On congruences of modular forms over imaginary quadratic fields

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Abstract

In this thesis, we consider the method of Harris-Soudry-Taylor et al. of attaching 2-dimensional l-adic Galois representations to cuspidal automorphic representations for $GL(2)$ over imaginary quadratic fields. In this process, one lifts an automorphic representation for $GL(2)$ over an imaginary quadratic field to an automorphic representation for $GSp(4)$ over the rationals; the latter automorphic representation has associated a 4-dimensional l-adic Galois representation, which turns out to be induced from a 2-dimensional representation of the absolute Galois group over the imaginary quadratic field. We aim in using this method to transfer level lowering results for $GSp(4)$ over the rationals to level lowering results for $GL(2)$ over an imaginary quadratic field.

Firstly, we study in detail the conductors of irreducible admissible non-supercuspidal and non-generic supercuspidal representations of $GSp(4)$ over a non-archimedean local field, and we obtain a result in the sense of Carayol and Livné on how the conductors degenerate modulo a prime number. In particular, when we have a corresponding mod l Galois representation and an l-adic lift of it, we list all the cases where the conductors differ.

Having this in our machinery, together with an explicit local theta correspondence between irreducible admissible representations of $GL(2, L)$ and irreducible admissible representations of $GSp(4, F)$ (here L is either a degree 2 field extension over F, or L is isomorphic to $F \times F$, with F a non-archimedean local field), we obtain a conditional result on level lowering for automorphic representations of $GL(2)$ over an imaginary quadratic field of prime discriminant. The result is conditional in the sense that we assume a level lowering result for representations of $GSp(4)$ over the rationals.

Finally, we prove a level lowering result by twisting particular automorphic representations over imaginary quadratic fields by grössencharacters.

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Chapter 1

Introduction

If π is a cuspidal automorphic representation for $GL(2)$ defined over $\mathbb{Q}$, Deligne in [12] and Deligne-Serre in [13] proved, using the theory of Eichler and Shimura, that one can attach 2-dimensional l-adic Galois representations to π. More generally, for F a totally real field, the analogous result for cuspidal automorphic representations of $GL(2)$ defined over F, was proved by Carayol, Taylor, and Jarvis, respectively in [6], [63], [33]. The problem of attaching an l-adic Galois representation to a cuspidal automorphic representation defined over an imaginary quadratic field K has been considered and solved mainly by Taylor in [65], Harris-Soudry-Taylor in [29], Berger-Harcos in [3], and Mok in [40]. The method they followed, which will be discussed in detail in this thesis, goes through automorphic representations for $GSp(4)$ over $\mathbb{Q}$ and the Galois representations attached to them (due to Weissauer [68] and Taylor [64]); the idea is to lift an automorphic representation π for $GL(2)$ defined over K to an automorphic representation for $GSp(4)$ over $\mathbb{Q}$, via the theory of the theta correspondence. We remark that this process requires the central character of π to be Galois invariant; nevertheless, in 2013, during working on this thesis, Harris-Lan-Taylor-Thorne in [28] and Scholze in [54] were able to associate l-adic Galois representations to cuspidal automorphic representations defined over imaginary quadratic fields (in fact over CM fields) without this assumption on the central character.

The initial goal of this thesis was to use known results of level lowering/raising congruences for modular Galois representations with image in $GSp(4,\bar{\mathbb{Q}}_l)$ and descend them to congruences for Galois representations attached to automorphic representations over an imaginary quadratic field. In particular, starting with a cuspidal automorphic representation π for $GL(2)$ over an imaginary quadratic field K, we construct the global theta lift Π to $GSp(4)$ over $\mathbb{Q}$. Write R for the l-adic Galois representation associated to Π, which turns out to be induced by a Galois representation ρ of the absolute Galois group of K; the latter Galois representation is the one attached to π. By knowing a congruence between R and some l-adic representation R', one may ask whether we can descend it back to a congruence between ρ and some ρ', where ρ' is the l-adic Galois representation associated to some automorphic representation π' for $GL(2)$ over the imaginary quadratic field K.

This idea faced some difficulties, and as a result we may only use it under certain hypotheses. More precisely, there is a theorem of Gee and Geraghty (Theorem 7.6.6 of [22]) that allows one to lift an automorphic mod l Galois representation with image in $GSp(4,\bar{\mathbb{F}}_l)$ to an automorphic l-adic Galois representation of $GSp(4,\bar{\mathbb{Q}}_l)$ in a way in which one may choose the ramification of this lift at the bad places. Unfortunately, this theorem is not directly applicable in the case we consider due to technical reasons, though one expects these technicalities to be removed in the future; this is something that we discuss in the last chapter. Another obstacle is that, even if one has an automorphic lift arising from an automophic representation, say Π', of $GSp(4,\mathbb{A}_{\mathbb{Q}})$, it is hard to tell whether Π' arises as a theta lift from some automorphic representation of $GL(2,\mathbb{A}_K)$, where K is an imaginary quadratic field; for this, we use a criterion which involves studying the poles of the degree 5 standard L-function of Π'. Despite this fact, we have developed a method, depending on some hypotheses, which allows one to obtain level lowering over an imaginary quadratic field with discriminant divided by only one prime p with $p \equiv 3 \bmod l$. In particular, in Section 6.2, under some hypotheses, we prove the following

Proposition. *Let l be a prime, and consider the imaginary quadratic field*

$K = \mathbb{Q}(\sqrt{-p})$ with $p \equiv 3 \bmod 4$. Let $\Sigma = \{\mathfrak{p}, \mathfrak{q}, \bar{\mathfrak{q}}\}$ be the set of primes such that $p\mathcal{O}_K = \mathfrak{p}^2$ and $q\mathcal{O}_K = \mathfrak{q}\bar{\mathfrak{q}}$. Let $\pi = \bigotimes_w \pi_w$ be a regular algebraic cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$, of Galois invariant central character, such that π is unramified outside Σ, and for places in Σ we have

1. *$\pi_\mathfrak{p} = (\mu| |_\mathfrak{p}^{1/2})St_{GL(2)}$, where μ is a character which is not Galois invariant and it is ramified, that does not degenerate modulo l;*

2. *$\pi_\mathfrak{q}$ is a supercuspidal representation which degenerates modulo l, and $\pi_{\bar{\mathfrak{q}}} = \chi \times \chi^{-1}\omega_{\pi_\mathfrak{q}}$, where χ is a ramified non-degenerate character.*

Then there exists an automorphic Galois representation ρ' which is isomoprhic modulo l to the Galois representation ρ attached to π; moreover, the conductor of ρ' is lower than the conductor of ρ.

As we discuss in the same section, one can find more such examples of level lowering congruences over imaginary quadratic fields, by using the same method. Moreover, we prove level lowering results by twisting particular L-parameters with a character that extends to a grössencharacter. In particular, we have the following two results (which require no extra hypotheses)

Theorem 6.3.1. *Suppose we have a modular mod l Galois representation*

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_l),$$

i.e., it has a lift ρ which arises from a regular algebraic cuspidal automorphic representation π. Assume that the component $\pi_\mathfrak{p}$ of π, at a prime $\mathfrak{p}$ which lies above a rational prime p that stays inert in K with $p \neq l$, is one of the following types:

1. *it is a principal series representation $\pi_\mathfrak{p} = \mu \times \nu$, with μ tamely ramified with unramified reduction such that it factors through the norm map, and ν ramified;*

2. *it is a twisted Steinberg representation $\pi_\mathfrak{p} = (\mu| |^{1/2})St_{GL(2)}$, with μ a tamely ramified with unramified reduction such that it factors through the norm map.*

Then $\bar{\rho}$ is modular of level lower than the level of π.

Theorem 6.3.2. *Suppose we have a modular mod l Galois representation*

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_l),$$

i.e., it has a lift ρ which arises from a regular algebraic cuspidal automorphic representation π. Let $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ with $p \neq l$, such that for the components[1] *$\pi_{\mathfrak{p}}$ and $\pi_{\bar{\mathfrak{p}}}$ of π at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively, we have that $a(\pi_{\bar{\mathfrak{p}}}) > 1$ and that $\pi_{\mathfrak{p}}$ is one of the following types:*

1. *principal series representation $\mu \times \nu$, with μ tamely ramified with unramified reduction, and ν ramified;*
2. *twisted Steinberg representation $(\mu| \,|^{1/2})St_{GL(2)}$, with μ tamely ramified with unramified reduction.*

Then $\bar{\rho}$ is modular of lower level than the level of π.

In addition, in this thesis, we study in great detail the conductors of non-supercuspidal irreducible admissible representations of $GSp(4, F)$ over a non-archimedean local field F. In particular, in Subsection 4.2.2, by considering the conductor of a mod l Galois representation of $\mathrm{Gal}(\bar{F}/F)$, we compare it with the conductor of an l-adic lift of this representation (here l is different from the residual characteristic of F), and we get a list of all the possible degenerations of conductors for non-supercuspidal representations of $GSp(4, F)$; this result is towards listing the possible levels of a modular mod l Galois representation with image in $GSp(4, \bar{\mathbb{F}}_l)$. In addition, in Section 5.6, we consider the conductors of L-parameters of non-generic supercuspidal representations, we define the conductor on the automorphic side for such representations (and the generic supercuspidal representations that share the same L-packet), and we prove that they are equal.

This thesis starts with some general theory for the group $GSp(4)$, and some basic notions of representation theory; this is presented in Chapter 2.

[1] Note that $\pi_{\mathfrak{p}}$ and $\pi_{\bar{\mathfrak{p}}}$ have equal central characters in this situation.

Moreover, there is an explicit description of the non-supercuspidal irreducible admissible representations of $GSp(4, F)$, where F is a non-archimedean local field, following Roberts and Schmidt ([48]).

In Chapter 3, we define some Galois theoretic notions which will be useful in the rest of this thesis. We also discuss the local Langlands correspondence for the groups of our interest; namely, we give a list of the L-parameters corresponding to infinite dimensional irreducible admissible representations of $GL(2, F)$ and $GSp(4, F)$, where F is a non-archimedean local field. For these representations, we also present a list of the local degree 4 and 5 L-factors as in [48]. None of the results in this chapter is new.

In Chapter 4, we consider the notion of the conductor, which measures the ramification behaviour of a representation. After presenting a list (see Table 4.1) for the conductors of the L-parameters of non-supercuspidal irreducible admissible representations for $GSp(4, F)$ (although in [48], this list is given for representations with trivial central character, our list is for general central character), we compare them with the conductors of their mod l reductions. This provides a classification of the degeneration of conductors for $GSp(4)$, in the same fashion as Carayol and Livné did in 1989 for $GL(2)$, extending this result in our situation. This classification has not appeared in the literature, and can be found in Subsection 4.2.2.

In Chapter 5, we lift a cuspidal automorphic representation of $GL(2)$ defined over an imaginary quadratic field to a cuspidal automorphic representation of $GSp(4)$ over $\mathbb{Q}$ as mentioned above; conversely, we give a criterion for an automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ to be a non-zero theta lift from such a representation (this depends on the poles of the degree 5 L-function, and is due to Kudla-Rallis-Soudry). For this, we need the theory of the theta correspondence, and thus we discuss it in some detail; at the same time we provide an explicit description of the local theta correspondence between irreducible admissible representations of $GL(2)$ and irreducible admissible representations of $GSp(4)$. Furthermore, we describe how one attaches Galois representations to cuspidal automorphic representations for $GL(2)$ defined over imaginary quadratic fields. The theta correspondence also helps us define the conductors for some supercuspidal representations of $GSp(4)$, namely the non-generic ones (and evidently the generic ones that share the same L-packet), and prove that these conductors and the conductors of the associated L-parameters are equal. The discussion in Section 5.6 (particularly, Definition 5.6.3, and Theorems 5.6.4 and 5.6.5) is the only new outcome in this chapter.

In Chapter 6, we study the level of a modular mod l Galois representation of the absolute Galois group of an imaginary quadratic field. After writing down the possible levels that such a representation can have, we present a method for conducting level lowering over imaginary quadratic fields with prime discriminants, by using level lowering results for $GSp(4)$; we do this by assuming the existence of congruences between l-adic Galois representations with image in $GSp(4)$, and we discuss how far this hypothesis is from being proved. In addition to that, we prove the existence of some congruences over imaginary quadratic fields, by twisting with a grössencharacter. Finally, we present some examples which indicate that in the case of an imaginary quadratic field, we do not have all the congruences that one expects, as we do in the classical case or in the totally real case. The results in this chapter (in particular Section 6.2 and Theorems 6.3.1 and 6.3.2) are the main outcomes of this thesis.

Chapter 2

Representation theory of $GSp(4, F)$

In this chapter, we introduce the reader to the similitude symplectic group $GSp(4)$, over a non-archimedean local field F. Our goal is to describe the representation theory concerning that group. Firstly, we define the group $GSp(4, F)$ and some important subgroups for our theory. After that, we discuss some aspects of representation theory for a general reductive algebraic group. Finally, following [48], we list and classify all the irreducible admissible non-supercuspidal representations of $GSp(4, F)$.

2.1 Generalities

To begin, we will set up some notation. As mentioned, we are going to consider $GSp(4)$ over a non-archimedean local field F. Let $\mathcal{O}_F$ be its ring of integers, $\mathfrak{p}_F$ the unique maximal ideal of $\mathcal{O}_F$ and q the number of elements of $\mathcal{O}_F/\mathfrak{p}_F$; denote by p the characteristic of the residue field $\mathcal{O}_F/\mathfrak{p}_F$. We also fix a generator ϖ for the ideal $\mathfrak{p}_F$ (i.e. $\mathfrak{p}_F = \varpi\mathcal{O}_F$), and if $x \in F^\times$ we denote the normalized absolute value of x by $|x|$, normalized in the sense that $|\varpi| = q^{-1}$. That is

$$\mathcal{O}_F = \{a \in F : |a| \leq 1\},$$

$$\mathfrak{p}_F = \{a \in F : |a| < 1\},$$

and

$$\mathcal{O}_F^\times = \{a \in F : |a| = 1\}.$$

Let us now consider the group $GSp(4, F)$. We define $GSp(4, F)$ to be the group consisting of all matrices $g \in GL(4, F)$ such that

$${}^t g J g = \lambda(g) J$$

where $\lambda : GSp(4, F) \to F^\times$ is the *similitude character* of $GSp(4, F)$, and J is the matrix

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Note that when λ is the trivial character, we get the group $Sp(4, F)$.

Now we are going to describe some important subgroups of $GSp(4, F)$; for more information on this, the reader may consult Section 2.1 of [48]. A more general description of the theory can be found in [53].

2.1.1 The parabolic subgroups

The algebraic group $GSp(4)$ has three proper parabolic subgroups up to conjugacy; these are respectively the *Borel* parabolic, the *Klingen* parabolic, and the *Siegel* parabolic subgroup.

Let us consider first the (F-rational points of the) Borel parabolic subgroup, denoted by B. It consists of all upper triangular matrices in $GSp(4, F)$, which we may view as follows

$$B = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\},$$

where $*$ denotes an arbitrary element in F. One may write an element g in B in the following form

$$g = \begin{pmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

where $a, b, c \in F^\times$ and $x, \lambda, \mu, \kappa \in F$; this is the so-called *Levi decomposition*, which is a property of parabolic subgroups. They can be represented as a semi-direct product of the so-called *unipotent radical* U and the *Levi subgroup* M. From the above decomposition of an element $g \in B$, we see that the unipotent radical U consists of all matrices of the form $\begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix}$, and the Levi subgroup M consists of all diagonal matrices of the form $\begin{pmatrix} a & & & \\ & b & & \\ & & cb^{-1} & \\ & & & ca^{-1} \end{pmatrix}$. Finally, it is not difficult to see that for $g \in B$ we have that the similitude character is $\lambda(g) = c$.

Now we consider the (F-rational points of the) Klingen parabolic, denoted by Q. One defines the Klingen parabolic as the subgroup of $GSp(4, F)$

$$Q = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \right\}.$$

From the Levi decomposition we have that every element q of Q can be written as

$$q = \begin{pmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \Delta t^{-1} \end{pmatrix} \begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

where $\Delta = ad - bc \in F^\times, t \in F^\times$ and $\lambda, \mu, \kappa \in F$. The unipotent radical consists of all upper triangular matrices of the form $\begin{pmatrix} 1 & \lambda & \mu & \kappa \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix}$, and the Levi subgroup consists of elements of the form $\begin{pmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \Delta t^{-1} \end{pmatrix}$. By definition, one may compute that the similitude character of an element q of the Klingen parabolic subgroup is $\lambda(q) = \Delta$.

Finally we consider the (F-rational points of the) Siegel parabolic subgroup, denoted by P. It is defined as follows

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \right\},$$

and every element p of P can be written as

$$p = \begin{pmatrix} a & b & & \\ c & d & & \\ & & \lambda a/\Delta & -\lambda b/\Delta \\ & & -\lambda c/\Delta & \lambda d/\Delta \end{pmatrix} \begin{pmatrix} 1 & & \mu & \kappa \\ & 1 & x & \mu \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where $\Delta = ad - bc \in F^\times, \lambda \in F^\times$, and $x, \mu, \kappa \in F$. Note that the Levi subgroup is the group of elements of the form $\begin{pmatrix} a & b & & \\ c & d & & \\ & & \lambda a/\Delta & -\lambda b/\Delta \\ & & -\lambda c/\Delta & \lambda d/\Delta \end{pmatrix}$.

We may write an element of the Levi subgroup in a more compact notation; let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, F)$, and set

$$A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^tA^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

In this notation, an element of the Levi subgroup may be written as $\begin{pmatrix} A & \\ & \lambda A' \end{pmatrix}$. Finally, we have that $\lambda(p) = \lambda$ for $p \in P$.

Levi subgroups are important since later we are going to induce representations from the parabolic subgroups which are trivial on the unipotent radical, to get the non-supercuspidal admissible representations of $GSp(4, F)$.

2.2 Representation Theory

In this section we are going to talk about some important notions of the theory of admissible representations for reductive groups in general.

Let F be a non-archimedean local field and denote by G the group of F-rational points of a reductive algebraic group defined over F.

Definition 2.2.1. An *admissible* representation of G is a pair (π, V), where V is a complex vector space and

$$\pi : G \to \mathrm{Aut}(V)$$

is a group homomorphism, such that

1. every vector in V is fixed by an open compact subgroup K of G; these representations are called *smooth*;

2. for any open compact subgroup K, the space V^K of K-fixed vectors is finite dimensional.

An admissible representation (π, V) of G is called *irreducible* when there are no proper G-stable subspaces of V. An *irreducible constituent* (sometimes called *irreducible subquotient*) of an admissible representation (π, V) is an irreducible representation isomorphic to W/W', where $W' \subset W \subset V$ are G-fixed subspaces of V. The *length* of a representation will be the number of its composition factors in its Jordan-Hölder series; in this thesis, we will consider representations of finite length, which means that they will have a finite number of irreducible subrepresentations and irreducible subquotients.

A representation (π, V) of G is called *unitary*, when there exists a non-degenerate Hermitian form on V which is G-invariant. A *character* of G is a smooth 1-dimensional representation of G, i.e., a continuous homomorphism from G to $\mathbb{C}^\times$. If (π, V) is a smooth representation and the centre of G acts as a character on V, we call this character the *central character* of π. If such a character exists, we denote it by ω_π. Note that an irreducible admissible representation admits a central character; this is a consequence of Schur's lemma, Proposition 4.2.4 of [4]. Moreover, if (π, V) is a representation of G and χ is a character of G, we denote by $(\chi\pi, V)$ the representation with representation space V, defined via $\chi\pi(g) = \chi(g)\pi(g)$; we call this representation the *twist* of π by the character χ.

2.2.1 Induction from the parabolic subgroups

If G is a locally compact group[1], then there is on G a left (resp. right) translation-invariant Borel measure, unique up to constant multiple, which we denote by $\int_G \, dg$; this is called the left (resp. right) *Haar measure*. A *unimodular* G has equal right and left Haar measures.

Let G be unimodular. Let M and U be closed subgroups of G, such that M normalizes U, $M \cap U = 1$, $P = MU$ is closed in G, U is also unimodular,

[1] Note that the group of F-rational points of a reductive algebraic group defined over F is locally compact.

and $P\backslash G$ is compact. One can take for P a parabolic subgroup of G and $P = MU$ its Levi decomposition. Fix a Haar measure du on U. Now, for $p \in P$ let $\delta_P(p)$ be the positive number such that for all locally constant compactly supported complex valued functions f on U, we have

$$\int_U f(p^{-1}up)du = \delta_P(p) \int_U f(u)du.$$

The character $\delta_P : P \to \mathbb{C}^\times$ is called the *modular character* of P. This character can also be defined as the ratio of the left and right Haar measures on P; note that the parabolic subgroups P are the only important non-unimodular groups for the theory. Now we may introduce the normalized induction from P to G.

Let (σ, W) be an admissible representation of P trivial on U (i.e., a representation of M). We define $\mathrm{Ind}_P^G\sigma$ to be the representation of G with representation space the space of locally constant functions f on G with values in W, such that

$$f(pg) = \delta_P(p)^{1/2}\sigma(p)f(g),$$

where $p \in P$, and $g \in G$. Moreover, the action of G on this space is by right translation, i.e.,

$$(\mathrm{Ind}_P^G\sigma)(h)f(g) = f(gh),$$

for $h, g \in G$. The representation $\mathrm{Ind}_P^G\sigma$ is called the *normalized induction*[2] from P to G of the representation σ. As we will see in the following proposition the admissibility is transferred by normalized induction; the induction is normalized by the modular character in order to preserve also unitaricity of the representation σ (see Proposition 3.1.4 of [9]).

Proposition 2.2.2. *With notation as above, if (σ, W) is an admissible rep-*

[2]Note that $\mathrm{Ind}_P^G\delta_P^{-1/2}\sigma$ will be the unnormalized induction, which has representation space consisting of locally constant functions $f : G \to W$ such that

$$f(pg) = \sigma(p)f(g).$$

resentation of P, then $\mathrm{Ind}_P^G\sigma$ *is an admissible representation of* G.

Proof. Let K be an open compact subgroup of G, X a finite subset of G, and W_0 a finite dimensional subspace of W. Then the space

$$I(K, X, W_0) = \{f \in (\mathrm{Ind}_P^G\sigma)^K : f(X) \subset W_0, f \text{ has support in } PXK\}$$

is finite dimensional.

Assume that (σ, W) is admissible. As before, let K be an open compact subgroup of G, and let X be a finite subset of G such that $PXK = G$; we can do this since $P\backslash G$ is compact. Moreover, let $L = \bigcap_{x\in X} xKx^{-1}$ and take $W_0 = W^{L\cap P}$ which is finite dimensional since σ is admissible. Then, with these choices of X and W_0, we have $(\mathrm{Ind}_P^G\sigma)^K = I(K, X, W_0)$, which is finite dimensional, that is $\mathrm{Ind}_P^G\sigma$ is admissible. For more details, the reader may also consult [9], Theorem 2.4.1. □

One representation that we are going to use quite often in our theory, and is defined in terms of normalized induction, is the so-called Steinberg representation and is described below.

Definition 2.2.3. Let G be unimodular and let $P_\emptyset$ be its minimal parabolic subgroup[3]. For each parabolic subgroup P, define the representation π_P to be the representation $\mathrm{Ind}_P^G\delta_P^{-1/2}$. Note that π_G is the trivial representation contained in all other π_P. We define the *Steinberg representation* of G to be the representation

$$St_G = \pi_{P_\emptyset} \Big/ \sum_{P \neq P_\emptyset} \pi_P \,.$$

The Steinberg representation St_G is an irreducible admissible representation of G. The irreducibility follows from Theorem 8.1.3 of [9], and admissibility from Proposition 2.2.2 since the modular character is an admissible representation.

[3]As mentioned in the beginning of this section, G denotes the F-rational points of a reductive algebraic group defined over F; this is known to be locally compact.

Our next goal is to understand the notion "non-supercuspidal". In order to do so, we are going to discuss a form of Frobenius reciprocity; this will imply that irreducible admissible representations come in two classes, namely the non-supercuspidals and the supercuspidals.

Let P, M, U be subgroups of G as defined above, and let (π, V) be a smooth representation of G. Consider the subspace $V(U)$ of V, generated by the vectors $v - \pi(u)v$, for $v \in V$ and $u \in U$. The *normalized Jacquet module* $r_U(\pi)$ of π is the smooth representation of M defined by $r_U(\pi) = \pi_U \otimes \delta_P^{-1/2}$, where π_U is the representation with representation space $V_U = V/V(U)$ and such that $\pi_U(m)(v + V(U)) = \pi(m)v + V(U)$; that is, for $m \in M$ and $v \in V$, we have

$$(r_U(\pi))(m)(v + V(U)) = \delta_P^{-1/2}(m)\pi(m)v + V(U).$$

The form of Frobenius reciprocity that we want is

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G\sigma) \cong \mathrm{Hom}_M(r_U(\pi), \sigma),$$

where σ is a representation of P, trivial on U. This is implied by the following lemmata, which can be found in [9].

Lemma 2.2.4. *Let M be a closed subgroup of G, and (σ, W) a smooth representation of M. If (π, V) is a smooth representation of G, then we have an isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G\sigma) \cong \mathrm{Hom}_M(\pi, \delta_P^{1/2}\sigma).$$

Proof. Consider the map[4] $\Lambda : \mathrm{Ind}_P^G\sigma \to W$, defined via $f \mapsto f(1)$. If we have any morphism F in $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G\sigma)$ and we compose it with Λ, then we have a morphism $\Lambda \circ F \in \mathrm{Hom}_M(\pi, \delta_P^{1/2}\sigma)$; note that as $\delta_P^{1/2}\sigma$ is a twist of σ by $\delta_P^{1/2}$, it has as representation space the space W.

To define an inverse map, let $f : V \to W$ be in $\mathrm{Hom}_M(\pi, \delta_P^{1/2}\sigma)$. Define a morphism $\Phi : V \to \mathrm{Ind}_P^G\sigma$ via $v \mapsto \Phi_v$, where $\Phi_v(g) = f(\pi(g)v)$. Now we

[4] This map is a P-intertwiner according to Theorem 2.4.1(b) of [9].

have that $\Phi_v \in \mathrm{Ind}_P^G \sigma$ since π is smooth and for $p \in P$ and $g \in G$ we have

$$\Phi_v(pg) = f(\pi(pg)v) = \delta_P^{1/2}(p)\sigma(p)f(\pi(g)v) = \delta_P^{1/2}(p)\sigma(p)\Phi_v(g),$$

as $f \in \mathrm{Hom}_M(\pi, \delta_P^{1/2}\sigma)$. To see that the map $f \mapsto \Phi$ is the inverse of the previous map, one has to check that

$$\Lambda(\Phi_v) = \Phi_v(1) = f(\pi(1)v) = f(v).$$

□

Lemma 2.2.5. *If* $0 \to X \to V \to W \to 0$ *is an exact sequence of smooth* U*-representations, then the sequence* $0 \to X_U \to V_U \to W_U \to 0$ *is exact.*

Proof. This is straightforward. From the fact that for a space V, $V(U)$ is generated by the vectors $v - \pi(u)v$ for $u \in U$ and $v \in V$ (where π is the representation realized in V), we get that

$$X_U \to V_U \to W_U \to 0$$

is exact. The fact that $X_U \to V_U$ is injective follows from the equality $X(U) = X \cap V(U)$. □

Proposition 2.2.6. *If* (π, V) *is a smooth representation of* G *and* (σ, W) *is a smooth representation of* M*, we have*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_M(r_U(\pi), \sigma). \tag{2.1}$$

Proof. This is Theorem 3.2.4 of [9], but here we sketch a proof. Note that by Lemma 2.2.4, we have

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_M(\pi, \delta_P^{1/2}\sigma).$$

If one has an element $T \in \mathrm{Hom}_M(\pi, \delta_P^{1/2}\sigma)$ (i.e., a morphism $T : V \to W$), then by Lemma 2.2.5 one can get a morphism $T_U : V_U \to W$; note that $W_U = W$ since U acts trivially on W. So we have obtained an element

$T_U \in \mathrm{Hom}_M(r_U(\pi), \sigma)$. Conversely, a morphism in $\mathrm{Hom}_M(r_U(\pi), \sigma)$ can always be lifted to a morphism in $\mathrm{Hom}_M(\pi, \delta_P^{1/2}\sigma)$. This proves the result. □

Corollary 2.2.7. *If there exists a non-zero G-morphism from (π, V) to $\mathrm{Ind}_P^G\sigma$, then $V_U \neq 0$.*

Proof. This follows directly from Proposition 2.2.6. □

Equation (2.1) is the required form of Frobenius reciprocity that we need in order to define the two classes of irreducible admissible representations we mentioned before. Suppose that we have an irreducible admissible representation (π, V) of G such that there exists a G-morphism from (π, V) to $\mathrm{Ind}_P^G\sigma$ for some proper subgroup P as defined above; then we call π a *non-supercuspidal* representation of G. The irreducible admissible representation (π, V) will be called *supercuspidal* if and only if $V_U = 0$ for all unipotent radicals U as subgroups of all possible proper parabolics P. That is, supercuspidal representations do not arise as subrepresentations or subquotients of representations obtained by normalized induction from some proper parabolic subgroup. For more information on the notion of supercuspidal representations see Section 5 of [9].

One more remark about the Jacquet module is the fact that if (π, V) is an admissible (resp. smooth) representation of G then the normalized Jacquet module $r_U(\pi)$ is an admissible (resp. smooth) representation of M. This result is proved in Theorem 3.3.1 of [9].

2.2.2 Langlands' parameterization data

In this subsection, we will set up some data that parameterize the irreducible admissible representations of $GSp(4, F)$. We begin with some definitions.

If (π, V) is an irreducible admissible representation of a unimodular locally compact group G, a *matrix coefficient* of π is a function of the form

$$g \mapsto \langle \pi(g)v, \hat{v} \rangle,$$

where $g \in G$, $v \in V$, and $\hat{v} \in \hat{V}$. Here $\hat{V}$ denotes the contragredient representation of V and if $\hat{v} : V \to \mathbb{C}$ is a linear functional, we write $\langle v, \hat{v} \rangle$ for $\hat{v}(v)$, for $v \in V$.

We remind the reader the definition of the spaces $L^p(G)$, for $1 \leq p < \infty$. Such spaces consist of measurable functions $f : G \to \mathbb{C}$, such that

$$\int_G |f(g)|^p dg < \infty.$$

Definition 2.2.8. Let G be a unimodular locally compact group.

1. A representation of G is called *square integrable* if it has a basis whose matrix coefficients are in the space $L^2(G)$.

2. A representation of G is called *essentially square integrable* if it becomes square integrable modulo the center of G, after twisting with a suitable character of G.

An example of an essentially square integrable representation is the Steinberg representation which we discussed in Definition 2.2.3. This result can be found in Theorem 8.1.3 of [9]. A second example of essentially square integrable representations are the supercuspidal representations mentioned above. The supercuspidal representations of G are characterized by the fact that they have matrix coefficients with compact support modulo the center of G (see Theorem 5.2.1 of [9]). As a result, these representations are essentially square integrable; in addition, this property implies (see Corollary 5.2.3 of [9]) that if a supercuspidal representation has unitary central character, then it is a unitary representation.

Definition 2.2.9. Let G be a unimodular locally compact group.

1. A representation of G is called *tempered* if it has a basis whose matrix coefficients lie in the space $L^{2+\epsilon}(G)$ for any $\epsilon > 0$.

2. A representation of G is called *essentially tempered* if it is tempered modulo the center of G, after twisting with a suitable character of G.

Now we describe the Langlands' parameterization for $GSp(4, F)$. Let D denote the set of all equivalence classes of the irreducible essentially square integrable representations of $GL(k, F)$ for $k = 1, 2$. For $\delta \in D$, there exists a unique real number $e(\delta)$ and a unique unitarizable representation $\delta^u \in D$ such that

$$\delta = |\det|^{e(\delta)}\delta^u.$$

Consider also the sets $D_+ = \{\delta \in D : e(\delta) > 0\}$ and $M(D_+)$, where the latter is the set of all finite multisets in D_+. Finally denote by $T(G)$ the set of all equivalence classes of the irreducible essentially tempered smooth representations of $GSp(n, F)$ for $n = 1, 2, 4$. Take $t = ((\delta_1, \ldots, \delta_r), \tau) \in M(D_+) \times T(G)$; this t will be called the *Langlands' parameterization data*, and for such t let

$$\delta_1 \times \delta_2 \times \cdots \times \delta_r \rtimes \tau$$

be the normalized induction[5] of t to $GSp(4, F)$. Choose a permutation p of the set $\{1, 2, \ldots, r\}$ such that

$$e(\delta_{p(1)}) \geq e(\delta_{p(2)}) \geq \cdots \geq e(\delta_{p(r)}).$$

Then the representation

$$\delta_{p(1)} \times \delta_{p(2)} \times \cdots \times \delta_{p(r)} \rtimes \tau$$

has a unique irreducible quotient which is an irreducible admissible representation of $GSp(4, F)$, and it will be denoted by $L(t)$. The mapping

$$t \mapsto L(t)$$

is a one-to-one parameterization of all irreducible admissible representations of $GSp(4, F)$ by the Langlands' parameterization data. This is the so-called Langlands quotient theorem, and has been proved by Silberger for the case

[5] In our case, t will be a representation of $F^\times \times F^\times \times F^\times$, $F^\times \times GSp(2, F)$, or $GL(2, F) \times F^\times$; as we are going to see in the next section, these are the Levi subgroups of the Borel, the Klingen, and the Siegel parabolic respectively.

of p-adic groups; it is Theorem 4.1 in [59]. We will call the quotient $L(t)$ the *Langlands quotient*.

2.3 The non-supercuspidal representations of $GSp(4, F)$

In this section, we are going to apply the normalized induction discussed above, in the case of $G = GSp(4, F)$. This will give a description of the non-supercuspidal representations of this group. The classification of the irreducible admissible non-supercuspidal representations of $GSp(4, F)$ was accomplished by Sally and Tadić in [53], following the unramified case which was finished by Rodier in [50] and [51].

Let us first consider the Borel parabolic case. We are going to induce a representation of the Levi subgroup of B to a representation of $GSp(4, F)$; in fact, we are inducing a representation of B which is trivial on the unipotent radical of B. Let χ_1, χ_2, and σ be characters of $F^\times$; we consider the character of B given by

$$\begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} \mapsto \chi_1(a)\chi_2(b)\sigma(c).$$

We denote by $\chi_1 \times \chi_2 \rtimes \sigma$ the representation of $GSp(4, F)$ obtained by normalized parabolic induction of the above character of B. The modular character[6] of B is

$$\delta_B : \begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} \mapsto |a|^4|b|^2|c|^{-3},$$

[6]For the modular characters of the parabolic subgroups of $GSp(4, F)$ see [48], Section 2.2.

and the representation space of $\chi_1 \times \chi_2 \rtimes \sigma$ consists of the locally constant functions $f : GSp(4, F) \to \mathbb{C}$ with the property

$$f(pg) = |a|^2|b||c|^{-3/2}\chi_1(a)\chi_2(b)\sigma(c)f(g),$$

where $p = \begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & cb^{-1} & * \\ & & & ca^{-1} \end{pmatrix} \in B$, and $g \in GSp(4, F)$. The group $GSp(4, F)$ acts on this space by right translation.

The Klingen parabolic Q has Levi subgroup isomorphic to $GL(1, F) \times GSp(2, F) \cong F^\times \times GL(2, F)$ (which is explained in §1 of [53]). If χ is a character of $F^\times$ and (π, V) is an admissible representation of $GL(2, F)$, we denote by $\chi \rtimes \pi$ the admissible representation of $GSp(4, F)$ obtained by normalized induction from the representation of Q on V (trivial on the unipotent radical of Q) defined by

$$\begin{pmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{pmatrix} \mapsto \chi(t)\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}),$$

where $\Delta = ad - bc$. The representation space of $\chi \rtimes \pi$ consists of all locally constant functions $f : GSp(4, F) \to V$ with the property

$$f(pg) = |t|^2|ad - bc|^{-1}\chi(t)\pi(\begin{pmatrix} a & b \\ c & d \end{pmatrix})f(g),$$

where $p = \begin{pmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{pmatrix} \in Q$, and $g \in GSp(4, F)$. Note that the modular character of Q is given by

$$\delta_Q : \begin{pmatrix} t & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & \Delta t^{-1} \end{pmatrix} \mapsto |t|^4 |ad - bc|^{-2}.$$

The group $GSp(4, F)$ acts on this space by right translation.

Finally, let us consider normalized induction from the Siegel parabolic P, which has Levi subgroup isomorphic to $GL(2, F) \times F^\times$ (see §1 of [53]). Let (π, V) be an admissible representation of $GL(2, F)$ and σ a character of $F^\times$. We denote by $\pi \rtimes \sigma$ the representation of $GSp(4, F)$ which is obtained by normalized induction from the representation of P on V given by

$$\begin{pmatrix} A & * \\ & cA' \end{pmatrix} \mapsto \sigma(c)\pi(A);$$

recall the notation from Subsection 2.1.1. This representation has representation space consisting of all locally constant functions $f : GSp(4, F) \to V$ such that

$$f(pg) = |\det(A)|^{3/2} |c|^{-3/2} \sigma(c)\pi(A) f(g)$$

for all $p = \begin{pmatrix} A & * \\ & cA' \end{pmatrix} \in P$ and $g \in GSp(4, F)$. Here the modular character of P is given by

$$\delta_P : \begin{pmatrix} A & * \\ & cA' \end{pmatrix} \mapsto |\det(A)|^3 |c|^{-3}.$$

Finally, $GSp(4, F)$ acts on this space by right translation.

For the representations $\chi_1 \times \chi_2 \rtimes \sigma$, $\chi \rtimes \pi$, and $\pi \rtimes \sigma$, the centre $F^\times$ of $GSp(4, F)$ is acting via the characters $\sigma^2 \chi_1 \chi_2$, $\chi \omega_\pi$, and $\sigma^2 \omega_\pi$ respectively; here ω_π is the central character of π (see Section 2.2 of [48]).

Let ψ be a character of $F^\times$. If one has a representation (π, V) of the general linear group $GL(k, F)$, then one can get another representation $(\psi\pi, V)$

defined by

$$\psi\pi(g) = \psi(\det(g))\pi(g).$$

For the similitude symplectic group $GSp(4, F)$, if ψ is a character of $F^\times$ and (π, V) a representation of $GSp(4, F)$, one can get a representation $\psi\pi$ of $GSp(4, F)$ which has the same representation space V as π, via

$$\psi\pi(g) = \psi(\lambda(g))\pi(g),$$

where λ is the similitude character of $GSp(4, F)$. This new representation is called the *twist* of π by the character ψ. For the representations induced from the parabolics B, Q, P of $GSp(4, F)$ one has

$$\psi(\chi_1 \times \chi_2 \rtimes \sigma) \cong \chi_1 \times \chi_2 \rtimes \psi\sigma,$$

$$\psi(\pi \rtimes \sigma) \cong \pi \rtimes \psi\sigma,$$

$$\psi(\chi \rtimes \pi) \cong \chi \rtimes \psi\pi$$

respectively.

2.3.1 Classification of the non-supercuspidal representations of $GSp(4, F)$

Now we are going to list the admissible irreducible non-supercuspidal representations of $GSp(4, F)$ as in [48]. The idea is that every irreducible admissible and non-supercuspidal representation of $GSp(4, F)$ is an irreducible constituent of a parabolically induced representation, induced from a direct product of supercuspidal representations of groups of lower rank; these direct products will be called *supercuspidal inducing data.* Roberts and Schmidt in [48], following Sally and Tadić, classify the supercuspidal inducing data that one needs to get all irreducible admissible non-supercuspidal representations in eleven classes, which will be called “types”. If an irreducible admissible non-supercuspidal representation is an irreducible constituent of a representation induced from B, Q, or P, then we will say that it is *supported* in B, Q, or P respectively. Representations contained in one of the first six types are supported in the Borel parabolic B; representations contained in types VII, VIII, IX are supported in the Klingen parabolic Q; and finally, representations supported in the Siegel parabolic P are contained in the last two types. Before we proceed with the list of irreducilbe admissible non-supercuspidal representations, we need to explain the notion of the Grothendieck group.

Definition 2.3.1. Let $\mathcal{A}$ be the category of all smooth representations of finite length of $GSp(4, F)$. The *Grothendieck group* of $\mathcal{A}$ is the abelian group generated by isomorphism classes $\langle \pi \rangle$ of objects π in $\mathcal{A}$, modulo the relations

$$\langle \pi_2 \rangle = \langle \pi_1 \rangle + \langle \pi_3 \rangle,$$

for all short exact sequences $\pi_1 \hookrightarrow \pi_2 \twoheadrightarrow \pi_3$ in $\mathcal{A}$.

To make notation easier, for elements in the Grothendieck group, we are going to omit "$\langle$" and "$\rangle$". For instance, when we have the (isomorphism class of the) representation π of $GSp(4, F)$, which has ρ as an irreducible subrepresentation and τ as an irreducible quotient, we will write

$$\pi = \rho + \tau.$$

Below we list the irreducible admissible non-supercuspidal representations of $GSp(4, F)$.

Type I. Let χ_1, χ_2, and σ be characters of $F^\times$. The representation $\chi_1 \times \chi_2 \rtimes \sigma$ is irreducible if and only if $\chi_1 \neq |\ |^{\pm 1}$, $\chi_2 \neq |\ |^{\pm 1}$, and $\chi_1 \neq |\ |^{\pm 1} \chi_2^{\pm 1}$. This is Lemma 3.2 of [53]. Type I consists of irreducible representations of the form

$$\chi_1 \times \chi_2 \rtimes \sigma.$$

Type II. Let χ be a character of $F^\times$ such that $\chi \neq |\ |^{\pm 3/2}$ and $\chi^2 \neq |\ |^{\pm 1}$. Type II consists of representations of the form

$$|\ |^{1/2} \chi \times |\ |^{-1/2} \chi \rtimes \sigma.$$

By Lemmata 3.3 and 3.7 of [53] we have that $|\ |^{1/2}\chi \times |\ |^{-1/2}\chi \rtimes \sigma$ decomposes into two irreducible constituents

$$\text{IIa. } \chi St_{GL(2)} \rtimes \sigma \quad \text{and} \quad \text{IIb. } \chi 1_{GL(2)} \rtimes \sigma.$$

The representation IIa is a subrepresentation and the representation IIb is a quotient of $|\ |^{1/2}\chi \times |\ |^{-1/2}\chi \rtimes \sigma$. One can write these representations as Langlands quotients as follows (see Lemmata 3.3 and 3.7 of [53])

$$\chi St_{GL(2)} \rtimes \sigma = \begin{cases} L(\chi St_{GL(2)} \rtimes \sigma), & \text{if } e(\chi) = 0 \\ L(\chi St_{GL(2)}, \sigma), & \text{if } e(\chi) > 0 \end{cases}$$

and

$$\chi 1_{GL(2)} \rtimes \sigma = \begin{cases} L(|\ |^{1/2}\chi, |\ |^{1/2}\chi^{-1}, |\ |^{-1/2}\chi\sigma), & \text{if } 0 \le e(\chi) < 1/2 \\ L(|\ |^{1/2}\chi, |\ |^{-1/2}\chi \rtimes \sigma), & \text{if } e(\chi) = 1/2 \\ L(|\ |^{1/2}\chi, |\ |^{-1/2}\chi, \sigma), & \text{if } e(\chi) > 1/2. \end{cases}$$

Type III. Let χ be a character of $F^\times$ such that $\chi \neq 1_{F^\times}$ and $\chi \neq |\ |^{\pm 2}$. Type III consists of representations of the form

$$\chi \times |\ | \rtimes |\ |^{-1/2}\sigma.$$

By Lemmata 3.4 and 3.9 of [53], the representation $\chi \times |\ | \rtimes |\ |^{-1/2}\sigma$ decomposes into the following two irreducible constituents

$$\text{IIIa. } \chi \rtimes \sigma St_{GSp(2)} \quad \text{and} \quad \text{IIIb. } \chi \rtimes \sigma 1_{GSp(2)}.$$

The representation IIIa is a subrepresentation and the representation IIIb is a quotient of $\chi \times |\ | \rtimes |\ |^{-1/2}\sigma$. One can write these representations as Langlands quotients as follows (see Lemmata 3.4 and 3.9 of [53])

$$\chi \rtimes \sigma St_{GSp(2)} = \begin{cases} L(\chi \rtimes \sigma St_{GSp(2)}), & \text{if } e(\chi) = 0 \\ L(\chi, \sigma St_{GSp(2)}), & \text{if } e(\chi) > 0 \end{cases}$$

and

$$\chi \rtimes \sigma 1_{GSp(2)} = \begin{cases} L(|\ |, \chi \rtimes |\ |^{-1/2}\sigma), & \text{if } e(\chi) = 0 \\ L(\chi, |\ |, |\ |^{-1/2}\sigma), & \text{if } e(\chi) > 0. \end{cases}$$

Type IV. Representations of this type are of the form

$$|\ |^2 \times |\ | \rtimes |\ |^{-3/2}\sigma,$$

where σ is an arbitrary character of $F^\times$. By Lemma 3.5 of [53] this representation decomposes as

$$|\ |^{3/2} St_{GL(2)} \rtimes |\ |^{-3/2}\sigma + |\ |^{3/2} 1_{GL(2)} \rtimes |\ |^{-3/2}\sigma$$

$$= |\ |^2 \rtimes |\ |^{-1}\sigma St_{GSp(2)} + |\ |^2 \rtimes |\ |^{-1}\sigma 1_{GSp(2)}.$$

Each of these representations is reducible, and decomposes into irreducible constituents as follows

$$\begin{aligned} |\ |^{3/2} St_{GL(2)} \rtimes |\ |^{-3/2}\sigma &= \sigma St_{GSp(4)} + L(|\ |^{3/2} St_{GL(2)}, |\ |^{-3/2}\sigma), \\ |\ |^{3/2} 1_{GL(2)} \rtimes |\ |^{-3/2}\sigma &= L(|\ |^2, |\ |^{-1}\sigma St_{GSp(2)}) + \sigma 1_{GSp(4)}, \\ |\ |^2 \rtimes |\ |^{-1}\sigma St_{GSp(2)} &= \sigma St_{GSp(4)} + L(|\ |^2, |\ |^{-1}\sigma St_{GSp(2)}), \\ |\ |^2 \rtimes |\ |^{-1}\sigma 1_{GSp(2)} &= L(|\ |^{3/2} St_{GL(2)}, |\ |^{-3/2}\sigma) + \sigma 1_{GSp(4)}. \end{aligned}$$

To sum up, the representation $|\ |^2 \times |\ | \rtimes |\ |^{-3/2}\sigma$ has four irreducible constituents, which are the following

IVa. $\sigma St_{GSp(4)}$
IVb. $L(|\ |^2, |\ |^{-1}\sigma St_{GSp(2)})$
IVc. $L(|\ |^{3/2} St_{GL(2)}, |\ |^{-3/2}\sigma)$
IVd. $\sigma 1_{GSp(4)}$.

The representation IVa is the unique subrepresentation of $|\ |^2 \times |\ | \rtimes |\ |^{-3/2}\sigma$ (see Proposition 1(ii) of [50]); this subrepresentation is essentially square integrable.

Type V. Representations of this type are of the form

$$|\ |\xi \times \xi \rtimes |\ |^{-1/2}\sigma,$$

where ξ is a non-trivial quadratic character of $F^\times$ and σ is any character of $F^\times$. By following Lemma 3.6 of [53], one gets that the above representation has two subrepresentations and two subquotients, which in the Grothendieck group notation can be written as

$$|\ |^{1/2}\xi St_{GL(2)} \rtimes |\ |^{-1/2}\sigma + |\ |^{1/2}\xi 1_{GL(2)} \rtimes |\ |^{-1/2}\sigma$$

$$= |\ |^{1/2}\xi St_{GL(2)} \rtimes \xi|\ |^{-1/2}\sigma + |\ |^{1/2}\xi 1_{GL(2)} \rtimes \xi|\ |^{-1/2}\sigma.$$

Each of these representations decomposes into irreducible constituents as follows

$$\begin{aligned}
|\ |^{1/2}\xi St_{GL(2)} \rtimes |\ |^{-1/2}\sigma &= \delta([\xi, |\ |\xi], |\ |^{-1/2}\sigma) + L(|\ |^{1/2}\xi St_{GL(2)}, |\ |^{-1/2}\sigma),\\
|\ |^{1/2}\xi 1_{GL(2)} \rtimes |\ |^{-1/2}\sigma &= L(|\ |^{1/2}\xi St_{GL(2)}, |\ |^{-1/2}\xi\sigma) + L(|\ |\xi, \xi \rtimes |\ |^{-1/2}\sigma),\\
|\ |^{1/2}\xi St_{GL(2)} \rtimes \xi|\ |^{-1/2}\sigma &= \delta([\xi, |\ |\xi], |\ |^{-1/2}\sigma) + L(|\ |^{1/2}\xi St_{GL(2)}, \xi|\ |^{-1/2}\sigma),\\
|\ |^{1/2}\xi 1_{GL(2)} \rtimes \xi|\ |^{-1/2}\sigma &= L(|\ |^{1/2}\xi St_{GL(2)}, |\ |^{-1/2}\sigma) + L(|\ |\xi, \xi \rtimes |\ |^{-1/2}\sigma).
\end{aligned}$$

To sum up, the representation $|\ |\xi \times \xi \rtimes |\ |^{-1/2}\sigma$ has four irreducible constituents, which are listed below

Va. $\delta([\xi, |\ |\xi], |\ |^{-1/2}\sigma)$
Vb. $L(|\ |^{1/2}\xi St_{GL(2)}, |\ |^{-1/2}\sigma)$
Vc. $L(|\ |^{1/2}\xi St_{GL(2)}, \xi|\ |^{-1/2}\sigma)$
Vd. $L(|\ |\xi, \xi \rtimes |\ |^{-1/2}\sigma)$.

