APPLICABILITY, IDEALIZATION, AND MATHEMATIZATION

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Abstract

In this thesis I provide a study of the applicability of mathematics. My starting point is the account of applicability offered in Hartry Field's book *Science without numbers* and arising from the nominalistic project carried out therein. By examining the limitations and shortcomings of Field's account, I develop a new one. My account retains the advantages and insights of Field's and avoids its difficulties, which are essentially due to its being incomplete and too restrictive.

Field's account is incomplete because it does not deal with the nature and use of idealization in science. Field only describes how mathematics is applied to highly idealized physical theories (e.g. ones containing postulates which are untestable or contradicted by experiment) but he does not explain how idealization arises and why idealized theories are relevant to the actual experimental investigation of empirical phenomena. I offer such an explanation for an elementary scientific theory to which the more complex examples discussed by Field can be reduced.

Even in presence of an analysis of idealization, Field's account of applicability remains problematic. The reason is that it characterizes the role of mathematics in applications in a very restrictive way, which neglects some of its most important uses. I show this by looking at several examples of applications. I then employ the resulting analysis of how mathematics enters them to give a characterization of applicability which does not suffer of the restrictiveness affecting Field's. This characterization encompasses Field's but also extends to applications he cannot adequately describe.

I thus complete and extend Field's account of applicability, reaching a more comprehensive and realistic alternative.

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INTRODUCTION

The applicability of mathematics has received a renewed interest in the recent philosophical debate (see for instance Bueno & Colyvan 2008, Colyvan 2001, Pincock 2004a-b, 2007 and Steiner 1998, 2005): nevertheless, the most basic philosophical problems posed by it, concerning how mathematics is applied and the nature of its role in applications, stand as yet in need of a detailed analysis.

My thesis is devoted to providing, at least for a wide family of applications, one such analysis. To this end, I consider what I take to be one of the best and most penetrating accounts of applicability offered in the philosophical literature of the last decades, namely that presented in Hartry Field's influential book *Science without numbers* (Field 1980). The wide literature on Field's book has mostly discussed his nominalistic project, while not paying comparable attention to his analysis of applicability (apart from acknowledging its importance).

My objective is to look at the latter analysis, identify the answers Field gives to the questions about how and why mathematics is applied, consider their shortcomings, and finally work toward a new account of applicability which retains the benefits of Field's account while avoiding its drawbacks.

Field shows for several scientific theories that mathematics is applicable to them because it reflects the structure of an underlying empirical setting. Thus, he can explain how mathematics is applied, namely in virtue of a structural similarity between empirical phenomena and a corresponding mathematical model.

On the basis of this conclusion, Field can also answer the question why mathematics is applied. According to him, the reason is twofold: mathematics is either applied to formulate scientific theories, because it is easier to do so mathematically than empirically, or it is used to study in a convenient way the empirical consequences of empirical theories.

Despite its clarity and plausibility, this analysis of applicability has two main shortcomings, which I will overcome in my account. As for the first one, whenever Field shows that mathematics can be applied to empirical settings, he in fact considers idealized empirical settings, satisfying properties which are untestable or contradicted by experimental results. Although this is not a problem for Field *qua* nominalist, it is a problem for his account of applicability, which only explains how mathematics is applied to idealized settings, but does not explain why this is relevant to the understanding of actual empirical settings. Only by clarifying this point can a satisfactory account of applicability be given: I supply the needed clarification in part 1 of this thesis (while also showing, in part 2, why idealizations are important in science).

The second shortcoming of Field's analysis is the restrictiveness of his characterization of applied mathematics. Since Field thinks that scientific theories can be reformulated in a non-mathematical way, he takes all uses of mathematics in science to be, at least in principle, eliminable and thus not of a substantial importance. Using several examples from a variety of experimental disciplines (in particular psychology, archaeology, genetics and economics), I will show that the importance of mathematics in applications is greater than Field thinks and, on this basis, I will provide a more realistic characterization of its applicability. This is done in part 2 of this thesis.

The main consequences of my critical revision of Field's analysis of applied mathematics are two: on the one hand, I provide an account of idealization which can be added to Field's account of applicability to integrate it; on the other hand, I extend Field's account by offering a more realistic characterization of the role mathematics plays in the sciences. In sum, I develop a new account of applicability which supplements and extends the one proposed by Field.

PART I: APPLICABILITY AND IDEALIZATION

CHAPTER 1: THE APPLICABILITY OF MATHEMATICS AND THE PROBLEM OF IDEALIZATION

1. The scientific use of mathematics

The development of many among the experimental sciences is characterized by a pervasive use of mathematics. An obvious example is physics, where quantities like masses, lengths or velocities are studied by means of their numerical measures or functions of these measures, to which various numerical operations (arithmetical ones, the operations of the calculus etc.) are applied.

Sometimes mathematics is so deeply ingrained in scientific practice that its employment may seem relatively free. However, one of the main reasons for applying mathematics is that it should help studying empirical phenomena and making predictions concerning them. Because of this, its use has to be directed and constrained by underlying empirical conditions, lest it turns out to be empirically uninformative or meaningless.

To give a concrete instance of the relationship between applied mathematics and the empirical conditions upon which it rests, a simple example can be conveniently used, concerning the measurement of the hardness of minerals. One way of establishing hardness measures is by means of a scratch test: if mineral sample *a* scratches mineral sample *b*, then *a* is said to be harder than *b*.

Now let the binary relation 'scratches' be symbolized by S. It is plausible to assume, on the basis of experimental evidence, that S is a transitive relation (i.e. S(*a*, *b*) and S(*b*, *c*) imply S(*a*, *c*)[1]) and also that it is connected (in other words, S(*a*, *b*) or S(*b*, *a*) has to be the case, for any mineral samples *a*, *b* under consideration[2]).

A finite collection of mineral samples may be formalized as a set X, on which the relation S is defined. Usually one talks about the *empirical structure* **S** = ⟨X, S⟩. If S is transitive and connected, then **S** can be described numerically. Just take any *x* in X

[1] The example I am using here is an idealized and simplified version of actual hardness measurement. In practice, hardness is evaluated by means of a scratch test to identify minerals. A knife-blade or the broken edge of a file are used against different minerals to check whether they are scratched by it equally easily. Such a criterion is relatively vague and only in conjunction with others (inspection of the minerals' colour and shape, for instance) is it sufficient to identify mineral samples. However, to some approximation, the scratch relation can be understood as an ordering of minerals, so my example is sufficiently realistic (for a detailed discussion of hardness measurement in a context which differs from the one being illustrated here see Adams 1966).

[2] It is possible that both alternatives hold, i.e. *a* scratches *b* and *b* scratches *a*. A situation in which none of the samples scratches the other may be considered equivalent to one where each sample scratches the other, indicating equal hardness.

and look at how many elements of X it scratches, including itself (different samples of the same mineral may be used): if n, assign to x the number n. Through repeated application of this procedure, a measure h of hardness can be obtained for the samples in X. This is a function satisfying the following biconditional:

$$\text{(i) } S(a, b) \text{ iff } h(a) \geq h(b).$$

What (i) says is that **S** is related by h to a finite set of natural numbers, on which order is defined. More formally, **S** is related to the *numerical structure* $\mathbf{N} = \langle \Sigma, \geq \rangle$, where Σ is the set of the first n positive integers, if the elements of X which differ in hardness form n classes[3].

Thanks to (i), the behaviour of S on X is kept track of by the numerical order relation on Σ. For example, consider $h(x)$, $h(y)$, $h(z)$, in the range of h: since these are numbers, $\geq$ must hold between any two of them in some order. Suppose $h(x) \geq h(y)$ and $h(y) \geq h(z)$. By numerical transitivity, $h(x) \geq h(z)$. Then (i) yields $S(x, y)$. Thus, numerical relations can be used to make inferences about the hardness of minerals. This depends on the fact that the formal properties of transitivity and connectedness characterize simultaneously the behaviour of mineral samples relative to the scratch relation and the behaviour of numbers, in particular positive integers, with respect to their usual ordering. The connection between **S** and **N** depends on their being of the same type and sharing certain fundamental formal properties, which can be carried back and forth between them.

This example shows, in terms of functions connecting empirical and mathematical structures, what it means that the applicability of mathematics is constrained by underlying empirical conditions. When a given empirical situation can be model-theoretically described as a particular type of empirical structure[4] satisfying suitable formal properties, it may be possible to 'translate' its behaviour into mathematical terms. This is done by means of a function which preserves the structural properties of an empirical setting and carries them into a mathematical model: the latter's properties are empirically meaningful because they correspond to empirical properties. Thus, the presence of function h in the above example shows that a certain family of numerical inequalities can be applied or, equivalently, that it is empirically meaningful. Any numerical inequality is related to the physical

[3] To each object in the same class the same numerical measure is assigned.

[4] One talks sometimes of impure sets in these cases, i.e. sets of physical objects (cf. for instance Burgess & Rosen 1997: 70), which may be endowed with operations and relations.

behaviour of mineral samples by h. In general, functions which preserve the structural properties of empirical settings and carry them into mathematical (typically numerical) models establish the applicability of these models.

More precisely, the presence of certain empirical properties (e.g. the transitivity and connectedness of the scratch relation) and structure-preserving functions (e.g. h) lies at the basis of the applicability of mathematics and explains why certain mathematical or numerical results are empirically informative.

There is thus an important correlation between the applicability of mathematical models and the empirical properties of the settings to which they are applied. It is these last properties which govern the possibility of meaningfully using mathematics to investigate certain natural phenomena.

2. Field 1980

The previous discussion shows that structure-preserving functions may be used to characterize the applicability of certain mathematical structures to empirical settings. Moreover, the mineral hardness example just discussed suggests that the existence of structure-preserving functions depends on empirical properties satisfied by a type of empirical setting. The role played by these formal properties is crucial. For example, a hardness scale is never needed just on a single set of mineral samples, but has to work for ideally any such set. Thus, it is important to look for the formal properties which *any* structure of the type of **S** has to satisfy in order for it to have a numerical model. It is these formal properties which lead to structure-preserving mappings like h and explain the applicability of mathematical models.

Thus, the applicability of hardness measures to minerals can be established by providing a set of axioms for structures of the type of **S** such that any model of these axioms is mapped into a numerical structure like **N** by some function h. In general, an account of the applicability of certain mathematical models to a class of empirical settings can be obtained along the same lines, i.e. by isolating a system of empirical axioms whose models are related to the mathematical models *via* suitable functions.

Achieving this kind of result is philosophically significant. Philosophical significance can best be seen if one considers scientific theories which have traditionally received a numerical treatment and whose ordinary development is carried out numerically, e.g. classical mechanics. Showing that the numerical treatment of these theories relies on a set of empirical axioms means to explicitly identify the empirical conditions under which a certain fragment of mathematics can be applied.

This is because, when a model of the empirical axioms is given, it follows from them that it can be described by a suitable numerical system (e.g. **S** can be described by **N**): thus it is made clear that the structure of the empirical setting can be investigated using numerical properties and reasoning on them. A clear analysis of the applicability of mathematics is therefore available, and this is a philosophically significant result.

Hartry Field's book *Science without Numbers* (Field 1980) contains a treatment of certain scientific theories, in particular a detailed discussion of the classical theory of the gravitational field, which is developed by means of structure-preserving mappings in the way just illustrated.

Exploiting results in the foundations of geometry and measurement, Field gives a system of empirical axioms for gravitation theory. In essence, he uses axioms describing the physical geometry of space-time and the quantity called gravitational potential[5], varying over it. He also shows that the laws governing the classical gravitational field[6], which are usually formulated numerically as partial differential equations, can be expressed in purely geometrical terms, as physical (as opposed to numerical) relations.

These results are obtained in the context of a nominalistic project, which is the main focus of Field 1980, and is directed toward the elimination of mathematical entities from scientific theories. In this context, Field provides an empirical axiom system for gravitation theory with the objective of showing that this theory can be formulated without making any use of mathematical references but only relying on physical ones. Moreover, the empirical axioms make it possible to prove a *representation theorem*, stating that any of their models is related to a numerical structure via a structure-preserving mapping. Field does this within a set-theoretical framework and also sketches a proof to the effect that, assuming the consistency of set theory, any empirically meaningful statement following from his axioms of

[5] He also briefly discusses mass-density.

[6] Poisson's law and the vector equation stating that the gravitational field is the gradient of the scalar field of potential.

gravitation theory plus the axioms of set theory[7] is already a consequence of the axioms of gravitation theory alone (for details, see Field 1980: 16–19).

The main reason why Field needs both the representation theorem and the last result to eliminate mathematical entities from scientific theories can be briefly illustrated. A representation theorem shows that suitable physical, non-numerical axioms are sufficient to characterize physical structures which are ordinarily dealt with numerically. Thus, the numerical formulation of the theories describing these structures can be replaced by a purely non-numerical one, containing only physical axioms and thus eliminating all mathematical references.

The other result Field obtains, showing that the empirical axioms plus set theory have the same empirical consequences as the empirical axioms alone, serves the purpose of showing that mathematics does not generate any new empirical content but is only used to derive empirical facts which are already constrained by the empirical axioms. This result is relevant to the elimination of mathematics from scientific theories when suitably strengthened.

The strengthening says that any empirical statement *provable* from a set of empirical axioms plus set theory can already be proved from the empirical axioms alone: if this strengthening could be established, then it could be concluded that mathematics can be eliminated not only from the formulation of scientific theories but also from their deductive development (i.e. non-mathematical axioms suffice to obtain all proofs of empirical facts). Unfortunately, this conclusion cannot be reached, as has been shown in Shapiro 1983 (this point is discussed in slightly more detail in chapter 1 of part 2). As a result, Field can only conclude that certain scientific theories can be satisfactorily characterized by non-mathematical axioms, while the use of mathematics in them does not generate any new empirical content, even though mathematics may be necessary to prove some of their empirical consequences. Nonetheless, as far as gravitation theory is concerned, Field is able to show that the differential calculus needed for it can be reconstructed in non-numerical terms: in this sense, he partially recovers the possibility of eliminating mathematics from several proofs and inferences needed in order to develop the main scientific theory he investigates.

[7] Field uses ZFC with urelements.

As I remarked above, the motivation for these results comes from an eliminative project aimed at dispensing with mathematics in scientific theories. However, the very same results generate a detailed account of the applicability of mathematics.

For example, Field proves representation theorems. This means that he gives empirical axioms and proves that the models of these axioms are related by structure-preserving functions to numerical structures. But this is an account of the applicability of the relevant numerical models, which isolates the empirical conditions (empirical axioms) under which they can be introduced to study certain empirical phenomena.

In the same vein, when Field shows that the numerical techniques of the calculus can be reconstructed in non-numerical terms, his nominalistic objective is to show that the numerical proofs involving these techniques can be eliminated in favour of purely non-numerical proofs exploiting physical relations only. However, this result can be naturally read as an explanation of the applicability of the calculus: its use makes empirical sense because its concepts and results correspond to forms of reasoning based on empirical relations.

For this reason, it is quite straightforward to think of Field 1980 as a study of the applicability of mathematics: in particular, the nominalistic project carried out in the book generates an account of applicability. For this reason, it becomes interesting to look at the philosophical relevance of Field 1980 from this particular point of view.

3. Two readings of Field

The foregoing discussion has implicitly made clear that it is possible to emphasize two different aspects of Field 1980, depending on whether the emphasis is put on Field's nominalistic project or his concern with explaining the applicability of mathematics (stated in Field 1980: 6). Accordingly, Field's results can be read in two ways. However, only one reading, focusing on nominalism and the eliminability of mathematics from science, has really received extensive attention in the philosophical literature. The other reading of Field, focusing on applicability, has been much less debated and it is on this that I wish to concentrate my attention.

Since the two readings of Field 1980 are related, it is possible to associate to the problems arisen from the nominalistic reading corresponding problems of relevance to the other reading, revolving around applicability. For this reason I will now briefly look at two major issues emerged in the discussion of Field's nominalism and use them as an aid to identifying corresponding, important issues which have to do with applicability. It is these issues that I will study in detail.

As already noticed, Field's nominalistic approach seeks to eliminate mathematical references from scientific theories: thus, the proofs of representation theorems and the other technical results he obtains may be seen as instrumental to reaching a nominalistic objective. On this understanding of Field's results, two crucial problems arise: they consist in clarifying the extent to which his eliminative project is successful and the extent to which it is faithful to nominalism (i.e. whether the resources used to carry out the project are really acceptable to a nominalist).

The first kind of problem has been discussed in Malament 1985, Hellman 1989: 135–144, Burgess & Rosen 1997: 117-118 and Balaguer 1998: ch.6. These works critically investigate the possibility of extending Field's approach, which is based on representation theorems, to theories like quantum or statistical mechanics (Dirac 1981: ch.3 happens to be relevant to this debate as well). It is at present unclear how general Field's approach really is[8].

The second kind of problem relates to the nature of the resources adopted by Field to carry out his project, in particular the question whether Field appeals to abstract entities of any kind or not. This point has been discussed for instance in Resnik 1983, 1985a-b and is touched upon in Shapiro 2000: 232. Two of the main objections to Field in this context concern the problematic status of space-time points, which Field considers nominalistically acceptable, and the fact that he formally characterizes space-time precisely as a numerical structure based on the reals (the vector space $\mathbf{R}^4$)[9], so that the difference between space-time and its abstract, numerical counterpart looks thin[10].

Now let me turn to the less well-investigated reading of Field's project: on this reading, Field's technical results are not primarily understood as accomplishing the elimination of mathematics from scientific theories, but as delivering *an account of the applicability of mathematics* to these theories. The above problem concerning how much Field's approach can be extended corresponds in this context to the problem whether all applications of mathematics can be realistically characterized in

[8] For example, Malament 1985 is sceptical about its adequacy to deal with quantum mechanics while Balaguer 1998 sketches a way of responding to Malament's scepticism.

[9] Except, roughly, for the fact that there is no way of singling out two privileged points in space-time corresponding to the real numbers 0 and 1.

[10] Field 1984 and Burgess 1991 contain attempts at the elimination of space-time points in favour of material objects, which may to some extent be read as a response to this kind of objection.

terms of structure-preserving mappings and representation theorems: if this is not the case, then Field's account of applicability is too restrictive and overlooks certain uses of mathematics in the sciences. I will make a case for this objection and discuss it in detail in part 2 of this thesis.

As for the issue of how much Field is faithful to nominalistic standards, this corresponds, from the point of view of applicability, to the problem of how much Field makes use of idealization. For instance, assuming many structural properties of the real numbers to be satisfied by space-time points means to give a strongly idealized characterization of space-time, purely theoretical and far removed from observation. The issue in this case is to understand whether idealization is legitimate and on which grounds it can be allowed. This is a problem I mainly discuss in part 1 of this thesis, but which will also be important in part 2.

My task in the following chapters is to provide a critical discussion of the two problems I have just highlighted: my objective is to show how the benefits of Field's analysis of applicability can be retained within an extended analysis which can successfully tackle the shortcomings of his (to be identified in the following chapters), proving more comprehensive and more realistic. As a starting point of this analysis, let me consider the problem of idealization as it arises in Field 1980.

4. Applicability and idealization

Any satisfactory account of applicability should explain how mathematics is used in describing or studying certain features or portions of the actual world. Thus, it has to provide an explicit link between mathematical theories and experiment. Field's interests, however, are mainly concentrated on the link between mathematics and theoretical, as opposed to experimental, physics. While this restricted focus does not necessarily affect his nominalistic project [11], it is in my opinion decidedly problematic if a fully satisfactory account of the applicability of mathematics is sought. The reason is that applicability relates to experiment and is not confined to the theoretical construction of formal theories. For instance, numbers are important in science because they are used to make predictions, but this means that numbers computed within a theoretical framework have to be compared with the outcomes of

[11] Unless one insists that theoretical physics makes use of idealizations, in particular ideal entities (e.g. space-time points), and that these count as *abstracta*, in which case a nominalist should not make use of idealizations.

experimentally obtained measurements. In general, the two numerical results will disagree. Actual measurement procedures exhibit a degree of variability incompatible with the exactness of purely mathematical results. The latter results often presuppose the availability of an infinitely precise or *idealized* measurement procedure, and yet they are used in actual scientific practice. The problem is then to explain the place and role of idealization in this context, relative to the investigation of actual, experimentally controlled, phenomena. In particular, two related questions arise, concerning the grounds on which idealizing assumptions are introduced and the reason why they are introduced.

Such twofold explanation is missing in Field 1980 or later works (see in particular the contributions collected in Field 1989). In essence, Field shows how numbers are applied to a scientific theory involving *idealization*. He does not show why this account of applicability is relevant to the description of the actual world, because he does not show how it relates to experiment. This of course does not imply a rejection of Field's approach, but certainly requires extending his account of applicability to cover idealization.

I think this can be done by starting from an idealized empirical theory like the ones Field considers and showing in which precise sense this theory can be thought of as emerging from weaker theories which take into account the limitations of experimental practice.

My task in the next four chapters is to do this for a scientific theory which is simpler than Field's. However my discussion can be in principle extended to Field's example of gravitation theory. In the next section I briefly explain why this is the case.

5. From Field to measurement

When establishing the representation theorems he needs to articulate his treatment of gravitation theory, Field explicitly refers (cf. Field 1980: 58) to the mathematical studies of measurement contained in the first volume of *Foundations of Measurement* (Krantz et al.1971).

My plan here is to look at these foundational studies and related works, in order to give an account of the applicability of numerical scales to measurement which, while based on the same approach used by Field to deal with scientific theories (i.e. on proving representation theorems), addresses in addition the problem of idealization as outlined above.

I restrict my attention to the theory of extensive or additive measurement[12], which is usually adopted to characterize the measurement of such fundamental, additive physical quantities as mass or length. The reason for this restriction is twofold: on the one hand, extensive measurement has been subjected to a particularly thorough study in the last five decades, and so many formal results are available to provide a solid basis for an account of idealization in this context; on the other hand, it is in principle possible to reduce the axiomatic theories discussed in Field 1980 to extensive measurement. By this clue, all observations which can be made concerning extensive measurement are naturally transferred to Field 1980. Therefore, a detailed discussion of extensive measurement, besides being interesting in itself, says something of relevance about the account of applicability in Field 1980.

The reason why Field's example of gravitation theory is in essence reducible to the theory of extensive measurement can be concisely illustrated here: Field essentially looks at three different theories, namely affine space-time geometry and the measurement theories of gravitational potential and mass-density. His non-numerical account of gravitation is based on a combination of these theories.

Gravitational potential, as Field axiomatizes it, can be treated in the framework of the theory of extensive measurement, as explained in chapter 4 of Krantz et al. 1971. Mass-density, if understood as a function of mass and volume, depends on the measurement of two extensive quantities and thus can be dealt with by means of them.

Moreover, the axiomatization of affine geometry[13] Field uses to characterize space-time is strong enough to ensure that, once an origin is arbitrarily fixed, any space-time point can be reached from that origin by a unique sum of translations[14] or 'displacements' (for a proof of this fact see Artin 1957, 1965). But parallel translations, intuitively, sum like lengths and thus can be treated by means of the

[12] I.e. measurement relying on the presence of a concrete concatenation operation that behaves formally like arithmetical addition: for instance the operation of length-addition which is obtained by collinearly juxtaposing e.g. two rigid rods.

[13] The relevant axioms are those given in Tarski & Sczerba 1964, but Field strengthens them to obtain isomorphism between space-time and a corresponding real coordinate system. The strengthening is mentioned in Field 1980: 37 and explicitly formulated in Burgess 1985: 383.

[14] Understood as structure-preserving transformations from space-time into itself.

theory of extensive measurement[15]. Because of this, an analysis of the problem of idealization for the latter theory suffices to illustrate how, in principle, the same problem can be solved in the context of Field 1980.

In the next sections of this chapter I present the classical theory of extensive measurement and show how the isolation of its axioms leads to an explanation of the applicability of numbers to certain empirical structures in terms of structure-preservation, in line with Field's approach.

Subsequently, I identify among the axioms the idealizing hypotheses, i.e. those assumptions in the classical theory of extensive measurement which either involve unobservable properties or postulate a greater regularity than actually exhibited by experiment. This illustrates the particular form in which the problem of idealization arises in extensive measurement (and the theories reducible to it). Although idealization is widely discussed in philosophy of science, its analysis in measurement has been largely overlooked and it cannot in many respects be reduced to the traditional philosophical account (this is just a passing remark; I will offer evidence for it in chapter 3). For this reason, an account of idealization for measurement does not amount to a mere repetition of well-known ideas. Once this account is in place, it can be used to integrate Field's own account of applicability.

The next chapters are devoted to dealing with idealization. The task of the remaining sections of this chapter is to introduce the theory of extensive measurement and identify its main idealizations.

6. Preliminaries to extensive measurement

I begin with a simple example, which will serve the purpose of introducing some of the axioms of extensive measurement and the problem of idealization for them. Consider a set of perfectly rigid rods which can be laid against one another and thus compared with respect to length. When we are given two rods a and b, we can check by comparison whether b is or is not longer than a. This can be done by putting corresponding endpoints of a and b in contact, while the rods lie along the same direction, and seeing what happens to their other endpoints. If one of them lies strictly between the endpoints of the other rod, then the latter rod is longer than the former. For instance, we can observe, by comparing two rods a and b, that a is longer

[15] Non-parallel translations are dealt with separately.

than b. If this is not the case, we conclude that a is not longer than b: we can denote the latter situation symbolically by $a \leq_L b$, where the binary relation '$\leq_L$' refers to the relation 'not longer than'. We usually understand this relation as an alternative: either b is longer than a or the endpoints of a and b can be brought simultaneously to exact coincidence.

A problem already arises at this stage: for the procedure of comparison I have just described presupposes that we can actually identify the endpoints of rigid rods and bring them to exact coincidence. Equivalently, we are assuming that we are able to decide whether rods a and b have exactly the same length: this means that, if we were to measure them, we would be able to decide whether they have the same numerical measure (relative to a fixed unit).

This, however, causes problems. The reason can be illustrated as follows: consider the case in which b is longer than a, and suppose we also have a rod c, longer than b. We can physically observe that c is longer than a and this, under normal experimental conditions, will happen for any a, b, c. The relation 'longer than' is therefore a transitive relation and it can also be seen that it is asymmetric (if a is longer than b, then it can't be the case that b is longer than a), while it can be taken to be irreflexive by definition, if it is assumed that it doesn't make sense to say that a is longer than itself.

Thus, 'longer than' satisfies certain formal properties, namely irreflextivity, asymmetry and transitivity, which hold e.g. in the real numbers with respect to their strict ordering according to magnitude. The possibility suggests itself that we can exploit this correspondence between 'longer than' and 'numerical ordering' to attach numbers to rods. For instance, if we have finitely many rods, a is the shortest of them and b is longer than a, we require the measure of b to be greater than that of a. By the same clue, we might suppose that, if b is not longer than a, then the measure of b must be smaller than or equal to that of a. Writing $\mu(x)$ for the measure of x, we have:

$$\text{(i) } \mu(b) \leq \mu(a)$$

and the consequence that forces itself upon us on the basis of this requirement is that, whenever we observe a and b to have the same length, we put:

$$\text{(ii) } \mu(b) = \mu(a).$$

Under this condition, suppose we find, as frequently happens in experimental practice, that *a* and *b* are observably of equal length and so are *b* and *c*, while *a* and *c* are not. By a previous assumption, we are able to decide whether rigid rods are equally long and thus whether they should be assigned the same measure or not. This leads us to the following set of measures:

(iii) $\mu(b) = \mu(a)$ and $\mu(b) = \mu(c)$ and $\mu(a) \neq \mu(c)$.

