley-Lieb algebras as two separate families of algebras we deduce the following.

Corollary 3.3.2. *For any $0 \leq p \leq n$ such that $n - p \in 2\mathbb{Z}$ we have*

$$C^{\pm}_{n,p} \uparrow_{OTL_{n+2}} \cong G_n(C^{\pm}_{n,p}) \downarrow_{OTL_{n+2}} . \tag{3.16}$$

Corollary 3.3.3. *For any $0 \leq p \leq n$ such that $n - p \in 2\mathbb{Z}$ and $q \in k$ such that $q^{2(p+1)} \neq \pm 1$, $q^{2(p+2)} \neq 1$ with $\delta = q + q^{-1}$, we have,*

$$C^{+}_{n,p} \uparrow_{OTL_{n+2}} \cong \begin{cases} C^{+}_{n+2,0} \oplus C^{+}_{n+2,2}, & p = 0, \\ C^{+}_{n+2,p-2} \oplus C^{+}_{n+2,p}, \oplus C^{-}_{n+2,p} \oplus C^{+}_{n+2,p+2} & p > 0, \end{cases} \tag{3.17}$$

$$C^{-}_{n,p} \uparrow_{OTL_{n+2}} \cong \begin{cases} C^{-}_{n+2,0} \oplus C^{-}_{n+2,2}, & p = 0, \\ C^{-}_{n+2,p-2} \oplus C^{+}_{n+2,p} \oplus C^{-}_{n+2,p}, \oplus C^{-}_{n+2,p+2}, & p > 0. \end{cases} \tag{3.18}$$

Proof. Corollary 3.3.2 follows from Lemma 3.3.1 by taking the idempotents e_n to be the idempotents E_n as defined in proposition 3.2.1. The result then follows from corollary 3.3.2 by applying the globalisation functor defined in 3.2.3 alongside the restriction rules given by theorem 3.1.15 □

Figures 3.9 and 3.10 represent the restriction and induction rules in the form of a Bratelli diagram. Each number at the $(n, \pm p)^{th}$ entry is the dimension of the module $C^{\pm}_{n,p}$. The arrows extending from these dimensions point to the summands of the modules $C^{\pm}_{n,p} \uparrow_{OTL_{n+2}}$.

Figure 3.9: Bratelli diagram for $OTL_n(\delta)$ when n is even

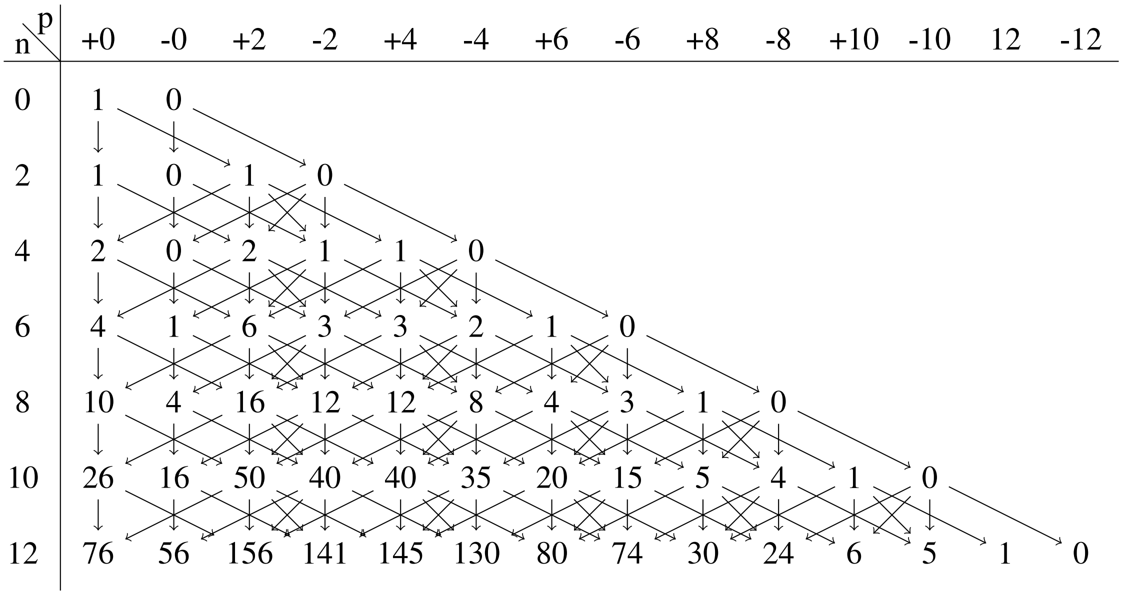
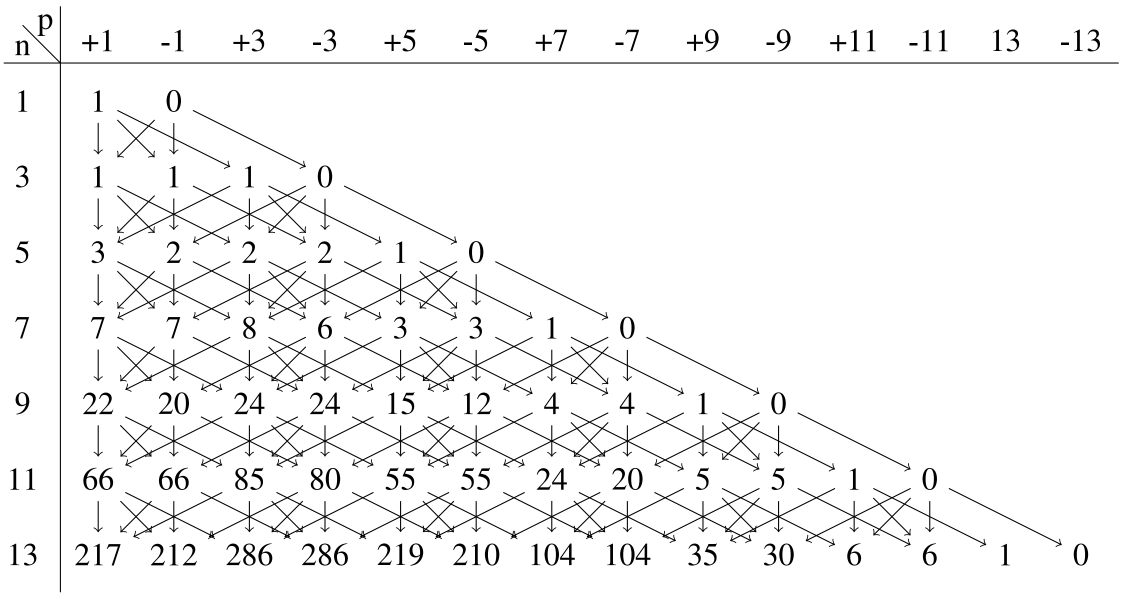


Figure 3.10: Bratelli diagram for $OTL_n(\delta)$ when n is odd



Chapter 4

Non semi-simple cases

When the Outer-Temperley-Lieb algebra is semi-simple we know, from chapter 2, the complete set of its irreducible modules and their multiplicities within the decomposition of the regular module. When $OTL_n(\delta)$ is not semi-simple, which by Theorem 2.6.6 is exactly when $TL_n(\delta)$ is not semi-simple, we still know by Lemmas 2.3.4 and 2.3.5 the basis and dimensions of the standard modules and that they become irreducible when we quotient by their radicals. In this chapter our main aim is to obtain a conjecture for the blocks of the Outer-Temperley-Lieb algebra, as we have done for the regular Temperley-Lieb algebra in section 1.7. As described in the afore mentioned section, we shall attempt to deduce the blocks of $OTL_n(\delta)$ by determining the homomorphisms between the indecomposable $OTL_n(\delta)$-modules. We begin in section 4.1 by looking for homomorphisms between standard $OTL_n(\delta)$-modules when n is even. While we are able to find $OTL_n(\delta)$-module homomorphisms in this case, we will not be able to determine if they are the complete set. In section 4.2 we give Conjecture 4.2.1 which claims that the homomorphisms found between the standard $OTL_n(\delta)$-modules in the first section are the only homomorphisms that can exist. We also discuss the extra difficulties that arise when we consider the case n odd. In the final section we assume true Conjecture 4.2.1 to give Conjecture 4.3.2 in which we claim how we can label the blocks of the Outer-Temperley-Lieb algebra.

In what follows we assume $k = \mathbb{C}$ and that $q \in k$ is a root of unity with $l \in \mathbb{Z}^+$ minimum such that $q^l = 1$. We shall also write TL_n for $TL_n(\delta)$ and OTL_n for $OTL_n(\delta)$, assuming that $\delta = q + q^{-1}$ unless otherwise stated.

4.1 Maps between modules for n even

In this section we attempt to determine some of the homomorphisms between the standard modules of OTL_n when n is even. In section 3.2 we used the idempotent $E_n = e_1 e_{n-1} \in TL_n(\delta)$ to define globalisation and localisation functors between OTL_n and OTL_{n-4}. To begin our study at q a root of unity, we shall use an alternative idempotent to define a new pair of globalisation and localisation functors between OTL_n and OTL_{n-2}. Using the decomposition of TL_n-modules when restricted to OTL_n as demonstrated in Lemma 2.6.4 we show how we can construct OTL_n-module homomorphisms by restricting TL_n-module homomorphisms. We use these ideas to begin drawing the homomorphisms we know we can construct onto a Bratelli diagram for $OTL_n(\sqrt{2})$. It will however become clear that as n increases, deciding exactly how the homomorphisms of the TL_n-modules restrict to OTL_n becomes more challenging. We therefore conclude this section by stating and proving Theorem 4.1.8 and Corollary 4.1.9 which describe for any n how we can restrict homomorphisms from the trivial TL_n-modules to OTL_n to provide homomorphisms from the new OTL_n-modules appearing in each layer of the Bratelli diagram. We shall, for what follows, assume n to be even, n odd is discussed in section 4.2.