By results of Rodier (see Propositions 1 and 5 in [50]), the representation $|\ |\xi \times \xi \rtimes |\ |^{-1/2}\sigma$ has a unique essentially square integrable subrepresentation;

this is exactly the representation[7] $\delta([\xi, |\ |\xi], |\ |^{-1/2}\sigma)$. For more details the reader should consult [50]. The representations Vb and Vc are subquotients, and Vd is the Langlands quotient of $|\ |\xi \times \xi \rtimes |\ |^{-1/2}\sigma$.

Type VI. This type consists of representations of the form

$$|\ | \times 1_{F^\times} \rtimes |\ |^{-1/2}\sigma,$$

where σ is an arbitrary character of $F^\times$. By Lemma 3.8 of [53], the representation $|\ | \times 1_{F^\times} \rtimes |\ |^{-1/2}\sigma$ has two subrepresentations and two subquotients, which in the Grothendieck group notation can be written as

$$|\ |^{1/2} St_{GL(2)} \rtimes |\ |^{-1/2}\sigma + |\ |^{1/2} 1_{GL(2)} \rtimes |\ |^{-1/2}\sigma$$

$$= 1_{F^\times} \rtimes \sigma St_{GSp(2)} + 1_{F^\times} \rtimes \sigma 1_{GSp(2)}.$$

Each of the reducible constituents decomposes as follows

$$\begin{aligned}
|\ |^{1/2} St_{GL(2)} \rtimes |\ |^{-1/2}\sigma &= \tau(S, |\ |^{-1/2}\sigma) + L(|\ |^{1/2} St_{GL(2)}, |\ |^{-1/2}\sigma),\\
|\ |^{1/2} 1_{GL(2)} \rtimes |\ |^{-1/2}\sigma &= \tau(T, |\ |^{-1/2}\sigma) + L(|\ |, 1_{F^\times} \rtimes |\ |^{-1/2}\sigma),\\
1_{F^\times} \rtimes \sigma St_{GSp(2)} &= \tau(S, |\ |^{-1/2}\sigma) + \tau(T, |\ |^{-1/2}\sigma),\\
1_{F^\times} \rtimes \sigma 1_{GSp(2)} &= L(|\ |^{1/2} St_{GL(2)}, |\ |^{-1/2}\sigma) + L(|\ |, 1_{F^\times} \rtimes |\ |^{-1/2}\sigma).
\end{aligned}$$

The representations[8] $\tau(S, |\ |^{-1/2}\sigma)$ and $\tau(T, |\ |^{-1/2}\sigma)$ are essentially tempered but not square integrable, and they are not equivalent. To sum up, we have

[7] Here, we follow the notation of [48]. In general the notation "$\delta(\dots)$" means that the subrepresentation is essentially square integrable.

[8] Again here, we follow the notation of [48]. Denoting the subrepresentation of type VIa by $\tau(S, \dots)$ and the subquotient of type VIb by $\tau(T, \dots)$, we mean that the representations are essentially tampered.

the following irreducible constituents of $|\ | \times 1_{F^\times} \rtimes |\ |^{-1/2}\sigma$.

VIa. $\tau(S, |\ |^{-1/2}\sigma)$
VIb. $\tau(T, |\ |^{-1/2}\sigma)$
VIc. $L(|\ |^{1/2}St_{GL(2)}, |\ |^{-1/2}\sigma)$
VId. $L(|\ |, 1_{F^\times} \rtimes |\ |^{-1/2}\sigma)$.

Type VII. These are the irreducible representations of the form

$$\chi \rtimes \pi,$$

where χ is a character of $F^\times$ and π a supercuspidal representation of $GL(2, F)$. By results of Waldspurger and Shahidi (see Proposition 5.1 of [69] and Section 8 of [57]), we have that $\chi \rtimes \pi$ is irreducible if and only if $\chi \neq 1_{F^\times}$ and $\chi \neq \xi|\ |^{\pm 1}$, where ξ is a quadratic character such that $\xi\pi \cong \pi$.

Type VIII. Representations of this type are of the form

$$1_{F^\times} \rtimes \pi,$$

where π is a supercuspidal representation of $GL(2, F)$. Such representations decompose into a direct sum of two irreducible constituents

$$\text{VIIIa. } \tau(S, \pi) \quad \text{and} \quad \text{VIIIb. } \tau(T, \pi).$$

By Proposition 4.8 of [53], the representations $\tau(S, \pi)$ and $\tau(T, \pi)$ are essentially tempered and they are not equivalent.

Type IX. These are representations of the form

$$|\ |\xi \rtimes |\ |^{-1/2}\pi,$$

where ξ is a non-trivial quadratic character of $F^\times$ and π is a supercuspidal representation of $GL(2, F)$ such that $\xi\pi \cong \pi$. This representation decomposes into two irreducible constituents

$$\text{IXa. } \delta(|\ |\xi, |\ |^{-1/2}\pi) \quad \text{and} \quad \text{IXb. } L(|\ |\xi, |\ |^{-1/2}\pi).$$

The representation $\delta(|\ |\xi, |\ |^{-1/2}\pi)$ is an essentially square integrable representation, and the quotient $L(|\ |\xi, |\ |^{-1/2}\pi)$ is a non-tempered representation (Proposition 4.8 of [53]).

Type X. These are the irreducible representations of the form

$$\pi \rtimes \sigma,$$

where π is a supercuspidal representation of $GL(2, F)$ and σ is a character of $F^\times$. By results of Shahidi (see Section 6 of [58]), $\pi \rtimes \sigma$ is irreducible if and only if $\pi \neq |\ |^{\pm 1/2}\rho$ with ρ a supercuspidal representation of $GL(2, F)$ with trivial central character.

Type XI. Representations of this type are of the form

$$|\ |^{1/2}\pi \rtimes |\ |^{-1/2}\sigma,$$

where π is a supercuspidal representation of $GL(2, F)$ of trivial central character, and σ is a character of $F^\times$. The representation decomposes into the two irreducible constituents

$$\text{XIa. } \delta(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma) \quad \text{and} \quad \text{XIb. } L(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma).$$

Note that $\delta(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ is an irreducible and essentially square integrable subrepresentation, and $L(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ is a non-tempered quotient (see Proposition 4.6 of [53]).

The following main result concerning the classification of irreducible admissible non-supercuspidal representations of $GSp(4, F)$, where F is an non-archimedean local field, is due to Sally and Tadić.

Theorem 2.3.2. *Let π be an irreducible admissible non-supercuspidal representation of $GSp(4, F)$. Then π is of type* I, II, III, IV, V, VI, VII, VIII, IX, X *or* XI. *Furthermore, the representation* π *belongs to only one type of the above.*

Proof. This is the main result of [53]. $\square$

2.3.2 Generic representations

We finish this chapter by discussing the notion of a generic representation. These are representations which act on particular spaces of functions, which are more concrete and easier to understand.

We fix a non-trivial additive character ψ of F. Every other such character is of the form $x \mapsto \psi(cx)$ for a uniquely determined element $c \in F$ (see Exercise 3.1.1 in [4]). For $c_1, c_2 \in F^\times$, we define a character ψ_{c_1,c_2} of the unipotent radical U of the Borel subgroup of $GSp(4, F)$ as follows

$$\psi_{c_1,c_2}(\begin{pmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{pmatrix}) = \psi(c_1 x + c_2 y).$$

If we have a function $W : GSp(4, F) \to \mathbb{C}$ that satisfies the transformation property

$$W(\begin{pmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{pmatrix} g) = \psi(c_1 x + c_2 y) W(g),$$

for all $g \in GSp(4, F)$, it will be called a *Whittaker function* with respect to ψ_{c_1,c_2}.

Let π be an irreducible admissible representation of $GSp(4, F)$. Suppose that we can realize the representation space of π as a space of Whittaker functions $W : GSp(4, F) \to \mathbb{C}$ with respect to a character ψ_{c_1,c_2}, such that $GSp(4, F)$ acts on this space by right translation. Then we say that π has a *Whittaker model* with respect to ψ_{c_1,c_2}. The space of such functions is denoted by $\mathcal{W}(\pi, \psi_{c_1,c_2})$. It is a result of Rodier (see [49]) that if a Whittaker model exists, then it is unique.

Definition 2.3.3. If an irreducible admissible representation of $GSp(4, F)$ has a Whittaker model, then it will be called a *generic* representation.

According to Table A.1 of [48], each of the types of representations of $GSp(4, F)$ that we listed above has a generic irreducible (sub)representation. These are the representations of type I, IIa, IIIa, IVa, Va, VIa, VII, VIIIa, IXa, X, and XIa.

Chapter 3

The local Langlands correspondence

In this chapter we recall some essential definitions coming from Galois theory, so that we are able to introduce the reader to the local Langlands correspondence; this correspondence is concerned with relating admissible representations of $GSp(4, F)$ to representations of the Galois group $\mathrm{Gal}(\bar{F}/F)$ with image in $GSp(4, \bar{\mathbb{Q}}_l)$ (where l is some prime different from the residual characteristic of F). For the next part of this chapter, we provide a list for the L-parameters associated to irreducible admissible representations of $GL(2, F)$ and for the L-parameters associated to irreducible admissible non-supercuspidal representations of $GSp(4, F)$. Finally, we consider the local L-factors that one can define for admissible representations of $GSp(4, F)$; later on, these will constitute the Euler products of the two L-functions attached to an automorphic representation for $GSp(4)$.

3.1 Galois theory

We begin by setting some notation. Firstly, let F be a non-archimedean local field, which will be a finite extension of the field $\mathbb{Q}_p$, for a prime number p. As before, let $\mathcal{O}_F$ be its ring of integers, $\mathfrak{p}_F$ the unique maximal ideal of

$\mathcal{O}_F$, and q the number of elements of $k_F = \mathcal{O}_F/\mathfrak{p}_F$. The prime p is the characteristic of the residue field k_F. Moreover, if ϖ is a generator of $\mathfrak{p}_F$, the normalized absolute value on F will be denoted by $| \ |$ and it is the unique extension of the corresponding absolute value on $\mathbb{Q}_p$; it holds that $|\varpi| = q^{-1}$. Finally, we will denote the discrete valuation on F by $v_F : F \to \mathbb{Z} \cup \{\infty\}$, with $|x| = q^{-v_F(x)}$.

We consider the groups

$$U^{(n)} = 1 + \varpi^n \mathcal{O}_F = \left\{ a \in F^\times : |1 - a| < \frac{1}{q^{n-1}} \right\},$$

for n a positive integer, with the convention that $U^{(0)} = 1 + \varpi^0 \mathcal{O}_F$ is the group of units $\mathcal{O}_F^\times$. This is a basis of neighbourhoods of the element $1 \in F^\times$, and

$$\mathcal{O}_F^\times \supset 1 + \varpi^1 \mathcal{O}_F \supset 1 + \varpi^2 \mathcal{O}_F \supset \dots$$

We have $U^{(0)}/U^{(1)} \cong (\mathcal{O}_F/\mathfrak{p}_F)^\times$, and for $n \geq 1$, $U^{(n)}/U^{(n+1)} \cong \mathcal{O}_F/\mathfrak{p}_F$ (see Kapitel II, Satz 3.10 of [41]). The next definition contains one of the basic notions of this thesis. That is, the notion of the conductor of a character of $F^\times$.

Definition 3.1.1. Let $\chi : F^\times \to \mathbb{C}^\times$ be a continuous homomorphism.

1. The smallest non-negative integer n such that $\chi(1 + \varpi^n \mathcal{O}_F) = 1$ is called the *conductor* of χ. We will denote the conductor of χ by $a(\chi)$.

2. The homomorphism χ will be called *unramified* if $\chi(\mathcal{O}_F^\times) = 1$, and *tamely ramified* if $\chi(1 + \varpi \mathcal{O}_F^\times) = 1$ but $\chi(\mathcal{O}_F^\times) \neq 1$.

We will return to this notion later, when we consider conductors of more general representations.

3.1.1 The Weil-Deligne group

If L/F is a field extension, we define the *Galois group* $\mathrm{Gal}(L/F)$ of this extension to be the group of all automorphisms of L which fix F. If in addition, the extension L/F is an infinite Galois extension, then one can equip

Gal(L/F) with the so-called Krull topology; this topology makes Gal(L/F) into a profinite group, i.e., we may write

$$\mathrm{Gal}(L/F) = \varprojlim_{M/F \text{ finite}} \mathrm{Gal}(M/F),$$

and the fundamental theorem of Galois theory holds for Gal(L/F). Since we are not going to need a lot of information on the topology of Gal(L/F), we refer the reader to [25], §1.

Let $\bar{F}$ (respectively $\bar{k}_F$) be the algebraic closure of F (respectively k_F); then we write $G_F = \mathrm{Gal}(\bar{F}/F)$ (respectively $G_{k_F} = \mathrm{Gal}(\bar{k}_F/k_F)$). By Satz 9.9 in Kapitel II of [41], there exists a surjective map $G_F \twoheadrightarrow G_{k_F}$ that takes $\sigma \in G_F$ to an automorphism $\bar{\sigma} \in G_{k_F}$ which is defined via

$$\bar{\sigma} : x \bmod \mathfrak{p}_{\bar{F}} \mapsto \sigma x \bmod \mathfrak{p}_{\bar{F}}.$$

Definition 3.1.2. We define the *inertia subgroup* I_F of G_F to be the kernel of the surjective map $G_F \twoheadrightarrow G_{k_F}$; that is

$$I_F = \ker(G_F \to G_{k_F}) = \{\sigma \in G_F : \sigma x \equiv x \bmod \mathfrak{p}_{\bar{F}}, \text{ for all } x \in \mathcal{O}_{\bar{F}}\}.$$

The fixed field of the inertia subgroup, i.e., $\bar{F}^{I_F} = F^{nr}$, will be called the *maximal unramified extension* of F, while an extension L/F will be called *unramified* if L is a subfield of F^{nr}. Thus we have $I_F = \mathrm{Gal}(\bar{F}/F^{nr})$ and $G_{k_F} \cong \mathrm{Gal}(F^{nr}/F)$. Generalizing the definition of the inertia group for the extension $\bar{F}/F$, we define the *inertia subgroup* of an extension L/F as

$$I_{L/F} = \{\sigma \in \mathrm{Gal}(L/F) : \sigma x \equiv x \bmod \mathfrak{p}_L, \text{ for all } x \in \mathcal{O}_L\}.$$

Consider the *Frobenius automorphism* Frob : $x \mapsto x^q$ which lies in G_{k_F}. The infinite cyclic subgroup of G_{k_F} generated by the Frobenius automorphism will be called the *Weil group* of k_F, and will be denoted by W_{k_F}. Note that sometimes it is the inverse of the Frobenius automorphism that is considered as the canonical generator of W_{k_F}; i.e., the map φ defined by $(\varphi(x))^q = x$.

We can make W_{k_F} a topological group by giving it the discrete topology. Now, as mentioned earlier, one has the following exact sequence

$$1 \to I_F \hookrightarrow G_F \twoheadrightarrow G_{k_F} \to 1.$$

The inverse image of the inverse Frobenius automorphism φ under the surjection $G_F \twoheadrightarrow G_{k_F}$ is the set ϕI_F, where ϕ is an element lying above φ. That is, the inverse image of the group W_{k_F} under the above surjection is $\bigcup_{n\in\mathbb{Z}} \phi^n I_F$.

Definition 3.1.3. We define the *Weil group* W_F of F to be the inverse image of W_{k_F} under the surjection $G_F \twoheadrightarrow G_{k_F}$. That is,

$$W_F = \bigcup_{n\in\mathbb{Z}} \phi^n I_F,$$

or equivalently,

$$W_F = \{\sigma \in G_F : \sigma|_{F^{nr}} = \phi^n, \text{ for some } n \in \mathbb{Z}\}.$$

Thus we get the exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_F & \longrightarrow & G_F & \longrightarrow & G_{k_F} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & I_F & \longrightarrow & W_F & \longrightarrow & W_{k_F} & \longrightarrow & 1 \end{array}$$

where the vertical arrows are inclusions. The Weil group W_F has a large subgroup I_F which is profinite (as it is a Galois group), but W_F is not a profinite group. We define a topology on W_F by requiring that I_F be open in W_F, that the subspace topology on I_F from W_F coincide with the subspace topology from G_F, and that left multiplication by ϕ be a homeomorphism.

Now we are going to consider a larger group, the so-called Weil-Deligne group. There is a continuous homomorphism

$$\| \ \| : W_F \to \mathbb{Q}^\times$$

which is defined by being trivial on the inertia subgroup, that is $\|I_F\| = \{1\}$, and such that $\|\phi\| = q^{-1}$, where ϕ is an element lying above the inverse Frobenius φ.

Definition 3.1.4. We define the *Weil-Deligne group* W'_F to be the semi-direct product

$$W'_F = \mathbb{C} \rtimes W_F,$$

where the action of W_F on $\mathbb{C}$ is (for $w \in W_F$ and $z \in \mathbb{C}$)

$$wzw^{-1} = \|w\|z. \tag{3.1}$$

Thus, the multiplication on this semi-direct product is

$$(z, w)(z', w') = (z + \|w\|z', ww'), \tag{3.2}$$

where $z, z' \in \mathbb{C}$ and $w, w' \in W_F$.

We give W'_F the product topology, corresponding to its set-theoretic structure as a cartesian product.

Before we move to the next paragraph, we introduce some useful notions. In particular, we consider the abelian extensions over F; that is extensions such that the corresponding Galois group is abelian. We define $\mathrm{Gal}(\bar{F}/F)^{ab}$ to be the maximal abelian continuous image of $\mathrm{Gal}(\bar{F}/F)$; i.e., it is the quotient of $\mathrm{Gal}(\bar{F}/F)$ modulo the closure of its commutator subgroup. Let F^{ab} be the algebraic extension of F such that $\mathrm{Gal}(F^{ab}/F) = \mathrm{Gal}(\bar{F}/F)^{ab}$. Then an extension L/F will be called *abelian* if $L \subset F^{ab}$. In the same fashion, one defines W_F^{ab} as W_F modulo the closure of its commutator subgroup. Finally, note that

$$W_F^{ab} \cong (W'_F)^{ab}. \tag{3.3}$$

3.1.2 Ramification filtration

We now introduce the reader to the so-called *ramification subgroups*, which measure how ramified our Galois representations are going to be. We first consider finite Galois extensions L/F. Note that by §9, Proposition 1 in [16], there is an α such that $\mathcal{O}_L = \mathcal{O}_F[\alpha]$. We define the following filtration on the Galois group $\mathrm{Gal}(L/F)$:

$$\begin{aligned}
\mathrm{Gal}(L/F)_{-1} &= \mathrm{Gal}(L/F);\\
\mathrm{Gal}(L/F)_0 &= \{\sigma \in \mathrm{Gal}(L/F) : \sigma x \equiv x \bmod \mathfrak{p}_L, \text{ for all } x \in \mathcal{O}_L\}\\
&= \{\sigma \in \mathrm{Gal}(L/F) : \sigma\alpha \equiv \alpha \bmod \mathfrak{p}_L\}\\
&= \{\sigma \in \mathrm{Gal}(L/F) : v_L(\sigma\alpha - \alpha) \geq 1\};\\
\mathrm{Gal}(L/F)_i &= \{\sigma \in \mathrm{Gal}(L/F) : \sigma x \equiv x \bmod \mathfrak{p}_L^{i+1}, \text{ for all } x \in \mathcal{O}_L\}\\
&= \{\sigma \in \mathrm{Gal}(L/F) : \sigma\alpha \equiv \alpha \bmod \mathfrak{p}_L^{i+1}\}\\
&= \{\sigma \in \mathrm{Gal}(L/F) : v_L(\sigma\alpha - \alpha) \geq i+1\}.
\end{aligned}$$

Note that for sufficiently large n, $\mathrm{Gal}(L/F)_n$ is going to be the trivial subgroup. Moreover, the subgroup $\mathrm{Gal}(L/F)_1$ is called the *wild inertia subgroup*, and note that $\mathrm{Gal}(L/F)_0$ is the inertia subgroup $I_{L/F}$. By Theorem 1(ii) of §8 in [16], we have that the wild inertia subgroup is the unique Sylow p-subgroup of the inertia subgroup. Moreover, we say that a finite extension L/F^{nr} is *tamely ramified* if $\mathrm{Gal}(L/F^{nr})_1$ is trivial. By Corollary 2 in §8 of [16], the composite of tamely ramified extensions is again tamely ramified. The *maximal tamely ramified extension* F^{tr} of F is the union of all tamely ramified extensions in $\bar{F}$.

It is clear to see from the way we defined the ramification subgroups that the filtration passes to subgroups. That is, if we have the finite extensions L/M and M/F, then

$$\mathrm{Gal}(L/M)_i = \mathrm{Gal}(L/F)_i \cap \mathrm{Gal}(L/M).$$

We would like to be able to define a similar filtration for infinite Galois extensions in order to use it for the group $\mathrm{Gal}(\bar{F}/F)$. A way to do this, is to use the fact that for an infinite extension L/F, the Galois group $\mathrm{Gal}(L/F)$

is profinite. Then we may write it as an inverse limit

$$\mathrm{Gal}(L/F) = \varprojlim_{M/F \text{ finite}} \mathrm{Gal}(M/F)$$

and transfer the filtration from the quotient groups $\mathrm{Gal}(M/F)$ to the group $\mathrm{Gal}(L/F)$. One problem that we face is that the so-called *lower numbering* that we have just defined for our filtration does not pass to quotients. Hence we have to modify it slightly to get the so-called *upper numbering* for our filtration.

To make our notation easier, we denote by g_i the order of the group $\mathrm{Gal}(L/F)_i$. Consider a real variable $x \in [-1, \infty)$ and write $\mathrm{Gal}(L/F)_x = \mathrm{Gal}(L/F)_i$, where i is the least integer greater than or equal to x. We define a function $\phi_{L/F}$ via

$$\phi_{L/F}(x) = \begin{cases} x, & \text{if } -1 \leq x \leq 0 \\ \dfrac{g_1}{g_0} + \cdots + \dfrac{g_m}{g_0} + \dfrac{(x-m)g_{m+1}}{g_0}, & \text{if } x \geq 0 \end{cases}$$

where m is the integer with $m \leq x < m+1$. The function $\phi_{L/F}$ is the so-called *Herbrand function*; it is continuous and strictly increasing and therefore it has an inverse function $\psi_{L/F}(y)$, for $y \in [-1, \infty)$, which is continuous and strictly increasing. In addition, one can see from the definition that $\phi_{L/F}$ is linear in the interval $[m, m+1]$, for m an integer ≥ -1. The new numbering for the ramification groups is given by

$$\mathrm{Gal}(L/F)^y = \mathrm{Gal}(L/F)_x, \tag{3.4}$$

where $y = \phi_{L/F}(x)$. Our next goal is to explain why the new upper numbering has nice compatibility properties when passing to quotients.

Suppose we have the Galois extensions L/M and M/F and the Galois groups $\Delta = \mathrm{Gal}(L/M)$ and $\Gamma = \mathrm{Gal}(L/F)$, so that $\mathrm{Gal}(M/F) \cong \Gamma/\Delta$. We have the following lemma.

Lemma 3.1.5. *For the Galois extensions L/M and M/F, the following hold:*

i. for the Herbrand function we have $\phi_{L/F}(x) = \phi_{M/F}(\phi_{L/M}(x))$*;*

ii. for $y = \phi_{L/M}(x)$ *we have* $(\Gamma/\Delta)_y = \Gamma_x\Delta/\Delta$.

Proof. The reader may refer to Theorem 2 of §9 in [16], and its proof. □

Now we are ready to see how the upper numbering passes to quotients.

Corollary 3.1.6. *For Galois extensions* $L/M/F$ *one has*

$$(\Gamma/\Delta)^z = \Gamma^z\Delta/\Delta, \tag{3.5}$$

Proof. By Equation (3.4) we have

$$(\Gamma/\Delta)^z = (\Gamma/\Delta)_y,$$

where $z = \phi_{M/F}(y)$. By using now Lemma 3.1.5 one has

$$(\Gamma/\Delta)_y = \Gamma_x\Delta/\Delta,$$

where $y = \phi_{L/M}(x)$. Finally, by applying Equation 3.4 on Γ and considering that $\phi_{L/F}(x) = \phi_{M/F}(\phi_{L/M}(x))$, one gets

$$\Gamma_x\Delta/\Delta = \Gamma^z\Delta/\Delta,$$

where $z = \phi_{M/F}(y) = \phi_{L/F}(x)$. Thus we have our result. □

Equation (3.5) indicates how one can define ramification filtration on quotient groups. Now that one has the filtration of ramification subgroups on quotients, one may define ramification subgroups on infinite Galois extensions by using the inverse limit formula

$$\mathrm{Gal}(L/F)^z = \varprojlim_{M/F \text{ finite}} \mathrm{Gal}(M/F)^z,$$

where $\mathrm{Gal}(M/F)$ are quotients of the Galois group $\mathrm{Gal}(L/F)$.

Note that if $\bar{F}$ is an algebraic closure of F, then $\mathrm{Gal}(\bar{F}/F)^0 = I_F$ is the inertia subgroup of $\mathrm{Gal}(\bar{F}/F)$. Recall that the wild inertia subgroup for finite extensions L/F is the unique Sylow p-subgroup of $I_{L/F}$; for infinite extensions such as $\bar{F}/F$, the wild inertia subgroup will be the maximal pro-p-subgroup of the inertia group.

In the beginning of this section we defined a filtration on $F^\times$ by the subgroups $U^{(n)} = 1 + \varpi^n \mathcal{O}_F$, where $\mathcal{O}_F^\times = U^{(0)}$. We extend this filtration to real exponents by $U^{(z)} = U^{(n)}$ if $n-1 < z \leq n$. We are going to use the local reciprocity map from local class field theory in order to relate the filtrations on $F^\times$ and on $\mathrm{Gal}(\bar{F}/F)$. Let us recall the local reciprocity map. The *local reciprocity map* is a map

$$r_F : F^\times \to \mathrm{Gal}(\bar{F}/F)^{ab}$$

such that

i. $r_F(\varpi) \in \phi I_{F^{ab}/F}$;

ii. $\mathcal{O}_F^\times$ is mapped onto $I_{F^{ab}/F} = \mathrm{Gal}(F^{ab}/F)^0$.

For a detailed description the reader should refer to §2 of [55]. By the definition of the Weil group of F and by the fact that every element a in $F^\times$ can be written uniquely as $a = \varpi^m u$ for $m \in \mathbb{Z}$ and $u \in \mathcal{O}_F^\times$ (see §1 of [16]), one has that the local reciprocity map induces an isomorphism

$$F^\times \cong W_F^{ab}. \tag{3.6}$$

Theorem 3.1.7. *The local reciprocity map*

$$r_F : F^\times \to Gal(\bar{F}/F)^{ab}$$

maps the groups $U^{(z)}$ onto the groups $Gal(F^{ab}/F)^z$ for all $z \geq 0$.

Proof. See Theorem 1 of §4 in [55]. □

Recall Definition 3.1.1, where we defined a character on $F^\times$ to be unramified when $\chi(\mathcal{O}_F^\times) = 1$. Seeing this character as a character of W_F via the local reciprocity map, we say that χ is *unramified* if it is trivial on the inertia subgroup I_F.

3.2 Local Langlands correspondence

In this section we discuss the local Langlands correspondence for $GSp(4, F)$ which was proved by Gan and Takeda in [18], and asserts the following. If $\mathrm{Irr}(GSp(4, F))$ is the set of irreducible admissible representations of $GSp(4, F)$ and $\Phi(GSp(4, F))$ is the set of equivalence classes of admissible continuous homomorphisms

$$W_F' \to GSp(4, \mathbb{C}),$$

then there is a surjective map

$$L : \mathrm{Irr}(GSp(4, F)) \to \Phi(GSp(4, F)).$$

If π is a representation in $\mathrm{Irr}(GSp(4))$, then $L(\pi)$ will be called the *L-parameter* of π. This map is finite-to-one and Gan and Takeda prove that in fact the fibers of the map, which are called *L-packets*, contain either one or two elements; in the case where an L-packet contains two elements, exactly one is a generic representation of $GSp(4, F)$. This is the Main Theorem of [18].

Below we are going to talk about the representations of the Weil-Deligne group, which will lead us to the formal definition of L-parameters. After that, we are going to describe explicitly the L-parameters which are attached to irreducible admissible non-supercuspidal representations of $GSp(4, F)$.

3.2.1 Representations of the Weil-Deligne group

Let us consider first representations of the Weil group. By a representation of W_F we mean a continuous homomorphism $\rho_0 : W_F \to GL(V)$, where V is a finite dimensional complex vector space.

Definition 3.2.1. If ρ_0 is a representation of W_F, we say that ρ_0 is *unramified* when $\rho_0|_{I_F}$ is trivial. Otherwise we say that ρ_0 is *ramified.*

By the local reciprocity map, we saw that there is an isomorphism

$$F^\times \cong W_F^{ab}.$$

Since any one-dimensional representation of W_F factors through W_F^{ab}, one may identify characters of W_F with characters of $F^\times$. From Equation (3.3) we have $F^\times \cong (W_F')^{ab}$ and characters of $F^\times$ may also be identified with characters of $(W_F')^{ab}$. In this thesis, we are going to freely identify characters of $F^\times$ with characters of the Weil (resp. Weil-Deligne) group without mentioning it every time.

We required that a representation of W_F is a continuous homomorphism. The next proposition gives an alternative way to think of continuity in this concept (see Section 2 of [52]).

Proposition 3.2.2. *A homomorphism $\rho_0 : W_F \to GL(V)$ is continuous if and only if ρ_0 is trivial on an open subgroup of the inertia group I_F.*

Proof. Suppose first that ρ_0 is a continuous homomorphism. By a standard property of complex Lie groups, there is an open neighbourhood $\mathcal{U}$ of the identity in $GL(V)$ which contains no non-trivial subgroups of $GL(V)$. Since ρ_0 is continuous, $\rho_0(\mathcal{U})^{-1}$ is an open neighbourhood of the identity in W_F and as a result, it contains an open subgroup J of I_F. Then $\rho_0(J)$ is a subgroup of $GL(V)$ contained in $\mathcal{U}$, so it is the trivial subgroup.

Conversely, if $\rho_0 : W_F \to GL(V)$ is a homomorphism which is trivial on an open subgroup J of the inertia, then it factors through W_F/J. That is, any open subgroup of $GL(V)$ has inverse image which is a union of cosets of the open subgroup J, hence it is open. Therefore ρ_0 is continuous. $\square$

As a result, a continuous representation of the Weil group is characterized by the fact that it is trivial on an open subgroup of the inertia group.

In the local Langlands correspondence some of the representations that we need to consider do not have this property. Therefore, we need to discuss representations of the Weil-Deligne group instead.

By a representation of W'_F we mean a finite-dimensional complex vector space V and a continuous homomorphism

$$\rho'_0 : W'_F \to GL(V)$$

such that the restriction of ρ'_0 to $\mathbb{C}$ is complex analytic.

Proposition 3.2.3. *Representations ρ'_0 of the Weil-Deligne group W'_F acting on the space V can be identified with pairs (ρ_0, N), where ρ_0 is a representation of W_F on V and N is a nilpotent endomorphism of V such that*

$$\rho_0(g) N \rho_0(g)^{-1} = \|g\| N, \tag{3.7}$$

for $g \in W_F$.

Proof. If we have a pair (ρ_0, N) we can get ρ'_0 by

$$\rho'_0((z, w)) = \rho_0(w) \exp(zN),$$

for $w \in W_F$ and $z \in \mathbb{C}$. To see that this is indeed a representation of the Weil-Deligne group, one uses Equations (3.2) and (3.7).

If we have a representation ρ'_0 of the Weil-Deligne group, we can get a pair (ρ_0, N) by

$$\rho_0 = \rho'_0|_{W_F}$$

and

$$N = (\log \rho'_0(z))/z,$$

and this is independent of $z \in \mathbb{C}$. We need to prove that N as given is nilpotent, and independent of z. The endomorphism N is nilpotent when $\log \rho'_0(z)$ is; since $\log \rho'_0(z)$ is given as power series in $\rho'_0(z) - 1$, we need to show that $\rho'_0(z)$ is unipotent. First note that if we substitute ϕ^{-1} in

Equation (3.1) and then apply ρ_0', we get

$$\rho_0'(\phi^{-1})\rho_0'(z)\rho_0'(\phi) = \rho_0'(z)^q;$$

i.e., $\rho_0'(z)$ and $\rho_0'(z)^q$ are similar and thus they have the same eigenvalues. If λ is an eigenvalue of $\rho_0'(z)$, by iteration λ^{q^n} is also an eigenvalue of $\rho_0'(z)$ for all $n \geq 0$. Since the number of distinct eigenvalues of $\rho_0'(z)$ is at most the dimension d of ρ_0', there are integers $0 \leq m_0 < n_0 \leq d$, such that

$$\lambda^{q^{m_0}} = \lambda^{q^{n_0}}.$$

We define a positive integer

$$r = \prod_{0 \leq m < n \leq d} (q^n - q^m),$$

which is independent of z. Note that every eigenvalue of $\rho_0'(z)$ is an r-th root of unity. If one applies this argument on $\rho_0'(z/r)$, then $\rho_0'(z) = \rho_0'(z/r)^r$ has every eigenvalue equal to 1; this proves that $\rho_0'(z)$ is unipotent, therefore N is nilpotent.

For the fact that N is independent of z as given above, the reader should refer to §3 in [52]. □

From now on, we may denote a representation of the Weil-Deligne group by a pair (ρ_0, N), where ρ_0 is a representation of the Weil group and N is a nilpotent matrix satisfying Equation (3.7). Such representations will be called *Weil-Deligne representations*.

Definition 3.2.4. We say that a representation (ρ_0, N) of W_F' is *unramified* when ρ_0 is unramified as a representation of W_F and $N = 0$. Otherwise we say that (ρ_0, N) is *ramified*.

Definition 3.2.5. A Weil-Deligne representation (ρ_0, N) is called *admissible* (or *F-semisimple*) when ρ_0 is semisimple.

Note that if (ρ_0, N) is a representation of W_F' with representation space V, then a subspace of V is invariant under W_F' if and only if it is invariant under both W_F and N. A representation of W'_F is irreducible when it has no non-trivial proper subspaces which are invariant under W'_F. By Equation (3.7) the kernel of N is stable under W_F, hence if a representation (ρ_0, N) is irreducible we must have $N = 0$. As a result, the irreducible Weil-Deligne representations are simply the irreducible representations of W_F.

If (ρ_0, N) is a Weil-Deligne representation, and χ is a character of W_F, then we define the *twist* of (ρ_0, N) by χ as $\chi(\rho_0, N) = (\chi\rho_0, N)$. This is a representation of W'_F, and it is straightforward to see that admissibility is preserved under twisting.

There is a bijection between complex representations $\rho'_0 = (\rho_0, N)$ of the Weil-Deligne group and λ_E-adic representations $\rho : G_F \to GL(n, E)$, where E is a finite extension over $\mathbb{Q}_l$ and λ_E the maximal ideal of the ring of integers of E. Here $\lambda_E \mid l$, and l is a prime different from the characteristic p of k_F. The construction of this bijection is due to Deligne and Grothendieck.

In order to define this bijection, we need to make two choices. Firstly, we fix a lift ϕ of the Frobenius automorphism (in fact, a lift of the inverse Frobenius automorphism). Secondly, we fix a non-trivial continuous homomorphism

$$t_l : I_F \to \mathbb{Q}_l,$$

which is defined as the composition

$$I_F \to I_F/\mathrm{Gal}(\bar{F}/F^{tr}) \cong \prod_{r \nmid p} \mathbb{Z}_r \to \mathbb{Z}_l.$$

Here F^{tr} is the maximal tamely ramified extension over F^{nr}, and the second map is the projection on the factor $\mathbb{Z}_l$. The isomorphism

$$I_F/\mathrm{Gal}(\bar{F}/F^{tr}) \cong \prod_{r \nmid p} \mathbb{Z}_r$$

is described in Corollary 3 of §8 in [16]. For more information on t_l, the reader can refer to §4 in [52].

Theorem 3.2.6. *Suppose l is a prime different from p. Let*

$$\rho : G_F \to GL(V_l)$$

be a λ_E-adic representation, where V_l is an E-vector space of finite dimension. Then:

i. There is a unique nilpotent endomorphism N_l of V_l such that

$$\rho(g_0) = \exp(t_l(g_0)N_l)$$

for g_0 in some open subgroup of I_F. Furthermore

$$\rho(g)N_l\rho(g)^{-1} = \|g\|N_l$$

for $g \in W_F$. We have $N_l = 0$ if and only if ρ is trivial on an open subgroup of I_F.

ii. The function $\rho_{0,l} : W_F \to GL(V_l)$ defined by

$$\rho_{0,l}(g) = \rho(g)\exp(-t_l(g_0)N_l),$$

where $g = \phi^m g_0$, for $g_0 \in I_F$ and $m \in \mathbb{Z}$, is a homomorphism and is trivial on an open subgroup of I_F (i.e., $\rho_{0,l}$ is a representation of W_F).

iii. For $g \in W_F$ we have

$$\rho_{0,l}(g)N_l\rho_{0,l}(g)^{-1} = \|g\|N_l.$$

Conversely, for each pair (ρ_0, N) there is a unique λ_E-adic representation

$$\rho : G_F \to GL(V_l).$$

Proof. See Proposition in §4 of [52] and the generalization remark after it. □

3.2.2 Local Langlands correspondence for $GL(2,F)$

In this subsection, we remind the reader the theory of irreducible admissible representations of $GL(2,F)$. After that, we attach L-parameters to these representations. In our description of the irreducible admissible representations of $GL(2,F)$ we will use the same notation as the one we used when we listed the irreducible admissible non-supercuspidal representations for $GSp(4,F)$; thus we need the notion of the Grothendieck group (see also Definition 2.3.1). That is, if $\mathcal{A}$ is the category of smooth representations of finite length of $GL(2,F)$, the *Grothendieck group* of $\mathcal{A}$ is the abelian group generated by isomorphism classes π (for simplicity we skip the notation "$\langle$" and "$\rangle$") of objects in $\mathcal{A}$, modulo the relations

$$\pi_2 = \pi_1 + \pi_3,$$

for all short exact sequences $\pi_1 \hookrightarrow \pi_2 \twoheadrightarrow \pi_3$ in $\mathcal{A}$.

The first thing to say is that the finite-dimensional irreducible admissible representations of $GL(2,F)$ are one-dimensional. In fact, they are of the form $\chi \circ \det$ for some character χ of $F^\times$. This is Proposition 2.7(a) of [32].

We consider now infinite dimensional irreducible admissible representations of $GL(2,F)$. The minimal parabolic subgroup of $GL(2,F)$ is the Borel subgroup B consisting of all upper triangular matrices; that is

$$B = \left\{ \begin{pmatrix} t_1 & x \\ & t_2 \end{pmatrix} \right\}.$$

This is in fact the unique proper parabolic subgroup of $GL(2,F)$, up to conjugation. If χ_1 and χ_2 are two characters of $F^\times$, define the character

$$\begin{pmatrix} t_1 & x \\ & t_2 \end{pmatrix} \mapsto \chi_1(t_1)\chi_2(t_2)$$

of the Borel. We apply normalized induction on this character, to get a representation of $GL(2,F)$ with representation space consisting of all locally constant functions f on $GL(2,F)$ such that

$$f(\begin{pmatrix} t_1 & x \\ & t_2 \end{pmatrix} g) = \left|\frac{t_1}{t_2}\right|^{1/2} \chi_1(t_1)\chi_2(t_2)f(g),$$

for $\begin{pmatrix} t_1 & x \\ & t_2 \end{pmatrix} \in B$ and $g \in GL(2,F)$. The group $GL(2,F)$ acts on this space by right translation. We denote this representation by $\chi_1 \times \chi_2$. The following theorem lists the non-supercuspidal irreducible admissible representations of $GL(2,F)$. Note that by Proposition 2.2.2, the representation $\chi_1 \times \chi_2$ is admissible.

Theorem 3.2.7. *For the parabolically induced representation $\chi_1 \times \chi_2$ of $GL(2,F)$, we have the following:*

i. The representation $\chi_1 \times \chi_2$ is irreducible when $\chi_1\chi_2^{-1} \neq |\ |^{\pm 1}$. In this case we have $\chi_1 \times \chi_2 \cong \chi_2 \times \chi_1$.

ii. If $\chi_1\chi_2^{-1} = |\ |$, we have that there is an irreducible subrepresentation denoted by $(\chi_2|\ |^{1/2})St_{GL(2)}$ and an irreducible quotient $(\chi_2|\ |^{1/2}) \circ det$ of dimension one. Considering these representations as elements in the Grothendieck group, we may write

$$\chi_1 \times \chi_2 = (\chi_2|\ |^{1/2})St_{GL(2)} + (\chi_2|\ |^{1/2}) \circ det.$$

iii. If $\chi_1\chi_2^{-1} = |\ |^{-1}$, we have that there is an irreducible one-dimensional subrepresentation $(\chi_1|\ |^{1/2}) \circ det$ and an irreducible quotient denoted by $(\chi_1|\ |^{1/2})St_{GL(2)}$. Considering these representations as elements in the Grothendieck group, we write

$$\chi_1 \times \chi_2 = (\chi_1|\ |^{1/2}) \circ det + (\chi_1|\ |^{1/2})St_{GL(2)}.$$

Proof. This is Theorem 3.3 in [32]. □

If the representation $\chi_1 \times \chi_2$ is irreducible, we say that it is a *principal series representation.* Moreover, note that $St_{GL(2)}$ is the Steinberg representation of Definition 2.2.3. In order to have all irreducible admissible representations of $GL(2,F)$, we also need to consider the representations which do not arise as subrepresentations or subquotients of representations obtained by normalized induction from the Borel parabolic; these are the supercuspidal representations which we briefly describe below.

Supercuspidal representations are in general difficult to describe, but if we assume that the residual characteristic p is odd, things become easier. Let L/F be a quadratic extension and ψ an admissible character of $L^\times$ such that $\psi \circ \sigma \neq \psi$. Here σ is the non-trivial element of $\mathrm{Gal}(L/F)$. Then one gets an irreducible admissible supercuspidal representation of $GL(2,F)$ from ψ, called a *base change.* We denote this supercuspidal representation by $BC(L/F,\psi)$. This construction can be found in more detail in Theorem 4.6 of [32]. If $p \neq 2$ then every supercuspidal representation of $GL(2,F)$ arises that way. The extra irreducible admissible representations that one gets if $p=2$ are called *extraordinary* representations.

Definition 3.2.8. An *L-parameter* for the group $GL(2,F)$ is an equivalence class of admissible representations of the Weil-Deligne group W_F'. Denote by $\Phi(GL(2,F))$ the set of L-parameters.

Let us now describe the L-parameters which are attached to the above representations (at least when p is odd). Below, we will be seeing characters of $F^\times$ as characters of W_F and vice versa, without distinguishing the notation.

i. To the principal series representations $\chi_1 \times \chi_2$, we attach L-parameters (ρ_0, N), with semisimple part $\rho_0 : W_F \to GL(2,\mathbb{C})$ given by

$$\rho_0 : w \mapsto \begin{pmatrix} \chi_1(w) & \\ & \chi_2(w) \end{pmatrix},$$

and nilpotent part $N=0$.

ii. A twisted Steinberg representation $(\chi|\ |^{1/2})St_{GL(2)}$ has attached the

L-parameter (ρ_0, N), with $\rho_0 : W_F \to GL(2, \mathbb{C})$ defined via

$$\rho_0 : w \mapsto \begin{pmatrix} \chi(w)|w| & \\ & \chi(w) \end{pmatrix}.$$

Here the nilpotent part is non-trivial; in fact $N = \begin{pmatrix} & 1 \\ & \end{pmatrix}$.

iii. To a supercuspidal representation $BC(L/F, \psi)$ we attach the L-parameter (ρ_0, N) with trivial nilpotent endomorphism N, and $\rho_0 = \mathrm{ind}_{W_L}^{W_F} \psi$.

We will usually write ϕ_π for the L-parameter of a representation π of $GL(2, F)$.

The local Langlands correspondence for $GL(2, F)$ has been proved by Kutzko in [36] (and more generally for $GL(n, F)$ by Harris and Taylor in [30] and Henniart in [31]). The supercuspidal representations are in bijection with irreducible 2-dimensional representations of the Weil-Deligne group (i.e., irreducible representations of the Weil group as we have seen above), the principal series representations correspond with semisimple representations of W'_F which are direct sums of two characters, and the twisted Steinberg representations correspond with reducible indecomposable representations of W'_F. It is now clear that the consideration of twisted Steinberg representations is forcing us to choose the Weil-Deligne group instead of the Weil group in the correspondence. Furthermore, the central characters of irreducible admissible representations are the determinants of the correpsponding Weil-Deligne representations; in particular, we have

$$\begin{aligned} \omega_{\chi_1 \times \chi_2} &= \chi_1 \chi_2; \\ \omega_{(\chi| \ |^{1/2}) St_{GL(2)}} &= \chi^2 | \ |; \\ \omega_{BC(L/F, \psi)} &= \psi|_{F^\times} \epsilon_{L/F}. \end{aligned}$$

Here $\epsilon_{L/F}$ is the quadratic character corresponding to the quadratic extension L/F; for $x \in F^\times$ we define $\epsilon_{L/F}(x) = 1$ if x is such that there is a $x' \in L^\times$ with $N_{L/F}(x') = x$, and $\epsilon_{L/F}(x) = -1$ otherwise. Here $N_{L/F}$ denotes the norm map of the extension L/F.

For completeness, we shall discuss briefly the archimedean L-parameters. For this we will consider only the complex archimedean place since this will be the case of interest to us. In this case, the Weil group is $W_{\mathbb{C}} = \mathbb{C}^{\times}$, and the definition of an archimedean L-parameter is analogous to the non-archimedean case (see Definition 3.2.8). For integers $n \geq 0$ and w with $n \equiv w + 1 \bmod 2$, define $\phi_{w,n} : W_{\mathbb{C}} \to GL(2, \mathbb{C})$ via

$$z \mapsto |z|^{-w} \begin{pmatrix} (z/\bar{z})^{n/2} & \\ & (z/\bar{z})^{-n/2} \end{pmatrix}.$$

According to §3.1 of [40], such a representation corresponds to an irreducible admissible representation of $GL(2, \mathbb{C})$ (the latter are described in §6 of [32]).

3.2.3 L-parameters for representations of $GSp(4, F)$

Now that we have seen the L-parameters attached to irreducible admissible representations of $GL(2, F)$, we are ready to list the L-parameters attached to irreducible admissible non-supercuspidal representations of $GSp(4, F)$. We follow Section 2.4 of [48].

Definition 3.2.9. An *L-parameter* for the group $GSp(4, F)$ is an equivalence class of continuous homomorphisms

$$\rho_0' : W_F' \to GSp(4, \mathbb{C}),$$

such that

- $\rho_0'|_{W_F}$ is semisimple;
- $\rho_0'|_{\mathbb{C}}$ is complex analytic.

The set of L-parameters for $GSp(4, F)$ will be denoted by $\Phi(GSp(4, F))$.

By Proposition 3.2.3, these are identified with pairs (ρ_0, N), where $\rho_0 : W_F \to GSp(4, \mathbb{C})$ is a semisimple representation of the Weil group, and N

is a nilpotent endomorphism of $\mathbb{C}^4$ such that $\exp(N) \in GSp(4,\mathbb{C})$. Below we describe the L-parameters for irreducible admissible non-supercuspidal representations of $GSp(4,F)$.

Type I To an irreducible representation of the form $\chi_1 \times \chi_2 \rtimes \sigma$ we attach the L-parameter (ρ_0, N), with $N = 0$ and $\rho_0 : W_F \to GSp(4,\mathbb{C})$ defined via

$$w \mapsto \begin{pmatrix} (\chi_1\chi_2\sigma)(w) & & & \\ & (\chi_1\sigma)(w) & & \\ & & (\chi_2\sigma)(w) & \\ & & & \sigma(w) \end{pmatrix}.$$

Type II In this case we have characters χ and σ of $F^\times$ such that $\chi^2 \neq |\ |^{\pm 1}$ and $\chi \neq |\ |^{\pm 3/2}$. Then $|\ |^{1/2}\chi \times |\ |^{-1/2}\chi \rtimes \sigma$ has two irreducible constituents, namely $\chi St_{GL(2)} \rtimes \sigma$ (type IIa) and $\chi 1_{GL(2)} \rtimes \sigma$ (type IIb). The L-parameters attached to these two irreducible representations have the same semisimple part $\rho_0 : W_F \to GSp(4,\mathbb{C})$ defined by

$$w \mapsto \begin{pmatrix} (\chi^2\sigma)(w) & & & \\ & |w|^{1/2}(\chi\sigma)(w) & & \\ & & |w|^{-1/2}(\chi\sigma)(w) & \\ & & & \sigma(w) \end{pmatrix}.$$

Moreover, $\chi St_{GL(2)} \rtimes \sigma$ has attached nilpotent part

$$N_1 = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

while $\chi 1_{GL(2)} \rtimes \sigma$ has attached nilpotent part $N = 0$.