This set is inconsistent, because numerical equality is transitive and so the first two conditions numerically entail the negation of the third. The conclusion is that, if we try to exploit certain formal similarities between the ordering on rods with respect to length and the usual ordering of real numbers for the purpose of numerically portraying rod comparisons with respect to length, we are bound to end up with inconsistent numerical assignments. Should we then forsake the possibility of scale construction?

7. Order on length

When, in experimental practice, triples of objects giving rise to (iii) of previous section are measured, what is sometimes done (cf. Eisenhart 1963: 164) is to so adjust their measures as to make them consistent. This is not merely a sleight of hand: one might think that the measurements obtained are affected by error and decide to uniformly distribute the error on all measures. Nonetheless, this move, besides presupposing a theory of error, disregards the possibility that the objects being measured are in fact quantitatively different, although not detectably so. In this case a conventional assignment of error-adjustments may be inconsistent with the actual differences in length existing between the objects being measured.

In order for this trouble to be avoided and the correspondence between numerical equality and length-equality to be preserved despite its failure in practice, it can then be assumed *that there is an ideal length-order relation* $\leq_L$ *which behaves like numerical order*. Its associated length-equality relation $=_L$ will then behave like numerical equality: in particular, it will satisfy transitivity, thus ruling out the possibility of inconsistencies like (iii).

This move solves the problem arisen in section 6, but only for an idealized counterpart of experimental practice: the solution requires the cost of choosing an account of measurement which is now removed from experience and, as it were, replaces it by a more regular counterpart, satisfying the formal property of transitivity[16]. The philosophical problem idealization poses stems from this kind of situation: when we make strong theoretical hypotheses, involving the concept of infinity or assuming, as in this case, a more regular empirical behaviour than exhibited by experiment[17], we construct scientific theories which are to some extent removed from the settings we actually intend to investigate. The mathematical treatment of scientific theories often involves idealization[18] and thus, as already observed, any satisfactory account of the applicability of mathematics must deal with it, explaining how it is introduced and why. This problem and its nature will be discussed in the last section of this chapter, once other ideal assumptions of the theory of extensive measurement are isolated.

For now, let us simply assume we have an ideal ordering of rods according to length which behaves like numerical order, and let us see where this presupposition leads us. First of all, if it is not the case that a is longer than b, then $a \leq_L b$ means that either b is longer than a or they have the same length. Thus, it is plausible to say that any a is not longer than itself, and write:

1) for any a, $a \leq_L a$.

For any two different rods a, b we $a \leq_L b$ determines, ideally, a physical alternative: for it may or may not be the case that b is not longer than a, i.e. $b \leq_L a$: if not, then we can write $a <_L b$. On the contrary, if both $a \leq_L b$ and $b \leq_L a$, then a and b have exactly the same length: we can write $a =_L b$ and say that:

2) $a \leq_L b$ and $b \leq_L a$ entail $a =_L b$.

These informal observations depend on the assumption of an ideal length-order, which makes it possible to always determine equality in length. Since the latter relation is transitive and it was observed earlier that $<_L$ is also transitive, it follows that $\leq_L$ is. That is:

3) for any rods a, b, c: $a \leq_L b$ and $b \leq_L c$ entail $a \leq_L c$.

Again, relying on physical intuition, we see that, for any two rods, they must either have different lengths or have equal lengths. Both alternatives cannot occur, because there is no physical configuration of rods realizing them simultaneously: on the other

[16] For a discussion of this point see Kyburg 1983: 69–72

[17] Here we assume length-equality to be a transitive relation, against the fact that we may have rods a, b, c such that a, b and b, c have observably equal length while a and c differ in length.

[18] Indeed, as I will show at the beginning in part 2, there are cases in which idealization is necessary to construct a scientific theory.

hand, one of them must occur. This property can be expressed formally by saying that:

4) for any rods a, b exactly one of the following holds: $a <_L b$ or $b <_L a$ or $a =_L b$.

The formal properties of length-order (1) to (4) have a physical motivation, but only under an ideal hypothesis, namely the transitivity of length-equality. Thus, overall, they characterize an idealized comparison procedure. They can be formalized in the usual way:

1. Reflexivity: $\forall x\ (x \leq_L x)$;
2. Antisymmetry: $\forall x \forall y\ (x \leq_L y \wedge y \leq_L x \rightarrow x =_L y)$;
3. Transitivity: $\forall x \forall y \forall z\ (x \leq_L y \wedge y \leq_L z \rightarrow x \leq_L z)$;
4. Trichotomy: $\forall x \forall y\ (x <_L y \vee y <_L x \vee x =_L y)$[19].

The above list axiomatizes length-comparison: trivially, the axioms also hold in the ordered reals. Thus, *via* idealization, it is possible to link the numerical ordering of the reals and the length-comparison relation between rods. This correlation, i.e. the fact that in both the physical and numerical case axioms 1 to 4 hold, makes it possible to find a way of assigning numbers to rods, if there are only finitely many of them.

In general, we can order the given rods from shortest to longest and assign to them a finite, increasing sequence of real numbers, or, if there are n rods, at most the first n positive integers (with the possible exception that rods having the same length are assigned the same positive integer). Calling μ the correspondence between rods and integers just described, we see that it satisfies the condition that:

(O) $a \leq_L b$ if and only if $\mu(a) \leq \mu(b)$

where $\mu(a)$ and $\mu(b)$ are numbers (this is formally the same result obtained for mineral hardness measurement in section 1). This condition tells us that we can freely move from the ordering of rods' measures to the ordering of rods and viceversa: we can, in other words, translate physical statements about the ordering of lengths into numerical statements about their measures and also do the converse. Whenever we are given the numerical statement $\mu(a) \leq \mu(b)$, we can conclude that rod a is not longer than rod b and thus use numbers to study a physical relation.

This possibility depends on axioms 1 to 4 and the fact that there are finitely many rods (in fact, axiom 3 and 4 and finiteness are enough). A statement to the effect that

[19] That the disjunction is exclusive follows from the other axioms.

axioms 1 to 4 plus finiteness ensure that condition (O) holds is a *representation theorem* for length-comparison and the function μ is called a *representation.*

Note that axioms 1 to 4 are purely empirical and they do not contain any reference to numbers or numerical relations: if moreover the number of rods is fixed and given, then the theory of length-comparison can be entirely formulated in explicitly empirical terms (in Field's terminology it is 'nominalistically acceptable'). Also observe that, in view of (O), measurement is not just the assignment of numbers to certain objects, but a structure-preserving assignment, which makes it possible to systematically keep track of certain relational properties of the objects being measured by means of corresponding relational properties of a numerical domain.

The representation theorem obtained for length-comparison can be strengthened and enriched. This depends on the well known fact that, if a rod is longer than another, we are able to say not only that it is but also how much longer it is. This is possible by exploiting, together with the physical relation $\leq_L$, a physical operation of collinear juxtaposition, by means of which two or more rods can be concatenated along the same direction to form a longer rod. I will symbolize this operation by '$+_L$': it is through its application that it is possible to evaluate differences in length between rods. Suitable axioms for collinear juxtaposition and its interaction with length-comparison give rise to the classical theory of extensive measurement.

8. Length-concatenation

Let us again consider a set of perfectly rigid rods, together with the binary relation $\leq_L$ and the binary operation $+_L$. Now suppose we pick a rod a and take it as our unit of measure: this means that we assign to it the number 1, by stipulation.

If rod b is longer than rod a, we can assign a uniquely determined real number to b exploiting the physical properties of $+_L$, exactly as the (idealized) properties of $\leq_L$ were exploited to obtain a scale of measurement for length-comparisons. These properties, as a consequence, can be isolated by describing a procedure fixing the numerical assignment for b. First of all, b is longer than a. Let us suppose it is also shorter than $a +_L a$: then consider a rod a' that, concatenated to a, is exactly as long as b. We have, in symbols:

$$a +_L a' =_L b$$

By taking x such that the concatenation of ten copies of x equals a in length, suppose we find that a' is longer than x. Then either the concatenation of a finite number (smaller than 10) of copies of x covers a' exactly, or it doesn't. If not, suppose a' is longer than 6 copies of x but shorter than 7 copies of x. Then that b is longer than $1a$ concatenated to $6x$. Now, again, we may take a'', the rod such that: $6x +_{\mathrm{L}} a'' =_{\mathrm{L}} a'$.

Once y has been found, such that the concatenation of ten copies of y equals x, a comparison entirely analogous to the one illustrated above can be made: suppose it shows that a'' equals the fourth concatenation of y. Then we conclude that b is as long as $1a$ concatenated to $6x$ concatenated to $4y$. We may *define* $6x$ to be $.6a$ and $4y$ to be $.04a$: in this case it seems natural to conclude that the length of b is $1.64a$. But such a conclusion says something substantial about the properties of the concatenation $+_{\mathrm{L}}$: in particular it says that it determines on the above numerical indices an operation which behaves like arithmetical addition. Similar conclusions can be reached for order. To see how, note that the numerical constants used above need not be assumed to be numbers, since they can be defined in terms of $+_{\mathrm{L}}$. For instance, $6x$ is:

$$x +_{\mathrm{L}} x +_{\mathrm{L}} x +_{\mathrm{L}} x +_{\mathrm{L}} x +_{\mathrm{L}} x.$$

By the same clue $.4a$ is the object z such that the fourth concatenation of z is as long as a. Other cases are dealt with analogously. Now, we can define a binary operation '+' on the constants just introduced, whose behaviour follows from the behaviour of concatenation. For instance:

$$\text{(s) } (2 + 3)x = 2x +_{\mathrm{L}} 3x \text{ leads to } 2 + 3 = 5.$$

The same is possible for order since, e.g.:

$$\text{(o) } 2x <_{\mathrm{L}} 3x \text{ leads to } 2 < 3.$$

Because of the behaviour of idealized length-comparison, indices are linearly ordered. At the same time, testable properties of length-concatenation like commutativity and associativity are transferred to index-concatenation. This suggests that a measurement procedure, based on the manipulation of rigid rods, may generate a system of indices which is linearly ordered and well-behaved algebraically[20]: if the empirical structure on rods is sufficiently strong, these indices may be identified with numerical measures on a suitable mathematical model, in particular the positive real numbers with order and addition. The axioms of classical extensive measurement can

[20] For a thorough exploration of this fact see Huntington 1902 and Whitney 1968.

be isolated as soon as the formal properties (testable and untestable) of concatenation and order needed to construct these measures are made explicit.

9. The axioms of extensive measurement

By looking in greater detail at the measurement procedure described above, it is possible to identify the properties of length-concatenation which are needed for a scale of length-measurement to be set up. Suppose we have a rod b longer than a. I assumed above that there was another rod a' such that its concatenation with a was as long as b. If we want this step of the procedure to be always feasible (and thus we do not impose any restriction on the physical resources we may help ourselves to), whichever a and b are given, we must assume:

Solvability for any a, b: if $a <_L b$, then there is c such that $a +_L c =_L b$.

The commutativity of length-concatenation ensures that $c +_L a =_L b$ as well. Concatenation is also associative. Here it suffices to assume associativity only:

Associativity for any a, b, c: $a +_L (b +_L c) =_L (a +_L b) +_L c$.

Now, when we concatenate several copies of x, we want to be sure that they will eventually cover a', and one property of concatenation required to ensure this says that the concatenation of two rods is strictly longer than the components:

Positivity for any a, b: $a +_L b >_L a, b$.

Moreover, we want to be sure that a finite number of rods like x will cover a', otherwise our approximation process to determine its length in terms of x does not terminate. We thus assume:

Archimedeanity for any a, b there is a positive integer n such that $na >_L b$[21].

At the same time we want to be able to choose x to be one tenth of a or, for that matter, any integer submultiple of a. This leads to:

Divisibility for any a and any positive integer n, there is x such that $nx =_L a$.

Looking back once again at the example in the previous section, we see that we were brought to choose y such that $100y =_L a$: this move was sufficient to assign an index to rod b, but, in general, we may be led to look for a z such that $1000z =_L a$ or even smaller objects. Note that $a >_L y >_L z$, and so we have a chain of decreasing elements: in general, it is best to have an infinite chain of this kind, because arbitrarily small

[21] na is defined recursively by the two clauses: (i) $1a = a$; (ii) $na = (n-1)a +_L a$.

elements may be needed to approach the length of the objects to be measured. Thus it is necessary to require:

No minimal elements for any a there is b such that $a >_L b$.

Finally, since the arbitrarily small elements whose existence is granted by the last axioms may be used to form arbitrarily large concatenations (like $100y$ or $1000z$ etc.) we require concatenation to always be defined. To this end, we assume:

Closure for any a, b there is a rod as long as $a +_L b$.

So far we have isolated the properties of Associativity, Closure, Positivity, Solvability, No minimal elements, Divisibility and Archimedeanity: they all stem quite naturally from the analysis of a basic measurement procedure for length-measurement, supposed to have general applicability.

General applicability led in particular to No minimal elements, granting the indefinite extensibility of decreasing chains, and to Closure, granting the indefinite iterability of length-concatenation. There is in fact a close connection between generality requirements and idealization, which will be explored later. For the moment, I only wish point out that generality requirements lead to a further empirical assumption, besides the ones already listed. To see why, suppose rods a and b are as long as the side and diagonal of a given square respectively. If a is the unit of measure, b is longer than a but shorter than $2a$ and we can proceed to consider the difference between a and b (by solvability) to see how many x's cover it (where x is the tenth part of a by divisibility). The difference a' between a and b lies between $4x$ and $5x$ so we take the new difference between $4x$ and a' and try to cover it using the tenth part of x, call it y. Again, we cannot cover it exactly and so we need to consider a new, smaller difference and try to cover it using the tenth part of y. This process has no end, because it is known, and it could be proved geometrically, that the side and diagonal of a square are not rational multiples of the same unit of measure, i.e. they are incommensurable. So when we use a to measure the length of b we generate an infinite sequence of approximations: equivalently, the measure of b with respect to a can only be established by a countable infinity of comparisons.

On the basis of this fact we may assume that, given an arbitrary b, an approximating procedure based on a unit a either determines a measure for b after finitely many comparisons, or it determines one such measure ideally, as the limit of an infinity of comparisons. In other words we assume to have a generally applicable measurement procedure which either terminates in finitely many steps or has an ideal termination at infinity. If we want to describe this ideal termination as part of an (idealized) empirical procedure we must assume the existence of limit rods and their uniqueness. Both conditions can be expressed by the following statement (formally corresponding to a sentence formulated in a second-order language):

Dedekind Completeness for any increasing sequence of rods $s = \{a_i\}_{i\in N}$ if s is bounded above, then s has a least upper bound (in short: a lub).

That s is bounded above means that there is a rod b such that it is longer than any rod in s: whenever this happens, the axiom of Dedekind Completeness tells us that there also is a rod which is longer than any rod in s and is the smallest one (so the unique one) to satisfy this property. This rod can be called lim s[22].

If Dedekind Completeness is assumed, together with the axioms of Associativity, Closure, Positivity and Solvability, and the Trichotomy of length-comparison, i.e. the fourth axiom of section 8, then both Divisibility and Archimedeanity follow (this is proved e.g. in Hölder 1996a: 239, 240–241).

Collecting together all the above observations, we can formulate an axiom system for extensive measurement, which depends upon the analysis of the measurement procedure described in this section and coincides with the system of axioms firstly presented in Hölder 1901[23] (whose original article has been recently translated in English as Hölder 1996a-b):

H1: *Trichotomy* $\forall x\forall y\,(x <_L y \vee y <_L x \vee x =_L y)$;

H2: *Closure* $\forall x\forall y\exists z(z =_L x +_L y)$;

H3: *Associativity* $\forall x\forall y\forall z((x +_L y) +_L z =_L x +_L (y +_L z))$;

H4: *Solvability* $\forall x\forall y(x <_L y \rightarrow \exists z\exists w(z +_L x =_L y \wedge x +_L w =_L y)$;

H5: *Positivity* $\forall x\forall y(x +_L y >_L x \wedge x +_L y >_L y)$;

H6: *No minimal elements* $\forall x\exists y(y <_L x)$;

H7: *Dedekind Completeness* Any bounded above, increasing sequence has a lub[24].

[22] In presence of the axioms it can be shown that the concept of least upper bound (or greatest lower bound, if we were considering decreasing sequences that are bounded below) and that of limit are equivalent. This legitimizes the notation I have just introduced.

[23] A discussion of the motivation of Hölder's system can be found in the introduction to Hölder1996a-b. As a matter of fact, the same axioms are, although informally, isolated in a 1887 paper by Helmholtz, available in English in Helmholtz 1977. A clear and thorough study of Helmholtz's contributions to measurement and the theory of magnitudes can be found in Darrigol 2003, while a comparative discussion of Helmholtz's and Hölder's works which also includes Russell's views on measurement can be found in Michell 1993 (see also Michell 1997, 2003b).

[24] Hölder uses a different formulation of Dedekind completeness, which is however equivalent to the one I have given here, in presence of the other axioms. This equivalent form is part of the axiomatization of extensive measurement in Huntington 1902.

Before discussing the main properties of this system of axioms, I wish to point out a few things about it. The commutativity of the binary operation of length-concatenation has not been assumed because it can be proved from the axioms (as shown in Hölder 1996a: 239). Similarly, the transitivity, asymmetry and irreflexivity of the strict ordering $<_L$ together with the monotonicity of addition with respect to order (i.e. the fact that, if $x <_L y$, then $x +_L z <_L x +_L z$) are theorems of Hölder's theory. The first six axioms can be formulated in a first-order language, whereas the last axiom cannot, because it quantifies over infinite sequences: the fact that it does strengthens the theory considerably, as was already indirectly shown by the fact that, in presence of Dedekind Completeness, neither Divisibility nor Archimedeanity need being assumed. The strength of Dedekind Completeness appears even more strikingly from the fact that all models of Hölder's theory must be isomorphic[25]. This result will be given and explained in the next section.

10. Representation

With an empirical axiom system for extensive measurement[26] in place it is possible to prove a representation theorem, which is much stronger than the one given in section 8 for the measurement of length-order. I will only give an outline of the proof in this section, in order to reach a precise formulation of the theorem itself.

Let a model of Hölder's axioms be given: this can be conceived of as an empirical structure whose domain contains rigid rods and on which a binary order relation and a binary concatenation operation are defined. We may denote this structure by:

$$\mathbf{L} = \langle \mathrm{L}, <_L, +_L \rangle$$

where L is a set of rods (necessarily infinitely large, by the axioms). We may pick an arbitrary rod from L, call it a, and take it as the unit of measure. Thus $\mu(a) = 1$. Suppose we want to assign a measure to b. Then, let us compare iterated concatenations of a and b and isolate the concatenations which give rise to the following type of inequality:

$$\text{(i)} \quad ma <_L nb,$$

where ma is the concatenation of m copies of a and nb is interpreted similarly.

[25] This is true under the assumption, which is made by Hölder, that rods having the same length are identical, i.e. that the relation '$=_L$' is treated as if it were the identity relation '='. I will provide a justification for this assumption below: empirically, it amounts to presupposing that different rods always differ in length.

[26] There is in fact a trouble with taking the system of axioms to be empirical, even allowing for idealization: this will be discussed at length in the next chapter.

To each inequality like (i) the positive rational number m/n can be associated. It follows from the axioms that, for any m/n we can find p/q such that:

$$\text{(ii) } p/q > m/n \text{ and } pa <_L qb$$ [27].

Iterating the application of (ii) one generates an increasing sequence of rational numbers p/q, which in particular is bounded above, by Archimedeanity. By the Dedekind completeness of the positive reals, this sequence has a limit: we define $\mu(b)$ to be this limit and write:

$$\mu(b) = \lim(m/n)_{a,b}$$

Because the limit is unique, we have found a way of assigning to each rod in L a uniquely determined positive real number. Now, note that the resulting assignments also preserve the structure of **L**, i.e. they 'translate' order and concatenation on rods into numerical order and arithmetical addition. In particular, for any two rods b, c in L we have:

$$(\alpha)\ \mu(b +_L c) = \mu(b) + \mu(c)$$ [28].

Also:

$$(\beta)\ \mu(b) < \mu(c) \text{ if and only if } b <_L c.$$

Intuitively, (β) depends on the fact that, as soon as the limits corresponding to the measures of b and c are distinct, the lub of the sequence of rationals associated to one object cannot also be the lub of the sequence associated to the other. Thus they are two different real numbers for which a strict order relation holds. This fact has an important consequence, namely that μ is a one-to-one function or an injection. Additional proofs establish:

(γ) μ is a bijection, i.e. a one-to-one and onto function[29].

Collecting all these observations together, we have the:

Representation Theorem for Extensive Measurement:

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[27] This is because solvability applied to (i) yields:

$$ma +_L x =_L nb, \text{ for some } x.$$

By trichotomy we have three cases:

Case 1: $a <_L x$. Then $(m+1)a <_L ma +_L x =_L nb$. Using monotonicity and the transitivity of $<_L$ we obtain $(m+1)a <_L nb$ and so we may choose p/q in (ii) to be $(m + 1)/n > m/n$.

Case 2: $x <_L a$. It then follows from Archimedeanity and closure that there is k such that $a <_L kx$. Using several properties of concatenation we deduce: $kma +_L kx =_L knb$. Applying Case 1, to this and $a <_L kx$, we see that p/q in (ii) can be chosen to be equal to $(km + 1)/kn > m/n$.

Case 3: $a =_L x$. In this case (i) $(m+1)a =_L nb$. Then choose p/q to be equal to $(2m+1)/2n$, the mean of (m/n) and $(m+1)/n$.

[28] For a proof of this fact see Hölder 1996a: 243.

[29] I omit a proof this fact, which can be found in Hölder 1996a: 244.

If **L** is a model of Hölder's axioms, then there is a scale of measurement μ from **L** onto the additive, ordered, positive reals which satisfies conditions (α) to (γ).

Denoting the additive, ordered positive reals by $\mathbf{R} = \langle R^+, <, +\rangle$, the previous theorem says that for any model **L** of Hölder's axioms, there is an isomorphism μ from **L** onto **R** (an isomorphism being in this case precisely a function which satisfies (α) to (γ)). Since any model of the axioms is isomorphic to **R**, all models of the axioms are isomorphic: studying **R** suffices to study their totality.

With the representation theorem for extensive measurement in place it is possible to conclude that any physical setting satisfying Hölder's axioms behaves exactly as the positive reals with order and addition. A theorem of logic ensures, in addition, that any first or second-order sentence that can be formulated in the language of Hölder's axioms is true in **L** if and only if it is true in **R**[30]. This means that, as far as length-comparison or length-concatenation are concerned, any true numerical sentence in the fragment of real arithmetic which is determined by the structure **R** has a physical content, because it describes something which physically is the case in **L**. It is thereby clear why the arithmetic of order and addition is applicable to measurement. This conclusion is reached by subjecting extensive measurement to the same treatment to which Field subjects scientific theories. Non-numerical axioms are isolated and a representation theorem is proved on their basis. This shows that the applicability of numbers (more precisely, of **R**) in extensive measurement can be explained using Field's approach to scientific theories. However, the explanation offered so far rests on the use of idealization: I will now clarify this point[31].

11. Idealization

Let us now take a closer look at Hölder's theory of extensive measurement. Here are the axioms again:

H1: *Trichotomy* $\forall x \forall y(x <_L y \vee y <_L x \vee x =_L y)$;

H2: *Closure* $\forall x \forall y \exists z(z =_L x +_L y)$;

H3: *Associativity* $\forall x \forall y \forall z((x +_L y) +_L z =_L x +_L (y +_L z))$;

[30] The same goes for formulas and satisfiability, under suitably corresponding assignments.

[31] Because, by previous remarks, the theories studied in Field 1980 can be reduced to extensive measurement, the isolation of idealizations in extensive measurement is relevant to the understanding of idealization in Field 1980.

H4: *Solvability* $\forall x \forall y(x <_L y \rightarrow \exists z \exists w(z +_L x =_L y \wedge x +_L w =_L y)$;

H5: *Positivity* $\forall x \forall y(x +_L y >_L x \wedge x +_L y >_L y)$;

H6: *No minimal elements* $\forall x \exists y(y <_L x)$;

H7: *Dedekind Completeness* Any increasing sequence which is bounded above has a lub.

The problem is to clarify why these axioms determine an idealized theory of measurement. I won't start from a definition of idealization but rather extract it from certain observations on some notable consequences (theoretical or metatheoretical) of the axioms.

First of all, in view of the representation theorem of extensive measurement, any (nonempty) model of the above theory must have as many elements as there are positive reals and so uncountably many. This implies that an account of extensive measurement relying on the axioms characterizes scale construction only for uncountably large empirical settings. No concrete instance of a measurement procedure is really described by the whole set of axioms.

Besides this fact, which crucially depends on H7, it follows from H2 and H5 that there are arbitrarily large objects, e.g. arbitrarily long rigid rods, in any (nonempty) model of the axioms. These can be formed by iterating concatenation on any fixed object, to obtain larger and larger ones. Axiom H6 is also a principle of infinity, but it is directed 'downward', as it were, to progressively smaller elements, while H2 and H5 are directed 'upward'.

Thus, even if H7 were dropped, the remaining assumptions would still be strong enough to characterize exclusively infinitely large measurement settings: these must be uncountably large if H7 is assumed or may be countably infinite if it is not assumed.

A further point concerning the discrepancy between Hölder's theory and experiment relates to the implicit assumption I have only quickly mentioned at the end of section 10 (see fn.25). The assumption amounts to the fact that distinct rods having the same length should be identical, i.e. that $=_L$ should be treated as if it were $=$. The intuitive motivation for this is that rigid rods having the same length would be assigned the same measure by the procedure described above and so they can be quantitatively identified. This intuitive motivation can be mathematically unpacked: we may start from a model **L** of Hölder's theory and allow for the possibility that there are distinct rods of equal length, i.e. ones related by $=_L$. In this case we can construct a measurement scale μ as we did above, but this won't be a one-to-one function, because we can have rods which are different but whose measure is the same. Holder 1901 implicitly assumes that different rods always differ in length and thus avoids the above possibility.

We can however take into account and recover a one-to-one scale of measurement μ if we construct it not for the single elements of L, but for the classes of elements of L which have the same length. For this construction to be consistent we need two things to happen:

i) That the classes we form are disjoint and exhaust L;

ii) That, for any a, b such that $a =_L b$, if a is substituted for b in an atomic formula ϕ in a first-order language for the models of Hölder's theory, then ϕ holds before the substitution is performed if and only if it holds after it is performed. For instance, if ϕ is the atomic sentence '$a +_L c =_L d$', we must be able to show that, under the assumption that $a =_L b$, it is the case that $b +_L c =_L d$, and viceversa[32].

Condition (i) holds if $=_L$ is an *equivalence relation*, i.e. a reflexive, symmetric and transitive relation on L. If this is the case, the axioms ensure that condition (ii) is also satisfied, which technically means that $=_L$ is also a *congruence* on **L** (a congruence is a structure-preserving equivalence relation). When (i) and (ii) hold we can prove the existence of an isomorphism from the equivalence classes of equally long rods in L onto the positive reals.

In order for this to be possible, the transitivity of $=_L$ is essential: this has already been assumed among the axioms for length-order. But there I pointed out that such an assumption is of an ideal character. In particular, it is not confirmed by experimental practice: when measuring, we are always bound to obtain instances of the failure of transitivity (e.g. triples of rods a, b, c such that a, b and b, c are observably of equal length while a and c are not).