It is clear that, as a symmetric element, the idempotent $e_{\frac{n}{2}} = \frac{1}{\delta} u_{\frac{n}{2}} \in TL_n$ for n even, is also an idempotent in OTL_n. Thus in a similar way to Proposition 3.2.2 we obtain an isomorphism

$$\Theta : OTL_{n-2} \longrightarrow e_{\frac{n}{2}} OTL_n e_{\frac{n}{2}}. \tag{4.1}$$

Definition 4.1.1. Given the above isomorphism we can define new globalisation and localisation functors,

$$F : \text{mod}-OTL_n \longrightarrow \text{mod}-OTL_{n-2}, \tag{4.2}$$

$$M \mapsto e_{\frac{n}{2}} M, \tag{4.3}$$

$$G : \text{mod}-OTL_{n-2} \longrightarrow \text{mod}-OTL_n, \tag{4.4}$$

$$N \mapsto OTL_n e_{\frac{n}{2}} \otimes_{OTL_{n-2}} N. \tag{4.5}$$

The convenience of the new functors over those defined in Definition 3.2.3 is that they allow us to pass the module structure between all sets of evenly ranked algebras instead of from n to $n+4$. However, we cannot use these functors when n is odd since the idempotent $e_{\frac{n}{2}}$ only exists when n is even.

Lemma 4.1.2. *Let $C^{\pm}_{n,p}$ denote the standard modules of OTL_n. Then,*

$$G(C^{\pm}_{n-2,p}) = C^{\pm}_{n,p}.$$

Proof. This follows from [21, Proposition 4] □

Lemma 4.1.3. *For any $0 \leq p \leq r \leq n \in \mathbb{N}$ with $n-p, n-r \in 2\mathbb{Z}$, if there exists a TL_n-module homomorphism $\theta_{r,p} : C_{n,r} \longrightarrow C_{n,p}$ then there exists an OTL_n-module homomorphism,*

$$\phi_{r,\pm p} : C^{+}_{n,r} \oplus C^{-}_{n,r} \longrightarrow C^{+}_{n,p} \oplus C^{-}_{n,p}. \tag{4.6}$$

Proof. By Lemma 2.6.4 we can define $\phi_{r,\pm p}$ to be the restriction of $\theta_{r,p}$ to OTL_n. □

One of the more useful cases of the lemma is when $r = n$. In this case, since $C^{-}_{n,n}$ does not exist and since $C_{n,n}$ is the trivial module, the image of $\theta_{n,p}(C_{n,n})$ can only be one-dimensional. Thus the restriction of $\theta_{n,p}$ to OTL_n must be a module homomorphism from $C^{+}_{n,n}$ to one of $C^{+}_{n,p}$ or $C^{-}_{n,p}$.

Figure 4.1: Bratelli diagram representing the maps between standard modules $C_{n,p}$ of $TL_n(\sqrt{2})$ for n even.

$n \backslash p$	0	2	4	6	8	10	12
0	1						
2	1	1					
4	2	3	1				
6	5	9	5	1			
8	14	28	20	7	1		
10	22	90	75	35	9	1	
12	132	297	275	154	54	11	1

Example 4.1.4. When $\delta = q + q^{-1} = \sqrt{2}$, q is an 8^{th} root of unity, hence $l = 8$. We can represent the homomorphisms between the standard $TL_n(\sqrt{2})$ modules on a Brattelli diagram as in figure 4.1. Here, the numbers on the left indicate the rank n while those on the top line indicate the module labelling index $p \in \Delta$. The number at each $(n,p)^{th}$ point is then the dimension of the module $C_{n,p}$ and an arrow between two of these dimensions represents a homomorphism between those modules.

We can draw a similar diagram for $OTL_n(\sqrt{2})$. As we already know the dimensions of the standard modules from figure 3.9, we only need to work out where we need to draw the arrows. To begin, we consider the row where $n = 2$. This line contains only two modules $C^+_{2,2}$ and $C^+_{2,0}$ both of which are one-dimensional, hence irreducible OTL_n-modules. Since, by Proposition 2.5.2, $OTL_n(\sqrt{2})$ is quasi-hereditary and both these modules are irreducible, there can be no homomorphism between them, as any such homomorphism would imply the head of $C^+_{2,2}$ appearing in $C^+_{2,0}$ which is a contradiction. The localisation map in equation (4.2) therefore implies there can be no homomorphism $C^+_{n,2} \longrightarrow C^+_{n,0}$ for any even n since any such map would imply a map from $C^+_{2,2} \longrightarrow C^+_{2,0}$, a contradiction.

We next consider the row $n = 4$, here we have four $OTL_4(\sqrt{2})$ indecomposable modules two of which are one-dimensional and therefore irreducible. The Bratelli diagram for $TL_n(\sqrt{2})$ (figure 4.1) implies that there is a TL_4-module homomorphism,

$$\theta_{4,2} : C_{4,4} \longrightarrow C_{4,2},$$

thus by Lemma 4.1.3 there must exist a non-zero OTL_4-module homomorphism,

$$\phi_{4,2} : C^+_{4,4} \longrightarrow C^+_{4,2} \oplus C^-_{4,2}.$$

If there were to exist a non-zero homomorphism $\phi_{4,-2} : C^+_{4,4} \to C^-_{4,2}$, the head of $C^+_{4,4}$ must appear in $C^-_{4,2}$ since $OTL_n(\sqrt{2})$ is quasi-hereditary. However, $C^-_{4,2}$ is a one dimensional, hence irreducible module which would lead to a contradiction. Thus $\theta_{4,2}$ restricts to the homomorphism $\phi_{4,2} : C^+_{4,4} \longrightarrow C^+_{4,2}$. Applying the globalisation functor, equation (4.4), we can now say for any even $n \geq 4$ there exists an OTL_n module homomorphism $\phi_{4,2} : C^+_{n,4} \longrightarrow C^+_{n,2}$. It remains to determine, for $n = 4$, if there exists a homomorphism from $C^-_{4,2}$ to any of $C^+_{4,0}$ or $C^+_{4,2}$. A simple Gram matrix calculation can be done to show $C^+_{4,0}$ is irreducible so we cannot have any homomorphism into $C^+_{4,0}$. As for $C^+_{4,2}$, since it is two dimensional and already contains $C^+_{4,4}$ as a sub-module we only need check if the action of the quotient module $C^+_{4,2}/C^+_{4,4}$ is isomorphic to that of $C^-_{4,2}$. It turns out this is not true since $C^-_{4,2}$ sends all symmetric diagrams to zero, while the quotient does not. Thus we have the complete set of homomorphisms between indecomposable modules of $OTL_4(\sqrt{2})$.

For the row $n = 6$ the Bratelli diagram for $TL_n(\sqrt{2})$ tells us there exists a TL_6-module homomorphism $\theta_{6,0} : C_{6,6} \longrightarrow C_{6,0}$ and hence an OTL_6-module homomorphism $\phi_{6,0} : C^+_{6,6} \longrightarrow C^+_{6,0} \oplus C^-_{6,0}$. However, $C^-_{6,0}$ is one-dimensional and hence an irreducible module, so we may argue as in the $\phi_{4,2}$ case to determine that for all even $n \geq 6$ there exists an OTL_n-module homomorphism $\phi_{6,0} : C^+_{6,6} \longrightarrow C^+_{6,0}$. We also know from the Bratelli diagram that there is a TL_6-module homomorphism $\theta_{4,2} : C_{6,4} \longrightarrow C_{6,2}$ and that by globalisation of the $n = 4$ case there exists an OTL_6-module homomorphism $\phi_{4,2} : C^+_{6,4} \longrightarrow C^+_{6,2}$ which is consistent with the restriction of $\theta_{4,2}$ to $OTL_6(\sqrt{2})$. To determine whether the module $C^-_{6,4}$ maps into either of modules $C^+_{6,2}$ or $C^-_{6,2}$, we can construct the Gram matrix of $C^+_{6,2}$ to get,

$$\det(G^+_{6,2}) = \det \begin{pmatrix} 2\delta^2+2 & 2\delta & 2 & 0 & 2 & 2\delta \\ 2\delta & 2\delta^2 & 2\delta & 0 & 2\delta & 2 \\ 2 & 2\delta & 2\delta^2+2 & 2\delta & 2 & 2\delta \\ 0 & 0 & 2\delta & \delta^2 & 0 & 1 \\ 2 & 2\delta & 2 & 0 & \delta^2 & \delta \\ 2\delta & 2 & 2\delta & 1 & \delta & \delta^2 \end{pmatrix}$$
$$= 8\delta^{12} - 72\delta^{10} + 248\delta^8 - 400\delta^6 + 288\delta^4 - 64\delta^2 = 0 \quad \text{at } \delta = \sqrt{2}.$$

Therefore $C^+_{6,2}$ is not irreducible at $\delta = \sqrt{2}$ and so we would expect to see a homomorphism into it from one of $C^+_{6,4}$ or $C^-_{6,4}$. In fact we can deduce that this must be a module homomorphism $\phi_{4,-2} : C^-_{6,4} \longrightarrow C^-_{6,2}$ since a module homomorphism from $C^+_{6,4}$ to $C^-_{6,2}$ would imply, by localisation, a module homomorphism $C^+_{4,4}$ to $C^-_{4,2}$ which would be a contradiction. We do not yet have enough information to determine whether or not there is a homomorphism $C^-_{6,4}$ to any other standard OTL_4-module or indeed from $C^-_{6,0}$ or $C^+_{6,6}$ to any other standard OTL_4-module.