Type III In this case, we assume that for the character χ of $F^\times$ we have $\chi \neq 1$ and $\chi \neq |\ |^{\pm 2}$. Then the induced from the Borel representation $\chi \times |\ | \rtimes |\ |^{-1/2}\sigma$ has two irreducible constituents, namely $\chi \rtimes \sigma St_{GSp(2)}$ (type IIIa) and $\chi \rtimes \sigma 1_{GSp(2)}$ (type IIIb). The L-parameters attached to each of these two irreducible representations will have the same semisimple part $\rho_0 : W_F \to GSp(4, \mathbb{C})$ that takes $w \in W_F$ to

$$\begin{pmatrix} |w|^{1/2}(\chi\sigma)(w) & & & \\ & |w|^{-1/2}(\chi\sigma)(w) & & \\ & & |w|^{1/2}\sigma(w) & \\ & & & |w|^{-1/2}\sigma(w) \end{pmatrix}.$$

To $\chi \rtimes \sigma St_{GSp(2)}$ we attach the L-parameter (ρ_0, N_4) with

$$N_4 = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix},$$

while $\chi \rtimes \sigma 1_{GSp(2)}$ has nilpotent part $N = 0$.

Type IV If σ is a character of $F^\times$, the representation $|\ |^2 \times |\ | \rtimes |\ |^{-3/2}\sigma$ has four irreducible constituents. The L-parameters attached to each of these will have the same semisimple part $\rho_0 : W_F \to GSp(4, \mathbb{C})$, with $w \in W_F$ taken to

$$\begin{pmatrix} |w|^{3/2}\sigma(w) & & & \\ & |w|^{1/2}\sigma(w) & & \\ & & |w|^{-1/2}\sigma(w) & \\ & & & |w|^{-3/2}\sigma(w) \end{pmatrix}.$$

To the irreducible constituent $\sigma St_{GSp(4)}$ (type IVa) we attach (ρ_0, N_5) with

$$N_5 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & -1 \\ & & & 0 \end{pmatrix},$$

to $L(|\ |^2, |\ |^{-1}\sigma St_{GSp(2)})$ (type IVb) we attach (ρ_0, N_4), to the subquotient $L(|\ |^{3/2} St_{GL(2)}, |\ |^{-3/2}\sigma)$ (type IVc) we attach (ρ_0, N_1), and to the irreducible quotient $\sigma 1_{GSp(4)}$ (type IVd) we attach (ρ_0, N) with $N = 0$.

Type V In this case we have a non-trivial quadratic character ξ and σ any character of $F^\times$. Then $|\,|\xi \times \xi \rtimes |\,|^{-1/2}\sigma$ decomposes into four irreducible constituents. The L-parameters attached to these four representations have the same semisimple part $\rho_0 : W_F \to GSp(4, \mathbb{C})$, that takes $w \in W_F$ to

$$\begin{pmatrix} |w|^{1/2}\sigma(w) & & & \\ & |w|^{1/2}(\xi\sigma)(w) & & \\ & & |w|^{-1/2}(\xi\sigma)(w) & \\ & & & |w|^{-1/2}\sigma(w) \end{pmatrix}.$$

To the irreducible subrepresentation $\delta([\xi, |\,|\xi], |\,|^{-1/2}\sigma)$ (type Va) we attach (ρ_0, N_3) where

$$N_3 = \begin{pmatrix} 0 & & & 1 \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

to the subquotient $L(|\,|^{1/2}\xi St_{GL(2)}, |\,|^{-1/2}\sigma)$ (type Vb) we attach (ρ_0, N_1), to the subquotient $L(|\,|^{1/2}\xi St_{GL(2)}, \xi|\,|^{-1/2}\sigma)$ (type Vc) we attach (ρ_0, N_2) where

$$N_2 = \begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix},$$

to the irreducible quotient $L(|\,|\xi, \xi \rtimes |\,|^{-1/2}\sigma)$ (type Vd) we attach (ρ_0, N) with $N = 0$.

Type VI For a character σ of $F^\times$ we consider $|\,| \times 1_{F^\times} \rtimes |\,|^{-1/2}\sigma$, which has four irreducible constituents. The L-parameters attached to them, all have the same semisimple part $\rho_0 : W_F \to GSp(4, \mathbb{C})$ which takes $w \in W_F$

to

$$\begin{pmatrix} |w|^{1/2}\sigma(w) & & & \\ & |w|^{1/2}\sigma(w) & & \\ & & |w|^{-1/2}\sigma(w) & \\ & & & |w|^{-1/2}\sigma(w) \end{pmatrix}.$$

To the irreducible constituents $\tau(S, |\ |^{-1/2}\sigma)$ and $\tau(T, |\ |^{-1/2}\sigma)$ (type VIa and type VIb respectively) we attach the L-parameter (ρ_0, N_3), to $L(|\ |^{1/2}St_{GL(2)}, |\ |^{-1/2}\sigma)$ (type VIc) we attach the L-parameter (ρ_0, N_1), and to $L(|\ |, 1_{F^\times} \rtimes |\ |^{-1/2}\sigma)$ (type VId) we attach the L-parameter (ρ_0, N) with $N = 0$.

Type VII Let χ be a character of $F^\times$ and π a supercuspidal representation of $GL(2, F)$. Let $\chi \rtimes \pi$ be irreducible (which is the case for type VII representations). We attach to $\chi \rtimes \pi$ the L-parameter (ρ_0, N) with $N = 0$ and $\rho_0 : W_F \to GSp(4, \mathbb{C})$ defined by

$$w \mapsto \begin{pmatrix} \chi(w)\det(\phi_\pi(w))\phi_\pi(w)' & \\ & \phi_\pi(w), \end{pmatrix}$$

where $\phi_\pi : W_F \to GL(2, \mathbb{C})$ is the L-parameter attached to the supercuspidal representation π. Here recall the notation A' for a matrix A from Subsection 2.1.1.

Type VIII In this case we have representations of the form $1 \rtimes \pi$ with π a supercuspidal representation of $GL(2, F)$. This has two irreducible constituents, namely $\tau(S, \pi)$ (Type VIIIa) and $\tau(T, \pi)$ (Type VIIIb), and to both of them we associate the L-parameter (ρ_0, N) with $N = 0$ and $\rho_0 : W_F \to GSp(4, \mathbb{C})$ defined via

$$w \mapsto \begin{pmatrix} \det(\phi_\pi(w))\phi_\pi(w)' & \\ & \phi_\pi(w) \end{pmatrix},$$

where $\phi_\pi : W_F \to GL(2, \mathbb{C})$ is the L-parameter attached to the supercuspidal representation π.

Type IX Let π be a supercuspidal representation of $GL(2,F)$ and ξ a non-trivial quadratic character of $F^\times$ such that $\xi\pi = \pi$. The representation $|\ |\xi \rtimes |\ |^{-1/2}\pi$ splits into two irreducible constituents, namely $\delta(|\ |\xi, |\ |^{-1/2}\pi)$ (type IXa) and $L(|\ |\xi, |\ |^{-1/2}\pi)$ (type IXb). Both of them have L-parameters with semisimple part $\rho_0 : W_F \to GSp(4,\mathbb{C})$ defined by

$$w \mapsto \begin{pmatrix} \xi(w)|w|^{1/2}\det(\phi_\pi(w))\phi_\pi(w)' & \\ & |w|^{-1/2}\phi_\pi(w) \end{pmatrix},$$

where $\phi_\pi : W_F \to GL(2,\mathbb{C})$ is the L-parameter attached to π. The irreducible subrepresentation $\delta(|\ |\xi, |\ |^{-1/2}\pi)$ has L-parameter with nilpotent part the matrix N_6, and the irreducible quotient $L(|\ |\xi, |\ |^{-1/2}\pi)$ has L-parameter with nilpotent part the matrix $N = 0$.

Here $N_6 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S$ and S is the symmetric matrix from Lemma 2.4.1 of [48]; i.e., S is such that

$${}^t\phi_\pi(w) S \phi_\pi(w) = \xi(w)\det(\phi_\pi(w)) S$$

for all $w \in W_F$.

Type X Let π be a supercuspidal representation of $GL(2,F)$ and σ a character of $F^\times$. We have the representation $\pi \rtimes \sigma$, which is induced from the Siegel parabolic subgroup, and we assume that $\omega_\pi \neq |\ |^{\pm 1}$, so that $\pi \rtimes \sigma$ is irreducible. If $\phi_\pi : W_F \to GL(2,\mathbb{C})$ is the L-parameter attached to π, then we attach to $\pi \rtimes \sigma$ the L-parameter (ρ_0, N) where $N = 0$ and $\rho_0 : W_F \to GSp(4,\mathbb{C})$ is defined via

$$w \mapsto \begin{pmatrix} \sigma(w)\det(\phi_\pi(w)) & & \\ & \sigma(w)\phi_\pi(w) & \\ & & \sigma(w) \end{pmatrix}.$$

Type XI Finally, let π be a supercuspidal representation of $GL(2,F)$ with trivial central character and σ a character of $F^\times$. The representation $|\ |^{1/2}\pi \rtimes |\ |^{-1/2}\sigma$ decomposes into the irreducible constituents $\delta(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$

(type XIa) and $L(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ (type XIb). To $\delta(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ we attach the L-parameter (ρ_0, N_2), and to $L(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ we attach the L-parameter (ρ_0, N) with $N = 0$. The semisimple part is the same for both, i.e., it is $\rho_0 : W_F \to GSp(4, \mathbb{C})$ defined by

$$w \mapsto \begin{pmatrix} \sigma(w)|w|^{1/2} & & \\ & \sigma(w)\phi_\pi(w) & \\ & & \sigma(w)|w|^{-1/2} \end{pmatrix}.$$

Here ϕ_π is again the L-parameter attached to π.

As mentioned in the beginning of this section, Gan and Takeda proved the local Langlands conjecture for $GSp(4, F)$. In fact they prove that there is a surjective map

$$L : \mathrm{Irr}(GSp(4, F)) \to \Phi(GSp(4, F))$$

such that for an L-parameter ϕ, the fiber of ϕ consists of a finite number of elements, and it is called an *L-packet*. Let us state some properties of this correspondence, which can be found in [18].

Let $\phi \in \Phi(GSp(4, F))$. We consider the group

$$\mathcal{C}(\phi) = \mathrm{Cent}(\phi)\big/\mathrm{Cent}(\phi)^0\mathbb{C}^\times ,$$

where $\mathrm{Cent}(\phi)$ is the centralizer of the image of ϕ in $GSp(4, \mathbb{C})$, $\mathrm{Cent}(\phi)^0$ is the identity component of $\mathrm{Cent}(\phi)$, and $\mathbb{C}^\times$ denotes the center of $GSp(4, \mathbb{C})$. It is proved in the Main Theorem of [18] that the order of the group $\mathcal{C}(\phi)$ is either 1 or 2, for all $\phi \in \Phi(GSp(4, F))$. Moreover, Gan and Takeda show that for an L-parameter ϕ, the order of the group $\mathcal{C}(\phi)$ is equal to the size of the L-packet of ϕ; thus an L-packet consists either of one representation or two representations of $GSp(4, F)$. If the L-packet of ϕ consists of two irreducible admissible representations, then exactly one of them is generic. Examples of L-packets consisting of two elements are $\{\tau(S, |\ |^{-1/2}\sigma), \tau(T, |\ |^{-1/2}\sigma)\}$, and $\{\tau(S, \pi), \tau(T, \pi)\}$; the former L-packet consists of types VIa and VIb, and the latter L-packet consists of types VIIIa and VIIIb. Moreover, as one can see from Table A.7 in [48], the generic representations $\delta([\xi, |\ |\xi], |\ |^{-1/2}\sigma)$ (type Va) and $\delta(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ (type XIa) also belong to L-packets consisting of two elements, since the corresponding group $\mathcal{C}(\phi)$ is of order 2. As a result, we know that there exist non-generic supercuspidal representations, which we denote $\delta^*([\xi, |\ |\xi], |\ |^{-1/2}\sigma)$ (say of type Va*) and $\delta^*(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ (say of type XIa*), such that they have the same L-parameters with $\delta([\xi, |\ |\xi], |\ |^{-1/2}\sigma)$ and $\delta(|\ |^{1/2}\pi, |\ |^{-1/2}\sigma)$ respectively.

Another property of the local Langlands correspondence for $GSp(4, F)$ is that if π is an irreducible admissible representation of $GSp(4, F)$ and $L(\pi) = (\rho_0, N)$ is the corresponding L-parameter, we have that $\lambda(\rho_0)$ is equal to the central character of π. Here λ is the similitude character of $GSp(4, \mathbb{C})$.

Moreover, the L-parameters are compatible with twisting; i.e., if $\phi = (\rho_0, N)$ is the L-parameter attached to an irreducible admissible representation π, then the twist $\chi\pi$ of π by the character χ has L-parameter $(\chi \otimes \rho_0, N)$.

3.3 The archimedean L-parameters

In this section, we will give some information on the archimedean L-parameters, which are defined as in the non-archimedean case (see Definition 3.2.9). We consider only the real archimedean place ∞. The Weil group of $\mathbb{R}$ is $W_{\mathbb{R}} = \mathbb{C}^\times \cup \mathbb{C}^\times j$, where $j^2 = -1$ and $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^\times$. The local reciprocity map in this case induces an isomorphism

$$W_{\mathbb{R}}^{\mathrm{ab}} \cong \mathbb{R}^\times,$$

which is given by

$$z \mapsto |z|_{\mathbb{C}} = |z|^2,$$

for $z \in \mathbb{C}^\times$, and

$$j \mapsto -1.$$

Let w, m_1, m_2 be integers with $m_1 > m_2 \geq 0$ and $m_1 + m_2 \equiv w + 1 \bmod 2$.

Then we define an L-parameter

$$\phi_{(w;m_1,m_2)} : W_{\mathbb{R}} \to GSp(4,\mathbb{C}),$$

that sends the element $z \in \mathbb{C}^\times$ to

$$|z|^{-w}\begin{pmatrix} (z/\bar{z})^{(m_1+m_2)/2} & & & \\ & (z/\bar{z})^{(m_1-m_2)/2} & & \\ & & (z/\bar{z})^{-(m_1-m_2)/2} & \\ & & & (z/\bar{z})^{-(m_1+m_2)/2} \end{pmatrix},$$

and

$$j \mapsto \begin{pmatrix} & & & 1 \\ & & 1 & \\ & (-1)^{w+1} & & \\ (-1)^{w+1} & & & \end{pmatrix}.$$

The archimedean L-packet that corresponds to the above L-parameter consists of two elements $\Pi^W_{(w;m_1,m_2)}$ and $\Pi^H_{(w;m_1,m_2)}$. The former one is a generic representation, and the latter is non-generic. For $m_2 \geq 1$, the representations $\Pi^W_{(w;m_1,m_2)}$ and $\Pi^H_{(w;m_1,m_2)}$ belong to the discrete series representations of $GSp(4,\mathbb{R})$, with $\Pi^W_{(w;m_1,m_2)}$ being generic discrete series and $\Pi^H_{(w;m_1,m_2)}$ being holomorphic discrete series. When $m_2 = 0$, $\Pi^W_{(w;m_1,m_2)}$ and $\Pi^H_{(w;m_1,m_2)}$ are limits of discrete series representations of $GSp(4,\mathbb{R})$. For more information, see for instance §3.1 of [40].

In the holomorphic case, it is more common to use the Blattner parameter by setting $k_1 = m_1 + 1$, and $k_2 = m_2 + 2$. In this notation, we will denote the L-parameter by $\phi_{(w;k_1,k_2)}$, with $k_1 \geq k_2 \geq 2$. Then the representation is holomorphic discrete series when $k_1 \geq k_2 \geq 3$. Note that the pair (k_1, k_2) corresponds to the minimal K-type of the representation $\Pi^H_{(w;m_1,m_2)}$ (here $K = U(2)$ is the maximal compact subgroup of $GSp(4,\mathbb{R})$), which is given by the representation

$$\mathrm{Sym}^{k_1-k_2}\mathbb{C}^2 \otimes \det{}^{\otimes k_2}.$$

Definition 3.3.1. Let Π be an automorphic representation of $GSp(4,\mathbb{A}_{\mathbb{Q}})$.

Suppose that the representation Π_∞ at the archimedean place is holomorphic or a holomorphic limit of discrete series representation of $GSp(4,\mathbb{R})$, which corresponds to an L-parameter $\phi_{(w;k_1,k_2)}$. The pair (k_1,k_2) $(k_1 \geq k_2 \geq 2)$ is called the *weight* of Π. If in addition Π_∞ is a holomorphic discrete series representation, i.e., $k_2 \geq 3$, then we will say that Π is a *cohomological* representation.

3.4 L-factors for representations of $GSp(4,F)$

Having the L-parameters in our machinery, we are going to consider the L-factors associated to them. There are two kinds of L-factors for representations of $GSp(4,F)$; the degree 4 and the degree 5. We first consider the degree 4 L-factors.

3.4.1 The degree 4 L-factors

We define the L-factors for the L-parameters of irreducible admissible representations of $GSp(4,F)$ following [48]. Let $\phi = (\rho_0, N)$ be a representation acting on a space V. Moreover, we let Frob be an element lying above the inverse Frobenius automorphism[1]. If V_N is the kernel of N, $V^{I_F} = \{v \in V : \rho_0(g)v = v \text{ for all } g \in I_F\}$, and $V_N^{I_F} = V^{I_F} \cap V_N$, we define the degree 4 L-factor of ϕ by

$$L(\phi,s) = \det(1 - q^{-s}\rho_0(\mathrm{Frob})|V_N^{I_F})^{-1}.$$

Now we consider the degree 4 L-factors attached to irreducible admissible representations of $GSp(4,F)$. The L-factors for generic representations and non-generic non-supercuspidal representations are defined in Section 4 of [18]; the L-factors for non-generic supercuspidal representations are studied in [11]. The degree 4 L-factors for irreducible admissible non-supercuspidal representations of $GSp(4,F)$ are given in detail in Table A.8 of [48]. We

[1] In this thesis, we usually denote this element by ϕ. Here we denote it by Frob so that we do not confuse it with the notation of the L-parameter.

present them in Table 3.1 for completion. For a character χ of $F^\times$, we will write

$$L(\chi, s) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1}, & \text{if } \chi \text{ is unramified,} \\ 1, & \text{if } \chi \text{ is ramified.} \end{cases}$$

Remark 3.4.1. The L-factor of an irreducible admissible representation of $GSp(4, F)$ is equal to the L-factor of the corresponding L-parameters. This is part of the local Langlands correspondence for $GSp(4, F)$, and it is proved in [18] for generic representations or non-generic non-supercuspidal representations, and in [11] for non-generic supercuspidal representations.

Table 3.1: Degree 4 L-factors for non-supercuspidal representations of $GSp(4, F)$

Type	Representation	$L(\phi, s)$
I	$\chi_1 \times \chi_2 \rtimes \sigma$	$L(\chi_1\chi_2\sigma, s)L(\chi_1\sigma, s)L(\chi_2\sigma, s)L(\sigma, s)$
IIa	$\chi St_{GL(2)} \rtimes \sigma$	$L(\chi^2\sigma, s)L(\| \|^{1/2}\chi\sigma, s)L(\sigma, s)$
IIb	$\chi 1_{GL(2)} \rtimes \sigma$	$L(\chi^2\sigma, s)L(\| \|^{1/2}\chi\sigma, s)L(\| \|^{-1/2}\chi\sigma, s)L(\sigma, s)$
IIIa	$\chi \rtimes \sigma St_{GSp(2)}$	$L(\chi\| \|^{1/2}\sigma, s)L(\| \|^{1/2}\sigma, s)$
IIIb	$\chi \rtimes \sigma 1_{GSp(2)}$	$L(\chi\| \|^{1/2}\sigma, s)L(\chi\| \|^{-1/2}\sigma, s)L(\| \|^{1/2}\sigma, s)L(\| \|^{-1/2}\sigma, s)$
IVa	$\sigma St_{GSp(4)}$	$L(\| \|^{3/2}\sigma, s)$
IVb	$L(\| \|^2, \| \|^{-1}\sigma St_{GSp(2)})$	$L(\| \|^{3/2}\sigma, s)L(\| \|^{-1/2}\sigma, s)$
IVc	$L(\| \|^{3/2}St_{GL(2)}, \| \|^{-3/2}\sigma)$	$L(\| \|^{3/2}\sigma, s)L(\| \|^{1/2}\sigma, s)L(\| \|^{-3/2}\sigma, s)$
IVd	$\sigma 1_{GSp(4)}$	$L(\| \|^{3/2}\sigma, s)L(\| \|^{1/2}\sigma, s)L(\| \|^{-1/2}\sigma, s)L(\| \|^{-3/2}\sigma, s)$
Va	$\delta([\xi, \| \|\xi], \| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)L(\xi\| \|^{1/2}\sigma, s)$
Vb	$L(\| \|^{1/2}\xi St_{GL(2)}, \| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)L(\xi\| \|^{1/2}\sigma, s)L(\| \|^{-1/2}\sigma, s)$
Vc	$L(\| \|^{1/2}\xi St_{GL(2)}, \xi\| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)L(\xi\| \|^{1/2}\sigma, s)L(\xi\| \|^{-1/2}\sigma, s)$
Vd	$L(\| \|\xi, \xi \rtimes \| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)L(\xi\| \|^{1/2}\sigma, s)L(\xi\| \|^{-1/2}\sigma, s)L(\| \|^{-1/2}\sigma, s)$
VIa	$\tau(S, \| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)^2$
VIb	$\tau(T, \| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)^2$
VIc	$L(\| \|^{1/2}St_{GL(2)}, \| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)^2L(\| \|^{-1/2}\sigma, s)$
VId	$L(\| \|, 1_{F^\times} \rtimes \| \|^{-1/2}\sigma)$	$L(\| \|^{1/2}\sigma, s)^2L(\| \|^{-1/2}\sigma, s)^2$

VII	$\chi \rtimes \pi$	1
VIIIa	$\tau(S, \pi)$	1
VIIIb	$\tau(T, \pi)$	1
IXa	$\delta(\vert\ \vert\xi, \vert\ \vert^{-1/2}\pi)$	1
IXb	$L(\vert\ \vert\xi, \vert\ \vert^{-1/2}\pi)$	1
X	$\pi \rtimes \sigma$	$L(\sigma, s)L(\omega_\pi\sigma, s)$
XIa	$\delta(\vert\ \vert^{1/2}\pi, \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert^{1/2}\sigma, s)$
XIb	$L(\vert\ \vert^{1/2}\pi, \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert^{1/2}\sigma, s)L(\vert\ \vert^{-1/2}\sigma, s)$

3.4.2 The degree 5 L-factors

One knows that there is an isomorphism between the projective group $PGSp(4)$ and $SO(5)$ as algebraic groups; this is described over a field of characteristic not equal to 2 in Section A.7 of [48]. Here

$$SO(n) = \{g \in SL(n) : {}^t g J_n g = J_n\},$$

where J_n is the $n \times n$ matrix with 1 on the anti-diagonal, and 0 everywhere else.

Thus, there is a homomorphism

$$\rho_5 : GSp(4, \mathbb{C}) \to SO(5, \mathbb{C})$$

which we compose with an L-parameter $\phi = (\rho_0, N)$ of an irreducible admissible representation of $GSp(4, F)$ to get $\rho_5 \circ \phi$, which is a 5-dimensional representation of the Weil-Deligne group W'_F. This homomorphism is analogous to the homomorphism

$$\mathrm{Ad} : GL(2, \mathbb{C}) \to SO(3, \mathbb{C}),$$

defined via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{ad - bc} \begin{pmatrix} a^2 & -ab & -b^2/2 \\ -2ac & ad + bc & bd \\ -2c^2 & 2cd & d^2 \end{pmatrix}.$$

The degree 5 L-factors $L(\rho_5 \circ \phi, s)$ of the L-parameters of irreducible admissible representations of $GSp(4, F)$ are given in Table A.10 of [48], and we present them in Table 3.2. These local factors are important for this thesis, since the partial degree 5 L-function (which will be defined later) gives a criterion for an automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ to be a theta lift (a notion that will be discussed in a later chapter).

Note that in Table 3.2 we denote by ϕ_π the L-parameter in $GL(2, \mathbb{C})$ of the supercuspidal representation π of $GL(2, F)$.

Table 3.2: Degree 5 L-factors for non-supercuspidal representations of $GSp(4,F)$

Type	Representation	$L(\rho_5 \circ \phi, s)$
I	$\chi_1 \times \chi_2 \rtimes \sigma$	$L(\chi_1, s)L(\chi_1^{-1}, s)L(\chi_2, s)L(\chi_2^{-1}, s)L(1, s)$
IIa	$\chi St_{GL(2)} \rtimes \sigma$	$L(\vert\ \vert^{1/2}\chi, s)L(\vert\ \vert^{1/2}\chi^{-1}, s)L(1, s)$
IIb	$\chi 1_{GL(2)} \rtimes \sigma$	$L(\vert\ \vert^{1/2}\chi, s)L(\vert\ \vert^{-1/2}\chi, s)L(\vert\ \vert^{1/2}\chi^{-1}, s)L(\vert\ \vert^{-1/2}\chi^{-1}, s)L(1, s)$
IIIa	$\chi \rtimes \sigma St_{GSp(2)}$	$L(\chi, s)L(\chi^{-1}, s)L(\vert\ \vert, s)$
IIIb	$\chi \rtimes \sigma 1_{GSp(2)}$	$L(\chi, s)L(\chi^{-1}, s)L(\vert\ \vert, s)L(\vert\ \vert^{-1}, s)L(1, s)$
IVa	$\sigma St_{GSp(4)}$	$L(\vert\ \vert^2, s)$
IVb	$L(\vert\ \vert^2, \vert\ \vert^{-1}\sigma St_{GSp(2)})$	$L(\vert\ \vert^2, s)L(\vert\ \vert, s)L(\vert\ \vert^{-2}, s)$
IVc	$L(\vert\ \vert^{3/2} St_{GL(2)}, \vert\ \vert^{-3/2}\sigma)$	$L(\vert\ \vert^2, s)L(1, s)L(\vert\ \vert^{-1}, s)$
IVd	$\sigma 1_{GSp(4)}$	$L(\vert\ \vert^2, s)L(\vert\ \vert, s)L(1, s)L(\vert\ \vert^{-1}, s)L(\vert\ \vert^{-2}, s)$
Va	$\delta([\xi, \vert\ \vert\xi], \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert\xi, s)L(\xi, s)L(1, s)$
Vb	$L(\vert\ \vert^{1/2}\xi St_{GL(2)}, \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert\xi, s)L(\xi, s)L(1, s)$
Vc	$L(\vert\ \vert^{1/2}\xi St_{GL(2)}, \xi\vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert\xi, s)L(\xi, s)L(1, s)$
Vd	$L(\vert\ \vert\xi, \xi \rtimes \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert\xi, s)L(\vert\ \vert^{-1}\xi, s)L(\xi, s)^2 L(1, s)$
VIa	$\tau(S, \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert, s)L(1, s)^2$
VIb	$\tau(T, \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert, s)L(1, s)^2$
VIc	$L(\vert\ \vert^{1/2} St_{GL(2)}, \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert, s)L(1, s)^2$
VId	$L(\vert\ \vert, 1_{F^\times} \rtimes \vert\ \vert^{-1/2}\sigma)$	$L(\vert\ \vert, s)L(\vert\ \vert^{-1}, s)L(1, s)^3$

VII	$\chi \rtimes \pi$	$L(\chi, s)L(\chi^{-1}, s)L(\mathrm{Ad} \circ \phi_\pi, s)$
VIIIa	$\tau(S, \pi)$	$L(1, s)^2 L(\mathrm{Ad} \circ \phi_\pi, s)$
VIIIb	$\tau(T, \pi)$	$L(1, s)^2 L(\mathrm{Ad} \circ \phi_\pi, s)$
IXa	$\delta(\| \ \|\xi, \| \ \|^{-1/2}\pi)$	$L(\| \ \|\xi, s)L(\mathrm{Ad} \circ \phi_\pi, s)L(\xi, s)^{-1}$
IXb	$L(\| \ \|\xi, \| \ \|^{-1/2}\pi)$	$L(\| \ \|\xi, s)L(\| \ \|^{-1}\xi, s)L(\mathrm{Ad} \circ \phi_\pi, s)$
X	$\pi \rtimes \sigma$	$L(\phi_\pi, s)L(\det(\phi_\pi)^{-1}\phi_\pi, s)L(1, s)$
XIa	$\delta(\| \ \|^{1/2}\pi, \| \ \|^{-1/2}\sigma)$	$L(\| \ \|^{1/2}\phi_\pi, s)L(1, s)$
XIb	$L(\| \ \|^{1/2}\pi, \| \ \|^{-1/2}\sigma)$	$L(\| \ \|^{1/2}\phi_\pi, s)L(\| \ \|^{-1/2}\phi_\pi, s)L(1, s)$

Chapter 4

On the ramification of representations of $GSp(4, F)$

In this chapter we are going to study the ramification of the irreducible admissible non-supercuspidal representations of $GSp(4, F)$, where F is a non-archimedean local field. In particular, we will talk about the notion of the conductor of the L-parameter attached to an admissible representation of $GSp(4, F)$, and discuss our result on how the conductor degenerates if we consider the representation modulo a prime l; this is proved for non-supercuspidal representations of $GSp(4, F)$. Our result is a generalization of a known result on the degeneration of conductors for representations of $GL(2, F)$, which was obtained independently by Carayol and Livné in 1989 (see [7] and [38] respectively).

4.1 The ramification of L-parameters

We start our discussion by defining the notion of *ramification* of representations. In particular we will define what a conductor is for the Weil-Deligne representations, and then we will list the conductors of the L-parameters associated to irreducible admissible representations of $GL(2, F)$ and $GSp(4, F)$.

4.1.1 Conductors of Weil-Deligne representations

Let $\rho_0' = (\rho_0, N)$ be an admissible representation of W_F' acting on a vector space V. Denote by V_N the kernel of N, and if $V^{I_F} = \{v \in V : \rho_0(g)v = v \text{ for all } g \in I_F\}$, let $V_N^{I_F} = V^{I_F} \cap V_N$. Set

$$b(\rho_0') = \dim V^{I_F} - \dim V_N^{I_F}.$$

Another quantity that we need to define is the following, which involves the ramification groups discussed in Subsection 3.1.2. Let L/F^{nr} be a finite Galois extension such that ρ_0 is trivial on the subgroup $\mathrm{Gal}(\bar{F}/L)$ of I_F. Set $G = \mathrm{Gal}(L/F^{nr})$. Then we define

$$a(\rho_0) = \sum_{i=0}^{\infty} \left(\dim V - \dim V^{\rho_0(G_i)}\right) \frac{g_i}{g_0},$$

where the G_i's are the ramification groups of the lower numbering, and g_i is the order of the group G_i. Note that since G_i is trivial for sufficiently large i, all but finitely many terms in the sum are zero. We present some facts about the quantity we just defined.

Proposition 4.1.1. *The following hold for $a(\rho_0)$:*

1. *The definition of $a(\rho_0)$ is independent of the choice of L.*
2. *$a(\rho_0)$ is a non-negative integer.*
3. *If we have a short exact sequence*

$$\rho_1 \hookrightarrow \rho_0 \twoheadrightarrow \rho_2$$

   we have that $a(\rho_0) = a(\rho_1) + a(\rho_2)$.

4. *If K is a finite extension of F in $\bar{F}$ and σ_0 is a representation of W_K, we have that*

$$a(\mathrm{ind}_{W_K}^{W_F} \sigma_0) = \dim(\sigma_0) d(K/F) + f(K/F) a(\sigma_0).$$

Here $f(K/F)$ is the residue degree $[k_K : k_F]$, and $d(K/F)$ is such that $\varpi^{d(K/F)}\mathcal{O}_F$ is the relative discriminant of K/F.

5. *Let K/F be a finite extension and χ be a character of $K^\times$ (identified with a character of W_K). Then $a(\chi)$ is equal to the conductor of χ as defined in Definition 3.1.1. That is, χ is unramified if and only if $a(\chi) = 0$.*

In addition, having a function a satisfying properties 3, 4, and 5, one may define $a(\rho_0)$ for all representations ρ_0 of W_F.

Proof. The reader may consult §10 of [52]. □

One can also write $a(\rho_0)$ with respect to the upper numbering. This can be done as follows:

$$\begin{aligned} a(\rho_0) &= \sum_{i=0}^{\infty} \left(\dim V - \dim V^{\rho_0(G_i)}\right) \frac{g_i}{g_0} \\ &= \sum_{i=0}^{\infty} \left(\dim V - \dim V^{\rho_0(G_i)}\right) \left(\phi_{L/F^{nr}}(i) - \phi_{L/F^{nr}}(i-1)\right) \\ &= \sum_{i=0}^{\infty} \int_{\phi_{L/F^{nr}}(i-1)}^{\phi_{L/F^{nr}}(i)} \left(\dim V - \dim V^{\rho_0(G^u)}\right) du. \end{aligned}$$

As a result, one has

$$a(\rho_0) = \int_{-1}^{\infty} \left(\dim V - \dim V^{\rho_0(G^u)}\right) du. \tag{4.1}$$

Definition 4.1.2. We define the *conductor* of a Weil-Deligne representation $\rho_0 = (\rho_0, N)$ via

$$a(\rho_0') = a(\rho_0) + b(\rho_0').$$

Moreover, we say that $a(\rho_0)$ is the *conductor* of the representation ρ_0 of the Weil group W_F.

Note that by Definition 3.2.4 one has that ρ_0' is unramified if and only if $b(\rho_0') = 0$ and $a(\rho_0) = 0$; that is, when $a(\rho_0') = 0$. From Proposition 4.1.1

one has the following properties for the conductor of a Weil-Deligne representation.

Proposition 4.1.3. *For the conductor of a Weil-Deligne representation ρ_0' we have the following:*

1. $a(\sigma_0' \oplus \tau_0') = a(\sigma_0') + a(\tau_0')$.

2. *If K is a finite extension of F in $\bar{F}$ and σ_0' is a representation of W_K', we have that*

$$a(\mathrm{ind}_{W_K}^{W_F} \sigma_0') = \dim(\sigma_0') d(K/F) + f(K/F) a(\sigma_0').$$

   Here $f(K/F)$ and $d(K/F)$ are as before.

3. *If $\dim \rho_0' = 1$, so that $\rho_0' = (\rho_0, 0)$, then $a(\rho_0') = a(\rho_0)$ which is equal to the conductor of Definition 3.1.1.*

Proof. See §10 of [52]. □

4.1.2 Conductors of irreducible admissible representations of $GL(2, F)$

Before we write down the conductors of the L-parameters of irreducible admissible representations of $GL(2, F)$, we need the following lemma in which we compute the conductor of the quadratic character $\epsilon_{L/F}$ of a quadratic extension L/F.

Lemma 4.1.4. *Let L/F be a quadratic extension of non-archimedean local fields and v_F the discrete valuation on F. Moreover, suppose $\epsilon_{L/F}$ is the quadratic character of this extension, which we may view as a representation of the Weil group W_F via the reciprocity map of local class field theory. Then the following hold:*

a) if L/F is unramified, we have

$$a(\epsilon_{L/F}) = d(L/F) = 0;$$

b) if L/F is ramified, we have

$$a(\epsilon_{L/F}) = d(L/F) = 1 + 2v_F(2).$$

Proof. We consider the two cases separately:

a) If L/F is unramified, then $\mathcal{O}_F^\times = N_{L/F}(\mathcal{O}_L^\times)$. Then, by definition of $\epsilon_{L/F}$ we have that $\epsilon_{L/F}(\mathcal{O}_F^\times) = 1$, so that is unramified. Moreover, by Corollary 1 in §5 of Chapter III of [56], we have $d(L/F) = 0$.

b) Suppose now that L/F is ramified. In this case $L = F(\sqrt{\varpi})$ and $\mathcal{O}_L = \mathcal{O}_F[\sqrt{\varpi}]$. Moreover, the conductor of $\epsilon_{L/F}$ as a representation of the Weil group, can be written as

$$a(\epsilon_{L/F}) = \sum_{i=0}^{\infty}(1 - \dim V^{\epsilon_{L/F}(G_i)})\frac{g_i}{g_0}, \tag{4.2}$$

where

$$G_i = \{\sigma \in \mathrm{Gal}(L/F) : \sigma(\sqrt{\varpi}) \equiv \sqrt{\varpi} \bmod \mathfrak{p}_L^{i+1}\}.$$

Since L/F is ramified quadratic, we get $g_0 = 2$; furthermore, when $\epsilon_{L/F}(G_i)$ is trivial, we get no contribution to Equation (4.2) for this particular i. If $\epsilon_{L/F}(G_i)$ is non-trivial, we have $g_i = 2$ (as a non-trivial subgroup of the inertia) and $\dim V^{\epsilon_{L/F}(G_i)} = 0$; thus we get a contribution of 1 in the sum (4.2) in this case. As a result, we get

$$a(\epsilon_{L/F}) = i_0,$$

where i_0 is the first value such that $g_{i_0} = 1$. Write $\mathrm{Gal}(L/F) = \{1, \tau\}$, we have $G_{i_0-1} = \{1, \tau\}$. This means that $\tau(\sqrt{\varpi}) \equiv \sqrt{\varpi} \bmod \sqrt{\varpi}^{i_0}$, or equivalently, that $-2\sqrt{\varpi} \in \sqrt{\varpi}^{i_0}\mathcal{O}_L$. Hence, $v_L(2) = i_0 - 1$, or equivalently, $i_0 = 1 + v_L(2) = 1 + 2v_F(2)$.

For the relative discriminant of L/F we have that it is equal to

$$\varpi^{d(L/F)}\mathcal{O}_F = \left(\det\begin{pmatrix} 1 & \sqrt{\varpi} \\ 1 & -\sqrt{\varpi}\end{pmatrix}\right)^2 \mathcal{O}_F,$$

i.e., $\varpi^{d(L/F)}\mathcal{O}_F = 2^2\varpi\mathcal{O}_F$. Also note that $2v_F(2) = i_0 - 1$, so that

$$\varpi^{d(L/F)}\mathcal{O}_F = \varpi^{i_0}\mathcal{O}_F.$$

As a result,

$$a(\epsilon_{L/F}) = d(L/F) = 1 + 2v_F(2).$$

□

Proposition 4.1.5. *The conductors of the L-parameters of irreducible admissible representations of $GL(2, F)$ are the following:*

i. The principal series representations $\chi_1 \times \chi_2$, where $\chi_1\chi_2^{-1} \neq |\ |^{\pm 1}$, have L-parameter of conductor

$$a(\chi_1 \times \chi_2) = a(\chi_1) + a(\chi_2).$$

ii. The twisted Steinberg representations $(\chi|\ |^{1/2})St_{GL(2)}$ have L-parameter of conductor

$$a((\chi|\ |^{1/2})St_{GL(2)}) = \begin{cases} 1, & \text{if } a(\chi) = 0; \\ 2a(\chi), & \text{if } a(\chi) > 0. \end{cases}$$

iii. The supercuspidal representations $BC(L/F, \psi)$ have L-parameter of conductor

$$a(BC(L/F, \psi)) = \begin{cases} 2a(\psi), & \text{if } L/F \text{ is unramified;} \\ a(\epsilon_{L/F}) + a(\psi), & \text{if } L/F \text{ is ramified.} \end{cases}$$

Proof. For the principal series representations it is a straightforward application of property 1 of Proposition 4.1.3.

For the twisted Steinberg representations we have that, for χ unramified, $\dim V^{I_F} = 2$ and $\dim V_N^{I_F} = 1$, so that the conductor is 1. On the other hand, for ramified χ, $\dim V^{I_F} - \dim V_N^{I_F} = 0$, so that by property 1 of Proposition 4.1.3, we get $a((\chi|\ |^{1/2})St_{GL(2)}) = 2a(\chi)$.

Finally, for the supercuspidal representations we will use property 2 of Proposition 4.1.3. If ψ is the character of W_L such that $\rho_0 = \mathrm{ind}_{W_L}^{W_F}\psi$, we have the following two cases:

a) If L/F is unramified, we have $d(L/F) = 0$ and $f(L/F) = 2$, so that

$$a(BC(L/F, \psi)) = a(\mathrm{ind}_{W_L}^{W_F}\psi) = 2a(\psi).$$

b) Suppose now that L/F is ramified. Then $f(L/F) = 1$ and by Lemma 4.1.4 we get $d(L/F) = a(\epsilon_{L/F})$. This implies

$$a(BC(L/F, \psi)) = a(\mathrm{ind}_{W_L}^{W_F}\psi) = a(\epsilon_{L/F}) + a(\psi).$$

□

There is also a notion of a conductor for the representations of $GL(2, F)$, which is obtained independently from the conductor of the attached L-parameter.

Definition 4.1.6. Consider the open compact subgroup

$$\Gamma_1(\mathfrak{p}_F^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathcal{O}_F) : c \in \mathfrak{p}_F^n, d \in 1 + \mathfrak{p}_F^n \right\}$$

of $GL(2, F)$. If π is an infinite dimensional irreducible admissible representation of $GL(2, F)$ and $\pi^{\Gamma_1(\mathfrak{p}_F^n)} \neq 0$ for some integer n, then we let $a(\pi)$ be the smallest such n. The integer $a(\pi)$ is called the *conductor* of π.

Remark 4.1.7. It is a consequence of the local Langlands correspondence for $GL(2, F)$ that the conductor of an irreducible admissible representation π as defined above, is equal to the conductor of its L-parameter ϕ_π. So the two definitions coincide. This is implied from the fact that the local Langlands correspondence preserves the so-called ε-factors of π and ϕ_π; for more details on ε-factors see [24] and Theorem 1.4.1 of [36].

4.1.3 Conductors of irreducible admissible representations of $GSp(4, F)$

Now we are going to write down the conductors for the L-parameters of irreducible admissible non-supercuspidal representations of $GSp(4, F)$; our list will include the two supercuspidal representations of types Va* and XIa* that we mentioned in Subsection 3.2.3.

Recall the nilpotent matrices N_1, N_2, N_3, N_4, N_5, and N_6, from Subsection 3.2.3. For the dimension of the kernel V_N of N we have that when $N = 0$, $\dim V_N = \dim V = 4$. Moreover we get $\dim V_{N_1} = \dim V_{N_2} = 3$, $\dim V_{N_3} = \dim V_{N_4} = 2$, and $\dim V_{N_5} = 1$. Finally, we have $\dim V_{N_6} \geq 2$, which together with the fact that a supercuspidal representation for $GL(2, F)$ is always ramified, implies that $\dim V^{I_F} - \dim V_{N_6}^{I_F} = 0$ for the type IXa representations. One can obtain the conductors of the L-parameters by using property 1 of Proposition 4.1.3. In Table 4.1 we list the irreducible admissible non-supercuspidal representations of $GSp(4, F)$ together with the two supercuspidal representations of types Va* and XIa* with corresponding L-parameter (ρ_0, N), and their conductors. From this table we see that the representations of $GSp(4, F)$ with unramified L-parameters are exactly the following:

- type I with χ_1, χ_2, σ unramified;
- type IIb with χ, σ unramified;
- type IIIb with $\chi\sigma, \sigma$ unramified;
- type IVd with σ unramified;
- type Vd with $\xi\sigma, \sigma$ unramified;
- type VId with σ unramified.

Remark 4.1.8. One can define the so-called *ϵ-factors* for the L-parameters, as in §11 of [52]. The conductor of the L-parameter of an irreducible admissible representation of $GSp(4, F)$ appears in the exponent of q in the ϵ-factor, as in Proposition 2.4.2, i), of [48].

On the other hand, if π is an irreducible admissible representation of $GSp(4, F)$, one can define an ϵ-factor for π. For the definition of ϵ-factors for generic representations see Section 7 of [57]; Gan and Takeda extend this definition, covering also the case of non-supercuspidal representations, in Section 4 of [18]; finally, Danisman in his thesis [11], extends the definition of ϵ-factors to non-generic supercuspidal representations. A similar expression in the exponent of q of the ϵ-factor of π can be obtained for the conductor of generic representations with trivial central character in Proposition 2.6.6 or Corollary 7.5.5 of [48].

Note that the local Langlands correspondence preserves the ϵ-factors (see Main Theorem of [18] and Proposition 6.1 of [11]), so that at least for a generic irreducible admissible representation π of $GSp(4, F)$ with trivial central character, we have that the conductor of the L-parameter of π coincides with the *minimal paramodular level* of π, as defined in [48]. This can be obtained by comparing Proposition 2.4.2 and Corollary 7.5.5 of [48]. This is something analogous to the $GL(2, F)$ case as in Remark 4.1.7.

Table 4.1: Conductors of irreducible admissible non-supercuspidal representations of $GSp(4,F)$

Type	Representation	Inducing data	$a(\rho_0)$	$\dim V^{I_F} - \dim V_N^{I_F}$
I	$\chi_1 \times \chi_2 \rtimes \sigma$		$a(\chi_1\chi_2\sigma) + a(\chi_1\sigma) + a(\chi_2\sigma) + a(\sigma)$	0
IIa	$\chi St_{GL(2)} \rtimes \sigma$	χ, σ unr	0	1
		$\chi\sigma$ unr	$a(\chi) + a(\sigma)$	1
		otherwise	$a(\chi^2\sigma) + 2a(\chi\sigma) + a(\sigma)$	0
IIb	$\chi 1_{GL(2)} \rtimes \sigma$		$a(\chi^2\sigma) + 2a(\chi\sigma) + a(\sigma)$	0
IIIa	$\chi \rtimes \sigma St_{GSp(2)}$	$\chi\sigma$, σ unr	0	2
		$\chi\sigma$ unr	$2a(\sigma)$	1
		σ unr	$2a(\chi)$	1
		otherwise	$2a(\chi\sigma) + 2a(\sigma)$	0
IIIb	$\chi \rtimes \sigma 1_{GSp(2)}$		$2a(\chi\sigma) + 2a(\sigma)$	0
IVa	$\sigma St_{GSp(4)}$	σ unr	0	3
		otherwise	$4a(\sigma)$	0
IVb	$L(\lvert\ \rvert^2, \lvert\ \rvert^{-1}\sigma St_{GSp(2)})$	σ unr	0	2
		otherwise	$4a(\sigma)$	0
IVc	$L(\lvert\ \rvert^{3/2} St_{GL(2)}, \lvert\ \rvert^{-3/2}\sigma)$	σ unr	0	1
		otherwise	$4a(\sigma)$	0
IVd	$\sigma 1_{GSp(4)}$		$4a(\sigma)$	0
Va	$\delta([\xi, \lvert\ \rvert\xi], \lvert\ \rvert^{-1/2}\sigma)$	$\xi\sigma$, σ unr	0	2
		$\xi\sigma$ unr	$2a(\sigma)$	1
		σ unr	$2a(\xi)$	1
		otherwise	$2a(\sigma) + 2a(\xi\sigma)$	0

Vb	$L(\lvert\ \rvert^{1/2}\xi St_{GL(2)}, \lvert\ \rvert^{-1/2}\sigma)$	$\xi\sigma$, σ unr	0	1
		$\xi\sigma$ unr	$2a(\sigma)$	1
		σ unr	$2a(\xi)$	0
		otherwise	$2a(\sigma)+2a(\xi\sigma)$	0
Vc	$L(\lvert\ \rvert^{1/2}\xi St_{GL(2)}, \xi\lvert\ \rvert^{-1/2}\sigma)$	$\xi\sigma$, σ unr	0	1
		$\xi\sigma$ unr	$2a(\sigma)$	0
		σ unr	$2a(\xi)$	1
		otherwise	$2a(\sigma)+2a(\xi\sigma)$	0
Vd	$L(\lvert\ \rvert\xi, \xi \rtimes \lvert\ \rvert^{-1/2}\sigma)$		$2a(\sigma)+2a(\xi\sigma)$	0
VIa	$\tau(S, \lvert\ \rvert^{-1/2}\sigma)$	σ unr	0	2
		otherwise	$4a(\sigma)$	0
VIb	$\tau(T, \lvert\ \rvert^{-1/2}\sigma)$	σ unr	0	2
		otherwise	$4a(\sigma)$	0
VIc	$L(\lvert\ \rvert^{1/2}St_{GL(2)}, \lvert\ \rvert^{-1/2}\sigma)$	σ unr	0	1
		otherwise	$4a(\sigma)$	0
VId	$L(\lvert\ \rvert, 1_{F^\times} \rtimes \lvert\ \rvert^{-1/2}\sigma)$		$4a(\sigma)$	0
VII	$\chi \rtimes \pi$		$a(\chi\omega_\pi\phi'_\pi)+a(\phi_\pi)$	0
VIIIa	$\tau(S, \pi)$		$2a(\phi_\pi)$	0
VIIIb	$\tau(T, \pi)$		$2a(\phi_\pi)$	0
IXa	$\delta(\lvert\ \rvert\xi, \lvert\ \rvert^{-1/2}\pi)$		$2a(\phi_\pi)$	0
IXb	$L(\lvert\ \rvert\xi, \lvert\ \rvert^{-1/2}\pi)$		$2a(\phi_\pi)$	0
X	$\pi \rtimes \sigma$		$a(\sigma\omega_\pi)+a(\sigma\phi_\pi)+a(\sigma)$	0
XIa	$\delta(\lvert\ \rvert^{1/2}\pi, \lvert\ \rvert^{-1/2}\sigma)$	σ unr	$a(\phi_\pi)$	1
		otherwise	$2a(\sigma)+a(\sigma\phi_\pi)$	0
XIb	$L(\lvert\ \rvert^{1/2}\pi, \lvert\ \rvert^{-1/2}\sigma)$		$2a(\sigma)+a(\sigma\phi_\pi)$	0

Va*	$\delta^*([\xi, \lvert\ \rvert \xi], \lvert\ \rvert^{-1/2}\sigma)$	$\xi\sigma$, σ unr	0	2
		$\xi\sigma$ unr	$2a(\sigma)$	1
		σ unr	$2a(\xi)$	1
		otherwise	$2a(\sigma) + 2a(\xi\sigma)$	0
XIa*	$\delta^*(\lvert\ \rvert^{1/2}\pi, \lvert\ \rvert^{-1/2}\sigma)$	σ unr	$a(\phi_\pi)$	1
		otherwise	$2a(\sigma) + a(\sigma\phi_\pi)$	0

4.2 Degeneration of conductors

In this section, we will consider l-adic Galois representations and their reductions mod l. In fact, we will compare the conductor of an l-adic Galois representation ρ and the conductor of the mod l reduction of ρ. For the whole section, we will assume that the prime l is different from the residual characteristic p of F.