[32] Suppose $a +_L c =_L d$ but not $b +_L c =_L d$. Then trichotomy implies two cases: to establish the result we need to prove a contradiction in each case. Case 1: $b +_L c <_L d$. Solvability yields the existence of e such that $b +_L c +_L e =_L d$. The transitivity of $=_L$ implies $b +_L c +_L e =_L a +_L c$ and, by monotonicity properties we have $b +_L e =_L a$. Now, positivity entails $b +_L e >_L b =_L a$ and $b +_L e >_L a$: this contradicts trichotomy. Case 2: $b +_L c >_L d$. Then $b +_L c >_L a +_L c$ and, by monotonicity, $b >_L a$, contradicting trichotomy again. If we assume not $a +_L c =_L d$ and $b +_L c =_L d$ a similar reasoning establishes the equivalence of the two equalities.

Putting together this observation about the transitivity of $=_L$ and the previous remarks on infinity, we see that Hölder's theory characterizes extensive measurement and accounts for the applicability of numbers to it only in case:

I_1) The existence of infinitely many objects to be measured is assumed;

I_2) The relation of empirical equality $=_L$ is assumed to be transitive.

If these assumptions are accepted, extensive measurement can be subjected to the treatment of scientific theories developed in Field 1980, based on empirical axiomatization and representation. The problem is that, from the point of view of applicability, this treatment is not enough, as it does not fully explain why an idealized theory of measurement is relevant to the investigation of actual physical phenomena. Applicability is here at most explained relative to an idealized empirical setting, as opposed to an actual experimental setting.

By *an idealized empirical setting* I mean the model of an empirical theory including, together with hypotheses which may be verified experimentally, further hypotheses that either postulate a greater regularity than exhibited by experiment (like I_2, since transitivity is only satisfied by idealized, as opposed to actual, empirical equality) or are unobservable or untestable (like the existence of infinitely many objects in I_1). I call *regularity idealizations* the hypotheses of the first type and *infinitary idealizations* the hypotheses of the second type. Later (in chapter 5) I will identify a further type of idealizing hypothesis, namely, *structural idealization*, which is best discussed only after the first two types have been dealt with.

The presence of idealizations makes it impossible to conclude that Hölder's theory provides an account of the applicability of a fragment of real arithmetic to extensive measurement *as it is actually performed.* But I remarked earlier that the theories Field studies in Field 1980 presuppose or are reducible to extensive measurement as here characterized (since Field seeks isomorphism onto certain real structures, he must have second order axioms similar to H7[33]).

Given these observations, it can be seen that Field's account of applicability is problematic, insofar as Hölder's theory proves to be so as an explanation of actual extensive measurement: both Field's account and Hölder's theory only deal with

[33]It should be pointed out that Field 1985: 248–249 contains a weakened representation theorem dropping a completeness axiom like H7, but relative to algebraic structures more general than the reals. The reason why this result cannot be read as a step toward an account of idealization is that the reals are in fact used in scientific practice (if only at a theoretical level), so it is necessary to explain how they, rather than more general structures, arise as idealizations related to applications.

applicability to idealized settings and do not explain applicability with direct reference to actual experimental settings.

Now, if it were possible to show that Hölder's theory can be weakened and also to make explicit the logical relation between its weakened, more realistic versions and the full theory, a way might be opened to explain how idealizations relate to experimental practice. In particular, assumptions I_1 to I_2 might be justified as generalizations arising from it. This would provide an account of the applicability of the theory of **R** to extensive measurement, which would integrate Field's account. I will obtain the desired integration in the next four chapters, by illustrating how Hölder's theory can be seen to arise from weakened versions thereof which do not include its idealizations.

CHAPTER 2: MEASUREMENT BY TYPES AND THE APPLICABILITY OF NUMBERS

1. A problem

Before entering a detailed discussion of idealization in extensive measurement, I wish to consider a possible objection against the fact that Hölder's theory (see the previous chapter, section 10) can be used to give a satisfactory account of the applicability of numbers to extensive structures, even if one freely accepts the use of idealization.

As shown in chapter 1, Hölder's theory can be used to prove a representation theorem. If its axioms are accepted as empirical, then the representation theorem provides an explanation of the applicability of the positive real numbers to extensive structures. This is because any extensive structure is, *via* representation, isomorphic to the positive reals with order and addition: the axioms of Hölder's theory then correspond to the idealized empirical conditions under which it is meaningful to study an extensive structure by means of a specified numerical model.

This conclusion provides a satisfactory account of the applicability of numbers to extensive structures only if the axioms of Hölder's theory and their consequences used in proving the representation theorem are *empirical* conditions. Only in this case the applicability of numerical models to extensive measurement can be explained by exhibiting its empirical grounds.

This is no longer possible if some of the axioms of Hölder's theory or their consequences are numerical conditions. In this case it may be objected that a representation theorem does not explain the applicability of numbers to an extensive structure, since numerical conditions are already present among the statements assumed to establish that very theorem. Thus, the applicability of these numerical conditions stands in need of an explanation, while the applicability of a numerical model established by the representation theorem cannot be reduced to a purely empirical basis.

Now, it turns out that this trouble arises for Hölder's theory because it entails what is commonly known as Archimedes' axiom[1] (see chapter 1, section 9, where I referred to the property of Archimedeanity). This axiom contains an existential

[1] The axiom says that for any x, y in **L** (a model of Hölder's axioms) there is a positive integer n such that $y <_L nx$.

quantification over the positive integers and it is needed to determine the numerical measures of the objects within an extensive structure: in fact, the representability of an extensive structure on the additive, ordered reals entails that it satisfies Archimedes' axiom.

In short, the proof of representation for extensive structures relies upon a numerical condition. For this reason, it looks as though a satisfactory account of the applicability of the positive reals to extensive measurement could not be obtained using Hölder's theory. In this chapter I show that this is not the case.

It is noteworthy that the problem of dealing with Archimedes' axiom in the context of an empirical theory does not only affect extensive measurement. The measurement theories used in Field 1980, being taken from Krantz et al. 1971: ch. 4 with minor changes, all contain (or entail) an Archimedean axiom similar to that of extensive measurement and playing a similar role in delivering the corresponding representation theorems. Since Field accepts as empirical axioms only the statements which are free of occurrences of mathematical references, the presence of Archimedean axioms within the theories he adopts to describe physical quantities must constitute a problem for both his account of applicability and his nominalistic project (which is based on eliminating mathematical references from scientific theories).

The task of the following sections is to illustrate an approach to representation theorems which solves this problem, troubling both Field's nominalism and Hölder's theory. The strategy I will pursue yields an explicit analysis of the empirical content of Archimedes' axiom, as a consequence of a broader analysis of the empirical content of numerical measures.

For this reason I am going to start from a discussion of numerical measures and, relying upon it, I will reach a characterization of Archimedes' axiom as a purely empirical condition. The approach I adopt here, which was originally developed in Niederée 1987, 1992, can be extended to the measurement theories adopted in Field 1980 and the weak forms of extensive measurement I will consider in the next chapters.

It is important to have this approach in place before a discussion of idealization, because only on its basis can one safely take extensive measurement to be based on empirical, possibly idealized, axioms.

2. Atomic formulas

In order to tackle the problem raised in the previous section, I need to present a particular treatment of extensive measurement, which requires looking at the representation theorem for extensive structures from a novel point of view. Only as a consequence of this change of perspective will it become possible to characterize Archimedes' axiom in empirical terms.

I now start from some preliminary observations, which are necessary to introduce the treatment of extensive measurement I want to describe. These preliminaries will be used in the next section to identify the empirical basis of numerical measures. Once this is accomplished, I will move on to deal with Archimedes' axiom.

Consider a model **L** of Hölder's axioms and a first-order formal language for it: this language includes in its non-logical vocabulary a relational constant for length-order and a functional constant for length-concatenation. Assume, in addition, that the language is expanded to one containing as many constants *a*, *b*, *c*, *d*, … as there are objects in L, the domain of **L**. If L is a set of rigid rods, the constants work as labels for the rods. In this case an atomic sentence like:

(i) $a <_L b$

may be taken to denote, in the idealized empirical setting **L** we are considering, a certain observation, from which it is concluded that the rod denoted by *b* is longer than the rod denoted by *a*. In view of (i), the atomic formula:

(ii) $a <_L x$

where *x* is a free variable, is satisfied under the assignment of the rod called *b* to *x*. More generally, this formula is satisfied by any assignments that maps *x* into a rod longer than *a*. Atomic formulas may be made structurally more complex by introducing the operation symbol for length-concatenation. For instance, in view of (i) and because monotonicity holds, we can write:

(iii) $a +_L a <_L b +_L b$

which again denotes an observation involving rods **a**, **b**[2]. Atomic sentences like (i) and (iii) may be understood as empirical records, describing the outcomes of certain experimentally tested interactions involving rigid rods. Empirical records are formulas and the information they convey is limited to the individual observation they denote.

[2] From now on I use bold for rods and italics for the constants denoting them.

As a consequence, if detailed study of the physical behaviour of certain objects is required, more than one observation will be needed to carry it out: to the corresponding family of observations a family of empirical records or atomic formulas can be associated. Consider, as a toy example, the set A of atomic formulas:

$$a <_L x; \qquad a +_L a <_L x; \qquad a +_L a +_L a >_L x.$$

The formulas in A are simultaneously satisfied when x is interpreted on any rod whose length lies between that of 2**a** and that of 3**a**. Note that the first formula in A is satisfied by any **b** longer than **a**, while this is not true of the whole set of formulas A. The point is that, by a suitable choice of atomic formulas, it is possible to restrict the family of objects which satisfy them. As the empirical information carried by the formulas increases, the number of objects satisfying them decreases. Reasoning along these lines, it may be conjectured that, if sufficiently many atomic formulas are available, one can restrict the number of objects satisfying them to only one.

Thus, it would be desirable to identify, for any rod in L, a set A′ of atomic formulas which are simultaneously satisfied by exactly that rod. If this could be done, then A′ would contain exhaustive empirical information about the empirical behaviour of a specific rod. If, in addition, the formulas in A′ only contained a constant a, denoting a unit of measure, then intuitively A′ could be identified with the measure of a certain rod in **a** units: A′ would exhaustively describe the interactions between a given rod and a fixed unit of measure.

It turns out that a set A′ with the above properties can be associated to each rod in L. This means that, for any element of an extensive structure, it is possible to attach to it a uniquely determined set of atomic formulas. The precise reason why this set of atomic formulas can be identified with a measure will be clarified in the next section.

3. How to describe numerical measures by sets of atomic formulas

One way of determining a unique set of atomic formulas for any rod in an extensive structure **L** consists in picking a sufficiently large set T′ of atomic formulas and, subsequently, providing a condition which isolates from it a unique subset for each rod in **L**. To this effect, it suffices to consider the set T′ containing all atomic formulas of the form:

$$Ma <_L Nx$$

where the positive integers M and N only abbreviate terms containing M and N occurrences of the functional symbol $+_L$ respectively.

Now, given an arbitrary rod **b**, consider the subset of T′ containing all the atomic formulas in T′ which are satisfied when **b** is assigned to the free variable x. As I remarked in the previous section, this subset can be directly taken as the measure of **b** in **a** units. To see why it makes sense to do so (which will also show the relevant subset to be uniquely determined), suppose that for some fixed positive integers m, n and a rod **b**:

$$ma <_L nb$$

holds (ma and nb denote the m-th and n-th iteration of concatenation on **a**, **b** respectively). Then it was proved in the previous chapter that there are p, q such that:

(ii) $p/q > m/n$ and $pa <_L qb$.

As a consequence, the choice of **b** isolates from T′ both the atomic formula $ma <_L nx$ and the atomic formula $pa <_L qx$. Reapplying (ii) we can find positive integers r, s such that **b** also isolates the atomic formula $ra <_L sb$ and so on.

In view of the results of chapter 1, we are therefore able to isolate from T′ a set of atomic formulas which correspond to an increasing sequence of positive rationals (m/n, p/q etc.) with an upper bound and thus a limit in the reals, which is the measure of **b** in **a** units. But to each rational in the increasing sequence an atomic formula can be attached, which is isolated by **b** from T′: thus, the totality S of the atomic formulas **b** isolates from T′ contains all the information needed to determine the unique measure of **b** in **a** units. More precisely, S contains the empirical records which describe all approximations of **b** from below in terms of **a**. But the set of these approximations can be identified with the real measure of **b** in **a** units and so S can be identified with this measure. S is also uniquely determined.

This move makes the empirical character of numerical measures apparent: they describe the totality of observations needed to uniquely express the length of a rod **b** as a function of the length of **a** using order and concatenation on an extensive structure.

In view of all this, it is plausible to conclude that extensive numerical measures are just codes of empirical information, identifiable with sets of atomic formulas denoting empirical records. It then follows that any model of Hölder's theory *generates* its own system of measures: to obtain the measure of **b** in **a** units one can, ideally, physically compare the multiples of **b** and the multiples of **a** and 'write down' the atomic formulas describing these comparisons. These atomic formulas can be used to isolate a set from T′, which is the required measure of **b**.

The interesting fact is that we can take measures to literally arise from extensive structures. This perspective on measures is important because it reduces them to a particular linguistic way of recording empirical information. In short, applied numbers can be reduced to an empirical basis. This point of view can be extended to Archimedes' axiom: the numerical quantification occurring in it can be reduced to an empirical basis, by means of atomic formulas. Before showing this explicitly in section 5, I wish to make a short digression, to illustrate how the particular characterization of measures developed so far can be used to provide an alternative formulation of the representation theorem for classical extensive measurement (cf. chapter 1, section 11).

4. Measures, types and type-structures

Having identified numerical measures with sets of atomic formulas, it is quite natural to think that a measurement structure for **L** should be constructible from them.

To this end, consider a function *f* such that, for a fixed model **L** of Hölder's theory and a fixed unit of measure **a**, it sends any **b** into the subset of T′ corresponding to its numerical measure relative to **a**. Now it can be checked that, when:

$$\mathbf{b} <_{\mathrm{L}} \mathbf{c},$$

the subset of T′ determined by **b** is *strictly included* into the subset of T′ determined by **c**[3]. As a consequence, the total ordering on **L** is isomorphically mapped by *f* onto the relation of inclusion between certain subsets of T′. It is not difficult to see that the subsets of T′ corresponding to numerical measures for the elements of **L** are linearly ordered by inclusion. Thus far, *f* preserves order on **L** and maps it into the relation of inclusion on certain sets of atomic formulas. In order for *f* to preserve concatenation on **L** and map it on an operation ⊕ over the same sets of atomic formulas, it suffices to define the relevant operation by the stipulation:

$$\text{(i) } f(x) \oplus f(y) = f(x +_{\mathrm{L}} y).$$

The operation ⊕ structurally behaves like arithmetical addition, because it is isomorphic to concatenation in **L**, which is in turn isomorphic to numerical addition by the representation theorem of extensive measurement.

Collecting together the last observations, we can determine a measurement scale and a measurement structure for models of Hölder's theory, both based on atomic

[3] It follows that, as long as two objects are not quantitatively equivalent, they must be associated to different subsets of T′.

formulas. The measurement scale is the function f above, while the measurement structure is simply a structure of sets of atomic formulas, ordered by inclusion and possessing an operation induced by extensive concatenation. It can be shown that the measurement structure thus defined codifies the same empirical information as **R**, because it is isomorphic to it. Indeed, it is possible to identify it with **R**. The same conclusion can be reformulated in a slightly more technical way, as follows. Given a first-order language for models of Hölder's theory, consider the set T of all atomic formulas of the form:

$$Mx < Ny;$$

Moreover, call σ the assignment that maps x into **a** and y into **b**.

Definition: the T-type of **b** in L with respect to **a** is the subset of T containing all the formulas of T which are satisfied under σ.

Denote the T-type of **a** with respect to **b** by $T(a, b)$. Then define f_a to be the function which assigns to any **b** in the domain of **L** its T-type with respect to **a**. It can be shown that:

$$\text{(ii) } b <_L c \text{ if and only if } T(a, b) \subset T(a, c).$$

As a result, f_a is an injective (one-to-one) function from the domain of **L** onto the T-types associated to the elements of L. Then f_a is a bijection between L and a certain family of subsets of T, call it T_L.

Using (i) and (ii) above, we can conclude that f_a is an isomorphism between **L** and **T**, where **T** is the structure:

$$\mathbf{T} = \langle T_L, \subset, \oplus \rangle.$$

Structure **T** may be called a type-structure for **L**: by previous remarks, it may be assimilated to the measurement structure **R** (due to structural identity). The representation theorem for extensive measurement can eventually be reformulated, in a way which makes the empirical basis of numerical measure completely explicit:

Representation theorem for extensive measurement:

Any model **L** of Hölder's axioms is isomorphic to the type-structure **T** it generates.

Ultimately, the system of numerical measures for **L** can be identified with the type-structure generated by **L**.

5. Archimedean axioms and separability

With the type-approach to extensive measurement in place, it is finally possible to address and solve the problem posed by Archimedes' axiom I pointed out in section 1. Archimedes' axiom contains a numerical quantification and thus it is problematic to have it within a supposedly empirical theory. This is a trouble for both Field's nominalistic point of view on scientific theories and for the development of a satisfactory account of applicability.

The reconstruction of extensive measurement based on types can be used to show that the extensive Archimedean axiom is equivalent to an empirically meaningful condition and thus it may either be accepted, because it is possible to spell out its empirical basis, or replaced by the equivalent condition, which does not quantify over the positive integers. The same goes for the Archimedean axioms, of a similar form, which are employed in the theories of measurement assumed by Field.

To see why this is the case, consider Archimedes' axiom for extensive measurement, i.e. the statement saying that, for any x, y in **L** (a model of Hölder's axioms) there is n such that $y <_L nx$. The empirical content of this axiom depends on the biconditional (ii) in the previous section, namely:

$$b <_L c \text{ if and only if } T(a, b) \subset T(a, c)$$

This says that, if **b** and **c** are different with respect to their quantitative behaviour, then their T-types are one strictly included into the other, and conversely. This means that a quantitative difference between **b** and **c** corresponds to the fact that there is at least one an atomic formula in T which is satisfied by **b** but not by **c**, relative to the unit **a**. One may say that **b** and **c** are *separated by* T: this set suffices to discriminate them, because it contains enough formulas to determine two different assignments of T-types to **b** and **c**[4]. In view of these remarks, the following definition can be introduced:

Definition: A model **L** of Hölder's axioms is said to be T-*separable*, if any two elements of its domain are separated by T[5].

[4] Once again, for convenience, we may take as identical the objects which are related by $=_L$ because, even though they isolate the same type, we anyway want them to be assigned the same measure: this was implicitly done above.

[5] This definition could be considerably generalized: for instance, given a formal language S, a set of formulas T in the language S containing one free variable, and a structure **S** for S, **S** is said to be T-separable if any two elements **a** and **b** in the domain of **S** are separated by T. The same could be stated for n-tuples of elements in the domain of **S**. For a precise discussion of separability within a general context see Niederée 1992: 263–268.

With this definition in place, a theorem proved by Niederée (see Niederée 1992: 259-260) follows:

In presence of the first-order axioms H1 to H6 of Hölder's theory, Archimedeanity is equivalent to T-separability[6].

This result is not surprising. Archimedeanity entails, for any distinct **a** and **b** in L, the existence of n such that the following formula holds:

$$\text{(i)}\ na \leq_{\mathrm{L}} b <_{\mathrm{L}} (n+1)a.$$

Exploiting this fact, it is possible to consider progressively greater multiples of **a** and **b** to obtain better and better approximations of **b** in **a** units. In other words, Archimedeanity governs the process of passage to the limit which lies at the basis of scale construction for extensive measurement. In presence of the other axioms, when $b <_{\mathrm{L}} c$ holds, we can always find a **z** so small[7] that we can bracket **b** within an interval of width **z** and not containing **c**[8] (that we can always find the relevant **z** means that the fact just stated can be proved from H1 to H6 plus Archimedeanity).

This is to say that there is always a stage in the approximation process used in extensive measurement at which, if **b** and **c** are different with respect to their empirical ordering, their approximations will diverge: this gives separability. T-separability in particular follows from the fact that the process of approximation can be carried out by means of operations denoted by atomic formulas alone, like those occurring in (i) above. The important feature of these formulas, apart from their being quantifier-free, is that they only describe finite iterations of concatenation: in other words, T-separability is a discriminability property based on arbitrary finite iterations of a physical operation. The last fact is compactly expressed by the numerical quantification occurring in Archimedes' axiom. However, while the axioms refers to the positive integers, the condition of T-separability only involves types, i.e. sets of observational records, and thus has a clear empirical meaning[9]. The

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[6] If Hölder's axioms with Dedekind Completeness replaced by Archimedeanity hold, then T-separability follows from them: I omit a proof, which requires long and tedious calculations. Conversely, if H1 to H6 hold and a model of them is T-separable, then it is Archimedean: this can be shown by contraposition assuming a non-Archimedean model. If, without loss of generality, there exists an infinitely large element **b**, for all positive integers m and a finite unit **a** we have that $y >_{\mathrm{L}} mx$ is satisfied when y is interpreted on **b** and x on **a**. *A fortiori* $ny >_{L} mx$ is satisfied for all m, n: the T-type of **b** with respect to **a** is then T. By closure **b** $+_{\mathrm{L}}$ **b** exists and is strictly larger than **b**. Its T-type relative to **a** is however T, as can be quickly verified. As a result, **b** and **b** $+_{\mathrm{L}}$ **b** are not T-separable.

[7] **z** may be understood as the difference between suitably large multiples of **a** and **b**.

[8] Technically, this condition is a version of the Hausdorff property for a topological space.

[9] Note that, even if we are talking about sets, set-membership here is equivalent to the fact that certain physical relations hold, and thus can be reduced to an explicit empirical basis.

only reason why Archimedeanity is needed is that it provides T-separability, which is implicit in the approximation process it directs: conversely, the numerical quantification found in this axiom codifies the notion of arbitrary finite iteration of a physical operation, and thus it has an underlying empirical motivation.

It is therefore possible to conclude that the role played by Archimedes' axiom in characterizing a measurable structure like **L** can be spelled out in terms of the empirical condition of T-separability. The quantification over the positive integers within Archimedes' axiom works, loosely speaking, as a 'bounded search' condition, saying that any two quantitatively different objects in the domain of **L** can be discriminated in a finite number of steps. In the light of these remarks, it may be concluded that the use of an Archimedean axiom in order to obtain a representation theorem can be justified on empirically acceptable grounds (allowing idealization). This solves the problem posed in section 1. The same conclusion can be reached for the empirical measurement structures described in Field 1980, which are reducible to extensive ones. It also generalizes to the measurement theories to be surveyed in the following chapters.

7. Some remarks on representation and types

The main formal results examined in this chapter have a particular importance for the analysis of the applicability of mathematics within a representational context. Firstly, the type-based representation theorem for extensive measurement brings to the fore the fact that both the measurement scale and the measurement structure for a model of Hölder's theory can be seen as if they were generated by the model itself. The same conclusion extends to all the theories reducible to Hölder's and thus it encompasses the geometrical and measurement theories employed in Field 1980. Secondly, the Archimedean axiom involved in the ordinary numerical proof of representation is equivalent to the condition of T-separability on T-types: this condition specifies the kind of procedure which suffices to distinguish quantitatively different objects.

Thirdly, numerical measures can be seen as codes of empirical information concerning certain interactions of a fixed object and a unit of measure, while Archimedean axioms say that these interactions can be used to make quantitative discriminations and always suffice to that end.

The resulting picture of extensive measurement reveals a tight interconnection between the models of Hölder's theory and their scales and measurement structures. The latter, being describable in terms of types, can be conceived as constituted by sets of formulas: each of these sets lists all the information necessary to locate the order of magnitude of an object on a quantitative continuum. In other words, T-types chart the quantitative relationships between the elements of an extensive structure and provide a reference frame for them, where the distance of any object from the unit of measure can be determined.

The important fact is that this charting by atomic formulas is possible because of the physical properties possessed by an extensive structure: the latter can give rise to its own reference frame and the mathematics which applies to it. For this reason, the properties of the positive, additive reals ordinarily used to deal with extensive structures numerically can be reduced to an empirical basis, as properties of type-structures, which in turn are simply obtained from the physical interactions taking place in the extensive structures themselves.

When a reference frame is generated by an empirical structure, a transition from physical manipulations to abstract reasoning takes place: instead of acting directly on physical objects, one considers the structured information obtained from them and makes inferences on it. Thus, the mathematics of extensive measurement can be understood as a particular way of reasoning about extensive structures in an indirect fashion, namely using families of formulas denoting observations. This conclusion suggests a precise characterization of the applicability of numerical models to mathematics. These models are applicable because they emerge from empirical settings generating them. This view on applied mathematics recommends itself at least for extensive measurement and the theories reducible to it because it explicates the applicability of mathematical models in a way which is deeper than that afforded by a numerical representation theorem (which does not contain any notion of generability) and also solves the difficulties posed by the occurrence of a numerical quantification within Archimedean axioms.

8. Back to Idealization

The type-approach of measurement expounded in this chapter, which was technically developed in Niederée 1987, 1992, has been presented here only for a strongly idealized theory of measurement, i.e. one whose axioms imply, as observed in chapter 1, that:

I_1) There are infinitely many, indeed uncountably many, objects;

I_2) The relation of empirical equality $=_L$ is transitive (contrary to experiment).

The next chapter will examine weaker theories of extensive measurement, where I_1 doesn't hold, while chapters 4 will consider measurement theories where I_2 does not. The type approach described here applies, with minor modifications, to these theories: in particular, T-separability holds for them and thus their Archimedean axioms can be understood as empirical conditions precisely along the lines of the discussion of section 5. This means that all the axiom systems for extensive measurement I will consider in the following chapters, which contain an Archimedean axiom, can be safely considered empirical theories.

CHAPTER 3: MEASUREMENT AND INFINITARY IDEALIZATIONS

1. Idealizations

In chapter 1 explained how Hölder's theory of extensive measurement can be used to prove a representation theorem establishing the existence of a measurement scale for any of its models. The discussion of chapter 2 has made clear that Hölder's theory can be accepted as a purely empirical theory, despite the fact that it entails a numerically formulated condition, namely Archimedes' axiom, which is used in an essential way to obtain a proof of representation.

Because of this, it makes sense to say that Hölder's axioms determines a set of empirical conditions which suffice for the applicability of the positive real numbers to measurement. The problem which arises at this point is that some of these empirical conditions are idealizations, i.e. assumptions which are either untestable or contradicted by experimental results. For this reason, the formal account of measurement provided by Hölder's theory cannot be said to describe experimental practice or just the observable behaviour of physical objects, but only an idealization thereof. Thus, in order to give an account of the applicability of numerical models to measurement as actually performed, it is not enough to have Hölder's axioms and a representation theorem depending on them. It is also necessary to relate Hölder's axioms to more realistic assumptions, which provide a less idealized description of extensive structures. In short, it is necessary to describe the relationship between idealized measurement and non-idealized measurement.

This is what I am going to do especially in the next two chapters. The idealizations I am going to look at are those of Hölder's theory which, as I pointed out in chapter 1, are of two kinds, namely *infinitary idealizations* and *regularity idealizations* (a further type of idealization is separately discussed in chapter 5). The availability of arbitrarily many copies of any object in an extensive structure and the non-existence of minimal elements are examples of infinitary idealizations.

On the other hand, the assumption that the relation of empirical equality or empirical equivalence denoted by $=_L$ is transitive exemplifies a regularity idealization, since the empirical relation of equality is experimentally only symmetric and presumably reflexive by definition, but not transitive, and it is replaced in Hölder's theory by an ideal relation whose formal properties include the above two plus the further property of transitivity. In this sense the assumed relation is more regular (it obeys a greater number of formal laws) than the observable one.

