Algebraic models for rational G - spectra

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To my brother and my parents.

ABSTRACT

In this thesis we present two themes. Firstly, for a compact Lie group G, we work with the category of Continuous Weyl Toral Modules (CWTM_G), where objects are sheaves of $\mathbb{Q}$ modules over a G topological category TC_G whose object space consists of the closed subgroups of G. It is believed that an algebraic model for rational G equivariant spectra (for any compact Lie group G) will be of the form of CWTM_G with some additional structure. We establish a very well behaved monoidal model structure on categories like CWTM_G allowing one to do homotopy theory there. We do this by using the fact that there is an injective model structure on the category of chain complexes in a Grothendieck category.
Secondly, we provide an algebraic model for rational $SO(3)$ equivariant spectra by using an extensive study of interaction between the restriction – coinduction adjunction and left and right Bousfield localisation. We start by splitting the category of rational $SO(3)$ equivariant spectra into three parts: exceptional, dihedral and cyclic. This splitting allows us to treat every part seperately. Our passage for the exceptional part is monoidal and it is applied to provide a monoidal algebraic model for G rational spectra for any finite G. The passage for the cyclic part is monoidal except for the last Quillen equivalence which simplifies the algebraic model.

Contents

Introduction

Rational equivariant cohomology theories

Cohomology theories are very important in algebraic topology as they are invariants for topological spaces. A cohomology theory E^* is a functor on spaces to the category of graded abelian groups. Moreover, it has to satisfy Eilenberg- Steenrod axioms, except for the dimension axiom.

Every cohomology theory E^* is represented on the homotopy level by an object called spectrum and denoted by E. This means that for any topological space X, $E^*(X) = [\Sigma^\infty X, E]^*$ where $\Sigma^\infty X$ denotes the suspension spectrum on X and square brackets denote homotopy classes of maps. Therefore we might study cohomology theories by studying the corresponding spectra. What is more, on the level of spectra the information we are interested in is up to homotopy, that is we want to work with the homotopy category of spectra.

Unfortunately, the category of spectra is very complicated to work with and even the homotopy level of information is very rich, mainly because of $\mathbb{Z}$–torsion groups. To make it simpler and be able to work with the category of spectra we rationalize it, to get rid of these complications. We obtain a category of spectra which captures the information about rational cohomology theories, i.e. those with values in $\mathbb{Q}$–modules. It turns out that this category of rational spectra is much easier, but still very useful.

Naturally, if we want to work with G–spaces (for G a compact Lie group) instead of just spaces, ordinary cohomology theories will not capture the G–action on the space. However, we can define G–equivariant cohomology theories which will take into account the G action on spaces.

It turns out that G–equivariant cohomology theories are also representable on the homotopy level analogously to the non–equivariant setting and the representing objects are called G–spectra. Therefore, instead of working with G–cohomology theories we might as well work with G–spectra which represent them.

Very often topologists use the term "spectra" without mentioning which category exactly they have in mind. This happens because we are interested in the homotopy level of information and all categories of spectra that we might want to consider have equivalent homotopy categories (see [SS03a, Section 7]).

Modelling the category of rational equivariant spectra

The category of G–spectra is important for algebraic topologists as its homotopy category is a good place to study G–equivariant cohomology theories. However, it is relatively difficult to work with. Therefore we try to find a purely algebraic description of it, i.e. an algebraic category which would be Quillen equivalent to the category of G–spectra. The nice part of this approach is the fact that the conditions on the adjoint pair of functors to form a Quillen equivalence are relatively easy to check.

The first idea is to rationalize the category of G–spectra, following the non-equivariant ideas, as we want to get rid of the $\mathbb{Z}$ torsions (and moreover this would make some tools work). Recall that a spectrum X is rational if its homotopy groups $\pi_*(X)$ are rational. However, what we mean by a category of rational G–spectra is the same category as the category of G–spectra, but with a model structure being a left Bousfield localisation of the stable model structure. New weak equivalences are maps which give isomorphisms after applying a rational homotopy group functor, i.e. $\pi_*(-) \otimes \mathbb{Q}$.

Therefore we want to find a small algebraic category in which calculations would be possible, equipped with a model structure which would be Quillen equivalent to the category of rational G–spectra. Note, that the level of accuracy we would like to get is "up to homotopy", therefore we don't need equivalences of categories (which is usually difficult to get), but (possibly a chain of) Quillen equivalences. If we find such a chain of Quillen equivalences between the category of rational G–spectra and some algebraic category we say that we found an "algebraic model" for rational G-spectra. Now we can work and perform constructions in this new setting to get true results for the homotopy category of rational G–spectra.

Existing work

It is expected, that for any compact Lie group G there exist an algebraic category $\mathcal{A}(G)$ which is Quillen equivalent to the rational G equivariant spectra.

There are many partial results, or examples for specific Lie groups G for which an algebraic model has been given. Schwede and Shipley provided in [SS03b, Example 5.1.2] an algebraic model for rational G equivariant spectra for finite G. Greenlees and Shipley presented in [GS] an algebraic model for rational torus equivariant spectra. Also, an algebraic model for the free rational G spectra was given in [GS14a] for any compact Lie group G.

However, there is no algebraic model known for the whole category of rational G-spectra for an arbitrary compact Lie group G. There is however, a very important result which gives an algebraic model for the category of non- equivariant rational spectra, which we refer to as a Shipley's theorem and which we state next. This theorem was used in proving some equivariant results.

Shipley's theorem

In her paper [Shi07] Shipley proved the following:

Theorem 0.0.1. *[Shi07, Theorem 1.1] Let R be a discrete commutative ring. Then the projective model category of unbounded differential graded $R-$algebras and stable model category of $HR-$algebra spectra are Quillen equivalent.*

The proof of this result goes by constructing a zig–zag of three monoidal Quillen equivalences between the stable model category of HR modules and projective model category of unbounded chain complexes of modules over R. The definition of the two intermediate categories uses the notion of a category of symmetric spectra in an arbitrary monoidal category, which again is a monoidal category and can be equipped with a compatible model structure. Because all categories in the zig–zag are monoidal and the three adjoint pairs of functors between them are weak monoidal Quillen equivalences they lift to the level of algebras, again as Quillen equivalences.

It was proved earlier by Schwede and Shipley in [SS03b] that the categories of $R-$modules and H$R-$modules are Quillen equivalent, however the functors used in this result did not preserve the monoidal structure, therefore the Quillen equivalence could not be lifted to the level of algebras.

The above theorem gives an algebraic model in particular for $H\mathbb{Q}-$modules. As the homotopy category of $H\mathbb{Q}-$modules is equivalent to the homotopy category of rational spectra we get an algebraic model in non–equivariant case.

Shipley's result is used for example in providing an algebraic model for rational torus equivariant spectra, as an intermediate step in the proof.

Rational Mackey functors vs sheaves

In the paper [Gre98a] Greenlees showed that the category of rational Mackey functors for any compact Lie group G is equivalent to the category of continuous Weyl toral modules, which denotes the category of sheaves with Weyl group actions over a specific topological category TC_G. The main advantage of this result is that we replaced a functor from a complicated Burnside category B_G by a functor from much easier category TC_G.

The category TC_G is a special subcategory of all closed subgroups of G and inclusions, which has very few morphisms and all of them increase the size of subgroups. TC_G is equipped with a special topology, so that we obtain the topological category. We discuss this topological category in more details and give several examples in Section 5.2.

An algebraic model for rational torus equivariant spectra

Suppose G is a torus, i.e. a compact, connected, abelian Lie group. In [GS] Greenlees and Shipley proved that the category of rational G–spectra is Quillen equivalent to the category of differential graded objects in some abelian category $\mathcal{A}(G)$. The objects of the category $\mathcal{A}(G)$ are sheaves of graded $\mathbb{Q}$–modules with some additional structure. These sheaves are over the space of closed subgroups of G denoted by $\mathrm{Sub}(G)$ and have the property that the fibre over a closed subgroup H gives information about the H–fixed points.

The proof of this result goes by constructing a zig–zag of several Quillen equivalences. It uses the ideas of cellularization described in Section 1.2 and model structures on diagram categories described in Section 1.4. One of the Quillen equivalences in the zig–zag comes from Shipley's theorem mentioned above. Another one uses the idea of formality (or rigidity), which under some assumptions allows one to get a Quillen equivalence between categories of modules over two weakly equivalent rings.

The idea of this proof gives directions to prove more general result – the analogous theorem for an arbitrary compact Lie group G.

Morita equivalences

Schwede and Shipley in [SS03b] provided a quite general tool for establishing a Quillen equivalence between a spectral model category with a set of (homotopically) compact, cofibrant and fibrant generators and a certain category of modules over a ring with possibly many objects. This machinery can be used in a case of G–equivariant spectra, where we obtain a model for it, namely the cateory of modules over a ring with many objects, however the ring itself is so complicated that this category is not useful to work with. Even if their machinery is applied to the category of rational G–equivariant spectra the outcome is still very complicated.

The idea mentioned above is discussed in details in Section 3.2.

An algebraic model for rational G–equivariant spectra for finite G

In [SS03b, Example 5.1.2] Schwede and Shipley provided an algebraic model for rational G–equivariant spectra when G is finite. In [Bar09b] Barnes provided much simpler algebraic model for rational G–equivariant spectra when G is finite. He was working with the category of G–equivariant EKMM S–modules to start with. The idea was to split it into separate parts using the idempotents of the rational Burnside ring, obtaining localised categories. Then he showed that each of these localised categories is Quillen equivalent to a category of modules over a commutative ring. The whole point of that approach was to use the monoidal version of Morita equivalence. However, a work of McClure recently redeveloped by Hill and Hopkins [HH13] shows that great care is needed when one deals with commutative rings. We present a different approach.

An algebraic model for rational $O(2)$–equivariant spectra

Barnes provided an algebraic model for rational $O(2)$–equivariant spectra in [Bar13]. He used the same tactics starting with splitting the category into two parts: cyclic and dihedral. He used monoidal Morita equivalences for dihedral part and he followed Greenlees and Shipley's approach for the cyclic part. However, again in both cases great care needs to be taken when one deals with commutative rings. We present slightly different approach with the cyclic part and dihedral part, however we were not able to provide a monoidal comparison for the dihedral part.

Contents of this thesis

Homotopy theory of sheaves

The first results of this thesis provide tools to study homotopy theory of sheaves of chain complexes of $\mathbb{Q}$ modules over a topological category using results from topos theory by Moerdijk in [Moe88] and [Moe90]. We put a well behaved model structure on this category as well as on the corresponding one with an action of a G, where G is any compact Lie group. Next, we restrict attention to the category introduced by Greenlees in [Gre98a] of continuous Weyl toral modules and put a well behaved model structure there.

We present the main theorems below (this is Corollary 6.2.1 and Theorem 8.2.9 respectively):

Theorem 0.0.2. *Suppose* $\mathbf{X}$ *is a topological category, then there exist a proper, cofibrantly generated model structure on the category of* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$ *of chain complexes of sheaves of* $\mathbb{Q}$*-modules over a topological category* $\mathbf{X}$ *where*

- *the cofibrations are the injections,*
- *the weak equivalences are the homology isomorphisms and*
- *the fibrations are maps which have the right lifting property with respect to trivial cofibrations.*

Theorem 0.0.3. *Suppose* G *is a compact Lie group. Then there is a proper, stable, cofibrantly generated, monoidal model structure on the category of chain complexes of continuous Weyl toral modules for* G*, where weak equivalences are homology isomorphisms and cofibrations are monomorphisms.*

These results provide the foundations for establishing an algebraic model for rational G equivariant spectra, where G is any compact Lie group, as it is believed that a model will be of the form of a category consisting of continuous Weyl toral modules with additional structure.

As one of the examples, we show that the algebraic model for the dihedral part of $SO(3)$ spectra is equivalent to the category of continous Weyl toral modules restricted to the dihedral part (see Example 8.2.3) and moreover that this equivalence of categories is compatible with the model structures (see Remark 8.2.10).

Induction – restriction and restriction – coinduction adjunctions

Suppose we have an inclusion $i : H \longrightarrow G$ of a subgroup H in a group G. This gives a pair of adjoint functors at the level of orthogonal spectra (see for example [MM02]), namely induction, restriction and coinduction as below (the left adjoint is above the corresponding right adjoint)

$$\begin{tikzcd} G-\mathcal{IS} \arrow[r, "i^*"] & H-\mathcal{IS} \arrow[l, bend right, "G_+\wedge_H -"'] \arrow[l, bend left, "F_H(G_+,-)"] \end{tikzcd}$$

These two pairs of adjoint functors are Quillen pairs and restriction as a right adjoint is used for example when we want to take H fixed points of G spectra, where H is not a normal subgroup of G. The first step then is to restrict to N_GH spectra and then take H fixed points.

It is natural to ask when the above pair of adjunctions passes to the localised categories. The answer is surprisingly complex and is studied in Section 9.2. It turns out that induction – restriction adjunction does not always pass to the Quillen adjuntion at the localised categories, even if we choose the idempotents on both sides to be "corresponding". However, with the "corresponding" choice of idempotents the restriction – coinduction adjunction pasess to Quillen adjunction at this level, for certain subgroups H. We discuss it below.

Restriction – coinduction adjunction

The main ingredient of our approach to an algebraic model is via studying the restriction – coinduction adjunction for rational orthogonal spectra: $L_{S_{\mathbb{Q}}}(G-\mathcal{IS})$ and $L_{S_{\mathbb{Q}}}(H-\mathcal{IS})$, where $H < G$ and G is any compact Lie group. It turns out, that if we Bousfield localise both of these categories of spectra with respect to idempotents related via restriction this adjunction is a Quillen adjunction (this is Lemma 9.2.6 which we state below)

Lemma 0.0.4. *Suppose G is any compact Lie group, $i : H \longrightarrow G$ is an inclusion of a subgroup and V is an open and closed set in $F(G)/G$, where $F(G)$ denotes the space of closed subgroups of G with finite index in their normalizer. Recall the rational Burnside ring $A(G) = C(F(G)/G, \mathbb{Q})$. Thus there is an idempotent e_V corresponding to the characteristic function on V. Then the adjunction*

$$\begin{tikzcd} i^* \ : \ L_{e_V S_{\mathbb{Q}}}(G-\mathcal{IS}) \arrow[r, shift left] & L_{e_{i^*V} S_{\mathbb{Q}}}(H-\mathcal{IS}) \ : \ F_H(G_+,-) \arrow[l, shift left] \end{tikzcd}$$

is a Quillen pair.

We then investigate the behaviour of this adjunction for every part of rational $SO(3)$ spectra seperately, as we discuss below.

Induction – restriction adjunction

The induction – restriction adjunction is not as well behaved with respect to localisation as the previous adjunction. We will show in Proposition 9.2.3 that when H is an exceptional subgroup of G which is N_GH–bad (the terminology is explained in Section 9.1) and we consider the category $L_{e_HS_{\mathbb{Q}}}(G-\mathcal{IS})$ of localised, rational G orthogonal spectra and $L_{e_HS_{\mathbb{Q}}}(N_GH-\mathcal{IS})$, this adjunction is not a Quillen adjunction.

However, it is a Quillen adjunction when considered for an exceptional subgroup H which is N_GH–good (see Proposition 9.2.2) and this result is used in the proof of Theorem 10.3.1. Also, it is a Quillen adjunction for dihedral parts of rational $SO(3)$ and $O(2)$ spectra (see Proposition 9.2.5), which is used in the proof of Theorem 11.3.1.

Application to $SO(3)$ case

We aim to give an algebraic model for the category of rational $SO(3)$ spectra $L_{S_{\mathbb{Q}}}SO(3)-\mathcal{IS}$. Our main tool will be the analysis of the restriction – coinduction adjunction, and to apply it we first split the category $L_{S_{\mathbb{Q}}}SO(3)-\mathcal{IS}$ using the result of Barnes [Bar09a] and idempotents of the rational Burnside ring $A(SO(3))$ as follows (see Section 9.2 for definitions of the idempotents):

$$\triangle\ :\ G-\mathcal{IS}_{\mathbb{Q}} \rightleftarrows L_{e_cS_{\mathbb{Q}}}(G-\mathcal{IS})\times L_{e_dS_{\mathbb{Q}}}(G-\mathcal{IS})\times L_{e_eS_{\mathbb{Q}}}(G-\mathcal{IS})\ :\Pi$$

where $G=SO(3)$. The above adjunction is a monoidal Quillen equivalence. We will call $L_{e_eS_{\mathbb{Q}}}(G-\mathcal{IS})$ the exceptional part, $L_{e_dS_{\mathbb{Q}}}(G-\mathcal{IS})$ the dihedral part and $L_{e_cS_{\mathbb{Q}}}(G-\mathcal{IS})$ the cyclic part of $L_{S_{\mathbb{Q}}}SO(3)-\mathcal{IS}$ and later we will consider them seperately.

Exceptional part of rational $SO(3)$ spectra

The first application of using restriction – coinduction adjunction in the passage towards an algebraic model is for exceptional part of rational $SO(3)$ spectra. This part can be split into finitely many parts, each of which corresponds to localisation of the category of rational $SO(3)$ spectra at an idempotent corresponding to one of the (conjugacy classes of the) exeptional subgroups. The splitting result allows us to deal with each exceptional subgroup seperately.

The first step towards an algebraic model for one such category $L_{e_HS_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$, corresponding to the subgroup H, is to use the restriction – coinduction adjunction to pass to the category $L_{e_HS_{\mathbb{Q}}}(N_GH-\mathcal{IS})$. The result is presented in Theorem 10.3.1 and Theorem 10.3.4 which we state below

Theorem 0.0.5. *Suppose H is an exceptional subgroup of $G=SO(3)$ and it is $N=N_GH$–good. Then the adjunction*

$$i^*\ :\ L_{e_HS_{\mathbb{Q}}}(G-\mathcal{IS}) \rightleftarrows L_{e_HS_{\mathbb{Q}}}(N-\mathcal{IS})\ :F_N(G_+,-)$$

is a strong monoidal Quillen equivalence, where e_H *on the right hand side denotes the idempotent of the rational Burnside ring* $A(N)$ *corresponding to the characteristic function of* $(H)_N$.

Theorem 0.0.6. *Suppose* H *is an exceptional subgroup of* $G = SO(3)$ *and* $N = N_G H$. *Then the composition of adjunctions*

$$\begin{tikzcd} L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS}) \arrow[r, shift left, "i^*"] & L_{i^*(e_H S_{\mathbb{Q}})}(N - \mathcal{IS}) \arrow[l, shift left, "F_N(G_+,-)"] \arrow[r, shift left, "\mathrm{Id}"] & L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS}) \arrow[l, shift left, "\mathrm{Id}"] \end{tikzcd}$$

is a strong monoidal Quillen equivalence, where e_H *on the further right hand side denotes the idempotent of the rational Burnside ring* $A(N)$ *corresponding to the characteristic function of* $(H)_N$. *Notice that if* H *is* N*-good then the right adjunction is trivial.*

This is the most important ingredient in our approach, which allows us to provide a monoidal algebraic model for the exceptional part of rational $SO(3)$ spectra.

It will appear in Section 10.3 that to use this approach we need to establish a non-monoidal passage via Morita equivalences first. This is used in the proof of Theorem 10.3.4. We also note, that Theorem 10.3.1 follows from Theorem 10.3.4, however the first one is stated with a self – contained proof, which is based on the good behaviour of the induction – restriction adjunction in this case, and thus we decided to present it.

The main result of this part is Theorem 10.3.13:

Theorem 0.0.7. *Suppose* H *is an exceptional subgroup of* $G = SO(3)$. *Then there is a zig-zag of monoidal Quillen equivalences from* $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ *to* $Ch(\mathbb{Q}[W] - \mathrm{mod})$ *where* $W = N_G H/H$.

Exceptional part of any compact Lie group G

The proof presented in Chapter 10 works for the exceptional part of any compact Lie group G as we state in Remark 10.3.14. Thus this new passage provides an algebraic model for an exceptional part of any compact Lie group G.

Finite G

As a special case of the exceptional part of any compact Lie group G we can restrict attention to G finite. In this case the whole category of rational G spectra is actually equal to its exceptional part, and thus our passage provides a monoidal algebraic model for rational G spectra when G is finite (this is Corollary 10.3.15, which we state below):

Corollary 0.0.8. *Suppose* G *is a finite group. Then there is a zig-zag of monoidal Quillen equivalences from* $L_{S_{\mathbb{Q}}}(G - \mathcal{IS})$ *to* $\prod_{(H),H \leq G} Ch(\mathbb{Q}[W_G H] - \mathrm{mod})$.

Dihedral part of rational $SO(3)$ spectra

Another application of the restriction – coinduction adjunction is in providing a passage towards the algebraic model for the dihedral part of rational $SO(3)$ spectra. This adjunction is used as a first step, to pass to the dihedral part of rational $O(2)$ spectra preserving monoidal structures (this is Theorem 11.3.1, which we state below)

Theorem 0.0.9. *Let $i : O(2) \longrightarrow SO(3)$ be an inclusion and note that $i^*(e_d S_{\mathbb{Q}}) = e_d S_{\mathbb{Q}}$. Then the following*

$$i^* \ : \ L_{e_d S_{\mathbb{Q}}}(SO(3) - \mathcal{IS}) \rightleftarrows L_{e_d S_{\mathbb{Q}}}(O(2) - \mathcal{IS}) \ : \ F_{O(2)}(SO(3)_+, -)$$

is a strong monoidal Quillen equivalence, where both idempotents correspond to the set of all dihedral subgroups of order greater than 4 and $O(2)$ (in $SO(3)$ and $O(2)$ respectively).

Unfortunatelly we were not able to provide a monoidal algebraic model for dihedral part of rational $O(2)$ spectra. However, when this is accomplished the result above will complete a proof of monoidal algebraic model for dihedral part of rational $SO(3)$ spectra.

We decided to provide a non-monoidal algebraic model for dihedral part of rational $SO(3)$ spectra, working with Morita equivalences. The idea of the proof is based on the one presented in [Bar13] and it is not monoidal.

The main result of this part is Theorem 11.2.16 (where the notation $\mathcal{A}(\mathcal{D})$ is explained in Section 11.1):

Theorem 0.0.10. *There exist a zig-zag of Quillen equivalences from $L_{e_d S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$ to $Ch(\mathcal{A}(\mathcal{D}))$.*

Cyclic part of rational $SO(3)$ spectra

Again, the first step in the passage towards the monoidal algebraic model for cyclic part of rational $SO(3)$ spectra is to use the restriction – coinduction adjunction and pass to the cyclic part of rational $O(2)$ spectra. This is Theorem 12.2.3, which we state below

Theorem 0.0.11. *Suppose K is the set of generators for $L_{e_c S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$ as established in Proposition 12.2.1 together with all their suspensions and desuspensions. Then the following*

$$i^* \ : \ L_{e_c S_{\mathbb{Q}}}(SO(3) - \mathcal{IS}) \rightleftarrows i^*(K) - \text{cell} - L_{e_c S_{\mathbb{Q}}}(O(2) - \mathcal{IS}) \ : F_{O(2)}(SO(3)_+, -)$$

is a strong monoidal Quillen equivalence, where the idempotent on the right hand side corresponds to the family of all cyclic subgroups of $O(2)$.

We decided to present a complete passage towards the monoidal algebraic model for the cyclic part of rational $O(2)$ spectra, which is based on the work of Greenlees, Shipley and Barnes. However, to make the passage monoidal we repeatedly use left Bousfield localisations (see Section 12.3).

The main results in this part are Theorems 12.3.24, 12.4.1 and 12.4.3:

Theorem 0.0.12. *There is a zig-zag of Quillen equivalences from $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$ to $d\mathcal{A}(O(2),c)$, where $d\mathcal{A}(O(2),c)$ is a category of differential objects in $\mathcal{A}(O(2),c)$ considered with the dualisable model structure (see Section 12.1).*

Theorem 0.0.13. *There is a zig-zag of Quillen equivalences between $L_{e_cS_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$ and $\mathrm{im}(K)-\mathrm{cell}-d\mathcal{A}(O(2),c)$, where $d\mathcal{A}(O(2),c)$ is considered with the dualisable model structure. Here $\mathrm{im}(K)$ denotes the derived image under the zig-zag of Quillen equivalences described in Section 12.3 of the set of cells K described in Proposition 12.2.1.*

Finally we present the result wich gives a much simpler algebraic model category as an algebraic model for the cyclic part of rational $SO(3)$ spectra. This is Theorem 12.4.3.

Theorem 0.0.14. *The adjunction*

$$\widetilde{F} \ : \ \mathrm{d}\mathcal{A}(SO(3),c) \rightleftarrows \mathrm{im}(K)-\mathrm{cell}-d\mathcal{A}(O(2),c) \ : \ \widetilde{R}$$

defined in the statement of the Theorem 12.1.28 is a Quillen equivalence, where both categories (before cellularisation on the right) are considered with the injective model structure. Here $\mathrm{im}(K)$ denotes the derived image under the zig-zag of Quillen equivalences described in Section 12.3 of the set of cells K described in Proposition 12.2.1.

We summarise the results for exceptional, dihedral and cyclic parts of rational $SO(3)$ spectra in the following

Theorem 0.0.15. *There is a zig-zag of Quillen equivalences between rational $SO(3)$ spectra $L_{S_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$ and the category*

$$\prod_{(H)\in\mathcal{E}} Ch(\mathbb{Q}[W_{SO(3)}H]-\mathrm{mod}) \times Ch(\mathcal{A}(\mathcal{D})) \times d\mathcal{A}(SO(3),c)$$

where $\mathcal{E}$ denotes the exceptional part of $SO(3)$.

Organisation of this thesis

This thesis consists of three parts. The first part provides a background definitions as well as an overview of major results used in Part II and Part III. Since, especially proofs presented in Part III, use a lot of existing results and constructions, we decided to state many of them, and where necessary also give proofs. However, there are no original results in this part and the proofs are given only for those statements, for which we didn't find references. In Chapter 1 we provide an overview on the model categories, in Chapter 2 we discuss spectra and G–spectra for a compact Lie group G. Then, in Chapter 3 we present some result on enriched categories and finally, in Chapter 4 we summarise briefly the theory of sheaves.

Part II provides an introduction to the theory of G-sheaves of $\mathbb{Q}$-modules over a G-topological category, where G is any topological group. We build up the theory starting with introducing topological categories in Chapter 5, then we proceed to sheaves over topological categories in Chapter 6. Next, in Chapter 7 we discuss the G-sheaves over G-topological spaces and finally in Chapter 8 we discuss the G-sheaves over G-topological categories and the category of Continuous Weyl Toral Modules.

Part III provides an algebraic model for $SO(3)$-rational spectra. Firstly, in Chapter 9 we discuss the general results for the group $SO(3)$, like the subgroup structure and some properties of adjunctions relating $SO(3)$ rational spectra. Then we use the splitting theorem, which allows us to split $SO(3)$ rational spectra into 3 parts: exceptional, dihedral and cyclic. We proceed to building a passage towards the algebraic model of $SO(3)$ rational spectra, dealing with each part seperately. We provide a monoidal algebraic model for the exceptional part in Chapter 10 using a completely new approach. Then we proceed to dihedral part in Chapter 11, however this passage, since it uses Morita equivalences, is not monoidal. In Chapter 12 we give a monoidal algebraic model for cyclic part, building on existing work of Greenlees and Shipley [GS] and Barnes [Bar13]. However, to keep it monoidal we make significant use of left Bousfield localisations. This gives a quite complicated monoidal algebraic model for rational cyclic $SO(3)$ spectra. In Section 12.1.2 we describe a relatively easy new category which is then shown to provide an algebraic model for the cyclic part of rational $SO(3)$ spectra. This last passage however, is not monoidal.

Further work

Monoidal algebraic model for dihedral part of rational $O(2)$ spectra

A natural direction of work would be to provide a monoidal algebraic model for the dihedral part of $O(2)$ spectra (and thus also $SO(3)$ spectra) and therefore complete the monoidal model for $O(2)$ and $SO(3)$ rational spectra. The idea is to try to split information into three parts, in a way analogous to the one presented in Section 12.3 for cyclic part.

Restriction – coinduction adjunction

We need to check how the new approach of looking at the restriction - coinduction adjunction works for more general examples and what homotopically valid information it brings. For instance, since for every compact Lie group G, the category of rational G equivariant spectra splits into a cyclic part and the rest (see Section 9.3) we would like to check when this adjunction preserves the homotopical information of this part (under some conditions it should be true, that the cyclic part of rational G spectra and the cyclic part of rational N spectra are Quillen equivalent, where N is the normalizer of the maximal torus in G).

Another direction involves investigating the relationship of cyclic part of rational G spectra and rational G_e spectra, for the identity component G_e of G from the view point of this adjunction.

Induction – restriction adjunction

It would be useful to track the behaviour of induction–restriction adjunction in cases of left Bousfield localisation or cellularization. As we have shown in Section 9.2, when H is an exceptional subgroup of G which is N_GH–bad this adjunction fails to be a Quillen pair. However, the same adjunction is a Quillen adjunction when considered for an exceptional subgroup H which is N_GH–good (see Proposition 9.2.2) or for dihedral parts of $SO(3)$ and $O(2)$ (see Proposition 9.2.5). We plan to investigate under what general conditions this adjunction is a Quillen adjunction at the level of localised or cellularised categories of rational G spectra.

Further groups

Several more examples of groups G and algebraic models for rational G equivariant spectra would help deduce more general results. The first group to investigate would be $SU(2)$ as it is a double cover of $SO(3)$. We expect its analysis to be an application of results for $SO(3)$.

Another related direction would involve looking at groups, for which the cotoral subgroups (and thus morphisms in the topological category of toral chains, see Section 5.2) appear not only in the cyclic part, i.e. subgroups of the maximal torus (up to conjugation). The simplest example being $C_2 \times SO(2)$.

Continuous Weyl Toral Modules

We plan to continue the work started in this thesis to investigate the relationship of the existing algebraic models and Continuous Weyl Toral Modules (see Section 8.2). For this, the following two paths may be taken. Firstly, we can work on relating the existing models with the respective Continuous Weyl Toral Modules in a way which preserves homotopical information.

Secondly, we may look for a passage, which can be generalised to the level of sheaves. However, the second approach presents some difficulties coming from the fact that we don't know yet how to relate a category of rational G–spectra to the category of G–sheaves over a G–topological category TC_G.

Notational conventions

We follow the general convention of writing the left adjoint arrow on top of the right one whenever we consider an adjoint pair.

When $\mathcal{C}$ is a triangulated category we use $[A, B]^{\mathcal{C}}$ for an abelian group of morphisms from A to B in $\mathcal{C}$ and $[A, B]_*^{\mathcal{C}}$ for a graded abelian group of morphisms from A to B in $\mathcal{C}$, i.e. $[A, B]_n^{\mathcal{C}} = [A[n], B]^{\mathcal{C}}$ where $A[n]$ is the n-fold shift of A.

Whenever we consider a suspension G–spectrum for a G–space X we continue to use notation X for it, instead of $\Sigma^{\infty} X$.

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Part I

Preliminaries

Chapter 1

Model categories

Suppose we have a category $\mathcal{C}$ and a class of maps W in $\mathcal{C}$ that we would like to formally invert. Moreover we would like to obtain a universal such category. This construction is called the localisation of $\mathcal{C}$ with respect to W and it may encounter set theoretical problems, as we may end up with a class of maps between two objects, instead of a set. However, the construction can be performed if W admits the calculus of left or right fractions.

Quillen in 1967, looking at the category of topological spaces Top and the class of weak homotopy equivalences, described in [Qui67] for a category $\mathcal{C}$ and a class of maps W the construction which lead to desired result. This construction can be performed provided one can define a structure on $\mathcal{C}$ called the model structure. A model structure is a choice of three classes of maps satisfying a list of axioms. One class is exactly those maps that we want to invert, we call them weak equivalences. Other two classes, fibrations and cofibrations, play auxiliary role. Quillen gives an explicit construction of localisation of $\mathcal{C}$ with respect to the class of weak equivalences and calls it the homotopy category $Ho(\mathcal{C})$.

If we have a model structure on a category $\mathcal{C}$ it gives us a tool to do "homotopy theory" there- we obtain a good category to work with homotopy limits and we get an action of homotopy category of simplicial sets $Ho(sSet)$ on a category $Ho(\mathcal{C})$. These results are described in detail in [Qui67] and [Hov99]. This point of view proved to be very fruitful allowing to do "homotopy theory" in categories far from topological.

1.1 Definitions

In this section we recall the basic definitions and properties of model categories. To get more information about it see [DS95] and [Hov99]. The first 5 definitions in this section are after [Hov99].

First note that given a category $\mathcal{C}$ we can consider a category Map$\mathcal{C}$ where objects are maps from $\mathcal{C}$ and morphisms are commutative squares.

Definition 1.1.1. Let $\mathcal{C}$ be a category. A map f in $\mathcal{C}$ is a retract of a map g in $\mathcal{C}$ if and only if there is a commutative diagram of the form

$$\begin{tikzcd}
X \arrow[rr, "Id_X" description] \arrow[dr] \arrow[ddd, "f"] & & X \arrow[ddd, "f"] \\
& Y \arrow[ur] \arrow[d, "g"] & \\
& Y' \arrow[dr] & \\
X' \arrow[rr, "Id_{X'}" description] \arrow[ur] & & X'
\end{tikzcd}$$

Definition 1.1.2. A functorial factorization in $\mathcal{C}$ is an ordered pair (α, β) of functors $\mathrm{Map}\mathcal{C} \longrightarrow \mathrm{Map}\mathcal{C}$ such that $f = \beta(f) \circ \alpha(f)$ for all f in $\mathrm{Map}\mathcal{C}$. In particular the domain of $\alpha(f)$ is the domain of f, the codomain of $\alpha(f)$ is the domain of $\beta(f)$ and the codomain of $\beta(f)$ is the codomain of f.

Definition 1.1.3. Suppose $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ are maps in a category $\mathcal{C}$. Then i has the left lifting property with respect to p and p has the right lifting property with respect to i if for every commutative diagram of the form

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "g"] & Y
\end{tikzcd}$$

there exist a **lift** in this diagram, i.e. a map $h : B \to X$ such that both triangles commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "g"] \arrow[ur, "h" description] & Y
\end{tikzcd}$$

Definition 1.1.4. A model structure on a category $\mathcal{C}$ consists of three subcategories of $\mathcal{C}$, called weak equivalences, fibrations and cofibrations, and two functorial factorisations (α, β) and (γ, δ) satisfying the following conditions:

1. (2 out of 3) If f and g are morphisms in $\mathcal{C}$ such that gf is defined and two out of $f,\ g,\ gf$ are weak equivalences then so is the third.

2. (Retracts) If f and g are morphisms in $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, fibration or cofibration then so is f.

3. (Lifting) Define a map to be a trivial cofibration if it is both a weak equivalence and a cofibration. Similarly, define a map to be a trivial fibration if it is both a weak equivalence and a fibration. Then trivial cofibrations have the left lifting property with respect to fibrations and cofibrations have the left lifting property with respect to trivial fibrations

4. (Factorisation) for any morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

Definition 1.1.5. A model category is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$

We sometimes call a trivial cofibration an acyclic cofibration and similarly for trivial fibrations, we sometimes call them acyclic fibrations.

As we mentioned in the introduction to this chapter, the model structure on a category $\mathcal{C}$ allows one to construct a homotopy category $Ho(\mathcal{C})$, with weak equivalences inverted. We summarise the construction presented in [Hov99, Section 1.2] in the following

Theorem 1.1.6. *Suppose $\mathcal{C}$ is a model category. Then there exists a category $Ho(\mathcal{C})$ together with a functor $\gamma : \mathcal{C} \longrightarrow Ho(\mathcal{C})$ with the property that $\gamma(f)$ is an isomorphism if and only if f is a weak equivalence in $\mathcal{C}$. Moreover, if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor that sends weak equivalences to isomorphisms then there exists a unique functor $Ho(F) : Ho(\mathcal{C}) \longrightarrow \mathcal{D}$ such that $Ho(F) \circ \gamma = F$.*

Usually checking all the axioms in the definition of a model category is difficult. However, very often model categories are cofibrantly generated, which means that the model structure is completely determined by a **set** of generating cofibrations and a **set** of generating trivial cofibrations. These two sets determine trivial fibrations and fibrations respectively in view of lifting condition in Definition 1.1.4. Knowing that a model structure is cofibrantly generated makes it easier to work with it.
To give the definition we first need some notation. The following notation and definition is from [SS00]. Having a cocomplete category $\mathcal{C}$ and a class of maps I, we denote:

- I-inj (I- injectives) the class of maps which have the right lifting property with respect to the maps in I

- I-cof (I- cofibrations) the class of maps which have the left lifting property with respect to the maps in I-inj

- I-cof$_{reg}$ (regular I- cofibrations) the class of possibly transfinite compositions of pushouts of maps in I. Hovey's notation for this is I-cell.

Definition 1.1.7. A model category $\mathcal{C}$ is called **cofibrantly generated** if it is bicomplete and there exist a set of cofibration I and a set of trivial cofibrations J such that:

- the fibrations are exactly J-inj

- the trivial fibrations are exactly I-inj

- the domain of each map in I (respectively J) is small relative to the class of I-cof$_{reg}$ (respectively in J-cof$_{reg}$).

Moreover (trivial) cofibrations are exactly the I (J)- cof.

To complete this definition we need to explain the notion of "smallness".
For κ a cardinal we say that an ordinal λ is κ-filtered if it is a limit ordinal and moreover $A \subseteq \lambda$ with $|A| \leq \kappa$ then sup$A < \lambda$.

Definition 1.1.8. [Hov99, Definition 2.1.3] Let $\mathcal{C}$ be a category with all small colimits, D a class of morphisms in $\mathcal{C}$, A an object of $\mathcal{C}$ and κ a cardinal. Then A is κ-small relative to D if for all κ-filtered ordinals λ and all λ- sequences of maps in D:

$$X_0 \longrightarrow X_1 \longrightarrow ... \longrightarrow X_\beta \longrightarrow ...$$

the canonical map of sets

$$colim_{\beta<\lambda}\mathcal{C}(A, X_\beta) \longrightarrow \mathcal{C}(A, colim_{\beta<\lambda}X_\beta)$$

is an isomorphism. A is **small** relative to D if there exists κ such that A is κ- small relative to D.

Working with cofibrantly generated model categories is much easier, and it is a notion useful for performing certain constructions on the model category, such as localisation.

Provided we have two model categories $\mathcal{C}$ and $\mathcal{D}$ we would like to have a notion of a functor between them which preserves the model structure, i.e. which induces a functor on the level of homotopy categories. It turns out that it is best to talk about adjoint pairs of functors between model categories.

Definition 1.1.9. Let $\mathcal{C}$ and D be model categories, and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a pair of adjoint functors. Suppose that F preserves cofibrations and G preserves fibrations (or equivalently F preserves cofibrations and trivial cofibrations, or equivalently G preserves fibrations and trivial fibrations). Then we call F a **left Quillen functor**, G a **right Quillen functor** and an adjoint pair (F, G) we call a **Quillen pair**.

Definition 1.1.10. [Hov99, Definition 1.3.6]
Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.

- If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a left Quillen functor then we define the **total left derived functor** $LF : Ho(\mathcal{C}) \longrightarrow Ho(\mathcal{D})$ as a composition:

$$Ho(\mathcal{C}) \xrightarrow{Ho(\hat{c})} Ho(\mathcal{C}) \xrightarrow{Ho(F)} Ho(\mathcal{D})$$

- If $G : \mathcal{D} \longrightarrow \mathcal{C}$ is a right Quillen functor then we define the **total right derived functor** $RG : Ho(\mathcal{D}) \longrightarrow Ho(\mathcal{C})$ as a composition:

$$Ho(\mathcal{D}) \xrightarrow{Ho(\hat{f})} Ho(\mathcal{D}) \xrightarrow{Ho(G)} Ho(\mathcal{C})$$

Where $\hat{c}$ and $\hat{f}$ denote the cofibrant and fibrant replacement functor respectively.

We have the following result:

Theorem 1.1.11. *[DS95, Theorem 9.7] Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a pair of adjoint functors, where F is a left Quillen functor and G is a right Quillen functor. Then the total derived functors*

$$LF : Ho(\mathcal{C}) \rightleftarrows Ho(\mathcal{D}) : RG$$

exist and form an adjoint pair.
If in addition for each cofibrant object A of $\mathcal{C}$ and fibrant object X of $\mathcal{D}$, a map $f : A \longrightarrow G(X)$ is a weak equivalence in $\mathcal{C}$ if and only if its adjoint $f^{\flat} : F(A) \longrightarrow X$ is a weak equivalence in $\mathcal{D}$, then LF and RG are inverse equivalences of categories. In this situation we call the adjoint pair of functors a **Quillen equivalence**.

We say that two model categories are **Quillen equivalent** if there is a finite zig–zag of Quillen equivalences between them.

The condition from Theorem 1.1.11 is not always the easiest one to check. Therefore we give a more useful criterion for checking when a given Quillen adjunction is a Quillen pair.

Proposition 1.1.12. *[Hov99, Corollary 1.3.16] Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen pair. The following are equivalent:*

1. *(F, G) is a Quillen equivalence*
2. *F reflects weak equivalences between cofibrant objects and, for every fibrant Y, the map $F\hat{c}GY \longrightarrow Y$ is a weak equivalence*
3. *G reflects weak equivalences between fibrant objects and, for every cofibrant X, the map $X \longrightarrow G\hat{f}FX$ is a weak equivalence*

Now we state a well known theorem, which is due to Kan and it allows to transfer a model structure through the right adjoint functor.

Theorem 1.1.13. *[Hir03, Theorem 11.3.2] Suppose $\mathcal{M}$ is a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J, $\mathcal{C}$ is a category closed under small limits and colimits and we have an adjoint pair*

$$F : \mathcal{M} \rightleftarrows \mathcal{C} : U$$

with F the left adjoint. Let $FI = \{F\alpha | \alpha \in I\}$ and similarly for FJ and suppose further that

- *both sets FI and FJ permit the small object argument*

- *U takes relative FJ-cell complexes to weak equivalences*

then there is a cofibrantly generated model structure on $\mathcal{C}$ with generating cofibrations FI and generating trivial cofibrations FJ. The weak equivalences and fibrations in $\mathcal{C}$ are these maps which U takes to weak equivalences or fibrations (respectively). F, U is a Quillen adjunction with respect to this model structure.

Later on we will work with W objects in a category $\mathcal{C}$, where W is a finite group. We denote this category by $\mathcal{C}[W]$. We can think of $\mathcal{C}[W]$ as a category of functors from W, which is a one object category with $Hom(*, *) = W$ to $\mathcal{C}$, also known as $\mathcal{C}^W$. It turns out that if $\mathcal{C}$ was a cofibrantly generated model category, then $\mathcal{C}[W]$ can be equipped with a model structure by appying Theorem 1.1.13 to the adjunction below:

$$Lan_U \ : \ \mathcal{C} \rightleftarrows \mathcal{C}^W \ : \ U$$

Where Lan_U is the left Kan extension along U. Recall that U has also a right adjoint, but we won't be using that much. It is a straightforward observation that U preserves cofibrations. Now we prove the following, well–known fact

Proposition 1.1.14. *Suppose*

$$F \ : \ \mathcal{C} \rightleftarrows \mathcal{D} \ : \ G$$

is a Quillen equivalence. Then this adjunction restricted to W–objects in $\mathcal{C}$ and $\mathcal{D}$ (with model structures transfered from that on $\mathcal{C}$ and $\mathcal{D}$ respectively) is a Quillen equivalence.

Proof. We have the following diagram

$$\begin{tikzcd}
\mathcal{C}[W] \arrow[r, shift left, "F_W"] \arrow[d, "U_{\mathcal{C}}"] & \mathcal{D}[W] \arrow[l, shift left, "G_W"] \arrow[d, "U_{\mathcal{D}}"] \\
\mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"]
\end{tikzcd}$$

where $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ functors commute with both left and right adjoints. Moreover $U_{\mathcal{C}}$ and $U_{\mathcal{D}}$ define weak equivalences and fibrations and they preserve cofibrant objects (they preserve cofibrations and initial objects) and fibrant objects. A check of the condition from Definition 1.1.11 for the adjunction (F_W, G_W) is just a diagram chaising.

□

Definition 1.1.15. A model category $\mathcal{C}$ is **stable** if it is pointed (i.e. the initial object is isomorphic to the terminal object) and the suspension functor $\sum$ is an equivalence of categories on the level of homotopy category of $\mathcal{M}$, $Ho(\mathcal{M})$.

Proposition 1.1.16. *[Hov99, Definition 7.1.1] A homotopy category of a stable model category is triangulated.*

The following are taken from [SS03b], Definition 2.1.2 and Lemma 2.2.1.

Definition 1.1.17. Let $\mathcal{C}$ be a triangulated category with infinite coproducts. A full triangulated subcategory of $\mathcal{C}$ (with shift and triangles induced from $\mathcal{C}$) is called localising if it is closed under coproducts in $\mathcal{C}$. A set $\mathcal{P}$ of objects of $\mathcal{C}$ is called a **set of generators** if the only localising subcategory of $\mathcal{C}$ containing objects of $\mathcal{P}$ is the whole $\mathcal{C}$. An object X in $\mathcal{C}$ is **(homotopically) compact** if for any family of objects $\{A_i\}_{i\in I}$ the canonical map

$$\bigoplus_{i\in I}[X, A_i]^{\mathcal{C}} \longrightarrow [X, \coprod_{i\in I} A_i]^{\mathcal{C}}$$

is an isomorphism. An object of a stable model category is called a (homotopically) compact generator if it is so when considered as an object of the homotopy category.

To check if a set of compact objects generates a triangulated category we have the following criterion

Proposition 1.1.18. *[SS03b, Lemma 2.2.1] Let $\mathcal{C}$ be a triangulated category with infinite coproducts and let $\mathcal{P}$ be a set of compact objects. Then $\mathcal{P}$ generates $\mathcal{C}$ in the sense of Definition 1.1.17 if and only if for any object X in $\mathcal{C}$, X is trivial if and only if there are no graded maps from objects of $\mathcal{P}$ to X, i.e. $[P, X]_* = 0$ for all $P \in \mathcal{P}$.*

There is an easy condition for a Quillen adjunction between stable model categories with sets of (homotopically) compact generators to be a Quillen equivalence

Lemma 1.1.19. *Suppose $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen pair between stable model categories, such that the right derived functor RU preserves coproducts. Then it is enough to check that a derived unit or counit condition from Proposition 1.1.12 is satisfied for the set of compact generators.*

Proof. This follows from the fact that the homotopy category of a stable model category is a triangulated category. As the derived unit and counit conditions are satisfied for a set of objects $\mathcal{K}$ then they are also satisfied for every object in the localising subcategory for $\mathcal{K}$. Since $\mathcal{K}$ consisted of generators the localising subcategory for $\mathcal{K}$ is the whole category.

Notice that LF always preserves coproducts, as it is left adjoint. □

Proposition 1.1.20. *Suppose $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen pair between stable model categories. Suppose further that K is a set of (homotopically) compact cofibrant generators for $\mathcal{C}$. If $F(K)$ is a set of compact objects in $\mathcal{D}$ then RU preserves coproducts.*

Proof. This is a well known fact, however we give here the proof of the statment.

As K is a set of compact generators for $\mathcal{C}$ it is enough to show that for every $k \in K$ and for every collection of objects A_i, $i \in I$ in $\mathcal{D}$ the following map

$$\oplus_{i\in I}[k, RU(A_i)]_*^{Ho(\mathcal{C})} \cong [k, \coprod_{i\in I} RU(A_i)]_*^{Ho(\mathcal{C})} \longrightarrow [k, RU(\coprod_{i\in I} A_i)]_*^{Ho(\mathcal{C})}$$

is an isomorphism. This map is isomorphic to the natural map

$$\oplus_{i\in I}[LF(k),A_i]_*^{Ho(\mathcal{D})} \longrightarrow [LF(k),\coprod_{i\in I}A_i]_*^{Ho(\mathcal{D})}$$

which is the map from the definition of $LF(k)=F(k)$ being a compact object. As we assumed it is, that finishes the proof. □

1.2 Localisation of model categories

The following definition will play an important role in localising homotopy categories. Material in this section is taken from [Hir03].

Definition 1.2.1. We say that a model structure on a category $\mathcal{C}$ is **proper** iff a pullback of a weak equivalence along a fibration is a weak equivalence and a pushout of a weak equivalence along a cofibration is a weak equivalence.
Sometimes we will refer to the category which have only the first mentioned property as a **right proper** (respectively **left proper** if it has only the second property).

This property of a model structure is required and mostly used when we want to further localize the homotopy category $Ho(\mathcal{C})$ with respect to some class of maps (see [Hir03].

Having a model category $\mathcal{C}$, there are several ways of constructing new model structure on $\mathcal{C}$. We explain two of them, called Bousfield localisation and "cellularization" which will be used later on.

In this section we are going to use some additional assumptions on the model category $\mathcal{C}$, therefore we give the following

Definition 1.2.2.

1. An object A in a category $\mathcal{C}$ is called **compact** iff the set of maps $Hom(A,\coprod_{t\in T}X_t)$ from it into a coproduct is in bijection with the coproduct of maps $\coprod_{t\in T}Hom(A,X_t)$.
2. A morphism $f:X\longrightarrow Y$ in a category $\mathcal{C}$ is called an effective monomorphism if:
   - the pushout $Y\bigcup_X Y$ exists
   - f is the equalizer of the pair of canonical maps $Y\rightrightarrows Y\bigcup_X Y$

Definition 1.2.3.
A model category $\mathcal{C}$ is **cellular** if it is a cofibrantly generated model category with a set of generating cofibrations I and a set of generating acyclic cofibrations J, such that:

- all domains and codomains of maps from I are compact
- the domains of maps from J are small relative to I
- the cofibrations are effective monomorphisms.

For example model categories on simplicial sets and Serre model category of topological spaces are cellular.

We start with the Bousfield localisation. In this part we will introduce several definitions after Hirschhorn:

Definition 1.2.4. [Hir03, Definition 3.1.1]
Suppose we have a model category $\mathcal{C}$ and a class of maps S. Then a **left localisation** of $\mathcal{C}$ with respect to S is a model category $L_S\mathcal{C}$ together with a left Quillen functor $j : \mathcal{C} \longrightarrow L_S\mathcal{C}$ such that the two conditions are satisfied:

- the total derived functor $Lj : Ho(\mathcal{C}) \longrightarrow Ho(L_S\mathcal{C})$ maps the images of S in $Ho(\mathcal{C})$ into isomorphisms
- if we have a model category N and a left Quillen functor $\phi : \mathcal{C} \longrightarrow N$ such that its total derived functor takes images of S in $Ho(\mathcal{C})$ into isomorphisms then there exists a unique left Quillen functor $\delta : L_S\mathcal{C} \longrightarrow N$ such that $\delta j = \phi$.

Similarly we can define a **right localisation** using everywhere right Quillen functors instead of left Quillen functors.

Definition 1.2.5. Suppose we have a model category $\mathcal{C}$ and a class of maps S.

- An object W in $\mathcal{C}$ is called S-local if W is fibrant and for every element $f : A \longrightarrow B$ in S the induced map of homotopy function complexes:

  $$\underline{map}(f, W) : \underline{map}(B, W) \longrightarrow \underline{map}(A, W)$$

  is a weak equivalence.

- A map $g : X \longrightarrow Y$ in $\mathcal{C}$ is called S-local equivalence if for every S-local object W the induced map of homotopy function complexes:

  $$\underline{map}(g, W) : \underline{map}(Y, W) \longrightarrow \underline{map}(X, W)$$

  is a weak equivalence.

Recall, that for any objects X and Y in a model category $\mathcal{C}$, a homotopy function complex $\underline{map}(X, Y)$ is a simplicial set of maps in the homotopy category $Ho(\mathcal{C})$ between these objects. If a model category $\mathcal{C}$ is stable then we can rewrite the above definition using the graded set of maps in the homotopy category of $\mathcal{C}$ instead of homotopy function complexes (see [BR14, Remark 3.5]): $g : X \longrightarrow Y$ is an S-local equivalence if and only if the map

$$[g, W]_*^{\mathcal{C}} : [Y, W]_*^{\mathcal{C}} \longrightarrow [X, W]_*^{\mathcal{C}}$$

is an isomorphism of graded abelian groups for every S-local object W.