Let K be a number field, and K_v be its completion at a place v of K. We will denote by $G_K = \mathrm{Gal}(\bar{K}/K)$ the absolute Galois group over K, and by G_v the decomposition groups $\mathrm{Gal}(\bar{K}_v/K_v)$.

Definition 4.2.1. Let

$$\rho : G_v \to GL(V)$$

be a Galois representation. We define the *Artin conductor* of ρ to be

$$a(\rho) = \int_{-1}^{\infty} \left(\dim V - \dim V^{\rho(G_v^u)}\right) du.$$

Here G_v^u are the ramification groups of G_v with respect to the upper numbering.

Recall that l-adic Galois representations are in a bijection with Weil-Deligne representations; this bijection is given by Theorem 3.2.6. We would like to see that the definition of the conductor of a Weil-Deligne representation coincides with the definition of the conductor of an l-adic Galois representation under this correspondence. Let

$$\rho : G_v \to GL(V)$$

be an l-adic Galois representation as in Theorem 3.2.6, and (ρ_0, N) a Weil-Deligne representation. By using Equation (4.1), one gets

$$a(\rho) - a(\rho_0) = \int_{-1}^{\infty} \left(\dim V^{\rho_0(G_v^u)} - \dim V^{\rho(G_v^u)}\right) du.$$

Let $u > 0$. We have that $\rho(\sigma) = \rho_0(\sigma)\exp(t_l(\sigma)N)$ for $\sigma \in G^u$; but G^u is a subgroup of $\mathrm{Gal}(\bar{K}_v/K_v^{tr})$, and on the latter group t_l is zero by definition. This means that for $u > 0$ we have that $\rho(G^u) = \rho_0(G^u)$, and

$$\begin{aligned} a(\rho) - a(\rho_0) &= \int_{-1}^{0} \left(\dim V^{\rho_0(G^u)} - \dim V^{\rho(G^u)}\right) du \\ &= \dim V^{\rho_0(I_{K_v})} - \dim V^{\rho(I_{K_v})} \\ &= \dim V^{I_{K_v}} - \dim V_N^{I_{K_v}}. \end{aligned}$$

This proves that

$$a(\rho) = a((\rho_0, N)),$$

and the two definitions coincide.

Suppose we have a Galois representation

$$\rho : G_v \to GL(V).$$

For the Artin conductor we have

$$\begin{aligned} a(\rho) &= \int_{-1}^{\infty} \left(\dim V - \dim V^{\rho(G^u)}\right) du \\ &= \left(\dim V - \dim V^{\rho(I_{K_v})}\right) + \int_{0}^{\infty} \left(\dim V - \dim V^{\rho(G^u)}\right) du. \end{aligned}$$

Definition 4.2.2. The quantity

$$\mathrm{sw}(\rho) = \int_{0}^{\infty} \left(\dim V - \dim V^{\rho(G^u)}\right) du$$

is the *Swan conductor* of the representation ρ.

4.2.1 The Carayol-Livné classification

Now we briefly discuss how the conductors of representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ degenerate when we consider them modulo the prime l, which is different from p. We may view these representations as representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ via the local Langlands correspondence and Theorem 3.2.6, and study them from this perspective. The main references for this theory are [7] and [38].

Let

$$\bar{\rho} : G_{\mathbb{Q}} \to GL(2, \bar{\mathbb{F}}_l)$$

be a modular mod l Galois representation. That is, $\bar{\rho}$ has a lift

$$\rho : G_{\mathbb{Q}} \to GL(2, \bar{\mathbb{Q}}_l)$$

which is attached to a modular form of some weight and some level N. If $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the decomposition group at p (with $p \neq l$), and $\bar{\rho}_p$ is the restriction of $\bar{\rho}$ to G_p, i.e.,

$$\bar{\rho}_p : G_p \to GL(2, \bar{\mathbb{F}}_l),$$

let

$$N(\bar{\rho}) = \prod_{p \neq l} p^{a(\bar{\rho}_p)}$$

be the (global) conductor of $\bar{\rho}$. Also, we denote by

$$\rho_p : G_p \to GL(2, \bar{\mathbb{Q}}_l),$$

the restriction of ρ to the decomposition group G_p.

Proposition 4.2.3. *For the Swan conductor of the representations ρ_p and $\bar{\rho}_p$, we have*

$$\mathrm{sw}(\rho_p) = \mathrm{sw}(\bar{\rho}_p).$$

Proof. See Section 1 of [38]. □

Below, we present the classification of degeneration of conductors for $GL(2, \mathbb{Q}_p)$; that is, we list the cases where we have $a(\rho_p) > a(\bar{\rho}_p)$. One can find more details on this classification in Section 1 of [7], and it relies on the fact that the Swan conductor is invariant under mod l reduction.

(i) Suppose that ρ_p corresponds to a principal series representation $\chi_1 \times \chi_2$.

Then we have

$$a(\rho_p) = a(\chi_1) + a(\chi_2),$$

while

$$a(\bar{\rho}_p) = a(\bar{\chi}_1) + a(\bar{\chi}_2).$$

In this case, we have that at least one of the characters, say χ_1, must be tamely ramified such that its conductor degenerates; that is, $a(\chi_1) = 1$ and $a(\bar{\chi}_1) = 0$. Then

$$\begin{aligned} a(\rho_p) &= 1 + a(\chi_2); \\ a(\bar{\rho}_p) &= a(\bar{\chi}_2). \end{aligned}$$

(ii) If ρ_p corresponds to the twisted Steinberg representation $(\chi| |^{1/2})St_{GL(2)}$, we have

$$a(\rho_p) = \begin{cases} 1, & \text{if } a(\chi) = 0; \\ 2a(\chi), & \text{if } a(\chi) > 0. \end{cases}$$

If χ is unramified, a degeneration occurs when $N = \begin{pmatrix} & 1 \\ & \end{pmatrix}$ reduces to zero modulo l. This can be achieved by conjugating N by the matrix $\begin{pmatrix} l & \\ & 1 \end{pmatrix}$. Thus we have

$$\begin{aligned} a(\rho_p) &= 1; \\ a(\bar{\rho}_p) &= 0. \end{aligned}$$

If χ is not unramified, then it must be tamely ramified with unramified reduction. If N degenerates modulo l (by conjugating as before), we have

$$\begin{aligned} a(\rho_p) &= 2; \\ a(\bar{\rho}_p) &= 0. \end{aligned}$$

If χ is not unramified, i.e., it is tamely ramified with unramified reduction, and N does not degenerate modulo l, we have

$$a(\rho_p) = 2;$$
$$a(\bar{\rho}_p) = 1.$$

(iii) Suppose that ρ_p is irreducible; such a ρ_p corresponds via the local Langlands correspondence to a supercuspidal representation $BC(L/\mathbb{Q}_p, \psi)$. By the results in Section 1 of [7], the only case where we have degeneration of the conductor of such a representation, is when $a(\rho_p) = 2$. This happens when $L/\mathbb{Q}_p$ is an unramified quadratic extension, and when $a(\psi) = 1$. In this case, ψ is a tamely ramified character of $L^\times$ with unramified reduction. We may have $a(\bar{\rho}_p) = 1$ or $a(\bar{\rho}_p) = 0$.

Remark 4.2.4. We remark the following:

1. Note that the quadratic character $\epsilon_{L/\mathbb{Q}_p}$ of an unramified quadratic extension $L/\mathbb{Q}_p$ is unramified. This implies that when the conductor of a supercuspidal representation degenerates modulo l, the determinant of the reduction is unramified.

2. If $BC(L/\mathbb{Q}_p, \psi)$ is a supercuspidal representation, where ψ is a character of W_L and $\mathrm{Gal}(L/\mathbb{Q}_p) = \{1, \sigma\}$, one has $\psi \neq \psi \circ \sigma$. This implies that ψ does not extend to a character of $W_{\mathbb{Q}_p}$. For the reduction $\bar{\psi}$ of ψ we have that it is possible to extend to a character of $W_{\mathbb{Q}_p}$. Suppose that $\bar{\psi}$ extends to a character of $W_{\mathbb{Q}_p}$; then according to Section 1 of [14], $\bar{\rho}_p$ is of the form $\begin{pmatrix} \bar{\psi}\epsilon_{L/\mathbb{Q}_p} & \\ & \bar{\psi} \end{pmatrix}$, $\begin{pmatrix} \bar{\psi}\epsilon_{L/\mathbb{Q}_p} & s \\ & \bar{\psi} \end{pmatrix}$, or $\begin{pmatrix} \bar{\psi} & s \\ & \bar{\psi}\epsilon_{L/\mathbb{Q}_p} \end{pmatrix}$, where $s|_{I_{\mathbb{Q}_p}} : I_{\mathbb{Q}_p} \twoheadrightarrow \bar{\mathbb{F}}_l$.

3. If χ is a character of $\mathbb{Q}_p^\times$ which is tamely ramified with unramified reduction, we have that (see 1.5 of [7])

$$p \equiv 1 \bmod l.$$

4. If we are in the case where the conductor of a supercuspidal representation $BC(L/\mathbb{Q}_p, \psi)$ degenerates, we get that ψ is a tamely ramified character of $L^\times$ with unramified reduction. By 1.5 of [7] we have

$$p \equiv -1 \bmod l.$$

By the classification above, we obtain all possible levels for the modular mod l representation $\bar{\rho}$.

Theorem 4.2.5. *Suppose that*

$$\bar{\rho} : G_{\mathbb{Q}} \to GL(2, \bar{\mathbb{F}}_l)$$

is a modular mod l Galois representation of weight $k \geq 2$ and level N coprime to l. Moreover, let $\phi_p \in W_{\mathbb{Q}_p}$ lie above the inverse of a Frobenius element. Then

$$N = N(\bar{\rho}) \prod_p p^{n(p)}$$

and for each p with $n(p) > 0$, one of the following holds:

1. $p \nmid N(\bar{\rho})$, $p(\mathrm{tr}\bar{\rho}_p(\phi_p))^2 \equiv (1+p)^2 \mathrm{det}\bar{\rho}_p(\phi_p) \bmod l$ *and* $n(p) = 1$.
2. $p \equiv -1 \bmod l$, *and one of the following holds:*
   (a) $p \nmid N(\bar{\rho})$, $\mathrm{tr}\bar{\rho}_p(\phi_p) \equiv 0 \bmod l$ *and* $n(p) = 2$;
   (b) $p \mid N(\bar{\rho})$ *but* $p^2 \nmid N(\bar{\rho})$, $\mathrm{det}\bar{\rho}_p$ *is unramified and* $n(p) = 1$.
3. $p \equiv 1 \bmod l$, *and one of the following holds:*
   (a) $p \nmid N(\bar{\rho})$ *and* $n(p) = 2$;
   (b) $p^2 \nmid N(\bar{\rho})$ *or the power of* p *dividing* $N(\bar{\rho})$ *is the same as the power dividing the conductor of* $\mathrm{det}\bar{\rho}_p$, *and* $n(p) = 1$.

Proof. It is a consequence of the classification of degeneration of conductors we described above. See also [15]. □

4.2.2 The classification for $GSp(4, F)$

Let

$$\bar{R} : G_{\mathbb{Q}} \to GSp(4, \bar{\mathbb{F}}_l)$$

be a modular mod l Galois representation. Assume that $\bar{R}$ has some lift

$$R : G_{\mathbb{Q}} \to GSp(4, \mathcal{O}_{\bar{\mathbb{Q}}_l})$$

which is attached to a Siegel modular form of some weight and some level N in the sense of Theorem I of [68]. Consider the restriction of $\bar{R}$ to the decomposition groups $G_p = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, which is denoted by

$$\bar{R}_p : G_p \to GSp(4, \bar{\mathbb{F}}_l).$$

If $\bar{V}$ is the representation space of $\bar{R}_p$, and G_p^u the ramification groups with respect to the upper numbering, we have that the conductor of $\bar{R}_p$ is

$$\begin{aligned} a(\bar{R}_p) &= \int_{-1}^{\infty} \left(\dim \bar{V} - \dim \bar{V}^{\bar{R}_p(G_p^u)} \right) du \\ &= \left(\dim \bar{V} - \dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} \right) + \mathrm{sw}(\bar{R}_p). \end{aligned}$$

Similarly, if we restrict R to the decomposition group G_p, we get the representation

$$R_p : G_p \to GSp(4, \bar{\mathbb{Q}}_l),$$

and if V is the representation space of R_p, then R_p has conductor

$$a(R_p) = \left(\dim V - \dim V^{R_p(I_{\mathbb{Q}_p})} \right) + \mathrm{sw}(R_p).$$

Proposition 4.2.6. *For the Swan conductor of the representations R_p and $\bar{R}_p$ we have*

$$\mathrm{sw}(R_p) = \mathrm{sw}(\bar{R}_p).$$

Proof. The crucial point here is that $l \neq p$. For a proof, the reader may refer to Section 1 of [38]. □

Now we are ready to start comparing the conductor of R_p with the conductor of its reduction mod l. From the fact that $\dim V = \dim \bar{V} = 4$ and by the equality of the Swan conductors, we get

$$a(R_p) = \left(4 - \dim V^{R_p(I_{\mathbb{Q}_p})}\right) + \mathrm{sw}(R_p),$$
$$a(\bar{R}_p) = \left(4 - \dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})}\right) + \mathrm{sw}(R_p),$$

and so $a(R_p) = a(\bar{R}_p)$ unless $\dim V^{R_p(I_{\mathbb{Q}_p})} \neq \dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})}$. As $\dim V^{R_p(I_{\mathbb{Q}_p})} \leq \dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})}$, we obtain $a(R_p) \geq a(\bar{R}_p)$ and we see that the conductors differ by at most 4; we are interested in which cases the inequality is strict. This happens in the following cases:

(1) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 3$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 4$; this means that $\bar{R}_p$ is unramified, so that $\mathrm{sw}(R_p) = 0$. We have

$$a(R_p) = 1 \text{ and } a(\bar{R}_p) = 0.$$

(2) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 2$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 4$; we have that $\bar{R}_p$ is unramified, so $\mathrm{sw}(R_p) = 0$. From this we get

$$a(R_p) = 2 \text{ and } a(\bar{R}_p) = 0.$$

(3) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 2$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 3$; then $a(R_p) = 2 + \mathrm{sw}(R_p)$ and $a(\bar{R}_p) = 1 + \mathrm{sw}(\bar{R}_p)$. This implies that

$$a(R_p) = 1 + a(\bar{R}_p).$$

(4) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 1$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 4$; we have that $\bar{R}_p$ is unramified, so $\mathrm{sw}(R_p) = 0$. From where we get

$$a(R_p) = 3 \text{ and } a(\bar{R}_p) = 0.$$

(5) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 1$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 3$; then $a(R_p) = 3 + \mathrm{sw}(R_p)$ and

$a(\bar{R}_p) = 1 + \mathrm{sw}(\bar{R}_p)$. This implies that

$$a(R_p) = 2 + a(\bar{R}_p).$$

(6) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 1$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 2$; then $a(R_p) = 3 + \mathrm{sw}(R_p)$ and $a(\bar{R}_p) = 2 + \mathrm{sw}(\bar{R}_p)$. This implies that

$$a(R_p) = 1 + a(\bar{R}_p).$$

(7) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 0$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 4$; this means that $\bar{R}_p$ is unramified, so that $\mathrm{sw}(R_p) = 0$. We have

$$a(R_p) = 4 \text{ and } a(\bar{R}_p) = 0.$$

(8) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 0$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 3$; then $a(R_p) = 4 + \mathrm{sw}(R_p)$ and $a(\bar{R}_p) = 1 + \mathrm{sw}(\bar{R}_p)$. So we have

$$a(R_p) = 3 + a(\bar{R}_p).$$

(9) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 0$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 2$; then $a(R_p) = 4 + \mathrm{sw}(R_p)$ and $a(\bar{R}_p) = 2 + \mathrm{sw}(\bar{R}_p)$. This implies that

$$a(R_p) = 2 + a(\bar{R}_p).$$

(10) $\dim V^{R_p(I_{\mathbb{Q}_p})} = 0$ and $\dim \bar{V}^{\bar{R}_p(I_{\mathbb{Q}_p})} = 1$; then $a(R_p) = 4 + \mathrm{sw}(R_p)$ and $a(\bar{R}_p) = 3 + \mathrm{sw}(\bar{R}_p)$. This implies that

$$a(R_p) = 1 + a(\bar{R}_p).$$

The notion of a "degenerating character" will be often used below, thus we give the following definition.

Definition 4.2.7. We say that a character χ *degenerates modulo* l, precisely when it is tamely ramified but with unramified reduction modulo l.

We present in Table 4.2 the classification of the Galois representations corresponding to the non-supercuspidal irreducible admissible representations of $GSp(4, \mathbb{Q}_p)$, in terms of the degeneration of their conductors. Note that we also include the two supercuspidal representations of type Va* and XIa*. In the table, by "cd" we mean that the degeneration occurs due to a degenerating character (in the sense of Definition 4.2.7); in this situation, by Remark 4.2.4, we have $p \equiv 1 \bmod l$. By "N_*" we mean that the matrix N_* degenerates modulo l, and by "sc" we mean that the conductor of a supercuspidal representation of $GL(2, \mathbb{Q}_p)$ degenerates modulo l; note that by Remark 4.2.4, the conductor of a supercuspidal degenerates only when $p \equiv -1 \bmod l$. In some cases where we write "$\bar{\omega}_\pi = 1$" we mean that the central character of π is trivial modulo l, and finally, by "•" we denote a case where none of the above happen but we still have degeneration of the conductor. The columns in Table 4.2 are the ten cases of strict inequality of conductors that we present above.

For representations of $GSp(4, \mathbb{Q}_p)$ which are induced from the Klingen or Siegel parabolic, in order to see how the conductor degenerates, one needs many properties of supercuspidal representations of $GL(2, \mathbb{Q}_p)$ which have degenerating conductor; the reader is advised to recall these from Subsection 4.2.1.

- **Type I.** For this type, the degeneration occurs only when the conductor of a tamely ramified character degenerates, i.e., we have $p \equiv 1 \bmod l$.

  Case (2) is obtained when χ_1 (resp. χ_2) degenerates and χ_2 (resp. χ_1) and σ are unramified.

  Case (4) occurs when χ_1 and χ_2 both degenerate and σ is unramified.

  Case (6) occurs when σ degenerates, $a(\chi_1\sigma) = 0$ and χ_2 is tamely ramified. Or more generally when σ is unramified, χ_1 degenerates, and $a(\chi_2) = a(\bar{\chi}_2)$ is a nonzero positive integer.

  Case (7) is obtained if all characters are degenerate, or if χ_1 and χ_2 are unramified and σ degenerates.

  Case (9) can happen if χ_1 is unramified, σ degenerates, and $a(\chi_2) =$

$a(\bar{\chi}_2)$ is a nonzero positive integer.

Case (10) occurs when χ_1, χ_2 and $\chi_1\chi_2$ are ramified and non-degenerate, but σ is degenerate.

- **Type IIa.** For this type, we have possible degeneration of conductors of characters (i.e., $p \equiv 1 \bmod l$) and possible degeneration of the nilpotent matrix N_1.

  Case (1) occurs when χ and σ are unramified, and N_1 reduces to 0 modulo l (conjugate[1] with the symplectic matrix $\begin{pmatrix} l & & & \\ & l & & \\ & & l^{-1} & \\ & & & l^{-1} \end{pmatrix}$).

  Case (2) can take place when σ is unramified and χ is the product of a ramified quadratic character with an unramified character. We also need that N_1 reduces to 0 modulo l, and the quadratic character to degenerate. This occurs only when $l = 2$.

  Case (4) is obtained when N_1 degenerates mod l, σ is unramified, and χ is degenerate. Another way is to take $\chi = \sigma^{-1}$, σ degenerate, and N_1 degenerate modulo l.

  Case (5) Similarly with the previous case, i.e., χ degenerates and σ unramified, but N_1 not congruent to 0 mod l (after conjugation).

  Case (6) occurs when σ is unramified, and χ is the product of the quadratic character $\epsilon_{L/\mathbb{Q}_p}$ with a degenerate character. Here L is a degree 2, ramified extension over $\mathbb{Q}_p$. Here N_1 may degenerate or not.

  Case (7) can take place if σ degenerates, χ is unramified, and N_1 is congruent to 0 modulo l.

  Case (8) happens for σ degenerating, χ degenerating or unramified, and N_1 non-degenerate.

  Case (9) can happen if χ is the product of a degenerate character with σ^{-1}, while σ is ramified but non-degenerate. We also need N_1 reducing

[1] Such an operation leaves the semisimple part ρ_0 unchanged.

to 0 modulo l.

Case (10) occurs when χ is ramified and non-degenerate, but σ is degenerate. N_1 can reduce to 0 modulo l or not.

- **Type IIb.** For this type, the degeneration occurs only when the conductor of a tamely ramified character degenerates, i.e., we have $p \equiv 1 \bmod l$.

  Case (2) is obtained for χ degenerate and $\sigma = \chi^{-1}$.

  Case (4) occurs when χ degenerates and σ is unramified.

  Case (6) occurs when σ is unramified, and χ is the product of the quadratic character $\epsilon_{L/\mathbb{Q}_p}$ with a degenerate character. Here L is a degree 2, ramified extension over $\mathbb{Q}_p$.

  Case (7) is obtained if all characters are degenerate, or if χ is unramified and σ degenerates.

  Case (9) can happen if χ is the product of a degenerate character with σ^{-1}, while σ is ramified but non-degenerate.

  Case (10) occurs when χ is ramified but non-degenerate but σ is degenerate.

- **Type IIIa.** Degeneration here occurs when a character degenerates ($p \equiv 1 \bmod l$) or when the matrix N_4 degenerates modulo l. Note that N_4 cannot reduce with only one nonzero entry.

  Case (2) takes place when σ and χ are unramified and $N_4 \equiv 0 \bmod l$ (this can be achieved when we conjugate the matrix with the symplectic matrix $\begin{pmatrix} l & & & \\ & 1 & & \\ & & 1 & \\ & & & l^{-1} \end{pmatrix}$).

  Case (4) occurs when χ degenerates, σ is unramified, and N_4 reduces to 0 modulo l.

Case (6) occurs when σ is unramified, $a(\chi) = a(\bar{\chi})$ is a nonzero positive integer, and N_4 reduces to zero modulo l. This case can also happen when σ is unramified, χ degenerates, and N_4 does not degenerate.

Case (7) takes place when χ is unramified, σ degenerates, and N_4 degenerates.

Case (9) occurs when σ degenerates, $a(\chi) = a(\bar{\chi})$ is a nonzero positive integer and N_4 degenerates. Also we may get this case by taking σ degenerate, χ unramified, and N_4 non-degenerate.

Case (10) occurs when σ is degenerate, $a(\chi) = a(\bar{\chi})$ is a nonzero positive integer, and N_4 non-degenerate.

- **Type IIIb.** For this type, the degeneration can take place when $p \equiv 1 \bmod l$, since we only have degeneration of characters.

  Case (2) occurs when σ is unramified and χ degenerates.

  Case (7) happens when σ degenerates and χ is unramified (or degenerating).

  Case (9) occurs when σ degenerates and $a(\chi) = a(\bar{\chi})$ is a nonzero positive integer.

- **Type IVa.** This is a more complicated type since we have as nilpotent matrix N_5 which might reduce to 0, to N_1, or to N_4 modulo l (by conjugation with one of the symplectic matrices $\begin{pmatrix} l^2 & & & \\ & l & & \\ & & l^{-1} & \\ & & & l^{-2} \end{pmatrix}$, $\begin{pmatrix} l & & & \\ & 1 & & \\ & & 1 & \\ & & & l^{-1} \end{pmatrix}$, or $\begin{pmatrix} l & & & \\ & l & & \\ & & l^{-1} & \\ & & & l^{-1} \end{pmatrix}$ respectively).

  Case (4) occurs when σ is unramified and N_5 reduces to 0.

  Case (5) occurs when σ is unramified and N_5 reduces to N_1.

  Case (6) occurs when σ is unramified and N_5 reduces to N_4.

Case (7) takes place when σ degenerates and N_5 reduces to 0.

Case (8) can happen when σ degenerates and N_5 reduces to N_1.

Case (9) occurs when σ degenerates and N_5 reduces to N_4.

Case (10) occurs when σ degenerates and N_5 is non-degenerate.

- **Type IVb.** Here, we can get degeneration by either a degeneration of a character ($p \equiv 1 \bmod l$) or by degeneration of the nilpotent matrix N_4.

  Case (2) occurs when σ is unramified and N_4 reduces to 0 modulo l.

  Case (7) occurs when σ degenerates and N_4 degenerates.

  Case (9) takes place when σ degenerates, but N_4 does not (i.e., it has two nonzero entries).

- **Type IVc.** The conductor of this type either degenerates by a degeneration of a character ($p \equiv 1 \bmod l$), or by degeneration of N_1.

  Case (1) takes place when σ is unramified and N_1 degenerates.

  Case (7) occurs when σ degenerates and N_1 reduces to 0.

  Case (8) can happen when σ degenerates, but N_1 is non-degenerate.

- **Type IVd.** For this type, we only have σ which degenerates (i.e., $p \equiv 1 \bmod l$) or not.

  Case (7) occurs when σ degenerates.

- **Type Va and type Va*.** Note that representations of type Va are in the same L-packet with the supercuspidal representations of type Va*, so that the conductors of their L-parameters degenerate in the same way. Here we have N_3 as the nilpotent matrix, and this may reduce to N_1, to N_2, or to 0 (conjugate by one of the symplectic matrices $\begin{pmatrix} l &&& \\ & 1 && \\ && 1 & \\ &&& l^{-1} \end{pmatrix}$, $\begin{pmatrix} 1 &&& \\ & l && \\ && l^{-1} & \\ &&& 1 \end{pmatrix}$, or $\begin{pmatrix} l &&& \\ & l && \\ && l^{-1} & \\ &&& l^{-1} \end{pmatrix}$ respectively).

Case (2) occurs when σ and ξ are unramified and N_3 reduces to 0 modulo l.

Case (3) occurs when σ and ξ are unramified and N_3 reduces to N_1 or to N_2 modulo l.

Case (4) takes place only for $l = 2$, since we need the quadratic character to degenerate and σ to be unramified. Moreover N_3 should reduce to 0.

Case (5) occurs when σ is unramified, N_3 reduces to N_1, and ξ degenerates, so only when $l = 2$.

Case (6) occurs when σ is unramified, ξ is ramified (non-degenerate), and N_3 reduces to 0.

Case (7) occurs when ξ is unramified, σ is degenerate, and N_3 reduces to 0.

Case (8) occurs when ξ is unramified, σ is degenerate, and N_3 reduces to N_1 or to N_2 modulo l.

Case (9) is obtained for ξ unramified, σ is degenerate, and N_3 is non-degenerate. We may also get this case by taking ξ to be ramified, σ degenerate, and N_3 reduce to 0 (or to N_1).

Case (10) is obtained when ξ is ramified, σ degenerates, and N_3 is non-degenerate (or reduces to N_2).

- **Type Vb.** Here, degeneration occurs when a character degenerates, or when N_1 degenerates.

  Case (1) can happen when σ and ξ are unramified, and N_1 degenerates.

  Case (2) may only happen for $l = 2$, since we need σ unramified, ξ degenerateing and N_1 degenerating.

  Case (3) may happen only for $l = 2$, since we again need ξ degenerating, along with σ being unramified and N_1 non-degenerating.

  Case (4) can happen when $l = 2$. We need σ degenerate and $\xi\sigma$ unramified. Also N_1 should degenerate.

Case (5) happens only when $l = 2$, since we need σ degenerate with $\xi\sigma$ unramified, and N_1 non-degenerate.

Case (6) occurs for $\xi\sigma$ unramified, $a(\sigma) = a(\bar{\sigma})$ a nonzero positive integer, and N_1 degenerate.

Case (7) occurs for σ degenerate, ξ unramified, and for N_1 degenerate.

Case (8) occurs when σ degenerates, ξ is unramified, and N_1 is non-degenerate.

Case (9) is obtained when σ degenerates, ξ is ramified (here N_1 can be degenerate or not).

Case (10) occurs when $a(\sigma) = a(\bar{\sigma})$ is a nonzero positive integer, σ is the product of ξ and a character that degenerates, and N_1 is non-degenerate.

- **Type Vc.** Degeneration occurs when a character degenerates or when N_2 reduces to 0 modulo l (by conjugation with the symplectic matrix $\begin{pmatrix} l & & & \\ & l & & \\ & & l^{-1} & \\ & & & l^{-1} \end{pmatrix}$).

  Case (1) is obtained by taking ξ and σ unramified and N_2 degenerating to 0.

  Case (2) may take place when N_2 reduces to 0, $\xi\sigma$ is unramified and σ degenerates (i.e., ξ degenerates). This can only happen when $l = 2$.

  Case (3) occurs when $\xi\sigma$ is unramified, σ degenerates (i.e., ξ degenerates), and N_2 is non-degenerate. This happens only if $l = 2$.

  Case (4) takes place when N_2 reduces to 0, σ is unramified, and ξ degenerates. Again this can only happen if $l = 2$.

  Case (5) occurs when σ is unramified, ξ degenerates, and N_2 is non-degenerate. Again this may happen only if $l = 2$.

  Case (6) occurs when σ is unramified and $a(\xi) = a(\bar{\xi})$ is a nonzero positive integer. Moreover N_2 must degenerate.

Case (7) takes place when ξ is unramified, σ degenerates, and N_2 reduces to 0.

Case (8) takes place when ξ is unramified, σ degenerates, and N_2 non-degenerate.

Case (9) occurs when ξ is ramified while σ and N_2 degenerate.

Case (10) is obtained when ξ is ramified, σ degenerates, but N_2 is non-degenerate.

- **Type Vd.** Here we have degeneration only when a character degenerates ($p \equiv 1 \bmod l$).

  Case (2) may take place if ξ degenerates and σ is unramified. This can only happen if $l = 2$ since ξ is a quadratic character.

  Case (7) is obtained if ξ is unramified and σ degenerates.

  Case (9) is obtained if ξ is ramified and σ degenerates.

- **Type VIa and type VIb.** These two types form a single L-packet, so they will degenerate in the same way. Here we have possible degeneration of the matrix N_3 to 0, to N_1, or to N_2, along with possible degeneration of a character.

  Case (2) occurs when σ is unramified and N_3 reduces to 0.

  Case (3) is obtained by taking σ to be unramified and N_3 to reduce to N_1 or to N_2.

  Case (7) is obtained by choosing σ degenerate and N_3 reducing to 0.

  Case (8) occurs when σ is degenerate and N_3 reduces to N_1 or to N_2.

  Case (9) occurs when σ is degenerate and N_3 is non-degenerate.

- **Type VIc.** Here we have the nilpotent matrix N_1 which possibly reduces to 0.

  Case (1) happens when σ is unramified and N_1 reduces to 0.

  Case (7) occurs when σ is degenerate and N_1 reduces to 0.

  Case (8) occurs when σ is degenerate and N_1 is non-degenerate.

- **Type VId.** The only possibility of degeneration is when σ degenerates ($p \equiv 1 \bmod l$).

  Case (7) is obtained when σ degenerates.

- **Type VII.** Here, the representation R_p depends on a character χ and a supercuspidal representation π of $GL(2, \mathbb{Q}_p)$. In the case where this degenerates, we have $p \equiv -1 \bmod l$.

  Case (7) occurs when χ is unramified, and $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 0$.

  Case (9) occurs when χ is unramified, and $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 1$. This case may also be obtained when $a(\chi) = a(\bar{\chi})$ is a nonzero positive integer and when $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 0$.

  Case (10) is obtained when $a(\chi) = a(\bar{\chi})$ is a nonzero positive integer and when $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 1$.

- **Type VIIIa and type VIIIb.** These two representations form a single L-packet. Again the conductor depends on the conductor of a supercuspidal representation of $GL(2, \mathbb{Q}_p)$, which will degenerate, so we have to assume that $p \equiv -1 \bmod l$.

  Case (7) occurs when $a(\phi_\pi) = 2$ and $a(\bar{\phi}_\pi) = 0$.

  Case (9) occurs when $a(\phi_\pi) = 2$ and $a(\bar{\phi}_\pi) = 1$.

- **Type IXa.** In this case, we have the non-trivial quadratic character ξ, for which we have $\xi\pi = \pi$; this means that ξ does not take part in the consideration of degeneration of conductors. This has attached an L-parameter with nilpotent part N_6, which can be reduced to 0 modulo l (by conjugation with the symplectic matrix $\begin{pmatrix} l & & & \\ & l & & \\ & & l^{-1} & \\ & & & l^{-1} \end{pmatrix}$). We also have that the supercuspidal representation of $GL(2, \mathbb{Q}_p)$ degenerates, so $p \equiv -1 \bmod l$.

  Case (7) is obtained with $a(\phi_\pi) = 2$ and $a(\bar{\phi}_\pi) = 0$, and N_6 reducing to 0.

Case (9) occurs with $a(\phi_\pi) = 2$ and $a(\bar{\phi}_\pi) = 1$, and N_6 reducing to 0.

- **Type IXb.** Similarly here, we have $\xi\pi = \pi$. When the supercuspidal representation of $GL(2, \mathbb{Q}_p)$ degenerates, we have $p \equiv -1 \bmod l$.

  Case (7) occurs when $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 0$.

  Case (9) occurs when $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 1$.

- **Type X.** We have possible degeneration of the character σ or of the supercuspidal $\pi = BC(L/\mathbb{Q}_p, \psi)$.

  Case (4) is obtained for σ unramified, and $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 0$; here we need $L/\mathbb{Q}_p$ to be an unramified quadratic extension so that $\epsilon_{L/\mathbb{Q}_p}$ is unramified.

  Case (5) is obtained for σ unramified, and $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 1$. Again we chose π such that $L/\mathbb{Q}_p$ is unramified, for we need $\epsilon_{L/\mathbb{Q}_p}$ to be unramified.

  Case (6) occurs when σ is unramified, and when the central character of π is trivial modulo l.

  Case (7) happens when σ degenerates, and $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 0$. So we have to assume $l = 2$ for this.

  Case (8) happens when σ degenerates, and $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 1$. So we have to assume $l = 2$ for this too.

  Case (9) occurs when σ is degenerate, and when the central character of π is trivial modulo l.

  Case (10) occurs for π non-degenerate and σ degenerate.

- **Type XIa and type XIa*.** The supercuspidal representations of type XIa* are in the same L-packet with XIa, so the conductors of their L-parameters degenerate in the same way. Here we have also the nilpotent matrix N_2 which might degenerate (under conjugation with $\begin{pmatrix} l & & & \\ & 1 & & \\ & & 1 & \\ & & & l^{-1} \end{pmatrix}$).

Case (4) occurs with σ unramified, N_2 degenerate, and $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 0$.

Case (5) occurs with σ unramified, N_2 degenerate, and $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 1$.

Case (6) is obtained for σ unramified, N_2 degenerate, and π non-degenerate.

Case (7) occurs with σ, N_2 and π degenerate; $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 0$. So we need $l = 2$.

Case (8) occurs with σ, N_2 and π degenerate; $a(\phi_\pi) = 2$ with $a(\bar{\phi}_\pi) = 1$. This happens for $l = 2$.

Case (9) occurs when σ and N_2 degenerate, but π not.

Case (10) occurs when σ degenerates, but N_2 and π are non-degenerate.

- **Type XIb.** Finally, we have possible degeneration of a character, or possible degeneration of the supercuspidal (i.e., $p \equiv 1$ or $-1 \bmod l$).

  Case (2) is obtained by choosing σ unramified, and $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 0$.

  Case (3) occurs when σ is unramified, and $a(\phi_\pi) = 2$ while $a(\bar{\phi}_\pi) = 1$.

  Case (7) can only happen when $l = 2$ since we need degeneration of both the supercuspidal and the character.

  Case (8) takes place again when $l = 2$ since we need degeneration of both the supercuspidal and the character.

  Case (9) occurs when σ degenerates, and π non-degenerate.

  Case (10) occurs when both the supercuspidal and the character are non-degenerate. Choose $\pi = BC(L/\mathbb{Q}_p, \psi)$ such that $L/\mathbb{Q}_p$ is unramified and that $\bar{\psi}$ extends to a character of $W_{\mathbb{Q}_p}$; in addition assume that the reduction of the associated Galois representation is of the form $\begin{pmatrix} \bar{\psi}\epsilon_{L/\mathbb{Q}_p} & s \\ & \bar{\psi} \end{pmatrix}$. Moreover, let $\bar{\sigma} = \bar{\psi}^{-1}$.

Table 4.2: Classification of degeneration of conductors for non-supercuspidal representations of $GSp(4,F)$

	Case (1)	Case (2)	Case (3)	Case (4)	Case (5)	Case (6)	Case (7)	Case (8)	Case (9)	Case (10)
Type I		cd		cd		cd	cd		cd	cd
Type IIa	N_1	$l=2$		N_1, cd	cd	cd	N_1, cd	cd	N_1, cd	cd
Type IIb		cd		cd		cd	cd		cd	cd
Type IIIa		N_4		N_4, cd		cd	N_4, cd		cd	cd
Type IIIb		cd					cd		cd	
Type IVa				N_5	N_5	N_5	N_5, cd	N_5, cd	N_5, cd	cd
Type IVb		N_4					N_4, cd		cd	
Type IVc	N_1						N_1, cd	cd		
Type IVd							cd			
Type Va		N_3	N_3	$l=2$	$l=2$	N_3	N_3, cd	N_3, cd	cd	cd
Type Vb	N_1	$l=2$	$l=2$	$l=2$	$l=2$	N_1	N_1, cd	cd	cd	cd
Type Vc	N_2	$l=2$	$l=2$	$l=2$	$l=2$	N_2	N_2, cd	cd	N_2, cd	cd
Type Vd		$l=2$					cd		cd	
Type VIa		N_3	N_3				N_3, cd	N_3, cd	cd	
Type VIb		N_3	N_3				N_3, cd	N_3, cd	cd	
Type VIc	N_1						N_1, cd	cd		
Type VId							cd			
Type VII							sc		sc	sc
Type VIIIa							sc		sc	
Type VIIIb							sc		sc	
Type IXa							N_6, sc		N_6, sc	
Type IXb							sc		sc	
Type X				sc	sc	$\bar{\omega}_\pi = 1$	$l=2$	$l=2$	cd, $\bar{\omega}_\pi = 1$	cd
Type XIa				N_2, sc	N_2, sc	N_2	$l=2$	$l=2$	N_2, cd	cd
Type XIb		sc	sc				$l=2$	$l=2$	cd	•
Type Va*		N_3	N_3	$l=2$	$l=2$	N_3	N_3, cd	N_3, cd	cd	cd
Type XIa*				N_2, sc	N_2, sc	N_2	$l=2$	$l=2$	N_2, cd	cd

Chapter 5

Lifting automorphic representations to $GSp(4)$

In this chapter, we will obtain automorphic representations for $GSp(4)$ over the rationals, which arise via the theory of the theta correspondence from automorphic representations for $GL(2)$ over an imaginary quadratic field. We will describe this process, and in particular, we will write down the local correspondence (namely the local theta lift) explicitly. This lifting of automorphic representations enables one to attach Galois representations to automorphic representations for $GL(2)$ over an imaginary quadratic field. This theory was developed by Harris, Soudry and Taylor in [29], Taylor in [65], Berger and Harcos in [3], and Mok in [40].

5.1 4-dimensional quadratic spaces

Let F be any field of characteristic different from 2. In this section, we will follow Roberts (§2 of [47]) in describing briefly the theory of 4-dimensional quadratic spaces over F. If X is such a space, this will lead us to a realization of the similitude group $GSO(X, F)$ via its even Clifford algebra. More details, such as information about proofs of results in this section, can be found in §2 of [47].

5.1.1 $GSO(X,F)$ and Clifford algebras

We begin with a 4-dimensional quadratic space X, with $(\cdot,\cdot)$ its associated symmetric and non-degenerate[1] bilinear form, defined over F.

Definition 5.1.1. We define the *similitude orthogonal group* of X, denoted by $GO(X,F)$, to be the group of all $g \in GL(X)$ such that

$$(gv, gw) = \nu(g)(v,w),$$

where $v, w \in X$ and $\nu(g) \in F^\times$. Here, $\nu : GO(X,F) \to F^\times$ is a multiplicative character called the *similitude character*. The connected component of the identity in $GO(X,F)$ will be denoted by $GSO(X,F)$, and is an index 2 subgroup of $GO(X,F)$. Finally, note that $O(X,F)$ is the kernel of ν.

Let x_1, x_2, x_3, x_4 be an orthogonal basis for X, and let d denote the discriminant[2] of X. Let $C = C(X)$ be the *Clifford algebra of* X; it is a unital, associative algebra which is generated by X, and roughly, it is the most possible free algebra generated by X, subject to the condition[3] $xy+yx = 2(x,y)$ for all $x, y \in X$. We have that $\dim(C) = 2^4 = 16$ over F. Now, we consider *the even Clifford algebra of* X *in* C, which is the linear combination of the even rank[4] elements of C. Its dimension over F is 8, and we denote it by $B = B(X)$. The centre of B is denoted by $E = E(X)$ and is of dimension 2 over F. Finally, let $C_1 = C_1(X)$ be the subspace spanned by elements of odd rank in C, which is of dimension 8. We consider the involution $*$ of C that takes a product of the x_i to the product of the same x_i in the reverse order. It is easy to see that $*$ preserves B and C_1. If $x \in B$, then it belongs to its center E if and only if $x^* = x$. If for $x \in C$ we set $N(x) = x^*x$, then $N(x) \in E$ when $x \in B$. Note that if we regard X as contained in C_1, we see

[1] That is, if $x \in F$ is nonzero, then there exists a $y \in F$ such that $(x,y) \neq 0$.

[2] The discriminant of X is defined as the determinant of the matrix $((x_i, x_j))$, and it is a number in $F^\times/F^{\times 2}$.

[3] Since F is not of characteristic 2.

[4] The rank of an element x of the Clifford algebra is the number of basis elements whose product is x. For instance, the even Clifford algebra is spanned by the elements $1, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1x_2x_3x_4$.

that X is the set of elements in C_1 such that $x^* = x$, and thus if $x \in X$, then $(x, x) = N(x)$.

The F-algebra E is called *the discriminant algebra of* X and has no non-zero nilpotent elements, i.e., we say it is *reduced*. It is characterised by the discriminant of X; i.e., if $d \neq 1$ then E is a field over F of degree 2, and if $d = 1$ then $E \cong F \times F$. Moreover, we set $\mathrm{Gal}(E/F) = \{1, \alpha\}$ and let $N_{E/F}(z) = z\alpha(z)$ and $T_{E/F}(z) = z + \alpha(z)$ be the *norm* and *trace* of E over F, respectively.

Let us now study the structures of B and E by introducing these notions in a more abstract manner.

Definition 5.1.2. Let B be an F-algebra with center E and involution $*$ which is the identity on E. We say that B is a *quadratic quaternion algebra over* F if E is 2-dimensional over F and reduced, and there exists a quaternion algebra D over F contained in B, such that the natural map

$$E \otimes_F D \to B,$$

defined via $z \otimes x \mapsto zx$, is an isomorphism of E-algebras and $*$ induces the canonical involution of D.

We may define a norm $N : B \to E$ and a trace $T : B \to E$ by $N(x) = xx^* = x^*x$ and $T(x) = x + x^*$, respectively. Furthermore, we define an E-bilinear form $(x, y) = T(xy^*)/2$, which is non-degenerate. The definition of a quadratic quaternion algebra B requires the existence of a particular quaternion algebra D; Proposition 2.1 of [47] implies that in fact we may consider any quaternion algebra over F in B. Moreover, we remark that if B is a quadratic quaternion algebra, there may be infinitely many non-isomorphic quaternion algebras D over F in B; but when $E \cong F \times F$, then $B \cong D \times D$, and any other quaternion algebra over F in B is isomorphic to D.

Proposition 5.1.3. *If* X *is a* 4*-dimensional quadratic space, then* $B = B(X)$ *is a quadratic quaternion algebra over* F*.*

Proof. See Proposition 2.2 of [47]. □

We may now use the notion of quadratic quaternion algebras to study the group $GSO(X, F)$. We first define a left action ρ of $F^\times \times B^\times$ on C_1 by

$$\rho(t, g)x = t^{-1}gxg^*,$$

which preserves X, and one can see that if $x \in X$ and $(t, g) \in F^\times \times B^\times$, then $N(\rho(t, g)x) = t^{-2}N_{E/F}(N(g))N(x)$. This can be regarded as

$$(\rho(t, g)x, \rho(t, g)x) = t^{-2}N_{E/F}(N(g))(x, x),$$

which means that $\rho(t, g) \in GO(X, F)$, with similitude factor $t^{-2}N_{E/F}(N(g))$. In fact, according to §2 of [47], if $(t, g) \in F^\times \times B^\times$ then $\rho(t, g) \in GSO(X, F)$. The following result determines $GSO(X, F)$ in terms of data from the Clifford algebra of X.

Proposition 5.1.4. *Let X be a 4-dimensional quadratic space over F of discriminant d, and write $B = B(X)$ and $E = E(X)$. Define an inclusion of $E^\times$ into $F^\times \times B^\times$ by $a \mapsto (N_{E/F}(a), a)$. Then the following sequence is exact:*

$$1 \to E^\times \to F^\times \times B^\times \xrightarrow{\rho} GSO(X, F) \to 1.$$

Proof. See Theorem 2.3 of [47]. □

Starting with a 4-dimensional quadratic space, we constructed a quadratic quaternion algebra B over F. Now we will start by a quadratic quaternion algebra B over F, and we will see which 4-dimensional quadratic spaces can be derived from it.

Definition 5.1.5. Let B be a quadratic quaternion F-algebra with center E, with $\mathrm{Gal}(E/F) = \{1, \alpha\}$. Then, a *Galois action* on B is an F-automorphism $a : B \to B$, such that $a^2 = 1$ and $a(zx) = \alpha(z)a(x)$ for $z \in E$ and $x \in B$.

Lemma 5.1.6. *Quaternion algebras over F contained in B are in 1-to-1 correspondence with Galois actions on B.*

Proof. We sketch a proof in order to indicate the bijection between Galois actions on B and quaternion algebras over F contained in B. For more information, the reader should consult §2 of [47]. If a is a Galois action, then the set of fixed points of a is a quaternion algebra over F contained in B; conversely, if D is a quaternion algebra over F contained in B, we may define a Galois action on B by letting $a : B \to B$ to be defined by $a(z \otimes x) = \alpha(z) \otimes x$ (where $z \in E$ and $x \in D$). These two maps are inverses of each other, and give a bijection between, on the one hand, quaternion algebras over F in B, and on the other hand, Galois actions on B. □

We are now able to explicitly construct 4-dimensional quadratic spaces, starting from a quadratic quaternion algebra over F equipped with a Galois action. Let B be a quadratic quaternion algebra over F with center E, $\mathrm{Gal}(E/F) = \{1, \alpha\}$, involution $*$, and $a : B \to B$ a Galois action on B. Let D be the quaternion algebra over F in B, which corresponds to the Galois action a (i.e., the fixed points of a as in Lemma 5.1.6). Define X_a to be the set of points $x \in B$, such that $a(x) = x^*$. According to §2 of [47], X_a is a 4-dimensional vector space over F, which can be equipped with the symmetric bilinear form induced by the norm of B to become a 4-dimensional quadratic space over F. We may define an action ρ_a of $F^\times \times B^\times$ on X_a by

$$\rho_a(t, g)x = t^{-1} g x a(g)^*,$$

and we have that $\rho_a(t, g) \in GSO(X_a, F)$ for $(t, g) \in F^\times \times B^\times$.