In the $n = 8$ we can again use a Gram matrix calculation to determine that the OTL_8-module $C^+_{8,6}$ is irreducible while $C^-_{8,6}$ is not. Thus the TL_8-module homomorphism $\theta_{8,6} : C_{8,8} \longrightarrow C_{8,6}$ restricts to an OTL_8-module homomorphism $\phi_{8,-6} : C^+_{8,8} \longrightarrow C^-_{8,6}$ to which we can then apply the globalisation functor,equation (4.4), as above. We can also deduce that $C^-_{8,0}$ is not irreducible at $\delta = \sqrt{2}$ therefore, by arguing as in $n = 6$, the restriction of the module homomorphism $C_{8,6} \longrightarrow C_{8,0}$ must restrict to a homomorphism $C^-_{8,6} \longrightarrow C^-_{8,0}$ since there can be no homomorphism $C^+_{6,6} \longrightarrow C^-_{6,0}$. As with $n = 6$ case, while we now know how the maps from $TL_n(\sqrt{2})$ restrict to the standard $OTL_6(\sqrt{2})$-modules, we do

Figure 4.2: Initial Bratelli diagram showing the maps between standard modules of $OTL_n(\sqrt{2})$.

$n \backslash p$	+0	-0	+2	-2	+4	-4	+6	-6	+8	-8	+10	-10	+12
0	1	0											
2	1	0	1	0									
4	2	0	2	1	1	0							
6	4	1	6	3	3	2	1	0					
8	10	4	16	12	12	8	4	3	1	0			
10	26	16	50	40	40	35	20	15	5	4	1	0	
12	76	56	156	141	145	130	80	74	30	24	6	5	1

not yet know what other homomorphisms might arise from the modules $C^+_{8,8}$ and $C^-_{8,6}$.

This is about as far as we can go working in this manner. While we can continue to apply the globalisation functor, equation (4.4), to draw the maps derived above for infinite layers, it is at this point it becomes difficult to determine what maps arise from the new modules introduced in each layer of the Bratelli diagram. Indeed, when $n = 10$ figure 4.1 tells us there exists a TL_n-module homomorphism $\theta_{10,4} : C_{10,10} \longrightarrow C_{10,4}$ and hence there must exist an OTL_{10}-module homomorphism $\phi_{10,4}$ mapping $C^+_{10,10}$ into one of $C^+_{10,4}$ or $C^-_{10,4}$. However to determine which of these the image of $\theta_{10,4}(C^+_{10,10})$ restricts to requires us to at least determine which of them are irreducible and given that these modules have dimension 40 and 35 respectivly, calculating the Gram matrix determinant of each module would take too long. Our progress so far can be seen in the Bratelli diagram in figure 4.2 where, as in figure 4.1, the numbers in each $(n, p)^{th}$ entry are the dimensions of the standard modules $C^+_{n,p}$ $C^-_{n,p}$. The blue lines show those maps whose existence was determined by restricting the $TL_n(\sqrt{2})$ maps and the green lines are those due to use of the globalisation functor.

To make progress with the above example, and other general cases, we need to first determine what OTL_n-module homomorphisms arise from the new standard OTL_n-modules, $C^+_{n,n}$ and $C^-_{n,n-2}$, that appear in each layer of the Bratelli diagram as n increases. The homomorphisms between the standard TL_n-modules have so far been useful in showing that OTL_n-module homomorphisms exists between direct sums of pairs of standard OTL_n-modules. To determine for a given pair p, s how exactly the homomorphism $\theta_{p,s}$ restricts to OTL_n we shall use the concrete construction of a TL_n-module homomorphism given in [5, Section 5]. Here the homomorphisms are given as the restriction of homomorphisms of modules of the *global Temperley-Lieb algebra*. This algebra, as defined following [5, Proposition 2.1], is the algebra given by setting $TL_\infty = \lim_{n\to\infty} TL_n$ and contains all finite k-linear combinations of Temperley-Lieb diagrams with infinitely many propagating lines added to the left of each diagram.

Definition 4.1.5. [5, page 599]. For any line r in a Temperley-Lieb diagram d we define *hook product* $h(r) = [a]$ where a is the number of lines, including r, to the left of r when the diagram is deformed to allow r to be a propagating line (if it is not already) and $[-]$ denotes the *box number* as defined in Definition A.1. We can then define the hook product of any diagram d with exactly p propagating lines as,

$$h(d) = \frac{[\frac{n+p}{2}]!}{\Pi_{r\in d} h(r)}\,. \tag{4.7}$$

An example of the deformation described in the definition which turns an arc into a propagating line can be seen in [5, Figure 9]. By adding infinitely many propagating lines to the left of the diagrams of the standard TL_n-modules $C_{n,p}$ we create the modules $C_{\infty,p}$ for TL_∞. Homomorphisms between these modules are then given by the following theorem.

Theorem 4.1.6. *[5, Theorem 5.1]. For all $p \geq s$ there is a non-zero TL_∞-module homo-*

morphism, $\theta^{\infty}_{p,s} : C_{\infty,p} \longrightarrow C_{\infty,s}$*, acting on each diagram* d *by,*

$$d \mapsto \sum_{e \in C_{\infty,n-p+s}} h(e)ed\,. \tag{4.8}$$

Where the composition ed *is as described at the beginning of [5, Section 5]. When* $p = n$ *this the is the unique such morphism of standard modules.*

As described following the proof of [5, Theorem 5.1] we can use $\theta^{\infty}_{p,s}$ to *engender* a TL_n-module homomorphism $\theta_{p,s} : C_{n,p} \longrightarrow C_{n,s}$ by restricting $\theta^{\infty}_{p,s}$ to the TL_n-module $C_{n,p}$ and projecting its image onto $C_{n,s}$. This leads to the following theorem of TL_n-modules.

Theorem 4.1.7. *[5, Theorem 5.3]. When* q *is an* l^{th} *root of unity the* TL_∞*-module homomorphism* $\theta^{\infty}_{p,s}$ *engenders a non-zero* TL_n*-module homomorphism,*

$$\theta_{p,s} : C_{n,p} \longrightarrow C_{n,s},$$

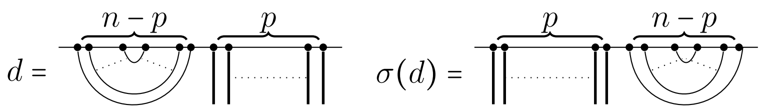
if and only if $p + s + 2 \equiv 0 \mod l$*, where* $0 \leq \frac{p+2}{2} \leq l$*.*

We shall now turn our attention to the TL_n-module homomorphisms, $\theta_{n,p} : C_{n,n} \longrightarrow C_{n,p}$. As described in the paragraph following Lemma 4.1.3 we must be able to restrict $\theta_{n,p}$ to a module homomorphism either $\phi_{n,p} : C^{+}_{n,n} \longrightarrow C^{+}_{n,p}$ or $\phi_{n,-p} : C^{+}_{n,n} \longrightarrow C^{-}_{n,p}$. Since $C_{n,n} \cong C^{+}_{n,n}$ when we restrict to OTL_n, the construction given in Theorem 4.1.6 shows that the identity diagram $I \in C^{+}_{n,n}$ is mapped by $\theta_{n,p}$ as,

$$\theta_{n,p}(I) = \sum_{e \in C_{n,p}} \frac{[\frac{n+p}{2}]!}{\Pi_{r \in e} h(r)} e\,. \tag{4.9}$$

We know $\theta_{n,p}(I)$ is non-zero and either a member of $C^{+}_{n,p}$ or $C^{-}_{n,p}$. As the sum in equation (4.9) runs over all top half n-diagrams of $C_{n,p}$, for every top half n-diagram e appearing in the linear combination $\theta_{n,p}(I)$, $\sigma(e)$ must also appear unless $e = \sigma(e)$. Thus we can determine which module the image is in by determining the coefficients of some diagram e in the linear combination and its reflected diagram $\sigma(e)$.

Figure 4.3: The diagrams d and $\sigma(d)$ of $C_{n,p}$.



We shall choose the pair of diagrams as shown in figure 4.3. For this choice of diagram the coefficient of $\sigma(d)$ in the linear combination is given by,

$$\frac{[\frac{n+p}{2}]!}{\Pi_{r\in\sigma(d)}h(r)} = 1\,.$$

The coefficient of d is then given by,

$$\frac{[\frac{n+p}{2}]!}{\Pi_{r\in d}h(r)} = \frac{[\frac{n+p}{2}]!}{[\frac{n-p}{2}]![p]!}\,,$$

where the $[p]!$ in the product $\Pi_{r\in d}h(r)$ comes from the product of $h(e)$ for each propagating line e in d and $[\frac{n-p}{2}]!$ in the product comes from the product of $h(f)$ for each arc f in d. By Theorem 4.1.7 we know the map $\theta_{n,p}$ can only exist if $n+p+2 \equiv 0 \mod l$, therefore $q^{n+p+2} = 1$ since q is a l^{th} root of unity. Let $r = \frac{n+p}{2}$, then $q^{2(r+1)} = 1$ and hence $q^{r+1} = \mp 1$. This implies,

$$[r+1] = \frac{q^{r+1} - q^{-(r+1)}}{q - q^{-1}} = 0\,.$$

Then, by Lemma A.3, $[r] = \pm 1$ and for any $a \leq r$ we have $[r-a] = \pm[a+1]$. As $r \geq p$ we can write,

$$\frac{[r]!}{[\frac{n-p}{2}]![p]!} = \frac{[r][r-1][r-2]\ldots[p+1]}{[\frac{n-p}{2}]!} = \frac{(\pm 1)^{r-p}[1][2][3]\ldots[r-p]}{[\frac{n-p}{2}]!}\,.$$

However $r - p = \frac{n+p}{2} - p = \frac{n-p}{2}$, hence,

$$\frac{(\pm 1)^{r-p}[1][2][3]\ldots[r-p]}{[\frac{n-p}{2}]!} = \frac{(\pm 1)^{\frac{n-p}{2}}[\frac{n-p}{2}]!}{[\frac{n-p}{2}]!} = (\pm 1)^{\frac{n-p}{2}}\,.$$

Where $q^{r+1} = -1$ we have $[r] = 1$ and $[r-a] = [a+1]$ hence the coefficient of d is 1. If $q^{r+1} = 1$ then we have $q^{\frac{n+p}{2}+1} = 1$, hence not only is $n+p+2 \equiv 0 \mod l$ but $n+p+2 \equiv 0 \mod 2l$. In this case the coefficient of d is -1 unless $\frac{n-p}{2}$ is even, which would imply $\frac{n+p}{2}+1$ is odd. However, since q is an l^{th} root of unity we can only have $q^{\frac{n+p}{2}+1} = 1$, in this case, if l is odd. The conclusion of the above is that we have proven following theorem.