Now we are ready to give the definition of the left Bousfield localisation:

Definition 1.2.6. Suppose $\mathcal{C}$ is a model category and S is a class of maps in $\mathcal{C}$. The **left Bousfield localisation** of $\mathcal{C}$ with respect to S (if it exists) is a model structure on $\mathcal{C}$ (denoted by $L_S\mathcal{C}$) such that:

- weak equivalences in $L_S\mathcal{C}$ are S-local equivalences
- cofibrations in $L_S\mathcal{C}$ are the same as cofibrations in $\mathcal{C}$
- fibrations in $L_S\mathcal{C}$ are the maps with the right lifting property with respect to all maps which are both cofibrations and S- local equivalences.

We are not saying that for every class of maps S in $\mathcal{C}$ there exists a left Bousfield localisation of $\mathcal{C}$ with respect to S. However, if it exists, it is a left localisation of $\mathcal{C}$ with respect to S as in Definition 1.2.4. The conditions under which left Bousfield localisation exists are given in the following

Theorem 1.2.7. *[Hir03, Theorem 4.1.1]*
Suppose $\mathcal{C}$ is a left proper, cellular model category and S is a set of maps in $\mathcal{C}$. Then the left Bousfield localisation of $\mathcal{C}$ with respect to S exists.

We give some properties of the left Bousfield localisation:

Proposition 1.2.8. *[Hir03, Proposition 3.3.3]*
Suppose we have a model category $\mathcal{C}$ and a class of maps S in $\mathcal{C}$. Then if $L_S\mathcal{C}$ is the left Bousfield localisation of $\mathcal{C}$ with respect to S then:

- *every weak equivalence in $\mathcal{C}$ is a weak equivalence in $L_S\mathcal{C}$*
- *trivial fibrations are the same in $\mathcal{C}$ and $L_S\mathcal{C}$*
- *every fibration in $L_S\mathcal{C}$ is a fibration in $\mathcal{C}$*
- *every trivial cofibration in $\mathcal{C}$ is a trivial cofibration in $L_S\mathcal{C}$*

From the above Proposition it immediately follows that if the left Bousfield localisation $L_S\mathcal{C}$ exists then the identity functors on $\mathcal{C}$ form a Quillen pair:

$$Id_{\mathcal{C}} \; : \; \mathcal{C} \rightleftarrows L_S\mathcal{C} \; : \; Id_{\mathcal{C}}$$

Now we state a result about Quillen adjunction and equivalences passing to the left Bousfield localised categories:

Theorem 1.2.9. *[Hir03, Theorem 3.3.20] Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and let $F : \mathcal{C} \leftrightarrows \mathcal{D} : R$ be a Quillen pair. Suppose S is a set of maps in $\mathcal{C}$, $L_S\mathcal{C}$ is the left Bousfield localisation of $\mathcal{C}$ with respect to S and $L_{\mathbf{L}FS}\mathcal{D}$ is the left Bousfield localisation of $\mathcal{D}$ with respect to $\mathbf{L}FS$, where $\mathbf{L}FS$ is the image of S under the left derived functor $\mathbf{L}F$. Then*

1. *(F, R) is a Quillen pair when considered as functors between localised categories $F : L_S\mathcal{C} \leftrightarrows L_{\mathbf{L}FS}\mathcal{D} : R$, and*

2. *if (F, R) was a Quillen equivalence between $\mathcal{C}$ and $\mathcal{D}$ then it is also a Quillen equivalence when considered as functors between localisations $F : L_S\mathcal{C} \leftrightarrows L_{\mathbf{L}FS}\mathcal{D} : R$*

We start discussing the second construction- "cellularization". First we need some definitions.

Definition 1.2.10. [Hir03, Definition 3.1.8]
Let K be a collection of objects of a model category $\mathcal{C}$ called "cells". A K- equivalence is a map $f : X \longrightarrow Y$ such that for every $A \in K$

$$\underline{map}(A, X) \longrightarrow \underline{map}(A, Y)$$

is a weak equivalence of homotopy function complexes.

Theorem 1.2.11. *[Hir03, Theorem 5.1.1]*
Let $\mathcal{C}$ be a right proper, cellular model category. Then there is a model structure K-cell-$\mathcal{C}$ on the same category $\mathcal{C}$, with the same fibrations as in $\mathcal{C}$ and weak equivalences - the K-equivalences. Cofibrations are defined by the left lifting property.

This is a universal construction, in the sense of right localisation form of Definition 1.2.4.

We get the following result, which states that cellularization preserves Quillen pairs and in some cases induces Quillen equivalences.

Theorem 1.2.12. *[GS, Proposition A.6.]*
Suppose we have a Quillen pair $F\ :\ \mathcal{M} \rightleftarrows \mathcal{N}\ :\ U$ between right proper, cellular stable model categories. Let K be a set of (homotopically) compact, cofibrant objects of $\mathcal{M}$. Then we have a Quillen pair:

$$F\ :\ (K - cell - \mathcal{M}) \rightleftarrows (F(K) - cell - \mathcal{N})\ :\ U$$

If moreover $A \longrightarrow U\hat{f}_{\mathcal{N}}F(A)$ is a weak equivalence for each $A \in K$ then the above Quillen pair is a Quillen equivalence. Recall that $\hat{f}_{\mathcal{N}}$ denotes fibrant replacement functor in N.

If in the above theorem we start with a Quillen equivalence, then we obtain a Quillen equivalence without additional assumptions.

From this theorem follows a first attempt to transferring the model structure through the left adjoint functor:

Corollary 1.2.13. *[GS, Proposition A.6.]*
If J is a set of (homotopically) compact, fibrant objects of $\mathcal{N}$ such that $F\hat{c}_{\mathcal{M}}U(B) \longrightarrow B$ is a weak equivalence for every $B \in J$ then there is a Quillen equivalence:

$$F\ :\ (U(J) - cell - \mathcal{M}) \rightleftarrows (J - cell - \mathcal{N})\ :\ U$$

Recall that $\hat{c}_{\mathcal{M}}$ is a cofibrant replacement functor in $\mathcal{M}$.

At the end of this section we give a brief characterization of the category $Ho(K-cell-\mathcal{M})$. First we need the following definition:

Definition 1.2.14. An object $W \in \mathcal{M}$ is K**- cellular** if $\underline{map}(W,f)$ is a weak equivalence of simplicial sets for any K- equivalence f.

Following this notation $Ho(K - cell - \mathcal{M})$ is a full subcategory of $Ho(\mathcal{M})$ with objects: K- cellular objects.

1.3 Monoidal model categories

Very often one has two structures on one category $\mathcal{C}$: a model structure and a closed symmetric monoidal structure $(\otimes, I)$. If we want to obtain the induced monoidal structure on the homotopy category $Ho(\mathcal{C})$ with induced unit, we need to make sure that those two structures on category $\mathcal{C}$ are compatible, i.e. satisfy two additional axioms, often called "pushout-product axiom" and the "unit axiom". All definitions and results in this section are from [SS00] and [SS03a].

Pushout- product axiom

Let $f : A \to B$ and $g : K \to L$ be cofibrations in $\mathcal{C}$. Then the map

$$f\Box g : A \otimes L \cup_{A\otimes K} B \otimes K \to B \otimes L$$

is also a cofibration. If in addition one of the maps f, g is a weak equivalence, then so is the $f\Box g$.

Unit axiom

Let $q : \hat{c}I \to I$ be a cofibrant replacement of the unit object I. Then for every cofibrant object A, the morphism $q \otimes Id : \hat{c}I \otimes A \to I \otimes A \cong A$ is a weak equivalence.

Notice that the unit axiom is redundant if the unit is cofibrant.

Definition 1.3.1. A model category is a **monoidal model category** if it is equipped with closed symmetric monoidal product $(\otimes, I)$ and satisfies the pushout–product axiom and the unit axiom.

Having a monoidal model category $\mathcal{C}$ one might want to work with the category of monoids in $\mathcal{C}$ (often denoted Mon$(\mathcal{C})$), the category of $R-$modules (for R - a monoid in $\mathcal{C}$) or a category of $R-$algebras (for R - a commutative monoid in $\mathcal{C}$). This raises questions: whether one can obtain induced model structures on those new categories and what conditions should be satisfied to get these. It turns out we need some more properties from the category $\mathcal{C}$:

Definition 1.3.2. A monoidal model category $\mathcal{C}$ satisfies the **monoid axiom** if every map obtained by cobase change and possibly transfinite compositions from maps

$$\{\text{trivial cofibrations}\} \otimes C$$

is a weak equivalence.
If I is a class of maps and $\mathcal{C}$ is a monoidal category, then $I \otimes C$ denotes the class of maps of the form $A \otimes Z \to B \otimes Z$, where $A \to B$ is in I and Z is an object in $\mathcal{C}$.

Remark 1.3.3. If, in particular, $\mathcal{C}$ is a cofibrantly generated model category then the pushout-product axiom holds if it holds for the set of generating cofibrations and generating trivial cofibrations. Similarly, the monoid axiom holds if it holds for the set of generating trivial cofibrations.

The following theorem gives the desired model structures on the category of Mon($\mathcal{C}$), $R-$modules and $R-$algebras:

Theorem 1.3.4. *[SS00, Theorem 3.1]*
Let $\mathcal{C}$ be a cofibrantly generated monoidal model category satisfying the monoid axiom. Assume moreover that every object in $\mathcal{C}$ is small relative to the whole category. A morphism in Mon($\mathcal{C}$), $R-$modules and $R-$algebras is defined to be a fibration or a weak equivalence if it is a fibration or a weak equivalence in the underlying category $\mathcal{C}$. A morphism is a cofibration if it has a left lifting property with respect to all trivial fibrations.

1. *Let R be a monoid in $\mathcal{C}$, then the category of left $R-$modules is a cofibrantly generated model category.*
2. *Let R be a commutative monoid in $\mathcal{C}$, then the category of $R-$modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*
3. *Let R be a commutative monoid in $\mathcal{C}$, then the category of $R-$algebras is a cofibrantly generated model category.*

Notice, that point 3 in the theorem gives model structure on Mon($\mathcal{C}$) if we take R to be the unit of tensor product I.

We will often say that in the cases presented in the above theorem, "forgetful functors create model structures" for monoids, $R-$modules and $R-$algebras. This is an example of use of the transfer theorem (see Theorem 1.1.13).

Now that we obtained model structures on the categories of $R-$modules and $S-$modules we would like them to give equivalent homotopy categories for weakly equivalent monoids R and S.

Definition 1.3.5. [SS03a, Definition 3.11]
Let $(\mathcal{C}, \otimes, I_C)$ be a monoidal model category such that there is an induced model structure on the level of modules over any monoid. We say that **Quillen invariance of modules holds** for $\mathcal{C}$ if for every weak equivalence of monoids in $\mathcal{C}$, $f : R \to S$, restriction and extension of scalars along f induce a Quillen equivalence between the respective module categories:

$$- \wedge_R S \ : \ R - Mod \rightleftarrows S - Mod \ : \ f^*$$

The following Theorem gives an easy to check condition on when the Quillen invariance of modules holds for a category $\mathcal{C}$:

Theorem 1.3.6. *[SS00, theorem 3.3]*
Assume that for any cofibrant left R–module N, $- \otimes_R N$ takes weak equivalences of right R–modules to weak equivalences in $\mathcal{C}$. If $R \to S$ is a weak equivalence of monoids, then the restriction and extension of scalars functors give a Quillen equivalence.

Similar result is true also for R and S - algebras.

Having two monoidal model categories $\mathcal{C}$ and $\mathcal{D}$ and model structures on the levels of monoids, modules and algebras in those two categories we can ask for conditions on the functors to preserve these new model structures. As the model structures on the categories $\mathrm{Mon}(\mathcal{C})$, $R-$modules and $R-$algebras in $\mathcal{C}$ are defined in terms of the model structure on the underlying category $\mathcal{C}$ it is natural to ask for the conditions on functors between $\mathcal{C}$ and D (to obtain induced functors on the levels of monoids, modules and algebras).

For this purpose we need the notion of a weak and strong monoidal Quillen pair. A weak monoidal Quillen pair gives necessary conditions for lifting the Quillen pair to the categories of monoids and modules. Almost all of the following definitions and results are from [SS03a].

Definition 1.3.7. [ML98, Chapter 7.] A weak monoidal functor between monoidal categories $(\mathcal{C}, \wedge, I_{\mathcal{C}})$ and $(\mathcal{D}, \otimes, I_{\mathcal{D}})$ is a functor $R : \mathcal{C} \to \mathcal{D}$ together with a morphism $v : I_{\mathcal{D}} \to R(I_{\mathcal{C}})$ and a natural transformation

$$\phi_{X,Y} : RX \otimes RY \to R(X \wedge Y)$$

which are associative and unital. A weak monoidal functor is strong monoidal if v and all $\phi_{X,Y}$ are isomorphisms.

A notion of a weak monoidal functor is the weakest condition on a functor to preserve monoids and modules.

Definition 1.3.8. [SS03a, Definition 3.6]
A **weak monoidal Quillen pair** between monoidal model categories $\mathcal{C}$ and $\mathcal{D}$ is a Quillen adjoint pair $(\lambda : \mathcal{D} \rightleftarrows \mathcal{C} : R)$ with the right adjoint a weak monoidal functor and such that the following two conditions holds:

- for all cofibrant objects A, B in D the comonoidal map

  $$\widetilde{\phi} : \lambda(A \otimes B) \to \lambda A \wedge \lambda B$$

  is a weak equivalence in $\mathcal{C}$, where the comonoidal map is adjoint to the map:

  $$\phi_{\lambda A, \lambda B} \circ \eta_A \otimes \eta_B : A \otimes B \to R(\lambda A \wedge \lambda B)$$

- for some (hence any) cofibrant replacement $q : \hat{c}I_{\mathcal{D}} \to I_{\mathcal{D}}$ of the unit object in $\mathcal{D}$, the composite map

  $$\widetilde{v} \circ \lambda(q) : \lambda(\hat{c}I_{\mathcal{D}}) \to \lambda(I_{\mathcal{D}}) \to I_{\mathcal{C}}$$

  is a weak equivalence in $\mathcal{C}$.

A **strong monoidal Quillen pair** is a weak monoidal Quillen pair for which comonoidal maps $\widetilde{\phi}, \widetilde{v}$ are isomorphisms. Note, that if $I_{\mathcal{D}}$ is cofibrant and λ is a strong monoidal functor then (λ, R) is a strong monoidal Quillen pair.

A weak (respectively strong) monoidal Quillen pair is a weak (respectively strong) monoidal Quillen equivalence if the underlying Quillen pair is a Quillen equivalence.

A weak monoidal Quillen pair induces a strong monoidal adjunction on the level of homotopy categories.

Notice that in the definition of a weak monoidal Quillen pair the right adjoint is assumed to be weak monoidal to induce a functor on the level of monoids and modules. However, no such assumptions are put on the left adjoint, which suggest that on the level of monoids and modules the induced right adjoint will have different left adjoint functor, as we can't induce the left adjoint on this levels. First we state the theorem and then we explain different left adjoint functors used in it:

Theorem 1.3.9. *[SS03a, Theorem 3.12] Let $R : \mathcal{C} \to \mathcal{D}$ be the right adjoint of a weak monoidal Quillen equivalence $(\lambda,\ R)$. Suppose that the unit objects in $\mathcal{C}$ and $\mathcal{D}$ are cofibrant.*

1. *Consider a cofibrant monoid B in $\mathcal{D}$ such that the forgetful functors create model structures for modules over B and modules over $L^{mon}B$. Then the adjoint functor pair*

$$L_B \ : \ B - Mod \rightleftarrows (L^{mon}B) - Mod \ : \ R$$

   is a Quillen equivalence.

2. *Suppose that Quillen invariance of modules holds in $\mathcal{C}$ and $\mathcal{D}$. Then for any fibrant monoid A in $\mathcal{C}$ such that the forgetful functors create model structures for modules over A and modules over RA, the adjoint functor pair*

$$L^A \ : \ RA - Mod \rightleftarrows A - Mod \ : \ R$$

   is a Quillen equivalence. If the right adjoint R preserves weak equivalences between monoids and the forgetful functors create model structures for modules over any monoid, then this holds for any monoid A in $\mathcal{C}$.

3. *If the forgetful functors create model structures for monoids in $\mathcal{C}$ and $\mathcal{D}$, then the adjoint functor pair*

$$L^{mon} \ : \ Mon(\mathcal{D}) \rightleftarrows Mon(\mathcal{C}) \ : \ R$$

   is a Quillen equivalence.

Note that we used the same symbol R for the right adjoint on the level of categories $\mathcal{C}$ and D as well as for the functor on more structured levels.

The various left adjoints, which were used in the above theorem are explained in [SS03a, Section 3.3].

Remark 1.3.10. If R is a right adjoint of a strong monoidal Quillen pair then $L^{mon} = \lambda$, $L_B = \lambda$ and $L^A(-) = \lambda(-) \wedge_{\lambda(RA)} A$.

1.4 Generalized diagrams

First we need to recall some definitions (after Greenlees and Shipley see [GS13b]). We begin with a diagram $\mathcal{M} : \mathbb{D} \longrightarrow \mathrm{CAT}$. It means that for every $d \in \mathbb{D}$ we have a category $\mathcal{M}(d)$ and for every morphism $\alpha : d \longrightarrow d' \in \mathbb{D}$ we have a functor $\alpha_* : M(d) \longrightarrow M(d')$. We then consider a category of $\mathcal{M}-$ modules (generalised diagrams) denoted by $\mathcal{M}-$ mod. An object in this category consist of an object $X(d) \in M(d)$ for every $d \in \mathbb{D}$ together with a transitive system of morphisms $\tilde{X}(\alpha) : \alpha_* X(d) \longrightarrow X(d')$, where $\alpha : d \longrightarrow d' \in \mathbb{D}$.

If each α_* have a right adjoint α^* then the above system of morphisms is equivalent to the adjoint system of morphisms consisting of $\hat{X}(\alpha) : X(d) \longrightarrow \alpha^* X(d')$, where $\alpha : d \longrightarrow d' \in \mathbb{D}$. We call the first *a left adjoint form* and the second *a right adjoint form.*

We call $\mathcal{M}$ a diagram of model categories if each $\mathcal{M}(d)$ is a model category and every α_* has a right adjoint α^* and α_*, α^* form a Quillen pair.

We have the following result establishing an injective model structure on $\mathcal{M}$–modules under some assumptions, which appeared first in [HR08, Theorem 3.3.5].

Theorem 1.4.1. *Let $\mathcal{M} : \mathbb{D} \longrightarrow \mathrm{CAT}$ be a diagram of model categories and left Quillen functors, where $\mathbb{D}$ is an inverse category. Then there is a model structure on the category of $\mathcal{M}$–modules, called injective model structure, where*

- *F is a weak equivalences if $\forall_{d\in\mathbb{D}}$ $F(d)$ is a weak equivalence in $\mathcal{M}(d)$*
- *F is a cofibration if $\forall_{d\in\mathbb{D}}$ $F(d)$ is a cofibration in $\mathcal{M}(d)$*
- *F is a fibration if it satisfies the RLP with respect to all acyclic cofibrations.*

Further on we will restrict our attention to the category of generalised diagrams indexed by the following category:

$$\begin{tikzcd} a \arrow[r, "\alpha"] & b & c \arrow[l, "\beta"'] \end{tikzcd}$$

i.e. we consider a diagram of model categories and adjoint Quillen pairs $\mathcal{M}$ as follows (we draw only left adjoints):

$$\begin{tikzcd} \mathcal{M}(a) \arrow[r, "\alpha_*"] & \mathcal{M}(b) & \mathcal{M}(c) \arrow[l, "\beta_*"'] \end{tikzcd}$$

The category of $\mathcal{M}$-modules will consists of quintuples (X, f, Y, g, Z) where $X \in \mathcal{M}(a)$, $Y \in \mathcal{M}(B)$, $Z \in \mathcal{M}(c)$, $f : \alpha_*(X) \longrightarrow Y$ and $g : \alpha_*(Z) \longrightarrow Y$, where both morphisms are in $\mathcal{M}(b)$. For the diagram of categories $\mathcal{M}$ as above we have the following

Lemma 1.4.2. *[Bar13, Section 4]Suppose for all $d \in \mathbb{D}$, $\mathcal{M}(d)$ is a monoidal category and all left Quillen functors in the diagram $\mathcal{M}$ are strong monoidal. Then the category of $\mathcal{M}$–modules is monoidal, with the tensor product defined objectwise, i.e. $\mathcal{M} \otimes \mathcal{M}'(a) := \mathcal{M}(a) \otimes \mathcal{M}'(a)$. This monoidal structure is closed.*

Lemma 1.4.3. *[Bar13, Lemma 4.1.3 and Lemma 4.1.5] Suppose for all $d \in \mathbb{D}$, $\mathcal{M}(d)$ is a proper, cellular, monoidal model category satisfying the monoid axiom. Suppose further that all left Quillen functors in the diagram $\mathcal{M}$ are strong monoidal Quillen pairs. Then the category of $\mathcal{M}$–modules with the weak equivalences the objectwise weak equivalences and the cofibrations the objectwise cofibrations is proper, cofibrantly generated, cellular, monoidal model and it satisfies the monoid axiom.*

To establish an algebraic model for cyclic part of $SO(3)$ spectra we will work with Quillen adjunctions between $\mathcal{M}$–modules and $\mathcal{N}$–modules (indexed by the same category $\mathbb{D}$) by working with Quillen adjunctions for every $d \in \mathbb{D}$. We restrict attention to the case of the category $\mathbb{D}$ specified above and state the conditions that allow us to deduce an adjunction between categories of generalised diagrams (resp. equivalence) from adjunctions for all $d \in \mathbb{D}$.

Suppose we have a following commutative diagram

$$\begin{tikzcd}
\mathcal{M}(a) \arrow[r, "\alpha_*^{\mathcal{M}}"] \arrow[d, "L_a"'] & \mathcal{M}(b) \arrow[d, "L_b"'] & \mathcal{M}(c) \arrow[l, "\beta_*^{\mathcal{M}}"'] \arrow[d, "L_c"'] \\
\mathcal{N}(a) \arrow[r, "\alpha_*^{\mathcal{N}}"] & \mathcal{N}(b) & \mathcal{N}(c) \arrow[l, "\beta_*^{\mathcal{N}}"']
\end{tikzcd}$$

where every L_i has a right adjoint R_i. If $L_b\alpha_*^{\mathcal{M}}$ is naturally isomorphic to $\alpha_*^{\mathcal{N}}L_a$ and $L_b\beta_*^{\mathcal{M}}$ is naturally isomorphic to $\beta_*^{\mathcal{N}}L_c$ then the functor (L_a, L_b, L_c) is a left adjoint from $\mathcal{M}$-modules to $\mathcal{N}$-modules, with the right adjoint (R_a, R_b, R_c).

Remark 1.4.4. If we work with generalised diagrams of model categories with the injective model structure and we have an adjunction of generalised diagrams which is a Quillen adjunction for every $d \in \mathbb{D}$ then it is automatically a Quillen adjunction with respect to injective model structure (see Theorem 1.4.1). The same is true for Quillen equivalences.

Chapter 2

Spectra and G-spectra

There are many constructions of categories of spectra equipped with model structures, such that the homotopy category is equivalent to the usual stable homotopy category. However, for thirty years mathematicians were trying to construct a category of spectra equipped with a strictly associative monoidal product compatible with model structure so that its homotopy category would be equivalent to the usual stable homotopy category with the smash product known in algebraic topology.

There are several categories having this property, like the category of symmetric spectra defined in [HSS00] and discussed in details in [Sch] or the category of orthogonal spectra defined and described in [MM02]. The construction of both categories is similar. First we construct a diagram of spaces (OR simplicial sets) indexed by some fixed category. Then we define a tensor product on the category of diagrams and choose a monoid S. We define spectra to be S-modules. Depending on the indexing category we get symmetric spectra or orthogonal spectra.

We briefly discuss the first category in Section 2.1 together with a monoidal model structure. Section 2.2 is devoted to the category of G–orthogonal spectra, where we present a monoidal model structure. In Section 2.3 we recall the original Bousfield localisation of spectra. We finish this chapter with the splitting result of Barnes in Section 2.4.

2.1 Symmetric spectra

Our standard reference for this chapter is [HSS00]. First we define a symmetric sequence in simplicial sets as follows.

Definition 2.1.1. A symmetric sequence X in simplicial sets is a collection of based simplicial sets $\{X_n\}_{n\geq 0}$ with the left action of Σ_n on X_n preserving the base point.

Definition 2.1.2. A symmetric spectrum X is a a symmetric sequence such that for every $n \geq 0$ there exist a structure map which is a base point preserving map $S^1 \wedge X_n \longrightarrow X_{n+1}$ such that the map obtained by the composition of above: $S^p \wedge X_n \longrightarrow X_{n+p}$ is $\Sigma_p \times \Sigma_n$–equivariant, where Σ_p permutes the factors of p-fold smash product $S^p = (S^1)^{\wedge p}$.

Naturally, a map of symmetric spectra is a sequence of base point preserving Σ_n–equivariant maps which commute with the structure maps. We use the notation Sp^{Σ} for the category of symmetric spectra.

Notice that if we want to model the category of G–equivariant spectra, for a compact Lie group G, then symmetric spectra cannot be used. We can however model G–equivariant spectra with symmetric spectra when G is finite.

Now let us discuss the monoidal structure on symmetric spectra. There is a tensor product of symmetric sequences defined as follows for every X, Y- symmetric sequences:

$$(X \otimes Y)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)$$

There is a sphere spectrum $\mathbf{S}$ such that $\mathbf{S}_n = S^n$. Every S^n has an action of Σ_n by permutation of factors in n-fold smash product. Notice that this is a monoid for this tensor product and that every symmetric spectrum is a left $\mathbf{S}$–module. Therefore we can define a monoidal product of symmetric spectra (which would be tensoring over $\mathbf{S}$) as a coequalizer:

$$X \otimes \mathbf{S} \otimes Y \rightrightarrows X \otimes Y \to X \otimes_{\mathbf{S}} Y$$

This is a very similar construction to one done in algebra with tensoring over some commutative ring two modules over this ring.

We will call this new monoidal product a smash product of spectra. Obviously $\mathbf{S}$ is the unit for the smash product.

The category of symmetric spectra with the smash product has a compatible model structure, i.e. it is a monoidal model category (see Section 1.3). Before we define it, we need to recall some definitions:

Definition 2.1.3.
1. A spectrum X is an Ω–spectrum iff the adjoint of every structure map $S^1 \wedge X_n \longrightarrow X_{n+1}$ is a weak equivalence.
2. A spectrum X is injective if for any monomorphism and a level equivalence $g : A \longrightarrow B$ and a map $f : A \longrightarrow X$ there is a map $h : B \longrightarrow X$ such that $hg = f$.

The stable model structure on symmetric spectra is as follows. A map of spectra $f : X \longrightarrow Y$ is a

- weak equivalence iff for every injective Ω–spectrum A the induced map

  $$[f, A] : [Y, A] \longrightarrow [X, A]$$

  on homotopy classes of maps of spectra is an isomorphism

- cofibration iff it has a left lifting property with respect to maps g which have the property that every g_n is an acyclic fibration of simplicial sets with respect to Serre's model structure
- fibration iff it has the right lifting property with respect to all acyclic cofibrations

We called the above weak equivalences stable equivalences. Fibrant objects are those symmetric spectra X which are Ω–spectra.

For the later purposes we mention the Eilenberg-MacLane functor $H : Ab \longrightarrow \mathrm{Sp}^{\Sigma}$. This is a functor which associates to an abelian group A the spectrum HA with the property that for every $n \geq 0$, HA_n is an Eilenberg-MacLane simplicial set of type (A, n) (i.e. a geometric realization of HA_n is an Eilenberg-MacLane space of type (A, n)).

It is important that the functor H can be made into a weak monoidal functor with respect to the tensor product of abelian groups and the smash product of spectra. This implies that H maps monoids into monoids, i.e. rings to ring spectra.

2.2 Orthogonal G-spectra

In this section we give the description of the category of orthogonal G–spectra after Chapter II of [MM02] and list some of its properties.

For a given compact Lie group G we choose some finite dimensional irreducible real representations of G and take U to be a direct sum of countably many copies of all these representations. We will call U a universe. A universe U is **complete** if it contains countably many copies of every isomorphism class of finite dimensional real representation. A universe U is **trivial** if it contains only copies of the trivial representation. Now let us take $V \subset U$ to be a finite dimensional sub–G–real inner product space in U and call it an indexing G–space in U.

Now we define a category $\mathcal{I}_G^U$ with objects real inner G–product spaces isomorphic to indexing G–spaces in U and morphisms non-equivariant linear isometric isomorphisms with a G action by conjugation. We can define a $\mathcal{I}_G^U$–space as a functor (enriched over based G–spaces) from the category $\mathcal{I}_G^U$ to based G–spaces. A morphism between two such functors is a G–space enriched natural transformation. That defines a category of $\mathcal{I}_G^U$–spaces.

To construct a monoidal product on this category we need some more work. There is a notion of an external smash product on this category, so for two $\mathcal{I}_G^U$–spaces X, Y we have

$$X \bar{\wedge} Y : \mathcal{I}_G^U \times \mathcal{I}_G^U \longrightarrow G - Top_*$$

We can now use the left Kan extension of this functor to get an internal smash product on the category of $\mathcal{I}_G^U$–spaces. We take a commutative monoid S with respect to this internal smash product (where S_V is S^V) and consider left S–modules.

The G–orthogonal spectra are the left S–modules and we use the notation $G-\mathcal{IS}$ for this category. The monoidal product on $G-\mathcal{IS}$ is defined as a smash product over S.

Before we present a model structure on $G-\mathcal{IS}$, we need to define stable homotopy groups π_* of spectra. First, notice that every G–orthogonal spectrum has an underlying G–prespectrum.

Definition 2.2.1. For a G–prespectrum X and $H \leqslant G$ we define an H–stable homotopy group of X to be

$$\pi_q^H(X) := \mathrm{colim}_V \pi_q^H(\Omega^V(X(V))) \quad \text{if } q \geqslant 0$$

where V runs over indexing spaces in U, and

$$\pi_{-q}^H(X) := \mathrm{colim}_{V\supset\mathbb{R}^q} \pi_0^H(\Omega^{V-\mathbb{R}^q}(X(V))) \quad \text{if } q > 0$$

A map of G–orthogonal spectra is a π_*–isomorphism if the map of underlying G–prespectra is.

The following choices of maps are proven to give a model structure on orthogonal G–spectra in [MM02, Theorem 4.2.]. A map of orthogonal spectra $f : X \longrightarrow Y$ is a:

- weak equivalence iff it is a π_*–isomorphism (i.e. it is a π_*^H–isomorphism for all $H \leqslant G$)
- cofibration iff it has a left lifting property with respect to the level acyclic fibrations, where a level acyclic fibration $g : A \longrightarrow B$ is a map which is a Serre acyclic fibration for every indexing space V: $g_V : A_V \longrightarrow B_V$.
- fibration iff it has the right lifting property with respect to the acyclic cofibrations.

We call this the stable model structure on $G-\mathcal{IS}$.

The stable model structure on the category of G–orthogonal spectra is a monoidal model category as it is shown in [MM02], Chapter III, Proposition 7.4 and 7.5.

Later on we will mainly work with G–orthogonal spectra, so we need some properties of the stable model structure. We summarise it in the following

Theorem 2.2.2. *The stable model structure on the category of G–orthogonal spectra is cofibrantly generated, proper and cellular.*

Proof. First two properties are proven in [MM02, Section III, Theorem 4.2]. The fact this is cellular is mentioned in [BR14, Section 2]. □

Lemma 2.2.3. *For any subgroup H in G, any orthogonal spectrum X and integers $p \geq 0$ and $q > 0$*

$$[\Sigma^p S^0 \wedge G/H_+, X]^G \cong \pi_p^H(X)$$

$$[F_q S^0 \wedge G/H_+, X]^G \cong \pi_{-q}^H(X)$$

where the left hand sides denote morphisms in the homotopy category of $G-\mathcal{IS}$ and F_q- is the left adjoint to the evaluation functor at $\mathbb{R}^q$, $Ev_{\mathbb{R}^q}(X) = X(\mathbb{R}^q)$.

Proof. This follows from [MM02, Chapter III, Theorem 4.16] and [Ada74, Part III, Proposition 2.8] □

2.3 Localisation of spectra

In this section we present a special class of localisations of spectra, namely localisations at objects. Our basic category to work with is the category $G-\mathcal{IS}$ of G–orthogonal spectra. Localisation is our main tool to make the category of G–spectra easier, by firstly rationalising it using the localisation at an object $S_{\mathbb{Q}}$, which is a rational sphere spectrum. Then, in case of $SO(3)$–spectra we localise it further to capture different behaviour of cyclic, dihedral and exceptional part, see Part III.

We start with some notation, then we state the result that the localisation at a cofibrant object exists. We finish with several useful properties of localised categories which will be used in Part III. Our standard reference for this chapter is [MM02].

Definition 2.3.1. [MM02, Chapter IV, Definition 6.2] Suppose E is a cofibrant object in $G-\mathcal{IS}$ or a cofibrant based G–space. Let X, Y, Z be objects in $G-\mathcal{IS}$. Then

1. A map $f: X \longrightarrow Y$ is an E–equivalence if $Id_E \wedge f : E \wedge X \longrightarrow E \wedge Y$ is a weak equivalence.
2. Z is E–local if $[f, Z]^G : [Y, Z]^G \longrightarrow [X, Z]^G$ is an isomorphism for all E–equivalences f.
3. An E–localisation of X is an E–equivalence $\lambda : X \longrightarrow Y$, where Y is an E–local object.
4. X is E–acyclic if the map $* \longrightarrow X$ is an E–equivalence.

Notice that an E–equivalence between E–local objects is a weak equivalence by [Bar08, Lemma 2.1.2].

The following result is [MM02, Chapter IV, Theorem 6.3]

Theorem 2.3.2. *Suppose E is a cofibrant object in $G-\mathcal{IS}$ or a cofibrant based G–space. Then there exists a new model structure on the category $G-\mathcal{IS}$, where a map $f : X \longrightarrow Y$ is*

- *a weak equivalence if it is an E–equivalence*
- *cofibration if it is a cofibration with respect to the stable model structure*
- *fibration if it has the right lifting property with respect to all trivial cofibrations.*

The E–fibrant objects are the E–local objects and E–fibrant approximation gives a Bousfield localisation $\lambda : X \longrightarrow L_E X$ of X at E.

We use the notation $L_E(G-\mathcal{IS})$ for this category.

It is easy to see that identity functor $Id : G-\mathcal{IS} \longrightarrow L_E(G-\mathcal{IS})$ is a left Quillen functor.

Definition 2.3.3. We say that a localisation with respect to E is smashing if for every spectrum X the map $Id_X \wedge^L \lambda : X \longrightarrow X \wedge^L L_E S$ is an E–localisation. We use notation $X \wedge^L -$ for the left derived functor of $X \wedge -$.

Barnes and Roitzheim showed in [BR14, Lemma 3.14] that if the localisation with respect to E is smashing then $L_E(G-\mathcal{IS})$ is the left Bousfield localisation with respect to a particular set of maps S_E, where

$$S_E = \{\Sigma^n\lambda : S^n \longrightarrow L_E S^n | n \in \mathbf{Z}\}$$

Localised model categories have the following properties

Lemma 2.3.4. *The model category $L_E(G-\mathcal{IS})$ is left proper, cofibrantly generated, monoidal and it satisfies the monoid axiom. If localisation with respect to E is smashing then the model category $L_E(G-\mathcal{IS})$ is cellular and right proper.*

Proof. Everything except for right properness and cellular was proven in [Bar08, Section 2.1]. Right properness and cellular follows from Lemma 3.14 and Theorem 3.11 in [BR14]. □

Proposition 2.3.5. *Let X be a cofibrant object in $G-\mathcal{IS}$. Then $X \wedge -$ is a left Quillen functor.*

Proof. It follows from the pushout-product axiom for $G-\mathcal{IS}$ that $X \wedge -$ preserves cofibrations. It preserves E–equivalences by the associativity of the smash product. □

We have a useful, but quite obvious observation:

Proposition 2.3.6. *Suppose E, F are cofibrant objects in $G-\mathcal{IS}$. Then $L_E(L_F(G-\mathcal{IS}))$, $L_F(L_E(G-\mathcal{IS}))$ and $L_{E\wedge F}(G-\mathcal{IS})$ are equal as model categories (i.e. all three classes of maps are equal).*

Note that $L_E(L_F(G-\mathcal{IS}))$ above means the category of G orthogonal spectra first localised at F and then localised further at E.

The category of rational G–equivariant orthogonal spectra can be constructed as a localised model structure on $G-\mathcal{IS}$ with respect to the rational sphere spectrum $S_{\mathbb{Q}}$, which is cofibrant. The construction of $S_{\mathbb{Q}}$ is presented in [Bar08, Section 1.5].

Notice that Proposition 2.3.6 implies that $L_{eS}(G-\mathcal{IS}_{\mathbb{Q}})$ is the same as $L_{eS_{\mathbb{Q}}}(G-\mathcal{IS})$.

Remark 2.3.7. localisation of $G-\mathcal{IS}$ with respect to $S_{\mathbb{Q}}$ or with respect to eS, where e is an idempotent of the Burnside ring (see next section) is smashing.

Once we have a Quillen adjunction between model categories it is interesting to see what happens to it after localisation at some objects. We have the following result:

Lemma 2.3.8. *Suppose that $F : \mathcal{C} \rightleftarrows \mathcal{D} : R$ is a Quillen adjunction of model categories where the left adjoint is strong monoidal. Suppose further that E is a cofibrant object in $\mathcal{C}$. Then*

$$F \ : \ L_E\mathcal{C} \rightleftarrows L_{F(E)}\mathcal{D} \ : R$$

is a strong monoidal Quillen adjunction. Moreover if the original adjunction was a Quillen equivalence then the one induced on the level of localised categories is as well.

Proof. Since the localisation didn't change the cofibrations, the left adjoint F still preserves them. To show that it also preserves acyclic cofibrations, take an acyclic cofibration $f : X \longrightarrow Y$ in $L_E\mathcal{C}$. By definition $f \wedge Id_E$ is an acyclic cofibration in $\mathcal{C}$. Since F was a left Quillen functor before localisation $F(f \wedge Id_E)$ is an acyclic cofibration in $\mathcal{D}$. As F was strong monoidal we have $F(f \wedge Id_E) \cong F(f) \wedge Id_{F(E)}$, so $F(f)$ is an acyclic cofibration in $L_{F(E)}\mathcal{D}$ which finishes the proof of the first part.

To prove the second part we use Part 2 from Proposition 1.1.12. Since F is strong monoidal and the original adjunction was a Quillen equivalence F reflects $F(E)$–equivalences between cofibrant objects. It remains to check that the derived counit is an $F(E)$–equivalence. $F(E)$–fibrant objects are fibrant in $\mathcal{D}$ and the cofibrant replacement functor remains unchanged by localisation. Thus this follows from the fact that F, R was a Quillen equivalence. □

We will use this result in several cases for the following two adjoint pairs of G–orthogonal spectra. Notice that both left adjoint are strong monoidal, thus the result follow from the Lemma above.

Corollary 2.3.9. *Let $i : N \longrightarrow G$ denote the inclusion of a subgroup and let E be a cofibrant object in $G - \mathcal{IS}$. Then*

$$i^* \ : \ L_E(G - \mathcal{IS}) \ \underset{\longleftarrow}{\overset{\longrightarrow}{}} \ L_{i^*(E)}(N - \mathcal{IS}) \ : F_N(G_+, -)$$

is a strong monoidal Quillen pair.

Corollary 2.3.10. *Let $\epsilon : N \longrightarrow W$ denote the projection of groups, where H is normal in N and $W = N/H$. Let E be a cofibrant object in $W - \mathcal{IS}$. Then*

$$\epsilon^* \ : \ L_E(W - \mathcal{IS}) \ \underset{\longleftarrow}{\overset{\longrightarrow}{}} \ L_{\epsilon^*(E)}(N - \mathcal{IS}) \ : (-)^H$$

is a strong monoidal Quillen pair.

2.4 Splitting the category of rational G–spectra

In this section we present a very useful result due to Barnes which allows us to split the model category of $G - \mathcal{IS}$ into a finite product of simpler model categories. The idea is to use the idempotents of the Burnside ring to do that. We know that idempotents of the Burnside ring split the homotopy category of G–spectra. Barnes' result allows us to perform a compatible splitting on the level of model categories.

Barnes' result works for both Burnside ring and rational Burnside ring (actually his theorem is stated in more generality than that), however we will use it in the context of rational Burnside ring only. Therefore we start with a short reminder about the rational Burnside ring.

2.4.1 Rational Burnside ring

The rational Burnside ring for a compact Lie group G is denoted $A(G)_{\mathbb{Q}}$ and defined as follows:

$$A(G)_{\mathbb{Q}} := [S^{\text{o}}, S^{\text{o}}]^G \otimes \mathbb{Q}$$

where for two G-spaces X, Y, $[X, Y]^G$ is an abelian group of maps in a stable G category from a suspension spectrum of X to a suspension spectrum of Y.

Recall that $\text{Sub}(G)$ is a space of all closed subgroups of a group G with Hausdorff metric defining the topology. We can consider the subspace $F(G)$ of those subgroups H of G which have finite index in the normalizer $N_G(H)$. Both $\text{Sub}(G)$ and $F(G)$ are equipped with a G action by conjugation. Therefore we can form quotient spaces (orbit spaces) $\text{Sub}(G)/G$ and $F(G)/G$.

The following result is due to tom Dieck [tD79, 5.6.4, 5.9.13]:

Proposition 2.4.1. *Let $C(X, \mathbb{Q})$ denote the ring of continuous functions from a space X to a discrete space $\mathbb{Q}$. There is a map*

$$\alpha : [S^0, S^0]^G \otimes \mathbb{Q} \longrightarrow C(\text{Sub}(G)/G, \mathbb{Q})$$

where $\alpha(f)(H)$ is the degree of the H-fixed point map $f^H : S^0 \longrightarrow S^0$. Moreover, the composite of α and restriction is an isomorphism:

$$\alpha : [S^0, S^0]^G \otimes \mathbb{Q} \longrightarrow C(F(G)/G, \mathbb{Q})$$

The above Proposition allows us to understand idempotents of the rational Burnside ring. Note that any idempotent $e \in A(G)_{\mathbb{Q}}$ corresponds to an open and closed subspace of $F(G)/G$ defined as its support $S(e) := \{(K) \in F(G)/G \mid \alpha(e)(K) = 1\}$ and every open and closed subset $U \subseteq F(G)/G$ defines an idempotent, i.e. the characteristic function of U, $e_U := \mathbb{1}_U$.

Another useful viewpoint on the rational Burnside ring is through the space $\text{Sub}_{\text{f}}(G)/_{\sim}$ where $\text{Sub}_{\text{f}}(G)$ is equipped with the f-topology (discussed further in Section 5.2) where $\sim$ denotes the equivalence relation generated by the following : two subgroups $H \leq N \leq G$ are related $H \sim N$ if N/H is a torus. Then $\text{Sub}_{\text{f}}(G)/_{\sim} \cong F(G)$.

By the above, idempotents correspond to a G invariant open and closed subspaces of $\text{Sub}_{\text{f}}(G)/_{\sim}$.

This is particularly useful when we want to consider a map on idempotents of rational Burnside rings induced from an inclusion of subgroup in a group $i : H \longrightarrow G$. If e is an idempotent in $A(G)_{\mathbb{Q}}$ with support $S \subseteq \text{Sub}_{\text{f}}(G)$ then $i^*(e)$ is an idempotent in $A(H)_{\mathbb{Q}}$ defined to correspond to the subspace $i^*(S) = \{K \leq H | K \in S\} \subseteq \text{Sub}_{\text{f}}(H)$. As a map $F(H)/H \longrightarrow \mathbb{Q}$ it is non zero on these $(K)_H$ that $K \in S$.

2.4.2 Splitting

The category of rational G-equivariant spectra can be split using the idempotents of the Burnside ring $\mathcal{A}(G)_{\mathbb{Q}}$. The following result is [Bar08, Corollary 3.2.5]. We state it in terms of G–orthogonal spectra $G-\mathcal{IS}_{\mathbb{Q}}$.

Theorem 2.4.2. *Suppose $\{e_i\}_{i\in I}$ is a finite set of idempotents of the rational Burnside ring $A(G)_{\mathbb{Q}}$, giving an orthogonal decomposition $1 = \bigoplus_{i\in I} e_i$. Then the following adjunction, where the left adjoint is a diagonal functor and the right one is a product*

$$\triangle \ : \ G-\mathcal{IS}_{\mathbb{Q}} \rightleftarrows \prod\nolimits_{i\in I} L_{e_i S}(G-\mathcal{IS}_{\mathbb{Q}}) \ : \Pi$$

is a strong symmetric monoidal Quillen equivalence with respect to the product model structure (see below). In particular, for any G–orthogonal spectra X, Y we have the following natural isomorphism

$$[X,Y]^G_{\mathbb{Q}} \cong \bigoplus_{i\in I} [e_i X, e_i Y]^G_{\mathbb{Q}}$$

Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are monoidal model categories. Then the product model structure on $\mathcal{M}_1 \times \mathcal{M}_2$ is defined as follows: a map (f_1, f_2) is a weak equivalence, a fibration or a cofibration if it its every factor f_i is so. Monoidal structure is defined as a monoidal structure on every factor, i.e. $(X_1, X_2) \otimes (Y_1, Y_2) = (X_1 \otimes Y_1, X_2 \otimes Y_2)$. Moreover if both $\mathcal{M}_1, \mathcal{M}_2$ are proper, cofibrantly generated, monoidal, or satisfy the monoid axiom then so is (does) $\mathcal{M}_1 \times \mathcal{M}_2$.

Chapter 3

Enriched categories

In this chapter we recall some basic definitions of enriched category theory. The introduction to the enriched category theory is in [Kel05]. Quite soon we will concentrate on $\mathcal{V}$–model categories, i.e. $\mathcal{V}$–enriched categories with a model structure, where these two structures are compatible. We give proofs of several well-known results which are going to be useful in Part III. In the last Section we move to Morita equivalences in different settings.

Definition 3.0.1. A category $\mathcal{C}$ is enriched over a symmetric monoidal category $(\mathcal{V}, \otimes, I)$ if for every pair of objects X, Y in $\mathcal{C}$ there is a morphism $\mathcal{V}$–object $\mathcal{C}_{\mathcal{V}}(X, Y)$, for every object X in $\mathcal{C}$ there is a map in $\mathcal{V}$ from the unit I to $\mathcal{C}_{\mathcal{V}}(X, X)$ and for every triple X, Y, Z of objects in $\mathcal{C}$ there is an associative and unital composition map in $\mathcal{V}$

$$\mathcal{C}_{\mathcal{V}}(Y, Z) \wedge \mathcal{C}_{\mathcal{V}}(X, Y) \longrightarrow \mathcal{C}_{\mathcal{V}}(X, Z)$$

Definition 3.0.2. A $\mathcal{V}$–enriched functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between two $\mathcal{V}$–enriched categories is given by a map on objects, i.e. for every object X in $\mathcal{C}$, $F(X)$ is an object of $\mathcal{D}$, together with morphisms in $\mathcal{V}$ for all pairs of objects X, Y in $\mathcal{C}$,

$$\mathcal{C}_{\mathcal{V}}(X, Y) \longrightarrow \mathcal{D}_{\mathcal{V}}(F(X), F(Y))$$

which are coherently associative and unital maps.

Definition 3.0.3. A $\mathcal{V}$–enriched category $\mathcal{C}$ is tensored over $\mathcal{V}$ if there is a bifunctor

$$-\otimes =: \mathcal{V} \times \mathcal{C} \longrightarrow \mathcal{C}$$

with a natural isomorphism $\mathcal{C}_{\mathcal{V}}(V \otimes X, Y) \cong \mathcal{V}(V, \mathcal{C}_{\mathcal{V}}(X, Y))$ for all $X, Y \in \mathcal{C}$ and $V \in \mathcal{V}$.

A $\mathcal{V}$–enriched category $\mathcal{C}$ is cotensored over $\mathcal{V}$ if there is a bifunctor

$$(-)^{(=)} : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$$

with a natural isomorphism $\mathcal{C}_{\mathcal{V}}(X, Y^V) \cong \mathcal{V}(V, \mathcal{C}_{\mathcal{V}}(X, Y))$ for all $X, Y \in \mathcal{C}$ and $V \in \mathcal{V}$.

3.1 Properties

In this section we discuss the properties of an enriched category and we start by stating a result from [Dug06, Lemma A.7].

Theorem 3.1.1. *Suppose we have an adjunction $F : \mathcal{V} \rightleftarrows \mathcal{U} : R$ between closed symmetric monoidal categories such that the left adjoint F is strong monoidal. Then any tensored, cotensored and enriched $\mathcal{U}$-category $\mathcal{C}$ becomes canonically enriched, tensored and cotensored over $\mathcal{V}$.*

Proof. Suppose $\mathcal{C}$ is a tensored and cotensored $\mathcal{U}$–category, i.e.

$$\mathcal{C}_{\mathcal{U}}(u \otimes c, d) \cong \mathcal{U}(u, \mathcal{C}_{\mathcal{U}}(c, d)) \cong \mathcal{C}_{\mathcal{U}}(c, d^u)$$

We define tensor, cotensor and internal hom over $\mathcal{V}$ as follows:

$$\begin{gathered} - \star = : \mathcal{V} \times \mathcal{C} \longrightarrow \mathcal{C} \ \text{ by } \ v \star c := F(v) \otimes c \\ \{-, =\} : \mathcal{V}^{op} \times \mathcal{C} \longrightarrow \mathcal{C} \ \text{ by } \ \{v, c\} := c^{F(v)} \\ \mathcal{C}_{\mathcal{V}}(-, =) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{V} \ \text{ by } \ \mathcal{C}_{\mathcal{V}}(c, d) := R\mathcal{C}_{\mathcal{U}}(c, d) \end{gathered}$$

Then it's just a check that we have required natural isomorphisms. □

Whenever we have two structures on a category we would like them to be compatible. In Section 1.3 we discussed monoidal model categories, which is just an example of a model structure on a category enriched over itself, where the two structures are compatible.

We want to generalise this idea, when a category $\mathcal{C}$ is enriched over some monoidal model category $\mathcal{V}$. We restrict attention to $\mathcal{V}$ being the category of symmetric spectra Sp^{Σ}. The following definition introduces a spectral version of a simplicial model category.

Definition 3.1.2. [SS03b, Definition 3.5.1] A spectral model category is a model category $\mathcal{C}$ which is tensored, cotensored and enriched (denoted here by $\mathrm{Hom}_{\mathcal{C}}$) over the category of symmetric spectra such that the analog of Quillen's SM7 holds:
For every cofibration $A \longrightarrow B$ and every fibration $X \longrightarrow Y$ in $\mathcal{C}$ the induced map

$$\mathrm{Hom}_{\mathcal{C}}(B, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, X) \times_{\mathrm{Hom}_{\mathcal{C}}(A,Y)} \mathrm{Hom}_{\mathcal{C}}(B, Y)$$

is a stable fibration of symmetric spectra. If in addition one of the above maps is a weak equivalence then the induced map is also a stable equivalence.
$K \wedge X$ and X^K denote tensor and cotensor for X in $\mathcal{C}$ and K in Sp^{Σ}.

Note that we can phrase the above condition in an adjoint form using tensors: for every cofibration $f : A \longrightarrow B$ and every fibration $g : X \longrightarrow Y$ in $\mathcal{C}$ the induced map

$$f \square g : A \otimes Y \cup_{A \otimes X} B \otimes X \longrightarrow B \otimes Y$$

is a cofibration in $\mathcal{C}$. If in addition one of the above maps is a weak equivalence then the induced map is also a weak equivalence in $\mathcal{C}$.

Lemma 3.1.3. *[SS03b, Lemma 3.5.2] A spectral model category $\mathcal{C}$ is in particular a simplicial and stable model category. For X cofibrant and Y fibrant in $\mathcal{C}$ there is a natural isomorphism of graded abelian groups*

$$\pi_*^s \mathrm{Hom}_{\mathcal{C}}(X,Y) \cong [X,Y]_*^{\mathrm{Ho}(\mathcal{C})}$$

We have the following, well known fact

Theorem 3.1.4. *The category $G-\mathcal{IS}$ of G-orthogonal spectra (with stable model structure) is a Sp^Σ-model category.*

Proof. We use Theorem 3.1.1 to show that the category $G-\mathcal{IS}$ of G-orthogonal spectra may be enriched, tensored and cotensored over Sp^Σ-model category. The proof is by constructing a strong symmetric monoidal Quillen adjunction with left adjoint mapping from Sp^Σ to $G-\mathcal{IS}$. We do that by composing a series of strong symmetric monoidal Quillen adjunctions as follows.