My aim in this and the next chapter is to provide an account of the relation of infinitary idealizations and regularity idealizations to experiment in extensive measurement. The resulting analysis will show that infinitary idealizations may be seen as plausible extrapolations from the data of experiment, while the regularity idealization of extensive measurement can be seen as the result of limit taking, i.e. it is satisfied by idealized extensive structures to which less idealized ones converge under suitable conditions. In this chapter I focus on infinitary idealizations alone.

2. Infinitary idealizations

Infinity arises from Hölder's theory in several ways (cf. chapter 1, section 11). First of all, H6[1] (non-existence of minimal elements) is a principle of infinity, because it implies that, given any x, it is possible to find y strictly smaller than x, z strictly smaller than y, and so on without end. Thus, all models of H1 to H6 must be infinitely large. In case they also satisfy H7 (Dedekind completeness), they are forced to be *uncountably* infinite (whereas H1 to H6 have countable models: for instance, the positive rationals satisfy them).

Finally, H2 (closure) is, in presence of positivity (axiom H5), an infinitary principle symmetrical to H6: while the latter requires an extensive structure to be unbounded below, axiom H2 forces it to be unbounded above. This is because, for any x, closure ensures the existence of $x +_{\mathrm{L}} x$ which, by positivity, is strictly larger than x. Iterating concatenation, we may reach arbitrarily large orders of magnitude: increasing, infinite sequences of larger and larger objects may thus be generated. From these remarks it can be quickly concluded that the infinitary idealizations in Hölder's system are H7 (Dedekind completeness), H6 (non existence of minimal elements) and H2 (closure, which is infinitary in presence of positivity).

A direct way of establishing that the idealizations corresponding to these axioms are extrapolations from experiment could consist in looking at theories of extensive measurement weaker than Hölder's and showing that their models can always be extended to models of Hölder's theory. If this were possible, it would be easy to conclude that realistic measurement settings, satisfying weak measurement theories,

[1] I refer to the axioms listed in chapter 1, section 11.

can actually be seen as subsystems of some model of Hölder's theory, from which the latter model can be recovered. Then the infinitary assumptions of Hölder's theory would simply depend upon the extensibility of weaker extensive structures.

It turns out that this strategy cannot be pursued in the exact form I have outlined, but a modified version thereof, involving additional plausible assumptions, can. I will use this modified strategy in my discussion of infinitary idealizations.

I begin with an immediate weakening of Hölder's theory, obtained by replacing axiom H7 by Archimedes' axiom (a possibility already explored in Hölder 1901). That a representation theorem for extensive measurement can be obtained from the resulting theory can be quickly seen. Dedekind completeness is only needed to ensure that the representation be *onto*. If one is content with representations *into* the reals, rather than onto them, then Archimedeanity is enough. The reason is that, for a measurement scale on the reals to be set up, one only needs as many inequalities of the form[2]:

$$\text{(i) } ma <_{\mathrm{L}} nb$$

as are guaranteed to exist by closure and Archimedes' axiom alone. The former ensures that all finite iterations of concatenation are defined, while the latter rules out the possibility that all multiples of the unit a are strictly smaller than b, thus granting their comparability with b by formulas of type (i). To determine unique real measures for objects like b, only the Dedekind Completeness *of the reals*, and not of extensive structures, is needed (so that one can find the least upper bound of classes of fractions m/n generated by the inequalities (i)).

If we call Archimedeanity H7′, then axioms H1 to H7′ (without H7) suffice for the proof of representation, since their models can be mapped into (not necessarily onto) a Dedekind complete structure, i.e. the positive, ordered reals.

3. Archimedeanity and Dedekind completion

The representation theorem for extensive measurement provable for the models of H1 to H7′ may be regarded as more informative than the representation theorem proved on the basis of the stronger theory of Hölder 1901. The reason is that, with the stronger assumptions in place, we can show identity in structure and cardinality

[2] From here on I use italics to talk about relations and objects in a certain extensive structure. I thus abandon the use of italics made in chapter 2, where it indicated syntactic items like formulas or constants.

between the models of Hölder's axioms and the arithmetical continuum. On the other hand, in presence of the weaker axioms H1 to H7′, representation says something more: it says that any model of the axioms can be *embedded* and, more importantly, *extended* to a Dedekind complete structure, *via* measurement. Each Archimedean extensive structure satisfying H1 to H6 can be enriched and extrapolated to a Dedekind complete one. This fact provides the key connection between Hölder's axioms and their weakened, Archimedean version.

To see this point more clearly, assume H1 to H7′. Let **L′** be a model of these axioms and x the unit of measure chosen from that model. In view of closure, concatenation on x generates larger and larger objects, without bound. At the same time, it is possible to find, for any positive integer n, some z in the domain of **L′** such that nz is strictly smaller than x (this is a weakened form of divisibility)[3]. Call any such z a *weak submultiple* of x.

Now consider any positive real r and an increasing sequence $\{q_i\}_{i\in N}$ of positive rationals converging to it. This sequence is bounded above and below, because it has a first element and a limit. Since both the successive concatenations of x and its weak submultiples are unbounded, their measures can be used to determine bounds for the sequence $\{q_i\}_{i\in N}$. Furthermore, for any two consecutive elements of the sequence q_j, q_{j+1}, it is possible to produce, by concatenation of multiples and weak submultiples of x, an object y whose measure $\mu(y)$ satisfies:

$$\text{(i) } q_j \leq \mu(y) \leq q_{j+1}.$$ [4]

This shows that, using multiples and weak submultiples of x, it is possible to approximate any positive real number (for it can be shown that the sequence of the $\mu(y)$ satisfying (i) and $\{q_i\}_{i\in N}$ have the same limit). But the positive reals with order and addition are a model of Hölder's theory so the conclusion is that, using multiples

[3] I show it for $n = 2$: the general result follows by iteration. Let x be given: by H6 there is y such that $y <_L x$. Calling z their difference, which exists by solvability, it follows that $z <_L x$ (for, if this were not the case, monotonicity and positivity would imply that $y +_L z \geq_L y +_L x >_L x$, contradicting $y +_L z =_L x$ because of trichotomy). By trichotomy, either $y =_L z$ or not. If not, it suffices to take the smaller of y, z, suppose it is z, to conclude: $2z =_L z +_L z <_L z +_L y =_L x$. If, on the other hand, y equals z, H6 provides w smaller than both, for which $2w <_L x$ holds.

[4] To prove this, consider q_j: since weak submultiples of x are unbounded below, one of them, call it z, has a measure $\mu(z)$ which is strictly smaller than q_j. Using Archimedeanity on the positive reals, it follows that there are integers k, $k+1$ such that $k(\mu(z)) \leq q_j < (k+1)(\mu(z))$. By the additivity of μ, the numbers bracketing q_j correspond to the measures of multiples of z. Now, if $(k+1)(\mu(z)) \leq q_{j+1}$ we are done. Otherwise we may repeat the same reasoning with a weak submultiple of x which is smaller than z: this procedure must terminate, because the difference $q_j - q_{j+1}$ is finite and thus larger than the measure of some weak submultiple of x.

and weak submultiples of x, it is possible to approximate a model of Hölder's theory starting from a model $\mathbf{L}'$ of H1 to H7′. This conclusion corresponds to the abstract fact that any model of H1 to H7′ admits a (unique) Dedekind completion, which extends it to a model of Hölder's theory

The concept of Dedekind completion links in this context Hölder's theory to its Archimedean version by connecting their models through an extensibility relation: each structure satisfying the weaker theory can not only be enlarged to a structure satisfying the stronger theory but also suffices to uniquely determine the enlargement.

The latter corresponds to a structure $\mathbf{L}$ where all the approximating sequences that are already constructible in $\mathbf{L}'$ have a limit: in other words, $\mathbf{L}$ takes into account all the possibilities which may emerge from the physical comparisons of objects whose order and concatenation obey an Archimedean axiom. We may think of this in relatively concrete terms. Suppose $\mathbf{L}'$ is given and we construct a measurement scale for it: then assume that, at a later time, a further object x, not contained in $\mathbf{L}'$, has to be measured (this requires that the approximating procedure which generated the scale for $\mathbf{L}'$ should be applicable to x). If x is not quantitatively equivalent to any of the elements of $\mathbf{L}'$, then its measure is a positive real which is not the measure of any of the elements of $\mathbf{L}'$: in the terminology of chapter 2, it is a new T-type.

Thus, as long as the same approximating procedure can be uniformly adopted to extend a scale of measurement when needed, its applications to new objects which may be added to $\mathbf{L}'$ build up an enlargement of $\mathbf{L}'$ which is isomorphic to its Dedekind completion.

We may therefore imagine a diachronic process starting from a model of H1 to H7′ and consisting in the progressive addition of new objects to this model: the termination of this process leads to an extensive structure in which H7 holds. When we replace H7′ with H7, it is as though we reverted from a diachronic process building up a Dedekind completion (up to isomorphism) to a synchronic one, in which no new objects can be added, in the sense that any further object to be measured must be quantitatively equivalent to some object in the model of H1 to H7[5].

[5] If there were new objects not quantitatively equivalent to any of the given objects, then Archimedeanity would fail. This can be seen in the numerical case by taking the real structure $\mathbf{R}$ and supposing that it is possible to add new objects to its domain without violating Archimedeanity. Technically this means that there is a proper extension of $\mathbf{R}$ which is Archimedean. Let x be an element in the extension but different from all positive reals: because Archimedes' axiom holds in the extension, for any positive real r there is k such that the following inequality holds: $kr \leq x <_{\mathrm{L}} (k+1)r$.

Thus Dedekind completeness, in presence of H1 to H6, describes a situation in which all possible approximation processes have been realized. What Dedekind completeness says is that, given Archimedes' axiom, it is possible to isolate the space of all outcomes of an approximating procedure based on successive concatenations.

In other words, Dedekind completeness forces the models of Hölder's theory to be exhaustive with respect to Archimedean approximations, and the fact that it does so may be seen to stem from a physical assumption about the universal applicability of a certain kind of approximating procedure. Axioms H1 to H7 reify this assumption by describing structures in which there are enough objects to realize all the possible applications of the approximating procedure. However, in view of the fact that the weakening of axioms H1 to H7′ is available, the use of Hölder's original theory and even its application in theoretical physics need not involve a commitment to the existence of uncountably many physical objects. This is because the purpose of Dedekind completeness is *not to establish their existence, but to express the full generality of an Archimedean approximating procedure*[6]. As long as we think that Archimedes' axiom holds and is always applicable to determine real measures, it is as though we were dealing with objects within an ideal, Dedekind complete extensive structure (in view of the results of chapter 2, this structure can be based on T-types).

To sum up the main point of this section: the general applicability of an approximating procedure is translated diachronically by the process of Dedekind completion and can be synchronically expressed by the condition of Dedekind completeness. This condition does not lead to a form of ontological commitment: what it does is to reify as an existential assumption a physical hypothesis about the scope of application of a measurement procedure. The reification determines an ideal enlargement of a more realistic measurement setting.

So this holds in particular when $r = 1/2^n$. As n increases, an infinite sequence of nested intervals including x is generated. By the Dedekind completeness of the reals, such sequence has a limit y *in the reals*, so y and x are different. The absolute value of their difference, in turn, is smaller than any fraction of the form $1/n$ (this is because, if it were bigger, then it would be possible to shrink the sequence of nested intervals around x to a point where it excludes y and y would not be a limit). We therefore have $|x - y| < 1/n$ or $n|x - y| < 1$. However, by solvability, $|x - y|$ is in the extension of **R** and, because the latter was assumed to be Archimedean, there must be n such that $n|x - y| > 1$, contradicting the previous inequality. It follows that x and y cannot be different and x is a positive real, which contradicts the assumption of a proper Archimedean extension of **R**. Via isomorphism, the same holds for any model of Hölder's theory (this proof is taken, with minor modifications, from Goffman 1974: 2–3).

[6] This situation differs from a situation in which one considers distinct states of a varying quantity, like temperature, where continuous variation would imply the existence of a continuum of states.

4. Minimality

There is one important difference between axioms H1 to H7′ and Hölder's axioms. The latter are categorical, the former are not. This means that the models of Hölder's theory are all isomorphic, while the models of H1 to H7′ are not (for instance, these axioms are satisfied by both the positive reals and the positive rationals with order and addition).

In particular, even though only infinitely large models are allowed by both theories, H1 to H7′ admit countable models, while Hölder's theory has only uncountable models. The slight weakening effected by assuming Archimedeanity in place of Dedekind completeness already suffices to drop the idealization identifying all extensive structures with the continuum of the positive reals. At the same time, dropping this idealization shows how it can be introduced as a generalization of the Archimedean behaviour of an extensive structure, *via* a particular extension process which is equivalent to Dedekind completion. This provides a justification for the assumption of H7 in terms of a weaker axiom system.

I will deal with the remaining infinitary idealizations and the regularity idealization of extensive measurement in a way similar to that just illustrated. In other words, I am going to consider weak theories of extensive measurement which do not contain certain idealizations and show that these idealizations can be recovered from the weak theories either by a suitable form of extrapolation or by generalizing certain features of a measurement procedure. Thus, idealizations are justifiable in the sense that they can be recovered from non-idealized theories. The models of these theories can be generalized to the models of idealized theories. In some cases, as with H7, this is straightforward, because all models of H1 to H7′ can be extended to structures satisfying H7. Sometimes, however, it is not possible to establish extensibility, in which case an idealization cannot be recovered by enlarging the models of a weaker theory but it has to be motivated as a plausible generalization.

This particular approach must be pursued to deal with H6, which entails the existence of infinitely many objects of progressively smaller orders of magnitude. It is possible to just drop H6 from the set of axioms H1 to H7′, when looking for a more realistic theory of extensive measurement from which this idealization can be recovered. If H6 is dropped, a representation theorem still goes through, except that it is now split into two cases, according as to whether H6 holds in the specific model considered or it does not.

If it holds, nothing has changed and we have a representation proof, in view of the observations of section 2. If H6 does not hold, then there exists a minimal element: in this case, axioms H1 to H5 and H7′ entail that any object is a multiple of the minimal element. The result follows quickly from the fact that, for any *a*, *b* there is a positive integer *n* for which:

$$na \leq_{\mathrm{L}} b <_{\mathrm{L}} (n+1)a.$$

Let *a* be the minimal element and *b* arbitrary. Since solvability holds, *b* is *na*, otherwise the difference between *b* and *na* would be smaller than *a*, against minimality. It can be seen that the proof of representation becomes trivial in this case because the axioms are so strong that, in presence of minimality, they reduce their models to sequences of concatenations of the minimal element. Even so, dropping H6 does not disturb the provability of representation and makes room for discrete models, thus weakening H1 to H7′ in a significant way.

What is however lost with this move is the possibility of treating all models of H1 to H5 and H7′ as extensive structures which can be extended to models of Hölder's axioms, even though they can be mapped into them and thus an embeddability relation still holds. There are nonetheless at least two reasons why H6 may be retained. One of them is that the representation theorem is trivialized, once H6 is dropped. The other reason is that it may make sense to assume that there are no physical reasons to deny that quantitative differences may be arbitrarily small.

This assumption is perhaps supported by the fact that, experimentally, it is hardly the case that one can find a unit of measure such that any other object turns out to correspond quantitatively exactly to some integer multiple of the unit: furthermore, at the abstract though still relatively intuitive level of geometry, we have examples of incommensurable magnitudes, which certainly cannot be multiples of the same unit of measure.

As a result, since in addition the possibility of refining a measurement is in principle always open[7], it is plausible to restrict one's attention to extensive

[7] Even in presence of uncertainty principles, this is not necessarily problematic. For what these principles forbid is an indefinite simultaneous refinement of measures of certain 'coupled' quantities (e.g. position and momentum or time and energy), not the indefinite refinement in the measurement of single quantities (although this may be achieved at the expense of precision in measuring some other quantity).

structures without minimal elements. If this move is accepted, then it is possible to introduce H6[8].

On the basis of this conclusion, an idealization like H7 can be introduced on the basis of a two-step reasoning. Firstly, even though in actual experimental settings arbitrarily small elements are not available, it is acceptable to assume that quantitative differences haven't got a lower limit. Secondly, it follows from this that an assumption like H6 can be included in a theory of extensive measurement and this assumption, in presence of H1 to H7′, determines a theory whose models can be extended to models of Hölder's axioms. This reasoning provides a motivation to introduce both H6 and H7.

It may now be asked whether a similar train of thought works for the last infinitary idealization to be discussed. In other words one may wonder whether it is possible to start from theories of measurement which do not assume closure (axiom H2), introduce H6 and show that, on the basis of a suitable set of extensive axioms without closure, the models of these axioms can be extended to closed structures and, subsequently, to models of Hölder's axioms. This question can be answered in the affirmative and is addressed in the remaining sections of this chapter.

5. Extensive measurement without closure

As previously remarked, the axiom H2 of closure implies (in presence of positivity, which is a relatively natural assumption in this context[9]) the existence of arbitrarily large objects in an extensive structure and so it is an infinitary idealization.

My strategy to justify its introduction is the same as the one adopted for Dedekind completeness. I start from a theory of extensive measurement which does not assume

[8] One may think that perhaps there are quantities which ultimately vary only discretely and so that assuming that there do not exist minimal elements, as H6 does, is problematic. I think the assumption is still plausible, at least locally, in view of the fact that the conceptualization of certain quantities like distance and time is almost inextricably related to the concept of continuous variation. It should also be born in mind how important the notion of continuous variation is for the mathematical treatment of quantities in general. Continuous variation requires H6, because it entails that quantitative differences may become arbitrarily small.

[9] Think of quantities like mass or length, for which positivity holds experimentally.

closure and show how, under plausible hypotheses (H6 among them), its models can be extended, firstly, to closed structures and, secondly, to models of Hölder's axioms.

If closure is not assumed, then the concatenation operation $+_L$, which is total by H2, becomes in general a partial one on an extensive domain. More clearly, H2 implies that, for any extensive structure with domain C, its concatenation operation is defined over the full Cartesian product C×C. Therefore, if closure is dropped, then concatenation is in general defined only on a nonempty subset B of C×C. In this case, the notation $(x,y) \in$ B may be introduced to signify that $x +_L y$ is defined or, equivalently, that objects x and y can be concatenated in the given model.

This treatment of concatenation has been developed in Luce & Marley 1969, which provides axioms for extensive measurement that do not include closure and entail a weak form of positivity (stating that, for any x, y, $x +_L y \geq_L x$). The axioms model structures like $\mathbf{C} = \langle$C, B, $\geq_L$, $+_L\rangle$, where B is the above subset of C×C and the other primitives are the usual ones of extensive measurement. The following is assumed:

Weak order: $\geq_L$ is a transitive and connected binary relation;

Right associativity of $+_L$: if $(x,y) \in$ B and $(x +_L y, z) \in$ B, then $(y,z) \in$ B and, in addition, $(x, y +_L z) \in$ B and $(x +_L y) +_L z \geq_L x +_L (y +_L z)$;

Right monotonicity: if $(x,z) \in$ B and $x \geq_L y$, then $(z,y) \in$ B and $x +_L z \geq_L z +_L y$;

Solvability: If $x <_L y$, there is $y - x \in$ C such that $(x, y - x) \in$ B and $y =_L x +_L (y - x)$[10];

Archimedes' axiom: if x, y are in C and y is non-maximal[11], the set of positive integers: $\{n$: $n \in N$ *and* $y \geq_L nx\}$ is finite (the integer multiples of a magnitude are recursively defined in the usual way[12] but may not exist).

The interest of this set of axioms lies in the fact that it can be proved for all of their models without a maximal element that they can be extended to closed structures. It then follows that, if these models do not have a minimal element, then they can be also extended to models of Hölder's theory. This means that the relatively realistic assumptions contained in the above list of axioms suffice, when minimality or

[10] The relation $=_L$ is defined here by the condition: $x =_L y$ if and only if $x \geq_L y$ and $y \geq x$.
[11] This qualification is necessary because positivity has been dropped and closure does not hold, so maximal elements are not automatically ruled out. In this context an element a is maximal if, whenever $(a, x) \in$ B, $a +_L x =_L a$ (see Luce & Marley 1969: 239).
[12] See chapter 1, section 10.

maximality constraints are waived, to legitimize an extrapolation of their models to closed, Dedekind complete ones, isomorphic to the real continuum. Before discussing this point any further, let me illustrate the relation between closed extensive structures and extensive structures without closure. Once this has been clarified, it will be easy to see what the relationship between non-closed structures and the models of Hölder's theory is.

6. Recovering closure

Closure can be 'recovered' from a non-closed structure satisfying the Luce & Marley axioms. This possibility revolves around a few crucial theorems. The first of them is the following:

(i) if $(x,y) \in$ B and $x \geq_L u$ and $y \geq_L v$, then $(u,v) \in$ B and $x +_L y \geq_L u +_L v$[13].

Theorem (i) shows that, as long as some concatenation is defined, the concatenations of any elements smaller than its arguments are defined. What is interesting for present purposes is an immediate consequence of (i), relying on a further theorem asserting the existence of at least one e in C, such that (e,e) is in B[14] (i.e., the existence of an element of C such that concatenation can be iterated on it at least once). Using this fact in (i) we obtain:

(ii) If $e \geq_L u$ and $e \geq_L v$ then $(u,v) \in B$.

where the distinct elements x and y in (i) are replaced by the single element e. This is to say that concatenation is defined below e. Equivalently, concatenation is defined for sufficiently small objects (i.e. smaller or equal to e).

Property (ii) is important because it entails that e can be used on C as a 'quasi-unit of measure', in the sense that any a in C can be expressed as a maximal finite concatenation of multiples of e plus an object x_a not larger than e. We have, for any a in C:

$$\text{(iii) } a =_L N_a e +_L x_a$$

[13] It is proved by repeated application of right monotonicity: just substitute in it y with u and z with y, to get from $(x, y) \in$ B and $x \geq_L u$, the consequence $(y, u) \in$ B and $x +_L u \geq_L y +_L u$; another application of right monotonicity, together with transitivity, leads to $(u, v) \in$ B and $y +_L u \geq_L u +_L v$.

[14] Since B is by assumption nonempty, there is some (x,y) in B, but if so, then, by connectedness, it must be the case that either x $\geq_L$ y or y $\geq_L$ x. In the first case, since $x \geq_L x$, from theorem (i) there follows that (x, x) is defined and $x +_L x \geq x +_L y$; in the second case it is sufficient to take $y \geq_L y$ to get a similar result.

where the constant N_a denotes the greatest positive integer such that $N_a e$ is defined and smaller than a and, in addition, $x_a \leq_L e$[15]. With (iii) in place we are in a position to uniformly describe all elements of C by means of the 'unit' e, up to small differences. In particular we can always attach a couple of parameters to any a in C, namely the ordered couple whose first element is the positive integer N_a and whose second element is the difference x_a.

These two parameters can be usefully exploited by observing that addition on the positive integers is a closed operation and concatenation between any two elements of C which are smaller than e is defined, in virtue of (ii). This allows the introduction of a defined concatenation operation on the ordered couples of form (N_a, x_a), by the following clauses:

(iv) $(N_a, x_a) + (N_b, x_b) = (N_a + N_b, x_a +_L x_b)$ in case $e \geq_L x_a +_L x_b$;

(v) $(N_a, x_a) + (N_b, x_b) = (N_a + N_b + 1, x_a +_L x_b - e)$ in case $e <_L x_a +_L x_b$[16].

The most interesting consequence of clauses (iv) and (v) is that, reinterpreting them in the light of (iii), they enable us not only to describe all elements a in C, but also all potential concatenations in a closed additive structure which includes **C** (and continues to satisfy the Luce & Marley axioms). To see this, suppose a finite extensive domain is given. Then there is a number N_a corresponding to the highest number of concatenations of e needed to express any element y of the domain by a couple (N_y, x_y). If the given finite domain is enlarged to include objects quantitatively greater than a, a greater number of concatenations of e needs to be available and thus a greater N_a.

We may thus consider stages in the hypothetical enlargement of a finite empirical domain: at each stage we only need finite sequences of multiples of e to reach all of its elements, thanks to Archimedeanity. Since each stage identifies a certain N_a, and these may progressively increase, it is natural to make the formal move of

[15] To see how this holds without closure, suppose $a >_L e$ (I skip the trivial case $a \leq_L$): then, the difference between them can be determined by solvability. If this difference is smaller than e we have: $a =_L N_a e +_L x_a$ where $N_a = 1$. If, on the contrary, $a =_L N_a e +_L x_a$ but $x_a >_L e$, then a contradiction follows, since it can be proved that $(N_a+1)e$ is defined and smaller than a (against the definition of N_a): thus x_a may be replaced by the difference $x_a - e$, which exists by solvability. If this difference is still greater than e we repeat the above reasoning: that it must terminate in a finite number of steps is a consequence of Archimedes' axiom.

[16] Clauses (iv) and (v) simply say that, to concatenate two elements approximated by e means to sum the multiples of e that are needed to 'reach' each of them and then add to this sum that of the differences between these multiples and the quantities to be reached. Clearly, if such differences add up to a quantity that exceeds e (as (v) says), one has to add a further multiple of e to the previously determined sum of multiples and then consider the difference between $x_a +_L x_b$ and e.

considering all positive integers as potential values of N_a, thus taking care of all finite extensions of an initially given finite domain.

More formally, we move from the set of couples (N_a, x_a), for a in C to the larger set of all couples of the form (n, x), where n is any natural number and x any object smaller than e with respect to the empirical ordering $\geq_L$. Technically this means that the set of couples (N_a, x_a) for a in C provides a *covering* of C, while its formal enlargement to the set of couples (n, x) extends this covering to a closed structure.

Instead of restricting attention to the indices N_a only, we study the totality of indices determined by the set of positive integers and the way they enter concatenations by means of conditions (iv) and (v). This enlarged domain of investigation exhausts all the empirical possibilities concerning multiple-formation[17] and thus leads to a closed concatenation operation.

The main consequence of this fact is that a representation theorem for a model of the Luce & Marley axioms can be obtained by embedding it into a closed structure satisfying an axiom system for which a representation into the reals exists[18]. Thus, any model of the Luce & Marley axioms without maximal elements can be extended to a closed extensive structure for which a representation theorem can be proved[19] and so the Luce & Marley axioms describe fragments of closed, measurable structures[20]. Then closure can be justified as a generalization of the behaviour of non-closed extensive structures (with a locally defined operation).

7. The theoretical role of Dedekind completeness and closure

Once Dedekind completeness is replaced by Archimedeanity, H6 is dropped and the extensive operation of concatenation is not assumed to be everywhere defined, one passes from Hölder's theory to the Luce & Marley axioms. Semantically, this transition from a stronger to a weaker theory corresponds to the transition from uncountably large extensive structures, isomorphic to the arithmetical continuum, to

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[17] The use of numerical indices made by Luce & Marley 1969 is not essential because a non numerical treatment of coverings and extensions to closed structures can be carried out in a purely algebraic fashion, as shown in Behrend 1956: 352–353.

[18] Luce & Marley 1969 uses the theory of closed extensive measurement presented in Suppes 1951.

[19] The models of the Luce & Marley axioms with maximal elements have a numerical representation as well. For details see Luce & Marley 1969: 246–247.

[20] Something stronger may be said. There is a proof of representation for a slight weakening of the Luce & Marley system which does not rely on their extension to a closed structure. In this sense the axiom system provides autonomously a proof of the existence of a scale of measurement (for details, see Krantz et al. 1971: 45–52).

a wider variety of extensive structures, possibly finite[21] and homomorphic[22] to the positive, ordered reals with addition. Since the Luce & Marley theory has finite models, the weakening of Hölder's theory thus reached appears significant, because it shows that a formal account of measurement (in particular, of scale construction) is possible, in which no infinitary idealizations are present.