Note that, in the beginning of this subsection, we started with a 4-dimensional quadratic space X over F, and we constructed a quadratic quaternion algebra $B(X)$ over F, with center $E(X)$; so, starting with X_a, we may construct $B(X_a)$ and $E(X_a)$.

Proposition 5.1.7. *Let B be a quadratic quaternion algebra over F with center E,* $\mathrm{Gal}(E/F) = \{1, \alpha\}$*, involution $*$, and $a : B \to B$ a Galois action on B. Then the sequence*

$$1 \to E^\times \to F^\times \times B^\times \xrightarrow{\rho_a} GSO(X_a, F) \to 1$$

is exact, where the inclusion of $E^\times$ *in* $F^\times \times B^\times$ *is defined by* $z \mapsto (N_{E/F}(z), z)$*. There exists a unique* F*-isomorphism* $B(X_a) \cong B$ *sending* $E(X_a)$ *onto* E*, so that the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E(X_a)^\times & \longrightarrow & F^\times \times B(X_a)^\times & \xrightarrow{\rho} & GSO(X_a, F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & E^\times & \longrightarrow & F^\times \times B^\times & \xrightarrow{\rho_a} & GSO(X_a, F) & \longrightarrow & 1 \end{array} \tag{5.1}$$

commutes, where the first two vertical arrows are isomorphisms.

Proof. See Proposition 2.7 of [47]. □

Propositions 5.1.4 and 5.1.7 give the characterization of $GSO(X, F)$ that we wanted, in terms of data from the Clifford algebra of X.

Let E be a 2-dimensional reduced algebra over F; this implies that E is either a quadratic extension of F, or $E \cong F \times F$. Let $\mathrm{Gal}(E/F) = \{1, \alpha\}$, and let D be a quaternion algebra over F with canonical involution $*$. Define $B_{D,E} = E \otimes_F D$, which we equip with the involution defined by $(z \otimes x)^* = z \otimes x^*$, and we see that $B_{D,E}$ is a quadratic quaternion algebra over F. Moreover, we define $a : B_{D,E} \to B_{D,E}$ by $a(z \otimes x) = \alpha(z) \otimes x$, and this is a Galois action of $B_{D,E}$. Finally, let $X_{D,E} = X_a$, i.e., the set of $z \otimes x \in B_{D,E}$ such that $a(z \otimes x) = (z \otimes x)^*$.

We may describe E in terms of a square-free element $d \in F^\times$, as in [47]. If $d \neq 1$, set $E_d = F(\sqrt{d})$, and if $d = 1$, set $E_d = F \times F$. Write $B_{D,E_d} = B_{D,d}$, and $X_{D,E_d} = X_{D,d}$. Evidently, d is the discriminant of $X_{D,d}$ (as mentioned before, we call E_d the discriminant algebra of $X_{D,d}$). When $d = 1$, we can say a bit more about the structure $B_{D,1}$. In particular, in that case, there is a canonical isomorphism of F-algebras $D \times D \cong B_{D,1}$, and a is given by $a(x, x') = (x', x)$, and $*$ is given by $(x, x')^* = (x^*, x'^*)$. This implies that $X_{D,1}$ is the set of pairs (x, x^*) for $x \in D$, that is, $X_{D,1}$ can be identified with D. With respect to the above identifications,

$$\rho_a(t, (g, g'))x = t^{-1} g x g'^*,$$

for $t \in F^\times$, $(g, g') \in D^\times \times D^\times$, and $x \in D$.

5.1.2 Explicit quadratic spaces for $GSO(X, F)$

In this subsection, we apply the above theory to algebras over the local field $\mathbb{Q}_p$. We start with a non-archimedean local field F, and later we will put $F = \mathbb{Q}_p$. Moreover, we will say some things about 4-dimensional quadratic spaces in the archimedean case $F = \mathbb{R}$.

Let $d \in F^\times/F^{\times 2}$. According to [47], up to isometry, there are two 4-dimensional quadratic spaces of discriminant d. One space is isometric to $X_{M_{2\times2},d}$, where $M_{2\times2} = M_{2\times2}(F)$ is the quaternion algebra of 2×2 matrices over F; the other is isometric to $X_{D_{\mathrm{ram}},d}$, where D_{ram} is the division quaternion algebra over F (or sometimes called the *non-split* quaternion algebra over F).

If $d = 1$, then $X_{M_{2\times2},1}$ is isometric to $M_{2\times2}(F)$ equipped with the determinant, and $X_{D_{\mathrm{ram}},1}$ is isometric to D_{ram} equipped with the norm. We also mentioned before, that $B_{D,1} \cong D \times D$ (for D either equal to $M_{2\times2}(F)$ or to D_{ram}). In this case, by Propositions 5.1.4 and 5.1.7, we have

$$GSO(D_{\mathrm{ram}}, F) \cong (D_{\mathrm{ram}}^\times \times D_{\mathrm{ram}}^\times)/F^\times$$

and

$$GSO(M_{2\times2}(F), F) \cong (GL(2, F) \times GL(2, F))/F^\times.$$

If $d \neq 1$, then $X_{M_{2\times2},d}$ and $X_{D_{\mathrm{ram}},d}$ are both isotropic[5]. We have that $B_{M_{2\times2},d}$ and $B_{D_{\mathrm{ram}},d}$ are both isomorphic to $M_{2\times2}(E_d)$. Now, we are going to describe the two quadratic spaces. Let δ be a representative for the non-trivial coset of $F^\times/N_{E_d/F}(E_d^\times)$, and α the non-trivial element of $\mathrm{Gal}(E_d/F)$.

[5] A quadratic vector space X equipped with a symmetric bilinear form $(\cdot,\cdot)$ is called *isotropic* when it contains at least one non-zero vector v such that $(v, v) = 0$; such a vector is called isotropic, otherwise it is called anisotropic.

Then we have

$$D_{\mathrm{ram}} = \left\{ \begin{pmatrix} e & f\delta \\ \alpha(f) & \alpha(e) \end{pmatrix} : e, f \in E_\delta \right\} \subset M_{2\times 2}(E_\delta).$$

The Galois actions a and a' on the quadratic quaternion algebra $M_{2\times 2}(E_d)$ corresponding to $M_{2\times 2}(F)$ and D_{ram}, are given by

$$a\left(\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = \begin{pmatrix} \alpha(e) & \alpha(f) \\ \alpha(g) & \alpha(h) \end{pmatrix} \text{ and } a'\left(\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right) = \begin{pmatrix} \alpha(h) & \delta\alpha(g) \\ \alpha(f)/\delta & \alpha(e) \end{pmatrix},$$

respectively. The 4-dimensional quadratic spaces are given by

$$X_{M_{2\times 2},d} = \left\{ \begin{pmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \alpha(e) \end{pmatrix} : e \in E_d, f, g \in F \right\}$$

and

$$X_{D_{\mathrm{ram}},d} = \left\{ \begin{pmatrix} f & -\delta e \\ \alpha(e) & g \end{pmatrix} : e \in E_d, f, g \in F \right\}.$$

Finally, note that $GSO(X_{M_{2\times 2},d}, F) \cong GSO(X_{D_{\mathrm{ram}},d}, F)$. The above results and the description of the quadratic spaces are discussed in §2 of [47].

Let K be a quadratic extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_K$, such that if v is a place of K, we let K_v be its completion at v. Fix a prime p of $\mathbb{Q}$; it is known that p either stays inert, or splits completely, or ramifies in K. The above discussion explains the situation when we have $F = \mathbb{Q}_p$. In particular, we have the following:

1. **The prime p either stays inert or ramifies in K:** $p\mathcal{O}_K = \mathfrak{p}$ or $p\mathcal{O}_K = \mathfrak{p}^2$. In this case, $K_\mathfrak{p}$ is a quadratic field over $\mathbb{Q}_p$, and

   $$K_\mathfrak{p} = \mathbb{Q}_p(\sqrt{d}) = E_d,$$

   i.e., $d \neq 1$. Then using the discussion from above and Proposition 5.1.4, we get

   $$GSO(X, \mathbb{Q}_p) \cong (GL(2, K_\mathfrak{p}) \times \mathbb{Q}_p^\times)/K_\mathfrak{p}^\times,$$

where X is either $X_{M_{2\times 2},d}$ or $X_{D_{\rm ram},d}$.

2. **The prime p splits in K:** $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. In this case, we have the discriminant algebra $E_d = \mathbb{Q}_p \times \mathbb{Q}_p$, so that $d = 1$. Using the remarks from above, and Proposition 5.1.4, we write

$$GSO(M_{2\times 2}(\mathbb{Q}_p), \mathbb{Q}_p) \cong (GL(2, \mathbb{Q}_p) \times GL(2, \mathbb{Q}_p))/\mathbb{Q}_p^{\times}.$$

Note that the degree of the extension $K_{\mathfrak{p}}/\mathbb{Q}_p$ is 1 while the inertia degree and the ramification index are trivial. The same holds for $K_{\bar{\mathfrak{p}}}/\mathbb{Q}_p$, so that we may identify $K_{\mathfrak{p}}$ and $K_{\bar{\mathfrak{p}}}$ with $\mathbb{Q}_p$. Thus, the discriminant algebra may be written as $E_d \cong K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}}$.

Finally, for the archimedean local field $\mathbb{R}$, up to isometry there are three 4-dimensional quadratic spaces of discriminant $d = 1$, which are of signature $(4, 0)$, $(2, 2)$, or $(0, 4)$ respectively. The quadratic space of signature $(4, 0)$ (respectively $(0, 4)$) is the space $X_{D_{\rm ram},1}$ (respectively $X_{D_{\rm ram},1}$, but with quadratic form multiplied by -1); here $D_{\rm ram}$ over $\mathbb{R}$ is the Hamiltonian quaternion algebra. The quadratic space with signature $(2, 2)$ is $X_{M_{2\times 2},1}$. For $d = -1$, up to isometry there are two 4-dimensional quadratic spaces of signatures $(1, 3)$ and $(3, 1)$; these are respectively $X_{M_{2\times 2},-1}$ and $X_{D_{\rm ram},-1}$.

5.2 Representation theory of $GO(X, F)$

Firstly, we set some notation. Let F be a non-archimedean local field, and if X is a 4-dimensional quadratic space of discriminant d we let E be the discriminant algebra of X. We mentioned in the previous section that if $d = 1$ we have $E = F \times F$, and if $d \neq 1$ we have $E = F(\sqrt{d})$. We will write $\mathrm{Irr}(GSO(X, F))$ for the set of irreducible admissible representations of $GSO(X, F)$. Moreover, we write $\mathrm{Irr}(GL(2, F))$ (resp. $\mathrm{Irr}(GL(2, E))$) for the set of irreducible admissible representations of $GL(2, F)$ (resp. $GL(2, E)$). When $d = 1$, we denote by $\mathrm{Irr}_f(GL(2, F) \times GL(2, F))$ the set of pairs of representations in $\mathrm{Irr}(GL(2, F))$ which have the same central character and by $\mathrm{Irr}_f(D_{\rm ram}^{\times} \times D_{\rm ram}^{\times})$ the set of pairs of irreducible admissible representations of

$D_{\mathrm{ram}}^{\times}$ with equal central characters; when $d \neq 1$, we denote by $\mathrm{Irr}_f(GL(2,E))$ the set of elements π in $\mathrm{Irr}(GL(2,E))$ such that $\omega_\pi^\alpha = \omega_\pi$, where α is the non-trivial element of $\mathrm{Gal}(E/F)$. Often in this section, we will denote representations in $\mathrm{Irr}(GSO(X,F))$ by $\tilde{\pi}$, and irreducible admissible representations of $GO(X,F)$ by $\hat{\pi}$.

We are going to study representations of $GO(X,F)$ via inducing them from the index 2 subgroup $GSO(X,F)$. In fact, we have the following definition.

Definition 5.2.1. Let $\tilde{\pi} \in \mathrm{Irr}(GSO(X,F))$. If the induced representation of $\tilde{\pi}$ to $GO(X,F)$ is irreducible, we call $\tilde{\pi}$ *regular*. In this case, we write

$$\hat{\pi}^+ = \mathrm{ind}_{GSO(X,F)}^{GO(X,F)} \tilde{\pi}.$$

If $\tilde{\pi}$ is not regular, we call $\tilde{\pi}$ *invariant*. In this case, the induced representation to $GO(X,F)$ is not irreducible, and we write

$$\mathrm{ind}_{GSO(X,F)}^{GO(X,F)} \tilde{\pi} = \hat{\pi}^+ \oplus \hat{\pi}^-,$$

where $\hat{\pi}^+$ and $\hat{\pi}^-$ are irreducible admissible representations of $GO(X,F)$.

We now introduce for invariant representations, the notion of a *distinguished* representation. Later in this thesis, we will consider the so-called *theta correspondence* for the groups $GO(X,F)$ and $GSp(4,F)$, and we will see that certain extensions of a distinguished representation of $GSO(X,F)$ cannot occur in the theta correspondence[6].

Definition 5.2.2. Let $\tilde{\pi}$ be in $\mathrm{Irr}(GSO(X,F))$. We say that $\tilde{\pi}$ is *generically distinguished*, if $\tilde{\pi}$ is invariant and there is an anisotropic vector $y \in X$ such that

$$\mathrm{Hom}_{SO(Y,F)}(\tilde{\pi}, 1) \neq 0,$$

[6] In particular, if F is non-archimedean and we have a distinguished representation $\tilde{\pi} \in \mathrm{Irr}(GSO(X,F))$, the representation $\hat{\pi}^-$ introduced in Definition 5.2.1 cannot occur in the theta correspondence (see Theorem 6.8 of [46]); we will discuss these results later in more detail.

where Y is the orthogonal complement of Fy in X. We will say that $\tilde{\pi}$ is *distinguished* if $\tilde{\pi}$ is generically distinguished or $d \neq 1$ and $\tilde{\pi}$ is invariant and 1-dimensional ("boundary" case).

Note that, according to Section 4 of [46], the group $SO(Y, F)$ can be considered to be a subgroup of $GSO(X, F)$; if $d = 1$, one can see $SO(Y, F)$ as the image of the subgroup $\{(g, (g^*)^{-1}) : g \in GL(2, F)\}$ or $\{(g, (g^*)^{-1}) : g \in D^\times_{\text{ram}}\}$ under ρ, and if $d \neq 1$ as the image of $\{(\det(g), g) : g \in GL(2, F)\}$ under ρ (recall ρ from Proposition 5.1.4, and the description of $GSO(X, F)$ in Subsection 5.1.2).

Finally, we remark that for this section F need not be only non-archimedean, for according to Section 6.1 of [62] we can choose it to be archimedean as well; the theory is analogous in this case.

5.2.1 The discriminant algebra is a field

In this subsection we consider $E = F(\sqrt{d})$, with $d \neq 1$. In this case, as we mentioned in Subsection 5.1.2, we have the two quadratic spaces $X_{M_{2\times 2},d}$ and $X_{D_{\text{ram}},d}$, such that the similitude groups $GSO(X_{M_{2\times 2},d}, F)$ and $GSO(X_{D_{\text{ram}},d}, F)$ are isomorphic. If we denote for simplicity both quadratic spaces by X, we have by Proposition 5.1.4 the isomorphism

$$GSO(X, F) \cong (GL(2, E) \times F^\times)/E^\times. \tag{5.2}$$

By Equation (5.2) we have that there is a 2-to-1 surjective map

$$\text{Irr}(GSO(X, F)) \to \text{Irr}_f(GL(2, E)),$$

that takes a representation $\tilde{\pi} \in \text{Irr}(GSO(X, F))$ to the representation $\pi \in \text{Irr}_f(GL(2, E))$, such that π maps g to $\tilde{\pi}(\rho(1, g))$ and π has the same representation space as $\tilde{\pi}$. Conversely, let us have a representation $\pi \in \text{Irr}(GL(2, E))$ such that $\omega_\pi^\alpha = \omega_\pi$ (here α is the non-trivial element of $\text{Gal}(E/F)$). Then the central character ω_π factors through the norm map $N_{E/F}$ of the extension E/F by exactly two characters χ and χ'; here χ' is the character χ twisted by the quadratic character $\epsilon_{E/F}$ of the quadratic extension E/F. Denote by (π, χ) and (π, χ') the elements of $\mathrm{Irr}(GSO(X, F))$ corresponding to π, which are defined by $(\pi, \chi)(\rho(t, g)) = \chi^{-1}\pi(g)$ and $(\pi, \chi')(\rho(t, g)) = \chi'^{-1}\pi(g)$ respectively. For more information on the 2-to-1 surjective map that we defined above, see Section 3 of [46]. As a result, we may think of representations in $\mathrm{Irr}(GSO(X, F))$ as pairs (π, χ), where π is a representation in $\mathrm{Irr}(GL(2, E))$ and χ is a character of $F^\times$ such that $\omega_\pi = \chi \circ N_{E/F}$.

The two following results characterize invariant and distinguished representations of $GSO(X, F)$ in the case where we have $E = F(\sqrt{d})$.

Proposition 5.2.3. *If $E = F(\sqrt{d})$ with $d \neq 1$, then a representation $\tilde{\pi} \in \mathrm{Irr}(GSO(X, F))$ is invariant if and only if $\tilde{\pi} = (\pi, \chi)$ for some representation $\pi \in \mathrm{Irr}(GL(2, E))$ which is obtained as a base change from a representation in $\mathrm{Irr}(GL(2, F))$. We will say that π is Galois invariant in this case.*

Proof. See Proposition 3.1 of [46]. □

Proposition 5.2.4. *If $E = F(\sqrt{d})$ with $d \neq 1$, then an invariant representation $\tilde{\pi} = (\pi, \chi) \in \mathrm{Irr}(GSO(X, F))$ is distinguished if and only if π is obtained as a base change from a representation in $\mathrm{Irr}(GL(2, F))$ with central character equal to $\chi\epsilon_{E/F}$. Here $\epsilon_{E/F}$ is the quadratic character associated to the quadratic extension E/F.*

Proof. See Proposition 4.1 and Theorem 5.3 of [46]. □

5.2.2 The discriminant algebra is split

We consider now the case where $d = 1$ so that $E = F \times F$. By Subsection 5.1.2 we have the two 4-dimensional quadratic spaces $M_{2\times2}(F)$ and D_{ram}, and the groups

$$GSO(M_{2\times2}(F), F) \cong (GL(2, F) \times GL(2, F))/F^\times, \tag{5.3}$$

and

$$GSO(D_{\mathrm{ram}}, F) \cong (D_{\mathrm{ram}}^\times \times D_{\mathrm{ram}}^\times)/F^\times, \tag{5.4}$$

respectively.

By the two isomorphisms (5.3) and (5.4) we get bijections

$$\mathrm{Irr}(GSO(M_{2\times 2}(F), F)) \to \mathrm{Irr}_f(GL(2,F) \times GL(2,F))$$

and

$$\mathrm{Irr}(GSO(D_{\mathrm{ram}}, F)) \to \mathrm{Irr}_f(D_{\mathrm{ram}}^\times \times D_{\mathrm{ram}}^\times),$$

which map a representation $\tilde{\pi}$ to the representation that sends (g, g') to $\tilde{\pi}(\rho(g, g'))$. Conversely, if (π, π') is a pair in $\mathrm{Irr}_f(GL(2,F) \times GL(2,F))$ or in $\mathrm{Irr}_f(D_{\mathrm{ram}}^\times \times D_{\mathrm{ram}}^\times)$ then the corresponding representation in $\mathrm{Irr}(GSO(X,F))$ (where $X = M_{2\times 2}(F)$ or D_{ram} respectively) has representation space $\pi \otimes_{\mathbb{C}} \pi'$ and is defined by sending $\rho(g, g')$ to $\pi(g) \otimes \pi'(g')$. Elements in $\mathrm{Irr}(GSO(X,F))$ (where $X = M_{2\times 2}(F)$ or D_{ram}) will be denoted as pairs (π, π') such that $\omega_\pi = \omega_{\pi'}$. For more information on the above bijections, the reader should refer to Section 3 of [46].

Proposition 5.2.5. *If $d = 1$ so that $E = F \times F$, we have the following:*

1. *if $\tilde{\pi} \in \mathrm{Irr}(GSO(X,F))$ (where $X = M_{2\times 2}(F)$ or D_{ram}) then $\tilde{\pi}$ is invariant if and only if $\tilde{\pi} = (\pi, \pi)$, where π is an irreducible admissible representation of either $GL(2,F)$ or $D_{\mathrm{ram}}^\times$.*
2. *if $\tilde{\pi}$ is an invariant representation in $\mathrm{Irr}(GSO(X,F))$ (where $X = M_{2\times 2}(F)$ or D_{ram}) then it is distinguished.*

Proof. For the first statement see Proposition 3.1 of [46]; for the second statement see Proposition 4.1 of [46]. □

5.2.3 Automorphic representations of $GO(X, \mathbb{A}_{\mathbb{Q}})$

Before we discuss the theta correspondence, we consider the relationship between irreducible cuspidal automorphic representations $\tilde{\pi}$ of $GSO(X, \mathbb{A}_{\mathbb{Q}})$ and irreducible cuspidal automorphic representations $\hat{\pi}$ of $GO(X, \mathbb{A}_{\mathbb{Q}})$.

We have that

$$GO(X, \mathbb{A}_{\mathbb{Q}}) \cong GSO(X, \mathbb{A}_{\mathbb{Q}}) \rtimes \{1, s\},$$

where s is the order 2 element of Proposition 2.5 of [47], which gives rise to an isomorphism by conjugation

$$s : GSO(X, \mathbb{A}_{\mathbb{Q}}) \to GSO(X, \mathbb{A}_{\mathbb{Q}})$$

which we denote again by s without confusion.

Let $\tilde{\pi}$ be an irreducible cuspidal automorphic representation of $GSO(X, \mathbb{A}_{\mathbb{Q}})$ realized in a space of cuspforms $V_{\tilde{\pi}}$. We define $\tilde{\pi}^s$ by taking

$$V_{\tilde{\pi}^s} = \{f \circ s : f \in V_{\tilde{\pi}}\};$$

then $\tilde{\pi}^s$ is an irreducible cuspidal automorphic representation of $GSO(X, \mathbb{A}_{\mathbb{Q}})$.

Let $\hat{\pi}$ be an irreducible cuspidal automorphic representation of $GO(X, \mathbb{A}_{\mathbb{Q}})$ realized in a space of cuspforms $V_{\hat{\pi}}$. We define

$$V_{\hat{\pi}}^{\circ} = \{f|_{GSO(X,\mathbb{A}_{\mathbb{Q}})} : f \in V_{\hat{\pi}}\}.$$

Then, by Lemma 2 of [29], either $V_{\hat{\pi}}^{\circ} = V_{\tilde{\pi}}$ for some irreducible cuspidal automorphic representation $\tilde{\pi}$ of $GSO(X, \mathbb{A}_{\mathbb{Q}})$ with $\tilde{\pi} = \tilde{\pi}^s$, or $V_{\hat{\pi}}^{\circ} = V_{\tilde{\pi}} \oplus V_{\tilde{\pi}^s}$ for some irreducible cuspidal automorphic representation $\tilde{\pi}$ of $GSO(X, \mathbb{A}_{\mathbb{Q}})$ with $\tilde{\pi} \neq \tilde{\pi}^s$.

Before stating the following result, we advise the reader to recall the notation and the defining terms from Definition 5.2.1.

Proposition 5.2.6. *Let*

$$\hat{\pi} = \bigoplus_i \sigma_i$$

where σ_i runs over all irreducible cuspidal automorphic representations of $GO(X, \mathbb{A}_{\mathbb{Q}})$ such that there is an irreducible cuspidal automorphic representation $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ of $GSO(X, \mathbb{A}_{\mathbb{Q}})$ with

$$V_{\sigma_i}^{\circ} = V_{\tilde{\pi}} \quad \textit{if } \tilde{\pi} = \tilde{\pi}^s,$$

or

$$V_{\sigma_i}^{\circ} = V_{\tilde{\pi}} \oplus V_{\tilde{\pi}^s} \quad \textit{if } \tilde{\pi} \neq \tilde{\pi}^s.$$

Then

$$\hat{\pi} = \bigoplus_{\delta} \otimes_v \hat{\pi}_v^{\delta(v)},$$

where δ runs over all maps from the set of all places of $\mathbb{Q}$ to $\{\pm\}$ with the property that $\delta(v) = +$ for almost all places v of $\mathbb{Q}$, $\delta(v) = +$ if $\tilde{\pi}_v$ is regular, and $\prod_v \delta(v) = +$ if $\tilde{\pi} = \tilde{\pi}^s$. Moreover, each $\otimes \hat{\pi}_v^{\delta(v)}$ is (isomorphic to) an irreducible cuspidal automorphic representation of $GO(X, \mathbb{A}_{\mathbb{Q}})$.

Proof. See Proposition 5.4 of [61]. □

As a consequence, if $\tilde{\pi}$ is an irreducible cuspidal automorphic representation of $GSO(X, \mathbb{A}_{\mathbb{Q}})$, then there is an irreducible cuspidal automorphic representation $\hat{\pi}$ of $GO(X, \mathbb{A}_{\mathbb{Q}})$ lying above $\tilde{\pi}$ such that

$$\hat{\pi} \cong \otimes_v \hat{\pi}_v^{\delta(v)}.$$

5.3 The theta correspondence

In this section, we will briefly introduce the reader to the so-called *theta correspondence*. The local theory provides a correspondence between irreducible admissible representations of two reductive groups. In particular, we are going to consider the local theta correspondence between the similitude groups $GO(X, F)$ and $GSp(4, F)$. The global theory associates to an automorphic representation of a reductive group (in our case $GO(X, \mathbb{A}_{\mathbb{Q}})$) an automorphic representation of a second reductive group (in our case $GSp(4, \mathbb{A}_{\mathbb{Q}})$). The reader may find it useful to consult [43] for the general theory, [39] for the local theory, and [45] and [46] for the theta correspondence for the similitude groups $GO(X, F)$ and $GSp(4, F)$.

After describing the theory, we are going to apply it in our situation. Let π be an automorphic representation of $GL(2, \mathbb{A}_{\mathbb{K}})$, where K is an imaginary quadratic field. By considering the isomorphisms (5.2) and (5.3) and the theta correspondence for similitudes, one may construct an automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, following [29].

5.3.1 Local theta lift: the non-archimedean case

We begin by setting some notation. For now, let F be a non-archimedean local field of characteristic zero and fix a non-trivial additive character ψ of F. Let also X be a 4-dimensional quadratic F-space of discriminant d, together with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Recall that $Sp(4,F)$ and $O(X,F)$ are the groups which contain the elements with trivial similitude character of $GSp(4,F)$ and $GO(X,F)$ respectively; the similitude characters are denoted by λ and ν for $GSp(4,F)$ and $GO(X,F)$ respectively. We will denote by χ_X the quadratic character of $F^\times$ which is defined by $\chi_X(t) = (t,d)_F$; here $(\cdot,\cdot)_F$ is the Hilbert symbol[7] of F.

To ψ and X we associate the *Weil representation* ω of $Sp(4,F)\times O(X,F)$. The representation ω acts on the space $\mathcal{S}(X^2)$ of locally constant, compactly supported functions ϕ on X^2, and is defined via:

- $\omega(1,h)\phi(x) = \phi(h^{-1}x)$;
- $\omega\left(\begin{pmatrix} a & \\ & {}^t a^{-1}\end{pmatrix}, 1\right)\phi(x) = \chi_X(\det(a))|\det(a)|^2\phi(xa)$;
- $\omega\left(\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}, 1\right)\phi(x) = \psi(\frac{1}{2}\mathrm{tr}(bx,x))\phi(x)$;
- $\omega(J,1)\phi(x) = \gamma\hat{\phi}(x)$.

We need to explain some notation on the action of ω. Firstly, for $\phi \in \mathcal{S}(X^2)$, we let $\hat{\phi}$ be its Fourier transform, which is defined by

$$\hat{\phi}(x) = \int_{X^2} \phi(x')\psi(\mathrm{tr}(x,x'))dx'.$$

If $x = (x_1,x_2) \in X^2$ and $h \in O(X,F)$, write $h^{-1}x = (h^{-1}x_1, h^{-1}x_2)$; for $a \in GL(2,F)$ and $b \in M_{2\times 2}(F)$ with ${}^t b = b$, we write $xa \in X^2$ and $bx = b^t x$.

[7] Recall that the Hilbert symbol for F is defined as follows: $(a,b)_F = 1$ if $z^2 = ax^2 + by^2$ has a non-zero solution $(x,y,z) \in F^3$, and $(a,b)_F = -1$ if not. It is defined in Kapitel V, §3 in [41], and by Satz 3.2 we have that if the discriminant algebra E of F is a quadratic extension $E = F(\sqrt{d})$, then $\epsilon_{E/F}(\cdot) = (\cdot,d)_F$; here $\epsilon_{E/F}$ is the quadratic character associated to the quadratic extension E/F.

Also, for $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$, write

$$(x, x') = \begin{pmatrix} (x_1, x'_1) & (x_1, x'_2) \\ (x_2, x'_1) & (x_2, x'_2) \end{pmatrix},$$

where $(\cdot, \cdot)$ in the entries of the matrix is the bilinear form of X. Finally, γ is a certain fourth root of unity, and $J = \begin{pmatrix} &&&1 \\ &&1& \\ &-1&& \\ -1&&& \end{pmatrix}$. For more information on the Weil representation, one can see Section 1 of [47].

Since we are interested in the similitude theta correspondence (i.e., the theta correspondence between the similitude groups $GSp(4, F)$ and $GO(X, F)$), we need to extend the Weil representation as in Section 1 of [46]. Define

$$R = \{(g, h) \in GSp(4, F) \times GO(X, F) : \lambda(g) = \nu(h)\}.$$

Then the Weil representation ω of $Sp(4, F) \times O(X, F)$ on $\mathcal{S}(X^2)$ extends to a unitary representation of R by

$$\omega(g, h)\phi = |\nu(h)|^{-1}\omega(g_1, 1)(\phi \circ h^{-1}),$$

where

$$g_1 = g \begin{pmatrix} 1 & \\ & \lambda(g) \end{pmatrix}^{-1} \in Sp(4, F).$$

This will be the *extended Weil representation*, which we still denote by ω.

Let $GSp(4, F)^+$ be the subgroup that contains elements g of $GSp(4, F)$ such that $\lambda(g) = \nu(h)$ for some $h \in GO(X, F)$. Note that the extended Weil representation involves only representations of $GSp(4, F)^+$; in fact, $GSp(4, F)^+$ is a subgroup of $GSp(4, F)$ of at most index 2. Moreover, we remark that there is a close relationship between $\mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0$ and $\mathrm{Hom}_{Sp(4,F) \times O(X,F)}(\omega, \pi_1 \otimes \sigma_1) \neq 0$ for π_1 and σ_1 irreducible constituents of $\pi|_{Sp(4,F)}$ and $\sigma|_{O(X,F)}$ respectively, due to Lemma 4.2 of [45]; this relationship indicates which are going to be the irreducible admissible representations of

$GSp(4,F)$ and of $GO(X,F)$ that participate in the local theta correspondence for similitudes. In particular, let $\mathcal{R}(Sp(4,F))$ be the set of irreducible admissible representations of $Sp(4,F)$ which are non-zero quotients of the Weil representation ω; analogously define $\mathcal{R}(O(X,F))$ as the set of irreducible admissible representations of $O(X,F)$ that are non-zero quotients of ω. Lemma 4.2 of [45] suggests that, for the similitude correspondence, we should consider the set $\mathcal{R}(GSp(4,F)^+)$ of irreducible admissible representations π of $GSp(4,F)^+$ such that $\pi|_{Sp(4,F)}$ is multiplicity free and has an irreducible constituent in $\mathcal{R}(Sp(4,F))$; for the same reasons, we also consider the set $\mathcal{R}(GO(X,F))$ of irreducible admissible representations σ of $GO(X,F)$ such that $\sigma|_{O(X,F)}$ is multiplicity free and has an irreducible constituent in $\mathcal{R}(O(X,F))$.

The following result gives a correspondence between the sets $\mathcal{R}(GSp(4,F)^+)$ and $\mathcal{R}(GO(X,F))$ for F a non-archimedean local field of odd residual characteristic. This is the so-called *Howe duality* for similitudes.

Theorem 5.3.1. *Let F be a non-archimedean local field of odd residual characteristic. The set*

$$\tilde{R} = \{(\pi,\sigma) \in \mathcal{R}(GSp(4,F)^+) \times \mathcal{R}(GO(X,F)) : \mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(GSp(4,F)^+)$ and $\mathcal{R}(GO(X,F))$.

Proof. See Section 4 of [45] or Theorem 1.2 of [46]. □

By Section 1 of [45], Theorem 5.3.1 essentially says that both of the following statements hold:

i. every representation $\pi \in \mathcal{R}(GSp(4,F)^+)$ occurs as the first entry of an element of $\tilde{R}$ and every $\sigma \in \mathcal{R}(GO(X,F))$ occurs as the second entry of an element of $\tilde{R}$;

ii. for all irreducible admissible representations π of $GSp(4,F)^+$ and σ_1, σ_2 of $GO(X,F)$, if (π,σ_1) and (π,σ_2) belong to $\tilde{R}$, then $\sigma_1 \cong \sigma_2$. The analogous result holds if the roles of $GSp(4,F)^+$ and $GO(X,F)$ are interchanged.

We also have the so-called *multiplicity preservation* for the correspondence between $\mathcal{R}(GSp(4,F)^+)$ and $\mathcal{R}(GO(X,F))$, which is the next result.

Proposition 5.3.2. *Let F be a non-archimedean local field of odd residual characteristic. For $\pi \in \mathcal{R}(GSp(4,F)^+)$ and $\sigma \in \mathcal{R}(GO(X,F))$ we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\omega, \pi \otimes \sigma) \leq 1.$$

Proof. This follows from Theorem 4.4 of [45]. □

We now briefly explain the correspondence in Theorem 5.3.1. Let $(\omega, \mathcal{S})$ be the extended Weil representation; here for simplicity we denote $\mathcal{S} = \mathcal{S}(X^2)$. Let σ be an irreducible admissible representation of $GO(X,F)$, and define

$$\mathcal{S}(\sigma) = \mathcal{S} \Big/ \bigcap_{t \in \mathrm{Hom}_{GO(X,F)}(\omega,\sigma)} \ker(t)\ ;$$

i.e., $\mathcal{S}(\sigma)$ is the maximal quotient of $\mathcal{S}$ on which $GO(X,F)$ acts as a multiple of σ. Via ω, $GO(X,F) \times GSp(4,F)^+$ acts on $\mathcal{S}(\sigma)$, and we denote this representation by $\omega(\sigma)$. By Lemma III.4 of Chapter 2 of [39], there is a smooth representation $\Theta(\sigma)$ of $GSp(4,F)^+$, unique up to isomorphism, such that

$$\omega(\sigma) \cong \sigma \otimes \Theta(\sigma)$$

as representations of $GO(X,F) \times GSp(4,F)^+$. The following result is the so-called *strong Howe duality* for the sets $\mathcal{R}(GO(X,F))$ and $\mathcal{R}(GSp(4,F)^+)$.

Theorem 5.3.3. *Let F be a non-archimedean local field of odd residual characteristic. For every $\sigma \in \mathcal{R}(GO(X,F))$, the representation $\Theta(\sigma)$ has a unique non-zero irreducible quotient $\theta(\sigma) \in \mathcal{R}(GSp(4,F)^+)$. In fact, there is a bijection*

$$\theta : \mathcal{R}(GO(X,F)) \to \mathcal{R}(GSp(4,F)^+),$$

given by Howe duality.

Proof. This is implied by Theorem 5.3.1, Proposition 5.3.2, and Proposition 1.1 of [45]. □

For a representation $\sigma \in \mathcal{R}(GO(X,F))$, we will call $\Theta(\sigma)$ the *big theta lift* and $\theta(\sigma)$ the *small theta lift* of σ.

Now let $\mathcal{R}(GSp(4,F))$ be the set of irreducible admissible representations π of $GSp(4,F)$ such that some irreducible constituent of $\pi|_{GSp(4,F)^+}$ is contained in $\mathcal{R}(GSp(4,F)^+)$. As we see in the next theorem, Roberts in [47], refines the above result to a strong Howe duality between the sets $\mathcal{R}(GSp(4,F))$ and $\mathcal{R}(GO(X,F))$.

Theorem 5.3.4. *Let F be a non-archimedean local field of odd residual characteristic. The set*

$$\{(\pi,\sigma) \in \mathcal{R}(GSp(4,F)) \times \mathcal{R}(GO(X,F)) : \mathrm{Hom}_R(\omega, \pi\otimes\sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(GSp(4,F))$ and $\mathcal{R}(GO(X,F))$. Moreover, we have that

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\omega, \pi\otimes\sigma) \leq 1$$

for $\pi \in \mathcal{R}(GSp(4,F))$ and $\sigma \in \mathcal{R}(GO(X,F))$.

Proof. See Theorem 1.8 of [47] (for F non-archimedean of odd residual characteristic; consider also that $\dim(X) = 4$ in our case). □

Remark 5.3.5. According to Section 2 of [19], when the discriminant d of the 4-dimensional quadratic space X is 1, we have that the similitude character ν of $GO(X,F)$ is surjective; this implies that $GSp(4,F)^+ = GSp(4,F)$. If $d \neq 1$, then $GSp(4,F)^+$ is an index 2 subgroup of $GSp(4,F)$; in that case, if $\sigma \in \mathcal{R}(GO(X,F))$, we will denote

$$\tilde{\Theta}(\sigma) = \mathrm{ind}_{GSp(4,F)^+}^{GSp(4,F)} \Theta(\sigma),$$

and

$$\tilde{\theta}(\sigma) = \mathrm{ind}_{GSp(4,F)^+}^{GSp(4,F)} \theta(\sigma).$$

Recall from Section 5.2 that when we have an invariant representation $\tilde{\pi} \in \mathrm{Irr}(GSO(X,F))$ and we induce it to a representation of $GO(X,F)$, we get

$$\mathrm{ind}_{GSO(X,F)}^{GO(X,F)} \tilde{\pi} = \hat{\pi}^+ \oplus \hat{\pi}^-,$$

where $\hat{\pi}^+$ and $\hat{\pi}^-$ are irreducible admissible representations of $GO(X,F)$. Roberts in [46] describes explicitly the set $\mathcal{R}(GO(X,F))$ using distinguished representations, as one can see from the following result.

Proposition 5.3.6. *Let F be a non-archimedean local field of odd residual characteristic. Let σ be an irreducible admissible representation of $GO(X,F)$. Then $\sigma \in \mathcal{R}(GO(X,F))$ if and only if σ is not of the form $\hat{\pi}^-$ for some distinguished representation $\tilde{\pi} \in \mathrm{Irr}(GSO(X,F))$.*

Proof. See Theorem 6.8 of [46]. □

Up to now, in the Howe duality theorems above we have assumed that F is a non-archimedean local field of odd residual characteristic. Until recently, for F being a local field of even residual characteristic, the Howe duality was only known for tempered representations. In particular, if we denote by $\mathcal{R}(GO(X,F))_{\mathrm{temp}}$ and $\mathcal{R}(GSp(4,F))_{\mathrm{temp}}$ the sets of tempered elements in $\mathcal{R}(GO(X,F))$ and $\mathcal{R}(GSp(4,F))$ respectively; then Roberts proves the following result.

Theorem 5.3.7. *Let F be a non-archimedean local field of even residual characteristic. The set*

$$\{(\pi,\sigma) \in \mathcal{R}(GSp(4,F))_{\mathrm{temp}} \times \mathcal{R}(GO(X,F))_{\mathrm{temp}} : \mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(GSp(4,F))_{\mathrm{temp}}$ and $\mathcal{R}(GO(X,F))_{\mathrm{temp}}$. Moreover, we have that

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\omega, \pi \otimes \sigma) \leq 1$$

for $\pi \in \mathcal{R}(GSp(4,F))_{\mathrm{temp}}$ and $\sigma \in \mathcal{R}(GO(X,F))_{\mathrm{temp}}$.

Proof. See part of Theorem 1.8 in [47]. □

For the similitude theta correspondence in the case of even residual characteristic, the temperedness assumption is needed because the same is true for the theta correspondence between the groups $Sp(4)$ and $O(X)$ (see Theorem 1.2 of [47]). Gan and Takeda in their 2014 papers [20] and [21], prove the Howe duality conjecture for the reductive dual pair $(Sp(4), O(X))$ without the temperedness assumption. In particular, they prove the following result.

Theorem 5.3.8. *Let F be a non-archimedean local field of characteristic different from 2, and arbitrary residual characteristic. The set*

$$\{(\pi, \sigma) \in \mathcal{R}(Sp(4, F)) \times \mathcal{R}(O(X, F)) : \mathrm{Hom}_{Sp(4,F)\times O(X,F)}(\omega, \pi \otimes \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(Sp(4, F))$ and $\mathcal{R}(O(X, F))$. Moreover, we have that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{Sp(4,F)\times O(X,F)}(\omega, \pi \otimes \sigma) \leq 1,$$

for $\pi \in \mathcal{R}(Sp(4, F))$ and $\sigma \in \mathcal{R}(O(X, F))$.

Proof. This is Theorem 1.3 of [20], or Theorem 1.2 of [21]. We present this theorem in this form following Proposition 1.1 of [45]. □

The following result, as far as we know, has not appeared in the literature. Though the arguments in the proof are essentially the ones Roberts uses to prove Theorem 1.8 of [47]. We only replace the use of Theorem 1.2 of [47] with Theorem 5.3.8.

Theorem 5.3.9. *Let F be a non-archimedean local field of even residual characteristic. The set*

$$\{(\pi, \sigma) \in \mathcal{R}(GSp(4, F)) \times \mathcal{R}(GO(X, F)) : \mathrm{Hom}_R(\omega, \pi \otimes \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(GSp(4, F))$ and $\mathcal{R}(GO(X, F))$. Moreover, we have that

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(\omega, \pi \otimes \sigma) \leq 1$$

for $\pi \in \mathcal{R}(GSp(4, F))$ and $\sigma \in \mathcal{R}(GO(X, F))$.

Proof. The first thing to notice is that in Section 4 of [45], the assumption that the residual characteristic is odd is needed only because Roberts uses the Howe duality for the reductive dual pair $(Sp(4), O(X))$, and this was known for odd residual characteristic; in particular the proof of Theorem 4.4 of [45] is independent of the residual characteristic. Thus, having Theorem 5.3.8, one gets a version of Theorem 4.4 of [45] which holds for arbitrary residual characteristic.

Consider first the case where the discriminant of X is 1. As we mentioned in Remark 5.3.5, in this case we have $GSp(4,F)^+ = GSp(4,F)$. Then, the result follows by Theorem 4.4 of [45] (the version for even residual characteristic).

Now, we consider the case where the discriminant of X is not 1, so that $[GSp(4,F) : GSp(4,F)^+] = 2$. The proof for this goes exactly as in Theorem 1.8 (2) of [47]; it only depends on results from Section 4 of [45], and on Kudla's theta dichotomy conjecture. The latter is proved in the case of interest to us in Lemma 1.4 of [47], and is residual characteristic independent. □

5.3.2 Local theta lift: the archimedean case

For completeness, we briefly mention some facts about the local theta correspondence for similitudes for the archimedean local field $F = \mathbb{R}$. As before, we fix a non-trivial unitary additive character ψ of $\mathbb{R}$. The Weil representation ω_∞ of $Sp(4,\mathbb{R}) \times O(X,\mathbb{R})$ is defined with respect to ψ as in the non-archimedean case (see Subsection 5.3.1), and is a smooth representation on $\mathcal{S}(X^2)$; the difference in the archimedean case is that we work with Harish-Chandra modules.

Let K_1 be a maximal compact subgroup of $Sp(4,\mathbb{R})$ and $\mathfrak{sp}(4,\mathbb{R})$ the Lie algebra of $Sp(4,\mathbb{R})$; furthermore, let J_1 be a maximal compact subgroup of $O(X,\mathbb{R})$ and $\mathfrak{o}(X,\mathbb{R})$ the Lie algebra of $O(X,\mathbb{R})$. By Section 1 of [47], $\mathcal{S}(X^2)$ is an $(\mathfrak{sp}(4,\mathbb{R}) \times \mathfrak{o}(X,\mathbb{R}), K_1 \times J_1)$-module under the action of ω_∞. Denote by $\mathcal{R}(O(X,\mathbb{R}))$ the set of irreducible $(\mathfrak{o}(X,\mathbb{R}), J_1)$-modules which are non-zero quotients of ω_∞, and similarly we define $\mathcal{R}(Sp(4,\mathbb{R}))$ as the set of irreducible

$(\mathfrak{sp}(4,\mathbb{R}), K_1)$-modules which are non-zero quotients of ω_∞. For the precise choices of maximal compact subgroups, one may refer to Section 1 of [47].

We consider now the extended Weil representation (denoted again by ω_∞), which extends, in the same way as in the non-archimedean case, to a unitary representation of

$$R = \{(g,h) \in GSp(4,\mathbb{R}) \times GO(X,\mathbb{R}) : \lambda(g) = \mu(h)\}.$$

The extended to R Weil representation also preserves $\mathcal{S}(X^2)$, but only at the level of Harish-Chandra modules. That is, if L_∞ is a maximal compact subgroup of R and $\mathfrak{r}_\infty$ is the Lie algebra of R, the space $\mathcal{S}(X^2)$ is closed under the action of ω_∞ restricted to L_∞ and $\mathfrak{r}_\infty$.

In order to state the main result which is due to Roberts, we need some more notation. Firstly, fix maximal compact subgroups K_∞ and J_∞ of $GSp(4,\mathbb{R})$ and $GO(X,\mathbb{R})$ respectively, while $\mathfrak{gsp}(4,\mathbb{R})$ and $\mathfrak{go}(X,\mathbb{R})$ are their Lie algebras. Let $\mathrm{Irr}(GO(X,\mathbb{R}))$ be the set of irreducible $(\mathfrak{go}(X,\mathbb{R}), J_\infty)$-modules, and $\mathrm{Irr}(GSp(4,\mathbb{R}))$ be the set of irreducible $(\mathfrak{gsp}(4,\mathbb{R}), K_\infty)$-modules. We denote by $\mathcal{R}(GO(X,\mathbb{R}))$ the set of $\sigma \in \mathrm{Irr}(GO(X,\mathbb{R}))$ such that $\sigma|_{O(X,\mathbb{R})} = \sigma|_{(\mathfrak{o}(X,\mathbb{R}),J_1)}$ has an irreducible constituent in $\mathcal{R}(O(X,\mathbb{R}))$; similarly we define $\mathcal{R}(GSp(4,\mathbb{R}))$ as the set of $\pi \in \mathrm{Irr}(GSp(4,\mathbb{R}))$ such that $\pi|_{Sp(4,\mathbb{R})} = \pi|_{(\mathfrak{sp}(4,\mathbb{R}),K_1)}$ has an irreducible constituent in $\mathcal{R}(Sp(4,\mathbb{R}))$.

Theorem 5.3.10. *The set*

$$\{(\pi,\sigma) \in \mathcal{R}(GSp(4,\mathbb{R})) \times \mathcal{R}(GO(X,\mathbb{R})) : \mathrm{Hom}_R(\omega_\infty, \pi \otimes \sigma) \neq 0\}$$

is the graph of a bijection between $\mathcal{R}(GSp(4,\mathbb{R}))$ and $\mathcal{R}(GO(X,\mathbb{R}))$.

Proof. In Theorem 1.8 of [47], the result is proved for X with signature (p,q) with p and q even. Lemma 4.2 of [61] does not have this assumption on the signature. □

Note that if π corresponds to σ as in Theorem 5.3.10, then π is unique only up to infinitesimal equivalence; the Casselman-Wallach canonical completion[8] of π is denoted by $\theta(\sigma)$.

5.3.3 Explicit local theta lift

Let $K/\mathbb{Q}$ be a quadratic extension, $\mathcal{O}_K$ the ring of integers of K, and for each place w of K, K_w the completion of K at w. Let $\pi = \bigotimes_w \pi_w$ be an automorphic representation of $GL(2, \mathbb{A}_K)$, where w runs through all places of K; we will assume that the central character ω_π of π is Galois invariant (i.e., $\omega_\pi^c = \omega_\pi$, where c is the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$). For each finite place $w = \mathfrak{p}$, $\pi_\mathfrak{p}$ is an irreducible admissible representation of $GL(2, K_\mathfrak{p})$. In this subsection, we will consider $\pi_\mathfrak{p}$ for $\mathfrak{p}$ lying above a rational prime p, and we will lift it to a representation of $GSp(4, \mathbb{Q}_p)$. We will consider two cases: p splits in K as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ (here $\bar{\mathfrak{p}}$ is the Galois conjugate of $\mathfrak{p}$), and p is non-split in K (i.e., $p\mathcal{O}_K = \mathfrak{p}$ or $p\mathcal{O}_K = \mathfrak{p}^2$).