Firstly we consider a Quillen equivalence on symmetric spectra in simplicial sets and topologial spaces induced by the geometric realisation and singular complex adjunction:

$$|-| \ : \ Sp^\Sigma \rightleftarrows Sp^\Sigma(\mathrm{Top}) \ : \mathrm{Sing}$$

Now we pass to orthogonal spectra $\mathcal{IS}^{\mathbb{R}^\infty}$ via the Quillen equivalence from [MMSS01, Theorem 10.4]. The notation $\mathbb{R}^\infty$ means that we use the universe $\mathbb{R}^\infty$. We will have to keep track of the universes when G is considered.

$$\mathbb{P} \ : \ Sp^\Sigma(\mathrm{Top}) \rightleftarrows \mathcal{IS}^{\mathbb{R}^\infty} \ : \mathbb{U}$$

The next step is to move to G orthogonal spectra indexed on the trivial universe $\mathbb{R}^\infty$ using the trivial action and G-fixed point adjunction, see [MM02, Chapter V, Section 3]

$$\epsilon^* \ : \ \mathcal{IS}^{\mathbb{R}^\infty} \rightleftarrows G-\mathcal{IS}^{\mathbb{R}^\infty} \ : \ (-)^G$$

Now we move to G orthogonal spectra indexed on a complete universe U. We use the notation i, where $i : \mathbb{R}^\infty \longrightarrow U$ is the inclusion, see [MM02, Chapter V, Proposition 3.4].

$$i_* \ : \ G-\mathcal{IS}^{\mathbb{R}^\infty} \rightleftarrows G-\mathcal{IS}^U \ : \ i^*$$

$G-\mathcal{IS}^U$ is a monoidal model category, all above left adjoints are strong monoidal so by Theorem 3.1.1 $G-\mathcal{IS}^U$ is enriched, tensored and cotensored over Sp_+^Σ.
What remains to be checked is the analog of Quillen's SM7. We use the tensor form of it and the notation F for the composite of all above left adjoints. Now suppose that $f : A \longrightarrow B$ is a (trivial) cofibration in Sp^Σ and $g : X \longrightarrow Y$ is a (trivial) cofibration in $G-\mathcal{IS}^U$. Then $f \square g := F(f) \square g$. As F is a left Quillen functor it preserves cofibrations and trivial cofibrations the result follows. □

Corollary 3.1.5. *Any localisation $L_E(G-\mathcal{IS})$ (if it exists) of G-orthogonal spectra is a Sp^Σ-model category.*

Proof. We use Theorem 3.1.4 and the strong symmetric monoidal Quillen adjunction:

$$Id \ : \ G-\mathcal{IS}^U \rightleftarrows L_E(G-\mathcal{IS}^U) \ : \ Id$$

The last step is to show that Quilen's SM7 holds. As cofibrations in $L_E(G-\mathcal{IS})$ and in $G-\mathcal{IS}^U$ are the same we just have to check the condition for $f : A \longrightarrow B$ a trivial cofibration in $G-\mathcal{IS}^U$ and $g : X \longrightarrow Y$ a (trivial) cofibration in $L_E(G-\mathcal{IS}^U)$. Recall that f is a weak equivalence in $L_E(G-\mathcal{IS}^U)$ iff $\mathrm{Id}_E \wedge f$ is a weak equivalence in $G-\mathcal{IS}^U$. $E \wedge -$ commutes with pushouts so $(\mathrm{Id}_E \wedge f)\Box g \cong \mathrm{Id}_E \wedge (f\Box g)$ and the last is a weak equivalence in $G-\mathcal{IS}^U$ by the pushout-product axiom, which means $f\Box g$ is a weak equivalence in $L_E(G-\mathcal{IS})$

□

3.2 Morita equivalences

Later we consider two cases when we apply Morita equivalences. The first one is in the algebraic setting, when categories are enriched over $Ch(\mathbb{Q})$ of chain complexes of $\mathbb{Q}$–modules and the second one is the topological setting when categories are enriched over symmetric spectra. First we develop the theory for the topological setting, however the algebraic one works analogously as discussed in [Bar09b, Section 4].

Schwede and Shipley proved in their paper [SS03b] that any spectral model category with a set of (homotopically) compact generators is Quillen equivalent to the category of modules over a ring with many objects. Before we are able to state the theorem we need some definitions. The following are taken from [SS03b].

In this section an adjective “spectra” means enriched over the category Sp^Σ of symmetric spectra. The following definition introduces the category of "modules over a ring with many objects".

Definition 3.2.1. Suppose $\mathcal{O}$ is a spectral category. A right $\mathcal{O}$–module M is a contravariant spectral functor from $\mathcal{O}$ to the category Sp^Σ of symmetric spectra, i.e. a symmetric spectrum $M(X)$ for every object X in $\mathcal{O}$ and for all pairs of objects X, Y in $\mathcal{O}$ coherently associative and unital maps of symmetric spectra

$$M(X) \wedge \mathcal{O}(Y, X) \longrightarrow M(Y)$$

A morphism of $\mathcal{O}$-modules $M \longrightarrow N$ consists of maps of symmetric spectra $M(X) \longrightarrow N(X)$ for all X in $\mathcal{O}$ strictly compatible with the action of $\mathcal{O}$. We use the notation $\mathrm{mod}-\mathcal{O}$ for the category of $\mathcal{O}$–modules. The representable module F_X is defined by $F_Y(X) := \mathcal{O}(X, Y)$

Notice that we already considered the category of enriched functors in the previous section and we denoted it $\mathrm{Fun}_{\mathcal{V}}(\mathcal{C}, \mathcal{V})$. According to that notation, $\mathcal{O}$–mod is the category $\mathrm{Fun}_{Sp^\Sigma}(\mathcal{O}^{op}, Sp^\Sigma)$.

The following theorem establishes the model structure on the category of $\mathcal{O}$-modules

Theorem 3.2.2. *[SS03b, Theorem A.1.1] Let $\mathcal{O}$ be a spectral model category. Then the category of $\mathcal{O}$-modules with the objectwise stable equivalences, objectwise stable fibrations and cofibrations defined by the lifting property is a cofibrantly generated spectral model category. Moreover the free modules $\{F_X\}_{X\in\mathcal{O}}$ form a set of (homotopically) compact generators for the homotopy category of $\mathcal{O}$-modules.*

Remark 3.2.3. If the category $\mathcal{O}$ satisfies the pushout-product or monoid axiom, then so does the category of $\mathcal{O}$–modules (see [Bar08, Theorem 5.3.9]) with respect to the model structure on $\mathcal{O}$–modules defined above.

Definition 3.2.4. A spectral functor $\Psi : \mathcal{O} \longrightarrow \mathcal{R}$ between spectral categories is a stable equivalence if it is a bijection on objects and for all objects $o, o' \in \mathcal{O}$ the map

$$\Psi_{o,o'} : \mathcal{O}(o,o') \longrightarrow \mathcal{R}(\Psi(o),\Psi(o'))$$

is a stable equivalence of symmetric spectra.

Theorem 3.2.5. *[SS03b, Theorem A.1.1] Let $\Psi : \mathcal{O} \longrightarrow \mathcal{R}$ be a stable equivalence of spectral categories. Then the restriction and extension of scalars along Ψ is a spectral Quillen equivalence of categories of modules. (Where restriction of scalars is given by precomposition with Ψ and extension of scalars is given by an enriched coend)*

Definition 3.2.6. [SS03b, Definition 3.9.1] Let $\mathcal{P}$ be a set of objects in a spectral model category $\mathcal{C}$. Let $\mathcal{E}(\mathcal{P})$ denote the full spectral subcategory of $\mathcal{C}$ on objects $\mathcal{P}$.
Define a functor $\mathrm{Hom}(\mathcal{P},-) : \mathcal{C} \longrightarrow \mathrm{mod}-\mathcal{E}(\mathcal{P})$ by $\mathrm{Hom}(\mathcal{P},Y)(P) := \mathrm{Hom}_{\mathcal{C}}(P,Y)$ and define a functor $-\wedge_{\mathcal{O}(\mathcal{P})}\mathcal{P} : \mathrm{mod}-\mathcal{E}(\mathcal{P}) \longrightarrow \mathcal{C}$ by an enriched coend formula, i.e.

$$X \wedge_{\mathcal{O}(\mathcal{P})} \mathcal{P} := \mathrm{coeq}[\bigvee_{P,P'\in\mathcal{P}} X(P') \wedge \mathcal{C}(P,P') \wedge P \rightrightarrows \bigvee_{P\in\mathcal{P}} X(P) \wedge P \ \ ]$$

Definition 3.2.7. [SS03b, Definition 3.9.2] A spectral Quillen pair between two spectral model categories $\mathcal{C}$ and $\mathcal{D}$ is a Quillen adjoint functor pair $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ together with a natural isomorphism of symmetric spectra

$$Hom_{\mathcal{C}}(A, RX) \cong Hom_{\mathcal{D}}(LA, X)$$

which on the vertices of the 0th level give the adjunction isomorphism.
A spectral Quillen pair is a spectral Quillen equivalence if the underlying Quillen functor pair is a Quillen equivalence.

There is a more general version of the theorem below (which doesn't assume that the category is a spectral model category), but we will only need the following

Theorem 3.2.8. *[SS03b, Theorem 3.9.3] Let $\mathcal{C}$ be a spectral model category and $\mathcal{P}$ a set of cofibrant and fibrant (homotopically) compact generators for $\mathcal{C}$. Then the pair of adjoint functors*

$$\mathrm{Hom}(\mathcal{P},-) \ : \ \mathcal{C} \ \underset{\longrightarrow}{\overset{\longleftarrow}{}} \ \mathrm{mod}-\mathcal{E}(\mathcal{P}) \ : \ -\wedge_{\mathcal{E}(\mathcal{P})}\mathcal{P}$$

is a spectral Quillen equivalence.

Remark 3.2.9. Theorem 3.2.8 is true for dg-model categories, i.e. model categories with the compatible enrichment in chain complexes of R–modules (see [Bar09b, Section 4]). The proof works similarly as in the spectral case, and the main point is that homotopy categories of both spectra and chain complexes are triangulated. In further chapters we will use both versions: spectral and dg.

Chapter 4

Sheaves

In this chapter we recall some basic properties of the category $Ch(\mathbb{Q})$ and the category of sheaves of $Ch(\mathbb{Q})$ over a space X. We will use them in Part II.

It is well known that the category $Ch(\mathbb{Q})$ is bicomplete. Moreover all limits and colimits are computed levelwise.

Definition 4.0.1. A differential graded algebra A is formal if it is quasi-isomorphic to its homology, regarded as a dga with trivial differential. A differential graded algebra B is said to be intrinsically formal if any other dga C with $H_*(B) \cong H_*(C)$ is quasi-isomorphic to B.

For example every algebra A in $Ch(\mathbb{Q})$ with $H_*(A)$ concentrated in degree zero is intrinsically formal. When c is in degree -2 then $\mathbb{Q}[c]$ with zero differentials is intrinsically formal.

Definition 4.0.2. A cocomplete abelian category $\mathcal{C}$ is called a Grothendieck category if it has a (categorical) generator P and filtered colimits are exact. A (categorical) generator for an abelian category is an object P, such that $\mathrm{Hom}(P, -)$ is a faithful functor.

In further chapters we will consider a topological group G and G–sheaves of $\mathbb{Q}$–modules over some G topological space (or category) X. As G has topology we should explain what we mean by continuous G action on $\mathbb{Q}$–modules. We give every $\mathbb{Q}$–module a discrete topology, therefore if G is not finite it acts through some finite quotient G/N, where N is a normal subgroup of finite index in G. We denote the category of $\mathbb{Q}$–modules with continuous G action by $\mathbb{Q}[G]$–modules.

4.1 Sheaves of $Ch(\mathbb{Q})$ over a space X

In this section we briefly recall the definition of a sheaf over a topological space and discuss an injective model structure on the category of chain complexes of sheaves. The standard reference for this section is [Bre97].

A sheaf is a preasheaf with some additional properties. There is an easiest definition of a sheaf when a codomain category has a forgetful functor to the category of $Sets$, but we chosen a more general approach:

Definition 4.1.1. Suppose C is a category with products. Then a sheaf F on a topological space X with values in the category C is a a contravariant functor from the category of open subsets of X with inclusions to C, satisfying the following "unique patching condition": for every open $U \subseteq X$ and for every cover $\{U_t\}_{t\in T}$, $\bigcup_{t\in T} U_t = U$, $F(U)$ is the equalizer:

$$F(U) \longrightarrow \prod_{t\in T} F(U_t) \rightrightarrows \prod_{(t,s)\in T\times T} F(U_t \cap U_s)$$

where two parallel morphisms are the unique extensions into product of the two maps (we present just one, in the other we use the projection to $F(U_s)$ and then restriction to $F(U_t \cap U_s)$): for every $t \in T$ projection on the t'th factor and then restriction:

$$\prod_{t\in T} F(U_t) \longrightarrow F(U_t) \longrightarrow F(U_t \cap U_s)$$

We have an important example of model structure on the category of chain complexes in an arbitrary Grothendieck category:

Theorem 4.1.2. *[Hov01, Theorem 2.2] Suppose A is a Grothendieck category. Then there exists a cofibrantly generated proper model structure on chain complexes in A: $Ch(A)$ where the cofibrations are injections, weak equivalences are quasi isomorphisms and the fibrations are those maps which have the right lifting property with respect to all injective weak equivalences. Moreover the homotopy category of this model structure is the derived category of A.*

We recall that $Shv(\mathbb{Q} - mod)/X$ is a Grothendieck category (see [Gro57]).

The category of sheaves of $\mathbb{Q}$–modules has some useful properties, which we now discuss.

Suppose we have a continuous map of topological spaces $f : X \longrightarrow Y$ and $Shv(\mathbb{Q})/X$ and $Shv(\mathbb{Q})/Y$ are categories of sheaves of $\mathbb{Q}$ modules over X and Y respectively. Then f induces an adjoint pair of functors:

$$f^* : \; Shv(\mathbb{Q})/Y \rightleftarrows Shv(\mathbb{Q})/X \; : \; f_*$$

We present the following easy observation, which follows from the étale definition of a sheaf

Proposition 4.1.3. *Pullback of a constant sheaf with discrete stalks is constant with the same stalks.*

A pullback functor along f commutes with colimits, as it is a left adjoint. Moreover it is also an exact functor, which means it commutes with all finite limits.

Lemma 4.1.4. *Suppose $f : X \to Y$ is a continuous map of topological spaces, $Shv(\mathbb{Q})/X$ and $Shv(\mathbb{Q})/Y$ are categories of sheaves of $\mathbb{Q}$ modules over X and Y respectively. Then $f^* : Shv(\mathbb{Q})/Y \longrightarrow Shv(\mathbb{Q})/X$ is a strong monoidal functor with respect to tensor product over constant sheaf $\mathbb{Q}_X$ and $\mathbb{Q}_Y$.*

Proof. Suppose $f : X \to Y$ is a continuous map of topological spaces, A, B are two $\mathbb{Q}$–module sheaves over Y, where $\mathbb{Q}$ is a constant sheaf over Y with every stalk $\mathbb{Q}$. First we show that we have a stalkwise isomorphism:

$$(f^*(A \otimes_{\mathbb{Q}} B))_x \cong (f^*(A) \otimes_{f^*\mathbb{Q}} f^*(B))_x$$

because

$$f^*(A \otimes_{\mathbb{Q}} B)_x \cong (A \otimes_{\mathbb{Q}} B)_{f(x)} \cong A_{f(x)} \otimes_{\mathbb{Q}_{f(x)}} B_{f(x)}$$

and

$$(f^*(A) \otimes_{f^*\mathbb{Q}} f^*(B))_x \cong (f^*(A))_x \otimes_{(f^*\mathbb{Q})_x} (f^*(B))_x \cong A_{f(x)} \otimes_{\mathbb{Q}_{f(x)}} B_{f(x)}$$

To get an isomorphism of sheaves we need a map of sheaves which will induce isomorphisms on stalks, i.e. we want a map

$$g : (f^*(A) \otimes_{f^*\mathbb{Q}} f^*(B)) \to (f^*(A \otimes_{\mathbb{Q}} B))$$

We will use here the *é*tale definition of sheaves. Notice that with this definition we want to obtain a map g into a pullback as in the diagram:

$$\begin{tikzcd}
(f^*(A) \otimes_{f^*\mathbb{Q}} f^*(B)) \arrow[dr, "\exists ! g"] \arrow[drr, bend left, "Id"] \arrow[ddr, bend right, "\overline{\pi}"'] & & \\
& (f^*(A \otimes_{\mathbb{Q}} B)) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A \otimes_{\mathbb{Q}} B \arrow[d, "\pi"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Where Id denotes a stalkwise identity map (but not an identity map of *é*tale spaces). To know that g as in the above diagram exists we need to show that Id is a continuous map, i.e. that the preimage of an open subset of *é*tale space $A \otimes_{\mathbb{Q}} B$ is open. We know that open subsets of *é*tale spaces are sections over an open set $U \subseteq Y$. Moreover it is enough to check this condition only for sections of the form $s_1 \otimes_{\mathbb{Q}} s_2 = \bigcup_{y \in U} s_{1_y} \otimes s_{2_y}$ over U because Id is a map of $\mathbb{Q}$–modules on stalks. The preimage of this section with respect to Id is $\bigcup_{x \in f^{-1}(U)} t_x$ where $t_x = s_{1_{f(x)}} \otimes s_{2_{f(x)}}$. We want to show this is a union of sections of $(f^*(A) \otimes_{f^*\mathbb{Q}} f^*(B))$ over open subsets of $f^{-1}(U)$. This will prove that the Id map is continuous and therefore there exists a map of sheaves $g : (f^*(A) \otimes_{f^*\mathbb{Q}} f^*(B)) \to (f^*(A \otimes_{\mathbb{Q}} B))$ which is a stalkwise isomorphism.

We will show that every germ t_x is locally represented by a section in the following way: $\forall_{x \in f^{-1}(U)}$ we have a section $\widetilde{t}_x$ over V_x–neighbourhood of x ($V_x \subseteq f^{-1}(U)$) such that $\forall_{p \in V_x}$ $(\widetilde{t}_x)_p = s_{1_{f(x)}} \otimes s_{2_{f(x)}}$.

Before we choose such local representation for every germ t_x, let us recall the inverse image functor on sheaves:

$f^*(A)(W) = \{s' | (s'_x)_{x \in W}$ where $s'_x \in A_{f(x)}$ such that $\forall_{x \in W}\ \exists_{U \ni f(x), V}\ x \in V \subset f^{-1}(U) \cap W$ and

$$s \in A(U) \text{ such that } s'_z = s_{f(z)} \ \forall_{z \in V}\}$$

By construction of the tensor product of sheaves and inverse image sheaf we can choose a neighbourhood W_1 of x such that there exists a section $\widetilde{s}_1$ over W_1 of $f^*(A)$ such that $\forall_{p \in W_1}$ $(\widetilde{s}_1)_p = s_{1_{f(p)}}$. Similarly we can get a section $\widetilde{s}_2$ of $f^*(B)$ over some W_2–neighbourhood of x. Now take $V_x = W_1 \cap W_2 \cap f^{-1}(U)$ and over this open neighbourhood of x we have a section $\widetilde{t}_x = [\widetilde{s}_1 \otimes \widetilde{s}_2]$ which satisfies the desired condition, i.e. $\forall_{p \in V_x}$ $(\widetilde{t}_x)_p = s_{1_{f(x)}} \otimes s_{2_{f(x)}}$. A union of these sections over $x \in f^{-1}(U)$ is the preimage of the section $s_1 \otimes_{\mathbb{Q}} s_2$ under the map Id, which finishes the proof.

□

Lemma 4.1.5. *Suppose $\mathcal{C}$ is an abelian category. Then $Ch(\mathcal{C})$ exists and if $\mathcal{C}$ was a Grothendieck category, so is $Ch(\mathcal{C})$.*

Proof. Suppose P is a generator for $\mathcal{C}$. Then the set of generators for $Ch(\mathcal{C})$ is defined to be a set of chain complexes $\{P_n\}_n \in Z$ where P_n is a chain complex which has P in degree n and $n+1$ with the identity differential and 0 everywhere else. Now a generator for $Ch(\mathcal{C})$ is the direct sum of all P_n's. Directed colimits are exact in $Ch(\mathcal{C})$, because they are exact in $\mathcal{C}$ and colimits and exactness is done levelwise in chain complexes. □

Corollary 4.1.6. *The category $Ch(Shv(\mathbb{Q})/X) \cong Shv(Ch(\mathbb{Q}))/X$ is a Grothendieck category.*

We have the following useful observations

Proposition 4.1.7. *[Hov01, Proposition 1.2] Every object in a Grothendieck category is small.*

At the end of this section we present this well-known fact, which will be used in Part II.

Proposition 4.1.8. *Open maps in* Top *are stable under pullbacks.*

Proof. Suppose we have a pullback diagram

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "q"] & Y \end{tikzcd}$$

where g is open. Since $A \subseteq B \times X$ the basis for topology on A is of the form $(U \times V) \cap A$ where $U \in B$ open and $V \in X$ open. To show that f is open it is enough to show that $f((U \times V) \cap A)$ is open. Since the above diagram is a pullback $(U \times V) \cap A = \{(u, v) \in U \times V | g(v) = q(u)\} = \{(u, v) \in U \times V | u \in q^{-1}g(v)\}$. This shows that $f((U \times V) \cap A) = U \cap q^{-1}g(V)$ and since g was open it is open. □

Part II

Sheaves

Chapter 5

Topological categories

In this chapter we first recall the notion of a topological category and then we give several examples. Next we proceed to G-topological categories, where G is any compact topological group. The most important family of examples for us is the one presented in Section 5.2. The introduction to internal categories is provided in [Bor94].

5.1 Definitions and examples

A topological category is an internal category in the category **Top** of topological spaces and continuous maps. We can also think about it as a simplicial object in **Top** truncated above level 2.

Definition 5.1.1. A topological category is a category C such that the set of objects C_0 is equipped with a topology, the set of morphisms C_1 is equipped with a topology and all four maps: domain (source) $s : C_1 \longrightarrow C_0$, codomain (target) $t : C_1 \longrightarrow C_0$, identity $s_0 : C_0 \longrightarrow C_1$ and composition $d_1 : C_1 \times_{C_0} C_1 \longrightarrow C_1$ are continuous with respect to this topologies. Notice that we have also projections $d_0 : C_1 \times_{C_0} C_1 \longrightarrow C_1$ and $d_2 : C_1 \times_{C_0} C_1 \longrightarrow C_1$ onto the second and first factor respectively.

The pullback above is formed over t and s respectively representing the set of pairs of composable arrows.

Moreover we require the categorical (simplicial) identities to hold, i.e.

- source and target of identity morphisms $s \circ s_0 = \mathrm{Id}_{C_0}$, $t \circ s_0 = \mathrm{Id}_{C_0}$,
- source and target of a composition $s \circ d_1 = s \circ d_2$, $t \circ d_1 = t \circ d_0$,
- associativity of the composition $d_1 \circ (\mathrm{Id} \times_{C_0} d_1) = d_1 \circ (d_1 \times_{C_0} \mathrm{Id})$
- unit laws for the composition $d_1 \circ (s_0 \times_{C_0} \mathrm{Id}) = d_2$, $d_1 \circ (\mathrm{Id} \times_{C_0} s_0) = d_0$

Two obvious examples of topological categories are the ones where part of the data is trivial

Example 5.1.2. A small category, considered as a topological category with discrete topology on the set of objects and the set of morphisms.

A special case of the above:

Example 5.1.3. Suppose C is a small groupoid, i.e. a category with all morphisms being isomorphisms. Then we can view it as a topological category with a discrete topology as above.

Example 5.1.4. A topological space X considered as a topological category $\mathbf{X}$ with the space of objects $X_0 = X$, the space of morphisms consisting only of identity maps on the objects, i.e. $X_1 = X$ with identity source, target and composition maps.

A more interesting family of examples comes from G–topological spaces, for G any compact Lie group (we can define it for G any topological group, but we will use it later on only for G compact Lie)

Example 5.1.5. Suppose X is a left G–topological space, i.e. there exists a continuous action map $G \times X \longrightarrow X$ which is associative and unital in the usual sense. Then we can form a topological category X_G as follows:

- the space of objects is X
- the space of morphisms is $G \times X$
- the identity map $s_0 : X \longrightarrow G \times X$ is given by $s_0(x) = (1, x)$ where 1 is the unit in G.
- the source map $s : G \times X \longrightarrow X$ is given by $s(g, x) = g^{-1}x$
- the target map $t : G \times X \longrightarrow X$ is given by $t(g, x) = x$, i.e. t is the projection on the second factor of the product.
- the composition map $d_1 : G \times G \times X \longrightarrow G \times X$ is given by $d_1(g_1, g_2, x) = (g_1 g_2, x)$

It is useful to think about a pair $(g, x) \in G \times X$ representing a morphism as about a point x and an arrow given by g with the codomain x.

Analogously, a G–topological category is an internal category in the category G-**Top** of G–topological spaces and G–equivariant, continuous maps.

A continuous functor between two topological categories $\mathbf{F} : \mathbf{X} \longrightarrow \mathbf{Y}$ consists of two continuous maps $F_0 : X_0 \longrightarrow Y_0$ and $F_1 : X_1 \longrightarrow Y_1$ such that all diagrams with source, targets, identity and composition maps commute.

Analogously, a continuous G–functor between two G–topological categories $\mathbf{F} : \mathbf{X} \longrightarrow \mathbf{Y}$ consists of two continuous G–maps $F_0 : X_0 \longrightarrow Y_0$ and $F_1 : X_1 \longrightarrow Y_1$ such that all diagrams with source, targets, identity and composition maps commute.

Notice that there is a forgetful functor from the category of topological categories to the category of topological spaces and it has both adjoints:

$$\begin{tikzcd}
\text{TopCat} \arrow[r, "U_0"] \arrow[r, leftarrow, bend left=50, "L"] \arrow[r, leftarrow, bend right=50, "R"'] & \text{Top}
\end{tikzcd}$$

where the left adjoint L is defined as follows: for any topological space X, $L(X) := \mathbf{X}$ where $\mathbf{X}$ is the topological category as in Example 5.1.4.

The right adjoint R is defined as follows: for a topological space X we put $R(X)$ to be a topological category with X as a space of objects and $X \times X$ as a space of morphisms with sourse and target maps being projections on the first and the second factor respectively. Composition is given by projection outside of the middle factor in trifold product $X \times X \times X \longrightarrow X \times X$. Identity map is the canonical map into the product.

These two functors present two universal ways of constructing a category on a set of objects: one is by viewing the set of objects as a discrete category with only identity maps and the other is by viewing it as a dense category with exactly one morphism between any two objects (which is then necessary an isomorphism).

5.2 Topological category of toral chains

The most important example of a topological category (or rather G–topological category) for us is a category of toral chains. The category of toral chains TC_G was defined in [Gre98a, Section 8] for any compact Lie group G as a subcategory of the category of all closed subgroups of G and inclusions (denoted by $S(G)$).

Definition 5.2.1. For a compact Lie group G the category TC_G of toral chains consists of:

- Objects: all closed subgroups of G
- Morphisms: one morphism $H \longrightarrow K$ whenever H is normal in K and K/H is a torus.

Identities exist in TC_G as one element group is a torus, but it is still not completely obvious that what we defined is a category, i.e. that there exist compositions. We get this by the following:

Proposition 5.2.2. *[Gre98a, Corollary 4.6]*
If H is normal in K and K/H is a torus and moreover K is normal in $\widehat{K}$ and $\widehat{K}/K$ is a torus then H is normal in $\widehat{K}$ and $\widehat{K}/H$ is a torus.

The category TC_G is equipped with topology on the space of objects called f- topology, which comes from the metric topology on the spaces $F(H)$ for all H closed subgroups in G. $F(H)$ denotes the subspace of all closed subgroups in H consisting of those subgroups K which have finite Weyl group $W_H(K) = N_H(K)/K$. This space inherits the topology from the space of all closed subgroups in H denoted by $S(H)$, which has topology given by the Hausdorff metric.

We need the following notations after [Gre98a] to give the description of generating open sets in f- topology.

For a closed subgroup H in G and $\epsilon > 0$ define a ball in $F(H)$:

$$O(H, \epsilon) = \{K \in F(H) \; | d(H, K) < \epsilon\}$$

Given also a neighbourhood A of the identity in G define a set:

$$O(H, A, \epsilon) = \bigcup_{a \in A} O(H, \epsilon)^a$$

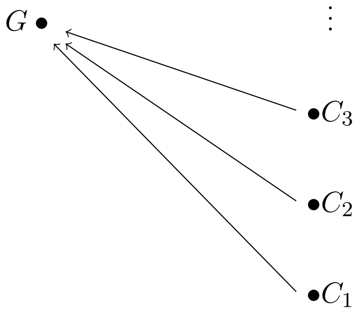
where $O(H, \epsilon)^a$ is the set of elements of $O(H, \epsilon)$ conjugated by a. We generate f- topology by sets $O(H, A, \epsilon)$ as H, A and ϵ vary.

We are ready to present some examples

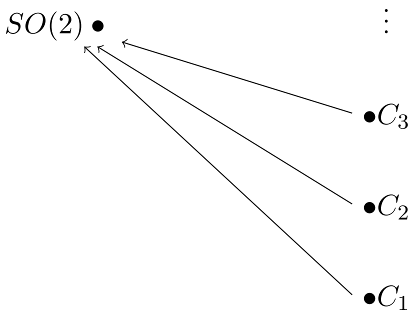
Example 5.2.3. Let G be a finite group. Then TC_G is a finite set of points with a discrete topology and only identity morphisms.

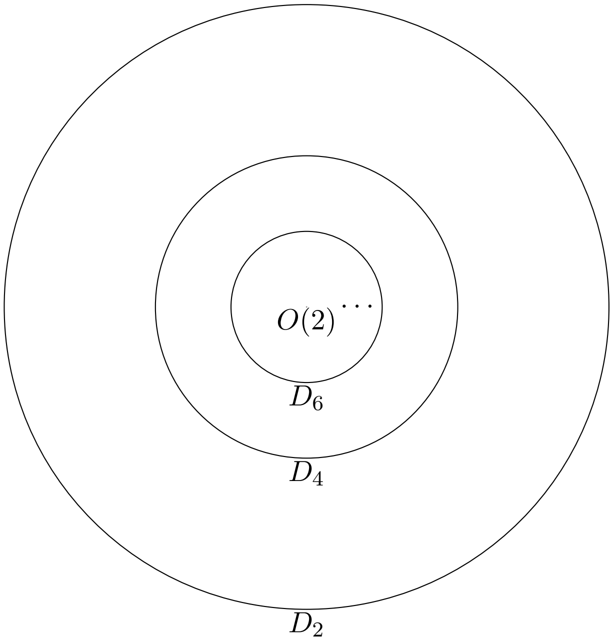
We present pictures illustrating examples of the categories TC_G for two more groups G:

Example 5.2.4. Let $G = SO(2)$. We get the following category TC_G:



Example 5.2.5. Let $G = O(2)$. We get the following category TC_G, where $D_{2n} \longrightarrow O(2)$:





Note that in the two given examples of TC_G above f–topology on the part related to $SO(2)$ is a discrete topology, but on the rest of $O(2)$ example the f–topology is as it is presented on the picture- the topology of a subspace of a plane with Euclidean metric. The picture presents infinitely many concentric circles with radiuses tending to 0 together with a central point.

We state some properties of f–topology which help to understand it better. In the following lemmas the subscript f was used to indicate that we consider the f–topology on the given space. Recall that the set of objects in TC_G and in $S(G)$ is the same, therefore we can consider the f–topology on $S(G)$.

Lemma 5.2.6. *[Gre98a, Lemma 8.6]*
TC_G with f-topology has the following properties:

1. *the action of G on TC_G by conjugation is continous*
2. *the inclusion map $F(H) \longrightarrow S_f(G)$ is continuous*
3. *the identity map $S_f(G) \longrightarrow S(G)$ is continuous*

Proposition 5.2.7. *Space of objects $S_f(G)$ inherits all separation properties (It is Hausdorff, regular, Tichonov, normal etc.) from the metric space $S(G)$.*

Proof. This follows from the continuity of the identity function in Part 3 of Lemma 5.2.6. □

Note that TC_G as a category, has at most one morphism between any two objects. This will make it easy to equip the space of morphisms with topology and check that a category defined that way and topology defined as f–topology are compatible giving a topological category.

Lemma 5.2.8. *[Gre98a, Lemma 8.7]*
Suppose objects of TC_G have f- topology and morphisms are topologized as a subspace of $S_f(G) \times S_f(G)$. Then TC_G is a topological category.

TC_G is a G–topological category as the source, target, identity and composition maps are G–equivariant with respect to the G action by conjugation.

At the end of this section we present a very useful observation

Lemma 5.2.9. *[Gre98a, Proposition 8.8]*
The source and target maps in TC_G are open maps.

The topological category TC_G has the following property

Lemma 5.2.10. *For every point H in the space of objects $S(G)$ there exists an open neighbourhood U_H of H with the property that $s^{-1}(U_H) \cap t^{-1}(U_H) = \{Id_K | K \in U_H\}$*

Proof. Set U_H to be $O(H, A, \epsilon)$ for $\epsilon > 0$ and $A = G$. Then suppose there is a non identity map in U_H, i.e. we have $K, L \in U_H$ such that $K \trianglelefteq L$ and L/K is a non-trivial torus. Then by definition of $O(H, A, \epsilon)$ $\exists g \in G$ such that $K^g \trianglelefteq L^g \leq H$. Moreover $N_H K^g / K^g$ is finite and L^g / K^g is infinite. Since $L^g / K^g \leq N_H K^g / K^g$ we get a contradiction. □

This tells us that around every point H there exists an open neighbourhood of H such that the only maps in the category with source and target in this neighbourhood are identity maps. So even though it is a categorically non-discrete topological category it is locally categorically discrete.

This property also implies that the inclusion of identity morphisms into the space of all morphisms is an open function. In other words, the subspace of the space of morphisms consisting of all identity morphisms is open.

When G is any torus we make the following observation about the topology of $S_f(G)$

Proposition 5.2.11. *Suppose G is a torus. Then $S(G)$ with f-topology is a discrete space.*

Proof. First, recall that f-topology is generated by open sets $O(H, A, \epsilon) = \bigcup_{a \in A} O(H, \epsilon)^a$, but since G is abelian, this is just $O(H, \epsilon)$.

Any closed subgroup of a torus is of the form $T \times F$ where T is a torus and F is a finite abelian group. Take $H = T \times F$. We want to show that if $K \leq H$, $K \neq H$ and H/K is finite then the Hausdorff distance between H and K is greater than 0. Since K is a subgroup of G it is of the form $T' \times F'$ and since H/K is finite $T' = T$. Since $K \neq H$ we have $F' \leq F$ and $F' \neq F$. Both F, F' are finite groups so $d(H, K) = d(F, F') > 0$. This shows that $S(G)$ is a discrete space and by Part 3 of Lemma 5.2.6, f topology for a torus is descrete.

□

Chapter 6

Sheaves over topological categories

Looking at contravariant functors on the category and sheaves over a topological space one might think about a generalisation of these two constructions viewed from the perspective of a topological category. This gives an idea of a sheaf over a topological category.

Roughly speaking, a sheaf F over a topological category C with values in a category D consists of the following data:

- A functor $F : C \longrightarrow D$
- A sheaf $\mathcal{F}$ on the topological space of objects of C denoted by C_0 such that for every object A in C the stalk at a point $A \in C_0$ is the same as the value of the functor F on A.

This states, that a sheaf over a topological category is a functor over it together with some continuity condition encoded in a sheaf structure. We give the precise definition using the language of sheaves:

Definition 6.0.1. A sheaf F over a topological category C is a sheaf F on the topological space of objects C_0 together with a map of sheaves over the topological space of morphisms C_1:

$$c : t^*F \longrightarrow s^*F$$

which satisfies the identity condition

$$s_0^*(c) = \mathrm{Id}$$

and the cocycle condition

$$d_1^*(c) = d_0^*(c)d_2^*(c)$$

(this codifies transitivity, i.e. $c(gf) = c(f)c(g)$ for two composable maps $f, g \in C_1$). In this definition we used the notation from Definition 5.1.1. We present the cocycle condition and the identity condition in a commutative diagrams below (where isomorphisms come from the categorical/simplicial identities):

$$\begin{tikzcd}
d_1^*t^*F \arrow[rrr, "d_1^*c"] \arrow[d, "\cong"] & & & d_1^*s^*F \arrow[d, "\cong"] \\
d_2^*t^*F \arrow[dr, "d_2^*c"'] & & & d_0^*s^*F \\
& d_2^*s^*F \arrow[r, "\cong"] & d_0^*t^*F \arrow[ur, "d_0^*c"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
s_0^*t^*F \arrow[r, "s_0^*c"] \arrow[d, "\cong"'] & s_0^*s^*F \arrow[d, "\cong"] \\
(ts_0)^*F \arrow[r, "\mathrm{Id}"] & (ss_0)^*F
\end{tikzcd}$$

If we have two sheaves over the same topological category we define a morphism between them as follows

Definition 6.0.2. Suppose we have two sheaves $(F, c), (H, \widetilde{c})$ over a topological category C. A morphism of sheaves $\nu : (F, c) \longrightarrow (H, \widetilde{c})$ consist of a morphism of sheaves over C_0: $\nu : F \longrightarrow H$ such that the following diagram of morphisms of sheaves over C_1 commutes:

$$\begin{tikzcd}
t^*F \arrow[r, "c"] \arrow[d, "t^*(\nu)"'] & s^*F \arrow[d, "s^*(\nu)"] \\
t^*H \arrow[r, "\widetilde{c}"] & s^*H
\end{tikzcd}$$

This gives a category of sheaves over a fixed topological category C.

The category of sheaves of sets over a topological groupoid $\mathbb{G}$ was studied for example in [Moe88] and [Moe90] (where he used a different, but equivalent definition to the one presented above and denoted the sheaves by étale $\mathbb{G}$-spaces). Another paper mentioning sheaves over topological categories is [Fri82].

6.1 Properties

Later on we restrict attention to the category of sheaves of $\mathbb{Q}$–modules over a topological category $\mathbf{X}$ and present formal properties of this category. But firstly, we make use of topos theory, i.e. we consider the category of sheaves of sets over a topological category $\mathbf{X}$, to deduce some of the properties for the category of sheaves of $\mathbb{Q}$–modules.

Firstly, we state an important

Theorem 6.1.1. *The category $Shv(\text{Sets})/\mathbf{X}$ of sheaves of sets over a topological category $\mathbf{X}$ is a Grothendieck topos.*

Proof. This is stated in [Moe90, Remark 1.6]. The proof follows the same pattern as the proof for sheaves of sets over a topological groupoid done in [Moe88, Section 2]. The idea is to show that small colimits in the bicategory of Grothendieck toposes exist and to present the category of sheaves of sets over a topological category as such a colimit. ☐

When s, t are open maps we can construct generators by hand following the method of [Moe90, Section 1.3], however we decided not to do that here, as we will only use the existence of the generators later.

Theorem 6.1.2. *Suppose $(\underline{\mathbb{Q}}, \mathrm{id})$ is a constant sheaf of sets over $\mathbf{X}$ (with the structure map being an identity) representing a commutative ring object (constant sheaf at $\mathbb{Q}$). The category of $(\underline{\mathbb{Q}}, \mathrm{id})$–modules in sheaves of sets over $\mathbf{X}$ is a Grothendieck category, i.e. it is an abelian category which has a (categorical) generator and where directed colimits are exact.*

Proof. This follows from [Joh77, Theorem 8.11 and remark before that], which says that a category of R modules in a Grothendieck topos (for R - a commutative ring object) is a Grothendieck category. ☐

Proposition 6.1.3. *The category of sheaves of $\mathbb{Q}$–modues over $\mathbf{X}$ is equivalent to the category of $(\underline{\mathbb{Q}}, \mathrm{id})$-modules in sheaves of sets over $\mathbf{X}$.*

Proof. The action map of $\underline{\mathbb{Q}}$ over an open set U is just and action of $\mathbb{Q}$ on sections over U. Since everything is compatible with restrictions this gives a sheaf of $\mathbb{Q}$ modules.

Since any pullback of a constant sheaf with discrete stalks is again constant with the same stalks, the action of the map id on the structure map requires this structure map to be a map of $\mathbb{Q}$-modules. ☐

From above theorem we deduce the following results

Corollary 6.1.4. *The category $Shv(\mathbb{Q} - mod)/\mathbf{X}$ of sheaves of $\mathbb{Q}$-modules over a topological category $\mathbf{X}$ is a Grothendieck category.*

Corollary 6.1.5. *The category $Shv(\mathbb{Q} - mod)/\mathbf{X}$ of sheaves of $\mathbb{Q}$-modules over a topological category $\mathbf{X}$ has all small colimits and all small limits.*

Proof. Every Grothendieck category is locally presentable by [Bek00, Proposition 3.10] and thus it is complete and cocomplete by [AR94, Remark 1.56]. ☐

Remark 6.1.6. If s, t, s_0 are open maps we can show directly that the category of sheaves of $\mathbb{Q}$ modules over the topological category $\mathbf{X}$ is a Grothendieck category. We can work with the definition to show that this category is abelian and that it has all limits and colimits. It is also a check that directed colimits are exact (since we can show that exactness is checked at the level of sheaves of $\mathbb{Q}$ modules over the topological space X_0). To obtain a generator we transfer it from the category of sheaves of sets over the topological category $\mathbf{X}$ via the left adjoint to the forgetful functor from $(\underline{\mathbb{Q}}, \mathrm{id})$-modules to Shv(Sets)/$\mathbf{X}$. (This is the free $(\underline{\mathbb{Q}}, id)$-module functor, and the transfer of a generator is a generator since the right adjoint is faithful).

Remark 6.1.7. For further references it is quite useful to note that all finite limits and all colimits are defined as underlying limits and colimits (respectively) in the category of sheaves over the space of objects equipped with the unique structure map which is also a finite limit or colimit of structure maps. This observation holds because to define structure maps we use pullback functors (s^* and t^*) and they are always exact and as left adjoints they commute with all colimits.

Remark 6.1.8. In case s, t, s_0 are open maps we present the construction of infinite products. They are defined as underlying infinite products of sheaves over topological space of objects, but the structure map is a bit more complicated.

Proposition 6.1.9. *Suppose* $\mathbf{X}$ *is a topological category with* s, t, s_0 *open maps. Then the infinite products exist and are defined as follows:*

$$\prod_i (A_i, \psi_i) := (\prod_i A_i, \widetilde{\prod_i \psi_i})$$

where $\widetilde{\prod_i \psi_i}$ *is the (unique) lift (dotted line) in the diagram:*

$$\begin{tikzcd}
t^*(\prod_i A_i) \arrow[r, dotted, "\widetilde{\prod_i \psi_i}"] \arrow[d, "!\beta"'] & s^*(\prod_i A_i) \arrow[d, "!\gamma"] \\
\prod_i t^* A_i \arrow[r, "\prod_i \psi_i"'] & \prod_i s^* A_i
\end{tikzcd}$$

where β, γ *are the unique maps into products induced by images of projections under functors* t^* *and* s^* *respectively.*

Proof. Firstly, we need to show that the lift in above diagram exists, that is:
1. γ is a monomorphism
2. $\mathrm{Im}(\prod_i \psi_i \circ \beta) \subseteq \mathrm{Im}(\gamma)$

Then we need to prove that the product constructed above:
3. has the universal property of a product.
4. satisfies cocycle condition and identity condition.

1. We want to show that

$$\gamma = \prod_i s^*(\pi_i) : s^*(\prod_i A_i) \longrightarrow \prod_i s^* A_i$$

is a monomorphism. Suppose there is an open set $U \subseteq X_0$ and two sections $p, r \in s^*(\prod_i A_i)(U)$ such that $\gamma(p) = \gamma(r)$. Section p comes from gluing compatible sections $p_j = (p_{ji})_i \in \prod_i A_i(s(U_j))$ for some open cover $\{U_j\}_j$ of U. Sections agree on the overlaps if they agree on the overlaps for all i. Analogously section r comes from gluing compatible sections $r_k = (r_{ki})_i \in \prod_i A_i(s(V_k))$ for some open cover $\{V_k\}_k$ of U.

Any map of sheaves is compatible with restrictions, i.e. $res_j(\gamma(p)) = ([p_{ji}])_i$ and $res_k(\gamma(r)) = ([r_{ki}])_i$.

For fixed i all sections $p_{ji} \in A_i(s(U_j))$ form a compatible family, so they give a section $([p_{*i}])_i \in \prod_i s^*A_i(U)$. Similarly, for fixed i all sections $r_{ki} \in A_i(s(U_k))$ form a compatible family, so they give a section $([r_{*i}])_i \in \prod_i s^*A_i(U)$ By unique gluing property for sheaves we have

$$([p_{*i}])_i = \gamma(p) = \gamma(r) = ([r_{*i}])_i$$

thus for all i $[p_{*i}] = [r_{*i}]$. That means for every i there exist a subcover of $\{U_j\}_j$ and $\{V_k\}_k$ call it $\{\widetilde{V_\alpha}\}_\alpha$ (depending on i) such that $p_{ji}|_{\widetilde{V_\alpha}} = r_{ki}|_{\widetilde{V_\alpha}}$. From the unique gluing property we get $[p_{ji}]|_{U_j \cap V_k} = [r_{ki}]|_{U_j \cap V_k}$ so we can choose subcover $\{\widetilde{V_\alpha}\}_\alpha$ to be non-empty intersections $U_j \cap V_k$. We can do it for all i and the subcover does not depend on i.

We get that $\{[(p_{ji})_i]\}_j$ forms a compatible family of sections which glues to give p where $[(p_{ji})_i] \in s^*(\prod_i A_i)(U_j)$ and similarly $\{[(r_{ki})_i]\}_k$ forms a compatible family of sections which glues to give r where $[(r_{ki})_i] \in s^*(\prod_i A_i)(V_k)$. We know those two compatible families are equal because their restrictions to the subcover $U_j \cap V_k$ are equal. Therefore $p = r \in s^*(\prod_i A_i)(U)$. That shows γ is a monomorphism of sheaves.

2.We will prove that $\mathrm{Im}(\prod_i \psi_i \circ \beta) \subseteq \mathrm{Im}(\gamma)$
From above result we know β is also a monomorphism. The image of γ will consist of all those sections of the product over U which come from gluing families of sections of A_i's compatible for the same cover of U for all i's. Similarly for the image of β.

Now suppose we have a section $([p_i])_i \in \prod_i t^*A_i(U)$ which is in the image of β, i.e. for all i it comes from gluing compatible family of sections $([p_{ji}])_i$ over U_j where $\{U_j\}_j$ is an open cover of U independent of i. All maps ψ_i are maps of sheaves, so they commute with restrictions:

$$\begin{tikzcd}
{[p_i]} \arrow[r, dotted] \arrow[d, "res_j"'] & \psi_i(U)([p_i]) \arrow[d, "res_j"] \\
p_{ji} \arrow[r] & \psi_i(U_j)(p_{ji})
\end{tikzcd}$$

where $p_{ji} \in A_i(sU_j)$. The family of sections $\{\psi_i(U_j)(p_{ji})\}_j$ is compatible because it is a restriction of a section. The cover $\{U_j\}_j$ does not depend on i, therefore the same argument works for all i. The family of sections $\{(\psi_i(U_j)(p_{ji}))_i\}_j$ is compatible and it glues back to the section $\prod_i \psi_i(([p_i])_i) = \prod_i \psi_i([p_i])$. This section is clearly in the image of γ, as it comes from gluing a family of sections of A_i's compatible for the same cover of U for all i's.

Now we have a lift, so that completes the construction of the product. We need to prove it has the universal property.

3. Suppose we have for every i a map $f_i : (Z, \psi_Z) \longrightarrow (A_i, \psi_i)$. Because product is defined as product of underlying sheaves over X_0 we get the unique map $f : Z \longrightarrow \prod_i A_i$ in the category of sheaves over X_0. It remains to show that this map commutes with structure maps, i.e. that the top square below commutes:

$$\begin{tikzcd}
t^*Z \arrow[rrr, "\psi_Z"] \arrow[dr, "t^*(f)"] \arrow[dd, "t^*(f_i)"'] \arrow[dddrr, bend left, "!\delta"] & & & s^*Z \arrow[dr, "s^*(f)"] \arrow[dd, "s^*(f_i)"'] \arrow[dddrr, bend left, "!\sigma"] & & \\
& t^*(\prod_i A_i) \arrow[rrr, "\widetilde{\prod_i \psi_i}"] \arrow[dl, "t^*(\pi_i)"] \arrow[ddr, "\beta"] & & & s^*(\prod_i A_i) \arrow[dl, "s^*(\pi_i)"] \arrow[ddr, "\gamma"] & \\
t^*A_i \arrow[rrr, "\psi_i"'] & & & s^*A_i & & \\
& & \prod_i t^*A_i \arrow[ull, "\widetilde{\pi}_i"] \arrow[rrr, "\prod_i \psi_i"] & & & \prod_i s^*A_i \arrow[ull, "\widetilde{\pi}_i"']
\end{tikzcd}$$

The top square commutes after postcomposition with γ. We know γ is a monomorphism, thus the top square commutes.

4.Now it remains to show that the map $\widetilde{\prod_i \psi_i}$ satisfies the cocycle and identity conditions if for all i maps ψ_i do.

We begin with the cocycle condition. Since for all i, ψ_i satisfies the cocycle condition, we can form the following commuting diagram:

$$\begin{tikzcd}
\prod_i d_1^*t^*A_i \arrow[rrr, "\prod_i d_1^*\psi_i"] \arrow[d, "\cong"] & & & \prod_i d_1^*s^*A_i \arrow[d, "\cong"] \\
\prod_i d_2^*t^*A_i \arrow[dr, "\prod_i d_2^*\psi_i"'] & & & \prod_i d_0^*s^*A_i \\
& \prod_i d_2^*s^*A_i \arrow[r, "\cong"] & \prod_i d_0^*t^*A_i \arrow[ur, "\prod_i d_0^*\psi_i"'] &
\end{tikzcd}$$

Now we relate the above to the diagram for $\prod_i A_i$ using unique maps into products, as follows:

$$\begin{tikzcd}
d_1^*t^*\prod_i A_i \arrow[rr, "d_1^*\widetilde{\prod_i \psi_i}"] \arrow[d] & & d_1^*s^*\prod_i A_i \arrow[d] & \\
d_1^*\prod_i t^*A_i \arrow[rr, "d_1^*\prod_i \psi_i"'] \arrow[dr] \arrow[dd] & & d_1^*\prod_i s^*A_i \arrow[dr] \arrow[dd] & \\
& d_1^*t^*A_i \arrow[rr, "d_1^*\psi_i"'] & & d_1^*s^*A_i \\
\prod_i d_1^*t^*A_i \arrow[rr, "\prod_i d_1^*\psi_i"] \arrow[ur] & & \prod_i d_1^*s^*A_i \arrow[ur, "!"] &
\end{tikzcd}$$

The front rectangle (the one relating $d_1^*t^*\prod_i A_i$, $d_1^*s^*\prod_i A_i$, $\prod_i d_1^*t^*A_i$ and $\prod_i d_1^*s^*A_i$) commutes after postcomposing with ! (for every i), so from the universal property of the product it commutes. This shows that the cocycle diagram for $\prod_i A_i$ commutes after postcomposing with a unique map

$$d_0^*s^*\prod_i A_i \longrightarrow \prod_i d_0^*s^*A_i$$

and since this map is a monomorphism (because both d_0, s are open maps), it commutes.