At the same time, because of the results discussed in this chapter, a mathematically explicit link can be established between settings without infinitary idealizations, e.g. the models of the Luce & Marley's axioms, and the models of Hölder's theory, exploiting Dedekind completion on the one hand, and the extension of a partial operation to a total one on the other hand, together with additional hypotheses on minimal and maximal elements. These additional hypotheses depend on the fact that it is not possible to start from a finite extensive structure and extend it to a model of Hölder's theory. The modified strategy which is however available to carry out this extension is one which allows the relevant hypotheses on grounds of plausibility: it suffices to think that, even though experimental settings are finite, their finiteness is not significant in the sense that it does not impose constraints on which orders of magnitude are physically possible (if we consider progressively smaller orders of magnitude, this means that H6 has to be assumed; if we consider progressively larger orders of magnitudes, it means that maximal elements should be ruled out[23]).

If this hypothesis is accepted, then a link between finite extensive structures and the continuum of the positive reals can be established, by the results illustrated above. There isn't a fundamental difference between the assumptions that there are no minimal or maximal elements and Dedekind completeness or closure. They all assert something concerning the full generality of a measurement procedure and the uniformity of behaviour of the objects in an extensive structure. Thus, it is really the need to generalize certain basic features of experimentally constrained extensive

[21] A finite model of the Luce & Marley axioms is exemplified in Luce & Marley 1969: 238.

[22] An isomorphism is a structure-preserving function which is injective and surjective. If the last two properties are dropped, we have in essence a homomorphism.

[23] Alternatively one might leave out the maximal elements from an extensive structure. In this case, it can be extended to a closed one. If this closed structure can be represented into an open interval of positive reals, then the maximal element can be re-introduced and represented on the least upper bound of this interval, in case the non-maximal elements tend to it asymptotically (this is done in Luce & Marley 1969: 237). Since the open interval is a continuum, it models Hölder's axioms. This means that, even if maximal elements are present, it is nonetheless possible (in the absence of minimal elements) to extend, up to maximal elements, the corresponding models of the Luce & Marley axioms to models of Hölder's theory.

measurement which motivates its infinitary idealizations. These, in turn, lead to empirical settings which can be seen as extensions or enrichments of experimentally given settings.

Infinitary idealizations arise to deal with arbitrary approximation processes (carried out in the arbitrarily small) or arbitrary iterations of concatenation (leading to the arbitrarily large): they result from the assumption that a certain measurement procedure is fully generalizable. In particular, approximation processes and multiple formation *via* concatenation are presupposed to be always possible: this implies that (a) all physical approximation procedures behave in the same manner and, ideally, identify a real measure; (b) there is no prescribed number of steps after which concatenation can never be applied. In fact (a) and (b) interact quite tightly because the definiteness of concatenation makes it possible to have arbitrarily large resources[24] to carry out comparisons of the form:

$$na \leq_{\mathrm{L}} b <_{\mathrm{L}} (n+1)a$$

to bracket an arbitrary b within ever smaller bounds and thus determine its measure.

The interaction of (a) and (b) indicates that the infinitary idealizations of extensive measurement depend upon a requirement to the effect that a measurement procedure leading to unique numerical assignments be uniformly applicable. This requirement motivates the infinitary idealizations of Hölder's theory and relates the latter theory to its weakened versions examined in this chapter.

Note that the weakened versions of extensive measurement I have considered so far are sufficient to prove a representation theorem, independently of their connection with Hölder's axioms. In other words, a formal account of extensive measurability which does not invoke the infinitary idealizations characteristic of Hölder's system can be given. It is therefore clear that idealizations are not needed to ensure measurability, but only to generalize it.

Because of this it is also possible to conclude that infinitary idealizations like closure and Dedekind completeness do not carry any intrinsic ontological commitment, despite being existential statements. They do not because non-idealized extensive structures satisfy properties which suffice to show their extensibility to idealized structures: in other words, infinitary idealizations characterize a totality of empirical possibilities, which are already constrained by weaker assumptions (at least

[24] That is, arbitrarily large defined concatenations.

in presence of additional hypotheses on minimality and maximality, which are themselves motivated solely on grounds of generality).

These possibilities do not have to be realized for measurement to be possible, but define the full extent of the empirical configurations that measurement on an Archimedean structure can give rise to. For this reason infinitary idealizations do not force strong ontological commitments, despite their being existential statements.

8. A revision of the traditional view of idealization

I wish to conclude this chapter by emphasizing the differences between the account of idealization I have offered so far and what might be called the traditional view of idealization, as it is found in the philosophy of science. The distinctions I make here also apply to the characterization of idealization I develop in the next chapters.

Traditionally, idealization has been understood as a form of simplification of complex phenomena, carried out by neglecting some of their aspects and focusing only on a selected few. For this reason, idealizations have generally been taken to introduce distortions in the description of empirical phenomena[25], thus leading to a systematically incorrect picture of their structure. One important consequence of this fact is that 'really powerful explanatory laws of the sort found in theoretical physics do not state the truth' (Cartwright 1983: 3).

For this reason, it has been thought that one natural way to deal with idealized models or theories is that of adding them the detail which was originally subtracted, in order to produce a sort of convergence between theory and experiment (cf. Nowak 1972: 537, McMullin 1985: 257, Laymon 1995: 356 and the comments in Batterman 2007: 267–268). In this sense, an idealized law or theory has only a transitional status, because it must lead to more accurate descriptions of reality.

In sum, idealization has on the traditional view a simplificatory role, consisting in the elimination of some details from a complex situation, and requires subsequent de-idealization, because its use is only instrumental to reaching a faithful description of reality.

The account of infinitary idealizations in extensive measurement I have just developed reverses the traditional view in that the introduction of idealizations does

[25] Take for instance the following quotations: '[idealization] may involve a distortion of the original or it can simply mean a leaving aside of some components in a complex in order to focus the better on the remaining ones' (McMullin 1985: 248) and '[....] I shall use idealization specifically to refer to the ignoring or misdescription of a cause' (Laymon 1995: 357).

not subtract detail from the axiomatic description of extensive structures but rather adds to them new properties. In a sense, 'realistic' measurement theories like the Luce & Marley system say less about extensive quantitites than Hölder's theory does. This is because the Luce & Marley axioms have many non-isomorphic models and so there are many properties of an extensive structure they do not constrain, whereas all models of Hölder's theory are isomorphic and so they share the same first and second-order properties. In addition, the models of Hölder's theory are in general richer than those of the Luce & Marley system because of the existential assumptions they satisfy (e.g. Dedekind completeness, divisibility etc.).

For this reason, the introduction of infinitary idealizations into the Luce & Marley system cannot appropriately be characterized as the elimination of detail from them: it may be understood as a simplificatory move, but not because it leads to neglecting certain features of a complex phenomenon. The simplification arises for example because it is easier to prove a representation theorem from strong axioms than it is from weaker ones.

Even so, the important point I want to emphasize is that the infinitary idealizations arising in extensive measurement cannot be correctly understood within the traditional framework offered in the philosophy of science to study idealization[26]. This follows from the observations made in the previous sections: infinitary idealizations emerge from the need to generalize the applicability of a measurement procedure or the physical behaviour of the objects involved in that procedure. The resulting generalizations lead to the concept of an additive continuum (a model of Hölder's theory) as a universal framework for extensive structures. Equivalently, infinitary idealizations lead from a local characterization of magnitudes to a global one, by providing a unified context in which any 'local' extensive structure (whose concatenation or convergence properties are only local, i.e. do not coincide with closure or Dedekind completeness) can be integrated.

[26] One may object to this that my discussion of measurement can be reduced to the traditional framework. To this effect, it suffices to take Hölder's theory to generate a misdescription of an actual measurement procedure and its weaker versions to generate less simplified, more realistic descriptions of measurement. The reason why this objection is not conclusive is that the theories of measurement weaker than Hölder's are such that their models can be extended to models of Hölder's theory. Thus it is incorrect to say that realistic measurement procedures described by weak theories are distorted in the passage to Hölder's theory: rather, they are generalized. Thus, it is not misdescription, but generalization which arises and the generalized theory is, furthermore, not dropped in favour of the weaker ones, but retained as a unitary framework for them.

This is the opposite of what happens with the traditional characterization of idealization, where some features of an actual empirical setting are detached from it. In the context of extensive measurement features are added: the main consequence of this is that de-idealization, at least in the traditional sense, cannot occur. An idealized extensive structure is already richer than a non-idealized one, so the latter cannot be reached by adding detail to the former.

The reason for this is that idealization in extensive measurement is introduced in order to provide a systematic treatment of a variety of different situations. It is not introduced by ignoring or misdescribing certain features of measurement, but only by assuming their generalizability to a unitary conceptual framework. What may occur, instead of de-idealization, is a change of conceptual framework, i.e. the possibility of idealizing in different directions or of finding different ways of extending the models of an empirical theory to richer structures: an example of this phenomenon, depending on the empirical status of Archimedes' axiom, will be discussed in chapter 5.

The upshot of the above discussion is that the traditional view of idealization neglects the particular form it takes at the fundamental level of measurement[27]. As a consequence, my account of idealization can be not only added to Field's account of applicability to improve it, but also proves interesting in its own right, because it describes a form of idealization which has been overlooked in the philosophical literature.

So far I have just described the nature of this kind of idealization, without explaining why it is useful. I postpone the discussion of this issue to chapters 2 and 3 of part 2: the main point can however be quickly stated. In the context of extensive measurement, infinitary idealizations provide a unified, rich framework in which the study of extensive structures can be carried out. Idealization is introduced precisely to make this unitary framework available: once it is in place, it becomes possible to identify global empirical principles characterizing the totality of extensive structures.

[27] The same conclusion can be reached for the philosophical literature on measurement, which is not conspicuous and neglects idealization. Leaving aside the discussions of operationalism (as formulated in Bridgman 1927), important philosophical works like Nagel 1931, Carnap 1966, Ellis 1960, 1961, 1966 mostly deal with the formal concept of measurement, conventional aspects of measurement procedures and with dimensional analysis, but do not discuss idealizations. More recent work has been concerned with the metaphysics of quantities (especially Mundy 1987 and Swoyer 1987) or technical issues (as can be seen by looking at most of the contributions collected in Ehrlich & Savage 1992) but a characterization of the idealizations peculiar to measurement or at least some theories of measurement has never, to the best of my knowledge, been discussed.

The isolation of these principles, which is possible precisely because of the presence of idealizing hypotheses, triggers new scientific developments, in particular the construction of new theories of measurement which generalize the extensive case.

How this is accomplished will be explained later, once a full discussion of the idealizations of Hölder's theory is in place.

CHAPTER 4: REGULARITY IDEALIZATION AND MEASUREMENT WITH THRESHOLDS

1. Empirical equivalence

The Luce & Marley theory of extensive measurement examined in the previous chapter axiomatizes an order relation $\geq_L$ which can be used to introduce the following definition:

(i) $x =_L y$ iff $x \geq_L y$ and $y \geq_L x$,

where the relation $=_L$ may be called empirical equivalence. Since the transitivity of $\geq_L$ is an axiom in the Luce & Marley system, it follows from (i) that empirical equivalence is transitive. This result contradicts common experimental facts, often revealing triples of objects $\langle x, y, z\rangle$ such that x is empirically equivalent to y and y to z, while x and z are not (this might happen e.g. with comparisons involving the relation 'as long as': see the discussion in chapter 1).

It follows from the Luce & Marley axioms that $=_L$ is represented on numerical equality[1]: since the latter is a transitive relation, so must be the former. Taking into account a non-transitive empirical equivalence would disrupt the representation theorem. This theorem is therefore kept in place only under the ideal assumption that $=_L$ is transitive, which I have called a regularity idealization and labelled I_2 in chapter 1.

It is more difficult to characterize the role of an idealization like I_2 than it is to explicate the status of the infinitary idealizations discussed in chapter 3. When the latter are dropped, a representation theorem for extensive structures can still be proved. Thus, there is a fundamental uniformity between axiomatic extensive measurement with infinitary idealizations and axiomatic extensive measurement without them.

The same uniformity is not present in the case of the regularity idealization I_2, precisely because if it is dropped, the representation theorem provable from the Luce & Marley system is no longer provable, given that it entails the transitivity of $=_L$. When representability holds for an extensive structure, it follows that it is embeddable into the positive, real continuum with order and addition. But embeddability is a necessary condition to justify an idealization as an extrapolation

[1] That is, calling μ the relevant measurement scale: $x =_L y$ if and only if $\mu(x) = \mu(y)$.

from a more realistic empirical setting[2]. If embeddability fails, as is the case without I_2, it becomes difficult to find such a justification, which was available for infinitary idealizations.

In fact, it might even be the case that no justification of I_2 could be possible, in which case its assumption would generate a gap between the actual behaviour of empirical equivalence and its theoretical characterization. This possibility has been considered in the philosophical literature, in particular in Körner 1968, which revolves around a discussion of the existence of a discontinuity between the empirical and theoretical level in the natural sciences. Although Körner's discussion is not restricted to measurement, it includes it and focuses on I_2 in that context (cf. Körner 1968: 151).

The reason why I am interested in Körner 1968 is its particular claim that there is a disconnection between the experimental practice and the formal account of measurement. If Körner is correct, then it is not possible to provide a satisfactory justification for the introduction of the regularity idealization I_2 and so, specifically, it is not possible to look at I_2 as an extrapolation from the actual behaviour of extensive structures.

My task in the next sections is to look at Körner's position in some detail and to show that, at least as far as extensive measurement (together with the scientific theories reducible to it) is concerned, his position can be rejected. My rejection is simultaneous with the development of an alternative account of the status of I_2, which I will characterize as an extrapolation to infinity of the behaviour of less idealized extensive structures. In order to reach this objective, I need to look first at Körner 1968, in order to explain his characterization of an idealizing hypothesis like I_2.

2. Körner and the relationship between experience and theory

It seems fair to say that Körner 1968 is almost entirely devoted to clarifying the relationship between abstract mathematical theories and the domains of experience to which they are applied, including the applicability of arithmetic to measurement.

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[2] If an idealization in extensive measurement can be obtained by extrapolating model of a weaker theory to models of a stronger theory, then the models of the weaker theory are embeddable into those of the stronger theory. The latter models may be assumed to be fragments of the positive real continuum with order and addition, i.e. a model of Hölder's theory, because all the relevant idealizations are satisfied in it. In this case the justification of idealization as extrapolation implies embeddability in the real continuum and so embeddability is a necessary condition for it.

In particular, Körner strives to make a case for what he calls the disconnection between experience and theory: his arguments are based on the idea that certain essential features characterizing the structure of experience are faithfully captured by a non-classical logic (of which I will say more below) rather than by classical logic. Therefore, in order for (classical) mathematics to be employed in the description of certain sections of the empirical world, the data coming from experience must undergo a preliminary modification, which makes it suitable for mathematical description. Because of this, mathematics is not, strictly speaking, applied to experimental settings, but only to their modified, or, in Körner's terminology, *idealized* counterpart (Körner's notion of idealization differs from the one I've been using so far in a way I will explain in a moment): the existence of a gap, or disconnection, between applied mathematics and empirical facts follows.

Idealization, in Körner's sense, is to be thought of as an essentially conceptual operation, consisting in the replacement of classes of non-mathematizable empirical data (e.g. concerning the structural features of some empirical system) with what might be called their mathematizable theoretical counterparts. Mathematical theories can be applied to the latter kind of data, while they can't be applied to the former: by the same token, mathematical models are always adopted to characterize theoretical objects (the theoretical, idealized counterparts of empirical data) and so these models do not correlate experience and theory but, on the contrary, mark a discontinuity between them, since they only apply to an idealized version of the data coming from experience.

In the case of measurement, this is precisely what happens with the failure of the transitivity of empirical equivalence. Experimentally gathered data (obtained e.g. by making measurements of length) typically show that the transitivity of empirical equivalence fails for certain triples of objects $\langle x, y, z\rangle$. If x is empirically equivalent to y and y to z, and moreover the candidate numerical relation to represent empirical equivalence is numerical equality, then no consistent assignment of measures is possible to x, y, and z, simply because in this case they have to be all assigned the same measure, while at the same time x and z must be assigned different measures.

If a measurement scale mapping empirical equivalence on numerical equality is sought, it is therefore necessary to make a move like the one described in chapter 1, and assume that there is an idealized counterpart of empirical equivalence which is transitive. Idealized empirical equivalence is understood by Körner as an *exact* relation, as opposed to empirical equivalence, which is an *inexact* relation. In general, it is the passage from inexactness to exactness to determine for Körner the shift from experience to its idealization, which is amenable to mathematical treatment: it is therefore necessary to explain what Körner means by this terminology.

3. Exactness and inexactness.

Let us get back to the triple $\langle x, y, z\rangle$ of section 1, exemplifying a violation of transitivity relative to empirical equivalence. Suppose we are asked to decide, for any two objects from the triple (with repetitions), whether we can assign them the same numerical measure or not.

If presented with couple (x, x), we would answer in the positive, whereas we would answer in the negative when presented with couple (x, z), since we can check that one of the relevant objects is quantitatively different from the other. The status of (x, y) and (y, z) is more problematic. To see this, observe that, once it is decided that x and y should be assigned the same measure, the same decision transfers to (y, z) since the same relation of empirical equivalence holds for both. However this decision leads to an inconsistent numerical assignment, because x and z must have different measures, while x, y and y, z (and so x, z, by the transitivity of numerical equality) must have the same measure.

This suggests that the inconsistency depends on the mistaken idea that it is always possible to decide in a 'yes-or-no' fashion whether the measures of two objects are equal or not. To see why this idea may be considered mistaken, suppose x and y are found to be empirically equivalent after an experimental test. Moreover, assume a more refined measurement procedure is applied at a later time to x and y and it reveals a quantitative difference between them. In this case we might conclude that, before refining the measurement procedure, there was not enough information available to decide whether the same numerical assignment should be given to x and y. So there are situations in which that question must remain unanswered and, consequently, empirical equivalence undecided.

Given this conclusion, three possible alternatives present themselves when we try, on the basis of evaluations of empirical equality, to decide whether any two objects x, y should be assigned the same numerical measure:

i) It can be ascertained that (x, y) must have the same numerical measure;

ii) It can be ascertained that (x, y) must have different numerical measures;

iii) It cannot be ascertained whether (x, y) must have the same numerical measure.

If we now detach cases (i) to (iii) from the example just given, concerning the refinement of a measurement procedure, and simply take them as defining the behaviour of a binary, empirical relation $\approx_L$, we see that this relation generates the alternatives:

I) $x \approx_L y$ holds;

II) $x \approx_L y$ does not hold;

III) $x \approx_L y$ is undetermined.

One such relation is suitably described by a characteristic function that takes three distinct values, rather than two, as in classical semantics. Values 0 and 1 correspond to cases (II) and (I) above respectively, while a third value, say 1/2, corresponds to (III). According to Körner, empirical equivalence is the relation $\approx_L$, which is semantically described in a many-valued fashion: this is called *inexact* because it admits borderline cases like (III), and not only clear-cut ones, as relations do when a two-valued semantics is in place[3]. The latter relations are thus called, by contrast, *exact*.

The gap existing between experience and theory depends, on Körner's view, on the fact that inexact relations or predicates occur in experience, while exact ones are used in theoretical investigations. In this vein, the application of mathematics (here in particular numerical models) requires a change of logic, corresponding to a 'discontinuous' transition from the inexact structure of experience to the exact structure of idealized experience, which can be treated by means of mathematical theory. Thus, for measurement to be possible, the empirical, inexact relation $\approx_L$ has to be replaced by the theoretical, exact relation of empirical equivalence found e.g. in the Luce & Marley axioms and denoted by '$=_L$'. This is what can be concluded by looking at how Körner's notion of idealization (conceived as the transition from inexact predicates to exact ones) works in the context of measurement (cfr. Körner 1968: ch.10).

[3] Here I avoid the technical details but the concept of inexactness developed by Körner, besides being formally explored in Körner 1960: 159–167 and Körner 1968: ch.2 (where the truth tables for inexact propositional logic are the same found in Kleene 1938: 153), has also been studied in Cleave 1974 and Cleave 1976 (the former article contains a proof of completeness for a sequent calculus associated to a many-valued first-order logic based on Körner's ideas).

Here I wish to stress that this treatment of inexactness and idealization in measurement is based on a particular presupposition, namely that empirical equivalence is the inexact counterpart of numerical equality. This presupposition is quite natural, because the application of numbers to measurement generally seeks to map empirical equivalence into numerical equality: however, it is not a necessary presupposition.

Indeed, one might think that the discontinuity between experience and theory pointed out by Körner arises only because he is only looking at the applicability of a *particular* numerical model to extensive structures. If empirical equivalence is required to be mapped into numerical equality, then its transitivity must be assumed. However, Körner does not rule out the possibility of numerical models which map non-transitive empirical equivalence on some other numerical relation, differing from numerical equality. If these models existed for extensive structures with a non-transitive equivalence, it would be possible to represent them numerically without effecting any change of logic. This would show that the applicability of numbers to measurement does not by itself require the transition from inexactness to exactness Körner takes to be crucial.

Moreover, Körner's proposal has the problem that it doesn't explain why a non-transitive relation should be mathematically modelled by a transitive one. Such a move may well be undesirable because it introduces a formal assumption distorting the description of certain empirical settings. The only reason given by Körner for the distortion is that it ensures the applicability of a particular mathematical model: this is not necessarily a good reason, e.g. if the distortion introduced is too heavy. In this case, the empirical meaningfulness of anything inferred using the assumption of transitivity would be at least doubtful.

This problem can be solved by taking an alternative approach to the problem of the non-transitivity of empirical equivalence. This approach consists of two steps: the first step amounts to showing that there is a deep connection between a weak order like the one axiomatized in Luce & Marley 1969 (determining a transitive empirical equivalence) and a generalized order with a non-transitive equivalence; the second step amounts to showing that it is possible to relate extensive structures with the generalized order to ones with the weak order, thus finding numerical representations for both. In particular, the numerical representation of the generalized order tends to the standard numerical representation of the weak order under suitable conditions.

From this point of view, the transitivity of empirical equivalence is justified as the limit to which non-transitive extensive structures converge. It can therefore be explained why the assumption of transitivity does not introduce too heavy a distortion in the description of an extensive domain: not only transitivity is not a distortion, but it is also approximated by non-transitive extensive structures. This provides a satisfactory justification for its introduction.

The remaining sections of this chapter are devoted to exploiting measurement-theoretic results to establish the points I have just made. Before doing this, I wish to state more explicitly why my approach to I_2 is preferable to Körner's.

4. An assessment of Körner's account of idealization and applicability

As remarked above, Körner's account of the applicability of mathematics is based on the idea that a process of idealization or transition from inexact predicates or relations to exact ones must be accomplished.

This approach faces one main difficulty, namely that of motivating idealization, as Körner qualifies it. In the case of measurement, idealization consists in replacing a non-transitive empirical relation with a transitive one. But this move introduces a false description of the empirical setting under investigation. Thus, the choice to idealize in the first place, in Körner's sense, appears problematic: it stands in need of a justification.

Körner does not really provide one, apart from pointing out that idealization is necessary for the introduction of mathematics to study empirical phenomena. But the use of mathematics in science is plausibly motivated only if mathematics captures certain essential features of the phenomena it is employed to describe: in other words, to be serviceable at all, mathematics has to reflect certain features of interest of the concrete facts it is used to study. This must happen also when idealization, in the sense of Körner, is introduced: otherwise it would be difficult to see why a systematic misdescription of inherently inexact phenomena should be at all adopted to investigate them. Thus, if the empirical adequacy of certain exact mathematical models is to be granted, Körner's idealization is only acceptable if the transition from the inexact to the exact it involves does not introduce any fundamental distortion in the mathematical description of the phenomena under investigation. In particular, the frequency with which borderline cases occur has to be negligible, at least relative to the purposes for which the mathematical model has been devised.

This must be the case, because the process of idealization in Körner's sense (i.e. the conversion from inexactness to exactness) is based on arbitrarily turning the borderline cases of an inexact predicate or relation (e.g. the relation $\approx_L$ of the previous section) into true or false cases, and so it can be carried out in a variety of different ways (Körner himself says that it just consists in the act of arbitrarily turning borderline cases into clear-cut ones. See Körner 1968: 162). The possibility of arbitrarily turning borderline cases into true or false ones may affect in heavily different ways the exact characterization of an empirical setting: thus, if the frequency with which borderline cases occur is very high, different idealizations may lead to considerably different exact structures, exhibiting quite different forms of behaviour. When this is the case, it seems to me that the reasonable conclusion is that, due to the high degree of uncertainty of the empirical information, there is no acceptable way of modelling it by means of an exact structure. On the contrary, when uncertainty is not overwhelming, it is possible to say that an exact structure is an idealization of an empirical setting, yet not one disconnected from it, insofar as the empirical setting approximates the idealized, exact structure well enough.

These remarks show that, whenever a mathematical model is applicable, this is because it is sufficiently similar to the empirical structure it models, in the sense that it approximates its behaviour. So it is worth pursuing a precise characterization of this approximation relation. I will do this for extensive measurement in the remainder of this chapter.

More precisely, I do it while developing an account of the role played by the regularity idealization I_2 in extensive measurement which is to be preferred to Körner's because of two reasons. Firstly, it does not require a change of logic (i.e. a transition from an inexact logic to an exact one) in the passage from empirical structures to mathematical models. Secondly, my account emphasizes the similarity, rather than the disconnection, between empirical settings and mathematical structures, and specifies it in terms of approximation. This makes it possible to understand why the use of theoretical idealizing assumptions like I_2 in empirical investigations is acceptable and does not merely interfere with a faithful description of the phenomena, a point which is difficult to explain in the context of Körner's framework.

5. An alternative approach

My plan, starting with this section, is to introduce an account of the role of I_2 in extensive measurement alternative to Körner's. My account exploits several measurement-theoretical results. I am going to articulate it in three steps.

a) Firstly (section 6), I am going to restrict, for ease of presentation, my attention to ordered structures, rather than full extensive structures. In this context, an axiom system for ordered structures with a non-transitive 'empirical equivalence' can be found, which leads to a numerical representation theorem (this result has been originally proved in Scott & Suppes 1958). This shows that there are ordered structures not satisfying transitivity but still representable on numerical models.

b) Secondly (sections 7, 8), I am going to extend the representation theorem of (a) to structures which are not only ordered but also possess an additive operation. It follows that there are extensive structures with a non-transitive empirical equivalence which can be modelled numerically (this fact has been originally proved in Krantz 1967). Mathematics can be applied to them without effecting any change of logic, like the one envisaged by Körner.

c) Thirdly (section 9), I am going to show how, within a suitable axiomatic framework, the scales of measurement for extensive structures with a non-transitive empirical equivalence converge to ordinary scales of measurement (the latter scales map a relation definable in terms of the order on the non-transitive extensive structures into numerical equality). This result (originally proved in Krantz 1967) links extensive measurement based on a non-transitive empirical equivalence to extensive measurement based on a transitive one. Thus, a convergence theorem can be used to justify the introduction of I_2. The same theorem also shows in which sense extensive structures with a non-transitive empirical equivalence approximate idealized structures satisfying transitivity. A connection between them can therefore be identified, against Körner's view, which emphasizes their disconnection.