Theorem 4.1.8. *Let $q \in k$ be an l^{th} root of unity such that $OTL_n(\delta)$ is quasi-hereditary. Suppose for some $0 \le p \le n \in \mathbb{N}$ with $n-p \in 2\mathbb{Z}$ there exists a TL_n-module homomorphism $\theta_{n,p} : C_{n,n} \longrightarrow C_{n,p}$. Then if $n+p+2 \equiv 0 \mod 2l$ and either l is even or l is odd and $\frac{n-p}{2}$ is odd, $\theta_{n,p}$ restricts to an OTL_n-module homomorphism $\phi_{n,-p} : C^+_{n,n} \longrightarrow C^-_{n,p}$. If $n+p+2 \not\equiv 0 \mod 2l$ or $n+p+2 \equiv 0 \mod 2l$ with l odd and $\frac{n-p}{2}$ even then $\theta_{n,p}$ restricts to an OTL_n-module homomorphism $\phi_{n,p} : C^+_{n,n} \longrightarrow C^+_{n,p}$.*

This theorem also allows us determine some of the maps arising from the new OTL_n-modules $C^-_{n,n-2}$ in each layer when there exists a TL_n-module homomorphism from $C_{n,n-2}$ to $C_{n,p}$ for some $0 \le p \le n$, $n-p \in 2\mathbb{Z}$.

Corollary 4.1.9. *Suppose for some $0 \le p \le n$ with $n-p \in 2\mathbb{Z}$, there exists a TL_n-module homomorphism $\theta_{n,p} : C_{n,n} \longrightarrow C_{n,p}$ which restricts to an OTL_n-module homomorphism $\phi_{n,\pm p} : C^+_{n,n} \longrightarrow C^\pm_{n,p}$. Then there exists an OTL_{n+2}-module homomorphism $\phi_{n+2,\mp p} : C^-_{n+2,n} \longrightarrow C^\mp_{n+2,p}$.*

Proof. By 4.1.6 and 4.1.7 there exists a TL_n-module homomorphism $\theta_{n+2,p} : C_{n+2,n} \longrightarrow C_{n+2,p}$ and by applying the globalisation functor, equation (4.4), to $\phi_{n,\pm p}$ there exists an OTL_n-module homomorphism $\phi_{n+2,\pm p} : C^+_{n+2,n} \longrightarrow C^\pm_{n+2,p}$. We shall consider the restriction of $\theta_{n+2,p}$ to $C^-_{n+2,n}$. Since $\theta_{n+2,p}$ is a homomorphism, for any $a - \sigma(a) \in C^-_{n+2,n}$ we have,

$$\theta_{n+2,p}(a - \sigma(a)) = \theta_{n+2,p}(a) - \theta_{n+2,p}(\sigma(a)), \qquad a \in C_{n+2,n} \text{ with } a \neq \sigma(a).$$

Let a be as in figure 4.4, then the homomorphism defined in Theorem 4.1.6 tells us

Figure 4.4: The element $a \in C_{n+2,n}$.

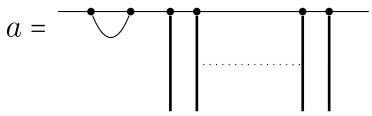
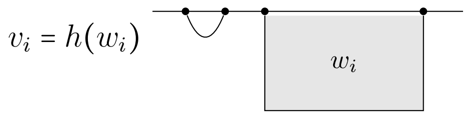


Figure 4.5: The elements v_i in the image of $\theta(a)$ where each w_i is a basis diagram of $C_{n,p}$.



that the image of $\theta_{n+2,p}(a)$ is a the sum of elements v_i of the form shown in figure 4.5 where each w_i is a basis diagram of $C_{n,p}$. The image of $\theta_{n+2,p}(\sigma(a))$ is therefore a sum of elements of the form given in figure 4.6. However, by Theorem 4.1.8 we know that $h(w_i) = \pm h(\sigma(w_i))$ for each $w_i \in C_{n,p}$, thus the image of $\theta_{n+2,p}(a - \sigma(a))$ is the sum of $h(w_i)v_i \mp h(w_i)\sigma(v_i)$ for all $w_i \in C_{n,p}$.

If $\theta_{n,p}$ restricts to $\phi_{n,-p}$ then $h(w_i) = -h(\sigma(w_i))$, thus

$$h(w_i)v_i \mp h(w_i)\sigma(v_i) = h(w_i)(v_i + \sigma(v_i)) \in C^+_{n+2,p}.$$

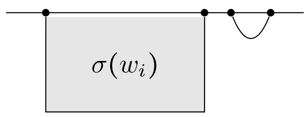
When $\theta_{n,p}$ restricts to $\phi_{n,p}$ then $h(w_i) = h(\sigma(w_i))$, thus

$$h(w_i)v_i \mp h(w_i)\sigma(v_i) = h(w_i)(v_i - \sigma(v_i)) \in C^-_{n+2,p}.$$

The same process can be applied to any other $b - \sigma(b) \in C^-_{n+2,n}$ since the diagram multiplication in equation (4.8), does not effect the single arc on the top of each $b \in C_{n+2,n}$. Thus the image of $\theta_{n+2,p}(C^-_{n+2,n})$ is contained entirely in $C^-_{n+2,p}$ when $\theta_{n,p}$ restricts to $\phi_{n,p}$, otherwise is contained entirely in $C^+_{n+2,p}$ as required. □

Using the above theorem we may now add to our Bratelli diagram in figure 4.2 to get the diagram as in figure 4.7. The green lines are the same as in figure 4.2, while the red lines represent the homomorphisms determined by the above theorem. We note that this may still not be a complete picture of all homomorphisms between the indecomposable

Figure 4.6: The elements $\sigma(v_i)$ in the image of $\theta(\sigma(a))$.

$\sigma(v_i) = h(\sigma(w_i))$ 

OTL_n-modules. Figures 4.8 and 4.9 show similar diagrams for $\delta = 1$ and $\delta = \sqrt{3}$ respectively. Again, these may not show the complete set of OTL_n-module homomorphisms.

We remark that when $\delta \neq \pm 1$ Theorem 4.1.8 and the above corollary are also true when n is odd. Since the index p must also be odd when n is odd and the diagrams used in their proofs (figures 4.3 and 4.4) still exist, the same proofs will still apply. We note that at $\delta = \pm 1$, when n is odd, OTL_n is not quasi-hereditary and Theorem 4.1.8 is not true. For example the theorem would imply the map $C_{3,3} \to C_{3,1}$ restricts to a map $C^+_{3,3} \to C^+_{3,1}$ when $\delta = 1$ hence $l = 6$, however it can be easily checked that in this case $C^-_{3,1} \cong C^+_{3,3}$ which is a contradiction. We shall discuss the n odd case further in the next section.

Figure 4.7: Second Bratelli diagram showing the homomorphisms between standard modules of $OTL_n(\sqrt{2})$.

$n \backslash p$	+0	-0	+2	-2	+4	-4	+6	-6	+8	-8	+10	-10	+12
0	1	0											
2	1	0	1	0									
4	2	0	2	1	1	0							
6	4	1	6	3	3	2	1	0					
8	10	4	16	12	12	8	4	3	1	0			
10	26	16	50	40	40	35	20	15	5	4	1	0	
12	76	56	156	141	145	130	80	74	30	24	6	5	1

Figure 4.8: Bratelli diagram showing the homomorphisms between standard modules of $OTL_n(1)$.

$n \backslash p$	+0	-0	+2	-2	+4	-4	+6	-6	+8	-8	+10	-10	+12	-12	+14
0	1	0													
2	1	0	1	0											
4	2	0	2	1	1	0									
6	4	1	6	3	3	2	1	0							
8	10	4	16	12	12	8	4	3	1	0					
10	26	16	50	40	40	35	20	15	5	4	1	0			
12	76	56	156	141	145	130	80	74	30	24	6	5	1	0	
14	232	197	518	483	511	496	329	308	140	133	42	35	7	6	1

Figure 4.9: Bratelli diagram showing the homomorphisms between standard modules of $OTL_n(\sqrt{3})$.