Now we proceed to show the identity condition. Since all ψ_i satisfy the identity condition we have the following commuting diagram

$$\begin{tikzcd}
\prod_i s_0^*t^*A_i \arrow[r, "\prod_i s_0^*\psi_i"] \arrow[d, "\cong"'] & \prod_i s_0^*s^*A_i \arrow[d, "\cong"] \\
\prod_i (ts_0)^*A_i \arrow[r, "\mathrm{Id}"] & \prod_i (ss_0)^*A_i
\end{tikzcd}$$

We relate this diagram with one for $\prod_i A_i$ as follows:

$$\begin{tikzcd}
s_0^*t^*\prod_i A_i \arrow[rr, "s_0^*\widetilde{\prod_i \psi_i}"] \arrow[d] & & s_0^*s^*\prod_i A_i \arrow[d] & \\
s_0^*\prod_i t^*A_i \arrow[rr, "s_0^*\prod_i \psi_i"'] \arrow[dd] \arrow[dr] & & s_0^*\prod_i s^*A_i \arrow[dd] \arrow[dr] & \\
 & s_0^*t^*A_i \arrow[rr, crossing over, "s_0^*\psi_i"'] & & s_0^*s^*A_i \\
\prod_i s_0^*t^*A_i \arrow[rr, "\prod_i s_0^*\psi_i"] \arrow[ur] & & \prod_i s_0^*s^*A_i \arrow[ur] &
\end{tikzcd}$$

Again, the same argument as before shows that the front rectangle commutes (the one relating $s_0^*t^*\prod_i A_i$, $s_0^*s^*\prod_i A_i$, $\prod_i s_0^*t^*A_i$ and $\prod_i s_0^*s^*A_i$), and thus the identity condition diagram commutes after postcomposing with the unique map

$$(ss_0)^*\prod_i A_i \longrightarrow \prod_i (ss_0)^*A_i$$

and since this map is a monomorphism (because both s_0, s are open maps), it commutes. This finishes the proof. □

From Remarks 6.1.7 and 6.1.8 we see that (under the assumption that s, t, s_0 are open maps) all limits and colimits are preserved by the forgetful functor to sheaves over the space of objects (which is forgetting the structure map)

$$U : Shv/\mathbf{X} \longrightarrow Shv/X_0$$

$$U(F, c) = F$$

This observation implies that U has both adjoints when viewed as a functor between sheaves of $\mathbb{Q}$ –modules and it follows from the Theorem below (Recall that a category is wellpowered if every object has a set of subobjects. The category $\mathcal{C}$ is well-copowered if $\mathcal{C}^{\mathrm{op}}$ is wellpowered).

Theorem 6.1.10. *Let $L : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between two Grothendieck categories. If L preserves colimits then it has a right adjoint. If in addition it preserves limits then it has a left adjoint.*

Proof. First part follows from the Freyd's special adjoint functor theorem (see [ML98, Chapter V, Section 8]) and the fact that every Grothendieck category is cocomplete, well-copowered and has a (categorical) generator. (The only non obvious point is about the category being well-copowered, but that follows from [AR94, Theorem 1.58]).

The second part follows from [AR94, Theorem 1.66]. □

When $\mathbf{X} = TC_G$, then by the property from Lemma 5.2.10 U is induced by an open inclusion of G–topological subcategory (with only identity morphisms) into G–topological category TC_G.

We can generalise this functor to get a restriction functor as follows: Suppose $\mathbf{X}$ is a topological category such that s, t are open maps. Then for every open set U in X_0 we have a restriction functor

$$(-)|_U : Shv(\mathbb{Q}-mod)/\mathbf{X} \to Shv(\mathbb{Q}-mod)/\mathbf{U}$$

where $\mathbf{U}$ denotes the full subcategory of $\mathbf{X}$ on the space of objects U. The space of morphisms in $\mathbf{U}$ is $U_1 = s^{-1}(U) \cap t^{-1}(U)$. Without confusion we can index this functors by open sets U in X_0.

The restriction functor is defined as follows: for any $V \subseteq U$ and any sheaf of $\mathbb{Q}$–modules (A, ψ_A) over $\mathbf{X}$

$$A|_U(V) := A(V)$$

and for any open set $W \subseteq U_1$ we define the structure map to be

$$\psi_A|_U(W) := \psi_A(W)$$

We can generalise this construction further. Suppose we have a continuous functor between topological categories: $\mathbf{F} : \mathbf{X} \longrightarrow \mathbf{Y}$. Then there is a pullback functor induced on the level of sheaves: $\mathbf{F}^* : Shv/\mathbf{Y} \longrightarrow Shv/\mathbf{X}$ defined as follows:

$$\mathbf{F}^*(A, \Psi_A) := (F_0^* A, F_1^*(\Psi_A) : t_Y^* F_0^* A \cong F_1^* t_X^* A \longrightarrow F_1^* s_X^* A \cong s_Y^* F_0^* A)$$

It is now an easy check that $\mathbf{F}^*$ preserves colimits since both F_0^* and F_1^* do and the respective structure maps are isomorphic by the same argument as in Lemma 6.1.15. Thus $\mathbf{F}^*$ has a right adjoint, which we call $\mathbf{F}_*$. In general $\mathbf{F}^*$ does not preserve infinite products, so it is not a right adjoint itself.

Recall that an open embedding $\mathbf{F}$ consists of a pair of continuous maps F_0, F_1, such that they induce homeomorphisms of X_0 with an open subspace of Y_0, and X_1 with an open subspace of Y_1. If $\mathbf{F} : \mathbf{X} \longrightarrow \mathbf{Y}$ is an open embedding then $\mathbf{F}^*$ has a left adjoint (and thus it has both adjoints) as both F_0^* and F_1^* have both adjoints (they are canonically naturally isomorphic to restriction functors, which have left adjoints - extensions by 0) and thus they both preserve all limits and colimits.

Remark 6.1.11. Every restriction $(-)|_U$ for U an open subset of X_0 is an example of a functor induced by an open inclusion and therefore it has both adjoints.

Now we can proceed to investigating chain complexes of $\mathbb{Q}$–modules.

Lemma 6.1.12. *There is an isomorphism of categories*

$$Ch(Shv(\mathbb{Q}-mod)/\mathbf{X}) \cong Shv(Ch(\mathbb{Q}-mod))/\mathbf{X}$$

that is the category of chain complexes in the category of sheaves of $\mathbb{Q}$*-modules over a topological category* $\mathbf{X}$ *is isomorphic to a category of sheaves of chain complexes of* $\mathbb{Q}$*-modules over the same topological category* $\mathbf{X}$*.*

Proof. We will construct two functors

$$G : Shv(Ch(\mathbb{Q}-mod))/\mathbf{X} \to Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$$

and

$$H : Ch(Shv(\mathbb{Q}-mod)/\mathbf{X}) \to Shv(Ch(\mathbb{Q}-mod))/\mathbf{X}$$

such that both compositions will give identities.

For every n there exists a functor $(-)_n : Ch(\mathbb{Q}-mod) \to \mathbb{Q}-mod$ which restricts to the n-th level of the chain complex.

Take $(B, \psi_B) \in Shv(Ch(\mathbb{Q}-mod))/\mathbf{X}$. For every open set $U \subseteq X_0$ we can apply the above functor to $B(U)$ getting $B(U)_n$. This gives a presheaf $B_n(U) := B(U)_n$. We need to check it is a sheaf. Recall that sheaf condition is in terms of products and equalizers, and as those are computed levelwise in $Ch(\mathbb{Q}-mod)$ and B was a sheaf we get a sheaf B_n. We define differentials as follows: d_n is a map of sheaves such that $d_n(U) = d_n$ from $B(U)$. ($B(U)$ is a chain complex, so we take its n-th differential). $d_i \circ d_{i+1} = 0$ as it equals zero on every open U.

It remains to show that every B_n can be equipped with a structure map obtained from ψ_B. For every open $V \subseteq X_1$ we can apply the $(-)_n$ functor to get the map $\psi_{B_n} : (t^*B)_n \to (s^*B)_n$. Moreover, as sheafification is done in terms of limits and colimits which are calculated levelwise in chain complexes we get the isomorphisms: $t^*(B_n) \cong (t^*B)_n$ and $s^*(B_n) \cong (s^*B)_n$. ψ_n composed with those isomorphisms is the required structure map for B_n.

It will satisfy the cocycle condition because ψ_B did and it will commute with d_i's because ψ_B did, so that d_i's become maps of sheaves over $\mathbf{X}$. We set $G((B, \psi_B)) := \{(B_n, \psi_{B_n}), d_n\}_n$

Any map $f : (B, \psi_B) \to (C, \psi_C)$ gives rise to a map of chain complexes, which on the level n is obtained by applying the functor $(-)_n$ for every open set of X_0: $f_n : B_n \to C_n$. It will commute with structure maps ψ_{B_n} and ψ_{C_n} because f commutes with ψ_B and ψ_C. Therefore we set $G(f) := \{f_n\}_n$.

We now construct the functor H as follows.
Take $\{(A_n, \psi_{A_n}), d_n\}_n \in Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$. We want to construct a sheaf of chain complexes. Let us put

$$A(U) := \qquad ... \xrightarrow{d_{n+1}(U)} A_n(U) \xrightarrow{d_n(U)} A_{n-1}(U) \xrightarrow{d_{n-1}(U)} ...$$

i.e. a new sheaf at open set $U \subseteq X_0$ will have the chain complex of (A_n, ψ_n) evaluated at U.

For every inclusion of open sets $V \subseteq U \subseteq X_0$ we get the restriction map - restriction of A_n on the level n. This map commutes with differential, because it did in the sheaf A. Moreover $d_i(U) \circ d_{i+1}(U) = d_i \circ d_{i+1}(U) = 0$, so we obtain a presheaf of chain complexes. It is a sheaf because every A_n was a sheaf and the sheaf condition is in terms of products and equalizers, which are done levelwise in chain complexes.

Now we need to equip the sheaf A with the structure map obtained from $\psi_{A_n} : t^*(A_n) \to s^*(A_n)$. Sheafification in chain complexes is done levelwise, so we have an isomorphism $t^*(A) \cong \{t^*(A_n)\}_n$, where $\{t^*(A_n)\}_n$ is a chain complex because $t^*(d_n) \circ t^*(d_{n+1}) = t^*(d_n \circ d_{n+1}) = t^*(0) = 0$. This allows us to put the structure map $\psi_A : t^*A \to s^*A$ over the open set $W \subseteq X_1$ to be $\psi(W)_n := \psi_{A_n}(W)$

Set $H(\{(A_n, \psi_{A_n}), d_n\}_n) := (A, \psi_A)$

Any map $\{f_n : (A_n, \psi_{A_n}) \to (B_n, \psi_{B_n})\}_n$ gives rise to a map $F : A \to B$ of sheaves of chain complexes, which for an open set $U \subseteq X_0$ is obtained by putting $F(U)_n = f_n(U) : A(U)_n \to B(U)_n$. It will commute with structure maps ψ_A and ψ_B because $\{f_n\}_n$ commutes with ψ_A and ψ_B. Therefore we set $H(\{f_n\}_n) := F$

It is easy to see that $G \circ H = \mathrm{Id}$ and $H \circ G = \mathrm{Id}$. ☐

We are interested in forming a monoidal model structure on the category $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$. We begin with the monoidal structure.

Lemma 6.1.13. *There is a symmetric monoidal structure on the category* $Shv(\mathbb{Q}-mod)/\mathbf{X}$ *of sheaves of* $\mathbb{Q}$*-modules over a topological category* $\mathbf{X}$ *constructed from a symmetric closed monoidal structure on the category of sheaves over topological space of objects* X_0 *of the topological category* $\mathbf{X}$.

Proof. We define the tensor product of two sheaves of $\mathbb{Q}$–modules over $\mathbf{X}$ as a tensor product of modules over a constant sheaf $(\underline{\mathbb{Q}}, \mathrm{id})$. This can be defined directly as follows:

$$(A, \psi_A) \otimes (B, \psi_B) := (A \otimes B, \psi_A \otimes \psi_B)$$

where, by abuse of notation we call $\psi_A \otimes \psi_B$ the following composite:

$$t^*(A \otimes B) \xrightarrow{\cong} t^*A \otimes t^*B \xrightarrow{\psi_A \otimes \psi_B} s^*A \otimes s^*B \xrightarrow{\cong} s^*(A \otimes B)$$

The two isomorphisms above are natural in both variables and follow from the fact that pullback functors are strong symmetric monoidal with respect to the tensor product of sheaves over X_0 defined as tensoring over the constant sheaf $\mathbb{Q}$ and with respect to the tensor product of sheaves over X_1 defined as tensoring over the constant sheaf $\mathbb{Q}$.

This tensor product is associative, because tensoring of sheaves over the constant sheaf $\mathbb{Q}$ is associative over X_0 and X_1. The unit is $(\underline{\mathbb{Q}}, \mathrm{id})$, i.e. a constant sheaf $\mathbb{Q}$ with identity as the structure map.

Moreover this tensor product is symmetric, i.e. for any two sheaves of $\mathbb{Q}$–modules over $\mathbf{X}$ $(A, \psi_A), (B, \psi_B))$ if $\gamma_{A,B} : A \otimes B \to B \otimes A$ is the symmetry isomorphism between the sheaves over X_0 then we need to show that this map commutes with structure maps ψ_A, ψ_B, but it's easy to see that it does stalkwise. ☐

Corollary 6.1.14. *There is a symmetric monoidal structure on the category* $Ch(Shv(\mathbb{Q} - mod)/\mathbf{X})$ *of chain complexes of sheaves of* $\mathbb{Q}$*-modules over a topological category* $\mathbf{X}$ *constructed from the one on the category* $Shv(\mathbb{Q} - mod)/\mathbf{X}$*.*

Proof. We extend the above tensor product to chain complexes in the usual way:

$$(X \otimes Y)_n = \oplus_{p+q=n} X_p \otimes Y_q$$

and the differential is given by the formula: $d(x \otimes y) = dx \otimes y + (-1)^p x \otimes dy$ where $p = |x|$. Unit is the unit for $Shv(\mathbb{Q} - mod)/\mathbf{X}$ concentrated in degree 0. Associativity follows from associativity for $Shv(\mathbb{Q} - mod)/\mathbf{X}$ and symmetry is induced by the symetry on $Shv(\mathbb{Q} - mod)/\mathbf{X}$ and a sign convention: $X \otimes Y \longrightarrow Y \otimes X$ where $x \otimes y \mapsto (-1)^{pq} y \times x$, $p = |x|, q = |y|$. ☐

Lemma 6.1.15. *Tensor product on* $Shv(\mathbb{Q}-mod)/\mathbf{X}$ *and on* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$ *commutes with colimits in both variables.*

Proof. Since the tensor product is symmetric, it is enough to show it commutes with colimits in the first variable. Thus we want to show that $((\mathrm{colim}_i A_i) \otimes B, (\mathrm{colim}_i \phi_i) \otimes \phi_B)$ is isomorphic to $(\mathrm{colim}_i (A_i \otimes B), \mathrm{colim}_i (\phi_i \otimes \phi_B))$. Since tensor of sheaves of $\mathbb{Q}$–modules over a space commutes with colimits, we see that $(\mathrm{colim}_i A_i) \otimes B \cong \mathrm{colim}_i (A_i \otimes B)$. It remains to show that the two structure maps are isomorphic. The first map is defined via commuting square

$$\begin{tikzcd}
t^*((\mathrm{colim}_i A_i) \otimes B) \arrow[r, dotted] \arrow[d, "\cong"] & s^*((\mathrm{colim}_i A_i) \otimes B) \arrow[d, "\cong"] \\
t^*(\mathrm{colim}_i A_i) \otimes t^*B \arrow[d, "\cong"] & s^*(\mathrm{colim}_i A_i) \otimes s^*B \arrow[d, "\cong"] \\
\mathrm{colim}_i t^*A_i \otimes t^*B \arrow[r, "(\mathrm{colim}_i \phi_i) \otimes \phi_B"] & \mathrm{colim}_i s^*A_i \otimes s^*B
\end{tikzcd}$$

The second map is defined via commuting square

$$\begin{tikzcd}
t^*(\mathrm{colim}_i(A_i \otimes B)) \arrow[r, dotted] \arrow[d, "\cong"] & s^*(\mathrm{colim}_i(A_i \otimes B)) \arrow[d, "\cong"] \\
\mathrm{colim}_i\, t^*(A_i \otimes B) \arrow[d, "\cong"] & \mathrm{colim}_i\, s^*(A_i \otimes B) \arrow[d, "\cong"] \\
\mathrm{colim}_i(t^*A_i \otimes t^*B) \arrow[r, "\mathrm{colim}_i(\phi_i \otimes \phi_B)"] & \mathrm{colim}_i(s^*A_i \otimes s^*B)
\end{tikzcd}$$

To show that these are isomorphic, first notice that we have the following commuting square

$$\begin{tikzcd}
\mathrm{colim}_i\, t^*A_i \otimes t^*B \arrow[r, "(\mathrm{colim}_i\, \phi_i) \otimes \phi_B"] \arrow[d, "\cong"] & \mathrm{colim}_i\, s^*A_i \otimes s^*B \arrow[d, "\cong"] \\
\mathrm{colim}_i(t^*A_i \otimes t^*B) \arrow[r, "\mathrm{colim}_i(\phi_i \otimes \phi_B)"] & \mathrm{colim}_i(s^*A_i \otimes s^*B)
\end{tikzcd}$$

since colimits commute with the tensor product of sheaves of $\mathbb{Q}$-modules over a topological space. It remains to show that the following diagram (and the respective one for s^*) commutes:

$$\begin{tikzcd}
\mathrm{colim}_i(t^*A_i \otimes t^*B) \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathrm{colim}_i\, t^*(A_i \otimes B) \arrow[d, "\cong"] \\
(\mathrm{colim}_i\, t^*A_i) \otimes t^*B \arrow[d, "\cong"] & t^*\,\mathrm{colim}_i(A_i \otimes B) \arrow[d, "\cong"] \\
t^*(\mathrm{colim}_i\, A_i) \otimes t^*B \arrow[r, "\cong"] & t^*((\mathrm{colim}_i\, A_i) \otimes B)
\end{tikzcd}$$

where maps are induced by the unique maps from colimits or come from t^* being strong monoidal or come from the fact that colimits commute with the tensor product of sheaves of $\mathbb{Q}$-modules over a topological space. Since these two paths are induced by the same map from every object $t^*A_i \otimes t^*B$, they are equal as the unique map from the colimit. □

Proposition 6.1.16. *The monoidal structures on $Shv(\mathbb{Q} - mod)/\mathbf{X}$ and on $Ch(Shv(\mathbb{Q} - mod)/\mathbf{X})$ are closed.*

Proof. By Lemma 6.1.15 both tensor products defined above commute with colimits of sheaves over $\mathbf{X}$ in both variables.

By Theorem 6.1.10 a functor which preserves colimits between Grothendieck categories has a right adjoint, thus internal hom functors exist and there is a usual adjunction giving a closed symmetric monoidal product. □

Lemma 6.1.17. *Suppose we have a continuous functor between two topological categories* $\mathbf{F} : \mathbf{X} \longrightarrow \mathbf{Y}$*. Then* F^* *is strong monoidal when viewed as a functor between categories of sheaves of* $\mathbb{Q}$*–modules over topological categories.*

Proof. We want to show that $\mathbf{F}^*(A \otimes B, \phi_A \otimes \phi_B) := (F_0^*(A \otimes B), F_1^*(\phi_A \otimes \phi_B))$ is isomorphic to $\mathbf{F}^*(A, \phi_A) \otimes \mathbf{F}^*(B, \phi_B) := (F_0^* A \otimes F_0^* B, F_1^* \phi_A \otimes F_1^* \phi_B)$. F_0^* and F_1^* are strong monoidal functors between sheaves over topological spaces, so $F_0^* A \otimes F_0^* B \cong F_0^*(A \otimes B)$. It remains to show that the two structure maps are isomorphic. This follows the same pattern as proof of Lemma 6.1.15, where the last diagram is replaced by the following commuting diagram

$$\begin{tikzcd}
F_1^* t^* A \otimes F_1^* t^* B \arrow[r, "\cong"] \arrow[d, "="] & F_1^* t^*(A \otimes B) \arrow[d, "="] \\
t^* F_0^* A \otimes t^* F_0^* B \arrow[r, "\cong"] & t^* F_0^*(A \otimes B)
\end{tikzcd}$$

Recall that $t^* F_0^* = F_1^* t^*$ since both functors are defined via pulling back using the same map $t \circ F_0 = F_1 \circ t$. ☐

6.2 Model structure

The following results allow us to give a description of a homotopy category of $Ch(Shv(\mathbb{Q} - mod)/\mathbf{X})$ with respect to weak equivalences being homology isomorphisms.

We establish a monoidal model structure on the category $Ch(Shv(\mathbb{Q} - mod)/\mathbf{X})$

Corollary 6.2.1. *Suppose* $\mathbf{X}$ *is a topological category, then there exist a proper, cofibrantly generated model structure on the category of* $Ch(Shv(\mathbb{Q} - mod)/\mathbf{X})$ *of chain complexes of sheaves of* $\mathbb{Q}$*-modules over a topological category* $\mathbf{X}$ *where*

- *the cofibrations are the injections,*
- *the weak equivalences are the homology isomorphisms and*
- *the fibrations are maps which have the right lifting property with respect to trivial cofibrations.*

Proof. This follows from the fact that $Shv(\mathbb{Q}-mod)/\mathbf{X}$ is a Grothendieck category (see Corollary 6.1.4) and from [Hov01, Theorem 2.2]. Properness is shown using a standard homological algebra argument, given for example in the proof of [CD09, Theorem 2.1].

☐

We will refer to the above model structure as the "injective" model structure.

Lemma 6.2.2. *The injective model structure on the category of* $Ch(Shv(\mathbb{Q} - mod)/\mathbf{X})$ *of chain complexes of sheaves of* $\mathbb{Q}$*-modules over a topological category* $\mathbf{X}$ *is a monoidal model structure, i.e. it satisfies the pushout-product axiom and the unit axiom.*

Proof. The pushout-product axiom follows from the fact that this model structure is exactly the same as the flat model structure in this category, i.e. a model structure given by a cotorsion pair (flat, cotorsion) which is the same as the injective cotorsion pair (all, injective), because everything is flat in $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$.

Recall that by [Gil06, Theorem 5.7] the flat model structure on the category of chain complexes of sheaves of $\mathbb{Q}$–modules over X_0 (X_1 respectively) is monoidal. Cofibrations, trivial cofibrations and tensor product are defined in the underlying category of chain complexes of sheaves of $\mathbb{Q}$–modules over X_0, so after forgetting to this category the pushout product axiom is satisfied. Pushouts are also preserved by the forgetful functor, thus we have the following situation:

Suppose $f_* : (A_*, \psi_{A_*}) \longrightarrow (B_*, \psi_{B_*})$ and $g_* : (C_*, \psi_{C_*}) \longrightarrow (D_*, \psi_{D_*})$ are cofibrations in $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$ then we have a diagram in $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$

$$\begin{tikzcd}
(A_*, \psi_{A_*}) \otimes (C_*, \psi_{C_*}) \arrow[rr, "f_* \otimes Id"] \arrow[dd, "Id \otimes g_*"] & & (B_*, \psi_{B_*}) \otimes (C_*, \psi_{C_*}) \arrow[dl] \arrow[dd, "Id \otimes g_*"] \\
& \bullet \arrow[dr, "h"] & \\
(A_*, \psi_{A_*}) \otimes (D_*, \psi_{D_*}) \arrow[ur] \arrow[rr, "f_* \otimes Id"] & & (B_*, \psi_{B_*}) \otimes (D_*, \psi_{D_*})
\end{tikzcd}$$

Where bullet denotes the pushout. After forgetting to the category $Ch(Shv(\mathbb{Q}-mod)/X_0)$ we get the following diagram:

$$\begin{tikzcd}
A_* \otimes C_* \arrow[rr, "f_* \otimes Id"] \arrow[dd, "Id \otimes g_*"] & & B_* \otimes C_* \arrow[dl] \arrow[dd, "Id \otimes g_*"] \\
& \bullet \arrow[dr, "h"] & \\
A_* \otimes D_* \arrow[ur] \arrow[rr, "f_* \otimes Id"] & & B_* \otimes D_*
\end{tikzcd}$$

where h is a cofibration because pushout product axiom holds in this category. It follows that h is a cofibration in $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$. Similar proof works for acyclic cofibrations. That shows the pushout product axiom is satisfied in $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$.

The unit axiom follows from the pushout-product axiom and the fact that every object in $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$ is cofibrant. ☐

Proposition 6.2.3. *The injective model structure on* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$ *is cellular.*

Proof. First we show that cofibrations are effective monomorphisms. By [Hir03, Proposition 10.9.4] it is enough to show that every cofibration is an equaliser of two parallel maps. Cofibrations are exactly monomorphisms and every monomorphism is a kernel of its cokernel, thus it is an equaliser of its cokernel and a parallel zero map.

If $\mathcal{A}$ is a Grothendieck category then $Ch(\mathcal{A})$ is as well and in every Grothendieck category all objects are small (see Proposition 4.1.7). ☐

Since we want to transfer the model structure on the category of modules over any monoid R and algebras over any commutative monoid R (in particular we will obtain the model structure on the category of monoids) we need to check that assumptions of [SS00, Theorem 3.1] are satisfied.

Proposition 6.2.4. *The monoid axiom is satisfied in* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$ *with the injective model structure.*

Proof. Since every object is cofibrant the monoid axiom follows from the pushout-product axiom. ☐

Now we are ready to transfer the model structure to R modules

Lemma 6.2.5. *There is a cofibrantly generated model structure on the category of left R–modules for any monoid R in* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$*, where the morphism is defined to be a fibration or a weak equivalence of left R–modules if it is a fibration or a weak equivalence in the underlying category* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$*.*

Proof. This is [SS00, Theorem 3.1 Part 1]. ☐

Lemma 6.2.6. *There is a cofibrantly generated, monoidal model structure on the category of R–modules for any commutative monoid R in* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$*, where the morphism is defined to be a fibration or a weak equivalence of left R–modules if it is a fibration or a weak equivalence in the underlying category* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$*. Moreover the monoid axiom is satisfied in the category of R–modules.*

Proof. This is [SS00, Theorem 3.1 Part 2]. ☐

Lemma 6.2.7. *Let R be a commutative monoid in* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$*. Then the category of R–algebras is a cofibrantly generated model category where a map is defined to be a fibration or a weak equivalence of R–algebras if it is a fibration or a weak equivalence in the underlying category* $Ch(Shv(\mathbb{Q}-mod)/\mathbf{X})$*.*

Proof. This is [SS00, Theorem 3.1 Part 3] ☐

Chapter 7

G-sheaves over G-topological spaces

For Chapter 7 and 8 we assume G is a topological group and the action of G is continuous.

In order to use some of the results from non–equivariant case, we show the equivalence of the usual definition of a G–sheaf of $\mathbb{Q}$-modules over a G–topological space X (Definition 7.0.1 below) and Definition 7.0.2 below.

Definition 7.0.1. A G–sheaf over a G–topological space X is a sheaf F over X together with a continuous action of G on the étale space $\overline{F}$ such that the projection map $\pi : \overline{F} \to X$ is G–equivariant. We call this an étale definition.

Definition 7.0.2. Suppose X is a G–topological space. Then we introduce the following continuous maps:

$$\begin{gathered}
e_0 : G \times G \times X \to G \times X \text{ where } e_0(g_1, g_2, x) = (g_2, g_1^{-1}x) \\
e_1 : G \times G \times X \to G \times X \text{ where } e_1(g_1, g_2, x) = (g_1 g_2, x) \\
e_2 : G \times G \times X \to G \times X \text{ where } e_2(g_1, g_2, x) = (g_1, x) \\
d_0 : G \times X \to X \text{ where } d_0(g, x) = g^{-1}x \\
d_1 : G \times X \to X \text{ where } d_1(g, x) = x \\
s_0 : X \to G \times X \text{ where } s_0(x) = (1, x) \text{ and 1 is the unit in } G.
\end{gathered}$$

A G–sheaf over a G–topological space X is a sheaf $\mathcal{E}$ together with an isomorphism of sheaves over $G \times X$:

$$\phi : d_1^*\mathcal{E} \longrightarrow d_0^*\mathcal{E}$$

satisfying the cocycle condition on $G \times G \times X$, i.e. $e_1^*\phi = e_0^*\phi \circ e_2^*\phi$ and the identity condition $s_0^*\phi = id_E$.

A G–equivariant morphism of G–sheaves $(\mathcal{E}, \phi)$, $(\mathcal{F}, \varphi)$ is a morphism $\alpha : \mathcal{E} \longrightarrow \mathcal{F}$ of sheaves such that $d_1^*\alpha \circ \phi = \varphi \circ d_0^*\alpha$.

Lemma 7.0.3. *Definitions 7.0.1 and 7.0.2 are equivalent.*

Proof. First, I will sketch the implication Definition 7.0.1 $\Longrightarrow$ Definition 7.0.2.

Suppose F is a G–equivariant sheaf on X with respect to Definition 7.0.1 and let $\pi : \overline{F} \longrightarrow X$ be the corresponding ètale space. Then d_0^*F is an ètale space $G \times \overline{F}$ with projection map $\pi_0 : G \times \overline{F} \longrightarrow G \times X$ such that $\pi_0(g, f) = (g, g\pi(f))$. Similarly, d_1^*F is an ètale space $G \times \overline{F}$ with projection map $\pi_1 : G \times \overline{F} \longrightarrow G \times X$ such that $\pi_1(g, f) = (g, \pi(f))$. Now the isomorphism of sheaves $\phi : d_1^*F \longrightarrow d_0^*F$ corresponds to an isomorphism of ètale spaces over $G \times X$, i.e. a homeomorphism which commutes with projection maps π_0, π_1. Let us define $\phi(g, f) := (g, g^{-1}f)$. This is clearly a homeomorphism and it commutes with projection maps. What remains to show is that it satisfies the identity and cocycle conditions, but this is a straightforward computation.

For the other implication:
Suppose we have a G–sheaf E in terms of Definition 7.0.2, i.e. we have a sheaf E over X together with an isomorphism $\psi : d_1^*E \to d_0^*E$ satisfying cocycle and identity conditions. This gives an étale space $\overline{E}$ with the projection $\pi : \overline{E} \to X$ and the isomorphism ψ gives an isomorphism of ètale spaces $\psi : G \times \overline{E} \to G \times \overline{E}$ over $G \times X$. Composing ψ with the second projection π_2 we get a map $\phi : G \times \overline{E} \to \overline{E}$ such that for all $(g, f) \in G \times \overline{E}$ we get $g^{-1}\pi(f) = \pi(\phi(g, f))$. Let us define $\widetilde{\phi}(g, f) = \phi(g^{-1}, f)$, then we get an "operation" of G on $\overline{E}$ such that π is G–equivariant. We need to check that this "operation" is a continuous action of G on $\overline{E}$, i.e $\forall f \in \overline{E}, g, h \in G$ we have $\widetilde{\phi}(1, f) = f$ and $\widetilde{\phi}(g, \widetilde{\phi}(h, f)) = \widetilde{\phi}(gh, f)$.

For all f in $\overline{E}$, $\widetilde{\phi}(1, f) = \phi(1, f) = \pi_2 \circ \psi(1, f) = \pi_2(1, f) = f$ as we know $\psi(1, f) = (1, f)$ from the identity condition $s_0^*\psi = id_E$ (note that s_0 is an inclusion, therefore s_0^* is a restriction of ψ to the $\{1\} \times \overline{E}$).

To show $\widetilde{\phi}(g, \widetilde{\phi}(h, f)) = \widetilde{\phi}(gh, f)$ we assume f is over x and we will recall what the cocycle condition means stalkwise:

$$e_1^*\psi_{(g,h,x)} = \psi_{(gh,x)}$$

$$e_0^*\psi_{(g,h,x)} = \psi_{(h,g^{-1}x)}$$

$$e_2^*\psi_{(g,h,x)} = \psi_{(g,x)}$$

$$e_1^*\psi_{(g,h,x)} = e_0^*\psi_{(g,h,x)} \circ e_2^*\psi_{(g,h,x)}$$

$$\psi_{(gh,x)} = \psi_{(h,g^{-1}x)} \circ \psi_{(g,x)}$$

We have the following equalities:

$$\widetilde{\phi}(g, \widetilde{\phi}(h, f)) = \phi(g^{-1}, \phi(h^{-1}, f)) = \pi_2 \circ \psi(g^{-1}, \pi_2 \circ \psi(h^{-1}, f)) =$$

$$\pi_2 \circ \psi(g^{-1}, \psi_{(h^{-1},x)}(f)) = \psi_{(g^{-1},hx)} \circ \psi_{(h^{-1},x)}(f)$$

and

$$\widetilde{\phi}(gh, f) = \phi(h^{-1}g^{-1}, f) = \pi_2 \circ \psi(h^{-1}g^{-1}, f) = \psi_{(h^{-1}g^{-1},x)}(f)$$

By cocycle condition they are equal, so we get an action of G on $\overline{E}$.

Continuity of this action follows from the fact that $\widetilde{\phi}$ is a composition of three continuous maps $\widetilde{\phi} = i \circ \pi_2 \circ \psi$ where $i : G \times \overline{E} \to G \times \overline{E}$, such that $i(g, f) = (g^{-1}, f)$.

□

Since Definitions 7.0.1 and 7.0.2 agree, we use the common terminology of G-sheaves over a G-topological space to denote these objects (we will not distinguish between étale and non-étale definition). Later on it will come useful to have an étale description as well as the formulation using topological categories.

Note, that both d_0, d_1 maps in Definition 7.0.2 are open and this definition is a particular case of the definition of a sheaf over a topological category with structure map being an isomorphism.

Next we present three obvious examples of categories of G–sheaves over G–topological spaces

Example 7.0.4. Suppose X is a topological space with a trivial G action. Then the category of sheaves over topological category X_G is equivalent to the category of sheaves over topological space X with G action on stalks. If G is finite and we consider sheaves of $\mathbb{Q}$ modules over X_G then it is equivalent to the category of sheaves of $\mathbb{Q}[G]$-modules over X.

Example 7.0.5. Suppose Y is a topological space with transitive G action. Then the category of sheaves over Y_G is equivalent to the category of sheaves over a point with an action of a stabiliser of a point $y \in Y$ on the stalk.

Example 7.0.6. Suppose Z is a topological space with a free G action. Then the category of sheaves over Z_G is equivalent to the category of sheaves over the orbit space Z/G.

This equivalence of definitions above allows us to deduce the following results for the category of G–sheaves of $\mathbb{Q}$–modules over a G–topological space X from analogous results for sheaves over a topological category:

Lemma 7.0.7. *For a topological group G the category $G - Shv(\mathbb{Q} - mod)/X$ of G–sheaves of $\mathbb{Q}$–modules over a G–topological space X is a Grothendieck category.*

Corollary 7.0.8. *For a topological group G the category $G - Shv(\mathbb{Q} - mod)/X$ of G–sheaves of $\mathbb{Q}$–modules over a G–topological space X has all small limits and colimits.*

Lemma 7.0.9. *For a topological group G there is an isomorphism of categories*

$$Ch(G - Shv(\mathbb{Q} - mod)/X) \cong G - Shv(Ch(\mathbb{Q} - mod))/X$$

that is the category of chain complexes in the category of G–sheaves of $\mathbb{Q}$-modules over a G–topological space X is isomorphic to a category of G–sheaves of chain complexes of $\mathbb{Q}$-modules over the same G–topological space X.

Now we mention restriction and extension by zero functors. It turns out that the restriction to any G–invariant open subset $U \subseteq X$ and extension by 0 from U to X functors form an adjoint pair.

Restriction of a G–sheaf B over X to U is defined as $B|_U(V) := B(V)$ for all open V in U. It is clearly a G–sheaf over U. Extension by zero from a G–sheaf A_U over U to X is defined as the sheafification of the following presheaf

$$E(A_U)(V) := \begin{cases} A(V), & \text{if } V \subseteq U \text{ open.} \\ 0, & \text{otherwise.} \end{cases}$$

It is clearly a G–sheaf over X and we use the notation $E(A_U)$ for it. We have a bijection natural in both variables

$$Hom_{G-Shv/X}(E(A_U), B) \cong Hom_{G-Shv/U}(A_U, B|_U)$$

defined in the same way as for non–equivariant sheaves (restriction of a map and extension of a map by 0).

Now we are ready to put a model structure on the category $Ch(G - Shv(\mathbb{Q} - mod)/X)$

Corollary 7.0.10. *Let G be a topological group. There exists a proper, stable, cofibrantly generated model structure on the category of $Ch(G - Shv(\mathbb{Q} - mod)/X)$ of chain complexes of G–sheaves of $\mathbb{Q}$-modules over a G–topological space X where the cofibrations are the injections, the weak equivalences are the homology isomorphisms and the fibrations are maps which have the right lifting property with respect to trivial cofibrations.*

Proof. This is a Grothendieck category so it follows from [Hov01, Theorem 2.2]. □

We need a notion of a tensor product on this category, so we define it as in the previous section and we get the following:

Lemma 7.0.11. *For any topological group G there is a closed symmetric monoidal structure on the category $G - Shv(\mathbb{Q} - mod)/X$ of G–sheaves of $\mathbb{Q}$-modules over a G–topological space X constructed from one on the category of sheaves over X.*

It seems that the condition for structure maps to be isomorphism is strong enough to allow us to prove the following directly:

Theorem 7.0.12. *The symmetric monoidal structure on G–sheaves of $\mathbb{Q}$-modules over a G–topological space X is closed.*

Proof. We can prove this theorem by the existence of the right adjoint. However, in this case we can give an explicit construction of the internal hom. To do this we use the equivalence of two definitions of G–sheaves of $\mathbb{Q}$-modules over a G–topological space X. We prove it while working with Definition 7.0.1.

The natural candidate for internal hom object from sheaf A over X to sheaf B over X is the usual internal hom $\underline{hom}_{Shv/X}(A,B)$: an ètale space $\overline{\underline{hom}_{Shv/X}(A,B)}$ together with the usual projection π to X and equipped with a continuous G action such that this projection is G–equivariant.

Recall that for $U \subseteq X$ open we have

$$\underline{hom}_{Shv/X}(A,B)(U) := Hom_{Shv/U}(A|_U, B|_U)$$

Let us define an action of G on $\overline{\underline{hom}_{Shv/X}(A,B)}$ by:

$$\psi : G \times \overline{\underline{hom}_{Shv/X}(A,B)} \longrightarrow \overline{\underline{hom}_{Shv/X}(A,B)}$$

$$\psi(g,f) = [g \circ f \circ g^{-1}, gU]$$

where f is a point over x in $\overline{\underline{hom}_{Shv/X}(A,B)}$, i.e. it is represented by a section over an open neighbourhood of x: $[f,U]$. This is well defined (by definition of germs), continuous action and the projection map is G–equivariant. It also extends to an action of G on the sections, as follows: for $f \in \underline{hom}_{Shv/X}(A,B)(U)$ we have $\psi(g,f) = g \circ f \circ g^{-1} \in \underline{hom}_{Shv/X}(A,B)(gU)$. That shows clearly that the action is continuous. We want to show that preimage of an open set is open. It's enough to check it for basic open sets: suppose we have $s \in \underline{hom}_{Shv/X}(A,B)(U)$ then $\psi^{-1}(s) = \cup_{g\in G}\{g\} \times gsg^{-1} \cong G \times s$ which is open.

It remains to show that $\overline{\underline{hom}_{Shv/X}(A,B)}$ together with the action ψ has the universal property (We denote the internal hom from A to B in the category of G–sheaves over X by $\underline{hom}_{G-Shv/X}(A,B)$):For all G–sheaves A, B, C over X we wan to show:

$$Hom_{G-Shv/X}(A \otimes B, C) \cong Hom_{G-Shv/X}(A, \underline{hom}_{G-Shv/X}(B,C))$$

In the underlying category of sheaves over X this is true. We need to show that the adjoint to a G–equivariant map of sheaves $\alpha : A \otimes B \longrightarrow C$ is again G–equivariant and that the adjoint to a G–equivariant map of sheaves $\beta : A \longrightarrow \underline{hom}_{G-Shv/X}(B,C)$ is again G–equivariant. To show that we will prove that the unit and counit of the adjunction on the level of sheaves over X are G–equivariant maps. First recall that the action of G on $A \otimes B$ is diagonal, i.e. $(g, a \otimes b) \mapsto ga \otimes gb$.

The unit map $\eta : A \longrightarrow \underline{hom}_{G-Shv/X}(B, A \otimes B)$ is defined as follows:

$$\eta(U)(a)(V) := b \otimes a|_V$$

where $a \in A(U)$ and $b \in B(V)$ and $V \subseteq U$. Moreover $\eta(gU)(ga)(gV) := gb \otimes ga|_{gV} = g(b \otimes a|_V)$, as g defines an isomorphism from sections of B over V to sections of B over gV.

Suppose now we have a G–equivariant map of sheaves $\alpha : A \otimes B \longrightarrow C$. Then the map $\underline{hom}_{G-Shv/X}(B, \alpha)$ is again G–equivariant by direct calculation: take $\gamma \in \underline{hom}_{G-Shv/X}(B, A\otimes$

$B)(U)$, then $\underline{hom}_{G-Shv/X}(B,\alpha)(\gamma) = \alpha(U)\circ\gamma$. Similarly $\psi(g,\gamma) \in \underline{hom}_{G-Shv/X}(B, A\otimes B)(gU)$ and $\underline{hom}_{G-Shv/X}(B,\alpha)(gU)(\psi(g,\gamma)) = \alpha(gU)\circ\psi(g,\gamma) = g\alpha g^{-1}g\gamma g^{-1} = g\alpha\circ\gamma g^{-1}$.

Therefore the adjoint of α denoted by $\widetilde{\alpha}$ is G–equivariant as it is the composition of G–equivariant maps:

$$\begin{tikzcd}
A \arrow[r, "\eta"] \arrow[dr, "\widetilde{\alpha}"'] & \underline{hom}_{G-Shv/X}(B, A\otimes B) \arrow[d, "\underline{hom}_{G-Shv/X}(B,\alpha)"] \\
& \underline{hom}_{G-Shv/X}(B,C)
\end{tikzcd}$$

The counit map $\varepsilon : \underline{hom}_{G-Shv/X}(B,C)\otimes B \longrightarrow C$ is defined as follows: for $\gamma(U) : B|_U \longrightarrow C|_U$ and $b \in B(U)$ $\varepsilon(\gamma,b) := \gamma(U)(b)$. We have $\varepsilon(g(\gamma(U),b)) = \varepsilon(g\gamma(U)g^{-1}, gb) = g\gamma(U)g^{-1}(gb) = g\gamma(U)(b)$, which shows that counit is G–equivariant.

Suppose we have a G–equivariant map of sheaves $\beta : A \longrightarrow \underline{hom}_{G-Shv/X}(B,C)$. Then the map $\beta\otimes Id_B$ is again G–equivariant by direct calculations. For $a\otimes b$ a germ of $A\otimes B$ over x we have $\beta\otimes Id_B(a\otimes b) = \beta(a)\otimes b$ and $\beta\otimes Id_B(g(a\otimes b)) = \beta\otimes Id_B(ga\otimes gb) = \beta(ga)\otimes gb = g\beta(a)\otimes gb = g(\beta(a)\otimes b)$.

Therefore the adjoint of β denoted by $\widetilde{\beta}$ is G–equivariant as it is the composition of G–equivariant maps:

$$\begin{tikzcd}
A\otimes B \arrow[r, "\beta\otimes Id_B"] \arrow[dr, "\widetilde{\beta}"'] & \underline{hom}_{G-Shv/X}(B,C)\otimes B \arrow[d, "\varepsilon"] \\
& C
\end{tikzcd}$$

That finishes the proof of the universal property of internal hom from B to C $\underline{hom}_{G-Shv/X}(B,C)$. The construction is natural in both variables. □

From two above results we get in a standard way the closed monoidal structure on the category of chain complexes of G–sheaves of $\mathbb{Q}$-modules over a G–topological space X:

Lemma 7.0.13. *There is a closed symmetric monoidal structure on the category of chain complexes G–sheaves of $\mathbb{Q}$-modules over a G–topological space X constructed from one on the category of G–sheaves of $\mathbb{Q}$-modules over a G–topological space X.*

The injective model structure is monoidal, in particular it satisfies the pushout-product, unit and monoid axioms and the proof is again the same as for the category of sheaves over the topological category. Similarly, the proof of transfer theorem to the category of monoids and modules for any monoid follows directly from work by Schwede and Shipley [SS00] as in Chapter 6.

Chapter 8

G-sheaves over G-topological categories

Let us recall that for G a topological group, a G–topological category is an internal category in the category of G-equivariant topological spaces, i.e. topological spaces with a continuous G action.

Definition 8.0.1. A G–sheaf S over a G–topological category C is a G–sheaf S over the G–space C_0 and a G–equivariant map of G–sheaves over C_1: $c : t^*S \longrightarrow s^*S$ which satisfies the identity condition

$$s_0^*(c) = \mathrm{Id}$$

and the cocycle condition:

$$d_1^*(c) = d_0^*(c)d_2^*(c)$$

In this definition we used the notation from Definition 5.1.1.

8.1 Properties

To deduce all the necessary properties of the category of G sheaves over a G topological category, we will use the fact that it is equivalent to the category of sheaves over a topological category built from the original one by encoding the G action into the space of morphisms (in a similar way as in Example 5.1.5 for G–topological spaces). This reduces the situation to one described in Chapter 6. Knowing that, we can deduce all the properties that were true for the category of sheaves over a topological category. The following construction is from [Moe90, Section 3].

Definition 8.1.1. Suppose C is a G–topological category. Then we construct a topological category C_G as follows. The space of objects of C_G is the same as for C, i.e. $C_{G0} = C_0$. The space of morphisms of C_G is constructed as the following pullback in the category of spaces (and not G spaces):

$$\begin{tikzcd}
(G \times C_0) \times_{C_0} C_1 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & G \times C_0 \arrow[d, "\mathrm{ac}"] \\
C_1 \arrow[r, "s"] & C_0
\end{tikzcd}$$

Where ac denotes the action map.

Now we describe source, target, identity and composition maps.

$$\begin{tikzcd}
(G \times C_0) \times_{C_0} C_1 \arrow[d, shift right, "t \circ \pi_2"'] \arrow[d, shift left, "\mathrm{proj} \circ \pi_1"] \\
C_0
\end{tikzcd}$$

Source map is defined to be $\mathrm{proj} \circ \pi_1$, where proj is the projection from the product $G \times C_0$ to the second factor. Target map is defined to be $t \circ \pi_2$. Identity map is defined using the identity map $C_0 \longrightarrow C_1$ and identity element for G: $C_0 \longrightarrow G \times C_0$.

Now we define the composition map on elements:

$$((G \times C_0) \times_{C_0} C_1) \times_{C_0} ((G \times C_0) \times_{C_0} C_1) \longrightarrow ((G \times C_0) \times_{C_0} C_1)$$

$$((g, x), f), ((h, f(g(x))), \tilde{f}) \mapsto ((h \circ g, x), \tilde{f} \circ h(f))$$

The intuitive way of thinking about maps in C_G is via ordered pairs: an element of a group and a map from C_1. Since an element of a group doesn't have a specified domain and codomain, but it acts on the whole space of objects C_0 we have to encode a point at which we consider this element of the group (as a morphism).

The following picture presents the map $((g, x), f) \in C_{G1}$

$$\begin{tikzcd}
x \arrow[r, "{(g,x)}"] & g(x) \arrow[r, "f"] & f(g(x))
\end{tikzcd}$$

The composition of two maps $((h, f(g(x))), \tilde{f}) \circ ((g, x), f) \in C_{G1}$ is shown below. Notice that

to obtain the composition we need to use the G action on the map f as below, as we need the first map to be in $G \times C_0$ and the second to be in C_1.

$$\begin{tikzcd}
x \arrow[r, "{(g,x)}"] \arrow[dr, "{(h\circ g,x)}"'] & g(x) \arrow[r, "f"] \arrow[d, "{(h,g(x))}"] & f(g(x)) \arrow[d, "{(h,f(g(x)))}"] & \\
& h(g(x)) \arrow[r, "h(f)"] & h(f(g(x))) \arrow[r, "\tilde{f}"] & \tilde{f}(h(f(g(x))))
\end{tikzcd}$$

Where $(h, g(x)) \circ (g, x) = (h \circ g, x) \in G \times C_0$ and $\tilde{f} \circ h(f) \in C_1$. Since they are composable, they define the required composition.

In the above construction the G action on C_0 is encoded in the definition of the space of morphisms of C_G and the G action on C_1 is encoded in the composition map when we take $g = e$ and $\tilde{f} = \mathrm{Id}$. This is an intuitive reason for the following

Theorem 8.1.2. *[Moe90, Proposition 3.9] Suppose C is a G–topological category. Then the category $G{-}Shv(\mathrm{Sets})/C$ is equivalent to the category $Shv(\mathrm{Sets})/C_G$, where C_G is a topological category described in Definition 8.1.1.*

From the above theorem we can deduce the following

Corollary 8.1.3. *For a topological group G the category $G - Shv(\mathbb{Q} - mod)/C$ of G–sheaves of $\mathbb{Q}$–modules over a G–topological category C is a Grothendieck category, so in particular it is an abelian category.*

As before, several properties follow immediately

Corollary 8.1.4. *For a topological group G the category $G - Shv(\mathbb{Q} - mod)/C$ of G–sheaves of $\mathbb{Q}$–modules over a G–topological category C has all small limits and colimits.*

Corollary 8.1.5. *For a topological group G there is an isomorphism of categories*

$$Ch(G - Shv(\mathbb{Q} - mod)/C) \cong G - Shv(Ch(\mathbb{Q} - mod))/C$$

that is the category of chain complexes in the category of G–sheaves of $\mathbb{Q}$-modules over a G–topological category C is isomorphic to a category of G–sheaves of chain complexes of $\mathbb{Q}$-modules over the same G–topological category C.

Again, it follows from [Hov01, Theorem 2.2] that we have a model structure

Corollary 8.1.6. *For a topological group G there exist a proper, cofibrantly generated model structure on the category of $Ch(G - Shv(\mathbb{Q} - mod)/C)$ of chain complexes of G–sheaves of $\mathbb{Q}$-modules over a G–topological category C where the cofibrations are the injections, the weak equivalences are the homology isomorphisms and the fibrations are maps which have the right lifting property with respect to trivial cofibrations.*

We call this model structure the injective model structure.

We need a notion of a monoidal product on this category, so we define it as in Chapter 6.

$$(A, \phi_A) \otimes (B, \phi_B) := (A \otimes B, \phi_A \otimes \phi_B)$$

for $(A, \phi_A), (B, \phi_B) \in G - Shv(\mathbb{Q} - \mathrm{mod})/C$, where $A \otimes B$ denotes the tensor product of G sheaves over a G topological space C_0 as in Chapter 7.

Lemma 8.1.7. *For any topological group G the tensor product defined above gives a closed symmetric monoidal structure on the category $G{-}Shv(\mathbb{Q}{-}mod)/C$ of G–sheaves of $\mathbb{Q}$–modules over a G–topological category C.*

Proof. This follows from the fact that this is a Grothendieck category and tensor product preserves colimits in both variables (by an argument analogous to the one in Lemma 6.1.15), thus it has a right adjoint – the internal hom. □

Theorem 8.1.8. *The injective model structure on the category $Ch(G - Shv(\mathbb{Q} - mod)/C)$ is a monoidal model structure, i.e it satisfies both pushut-product and unit axioms.*

Proof. The proof works along the same lines as the one for Lemma 6.2.2. Since the forgetful functor commutes with pushouts and both cofibrations and trivial cofibrations are created by this forgetful functor it is enough to check the pushout-product axiom at the level of sheaves over C_0 (without G action). Since this holds, it holds also in the category of G sheaves oves G topological category C.

Since every object is cofibrant the unit axiom follows from the pushout-product axiom. □

At the end of this section we present that a continuous homomorphism between topological groups induces an adjoint pair between corresponding categories of sheaves of $\mathbb{Q}$–modules.

Proposition 8.1.9. *Suppose $f : H \longrightarrow G$ is a continuous homomorphism between topological groups. Suppose further that C is a G topological category. Then we can view C as an H topological category via restriction along f and we obtain the following forgetful (restriction) functor:*

$$F^* : G - Shv(\mathbb{Q} - \mathrm{mod})/C \longrightarrow H - Shv(\mathbb{Q} - \mathrm{mod})/C$$

which has a right adjoint.
Note that on the right C is viewed as an H topological category.

Proof. The right adjoint exists since F^* preserves colimits and both domain and codomain are Grothendieck categories. □

8.2 Continuous Weyl toral modules

Finally we consider the category we are most interested in. It is conjectured that an algebraic model for any compact Lie group G will be of the form of chain complexes of continuous Weyl toral modules for G with some additional structure.