To sum up, (a) shows that, in the restricted context of ordered structures, dropping transitivity does not compromise numerical representability and thus the applicability of mathematics. Then, (b) shows how to extend this result to richer structures with an additive operation. Finally, (c) shows that, in the limit, these structures generate ordinary extensive measurement scales on the positive reals with order, addition and numerical equality. With (a) to (c) in place, an account of I_2 and a justification for its introduction can be given, which are preferable to Körner's because they are set within a uniform logical background (no change of logic is needed) and they clearly explain why it makes sense to assume I_2 despite its being contradicted by experiment.

6. Semiorders

I now turn to establishing (a) of previous section. For the moment I am only interested in structures like $\mathbf{L} = \langle L, >_L, \approx_L \rangle$, where the relation $>_L$ is assumed to be asymmetric and transitive. The binary relation $\approx_L$ is to be understood as the *symmetric complement* of $>_L$. In other words, all the objects that do not bear the relation $>_L$ to one another stand in the relation $\approx_L$: it follows that $\approx_L$ is reflexive and symmetric.

If $\approx_L$ is also transitive, then we have a weak order structurally identical to the numerical relation 'greater or equal to'. This means (assuming a finite L) that $>_L$ can be represented on the numerical order $>$ and $\approx_L$ on numerical equality, symbolized by $=$. This is what happens to the order relation of an extensive structure when the usual axioms are in place.

If, on the other hand, the transitivity of $\approx_L$ is not assumed, the question arises whether $\mathbf{L} = \langle L, >_L, \approx_L \rangle$ can still have a numerical representation. More precisely, the problem is whether it is possible to find a system of axioms for structures of the type of $\mathbf{L}$ which (i) does not presuppose the transitivity of $\approx_L$ and for which (ii) a representation theorem can be proved.

One way of obtaining this result is to look for candidate numerical representing structures for $\mathbf{L}$, having a possibly non-transitive relation. An example is provided by the real structure: $\mathbf{R}_\delta = \langle R^+, >_\delta \rangle$, where R^+ is the set of positive reals, δ a nonnegative constant and the binary relation $>_\delta$ is defined, for any two reals a and b, by:

$$x >_\delta y \text{ if and only if } x > y + \delta.$$

The numerical relation $=_\delta$ can now be defined by putting:

$$x =_\delta y \text{ if and only if not}(x >_\delta y) \text{ and not } (y >_\delta x).$$

It can be checked that the relation $=_\delta$ fails to be transitive for positive δ. It is also relatively straightforward to regard $\mathbf{R}_\delta$ as modelling a particular kind of empirical situation. This is because the constant δ can be regarded as a *threshold.* Suppose, for instance, that δ is taken to represent the discriminating power characteristic of a certain measurement procedure. Then this procedure can distinguish between x and y only if $x >_\delta y$, i.e. only if they are farther apart than the discriminating power δ. Symmetrically, x and y are indistinguishable if their difference lies below threshold, i.e. when $x =_\delta y$. This possibility is compatible with the existence of non-transitive triples, since the conditions $x =_\delta y$ and $y =_\delta z$ are consistent with $x >_\delta z$.

One main consequence follows from these observations, namely that problems (i) and (ii) above can be reduced to the problem of finding axioms for **L** such that they establish its representability on $\mathbf{R}_\delta$, for some nonnegative δ. A system of axioms for which all this holds exists and is the following:

I) $x \approx_{\mathrm{L}} x$;

II) $\approx_{\mathrm{L}}$ is the symmetric complement of $>_{\mathrm{L}}$;

III) $x >_{\mathrm{L}} y$ *and* $y \approx_{\mathrm{L}} z$ *and* $z >_{\mathrm{L}} w$ *entails* $x >_{\mathrm{L}} w$;

IV) it is never the case that $x >_{\mathrm{L}} y$ and $y >_{\mathrm{L}} z$ while, for some w, $x \approx_{\mathrm{L}} w$ and $w \approx_{\mathrm{L}} z$.

A structure **L** satisfying (I) to (IV) is called a *semiorder*[4]. Note that $>_{\mathrm{L}}$ is irreflexive, transitive (this follows from the first three axioms) and asymmetric, while $\approx_{\mathrm{L}}$ is not necessarily transitive.

This result establishes (a) of section 6. It is also important because, given the intuitive interpretation of the representing structure $\mathbf{R}_\delta$, it suggests a fruitful way of extending the representation theorem for semiorders to extensive structures. Semiordered extensive structures can be understood as extensive structures with thresholds, whose effect is that of making quantitative differences discriminable only when they are sufficiently large (i.e. above threshold).

As a consequence, the usual form of extensive measurement with a transitive empirical equivalence may be understood as the limiting case of extensive measurement with thresholds, when the precision of measurement increases below any fixed threshold. This is, very roughly, the content of the convergence theorem I

[4] This concept has been originally introduced in Luce 1956.

will survey in section 9[5]. In order to establish it, it is necessary to find the axioms which make it possible to prove a representation theorem for semiordered extensive structures, which I will now do.

7. Extensive semiorders: preliminaries

The previous discussion of semiorders has suggested the possibility of isolating a system of axioms for extensive measurement which assumes the existence of thresholds. The presence of thresholds implies that measurement can be carried out only up to a fixed degree of precision. It is therefore desirable to formally translate this intuition into a set of axioms and obtain a representation theorem for extensive semiorders which gives rise to 'imprecise' measurement scales, i.e. ones based on a measurement procedure which cannot detect arbitrarily fine quantitative differences.

This result can be obtained on the basis of the axiomatic theory of extensive semiorders presented in Krantz 1967. My task in this and the next section is to describe the theory and illustrate the representation theorem for it, thus establishing point (b) of section 6.

The extensive structures studied in Krantz's theory are based on a nonempty set L and the related set X of all *finite* subsets of L. It is assumed that:

K1: $\langle X, >_L \rangle$ is a semiorder[6].

The strategy to extend this semiorder to an additive structure consists in taking the finite subsets of the domain X (containing finite subsets of L) and their unions $\bigcup Y$ as concatenations[7]. These unions, being finite subsets of L, are elements of X and therefore can be compared with respect to the semiorder relation.

The use of set-theoretical union in this context has two important consequences: the first is that, as long as the existence of disjoint copies of any finite subset of L is not assumed (as will be the case), union does not give rise to the infinitary consequences of closure[8] (in particular the union of a set with itself does not generate a larger one). The second consequence is that approximations ordinarily carried out in extensive measurement by taking exact copies of some given object x and

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[5] What this theorem shows is that, under suitable axioms for extensive measurement, it makes sense to impute the lack of transitivity of empirical equivalence to the limited discrimination power of a measurement procedure.

[6] The semiorder relation $\approx_L$ is definable in terms of $>_L$, as shown in section 6.

[7] The elements of one such union are the elements of the elements of Y, which determine a finite subset of L and thus an element of X.

[8] Because concatenation is usually taken to be positive, while union is idempotent, i.e. $a \cup a = a$.

concatenating them can now be substituted by *imprecise* approximations obtained by unions of objects which are sufficiently close in magnitude to x[9].

I will explain how this is done only in the next section. This is because the concept of 'closeness in magnitude' needed to talk about imprecise approximations is expressed using relations which are definable in terms of the semiorder. It is to them I now turn: these relations characterize the *unique weak order induced by the semiorder.* Ordinarily, extensive measurement assumes the existence of a weak order $\geq_L$ (a transitive and connected binary relation): here no such weak order is assumed, yet the semiorder suffices to determine one.

Nonetheless, this move does not amount to a mere sleight of hand, consisting in reintroducing a weak order which was not explicitly assumed initially. This is because the weak order involved is not, strictly speaking, as strong as the weak order of extensive measurement, being a function of the richness of the extensive domain over which it is defined (this will be explained in a moment). However, it can be proved using Krantz's axioms that the less regular weak order definable from the semiorder becomes the standard weak order of extensive measurement in presence of a sufficiently rich extensive domain: in this case, it is also possible to construct an extensive measurement scale of the standard type, like those which exist for extensive structures without thresholds. It is in this precise sense that extensive measurement with thresholds converges to ordinary extensive measurement (this fact, which establishes (c) of section 5, will be discussed later).

Now, the weak order W induced by the semiorder $>_L$ is defined as follows:

(i) $x\mathrm{W}y$ if and only if, for all z in L, $z >_L x$ implies $z >_L y$ and $y >_L z$ implies $x >_L z$[10].

The relation W is transitive (because of the transitivity of $>_L$) and connected, since it may be verified that, for any x, y in L, either $x\mathrm{W}y$ or $y\mathrm{W}x$. Thus W behaves like the familiar 'smaller or equal to' relation: for this reason, a strict order and a transitive 'equality' relation can be defined from it. The equality E is simply given by:

(ii) $x\mathrm{E}y$ if and only if $x\mathrm{W}y$ and $y\mathrm{W}x$.

Note that E is reflexive and symmetric, but also transitive, because of the transitivity of W. It can be proved that the equivalence relation E coincides with the equivalence relation of 'perfect substitute', defined by:

[9] It is by means of these approximations that a representation theorem leading to imprecise scales can be proved, as will be shown in the next section.

[10] Definition (i) says in essence that y is not larger than x, because anything larger than x is by the definition certainly larger than y, while, whenever y is larger than something, x is too.

(iii) xEy if and only if, for all z in L, $x \approx_L z$ if and only if $y \approx_L z$[11].

The fact that (ii) and (iii) define the same relation is quite significant and explains my previous remark about the difference between the standard weak order of extensive measurement and the one induced by a semiorder.

To see the difference, consider a fixed set L in which xEy holds. Reasoning intuitively in terms of thresholds, we see that, if any object in an extensive domain L lies sufficiently far away from both x and y, there won't be any objects indistinguishable from either. Thus, if x and y are below threshold from each other, it follows using (iii) that xEy. However, a subsequent enlargement of L, obtained by introducing an object z which is close to x but above threshold relative to y, will disrupt the relation E, which must cease to hold. Thus E is, in a sense, threshold-sensitive, whereas the standard equivalence relation $=_L$ of extensive measurement is not, since it holds when exact quantitative identity does. The convergence which can be proved from Krantz's axioms shows that E converges to $=_L$ as soon as the extensive domain under investigation contains arbitrarily small objects: this is just one way to algebraize the assumption that a measurement procedure capable of arbitrary refinement is ideally available.

My task in the next section is to introduce most of Krantz's axioms and prove a representation theorem for semiordered extensive structures, giving rise to an imprecise measurement scale. This scale, as precision increases, tends to an ordinary extensive scale.

8. Approximate measures

In order to deal with scale construction in the context of a semiorder, axioms are needed which ensure the possibility of using concatenations to obtain approximate measures. Krantz 1967 takes care of this requirement by introducing sets Y(x), defined as follows for any x in X:

> Y(x) is the set of all subsets of X such that: (i) they are finite and (ii) their elements are pairwise disjoint and each of them precedes x in the strict order P definable from W[12].

[11] For this result see Luce 1973: 42 and the bibliographical references therein, but also Roberts 1979: 257.

[12] By the condition: xPy iff xWy and not yWx.

Each element of Y(x) is made of a collection of objects strictly smaller than x: in other words, Y(x) is the set of all the finite concatenations which approximate from below exact iterations of concatenation on x. The idea behind the introduction of Y(x) is that one might take x as a unit of measure and use approximations of the iteration of concatenation on x to generate estimates of any other object z. In order to ensure this possibility, an axiom is needed, namely:

K2: For any x and z, there is Y such that Y is an element of Y(x) and $\bigcup Y >_L z$.

Axiom K2 says that, whenever a unit of measure x is chosen, it is possible to find an imprecise concatenation of its 'copies', such that, when they are taken all together, they strictly exceed any given object z. The condition asserted is a form of Archimedeanity.

Note that K2 is a rather strong condition, implying that no minimal elements exist (if x is minimal, then Y(x) must be empty, against the axiom): however, if this assumption is taken as a plausible one, as it may be according to the discussion of minimality in chapter 3, it can be included among the axioms as a plausible idealization. With the above definition and axiom in place, it is already possible to generate imprecise numerical measures.

To see how this can be done, call N(Y) the cardinality of Y: the smaller N(Y), the larger the elements of Y. In this case, the elements of Y are going to be sets of sufficiently small elements of L and thus to provide individually good approximations of the 'unit of measure' x. This suggests that the best approximation of a finite, imprecise sequence of multiples of x will be given by the Y in Y(x) satisfying K2 and whose cardinality, in addition, is minimal. The cardinality N(Y) then provides an upper bound for the measure of z in x units.

In an analogous manner, one may define another numerical estimate, giving a lower bound for the measure of z in x units, as follows:

$$\text{(i) } M(x, z) = \inf\{N(Y)\text{: Y is in Y}(x) \textit{ and } \bigcup Y \text{ W } z\}.$$

The existence of M(x, z) is ensured by K2 and provides a numerical index determining how long an approximate, imprecise sequence of elements constructed from x has to be in order for it to reach the lower threshold of z[13].

[13] Krantz considers other numerical indices, called N and n, which I won't discuss here. Results analogous to those obtained for M in the following sections can be proved for them.

Definition (i) lies at the basis of the construction of imprecise scales and thus leads to an imprecise representation theorem: in order to obtain it, two more axioms are needed. Nonetheless, even at this stage it is possible to show that the numerical estimates M essentially preserve the weak ordering W on X. In particular, it can be proved that:

$$\text{(i) } M(x, y) > M(x, z) \text{ implies } y\mathrm{P}z \text{ and } y\mathrm{W}z \text{ implies } M(y, x) \leq M(z, x).$$

(cf. Krantz 1967: 351). An ordinary representation theorem would state the existence of a scale μ satisfying:

$$x \leq_{L} y \text{ iff } \mu(x) \leq \mu(y).$$

In (i), M plays the role of μ and determines a couple of conditionals rather than one biconditional. This is because P and W are defined in terms of a semiorder relation and describe two different cases, one where the imprecise estimate M suffices to discriminate y and z, and the other in which y and z may be indistinguishable.

In order to obtain a full representation theorem, it is necessary to establish the imprecise counterpart of the familiar condition:

$$\mu(x +_{L} y) = \mu(x) + \mu(y).$$

In the context of Krantz's theory, this must be done relative to the operation of disjoint union, since set-theoretical union is used to describe concatenations. In a fashion analogous to that of the previous result about order-preservation, concatenation-preservation on an imprecise scale is not described by a single equality but by two inequalities involving the numerical function M.

In the ordinary representation theorem, concatenation is mapped into numerical addition: on the other hand, the imprecise representation theorem states that the number M assigns to the disjoint union of two objects z and z', relative to a unit x, is included between two numbers which are 'almost' the addition of the numbers M individually assigns to z and z'. In symbols, we have:

$$\text{(ii) } M(x, z) + M(x, z') - 1 \leq M(x, z \cup z') \leq M(x, z) + M(x, z')$$[14].

This result can only be proved in presence of K1, K2 and two further axioms, K3 and K4, expressing a condition on the richness of the extensive semiorder and a monotonicity property respectively[15].

[14] See Krantz 1967: 351–352 for a proof.

[15] K3: For any x, z in X, any integer m and any w in X, there is Y in Y(x) such that: $N(Y) \geq m$, $\bigcup Y$ is disjoint from w and any subset Y′ of Y for which $N(Y') \geq M(x, z)$ satisfies $\bigcup Y'$ W z.

Theorems (i) and (ii) above establish (b) of section 6: they show that thresholds (characterized by the semiorder axioms) can be exploited to define numerical estimates like M, which provide an imprecise scale of measurement for the semiordered set X with the additive operation of disjoint union defined on it. In short, extensive structures with thresholds are numerically representable, despite their being ordered by a relation which is weaker than the standard order of extensive measurement, which determines a transitive relation of empirical equivalence. As a consequence, we don't need a weak order with an associated transitive empirical equivalence to apply numbers to extensive structures.

The next section will establish (c) of section 5, i.e. the convergence of the imprecise measurement scales based on estimates like M to an exact scale μ. This result can only be obtained by adding an infinitary assumption to the ones already postulated.

9. Convergence

A standard representation theorem for extensive measurement can be seen, in the context of Krantz's theory, as the limit of theorems (i) and (ii) of the previous section, as the degree of precision of a measurement procedure tends to infinity. The assumption which suffices to establish this connection between imprecise measurement scales and exact ones is the last axiom of Krantz 1967, namely:

K5: For any y, z in X, if zPy, then there is x in X, such that x and y are disjoint and zP$x \cup y$.

It follows from K5, in presence of the other axioms, that an infinite sequence $\{x_i\}_{i\in N}$ exists, satisfying:

$$\lim_{n\to\infty} M(x_n, z) = \infty .$$

The last equality says that any element z can be approximated by arbitrarily small units of measure (see Krantz 1967: 354 for a proof). Intuitively, this is equivalent to the existence of arbitrarily small discriminable objects and thus to the possibility or arbitrarily refining the precision of measurement. Now a standard extensive measurement scale can be set up. This is because, once a certain n if fixed, one can form the ratio:

K4: For any x, x', y, y' such that x and x' are disjoint and so are y and y', if xWy and x'W y', then $x \cup y$ W$x' \cup y'$.

$$\frac{M(x_n, z)}{M(x_n, y)},$$

which expresses an imprecise measure of z in y units, when both y and z are approximated by the same object x_n.

As n becomes arbitrarily large, the above ratio becomes a less and less imprecise measure of y in z units, in the sense that the reference x_n tends to become smaller than any threshold. In particular, it follows from Krantz's axioms that the limit below exists:

$$\mu(z) = \lim_{n\to\infty} \frac{M(x_n, z)}{M(x_n, y)}.$$

This limit provides a unique real measure for any z in y units: it can be seen that it is a function of the imprecise estimate M, as n tends to infinity. Under the same condition, theorems (i) and (ii) of previous section tend to the corresponding conditions satisfied by standard extensive measurement scales.

Intuitively, it is not difficult to see that (i) of section 8 turns into a biconditional because, whenever yEz holds, the measure of z in y units must be the same as the measure of y in z units (otherwise there is some concatenation of approximate copies of x_n such that it discriminates z and y, so that yEz cannot hold). As a result, these measures are equal and E is mapped by μ on equality. This suffices to conclude that:

$$y\mathrm{W}z \text{ iff } \mu(y) \leq \mu(z).$$

Moreover, taking (ii) of section 8 and dividing it throughout by $M(x_n, y)$, one obtains:

$$\frac{M(x_n, z)}{M(x_n, y)} + \frac{M(x_n, z')}{M(x_n, y)} - \frac{1}{M(x_n, y)} \leq \frac{M(x_n, z \cup z')}{M(x_n, y)} \leq \frac{M(x_n, z)}{M(x_n, y)} + \frac{M(x_n, z')}{M(x_n, y)}$$

whose middle term is known to have a limit when n tends to infinity. Since, in this case, the left-hand tem and the right-hand term have the same limit, which is by definition $\mu(z) + \mu(z')$, it follows that, for disjoint unions:

$$\mu(z) + \mu(z') = \mu(z \cup z').$$

As a result, in presence of K5 and thus of an infinite decreasing sequence of discriminable elements, the imprecise scales of measurement for an extensive semiorder converge to a standard extensive scale. In particular, the semiorder generates a relation E of identical physical behaviour which in the limit can be represented on the transitive relation of numerical equality (even without taking the limit, E is an equivalence relation, but it can be falsified by further refinements of a measurement procedure, as remarked above). This conclusion establishes (c) of section 5.

10. Regularity idealization

Krantz's theory of semiordered extensive measurement can be used to address the problem posed by Körner and concerning the applicability of mathematics to experimental science. In the context of measurement this problem reduces to the question about the legitimacy of modelling the non-transitive relation of empirical equivalence on the numerical relation of equality. Körner admits this particular form of modelling but he regards it as an instance of a general phenomenon characterizing the applicability of mathematics: for the latter to be possible, inexact empirical structures must be turned into exact, idealized ones. In measurement, an inexact empirical relation of indiscriminability is turned into an exact equivalence relation, which is structurally identical to numerical equality.

In the previous sections I have shown by means of measurement-theoretic results that this is not what happens in extensive measurement. It is perfectly possible to start from an extensive structure on which an empirical ordering is defined whose associated empirical equivalence is in general non-transitive and yet has a numerical representation. This is just an extensive semiorder.

In presence of the axioms of Krantz 1967, it is possible both to construct approximate, imprecise measurement scales for extensive semiorders (where the relation of empirical equivalence is falsifiable by a further refinement of the measurement procedure) and to show that such approximate scales converge to a standard extensive scale, which represents a transitive empirical equivalence on numerical equality. It is therefore possible to take care of the fact that empirical equivalence is non-transitive and to show under what conditions (in this case, Krantz's axioms) it makes sense to assume transitivity. Furthermore, this property emerges from extensive semiorders asymptotically, i.e. as the result of limit-taking.

All of this can be shown within a classical environment, i.e. without having to make use of any non-classical logic, as Körner does to characterize the failure of transitivity of empirical equivalence.

In sum, no need to resort to a change of logic arises, a clear motivation (the convergence theorem of section 9) for assuming the transitivity of empirical equivalence can be provided and this particular idealization is seen to arise from an operation of passage to the limit. As a result, there is no disconnection between experience and theory, in the sense of Körner, but rather a connection between them which is based on approximation and the generalizability of certain features of measurement practice, like the possibility of refining the precision of a given measurement.

It is also noteworthy that, in view of the convergence theorm of section 9, the assumption of a transitive empirical equivalence is not to be read as merely a false assumption but as a property emerging from the enrichment of an empirical setting. This is because, as soon as K5, which expresses a density property, is introduced, a standard representation theorem can be proved for extensive semiorders.

Without K5 the transitive relation E is still definable from the primitives of the semiorder, but it is a fallible equivalence relation, in the sense that it depends on how rich a fixed model of K1 to K4 is, i.e. on how fine the quantitative discriminations possible within that model are. Once a sufficiently rich model is available, by the addition of K5, the relation E on it can be entirely assimilated to the familiar relation $=_L$ of ordinary extensive measurement.

The observations of the previous chapter concerning infinitary idealizations (section 7 of chapter 3) apply here as well: idealization does not consist in the introduction of false assumptions or the elimination of detail, but rather in the integration of a given empirical setting within a richer framework, where certain formal properties are satisfied, which emerge as limiting cases of analogous properties holding in weaker settings[16].

It only remains to show what the relation is between Krantz's theory of measurement and the theories of measurement previously considered, in order to provide a unitary treatment of the infinitary idealizations examined in chapter 3 and the regularity idealization discussed in this chapter.

[16] To some extent, the construction of imprecise measurement scales converging to an exact one accounts for the commonly made assumption that numerical measures are uniquely determined, which is at variance with the fact that, as a rule, repeated measurements of the same quantity yield different values. A comprehensive analysis of the relation between the scattered measurement values obtained in practice and the unique numerical measures determined by an axiomatic theory would require a discussion of error in measurement. Although this is a topic of great importance, I have decided not to include it in my discussion mainly because Kyburg 1983 (especially chapters 3 and 4) already contains a detailed treatment of this issue for extensive measurement (further observations on the same subject are also found in Kyburg 1992). Kyburg's discussion can be integrated within the framework I develop here.

11. Idealization as enrichment

Since K1 to K5 imply a standard representation theorem, it follows that all of their models are endowed with a weak ordering and an associative concatenation operation satisfying monotonicity. Furthermore, these models must be Archimedean, as this is a necessary condition for representability on the reals.

This means that all models of K1 to K5 satisfy at least four of the five axioms in Luce & Marley system, solvability being the problematic assumption. It is however clear that K5 is a weakened version of solvability, since it asserts that, given *y* strictly larger than *x*, it is possible to concatenate *x* with a distinct object *x′* in order to obtain a new object which is still strictly smaller than *y*. As a consequence, even though the difference between *y* and *x* is not assumed to exist, it can be approximated arbitrarily well: this is sufficient to obtain representability on the reals.

Because K5 is almost a solvability condition, it seems reasonable to look at models of Krantz's axioms as extensive structures which can directly be extrapolated to models of the Luce & Marley's axioms, since they satisfy nearly all of them while they very closely approximate solvability[17].

On the basis of these observations, it is possible to start from K1 to K4 as basic assumptions characterizing a concrete experimental setting for extensive measurement. These axioms are sufficient to construct imprecise scales. As soon as it is assumed that the precision of measurement can in principle always be refined, such an assumption can be translated into condition K5, which leads to extensive structures that are almost models of the Luce & Marley's axioms. By an immediate extrapolation one reaches models of the latter theory and, thanks to the results surveyed in chapter 3, it is known how to extend these to closed structures (since K1 to K5 rule out the existence of minimal elements while maximality is ruled out by the behaviour of disjoint union) and, finally, to models of Hölder's theory.

The route just illustrated leads from semiordered measurement settings for which neither the infinitary idealizations, except for non-minimality, nor the regularity idealization of extensive measurement hold, to extensive measurement settings satisfying all of these idealizations. Such a result is reached by postulating conditions on the generalizability of a measurement procedure, which are algebraically translated into existential axioms on the size or richness of an extensive domain.

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[17] In addition, the fact that K1 to K5 suffice to uniquely determine real measures for their model makes the type approach outlined in chapter 2 applicable to this theory as well.

The latter axioms can be regarded as the syntactic counterpart of a semantic process of extension of measurement settings which do not satisfy certain idealizing hypotheses to measurement settings which do.

A satisfactory account of idealization in extensive measurement can thus be given by looking, as I have done in the last two chapters, at weak axiom systems for it in order to show how, under plausible hypotheses, their models can be used to generate richer structures, which ultimately satisfy Hölder's theory.

While this analysis provides an account of how idealizing hypotheses emerge and thus elucidates the relationship between experiment and theory, it does not yet deliver a full account of idealization. The reason is that any such account must also explain why idealizations are used.

This problem is particularly evident in the case of extensive measurement: if the whole point of measurement were just the construction of numerical scales, idealization could certainly be dispensed with, since the existence of scales can be proved without it. Thus, if idealization is to play a scientifically relevant role, it must be found elsewhere. My answer to this problem (to be articulated later) is that idealizations are significant because only in their presence can certain results be proved, which play a crucial role in the process of scientific theory-construction and in orienting experimental researches.

The analysis of how idealizations arise provided so far addresses the incompleteness of Field's account of applicability, as characterized in chapter 1, and can be adjoined to it. On the other hand, since the explanation of the reason why idealizations are useful in science is related to the restrictiveness of Field's account, I postpone its discussion until part 2, which provides a detailed analysis of the limitations of Field's characterization of applicability. Before moving to it, I will discuss, in the next chapter, one more type of idealization present in extensive measurement.

CHAPTER 5: ARCHIMEDES' AXIOM AND STRUCTURAL IDEALIZATION

1. One further issue

In the previous chapters I have shown how Hölder's theory can be related to several weaker and less idealized theories, which provide more realistic descriptions of extensive measurement.

I focused on two different kinds of idealization, namely infinitary idealizations and regularity idealizations. By progressively weakening Hölder's theory, I have reached Krantz's theory of semiordered extensive structures: using this theory, it is possible to gradually introduce all the idealizations of Hölder's theory by suitable assumptions on the generalizability of a measurement procedure.

In short, the models of a rather weak and realistic theory of measurement can be extended to models of Hölder's theory. For this reason, the idealizations of this theory correspond to enrichments of more realistic empirical settings. The introduction of idealizing hypotheses is therefore suggested and to some extent justified by the properties of actual empirical settings and the need to generalize them: for example, by generalizing Archimedean approximations one obtains Dedekind completeness, and by generalizing the possibility of forming concatenations one obtains closure.