Let $K_\mathfrak{p} = \mathbb{Q}_p(\sqrt{d})$ be a field, with $d \neq 1$; this is the case when $p\mathcal{O}_K = \mathfrak{p}$ or $p\mathcal{O}_K = \mathfrak{p}^2$. Also write $|\ |_\mathfrak{p}$ for the normalized absolute value of $K_\mathfrak{p}$. Consider the irreducible admissible representation $\pi_\mathfrak{p}$ of $GL(2, K_\mathfrak{p})$, for which the central character factors through the norm map, i.e., $\omega_{\pi_\mathfrak{p}} = \chi_\mathfrak{p} \circ N_{K_\mathfrak{p}/\mathbb{Q}_p}$ for some character $\chi_\mathfrak{p}$ of $\mathbb{Q}_p^\times$. By Subsection 5.2.1, the pair $(\pi_\mathfrak{p}, \chi_\mathfrak{p})$ defines a representation of $GSO(X, \mathbb{Q}_p)$, where $X = X_{M_{2\times 2},d}$ or $X_{D_{\mathrm{ram}},d}$ as described in Subsection 5.1.2. For the next theorem, we advise the reader to recall notation and terminology from Section 5.2. Moreover, we will use c also for the non-trivial element of $\mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p)$, without confusing it with the global case.

Theorem 5.3.11. *Suppose $p\mathcal{O}_K = \mathfrak{p}$ or $p\mathcal{O}_K = \mathfrak{p}^2$, such that $K_\mathfrak{p}$ is a field. Let $\tilde{\pi}_\mathfrak{p} = (\pi_\mathfrak{p}, \chi_\mathfrak{p})$ be an irreducible admissible representation of $GSO(X, \mathbb{Q}_p)$. We have the following:*

1. *Let $\pi_\mathfrak{p}$ be a supercuspidal representation of $GL(2, K_\mathfrak{p})$. Then we have:*
   - *$\tilde{\pi}_\mathfrak{p}$ is regular; then $\pi_\mathfrak{p}$ is not a base change from $GL(2, \mathbb{Q}_p)$, and $\Theta(\hat{\pi}_\mathfrak{p}^+) = \theta(\hat{\pi}_\mathfrak{p}^+)$ is generic supercuspidal.*

[8] For a description of the Casselman-Wallach canonical completion, see [8]. Essentially, it is the extension of the Harish-Chandra module π to a representation of $GSp(4, \mathbb{R})$.

- *$\tilde{\pi}_{\mathfrak{p}}$ is invariant and distinguished; then $\pi_{\mathfrak{p}}$ is a base change from some supercuspidal representation π_p of $GL(2,\mathbb{Q}_p)$ with central character $\omega_{\pi_p} = \chi_{\mathfrak{p}}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. We get*

$$\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes \pi_p,$$

  i.e., a representation of type VII.

- *$\tilde{\pi}_{\mathfrak{p}}$ is invariant, but not distinguished; then $\pi_{\mathfrak{p}}$ is a base change from some supercuspidal representation π_p of $GL(2,\mathbb{Q}_p)$ with central character $\omega_{\pi_p} = \chi_{\mathfrak{p}}$. Then $\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+})$ and $\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{-}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{-})$ are both irreducible supercuspidal representations of $GSp(4,\mathbb{Q}_p)$. They lie in the same L-packet, of L-parameter $\phi_{\pi_p} \oplus \phi_{\pi_p}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+})$ is generic, while $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{-})$ is non-generic.*

2. *Let $\pi_{\mathfrak{p}} = (\mu|\ |_{\mathfrak{p}}^{1/2})St_{GL(2)}$. As the central character of $\pi_{\mathfrak{p}}$ factors through the norm map, we have $\mu^2|\ |_{\mathfrak{p}} = \chi \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. Then there is a (possibly trivial) quadratic character η of $\mathbb{Q}_p^{\times}$ such that $\mu^c/\mu = \eta \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. We obtain the following cases:*

   - *$\tilde{\pi}_{\mathfrak{p}}$ is regular; then $\mu^c \neq \mu$. Also we have $\eta \neq 1$ or $\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$, and*

$$\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \delta(|\ |\eta\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}, |\ |^{-1/2}BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \mu|\ |_{\mathfrak{p}}^{1/2})),$$

     i.e., a representation of type IXa.

   - *$\tilde{\pi}_{\mathfrak{p}}$ is invariant and distinguished; then $\mu = \mu' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$, for μ' a character of $\mathbb{Q}_p^{\times}$. Furthermore, we have $\eta = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ and $\chi_{\mathfrak{p}} = |\ |(\mu')^2\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. We have*

$$\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes (\mu'|\ |^{1/2})St_{GSp(2)}.$$

     That is, a representation of type IIIa.

   - *$\tilde{\pi}_{\mathfrak{p}}$ is invariant, but not distinguished; again we have $\mu = \mu' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ for μ' a character of $\mathbb{Q}_p^{\times}$, but this time $\eta = 1$ and $\chi_{\mathfrak{p}} =$*

$|\ |(\mu')^2$. *Then we get*

$$\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \delta([\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}, |\ |\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}], \mu'),$$

i.e., a representation of type Va, while $\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{-}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{-})$ *is the non-generic supercuspidal representation of type Va*. That is,* $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+})$ *and* $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{-})$ *form a single L-packet.*

3. *Let* $\pi_{\mathfrak{p}} = \chi_1 \times \chi_2$ *be a principal series representation of* $GL(2, K_{\mathfrak{p}})$*. Then we have:*

   - $\tilde{\pi}_{\mathfrak{p}}$ *is regular; in this case* $\chi_2 \neq \chi_2^c$*, and* χ_1 *is not equal to* $\chi_{\mathfrak{p}}$ *or* $\chi_{\mathfrak{p}}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ *(*χ_1 *seen as a character of* $\mathbb{Q}_p^{\times}$*). Then*

     $$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \left(\frac{\chi_1}{\chi_{\mathfrak{p}}}\right)^{-1} \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes \frac{\chi_1}{\chi_{\mathfrak{p}}} BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \chi_2^c),$$

     i.e., a representation of type VII, unless $\frac{\chi_1}{\chi_{\mathfrak{p}}} = |\ |^{-1}$ *or* $|\ |^{-1}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$*, in which case we have that* $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+})$ *is a representation of type IXb.*

   - $\tilde{\pi}_{\mathfrak{p}}$ *is invariant distinguished; in this case* $\chi_2 = \chi_2^c$*, or* $\chi_2 \neq \chi_2^c$ *and* $\chi_1 = \chi_{\mathfrak{p}}$ *(*χ_1 *seen as a character of* $\mathbb{Q}_p^{\times}$*). When* $\chi_2 = \chi_2^c$ *so that* $\chi_2 = \chi_2' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$*, we have*

     $$\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \chi_1'^{-2}\chi_{\mathfrak{p}}\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \times \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes \chi_1'^2\chi_2'\chi_{\mathfrak{p}}^{-1},$$

     i.e., a representation of type I. Here, the character χ_1' *is such that* $\chi_1 = \chi_1' \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$.
     If $\chi_2 \neq \chi_2^c$ *and* $\chi_1 = \chi_{\mathfrak{p}}$*, we have*

     $$\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^{+}) = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p} \rtimes BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \chi_2^c),$$

     i.e., a representation of type VII.

   - $\tilde{\pi}_{\mathfrak{p}}$ *is invariant, but not distinguished; in this case* $\chi_2 \neq \chi_2^c$ *and*

$\chi_1 = \epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}$ (χ_1 seen as a character of $\mathbb{Q}_p^\times$). Then

$$\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^+) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+) = \tau(S, BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \chi_2^c)),$$

and

$$\tilde{\Theta}(\hat{\pi}_{\mathfrak{p}}^-) = \tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^-) = \tau(T, BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \chi_2^c)).$$

That is, $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^+)$ and $\tilde{\theta}(\hat{\pi}_{\mathfrak{p}}^-)$ form a single L-packet containing representations of type VIIIa and VIIIb.

Proof. See Theorem A.11 of [17]; in addition to the latter theorem, we write the types of the theta lifts as explicit as possible. Note that in the principal series case, we use Lemma A.5 of [17] and the discussion in p. 384 of [29] to see how a principal series representation of $GL(2, K_{\mathfrak{p}})$ corresponds to a representation of $GO(X, \mathbb{Q}_p)$. Some important information also can be found in the proof of Theorem A.11 of [17]. □

Now, let us consider the case where we have a rational prime p which splits in K as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, where $\bar{\mathfrak{p}}$ is the Galois conjugate of $\mathfrak{p}$. In this case, note that the completions $K_{\mathfrak{p}}$ and $K_{\bar{\mathfrak{p}}}$ of K at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively, can be identified with $\mathbb{Q}_p$; this is because the ramification index and the inertia degree of the extensions $K_{\mathfrak{p}}/\mathbb{Q}_p$ and $K_{\bar{\mathfrak{p}}}/\mathbb{Q}_p$ are trivial. In this way, we may also identify representations in $\mathrm{Irr}_f(GL(2, \mathbb{Q}_p) \times GL(2, \mathbb{Q}_p))$ with representations in $\mathrm{Irr}_f(GL(2, K_{\mathfrak{p}}) \times GL(2, K_{\bar{\mathfrak{p}}}))$, where the latter set is the set of pairs of irreducible admissible representations of $GL(2, K_{\mathfrak{p}})$ and $GL(2, K_{\bar{\mathfrak{p}}})$ with equal central characters. Following the correspondence of Subsection 5.2.2 for the quadratic space $M_{2\times 2}(\mathbb{Q}_p)$ of discriminant $d = 1$, we may write representations in $\mathrm{Irr}(GSO(M_{2\times 2}(\mathbb{Q}_p), \mathbb{Q}_p))$ as pairs $(\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})$, where $\omega_{\pi_{\mathfrak{p}}} = \omega_{\pi_{\bar{\mathfrak{p}}}}$. By Proposition 5.2.5, recall that invariant irreducible admissible representations of $GSO(M_{2\times 2}(\mathbb{Q}_p), \mathbb{Q}_p)$ are characterized by the fact that $\pi_{\mathfrak{p}} \cong \pi_{\bar{\mathfrak{p}}}$, and that all invariant representations are distinguished.

Theorem 5.3.12. *Suppose $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, and consider a representation $(\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}}) \in \mathrm{Irr}(GSO(M_{2\times 2}(\mathbb{Q}_p), \mathbb{Q}_p))$. Then we have the following cases:*

1. *If* $\pi_{\mathfrak{p}} \cong \pi_{\bar{\mathfrak{p}}} \cong \pi$ *is a supercuspidal representation, then*

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \tau(S, \pi),$$

   i.e., a generic representation of type VIIIa.

2. *If* $\pi_{\mathfrak{p}} \ncong \pi_{\bar{\mathfrak{p}}}$ *are both supercuspidal representations, then* $\Theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+)$ *a generic supercuspidal representation of* $GSp(4, \mathbb{Q}_p)$ *with L-parameter* $\phi_{\pi_{\mathfrak{p}}} \oplus \phi_{\pi_{\bar{\mathfrak{p}}}}$*.*

3. *If* $\pi_{\mathfrak{p}} \cong \pi_{\bar{\mathfrak{p}}} \cong (|\ |^{1/2}\mu)St_{GL(2)}$*, then*

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \tau(S, \mu),$$

   i.e., a generic representation of type VIa.

4. *If* $\pi_{\mathfrak{p}} = (|\ |^{1/2}\mu_1)St_{GL(2)}$ *and* $\pi_{\bar{\mathfrak{p}}} = (|\ |^{1/2}\mu_2)St_{GL(2)}$ *with* $\mu_1 \neq \mu_2$ *but* $\mu_1^2 = \mu_2^2$*, then*

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \delta(\left[\frac{\mu_1}{\mu_2}, |\ |\frac{\mu_1}{\mu_2}\right], \mu_2),$$

   i.e., a generic essentially square integrable representation of type Va; note that $\frac{\mu_1}{\mu_2}$ *is a non-trivial quadratic character.*

5. *If* $\pi_{\mathfrak{p}}$ *supercuspidal and* $\pi_{\bar{\mathfrak{p}}} = (|\ |^{1/2}\mu)St_{GL(2)}$*, then*

$$\Theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+) = \delta(|\ |^{1/2}(|\ |^{-1/2}\mu^{-1})\pi_{\mathfrak{p}}, \mu) = \delta(\mu^{-1}\pi_{\mathfrak{p}}, \mu),$$

   is a generic essentially square integrable representation of type XIa; the central character of the supercuspidal representation $(|\ |^{-1/2}\mu^{-1})\pi_{\mathfrak{p}}$ *is trivial, as required by representations of this type.*

6. *If* $\pi_{\mathfrak{p}}$ *is supercuspidal and* $\pi_{\bar{\mathfrak{p}}} = \chi_1 \times \chi_2$*, then* $\theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+)$ *is the Langlands quotient of* $\chi_1^{-1}\pi_{\mathfrak{p}} \rtimes \chi_1$*, i.e. a representation of type X.*

7. *If* $\pi_{\mathfrak{p}} = (|\ |^{1/2}\mu)St_{GL(2)}$ *and* $\pi_{\bar{\mathfrak{p}}} = \chi_1 \times \chi_2$*, then* $\theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+)$ *is the Langlands quotient of* $(|\ |^{1/2}\frac{\mu}{\chi_1})St_{GL(2)} \rtimes \chi_1$*, where* $\frac{\mu}{\chi_1}$ *is not a quadratic*

or a trivial character[9]*. Thus,* $\theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+)$ *is a representation of type IIa or IVc.*

8. *If* $\pi_{\mathfrak{p}} = \chi_1 \times \chi_2$ *and* $\pi_{\bar{\mathfrak{p}}} = \chi_1' \times \chi_2'$*, then* $\theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+)$ *is the Langlands quotient of* $\frac{\chi_2'}{\chi_1} \times \frac{\chi_1'}{\chi_1} \rtimes \chi_1$*. Thus,* $\theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+)$ *is a representation of type*[10] *I or IIIb.*

Proof. See Theorem A.10 of [17]. In addition to the latter theorem, we write the types of the theta lifts as explicit as possible. □

Remark 5.3.13. Theorems A.10 and A.11 of [17] use a different notation than the one that Roberts and Schmidt use in [48]; in this thesis we follow the notation of Roberts and Schmidt. Subsequently, we found out that the paper of Johnson-Leung and Roberts [34] contains these results on the explicit local theta lift, which also verifies the types we list.

Finally, we briefly discuss the archimedean case when $K/\mathbb{Q}$ is imaginary quadratic (for the real quadratic case see [34], where one has to lift a pair of irreducible admissible representations of $GL(2,\mathbb{R})$, which correspond to the two real archimedean places). Suppose that π_∞ is an irreducible admissible representation of $GL(2,\mathbb{C})$ with associated L-parameter $\phi_{w,n} : W_{\mathbb{C}} \to GL(2,\mathbb{C})$, where

$$z \mapsto |z|^{-w} \begin{pmatrix} (z/\bar{z})^{n/2} & \\ & (z/\bar{z})^{-n/2} \end{pmatrix}$$

for integers[11] $n \geq 1$ and w, such that $n \equiv w + 1 \bmod 2$. The representation π_∞ is lifted via the archimedean theta correspondence to a representation Π_∞ of $GSp(4,\mathbb{R})$ which belongs the the archimedean L-packet with L-parameter $\phi_{(w;n,0)}$, and it is a limit of discrete series. This is Proposition 5.2 of [40].

[9] Since we require that $\chi_1\chi_2^{-1} \neq |\ |^{-1}$ and that the central characters of the pair $(\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})$ are equal, i.e., $\chi_1\chi_2 = |\ |\mu^2$.

[10] Note that the Langlands quotients of type IIb, IVd, Vd, and VId are not obtained, since the principal series condition $\chi_1'\chi_2'^{-1} \neq |\ |^{\pm 1}$ is not valid.

[11] We take $n \geq 1$ and not just $n \geq 0$, since the cuspidal automorphic representations of $GL(2,\mathbb{A}_K)$ that we consider in this thesis have Galois representations attached to them; as we will see later, for such representations one needs $n \geq 1$.

5.3.4 Global theta lift

We now consider the global theta correspondence for similitudes, which provides a correspondence between automorphic representations of $GO(X, \mathbb{A}_{\mathbb{Q}})$ and automorphic representations of $GSp(4, \mathbb{A}_{\mathbb{Q}})$; here X will be a 4-dimensional quadratic space over $\mathbb{Q}$. We follow Section 5 of [47], where the reader can find more details about the global theta correspondence for similitudes. For the general theory, one can refer also to Section 8 of [43].

Firstly, we need some notation. If A is a ring, consider the set

$$R(A) = \{(g, h) \in GSp(4, A) \times GO(X, A) : \lambda(g) = \nu(h)\},$$

and we denote by $GSp(4, A)^+$ the elements g in $GSp(4, A)$ such that $\lambda(g) = \nu(h)$ for some $h \in GO(X, A)$. For the infinite place of $\mathbb{Q}$, fix a maximal compact subgroup J_∞ of $GO(X, \mathbb{R})$, and let $\mathfrak{h}_\infty$ be the Lie algebra of $GO(X, \mathbb{R})$. Moreover, fix a maximal compact subgroup K_∞ of $GSp(4, \mathbb{R})$, and denote by $\mathfrak{g}_\infty$ the Lie algebra of $GSp(4, \mathbb{R})$. Finally, let L_∞ be a maximal compact subgroup of $R(\mathbb{R})$, and $\mathfrak{r}_\infty$ be the Lie algebra of $R(\mathbb{R})$.

In order to describe the global theta correspondence we will need a global version of the Weil representation. Recall that for non-archimedean local fields of odd residual characteristic, we defined the (extended) Weil representation in Subsection 5.3.1. The (extended) Weil representation for non-archimedean local fields of even residual characteristic and for the archimedean field $\mathbb{R}$ is defined in the same way (see Section 1 of [47]). If v is a place of $\mathbb{Q}$, we will denote the extended Weil representation of $R(\mathbb{Q}_v)$ by ω_v. As we mentioned before, for non-archimedean places p of $\mathbb{Q}$ (this time including the ones with even residual characteristic) ω_p is a representation of $R(\mathbb{Q}_p)$ acting on $\mathcal{S}(X(\mathbb{Q}_p)^2)$; for the archimedean place, ω_∞ is an $(\mathfrak{r}_\infty, L_\infty)$-module acting on the Schwartz-Bruhat space $\mathcal{S}(X(\mathbb{R})^2)$. Let x_1, x_2, x_3, x_4 be a basis of X over $\mathbb{Q}$. If $(g, h) \in R(\mathbb{A}_{\mathbb{Q}})$ then for almost all finite places p, $\omega_p(g_p, h_p)$ fixes the characteristic function of $\mathbb{Z}_p x_1 + \mathbb{Z}_p x_2 + \mathbb{Z}_p x_3 + \mathbb{Z}_p x_4$. Let

$$\mathcal{S}(X(\mathbb{A}_{\mathbb{Q}})^2) = \bigotimes_v \mathcal{S}(X(\mathbb{Q}_v)^2)$$

be the restricted direct product over all places v of $\mathbb{Q}$ of the complex spaces $\mathcal{S}(X(\mathbb{Q}_v)^2)$, restricted with respect to the characteristic function of $\mathbb{Z}_p x_1 + \mathbb{Z}_p x_2 + \mathbb{Z}_p x_3 + \mathbb{Z}_p x_4$ for $v = p$ finite. Then $\mathcal{S}(X(\mathbb{A}_\mathbb{Q})^2)$ is an $R(\mathbb{A}_{\mathbb{Q},f}) \times (\mathfrak{r}_\infty, L_\infty)$-module; here $\mathbb{A}_{\mathbb{Q},f}$ denotes the finite part of the adèle ring. Let $\phi = \bigotimes_v \phi_v \in \mathcal{S}(X(\mathbb{A}_\mathbb{Q})^2)$ and $(g,h) \in R(\mathbb{A}_\mathbb{Q})$. The function $\omega(g,h)\phi : X(\mathbb{A}_\mathbb{Q}^2) \to \mathbb{C}$ given by

$$(\omega(g,h)\phi)(x) = \prod_v (\omega_v(g_v,h_v)\phi_v)(x_v)$$

is well defined (see Section 5 of [47]).

We now define the global theta lift for similitudes. For $\phi \in \mathcal{S}(X(\mathbb{A}_\mathbb{Q})^2)$ and $(g,h) \in R(\mathbb{A}_\mathbb{Q})$, set

$$\theta(g,h;\phi) = \sum_{x \in X(\mathbb{Q})^2} \omega(g,h)\phi(x),$$

which is a series that converges absolutely and is left invariant under $R(\mathbb{Q})$. If f is a cuspidal automorphic form on $GO(X,\mathbb{A}_\mathbb{Q})$ and $\phi \in \mathcal{S}(X(\mathbb{A}_\mathbb{Q})^2)$, we define a function $\theta(f,\phi)$ on $GSp(4,\mathbb{Q})^+ \backslash GSp(4,\mathbb{A}_\mathbb{Q})^+$ by

$$\theta(f,\phi)(g) = \int_{O(X,\mathbb{Q})\backslash O(X,\mathbb{A}_\mathbb{Q})} \theta(g,h_1h;\phi) f(h_1h)\, dh_1,$$

where $h \in GO(X,\mathbb{A}_\mathbb{Q})$ is any element such that $(g,h) \in R(\mathbb{A}_\mathbb{Q})$. The measure dh_1 is as described in [27] (after formula 5.1.11). According to Section 2 of [29], this integral converges absolutely, is independent of the choice of h, and can be extended uniquely to a function on $GSp(4,\mathbb{Q})\backslash GSp(4,\mathbb{A}_\mathbb{Q})$ by insisting that it is left invariant under $GSp(4,\mathbb{Q})$ and that it has support in $GSp(4,\mathbb{Q})GSp(4,\mathbb{A}_\mathbb{Q})^+$. This extended function, which we also denote by $\theta(f,\phi)$, is an automorphic form on $GSp(4,\mathbb{A}_\mathbb{Q})$. If $\hat{\pi}$ is a $GO(X,\mathbb{A}_{\mathbb{Q},f}) \times (\mathfrak{h}_\infty, J_\infty)$-invariant subspace of the space of cuspidal automorphic forms on $GO(X,\mathbb{A}_\mathbb{Q})$, then we denote by $\Theta(\hat{\pi})$ the $GSp(4,\mathbb{A}_{\mathbb{Q},f}) \times (\mathfrak{g}_\infty, K_\infty)$-invariant subspace of the space of automorphic forms on $GSp(4,\mathbb{A}_\mathbb{Q})$, generated by all $\theta(f,\phi)$ for f running through all cuspidal automorphic forms in $\hat{\pi}$ and $\phi \in \mathcal{S}(X(\mathbb{A}_\mathbb{Q})^2)$.

Remark 5.3.14. Two questions might arise at this point. The first one has to do with whether $\Theta(\hat{\pi})$ is contained in the space of cuspidal automorphic forms of not. The second question to consider is if the global theta lift $\Theta(\hat{\pi})$ is zero or not; is there a criterion to tell in which cases $\Theta(\hat{\pi})$ is non-zero? We are going to answer these questions later in this thesis, in conjunction with the case of interest to us.

5.4 Associating Galois representations over imaginary quadratic fields

In this section, we describe a process for attaching 2-dimensional Galois representations to regular algebraic cuspidal automorphic representations of $GL(2, \mathbb{A}_K)$, where K is an imaginary quadratic field. The main references for that are [29], [3], and [40]. The cuspidal automorphic representations of $GL(2, \mathbb{A}_K)$ that we are using should have Galois invariant central character, since we are relating automorphic representations of the groups $GL(2, \mathbb{A}_K)$ and $GSO(X, \mathbb{A}_{\mathbb{Q}})$ through data from Clifford algebras, as in Proposition 5.1.4.

Definition 5.4.1. Let $\pi = \bigotimes_w \pi_w$ be a cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$. We say that π is *regular algebraic* if at the infinite place ∞, π_∞ has L-parameter $\phi_{w,n} : \mathbb{C}^\times \to GL(2, \mathbb{C})$ given by[12]

$$z \mapsto |z|^{-w} \begin{pmatrix} (z/\bar{z})^{n/2} & \\ & (z/\bar{z})^{-n/2} \end{pmatrix},$$

for some integers $n \geq 1$ and w, with $n \equiv w + 1 \bmod 2$. Note that for the infinite place, the Weil group is $W_{\mathbb{C}} = \mathbb{C}^\times$. Moreover, if we set $k = n + 1$, we will say that π is of *weight* $k \geq 2$.

We now write down two cases of cuspidal automorphic representations

[12] A different normalization for this L-parameter is the one on p. 394 of [29], or the one used in [3]; in these cases, the L-parameter is $\phi_{w,n}$ twisted by the character $|\ |^{-n}$. Also, in [3] they set $k = n + 1$, with $k \geq 2$.

over imaginary quadratic fields for which one can attach 2-dimensional l-adic Galois representations in a trivial way.

Proposition 5.4.2. *Suppose π is a regular algebraic cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$ with $\omega_\pi = \omega_\pi^c$, for c the non-trivial element of* $\mathrm{Gal}(K/\mathbb{Q})$. *Let Σ be the finite set of places that ramify in K, places where π or π^c is ramified, and places lying above l. If π falls into one of the following cases:*

1. *$\delta\pi \cong \pi$ for some non-trivial quadratic character δ of K;*
2. *$\mu\pi \cong (\mu\pi)^c$ for some finite order character of μ of K;*

then, for each prime l, there is a continuous irreducible l-adic Galois representation

$$\rho : G_K \to GL(2, \bar{\mathbb{Q}}_l)$$

such that, for all primes $\mathfrak{p}$ of K outside Σ, ρ is unramified at $\mathfrak{p}$ and the characteristic polynomial of $\rho(\phi_\mathfrak{p})$ agrees with the Hecke polynomial of π at $\mathfrak{p}$. Here $\phi_\mathfrak{p} \in W_{K_\mathfrak{p}}$ lies above the inverse of a Frobenius element.

Proof. The reader may refer to Remark 1.5 of [3]. Here we sketch a proof of the result.

For the first assertion, let L/K be the quadratic extension associated to the non-trivial quadratic character δ; then we have that π is the automorphic induction from an idèle class character ψ of L such that ψ is not isomorphic to its Galois conjugate under $\mathrm{Gal}(L/K)$. The Galois representation attached to π is the representation obtained as the induction of the character $\psi' : \mathrm{Gal}(\bar{K}/L) \to \bar{\mathbb{Q}}_l^\times$ to $\mathrm{Gal}(\bar{K}/K)$. Here ψ' is the associated Galois character to the idèle class character ψ.

For the second assertion we have that a twist of π is a base change from a cuspidal automorphic representation of $GL(2, \mathbb{A}_\mathbb{Q})$ with a Galois representation $\tau : G_\mathbb{Q} \to GL(2, \bar{\mathbb{Q}}_l)$ attached to it. Then $\tau|_{G_K}$ is the Galois representation associated to $\mu\pi$. □

5.4.1 Construction of Siegel cuspforms

From now on, we will assume that π does not fall in one of the two cases of Proposition 5.4.2. Let $K = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field, such that $\mathrm{Gal}(K/\mathbb{Q}) = \{id, c\}$. We start by choosing a suitable 4-dimensional quadratic space. Let $D = M_{2\times 2}(\mathbb{Q})$ be the split quaternion algebra over $\mathbb{Q}$, and let $B = K \otimes_{\mathbb{Q}} D = M_{2\times 2}(K)$ be a quadratic quaternion algebra over $\mathbb{Q}$ as in Definition 5.1.2. By Lemma 5.1.6, we may associate a Galois action $a : B \to B$ such that

$$a(z \otimes x) = z^c \otimes x,$$

for $z \in K$ and $x \in D$. Then we have a 4-dimensional quadratic space X consisting of $x \in B$ such that $x^c = {}^t x$; that is,

$$X = \{x \in M_{2\times 2}(K) : x = {}^t x^c\}$$

is the space of Hermitian matrices in $M_{2\times 2}(K)$. We equip X with the quadratic form $-\det : X \to \mathbb{Q}$ of discriminant $d \neq 1$. Locally at the non-archimedean places, for the split quaternion algebra $D_p = D \otimes_{\mathbb{Q}} \mathbb{Q}_p$, we have

- for $p\mathcal{O}_K = \mathfrak{p}$, the quadratic quaternion algebra becomes

  $$K_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} D_p = M_{2\times 2}(K_{\mathfrak{p}}),$$

  which gives the quadratic space $X_{M_{2\times 2},d}$ as in Subsection 5.1.2.

- for $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, the quadratic quaternion algebra becomes

  $$(\mathbb{Q}_p \times \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} D_p = M_{2\times 2}(\mathbb{Q}_p) \times M_{2\times 2}(\mathbb{Q}_p),$$

  which implies that the quadratic space in this case is $M_{2\times 2}(\mathbb{Q}_p)$ equipped with the quadratic form $-\det$ (see Subsection 5.1.2).

For the archimedean place ∞ of $\mathbb{Q}$, we choose $D_\infty = D \otimes_{\mathbb{Q}} \mathbb{R} = M_{2\times 2}(\mathbb{R})$ for the quaternion algebra over $\mathbb{R}$ which gives the quadratic quaternion algebra $\mathbb{C} \otimes_{\mathbb{R}} D_\infty = M_{2\times 2}(\mathbb{C})$. By Lemma 5.1.6, we consider the Galois action a :

$M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$ such that

$$a(z \otimes x) = \bar{z} \otimes x,$$

where $z \in \mathbb{C}$, $x \in M_{2\times 2}(\mathbb{R})$, and $\bar{z}$ is the complex conjugate of z. This gives the quadratic space $X_{M_{2\times 2},-1}$ consisting of Hermitian matrices; i.e., elements x in $X_{M_{2\times 2},-1}$ are such that $x = {}^t\bar{x}$. This quadratic space, equipped with $-\det$ as a quadratic form, has discriminant -1.

The first step in the method is to relate representations of $GL(2, \mathbb{A}_K)$ with representations of $GO(X, \mathbb{A}_\mathbb{Q})$. Analogously to the local case, by tensoring with the adèle ring $\mathbb{A}_\mathbb{Q}$, Propositions 5.1.4 and 5.1.7 hold for $\mathbb{A}_\mathbb{Q}$. That is, there is an exact sequence

$$1 \to \mathbb{A}_E^\times \to \mathbb{A}_\mathbb{Q}^\times \times B^\times(\mathbb{A}_\mathbb{Q}) \to GSO(X, \mathbb{A}_\mathbb{Q}) \to 1,$$

where $B(\mathbb{A}_\mathbb{Q}) = B \otimes_\mathbb{Q} \mathbb{A}_\mathbb{Q}$ and $\mathbb{A}_E = E \otimes_\mathbb{Q} \mathbb{A}_\mathbb{Q}$. For the choice $B = M_{2\times 2}(K)$ which we made above, we see that $E = K$ and the latter exact sequence becomes

$$1 \to \mathbb{A}_K^\times \to \mathbb{A}_\mathbb{Q}^\times \times GL(2, \mathbb{A}_K) \to GSO(X, \mathbb{A}_\mathbb{Q}) \to 1.$$

This implies the following result.

Proposition 5.4.3. *There is a 2-to-1 surjection between cuspidal automorphic representations $\tilde{\pi}$ of $GSO(X, \mathbb{A}_\mathbb{Q})$ and cuspidal automorphic representations π of $GL(2, \mathbb{A}_K)$ with central character ω_π that factors through*[13] *the idèle norm map.*

Proof. See Proposition 1 of [29]. □

Consider a regular algebraic cuspidal automorphic representation π of $GL(2, \mathbb{A}_K)$ of weight $k \geq 2$, and χ, χ' the two Hecke characters of $\mathbb{A}_\mathbb{Q}^\times$ such that $\omega_\pi = \chi \circ N_{K/\mathbb{Q}}$ and $\omega_\pi = \chi' \circ N_{K/\mathbb{Q}}$. If χ is the Hecke character such that $\chi_\infty(-1) = (-1)^k$, we choose this one for the correspondence. Now, according

[13] By exactly two Hecke characters.

to Subsection 5.2.3 we may lift $\tilde{\pi}$ to a cuspidal automorphic representation $\hat{\pi}$ of $GO(X, \mathbb{A}_{\mathbb{Q}})$.

The second step is to consider the theta correspondence between the groups $GO(X, \mathbb{A}_{\mathbb{Q}})$ and $GSp(4, \mathbb{A}_{\mathbb{Q}})$. Following Subsection 5.3.4, one can lift $\hat{\pi}$ to an automorphic representation $\Pi = \Theta(\hat{\pi})$ of $GSp(4, \mathbb{A}_{\mathbb{Q}})$. Now we can ask whether Π is non-zero, and if Π is cuspidal or not. The first question is answered by Theorem 1.3 of [62], and depends on the non-vanishing of the local theta lifts. In particular, we have the following result.

Proposition 5.4.4. *Let $\hat{\pi} = \bigoplus_v \hat{\pi}_v$ be an infinite-dimensional cuspidal automorphic representation of $GO(X, \mathbb{A}_{\mathbb{Q}})$. Then the global theta lift $\Theta(\hat{\pi})$ to $GSp(4, \mathbb{A}_{\mathbb{Q}})$ does not vanish if and only if each local constituent $\hat{\pi}_v$ has a non-zero theta lift to $GSp(4, \mathbb{Q}_v)$.*

Proof. See part (1) of Theorem 1.3 of [62]. □

Proposition 6.5 of [62] implies in general the non-vanishing of the theta lift of the local constituents $\hat{\pi}_v$ for all places v (archimedean and non-archimedean). In fact, we may obtain the following.

Proposition 5.4.5. *Let $\Theta(\hat{\pi})$ be the global theta lift of $\hat{\pi}$ as above. Then $\Theta(\hat{\pi})$ does not vanish.*

Proof. Firstly, for the non-archimedean places we have that the local theta lifts do not vanish by Theorems 5.3.11 and 5.3.12. For the archimedean place, since we have chosen the central character of π to factor through the norm map by the character χ with $\chi_\infty(-1) = (-1)^k$, by (part 3 of) Lemma 12 of [29] we get that the archimedean local theta lift does not vanish too. Finally, by Proposition 5.4.4 we obtain that the global theta lift $\Theta(\hat{\pi})$ does not vanish as the local ones do not vanish. □

Proposition 5.4.6. *The representation $\Theta(\hat{\pi})$ is a cuspidal automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ of weight $(k, 2)$, with $k \geq 2$.*

Proof. By our choice that the central character of π factors through the norm map by the character χ with $\chi_\infty(-1) = (-1)^k$, we have by (part 1 of) Lemma 12 and Lemma 5 of [29] that $\Theta(\hat{\pi})$ is cuspidal.

For the weight, note that if Π_∞ is the theta lift of $\hat{\pi}_\infty$, one can choose Π_∞ to be a holomorphic limit of discrete series representation of weight $(k, 2)$; this can be seen in Proposition 3 of [29] or Proposition 5.2 of [40]. □

To sum up, we have constructed a cuspidal automorphic representation Π of $GSp(4, \mathbb{A}_Q)$ of weight $(k, 2)$, from a regular algebraic cuspidal automorphic representation π of $GL(2, \mathbb{A}_K)$ of weight $k \geq 2$, where K is an imaginary quadratic form, by assuming that the central character of π factors through the norm map $N_{\mathbb{Q}}^K$.

5.4.2 Galois representations attached to Siegel cuspforms

Let l be a rational prime. We are interested in attaching an l-adic Galois representation to the cuspidal automorphic representation Π, and we do that by following [3]. The first thing to consider is the following result which is due to Weissauer, extending earlier work of Taylor.

Theorem 5.4.7. *Let Π be an irreducible cuspidal automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, such that Π_∞ belongs to the holomorphic discrete series representations of weight (k_1, k_2) with $k_1 \geq k_2 \geq 3$. Let S denote the union of $\{l\}$ with the set of the rational primes p where Π_p has a ramified L-parameter. Then there exists a continuous semisimple l-adic Galois representation*

$$R : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(4, \bar{\mathbb{Q}}_l)$$

such that if p is a prime outside S, then R is unramified at p and the characteristic polynomial of $R(\phi_p)$ agrees with the Hecke polynomial of Π at p. Here $\phi_p \in W_{\mathbb{Q}_p}$ lies above the inverse of a Frobenius element.

Proof. See Theorem 1 of [68]. □

The difficulty in applying Theorem 5.4.7 directly to our constructed cuspidal automorphic representation Π is that it requires Π_∞ to be of weight (k_1, k_2) with $k_1 \geq k_2 \geq 3$, while Π is of weight $(k, 2)$ with $k \geq 2$. Taylor proved in [64] that one can attach an l-adic Galois representation to a Siegel modular form of low weight when one knows how to attach an l-adic Galois representation to a Siegel modular form of higher weight, by using the theory of *pseudorepresentations*. If we consider Theorem 5.4.7 together with Theorem 2 of [64], we can obtain a continuous semisimple Galois representation

$$R : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(4, \bar{\mathbb{Q}}_l)$$

attached to the cuspidal automorphic representation Π of weight $(k, 2)$, $k \geq 2$. The representation R is attached to Π in the sense that if p is a prime outside S (as in Theorem 5.4.7), then R is unramified at p and the characteristic polynomial of $R(\phi_p)$ agrees with the Hecke polynomial of Π at p. In fact, we are going to see in the next subsection that R is an irreducible representation, and this is due to the fact that the central character of our initial π factors through the norm map.

Definition 5.4.8. Let Π be an automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ of weight (k_1, k_2), with $k_2 \geq 2$; let the archimedean L-parameter of Π be $\phi_{(w;m_1,m_2)}$ with $m_1 = k_1 - 1$, $m_2 = k_2 - 2$. Suppose R is the associated Galois representation. The *Hodge-Tate weights* of R are given by

$$\{\delta_\infty, \delta_\infty + k_2 - 2, \delta_\infty + k_1 - 1, \delta_\infty + k_1 + k_2 - 3\},$$

where

$$\delta_\infty = \frac{1}{2}(w + 3 - k_1 - k_2).$$

Remark 5.4.9. By Proposition 5.2 of [40], if the regular algebraic cuspidal automorphic representation π of $GL(2, \mathbb{A}_K)$ is of weight k, with archimedean L-parameter $\phi_{w,n}$ (here $k = n+1$), then the cuspidal automorphic representation Π has archimedean L-parameter $\phi_{(w;n,0)}$. In addition, by Definition 5.4.8, the *Hodge-Tate weights*[14] of the corresponding 4-dimensional Galois repre-

[14]As we will see later, we are interested in whether these numbers are all distinct or not; in our case they are not all distinct. This will be important in Chapter 6. A more detailed description of this notion, can be found for example in [2].

sentation R are given by

$$\{\delta, \delta, \delta + n, \delta + n\},$$

where $\delta = \frac{1}{2}(w - n)$.

5.4.3 2-dimensional Galois representations over imaginary quadratic fields

Now, by using the 4-dimensional representation R of $G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we are going to discuss how to attach a 2-dimensional representation of $G_K = \text{Gal}(\bar{K}/K)$ to the cuspidal automorphic representation π of $GL(2, \mathbb{A}_K)$, where K is an imaginary quadratic field. The first thing to consider is the following result.

Lemma 5.4.10. *Let $\epsilon_{K/\mathbb{Q}}$ be the quadratic character of $\mathbb{Q}$ associated to the quadratic extension $K/\mathbb{Q}$. Then for the Galois representation R we have*

$$R \otimes \epsilon_{K/\mathbb{Q}} \cong R. \tag{5.5}$$

Proof. See Section 4 of [3]. □

Lemma 5.4.10 enables us to get R by induction from a 2-dimensional Galois representation as in the following result.

Proposition 5.4.11. *There exists a continuous irreducible l-adic Galois representation*

$$\rho : \text{Gal}(\bar{K}/K) \to GL(2, \bar{\mathbb{Q}}_l)$$

such that

$$R = \text{ind}_{G_K}^{G_{\mathbb{Q}}} \rho.$$

Therefore,

$$R|_{G_K} = \rho \oplus \rho^c.$$

Proof. One can find the proof in Lemma 4.1 of [3], but we include a proof in order to indicate the importance of Equation (5.5).

Let us first assume that R is irreducible. By Equation (5.5) we get that

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\epsilon_{K/\mathbb{Q}}, \mathrm{End}(R)) \neq 0.$$

Since $\epsilon_{K/\mathbb{Q}}|_K$ is trivial, by Schur's Lemma we get that $R|_{G_K}$ is reducible; note that $\mathrm{End}(R) \cong R \otimes R^\vee$, where $R^\vee$ is the contragredient representation. Let ρ be an irreducible constituent of $R|_{G_K}$ of minimal dimension, i.e., $\dim(\rho) \leq 2$. Then, Frobenius reciprocity

$$\mathrm{Hom}_{G_K}(\rho, R|_{G_K}) \cong \mathrm{Hom}_{G_{\mathbb{Q}}}(\mathrm{ind}_{G_K}^{G_{\mathbb{Q}}} \rho, R),$$

together with the fact that R is irreducible, implies that

$$R = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\rho)$$

with $\dim(\rho) = 2$.

Now assume that R is reducible. By Equation (5.5), if λ (respectively τ) is a 1-dimensional (respectively 2-dimensional) subrepresentation of R, then $\lambda \otimes \epsilon_{K/\mathbb{Q}}$ (respectively $\tau \otimes \epsilon_{K/\mathbb{Q}}$) is also a subrepresentation of R. We consider the following cases:

1. $R \cong \tau \oplus (\tau \otimes \epsilon_{K/\mathbb{Q}})$. Then $R = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\tau|_{G_K})$.

2. $R \cong \tau \oplus \sigma$, where σ is also a 2-dimensional $\epsilon_{K/\mathbb{Q}}$-invariant representation. That is, $\tau = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\mu)$ and $\sigma = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\nu)$, where μ and ν are 1-dimensional. Then $R = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\mu \oplus \nu)$.

3. $R \cong \tau \oplus \lambda \oplus (\lambda \otimes \epsilon_{K/\mathbb{Q}})$. Since $\tau \cong \tau \otimes \epsilon_{K/\mathbb{Q}}$, we have $\tau = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\mu)$, where μ is 1-dimensional. Then $R = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\mu \oplus \lambda|_{G_K})$.

4. $R \cong \lambda \oplus (\lambda \otimes \epsilon_{K/\mathbb{Q}}) \oplus \mu \oplus (\mu \otimes \epsilon_{K/\mathbb{Q}})$. Then $R = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\lambda|_{G_K} \oplus \mu|_{G_K})$.

As far as the irreducibility of ρ is concerned, this is Proposition 5.9 of [40]. For this, we need our assumption that we are not in the first case of Proposition 5.4.2. □

Remark 5.4.12. By Proposition 5.10 of [40], since we have assumed that we are not in the second case of Proposition 5.4.2 (i.e., our cuspidal automorphic representation π is not a base change from $GL(2, \mathbb{A}_{\mathbb{Q}})$), we have that $\rho \not\cong \rho^c$. This implies, as we mentioned earlier, that $R = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}}(\rho)$ is in fact irreducible.

Let l be a rational prime. We fix embeddings $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$, and an isomorphism $\iota_l : \bar{\mathbb{Q}}_l \cong \mathbb{C}$ which is compatible with these two empbeddings. If w is a place of K and ρ_w is an l-adic Galois representation of $G_{K_w} = \mathrm{Gal}(\bar{K}_w/K_w)$, we denote by $\mathcal{WD}_w$ the Weil-Deligne functor that sends the Galois representation ρ_w to the associated Weil-Deligne representation as in Theorem 3.2.6. Moreover, we denote by $| \ |_w$ the normalized absolute value for K_w, and ϕ_{π_w} the L-parameter associated to π_w under the local Langlands correspondence for $GL(2, K_w)$. Then the Galois representation ρ is associated to the cuspidal automorphic representation π of $GL(2, \mathbb{A}_{\mathbb{Q}})$, together with a local-global compatibility statement up to semisimplification[15], as in the following theorem due to Mok.

Theorem 5.4.13. *Let π be a regular algebraic cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$, where K is an imaginary quadratic field, with central character that factors through the norm map $N_{K/\mathbb{Q}}$. Then there is a continuous irreducible l-adic Galois representation*

$$\rho : \mathrm{Gal}(\bar{K}/K) \to GL(2, \bar{\mathbb{Q}}_l)$$

such that, for each non-archimedean place w of K not dividing l, we have that

$$\iota_l(\mathcal{WD}_w(\rho|_{G_{K_w}}))^{\mathrm{ss}} \cong \phi^{\mathrm{ss}}_{\pi_w \otimes |\det|_w^{-1/2}}.$$

Furthermore, if π_w is not a twisted Steinberg representation, then we have the full local-global compatibility (up to Frobenius semisimplification[16]).

$$\iota_l(\mathcal{WD}_w(\rho|_{G_{K_w}}))^{F-\mathrm{ss}} \cong \phi_{\pi_w \otimes |\det|_w^{-1/2}}.$$

[15] The symbol "ss" as a superscript of a representation means that we consider the semisimplification of the representation.

[16] Which we denote with a superscript "F-ss". See Definition 3.2.5.

Proof. The process for attaching the Galois representation ρ to the cuspidal automorphic representation π has been described essentially in this section, and as we mentioned before is due to Harris-Soudry-Taylor (see [29]), Taylor (see [65]), and Berger-Harcos (see [3]). The local-global compatibility statements are due to Mok (see Theorem 1.1. of [40]). □

5.4.4 Inducing and conductors

Before we close this section, we discuss inducing the L-parameters of irreducible admissible representations of $GL(2, K_{\mathfrak{p}})$ to L-parameters of $GSp(4, \mathbb{Q}_p)$. The following can be found in [34].

Firstly, let $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. If $(\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}}) \in \mathrm{Irr}(GSO(M_{2\times 2}(\mathbb{Q}_p), \mathbb{Q}_p))$, with $\phi_{\mathfrak{p}}$ and $\phi_{\bar{\mathfrak{p}}}$ being the L-parameters attached to $\pi_{\mathfrak{p}}$ and $\pi_{\bar{\mathfrak{p}}}$ respectively, then $\theta((\pi_{\mathfrak{p}}, \pi_{\bar{\mathfrak{p}}})^+)$ has L-parameter $\phi_p : W'_{\mathbb{Q}_p} \to GSp(4, \mathbb{C})$ defined as

$$\phi_p(w) = \phi_{\mathfrak{p}}(w) \oplus \phi_{\bar{\mathfrak{p}}}(w),$$

as a symplectic direct sum.

Suppose now that $p\mathcal{O}_K = \mathfrak{p}$ or $p\mathcal{O}_K = \mathfrak{p}^2$. Let $\pi_{\mathfrak{p}}$ be an irreducible admissible representation of $GL(2, K_{\mathfrak{p}})$ with central character that factors through the norm map, which has associated L-parameter $\phi_{\mathfrak{p}} : W'_{K_{\mathfrak{p}}} \to GL(2, \mathbb{C})$. Then $\theta(\hat{\pi}_{\mathfrak{p}}^+)$ has associated L-parameter $\phi_p : W'_{\mathbb{Q}_p} \to GSp(4, \mathbb{C})$ the representation

$$\phi_p = \mathrm{ind}_{W'_{K_{\mathfrak{p}}}}^{W'_{\mathbb{Q}_p}} \phi_{\mathfrak{p}}.$$

In the cases where $\theta(\hat{\pi}_{\mathfrak{p}}^-)$ is non-zero, as it shares the same L-packet with $\theta(\hat{\pi}_{\mathfrak{p}}^+)$, it has the same L-parameter.

Proposition 5.4.14. *For the conductors of these representations, we have the following:*

1. *if $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, we have*

$$a(\phi_p) = a(\phi_{\mathfrak{p}}) + a(\phi_{\bar{\mathfrak{p}}});$$

2. if $p\mathcal{O}_K = \mathfrak{p}$, we have

$$a(\phi_p) = 2a(\phi_{\mathfrak{p}});$$

3. if $p\mathcal{O}_K = \mathfrak{p}^2$, we have

$$a(\phi_p) = 2d(K_{\mathfrak{p}}/\mathbb{Q}_p) + a(\phi_{\mathfrak{p}}).$$

Proof. We just use the properties in Proposition 4.1.3. □

5.5 The standard L-function and the theta correspondence

In this section, we define two kinds of L-functions for an automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$. We have already discussed the local factors which constitute the global L-functions in Section 3.4. These L-functions are defined directly as Euler products, and one of them will give us a criterion for when an automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, arises as a theta lift from some automorphic representation of $GO(X, \mathbb{A}_{\mathbb{Q}})$ for some 4-dimensional quadratic space X. We begin with the degree 4 L-function.