We begin this section with a definition of a continuous Weyl toral module, then we present some examples, and at the end we show that the category of chain complexes of continuous Weyl toral modules can be equipped with an injective model structure.

Definition 8.2.1. For G a compact Lie group, a continuous Weyl toral module is a G–sheaf of $\mathbb{Q}$–modules over the G–topological category TC_G with the additional property that for an object H in TC_G, H acts trivially on the stalk over H.

This notion was first introduced in [Gre98a]. For fixed G, continuous Weyl toral modules form a category which we denote $CWTM_G$. We present two examples.

Example 8.2.2. Suppose G is finite. Then the category $CWTM_G$ is equivalent to the category $\prod_{(H)\leq G} \mathbb{Q}[W_GH] - \mathrm{mod}$. This is an equivalence of categories of sheaves over two topological spaces induced by inclusion of a subspace (consisting of one point for every orbit) into a space. Note that TC_G in that case is a disjoint union of conjugacy classes of subgroups of G. For a single orbit orb$H$ we have the following equivalence of categories:

$$\mathrm{res} \ : \ G - Shv(\mathbb{Q} - \mathrm{mod})/\mathrm{orb}H \ \underset{\longrightarrow}{\longleftarrow} \ Shv(\mathbb{Q}[W_GH] - \mathrm{mod})/\{H\} \ : \ \mathrm{ext}$$

Notice that chain complexes in $\prod_{(H)\leq G} \mathbb{Q}[W_GH] - \mathrm{mod}$ (with weak equivalences = homology isomorphisms) give the algebraic model for G rational spectra.

Example 8.2.3. Since we know that $TC_{O(2)}$ splits into disjoint union of two pieces, cyclic $TC_{\mathcal{C}(O(2))}$ and dihedral $TC_{\mathcal{D}(O(2))}$ (see Example 5.2.5), we can consider continuous Weyl toral modules over the dihedral part $TC_{\mathcal{D}(O(2))}$. Note that we have the following equivalence of categories:

$$\mathrm{res} \ : \ G - Shv(\mathbb{Q} - \mathrm{mod})/TC_{\mathcal{D}(O(2))} \ \underset{\longrightarrow}{\longleftarrow} \ Shv(\mathbb{Q}[W] - \mathrm{mod})/\mathcal{D}(O(2)) \ : \ \mathrm{ext}$$

where the category on the right consists of sheaves with trivial W action over the orbit of $O(2)$ and the space $\mathcal{D}(O(2))$ on the right consists of $O(2)$ and one D_{2n} for every n coherently chosen (see below). Notice that both $TC_{\mathcal{D}(G)}$ and $\mathcal{D}(O(2))$ are just topological spaces (i.e. there are no non-identity morphisms). Also right adjoint restriction functor is induced by pulling back along the inclusion of space of the subgroups D_{2n} (where all D_{2n} subgroups are chosen to be the symmetry groups of standard regular n-gon on an xy-plane) and $O(2)$ into $TC_{\mathcal{D}(G)}$

Note that the category on the right is isomorphic to the category $\mathcal{A}(\mathcal{D})$ described in Section 11.1, and that $Ch(\mathcal{A}(\mathcal{D}))$ gives the algebraic model for the dihedral part of rational $O(2)$–equivariant spectra (where weak equivalences are homology isomorphisms).

We are interested in putting a monoidal model structure on the category of $Ch(CWTM_G)$ with weak equivalences the homology isomorphisms. One way to do this would be to show that the corresponding category of G sheaves of sets over TC_G with the restriction on the stalks has generators and thus is a Grothendieck topos. However, there is another way, which uses the following result from [Gre98a]

Theorem 8.2.4. *[Gre98a, Theorem A and B] Let G be a compact Lie group. There is an equivalence of categories between rational Mackey functors for G and continuous Weyl toral modules for G, $CWTM_G$.*

Remark 8.2.5. This is the composition of functors from Theorem A and Theorem B from [Gre98a]. Notice that the functors from Theorem A and Theorem B are not equivalences of categories, however its composition is. This is discussed in [Gre01].

Recall that a rational Mackey functor for a compact group G can be defined as enriched over $\mathbb{Q}$-modules contravariant functor from the stable orbit category $s\mathcal{O}_G^{\mathrm{op}}$ to $\mathbb{Q}$–modules (see for example [Gre98a, Section 3]). The stable orbit category is the enriched over $\mathbb{Q}$ category on objects G/H_+, for H closed subgroups of G and enriched homs

$$Hom_{\mathbb{Q}}(G/H_+, G/K_+) := [\Sigma^\infty G/H_+, \Sigma^\infty G/K_+] \otimes \mathbb{Q}$$

From the above theorem we can deduce the properties of $CWTM_G$. First of all, we know that the category of rational Mackey functors is an abelian category, and it has (categorical) generators, namely the representable functors (to show they are categorical generators we use enriched Yoneda lemma). Thus

Corollary 8.2.6. *The category of continuous Weyl toral modules is a Grothendieck category, in particular it is abelian.*

Proof. We need to show that filtered colimits are exact. Since they are in the category of $G - Shv(\mathbb{Q} - mod)/TC_G$ and they are calculated in the same way in the subcategory of $CWTM_G$, they are exact. □

Corollary 8.2.7. *The category of continuous Weyl toral modules has all colimits and all limits.*

We define a tensor product on the category of continuous Weyl toral modules in the same way as for G sheaves of $\mathbb{Q}$-modules over a G topological category. The only thing to check is whether the tensor of two objects is still a continuous Weyl toral module, i.e. we need to check the condition with the trivial action from Definition 8.2.1.

Suppose $(A, \phi_A), (B, \phi_B)$ are in $CWTM_G$. Then we want to show that

$$(A, \phi_A) \otimes (B, \phi_B) := (A \otimes B, \phi_A \otimes \phi_B)$$

is also in $CWTM_G$. We know that for any $H \in TC_G$, H acts trivially on A_H and B_H. Since the action on $A_H \otimes B_H$ is diagonal, H acts trivially on $A_H \otimes B_H$. Since we know that $(A \otimes B)_H \cong A_H \otimes B_H$, the tensor product of two continuous Weyl toral modules is a continuous Weyl toral module.

Proposition 8.2.8. *Tensor product in $CWTM_G$ defined above is a closed symmetric monoidal product.*

Proof. This follows from the same argument as for Lemma 8.1.7. ☐

Theorem 8.2.9. *Suppose G is a compact Lie group. Then there is a proper, stable, cofibrantly generated, monoidal model structure on the category of chain complexes of continuous Weyl toral modules for G, where the weak equivalences are the homology isomorphisms and the cofibrations are the monomorphisms.*

Proof. This follows from the same pattern as Corollary 8.1.6 and Theorem 8.1.8. ☐

Remark 8.2.10. The equivalence of categories presented in Example 8.2.3 is a Quillen equivalence when both categories are considered with the injective model structure. This follows since we can view any of these two injective model structures as a left induced transfer of the other and since this is an equivalence of categories we get a Quillen equivalence.

Part III

Algebraic model for SO(3) rational spectra

Chapter 9

General results for SO(3)

We start this chapter by considering closed subgroups of $SO(3)$ in Section 9.1. We discuss the space $F(G)/G$, which is the orbit space of all closed subgroups with finite index in their normaliser, where the topology is induced from the Hausdorff metric.

In Section 9.2 we discuss the idempotents of the rational Burnside ring $A(SO(3))$ and the induced splitting of rational $SO(3)$ equivariant orthogonal spectra. The main part of Section 9.2 consists of analysis of two adjunctions: the induction - restriction and restriction - coinduction with relation to the localisation of categories of spectra.

In Section 9.3 we present a general result showing that there is always an idempotent picking the cyclic (maximal torus) part of any compact Lie group G.

9.1 Closed subgroups of SO(3)

In this chapter we use a simplified notation. We set $G = SO(3)$, i.e. a group of rotations of $\mathbb{R}^3$. We choose a maximal torus in $SO(3)$ with rotation axis being the z-axis and we use the notation T for it.

We divide all closed subgroups of G into 3 parts: **exceptional $\mathcal{E}$, dihedral $\mathcal{D}$ and cyclic (toral)** $\mathcal{C}$.

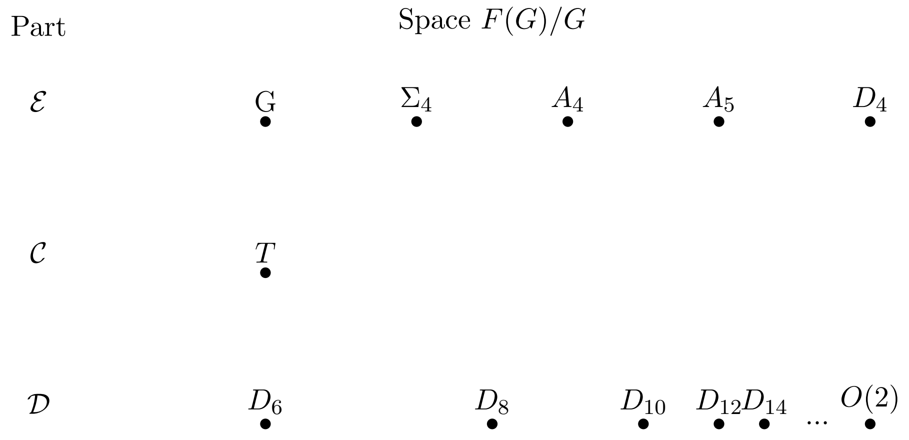
There are 5 conjugacy classes of subgroups which we call exceptional, namely G itself, the symmetry group Σ_4 of a cube, the symmetry group A_4 of a tetrahedron, the symmetry group A_5 of the dodecahedron and D_4 which is a dihedral group of order 4. Thus exceptional part consists of subgroups with finite Weyl group in $SO(3)$, i.e. $W_{SO(3)}H = N_{SO(3)}H/H$ is finite. Normalisers of these exceptional subgroups are as follows: Σ_4 is equal to its normaliser, A_5 is equal to its normaliser, the normaliser of A_4 is Σ_4. The Normaliser of D_4 is Σ_4.

The dihedral part consists of all dihedral subgroups D_{2n} of $SO(3)$ where n is greater than 2, together with all $O(2)$. Note that $O(2)$ is a normaliser for itself in $SO(3)$ and all dihedral subgroups of order $2n, n > 2$ are conjugate in $SO(3)$, i.e. there is one conjugacy class of dihedral subgroups for each order greater than 4. The normaliser of D_{2n} is D_{4n}.

The cyclic part consist of all tori in $SO(3)$ and all cyclic subgroups of those tori. Note that for any order there is one conjugacy class of the subgroups from the cyclic part of that order in G.

Remark 9.1.1. Note that an exceptional part consists of finitely many conjugacy classes of subgroups, so we could add first finitely many finite dihedral subgroups to the exceptional part (i.e. $D_6, D_8, ...$), but we cannot add all of them, as the splitting result does not allow us to split the category of spectra into infinitely many pieces.

We present the space $F(G)/G$ of conjugacy classes of subgroups of $SO(3)$ with finite index in their normalisers with topology induced by the Hausdorff metric, which we will use for choosing idempotents of the rational Burnside ring in the next section.



The topology on $\mathcal{E}$ is discrete, $\mathcal{C}$ consists of one point T and $\mathcal{D}$ forms a sequence of points converging to $O(2)$.

Before we go any further with the construction we cite the result of Greenlees which shows what the algebraic model for the homotopy category of rational $SO(3)$- spectra is.

Theorem 9.1.2. *[Gre01, Theorem 2.2] There is an equivalence of triangulated categories*

$$Ho(G-\text{spectra}) \simeq \prod_{(H),H\in\mathcal{E}} (\text{graded} - \mathbb{Q}[W_G H] - \text{modules}) \times D(\mathcal{A}(\mathcal{D})) \times \ D(\mathcal{A}(G,c))$$

where the first product is over conjugacy classes of exceptional subgroups, $\mathcal{A}(\mathcal{D})$ is described in Section 11.1 and $\mathcal{A}(G,c)$ is described in Section 12.1. For an abelian category $\mathcal{A}$, $D(\mathcal{A})$ denotes its derived category.

We devote the last part of this section for the discussion on *good* and *bad* subgroups.

Definition 9.1.3. [Gre01, Definition 6.3] Suppose H is a closed subgroup of G. We say that $K \subseteq N_G H$ is H–**bad** in G if there exist $g \notin N_G H$ such that $g^{-1}Kg \subseteq H$.
$K \subseteq N_G H$ is called H–**good** in G if it is not H–bad.

Remark 9.1.4. [Gre01, Remark 6.4] K is H–bad if and only if $(G/H)^K \neq N_G H/H$.

Next we state an obvious, but useful observation

Remark 9.1.5. Any subgroup H in a compact Lie group Γ is H–good in Γ.

For closed subgroups of $SO(3)$ we have the following

Lemma 9.1.6. *For exceptional subgroups in $G = SO(3)$ we have the following relation between H and its normaliser $N_G H$:*

1. *$SO(3)$ is $SO(3)$–good in $SO(3)$*
2. *A_5 is A_5–good in $SO(3)$*
3. *Σ_4 is Σ_4–good in $SO(3)$*
4. *A_4 is Σ_4–good in $SO(3)$*
5. *D_4 is Σ_4–bad in $SO(3)$*

Proof. Part 1 is trivial, Part 2 and 3 follow from the fact that A_5 is its own normaliser in $SO(3)$ and Σ_4 is its own normaliser in $SO(3)$. Part 4 follows from the fact that there is one conjugacy class of A_4 in Σ_4, as there is just one subgroup of index 2 in Σ_4. Part 5 follows from the observation that there are 2 subgroups of order 4 in D_8 (so also in Σ_4) and they are conjugate by an element $g \in D_{16}$, which is the generating rotation by 45 degrees (thus $g \notin D_8$ and thus $g \notin \Sigma_4$). □

9.2 Idempotents of the rational Burnside ring and splitting

By the result of tom Dieck (see Section 2.4.1) we know that idempotents of rational Burnside ring of $SO(3)$ correspond to open and closed subspaces of the space $F(G)/G$ above.

We use the following idempotents in the Burnside ring of $SO(3)$: e_c corresponding to the characteristic function of the cyclic part $\mathcal{C}$, i.e. the conjugacy class of the torus T, e_d corresponding to the characteristic function of the dihedral part $\mathcal{D}$ and e_e corresponding to the characteristic function of the exceptional part $\mathcal{E}$. $\mathcal{E}$ is a disjoint union of 5 points, so e_e is in fact a sum of 5 idempotents, one for every subgroup in the exceptional part: e_G, e_{Σ_4}, e_{A_4}, e_{A_5} and e_{D_4}.

The first step on the way towards the algebraic model is to split the category of rational G-equivariant spectra using the above idempotents of the Burnside ring $A(G)$. By Theorem 2.4.2 of Barnes this gives the monoidal Quillen Equivalence:

$$\triangle \; : \; G-\mathcal{IS}_{\mathbb{Q}} \rightleftarrows L_{e_c S_{\mathbb{Q}}}(G-\mathcal{IS}) \times L_{e_d S_{\mathbb{Q}}}(G-\mathcal{IS}) \times L_{e_e S_{\mathbb{Q}}}(G-\mathcal{IS}) \; : \Pi$$

where $G-\mathcal{IS}_{\mathbb{Q}}$ denotes the category of rational G orthogonal spectra.

We use the name *H–equivalence* for a weak equivalences in the category $L_{e_H S_{\mathbb{Q}}}(G-\mathcal{IS})$ (or $L_{e_H S_{\mathbb{Q}}}(N-\mathcal{IS})$, depending on a context) and *H–fibrant replacement* for the fibrant replacement there. Similarly, we use the name $\mathcal{D}$–equivalence for a weak equivalences in the category $L_{e_d S_{\mathbb{Q}}}(G-\mathcal{IS})$ and $\mathcal{C}$–equivalence for a weak equivalences in the category $L_{e_c S_{\mathbb{Q}}}(G-\mathcal{IS})$.

Now we consider each category from the right hand side product separately. We start with the exceptional part in the next chapter. Before we do that we state a very useful result which gives a characterisation of weak equivalences in each localised category of rational G–orthogonal spectra mentioned above.

Lemma 9.2.1. *A map f*

1. *between $e_H S_{\mathbb{Q}}$ - local objects is a weak equivalence in $L_{e_H S_{\mathbb{Q}}}(G-\mathcal{IS})$ iff $\pi_*^H(f)$ is an isomorphism.*
2. *between $e_d S_{\mathbb{Q}}$ - local objects is a weak equivalence in $L_{e_d S_{\mathbb{Q}}}(G-\mathcal{IS})$ iff $\pi_*^D(f)$ is an isomorphism for every subgroup $D \in \mathcal{D}$.*
3. *is a weak equivalence in $L_{e_c S_{\mathbb{Q}}}(G-\mathcal{IS})$ iff $\pi_*^C(f)$ is an isomorphism for every subgroup $C \in \mathcal{C}$.*

Proof. To prove part 1, first notice that a map f is, by definition, a weak equivalence in $L_{e_H S_{\mathbb{Q}}}(G-\mathcal{IS})$ iff $e_H S_{\mathbb{Q}} \wedge f$ is a π_*^K–isomorphism for every subgroup $K \leqslant G$. We know that this holds iff $\Phi^K(e_H S_{\mathbb{Q}} \wedge f)$ is a non-equivariant equivalence for all $K \leqslant G$. As geometric fixed point functor commutes with smash product that is equivalent to $\Phi^H(e_H S_{\mathbb{Q}} \wedge f)$ being a non-equivariant equivalence, i.e. $\pi_*(\Phi^H(e_H S_{\mathbb{Q}} \wedge f))$ being an isomorphism. Since f is between $e_H S_{\mathbb{Q}}$- local objects

$$\pi_*(\Phi^H(e_H S_{\mathbb{Q}} \wedge f)) \cong \pi_*^H(f)$$

that finishes the proof of part 1.

It also can be proven as Part 2 below.

To show Part 2 first note that $\mathcal{C}$ together with all subgroups D_4 in $SO(3)$ forms a family of subgroups (we use notation $\mathcal{C} \cup D_4$ for it) and so does the set of all conjugates to subroups of $O(2)$ together with all $O(2)$ (we use notation $[\leqslant_G O(2)]$ for it). Now, $L_{e_d S_{\mathbb{Q}}}(G-\mathcal{IS})$ is equivalent to $L_{\tilde{E}(\mathcal{C} \cup D_4)}(L_{E[\leqslant_G O(2)]}(G-\mathcal{IS}))$ as monoidal model categories. We know by [MM02, Chapter IV, Proposition 6.7] that f is a weak equivalence in $L_{E[\leqslant_G O(2)]}(G-\mathcal{IS})$ iff $\pi_*^H(f)$ is an isomorphism for every subgroup $H \in [\leqslant_G O(2)]$ (see for example [LMSM86, Definition 2.10] for $E\mathcal{F}$ and $\tilde{E}\mathcal{F}$).

Now if we further localise this category as above and consider f to be a map between local objects then by [MM02, Chapter IV, Theorem 6.13] π_*^H on the domain and codomain of f will be 0 for every $H \in \mathcal{C} \cup D_4$ thus we can conclude.

As $\mathcal{C}$ forms a family of subgroups, Part 3 follows from [MM02, Chapter IV, Proposition 6.7]. □

Since we will be interested in taking H fixed points of G spectra, we need to pass to N_GH spectra first. The natural choice of adjunction between G spectra and N_GH spectra would be the induction and restriction functors. However, this turns out to be a Quillen adjunction only in some cases relevant for us.

Proposition 9.2.2. *Suppose H is an exceptional subgroup of G which is $N = N_GH$-good in G. Then*

$$i^* \ : \ L_{e_HS_{\mathbb{Q}}}(G-\mathcal{IS}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} L_{e_HS_{\mathbb{Q}}}(N-\mathcal{IS}) \ : \ G_+ \wedge_N -$$

is a Quillen adjunction.

Proof. This was a Quillen adjunction before localisation by [MM02, Chapter V, Proposition 2.3] so the left adjoint preserves cofibrations. It preserves acyclic cofibrations as $G_+ \wedge_N -$ preserved acyclic cofibrations before localisation and we have a natural (in an N–spectrum X) isomorphism (see [MM02, Chapter V, Proposition 2.3]):

$$(G_+ \wedge_N X) \wedge e_HS_{\mathbb{Q}} \cong G_+ \wedge_N (X \wedge i^*(e_HS_{\mathbb{Q}}))$$

Note that, since H is N-good in G, $i^*(e_H) \cong e_H$ (see Section 2.4.1 for definition of $i^*(e_H)$) where the later is the idempotent corresponding to $(H)_N$ in $A(N)$.

□

Proposition 9.2.3. *Suppose H is an exceptional subgroup of G which is $N = N_GH$-bad in G. Then*

$$i^* \ : \ L_{e_HS_{\mathbb{Q}}}(G-\mathcal{IS}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} L_{e_HS_{\mathbb{Q}}}(N-\mathcal{IS}) \ : \ G_+ \wedge_N -$$

is not a Quillen adjunction.

It is worth mentioning that although the above adjunction does not behave well with respect to model structures, the one with restriction and coinduction does as it is shown in Corollary 9.2.7 below.

Proof. It is enough to show that $G_+ \wedge_N -$ does not preserve acyclic cofibrations. Firstly, since H is N-bad in G then there exists H' such that $(H)_G = (H')_G$ and $(H)_N \neq (H')_N$.

Take a map f to be the inclusion into coproduct $N/H_+ \longrightarrow N/H_+ \vee N/H'_+$. This is a weak equivalence since $\Phi^H(N/H_+) = \Phi^H(N/H_+ \vee N/H'_+)$. It is also a cofibration as a pushout of a cofibration $* \longrightarrow N/H_+$ along the map $* \longrightarrow N/H'_+$. Applying the left adjoint gives the map $G_+ \wedge_N f : G/H_+ \longrightarrow G/H_+ \vee G/H'_+$ which is an inclusion into the coproduct. Now $\Phi^H(G/H_+ \vee G/H'_+) = N/H_+ \vee N/H'_+ \neq N/H_+$ since $(H)_G = (H')_G$. Note that $N/H_+ \not\simeq S^0$ as N/H is a finite set of points and never just a point (as H is N-bad by assumption). □

Example 9.2.4. D_4 in $SO(3)$ is Σ_4-bad as discussed in Lemma 9.1.6.

Proposition 9.2.5. *Suppose e_d is the idempotent of $A(SO(3))$ corresponding to all dihedral supgroups of order greater than 4 and all $O(2)$. Then*

$$i^* \ : \ L_{e_d S_{\mathbb{Q}}}(SO(3)-\mathcal{IS}) \leftrightarrows L_{e_{i^*d} S_{\mathbb{Q}}}(O(2)-\mathcal{IS}) \ : \ G_+ \wedge_N -$$

is a Quillen adjunction, where e_{i^*d} is the idempotent for rational $O(2)$ spectra corresponding to all dihedral subgroups of order grater than 4 and the $O(2)$.

Proof. The proof follows the same pattern as the proof of Proposition 9.2.2. □

It turns out that the restriction and function spectrum adjunction gives a Quillen adjunction under general conditions, as we present at the end of this section

Lemma 9.2.6. *Suppose G is any compact Lie group, $i : H \longrightarrow G$ is an inclusion of a subgroup and V is an open and closed set in $F(G)/G$ such that i^*V is not empty in $F(H)/H$. Then the adjunction*

$$i^* \ : \ L_{e_V S_{\mathbb{Q}}}(G-\mathcal{IS}) \rightleftarrows L_{e_{i^*V} S_{\mathbb{Q}}}(H-\mathcal{IS}) \ : \ F_H(G_+, -)$$

is a Quillen pair. We use notation e_V here for the idempotent corresponding to the characteristic function on V.

Proof. Before localisations this was a Quillen pair by [MM02, Chapter V, Proposition 2.4]. It is a Quillen pair after localisation by Lemma 2.3.8, the fact that i^* is strong symmetric monoidal and the equivalence $S_{\mathbb{Q}} \cong i^*(S_{\mathbb{Q}})$. We use the notation i^*V for preimage of V under the inclusion on spaces of subgroups induced by i, i.e. $\mathrm{Sub_f}(H)/_{\sim} \longrightarrow \mathrm{Sub_f}(G)/_{\sim}$ (see Section 2.4.1). □

In the next chapters we will repeatedly use the above lemma, mainly in situations where after further localisation of the right hand side we will get a Quillen equivalence. To prepare for that, we state the following four cases

Corollary 9.2.7.

1. *Suppose H is an exceptional subgroup of G (any compact Lie group). Then*

   $$i^* \ : \ L_{e_H S_{\mathbb{Q}}}(G-\mathcal{IS}) \rightleftarrows L_{e_H S_{\mathbb{Q}}}(N-\mathcal{IS}) \ : \ F_N(G_+, -)$$

   is a Quillen adjunction.

2. *Let $\mathcal{D}$ denote dihedral part of $SO(3)$ and e_d denote the idempotent corresponding to it. Then*

   $$i^* \ : \ L_{e_d S_{\mathbb{Q}}}(SO(3)-\mathcal{IS}) \rightleftarrows L_{e_d S_{\mathbb{Q}}}(O(2)-\mathcal{IS}) \ : \ F_{O(2)}(SO(3)_+, -)$$

   is a Quillen adjunction. The idempotent on the right hand side e_d corresponds to the dihedral part of $O(2)$ excluding all subgroups D_2 and D_4.

Proof.

Part 1 for H which is $N = N_G H$–good follows from the fact that the idempotent on the right hand side $e_H = i^*(e_H) = e_{i^* H}$. For H which is $N = N_G H$–bad it is true since the left hand side is a further localisation of $L_{e_{i^* H} S_{\mathbb{Q}}}(N - \mathcal{IS})$ at the idempotent e_H:

$$\begin{tikzcd}
L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS}) \arrow[r, shift left, "i^*"] & L_{i^*(e_H) S_{\mathbb{Q}}}(N - \mathcal{IS}) \arrow[l, shift left, "F_N(G_+,-)"] \arrow[r, shift left, "Id"] & L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS}) \arrow[l, shift left, "Id"]
\end{tikzcd}$$

Note that since H is N–bad, $e_H \neq i^*(e_H)$ and $e_H i^*(e_H) = e_H$.

Part 2 follows since the idempotent on the right hand side $e_d = e_{i^* d}$. □

Similar result is also true for cyclic part, but we decided to state it in Chapter 12 as we will proceed with cellularisation of that part straight away.

9.3 Idempotent for the cyclic part

Suppose that G is any compact Lie group. We show that there is always an idempotent in the rational Burnside ring $A(G)$ corresponding to the cyclic part, i.e. to the characteristic function for all subgroups subconjugate to the maximal torus T.

Our standard reference for Lie groups is [BtD85]. Recall that a maximal torus in a compact Lie group G is a maximal, connected abelian subgroup T. Every compact Lie group G has at least one maximal torus. Since any two maximal tori in a compact connected Lie group G are conjugate we have the following observations

Lemma 9.3.1. *The subset U_T of $\mathcal{F}(G)/G$ consisting of all subconjugates of T is open and closed.*

Proof. We apply [Gre98a, Lemma 3.3] to the maximal torus T in G. □

Corollary 9.3.2. *Suppose G is any compact Lie group. Then there exists an idempotent e_T in the rational Burnside ring $A(G)$ corresponding to the cyclic part of G, i.e. corresponding to the characteristic function of U_T.*

Chapter 10

Exceptional part

The exceptional part of rational $SO(3)$ equivariant orthogonal spectra has the form of a finite product so this allows us to split it further and work with one exceptional subgroup at a time. Because almost every step is the same for all exceptional subgroups we will use the notation H for an exceptional subgroup of G, and we will work with one category $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$. We note that some steps of this general approach might become easier or even trivial for a particular subgroup H, however we choose to describe the comparison generally to show that the same approach works for an exceptional part of any compact Lie group. As a special case, this approach covers all finite groups G.

The proof of the non - monoidal algebraic model for the exceptional part follows the same steps as the proof of the algebraic model for any finite G from [Bar09b]. We need to stress that this proof as it stands is not monoidal. Since we are interested in preserving monoidal structures as well, we also construct a zig-zag of monoidal Quillen equivalences in Section 10.3, which will use in the proof of Theorem 10.3.4 the fact that we have a non-monoidal comparison already in place. We were not able to find a proof which works directly.

Below we present short sketches of steps in both comparisons to outline general ideas. Before we concentrate on the monoidal comparisons in Section 10.3 we present some details of the algebraic model in Section 10.1 and we give the details of non monoidal approach in Section 10.2.

Non - monoidal comparison:

Fix an exceptional subgroup H. The plan for this part is as follows.

First we choose the generator $\hat{f}G/H_+$ for the category $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ which is fibrant and cofibrant. Here $\hat{f}$ is the fibrant replacement functor in the localised category.

The second step is to show that the category $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ is a spectral model category, where the enrichment is in the category of symmetric spectra Sp^{Σ} with the stable model structure.

Now we use Morita equivalence to pass from $L_{e_H S_{\mathbb{Q}}}(G-\mathcal{IS})$ to a category $\mathrm{mod}-\mathcal{E}_{\mathrm{top}}^H$ of right modules over $\mathcal{E}_{\mathrm{top}}^H$, where $\mathcal{E}_{\mathrm{top}}^H$ is the enriched endomorphism symmetric spectrum of the generator $\hat{f}G/H_+$.

The next step is the passage from the topological world to the algebraic setting using the result of Shipley. This gives a change of the ring symmetric spectrum $\mathcal{E}_{\mathrm{top}}^H$ into a ring rational chain complex $\mathcal{E}_{\mathrm{t}}^H$. We follow the notation of Greenlees and Shipley and also Barnes of subindex top which suggest that the category is topological and subindex t which suggests that the category is algebraic but with the topological origin.

There is not much control over the ring $\mathcal{E}_{\mathrm{t}}^H$, however we can compute its homology. Since it's concentrated in degree zero, we get intrinsic formality of the ring $\mathcal{E}_{\mathrm{t}}^H$. Thus we obtain a Quillen equivalence between the category of $\mathcal{E}_{\mathrm{t}}^H$-modules and the category of $H_*(\mathcal{E}_{\mathrm{t}}^H)$-modules. That gives the algebraic model for $L_{e_H S_{\mathbb{Q}}}(G-\mathcal{IS})$.

In the proof of Theorem 10.2.8 we present a diagram which shows every step of this comparison. The reader may wish to refer to this diagram now, but the notation will be introduced as we proceed.

Monoidal comparison:

This proof will use several of the above steps, but in a different order. The main difference is in replacing Morita equivalence by the fixed point - inflation adjunction. The plan for this part is as follows. Fix an exceptional subgroup H.

First we move from the category $L_{e_H S_{\mathbb{Q}}}(G-\mathcal{IS})$ to the category $L_{e_H S_{\mathbb{Q}}}(N-\mathcal{IS})$ using the restriction - coinduction adjunction. Recall that N denotes the normalizer $N_G H$.

The second step is to use the fixed point - inflation adjunction between $L_{e_H S_{\mathbb{Q}}}(N-\mathcal{IS})$ and $L_{e_1 S_{\mathbb{Q}}}(W-\mathcal{IS})$, where W denotes the Weyl group N/H. Recall that W is finite, as H is an exceptional subgroup of G.

Next we use the identity adjunction and restriction of universe to pass from $L_{e_1 S_{\mathbb{Q}}}(W-\mathcal{IS})$ to free W orthogonal spectra indexed by a trivial W universe. This category is equivalent as a monoidal model category to the category $\mathcal{IS}[W]$ of orthogonal spectra with W action.

Now we pass to symmetric spectra with W action using the forgetful functor from orthogonal spectra. Next we move to $H\mathbb{Q}$-modules with W action in symmetric spectra. From here we use the result of Shipley to get to $Ch(\mathbb{Q})[W]$, the category of rational chain complexes with W action, which is equivalent as monoidal model category to $Ch(\mathbb{Q}[W])$, the category of chain complexes of $\mathbb{Q}[W]$-modules. That gives an algebraic model which is compatible with the monoidal product, i.e. the zig-zag of Quillen equivalences induces a strong monoidal equivalence on the level of homotopy categories.

After Theorem 10.3.13 we present a diagram which shows every step of this comparison. The reader may wish to refer to this diagram now, but the notation will be introduced as we proceed.

10.1 The category Ch($\mathbb{Q}[W]$ – mod)

Suppose W is a finite group. In this section we discuss briefly the category of chain complexes of $\mathbb{Q}[W]$ left modules.

Firstly, this can be equipped with the projective model structure, where weak equivalences are homology isomorphisms and fibrations are levelwise surjections. Cofibrations are levelwise split monomorphisms with cofibrant cokernel. This model structure is cofibrantly generated by [Hov99, Section 2.3].

Note that $\mathbb{Q}[W]$ is not generally a commutative ring, however it is a Hopf algebra with cocommutative coproduct given by $\Delta : \mathbb{Q}[W] \longrightarrow \mathbb{Q}[W] \otimes \mathbb{Q}[W]$, $g \mapsto g \otimes g$. This allows us to define an associative and commutative tensor product on Ch($\mathbb{Q}[W]$ – mod), namely tensor over $\mathbb{Q}$, where the action on the $X \otimes_{\mathbb{Q}} Y$ is diagonal. The unit is a chain complex with $\mathbb{Q}$ at the level 0 with trivial W action and zeros everywhere else and it is cofibrant in the projective model structure. Monoidal product defined this way is closed, where the internal hom is given by an internal hom over $\mathbb{Q}$ formula with W action by conjugation.

This category is equivalent to the category of W-objects in a category of Ch($\mathbb{Q}$-mod), and we can transfer the projective model structure on Ch($\mathbb{Q}$-mod) to the category of W objects there using Theorem 1.1.13.

It remains to show that this is a monoidal model category satisfying the monoid axiom. This is [Bar09b, Proposition 4.3]

10.2 Non - monoidal comparison

We start this section by establishing the background for the Morita equivalence.

Lemma 10.2.1. *G/H_+ is a (homotopically) compact, cofibrant generator for $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$.*

Proof. G/H_+ is cofibrant, because $* \longrightarrow G/H_+$ is a generating cofibration the for stable model structure on $G - \mathcal{IS}$ (see [MM02, Chapter III, Definition 1.1]).
We start with the calculations below (where X is an object in $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ and superscript $L_{e_H G}$ denotes the homotopy category of $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$).

$$[G/H_+, X]^{L_{e_H G}}_{\mathbb{Q}*} \cong [G/H_+, e_H S_{\mathbb{Q}} \wedge X]^G_{\mathbb{Q}*} \cong \pi^H_*(e_H S_{\mathbb{Q}} \wedge X)$$

and $\pi_*^K(e_H S_{\mathbb{Q}} \wedge X) \cong 0$ for all $K \subseteq G, K \neq H$. Now assume that $[G/H_+, X]_{\mathbb{Q}*}^{L_{e_H G}} = 0$, then $e_H S_{\mathbb{Q}} \wedge X \simeq *$ and since $e_H S_{\mathbb{Q}} \wedge X$ is H - equivalent to X we get that X is H - equivalent to $*$.
To show that G/H_+ is (homotopically) compact we work with the definition. Suppose we have a collection of objects Y_i in $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$. Then, since idempotents commute with coproducts, we have

$$[G/H_+, \bigvee_{i \in I} Y_i]_{\mathbb{Q}*}^{L_{e_H G}} \cong [G/H_+, e_H S_{\mathbb{Q}} \wedge (\bigvee_{i \in I} Y_i)]_*^G \cong [G/H_+, \bigvee_{i \in I} (e_H S_{\mathbb{Q}} \wedge Y_i)]^G$$
$$\cong \bigoplus_{i \in I} [G/H_+, Y_i]_{\mathbb{Q}*}^{L_{e_H G}} \quad (10.2.2)$$

Thus it follows that G/H_+ is a generator for $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$. □

Lemma 10.2.3. *The category $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ is a spectral model category, where the enrichment is in symmetric spectra* Sp^{Σ}.

Proof. Apply Corollary 3.1.5.

□

Now that we have a spectral model category and the cofibrant and fibrant (homotopically) compact generator $\hat{f}G/H_+$ we can proceed to Morita equivalence.

Define $\mathcal{E}_{\mathrm{top}}^H$ to be the enriched endomorphism symmetric spectrum of the generator $\hat{f}G/H_+$ in $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$, i.e.

$$\mathcal{E}_{\mathrm{top}}^H := \underline{\mathrm{Hom}}_{\mathrm{Sp}^{\Sigma}}(\hat{f}G/H_+, \hat{f}G/H_+)$$

We use the notation $\mathrm{mod} - \mathcal{E}_{\mathrm{top}}^H$ for the category of right modules over $\mathcal{E}_{\mathrm{top}}^H$, see Section 3.2.

Theorem 10.2.4. *The adjunction*

$$\mathrm{Hom}(\hat{f}G/H_+, -) \ : \ L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathrm{mod} - \mathcal{E}_{\mathrm{top}}^H \ : \ - \wedge_{\mathcal{E}_{\mathrm{top}}^H} \hat{f}G/H_+$$

is a Quillen equivalence.

Proof. This follows from the Morita equivalence [SS03b, Theorem 3.9.3], see also Section 3.2 Theorem 3.2.8. Refer to Definition 3.2.6 for constructions of both functors. □

To use the result of Shipley, $\mathcal{E}_{\mathrm{top}}^H$ needs to be a ring in $H\mathbb{Q}$-modules. Since $H\mathbb{Q} \wedge -$ is a spectral functor from $\mathcal{E}_{\mathrm{top}}^H$ to $H\mathbb{Q} \wedge \mathcal{E}_{\mathrm{top}}^H$ which induces a stable equivalence from $\mathcal{E}_{\mathrm{top}}^H$ to $H\mathbb{Q} \wedge \mathcal{E}_{\mathrm{top}}^H$ by [SS03b, Theorem A.1.1], the restriction and extension of scalars along $H\mathbb{Q} \wedge -$ forms a Quillen equivalence of module categories. To simpify the notation we keep using $\mathcal{E}_{\mathrm{top}}^H$ for $H\mathbb{Q} \wedge \mathcal{E}_{\mathrm{top}}^H$.

Morita equivalence puts all the equivariance into the ring over which we consider the modules. However we are still in the topological setting (the ring is the ring symmetric spectrum). The following theorem provides the passage to the algebraic world.

Theorem 10.2.5. *There is a zig-zag of Quillen equivalences between the following two categories: mod-$\mathcal{E}^H_{\text{top}}$ and mod-$\mathcal{E}^H_{\text{t}}$, where $\mathcal{E}^H_{\text{t}}$ is a ring rational chain complex with the property that $H_*(\mathcal{E}^H_{\text{t}}) \cong \pi_*(\mathcal{E}^H_{\text{top}})$.*

Proof. This follows from [Shi07, Part 2 of Corollary 2.15] . □

Let us now calculate $\pi_*(\mathcal{E}^H_{\text{top}})$. Note that S in the first expresion is a sphere symmetric spectrum.

$$\begin{aligned}\pi_*(\mathcal{E}^H_{\text{top}}) &\cong [S, \underline{\text{Hom}}_{\text{Sp}^\Sigma}(\hat{f}G/H_+, \hat{f}G/H_+)] \cong [S, \text{Sing}\mathbb{U}(i^*F(\hat{f}G/H_+, \hat{f}G/H_+))^H] \\ &\cong [S, \mathbb{U}(i^*F(\hat{f}G/H_+, \hat{f}G/H_+))^H] \cong [S, (i^*F(\hat{f}G/H_+, \hat{f}G/H_+))^H] \\ &\cong [S, i^*F(\hat{f}G/H_+, \hat{f}G/H_+)] \cong [S, F(\hat{f}G/H_+, \hat{f}G/H_+)] \\ &\cong [\hat{f}G/H_+, \hat{f}G/H_+] \cong [G/H_+, \hat{f}G/H_+] \cong \pi_*((\hat{f}G/H_+)^H) \qquad (10.2.6)\end{aligned}$$

Now we have to calculate $\pi_*((\hat{f}G/H_+)^H)$. We know that $\hat{f}G/H_+$ is H–weakly equivalent to $E\langle H\rangle \wedge G/H_+$, so we have isomorphisms $\pi_*((\hat{f}G/H_+)^H) \cong \pi_*((E\langle H\rangle \wedge G/H_+)^H) \cong \pi_*(\Phi^H(G/H_+))$ where Φ^H denotes geometric H–fixed points.

We know that geometric fixed point functor commutes with the suspension functor in the sense that $\Phi^H(\Sigma^\infty G/H_+) \cong \Sigma^\infty((G/H_+)^H)$ and $(G/H_+)^H \cong W_G H$ which, by assumption on H is a finite set. Thus rational stable homotopy groups of suspension spectrum of finite set of points is just $\mathbb{Q}[W_G H]$

The long calculation above follows from the derived adjunctions presented in Theorem 3.1.4 and the fact that S in the first line is cofibrant, i.e. derived left adjoint are the same as left adjoints. We skipped the notation suggesting in which homotopy category every part is, as that follows from the context.

As $H_*(\mathcal{E}^H_{\text{t}}) \cong \pi_*(\mathcal{E}^H_{\text{top}})$ we know that $H_*(\mathcal{E}^H_{\text{t}})$ is concentrated in degree zero we use methods of [Bar13, Theorem 7.5 and 7.6] to get the following

Lemma 10.2.7. *There is a zig-zag of Quillen equivalences between mod-$\mathcal{E}^H_{\text{t}}$ and mod-$H_*(\mathcal{E}^H_{\text{t}})$ induced by stable equivalences $\mathcal{E}^H_t \longleftarrow C_0\mathcal{E}^H_t \longrightarrow H_*\mathcal{E}^H_t$ where C_0 is the (-1)-connected cover.*

Now we can summarise the result of this section in the

Theorem 10.2.8. *There exist a zig-zag of Quillen equivalences from $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ to $Ch(\mathbb{Q}[W] - \text{mod})$.*

Proof. This follows from Lemma 10.2.1, Lemma 10.2.3, Theorem 10.2.4, Theorem 10.2.5 and Lemma 10.2.7.
To illustrate the whole path we sketch below a diagram showing every step discussed above.

$$\begin{tikzcd}
L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS}) \arrow[d, shift left, "\mathrm{Hom}(\hat{f}G/H_+,-)"] \\
\mathrm{mod} - \mathcal{E}^H_{\mathrm{top}} \arrow[u, shift left, "-\wedge_{\mathcal{E}^H_{\mathrm{top}}} \hat{f}G/H_+"] \arrow[d, leftrightarrow, "\text{zig--zag of}"', "\text{Quillen equivalences}"] \\
\mathrm{mod} - \mathcal{E}^H_{\mathrm{t}} \arrow[d, leftrightarrow, "\text{zig--zag of}"', "\text{Quillen equivalences}"] \\
\mathrm{mod} - H_*(\mathcal{E}^H_{\mathrm{t}}) \cong Ch(\mathbb{Q}[W])
\end{tikzcd}$$

□

We stress again that the above comparison is not monoidal. We present a monoidal comparison in the next section, which we believe is conceptually different and our proof relies on the fact that we have an algebraic model established already.

10.3 Monoidal comparison

At the beginning of this approach we would like to use the fixed point - inflation adjunction. However, as H is not necessary normal in G first we need to move to the category of N-orthogonal spectra, where $N = N_G H$. Notice that this passage needs to be monoidal.

There is an inclusion of a subgroup $i : N \longrightarrow G$ which induces two adjoint pairs between corresponding categories of orthogonal spectra. The first choice would be to work with the induction and pullback functor adjunction. However this is not always a Quillen adjunction as we discussed in details in Section 9.2. The pullback functor i^* is strong monoidal, so we choose to work with it as a left adjoint, where the right adjoint is the function spectrum functor. We showed in Section 9.2 that this is always a Quillen adjunction for localisations at an exceptional subgroup. Moreover it is automatically strong monoidal. We begin by proving the following

Theorem 10.3.1. *Suppose H is an exceptional subgroup of G and it is $N = N_G H$–good. Then the adjunction*

$$i^* \ : \ L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS}) \rightleftarrows L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS}) \ : F_N(G_+, -)$$

is a strong monoidal Quillen equivalence, where e_H on the right hand side denotes the idempotent of the rational Burnside ring $A(N)$ corresponding to the characteristic function of $(H)_N$.

Proof. Firstly, if H was an exceptional subgroup of G with an idempotent e_H then it is represented by a characteristic G–function for conjugates of H in the G–topological space $Sub(G)$. Inclusion of N into G induce an inclusion of N–spaces $i : Sub(N) \longrightarrow Sub(G)$. Now e_H in $A(N)$ is equal to $i^*(e_H)$.

This is an adjunction by [MM02, Chapter V, Proposition 2.4]. Moreover the same proposition shows that i^* preserves cofibrations, because they remain unchanged by the localisation. Clearly smash product also remains unchanged by the localisation, so i^* is still strong monoidal.

This is a Quillen adjunction by Part 1 of Corollary 9.2.7 .

We claim that i^* preserves all H–equivalences. Suppose $f : X \longrightarrow Y$ is an H–equivalence in $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$, i.e. $Id_{e_H S_{\mathbb{Q}}} \wedge f : e_H S_{\mathbb{Q}} \wedge X \longrightarrow e_H S_{\mathbb{Q}} \wedge Y$ is a π_*–isomorphism. As i^* is strong monoidal $i^*(Id_{e_H S_{\mathbb{Q}}} \wedge f) \cong Id_{i^*(e_H S_{\mathbb{Q}})} \wedge i^*(f) \cong Id_{e_H S_{\mathbb{Q}}} \wedge i^*(f)$ and i^* preserves π_*–isomorphisms we can conclude.

To show this is a Quillen equivalence we will use Part 2 from Proposition 1.1.12. It is easy to see that i^* reflects H–equivalences using the fact it is strong monoidal and the isomorphism $[N/H_+, i^*(X)]^N \cong [G/H_+, X]^G$.

As i^* preserves all H–equivalences the derived counit condition becomes the counit condition. Thus we have to check that for every fibrant $Y \in L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS})$ the counit map $\varepsilon_Y : i^* F_N(G_+, Y) \longrightarrow Y$ is an H–equivalence (in N–spectra), i.e it is a π_*^H–isomorphism of N–spectra.

First we check that domain and codomain have isomorphic stable H homotopy groups:

$$\begin{aligned}\pi_*^H(i^* F_N(G_+, Y)) \cong \pi_*^H(F_N(G_+, Y)) &\cong [G/H+, F_N(G_+, Y)]_*^G \\ &\cong [i^*(G/H_+), Y]_*^N \cong [N/H_+, Y]_*^N \cong \pi_*^H(Y) \quad (10.3.2)\end{aligned}$$

The second but last isomorphism follows from the fact that the map $N/H_+ \longrightarrow G/H_+$ (induced by inclusion $N \longrightarrow G$) is an H–equivalence in N–spectra, i.e an equivalence in $L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS})$.

Since $i^*(\hat{f}G/H_+)$ is H-equivalent to N/H_+ which is compact, we can use Proposition 1.1.20. It is therefore enough to check the counit condition for a generator. We will check it for the spectrum $i^*(\hat{f}G/H_+)$, which is a generator for localised N–spectra. Stable H homotopy groups of this generator is $\mathbb{Q}[W_G H]$ in degree 0, where $W_G H$ is the Weyl group for H in G, so in particular $\mathbb{Q}[W_G H]$ is a finite dimensional vector space.

Now it is enough to show that $[N/H_+, \varepsilon_{i^*(\hat{f}G/H_+)}]^N$ is surjective. One of the triangle identities on $i^*(\hat{f}G/H_+)$ for the adjunction is stating that the following diagram commutes

$$\begin{tikzcd}
i^*(\hat{f}G/H_+) \arrow[d, "i^*(\eta_{\hat{f}G/H_+})"'] \arrow[drr, "Id"] & & \\
i^*F_N(G_+, i^*(\hat{f}G/H_+)) \arrow[rr, "\varepsilon_{i^*(\hat{f}G/H_+)}"'] & & i^*(\hat{f}G/H_+)
\end{tikzcd}$$

Thus postcomposition with $\varepsilon_{i^*(\hat{f}G/H_+)}$ is surjective on the homotopy level. It follows that the counit map is an H–equivalence of N–spectra for every fibrant Y which finishes the proof. $\square$

Remark 10.3.3. The assumptions of Theorem 10.3.1 are satisfied for subgroups A_5, A_4 and Σ_4 of $SO(3)$.

If H is an exceptional subgroup of G which is N_GH–bad then we need to alter the above statement slightly. We would like to point out that the main point of the proof is different than in the proof of the previous theorem.

The argument above will not work in the context where i^* does not preserve fibrant replacements. However we found the proof amusing, so we decided to present it, even though the proof below can be applied also in the case where H is an exceptional N_GH–good subgroup of G.

Below we use the existence of a zig-zag of Quillen equivalences established already in Section 10.2 (non-monoidally), which gives us the algebraic model for its homotopy category, so in particular we know what a certain derived functor (namely Li^* below) induces on the homotopy level.

Theorem 10.3.4. *Suppose H is an exceptional subgroup of G. Then the composition of adjunctions*

$$\begin{tikzcd}
L_{e_HS_{\mathbb{Q}}}(G-\mathcal{IS}) \arrow[r, shift left, "i^*"] & L_{i^*(e_HS_{\mathbb{Q}})}(N-\mathcal{IS}) \arrow[l, shift left, "F_N(G_+,-)"] \arrow[r, shift left, "\mathrm{Id}"] & L_{e_HS_{\mathbb{Q}}}(N-\mathcal{IS}) \arrow[l, shift left, "\mathrm{Id}"]
\end{tikzcd}$$

is a strong monoidal Quillen equivalence, where e_H on the further right hand side denotes the idempotent of the rational Burnside ring $A(N)$ corresponding to the characteristic function of $(H)_N$. Notice that if H is N-good then the right adjunction is trivial.

Proof. First notice that if H is N–bad $i^*(e_HS_{\mathbb{Q}}) \not\simeq e_HS_{\mathbb{Q}}$ as localised N–spectra. The reason for that is that $(H)_G$ restricts to more than one conjugacy class of subgroups of N. That is why we need a further localisation - we only want to consider the $(H)_N$.

Above composition of adjunctions forms a Quillen adjunction by Part 1 of Corollary 9.2.7. We need to show that the composition is a Quillen equivalence. We use Part 3 from Proposition

1.1.12. Firstly $F_N(G_+, -)$ preserves and reflects weak equivalences between fibrant objects. Let X be a fibrant object in $L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS})$. Then $F_N(G_+, X)$ is also fibrant and

$$\begin{aligned}[G/H_+, e_H F_N(G_+, X)]^G &\cong [G/H_+, F_N(G_+, X)]^G \\ &\cong [i^*(G/H_+), X]^N \cong [e_H i^*(G/H_+), e_H X]^N \cong [N/H_+, e_H X]^N \end{aligned} \quad (10.3.5)$$

Now we need to show that the derived unit is a weak equivalence on the cofibrant generator for $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$, which is $e_H G/H_+$. This is

$$e_H G/H_+ \longrightarrow F_N(G_+, \hat{f} i^*(e_H G/H_+))$$

To check that this is a weak equivalence in $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ it is enough to check that on the homotopy level the induced map

$$[G/H_+, e_H G/H_+]^G \longrightarrow [G/H_+, F_N(G_+, \hat{f} i^*(e_H G/H_+))]^G$$

is an isomorphism. This map fits into a commuting diagram below

$$\begin{tikzcd}
{[G/H_+, e_H G/H_+]^G} \arrow[d] \arrow[rrd, "Li^*"] & & \\
{[G/H_+, F_N(G_+, \hat{f} i^*(e_H G/H_+))]^G} & & {[i^* G/H_+, \hat{f} i^*(e_H G/H_+)]^N} \arrow[ll, "\cong"']
\end{tikzcd}$$

Since the horizontal map is an isomorphism it is enough to show that Li^* is an isomorphism. Now we know that H is an exceptional subgroup in both N and G, thus from the section 10.2 we know that the algebraic model for G spectra localised at e_H is isomorphic to the algebraic model for the N spectra localised at e_H. Moreover we know that the Li^* is induced by the map of rings $id : \mathbb{Q}[W_N H] \longrightarrow \mathbb{Q}[W_G H]$, which means it is the identity map. Thus Li^* is an isomorphism on the hom sets at the homotopy level which finishes the proof.

□

Now we use the fixed point - inflation adjunction. Recall that W below denotes the Weyl group N/H and by assumption on H it is finite. Moreover there is a projection map $\epsilon : N \longrightarrow W$ which induces the left adjoint below.