In this context idealization cannot be reduced to the assumption of false empirical hypotheses or a selective (and thus incomplete) description of phenomena: on the contrary, it consists in a process of extension and enrichment, as I pointed out in chapter 3, section 8. There I also remarked that extension may not be possible in a unique way, i.e. there may exist alternative ways of enriching an empirical setting.

This phenomenon can be concretely exemplified by looking at Archimedes' axiom. All the theories of extensive measurement I have considered so far assume this kind of axiom (Krantz 1967 does not formulate one in the usual manner but all models of axioms K1 to K5 are Archimedean), which is crucial to formally characterize the construction of a measurement scale into the positive reals (this point has already been stressed in chapter 2). The same condition also plays an important role in carrying out the extensions of the models of weak theories of extensive measurement to models of Hölder's theory: this is because both Dedekind completion and the extension of non-closed structures to closed ones illustrated in chapter 3 require Archimedes' axiom.

As a consequence, the generalization of weak theories of extensive measurement to Hölder's theory becomes problematic if the possibility of dropping the Archimedean axiom is taken seriously. This possibility may be motivated by the idea that this axiom is an idealization. If so, its introduction has to be justified by showing that it arises, e.g. as a plausible generalization, from a theory which does not include it (this strategy of justification was adopted for Dedekind completeness, closure and the transitivity of empirical equivalence). If this cannot be done, then alternative enrichments of an extensive structure may become available, according to whether Archimedes' axiom is used or not.

Before looking at this possibility, however, it is necessary to clarify why it is plausible to take Archimedes' axiom as an idealization and why it is difficult to introduce it through generalization, as was done with the other idealizing hypotheses of extensive measurement. First of all, note that Dedekind completeness entails Archimedes' axiom: as long as one is willing to classify the former property among idealizing hypotheses, it seems possible to do so with the latter, since it is not testable. In short, it seems reasonable to take the untestable consequence of an idealization as an idealization. The untestability of Archimedes' axiom is established in the next section.

2. The Archimedean axiom and testability

One reason why testing the Archimedean axiom is problematic is that it is impossible to experimentally check whether it fails. Intuitively, an Archimedean axiom (in presence of further assumptions like H1 to H6 of Hölder's theory) says that, no matter how small x and large y, there always is an integer n such that $nx >_L y$. If this failed, x would be smaller than the n-th part of y, for any n, which means by definition that x would be an infinitesimal relative to y: by closure $x +_L y$ exists, and it is infinitely close to y, i.e. it differs from it by an infinitesimal amount. In order to check that this never happens, one should be able to make comparisons involving arbitrarily small orders of magnitude. This is an unfeasible task and thus it is not possible to decide by experiment whether Archimedes' axiom is contradicted by experimental fact.

Besides this point, which depends directly on the statement of the axiom, two measurement-theoretical results can be used to reach a strong and subtle conclusion, establishing the untestability of the Archimedean property. The first result can be found in Suppes 1969: 4–8 and is an axiomatization of equally-spaced[1], finite extensive structures which does not include any Archimedean axiom but suffices to prove representability into the additive reals. As a consequence, there are finite extensive structures whose measurability can be established without relying on the Archimedean axiom. This is not directly a result about its testability, but it suggests that all ordinary practices of equally-spaced measurement carried out with suitable instruments and taking place, as they do, within finite settings, can go through independently of whether an Archimedean condition is assumed.

A second result which identifies circumstances under which no Archimedean conditions have to be assumed for extensive measurability has been proved in Adams et al. 1970: this result is particularly interesting because it also shows a way of setting up an extensive measurement scale which does not require the use of successive concatenations in order to determine numerical measures.

In order to see this, it is convenient to consider length measurement performed on rigid rods and to assume as given a family of rods which determines an extensive structure **L** satisfying Hölder's axioms H1 to H6 plus a condition stating the commutativity of concatenation. From these axioms no Archimedean condition follows, so their models may or may not be Archimedean.

What can be done experimentally on **L** consists in examining only finitely many objects in its domain L and making comparisons between them. If these comparisons were written down in a formal first-order language containing constants for the objects in L, experimental observations could be denoted by atomic formulas like:

$$a +_{\mathrm{L}} b <_{\mathrm{L}} c +_{\mathrm{L}} d.$$

Formulas of this type say that certain concatenations are quantitatively smaller than others. A series of experiments on **L** would then produce a finite number of these formulas, involving only a finite subset L′ of L.

Thus we have a finite empirical domain L′, which is described by a (finite) system of inequalities. Viewing this situation in slightly more abstract terms, we have a

[1] In this case the 'equal spaces' are the differences between the elements of an extensive structure. Ruler measurement is a concrete example of equally-spaced extensive measurement (in which case the axioms characterize the arrangement of the marks on the ruler).

system of linear inequalities and we can therefore pose the problem of whether it has a solution in the positive reals, in case $+_L$ and $<_L$ are interpreted as numerical order and addition.

To solve this problem means to find an extensive scale for L′: the crucial point is that setting up this scale is equivalent to solving a system of linear inequalities, which can be done without making use of the usual approximating procedure based on successive concatenations of a unit of measure (results from convex analysis, described in Krantz et al. 1971: 59–67 give rise to algorithms which solve the relevant systems of inequalities). Since Archimedes' axiom is used to determine numerical measures precisely when successive concatenations are employed, it looks as though this axiom could be dispensed with, as long as there is a method of constructing real measures not relying on the method of successive concatenations: in fact, it follows from a result of Adams et al.1970[2] that the real measurability of L′ is equivalent to **L** satisfying a weakening of H1 to H6 plus commutativity.

Thus, the extensive scalability of a finite extensive domain on the basis of a finite amount of experimental information is independent of the Archimedean axiom. This means that the measurability of finite domains on the reals does not confirm Archimedes' axiom, as it can be obtained without assuming it.

For this reason, one may even think that real measurability is a consequence of finiteness, which needs not hold if rich, infinitely large extensive domains are considered. The upshot of all this is that Archimedeanity does not seem to be an assumption which is forced or suggested by experience, since it is not needed in the measurement of finite settings. It is therefore natural to qualify it as an idealization, not unlike Dedekind completeness, from which it follows. However, Archimedeanity does not behave like the other idealizations of extensive measurement.

3. Idealization and alternative extensions

Assume axioms H1 to H6 of Hölder's theory[3]. Any Archimedean model of these axioms can be extended to a model of Hölder's theory. If we consider Archimedes' axiom as an idealization, we may want to justify its introduction on the basis of H1 to H6. A difficulty arises at this point, in view of the discussion of section 2.

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[2] The original result, for which cf. Adams et al.1971: 397–398, is formulated in a different manner and for a different axiom system (namely, a modified version of the axioms of Suppes 1951).

[3] The following discussion can be generalized to weaker sets of axioms, but this requires technical digressions which are best avoided here.

To see why, contrast this case with that of the other idealizations of extensive measurement. In the case of closure, it is possible to start from non-closed extensive structures, prove that they are locally defined and take closure to be a generalization of a local property which they already possess. Dedekind completeness can be dealt with similarly, as a generalization of the Archimedean behaviour of non-complete extensive structures. Finally, something analogous is true of empirical equivalence, whose transitivity arises asymptotically from non-transitive extensive structures.

With Archimedeanity the situation is different because one may insist that the impression that something like a local form of Archimedeanity holds, and can thus be generalized to the full idealizing hypothesis, is only produced by the fact that in practice we only have to do with finite extensive structures. As a consequence, if we were to provide a theoretical description of infinitely large structures, Archimedes' axiom might perfectly well fail for them[4].

For this reason Archimedes' axiom can be taken as a conjecture concerning the global properties of extensive structures, rather than a generalization of their behaviour, as happens with the previously examined idealizing hypotheses.

Dropping the latter hypotheses does not change, in presence of the Archimedean axiom, the axiomatic characterization of extensive structures. Furthermore, this move does not interfere with their numerical measurability. On the contrary, dropping or retaining the Archimedean axiom means to drop or retain a conjecture about the structure of extensive systems. As a result, according to whether the conjecture is accepted or not, one obtains different characterizations of extensive structures and of their measurability.

If Archimedes' axiom is not presupposed, a non-classical concept of extensive magnitude can be developed, which is consonant with the habit, not infrequent in theoretical physics, of assuming that certain quantities may vary by infinitesimal amounts (examples may be found e.g. in Segel 1991).

To clarify this point a bit, note that, if Archimedeanity is dropped, a unique real scale for a whole extensive structure will not in general be available: for instance, if there are infinitely large objects (which may happen if Archimedes' axiom is dropped), there are no real numbers to measure them, while the same holds for infinitesimals. However, it is possible to have 'local' real scales, measuring only the

[4] A defence of the theoretical possibility of taking even classical quantities like mass to be non-Archimedean can be found in Bernadete 1968 (see Kyburg 1968 for critical comments).

objects which are not infinitesimal or infinitely large relatively to each other: objects which are so behaved may be intuitively understood as 'orders of infinity' within a non-Archimedean structure (a rigorous formulation of these ideas can be found in Narens 1974: 381–386). Even in presence of local real scales, though, objects which are infinitely close quantitatively are identified in the sense that they are assigned the same real measure. In this sense a real measurement scale does not precisely reflect the features of the underlying extensive structure, because it identifies quantitatively different objects when their difference becomes 'negligible'.

For this reason, if Archimedeanity is dropped there arises the theoretical possibility that real scales only capture certain features of extensive domains, but miss others. Without Archimedeanity, a discrepancy arises between extensive structures and their real measures, since the latter, endowed with order and addition, must be Archimedean. This discrepancy shows that there are different ways in which an empirical setting may be idealized, i.e. there are idealizations selecting alternative frameworks within which empirical settings can be embedded.

These idealizations determine a space of possible theoretical choices, depending on whether they are postulated or not. Archimedeanity is a case in point. The notion of idealization as enrichment is not disturbed by this conclusion, since the concept of Dedekind completeness for ordinary extensive structures has its non-Archimedean counterpart, called Archimedean completeness[5], which was originally formulated by Hans Hahn (for a clear an informative discussion of the main ideas and results obtained by Hahn and their subsequent generalizations see Ehrlich 1995).

4. Choice of extension

The foregoing discussion of Archimedes' axiom has shown that certain idealizations constrain in a particular way the empirical concepts they are used to characterize. With the Archimedean axiom in place, extensive structures can be exactly described

[5] This definition can be formulated for ordered Abelian groups, of which extensive structures are the positive part. An ordered Abelian group is said to be Archimedean complete if it does not admit any proper Archimedean extension. Given a group G, a proper Archimedean extension of G is a group H such that there is some x in the domain of H but not in that of G and for which there exists some y in G such that $ny > x$. In other words, Archimedean extensions add elements which are finitely comparable with some element of the structure they extend. Archimedean completeness then says that there is no way of extending a structure other than by adding to it new orders of infinity, i.e. elements which are either infinitesimal or infinitely large relative to any element of the initially given structure. If one partitioned a structure into classes of finitely comparable elements, Archimedean completeness would amount to the fact that each class is Dedekind complete.

by the reals, in the sense that quantitatively different objects are assigned different numerical measures. Without this idealization, quantitatively different objects which are infinitely close are assigned the same numerical measure and thus they are not distinguished by means of the real numbers. The resulting numerical description is 'inexact' in the sense that it does not preserve certain quantitative differences.

Nonetheless, with or without an Archimedean axiom, an extensive domain can be embedded into a richer one, with the difference that in one case the embedding leads to a system isomorphic to the positive reals with order and addition, while in the other case it may lead to a different kind of structure, technically an Archimedean complete, non-Archimedean structure.

For this reason a problem arises concerning the choice of an Archimedean or a non-Archimedean framework as the setting of extensive measurement: to each choice corresponds a distinctive ideal characterization of extensive structures. Since the Archimedean axiom is untestable, the choice between assuming it or not won't be solved by purely experimental means. It is also doubtful that it can be motivated by the actual use of approximation procedures which are consistent with Archimedeanity and have not so far revealed any violation of that property. Such a motivation, as it stands, is not particularly strong, in the light of the results of section 2. One could always deal with finite extensive measurement by means of systems of linear inequalities, thereby avoiding any appeal to Archimedes' axiom.

The assumption of this axiom as an acceptable conjecture can nevertheless be supported in a subtler and more convincing way. It was pointed out in chapter 2 that Archimedes' axiom is equivalent, in presence of the other axioms of extensive measurement, to T-separability, i.e. to the sufficiency of atomic formulas to discriminate any two quantitatively different objects. This property fails for non-Archimedean structures.

To see why, consider an extensive structure for which Archimedes' axiom fails and which, in particular, contains infinitesimals. Suppose x is taken as the unit of measure. Then, if a is infinitesimal relative to b, this means that, for any positive integer n, $na <_L b$. In this case, if we try to form the T-type of b relative to a, this is going to contain the whole T[6]. By closure, $b +_L b$ exists and it can be shown that its T-type is T as well. So b and $b +_L b$, are different but they cannot be discriminated by

[6] A proof of this fact can be found in fn.5 to chapter 2.

atomic formulas. A larger set of formulas is needed in order to achieve this objective for extensive structures which are not Archimedean.

This means that Archimedeanity is equivalent to the fact that a minimal set of formulas suffices to make quantitative discriminations (for, when it is dropped, more formulas are required to the same end).

From this point of view, Archimedeanity may be assumed because it is equivalent to the possibility of determining quantitative differences in the simplest possible way, i.e. using a 'minimum' of empirical information[7]. Thus, to assume Archimedes' axiom amounts to deciding to rely on a minimum of empirical information to describe an extensive domain, unless experiment itself or independent physical hypotheses, possibly of a theoretical character, suggest otherwise. From this point of view it is plausible to characterize extensive structures as Archimedean.

5. Types of idealization

In this chapter I have discussed the possibility of dropping the Archimedean axiom from the theory of extensive measurement, showing that such a move would open the way to characterizing extensive magnitudes in a non-classical fashion, even though one compatible with the existence of real scales. If, in view of its untestability and of the fact that it approximates Dedekind completeness, the Archimedean axiom is taken as an idealization, then retaining or dropping this idealization influences the type of concept of extensive magnitude one may assume.

It is not so for the infinitary idealizations of extensive measurement, like closure, which only have to do with the richness of a certain empirical domain but not with its structural features. A slightly different situation arises for regularity idealizations, which have to do with the structural features of an empirical domain but do not generate alternative conceptualizations thereof, insofar as the structural features they postulate may be seen to arise, by a limiting process, from less idealized empirical domains.

[7] This is the case of classical quantities like mass or length, whereas for example in special relativity velocity violates Archimedeanity, since the velocity of light c behaves like an infinitely large quantity, which is only asymptotically approached by the other velocities. Given a velocity $v < c$, nv is at most c, since c is maximal, and thus there is no n such that $nv > c$, against Archimedeanity.

These distinctions suggest a classification of idealizing hypotheses into different types, according to their role: two of these types, namely infinitary and regularity idealizations, have already been exemplified in the previous chapters.

In general, infinitary idealizations extend local or partial properties of an empirical setting to total ones. For example, closure in extensive measurement arises from non-closed structures and extends a locally defined operation to a totally defined one. Similarly, Dedekind completeness arises from Archimedean structures and extends a partial property (i.e. the convergence of Archimedean approximations) to a total one. The extensions thus performed are accompanied by a form of enrichment, since the properties of closure and Dedekind completeness are satisfied by enlargements of non-closed, Archimedean extensive structures.

Distinct from infinitary idealizations are regularity idealizations, which emerge from the asymptotic behaviour of empirical structures. The transitivity of empirical equivalence in extensive measurement provides an example. This property arises as extensive measurement with thresholds tends to exact extensive measurement.

Note that, in view of the discussion in chapter 4, section 6, transitivity is not really a partial property of extensive structures which becomes total as idealization is introduced. This is because, as the convergence process leading to the idealization is carried out, certain objects which were empirically equivalent in presence of sufficiently large thresholds cease to be so. Thus, the relation of empirical equivalence is not in general preserved under convergence (unlike e.g. local definiteness when it is extended to total definiteness in the case of closure): nevertheless, it ultimately, i.e. in the limit, turns into a transitive relation.

To regularity idealizations and infinitary idealizations it is possible to add, in view of the remarks made in this chapter, a third type of idealization, which may be called *structural idealization*. A structural idealization is one constraining the global properties of an empirical setting and thus the structure of its enrichments. Archimedes' axiom exemplifies this kind of idealization because it can be understood as an assumption about the global behaviour of extensive structures (concerning the finite comparability of their elements) which forces them, when assumed, to have enrichments which are isomorphic to the real continuum. On the other hand, if Archimedes' axiom is dropped, the global behaviour of extensive structures is no longer constrained by it, and they have enrichments which are not isomorphic to the real continuum.

Each of the three types of idealization I have described can be introduced within a weaker empirical theory on the basis of a distinctive motivation. Infinitary idealizations are introduced as generalizations of partial properties, regularity idealizations are introduced as asymptotic extrapolations while structural idealizations can be introduced on grounds of simplicity.

Using the classification of idealizing hypotheses I have just given, it is possible to describe idealization as a process of enrichment, which is performed in stages on some hypothetical empirical settings. Structural idealizations are added to it to constrain its global properties; subsequently, infinitary idealizations generalize some of its local features; finally, regularity idealizations add to it asymptotic properties.

This picture delivers an exhaustive characterization of idealization not only for extensive measurement but also for the theories which can be reduced to it, and thus in particular the ones studied by Field.

The open question which remains to be addressed is now not one about the nature of idealizations but one about the reason why idealizations are used. This question will be addressed in the next chapters and will complete the account of idealization I set out to develop.

As I already remarked, the reason why I have deferred a discussion of this aspect of idealization is that it is closely connected with uses of applied mathematics which show the restrictiveness of Field's account of applicability. The motivations for idealization explored so far can simply be added to Field's account of applicability in order to complete it and supply it with the missing connection between idealized scientific theories and experiment. On the other hand, the explanation of the importance of idealization in science I am going to provide leads to considering ways of applying mathematics which call for a revision of Field's account. The next chapters are devoted to providing this revision.

PART II: APPLICABILITY AND MATHEMATIZATION

CHAPTER 1: THE NATURE OF APPLIED MATHEMATICS

1. The issue

In the previous chapters I have given an account of the way idealizations arise in extensive measurement, which in principle generalizes to the scientific theories discussed in Field 1980.

On this account idealization can be seen as a certain form of integration of empirical settings into richer frameworks extending them. I have already mentioned that, in order for an account of idealization to be complete, it is necessary to explain why idealizations are used in science, not only how they can be justified and how they arise. This is a problem I will address in the following chapters, especially the next one. In short, my view is that idealizations are used mainly because certain results of importance for empirical investigations can be proved in presence of them but not without them.

For example, the introduction of idealizations in extensive measurement makes it possible to think of extensive structures as additive continua, isomorphic to the positive reals with order and addition. Once idealizations are in place, certain strong mathematical properties of continua, like Dedekind completeness, become available to study measurement settings. It can be shown (as I will do in chapter 2 of this part) that strong, ideal properties of this kind are needed to obtain a generalization of the concept of measurability beyond the extensive case and to isolate axioms which can be used to construct theories of measurement for non-extensive settings (this is illustrated in chapter 3 of this part). These results, as I will make clear, are possible precisely when idealizing hypotheses are present, but cannot be established without them. Therefore, the importance of idealizations lies in their function as a guide to theoretical developments in the sciences.

This function is often performed in a rather abstract fashion. In the case of extensive measurement, extensive structures are assimilated, under idealization, to certain mathematical structures (i.e. additive continua): as a consequence, their general properties are investigated by mathematical means. In this context, the use of mathematics appears to be crucial: without it, a general analysis of the properties of extensive structures would not be available.

Such a conclusion conflicts, as I will explain in what follows, with Field's characterization of the role played by mathematics in applications, which depends on his nominalistically motivated treatment of scientific theories. This treatment leads to the elimination of mathematics from these theories (through their reformulation in non-mathematical terms) and is used by Field to show that the role of mathematics in applications is not essential.

The account of the fruitfulness of idealization I am proposing here leads to a different view. Idealizations are important because they make scientifically fruitful mathematical investigations possible: the relevant mathematics has to be in place and cannot be eliminated if these investigations are to be carried out (I will say more on this point later). Thus, my perspective is in contrast with Field's view concerning the actual importance of mathematics in applications.

If the point of view I propose is correct, then Field does not offer a realistic account of the applicability of mathematics. In the following chapters I will offer evidence for this claim, by looking at several examples from actual scientific practice. Once I have shown that Field's account of applicability is not adequate, I will proceed to develop an alternative one which generalizes his and proves more realistic (this is articulated in chapter 5). In this chapter I focus on illustrating Field's characterization of applied mathematics, in order to identify in outline its main limitations, which I will subsequently proceed to discuss in detail.

2. Field's characterization of applied mathematics

In a nutshell, Field 1980 identifies two main functions of applied mathematics:

1) A linguistic one: mathematical theories are used to formulate scientific theories (Field 1980: 24, 33, 42)[1];
2) A deductive one: mathematical theories are used to speed up deductions within scientific theories, which should be in principle possible without them (see Field 1980: 11, 15, 28[2]).

Field's nominalistic program may be seen as an attempt to show that mathematics is dispensable (and thus eliminable) from science by showing that neither of its scientific uses is really essential to the development of a scientific theory. Since the nominalistic project of eliminating mathematics guides Field's account of

[1] 'abstract entities are useful because we can use them to formulate abstract counterparts of concrete statements' (Field 1980: 24). And later: '[physical theories] Insofar as they've been rigorously formulated at all, they've been formulated mathematically, for it is *easier* to formulate a theory that way when one has a sufficiently developed mathematics' (Field 1980: 42, italics in the original).

[2] Especially Field 1980: 11 reads: '*the conclusions we arrive at by these* [mathematical] *means are not genuinely new, because they are already derivable in a more long-winded fashion from the premises, without recourse to the mathematical entities*' (italics in the original).

applicability, the latter ends up attaching a rather limited importance to mathematics in applications. More explicitly, because Field's nominalistic objective is to eliminate mathematics from scientific theories, he constrains the uses of mathematics in science to be only eliminable ones. As a consequence, mathematics cannot play any fundamental role in directing empirical investigations.

Uses (1) and (2) above fit this picture, once one bears in mind that the objective of Field is to reformulate scientific theories in non-mathematical terms (so that (1) is shown to be an eliminable use of mathematics) and to reconstruct in non-mathematical terms the main mathematical, deductive techniques applied to those theories or to prove that they can in general be replaced by non-mathematical deductions (so that (2) is shown to be an eliminable use of mathematics).

What I will claim in this chapter and establish in the next four is that there are uses of mathematics which are crucial for empirical investigations, against Field's tendency to regard them as eliminable[3]. In the next sections I will make the last remarks more precise. In particular, sections 4 to 6 are devoted to explaining why I think there are uses of mathematics which cannot be eliminated from science[4].

Moreover, in sections 7 and 8 I outline some examples of applications to which the representational approach adopted by Field (i.e. based on proving representation theorems) does not apply[5]. Since Field describes the applicability of mathematical models using representation theorems, the fact that mathematics may be applied without them implies that his explanation of the applicability of mathematics is too restrictive. As a result, his account rules out both certain forms of applications (the ones not based on representation theorems) and certain scientific uses of mathematics

[3] I need to point out in this connection that my focus here is only on Field's account of applicability, so I am not trying to establish a form of indispensability by pointing to uses of mathematics which Field's strategy cannot eliminate. It will become clear in the next chapters that my remarks on the uses of applied mathematics are in general compatible with nominalism, although they require a revision of Field's nominalistic strategy.

[4] A similar conclusion is reached in Lyon & Colyvan 2008 (which formulates objections to Field) about the explanatory power of the mathematics of phase-spaces. Eliminating the mathematics would decrease the explanatory power of the physical theory to which they are applied, even though the physical theory can be nominalized. Especially in chapter 3 I reach conclusions that are similar to those of Colyvan & Lyon in some respects, but stronger. This is because I show that there are situations in which an elimination of mathematics would imply the elimination of the very empirical theory one is trying to construct. In this context Field's eliminative strategy becomes highly problematic.

[5] These facts do not necessarily provide support for mathematical realism. This is because they do not necessarily entail ontological commitment to mathematical entities (for instance the mathematical concepts involved may have a fundamental role in the metatheory of some scientific theory, but one of a purely syntactic, deductive nature. Note that Field does not eliminate mathematical deductions at the metatheoretical level). They only show that Field's treatment of scientific theories does not generate a realistic account of applicability.

(the ones which are not eliminable) which are nonetheless present and important in scientific practice. This calls for a revision of Field's account of applicability, which I will elaborate after I have given, in this chapter, an overview of the problems affecting it.

Before doing all this, however, I need to refine the characterization of the uses of applied mathematics I have ascribed to Field at the beginning of this section and also to provide a more detailed account of the strategy he pursues to establish their eliminability. Then I will show that even the refined version of Field's analysis of applicability faces the difficulties I have just pointed out.

3. Field's account in greater detail

Field's analysis of the scientific use of mathematics is actually more complex than the one summarized in (1) and (2) above. To see why, it is necessary to look in some detail at the structure of his nominalistic strategy.

As I have already observed, this strategy establishes the inessential nature of (1) by axiomatizing the relevant scientific theories in such a way that the axioms do not contain any mathematical reference. In addition, Field shows that the non-mathematical axioms are such that their models can be treated in the usual mathematical fashion. Thus, the axioms are adequate to replace the ordinary mathematical characterization of the empirical structures they describe: this result is obtained by means of representation theorems. Thus, the eliminability of (1) is based on the axiomatization of an empirical theory and the proof of a representation theorem. On the other hand, one strong way of establishing the inessential nature of (2) is to prove that mathematical theories are *deductively conservative* over the scientific theories they are applied to. This means that, given a nominalistically formulated scientific theory T (i.e. a set of empirical axioms) and a mathematical theory M, any statement in the vocabulary of T which can be proved using T plus M can in fact be proved from T alone.

However, this notion of conservativeness is not the one ultimately embraced by Field and thus (2) above must be modified accordingly. The reason is that the property of deductive conservativeness fails for the scientific and mathematical theories in Field 1980. This has been shown in Shapiro 1983. The alternative, if these theories have to be retained, is to resort to *semantic conservativeness*: this property amounts to the fact that any nominalistically formulated (i.e. empirical) consequence of M plus T is a consequence of T alone[6]. A proof of this form of conservativeness is sketched in Field 1980 (appendix to chapter 1) and it is this semantic property which Field has embraced on the basis of the criticisms of Shapiro 1983. In fact Field 1985 explicitly remarks that:

> 'mathematics is useful because it is often easier to see that a nominalistic claim follows from a nominalistic theory plus mathematics than to see that it follows from the nominalistic theory alone' (Field 1985: 241).

As a result, (2) of section 2 should be modified as follows:

> 2′) Mathematics has a deductive use in applications: this consists either in shortening proofs which could be carried out without it, or in deducing consequences of nominalistically formulated theories which follow from these theories alone, although they may not be provable from them without resorting to mathematics (this is precisely because deductive conservativeness fails, while semantic conservativeness holds: mathematics does not extend the empirical content of nominalistically formulated theories, but may be necessary to make it explicit by providing proofs).

Note that (2′) is stronger than (2), since it includes it. When semantic conservativeness holds but deductive conservativeness fails mathematics is not proof-theoretically dispensable because it may be necessary to actually carry out certain deductions. This may be a concern for Field's nominalistic project, insofar as it points to a difficulty in fully eliminating mathematics from scientific theories. Such a concern is not however the focus of my attention. What I am interested in is the particular account of applicability and of the scientific uses of mathematics Field develops. As I will point out in the next sections and show in detail in the following chapters, Field characterizes the role of mathematics in applications in an unnecessarily restrictive way, because he reduces it to (1) and (2′). There are uses of applied mathematics which are of significance and cannot be reduced to any of (1) and (2′): I will now say more about this point.