$n \backslash p$	+0	-0	+2	-2	+4	-4	+6	-6	+8	-8	+10	-10	+12	-12	+14
0	1	0													
2	1	0	1	0											
4	2	0	2	1	1	0									
6	4	1	6	3	3	2	1	0							
8	10	4	16	12	12	8	4	3	1	0					
10	26	16	50	40	40	35	20	15	5	4	1	0			
12	76	56	156	141	145	130	80	74	30	24	6	5	1	0	
14	232	197	518	483	511	496	329	308	140	133	42	35	7	6	1

4.2 Maps between modules for all n

The results of the previous section determine all OTL_n-module homomorphisms which arise by restricting known TL_n-modules homomorphisms for n even. However, as previously mentioned, we still do not know if these are the only OTL_n-module homomorphisms that exist. There is some evidence to suggest that no other homomorphisms exist. We note first that it is true in the low rank cases that we can calculate by hand. Example 4.1.4 shows this for $\delta = \sqrt{2}$ up to $n = 4$ and we can make similar calculations for other non-generic choices of δ in these low rank cases. In the $n \geq 6$ cases, the globalisation and localisation functors imply we only need to determine whether any new maps, besides those we have deduced from Theorem 4.1.8 and Corollary 4.1.9, come from the modules $C^{+}_{n,n}$, $C^{-}_{n,n-2}$ in each layer and $C^{-}_{6,0}$ in the $n = 6$ layer.

One way to check no other maps might exist would be to consider how any homomorphism between two irreducible OTL_n-modules induces to the TL_n-modules. Since we know all the TL_n-module homomorphisms, showing that an OTL_n-module map induces a TL_n-module homomorphism means the only OTL_n-modules homomorphisms that can exist are those derived from TL_n-module homomorphisms in the first place. However, this is not easy to show and at this time we shall simply make the following conjecture.

Conjecture 4.2.1. *When OTL_n is not semi-simple the only maps arising between any pair of OTL_n-modules are those formed by restricting the maps between pairs of TL_n-modules.*

In the case of n even, Conjecture 4.2.1 being true allows us to determine all OTL_n standard module homomorphisms using Theorem 4.1.8 and the globalisation functor in equation (4.4). The process for determining these homomorphisms is similar to example 4.1.4 where, in the low rank cases, we can calculate explicitly how the TL_n-module homomorphisms restrict to OTL_n, we then apply the globalisation functor to the result. Since the only two new standard OTL_n-modules in a layer are $C^{+}_{n,n}$ and $C^{-}_{n,n-2}$, Theorem

4.1.8 and Corollary 4.1.9 will tell us any possible module homomorphisms arising from these modules to which, again, we can apply the globalisation functor to obtain module homomorphisms for higher rank n.

Finding the maps between the standard OTL_n-modules for n odd is more complex. As remarked after Corollary 4.1.9, Theorem 4.1.8 and Corollary 4.1.9 still apply for n odd as long as $\delta \neq \pm 1$. Thus we can still determine where the TL_n-module homomorphisms, from the trivial module $C_{n,n}$, are restricted to when considered as OTL_n-module homomorphisms. We can also work in the same way as example 4.1.4 to restrict TL_n-module homomorphisms to OTL_n in the lower rank cases. Our remaining problem in this case is that the idempotent $e_{\frac{n}{2}}$ does not exists in OTL_n when n is an odd, thus we are unable to use the functors from Definition 4.1.1 to globalise these homomorphisms between modules to higher rank n. We can attempt to find an idempotent that would work for us as follows.

Any non-trivial idempotent E in OTL_n which allows us to define an isomorphism between OTL_n and $EOTL_{n-2}E$ can only be represented by a sum of diagrams containing $n-2$ propagating lines. Thus we cannot represent E by a single diagram as any such diagram must have at least two arcs to be in OTL_n. We shall instead attempt to work with the following candidate.

Lemma 4.2.2. *Let $e_{\frac{n-1}{2}}$ and $e_{\frac{n+1}{2}}$ be the usual idempotents $\frac{1}{\delta}u_{\frac{n-1}{2}}$ and $\frac{1}{\delta}u_{\frac{n+1}{2}}$ of TL_n. Then the element,*

$$E = \frac{1}{4}\left(e_{\frac{n-1}{2}} + e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}\right), \tag{4.10}$$

is an idempotent of OTL_n.

Proof. First we note that $\sigma(e_{\frac{n-1}{2}}) = e_{\frac{n+1}{2}}$ therefore $E \in OTL_n$. To show E is an idempotent we calculate,

$$\begin{aligned}E^2 = \frac{1}{16}(\quad & e_{\frac{n-1}{2}}e_{\frac{n-1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} \\ & + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} \\ & + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}}e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} \\ & + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}e_{\frac{n-1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}).\end{aligned}$$

Since $e_{\frac{n-1}{2}}$ and $e_{\frac{n+1}{2}}$ are idempotent in TL_n and by the original definition of TL_n they satisfy $e_{\frac{n-1}{2}}e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} = e_{\frac{n-1}{2}}$, $e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} = e_{\frac{n+1}{2}}$ and we can cancel to get,

$$\begin{aligned}E^2 = \frac{1}{16}(\quad & e_{\frac{n-1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}} \\ & + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} + e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} \\ & + e_{\frac{n-1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}}e_{\frac{n+1}{2}} + e_{\frac{n-1}{2}} \\ & + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}} + e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}} + e_{\frac{n+1}{2}}e_{\frac{n-1}{2}}) = E,\end{aligned}$$

as required. □

Since E is an idempotent we can define globalisation and localisation functors,

$$\begin{aligned}F : \mathrm{mod}-OTL_n &\longrightarrow \mathrm{mod}-EOTL_nE, \\ M &\mapsto EM, \\ G : \mathrm{mod}-EOTL_nE &\longrightarrow \mathrm{mod}-OTL_n, \\ N &\mapsto OTL_nE \otimes_{EOTL_nE} N,\end{aligned}$$

however, to progress further we need to show we can find some isomorphism

$$\Theta' : OTL_{n-2}(\delta) \longrightarrow EOTL_nE.$$

Such a calculation is lengthy and at this time beyond the scope of this thesis. Instead we can get some partial information by considering the idempotent E_n as defined in section 3.2. By Proposition 3.2.2 we get an isomorphism $\Theta : OTL_{n-4} \longrightarrow E_nOTL_nE_n$ and hence the globalisation and localisation functors as defined in 3.2.3. Thus for $\delta \neq \pm 1$ we

Figure 4.10: Bratelli diagram showing the homomorphisms between standard modules of $OTL_n(\sqrt{2})$ for n odd.

n \ p	+1	-1	+3	-3	+5	-5	+7	-7	+9	-9	+11	-11	+13	-13	+15
1	1	0													
3	1	1	1	0											
5	3	2	2	2	1	0									
7	7	7	8	6	3	3	1	0							
9	22	20	24	24	15	12	4	4	1	0					
11	66	66	85	80	55	55	24	20	5	5	1	0			
13	217	212	286	286	219	210	104	104	35	30	6	6	1	0	
15	715	715	1008	994	819	819	462	448	175	175	48	43	7	7	1

can use Theorem 4.1.8 and Corollary 4.1.9 to construct at least some of the maps between the indecomposable OTL_n modules. For example figure 4.10 shows some of the maps between OTL_n-modules when $\delta = \sqrt{2}$, where the line colours in the diagram have the same meanings as those in figure 4.7.

4.3 The blocks of the Outer-Temperley-Lieb algebra

We conclude in this section by making a conjecture about the labelling set of the blocks of the Outer-Temperley-Lieb algebra when is it not semi-simple. Under the assumption that Conjecture 4.2.1 is true we can use the maps constructed in section 4.1 to assist in determining the blocks of $OTL_n(\delta)$ for n even. We shall therefore throughout this section assume that n is even, $\delta \neq 0$ and Conjecture 4.2.1 is true.

Denote by Δ the labelling set of the standard modules of TL_n. Under these assumptions, we can deduce which simple modules sit in a block of their own. Recall that a critical TL_n-module, as defined in sections 1.6 and 1.7, is simple and is not a composition factor of any other standard TL_n-module. It then follows that if $C_{n,p}$ is a critical TL_n-module, the OTL_n-modules $C^+_{n,p}$, $C^-_{n,p}$ are simple OTL_n-modules for if one of $C^+_{n,p}$, $C^-_{n,p}$ were not simple, it would have a Gram matrix with determinant zero with respect to the forms $[-,-]^{\pm}$ which by Corollary 2.6.5 would mean $C_{n,p}$ has a zero determinant Gram matrix with respect to the form $[-,-]$, a contradiction. Furthermore, these simple OTL_n-modules cannot be composition factors of any other standard, reducible, OTL_n-module $C^{\pm}_{n,s}$, since this would require there to exist a map $C^{\pm}_{n,p} \longrightarrow C^{\pm}_{n,s}$ which by Conjecture 4.2.1 implies that there must exist a map $C_{n,p} \longrightarrow C_{n,s}$ which is a contradiction. Thus since simple module of an algebra can only belong to exactly one of its blocks we can make the following conjecture.

Conjecture 4.3.1. *For any $p \in \Delta$ such that $C_{n,p}$ is a critical TL_n-standard modules, the labels $+p$ and $-p$ of the standard OTL_n-modules $C^+_{n,p}$ and $C^-_{n,p}$ respectively, label a unique block of OTL_n containing only $C^+_{n,p}$ and $C^-_{n,p}$ respectively.*

To prove this only requires that Conjecture 4.2.1 is true. We can use this to give the following conjecture describing all the blocks of OTL_n. Some comments on our reasoning for this will be given along with an example after the statement.