Theorem 10.3.6. *The adjunction*

$$\epsilon^* \ : \ L_{e_1 S_{\mathbb{Q}}}(W - \mathcal{IS}) \rightleftarrows L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS}) \ : (-)^H$$

is a strong monoidal Quillen equivalence.
Here e_1 is the idempotent of the rational Burnside ring $A(W)$ corresponding to the characteristic function for the trivial subgroup.

Proof. This is an adjunction by [MM02, Chapter V, Proposition 3.10] and by the same result the left adjoint preserves cofibrations. As the localisation didn't change the smash product, ϵ^* is still a strong monoidal functor.
To prove this is a Quillen pair we refer to [GS14b, Proposition 3.2] which states that (in notation adapted to our case):

$$\epsilon^* \ : \ (W-\mathcal{IS}) \rightleftarrows L_{\widetilde{E}[\not\supseteq H]}(N-\mathcal{IS}) \ : (-)^H$$

is a Quillen equivalence. Now we localise this result further at the $e_1 S_{\mathbb{Q}}$ on the side of W spectra and $e_H S_{\mathbb{Q}}$ on the side of N spectra. It follows from Lemma 2.3.8 that this is a Quillen adjunction and in fact a Quillen equivalence. The right hand side after this localisation is equivalent to $L_{e_H S_{\mathbb{Q}}}(N-\mathcal{IS})$. □

Next we move from $L_{e_1 S_{\mathbb{Q}}}(W-\mathcal{IS})$ to free W rational orthogonal spectra by the identity functor. First recall that a free model structure on W rational orthogonal spectra is defined as follows:

- A map f is a weak equivalence in free$-W-\mathcal{IS}_{\mathbb{Q}}$ iff $\pi^1_{*\mathbb{Q}}(f)$ is an isomorphism. 1 denotes the trivial subgroup in W. (equivalently, f is a weak equivalence in free $-\, W-\mathcal{IS}_{\mathbb{Q}}$ iff $EW_+ \wedge f$ is a π_* rational isomorphism)
- a cofibration is a map obtained from the original generating cofibrations by restricting to the orbit W_+
- Fibrations are defined via lifting property.

free $-\, W-\mathcal{IS}_{\mathbb{Q}}$ is a cofibrantly generated, proper, monoidal model category.

Lemma 10.3.7. *The adjunction*

$$Id \ : \ \text{free}-W-\mathcal{IS}_{\mathbb{Q}} \rightleftarrows L_{e_1 S_{\mathbb{Q}}}(W-\mathcal{IS}) \ : Id$$

is a strong monoidal Quillen equivalence. We use the subindex $\mathbb{Q}$ to denote the category localised at $S_{\mathbb{Q}}$.

Proof. This is [MM02, Chapter IV, Theorem 6.9]. It is obvious this is monoidal. □

Next we restrict W-universe from the complete to the trivial one. We adapt slightly the notation from Chapter V in [MM02].

Lemma 10.3.8. *The adjunction*

$$\mathrm{I}^c_t \ : \ \text{free}-W-\mathcal{IS}_{\mathbb{Q}_t} \rightleftarrows \text{free}-W-\mathcal{IS}_{\mathbb{Q}_c} \ : \mathrm{I}^t_c = \mathrm{res}$$

is a strong monoidal Quillen equivalence. We use the subindex t to remind us that we consider the trivial W-universe and I^t_c to denote the restriction (denoted also res *above) from the complete W–universe to the trivial one. I^c_t denotes the extension from the trivial W–universe to the complete one.*

Proof. This is a strong monoidal adjunction by [MM02, Chapter V, Theorem 1.5].
First we note that the left adjoint preserves generating cofibrations and generating acyclic cofibrations, since $\mathrm{I}_t^c F_V \cong F_V$ by [MM02, Chapter V, 1.4].

The right adjoint res preserves and reflects all weak equivalences since in both model structures they are defined as those maps which after forgeting to non equivariant spectra are $\pi_{*\mathbb{Q}}$ isomorphisms. The derived unit for the cofibrant generator W_+ (in this case categorical unit is also the derived unit) is an isomorphism and this follows from [MM02, Chapter V, Theorem 1.5], and thus for any cofibrant object it is an weak equivalence. By Part 3 of Proposition 1.1.12 this is a Quillen equivalence. ☐

We know that free $- W - \mathcal{IS}_{\mathbb{Q}_t}$ is equivalent as a monoidal category to the category $\mathcal{IS}_{\mathbb{Q}}[W]$ of rational orthogonal spectra with W action:

Lemma 10.3.9. *There is an equivalence of monoidal model categories between* free$-W-\mathcal{IS}_{\mathbb{Q}_t}$ *and* $\mathcal{IS}_{\mathbb{Q}}[W]$.

Proof. These two categories are equivalent as monoidal categories. However we had to take free model structure on the left hand side to make it into an equivalence of monoidal model categories. Suppose X is an object of free $- W - \mathcal{IS}_{\mathbb{Q}_t}$, i.e. for every indexing space $\mathbb{R}^n$ (trivial W–space), $X(V)$ has a W action compatible with the suspension. That gives a W action on the whole X considered as an object of $\mathcal{IS}$. Going in the other direction we extract W–action for every indexing space $\mathbb{R}^n$ from one W action on X. By slight abuse of notations we can consider these two functors to be identities.

We need to show that this two categories have the same model structure, i.e. all three classes of maps are the same. Firstly, weak equivalences on both sides are just non-equivariant $\pi_{*\mathbb{Q}}$–isomorphisms. Secondly, generating cofibrations in both model structures are the same, so we can deduce that these model categories are equivalent. ☐

We removed all dificulties comming from the equivariance with respect to a topological group. What is left now is a finite group action on the rational orthogonal spectra. It follows from Shipley's result that rational chain complexes model rational orthogonal spectra. Thus we apply Proposition 1.1.14 to the zig-zag of known Quillen equivalences below to get that W-objects in $\mathcal{IS}_{\mathbb{Q}}$ are modelled by W-objects in rational chain complexes, i.e. chain complexes of $\mathbb{Q}[W]$-modules.

To apply the resut of Shipley we need to work with rational symmetric spectra of the form of H$\mathbb{Q}$-modules. We pass to this category using the next two lemmas.

First we pass to symmetric spectra with a W action using the composition of forgetful functor and the functor induced by singular complex:

Lemma 10.3.10. *The adjunction*

$$\mathbb{P} \circ |-| \;:\; Sp^{\Sigma}_{\mathbb{Q}}[W] \rightleftarrows \mathcal{IS}_{\mathbb{Q}}[W] \;: \mathrm{Sing} \circ \mathbb{U}$$

is a strong monoidal Quillen equivalence.

Proof. This pair of functors give a monoidal Quillen equivalence on the categories without W action (see for example Section 7 in [SS03a] and proof of Theorem 3.1.4). Thus by Proposition 1.1.14 it is a Quillen equivalence when restricted to W–objects. ☐

Next we move to $H\mathbb{Q}$-modules in symmetric spectra with W action.

Lemma 10.3.11. *The adjunction*

$$H\mathbb{Q} \wedge - \ : \ Sp^{\Sigma}_{\mathbb{Q}}[W] \ \underset{\longleftarrow}{\overset{\longrightarrow}{}} \ (H\mathbb{Q} - \mathrm{mod})[W] \ : U$$

is a strong monoidal Quillen equivalence. U denotes forgetful functor and the model structure on $(H\mathbb{Q} - \mathrm{mod})[W]$ is the one created from $Sp^{\Sigma}[W]$ by the right adoint U.

Proof. As the model structure on the right hand side is created by U this is a Quillen adjunction. It is a Quillen equivalence when considered as the adjunction between categories without W–action since $H\mathbb{Q}$ is weakly equivalent to $S_{\mathbb{Q}}$. Thus by Proposition 1.1.14 it is a Quillen equivalence when restricted to W–objects.

Notice that the smash product on the right is over $H\mathbb{Q}$ which makes the left adjoint strong monoidal. ☐

From here we use the result of Shipley to get to $Ch(\mathbb{Q})[W]$ with the projective model structure, which is equivalent as a monoidal model category to $Ch(\mathbb{Q}[W])$ with the projective model structure (see Section 10.1).

Lemma 10.3.12. *There is a zig-zag of monoidal Quillen equivalences between the category $(H\mathbb{Q} - \mathrm{mod})[W]$ and the category $Ch(\mathbb{Q}[W])$.*

Proof. Apply Proposition 1.1.14 to the zig zag of Quillen equivalences from the Theorem 0.0.1 in case $R = \mathbb{Q}$. ☐

We can sumarise the results of this section in the theorem below.

Theorem 10.3.13. *There is a zig-zag of monoidal Quillen equivalences from $L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS})$ to $Ch(\mathbb{Q}[W] - \mathrm{mod})$.*

To illustrate the whole path we sketch below a diagram showing every step of the zig-zag. We put left Quillen functors on the left.

$$\begin{tikzcd}
L_{e_H S_{\mathbb{Q}}}(G - \mathcal{IS}) \arrow[d, shift right, "i^*"'] \\
L_{e_H S_{\mathbb{Q}}}(N - \mathcal{IS}) \arrow[u, shift right, "F_N(G_+, -)"'] \arrow[d, shift left, "(-)^H"] \\
L_{e_1 S_{\mathbb{Q}}}(W - \mathcal{IS}) \arrow[u, shift left, "\epsilon^*"] \arrow[d, shift left, "Id \text{ change of model structure}"] \\
\text{free} - W - \mathcal{IS}_{\mathbb{Q}_c} \arrow[u, shift left, "Id"] \arrow[d, shift left, "\mathrm{I}_c^t \quad \text{change of universe}"] \\
\text{free} - W - \mathcal{IS}_{\mathbb{Q}_t} \arrow[u, shift left, "\mathrm{I}_t^c"] \arrow[d, leftrightarrow, "\cong"', "\text{of monoidal model categories}"] \\
\mathcal{IS}_{\mathbb{Q}}[W] \arrow[d, shift left, "\mathrm{Sing} \circ \mathbb{U}"] \\
Sp^{\Sigma}_{\mathbb{Q}}[W] \arrow[u, shift left, "\mathbb{P} \circ |-|"] \arrow[d, shift right, "H\mathbb{Q} \wedge -"'] \\
(H\mathbb{Q} - \mathrm{mod})[W] \arrow[u, shift right, "U"'] \arrow[d, leftrightarrow, "\text{zig-zag of}"', "\text{Quillen equivalences}"] \\
Ch(\mathbb{Q}[W])
\end{tikzcd}$$

Remark 10.3.14. Note that we can do all of the above steps for any subgroup H of any compact

Lie group G, provided that $W_G(H)$ is finite, H does not contain any subgroups cotoral in H and there exist an idempotent e_H corresponding to the conjugacy class of H in G.

For any such H the forgetful functor from H part of $G - \mathcal{IS}$ to H part of $N_G H - \mathcal{IS}$ is a left adjoint of a Quillen equivalence. The rest of the construction follows the same pattern. That result provides a monoidal algebraic model for the exceptional part of rational G- spectra for any compact Lie group G. We define an exceptional part here to consists of finitely many conjugacy classes of exceptional subgroups.

If G is finite then every subgroup of G is exceptional and the category of G rational spectra splits as a finite product of categories, each localised at a conjugacy class of a subgroup of G. This observation allows us to deduce the following

Corollary 10.3.15. *Suppose G is a finite group. Then there is a zig-zag of monoidal Quillen equivalences from $L_{S_{\mathbb{Q}}}(G - \mathcal{IS})$ to $\prod_{(H), H \leq G} Ch(\mathbb{Q}[W_G H] - \mathrm{mod})$.*

Proof. This follows from the splitting result [Bar09a] and Theorem 10.3.13. □

Chapter 11

Dihedral part

The algebraic model for the dihedral part is almost identical to the algebraic model of the dihedral part for $O(2)$. The difference comes from two things.

Firstly, in $SO(3)$ every dihedral subgroup of order 2, namely D_2 is conjugate to cyclic subgroups C_2. Secondly, the normaliser of D_4 in $SO(3)$ is a subgroup Σ_4. That excludes subgroups conjugate to D_2 and subgroups conjugate to D_4 from our dihedral part $\mathcal{D}$.

We know from [Gre01] that the algebraic model for the homotopy category of the dihedral part of $SO(3)$ - equivariant rational spectra will be of the form of certain sheaves over an orbit space for $\mathcal{D}$. We use the notation $\mathcal{A}(\mathcal{D})$ for it and we devote Section 11.1 to this category.

We describe a zig-zag of non-monoidal Quillen equivalences, because we are not able to replace non -monoidal Morita equivalence by a monoidal one - we encounter the same problem as for the exceptional part of the model. We follow the approach from [Bar13], however we change the original proof slightly to avoid working with commutative ring spectra. We also choose to work with orthogonal spectra rather than S–modules [EKMM97].

The plan for this part is as follows. Let $Ch(\mathcal{A}(\mathcal{D}))$ denote the category of chain complexes in $\mathcal{A}(\mathcal{D})$.

First we choose a set of generators $\mathcal{G}$ for $L_{e_d S_{\mathbb{Q}}}(G-\mathcal{IS})$ which consists of one element $\hat{f}(e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n})_+$ for every conjugacy class of dihedral subgroups D_{2n} from $\mathcal{D}$ and $\hat{f}(e_d S_{\mathbb{Q}} \wedge SO(3)/O(2))_+$. Recall that $\hat{f}$ denotes fibrant replacement and the idempotent $e_{D_{2n}}$ is an idempotent of the Burnside ring $A(SO(3))$ corresponding to the characteristic function on the conjugacy class of the subgroup D_{2n}.

Now we show that $L_{e_d S_{\mathbb{Q}}}(G-\mathcal{IS})$ is a spectral model category, where the enrichment is over the symmetric spectra.

Using Morita equivalence we pass to the category $\text{mod}-\mathcal{E}_{\text{top}}$ of modules over $\mathcal{E}_{\text{top}}$, where $\mathcal{E}_{\text{top}}$ is the enriched endomorphism symmetric spectrum on the set of generators $\mathcal{G}$.

The next step is to use the result of Shipley (see [Shi07]) to pass from the category enriched over symmetric spectra to one enriched over rational chain complexes mod $-\,\mathcal{E}_{\mathrm{t}}$.

Now we calculate the homology groups of the chain complexes enrichment of $\mathcal{E}_{\mathrm{t}}$. As they all are concentrated in degree zero we can replace the category mod $-\,\mathcal{E}_{\mathrm{top}}$ by a Quillen equivalent category mod $-\,H_*(\mathcal{E}_{\mathrm{t}})$.

The last two steps are obtained starting from the algebraic end. We do the Morita equivalence for the category $Ch(\mathcal{A}(\mathcal{D}))$ choosing a set of generators $\mathcal{G}_{\mathrm{a}}$. That gives us a category mod $-\,\mathcal{E}_{\mathrm{a}}$, where $\mathcal{E}_{\mathrm{a}}$ is the enriched endomorphism chain complex category on the set of generators $\mathcal{G}_{\mathrm{a}}$.

In the last part we find the comparison between the enrichmed $\mathrm{Ch}(\mathbb{Q})$ categories $\mathcal{E}_{\mathrm{a}}$ and $H_*(\mathcal{E}_{\mathrm{t}})$. That leads to the final Quillen equivalence at the level of module categories and the algebraic model for the dihedral part.

After Theorem 11.2.16 we present a diagram which shows every step of this comparison. The reader may wish to refer to this diagram now, but the notation will be introduced as we proceed.

11.1 The category $Ch(\mathcal{A}(\mathcal{D}))$

First we recall the construction of $\mathcal{A}(\mathcal{D})$ (see [Gre01]) , then we concentrate on the model structure on $Ch(\mathcal{A}(\mathcal{D}))$ and show it is a dg model structure, i.e. it is a category enriched, tensored and cotensored in $Ch(\mathbb{Q}-\mathrm{mod})$ and satisfying the analogue of the pushout-product axiom or Quillen's SM7, where simplicial sets are replaced by symmetric spectra (see Definition 3.1.2).

Material in this section is based on [Bar13, Section 6.1], however we had to slightly adjust the definition of $\mathcal{A}(\mathcal{D})$ presented there. We define it below. Let W be a group of order two.

Definition 11.1.1. Define a category $\mathcal{A}(\mathcal{D})$ as follows.

An object M consists of a $\mathbb{Q}$–module M_∞, a collection $M_k \in \mathbb{Q}[W]$–mod for $k \geqslant 3$ and a map (called the germ map) of $\mathbb{Q}[W]$–modules $\sigma_M : M_\infty \longrightarrow \mathrm{colim}_{n>2} \prod_{k\geqslant n} M_k$, where W action on M_∞ is trivial.

A map $f : M \longrightarrow N$ in $\mathcal{A}(\mathcal{D})$ consists of a map $f_\infty : M_\infty \longrightarrow N_\infty$ of $\mathbb{Q}$–modules and a collection of maps of $\mathbb{Q}[W]$–modules $f_k : M_k \longrightarrow N_k$ which commute with germ maps σ_M and

σ_N:

$$\begin{tikzcd}
M_\infty \arrow[r, "\sigma_M"] \arrow[d, "f_\infty"] & \operatorname{colim}_{n>2} \prod_{k\geqslant n} M_k \arrow[d, "\operatorname{colim}_{n>2} \prod_{k\geqslant n} f_k"] \\
N_\infty \arrow[r, "\sigma_N"] & \operatorname{colim}_{n>2} \prod_{k\geqslant n} N_k
\end{tikzcd}$$

Definition 11.1.2. Define a category $Ch(\mathcal{A}(\mathcal{D}))$ to be the category of chain complexes in $\mathcal{A}(\mathcal{D})$ and $\mathrm{g}\mathcal{A}(\mathcal{D})$ to be a category of graded objects in $\mathcal{A}(\mathcal{D})$.

An object M of $Ch(\mathcal{A}(\mathcal{D}))$ consists of rational chain complex M_∞, a collection of chain complexes of $\mathbb{Q}[W]$–modules M_k for $k \geqslant 3$ and a germ map of chain complexes of $\mathbb{Q}[W]$–modules $\sigma_M : M_\infty \longrightarrow \operatorname{colim}_{n>2} \prod_{k\geqslant n} M_k$, where W action on M_∞ is trivial.

The following definition is from [Bar13] (after [Gre98b]) and describes a functor from the homotopy category of rational $O(2)$ spectra to $\mathrm{g}\mathcal{A}(\mathcal{D})$ called $\pi_*^{\mathcal{D}}$:

Definition 11.1.3. Let X be an $O(2)$–spectrum with rational homotopy groups. We define an object $\pi_*^{\mathcal{D}}(X)$ of $\mathrm{g}\mathcal{A}(\mathcal{D})$ as follows:

$$\pi_*^{\mathcal{D}}(X)_k = e_{D_{2k}}\pi_*^{D_{2k}}(X)$$

where $e_{D_{2k}} \in A(O(2))$ and

$$\pi_*^{\mathcal{D}}(X)_\infty = \operatorname{colim}_{n>2}(f_n\pi_*^{O(2)}(X))$$

where $f_n = e_{\mathcal{D}} - \sum_{k=1}^{n-1} e_k$.
If $k \geqslant n$ there is a map

$$f_n\pi_*^{O(2)}(X) \longrightarrow e_{D_{2k}}\pi_*^{D_{2k}}(X)$$

induced by the inclusion $D_{2k} \longrightarrow O(2)$ and the application of $e_{D_{2k}}$, so we get a map

$$f_n\pi_*^{O(2)}(X) \longrightarrow \prod_{k\geqslant n} e_{D_{2k}}\pi_*^{D_{2k}}(X)$$

and thus a germ map

$$\operatorname{colim}_{n>2} f_n\pi_*^{O(2)}(X) \longrightarrow \operatorname{colim}_{n>2} \prod_{k\geqslant n} e_{D_{2k}}\pi_*^{D_{2k}}(X)$$

This defines a functor $\pi_*^{\mathcal{D}} : Ho(O(2) - \mathrm{Spectra}_{\mathbb{Q}}) \longrightarrow \mathrm{g}\mathcal{A}(\mathcal{D})$.

To obtain a functor from $Ho(SO(3) - \mathrm{Spectra}_{\mathbb{Q}})$ to $\mathrm{g}\mathcal{A}(\mathcal{D})$ we need to precompose with the pullback functor along the inclusion $i : O(2) \longrightarrow SO(3)$, i.e.

$$i^* : Ho(SO(3) - \mathrm{Spectra}_{\mathbb{Q}}) \longrightarrow Ho(O(2) - \mathrm{Spectra}_{\mathbb{Q}})$$

We don't need to take the derived functor, because i^* preserves all weak equivalences.

It is useful to consider several adjoint pairs involving the category $Ch(\mathcal{A}(\mathcal{D}))$:

Definition 11.1.4. [Bar13, Definition 6.1.7] Let $A \in Ch(\mathbb{Q})$, $X \in Ch(\mathbb{Q}[W])$ and $M \in Ch(\mathcal{A}(\mathcal{D}))$. We define the following functors:

- $i_k : Ch(\mathbb{Q}[W]) \longrightarrow Ch(\mathcal{A}(\mathcal{D}))$ by $(i_k(X))_\infty = 0$ and $(i_k(X))_n = 0$ for $n \neq k$ and $(i_k(X))_k = X$.
- $p_k : Ch(\mathcal{A}(\mathcal{D})) \longrightarrow Ch(\mathbb{Q}[W])$ by $p_k(M) = M_k$
- $c : Ch(\mathbb{Q}) \longrightarrow Ch(\mathcal{A}(\mathcal{D}))$ by $(cA)_k = A$, $(cA)_\infty = A$ and σ_{cA} to be the diagonal map into the product.

Then (i_k, p_k), (p_k, i_k) and $(c, \pitchfork^W)$ form adjoint pairs, where the functor $\pitchfork^W$ is defined below.

The functor $\pitchfork^W$ is defined as follows: let $M \in Ch(\mathcal{A}(\mathcal{D}))$. Then $\pitchfork^W M$ is defined as the W fixed points of the pullback of the diagram below in the category $Ch(\mathbb{Q}[W])$–modules:

$$\begin{tikzcd}
 & \prod_{k\geqslant 3} M_k \arrow[d] \\
M_\infty \arrow[r, "\sigma_M"] & \operatorname{colim}_n \prod_{k\geqslant n} M_k
\end{tikzcd}$$

where W–action is trivial on M_∞ and the vertical arrow is the map into the colimit induced by the inclusion of the first term.

We should think about the functor $\pitchfork^W$ as a global sections functor: we pick an element of the stalk at ∞ point and elements of all the other stalks in a compatible way, which gives us a global section. Compatibility condition is captured by the pullback square.

There is a closed monoidal structure on the category $Ch(\mathcal{A}(\mathcal{D}))$ defined as follows: Let M, N be objects in $Ch(\mathcal{A}(\mathcal{D}))$, then $(M \otimes N)_k := M_k \otimes_{\mathbb{Q}} N_k$ and $(M \otimes N)_\infty := M_\infty \otimes_{\mathbb{Q}} N_\infty$ and the germ map is given by the following composite:

$$M_\infty \otimes N_\infty \longrightarrow \operatorname{colim}_n \prod_{k\geqslant n} M_k \otimes \operatorname{colim}_n \prod_{k\geqslant n} N_k \longrightarrow \operatorname{colim}_n (\prod_{k\geqslant n} M_k \otimes \prod_{k\geqslant n} N_k) \longrightarrow \operatorname{colim}_n \prod_{k\geqslant n} (M_k \otimes N_k)$$

where the second map is an isomorphism.

The internal hom is defined for k–part as follows:

$$\underline{Hom}(M, N)_k := Hom_{\mathbb{Q}}(M_k, N_k)$$

Internal hom over ∞–part is defined to be a W fixed points of a stalk over $(O(2))$ of an internal hom of sheaves of chain complexes of $\mathbb{Q}[W]$–modules M, N over a space $\mathcal{D}$. (Every object M of $Ch(\mathcal{A}(\mathcal{D})))$ can be viewed as a sheaf of chain complexes of $\mathbb{Q}[W]$–modules over $\mathcal{D}$.) Recall that $\mathcal{D}$ denotes conjugacy classes of subgroups from dihedral part of $SO(3)$.

The category $Ch(\mathcal{A}(\mathcal{D}))$ is bicomplete by [Bar13, Lemma 6.1.6] so we can proceed to defining a model structure on it:

Proposition 11.1.5. *[Bar13, Proposition 6.1.10] There exists a symmetric monoidal model structure on the category $Ch(\mathcal{A}(\mathcal{D}))$ where f is a weak equivalence or fibration if f_∞ and each of f_k are weak equivalences or fibrations respectively. This model structure is cofibrantly generated, proper and satisfies the monoid axiom.*

We call the above model structure a **projective model structure** on $Ch(\mathcal{A}(\mathcal{D}))$. The generating cofibrations are of the form $cI_\mathbb{Q}$ and $i_k I_{\mathbb{Q}[W]}$ for $k \geqslant 3$ and generating acyclic cofibrations are of the form $cJ_\mathbb{Q}$ and $i_k J_{\mathbb{Q}[W]}$ for $k \geqslant 3$. $I_\mathbb{Q}$ and $J_\mathbb{Q}$ denote generating cofibrations and generating trivial cofibrations (respectively) for the projective model structure on $Ch(\mathbb{Q})$, and $I_{\mathbb{Q}[W]}$, $J_{\mathbb{Q}[W]}$ denote generating cofibrations and generating trivial cofibrations (respectively) for the projective model structure on $Ch(\mathbb{Q}[W])$.

With the above definition of monoidal model structure the adjunctions from Definition 11.1.4 are strong symmetric monoidal Quillen pairs.

We finish this chapter by stating that $Ch(\mathcal{A}(\mathcal{D}))$ is a dg–model category

Proposition 11.1.6. *[Bar13, Corollary 6.1.12] The construction $\boxplus^W \underline{Hom}(-,=)$ defines an enrichment of $Ch(\mathcal{A}(\mathcal{D}))$ in $Ch(\mathbb{Q})$ compatible with the model structure on $Ch(\mathcal{A}(\mathcal{D}))$.*

Lemma 11.1.7. *[Bar13, Lemma 6.1.13] The collection $i_k\mathbb{Q}[W]$ for $k \geqslant 3$ and $c\mathbb{Q}$ form a set of (homotopically) compact, cofibrant and fibrant generators for the category $Ch(\mathcal{A}(\mathcal{D}))$ with a projective model structure.*

11.2 Comparison

We begin by establishing the necessary conditions for the first step, Morita equivalence. We start from the topological end.

Lemma 11.2.1. *Suppose we have a Quillen adjunction between two stable model categories*

$$F \ : \ \mathcal{C} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathcal{D} \ : U$$

such that there is a set $\{\sigma_i\}_{i\in I}$ of cofibrant, (homotopically) compact generators (in the sense of Definition 1.1.17) for $\mathcal{C}$, U reflects weak equivalences between fibrant objects and the derived functor of U (also denoted RU) commutes with coproducts. Then the images under F of $\{\sigma_i\}_{i\in I}$ form a set of (homotopically) compact, cofibrant generators for $\mathcal{D}$.

Proof. Suppose $\{\sigma_i\}_{i\in I}$ is a set of cofibrant, (homotopically) compact generators for $\mathcal{C}$. By Proposition 1.1.18 this is equivalent to the condition that for any object $X \in \mathcal{C}$, X is trivial iff $[\sigma_i, X]_* = 0$ for all $i \in I$.

$\{F(\sigma_i)\}_{i\in I}$ is a set of (homotopically) compact objects by the same argument as in Proposition 1.1.20 since RU commutes with coproducts. We use the above criterion for checking that the set of (homotopically) compact objects is a set of generators. Take Y in $\mathcal{D}$, and assume $\forall_{i\in I}[F(\sigma_i), Y]^{Ho(\mathcal{D})} = 0$ then

$$[F(\sigma_i), Y]^{Ho(\mathcal{D})} \cong [\sigma_i, U\hat{f}(Y)]^{Ho(\mathcal{C})}$$

where $\hat{f}$ denotes fibrant replacement functor in $\mathcal{D}$. Since $\{\sigma_i\}_{i\in I}$ was a set of cofibrant, (homotopically) compact generators for $\mathcal{C}$ we get that $U\tilde{f}(Y)$ is trivial in $\mathcal{C}$. Since U reflects weak equivalences between fibrant objects $\tilde{f}(Y)$ is trivial in $\mathcal{D}$ and thus Y is trivial in $\mathcal{D}$ which finishes the proof. □

Lemma 11.2.2. *The set $\hat{\mathcal{G}}$ consisting of one spectrum $e_{D_{2n}}SO(3)/D_{2n+}$ for every conjugacy class of dihedral subgroups D_{2n} from $\mathcal{D}$ and $SO(3)/O(2)_+$ is a set of (homotopically) compact, cofibrant generators for $L_{e_dS_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$.*

Proof. We can check the required conditions directly, but instead we use Lemma 11.2.1 for the adjunction $(SO(3)_+\wedge_{O(2)}-, i^*)$ and result of [Bar13] for the $O(2)$-spectra. First notice that i^* satisfies both conditions of Lemma 11.2.1. It preserves all weak equivalences because it is both left and right Quillen functor by Corollary 9.2.7 and Proposition 9.2.5. Therefore $Ri^*=i^*$ and since it is a left adjoint it preserves coproducts.

i^* reflects all weak equivalences by the observation: $i^*(f)$ is a weak equivalence iff $i^*(f)\wedge e_dS_{\mathbb{Q}}\cong i^*(f\wedge e_dS_{\mathbb{Q}})$ is a π_* isomorphism. Since, by adjunction on the homotopy level $\pi_*^H(i^*(f\wedge e_dS_{\mathbb{Q}}))\cong\pi_*^H(f\wedge e_dS_{\mathbb{Q}})$ we can conclude.

By [Bar13, Lemma 6.2.1] compact generators for the dihedral part of $O(2)$–spectra are of the form $e_{D_{2n}}O(2)/D_{2n+}$ and e_dS. Applying $SO(3)_+\wedge_{O(2)}-$ to them finishes the proof:

$$SO(3)_+\wedge_{O(2)}(e_{D_{2n}}O(2)/D_{2n+})\cong e_{D_{2n}}SO(3)/D_{2n+}$$

by Proposition 2.3 from Chapter V in [MM02], as below:

$$\begin{aligned}SO(3)_+\wedge_{O(2)}(e_{D_{2n}}O(2)/D_{2n+})&\cong SO(3)_+\wedge_{O(2)}(O(2)/D_{2n+}\wedge e_{D_{2n}}S)\\&\cong SO(3)/D_{2n+}\wedge e_{D_{2n}}S\cong e_{D_{2n}}SO(3)/D_{2n+}\end{aligned}\tag{11.2.3}$$

We used here that $i^*(e_{D_{2n}})=e_{D_{2n}}$ and that $i^*(e_d)=e_d$ where last idempotent corresponds to the set of all dihedral subgroups of $O(2)$ with order greater than 4 and $O(2)$. □

To finish discussion about generators, we present the following result about objects in $L_{e_dS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$, namely that i^* preserves generators up to weak equivalence.

Proposition 11.2.4.

1. *The map $f:O(2)/O(2)_+\longrightarrow i^*(SO(3)/O(2)_+)$ induced by inclusion $O(2)\longrightarrow SO(3)$ is a weak equivalence in $L_{e_dS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$.*
2. *The map $f_{2n}:e_{D_{2n}}O(2)/D_{2n+}\longrightarrow i^*(e_{D_{2n}}SO(3)/D_{2n+})$ for $n>2$ induced by inclusion $O(2)\longrightarrow SO(3)$ is a weak equivalence in $L_{e_dS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$*

Proof. For Part 1 to show that the map $f : O(2)/O(2)_+ \longrightarrow i^*(SO(3)/O(2)_+)$ is a weak equivalence, by definition we need to show that $e_d f$ is an equivariant rational π_* isomorphism. To do that we need to check that for all subgroups $H \leq O(2)$ the H-geometric fixed points

$$\Phi^H(e_d f) : \Phi^H(e_d O(2)/O(2)_+) \longrightarrow \Phi^H(e_d i^*(SO(3)/O(2)_+))$$

is a non-equivariant rational π_*-isomorphism.

Since geometric fixed points commute with smash product and suspensions, for every subgroup $H \notin \mathcal{D}$, $\Phi^H(e_d f)$ is a trivial map between trivial objects.
For $H = O(2)$ it is an identity on S^0 since $O(2)$ is its own normaliser in $SO(3)$ and $O(2)$ is $O(2)$–good in $SO(3)$.
For $H = D_{2n}$ it is also an identity on S^0 since every D_{2n} is $O(2)$–good in $SO(3)$ (there is just one conjugacy class for every n of D_{2n} subgroups in $O(2)$).

Part 2 follows the same pattern, however the domain and codomain of the map f_{2n} are already $\mathcal{D}$-local, so $f \cong e_d f$. Since the idempotent used is $e_{D_{2n}}$ the only non-trivial geometric fixed points will be for the subgroup $H = D_{2n}$. The result follows from simple observation: every D_{2n} is D_{2n}–good in $SO(3)$ and $O(2)$. Thus the map on geometric fixed points for D_{2n} is the identity on D_{4n}/D_{2n+}. That finishes the proof. □

We establish the enrichment necessary to proceed with Morita equivalence

Lemma 11.2.5. *$L_{e_d S_{\mathbb{Q}}}(G - \mathcal{IS})$ is a spectral model category, where the enrichment is in symmetric spectra* Sp^Σ.

Proof. Apply Corollary 3.1.5. □

Now that we have a spectral model category and the set of cofibrant and fibrant generators $\mathcal{G}$ consisting of fibrant replacements of elements from $\hat{\mathcal{G}}$, we can proceed to Morita equivalence.

Define $\mathcal{E}_{\mathrm{top}}$ to be the enriched endomorphism symmetric spectrum subcategory of $L_{e_d S_{\mathbb{Q}}}(G - \mathcal{IS})$ on the set of generators $\mathcal{G}$. We use the notation $\mathrm{mod} - \mathcal{E}_{\mathrm{top}}$ for the category of modules over $\mathcal{E}_{\mathrm{top}}$.

Theorem 11.2.6. *The adjunction*

$$\mathrm{Hom}(\mathcal{G}, -) \; : \; L_{e_d S_{\mathbb{Q}}}(G - \mathcal{IS}) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathrm{mod} - \mathcal{E}_{\mathrm{top}} \; : \; - \wedge_{\mathcal{E}_{\mathrm{top}}} \mathcal{G}$$

is a Quillen equivalence.

Proof. This follows from [SS03b, Theorem 3.9.3], see also Theorem 3.2.8. □

Similarly as for non-monoidal exceptional comparison, to apply result of Shipley, $\mathcal{E}_{\mathrm{top}}$ needs to be in $H\mathbb{Q}$-modules. The same argument as for exceptional part (see discussion below Theorem 10.2.4) works and we use the notation $\mathcal{E}_{\mathrm{top}}$ for $H\mathbb{Q} \wedge \mathcal{E}_{\mathrm{top}}$.

Morita equivalence puts all the equivariance into the ring (with many objects) over which we consider the modules. However we are still in the topological setting (the ring is the ring symmetric spectrum). The following theorem provides the passage to the algebraic world.

Theorem 11.2.7. *There is a zig-zag of Quillen equivalences between the following two categories: mod–$\mathcal{E}_{\mathrm{top}}$ (enriched over $Sp^{\Sigma}_{\mathbb{Q}}$) and mod–$\mathcal{E}_{\mathrm{t}}$ enriched over Ch($\mathbb{Q}$–mod), where $\mathcal{E}_{\mathrm{t}}$ is a ring rational chain complex with the property that there is an isomorphism of categories enriched over graded $\mathbb{Q}$ modules $H_*(\mathcal{E}_{\mathrm{t}}) \cong \pi_*(\mathcal{E}_{\mathrm{top}})$.*

Proof. This follows from the result of Shipley where we use a version for rings with many objects mentioned in [Shi07, Corollary 2.16] and described also in [Bar09b, Theorem 6.5]. □

Let us now calculate $\pi_*(\mathcal{E}_{\mathrm{top}})$. By construction, for any two objects $a, b \in \mathcal{G}$, $\pi_*(\mathcal{E}_{\mathrm{top}})(a, b)$ is naturaly isomorphic to $[a, b]^{L_{e_d S_{\mathbb{Q}}}(SO(3)-\mathcal{IS})}$ (this was explicitly calculated in 10.2.6). Thus we have the results below, where the second isomorphism in every line follows from the calculations from Proposition 11.2.4 and [Bar13, Proposition 6.2.9 and 6.2.10]

$$\begin{aligned}&\pi_*(\mathcal{E}_{\mathrm{top}})(e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+, e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+)) \cong \\ &\qquad [e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+, e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+]_* \cong [c\mathbb{Q}, c\mathbb{Q}]^{g\mathcal{A}(\mathcal{D})} \cong e_d A(O(2))_{\mathbb{Q}}\end{aligned} \tag{11.2.8}$$

and for every $n > 2$

$$\begin{aligned}&\pi_*(\mathcal{E}_{\mathrm{top}})(e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}, e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+})) \cong \\ &\qquad [e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}, e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}]_* \cong [i_{2n}\mathbb{Q}[W], i_{2n}\mathbb{Q}[W]]^{g\mathcal{A}(\mathcal{D})} \cong \mathbb{Q}[W]\end{aligned} \tag{11.2.9}$$

$$\begin{aligned}&\pi_*(\mathcal{E}_{\mathrm{top}})(e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+, e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+})) \cong \\ &\qquad [e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+, e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}]_* \cong [c\mathbb{Q}, i_{2n}\mathbb{Q}[W]]^{g\mathcal{A}(\mathcal{D})} \cong \mathbb{Q}\end{aligned} \tag{11.2.10}$$

$$\begin{aligned}&\pi_*(\mathcal{E}_{\mathrm{top}})(e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}, e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+)) \cong \\ &\qquad [e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}, e_d S_{\mathbb{Q}} \wedge SO(3)/O(2)_+]_* \cong [i_{2n}\mathbb{Q}[W], c\mathbb{Q}]^{g\mathcal{A}(\mathcal{D})} \cong \mathbb{Q}\end{aligned} \tag{11.2.11}$$

And finally, for $n \neq k$ we have

$$\begin{aligned}&\pi_*(\mathcal{E}_{\mathrm{top}})(e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}, e_{D_{2k}} S_{\mathbb{Q}} \wedge SO(3)/D_{2k+})) \cong \\ &[e_{D_{2n}} S_{\mathbb{Q}} \wedge SO(3)/D_{2n+}, e_{D_{2k}} S_{\mathbb{Q}} \wedge SO(3)/D_{2k+}]_* \cong [i_{2n}\mathbb{Q}[W], i_{2k}\mathbb{Q}[W]]^{g\mathcal{A}(\mathcal{D})} \cong 0\end{aligned} \tag{11.2.12}$$

As $H_*(\mathcal{E}_{\mathrm{t}})$ is isomorphic to $\pi_*(\mathcal{E}_{\mathrm{top}})$, it is concentrated in degree zero so we apply methods from [Bar09b, Theorem 7.5 and 7.6] to get the following

Lemma 11.2.13. *There is a zig zag of Quillen equivalences of Ch($\mathbb{Q}$) model categories*

$$\mathrm{mod}-\mathcal{E}_{\mathrm{t}} \rightleftarrows \mathrm{mod}-C_0\mathcal{E}_{\mathrm{t}} \rightleftarrows \mathrm{mod}-H_*\mathcal{E}_{\mathrm{t}}$$

induced by the zig-zag of quasi-isomorphisms of Ch($\mathbb{Q}$)-categories

$$\mathcal{E}_{\mathrm{t}} \longleftarrow C_0\mathcal{E}_{\mathrm{t}} \longrightarrow H_*\mathcal{E}_{\mathrm{t}}$$

where C_0 is the (-1)–connected cover functor on Ch($\mathbb{Q}$). The first map above is the inclusion and the second one is the projection on enriched homs.

Next we start work from the algebraic side. The idea is to perform a Morita equivalence in the algebraic setting for the category $Ch(\mathcal{A}(\mathcal{D}))$ which is a Ch($\mathbb{Q}$) model category by Proposition 11.1.6. We use the notation $\mathcal{G}_{\mathrm{a}}$ for the set of generators for $Ch(\mathcal{A}(\mathcal{D}))$ from Lemma 11.1.7.

Define $\mathcal{E}_{\mathrm{a}}$ to be the enriched endomorphism rational chain complex subcategory of $Ch(\mathcal{A}(\mathcal{D}))$ on the set of generators $\mathcal{G}_{\mathrm{a}}$. We use the notation mod $-\ \mathcal{E}_{\mathrm{a}}$ for the category of modules over $\mathcal{E}_{\mathrm{a}}$.

We proceed to algebraic version of Morita equivalence:

Theorem 11.2.14. *The adjunction*

$$\mathrm{Hom}(\mathcal{G}_{\mathrm{a}},-)\ :\ Ch(\mathcal{A}(\mathcal{D})) \rightleftarrows \mathrm{mod}-\mathcal{E}_{\mathrm{a}}\ :\ -\wedge_{\mathcal{E}_{\mathrm{a}}}\mathcal{G}_{\mathrm{a}}$$

is a Quillen equivalence.

Proof. This is enriched in chain complexes of $\mathbb{Q}$ modules version of Morita equivalence [SS03b, Theorem 3.9.3] □

The last missing link is the comparison between mod $-\ H_*\mathcal{E}_{\mathrm{t}}$ and mod $-\ \mathcal{E}_{\mathrm{a}}$ as model categories. This is shown in

Proposition 11.2.15. *There is an isomorphism of categories enriched over Ch($\mathbb{Q}$) given by $i^*\circ\pi_*^{\mathcal{D}}\ :\ H_*\mathcal{E}_{\mathrm{t}} \longrightarrow \mathcal{E}_{\mathrm{a}}$ (where $\pi_*^{\mathcal{D}}$ is described in Definition 11.1.3), which gives an isomorphism of module categories*

$$\mathrm{mod}-H_*\mathcal{E}_{\mathrm{t}} \longrightarrow \mathrm{mod}-\mathcal{E}_{\mathrm{a}}$$

Proof. This follows from [Bar13, Proposition 6.2.8 and 6.2.9].
First note that $\pi_*^{\mathcal{D}}\circ i^*(SO(3)/O(2)_+) = c\mathbb{Q}$ and $\pi_*^{\mathcal{D}}\circ i^*(e_{D_{2n}}SO(3)/D_{2n+}) = i_{2n}\mathbb{Q}[W]$ and that $\pi_*^{\mathcal{D}}\circ i^*$ induces isomorphisms on enriched homs on the generators:

Firstly, it follows from Proposition 11.2.4 that after applying i^* we get the generators for dihedral part of $O(2)$ from [Bar13]. Secondly, by Theorem 11.3.1 i^* is an equivalence on homotopy categories (dihedral parts), thus it induces isomorphisms on enriched homs on the generators. Calculations from [Bar13, Proposition 6.2.8] finish the proof. □

Now we summarise the result of this section in the following

Theorem 11.2.16. *There exist a zig-zag of Quillen equivalences from* $L_{e_dS_{\mathbb{Q}}}(G-\mathcal{IS})$ *to* $Ch(\mathcal{A}(\mathcal{D}))$.

To illustrate the whole path we present below a diagram showing every step of the zig-zag. Left adjoints are placed on the left.

$$\begin{tikzcd}
L_{e_dS_{\mathbb{Q}}}(G-\mathcal{IS}) \arrow[d, shift left, "\mathrm{Hom}(\mathcal{G},-)"] \\
\mathrm{mod}-\mathcal{E}_{\mathrm{top}} \arrow[u, shift left, "-\wedge_{\mathcal{E}_{\mathrm{top}}}\mathcal{G}"] \arrow[d, leftrightarrow, "\text{zig$-$zag of}"', "\text{Quillen equivalences}"] \\
\mathrm{mod}-\mathcal{E}_{\mathrm{t}} \arrow[d, leftrightarrow, "\text{zig$-$zag of}"', "\text{Quillen equivalences}"] \\
\mathrm{mod}-H_*(\mathcal{E}_{\mathrm{t}}) \arrow[d, leftrightarrow, "\text{isomorphism of}"', "\text{model categories}"] \\
\mathrm{mod}-\mathcal{E}_{\mathrm{a}} \arrow[d, shift right, "-\wedge_{\mathcal{E}_{\mathrm{a}}}\mathcal{G}_{\mathrm{a}}"'] \\
Ch(\mathcal{A}(\mathcal{D})) \arrow[u, shift right, "\mathrm{Hom}(\mathcal{G}_{\mathrm{a}},-)"']
\end{tikzcd}$$

We stress again that the above comparison is not monoidal.

11.3 Monoidal comparison

To obtain monoidal comparison for the dihedral part of $SO(3)$ spectra it is enough to get the monoidal comparison for the dihedral part of $O(2)$ spectra, as the restriction functor is a monoidal Quillen equivalence between these dihedral parts:

Theorem 11.3.1. *Let* $i : O(2) \longrightarrow SO(3)$ *be an inclusion and note that* $i^*(e_dS_{\mathbb{Q}}) = e_dS_{\mathbb{Q}}$. *Then the following*

$$i^* \ : \ L_{e_dS_{\mathbb{Q}}}(SO(3)-\mathcal{IS}) \rightleftarrows L_{e_dS_{\mathbb{Q}}}(O(2)-\mathcal{IS}) \ : \ F_{O(2)}(SO(3)_+,-)$$

is a strong monoidal Quillen equivalence, where the idempotent on the right hand side corresponds to the set of all dihedral subgroups of order greater than 4 and $O(2)$.

Proof. This is a Quillen adjunction by Corollary 9.2.7 and moreover i^* is also a right Quillen functor by Proposition 9.2.5 since $i^*(e_d S_{\mathbb{Q}}) = e_d S_{\mathbb{Q}}$. The left adjoint is strong monoidal as localisation didn't change the monoidal structures.

To show that this is a Quillen equivalence we use similar argument to the one for the general exceptional N subgroup (see Theorem 10.3.4).

Firstly, $F_{O(2)}(SO(3)_+, -)$ preserves and reflects weak equivalences between fibrant objects by the same argument as in Theorem 10.3.4.

Now we need to show that the derived unit is a weak equivalence on cofibrant objects in $L_{e_d S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$. This is

$$Y_+ \longrightarrow F_{O(2)}(SO(3)_+, \hat{f}i^*(Y))$$

To check that this is a weak equivalence in $L_{e_d S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$ it is enough to check that on the homotopy level the induced map

$$[X, Y]^{L_d SO(3)} \cong [X, e_d Y]^{SO(3)} \longrightarrow [X, F_{O(2)}(SO(3)_+, \hat{f}i^*(e_d Y))]^{SO(3)}$$

is an isomorphism for every generator X of $L_{e_d S_{\mathbb{Q}}}(SO(3){-}\mathcal{IS})$. This map fits into a commuting diagram below

$$\begin{tikzcd}
{[X, e_d Y]^{SO(3)}} \arrow[d] \arrow[dr, "i^*"] & \\
{[X, F_{O(2)}(SO(3)_+, \hat{f}i^*(e_d Y))]^{SO(3)}} & {[i^* X, \hat{f}i^*(e_d Y)]^{O(2)}} \arrow[l, "\cong"']
\end{tikzcd}$$

Since the horizontal map is an isomorphism it is enough to show that i^* is an isomorphism on hom sets, where the domain is a generator for $L_{e_d S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$. We do this by using the second Quillen adjunction between these two categories, namely $(SO(3)_+ \wedge_{O(2)} -, i^*)$.

We have the following commuting diagram

$$\begin{tikzcd}
{[e_H SO(3)/H_+, e_d Y]^{SO(3)}} \arrow[d, "\cong"'] \arrow[dr, "i^*"] & \\
{[e_H O(2)/H_+, i^*(e_d Y)]^{O(2)}} & {[i^*(e_H SO(3)/H_+), i^*(e_d Y)]^{O(2)}} \arrow[l, "- \circ \eta"']
\end{tikzcd}$$

where H above denotes the finite dihedral subgroup or $O(2)$ (When H is $O(2)$ we understand e_H as e_d).

η above denotes the categorical unit on cofibrant generators, which is the map

$$\eta_{e_H O(2)/H_+} : e_H O(2)/H_+ \longrightarrow e_H i^*(SO(3)/H_+)$$

induced by an inclusion $O(2) \longrightarrow SO(3)$.

By Proposition 11.2.4 this is a weak equivalence in $L_{e_d S_{\mathbb{Q}}}(O(2) - \mathcal{IS})$ for all H in $\mathcal{D}$ and thus $- \circ \eta$ is an isomorphism on homotopy level. From this it follows that i^* is an isomorphism on hom sets and thus the derived unit of the adjunction where i^* is the left adjoint is a weak equivalence in $L_{e_d S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$ and this adjunction is a Quillen equivalence. $\square$

Chapter 12

Cyclic part

In this chapter we follow the approach of [Bar13] and [GS] to find an algebraic model for the cyclic part of $SO(3)$-spectra.

We begin by describing the category $d(\mathcal{A}(SO(3), c))$ in Section 12.1. Then we proceed to establishing the comparison between the cyclic part of rational $SO(3)$ equivariant orthogonal spectra and $d(\mathcal{A}(SO(3), c))$. This comparison is monoidal up to the category cell – $\mathcal{A}(O(2), c)$. We simplify this category and get $d(\mathcal{A}(SO(3), c))$, however this last comparison is not monoidal.

12.1 The category $d(\mathcal{A}(SO(3), c))$

Before we are ready to describe the category $\mathcal{A}(SO(3), c)$ we have to introduce the category $\mathcal{A}(O(2), c)$. We give a description of $\mathcal{A}(O(2), c)$ as a category on the objects of $\mathcal{A}(SO(2))$ with W action. Recall that W is a group of order 2. We later pass to $\mathcal{A}(SO(3), c)$ using the adjunction described in Theorem 12.1.28.

12.1.1 The category $\mathcal{A}(O(2), c)$

Material in this section is based on [Gre99] and [Bar13, Section 3].

Firstly, we need some definitions

Definition 12.1.1. Let $\mathcal{F}$ denote the family of all finite cyclic subgroups in $O(2)$. Then we define a ring in $\mathbb{Q}[W]$ modules

$$\mathcal{O}_{\mathcal{F}} := \prod_{H \in \mathcal{F}} \mathbb{Q}[c_H]$$

where each c_H has degree -2 and w acts on each c_H by -1. For simplicity in further notation we set $c := c_1$.

We use the notation $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ for the following colimit of localisations:

$$\operatorname*{colim}_k \mathcal{O}_{\mathcal{F}}[c^{-1}, c_{C_2}^{-1}, ..., c_{C_k}^{-1}]$$

where the maps in the colimit are the inclusions. Notice that we can do the similar construction on the ring $\widetilde{\mathcal{O}_{\mathcal{F}}} := (1-e_1)\mathcal{O}_{\mathcal{F}}$ and call it $\widetilde{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}$, where e_1 is the projection on the first factor in the ring $\mathcal{O}_{\mathcal{F}}$. Then another way to define $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is as $\mathbb{Q}[c, c^{-1}] \times \widetilde{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}$.

This last description of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ will be useful when we compare this model to the one for cyclic part of $SO(3)$ spectra.

Definition 12.1.2. An object of $\mathcal{A}(O(2), c)$ consists of a triple (M, V, β) where M is an $\mathcal{O}_{\mathcal{F}}$ module in $\mathbb{Q}[W]$ modules, V is a graded rational vector space with a W action and β is a map of $\mathcal{O}_{\mathcal{F}}$ modules (in $\mathbb{Q}[W]$ modules)

$$\beta : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$$

such that $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} \beta$ is an isomorphism of $\mathcal{O}_{\mathcal{F}}$ modules.

A morphism between two such objects $(\alpha, \phi) : (M, V, \beta) \longrightarrow (M', V', \beta')$ consists of a map of $\mathcal{O}_{\mathcal{F}}$ modules $\alpha : M \longrightarrow M'$ and a map of graded $\mathbb{Q}[W]$-modules such that the relevant square commutes.

Notice that instead of modules over $\mathcal{O}_{\mathcal{F}}$ in $\mathbb{Q}[W]$ modules we can consider modules over $\mathcal{O}_{\mathcal{F}}[W]$ in $\mathbb{Q}$ modules, where $\mathcal{O}_{\mathcal{F}}[W]$ is a group ring with a twisted W action (namely W acts on each c_H by -1). We will use this description in the next section.