Definition 5.5.1. Let $\Pi = \bigotimes_v \Pi_v$ be an automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ and $\chi = \bigotimes_v \chi_v$ a Dirichlet character of finite order. Let S be a finite set of places of $\mathbb{Q}$ containing the archimedean place and the places v for which Π_v is ramified. The *partial spinor L-function* of Π twisted by χ is defined as the Euler product

$$L^S(\Pi, \chi, s) = \prod_{p \notin S} L(\Pi_p, \chi_p, s),$$

where $L(\Pi_p, \chi_p, s)$ is the degree 4 local factor (as defined in Section 3.4) twisted by χ_p. That is,

$$L(\Pi_p, \chi_p, s) = \prod_{j=1}^{4} (1 - \chi_p(p)\alpha_{p,j} p^{-s})^{-1},$$

where $\alpha_{p,j}$ $(j = 1, \ldots, 4)$ are the Satake parameters associated to the unramified representation Π_p.

Now, we define the degree 5 L-function.

Definition 5.5.2. Let $\Pi = \bigotimes_v \Pi_v$ be an automorphic representation of $GSp(4, \mathbb{A}_\mathbb{Q})$ and $\chi = \bigotimes_v \chi_v$ a Dirichlet character of finite order. Let S be a finite set of places of $\mathbb{Q}$ containing the archimedean place and the places v for which Π_v is ramified. The *partial standard L-function* of Π twisted by χ is defined as the Euler product

$$\zeta^S(\Pi, \chi, s) = \prod_{p \notin S} \zeta(\Pi_p, \chi_p, s),$$

where $\zeta(\Pi_p, \chi_p, s)$ is the degree 5 local factor (as defined in Section 3.4) twisted by χ_p. Namely,

$$\zeta(\Pi_p, \chi_p, s) = \prod_{i=1}^{5} (1 - \chi_p(p) c_{p,i} p^{-s})^{-1},$$

where $c_{p,i}$ $(i = 1, \ldots, 5)$ are the constants $\{1, \frac{\alpha_{p,1}}{\alpha_{p,3}}, \frac{\alpha_{p,3}}{\alpha_{p,1}}, \frac{\alpha_{p,1}}{\alpha_{p,2}}, \frac{\alpha_{p,2}}{\alpha_{p,1}}\}$, with $\alpha_{p,j}$ $(j = 1, \ldots, 4)$ the Satake parameters associated to the unramified representation Π_p.

In this thesis, we are mostly interested in the degree 5 L-function, as it gives a criterion for an automorphic representation of $GSp(4, \mathbb{A}_\mathbb{Q})$ to arise as a theta lift from a representation of $GO(X, \mathbb{A}_\mathbb{Q})$, for some 4-dimensional quadratic space X. We see that, in the following result.

Theorem 5.5.3. *Suppose* Π *is a cuspidal automorphic representation of* $GSp(4, \mathbb{A}_\mathbb{Q})$*,* S *is a sufficiently large finite set of places of* $\mathbb{Q}$*, and* χ_0 *a unitary character. If the partial standard L-function* $\zeta^S(\Pi, \chi_0, s)$ *has a pole at* $s = 1$*, then*

1. *the pole of* $\zeta^S(\Pi, \chi_0, s)$ *at* $s = 1$ *is simple;*

2. *the restriction of* Π *to* $Sp(4, \mathbb{A}_{\mathbb{Q}})$ *contains a theta lift from some automorphic representation of the orthogonal group* $O(X, \mathbb{A}_{\mathbb{Q}})$*, where* X *is a 4-dimensional quadratic space.*

Proof. See Theorem 7.1 of [35]. □

Note that according to Remark 4.3 of [68], a unitary character χ_0 such that $\zeta^S(\Pi, \chi_0, s)$ has a pole at $s = 1$, must be quadratic. If χ_0 is the quadratic character such that the standard L-function of Π has a pole at $s = 1$, let us make some remarks on the quadratic space X. If L is the discriminant algebra of X over $\mathbb{Q}$ and d is the discriminant of X, then χ_0 is the quadratic Hecke character $\chi_0 : \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \to \{\pm 1\}$ attached to the discriminant d, in the sense that at every place v of $\mathbb{Q}$, $\chi_{0,v}(t) = (t, d)_v$. Here $(\cdot, \cdot)_v$ is the quadratic Hilbert symbol of the completion of $\mathbb{Q}$ at v. In fact, if $L/\mathbb{Q}$ is a quadratic extension, χ_0 is the quadratic character attached to this extension, and if $L = \mathbb{Q} \oplus \mathbb{Q}$, then χ_0 is trivial. For a more precise description of the space X, see Section 7 of [35].

Suppose now that we have an irreducible automorphic representation Π of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, such that for a sufficiently large finite set S of places of $\mathbb{Q}$, $\zeta^S(\Pi, \chi_0, s)$ has a pole at $s = 1$. Theorem 5.5.3 tells us that the restriction of Π to $Sp(4, \mathbb{A}_{\mathbb{Q}})$ contains an irreducible constituent Π^0 which is in correspondence under the theta correspondence with an irreducible automorphic representation $\hat{\pi}^0$ of $O(X, \mathbb{A}_{\mathbb{Q}})$. By Section 5 of [68], $\hat{\pi}^0$ extends to an irreducible automorphic representation $\hat{\pi}$ of $GO(X, \mathbb{A}_{\mathbb{Q}})$, such that $\hat{\pi}_v$ is in correspondence via the extended theta correspondence with Π'_v (v runs through all places in $\mathbb{Q}$); here Π'_v is the irreducible admissible representation at v of an irreducible representation $\Pi' = \Pi \otimes \chi$ of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, where χ is a quadratic character of $\mathbb{A}_{\mathbb{Q}}^{\times}$. Also χ is unramified at almost all places of $\mathbb{Q}$.

5.6 On the conductors of non-generic supercuspidal representations of $GSp(4,F)$

Let F be a non-archimedean local field of odd residual characteristic. Since we have discussed some notions of the local theta correspondence, we are able to study non-generic supercuspidal representations of $GSp(4,F)$, which can be described as in Theorem 5.6.1. In particular, we are going to define the conductor of a non-generic supercupidal representation, and we will prove that it agrees with the conductor of the L-parameter attached to it.

First we describe non-generic supercuspidal representations of $GSp(4,F)$ in terms of the theta lift, as in [19].

Theorem 5.6.1. *Let π be an irreducible admissible representation of $GSp(4,F)$. Then one has the following two mutually exclusive possibilities:*

1. *π arises as a theta lift from*

$$GSO(D_{ram},F) \cong \left(D_{ram}^\times \times D_{ram}^\times\right)/F^\times;$$

2. *π arises as a theta lift from*

$$GSO(V,F) \cong \left(GL(4,F)\times F^\times\right)/\{(z,z^{-2}) : z\in F^\times\},$$

where V is the 6-dimensional split quadratic space $V=\mathbb{H}\oplus\mathbb{H}\oplus\mathbb{H}$. Here $\mathbb{H}$ is the hyperbolic plane.

Proof. See Theorem 5.2 of [18]. □

In fact, in [19] it is proved that the supercuspidal representations that arise as a theta lift from $GSO(D_{\text{ram}},F)$ are precisely the non-generic ones. That is, a non-generic supercuspidal representation of $GSp(4,F)$ is the theta lift $\theta(\tau)$ of an irreducible representation $\tau=(\tau_1,\tau_2)$ of $GSO(D_{\text{ram}},F)\cong(D_{\text{ram}}^\times\times D_{ram}^\times)/F^\times$, where τ_1,τ_2 are non-isomorphic irreducible representations of $D_{\text{ram}}^\times$ with equal central characters.

As D_{ram} is the division quaternion algebra over F, by the local Jacquet-Langlands correspondence there is an injection

$$\sigma \mapsto JL(\sigma),$$

from irreducible admissible representations of $D_{\text{ram}}^{\times}$ to irreducible admissible representations of $GL(2, F)$. The image of JL consists of all twisted Steinberg representations and all supercuspidal representations of $GL(2, F)$ (i.e., the essentially square integrable ones); the precise definition of JL can be found in §2 of [14]. Now, for an irreducible representation $\tau = (\tau_1, \tau_2)$ of $GSO(D_{\text{ram}}, F)$, by the local Jacquet-Langlands correspondence, τ_1 and τ_2 define irreducible admissible representations of $GL(2, F)$ denoted respectively by $JL(\tau_1)$ and $JL(\tau_2)$. In this way, one may get a representation $\tau' = (JL(\tau_1), JL(\tau_2))$ of $GSO(M_{2\times 2}(F), F)$ which produces a non-zero theta lift $\theta(\tau')$ to $GSp(4, F)$ according to Theorem 5.6 (ii) of [18].

By the local Langlands correspondence we may attach to $JL(\tau_1)$ and to $JL(\tau_2)$ the L-parameters ϕ_1 and ϕ_2 respectively; evidently, we have $\det(\phi_1) = \det(\phi_2)$. By Section 7 of [18], if $\tau = (\tau_1, \tau_2)$ is an irreducible representation of $GSO(D_{\text{ram}}, F)$ and $\tau' = (JL(\tau_1), JL(\tau_2))$ is the associated representation of $GSO(M_{2\times 2}(F), F)$, then $\{\theta(\tau'), \theta(\tau)\}$ is an L-packet of the L-parameter $\phi_1 \oplus \phi_2$. Finally, we note that $\theta(\tau')$ is a generic representation of $GSp(4, F)$, while $\theta(\tau)$ is the non-generic supercuspidal that we considered initially.

Remark 5.6.2. Having the L-parameter of a non-generic supercuspidal representation, we may go further in our consideration of the degeneration of conductors of Subsection 4.2.2. As we saw above, the L-parameter of a non-generic supercuspidal representation $\theta(\tau)$ of $GSp(4, F)$ is of the form $\phi = \phi_1 \oplus \phi_2$, where ϕ_1 and ϕ_2 are L-parameters of irreducible admissible representations of $GL(2, F)$ (in particular they are twisted Steinberg representations or supercuspidal representations). We have studied the degeneration of conductors of irreducible admissible representations of $GL(2, F)$ in Subsection 4.2.1, which we may apply to get the degeneration of the conductor of the non-generic supercuspidal representation $\theta(\tau)$ of $GSp(4, F)$. The same applies to the generic representation $\theta(\tau')$, which belongs to the same

L-packet with $\theta(\tau)$. Note that after this, the only representations left to be considered for degeneration of conductors, are the ones which are generic supercuspidal.

Now, we write down a definition for the conductor (on the automorphic side) of a non-generic supercuspidal representation; this also applies to the generic supercuspidal which shares the same L-packet with it.

Definition 5.6.3. Let $\theta(\tau)$ be a non-generic supercuspidal representation of $GSp(4, F)$, where $\tau = (\tau_1, \tau_2)$ is as above. We define the *conductor* $a(\theta(\tau))$ of $\theta(\tau)$ to be

$$a(\theta(\tau)) = a(JL(\tau_1)) + a(JL(\tau_2)),$$

where $a(JL(\tau_i))$ is defined in Definition 4.1.6, for $i = 1, 2$.

In his thesis [11], Danisman defines ε-factors $\varepsilon(s; \theta(\tau), \psi)$, where ψ is an additive character of F, for non-generic supercuspidal representations $\theta(\tau)$ of $GSp(4, F)$, and he proves that they are preserved under the local Langlands correspondence for $GSp(4, F)$ (see Proposition 6.1 of [11]). Below, we are going to give an identity which is a stepping stone in proving the equality of the conductor of an irreducible admissible representation and the conductor of its L-parameter.

Theorem 5.6.4. *If $\theta(\tau)$ is a non-generic supercuspidal representation of $GSp(4, F)$, for the ε-factor attached to it, we have*

$$\varepsilon(s; \theta(\tau), \psi) = \varepsilon(0; \theta(\tau), \psi) q^{-s(4n(\psi)+a(\theta(\tau)))}. \tag{5.6}$$

Here $n(\psi)$ is the largest integer n such that ψ is trivial on $\varpi^{-n}\mathcal{O}_F$.

Proof. In [11] it is proved that

$$\varepsilon(s; \theta(\tau), \psi) = \varepsilon(s; \tau_1, \psi)\varepsilon(s; \tau_2, \psi). \tag{5.7}$$

The local Jacquet-Langlands correspondence preserves ε-factors (see §14 of

[32], or [67]), so that

$$\varepsilon(s; \tau_i, \psi) = \varepsilon(s; JL(\tau_i), \psi),$$

for $i = 1, 2$. But by [5] we have

$$\varepsilon(s; JL(\tau_i), \psi) = \varepsilon(0; JL(\tau_i), \psi) q^{-s(2n(\psi)+a(JL(\tau_i)))},$$

for $i = 1, 2$. This implies the required result. □

In the following theorem we prove that the conductor of a non-generic supercuspidal representation is equal to the conductor of its L-parameter; of course this also holds for the generic supercuspidal which lies in the same L-packet with the former. We can do that either by using Definition 5.6.3 and the fact that the local Langlands for $GL(2, F)$ respects the conductors, or by using Equation (5.6).

Theorem 5.6.5. *If $\theta(\tau)$ is a non-generic supercuspidal representation of $GSp(4, F)$ and $\phi = \phi_1 \oplus \phi_2$ is its associated L-parameter, then we have*

$$a(\theta(\tau)) = a(\phi).$$

Proof. If $\theta(\tau)$ is a non-generic supercuspidal representation of $GSp(4, F)$, we know that $\tau = (\tau_1, \tau_2)$ is an irreducible representation of $GSO(D_{\text{ram}}, F)$. As above, we associate to τ_i, the representation $JL(\tau_i)$ via the local Jacquet-Langlands correspondence, for $i = 1, 2$. The L-parameters attached to $JL(\tau_i)$ are ϕ_i, and since the local Langlands correspondence for $GL(2, F)$ respects the conductors, we have

$$a(JL(\tau_i)) = a(\phi_i),$$

for $i = 1, 2$. As a result, we have

$$\begin{aligned} a(\theta(\tau)) &= a(JL(\tau_1)) + a(JL(\tau_2)) \\ &= a(\phi_1) + a(\phi_2) \\ &= a(\phi). \end{aligned}$$

Note that one may also obtain $a(\theta(\tau)) = a(\phi)$ by compairing the ε-factors of $\theta(\tau)$ and of ϕ. For the representation $\theta(\tau)$, we have by Theorem 5.6.4 the expression

$$\varepsilon(s;\theta(\tau),\psi) = \varepsilon(0;\theta(\tau),\psi)q^{-s(4n(\psi)+a(\theta(\tau)))};$$

on the other hand, we have by §11 of [52], that

$$\varepsilon(s;\phi,\psi) = \varepsilon(0;\phi,\psi)q^{-s(4n(\psi)+a(\phi))}.$$

Finally, the local Langlands correspondence for $GSp(4,F)$ preserves the ε-factors of non-generic supercuspidal representations by Proposition 6.1 of [11]. From this, the preservation of the conductors follows. □

As far as we know Theorems 5.6.4 and 5.6.5 have not appeared in the literature. The assumption that the residual characteristic of F is not 2 is crucial for Equation (5.7); Danisman uses it in the proof of Theorem 4.4 (see also the proof of Lemma 5.1 in [11]).

There is one more case where we know that the conductor of an irreducible admissible representation of $GSp(4,F)$ is defined and is equal to the conductor of its associated L-parameter; in this case F can have even residual characteristic. Let π be a generic irreducible admissible representation of $GSp(4,F)$ with trivial central character. Then the conductor of π is defined as the so-called *paramodular level*, which is denoted by N_π and defined in Theorem 7.5.4 in [48]. Let $a(\phi_\pi)$ be the conductor of the L-parameter ϕ_π associated to π by the local Langlands correspondence for $GSp(4,F)$. Then by Proposition 2.4.2 of [48], Corollary 7.5.5 of [48], and the preservation of ε-factors under the local Langlands correspondence for generic representations (see Main Theorem of [18]) we obtain

$$N_\pi = a(\phi_\pi).$$

Chapter 6

Congruences over imaginary quadratic fields

The purpose of this chapter is to study whether there are congruences between modular forms over imaginary quadratic fields. One would hope to use the lifting to $GSp(4, \mathbb{A}_{\mathbb{Q}})$ we described in order to get some congruences, though it seems that most of the time this does not produce any congruences between automorphic representations for $GL(2)$ over an imaginary quadratic field. We will discuss the reasons below.

Firstly, we write down all posible levels for a modular form over an imaginary quadratic field. After that, we examine whether one can prove the existence of congruences over imaginary quadratic fields by assuming the existence of congruences for automorphic forms for $GSp(4)$, and we prove a conditional result for imaginary quadratic fields of prime discriminant; in addition, we discuss the reasons why our method does not work in general. On the other hand, we prove an unconditional result of level lowering by twisting automorphic representations by a grössencharacter. Finally, we are able to give some potential examples of level lowering/raising congruences by looking at some tables of rational cuspforms over imaginary quadratic fields.

Throughout this chapter, l will be a rational prime, K will be an imaginary quadratic field, and if F is a field, we write $G_F = \mathrm{Gal}(\bar{F}/F)$. Moreover, if w is a place of K then K_w will be the completion of K at w, and if $\mathfrak{p}$ is a finite place of K, we denote by $N_{K/\mathbb{Q}}(\mathfrak{p})$ the norm of the ideal $\mathfrak{p}$ and by $\varpi_{\mathfrak{p}}$ a uniformizer of $K_{\mathfrak{p}}$. Finally, we write $\mathcal{O}_{K_{\mathfrak{p}}}$ and $k_{\mathfrak{p}}$ for the ring of integers and the residue field of $K_{\mathfrak{p}}$.

6.1 The possible levels over imaginary quadratic fields

Let

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_l)$$

be a continuous mod l representation, where $\bar{\mathbb{F}}_l$ is an algebraic closure of the finite field $\mathbb{F}_l$ of order l. This representation will be assumed to be modular[1] in the sense of Theorem 5.4.13, of level $\mathfrak{N}$; i.e., it has a lift

$$\rho : G_K \to GL(2, \bar{\mathbb{Q}}_l)$$

that comes from a regular algebraic cuspidal automorphic representation π of level $\mathfrak{N}$ and Galois invariant central character. Note that there is no odd/even distinction for representations of G_K, for there is no complex conjugation in G_K.

Recall in Definitions 4.2.1 and 4.2.2 how the Artin conductor and the Swan conductor of a Galois representation are defined; if $\mathfrak{p}$ is a prime in K with $\mathfrak{p} \nmid l$ and

$$\bar{\rho}_{\mathfrak{p}} : G_{K_{\mathfrak{p}}} \to GL(2, \bar{\mathbb{F}}_l)$$

is the restriction of $\bar{\rho}$ to the decomposition group $G_{K_{\mathfrak{p}}}$, we denote the Artin conductor by $a(\bar{\rho}_{\mathfrak{p}})$. The global conductor of $\bar{\rho}$ is defined as the product

$$\mathfrak{N}(\bar{\rho}) = \prod_{\mathfrak{p} \nmid l} \mathfrak{p}^{a(\bar{\rho}_{\mathfrak{p}})}.$$

[1]One can find a class of examples of representations $\bar{\rho}$ of G_K which are modular by considering representations of G_K obtained by restriction from representations of $G_{\mathbb{Q}}$; if the representation of $G_{\mathbb{Q}}$ is odd, then it is modular, so $\bar{\rho}$ is modular by base change. If the representation of $G_{\mathbb{Q}}$ is even then it is not modular.

We also let

$$\rho_{\mathfrak{p}} : G_{K_{\mathfrak{p}}} \to GL(2, \bar{\mathbb{Q}}_l)$$

be the restriction of ρ to $G_{K_{\mathfrak{p}}}$, for $\mathfrak{p} \nmid l$.

Proposition 6.1.1. *For the Swan conductors of the representations $\bar{\rho}_{\mathfrak{p}}$ and $\rho_{\mathfrak{p}}$ we have*

$$\mathrm{sw}(\rho_{\mathfrak{p}}) = \mathrm{sw}(\bar{\rho}_{\mathfrak{p}}).$$

Proof. See Section 1 of [38]. □

Note that the classification of degeneration of conductors for a Galois representation over the imaginary quadratic field K is a straightforward generalization of our discussion of Subsection 4.2.1; the only differences are highlighted in the following lemmata.

Lemma 6.1.2. *Let $\chi : G_{K_{\mathfrak{p}}} \to \bar{\mathbb{Q}}_l^{\times}$ be a tamely ramified character, with unramified reduction mod l. Then we have*

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv 1 \bmod l.$$

Proof. By Lemma 6 of [64], the character χ can be seen as a character $\chi : G_{K_{\mathfrak{p}}} \to E^{\times}$ for some finite extension $E/\mathbb{Q}_l$. This character has abelian image, so that it factors through $G_{K_{\mathfrak{p}}}^{\mathrm{ab}}$. If we restrict to the Weil group, χ factors through $W_{K_{\mathfrak{p}}}^{\mathrm{ab}} \cong K_{\mathfrak{p}}^{\times}$. By restricting to the units $\mathcal{O}_{K_{\mathfrak{p}}}^{\times}$, we get a character (for simplicity of notation we still denote it by χ)

$$\chi : \mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1 + \varpi_{\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p}}}) \to \mathcal{O}_E^{\times},$$

since $\chi(1 + \varpi_{\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p}}}) = 1$ as χ is tamely ramified; note that the image can be taken to be in the ring of integers $\mathcal{O}_E^{\times}$ as this character is of finite order. We have seen that $\mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1 + \varpi_{\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p}}}) \cong k_{\mathfrak{p}}^{\times}$, where $k_{\mathfrak{p}}$ is the residue field of $K_{\mathfrak{p}}$, and as χ was chosen to have unramified reduction modulo l, we have found a non-trivial character

$$\chi : k_{\mathfrak{p}}^{\times} \to \ker(\mathcal{O}_E^{\times} \twoheadrightarrow k_E^{\times}),$$

where k_E is the residue field of E. The group $\ker(\mathcal{O}_E^\times \twoheadrightarrow k_E^\times)$ is a pro-l group, and $k_\mathfrak{p}^\times$ has order $N_{K/\mathbb{Q}}(\mathfrak{p}) - 1$. In order that a non-trivial character exists of this form, we must have

$$l \mid N_{K/\mathbb{Q}}(\mathfrak{p}) - 1.$$

□

Lemma 6.1.3. *Suppose that $\rho_\mathfrak{p}$ is the Galois representation associated to a supercuspidal representation $\pi_\mathfrak{p} = BC(L/K_\mathfrak{p}, \psi)$ of $GL(2, K_\mathfrak{p})$. If the conductor $a(\rho_\mathfrak{p})$ degenerates, then*

$$N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv -1 \bmod l.$$

Proof. Recall that a supercuspidal representation has conductor that degenerates only when $L/K_\mathfrak{p}$ is an unramified quadratic extension (see Subsection 4.2.1). As $L/K_\mathfrak{p}$ is an unramified quadratic extension, we have that $\mathcal{O}_L^\times/\mathcal{O}_{K_\mathfrak{p}}^\times \cong k_L^\times/k_\mathfrak{p}^\times$ and $[k_L : k_\mathfrak{p}] = 2$ (where $\mathcal{O}_L$ and k_L are the ring of integers and the residue field of L), and so the order of $\mathcal{O}_L^\times/\mathcal{O}_{K_\mathfrak{p}}^\times$ is $N_{K/\mathbb{Q}}(\mathfrak{p}) + 1$. Moreover, when we have degeneration of the conductor of the supercuspidal representation modulo l, the character $\psi : L^\times \to E^\times$ is tamely ramified with unramified reduction; here E is a finite extension of $\mathbb{Q}_l$; denote also by $\mathcal{O}_E$ the ring of integers and by k_E the residue field of E. Notice also that as $\pi_\mathfrak{p}$ is a supercuspidal representation, we have that $\psi \circ \sigma \not\cong \psi$, where σ is the non-trivial element of $\mathrm{Gal}(L/K_\mathfrak{p})$.

We consider the non-trivial character

$$\frac{\psi \circ \sigma}{\psi} : L^\times \to E^\times.$$

This character factors through $L^\times/K_\mathfrak{p}^\times$ since it is trivial on $K_\mathfrak{p}^\times$ but not on $L^\times$. We now restrict the latter to $\mathcal{O}_L^\times/\mathcal{O}_{K_\mathfrak{p}}^\times$ and obtain a character

$$\frac{\psi \circ \sigma}{\psi}|_{\mathcal{O}_L^\times/\mathcal{O}_{K_\mathfrak{p}}^\times} : \mathcal{O}_L^\times/\mathcal{O}_{K_\mathfrak{p}}^\times \to \mathcal{O}_E^\times.$$

Note that since the mod l reduction $\bar{\psi}$ is unramified, it is Galois invariant[2], i.e.,

$$\bar{\psi} \circ \sigma \cong \bar{\psi};$$

furthermore, since the uniformizers ϖ_L and $\varpi_{\mathfrak{p}}$ of L and $K_{\mathfrak{p}}$ respectively are equal (as $L/K_{\mathfrak{p}}$ is unramified), and $\psi \circ \sigma \not\cong \psi$, the above character is non-trivial. As a result, we have a non-trivial character (with trivial mod l reduction)

$$\frac{\psi \circ \sigma}{\psi}|_{\mathcal{O}_L^\times/\mathcal{O}_{K_{\mathfrak{p}}}^\times} : \mathcal{O}_L^\times/\mathcal{O}_{K_{\mathfrak{p}}}^\times \to \ker(\mathcal{O}_E^\times \twoheadrightarrow k_E^\times)$$

such that $\mathcal{O}_L^\times/\mathcal{O}_{K_{\mathfrak{p}}}^\times$ has order $N_{K/\mathbb{Q}}(\mathfrak{p}) + 1$, and as $\ker(\mathcal{O}_E^\times \twoheadrightarrow k_E^\times)$ is a pro-l group, we obtain

$$l \mid N_{K/\mathbb{Q}}(\mathfrak{p}) + 1.$$

□

Theorem 6.1.4. *Suppose that*

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_l)$$

is a mod l Galois representation of global conductor $\mathfrak{N}$ coprime to l. If $\mathfrak{p}$ is a finite prime of K, write $\phi_{\mathfrak{p}} \in W_{K_{\mathfrak{p}}}$ for a lift of the inverse of a Frobenius element. Then

$$\mathfrak{N} = \mathfrak{N}(\bar{\rho}) \prod_{\mathfrak{p}} \mathfrak{p}^{n(\mathfrak{p})},$$

and for each $\mathfrak{p} \nmid l$ with $n(\mathfrak{p}) > 0$, one of the following holds:

1. $\mathfrak{p} \nmid \mathfrak{N}(\bar{\rho})$, $N_{K/\mathbb{Q}}(\mathfrak{p})(\mathrm{tr}\bar{\rho}_{\mathfrak{p}}(\phi_{\mathfrak{p}}))^2 \equiv (1 + N_{K/\mathbb{Q}}(\mathfrak{p}))^2 \det\bar{\rho}_{\mathfrak{p}}(\phi_{\mathfrak{p}}) \bmod l$ *and* $n(\mathfrak{p}) = 1$.
2. $N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv -1 \bmod l$, *and one of the following holds:*
   (a) $\mathfrak{p} \nmid \mathfrak{N}(\bar{\rho})$, $\mathrm{tr}\bar{\rho}_{\mathfrak{p}}(\phi_{\mathfrak{p}}) \equiv 0 \bmod l$ *and* $n(\mathfrak{p}) = 2$;
   (b) $\mathfrak{p} \mid \mathfrak{N}(\bar{\rho})$ *but* $\mathfrak{p}^2 \nmid \mathfrak{N}(\bar{\rho})$, $\det\bar{\rho}_{\mathfrak{p}}$ *is unramified and* $n(\mathfrak{p}) = 1$.

[2] If x is in the kernel of the norm map $N_{L/K_{\mathfrak{p}}}$ of the extension $L/K_{\mathfrak{p}}$, then $x \in \mathcal{O}_L^\times$, i.e., $\bar{\psi}(x) = 1$ as $\bar{\psi}$ is unramified. This implies that the kernel of the norm map is contained in the kernel of $\bar{\psi}$, and as a result $\bar{\psi}$ factors through the norm map $N_{L/K_{\mathfrak{p}}}$.

3. $N_{K/\mathbb{Q}}(\mathfrak{p}) \equiv 1 \bmod l$*, and one of the following holds:*

(a) $\mathfrak{p} \nmid \mathfrak{N}(\bar{\rho})$ *and* $n(\mathfrak{p}) = 2$*;*

(b) $\mathfrak{p}^2 \nmid \mathfrak{N}(\bar{\rho})$ *or the power of* $\mathfrak{p}$ *dividing* $\mathfrak{N}(\bar{\rho})$ *is the same as the power dividing the conductor of* $\det\bar{\rho}_{\mathfrak{p}}$ *and* $n(\mathfrak{p}) = 1$*.*

Proof. This is a straightforward generalization of the degeneration of conductors described in Subsection 4.2.1, and the statement is written as in [15]. The consideration of how the conductors of representations of $G_{K_{\mathfrak{p}}}$ degenerate modulo l, which is a prime different from the residual characteristic of $K_{\mathfrak{p}}$, is the same as in [7] and [38]. In this case, as before, we have that the Swan conductor stays invariant modulo l (Proposition 6.1.1), and statements 2 and 3 follow by Lemmata 6.1.3 and 6.1.2. □

6.2 On level lowering via automorphic representations for $GSp(4)$

In this section, we are investigating whether one can transfer congruences between automorphic representations for $GSp(4)$ over $\mathbb{Q}$, to congruences between automorphic representations for $GL(2)$ over an imaginary quadratic field.

Let us fix some notation that we are going to use throughout this section. We start by considering a regular algebraic cuspidal automorphic representation π of $GL(2, \mathbb{A}_K)$ with Galois invariant central character, where K is an imaginary quadratic field. Let Π be the cuspidal automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ which is obtained via the theta correspondence with π; this is in the sense of Subsection 5.4.1. Moreover, R will be the 4-dimensional irreducible l-adic Galois representation attached to Π, which we know is the induction of the 2-dimensional representation ρ of G_K attached to π; we assume that the mod l reduction $\bar{\rho}$ of ρ is irreducible. Finally, having the representation

$$R : G_{\mathbb{Q}} \to GL(4, \bar{\mathbb{Q}}_l),$$

one may conjugate it to have image in $GL(4, \mathcal{O}_{\bar{\mathbb{Q}}_l})$. Then we reduce modulo the maximal ideal to get a reduction

$$\bar{R} : G_{\mathbb{Q}} \to GL(4, \bar{\mathbb{F}}_l),$$

which is well defined if it is irreducible, and if not, we take the semisimplification of the reduction.

Also, we remark that the unramified irreducible admissible representations Π_p of the representation $\Pi = \bigotimes_v \Pi_v$, are either of type I or IIIb. This is derived from Subsection 5.3.3.

6.2.1 Automorphic lifts of the mod l Galois representation

As we are interested in proving congruences over imaginary quadratic fields via congruences between automorphic representations of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, we need to know what kind of level lowering results already exist for $GSp(4)$. The main delegate is a result of Gee and Geraghty (Theorem 7.6.6 of [22]) which is in the same fashion of Theorem 1 of [14]. Inspired by the latter theorem of Diamond and Taylor, Gee proved results in this sense for various algebraic groups, where one can choose the local behaviour of the automorphic lifts of an automorphic mod l Galois representation. We will explain such a result below, but first we need to make the notion "choosing the local behaviour" more precise via the following definition.

Definition 6.2.1. Let F be a non-archimedean local field of residual characteristic p different from l. If τ is a representation of I_F with open kernel which extends to a representation of G_F, then we call τ an *inertial type*. Furthermore, we say that an l-adic representation ρ of G_F is *of inertial type* τ, when the restriction of the corresponding Weil-Deligne representation to I_F is equivalent to τ.

The general form of a theorem which implies the existence of automorphic lifts with prescribed ramification is as follows: Suppose we have an irreducible mod l Galois representation $\bar{\rho}$ of $G_{\mathbb{Q}}$ which is the reduction of some automorphic l-adic Galois representation ρ. Let S be a finite set of non-archimedean places ($l \notin S$) which contains the places where ρ is ramified. For each $v \in S$ choose an inertial type τ_v, such that τ_v is a lift of $\bar{\rho}|_{I_v}$. Then, there should exist an l-adic automorphic lift ρ' of $\bar{\rho}$, such that $\rho'|_{I_v}$ is equivalent to τ_v for each $v \in S$.

These kind of propositions also have some technical requirements for the representation $\bar{\rho}$; we see below the formal statement of the result we wish to apply in our situation, together with these technical assumptions. Before that, we need some notation and terminology.

We denote by ϵ_l the l-adic cyclotomic character of $G_{\mathbb{Q}}$, and by ζ_l a primitive l-th root of unity. We fix an isomorphism $\iota : \bar{\mathbb{Q}}_l \cong \mathbb{C}$. We now describe a notion of functoriality between the groups $GL(4)$ and $GSp(4)$, which allows one to transfer properties from automorphic representations of $GL(4, \mathbb{A}_{\mathbb{Q}})$ to automorphic representations of $GSp(4, \mathbb{A}_{\mathbb{Q}})$.

Theorem 6.2.2. *There is an injective map from the set of globally generic cuspidal representations Π of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ to the set of pairs consisting of a globally generic automorphic representation $\tilde{\Pi}$ of $GL(4, \mathbb{A}_{\mathbb{Q}})$ and an automorphic representation θ of $GL(1, \mathbb{A}_{\mathbb{Q}})$. This map satisfies the following properties:*

1. *the central character of Π is $\omega_{\Pi} = \theta$, and the central character of $\tilde{\Pi}$ is $\omega_{\tilde{\Pi}} = \omega_{\Pi}^2$;*

2. *$\tilde{\Pi} \cong \tilde{\Pi}^{\vee} \otimes \omega_{\Pi}$, where $\tilde{\Pi}^{\vee}$ denotes the contragredient representation;*

3. *by considering $GSp(4)$ as a subgroup of $GL(4)$, for each place v of $\mathbb{Q}$, there is an equality of the corresponding Weil-Deligne representations of Π_v and $\tilde{\Pi}_v$;*

4. *if the pair $(\tilde{\Pi}, \theta)$ is such that $\tilde{\Pi}$ is cuspidal, then $(\tilde{\Pi}, \theta)$ is in the image of the map if and only if the partial exterior square L-function $L^S(s, \tilde{\Pi}, \bigwedge^2 \otimes \theta^{-1})$ has a pole at $s = 1$ (here S is a finite set of places of $\mathbb{Q}$).*

Proof. This is due to Asgari and Shahidi (see [1]), and Gan and Takeda (see Theorem 12.1 of [18]). In particular, Asgari and Shahidi prove assertions (1), (2) and (4), while Gan and Takeda prove assertion (3). $\square$

Remark 6.2.3. Note that the representations in Theorem 6.2.2 need to be globally generic. Having a regular algebraic cuspidal automorphic representation π of $GL(2, \mathbb{A}_K)$, when we take the global theta lift to a cuspidal automorphic representation Π of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, we have that at the archimedean place ∞, Π_∞ is holomorphic and non-generic. This means that Π is not globally generic. A first step towards removing the "globally generic" assumption can be found in Theorem 5.1.2 of [42], which concerns cuspidal automorphic representations with full level structure. Pitale, Saha and Schmidt define the irreducible admissible representations $\tilde{\Pi}_v$ for all places v, and they use a converse theorem to prove that $\tilde{\Pi} = \bigotimes_v \tilde{\Pi}_v$ is an automorphic representation of $GL(4, \mathbb{A}_{\mathbb{Q}})$. In general, for all cuspidal automorphic representations, the functorial transfer from $GSp(4)$ to $GL(4)$ is expected to follow from Arthur's work via the trace formula.

For our global theta lift, which is non-generic, one can get functoriality in the sense of Theorem 6.2.2 by the endoscopic classification of Arthur; in particular, the lifting $\tilde{\Pi}$ (an automorphic representation of $GL(4, \mathbb{A}_{\mathbb{Q}})$) of Π is given by Arthur's global parameter, since Π and $\tilde{\Pi}$ share the same global parameter. This lift is described by Mok in [40]; in particular see the proof of Theorem 3.1 of [40].

Furthermore, we say that an automorphic representation Π of $GSp(4, \mathbb{A}_{\mathbb{Q}})$ is *ι-ordinary*, if the functorial lift $\tilde{\Pi}$ of Π to $GL(4, \mathbb{A}_{\mathbb{Q}})$ is. We do not include the definition for ι-ordinary, since it is a bit technical and we do not need all the details in this thesis. The reader can find a detailed definition in §4.1 of [22]. Moreover, in the statement of the theorem of Gee and Geraghty, the notion of "crystalline representation" is mentioned; for this, there is a detailed definition in [2].

Definition 6.2.4. A continuous irreducible representation

$$R : G_{\mathbb{Q}} \to GSp(4, \bar{\mathbb{Q}}_l)$$

is called

1. *$GSp(4)$-ordinarily automorphic*, if R arises from some ι-ordinary automorphic representation Π of $GSp(4, \mathbb{A}_{\mathbb{Q}})$;

2. *$GSp(4)$-ordinarily automorphic and holomorphic*, if in addition Π_∞ belongs to the holomorphic discrete series;

3. *$GSp(4)$-ordinarily automorphic and generic*, if Π can be chosen to be globally generic;

Note that, in the above definition, R can be simultaneously holomorphic and generic, due to different choices of Π in the same global L-packet.

We are ready to state the theorem of Gee and Geraghty. In fact, since we are interested in the level part of the theorem, we write a simplified version in the sense that we do not include statements about the weights.

Theorem 6.2.5. *Let $l \geq 5$ be a prime such that $[\mathbb{Q}(\zeta_l) : \mathbb{Q}] > 2$, and fix an isomorphism $\iota : \bar{\mathbb{Q}}_l \cong \mathbb{C}$. Suppose that*

$$\bar{R} : G_{\mathbb{Q}} \to GSp(4, \bar{\mathbb{F}}_l)$$

is an irreducible representation, and let n be an integer such that the character $\bar{\epsilon}_l^n$ of $G_{\mathbb{Q}}$ is unramified. Suppose that $\bar{R}$ satisfies the following:

1. *There are finite fields $\mathbb{F}_l \subset k \subset k'$ such that*

$$Sp(4, k) \subset \bar{R}(G_{\mathbb{Q}}) \subset (k')^{\times} GSp(4, k). \tag{6.1}$$

2. *The representation $\bar{R}$ has a lift $R = R_\Pi$, which is $GSp(4)$-ordinarily automorphic arising from some automorphic representation Π, and generic of level prime to l. Let ψ be the similitude factor of R_Π.*

3. *Define $\psi_n := \psi \epsilon^n \tilde{\omega}^{-n}$, where $\tilde{\omega}$ is the Teichmüller lift of the mod l cyclotomic character (so that $\bar{\psi}_n = \bar{\psi}$, and ψ_n is crystalline). Then $\bar{R}|_{G_{\mathbb{Q}_l}}$ has an ordinary crystalline symplectic lift of similitude factor ψ_n.*

Then $\bar{R}$ has an ordinary crystalline symplectic lift R' of similitude factor ψ_n, which is $GSp(4)$-ordinarily automorphic of level prime to l, generic, and holomorphic. Moreover, given any finite set of non-archimedean rational places S containing the places where R is ramified, and an inertial type τ_v for each $v \in S$ ($v \neq l$) such that $\bar{R}|_{G_{\mathbb{Q}_v}}$ has a symplectic lift of type τ_v and similitude factor ψ_n, R' can be chosen to have inertial type τ_v for all $v \in S$, $v \neq l$.

Proof. See Theorem 7.6.6 of [22]. □

Remark 6.2.6. For proving the theorem above, Gee and Geraghty use an analogous result for automorphic representations of $GL(4, \mathbb{A}_{\mathbb{Q}})$ (see Theorem 7.5.2 of [22]) and the functorial transfer from $GL(4)$ to $GSp(4)$ (see Theorem 6.2.2). Note that the automorphic representations in Theorem 7.5.2 of [22] are required to have corresponding Galois representation with regular Hodge-Tate weights; this is stated in §7.5 of [22]. The same holds for the Galois representation R appearing in Theorem 6.2.5. For the notion "regular Hodge-Tate weights" the reader may consult §4.3 of [10].

One would hope to apply Theorem 6.2.5 in order to get level lowering/raising for the representation Π. Such a result allows one to get a congruence modulo l between R and an automorphic Galois representation R' (arising from some automorphic representation Π' of $GSp(4, \mathbb{A}_{\mathbb{Q}})$), by choosing inertial types τ_p for R', for finitely many primes p. Theorem 6.2.5 is not directly applicable in our situation for various reasons which we discuss below.

Firstly, since the representation R does not always have regular Hodge-Tate weights (see Remark 5.4.9), we are not able to apply Theorem 6.2.5 to get level lowering/raising for our automorphic representation Π, as such a result requires the Hodge-Tate weights to be regular. We understand that Calegari and Geraghty are working on modularity lifting results for low weight Siegel modular forms, which may allow an extension of Theorem 6.2.5 applicable to weights occuring in our situation; although we have not seen a preprint yet.

The second thing to consider is, as we mentioned above, that our Galois representation R is associated to an automorphic representation which is not globally generic. Assumption (2) of Theorem 6.2.5 requires this representation to be globally generic. The genericity in Theorem 6.2.5 is important since Gee and Geraghty use the functoriality between $GL(4)$ and $GSp(4)$ (see Theorem 6.2.2) in the proof of their result. However, as we remarked in Remark 6.2.3, it is very likely that, in the near future, genericity will not be needed for the functorial transfer due to Arthur's trace formula. Nevertheless, in our situation, Mok ([40]) writes down an automorphic representation $\tilde{\Pi}$ of $GL(4, \mathbb{A}_{\mathbb{Q}})$ which shares the same global L-parameter with the automorphic representation Π of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, the latter being non-generic.

Finally, we need to consider also assumption (1) of Theorem 6.2.5. This assumption asserts that the image of the mod l Galois representation $\bar{R}$ is big enough to contain $Sp(4, k)$ for some finite field k such that $\mathbb{F}_l \subset k$. This does not happen for representations which we consider, since these are induced from a 2-dimensional representation of an index 2 subgroup; recall that we have $R = \mathrm{ind}_{G_K}^{G_{\mathbb{Q}}} \rho$. Assumption (1) implies[3] that the image of the mod l Galois representation $\bar{R}$ is *big* in the sense of Definition 2.5.1 of [10]. Results of Thorne (see [66]) imply that the "big" image condition can be relaxed to a so-called "*adequate*" image condition; for this notion see Definition 2.3 of [66]. In addition, Guralnick-Herzig-Taylor-Thorne in Theorem 9 of [26], proved that if $\bar{R}$ is (absolutely) irreducible, then it is adequate for l big enough; in our case for $l \geq 11$.

Gee informed us (personal communication) that most of the technical assumptions in Theorem 6.2.5 could be removed after the work of Thorne. Despite the fact that we cannot directly apply Theorem 6.2.5, in the next subsection, we are going to assume such a result and see what one might obtain by choosing inertial types at the bad places.

[3] See Lemma 2.5.5 of [10]

6.2.2 Level lowering by choosing inertial types

In this subsection, we are going to provide a level lowering result under some hypotheses. The idea is to begin with a regular algebraic cuspidal automorphic representation π of $GL(2, \mathbb{A}_K)$, where K is an imaginary quadratic field, and consider its theta lift Π to $GSp(4, \mathbb{A}_{\mathbb{Q}})$. For the l-adic Galois representation R which is attached to Π, we will consider its mod l reduction $\bar{R}$ (for which we assume that it has image in $GSp(4, \bar{\mathbb{F}}_l)$), and we will choose inertial types for all places in a finite set S. After that, we will apply a stronger version of Theorem 6.2.5 to $\bar{R}$, and we will get an l-adic automorphic lift R' of $\bar{R}$ for which the chosen inertial types will read its local behaviour. One critical point is to suppose that R' is attached to some automorphic representation Π' which is a global theta lift from some automorphic representation π' of $GL(2, \mathbb{A}_L)$ for some quadratic field L (this can be achieved by testing whether the twisted standard L-function of Π' has a pole at $s = 1$; see Theorem 5.5.3). In general, we want to see when one can descend the congruence between R and R', to a congruence between the Galois representations attached to π and π' (or, as we will see, a twist of π').

One thing to consider here is whether the quadratic fields K and L are the same or not. We will choose our representation π and the inertial types of $\bar{R}$ carefully enough to ensure that $K = L$. Although, there can be situations where π and π' are automorphic representations over different quadratic fields, but their theta lifts are congruent. This might be interesting too for future work.

The following hypothesis will enable us to obtain an automorphic lift of $\bar{R}$, after we choose the inertial types.

Hypothesis 6.2.7. *Suppose that the mod l reduction $\bar{R}$ of the Galois representation R associated to Π is irreducible and valued in $GSp(4, \bar{\mathbb{F}}_l)$. Suppose S is a finite set of primes including the primes where R is ramified and the primes which ramify in K. For each $p \in S$, $p \neq l$, choose an inertial type τ_p lifting $\bar{R}|_{I_p}$. Then there is a Galois representation*

$$R' : G_{\mathbb{Q}} \to GSp(4, \bar{\mathbb{Q}}_l)$$

associated to some automorphic representation Π' *of* $GSp(4, \mathbb{A}_{\mathbb{Q}})$*, such that* $R'|_{I_p} \cong \tau_p$*, for all* $p \in S$*.*

This is yet unproved, but as we discussed in Subsection 6.2.1, one hopes that Theorem 6.2.5 will be strengthened sufficiently to provide a usable version of the hypothesis above. Now we will use this hypothesis to try to descend congruences from $GSp(4)$ to $GL(2)$ over an imaginary quadratic field. We will just give one example of how this works; it is likely that the method would allow us to prove more results on congruences, applying to more general fields, and to more general automorphic representations.

Remark 6.2.8. In the following, we will need a local-global compatibility result between our non-cohomological representation Π and its associated Galois representation R. For cohomological cuspidal automorphic representations with irreducible associated Galois representation, this is a result of Sorensen (see [60]) and Mok (see Theorem 3.1 of [40]). For a non-cohomological representation Π which is a theta lift, the required local-global compatibility result exists up to semisimplification; this is Theorem 4.11 of [40]. In the example below, we will assume the full local-global compatibility whenever it is needed, however this has not been proven yet.

We now make our choices. Let l be a prime[4]. Take the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-p})$ with $p \equiv 3 \bmod 4$, thus of discriminant $-p$; also, denote by c the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$. Let $\Sigma = \{\mathfrak{p}, \mathfrak{q}, \bar{\mathfrak{q}}\}$ and $S = \{p, q\}$, with $p\mathcal{O}_K = \mathfrak{p}^2$ and $q\mathcal{O}_K = \mathfrak{q}\bar{\mathfrak{q}}$. Let $\pi = \bigotimes_w \pi_w$ be a regular algebraic cuspidal automorphic representation of $GL(2, \mathbb{A}_K)$, of Galois invariant central character. Assume that π is unramified outside Σ, and for places in Σ we have:

1. $\pi_{\mathfrak{p}} = (\mu| \ |_{\mathfrak{p}}^{1/2}) St_{GL(2)}$ with $\tilde{\pi}_{\mathfrak{p}}$ regular (i.e., $\mu^c \neq \mu$), and μ a ramified character that does not degenerate modulo l.

[4]This prime should be big enough. For instance, in order to use Theorem 6.2.5, we need $l \geq 5$. As we mentioned, Theorem 6.2.5 might be strengthened enough to apply in our situation; after Thorne's work, one may require $l \geq 11$ in order to avoid the "big image" assumption (assumption (1) of Theorem 6.2.5). For this, recall the discussion at the end of the previous subsection.

2. $\pi_{\mathfrak{q}} = BC(E/K_{\mathfrak{q}}, \psi)$ is a supercuspidal representation which degenerates modulo l (here $E/K_{\mathfrak{q}}$ is an unramified quadratic extension), and $\pi_{\bar{\mathfrak{q}}} = \chi \times \chi^{-1}\omega_{\pi_{\mathfrak{q}}}$, where χ is a ramified non-degenerate character and $\omega_{\pi_{\mathfrak{q}}}$ is the central character of $\pi_{\mathfrak{q}}$. In this case, we have

$$N_{K/\mathbb{Q}}(\mathfrak{q}) \equiv -1 \bmod l.$$

In this set up, the theta lift $\Pi = \bigotimes_v \Pi_v$ of π is a cuspidal automorphic representation of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, such that it is unramified outside $S \cup \{l\}$, and for the primes in S we have:

1. According to Theorem 5.3.11, Π_p is a representation of type IXa. In fact, we have

$$\Pi_p = \delta(|\ |\eta\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}, |\ |^{-1/2} BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \mu|\ |_{\mathfrak{p}}^{1/2})),$$

where η is the quadratic character of $\mathbb{Q}_p^\times$ such that $\mu^c/\mu = \eta \circ N_{K_{\mathfrak{p}}/\mathbb{Q}_p}$. This representation has L-parameter $\Phi_p = (\rho_0, N_6)$ with

$$\rho_0 : w \mapsto \begin{pmatrix} \eta\epsilon_{K_{\mathfrak{p}}/\mathbb{Q}_p}\det(\phi)\phi'(w) & \\ & |w|^{-1/2}\phi(w) \end{pmatrix},$$

where ϕ is the L-parameter attached to $BC(K_{\mathfrak{p}}/\mathbb{Q}_p, \mu|\ |_{\mathfrak{p}}^{1/2})$. Let R_p be the associated Galois representation to Φ_p.