Conjecture 4.3.2. *For n even the labels of the blocks of OTL_n are given by the set*

$$\bigcup_{\substack{k\in\mathbb{Z}^+\setminus\{0\}\\ \frac{k}{2}l-1\,\in\Delta}} \left\{+\left(\frac{k}{2}l-1\right), -\left(\frac{k}{2}l-1\right)\right\}, \tag{4.11}$$

along with, for each $p \in \Delta$ such that $2(p+1) < l$, the sets

$$\begin{aligned}\{+p\} \cup \bigcup_{\substack{k\in\mathbb{Z}^+\setminus 2\mathbb{Z}^+\\ kl-p-2\,\in\Delta}} \{+(kl-p-2)\} \cup \bigcup_{\substack{k\in\mathbb{Z}^+\setminus 2\mathbb{Z}^+\\ kl+p\,\in\Delta}} \{-(kl+p)\} \\ \cup \bigcup_{\substack{k\in2\mathbb{Z}^+\\ kl-p-2\,\in\Delta}} \{-(kl-p-2)\} \cup \bigcup_{\substack{k\in2\mathbb{Z}^+\\ kl+p\,\in\Delta}} \{+(kl+p)\},\end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \{-p\} \cup \bigcup_{\substack{k\in\mathbb{Z}^+\setminus 2\mathbb{Z}^+\\ kl-p-2\,\in\Delta}} \{-(kl-p-2)\} \cup \bigcup_{\substack{k\in\mathbb{Z}^+\setminus 2\mathbb{Z}^+\\ kl+p\,\in\Delta}} \{+(kl+p)\} \\ \cup \bigcup_{\substack{k\in 2\mathbb{Z}^+\\ kl-p-2\,\in\Delta}} \{+(kl-p-2)\} \cup \bigcup_{\substack{k\in 2\mathbb{Z}^+\\ kl+p\,\in\Delta}} \{-(kl+p)\}\,. \end{aligned} \tag{4.13}$$

We can draw a weight diagram to represent the blocks of OTL_n as we did in figure 1.8 for $TL_n(\sqrt{2})$. Figure 4.11 represents the blocks of $OTL_n(\sqrt{2})$ for n even deduced from the Bratelli diagram 4.7. In this diagram we choose to draw the integer line for the negative indices below the positive one. The symmetry between the positive and negative lines follows from Corollary 4.1.9 which is why in the above conjecture we require both sets in equations (4.12) and (4.13). The alternating nature of the labels within the blocks involving the non-critical modules appears to follow from Theorem 4.1.8, although this is not entirely obvious. For our $\delta = 1$ and $\delta = \sqrt{3}$ case we can draw similar diagrams, see figures 4.12 and 4.13, using the Bratelli diagrams in figures 4.8 and 4.9 respectively. These also have the alternating positive, negative, positive, negative structure for the labels in the blocks involving the non-critical modules which corresponds with the way we have defined the sets labelling the blocks in equations (4.12) and (4.13).

We end by noting a way in which we can use the blocks of TL_n to help in writing down the blocks of OTL_n. Denote by $X = \{a_1, a_2, \dots, a_n\}$ the set of primitive central, orthogonal idempotents of $TL_n(\delta)$. Then as described in section 1.7 the blocks of TL_n are given by $B_i = TL_n a_i$.

Lemma 4.3.3. *Each $a_i \in X$ satisfies $\sigma(a_i) = a_i$, that is $a_i \in OTL_n$ for all i.*

Proof. Note that each block of TL_n contains a unique simple $S_{n,p}$ indexed by the number of propagating lines $p \in \Delta$. Since σ, as defined, is an involutive automorphism it does not change the number of propagating lines and we must have $\sigma(S_{n,p}) = S_{n,p}$.

Let $a_i \in X$, then it is clear that $\sigma(a_i)$ is also a primitive central idempotent of TL_n since σ is an automorphism, and that the set $\sigma(X) = \{\sigma(a_1), \sigma(a_2), \dots, \sigma(a_n)\}$ is a set of primitive central orthogonal idempotents of TL_n. Since the blocks are unique, up to isomorphism, there must exist a j such that each $\sigma(a_i) = a_j \in X$. However, since $\sigma(S_{n,p}) = S_{n,p}$ and each block contains a unique simple module we must have $B_i = \sigma(B_i)$, hence $\sigma(a_i) = a_i$ as required. □

Thus each $a_i \in X$ is now a central orthogonal idempotent of OTL_n, however it is not necessarily primitive. We can however write $a_i = b_{i_1} + b_{i_2} + \cdots$ where each b_{i_j} is a primitive central orthogonal idempotent of OTL_n. The blocks of OTL_n can thus be written as,

$$OTL_n b_{1_1} \oplus OTL_n b_{1_2} \oplus \cdots \oplus OTL_n b_{2_1} \oplus OTL_n b_{2_2} \oplus \cdots \oplus OTL_n b_{n_1} \oplus OTL_n b_{n_2} \oplus \cdots$$

Note that this construction shows that the blocks of TL_n are unions of the blocks of OTL_n. We would expect by the homomorphisms between standard modules which we are able to determine that the idempotents a_i split into at least two parts. Assuming true conjuncture 4.2.1 we would expect a_i to split into exactly two parts in the case where $C_{n,i}$ is a critical module by Conjecture 4.3.1.

Figure 4.11: Weight diagram representing the blocks of $OTL_n(\sqrt{2})$.

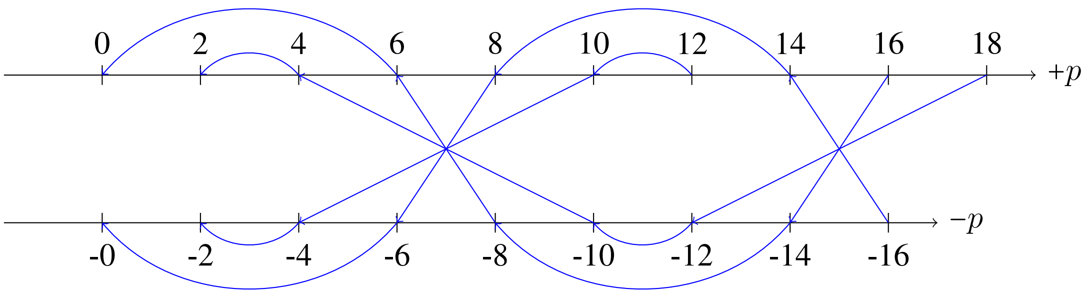


Figure 4.12: Weight diagram representing the blocks of $OTL_n(1)$.

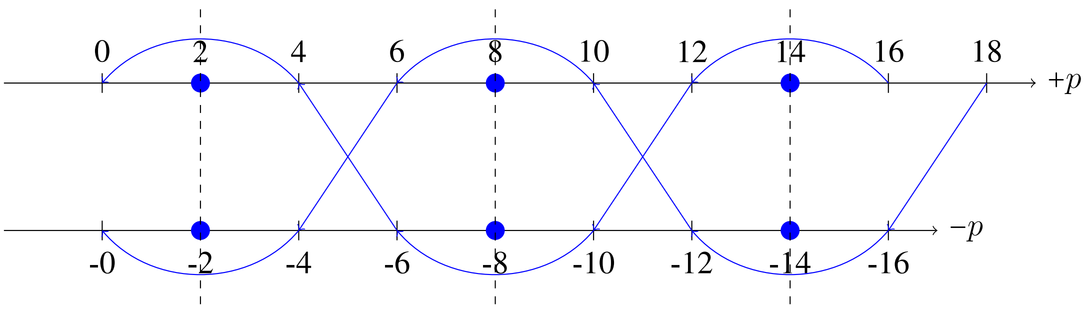
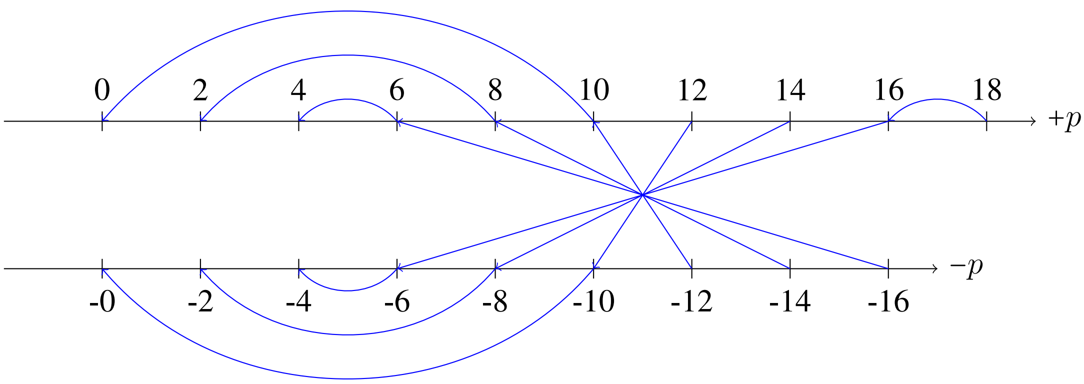


Figure 4.13: Weight diagram representing the blocks of $OTL_n(\sqrt{3})$.



4.4 Summary and Applications

We have defined the Outer-Temperley-Lieb algebra, $OTL_n(\delta)$, as a fixed ring of the Temperley-Lieb algebra with respect to an automorphism reflecting diagrammatic representations of the Temperley-Lieb algebra in the vertical line. We have constructed a diagram calculus for the Outer-Temperley-Lieb algebra using that of the Temperley-Lieb algebra. We have shown that we can use the cell modules of the Temperley-Lieb algebra to construct modules of the $OTL_n(\delta)$ which themselves turn out to be cell modules for the Outer-Temperley-Lieb algebra, that is $OTL_n(\delta)$ is also a cellular algebra. This was followed by using a relation between cellular and quasi-hereditary algebras to show that the Outer-Temperley-Lieb algebra is also a quasi-hereditary algebra for all but a few choices of the parameter δ.