Definition 12.1.3. An object of $d(\mathcal{A}(O(2), c))$ is an object of $\mathcal{A}(O(2), c)$ equipped with a differential and a morphism in this category is a morphism in $\mathcal{A}(O(2), c)$ which commutes with the differentials.

Next we proceed to discussing the properties of the category $d(\mathcal{A}(O(2), c))$.

Proposition 12.1.4. *All limits and colimits exist in $d(\mathcal{A}(O(2), c))$.*

Proof. This follows from [Bar13, Definition 3.2.4]. □

Definition 12.1.5. [Bar13, Definition 3.2.5] There is a symmetric monoidal product in $d(\mathcal{A}(O(2), c))$, which for two objects (M, V, β) and (M', V', β') is defined as the following composite:

$$\beta \otimes \beta' : M \otimes_{\mathcal{O}_{\mathcal{F}}} M' \longrightarrow (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) \otimes_{\mathcal{O}_{\mathcal{F}}} (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V') \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (V \otimes_{\mathbb{Q}} V')$$

where W action is diagonal.

The tensor product defined above is closed and the function object construction is given by the same construction as in [Bar13, Definition 3.2.7].

Now we proceed to discuss model structures and the following follows from [Gre99, Appendix A]

Theorem 12.1.6. *There is a stable, proper, monoidal, model structure on the category $d(\mathcal{A}(O(2), c))$ where the weak equivalences are homology isomorphisms. The cofibrations are the injections and the fibrations are defined via the right lifting property. We call this model structure the injective model structure.*

Theorem 12.1.7. *[Bar13, Theorem 3.4.2] There is a cofibrantly generated, stable, proper, monoidal model structure on the category $d(\mathcal{A}(SO(2)))$ with weak equivalences the homology isomorphisms. The generating cofibrations have the form*

$$S^{n-1} \otimes P \longrightarrow D^n \otimes P$$

for $P \in \mathcal{P}$ and $n \in \mathbb{Z}$ where $\mathcal{P}$ is the set of representatives for the isomorphisms classes of dualisable objects. We call this model structure the dualisable model structure.

Theorem 12.1.8. *[Bar13, Theorem 3.5.4] There is a cofibrantly generated, stable, proper, monoidal model structure on the category $d(\mathcal{A}(O(2), c))$ with weak equivalences the homology isomorphisms. Fibrations are these maps which forget to fibrations in the dualisable model structure on $d(\mathcal{A}(SO(2)))$. We call this a dualisable model structure.*

12.1.2 The category $\mathcal{A}(SO(3), c)$

Looking at the cyclic part of the spaces of subgroups of $SO(3)$ and $O(2)$ we see that the stabiliser of the trivial subgroup is connected in $SO(3)$, while it is not in $O(2)$. This seems to be the main ingredient playing a role in capturing the difference between these algebraic models.

Let us denote by $\mathcal{F}_{SO(3)}$ the family of all finite cyclic subgroups in $SO(3)$. Then we use the notation $\mathcal{O}_{\overline{\mathcal{F}}} := \mathcal{O}_{\mathcal{F}_{SO(3)}}$, by which we mean a graded ring $\prod_{(H)\leq \mathcal{F}_{SO(3)}} \mathbb{Q}[c_{(H)}]$ where c_1 is in degree -4 and all other $c_{(H)}$ are in degree -2. W acts on it by fixing c_1 and sending $c_{(H)}$ to $-c_{(H)}$ for all subgroups $H \leq SO(3)$, $H \neq 1$. For simplicity we will use the notation $d := c_1$ in that ring. We define the ring $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ as a group ring with the twisted W action away from e_1 and $e_1\mathcal{O}_{\overline{\mathcal{F}}}[W] = \mathbb{Q}[d]$.

We define the category $\mathcal{A}(SO(3), c)$ as follows

Definition 12.1.9. An object in $\mathcal{A}(SO(3), c)$ consists of a triple (M, V, β) where M is an $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ module in $\mathbb{Q}$ modules, V is a graded rational vector space with a W action and β is a map of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ modules

$$\beta : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$$

such that the adjoint $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ is an isomorphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]$ modules.

A morphism between two such objects $(\alpha, \phi) : (M, V, \beta) \longrightarrow (M', V', \beta')$ consists of a map of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ modules $\alpha : M \longrightarrow M'$ and a map of graded $\mathbb{Q}[W]$-modules such that the relevant square commutes.

Notice that the condition on the map β implies that the image of $e_1 M$ lies in $(\mathbb{Q}[c, c^{-1}] \otimes V)^W$.

We define the category $\hat{A}(SO(3), c)$ to consist of triples (M, V, β) where M is an $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ module in $\mathbb{Q}$ modules, V is a graded rational vector space with a W action and β is a map of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ modules

$$\beta : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$$

A morphism between two such objects $(\alpha, \phi) : (M, V, \beta) \longrightarrow (M', V', \beta')$ consists of a map of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ modules $\alpha : M \longrightarrow M'$ and a map of graded $\mathbb{Q}[W]$-modules such that the relevant square commutes.

Now we construct a functor Γ which will be a right adjoint to the inclusion functor $i : \mathcal{A}(SO(3), c) \longrightarrow \hat{\mathcal{A}}(SO(3), c)$.

Suppose $X = (\gamma : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ is an object in $\hat{\mathcal{A}}(SO(3), c)$. Then we define $\Gamma(X)$ in two steps.

Step 1. We start by adjoining γ:

$$\overline{X} = (\overline{\gamma} : \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

Note that this is an object of $\hat{\mathcal{A}}(O(2), c)$. Now we apply Γ for the $O(2)$ case to it, which by construction is the following pullback (see [Gre99, Definition 20.2.2] for definition of V' and $\overline{\gamma}'$):

$$\begin{tikzcd}
(\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M)' \arrow[r, "\overline{\gamma}'"] \arrow[d] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V' \arrow[d] \\
\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M \arrow[r] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M
\end{tikzcd}$$

where we use the notation ' for the result of Γ in $O(2)$ case.

Step 2. There is a unit map $M \longrightarrow \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M$, so we form the second pullback square

$$\begin{tikzcd}
M'' \arrow[r] \arrow[d] & (\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M)' \arrow[r, "\overline{\gamma}'"] \arrow[d] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V' \arrow[d] \\
M \arrow[r, "\text{unit}"] & \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M \arrow[r] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M
\end{tikzcd}$$

$\Gamma(X)$ is defined to be the top row of the above diagram, i.e. $\gamma'' : M'' \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V'$.

We need to show that this is an object of $\mathcal{A}(SO(3), c)$. It is clear that the second pullback construction is the identity away from the idempotent e_1.

Firstly we define $M^\dagger$ by the following pullback:

$$\begin{tikzcd}
M^\dagger \arrow[r] \arrow[d] & \mathcal{E}^{-1}\mathcal{O}_{\overline{\mathcal{F}}} \otimes V' \arrow[d] \\
M \arrow[r] & \mathcal{E}^{-1}\mathcal{O}_{\overline{\mathcal{F}}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M
\end{tikzcd}$$

where $\mathcal{E}^{-1}\mathcal{O}_{\overline{\mathcal{F}}} = \mathbb{Q}[d,d^{-1}] \times \widetilde{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}$. Now we get

$$\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M^\dagger \cong (\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M)'$$

Using these observations, we see that the second pullback diagram used for defining Γ above, has the following form:

$$\begin{tikzcd}
M^\dagger \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M^\dagger \arrow[d] \\
M \arrow[r, "\text{unit}"] & \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M
\end{tikzcd}$$

since the right vertical map is of the form $\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} -$ applied to a map $M^\dagger \longrightarrow M$ the top map is also a unit. Thus γ'' is the following composite

$$M^\dagger \xrightarrow{\text{unit}} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M^\dagger \xrightarrow{\overline{\gamma}'} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} \mathcal{E}^{-1}\mathcal{O}_{\overline{\mathcal{F}}} \otimes V' = \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V'$$

Recall that by the construction the adjoint of $\overline{\gamma}'$, i.e. a map

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M^\dagger \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V'$$

is an isomorphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]$ modules. Since the adjoint of γ'' is the same as adjoint of $\overline{\gamma}'$ we can conclude.

Remark 12.1.10. Notice that the construction above is the same as defining Γ as the composite $G \circ \tilde{\Gamma} \circ \hat{F}$ from the diagram below, where $\tilde{\Gamma}$ is the right adjoint to the inclusion for cyclic part of $O(2)$ algebraic model. The horizontal adjunctions are defined after Definition 12.1.26. Notice that the diagram of left adjoints commutes.

$$\begin{tikzcd}
\mathcal{A}(SO(3),c) \arrow[r, shift left, "F=\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\overline{\mathcal{F}}}}-"] \arrow[d, shift right, "i"'] & \mathcal{A}(O(2),c) \arrow[l, shift left, "G=(-)^W\times \mathrm{Id}"] \arrow[d, shift right, "\tilde{i}"'] \\
\hat{\mathcal{A}}(SO(3),c) \arrow[r, shift left, "\hat{F}=\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\overline{\mathcal{F}}}}-"] \arrow[u, shift right, dotted, "\Gamma"'] & \hat{\mathcal{A}}(O(2),c) \arrow[l, shift left, "\hat{G}=(-)^W\times \mathrm{Id}"] \arrow[u, shift right, "\tilde{\Gamma}"']
\end{tikzcd}$$

Lemma 12.1.11. Γ *constructed above is the right adjoint to the inclusion functor*

$$i : \mathcal{A}(SO(3),c) \longrightarrow \hat{\mathcal{A}}(SO(3),c)$$

Proof. By the remark above we get a natural bijection:

$$\mathrm{Hom}_{\mathcal{A}(SO(3),c)}(X, G\circ\tilde{\Gamma}\circ\hat{F}(Y)) \cong \mathrm{Hom}_{\hat{\mathcal{A}}(O(2),c)}(\tilde{i}\circ F(X), \hat{F}(Y))$$

and by commutativity of the square of left adjoints we get

$$\mathrm{Hom}_{\hat{\mathcal{A}}(O(2),c)}(\tilde{i}\circ F(X), \hat{F}(Y)) \cong \mathrm{Hom}_{\hat{\mathcal{A}}(O(2),c)}(\hat{F}\circ i(X), \hat{F}(Y))$$

Since $\hat{F}$ is faithful that is $\mathrm{Hom}_{\hat{\mathcal{A}}(SO(3),c)}(i(X),(Y))$, which finishes the proof.

□

Proposition 12.1.12. *All small limits and colimits exist in* $\mathcal{A}(SO(3),c)$.

Proof. Suppose we have a diagram of objects $M_i \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes V_i$ in $\mathcal{A}(SO(3),c)$ indexed by a category I. The colimit of this diagram is

$$\operatorname*{colim}_i M_i \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes(\operatorname*{colim}_i V_i)$$

The limit is formed in a category $\hat{\mathcal{A}}(SO(3),c)$ first, and then we apply Γ. The limit of the above diagram in $\hat{\mathcal{A}}(SO(3),c)$ is $f : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes(\lim_i V_i)$ constructed using the following pullback:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "f"] & \lim_i M_i \arrow[d] \\
\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes(\lim_i V_i) \arrow[r] & \lim_i(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes V_i)
\end{tikzcd}$$

□

Now we define two functors which will be used in the next Lemma.
Let $e : \mathbb{Q}[W] - \mathrm{mod} \longrightarrow \mathcal{A}(SO(3), c)$, where $\mathbb{Q}[W] - \mathrm{mod}$ denotes the category of graded $\mathbb{Q}[W]$ modules, be defined by

$$e(V) := (P \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

where $e_1 P = \mathbb{Q}[d, d^{-1}] \otimes V^+ \oplus \Sigma^2 \mathbb{Q}[d, d^{-1}] \otimes V^-$ and $(1 - e_1)P = (1 - e_1)\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$. Here V^+ is the W-fixed part of V and V^- is -1 eigenspace.

Let $f : \mathrm{tors} - \mathcal{O}_{\overline{\mathcal{F}}}[W]^f - \mathrm{mod} \longrightarrow \mathcal{A}(SO(3), c)$, where $\mathrm{tors} - \mathcal{O}_{\overline{\mathcal{F}}}[W]^f - \mathrm{mod}$ denotes the category of $\overline{\mathcal{F}}$-finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$-modules, be defined by

$$f(N) := (N \longrightarrow 0)$$

Recall that an $\mathcal{O}_{\overline{\mathcal{F}}}[W]$-module M is $\overline{\mathcal{F}}$-finite if it is a direct sum of its submodules $e_{(H)}M$:

$$M = \bigoplus_{(H) \in \overline{\mathcal{F}}} e_{(H)}M$$

Proposition 12.1.13. *For any object* $X = (\gamma : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ *in* $\mathcal{A} = \mathcal{A}(SO(3), c)$*, any* V *in* $\mathbb{Q}[W] - \mathrm{mod}$ *and any* N *in* $\mathrm{tors} - \mathcal{O}_{\overline{\mathcal{F}}}[W]^f - \mathrm{mod}$ *we have natural isomorphisms:*

$$Hom_{\mathcal{A}}(X, e(V)) = Hom_{\mathbb{Q}[W]}(U, V)$$

$$Hom_{\mathcal{A}}(X, f(N)) = Hom_{\mathcal{O}_{\overline{\mathcal{F}}}[W]}(M, N)$$

Remark 12.1.14. The above proposition implies that an object $e(V)$ is injective for any V and if N is an injective $\overline{\mathcal{F}}$-finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$-module then $f(N)$ is also injective.

Lemma 12.1.15. *The category* $\mathcal{A}(SO(3), c)$ *is a (graded) abelian category of an injective dimension 1. Moreover it is split, i.e. every object* X *of* $\mathcal{A}(SO(3), c)$ *has a splitting* $X = X_+ \oplus X_-$ *so that* $Hom(X_\delta, Y_\epsilon) = 0$ *and* $Ext(X_\delta, Y_\epsilon) = 0$ *if* $\delta \neq \epsilon$ *and* $(\Sigma X)_\delta = \Sigma(X_{\delta+1})$

Proof. This category is enriched in abelian groups and by construction (and existence) of all limits and colimits we can conclude that it is an abelian category.

For an object $X = (\gamma : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ we construct the injective resolution of length 1 as follows: let $TM := \ker\gamma$, which is torsion, and thus there is an injective resolution

$$0 \longrightarrow TM \longrightarrow I' \longrightarrow J' \longrightarrow 0$$

where I', J' are injective $\overline{\mathcal{F}}$ finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}[W]$ modules, since $\mathbb{Q}[d]$ and all $\mathbb{Q}[c_{(H)}][W]$ are of injective dimension 1.

Let us use a simplified notation below. Let P denote the $\mathcal{O}_{\overline{\mathcal{F}}}$ module from the definition of $e(V)$ presented above.

If Q is the image of γ then $J'' = P/Q$ is divisible and an $\overline{\mathcal{F}}$ finite torsion $\mathcal{O}_{\overline{\mathcal{F}}}$ module and hence injective. We form a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & TM \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I' \arrow[r] \arrow[d] & I' \oplus P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & J' \arrow[r] \arrow[d] & J' \oplus J'' \arrow[r] \arrow[d] & J'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and hence a diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & I' \oplus P \arrow[r] \arrow[d] & J' \oplus J'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \arrow[r] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

which is the required resolution.

Finally, the splitting is given by taking even graded part and odd graded part and this satisfies the required conditions since the resolution above of an object X_δ is entirely in parity δ. □

By the [Gre99, Proposition 4.1.3] we can construct the derived category of $\mathcal{A}(SO(3), c)$ by taking differential objects in $\mathcal{A}(SO(3), c)$ and inverting the homology isomorphisms.

Now we define a closed symmetric monoidal product

Definition 12.1.16. There is a symmetric monoidal product on $\mathcal{A}(SO(3), c)$ defined analogously to the one presented in Definition 12.1.5. For two objects (M, V, β) and (M', V', β') it is defined as the following composite:

$$\begin{aligned}\beta \otimes \beta' : M \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M' \longrightarrow (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V) \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V') \cong \\ (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}) \otimes (V \otimes V') \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes (V \otimes V') \quad (12.1.17)\end{aligned}$$

where W action is diagonal and the second arrow is induced by the counit on $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$. The unit for this tensor product is $\mathcal{O}_{\overline{\mathcal{F}}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}$.

Lemma 12.1.18. *The tensor product defined above is closed.*

Proof. Suppose $A = (\gamma : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ and $B = (\gamma' : M' \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')$ are two objects in $\mathcal{A}(SO(3), c)$. Then the internal hom $\underline{Hom}(A, B)$ is constructed by applying Γ to the object $\delta : Q \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes Hom_{\mathbb{Q}}(U, U')$ defined by the pullback square below:

$$\begin{tikzcd}
Q \arrow[r, "\delta"] \arrow[dd] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes Hom_{\mathbb{Q}}(U, U') \arrow[d] \\
 & Hom_{\mathcal{O}_{\overline{\mathcal{F}}}}(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U') \arrow[d] \\
Hom_{\mathcal{O}_{\overline{\mathcal{F}}}}(M, M') \arrow[r] & Hom_{\mathcal{O}_{\overline{\mathcal{F}}}}(M, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')
\end{tikzcd}$$

☐

Definition 12.1.19. An object of $d(\mathcal{A}(SO(3), c)$ consists of $\mathcal{O}_{\overline{\mathcal{F}}}[W]$-module M equipped with a differential and a chain complex of $\mathbb{Q}[W]$ modules V together with a map $\gamma : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V$ which commutes with differentials. A differential on a $\mathcal{O}_{\overline{\mathcal{F}}}[W]$-module M consists of maps $d_n : M_n \longrightarrow M_{n-1}$ such that $d_{n-1} \circ d_n = 0$ and $cd_n = d_{n-2}c$ where c is the sum of all elements $c^2_{(H)}$ for varying $(H) \in \overline{\mathcal{F}}, H \neq 1$ and d.

A morphism in this category is a morphism in $\mathcal{A}(SO(3), c)$ which commutes with the differentials.

Notice that all above constructions (Γ, limits and colimits, tensor product and internal hom) pass to the category $d(\mathcal{A}(SO(3), c)$.

Theorem 12.1.20. *There is an injective model structure on the category $d(\mathcal{A}(SO(3), c)$.*

Proof. Since the category $\mathcal{A}(SO(3), c)$ is abelian of injective dimension 1 we can use the construction from [Gre99, Appendix A]. ☐

To show that this model structure is right proper we need to show that there are enough "wide spheres" in $\mathcal{A}(SO(3), c)$

Definition 12.1.21. We define d^n to be an element of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ of the form $(c^{2n}, c^{2n}, c^{2n}, c^{2n}, ...)$ and we call it an Euler class. Notice, that we can view an Euler class as an element of $\mathcal{O}_{\overline{\mathcal{F}}}$ of the form $(d^n, c^{2n}, c^{2n}, c^{2n}, ...)$, hence the name.
We define c^n to be an element of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ of the form $(c^{2n+1}, c^{2n+1}, c^{2n+1}, c^{2n+1}, ...)$ and we call it a c-Euler class.

Definition 12.1.22. A wide sphere in $\mathcal{A}(SO(3), c)$ is an object $P = (S \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes T)$ where T is a graded $\mathbb{Q}[W]$-module, which is finitely generated as a $\mathbb{Q}$-module on elements $t_1, ..., t_d$, where each t_i is either W-fixed or W acts on t_i by -1 and $deg(t_i) = k_i$. S is an $\mathcal{O}_{\overline{\mathcal{F}}}$ submodule of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes T$ generated by elements $\tilde{c}^{a_i} \otimes t_1, ..., \tilde{c}^{a_d} \otimes t_d$ where $\tilde{c}^{a_i}$ is either an Euler classes if t_i is W fixed or a c-Euler class if W acts on t_i by -1, and an element $\sum_{i=1}^{d} \sigma_i \otimes t_i$ of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes T$. It is also required that the structure map is the inclusion.

Proposition 12.1.23. *There are enough wide spheres in $\mathcal{A}(SO(3), c)$*

Proof. We need to show that for any object $X = (\beta : N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ in $\mathcal{A}(SO(3), c)$ and any $n \in N$ there exists a wide sphere P and a map $P \longrightarrow X$ such that n is in the image and for any $u \in U$ there exists a wide sphere $\overline{P}$ and a map $\overline{P} \longrightarrow X$ such that u is in the image. Since adjoint of β is an isomorphism it is enough to show the above condition for any $n \in N$.

Take $X = (\beta : N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ in $\mathcal{A}(SO(3), c)$ and $n \in N$. Then $\beta(n) = \sum_{i=1}^{d} \sigma_i \otimes t_i$. We may assume that every t_i is either W fixed or W acts on t_i by -1. Then notice that since $e_1\beta(n)$ is W fixed $e_1\sigma_i$ will be of the form c^{2k} if t_i was W fixed or c^{2k+1} if W acts on t_i by -1 (k is any integer).

For each i there exist $p_i \in N$ s.t. $\beta(p_i) = d^{b_i} \otimes t_i$ (d^{b_i} is an Euler class) if t_i was W fixed or $\beta(p_i) = c^{b_i} \otimes t_i$ (c^{b_i} is a c-Euler class) if W acts on t_i by -1. Set $f = d^{b_1+\ldots+b_d}$. We may assume that b_i's were chosen so that $\sigma_i d^{b_1+\ldots+b_d}/d^{b_i}$ is in $\mathcal{O}_{\overline{\mathcal{F}}}$ if t_i was W fixed and $\sigma_i c^{-1} d^{b_1+\ldots+b_d}/d^{b_i}$ is in $\mathcal{O}_{\overline{\mathcal{F}}}$ if W acts on t_i by -1.

Now we have

$$\beta(\overset{+}{\sum} \sigma_i d^{b_1+\ldots+b_d}/d^{b_i} p_i + \overset{-}{\sum} \sigma_i c^{-1} d^{b_1+\ldots+b_d}/d^{b_i} p_i) = \sum_{i=1}^{d} \sigma_i f \otimes t_i = \beta(fn)$$

where $\sum^{+}$ denotes the sum over all t_i which are W fixed and $\sum^{-}$ denotes the sum over all the others.

Since adjoint of β is an isomorphism there exists an Euler class d^b such that

$$d^b(\overset{+}{\sum} \sigma_i d^{b_1+\ldots+b_d}/d^{b_i} p_i + \overset{-}{\sum} \sigma_i c^{-1} d^{b_1+\ldots+b_d}/d^{b_i} p_i) = d^b f n$$

We take d^b to be the smallest such Euler class.

We take a wide sphere $P = (S \longrightarrow \mathcal{E}_{\mathcal{O}_{\mathcal{F}}}^{-1} \otimes T)$ where T is a $\mathbb{Q}$ vector space generated by t_i for $i = 1, ..., d$, $deg(t_i) = k_i$ and S is an $\mathcal{O}_{\overline{\mathcal{F}}}$ submodule of $\mathcal{E}_{\mathcal{O}_{\mathcal{F}}}^{-1} \otimes T$ generated by $\sum_{i=1}^{d} \sigma_i \otimes t_i$ and $d^b f \otimes t_i$ if t_i is W fixed and $d^b c^{-1} f \otimes t_i$ if W acts on t_i by -1. The structure map is the inclusion.

To finish the proof we set a map from P to X by sending $\sum_{i=1}^{d} \sigma_i \otimes t_i$ to n and $d^b f \otimes t_i$ to $d^b d^{b_1+\ldots+b_d}/d^{b_i} p_i$ if t_i is W fixed and $d^b c^{-1} f \otimes t_i$ to $d^b c^{-1} d^{b_1+\ldots+b_d}/d^{b_i} p_i$ if W acts on t_i by -1.

The Euler classes d^b and f are needed to ensure that the relation between n and p_i's after applying β is replicated in the wide sphere.

□

Proposition 12.1.24. *The injective model structure on* $d(\mathcal{A}(SO(3),c))$ *is proper.*

Proof. Since cofibrations are monomorphism it is left proper. To show that it is right proper notice that among trivial cofibrations there are maps $0 \longrightarrow D^n \otimes P$, for any $P \in \mathcal{P}$, where $D^n \otimes P$ denotes an object built from P and ΣP with the differential being the identity map from suspension of P to P and $\mathcal{P}$ denotes the set of isomorphism classes of wide spheres. Thus fibrations are in particular surjections. Right properness follows from the fact that in $d(\hat{\mathcal{A}}(SO(3),c))$ pullbacks along surjections of homology isomorphisms are homology isomorphisms and Γ preserves homology isomorphisms by [Bar13][Proposition 3.4.6]. ☐

Corollary 12.1.25. *The category* $d(\mathcal{A}(SO(3),c))$ *is a Grothendieck category.*

Proof. Since directed colimits are exact in $d(\mathcal{A}(SO(3),c))$ it remains to show that there is a (categorical) generator. We take $J := \bigoplus_{P\in\mathcal{P}} P$ where $\mathcal{P}$ is a set of all wide spheres. By Proposition 12.1.23 $Hom(J,-)$ is faithful and thus J is a categorical generator. ☐

Next we define a set of objects which will be (homotopical) generators for the category $d(\mathcal{A}(SO(3),c))$ with the injective model structure

Definition 12.1.26. We define a set $\mathcal{K}$ in $d(\mathcal{A}(SO(3),c))$ to consist of all suspensions and desuspensions of the following objects:

$$\mathbb{Q}_1 \longrightarrow 0$$

$$\{\mathbb{Q}[W]_{(H)} \longrightarrow 0\}_{H\in\overline{\mathcal{F}}-\{1\}}$$

where $\mathbb{Q}[W]$ is on a place corresponding to (H) and an object

$$M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]$$

where e_1M consists of $\mathbb{Q}[d] \oplus \Sigma^2\mathbb{Q}[d]$, $(1-e_1)M = (1-e_1)\mathcal{O}_{\overline{\mathcal{F}}} \otimes \mathbb{Q}[W]$ and the map is the inclusion.

We proceed to study the adjunction relating $d(\mathcal{A}(SO(3),c))$ and $d(\mathcal{A}(O(2),c))$.

The map of rings $\mathbb{Q}[d] \longrightarrow \mathbb{Q}[c]$ defined by $d \mapsto c^2$ extends to the map of rings

$$f : \mathcal{O}_{\overline{\mathcal{F}}} \longrightarrow \mathcal{O}_{\mathcal{F}}$$

(with the identity on all the factors except for the first one). This induces the following adjunction:

$$F = \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} - \ : \ d(\mathcal{A}(SO(3),c)) \rightleftarrows d(\mathcal{A}(O(2),c)) \ : \ R = (-)^W \times \mathrm{Id}$$

where the left adjoint is defined as follows.
Take $X = (\gamma : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$ in $d(\mathcal{A}(SO(3),c))$. Then

$$F(X) := (\overline{\gamma} : \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

It is easy to see that this construction gives an object in $d(\mathcal{A}(O(2),c))$, since the adjoint of $\overline{\gamma}$ is the same as adjoint of γ and thus it is an isomorphism.

The right adjoint is defined as follows.
Take $Y = (\delta : N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ in $d(\mathcal{A}(O(2),c))$. Then

$$R(Y) := (\delta \circ i : (e_1 N)^W \times (1-e_1)N \longrightarrow N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$$

where i is the inclusion.

Now we show that the adjoint condition for $\delta \circ i$ holds. Thus we want to show that

$$\overline{\delta \circ i} : \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} ((e_1 N)^W \times (1-e_1)N) \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U$$

is an isomorphism of $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}[W]$ modules.

Notice that we have a natural map

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (\varepsilon_N) : \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} ((e_1 N)^W \times (1-e_1)N) \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (N)$$

where ε is the counit of the adjunction:

$$\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} - \ : \ \mathcal{O}_{\overline{\mathcal{F}}}[W] - \mathrm{mod} \rightleftarrows \mathcal{O}_{\mathcal{F}}[W] - \mathrm{mod} \ : \ (-)^W \times Id$$

After applying e_1 this map is an isomorphism for finitely generated modules $e_1 N$ and since every module is a colimit of finitely generated ones and $\otimes$ commute with colimits it is an isomorphism. Since it is an isomorphism away from e_1 it is an isomorphism. To complete the argument notice that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\overline{\mathcal{F}}}} ((e_1 N)^W \times (1-e_1)N) \arrow[r, "\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (\varepsilon_N)"] \arrow[dr, "\overline{\delta \circ i}"'] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} (N) \arrow[d, "\overline{\delta}"] \\
 & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U
\end{tikzcd}$$

It is easy to see that this is an adjoint pair as it is enough to check that we have a natural bijection of sets after applying e_1. Now we have the following

Lemma 12.1.27. *The above adjunction is a Quillen pair when we consider both categories with the injective model structures.*

Proof. The left adjoint is exact, so it preserves cofibrations and H_* isomorphisms. ☐

Theorem 12.1.28. *The composition*

$$\begin{tikzcd} d(\mathcal{A}(SO(3),c)) \arrow[r, shift left, "F"] & d(\mathcal{A}(O(2),c)) \arrow[l, shift left, "R"] \arrow[r, shift left, "-\otimes\widetilde{\mathbb{Q}}"] & d(\mathcal{A}(O(2),c)) \arrow[l, shift left, "-\otimes\widetilde{\mathbb{Q}}"] \end{tikzcd}$$

is a Quillen pair when we consider both model categories with the injective model structures (where F and R are defined above). Moreover the composite right adjoint preserves all weak equivalences and the unit is a weak equivalence for every element of $\mathcal{K}$, where $\mathcal{K}$ is given in Definition 12.1.26. Thus the following is a Quillen equivalence, where we denote the composite functors by $\widetilde{F}$ and $\widetilde{R}$ respectively:

$$\begin{tikzcd} \mathcal{K}-\mathrm{cell}-d(\mathcal{A}(SO(3),c)) \arrow[r, shift left, "\widetilde{F}"] & \widetilde{F}(\mathcal{K})-\mathrm{cell}-d(\mathcal{A}(O(2),c)) \arrow[l, shift left, "\widetilde{R}"] \end{tikzcd}$$

Proof. The only part which needs a proof is that the derived unit is an isomorphism for every element of $\mathcal{K}$. Since the right adjoint preserves all weak equivalences it is enough to show that the categorical unit is a weak equivalence. We apply $\widetilde{F}$ to cells from $\mathcal{K}$ (ignoring suspensions as they work in the same way in both categories):

$$\widetilde{F}(\mathbb{Q}_1 \longrightarrow 0) = \widetilde{\mathbb{Q}} \oplus \Sigma^{-2}\mathbb{Q} \longrightarrow 0$$

where c sends $\widetilde{\mathbb{Q}}$ to $\mathbb{Q}$

$$\widetilde{F}(\mathbb{Q}[W]_{(H)} \longrightarrow 0) = \mathbb{Q}[W]_H \longrightarrow 0$$

where the left $\mathbb{Q}[W]$ is in the place corresponding to (H) and the resulting $\mathbb{Q}[W]$ is in the place corresponding to H. This holds for all $(H) \in \overline{\mathcal{F}}$ except for $H = 1$.

$$\widetilde{F}(M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]) = \Sigma^2\widetilde{\mathbb{Q}} + \mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W] \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]$$

where c acts on $\widetilde{\mathbb{Q}}$ in degree 2 by sending it to $\mathbb{Q} \subseteq \mathbb{Q}[W]$ in degree 0 and the map is the inclusion.

Notice that for all cells $\sigma \in \mathcal{K}$ we have $\widetilde{R}\widetilde{F}(\sigma) = \sigma$ and thus the unit of this adjunction is a weak equivalence on all $\sigma \in \mathcal{K}$.

It follows from the Cellularisation Principle [GS, Proposition A.6.] (see also Theorem 1.2.12) that the adjunction $\widetilde{F}, \widetilde{R}$ between cellularised model categories is a Quillen equivalence. □

It remains to show that the model structure $\mathcal{K} - \mathrm{cell} - d(\mathcal{A}(SO(3),c))$ is equal to the injective model structure on $d(\mathcal{A}(SO(3),c))$. To do that we will show that the set of cells $\mathcal{K}$ is a set of generators for the injective model structure on $d(\mathcal{A}(SO(3),c))$.

Theorem 12.1.29. *The set of all suspensions and desuspensions of the following cells*

$$\sigma_1 := (\mathbb{Q}_1 \longrightarrow 0)$$

where $\mathbb{Q}$ *is at the place indexed by the trivial subgroup, for every* $H \in \overline{\mathcal{F}}$, $H \neq 1$

$$\sigma_{(H)} := (\mathbb{Q}[W]_{(H)} \longrightarrow 0)$$

where $\mathbb{Q}[W]$ *is at the place indexed by the conjugacy class of a subgroup* H, *and*

$$\sigma_T := (M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W])$$

where $e_1 M = \mathbb{Q}[d] \oplus \Sigma^2\mathbb{Q}[d]$ *and* $(1-e_1)M = (1-e_1)\mathcal{O}_{\mathcal{F}}$, *in* $d(\mathcal{A}(SO(3),c))$ *is the set of (homotopically) compact generators for the category* $d(\mathcal{A}(SO(3),c))$ *with the injective model structure.*

Proof. First note that $\sigma_T = (\mathcal{O}_{\overline{\mathcal{F}}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}) \oplus (N \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \widetilde{\mathbb{Q}})$, where $e_1 N = \Sigma^2\mathbb{Q}[d]$ and $(1-e_1)N = (1-e_1)\mathcal{O}_{\mathcal{F}} \otimes \widetilde{\mathbb{Q}}$ and both structure maps are inclusions. We will call the first summand S^0 and the second σ_T^-. Therefore it is enough to show that all suspensions and desuspensions of σ_1, σ_H, σ_T^-, S^0 for all $H \in \mathcal{F}$, $H \neq 1$ form a set of generators. We will call this set $\mathcal{L}$.

We need to show that these cells are homotopically compact objects. All cells are homotopically compact since they are compact and the fibrant replacement commutes with the direct sum.

We will show that if $[\sigma, X]^{\mathcal{A}}_* = 0$, for all $\sigma \in \mathcal{L}$ then $H_*(X) = 0$ and thus X is weakly equivalent to 0. By Lemma 12.1.15 and [Gre99, Lemma 4.2.4] we can use the following short exact sequence (Adams short exact sequence) to calculate the maps in the derived category of $\mathcal{A} = \mathcal{A}(SO(3),c)$ from X to Y in $d\mathcal{A}$:

$$0 \longrightarrow \mathrm{Ext}_{\mathcal{A}}(\Sigma H_*(X), H_*(Y)) \longrightarrow [X,Y]^{\mathcal{A}}_* \longrightarrow \mathrm{Hom}_{\mathcal{A}}(H_*(X), H_*(Y)) \longrightarrow 0$$

Let us make the following observation: for every $X \in d(\mathcal{A}(SO(3),c))$, where

$$X = (\gamma : P \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V)$$

we have the following fibre sequence

$$\widehat{X} \longrightarrow X \longrightarrow e(V)$$

where $e(V)$ is the functor described before Proposition 12.1.13 and $\widehat{X}$ is the fibre of the map $X \longrightarrow e(V)$.

By definition, a structure map in $e(V)$ is an inclusion, and thus it is a torsion free object. Let us simplify the notation: Let

$$E\mathcal{F}_+ = \bigoplus_{H \in \mathcal{F}, H \neq 1} ((\Sigma^{-2}\mathbb{Q}[c_H, c_H^{-1}]/\mathbb{Q}[c_H]) \longrightarrow 0) \oplus (\Sigma^{-2}\mathbb{Q}[d, d^{-1}]/\mathbb{Q}[d] \longrightarrow 0)$$

We call the H summand in the above by α_H

now

$$\widehat{X} := E\mathcal{F}_+ \otimes X$$

Now observe that every summand α_H in $E\mathcal{F}_+$ is built as a sequential colimit from suspensions of $\alpha_H^n = (\mathbb{Q}[c_H]/{c_H}^n \longrightarrow 0)$ and inclusions, or if it is the last summand α_1 it is built as a sequential colimit of $\alpha_1^n = (\mathbb{Q}[d]/d^n \longrightarrow 0)$ and inclusions, and thus

$$[\sigma_K, \widehat{X}]_*^{\mathcal{A}} = [\sigma_K, E\mathcal{F}_+ \otimes X]_*^{\mathcal{A}} \cong [\sigma_K, \bigoplus_H (\alpha_H \otimes X)]_*^{\mathcal{A}} \cong \bigoplus_i [\sigma_K, \alpha_H \otimes X]_*^{\mathcal{A}}$$

where the last isomorphism follows since σ_K is a (homotopically) compact object. For all H, α_H^n is a dualisable object (by [Gre99][Corollary 2.3.7 and Lemma 2.4.3]), and thus we can proceed:

$$[\sigma_K, \alpha_H \otimes X]_*^{\mathcal{A}} \cong [\sigma_K, \operatorname*{colim}_n \alpha_H^n \otimes X]_*^{\mathcal{A}} \cong \operatorname*{colim}_i [\sigma_K, Hom(D(\alpha_H^n), X)]_*^{\mathcal{A}} \cong \operatorname*{colim}_i [D(\alpha_H^n) \otimes \sigma_K, X]$$

since $D(\alpha_H^n) \otimes \sigma_K = 0$ if $K \neq H$ and every $D(\alpha_H^n) \otimes \sigma_H$ is finitely built from σ_H we have that $[D(\alpha_H^n) \otimes \sigma_H, X] = 0$ and thus $[\sigma_H, \widehat{X}]_*^{\mathcal{A}} = 0$ for all $H \in \mathcal{F}$.

Now take X to be an object in $d(\mathcal{A}(SO(3), c))$ and assume that $[\sigma, X]_*^{\mathcal{A}} = 0$ for all $\sigma \in \mathcal{L}$. By the above calculation it follows that $[\sigma_H, \widehat{X}]_*^{\mathcal{A}} = 0$ for all $H \in \mathcal{F}$.

From the Adams short exact sequence we get that

$$\mathrm{Hom}_{\mathcal{A}}(H_*(\sigma_H), H_*(\widehat{X})) = \mathrm{Hom}_{\mathcal{A}}(\sigma_H, H_*(\widehat{X})) = 0 = e_H H_*(\widehat{X}) = 0$$

Since $H_*(\widehat{X}) = \bigoplus_{H \in \mathcal{F}} e_H H_*(\widehat{X})$ we get that $\widehat{X}$ is weakly equivalent to 0 and thus $[S^0, \widehat{X}]_*^{\mathcal{A}} = 0$ and $[\sigma_T^-, \widehat{X}]_*^{\mathcal{A}} = 0$.

Now, by the fibre sequence and the fact that every fibre sequence induces a long exact sequence on $[E, -]$ we get that $[\sigma, e(V)]_*^{\mathcal{A}} = 0$ for every $\sigma \in \mathcal{L}$. From the Adams short exact sequence we get that

$$\mathrm{Hom}_{\mathcal{A}}(H_*(S^0), H_*(e(V))) = \mathrm{Hom}_{\mathcal{A}}(S^0, H_*(e(V))) = H_*^+(e(V)) = 0$$

$$\mathrm{Hom}_{\mathcal{A}}(H_*(\sigma_T^-), H_*(e(V))) = \mathrm{Hom}_{\mathcal{A}}(\sigma_T^-, H_*(e(V)) = H_*^-(e(V)) = 0$$

where $H_*^+(e(V))$ is the W fixed part of $H_*(e(V))$ and $H_*^-(e(V))$ denotes -1 eigenspace. Since $H_*(e(V)) = H_*^+(e(V)) \oplus H_*^-(e(V))$ we get that $e(V)$ is weakly equivalent to 0. Since the fibre sequence induces a long exact sequence in homology we conclude that $H_*(X) = 0$ and thus X is weakly equivalent to 0 which finishes the proof.

□

The following result follows from Theorem 12.1.28 and Theorem 12.1.29

Corollary 12.1.30. *The following is a Quillen equivalence, where $\mathcal{K}$ is given in Definition 12.1.26 and $d(\mathcal{A}(SO(3), c))$ is considered with the injective model structure:*

$$\begin{tikzcd} d(\mathcal{A}(SO(3), c)) \arrow[r, shift left, "\widetilde{F}"] & \widetilde{F}(\mathcal{K}) - \text{cell} - d(\mathcal{A}(O(2), c)) \arrow[l, shift left, "\widetilde{R}"] \end{tikzcd}$$

12.2 Restriction to the cyclic part of $O(2)$ rational spectra

One idea for the comparison is to restrict the cyclic part of the category of $SO(3)$–orthogonal spectra to the cyclic part of the category of $O(2)$–orthogonal spectra. It is important to use the functor i^* as a left adjoint, because the adjunction $(SO(3)_+ \wedge_{O(2)} -, i^*)$ is not a Quillen pair for the model categories localised at the cyclic part (by a similar argument as the one in Proposition 9.2.3).

Then we can proceed with the slight modification of the proof of the algebraic model for cyclic part of $O(2)$–spectra done in [Bar13] cellularizing the zig-zag of Quillen equivalences along the way. We modify the proof from [Bar13] to avoid working with commutative ring spectra. To do that we make more use of left Bousfield localisations.

Another method for comparison would be to apply the strategy of [Bar13] from the beginning and work with diagrams of $SO(3)$–spectra. General ideas would be the same, but subtle modifications would again be needed for the methods from [Bar13].

We choose to work using the first method. We start by establishing generators for the cyclic part of $SO(3)$ rational spectra.

Proposition 12.2.1. *A set K consisting of one $SO(3)$–spectrum $SO(3)/C_{n+}$ for every natural $n > 0$ and an $SO(3)$–spectrum $SO(3)/SO(2)_+$ is a set of cofibrant (homotopically) compact generators for the category $L_{e_c S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$.*

Proof. All objects in K are homotopically compact in $L_{e_c S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$ by the same calculation as 10.2.2. This is a set of generators for $L_{e_c S_{\mathbb{Q}}}(SO(3) - \mathcal{IS})$ by Proposition 1.1.18 and [MM02, Chapter IV, Proposition 6.7]. □

Now we restrict to the cyclic part of $O(2)$ rational spectra.

Proposition 12.2.2. *Let $i : O(2) \longrightarrow SO(3)$ be an inclusion. Then the following*

$$i^* \ : \ L_{e_c S_{\mathbb{Q}}}(SO(3) - \mathcal{IS}) \rightleftarrows L_{e_c S_{\mathbb{Q}}}(O(2) - \mathcal{IS}) \ : \ F_{O(2)}(SO(3)_+, -)$$

is a strong monoidal Quillen adjunction, where the idempotent on the right hand side corresponds to the family of all cyclic subgroups of $O(2)$.

Proof. This follows from Proposition 9.2.6 and the following composition of Quillen adjunctions:

$$\begin{tikzcd}
L_{e_c S_{\mathbb{Q}}}(SO(3) - \mathcal{IS}) \arrow[r, shift left, "i^*"] & L_{i^*(e_c S_{\mathbb{Q}})}(O(2) - \mathcal{IS}) \arrow[l, shift left, "F_{O(2)}(SO(3)_+,-)"] \arrow[r, shift left, "Id"] & L_{e_c S_{\mathbb{Q}}}(O(2) - \mathcal{IS}) \arrow[l, shift left, "Id"]
\end{tikzcd}$$

Note that $i^*(e_c S_{\mathbb{Q}})$ has non-trivial geometric fixed points not only for all cyclic subgroups of $O(2)$ but also for D_2, as D_2 is conjugate to C_2 in $SO(3)$. To get rid of that and take into account only cyclic part we use the further localisation. □

Theorem 12.2.3. *Suppose K is the set of generators for $L_{e_cS_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$ as established in Proposition 12.2.1 together with all their suspensions and desuspensions. Then the following*

$$i^* \ : \ L_{e_cS_{\mathbb{Q}}}(SO(3)-\mathcal{IS}) \rightleftarrows i^*(K)-\mathrm{cell}-L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS}) \ : F_{O(2)}(SO(3)_+,-)$$

is a strong monoidal Quillen equivalence, where the idempotent on the right hand side corresponds to the family of all cyclic subgroups of $O(2)$.

Proof. The fact this is a Quillen adjunction follows from Proposition 12.2.2 and a cellularization principle from [GS, Appendix A] for K and $i^*(K)$. Notice that since K was a set of generators for the category $L_{e_cS_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$ cellularization with respect to K will not change this model structure.

Firstly, all of the generators from K are (homotopically) compact and cofibrant by Proposition 12.2.1. Now we need to check that their images with respect to i^* are homotopically compact, i.e. suspension spectra of $SO(3)/C_{n+}$ for all n and $SO(3)/SO(2)_+$ as cyclic $O(2)$ spectra. By Lemma 11.2.1 it is enough to show they are homotopically compact as $O(2)$ spectra and this follows from the fact that a smooth, compact G-manifold admits a structure of a finite G-CW complex ([Ill00]) and a suspension spectrum of a finite G-CW complex is homotopically compact.

It remains to show that the derived unit maps on generators are weak equivalences. As in the proof for exceptional subgroup (Theorem 10.3.4) it is enough to check the induced map on the level of homotopy categories. This is equivalent to showing that the derived Li^* is an isomorphism on hom sets. This holds by [Gre01, Theorem 6.1] which states that if $X \cong e_cX$ then Li^* is an isomorphism:

$$[X,Y]^{SO(3)} \longrightarrow e_c[i^*X, i^*Y]^{O(2)}$$

which implies that

$$Li^* : [X,Y]^{L_{e_c}SO(3)} \cong [e_cX, e_cY]^{SO(3)} \longrightarrow e_c[i^*(e_cX), i^*(e_cY)]^{O(2)} \cong [i^*X, i^*Y]^{L_{e_c}O(2)}$$

is an isomorphism. From this it follows that the adjunction is a Quillen equivalence.

To show that this is a strong monoidal Quillen equivalence we refer to the monoidal version of cellularization principle from [Bar13, Section 5]. Firstly, we notice that K is a monoidal set of cofibrant objects and the unit is cofibrant in $L_{e_cS_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$. Since the left adjoint is strong monoidal, and this is a Quillen adjunction by non-monoidal cellularisation principle, the unit in $i^*(K)-\mathrm{cell}-L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$ is also cofibrant. Thus, by [Bar13, Proposition 5.2.7] this is a symmetric monoidal Quillen equivalence. □

It is enough now to give the monoidal algebraic model for the cyclic part of $O(2)$ rational spectra, since the above monoidal Quillen equivalence provides a link to the cyclic part of $SO(3)$ rational spectra. Below we present the slight modification of the construction from

[Bar13, Section 4]. To provide the algebraic model for $SO(3)$ rational spectra we need to cellularise every step of this passage with respect to derived images of $i^*(K)$ from Theorem 12.2.3. Notice, that by [Bar13, Proposition 5.2.7] the set $i^*(K)$ is a stable monoidal set of cofibrant objects, thus the resulted cellularisation will be monoidal.

12.3 Comparison for cyclic part of $O(2)$ rational spectra

12.3.1 Generalized diagrams

Let us recall that a family of subgroups of a Lie group G is a collection of subgroups closed under conjugation and taking subgroups. For every family $\mathcal{F}$ there is a universal G-space (G-CW complex) $E\mathcal{F}$ (see for example [LMSM86, Definition 2.10]) such that:

$$E\mathcal{F}^H \simeq \begin{cases} *, & \text{if } H \in \mathcal{F}. \\ \emptyset, & \text{otherwise.} \end{cases}$$

where $(-)^H$ denotes H fixed points and $\simeq$ means nonequivariantly equivalent.

Now let us take $\mathcal{F}$ to be the family consisting of the torus and its subgroups in $O(2)$. We take an $O(2)$-space $E\mathcal{F}_+$ together with a map to S^0 which sends basepoint to the basepoint and everything else to the other point $E\mathcal{F}_+ \longrightarrow S^0$. We can take a cofibre of this map and we call it $\tilde{E}\mathcal{F}$ (see for example [LMSM86, Definition 2.10]). Notice that

$$\tilde{E}\mathcal{F}^H \simeq \begin{cases} S^0, & \text{if } H \notin \mathcal{F}. \\ *, & \text{otherwise.} \end{cases}$$

We define a commutative ring spectrum $DE\mathcal{F}_+ := F(E\mathcal{F}_+, S)$, where $F(E\mathcal{F}_+, S)$ denotes the functional spectrum, i.e. right adjoint to smashing with an $O(2)$ space $E\mathcal{F}_+$ applied to S. There is a unit map $\lambda : S \longrightarrow DE\mathcal{F}_+$.

Definition 12.3.1. We define $\mathcal{M}$ to be the following diagram of model categories and adjoint Quillen pairs with left adjoints on the top:

$$\begin{tikzcd} L_{e_cS_{\mathbb{Q}}}(DE\mathcal{F}_+ - \text{mod}) \arrow[r, shift left, "Id"] & L_{e_cS_{\mathbb{Q}}\wedge\tilde{E}\mathcal{F}}(DE\mathcal{F}_+ - \text{mod}) \arrow[l, shift left, "Id"] \arrow[r, shift right, "\lambda^*"'] & L_{e_cS_{\mathbb{Q}}\wedge\tilde{E}\mathcal{F}}(O(2) - \mathcal{IS}) \arrow[l, shift right, "DE\mathcal{F}_+\wedge(-)"'] \end{tikzcd}$$

Localisation of $DE\mathcal{F}_+ -$mod at $e_cS_{\mathbb{Q}}$ is understood to be the localisation at $e_cS_{\mathbb{Q}} \wedge DE\mathcal{F}_+$. We decided to keep the shorter notation.

Now we consider the category $\mathcal{M}$–mod of generalised diagrams (see Section 1.4). The idea in that step is to separate the behaviour of finite cyclic subgroups from the behaviour of the torus. It is not possible to completely seperate these two parts, as we don't have idempotents of the rational Burnside ring to do this. However, the leftmost localised category of $DE\mathcal{F}_+$-mod captures the behaviour of finite cyclic subgroups. The rightmost category captures the behaviour of the torus. The middle category and the maps in the diagram show how the two previous categories interact.

Note that all three model categories are proper by [BR14, Proposition 3.7], stable and cofibrantly generated. Therefore by Theorem 1.4.1 there is an injective model structure on the category $\mathcal{M}$-mod which is cofibrantly generated, proper stable monoidal and satisfies the monoid axiom, see Lemma 1.4.3.

Notice further that all three categories are cellular, which means that the category of $\mathcal{M}$-modules is cellular.

Before we proceed, we need to describe the set of cofibrant generators for $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$

Lemma 12.3.2. *The set $\mathcal{J}$ consisting of one $O(2)$–spectrum $O(2)/C_{n+}$ for every natural n, $n > 0$ and a $O(2)$–spectrum $O(2)/SO(2)_+$ is a set of cofibrant (homotopically) compact generators for the category $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$.*

Proof. The same as proof of Proposition 12.2.1. ☐

Now we construct an adjoint pair between $\mathcal{M}$-mod and $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$. Suppose we have an object X of $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$. Then we construct an object of $\mathcal{M}$-mod, denoted $\mathcal{M}\wedge X$ as follows:

$$\mathcal{M}\wedge X := (DE\mathcal{F}_+\wedge X, \mathrm{Id}, DE\mathcal{F}_+\wedge X, \mathrm{Id}, X)$$

This functor has a right adjoint, called pb, which sends a generalized diagram $M=(m,\alpha,n,\beta,l)$ to the pullback of the following diagram (in $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$):

$$\begin{tikzcd}
 & l \arrow[d, "\lambda\wedge l"] \\
 & DE\mathcal{F}_+\wedge l \arrow[d, "\beta"] \\
m \arrow[r, "\alpha"'] & n
\end{tikzcd}$$

We get the following comparison between a category $\mathcal{M}$-mod and $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$, where we consider $\mathcal{M}$-mod with the injective model structure as discussed above.

Proposition 12.3.3. *There is a strong symmetric monoidal Quillen adjoint pair*

$$\mathcal{M}\wedge(-):\ L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS}) \rightleftarrows \mathcal{M}-\mathrm{mod}\ \ :\mathrm{pb}$$

Proof. It is clear that the left adjoint preserves cofibrations and acyclic cofibrations. ☐

Generators for the category $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$ are of the form presented in Lemma 12.3.2 and these are all (homotopically) compact objects.

Their images under the left adjoint $\mathcal{M}\wedge(-)$ are homotopically compact by the same argument as in the proof of [Bar13, Proposition 4.2.4]. We use the notation $\mathcal{J}_{\text{top}}$ for the image of the set of the generators $\mathcal{J}$ for $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$ under the functor $\mathcal{M}\wedge(-)$.