2. According to Theorem 5.3.12, Π_q is a representation of type X. In fact,

$$\Pi_q = \chi^{-1}\pi_{\mathfrak{q}} \rtimes \chi.$$

The L-parameter of this representation is $\Phi_q = (\rho_0, 0)$ with

$$\rho_0 : w \mapsto \begin{pmatrix} \chi^{-1}\omega_{\pi_{\mathfrak{q}}}(w) & & \\ & \phi_{\pi_{\mathfrak{q}}}(w) & \\ & & \chi(w) \end{pmatrix}.$$

Let R_q be the associated Galois representation to Φ_q. Here $\phi_{\pi_{\mathfrak{q}}}$ is the

L-parameter associated to $\pi_{\mathfrak{q}}$.

Here we use the full local-global compatibility assumption of Remark 6.2.8, since, for instance, the L-parameter corresponding to Π_p has non-zero monodromy operator N_6. Thus, if R is the 4-dimensional l-adic Galois representation attached to Π, we have

$$R_p \cong R|_{D_p}, \text{ and } R_q \cong R|_{D_q},$$

for D_p and D_q being the decomposition groups at p and q respectively. Now we make our choices for the inertial types for the primes in S.

1. For the prime p, we choose

$$\tau_p = R_p|_{I_p}. \tag{6.2}$$

   This extends by definition to a representation R'_p of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

2. For the prime q, we choose

$$\tau_q = \begin{pmatrix} \chi'^{-1} & & & \\ & 1 & t & \\ & & 1 & \\ & & & \chi' \end{pmatrix}, \tag{6.3}$$

   where χ' is a ramified character with the same ramification as χ that does not degenerate modulo l and such that $\chi' \equiv \chi \bmod l$, and $t : I_{\mathbb{Q}_q} \to \bar{\mathbb{Q}}_l$ is a non-trivial character. This inertial type extends to a representation R'_q of $\mathrm{Gal}(\bar{\mathbb{Q}}_q/\mathbb{Q}_q)$.

Since we have chosen the inertial types, we may apply Hypothesis 6.2.7 on R, and get an automorphic lift R' of $\bar{R}$, which is associated to an automorphic representation $\Pi' = \bigotimes_v \Pi'_v$ of $GSp(4, \mathbb{A}_{\mathbb{Q}})$, and such that $R'|_{D_p} \cong R'_p$ and $R'|_{D_q} \cong R'_q$. Moreover, R' is unramified outside $S \cup \{l\}$. Let us see now how the representation Π' looks like locally. For non-archimedean places outside $S \cup \{l\}$, Π' is unramified. For the places in S we have:

1. By our choice τ_p of inertial type at p, we get that R'_p corresponds to an L-parameter with nilpotent part N_6, and the latter corresponds to an irreducible admissible representation of type IXa. Such a representation may be a local theta lift only from a representation

   $$(\mu'| |^{1/2})St_{GL(2)}$$

   of $GL(2, F_p)$, where $F_p/\mathbb{Q}_p$ is a quadratic extension of local fields, and if c is the non-trivial element of $\mathrm{Gal}(F_p/\mathbb{Q}_p)$, we have that $\mu'^c \neq \mu'$ (for this, see Subsection 5.3.3).

2. For the prime q, we chose an inertial type τ_q which extends to a representation R'_q; the latter corresponds to an L-parameter with nilpotent part N_1. The representation Π'_q which corresponds to such an L-parameter is of type IIa, IVc, Vb, or VIc, and according to Subsection 5.3.3, the latter two cannot be theta lifts. A representation of type IIa or IVc, may be a theta lift only from a pair

   $$((\psi'| |^{1/2})St_{GL(2)}, \chi_1 \times \chi_2) \in \mathrm{Irr}_f(GL(2, \mathbb{Q}_q) \times GL(2, \mathbb{Q}_q)).$$

   Note that the L-parameter associated to the Galois representation of inertial type τ_q, when restricted to $I_{\mathbb{Q}_q}$, fixes only one basis vector. This never happens for the L-parameter associated to an irreducible admissible representation of type IVc or VIc.

Theorem 6.2.9. *Suppose that there exists a non-trivial quadratic Dirichlet character χ_0 of $\mathbb{A}_{\mathbb{Q}}^{\times}$, such that the partial standard L-function $\zeta^S(\Pi', \chi_0, s)$ of Π' has a pole at $s = 1$. Then, the automorphic representation Π' is a global theta lift from an automorphic representation $\pi' = \bigotimes_w \pi'_w$ of $GL(2, \mathbb{A}_K)$, where $K = \mathbb{Q}(\sqrt{-p})$, with $p \equiv 3 \bmod 4$.*

Proof. According to Section 5.5 and in particular Theorem 5.5.3, the automorphic representation Π' is a global theta lift from some automorphic representation $\pi' = \bigotimes_w \pi'_w$ of $GL(2, \mathbb{A}_L)$, where L is a quadratic field. We want to show that in our setting we have $L = K$.

As Π' is a global theta lift from $\pi' = \bigotimes_w \pi'_w$, we have that Π'_p is an irreducible admissible representation of type IXa which is a local theta lift from a representation

$$(\mu'|\ |^{1/2})St_{GL(2)}$$

of $GL(2, L_p)$, where $L_p/\mathbb{Q}_p$ is a quadratic extension of local fields. From this, we get that the prime p does not split in the quadratic extension $L/\mathbb{Q}$. Moreover, the representation Π'_q is a local theta lift; that is, it can only be a representation of type IIa, and as we mentioned above, it comes from a pair

$$((\psi'|\ |^{1/2})St_{GL(2)}, \chi_1 \times \chi_2) \in \mathrm{Irr}_f(GL(2, \mathbb{Q}_q) \times GL(2, \mathbb{Q}_q)).$$

This implies that q splits in the quadratic extension $L/\mathbb{Q}$. Finally, for v being a non-archimedean place outside S, we have that Π'_v is an unramified representation which is a local theta lift from an unramified principal series representation of $GL(2, L_w)$ for w dividing v and $L_w/\mathbb{Q}_v$ quadratic extension, or Π'_v is a local theta lift from a pair of unramified principal series representations that lies in $\mathrm{Irr}_f(GL(2, \mathbb{Q}_v) \times GL(2, \mathbb{Q}_v))$.

To sum up, we have that p (for which we have $p \equiv 3 \bmod 4$) either stays inert or ramifies in L, q splits in L, and all other primes either stay inert or split in L. As L is a quadratic extension of $\mathbb{Q}$, p is forced to ramify in L, and evidently is the only prime that ramifies in L. By our assumptions, 2 does not ramify in L, thus L cannot be the real quadratic field $\mathbb{Q}(\sqrt{p})$. The only possibility is that $L = \mathbb{Q}(\sqrt{-p})$. $\square$

Let ρ be the irreducible 2-dimensional l-adic Galois representation of G_K associated to π. Having that Π' is a global theta lift from an automorphic representation π' of $GL(2, \mathbb{A}_K)$, we get that the Galois representation R' associated to Π' is induced from a 2-dimensional Galois representation ρ' of G_K.

We now restrict the representations R and R' to G_K, and we obtain

$$R|_{G_K} = \rho \oplus \rho^c$$

and

$$R'|_{G_K} = \rho' \oplus \rho'^c.$$

Note that since we have assumed $\bar{\rho}$ to be irreducible, we have that $\bar{R}|_{G_K} = \bar{R}'|_{G_K}$ is the direct sum $\bar{\rho}' \oplus \bar{\rho}'^c$ of irreducible representations. As $\bar{\rho}'$ is irreducible, we get that ρ' is irreducible. Since R and R' are congruent modulo l, we get that either ρ and ρ' have isomorphic mod l Galois representations, or ρ and ρ'^c have isomorphic mod l Galois representations.

By Equation 6.2 and Proposition 5.4.14 we have that $\pi_{\mathfrak{p}}$ has the same ramification as $\pi'_{\mathfrak{p}} = (\mu'|\ |^{1/2})St_{GL(2)}$. Also, by the inertial type we choose for the prime q, we have that the supercuspidal representation $\pi_{\mathfrak{q}}$ is congruent to the twisted Steinberg representation $\pi'_{\mathfrak{q}} = (\psi'|\ |^{1/2})St_{GL(2)}$, where ψ' is unramified. This means that the level of the representation π' is lower than the level of π.

Remark 6.2.10. We make some remarks on the above result.

1. An important step in the argument for proving that the quadratic fields K and L are the same, is that the prime p was forced to be ramified in L since it was the only one that could be ramified. If the discriminant of K had two or more prime factors, or if we had that q stays inert in K, then we would not be able to tell for sure that $L = K$ by using the same argument.

2. One can extend the above result to automorphic representations π which are ramified at more primes. This can be done if one adds to S more primes that split in K and choose the local representations for the primes lying above primes in S to be representations which ramify. These representations should be such that we may choose inertial types which assure us that the primes in S split also in L, in the same way as above. For this, see another example in the next remark.

3. We can derive more examples of level lowering in the same fashion as above. For instance, consider the same setting for the representation π as we had before, but with $\pi_{\mathfrak{q}} = (\alpha|\ |^{1/2})St_{GL(2)}$ and $\pi_{\bar{\mathfrak{q}}} = \chi_1 \times \chi_2$,

with $\frac{| |^{1/2}\alpha}{\chi_1} \neq | |^{\pm 3/2}$ and χ_1 a tamely ramified character with unramified reduction. Keep the same inertial type for p, but for the prime q choose the inertial type

$$\tau_q = \begin{pmatrix} \alpha'^2 & & & \\ & \alpha' & t & \\ & & \alpha' & \\ & & & 1 \end{pmatrix}$$

with α' such that $\alpha' \equiv \alpha \bmod l$ and such that the conductor of α' is less than or equal to the conductor of α, and t a non-trivial character. The proof goes as before. However, we can treat this case by a different method without any hypotheses (it is a special case of Theorem 6.3.2 below).

6.3 Congruences by twisting

In this subsection we will prove a level lowering result for cuspidal automorphic representations over an imaginary quadratic field K, by adjusting an argument of Carayol that lowers the level by twisting the automorphic representation by a character. In particular, we will prove the following two results, for an inert prime and for a split prime in K respectively.

Theorem 6.3.1. *Suppose we have a modular mod l Galois representation*

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_l),$$

i.e., it has a lift ρ which arises from a regular algebraic cuspidal automorphic representation π. Assume that the component $\pi_{\mathfrak{p}}$ of π, at a prime $\mathfrak{p}$ which lies above a rational prime p that stays inert in K with $p \neq l$, is one of the following types:

1. *it is a principal series representation $\pi_{\mathfrak{p}} = \mu \times \nu$, with μ tamely ramified with unramified reduction such that it factors through the norm map, and ν ramified;*

2. *it is a twisted Steinberg representation* $\pi_{\mathfrak{p}} = (\mu|\ |^{1/2})St_{GL(2)}$, *with* μ *a tamely ramified with unramified reduction such that it factors through the norm map.*

Then $\bar{\rho}$ *is modular of level lower than the level of* π.

Proof. In both cases, μ is assumed to be a tamely ramified character with unramified reduction. Such a character can be decomposed as

$$\mu = \mu_{nr}\mu_r,$$

where μ_{nr} is an unramified character of $K_{\mathfrak{p}}^{\times}$, and μ_r is a tamely ramified character of $K_{\mathfrak{p}}^{\times}$ with trivial reduction (i.e., $a(\mu_r) = 1$ but $a(\bar{\mu}_r) = 0$), such that $\mu_r(\varpi_{\mathfrak{p}}) = 1$. Moreover, in both cases we have assumed that μ factors through the norm map and since μ_{nr} is unramified (i.e., its kernel contains the kernel of the norm map) we have that μ_r factors through the norm map too. This fact will enable us to extend μ_r to a grössencharacter $\tilde{\mu}_r$. As any element $x \in K_{\mathfrak{p}}^{\times}$ can be written as $x = \varpi_{\mathfrak{p}}^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_{K_{\mathfrak{p}}}^{\times}$ (and as μ_r is trivial on $\varpi_{\mathfrak{p}}$), we get that μ_r is a character of $\mathcal{O}_{K_{\mathfrak{p}}}^{\times}$ which (as a tamely ramified character) is trivial on $(1 + \varpi_{\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p}}})$; this means that μ_r is a character of $\mathcal{O}_{K_{\mathfrak{p}}}^{\times}/(1 + \varpi_{\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p}}}) \cong (\mathcal{O}_{K_{\mathfrak{p}}}/\varpi_{\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p}}})^{\times}$. As $\mathfrak{p} = p\mathcal{O}_K$ is a principal ideal, we have $(\mathcal{O}_{K_{\mathfrak{p}}}/\varpi_{\mathfrak{p}}\mathcal{O}_{K_{\mathfrak{p}}})^{\times} \cong (\mathcal{O}_K/\mathfrak{p})^{\times}$, so that

$$\mu_r : (\mathcal{O}_K/\mathfrak{p})^{\times} \to \mathbb{C}^{\times}.$$

That is, μ_r extends to a Dirichlet character for K of conductor $\mathfrak{p}$. Since μ_r factors through the norm map, and since $\mathfrak{p}$ has norm p^2 (as p is inert in K), we have that μ_r factors through $(\mathbb{Z}/p^2\mathbb{Z})^{\times}$; that is, if $N_{K/\mathbb{Q}}$ is the norm map of the extension $K/\mathbb{Q}$, we have

$$\mu_r : (\mathcal{O}_K/\mathfrak{p})^{\times} \xrightarrow{N_{K/\mathbb{Q}}} (\mathbb{Z}/p^2\mathbb{Z})^{\times} \xrightarrow{\phi} \mathbb{C}^{\times}.$$

By Proposition 3.1.2 of [4], the Dirichlet character ϕ of conductor p^2, extends to a grössencharacter

$$\tilde{\phi} : \mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{C}^{\times},$$

which we may compose with the idèle norm map

$$\tilde{N}_{K/\mathbb{Q}} : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times$$

to get a grössencharacter

$$\tilde{\mu}_r : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$$

that extends μ_r. Now we are able to proceed to the twisting argument.

1. Firstly we consider π with local component $\pi_\mathfrak{p} = \mu \times \nu$ at $\mathfrak{p}$, and attached Galois representation ρ. We twist π with $\tilde{\mu}_r^{-1}$ to get a cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ that has local component at $\mathfrak{p}$ the representation

$$\pi'_\mathfrak{p} = \mu_r^{-1}(\mu_{nr}\mu_r \times \nu) = \mu_{nr} \times \mu_r^{-1}\nu.$$

   As μ factors through the norm map, the central character of $\tilde{\mu}_r^{-1}\pi$ factors through the norm map, and as a result we may attach to it a Galois representation $\tilde{\mu}_r^{-1}\rho$. Then the conductor of $\pi'_\mathfrak{p}$ is

$$a(\pi'_\mathfrak{p}) = a(\mu_{nr}) + a(\mu_r^{-1}\nu)$$

   with

$$a(\mu_{nr}) = 0 < 1 = a(\mu)$$

   and

$$a(\mu_r^{-1}\nu) \leq a(\nu),$$

   since $a(\mu_r^{-1}) = 1$ and $a(\nu) \geq 1$. That is,

$$a(\pi'_\mathfrak{p}) < a(\pi_\mathfrak{p}).$$

   Moreover, the conductor of π at the other places is not getting bigger under the twisting since μ_r, as a Dirichlet character, has conductor $\mathfrak{p}$. Therefore, the power of $\mathfrak{p}$ dividing the conductor of $\tilde{\mu}_r^{-1}\rho$ is smaller than the power of $\mathfrak{p}$ dividing the conductor of ρ. The congruence occurs as $\tilde{\mu}_r^{-1}$ has trivial mod l reduction, i.e., ρ and $\tilde{\mu}_r^{-1}\rho$ are congruent mod l.

2. Now we consider π with local component $\pi_{\mathfrak{p}} = (\mu|\ |^{1/2})St_{GL(2)}$, and attached Galois representation ρ. We twist π with $\tilde{\mu}_r^{-1}$ to get a cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ with local component at $\mathfrak{p}$ the representation
$$\pi'_{\mathfrak{p}} = (\mu_{nr}|\ |^{1/2})St_{GL(2)}.$$
Again $\tilde{\mu}_r^{-1}\pi$ has central character that factors through the norm map, so that we may attach to it a Galois representation $\tilde{\mu}_r^{-1}\rho$. The conductor of $\pi'_{\mathfrak{p}}$ is $a(\pi'_{\mathfrak{p}}) = 1$ while $a(\pi_{\mathfrak{p}}) = 2a(\mu) = 2$. That is
$$a(\pi'_{\mathfrak{p}}) < a(\pi_{\mathfrak{p}}).$$
For the same reasons as before the conductors at the other places do not get bigger under twisting, and we have a level lowering congruence between ρ and $\tilde{\mu}_r^{-1}\rho$.

□

Theorem 6.3.2. *Suppose we have a modular mod l Galois representation*
$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_l),$$
i.e., it has a lift ρ which arises from a regular algebraic cuspidal automorphic representation π. Let $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$ with $p \neq l$, such that for the components[5] *$\pi_{\mathfrak{p}}$ and $\pi_{\bar{\mathfrak{p}}}$ of π at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ respectively, we have that $a(\pi_{\bar{\mathfrak{p}}}) > 1$ and that $\pi_{\mathfrak{p}}$ is one of the following types:*

1. *principal series representation $\mu \times \nu$, with μ tamely ramified with unramified reduction, and ν ramified;*
2. *twisted Steinberg representation $(\mu|\ |^{1/2})St_{GL(2)}$, with μ tamely ramified with unramified reduction.*

[5]Note that $\pi_{\mathfrak{p}}$ and $\pi_{\bar{\mathfrak{p}}}$ have equal central characters in this situation.

Then $\bar{\rho}$ is modular of lower level than the level of π.

Proof. The first thing to notice is that in the case where $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, we may fix isomorphisms $K_{\mathfrak{p}} \cong \mathbb{Q}_p$ and $K_{\bar{\mathfrak{p}}} \cong \mathbb{Q}_p$. That is, a character of $K_{\mathfrak{p}}^{\times}$ can essentially be thought of as a character of $\mathbb{Q}_p^{\times}$. As in the proof of Theorem 6.3.1, we may write $\mu : \mathbb{Q}_p^{\times} \to \mathbb{C}^{\times}$ as a product $\mu = \mu_{nr}\mu_r$, such that μ_{nr} is unramified, μ_r is tamely ramified with trivial reduction and $\mu_r(p) = 1$. Then, as before, we write μ_r as a character

$$\mu_r : \mathbb{Z}_p^{\times}/1 + p\mathbb{Z}_p \to \mathbb{C}^{\times},$$

and considering also that $\mathbb{Z}_p^{\times}/1 + p\mathbb{Z}_p \cong (\mathbb{Z}_p/p\mathbb{Z}_p)^{\times} \cong (\mathbb{Z}/p\mathbb{Z})^{\times}$, μ_r becomes a Dirichlet character of conductor p. By Proposition 3.1.2 of [4], μ_r extends to a grössencharacter, which we compose with the idèle norm map $\tilde{N}_{K/\mathbb{Q}}$ to get

$$\tilde{\mu}_r : K^{\times}\backslash\mathbb{A}_K^{\times} \xrightarrow{\tilde{N}_{K/\mathbb{Q}}} \mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}^{\times} \to \mathbb{C}^{\times}.$$

Now we consider the two cases of the theorem.

1. Suppose that $\pi_{\mathfrak{p}} = \mu \times \nu$, with ρ being the Galois representation attached to π. We consider the twist $\tilde{\mu}_r^{-1}\pi$, which has local component at $\mathfrak{p}$ the representation

   $$\pi'_{\mathfrak{p}} \cong \mu_r^{-1}(\mu \times \nu) = \mu_{nr} \times \mu_r^{-1}\nu$$

   with

   $$a(\mu_{nr}) + a(\mu_r^{-1}\nu) < a(\mu) + a(\nu).$$

   As $\tilde{\mu}_r^{-1}$ factors through the norm map, our new cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ has Galois invariant central character, and so we may attach to it a Galois representation $\tilde{\mu}_r^{-1}\rho$ which is congruent to ρ since μ_r is trivial modulo l. Therefore, the if $\mathfrak{p}^{\kappa}\bar{\mathfrak{p}}^{\lambda}$ divides exactly the conductor of ρ, then $\mathfrak{p}^{\kappa'}\bar{\mathfrak{p}}^{\lambda'}$ divides exactly the conductor of $\tilde{\mu}_r^{-1}\rho$, with $\kappa' + \lambda' < \kappa + \lambda$.

2. Now suppose that $\pi_{\mathfrak{p}} = (\mu|\ |^{1/2})St_{GL(2)}$, with ρ the Galois representa-

tion attached to π. The twist $\tilde{\mu}_r^{-1}\pi$ now has local component at $\mathfrak{p}$ the representation

$$\pi'_{\mathfrak{p}} \cong \mu_r^{-1}(\mu|\ |^{1/2})St_{GL(2)} = (\mu_{nr}|\ |^{1/2})St_{GL(2)}$$

with

$$a((\mu_{nr}|\ |^{1/2})St_{GL(2)}) = 1 < 2 = a((\mu|\ |^{1/2})St_{GL(2)}).$$

The cuspidal automorphic representation $\tilde{\mu}_r^{-1}\pi$ has attached a Galois representation $\tilde{\mu}_r^{-1}\rho$ which is congruent to ρ modulo l, for the same reasons as above. So again, the level is getting lower by twisting by $\tilde{\mu}_r^{-1}$.

Note that in both cases, the conductor of $\pi_{\bar{\mathfrak{p}}}$ cannot be raised by twisting with μ_r^{-1}, as we have assumed that $a(\pi_{\bar{\mathfrak{p}}}) > 1$. □

Remark 6.3.3. The assumption that the conductor of $\pi_{\bar{\mathfrak{p}}}$ is greater than 1 in Theorem 6.3.2 excludes the following phenomenon. Let $\pi_{\mathfrak{p}} = \mu_r^{-1}\mu_{nr} \times \mu_r\nu_{nr}$ and $\pi_{\bar{\mathfrak{p}}} = \mu_{nr} \times \nu_{nr}$ (which have equal central characters), where μ_r is tamely ramified with trivial mod l reduction, and μ_{nr}, ν_{nr} are unramified characters. After twisting the automorphic representation with $\tilde{\mu}_r^{-1}$ as in the Theorem, we get local components

$$\pi'_{\mathfrak{p}} = \mu_r^{-2}\mu_{nr} \times \nu_{nr}$$

and

$$\pi'_{\bar{\mathfrak{p}}} = \mu_r^{-1}\mu_{nr} \times \mu_r^{-1}\nu_{nr}.$$

This not only lowers the conductor of $\pi_{\mathfrak{p}}$, but at the same time might raise the conductor of $\pi_{\bar{\mathfrak{p}}}$. Something similar can take place when we have, for example, $\pi_{\bar{\mathfrak{p}}} = (\chi|\ |^{1/2})St_{GL(2)}$ with χ an unramified character.

6.4 Examples of congruences

Lingham in Sections 7.1 and 7.2 of his thesis [37], constructed tables of Hecke eigenvalues of rational cuspforms of weight 2, for the group $\Gamma_0(\mathfrak{N})$ (various level $\mathfrak{N}$), over the imaginary quadratic fields $\mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\sqrt{-31})$. We are going to use these tables in order to find some potential examples of level lowering/raising congruences. Note that the tables contain the first few eigenvalues of the cuspforms, and thus the congruences that we find provide only evidence for possible examples.

We are going to use the notation that Lingham uses in his thesis. Let K be one of the above imaginary quadratic fields. For a rational prime p, we write $p\mathcal{O}_K = \mathfrak{p}_p\bar{\mathfrak{p}}_p$ if the prime splits, and $p\mathcal{O}_K = \mathfrak{p}_p$ when the prime stays inert. Also, the following tables include all the cuspforms from Lingham's tables that we need; for a more complete consideration the reader is advised to look at Lingham's tables (Sections 7.1 and 7.2 of [37]). The rows represent the cuspforms; the first column is the level $\mathfrak{N}$, the second is the norm of the level, the third column is the name of the cuspform (following Lingham's names), and the rest of the columns are the first few Hecke eigenvalues.

Table 6.1: Rational Newforms for $\mathbb{Q}(\sqrt{-23})$

$\mathfrak{N}$	$N_{K/\mathbb{Q}}(\mathfrak{N})$		$\mathfrak{p}_2$	$\bar{\mathfrak{p}}_2$	$\mathfrak{p}_3$	$\bar{\mathfrak{p}}_3$	$\mathfrak{p}_{13}$	$\bar{\mathfrak{p}}_{13}$	$\mathfrak{p}_{23}$	$\mathfrak{p}_5$	$\mathfrak{p}_{29}$	$\bar{\mathfrak{p}}_{29}$	$\mathfrak{p}_{31}$	$\bar{\mathfrak{p}}_{31}$	$\mathfrak{p}_{41}$	$\bar{\mathfrak{p}}_{41}$	$\mathfrak{p}_{47}$	$\bar{\mathfrak{p}}_{47}$	$\mathfrak{p}_7$
$\mathfrak{p}_2\bar{\mathfrak{p}}_3$	6	f_1	1	1	0	-1	-2	2	-4	-2	6	6	0	-4	-2	2	8	0	-6
$\bar{\mathfrak{p}}_2\mathfrak{p}_{13}$	26	f_2	0	-1	-2	1	1	5	6	-1	0	-3	5	-4	12	9	9	6	-4
$\mathfrak{p}_2^2\bar{\mathfrak{p}}_2^3$	32	f_4	0	0	-1	-3	-1	3	-8	-2	-5	-1	-5	5	3	-5	-13	-3	-2
$\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_3^2$	36	f_5	-1	-1	0	-2	2	-4	-6	2	0	-6	-10	8	-6	-6	0	12	-4
$\mathfrak{p}_2^2\bar{\mathfrak{p}}_2\mathfrak{p}_3^2$	72	f_{13}	0	1	0	-2	-4	2	6	8	6	-6	2	8	6	0	0	0	2
$\bar{\mathfrak{p}}_2\bar{\mathfrak{p}}_3\mathfrak{p}_{13}$	78	f_{14^*}	0	-1	3	-1	1	0	1	4	10	2	-5	1	-3	-11	-11	6	-9
$\mathfrak{p}_2\bar{\mathfrak{p}}_3\bar{\mathfrak{p}}_{13}$	78	f_{16}	1	-2	-3	-1	-5	1	-4	1	-3	-6	6	5	7	-4	-10	3	-12
$\mathfrak{p}_2^2\bar{\mathfrak{p}}_2^3\mathfrak{p}_3$	96	f_{19}	0	0	1	2	4	-2	2	8	10	-6	-10	0	-2	0	-8	-8	-2

Table 6.2: Rational Newforms for $\mathbb{Q}(\sqrt{-31})$

$\mathfrak{N}$	$N_{K/\mathbb{Q}}(\mathfrak{N})$		$\mathfrak{p}_2$	$\bar{\mathfrak{p}}_2$	$\mathfrak{p}_5$	$\bar{\mathfrak{p}}_5$	$\mathfrak{p}_7$	$\bar{\mathfrak{p}}_7$	$\mathfrak{p}_3$	$\mathfrak{p}_{19}$	$\bar{\mathfrak{p}}_{19}$	$\mathfrak{p}_{31}$	$\mathfrak{p}_{41}$	$\bar{\mathfrak{p}}_{41}$	$\mathfrak{p}_{47}$	$\bar{\mathfrak{p}}_{47}$
$\mathfrak{p}_2\mathfrak{p}_5$	10	f_1	1	1	-1	0	2	4	-4	-6	0	4	-2	2	6	-8
$\mathfrak{p}_2\mathfrak{p}_7$	14	f_2	-1	0	0	3	1	-1	1	2	2	5	9	-6	-6	12
$\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5$	20	f_3	-1	-1	-1	0	-4	2	-2	2	-4	-4	0	-12	0	6
$\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5$	20	f_{4^*}	1	-1	-1	0	2	-4	4	2	8	-4	6	-6	6	0
$\mathfrak{p}_2^3\bar{\mathfrak{p}}_2^2$	32	f_5	0	0	-1	3	-3	-5	-2	-7	3	0	-9	-1	0	0
$\mathfrak{p}_2^4\bar{\mathfrak{p}}_2$	32	f_6	0	-1	2	2	-2	2	4	-4	4	0	-8	-8	-2	2
$\mathfrak{p}_2^4\bar{\mathfrak{p}}_2^2$	64	f_{10}	0	0	-1	-1	1	-1	-2	5	-5	0	-5	-5	-8	8
$\mathfrak{p}_2\mathfrak{p}_5\mathfrak{p}_7$	70	f_{11}	1	-2	-1	-3	1	1	-1	0	-6	1	-5	-4	-6	10
$\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5\mathfrak{p}_7$	140	f_{36^*}	-1	1	1	-4	-1	-2	-2	6	4	0	4	-12	-12	-2
$\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5\bar{\mathfrak{p}}_7$	140	f_{37}	1	1	-1	0	2	1	4	2	-4	-4	-6	6	-6	12
$\mathfrak{p}_2^3\bar{\mathfrak{p}}_2^2\mathfrak{p}_5$	160	f_{44^*}	0	0	-1	-2	2	0	-2	-2	-2	10	6	-6	10	0
$\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_7^2$	196	f_{51}	1	1	-2	2	0	2	4	4	-4	0	-8	8	2	-2

6.4.1 Level raising examples

Now we will present some potential examples of level raising congruences for rational cuspforms of weight 2 over the imaginary quadratic fields $\mathbb{Q}(\sqrt{-23})$ and $\mathbb{Q}(\sqrt{-31})$.

Before we proceed to the examples, we consider the nature of these level raising congruences; these are in the sense of Ribet's Theorem 1 in [44]. In particular one would expect that, if K is an imaginary quadratic field and

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_l)$$

is a modular mod l Galois representation arising from a regular algebraic cuspidal automorphic representation of level $\mathfrak{N}$, and $\mathfrak{p}$ is a prime such that $\mathfrak{p} \nmid l$, $(\mathfrak{p}, \mathfrak{N}) = 1$, which satisfies

$$N_{K/\mathbb{Q}}(\mathfrak{p})(\mathrm{tr}\bar{\rho}_{\mathfrak{p}}(\phi_{\mathfrak{p}}))^2 \equiv (1 + N_{K/\mathbb{Q}}(\mathfrak{p}))^2 \det\bar{\rho}_{\mathfrak{p}}(\phi_{\mathfrak{p}}) \bmod l, \tag{6.4}$$

then $\bar{\rho}$ is modular of level $\mathfrak{N}\mathfrak{p}$. Here $\phi_{\mathfrak{p}}$ lies above the inverse of a Frobenius element. We are looking for evidence for or against the above statement, and below we present some. It is interesting that over imaginary quadratic fields, this level raising statement might not hold in general.

Example 6.4.1. Let $K = \mathbb{Q}(\sqrt{-23})$ and we consider the cuspform f_2 which is of level $\mathfrak{N} = \bar{\mathfrak{p}}_2\mathfrak{p}_{13}$. If a representation

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_5)$$

is modular arising from f_2, we see that identity (6.4) holds for $\mathfrak{p} = \bar{\mathfrak{p}}_3$; thus we expect $\bar{\rho}$ to arise also from a cuspform of level $\mathfrak{N}\bar{\mathfrak{p}}_3$. Again this seems to be true, since f_{14^*} is of level $\mathfrak{N}\bar{\mathfrak{p}}_3$ and is congruent to f_2 modulo 5.

Example 6.4.2. Let $K = \mathbb{Q}(\sqrt{-23})$ and we consider the cuspform f_4 which is of level $\mathfrak{N} = \mathfrak{p}_2^2\bar{\mathfrak{p}}_2^3$. If a representation

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_5)$$

is modular arising from f_4, we see that identity (6.4) holds for $\mathfrak{p} = \mathfrak{p}_3$; thus we expect $\bar{\rho}$ to arise also from a cuspform of level $\mathfrak{N}\mathfrak{p}_3$. Again this seems to be true, since f_{19} is of level $\mathfrak{N}\mathfrak{p}_3$ and is congruent to f_4 modulo 5.

Example 6.4.3. Let $K = \mathbb{Q}(\sqrt{-23})$. We see that f_1 is a cuspform of level $\mathfrak{N} = \mathfrak{p}_2\bar{\mathfrak{p}}_3$. If a representation

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_3)$$

is modular arising from f_1, by considering the identity (6.4) for $\mathfrak{p} = \bar{\mathfrak{p}}_{13}$, we see that it holds mod 3, and we expect that $\bar{\rho}$ arises from some cuspform of level $\mathfrak{N}\bar{\mathfrak{p}}_{13}$. This seems to be true as we see that f_{16} is of level $\mathfrak{N}\bar{\mathfrak{p}}_{13}$ and that f_1 is congruent to f_{16} mod 3. Note that in this example, we have $\bar{\mathfrak{p}}_3 \mid 3$ and $(\mathfrak{N}, \bar{\mathfrak{p}}_3) \neq 1$.

For $K = \mathbb{Q}(\sqrt{-23})$ we did not find counter-examples for the level raising statement. Let us see what happens for $K = \mathbb{Q}(\sqrt{-31})$.

Example 6.4.4. Let $K = \mathbb{Q}(\sqrt{-31})$. The cuspform f_1 is of level $\mathfrak{p}_2\mathfrak{p}_5$, and let

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_3)$$

be a modular representation arising from f_1. The prime $\mathfrak{p} = \mathfrak{p}_7$ satisfies identity (6.4) modulo 3, so we expect that we can raise the level by $\mathfrak{p}_7$. Indeed, we see that f_{11} is of level $\mathfrak{p}_2\mathfrak{p}_5\mathfrak{p}_7$ and it is congruent to f_1 modulo 3.

Example 6.4.5. For $K = \mathbb{Q}(\sqrt{-31})$, if

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_5)$$

is a Galois representation attached to f_{21^*} of level $\mathfrak{p}_2\mathfrak{p}_7^2$, we see that $\bar{\mathfrak{p}}_2$ satisfies identity (6.4). Moreover, f_{21^*} seems to be congruent modulo 5 to the cuspform f_{51}, which is of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_7^2$. Thus, in this case the level raising statement might hold.

We now consider some examples which do not support the level raising statement. This phenomenon does not happen for totally real quadratic extensions over $\mathbb{Q}$.

Example 6.4.6. Let $K = \mathbb{Q}(\sqrt{-31})$ and take f_2 which is of level $\mathfrak{p}_2\mathfrak{p}_7$. Suppose that a modular mod 3 Galois representation

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_3)$$

arises from f_2. The prime $\mathfrak{p} = \mathfrak{p}_5$ satisfies identity (6.4) modulo 3, so we expect that there is a cuspform of level $\mathfrak{p}_2\mathfrak{p}_7\mathfrak{p}_5$; the only cuspform of this level that we can find in Lingham's tables is f_{11}, which is not congruent to f_2 modulo 3.

Example 6.4.7. Let $K = \mathbb{Q}(\sqrt{-31})$. The cuspform f_3 is of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5$, and we assume that a Galois representation

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_3)$$

arises from f_3. The prime $\mathfrak{p} = \mathfrak{p}_7$ satisfies identity (6.4), so we expect that there is a cuspform of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5\mathfrak{p}_7$ which gives rise to $\bar{\rho}$ as well. The only cuspform of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5\mathfrak{p}_7$ in Lingham's tables is f_{36^*}, but it does not give rise to $\bar{\rho}$ since it is not congruent to f_3 modulo 3.

Nevertheless, we may try to raise the level by $\mathfrak{p} = \bar{\mathfrak{p}}_7$ for which identity (6.4) still holds modulo 3. A cuspform of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5\bar{\mathfrak{p}}_7$ from Lingham's tables is f_{37} which seems to be congruent to f_3.

Example 6.4.8. If in the previous example we consider the other cuspform of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5$, i.e., f_{4^*}, we obtain that if we try to raise the level by the prime $\mathfrak{p}_7$ (which satisfies identity (6.4) modulo 3), we see in the tables of Lingham that there is no cuspform of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5\mathfrak{p}_7$ congruent to f_{4^*} mod 3.

As before, we see that f_{4^*} is probably congruent modulo 3 to the cuspform f_{37} of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_5\bar{\mathfrak{p}}_7$; note that $\bar{\mathfrak{p}}_7$ satisfies identity (6.4).

Example 6.4.9. Let $K = \mathbb{Q}(\sqrt{-31})$. Consider a Galois representation

$$\bar{\rho} : G_K \to GL(2, \bar{\mathbb{F}}_7)$$

attached to f_5 (which is of level $\mathfrak{p}_2^3\bar{\mathfrak{p}}_2^2$). The prime $\mathfrak{p} = \mathfrak{p}_5$ satisfies the identity (6.4) modulo 7, so we expect to be able to raise the level by $\mathfrak{p}_5$. This does not seem to happen since f_5 is not congruent to f_{44^*} modulo 7, and this is the only cuspform in Lingham's tables of level $\mathfrak{p}_2^3\bar{\mathfrak{p}}_2^2\mathfrak{p}_5$.

On the contrary, it seems that f_5 is congruent to f_{44^*} modulo 5; here the identity (6.4) holds for $\mathfrak{p}_5$ modulo 5. Note that in this case the prime $\mathfrak{p}_5$ divides 5.

6.4.2 Level lowering examples

We now try to find possible congruences by considering Theorem 6.1.4. Having considered examples which fall in case 1. of Theorem 6.1.4 in the previous subsection, we now look for examples of case 2.(b). Examples for the case 2.(a) could not be found in Lingham's tables; for case 3.(a), one has to look for cuspforms of higher level than the ones in Lingham's tables.

Example 6.4.10. Let $K = \mathbb{Q}(\sqrt{-23})$. We notice that f_{13} has level $\mathfrak{p}_2^2\bar{\mathfrak{p}}_2\mathfrak{p}_3^2$, and we see that $N_{K/\mathbb{Q}}(\mathfrak{p}_2) \equiv -1 \bmod 3$. Thus, one would expect that f_{13} is possibly supercuspidal at $\mathfrak{p}_2$ with conductor that degenerates modulo 3, and that f_{13} is congruent to a cuspform of level $\mathfrak{p}_2\bar{\mathfrak{p}}_2\mathfrak{p}_3^2$. This seems to be true, as the first eigenvalues of f_{13} are congruent to the ones of f_5 modulo 3. This possible level lowering is of the form 2.(b) of Theorem 6.1.4.

Example 6.4.11. Consider the quadratic field $K = \mathbb{Q}(\sqrt{-31})$, and the cuspform f_{10} of level $\mathfrak{p}_2^4\bar{\mathfrak{p}}_2^2$. If we want to lower the level at the prime $\bar{\mathfrak{p}}_2$, we first notice that

$$N_{K/\mathbb{Q}}(2) \equiv -1 \bmod 3.$$

Moreover, we have the cuspform f_6 of level $\mathfrak{p}^4\bar{\mathfrak{p}}_2$, which seems to be congruent to f_{10} modulo 3. So that we expect that at the prime $\bar{\mathfrak{p}}_2$ the associated Galois representations fall in the case 2.(b) of Theorem 6.1.4. To be more explicit, we expect f_{10} to be supercuspidal at the prime $\bar{\mathfrak{p}}_2$, whose conductor degenerates modulo 3.

Bibliography

[1] M. Asgari, F. Shahidi, *Generic transfer from $GSp(4)$ to $GL(4)$*, Compos. Math., 142(3) (2006), 541-550

[2] L. Berger, *An introduction to the theory of p-adic representations*, Geometric aspects of Dwork theory, Volume I (2004), 255-292

[3] T. Berger, G. Harcos, *l-adic representations associated to modular forms over imaginary quadratic fields*, Int. Math. Res. Not. **23** (2007)

[4] D. Bump, *Automorphic Forms and Representations*, Cambridge University Press (1998)

[5] C. Bushnell, G. Henniart, P. Kutzko, *Correspondance de Langlands locale pour GL_n et conducteurs de paires*, Ann. Scient. Éc. Norm. Sup. (1998), 537-560

[6] H. Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. Ec. Norm. Sup., 4^e série **19** (1986), 409-468

[7] H. Carayol, *Sur les représentations galoisiennes modulo l attachées aux formes modulaires*, Duke Math. J. **59** (1989), 785-801

[8] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Can. J. Math., Vol. XLI, No. 3 (1989), 385-438

[9] W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint, `http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf`

[10] L. Clozel, M. Harris, R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Pub. Math. IHES **108** (2008), 1-181

[11] Y. Danisman, *L-factors of supercuspidal representations of p-adic $GSp(4)$*, PhD Thesis, Ohio State University (2011)

[12] P. Deligne, *Formes modulaires et représentations l-adiques*, Sém. Bourbaki 355, Springer Lecture Notes **179** (1971), 139-172

[13] P. Deligne, J.-P. Serre, *Formes modulaires de poids* 1, Ann. Sci. Ec. Norm. Sup., 4^e série **7** (1974), 507-530

[14] F. Diamond, R. Taylor, *Lifting modular mod l representations*, Duke Math. **74** (1994), no. 2, 253-269

[15] F. Diamond, R. Taylor, *Non-optimal levels of mod l modular representations*, Invent. Math. **115** (1994), 435-462

[16] A. Fröhlich, *Local fields*, Algebraic Number Theory (edited by J. Cassels, A. Fröhlich), Academic Press (1967), 1-41

[17] W. T. Gan, A. Ichino, *On endoscopy and the refined Gross-Prasad conjecture for (SO_5, SO_4)*, J. Inst. Math. Jussieu **10** (2011), no. 2, 235-324

[18] W. T. Gan, S. Takeda, *The local Langlands conjecture for $GSp(4)$*, Ann. of Math. (2) **173** (2011), no. 3, 1841-1882

[19] W. T. Gan, S. Takeda, *Theta correspondences for $GSp(4)$*, Represent. Theory **15** (2011), 670-718

[20] W. T. Gan, S. Takeda, *On the Howe duality conjecture in classical theta correspondence*, preprint

[21] W. T. Gan, S. Takeda, *A proof of the Howe duality conjecture*, preprint

[22] T. Gee, D. Geraghty, *Companion forms of unitary and symplectic groups*, Duke Math. Journal **161** (2012), Number 2, 247-303

[23] S. S. Gelbart, *Automorphic Forms on Adele Groups*, Princeton University Press (1975)

[24] P. Gérardin, P. Kutzko, *Facteurs locaux pour $GL(2)$*, Annales scientifiques de l'École Normale Supérieure (4) **13** (1980), no. 3, 349-384

[25] K. Gruenberg, *Profinite groups*, Algebraic Number Theory (edited by J. Cassels, A. Fröhlich) Academic Press (1967), 116-128

[26] R. Guralnick, F. Herzig, R. Taylor, J. Thorne, *Appendix: adequate subgroups*, `http://math.harvard.edu/~thorne/appendix.pdf`

[27] M. Harris, S. Kudla, *Arithmetic automorphic forms for the non-holomorphic discrete series of $GSp(2)$*, Duke Math. J. **66(1)** (1992), 59-121

[28] M. Harris, K.-W. Lan, R. Taylor, J. Thorne, *On the rigid cohomology of certain Shimura varieties*, preprint

[29] M. Harris, D. Soudry, R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields I. Lifting to $GSp_4(\mathbb{Q})$*, Invent. Math. **112** (1993), 377-411

[30] M. Harris, R. Taylor, *The geometry and cohomology of certain simple Shimura varieties*, Annals of Math Studies **151**, Princeton University Press (2001)

[31] G. Henniart, *Une preuve simple des correspondance de Langlands pour $GL(n)$ sur un corps p-adic*, Invent. Math. **139** (2000), 439-455

[32] H. Jacquet, R. P. Langlands, *Automorphic forms on GL(2)*, Springer-Verlag Berlin Heidelberg (1970)

[33] F. Jarvis, *On Galois representations associated to Hilbert modular forms of low weight*, Crelle **491** (1997), 199-216

[34] J. Johnson-Leung, B. Roberts, *Siegel modular forms of degree two attached to Hilbert modular forms*, preprint

[35] S. S. Kudla, S. Rallis, D. Soudry, *On the degree* 5 *L-function for* $Sp(2)$, Invent. Math. **107** (1992), 483-541

[36] P. Kutzko, *The Langlands conjecture for* Gl_2 *of a local field*, Annals of Math. **112** (1980), 381-412

[37] M. Lingham, *Modular forms and elliptic curves over imaginary quadratic fields*, PhD Thesis, University of Nottingham (2005)

[38] R. Livné, *On the conductors of mod* l *representations coming from modular forms*, Number Theory **31** (1989), 133-141

[39] C. Moeglin, M.-F. Vigneras, J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics 1291, Springer-Verlag, Berlin-Heidelberg-New York (1987)

[40] C. P. Mok, *Galois representations associated to automorphic forms on* GL_2 *over CM fields and local-global compatibility up to semi-simplification*, preprint

[41] J. Neukirch, *Algebraische Zahlentheorie*, Springer-Verlag, Berlin-Heidelberg (2007)

[42] A. Pitale, A. Saha, R. Schmidt, *Transfer of Siegel cusp forms of degree* 2, Mem. Amer. Math. Soc., Vol. 232, no. 1090 (2014)

[43] D. Prasad, *Weil representation, Howe duality, and the theta correspondence*, CRM Proceedings and Lecture Notes, vol. 1 (1993), 105-127

[44] K. Ribet, *Raising the levels of modular representations*, Séminaire de Théorie des Nombres, Paris 1987-88, 259-271, Prog. Math. **81**, Birkhäuser Boston, Boston, MA (1990)

[45] B. Roberts, *The theta correspondence for similitudes*, Israel J. of Math. **94** (1996), 285-317

[46] B. Roberts, *The nonarchimedean theta correspondence for* $GSp(2)$ *and* $GO(4)$, Trans. AMS **351** (1999), 781-811

[47] B. Roberts, *Global L-packets for $GSp(2)$ and theta lifts*, Doc. Math. **6** (2001), 247-314

[48] B. Roberts, R. Schmidt, *Local Newforms for $GSp(4)$*, Springer Lecture Notes in Mathematics, vol. 1918 (2007)

[49] F. Rodier, *Whittaker models for admissible representations of reductive p-adic split groups*, Harmonic Analysis on Homogeneous Spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), Amer. Math. Soc. (1973), 425-430

[50] F. Rodier, *Décomposition de la série principale des groupes réductifs p-adiques*, Non commutative Harmonic Analysis and Lie groups, Lecture Notes in Math. 880, Springer-Verlag, Berlin (1981), 408-423

[51] F. Rodier, *Sur les représentations non ramifiées des groupes réductifs p-adiques; l'example de $GSp(4)$*, Bull. Soc. Math. France **116** (1988), 15-42

[52] D. Rohrlich, *Elliptic curves and the Weil-Deligne group*, Elliptic curves and related topics (edited by H. Kisilevsky, M. Ram Murty), CRM Proceedings and Lecture notes **4**, Amer. Math. Soc. (1994), 125-157

[53] P. Sally, M. Tadić, *Induced representations and classifications for $GSp(2,F)$ and $Sp(2,F)$*, Mémoires Soc. Math. de France **52** (1993), 75-133

[54] P. Scholze, *On torsion in the cohomology of locally symmetric varieties*, preprint

[55] J-P. Serre, *Local class field theory*, Algebraic Number Theory (edited by J. Cassels, A. Fröhlich), Academic Press (1967), 129-162

[56] J-P. Serre, *Local Fields*, Graduate Texts in Math., vol. 67, Springer-Verlag, New York (1979)

[57] F. Shahidi, *A proof of Langlands conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. **132** (1990), 273-330

[58] F. Shahidi, *Langlands' conjecture on Plancherel measures for p-adic groups*, Harmonic analysis on reductive groups (Brunswick, ME, 1989) of Progr. Math. **101** (1991), 277-295

[59] A. Silberger, *The Langlands quotient theorem for p-adic groups*, Math. Ann. **236** (1978), 95-104

[60] C. Sorensen, *Galois representations attached to Hilbert-Siegel modular forms*, Do. Math. **15** (2010), 623-670

[61] S. Takeda, *Some local-global non-vanishing results for theta lifts from orthogonal groups*, Trans. Amer. Math. Soc. **361** (2009), 5575-5599

[62] S. Takeda, *Some local-global non-vanishing results of the theta lifts for symplectic-orthogonal dual pairs*, J. Reine Angew. Math. **657** (2011), 81-111

[63] R. Taylor, *On Galois representations associated to Hilbert modular forms*, Invent. Math. **98** (1989), 265-280

[64] R. Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), 281-322

[65] R. Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields. II*, Invent. Math. **116** (1994), 619-643

[66] J. Thorne, *On the automorphy of l-adic Galois representations with small residual image*, Journal of the Inst. of Math. Jussieu, 11(4):855-920 (2012)

[67] J. Weinstein, *The local Jacquet-Langlands correspondence via Fourier analysis*, Journal de Théorie des Nombres de Bordeaux **22** (2010), 483-512

[68] R. Weissauer, *Four dimensional Galois representations*, Astérisque **302** (2005), 67-150

[69] J.-L. Waldspurger, *Un exercice sur* $GSp(4, F)$ *et les représentations de Weil*, Bull. Soc. Math. France **115** (1987), 35-69