One of the main results is Theorem 2.6.7 which states that over a field not of characteristic two, the Outer-Temperley-Lieb algebra is semi-simple if and only if the Temperley-Lieb algebra is semi-simple. This is proven by showing, for the right choice of bilinear form, that the Gram matrices for cell modules of the Temperley-Lieb algebra can be constructed from those of the Outer-Temperley-Lieb algebra, then applying known results relating such Gram matrices to irreducibility of a cell module. Two more major results, Theorem 3.1.15 and Corollary 3.3.3 are proven in chapter 3 which state the restriction and induction rules for the for the cell modules of the $OTL_n(\delta)$.

The final section begins the study of representation theory of the Outer-Temperley-Lieb algebra when it is not semi-simple, that is when q is a root of unity. While restricting to cases where OTL_n is quasi-hereditary, for n even we have shown how we can construct homomorphisms between certain indecomposable modules of the Outer-Temperley-Lieb algebra. This was used as a basis to state Conjecture 4.2.1 describing the complete set of homomorphisms between indecomposable modules of the Outer-Temperley-Lieb algebra in the non semi-simple case. For n even we have used this to make two further Conjectures, 4.3.1 and 4.3.2, both describing the structure of the blocks of the Outer-Temperley-Lieb algebra in the non semi-simple cases. We have also discussed the complexities that arise when trying to approach the n odd case in a similar way.

Although it is not covered here, the work in this thesis can be continued further by first attempting to prove the conjectures made in the previous sections for the blocks of the Outer-Temperley-Lieb algebra in the n even case. For the n odd there is further work to be done, as described in section 4.2, to find an idempotent E of OTL_n such that $EOTL_nE$ is isomorphic to OTL_{n-2}. Once this is found it can be used to construct the block of OTL_n for n odd.

Some additional questions to consider would be what happens to the construction of the blocks of the Outer-Temperley-Lieb algebra in the cases where it is not a quasi-hereditary algebra. We can also ask how the representation theory of the Outer-Temperley-Lieb algebra alters when we consider it as an algebra over a ring as opposed to a field. Given that the Outer-Temperley-Lieb algebra was defined as a fixed ring of the Temperlery-Lieb algebra we could also investigate the fixed rings of some of the generalisations of the Temperley-Lieb algebra, for example the affine Temperley-Lieb algebra.

The parent algebra of $OTL_n(\delta)$, the Temperley-Lieb is known to have applications in the field of statistical mechanics as described in the introduction. Unfortunately, the Outer-Temperley-Lieb algebra, does not appear to have any obvious applications in this area. There is however a possible application in the field of quantum computing. The Temperley-Lieb algebra is known to have links in this area, see for example [31, Chapter 1]. The basic idea stems from the notion that the diagrammatic representations of Temperley-Lieb algebra can be used to encode data for transmission, if an error occurs in this transmission the diagram will be deformed into another Temperley-Lieb diagram. However, by replacing the Temperley-Lieb algebra with the Outer-Temperley-Lieb algebra the encoding is now given by a symmetric pairs of diagrams. This allows us to detect errors in our transmission since if an error occurs and causes a deformation in a diagram the pair will no longer be symmetric and therefore no longer a member of the Outer-Temperley-Lieb algebra.

Appendices

A The box number and useful identities

In this section we define the *box number* both directly and recursively. The box number is used in chapter 4 to give a concrete construction of homomorphisms between indecomposable $TL_n(\delta)$ modules.

Definition A.1. Let k be a ring with $\operatorname{char} k \neq 2$. Let $q = e^{i\pi/r} \in k$ with r some positive integer. For $n \in \mathbb{N}$ the n^{th} *box number*, $[n]$, sometimes referred to as the *Gaussian coefficient* or q-number, is defined by the formula

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{A.1}$$

Note that $[0] = 0$ and $[1] = 1$.

Lemma A.2. *Let $\delta = q + q^{-1}$, then $\delta = [2]$ and we may define the box number using the recursion,*

$$[0] = 0, \quad [1] = 1, \quad [n+1] = \delta[n] - [n-1]. \tag{A.2}$$

Proof. We may write $[n+1]$ as,

$$\begin{aligned}[n+1] &= \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} = \frac{qq^n - q^{-1}q^{-n}}{q - q^{-1}} \\ &= \frac{(q + q^{-1})(q^n - q^{-n}) - q^{n-1} + q^{-(n-1)}}{q - q^{-1}} \\ &= \delta[n] - [n-1],\end{aligned}$$

as $\delta = q + q^{-1}$. $\square$

Lemma A.3. *Let* $q \neq \pm 1$. *Suppose* $r \in \mathbb{N}$ *such that* $[r+1] = 0$. *Then* $[r] = \pm 1$ *and for any* $a \leq r$, $[r-a] = \pm [a+1]$.

Proof. Since $[r+1] = 0$ we must have $q^{r+1} - q^{-(r+1)} = 0$, that is $q^{2(r+1)} = 1$. Therefore $q^{r+1} = \pm 1$ and we deduce

$$q^r = \pm q^{-1},$$
$$[r] = \frac{q^r - q^{-r}}{q - q^{-1}} = \frac{\pm q^{-1} \mp q^1}{q - q^{-1}} = \mp 1.$$

Now for any $a \leq r \in \mathbb{N}$ we have

$$[r-a] = \frac{q^r q^{-a} - q^{-r} q^a}{q - q^{-1}} = \frac{\pm q^{-(a+1)} \mp q^{a+1}}{q - q^{-1}} = \mp [a+1].$$

$\square$

B Gram matrices and bilinear forms

In this section we shall review some of the theory of *bilinear forms* and their related *Gram matrices*. Our main objective is to give enough background to show that for certain types of a finite dimensional algebra A over a field k, if M is a left A-module then the non-degeneracy of the Gram matrix of an appropriate bilinear form $\langle -,- \rangle : M \times M \longrightarrow k$ is enough to determine whether the module is simple or not. We note that this is not true for an algebra in general.

Definition B.1. Let k be a field and A a finite dimensional algebra over k. For any left A-module M we can define a *bilinear form* on M by a mapping $\langle -,- \rangle : M \times M \longrightarrow k$ which satisfies

$$\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$$
$$\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$$
$$\langle au, v \rangle = \langle u, av \rangle = a\langle u, v \rangle$$

for all $u, v, w \in M$, $a \in k$.

Definition B.2. Let $\{m_1, m_2, \ldots m_r\}$ be an ordered basis of M. For any such form we can associate a matrix $G(M)$ with entries given by,

$$G(M)_{i,j} = \langle m_i, m_j \rangle. \tag{B.1}$$

Such a matrix is known as the *Gram matrix* of M.

Definition B.3. Let t be an involutive anti-automorphism on A. A bilinear form as defined in definition B.1 is said to be *contravariant* (or *invariant*) if,

$$\langle am, m' \rangle = \langle m, a^t m' \rangle, \tag{B.2}$$

for all $a \in A$, $m, m' \in M$.

We denote by $R_{\langle -,- \rangle}$ the subset of M containing each $m \in M$ such that $\langle m, m' \rangle = 0$ for all $m' \in M$. More commonly, $R_{\langle -,- \rangle}$ is known as the *radical*, of M with respect to the bilinear form $\langle -,- \rangle$.

Lemma B.4. *Let $S = R_{\langle -,- \rangle}$. Then S is an A-submodule of M.*

Proof. For $s \in S$ we have $\langle as, m' \rangle = \langle s, a^t m' \rangle = 0$, as $a^t m' \in M$. Thus $as \in S$ as required. □

Corollary B.5. *Let A have involutive anti-automorphism t and let M be a left A-module. Suppose we have a bilinear form $\langle -,- \rangle : M \times M \to k$ which is contravariant with respect to t. Then if the Gram matrix $G(M)$ with respect to $\langle -,- \rangle$ has zero determinant, M is not irreducible.*

Proof. The determinant $\det G(M) \neq 0$ if and only if the rank of $\langle -,- \rangle$ is $\dim M$. Since rank$\langle -,- \rangle = \dim M - \dim R_{\langle -,- \rangle}$ we have $\dim R_{\langle -,- \rangle} = 0$ if and only if $\det G(M) \neq 0$. Thus when $\det G(M) = 0$, the radical of M is non-zero and hence it is a non-zero submodule of M. Thus M is not irreducible as required. □

The converse of the above corollary is not true in general, that is having a Gram matrix of an A-module M, with respect to a contravariant form, with non-zero determinant does not always imply that M is irreducible. The problem here is that the chosen form may not necessarily be unique. We shall now show that when we restrict the module to one of certain structure, there can only be one unique contravariant form. We first require some results regarding dualities.

Lemma B.6. *Let t be an involutive anti-automorphism on A. Any left A-module M can be considered as a right A-module with action $ma := a^t m$. Furthermore let A^{op} be the opposite algebra of A, that is if $a, b \in A^{op}$ and A^{op} has multiplication $\times$ then $a \times b = ba \in A^{op}$. Then any left A-module can be considered as a right A^{op}-module with action $ma := am$.*

Proof. For the first part, for any $a, b \in A^{op}, m \in M$ we have,

$$(ma)b = b^t(ma) = b^t a^t m = (ab)^t m = m(ab).$$

For the second part, for any $a, b \in A^{op}, m \in M$,

$$(ma)b = b(ma) = b(am) = (ba)m = m(a \times b),$$

as required. □

It follows from the second part of the above lemma that left A-module homomorphisms become become right A^{op}-module homomorphisms, hence we have a (covariant) functor F from the category of left A-modules to the category of right A^{op}-modules. We can similarly define a functor F' from the category of right A-modules to the category of left A^{op}-modules and likewise a functor F'' from the category of left A^{op}-modules to the category of right A-modules.

Definition B.7. For M a left A-module, we denote by $M^* := \mathrm{Hom}_A(M, k)$, the usual dual right A-module of M, see for example [7, Definition 43.5].