Proposition 12.3.4. *The following adjunction*

$$\mathcal{M}\wedge(-):\ L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS}) \rightleftarrows \mathcal{J}_{\text{top}}-\text{cell}-\mathcal{M}-\text{mod}\ :\text{pb}$$

is a strong symmetric monoidal Quillen equivalence.

Proof. This follows from the cellularisation principle [GS, Appendix A]. We need to show that the derived unit is a weak equivalence on the set $\mathcal{J}$ of generators for the left hand side.

Since all cells from $\mathcal{J}$ are cofibrant, the derived left adjoint is just a left adjoint $\mathcal{M}\wedge -$. Now the right derived functor on objects $\mathcal{M}\wedge j$ for $j\in\mathcal{J}$ is weakly equivalent to taking a homotopy pullback of the following diagram:

$$\begin{tikzcd}
 & e_cS_{\mathbb{Q}}\wedge\tilde{E}\mathcal{F}\wedge j \arrow[d, "\text{Id}\wedge\text{Id}\wedge\lambda\wedge\text{Id}"] \\
 & e_cS_{\mathbb{Q}}\wedge\tilde{E}\mathcal{F}\wedge DE\mathcal{F}_+\wedge j \arrow[d, "\text{Id}"] \\
e_cS_{\mathbb{Q}}\wedge DE\mathcal{F}_+\wedge j \arrow[r, "\text{Id}\wedge!\wedge\text{Id}\wedge\text{Id}"'] & e_cS_{\mathbb{Q}}\wedge\tilde{E}\mathcal{F}\wedge DE\mathcal{F}_+\wedge j
\end{tikzcd}$$

where the map ! is the map $S^0\longrightarrow\tilde{E}\mathcal{F}$. Since homotopy pullbacks commute with smash products, the pullback of the above is weakly equivalent to the homotopy pullback of

$$\begin{tikzcd}
 & \tilde{E}\mathcal{F} \arrow[d] \\
DE\mathcal{F}_+ \arrow[r] & DE\mathcal{F}_+\wedge\tilde{E}\mathcal{F}
\end{tikzcd}$$

(in the category $O(2)-\mathcal{IS}$) smashed with $e_cS_{\mathbb{Q}}\wedge j$. Since the homotopy pullback of the last diagram is just a sphere spectrum (see [GS, Section 4D]), the derived unit is a weak equivalence (in $L_{e_cS_{\mathbb{Q}}}(O(2)-\mathcal{IS})$) on the cells $j\in\mathcal{J}$.

We use conditions from [Bar13, Section 5] to deduce the adjunction is strong monoidal. Notice that left hand side is cellularised with respect to the set of generators K, which does not make any change to the model category, and since it is a monoidal category the set $\mathcal{J}$ is a monoidal set of cells. □

12.3.2 Inflation and T-fixed point adjunction

First let us recall the T-fixed point functor from $O(2)$ orthogonal spectra to W orthogonal spectra. Let T be a torus in $O(2)$ and let $\epsilon : O(2) \longrightarrow O(2)/T$ be the projection onto the quotient. This induces the left adjoint to the T–fixed points functor below (see [MM02, Chapter V, Proposition 3.10]):

$$\begin{tikzcd} W - \mathcal{IS} \arrow[r, shift left, "\epsilon^*"] & O(2) - \mathcal{IS} \arrow[l, shift left, "(-)^T"] \end{tikzcd}$$

where ϵ^* denotes the inflation functor.

This adjunction passes to the Quillen adjunction on the localised model categories and we have the following diagram

$$\begin{tikzcd} L_{e_1 S_{\mathbb{Q}}}(W - \mathcal{IS}) \arrow[r, shift left, "\epsilon^*"] & L_{e_c S_{\mathbb{Q}}}(O(2) - \mathcal{IS}) \arrow[l, shift left, "(-)^T"] \end{tikzcd}$$

As ϵ was a strong monoidal functor before localisation the above adjunction is compatible with the smash product after localisation. Right adjoint is weak monoidal.

By [SS03a, Theorem 3.12] for any fibrant monoid A in $O(2) - \mathcal{IS}$ we have the induced adjoint Quillen pair between A^T modules and A modules, where the left adjoint is denoted ϵ^{*A} see Section 1.3 for construction.

We consider several cases, when this adjunction for a certain ring, after further localisation becomes a Quillen equivalence.

We start with the following

Proposition 12.3.5. *The adjunction*

$$\epsilon^{*DE\mathcal{F}_+} : DE\mathcal{F}_+^T - \text{mod} \rightleftarrows DE\mathcal{F}_+ - \text{mod} : (-)^T$$

is a symmetric monoidal Quillen adjunction.

Now we localise the above adjunction at the $e_1 S_{\mathbb{Q}}$ on the left and $e_c S_{\mathbb{Q}}$ on the right hand side to obtain a Quillen equivalence

Proposition 12.3.6. *The adjunction*

$$\epsilon^{*DE\mathcal{F}_+} : L_{e_1 S_{\mathbb{Q}}}(DE\mathcal{F}_+^T - \text{mod}) \rightleftarrows L_{e_c S_{\mathbb{Q}}}(DE\mathcal{F}_+ - \text{mod}) : (-)^T$$

is a symmetric monoidal Quillen equivalence.

Proof. Firstly, a homotopically compact generator for the left hand side is $DE\mathcal{F}_+^T \wedge W_+$ (since the generators can be transfered through the left adjoint by Lemma 11.2.1) and it is cofibrant in $DE\mathcal{F}_+^T$-modules.

Secondly, a homotopically compact generator fot the right hand side is $DE\mathcal{F}_+ \wedge O(2)/T_+$. This follows from the fact that $DE\mathcal{F}_+ \wedge O(2)/C_{n+}$, for all n can be built up from $DE\mathcal{F}_+ \wedge O(2)/T_+$ by cofibre sequences and suspensions by representations. This in turn is a consequence of [GS, Lemma 13.6].

Since the right adjoint preserves all weak equivalences, instead of derived unit it is enough to show that the categorical unit is a weak equivalence. Notice also that, since $O(2)/T_+$ is T-fixed we have $(DE\mathcal{F}_+ \wedge O(2)/T_+)^T \cong DE\mathcal{F}_+^T \wedge W_+$.

Since ϵ^* was strong monoidal, the left adjoint on a generator is of the form

$$\begin{aligned} DE\mathcal{F}_+ \wedge_{\epsilon^*(DE\mathcal{F}_+)^T} \epsilon^*(DE\mathcal{F}_+^T \wedge W_+) &\cong DE\mathcal{F}_+ \wedge_{\epsilon^*(DE\mathcal{F}_+^T)} \epsilon^*(DE\mathcal{F}_+^T) \wedge O(2)/T_+ \\ &\cong DE\mathcal{F}_+ \wedge O(2)/T_+ \quad (12.3.7) \end{aligned}$$

It follows that the categorical unit on the generator is a weak equivalence.

By the arguments given above the categorical counit on the generator is also a weak equivalence. This shows that the derived adjunction is an equivalence of categories and thus the adjunction on the level of model categories is a Quillen equivalence.

□

We localise the adjunction further at a map $DE\mathcal{F}_+ \longrightarrow DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}$ on the right hand side (we call it f) and the corresponding derived right adjoint applied to it (we call it $(f)^T$)

Proposition 12.3.8. *The adjunction*

$$\epsilon^{*DE\mathcal{F}_+} : L_{\{(f)^T\}} L_{e_1 S_\mathbb{Q}}(DE\mathcal{F}_+^T - \text{mod}) \rightleftarrows L_{\{f\}} L_{e_c S_\mathbb{Q}}(DE\mathcal{F}_+ - \text{mod}) : (-)^T$$

is a symmetric monoidal Quillen equivalence.

Proof. Since it was a symmetric monoidal Quillen equivalence before this last localisation, it is still so after by [Hir03, Theorem 3.3.20, part 1b]. □

Proposition 12.3.9. *The adjunction*

$$\epsilon^* : W - \mathcal{IS} \rightleftarrows O(2) - \mathcal{IS} : (-)^T$$

is a symmetric monoidal Quillen adjunction.

Proof. This is [MM02, Section V, Proposition 3.10]. □

We localised the above to obtain a Quillen equivalence

Proposition 12.3.10. *The adjunction*

$$\epsilon^* : L_{e_1 S_{\mathbb{Q}}}(W - \mathcal{IS}) \rightleftarrows L_{e_c S_{\mathbb{Q}} \wedge \tilde{E}\mathcal{F}}(O(2) - \mathcal{IS}) : (-)^T$$

is a symmetric monoidal Quillen equivalence.

Proof. Since ϵ^* is strong monoidal and $\epsilon^*(e_1 S_{\mathbb{Q}}) = e_c S_{\mathbb{Q}}$ the above adjuncton is a composite of two adjunctions, the second being identity adjunction between $L_{e_c S_{\mathbb{Q}}}(O(2) - \mathcal{IS})$ and further localisation at $\tilde{E}\mathcal{F}$, namely $L_{e_c S_{\mathbb{Q}} \wedge \tilde{E}\mathcal{F}}(O(2) - \mathcal{IS})$.

To verify that this is a Quillen equivalence we will work with the derived unit and the derived counit on generators. The generator for the left hand side is W_+ and the generator for the right hand side is $O(2)/T_+$ (as all the other generators for $L_{e_c S_{\mathbb{Q}}}(O(2) - \mathcal{IS})$ are of the form $O(2)/H_+$ for $H \leq \mathcal{F}$ and they are weakly equivalent to a point in $L_{e_c S_{\mathbb{Q}} \wedge \tilde{E}\mathcal{F}}(O(2) - \mathcal{IS})$).

It is worth noticing that the right derived functor acts as a geometric fixed point functor, because $\phi^N(X) = (X \wedge \tilde{E}[\not\supseteq N])^N$. Now, W_+ is sent to $O(2)/T_+$ and then back to W_+. $O(2)/T_+$ is sent to W_+ and back to itself. By derived triangle identities postcomposition with the derived counit on $O(2)/T_+$ is surjective:

$$\varepsilon_{\epsilon^* W_+} \circ - : [O(2)/T_+, \epsilon^*(\epsilon^* W_+)^T] \longrightarrow [O(2)/T_+, \epsilon^* W_+]$$

and since both sides are finitely generated ($\mathbb{Q}[W]$) it is an isomorphism. Since $O(2)/T_+$ is the generator it follows that the $\varepsilon_{\epsilon^* W_+}$ is a weak equivalence. From the other triangle identity on the homotopy level it follows that also the derived unit on W_+ is an isomorphism. It follows that this adjunction is a Quillen equivalence. □

We can extend the functor $(-)^T$ to the level of generalized diagrams.

Definition 12.3.11. We define $\mathcal{M}_{\text{top}}$ to be the following diagram of model categories and adjoint Quillen pairs:

$$\begin{tikzcd} L_{e_1 S_{\mathbb{Q}}}(DE\mathcal{F}_+^T - \text{mod}) \arrow[r, shift left, "\text{Id}"] & L_{\{(f)^T\}} L_{e_1 S_{\mathbb{Q}}}(DE\mathcal{F}_+^T - \text{mod}) \arrow[l, shift left, "\text{Id}"] \arrow[r, shift right, "U"'] & L_{e_1 S_{\mathbb{Q}}}(W - \mathcal{IS}) \arrow[l, shift right, "DE\mathcal{F}_+^T \wedge -"'] \end{tikzcd}$$

where U denotes the forgetful functor.

Now we consider the category $\mathcal{M}_{\text{top}}$-mod of generalized diagrams and we have the following comparison

Proposition 12.3.12. *The adjunction*

$$\epsilon^{*\bullet} : \mathcal{M}_{\text{top}} - \text{mod} \rightleftarrows \mathcal{M} - \text{mod} : (-)^T$$

is a symmetric monoidal Quillen equivalence, where both categories are considered with the injective model structure.

Instead of the category $\mathcal{M} - \mathrm{mod}$ we want to consider $\mathcal{J}_{\mathrm{top}} - \mathrm{cell} - \mathcal{M} - \mathrm{mod}$, so we have to cellularise the above equivalence. We use the notation $\mathcal{J}_{\mathrm{top}}^T$ for the image of the set of generators $\mathcal{J}_{\mathrm{top}}$ under the derived functor of $(-)^T$. Recall that $\mathcal{J}_{\mathrm{top}}$ is the image of the set $\mathcal{J}$ from Lemma 12.3.2 under the functor $\mathcal{M} \wedge -$.

By the cellularisation principle (see [GS, Appendix A] and monoidal version see [Bar13, Proposition 5.2.8]) we have the following

Proposition 12.3.13. *The adjunction*

$$\epsilon^{*\bullet} : \mathcal{J}_{\mathrm{top}}^T - \mathrm{cell} - \mathcal{M}_{\mathrm{top}} - \mathrm{mod} \rightleftarrows \mathcal{J}_{\mathrm{top}} - \mathrm{cell} - \mathcal{M} - \mathrm{mod} : (-)^T$$

is a symmetric monoidal Quillen equivalence.

12.3.3 Passage to algebra

Now we apply four Quillen equivalences extended to the level of generalised diagrams. Firstly we restrict the W universe of all three categories in the diagram $\mathcal{M}_{\mathrm{top}}$ from complete to trivial, and since all three categories were localised at e_1 it will be a Quillen equivalence. Secondly, we change the category to W objects in non-equivariant orthogonal spectra and extend this to the generalised diagrams. Then we use the adjunction with the forgetful functor to pass to symmetric spectra with the W action and finally we apply $- \wedge H\mathbb{Q}$ to end in $H\mathbb{Q}$ modules. This is the category necessary for the next step, namely the passage to algebra.

Since all the above steps are Quillen equivalences, they will remain so after cellularising with respect to derived images of the set of cells $\mathcal{J}_{\mathrm{top}}$ along the way (see [GS13a, Corollary 2.7]).

Now we proceed as in the monoidal passage for exceptional subgroups (see Lemma 10.3.7) and change the model structure on all three categories into a free model structure. We keep the notation the same, $\mathcal{M}_{\mathrm{top}}$.

This step is needed for the following

Proposition 12.3.14. *The adjunction*

$$\mathrm{res}^\bullet : \mathcal{M}_{\mathrm{top}} - \mathrm{mod}_c \leftrightarrows \mathcal{M}_{\mathrm{top}} - \mathrm{mod}_t : \mathrm{inf}^\bullet$$

is a strong symmetric monoidal Quillen equivalence where the left adjoint inf *denotes a functor on the generalised diagrams which is inflation on the category $L_{e_1 S_{\mathbb{Q}}}(W - \mathcal{IS})$ and it's induced from that at the level of modules over $DE\mathcal{F}_+^T$.*

Proof. Since the right adjoint of a right vertical Quillen equivalence (see Lemma 10.3.8) is strong monoidal, we have the Quillen equivalence at the level of modules over $DE\mathcal{F}_+^T$ and $\mathrm{res}(DE\mathcal{F}_+^T)$ (which we still denote by $DE\mathcal{F}_+^T$ below).

$$\begin{tikzcd}
\text{free}(DE\mathcal{F}_+^T - \text{mod})_c \arrow[r, shift left, "\text{Id}"] \arrow[d, shift left, "\text{res}"] & L_{\{(f)^T\}}\text{free}(DE\mathcal{F}_+^T - \text{mod})_c \arrow[l, shift left, "\text{Id}"] \arrow[r, shift right, "U"'] \arrow[d, shift left, "\text{res}"] & \text{free}(W - \mathcal{IS})_c \arrow[l, shift right, "DE\mathcal{F}_+^T\wedge -"'] \arrow[d, shift left, "\text{res}"] \\
\text{free}(DE\mathcal{F}_+^T - \text{mod})_t \arrow[r, shift left, "\text{Id}"] \arrow[u, shift left, "\text{inf}"] & L_{\{(f)^T\}}\text{free}(DE\mathcal{F}_+^T - \text{mod})_t \arrow[l, shift left, "\text{Id}"] \arrow[r, shift right, "U"'] \arrow[u, shift left, "\text{inf}"] & \text{free}(W - \mathcal{IS})_t \arrow[l, shift right, "DE\mathcal{F}_+^T\wedge -"'] \arrow[u, shift left, "\text{inf}"]
\end{tikzcd}$$

The category in the middle is localised at the derived image of $\{f\}^T$ and the adjunction remains a Quillen equivalence. Thus the adjunction on the level of generalised diagrams is a Quillen equivalence. □

Now we remove the equivariance outside the spectrum level by an equivalence of categories (which is also a Quillen equivalence) below

Proposition 12.3.15. *There is an equivalence of categories (which is also a Quillen equivalence) between $\mathcal{M}_{\text{top}} - \text{mod}_t$ and W objects in generalised diagrams in a category of rational orthogonal spectra (which we denote by $\mathcal{M}[W]$-mod).*

Proof. Follows from Lemma 10.3.9. □

Since this is an equivalence of categories we keep the notation in the diagrams the same.

Now we pass to rational symmetric spectra with W action. First recall that $\mathbb{U}$ is the forgetful functor from orthogonal spectra to symmetric spectra and $\mathbb{P}$ is its left adjoint. We abuse the notation below and skip the geometric realisation and a singular simplicial set functors.

Proposition 12.3.16. *The adjunction*

$$\begin{tikzcd}
\mathbb{U}^{\bullet} : \mathcal{M}[W] - \text{mod} \arrow[r, shift right] & \mathbb{U}\mathcal{M}[W] - \text{mod} : \mathbb{P}^{\bullet} \arrow[l, shift right]
\end{tikzcd}$$

is a strong symmetric monoidal Quillen equivalence.

Proof. The diagram category $\mathcal{M}[W] - \text{mod}$ is as in Proposition 12.3.15, which we write as

$$\begin{tikzcd}
DE\mathcal{F}_+^T - \text{mod} \arrow[r, shift left, "\text{Id}"] & L_{\{(f)^T\}}(DE\mathcal{F}_+^T - \text{mod}) \arrow[l, shift left, "\text{Id}"] \arrow[r, shift right, "U"'] & \mathcal{IS}_{\mathbb{Q}}[W] \arrow[l, shift right, "DE\mathcal{F}_+^T\wedge -"']
\end{tikzcd}$$

and the diagram category $\mathbb{U}\mathcal{M}[W] - \text{mod}$ is

$$\begin{tikzcd}
\mathbb{U}DE\mathcal{F}_+^T - \text{mod} \arrow[r, shift left, "\text{Id}"] & L_{\{\mathbb{U}(f)^T\}}(\mathbb{U}DE\mathcal{F}_+^T - \text{mod}) \arrow[l, shift left, "\text{Id}"] \arrow[r, shift right, "U"'] & \text{Sp}^{\Sigma}_{\mathbb{Q}}[W] \arrow[l, shift right, "\mathbb{U}DE\mathcal{F}_+^T\wedge -"']
\end{tikzcd}$$

As in Proposition 12.3.14 we have a Quillen equivalence at the level of categories (vertical adjunction on the right hand side of the diagram below) and since left adjoint is strong monoidal, it is also a Quillen equivalence at the level of modules. Thus the middle one is a Quillen equivalence.

$$\begin{tikzcd}
DE\mathcal{F}_+^T - \mathrm{mod} \arrow[r, shift left, "\mathrm{Id}"] \arrow[d, shift left, "\mathbb{U}"] & L_{\{(f)^T\}}(DE\mathcal{F}_+^T - \mathrm{mod}) \arrow[l, shift left, "\mathrm{Id}"] \arrow[r, shift right, "U"'] \arrow[d, shift left, "\mathbb{U}"] & \mathcal{IS}_{\mathbb{Q}}[W] \arrow[l, shift right, "DE\mathcal{F}_+^T \wedge -"'] \arrow[d, shift left, "\mathbb{U}"] \\
\mathbb{U}DE\mathcal{F}_+^T - \mathrm{mod} \arrow[r, shift left, "\mathrm{Id}"] \arrow[u, shift left, "\mathbb{P}"] & L_{\{\mathbb{U}(f)^T\}}(\mathbb{U}DE\mathcal{F}_+^T - \mathrm{mod}) \arrow[l, shift left, "\mathrm{Id}"] \arrow[r, shift right, "U"'] \arrow[u, shift left, "\mathbb{P}"] & \mathrm{Sp}^{\Sigma}_{\mathbb{Q}}[W] \arrow[l, shift right, "\mathbb{U}DE\mathcal{F}_+^T \wedge -"'] \arrow[u, shift left, "\mathbb{P}"]
\end{tikzcd}$$

Thus we get a strong monoidal Quillen equivalence of the generalised diagrams. $\square$

The last step in this section is to pass from symmetric spectra to $H\mathbb{Q}$ modules.

Proposition 12.3.17. *The adjunction*

$$H\mathbb{Q} \wedge -^{\bullet} : \mathbb{U}\mathcal{M}[W] - \mathrm{mod} \rightleftarrows H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W] - \mathrm{mod} : \mathrm{U}^{\bullet}$$

is a strong symmetric monoidal Quillen equivalence, where U denotes the forgetful functor, right adjoint to $H\mathbb{Q} \wedge -$.

Proof. Since the left adjoint $H\mathbb{Q} \wedge -$ is strong monoidal, we get commuting diagrams (one with the left adjoints and the other with the right adjoints) as follows

$$\begin{tikzcd}
\mathbb{U}DE\mathcal{F}_+^T - \mathrm{mod} \arrow[r, shift left, "\mathrm{Id}"] \arrow[d, shift right, "H\mathbb{Q}\wedge -"'] & L_{\{\mathbb{U}(f)^T\}}(\mathbb{U}DE\mathcal{F}_+^T - \mathrm{mod}) \arrow[l, shift left, "\mathrm{Id}"] \arrow[r, shift right, "U"'] \arrow[d, shift right, "H\mathbb{Q}\wedge -"'] & \mathrm{Sp}^{\Sigma}_{\mathbb{Q}}[W] \arrow[l, shift right, "\mathbb{U}DE\mathcal{F}_+^T \wedge -"'] \arrow[d, shift right, "H\mathbb{Q}\wedge -"'] \\
H\mathbb{Q} \wedge \mathbb{U}DE\mathcal{F}_+^T - \mathrm{mod} \arrow[r, shift left, "\mathrm{Id}"] \arrow[u, shift right, "\mathrm{U}"'] & L_{\{H\mathbb{Q}\wedge\mathbb{U}(f)^T\}}(H\mathbb{Q} \wedge \mathbb{U}DE\mathcal{F}_+^T - \mathrm{mod}) \arrow[l, shift left, "\mathrm{Id}"] \arrow[r, shift right, "U"'] \arrow[u, shift right, "\mathrm{U}"'] & (H\mathbb{Q} - \mathrm{mod})[W] \arrow[l, shift right, "\mathbb{U}DE\mathcal{F}_+^T \wedge -"'] \arrow[u, shift right, "\mathrm{U}"']
\end{tikzcd}$$

The right vertical adjunction is a Quillen equivalence by Lemma 10.3.11, so by [SS03a, Theorem 3.12, part 1] the left adjunction is a Quillen equivalence. Thus the middle one is also a Quillen equivalence and the adjunction at the level of generalised diagrams is a Quillen equivalence.

$\square$

Now we are ready to use the results from [Shi07] to move from topology to algebra on generalised diagrams. The passage involves several Quillen equivalences, so we state it as follows

Proposition 12.3.18. *There is a zig-zag of Quillen equivalences between the category* $H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W] - \mathrm{mod}$ *and the category* $\theta(H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W]) - \mathrm{mod}$*, where* θ *denotes the derived functor described in [Shi07, Section 2.2].*

Proof. It follows from [Shi07, Corollary 2.15 part 2] that the zig-zag on the right hand side of the diagram (vertical adjunction) is a zig-zag of Quillen equivalences. The same is true for the left hand side of the diagram (vertical adjunction) and thus we still get Quillen equivalences after localising the middle category of each step at the derived image of the set of maps $\{H\mathbb{Q} \wedge \mathbb{U}(f)^T\}$. □

All Quillen equivalences presented above are still Quillen equivalences after cellularising at the derived images of the cells from the set $\mathcal{J}^T_{\mathrm{top}}$. We denote the derived images of cells from the set $\mathcal{J}^T_{\mathrm{top}}$ in $\theta(H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W]) - \mathrm{mod}$ by $\mathcal{J}^T_{\mathrm{t}}$.

12.3.4 Intrinsic formality

First notice that $\theta(H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W]) - \mathrm{mod}$ is already an algebraic category, however not a very explicit one. The only thing we know is the homology of the rings that we consider the modules to be over. This turns out to be enough, since each of this rings is weakly equivalent to its homology and that result allows us to replace every ring by its homology in a Quillen equivalent way.

Following the approach in [Bar13] we create a zig-zag of two Quillen equivalences linking our category of generalised diagrams obtained from Proposition 12.3.18 to a more approachable category of generalised diagrams.

For simplicity of the notation we set $R_1 := \theta(H\mathbb{Q} \wedge \mathbb{U}DE\mathcal{F}^T_+)$ and $R_2 := \theta S$. We know from [Bar13, Lemma 4.5.1] that $H_*(R_1) = \mathcal{O}_\mathcal{F}$ and $H_*(R_2) = \mathbb{Q}$ as rings in $\mathbb{Q}[W]$ modules. We recall from [Bar13, Section 4.5] an intermediate ring for R_1, call it P_1 together with the maps making the following diagrams commute :

$$\begin{tikzcd}
R_1 \arrow[d, "h"'] & R_2 \arrow[l, "\alpha"'] \arrow[d, "\mathrm{Id}"] \\
P_1 & R_2 \arrow[l, "\beta"'] \\
\mathcal{O}_\mathcal{F} \arrow[u, "g"] & \mathbb{Q} \arrow[l, "\text{unit map}"] \arrow[u, "\text{unit map}"']
\end{tikzcd}$$

where α is an image of the unit map under all the functors presented in comparison so far and β is the composition of α and h (we describe P_1 and h below).

P_1 is defined as follows: since $H_*(R_1) = \mathcal{O}_\mathcal{F}$, for every $H \in \mathcal{F}$ there is a cycle x_H inside R_1 representing e_H in homology. Since we want to choose a cycle representation for e_H which is W invariant we set $y_H = 1/2(x_H + wx_H)$. Now, $H_*(R_1[(y_H)^{-1}]) = e_H H_*(R_1)$ so we define $P_1 := \prod_{H\in\mathcal{F}} R_1[(y_H)^{-1}]$.

Map h in the diagram above is defined to be the canonical map into the product $R_1 \longrightarrow P_1$. It remains to define g from the above diagram. For every $H \in \mathcal{F}$ we choose a cycle representative a_H in $R_1[(y_H)^{-1}]$ for the homology class c_H. Since we want W to act as -1 on the representative for this class, we set $b_H := 1/2(a_H - wa_H)$. We have a map $\mathbb{Q}[c_H] \longrightarrow R_1[(y_H)^{-1}]$ which sends c_H to b_H. We define g to be the product over H of the above maps.

Notice that with these definitions, all vertical arrows in the above diagram are homology isomorphisms.

For further convinience we denote by $\mathcal{M}_{\hat{a}}$-mod the category of the generalised diagrams, where $\mathcal{M}_{\hat{a}}$ is:

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{F}} - \mathrm{mod} \arrow[r, shift left, "\mathrm{Id}"] & L_{\tilde{f}}(\mathcal{O}_{\mathcal{F}} - \mathrm{mod}) \arrow[l, shift left, "\mathrm{Id}"] \arrow[r, shift right, "U"'] & \mathbb{Q}[W] - \mathrm{mod} \arrow[l, shift right, "\mathcal{O}_{\mathcal{F}}\otimes -"']
\end{tikzcd}$$

Notice that $\{\tilde{f}\}$ denotes the derived image of the maps $\{\theta(H\mathbb{Q} \wedge \mathbb{U}(f)^T)\}$ via the functors of extension and restriction of scalars along h and g respectively. Notice further that this is equivalent to the set of maps $\{\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\}$, by which we mean all suspensions and desuspensions of the inclusion map $\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ (see [Bar13, Lemma 4.5.1]).

From the construction above we can deduce the following

Proposition 12.3.19. *The above maps of rings induce two adjoint pairs of extension and restriction of scalars between* $\theta(H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W]) - \mathrm{mod}$ *and* $\mathcal{M}_{\hat{a}} - \mathrm{mod}$ *which are symmetric monoidal Quillen equivalences.*
If $\mathcal{J}_{\hat{a}}$ *denotes the derived image of cells* $\mathcal{J}_{\mathrm{t}}^T$ *under the extension along* f *and restriction along* g *on the level of generalised diagrams, then the above pair of adjoin functors induce Quillen equivalences between* $\mathcal{J}_{\mathrm{t}}^T - \mathrm{cell} - \theta(H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W]) - \mathrm{mod}$ *and* $\mathcal{J}_{\hat{a}} - \mathrm{cell} - \mathcal{M}_{\hat{a}} - \mathrm{mod}$.

Now we notice that our middle category involves localisation, thus we present the following two results

Lemma 12.3.20. *Two model structures* $L_{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}(\mathcal{O}_{\mathcal{F}} - \mathrm{mod})$ *and* $L_{\{\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\}}(\mathcal{O}_{\mathcal{F}} - \mathrm{mod})$ *are the same. Notice that the last category is understood as localised at the specified map and all its suspensions.*

Proof. Both localisations keep cofibrations the same, so it is enough to show that the classes of weak equivalences in these model structures are equal. This is shown in [BR14, Lemma 3.14]. □

Lemma 12.3.21. *The adjunction induced by the inclusion of rings* $j : \mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ *is a Quillen adjunction*

$$\begin{tikzcd}
\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} - \ : \ \mathcal{O}_{\mathcal{F}} - \mathrm{mod} \arrow[r, shift left] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} - \mathrm{mod} \ : \ j^* \arrow[l, shift left]
\end{tikzcd}$$

where both categories are with projective model structures. Moreover it induces a Quillen equivalence

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} - \ : \ L_{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}(\mathcal{O}_{\mathcal{F}} - \text{mod}) \rightleftarrows \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} - \text{mod} \ : \ j^*$$

Proof. The adjunction is a Quillen pair after the left hand side is localised at $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$, since the cofibrations didn't change and new weak equivalences are exactly the maps f, such that $H_*(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} f)$ is an isomorphism. Thus the left adjoint preserves and reflects all "new" weak equivalences.

By Part 2 of Proposition 1.1.12 to prove this is a Quillen equivalence it is enough to show that the (categorical) counit map on a fibrant generator for $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}-\text{mod}$ is a H_* isomorphism. Since $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is a fibrant generator for $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}-\text{mod}$, the categorical counit is an isomorphism:

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$$

$\square$

Now we use one more Quillen equivalence to replace the middle category in the generalised diagram (which is a localisation) by one which is just modules over a different ring.

Quillen equivalence presented in Lemma 12.3.20 and Lemma 12.3.21 fit into the Quillen equivalence between generalised diagrams (where two other vertical adjunctions are identities and U below denotes forgetful functor):

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{F}} - \text{mod} \arrow[r, shift left, "\text{Id}"] \arrow[d, shift right, "\text{Id}"'] & L_{\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}}(\mathcal{O}_{\mathcal{F}} - \text{mod}) \arrow[l, shift left, "\text{Id}"] \arrow[r, shift right, "U"'] \arrow[d, shift right, "\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\mathcal{F}}}-"'] & \mathbb{Q}[W] - \text{mod} \arrow[l, shift right, "\mathcal{O}_{\mathcal{F}}\otimes -"'] \arrow[d, shift right, "\text{Id}"'] \\
\mathcal{O}_{\mathcal{F}} - \text{mod} \arrow[r, shift left, "\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\mathcal{F}}}-"] \arrow[u, shift right, "\text{Id}"'] & \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} - \text{mod} \arrow[l, shift left, "j^*"] \arrow[r, shift right, "U"'] \arrow[u, shift right, "j^*"'] & \mathbb{Q}[W] - \text{mod} \arrow[l, shift right, "\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes -"'] \arrow[u, shift right, "\text{Id}"']
\end{tikzcd}$$

We denote the bottom row by $\mathcal{M}_a$ and we summarise the above in the following

Proposition 12.3.22. *The adjunction (described above)*

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} - \ : \ \mathcal{M}_{\hat{a}} - \text{mod} \rightleftarrows \mathcal{M}_a - \text{mod} \ : \ j^*$$

is a symmetric monoidal Quillen equivalence, and thus the adjunction

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} - \ : \ \mathcal{J}_{\hat{a}} - \text{cell} - \mathcal{M}_{\hat{a}} - \text{mod} \rightleftarrows \mathcal{J}_a - \text{cell} - \mathcal{M}_a - \text{mod} \ : \ j^*$$

is a symmetric monoidal Quillen equivalence, where $\mathcal{J}_a$ is the derived image of $\mathcal{J}_{\hat{a}}$ under the left adjoint.

12.3.5 Algebraic model for cyclic O(2) rational spectra

Now we are ready to link the category $\mathcal{J}_a - \text{cell} - \mathcal{M}_a - \text{mod}$ with the category described in Section 12.1.1. Firstly we define an adjoint pair of functors between the category of generalised diagrams and $d\mathcal{A}(O(2), c)$ after [Bar13, Section 4.6] as follows.

We define the left adjoint l^* to send an object $\beta : M \longrightarrow \mathcal{E}^{-1}\mathcal{O}_\mathcal{F} \otimes V$ of $d\mathcal{A}(O(2), c)$ to the quintuple $(M, \mathrm{Id}, \mathcal{E}^{-1}M, (\mathcal{E}^{-1}\mathcal{O}_\mathcal{F} \otimes_{\mathcal{O}_\mathcal{F}} \beta)^{-1}, V)$, an object in the category $\mathcal{M}_a - \text{mod}$.

Now we define the right adjoint to be $\Gamma = \Gamma_h \circ \Gamma_v$, where Γ_h is the right adjoint to the inclusion $d\mathcal{A}(O(2), c) \longrightarrow d\widehat{\mathcal{A}}(O(2), c)$ (see [Bar13, Section 3]). Γ_v on an object $(a, \alpha, b, \gamma, c)$ is defined using the pullback of the following diagram (in the category of $\mathcal{O}_\mathcal{F}$-modules in $\mathbb{Q}[W]$-modules)

$$\begin{tikzcd} a \arrow[r] & \mathcal{E}^{-1}\mathcal{O}_\mathcal{F} \otimes_{\mathcal{O}_\mathcal{F}} a \arrow[r] & b & \mathcal{E}^{-1}\mathcal{O}_\mathcal{F} \otimes c \arrow[l] \end{tikzcd}$$

namely we get $\delta : P \longrightarrow \mathcal{E}^{-1}\mathcal{O}_\mathcal{F} \otimes c$, which is an object in $d\widehat{\mathcal{A}}(O(2), c)$.

Proposition 12.3.23. *The adjunction*

$$\Gamma \;:\; \mathcal{J}_a - \text{cell} - \mathcal{M}_a - \text{mod} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathrm{d}\mathcal{A}(O(2), c) \;:\; l^*$$

is a symmetric monoidal Quillen equivalence, where the category of generalised diagrams is considered with projective - injective model structure and the other category is considered with the dualizable model structure.

Proof. This is [Bar13, Proposition 4.6.2]. □

As a consequence of all the results presented in this section we get the following

Theorem 12.3.24. *There is a zig-zag of Quillen equivalences from $L_{e_c S_\mathbb{Q}}(O(2) - \mathcal{IS})$ to $d\mathcal{A}(O(2), c)$, where $d\mathcal{A}(O(2), c)$ is considered with the dualisable model structure.*

On the next page we present a diagram which summarises the passage from $L_{e_c S_\mathbb{Q}}(O(2) - \mathcal{IS})$ to $d\mathcal{A}(O(2), c)$

$$\begin{tikzcd}
L_{e_c S_{\mathbb{Q}}}(O(2) - \mathcal{IS}) \arrow[d, shift right, "\mathcal{M}\wedge -"'] \\
\mathcal{J}_{\mathrm{top}} - \mathrm{cell} - \mathcal{M} - \mathrm{mod} \arrow[u, shift right, "\mathrm{pb}"'] \arrow[d, shift left, "(-)^T"] \\
\mathcal{J}_{\mathrm{top}}^T - \mathrm{cell} - \mathcal{M}_{\mathrm{top}} - \mathrm{mod} \arrow[u, shift left, "\epsilon^{*\bullet}"] \arrow[d, shift left, "\mathrm{res}^{\bullet}"] \\
\mathcal{J}_{\mathrm{top}}^T - \mathrm{cell} - \mathcal{M}_{\mathrm{top}} - \mathrm{mod}_t \arrow[u, shift left, "\mathrm{inf}"] \arrow[d, leftrightarrow, "\text{equivalence}"', "\text{of monoidal model categories}"] \\
\mathcal{J}_{\mathrm{top}}^T - \mathrm{cell} - \mathcal{M}[W] - \mathrm{mod} \arrow[d, shift right, "\mathbb{P}"'] \\
\mathbb{P}(\mathcal{J}_{\mathrm{top}}^T) - \mathrm{cell} - \mathbb{U}\mathcal{M}[W] - \mathrm{mod} \arrow[u, shift right, "\mathbb{U}^{\bullet}"'] \arrow[d, shift right, "H\mathbb{Q}\wedge -"'] \\
H\mathbb{Q} \wedge \mathbb{P}(\mathcal{J}_{\mathrm{top}}^T) - \mathrm{cell} - H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W] - \mathrm{mod} \arrow[u, shift right, "U"'] \arrow[d, leftrightarrow, "\text{zig$-$zag of}"', "\text{Quillen equivalences}"] \\
\mathcal{J}_{\mathrm{t}}^T - \mathrm{cell} - \theta(H\mathbb{Q} \wedge \mathbb{U}\mathcal{M}[W]) - \mathrm{mod} \arrow[d, leftrightarrow, "\text{zig$-$zag of}"', "\text{Quillen equivalences}"] \\
\mathcal{J}_{\hat{\mathrm{a}}} - \mathrm{cell} - \mathcal{M}_{\hat{\mathrm{a}}} - \mathrm{mod}
\end{tikzcd}$$

and we continue:

$$\begin{tikzcd}
\mathcal{J}_{\hat{\mathrm{a}}} - \mathrm{cell} - \mathcal{M}_{\hat{\mathrm{a}}} - \mathrm{mod} \arrow[d, shift right, "\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\otimes_{\mathcal{O}_{\mathcal{F}}}-"'] \\
\mathcal{J}_{\mathrm{a}} - \mathrm{cell} - \mathcal{M}_{\mathrm{a}} - \mathrm{mod} \arrow[u, shift right, "j^*"'] \arrow[d, shift left, "\Gamma"] \\
\mathrm{d}\mathcal{A}(O(2), c) \arrow[u, shift left, "l^*"]
\end{tikzcd}$$

The above passage is monoidal.

12.4 Algebraic model for cyclic SO(3) rational spectra

To obtain an algebraic model for cyclic part of rational $SO(3)$ spectra we cellularise the above zig-zag of Quillen equivalences at the derived images of the cells described in Proposition 12.2.1, called K. This preserves Quillen equivalences and results in the following:

Theorem 12.4.1. *There is a zig-zag of Quillen equivalences from* $L_{e_cS_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$ *and* $\mathrm{im}(K)-\mathrm{cell}-d\mathcal{A}(O(2),c)$*, where* $d\mathcal{A}(O(2),c)$ *is considered with the dualisable model structure. Here* $\mathrm{im}(K)$ *denotes the derived image under the zig-zag of Quillen equivalences described in Section 12.3 of the set of cells* K *described in Proposition 12.2.1.*

The above result gives an algebraic model for the cyclic part of rational $SO(3)$ equivariant spectra. However, it is not easy to work with. We show that the above model is Quillen equivalent to the simpler, algebraic category described in Section 12.1.2. Unfortunately, the following two adjunctions are not monoidal.

Lemma 12.4.2. *The identity adjunction between* $\mathrm{im}(K)-\mathrm{cell}-d\mathcal{A}(O(2),c)$ *where* $d\mathcal{A}(O(2),c)$ *was considered with the dualisable model structure and* $\mathrm{im}(K) - \mathrm{cell} - d\mathcal{A}(O(2),c)$ *where* $d\mathcal{A}(O(2),c)$ *was considered with the injective model structure is a Quillen equivalence.*

Theorem 12.4.3. *The adjunction*

$$\widetilde{F}\ :\ \mathrm{d}\mathcal{A}(SO(3),c) \rightleftarrows \mathrm{im}(K)-\mathrm{cell}-d\mathcal{A}(O(2),c)\ :\ \widetilde{R}$$

defined in the statement of the Theorem 12.1.28 is a Quillen equivalence, where both categories (before cellularisation on the right) are considered with the injective model structure. Here $\mathrm{im}(K)$ *denotes the derived image under the zig-zag of Quillen equivalences described in Section 12.3 of the set of cells* K *described in Proposition 12.2.1.*

Firstly we will simplify the set $\mathrm{im}(K)$.

Take L to be the set of all suspensions and desuspensions of the following cells in the category $L_{e_c S_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$: $SO(3)/SO(2)_+$, $SO(3)_+$ and for every natural $n > 1$

$$\sigma_n = SO(3)_+ \wedge_{C_n} e_{C_n} S^0$$

We call L the set of basic cells.

Notice that cellularisation of $d\mathcal{A}(O(2), c)$ with respect to $\mathrm{im}(K)$ gives the same model structure as cellularisation of $d\mathcal{A}(O(2), c)$ with respect to $\mathrm{im}(L)$. Since

$$SO(3)/C_{n+} = \bigvee_{C_m \subseteq C_n} \sigma_m$$

which is a consequence of [Gre99, Lemma 2.1.5], the set L is a set of generators for K-cell-$L_{e_c S_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$ and thus K-cell-$L_{e_c S_{\mathbb{Q}}}(SO(3)-\mathcal{IS}) = L$-cell-$L_{e_c S_{\mathbb{Q}}}(SO(3)-\mathcal{IS})$.

Proof. of Theorem 12.4.3.
Now it is enough to show that $\mathrm{im}(L)$ consists of the same objects as $\widetilde{F}(\mathcal{K})$, where $\mathcal{K}$ is the set described in Definition 12.1.26. We show that in Lemma 12.4.4 below. The result follows then from Corollary 12.1.30.

□

Lemma 12.4.4. *The set* $\mathrm{im}(L)$ *consists of the same objects as* $\widetilde{F}(\mathcal{K})$*, where* $\mathcal{K}$ *is the set described in Definition 12.1.26 and* $\mathrm{im}(L)$ *denotes the derived image under the zig-zag of Quillen equivalences described in Section 12.3 of the set of cells L described after Theorem 12.4.3.*

Proof. Firstly, notice that for every $n > 1$ σ_n is weakly equivalent in $L_{e_c S_{\mathbb{Q}}}(O(2)-\mathcal{IS})$ to $O(2) \wedge_{C_n} e_{C_n} S^0$. The map is induced by the inclusion of $O(2)$ into $SO(3)$ and we will show that it induces an isomorphism on all π_*^H for $H \in \mathcal{C}$. We will use the notation $N = O(2)$ and $G = SO(3)$ below.

$$\pi_*^H(N \wedge_{C_n} e_{C_n} S^0) = [N/H_+, F_{C_n}(N_+, S^{L_N(C_n)} \wedge e_{C_n} S^0)]^N = [N/H_+, S^{L_N(C_n)} \wedge e_{C_n} S^0]^{C_n}$$

since the codomain has only geometric fixed points for $H = C_n$ we get a non zero result only for $H = C_n$:

$$[\Phi^{C_n}(N/C_{n+}), \Phi^{C_n}(S^{L_N(C_n)})]^1 = [S^1 \vee S^1, S^1] = \Sigma(\mathbb{Q}[W])$$

where $L_N(C_n)$ is the tangent C_n representation at the identity coset of N/C_n and thus it is 1 dimensional trivial representation. and similarly:

$$\pi_*^H(G \wedge_{C_n} e_{C_n} S^0) = [G/H_+, F_{C_n}(G_+, S^{L_G(C_n)} \wedge e_{C_n} S^0)]^G = [G/H_+, S^{L_G(C_n)} \wedge e_{C_n} S^0]^{C_n}$$

since the codomain has only geometric fixed points for $H = C_n$ we get non zero result only for $H = C_n$:

$$[\Phi^{C_n}(G/C_{n+}), \Phi^{C_n}(S^{L_G(C_n)})]^1 = [S^1 \vee S^1, S^1] = \Sigma(\mathbb{Q}[W])$$

since $L_G(C_n)$ is 3 dimensional, but it has one dimensional C_n fixed subspace.

The images of the cells in $\mathcal{A}(O(2), c)$ are therefore

$$\mathrm{im}(G \wedge_{C_n} e_{C_n} S^0) = \mathrm{im}(N \wedge_{C_n} e_{C_n} S^0) = (\Sigma\mathbb{Q}[W]_{C_n} \longrightarrow 0)$$

by [Gre99, Example 5.8.1] where $\Sigma\mathbb{Q}[W]$ is in the place C_n.

Now we will use the functors $\pi_*^{\mathcal{A}}$ described in [Gre99]. Since $SO(3)_+$ is free we get

$$\begin{aligned}\pi_*^{\mathcal{A}}(SO(3)_+) = (\pi_*^T(SO(3)_+) \longrightarrow 0) = (\pi_*(\Sigma SO(3)/T_+) \longrightarrow 0) = \\ (\pi_*(\Sigma S(\mathbb{R}^3)_+) \longrightarrow 0) = (\Sigma^3\widetilde{\mathbb{Q}} \oplus \Sigma\mathbb{Q} \longrightarrow 0) \quad (12.4.5)\end{aligned}$$

where $\Sigma^3\widetilde{\mathbb{Q}} \oplus \Sigma\mathbb{Q}$ is in the place corresponding to the trivial subgroup 1 and c sends $\tilde{\mathbb{Q}}$ in degree 3 to $\mathbb{Q}$ in degree 1.

And finally $SO(3)/T_+ = S(\mathbb{R}^3)_+$ as an $O(2)$ space is built from the following cells:

$$N/T_+ \vee N/D_{2+} \cup N_+ \wedge e^1$$

Thus the cofibre sequence

$$N_+ \longrightarrow N/T_+ \vee N/D_{2+} \longrightarrow G/T_+$$

gives the long exact sequence

$$(\Sigma\mathbb{Q}[W] \longrightarrow 0) \longrightarrow (\mathcal{O}_{\mathcal{F}}[W] \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W]) \oplus (\Sigma\mathbb{Q} \longrightarrow 0) \longrightarrow \mathrm{im}(G/T_+)$$

and hence

$$(1 - e_1)\mathrm{im}(G/T_+) = (1 - e_1)(\mathcal{O}_{\mathcal{F}}[W] \longrightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q}[W])$$

and

$$e_1\mathrm{im}(G/T_+) = \Sigma^2\widetilde{\mathbb{Q}} \oplus \mathbb{Q}[c][W]$$

where c acts on $\widetilde{\mathbb{Q}}$ in degree 2 by sending it to $\mathbb{Q}$ in degree 0.

Notice that these images are exactly the cells (up to suspension) in $\widetilde{F}(\mathcal{K})$ which finishes the proof. □

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Bibliography

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.

[AR94] Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*, volume 189 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1994.

[Bar08] D. Barnes. Rational equivariant spectra. *http://arxiv.org/abs/0802.0954*, 2008.

[Bar09a] D. Barnes. Splitting monoidal stable model categories. *J. Pure Appl. Algebra*, 213(5):846–856, 2009.

[Bar09b] David Barnes. Classifying rational G-spectra for finite G. *Homology, Homotopy Appl.*, 11(1):141–170, 2009.

[Bar13] D. Barnes. Rational O(2)-equivariant spectra. *http://arxiv.org/abs/1201.6610*, 2013.

[Bek00] Tibor Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000.

[Bor94] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1994. Basic category theory.

[BR14] David Barnes and Constanze Roitzheim. Stable left and right Bousfield localisations. *Glasg. Math. J.*, 56(1):13–42, 2014.

[Bre97] Glen E. Bredon. *Sheaf theory*, volume 170 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1997.

[BtD85] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1985.

[CD09] Denis-Charles Cisinski and Frédéric Déglise. Local and stable homological algebra in Grothendieck abelian categories. *Homology, Homotopy Appl.*, 11(1):219–260, 2009.

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[Dug06] Daniel Dugger. Spectral enrichments of model categories. *Homology, Homotopy Appl.*, 8(1):1–30, 2006.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[Fri82] Eric M. Friedlander. *Étale homotopy of simplicial schemes*, volume 104 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[Gil06] James Gillespie. The flat model structure on complexes of sheaves. *Trans. Amer. Math. Soc.*, 358(7):2855–2874, 2006.

[Gre98a] J. P. C. Greenlees. Rational Mackey functors for compact Lie groups. I. *Proc. London Math. Soc. (3)*, 76(3):549–578, 1998.

[Gre98b] J. P. C. Greenlees. Rational O(2)-equivariant cohomology theories. In *Stable and unstable homotopy (Toronto, ON, 1996)*, volume 19 of *Fields Inst. Commun.*, pages 103–110. Amer. Math. Soc., Providence, RI, 1998.

[Gre99] J. P. C. Greenlees. Rational S^1-equivariant stable homotopy theory. *Mem. Amer. Math. Soc.*, 138(661):xii+289, 1999.

[Gre01] J. P. C. Greenlees. Rational SO(3)-equivariant cohomology theories. In *Homotopy methods in algebraic topology (Boulder, CO, 1999)*, volume 271 of *Contemp. Math.*, pages 99–125. Amer. Math. Soc., Providence, RI, 2001.

[Gro57] Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J. (2)*, 9:119–221, 1957.

[GS] J. P. C. Greenlees and B. Shipley. An algebraic model for rational torus-equivariant spectra. *http://arxiv.org/abs/1101.2511*.

[GS13a] J. P. C. Greenlees and B. Shipley. The cellularization principle for Quillen adjunctions. *Homology Homotopy Appl.*, 15(2):173–184, 2013.

[GS13b] J. P. C. Greenlees and B. Shipley. Homotopy theory of modules over diagrams of rings. *http://arxiv.org/abs/1309.6997*, 2013.

[GS14a] J. P. C. Greenlees and B. Shipley. An algebraic model for free rational G-spectra. *Bull. Lond. Math. Soc.*, 46(1):133–142, 2014.

[GS14b] J P C Greenlees and Brooke Shipley. Fixed point adjunctions for equivariant module spectra. *Algebr. Geom. Topol.*, 14(3):1779–1799, 2014.

[HH13] M. Hill and M. Hopkins. Equivariant multiplicative closure. *http://arxiv.org/abs/1303.4479*, 2013.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Hov01] Mark Hovey. Model category structures on chain complexes of sheaves. *Trans. Amer. Math. Soc.*, 353(6):2441–2457 (electronic), 2001.

[HR08] T. Hüttemann and O. Röndigs. Twisted diagrams and homotopy sheaves. *http://arxiv.org/abs/0805.4076*, 2008.

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[Ill00] Sören Illman. Existence and uniqueness of equivariant triangulations of smooth proper G-manifolds with some applications to equivariant Whitehead torsion. *J. Reine Angew. Math.*, 524:129–183, 2000.

[Joh77] P. T. Johnstone. *Topos theory*. Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1977. London Mathematical Society Monographs, Vol. 10.

[Kel05] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, (10):vi+137, 2005. Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

[LMSM86] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[MM02] M. A. Mandell and J. P. May. Equivariant orthogonal spectra and S-modules. *Mem. Amer. Math. Soc.*, 159(755):x+108, 2002.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001.

[Moe88] Ieke Moerdijk. The classifying topos of a continuous groupoid. I. *Trans. Amer. Math. Soc.*, 310(2):629–668, 1988.

[Moe90] Ieke Moerdijk. The classifying topos of a continuous groupoid. II. *Cahiers Topologie Géom. Différentielle Catég.*, 31(2):137–168, 1990.

[Qui67] Daniel G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Sch] S. Schwede. An untitled book project about symmetric spectra. *http://www.math.uni-bonn.de/ schwede/SymSpec.pdf.*

[Shi07] Brooke Shipley. HZ-algebra spectra are differential graded algebras. *Amer. J. Math.*, 129(2):351–379, 2007.

[SS00] Stefan Schwede and Brooke Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[SS03a] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334 (electronic), 2003.

[SS03b] Stefan Schwede and Brooke Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.

[tD79] Tammo tom Dieck. *Transformation groups and representation theory*, volume 766 of *Lecture Notes in Mathematics.* Springer, Berlin, 1979.