Definition B.8. Let t be an involutive anti-automorphism on A. By considering M^* as a left A^{op}-module we can define a right A-module $M^\circ := F''(M^*)$. Note by the construction of F'', M° is a right A-module and hence can be considered as a left A-module using the involution t. The map $M \mapsto M^\circ$ is sometimes called a *contravariant duality*.

Lemma B.9. *Assume that the A-module M has a simple head L, which only appears once as a composition factor of M and that the dual of L, L° satisfies $L \cong L^\circ$. If the algebra A has a involutive anti-automorphism t and M' is a maximal proper submodule of the A-module M, so $L = M/M'$, then* $\dim \mathrm{Hom}_A(M, M^\circ) = 1$ *and the rank of* $f \in \mathrm{Hom}_A(M, M^\circ)$ *is equal to* $\dim L$.

Proof. We first note there exists a map $f \in \mathrm{Hom}_A(M, M^\circ)$ which kills M'. Suppose there exists $\theta \in \mathrm{Hom}_A(M, M^\circ)$ with $\ker\theta \neq M'$. Then for some $C \neq 0$ there exists a short exact sequence,

$$0 \longrightarrow C \longrightarrow \operatorname{im}\theta \longrightarrow L \longrightarrow 0.$$

Since C is a sub-module of $\operatorname{im}\theta$ it is a sub-module of M°. Thus C° is a quotient module of M. However, as L is the simple head of M it must be a composition factor of C°, and thus a composition factor of C by the duality. This contradicts the requirement that L only appears once as a composition factor in M, hence no such map θ can exist. □

Proposition B.10. *Let A have involutive anti-automorphism t. For M any left A module, we can associate an element $g \in \operatorname{Hom}_A(M, M^{\circ})$ for each contravariant form $\langle -,- \rangle : M \times M \to k$, by $g(m)(m') = \langle m, m' \rangle$, $m, m' \in M$. This gives a bijective correspondence between contravariant forms and morphisms. Furthermore if M satisfies the same conditions as in lemma B.9 then the form $\langle -,- \rangle : M \times M \to k$ is unique up to scalars and non-singular if and only if the associated map g is a isomorphism.*

Proof. We show g is indeed a left A-module homomorphism. For $m, m' \in M$ we have $g(m) \in \operatorname{Hom}_k(M, k)$, thus

$$g(am)(m') = \langle am, m' \rangle = \langle m, a^t m' \rangle = g(m)(a^t m') = (g(m)a^t)(m') = (ag(m))(m'),$$

since the action of a on the left of $g(m)$ is given by the action of a^t on the right. As for the bijective correspondence, for any $g \in \operatorname{Hom}_A(M, M^{\circ})$ we may define a form via $\langle m_1, m_2 \rangle = g(m_1)(m_2)$. The final statement follows from noting that the rank of the image under $g \in \operatorname{Hom}_A(M, M^{\circ})$ is the rank of $\langle -,- \rangle$. □

Corollary B.11. *Let A have involutive anti-automorphism t and let M be a left A-module satisfying the same conditions as in lemma B.9. Then if the Gram matrix $G(M)$ with respect to the unique form $\langle -,- \rangle : M \times M \to k$ has zero non-determinant, M is irreducible.*

Proof. By the above theorem, the form $\langle -,- \rangle$ on M is unique. Thus M has a unique radical $R = R_{\langle -,- \rangle}$ and a unique Gram matrix $G(M)$. The determinant $\det G(M) \neq 0$ if and only if the rank of $\langle -, - \rangle$ is $\dim M$. Since the rank$\langle -, - \rangle = \dim M - \dim R$ we have $\dim R = 0$ if and only if $\det G(M) \neq 0$, that is the unique radical of M is zero. It follows that M is irreducible as it can have no possible sub-module. □

It remains to ask if we know of any type of module which satisfies the criteria of lemma B.9.

Lemma B.12. *Let A be a finite dimensional algebra over a field k. Suppose that A is a cellular algebra as in definition 1.3.7 with respect to a involutive anti-automorphism t. For any cell module M of A, if L is the head of M then L only appears once as a composition factor of M and $L \cong L^{\circ}$, that is to say M has a simple preserving duality.*

Proof. It follows by the definition of a cellular algebra that L only appears once as a composition factor of M. The second statement is given as an exercise in [24, Chapter 2, Exercise 7, iii)]. □

C Some basic homological algebra

In this section we shall recall some well known ideas of homological algebra. These are mostly used in chapter 3 to help prove the induction and restriction rules of the Outer-Temperley-Lieb algebra. Most of the statements we shall give can be found and are proven in [28] but they can be found in many homological algebra texts, see for example [3] or [1, Appendix A]. In all that follows we shall assume k to be a field and A a finite dimensional k-algebra.

Definition C.1. [28, Chapter 2]. Let M, M' be A-modules. The map $f : M \longrightarrow M'$ is said to be *monic* or a *monomorphism* if it is one-one. It is said to be *epic* or a *epimorphism* if it is onto.

Definition C.2. [28, Chapter 2]. Let M, M', M'' be A-modules. Then a pair of maps f, g such that,

$$M' \xrightarrow{f} M \xrightarrow{g} M'',$$

are said to be exact at M if $\operatorname{im} f = \ker g$. If we have a, possibly infinite, sequence of maps,

$$\cdots \longrightarrow M_{n-1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \cdots,$$

then if each pair of adjacent maps is exact the sequence itself is said to be exact.

Definition C.3. [28, Chapter 2] For A-modules M, M', M'' and maps f, g, if the sequence,

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0, \tag{C.1}$$

is exact then it is called a short exact sequence. Such a short exact sequence is said to be *split* if there exists a map $v : M'' \longrightarrow M$ such that gv is equal to the identity map on in M''.

Lemma C.4. *[28, Theorem 2.7, Exercise 2.22]. If the short exact sequence (C.1) is split, M' and M'' are summands of M.*

Definition C.5. [28, Chapter 3]. A module P is called a projective module if given the diagram,

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "v"'] & \\ M \arrow[r, "g"'] & M' \arrow[r] & 0 \end{tikzcd},$$

such that $g : M \longrightarrow M'$ is an epimorphism, for any map $f : P \longrightarrow M'$ there exists a map $v : P \longrightarrow M$ with $f = gv$.

Definition C.6. [28, Chapter 3]. A module E is called an injective module if given the diagram,

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & M \arrow[u, "g"] \arrow[r, "f"'] & M' \arrow[ul, dashed, "u"'] \end{tikzcd},$$

such that $f : M \longrightarrow M'$ is an monomorphism, for any map $g : M \longrightarrow E$ there exists a map $u : M' \longrightarrow E$ with $g = uf$.

Definition C.7. [28, Chapter 3]. Let M be an A module. An injective resolution of M is an exact sequence,

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_n \longrightarrow E_{n+1} \longrightarrow \cdots,$$

such that each E_i is injective. Similarly we can define a projective resolution of M to be the exact sequence,

$$0 \longrightarrow M \longrightarrow P_0 \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_n \longrightarrow P_{n+1} \longrightarrow \cdots,$$

such that each P_i is projective.

For any two A modules M, M' we denote by $\mathrm{Hom}_A(M, M')$ the set of all maps $M \longrightarrow M'$.

There is a lot that can and has been said about this topic but we shall not repeat it all here as it is not required for this thesis. We shall simply conclude this section by defining and stating a theorem about the usual m^{th} extension functor, Ext_A^m, which we shall use to help complete the proof of theorem 3.1.15

Definition C.8. [1, Appendix A.4] Denote by $\mathrm{Mod}\,A$ the category of all right A-modules. A *cochain complex* in $\mathrm{Mod}\,A$ is a sequence,

$$C^\bullet: \quad 0 \xrightarrow{d^0} C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} C^{n-1} \xrightarrow{d^n} C^n \xrightarrow{d^{n+1}} C^{n+1} \xrightarrow{d^{n+2}} \cdots,$$

of A-modules connected by homomorphisms such that $d^{n+1}d^n = 0$ for all $n \in \mathbb{N}$. The n^{th} cohomology A-module of $C^\bullet$ is the quotient module,

$$H^n(C^\bullet) = \frac{\ker d^{n+1}}{\operatorname{im} d^n}.$$

Definition C.9. [1, Appendix A.4] We define the m^{th} extension bifunctor,

$$\mathrm{Ext}_A^m : (\mathrm{Mod}\,A)^{op} \times \mathrm{Mod}\,A \longrightarrow \mathrm{Mod}\,K,$$

for any two modules $M, N \in \mathrm{Mod}\,A$ by taking a projective resolution $P_\bullet$ of M and constructing the induced cochain complex,

$$\begin{aligned} \mathrm{Hom}_A(P_\bullet, N) : 0 \longrightarrow \mathrm{Hom}_A(P_0, N) &\xrightarrow{\mathrm{Hom}_A(h_1, N)} \mathrm{Hom}_A(P_1, N) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Hom}_A(P_m, N) &\xrightarrow{\mathrm{Hom}_A(h_{m+1}, N)} \mathrm{Hom}_A(P_{m+1}, N) \longrightarrow \cdots, \end{aligned}$$

of k-vector spaces. Then Ext_A^m is the m^{th} cohomology,

$$H^m(\mathrm{Hom}_A(P_\bullet, N)) = \ker \mathrm{Hom}_A(h_{m+1}, N) / \operatorname{im} \mathrm{Hom}_A(h_m, N).$$

Definition C.10. [28, Chapter 7]. For two A-modules M', M'' an extension of M' by M'' is a short exact sequence,

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

Theorem C.11. *[28, Theorem 7.11]. If* $\mathrm{Ext}^1_A(M'', M') = 0$ *then every extension of* M' *by* M'' *is split.*

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