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[6] Deductive conservativeness follows from semantic conservativeness if a completeness theorem holds. However, the logical apparatus employed in Field 1980 does not satisfy completeness and thus the equivalence of the two forms of conservativeness does not follow.

4. A first limitation of Field's account

From the point of view of the analysis of applicability[7], Field's nominalistic strategy has a distinctive advantage. This advantage has been already explored in part 1 and it is related to the representational approach Field pursues. Thanks to it, Field is able to reformulate scientific theories in such a way that their empirical basis becomes perspicuous. At the same time, the fact that mathematics can be meaningfully applied to these theories becomes transparent. This is because their models can be shown to be structurally similar to certain mathematical models by means of a representation theorem. Structural similarity implies that it is possible to study the formal properties of certain mathematical models in order to gain information about corresponding empirical settings. This perspective, although mainly motivated by nominalistic concerns, provides some important insights into the applicability of mathematics.

At the same time, another aspect of Field's nominalistic strategy proves disadvantageous when the problem of delivering a realistic account of applicability is taken as fundamental. The disadvantage is that Field seeks to systematically and uniformly eliminate mathematical references from scientific theories. Thus, for instance, an appropriate nominalistic description of a scientific theory must not, according to him, contain any reference to functions or numbers[8].

The problem is that sometimes the application of mathematical concepts like that of function consists in finding for them an empirical, possibly idealized, interpretation, so that their mathematical properties can be used directly to characterize the empirical setting on which they are interpreted.

In other words, there are situations in which mathematical concepts are used to model empirical settings. As long as they refer to features of these settings, they are not harmful to the mathematical nominalist. However, a nominalistic strategy based on the elimination of mathematical references as a general methodological principle is almost forced to ignore or fail to recognize these cases. In particular, it fails to

[7] As a separate problem from mathematical anti-realism, which I am not discussing here.

[8] This requirement may be related to the fact that, as Shapiro observes, 'Hartry Field takes mathematical language at face value' (Shapiro 2000: 226). If this is the case, then Field must think that any occurrence of a mathematical reference within a scientific theory should actually be understood to denote a particular kind of abstract object, e.g. a set. Such a standpoint motivates, from a nominalistic point of view, the systematic elimination of mathematical references. However, as far as my discussion is concerned, the important thing is that (independently of what the motivation for it is) Field does in fact try to achieve the systematic elimination of mathematical references from scientific theories. It is this feature of his nominalistic strategy which proves problematic, as far as the problem of describing the applicability of mathematics is concerned. Because of the particular kind of strategy Field pursues, he is led to neglecting certain applicative uses of mathematics of importance.

acknowledge that mathematical references are sometimes eliminable and at other times interpreted over (possibly idealized) empirical settings and thus not eliminable.

For example, in presence of references to a family of continuous functions, endowed with some additional algebraic structure, Field's strategy requires their elimination[9]: however there are empirical situations which are described precisely by means of families of functions like the ones just mentioned. I will discuss this point in detail in chapter 3, which illustrates how certain groups of continuous functions prove crucial in psychophysical measurement to describe the responses of an experimental subject. For this very reason it does not make sense to eliminate them, because they are an integral part of the characterization of a type of empirical setting[10]. This means that, in certain situations, the issue of eliminating mathematical references cannot meaningfully arise.

Because however it always arises when Field's strategy is in place, it follows that the latter cannot take into account certain meaningful applications of mathematics. In other words, Field's understanding of applied mathematics is, as it stands, too restrictive.

5. Mathematics and scientific theory-construction

As I have observed in the previous section, there are applications in which mathematical concepts receive an empirically meaningful interpretation and are used as essential ingredients in the construction of scientific theories. The precise way in which this happens will only be explored later: here, however, I wish to give a general impression of it. This will serve the purpose of clarifying the fact there exist scientific uses of mathematics which considerably differ from the ones isolated by Field.

A good example to illustrate my point is provided by measurement in psychophysics. In this context the results of certain experiments are ordinarily described numerically, while it is not clear whether the numbers produced by the

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[9] This is stated in general in Field 1980: 1, where functions are classified among the abstract entities the nominalist seeks to eliminate.

[10] Note that Field eliminates some references to functions, e.g. to the scalar field describing the gravitational potential in Field 1980: 63 and ff. The fact that there are references to functions which cannot be eliminated causes a difficulty for Field's strategy, although not necessarily for mathematical nominalism. This is because these references may be endowed with an empirical interpretation. If so, however, the concept 'function' cannot be classified as one to be eliminated from scientific theories, as Field does. As a result, his strategy cannot be uniformly applied and may prove unnecessary when the application of certain mathematical concepts consists in using them to model an empirical setting. I will get back to these points more explicitly in chapter 3.

experiments measure anything and, if so, how empirically informative they are. The problem is to justify the belief that we can capture certain features of psychophysical experimental settings by means of numerical models. One way to solve this problem consists in proving a representation theorem determining a correspondence between a psychophysical setting and a numerical one.

To this end, it is necessary to determine empirical axioms delivering the desired representation theorem, i.e. to build up a scientific theory which is sufficient to establish it. We thus have a theory-construction problem, which is solved by isolating a suitable system of axioms. Mathematics has been used in psychophysics precisely to solve this problem, namely to identify empirical assumptions establishing a certain form of psychophysical measurement. In particular, this has been done through the integration of suitable mathematical concepts and properties into an empirical theory.

This use of mathematics differs from the (eliminable) ones Field considers. It is different from (1) of section 2 because the problem is not that of mathematically formulating a scientific theory but that of determining the axioms of an *empirical* theory. It also differs from (2′) of section 3 because the problem is not that of proving theorems from a scientific theory but that of isolating axioms, which are used to establish a metatheorem (i.e. a representation theorem).

Psychophysical axioms have been found by means of mathematics, in presence of idealization: for this reason, the construction of a psychophysical theory of measurement provides an example of the main reason why idealization is scientifically fruitful. It is on its basis that certain theories can be articulated and certain applicative problems solved. In particular, idealizing hypotheses make certain essential (i.e. non-eliminable) applicative uses of mathematics possible.

This point can be briefly illustrated for psychophysical measurement. In order to find empirical axioms for it, a three-stage strategy has been pursued[11]: firstly, in presence of the idealizations of Hölder's theory, (i) certain abstract mathematical properties of extensive scales of measurement have been isolated. Subsequently, it has been proved that (ii) the mathematical properties so isolated are satisfied by certain mathematical models (groups of functions) which have numerical representations. Finally, a psychophysical axiom system has been constructed, based

[11] To be described in detail in the next two chapters.

on the mathematical analysis of (i) and on the possibility of finding an empirical interpretation for the mathematical models isolated in (ii).

Stage (i) relies on an analysis of the mathematical properties characteristic of an idealized theory: some of their consequences are studied mathematically in (ii) and finally the properties themselves are 'immersed' into an empirical theory. This process results in the formulation of an axiom system, which is then used to investigate a type of empirical setting. What is crucial is that axiomatization, together with the corresponding proof of representation, is based on the mathematical concepts extracted from (i) and (ii): these concepts are directly employed to produce an empirical theory and to identify the provability conditions of its related representation theorem. Since some axioms of the empirical theory are untestable, they would not be reached without resorting to mathematical means. In addition, because they incorporate properties of a mathematical character (they contain references to functions and groups), an eliminative approach to scientific theories would get rid of these axioms, and thus of the empirical theory they articulate.

The example from psychophysics just outlined will later be discussed at length, because it provides a simple but striking instance of the conceptual centrality of mathematics in applications. This example shows that Field is overlooking, when developing his account of applicability out of his nominalistic, eliminative program, some important scientific uses of mathematics, typically empirical theory-construction. It is noteworthy that Field himself extensively uses mathematics to carry out theory-construction in Field 1980, in a way entirely similar to the one just described.

Field uses mathematical concepts and theorems in a creative way to frame his axiomatic reformulation of gravitation theory. He starts from abstract mathematical properties and, in view of their logical connections, endows them with an empirical interpretation over space-time, in order to find the nominalistic axioms he needs. The use of mathematics involved in this process differs from the ones Field describes in the same way in which the use of mathematics in psychophysics surveyed above did. I will now provide some evidence for the fact that Field actually employs mathematics in the way I have just described.

6. Theory-construction in Field 1980: an example

In order to nominalize gravitation theory, Field lays down a system of non-mathematical axioms characterizing space-time, together with a system of non-numerical, measurement-theoretic axioms characterizing a quantity, the gravitational potential, varying over space-time. From these axioms he can prove two representation theorems showing respectively that numerical coordinates can be introduced over space-time, and that a scale of measurement can be set up for the gravitational potential. In view of these two representation theorems, it is possible to establish the existence of a numerical function which maps numerical coordinates into numerical measures: this function is called a scalar field and it essentially measures the intensity of the potential at each different position in space-time.

Usually, such a function is assumed to be continuous[12]. Because of this, Field needs to be able to formulate this continuity requirement in terms of the primitives of his nominalistic theories of space-time and measurement. More precisely, he needs to formulate a condition, in terms of these empirical primitives, which is sufficient to strengthen the representation theorem mentioned above. The reason is that a proof of the fact that geometry can be coordinatized and the gravitational potential scaled is already available: this proof implies that a scalar field is defined. What is needed is some additional nominalistic condition which shows it to be, in addition, continuous. In order to achieve this result, Field has to make crucial use of mathematical concepts, in particular topological ones.

To see why, consider a scalar field on three-dimensional space (the generalization to n dimensions is straightforward), that is, a three-place numerical function f such that, if x, y, z are the real numbers determining the coordinates of a certain point in space, we have $f(x, y, z) = r$, where r measures a quantity, e.g. the gravitational potential, at that point. Scalar fields like f may be assumed, as in the case which interests Field, to be continuous functions of their arguments. A way of formulating continuity for a many-place or multivariate function can be seen quite clearly if one starts from the classical case of a function of one real variable. In that case, Weierstrass' definition can be used, stating that the function $g(x)$ is continuous at point a when the following holds:

[12] This is due to the fact that one of the laws governing the gravitational field requires the differentiability of the scalar field and thus its continuity.

(C) for any number $\varepsilon > 0$ there exists a number δ such that, when $|x - a| < \delta$, it is also the case that $|g(x) - g(a)| < \varepsilon$.

If g is continuous at any point, then it is simply said to be continuous. The above definition asserts, intuitively, that, whenever x is sufficiently close to a, the corresponding values of the function g, namely $g(x)$ and $g(a)$, get correspondingly close. Alternatively, one might say that, as the difference $x - a$ tends to zero, the difference $g(x) - g(a)$ tends to zero as well. But this can be expressed in terms of limits as:

(C_L) g is continuous at a if the limit of $g(x)$ as x tends to a is $g(a)$[13].

Note that the definition (C) of continuity is metrical, i.e. it makes use of a concept of distance, because it presupposes the possibility of specifying how far apart x and a, or $g(x)$ and $g(a)$, are: the absolute values of their differences measure their distance. Now, if the continuity of a function like $f(x, y, z)$ at point (a, b, c) is to be defined, (C) above can be adapted to it by simply stating that:

(C_3) for any $\varepsilon > 0$, there are δ_1, δ_2, δ_3 such that, whenever $|x - a| < \delta_1$, $|y - b| < \delta_2$ and $|z - c| < \delta_3$, also $|f(x, y, z) - f(a, b, c)| < \varepsilon$.

Considering $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and taking the Euclidean[14] metric d on R^3, it is possible to rewrite (C_3) in the same form as (C). It suffices to say that, for any $\varepsilon > 0$, $d((x, y, z), (a, b, c)) < \delta$ implies $|f(x, y, z) - f(a, b, c)| < \varepsilon$. This definition is the three-dimensional analogous of (C) and the four-dimensional case for space-time is treated in an entirely similar manner. The definitions used so far are metrical and this poses a problem for Field, because the only primitive concept he uses in his space-time theory and in the theory of measurement which describes the gravitational potential is the non-metrical concept of betweenness[15].

Thus, the question which arises is whether it is possible to formulate a continuity requirement using a physical geometry and a theory of measurement without metrical primitives. An affirmative answer to this question can be given, and Field exploits it to reach a non-numerical axiom equivalent to the continuity condition (C) (or its analogous, higher-dimensional counterparts) but independent of any metrical concept. In order to understand how Field reaches this result, it is necessary to

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[13] Technically, the definitions (C) and (C_L) are equivalent in presence of the countable axiom of choice.

[14] That is, based on Pythagoras' theorem.

[15] Geometrically betweenness is a ternary relation holding among triples of points if they are collinear and one of them lies between the other. For example betweenness holds of a, b, c if b lies on segment ac.

consider the mathematical concepts and theorems he implicitly makes essential use of.

First of all, the continuity condition (C) can be proved to be equivalent to a non-metrical, numerical condition (C_T). To see what (C_T) is, consider the statement of (C)[16] and, in particular, the inequality: $|x - a| < \delta_1$. This is equivalent to:

$$a - \delta_1 < x < a + \delta_1,$$

i.e. to x belonging to the open interval Y = $(a - \delta_1; a + \delta_1)$.

By the same clue, the inequality $|g(x) - g(a)| < \varepsilon$ says that $g(x)$ lies in the open interval X = $(g(a) - \varepsilon; g(a) + \varepsilon)$. The upshot of all this is that (C) can be seen as a condition relating open intervals. Note that, reasoning geometrically, if $f(x)$ is plotted against x, Y is an open interval on the x-axis and X an open interval on the $f(x)$-axis. Now it can be proved[17] that (C) is equivalent to:

(C_T) for any open interval X on the $f(x)$-axis, f^{-1}(X) is an open interval on the x-axis.

Here f^{-1} is not the inverse of f, which may not exist if f is not injective, but only the set of all x in Y such that $f(x)$ is in X.

Definition (C_T) is as strong as (C) but can be formulated without a reference to metrical notions, for it only refers to the concept of open interval (an open interval (a, b) is just the set of points greater than a and smaller than b: order suffices to define it). Even though (C_T) is a numerical definition, it can be generalized to the topological definition of continuity, which involves functions joining what intuitively may be seen as two systems of open sets, called topological spaces[18]. Thus, given two topological spaces Σ and Σ', and a function f from Σ to Σ', f is said to be continuous if:

(C_Σ) for any open set X of Σ', f^{-1}(X) is an open set of Σ[19].

The definition (C_Σ) of continuity is a formal, non-numerical notion ('open set' may be taken as a primitive concept in the axioms for a topological space), and thus it can be interpreted on the ordered empirical structures Field works with, since their order can be used to define open regions of space-time, i.e. the open sets of a topology.

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[16] An entirely analogous reasoning goes through for the multivariate case, with the difference that it is based on the metric d rather than on the absolute value metric.

[17] See appendix 1 for a proof.

[18] A topological space is semantically defined as a set S, taken together with a family of subsets thereof, called open sets and assumed to contain the empty set, the set S, and to be closed under finite intersection and union.

[19] This definition does not presuppose open sets to have a prescribed dimension, so it can be used as the topological counterpart of numerical continuity in the multivariate case.

More precisely, Field uses a geometrical relation of betweenness to define one kind of open set on space-time and a quantitative relation of betweenness to define another kind of open set over space-time, i.e. one determined by the intensity of the gravitational potential.

It is crucial that both kinds of open sets, and so two different topologies, are defined on the same domain. This makes it possible for Field to use (C_Σ) to give his nominalistic definition of continuity without even having to refer to a function *f*. The nominalistic continuity condition Field reaches is that, for any space-time point *x*, any quantitative open region of space-time containing *x* includes some geometrical open region of space-time which contains *x*[20].

The point to be emphasized here is that the last nominalistic definition of continuity can only be reached because an abstract characterization of continuity, in topological terms, is available and because the concept of topological space is interpreted over space-time, i.e. integrated into it[21]. It is the presence of this mathematical concept which makes it possible for Field to find the empirical condition he needs.

A further point can be made to stress the essential role played by mathematics to yield this result. Even when (C_Σ) is in place and Field can exploit it to define continuity nominalistically, there is still no assurance that his definition is, as desired, equivalent to numerical continuity, *via* a suitable representation theorem.

To see why, consider (C) and (C_T) above: the latter is a numerical definition of continuity in terms of open sets. Its equivalence to (C) can be proved, but only in presence of the additional algebraic properties of the reals, in particular the metrical ones occurring in the definition of (C). On the other hand, the general topological definition of continuity (C_Σ) is essentially a formal analogon of (C_T): it is not obvious that, in presence of a numerical representation of the topological spaces described by (C_Σ), this condition should hold if and only if (C) holds on the numerical representation.

[20] To see why this corresponds to the topological definition of continuity, just look at (C_Σ) and fix x in an open set X. It follows that $f^{-1}(x)$ is in an open set of Σ. If Σ and Σ' have the same domain, we can avoid referring to functions and say that the fact that x is in an open set of Σ' implies that x is in an open set of Σ. Furthermore, since the image under f of the open set $f^{-1}(X)$ is included in X, we can require for topologies over the same domain that an open set of Σ' containing x includes an open set of Σ containing x. This is Field's nominalistic counterpart of continuity.

[21] Since what matters here is the topological structure of space-time, a structuralist like Resnik or Shapiro could discern here an essential use of mathematics to study an empirical setting.

This is in fact the case in a strong sense, because certain suitable conditions are satisfied by the topologies Field considers over space time[22]: when these are in place, (C_Σ) is provably equivalent to a condition similar to (C_L)[23], which is in turn provably equivalent to (C) (if countable choice is assumed). In fact, it follows in general that (C_Σ) implies a sequential continuity condition similar to (C_L), and this[24] is enough to constrain the behaviour of limits over the relevant topologies in such a way that it reflects that determined by the ordinary definition of continuity.

These remarks show that Field makes a substantial use of mathematical concepts (theorems about numerical continuity, theorems from topology) to isolate the axioms he needs to construct a theory of gravitation which is wholly non-numerical. The problem he has to face, i.e. finding a nominalistically acceptable counterpart of a given numerical property, is itself of a mathematical nature, because here 'counterpart' means that the equivalence between the nominalistic and numerical notion has to be proved *via* a representation theorem. Thus the problem is to find a definition of continuity from which a prescribed conclusion can be proved. Such a problem, involves a mathematical property, the need to prove several theorems and the incorporation of the mathematical concept of topological space into an empirical theory. The appeal to mathematical notions and theorems is essential throughout.

Therefore Field uses mathematics substantially for empirical theory-construction (in this case, to frame a suitable empirical continuity axiom). But this use of mathematics is the same as that sketched in the previous discussion of psychophysical measurement. In general, it is present in applications whenever an axiomatization of a class of empirical settings is looked for, which can be used to show that they have suitable mathematical models, satisfying desired mathematical properties. Under these circumstances, it seems plausible to conclude that mathematics is used in an essential way[25], not to prove theorems or to formulate a

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[22] In particular the topological assumption called the first axiom of countability is needed. For the technical details see the appendix

[23] For a proof see e.g. Crossley 2005: 13.

[24] For a proof see e.g. Pfanzagl 1968: 65–66.

[25] Whether this may pose a challenge to a nominalist like Field is certainly an issue of interest, but it does not constitute the main focus of my discussion. In this connection, however, I wish to point out that certain essential uses of mathematics, like the one described in this section, are not necessarily harmful to the nominalist. In the present example a crucial use of topological concepts and theorems was involved. It is important to stress that what is applied are concepts and theorems, i.e. that mathematics is applied in a conceptual fashion, in particular it is used to determine the logical connection existing between distinct definitions of continuity. These logical connections can then be exploited to formulate an empirical continuity axiom, in which case they receive an empirical interpretation. Mathematics thus works as a formal methodology adopted to investigate the structure

scientific theory, but to isolate empirical axioms which are sufficient to establish target metatheorems. Such a use of mathematics differs from the ones Field identifies and yet is pervasive in his own treatment of scientific theories. This evinces in a rather evident way that the account of applicability Field proposes is too restrictive, for it even rules out certain uses of mathematics which are important for Field himself. These very uses are equally relevant to scientific practice, as I will illustrate in detail when I discuss psychophysical measurement.

7. A representation-independent applicative problem

The limitation of Field's characterization of applicability I have so far described illustrates one of the two main reasons why his account of the scientific uses of mathematics is not comprehensive enough. There is another, independent reason why the same conclusion can be reached, which I now wish to explore.

In short, it amounts to the fact that Field's strategy requires the proof of representation theorems and thus deals only with forms of application in which the relation of structure-preservation plays a crucial role. A representation theorem establishes, roughly speaking, some form of structural similarity between an empirical setting and a mathematical model. As soon as certain inferences are carried out in the mathematical model, it is possible by structure-preservation to interpret them back into the empirical setting. In the case of extensive measurement, for example, numerical inferences involving addition and order can be interpreted back into empirical facts involving the quantitative behaviour of physical objects.

This style of reasoning is typical of all forms of applications relying on the concept of representation. To apply mathematics means in this context to study by mathematical means the properties of some mathematically representable empirical system.

Although this characterization of applicability is in many respects satisfactory, it is too restrictive, because not all applications of mathematics can be reduced to a representational paradigm. In other words, there exist applicative problems which are not tackled by means of structure-preservation and the associated style of reasoning: their solution is not obtained by drawing mathematical inferences within a mathematical model and later interpreting them on some empirical structure. Even if this is not the case for several scientific theories, there are families of empirically

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of an idealized empirical setting like space-time. Thus, even if mathematics cannot be eliminated from theory construction, the role it plays in it does not necessarily generate a worry for the nominalist.

important problems for which it happens. This is not easy to see in the abstract. Therefore, let me provide one example, taken from economics, of a type of applicative problem whose solution does not rely on inferences made on a mathematical model introduced by a representation.

Suppose a bank has a certain capital C to be invested in securities S and loans L whose interest rates are, respectively 5% and 10%. An important practical problem is that of deciding which proportions of C have to be invested in loans and securities in order to maximize the return of interest. Note that this problem does not start with the axiomatic characterization of any empirical setting to be represented: we just have amounts of money to manage. In particular $C = L + S$ holds if the whole capital is to be invested. The return of interest from C is:

$$\text{(i) } 0.1L + 0.05S$$

and the problem is to choose L and S in such a way that (i) is a maximum. This will generally have to be achieved under certain constraints. An obvious one is:

$$\text{(ii) } L \geq 0 \text{ and } S \geq 0,$$

while a possible one is:

$$\text{(iii) } S \geq 1/4\,(L + S) \text{ or, equivalently, } L - 3S \geq 0.$$

This means that at least one fourth of the total investment has to be in securities. Conditions (i) to (iii) are all numerical and empirically meaningful because they formulate certain basic financial requirements. They do not arise from the introduction of a model for an empirical setting. The problem at hand consists in finding numerical values for L and S which solve the system of constraints (i) to (iii). Note that, once these values are found, the problem is solved and the relevant investment can be made. There is no need to have a structure-preserving mapping to interpret the solutions to (i) to (iii) into some hypothetical empirical structure. In short, no relevant role is played by structure-preservation. This example is an instance of a general type of applicative problem called a linear program: a comprehensive discussion of linear programs will be given in chapter 5. Here I simply wished to show that a typical linear program clearly exemplifies a form of application which does not rely on representation.

Although the particular problem I considered may be solved in a relatively easy way, the same is not true of linear programs in general. Mathematics is applied to them to study their solvability conditions. In particular, it is applied to this effect without relying on a structural correspondence between an empirical and a numerical system, which is foreign to the nature of the problems involved.

8. More on representation-independent problems

The example in the previous section describes a form of application which is not of the type discussed by Field mainly because it is formulated and dealt with in the absence of an axiomatic framework characterizing an empirical setting and a representation theorem. Field's account of applicability, on the other hand, is based on representations, whose existence depends on the presence of certain empirical axioms and structure-preserving mappings. What matters in his approach is the axiomatic theory. One does not look at the individual models of the theory but only at the formal properties each of them satisfies, in order to establish representation theorems. Once the theorems are in place, applicability can be explained and justified the way Field wants it to be. Since representation theorems depend on empirical axioms and hold for the totality of their models, proving them does not require looking at the features of the empirical models of the axioms taken individually. If representations exist, then it is clear that, in principle, whichever empirical structure one considers, it can be studied through a mathematical model, if only it satisfies the right axioms.

The previous section illustrated the restrictiveness of this view, which does not take into account the fact that what matters to certain applications is solving a particular kind of problem rather than studying the features of a given empirical structure. Now I want to show that, even when what matters is the structure of an empirical setting, there are forms of applications which do not work in the representational way Field presupposes. In particular this is related to the fact that certain applications investigate the structure of *particular* empirical settings, rather than the general, empirical conditions which ensure their representability.

A good example is offered by archaeology, in particular by the problem of ordering chronologically a group of deposits on the basis of information concerning the artefacts they contain. What is needed is to reconstruct a specific empirical ordering of deposits, using a set of data which generally says which types of artefacts are present in which deposits. Note that the objective is to say, for a given family of deposits, which ones are older and which ones are more recent. A particular ordering of this family of deposits is sought. It is not sufficient to know that, e.g., chronological order is a transitive and connected relation, which may well be assumed: this would be enough if the only problem was that of deciding their numerical representability on a finite linear order. Yet the problem here is the different one of finding the particular way in which the deposits are ordered. An explicit representation could in fact be established only if the ordering was known. Since, however, it is not known and the objective is precisely to recover it, no representation can be employed to describe it, and thus mathematics cannot be applied to the present problem in the usual representational form.

Nonetheless, mathematics can be used, in a non-representational fashion, to find a chronological ordering of deposits. A possible strategy to do this consists in using a particular presentation of the known data about the artefacts present in the deposits. Suitable mathematical theorems can be invoked to act on it to solve the problem at hand[26]. More explicitly, the data on artefacts is arranged into a matrix and it is proved that, under certain empirical assumptions, if the matrix can be reduced to a certain canonical form, then the order of its rows in the canonical form yields the desired chronology. In this context no representational techniques are used: the reason is that there is no explicitly given empirical structure on the deposits to be modelled numerically. Rather, the problem is to deal in a mathematically helpful way with the given information about the distribution of artefacts in the deposits.

9. Concluding remarks

Even though the issues touched upon in this chapter and concerning the restrictiveness of Field's account of applicability have yet to be discussed in detail, I hope I have given a sufficiently precise overview of them. In essence, the point is that there are uses of mathematics in applications which differ from those Field identifies.

One of them is theory-construction, which has been briefly explored in the cases of psychophysics and gravitation theory, as reconstructed in Field 1980. Field overlooks in his characterization of applicability one use of mathematics which is both crucial to mathematical modelling and to the success of his nominalistic strategy.

In addition, Field does not consider the uses of applied mathematics which are not based on the presence of a representation. When a representation exists, it links

[26] The explanation which follows is rather sketchy. A thorough discussion of this example will be provided in chapter 4.

an empirical setting to a mathematical model: in virtue of the representation it is possible to work on the mathematical model and use the conclusions thus reached to gain information about the empirical setting. If this were the only way in which mathematics is applied, it would follow that: (i) any application of mathematics should be directed toward getting some information about the properties of some empirical setting and (ii) any application of mathematics which does this should rely on the presence of a representation.

The discussion of the linear program I gave in section 7 provides a counterexample to (i), while the discussion of the archaeological example I gave in section 8 provides a counterexample to (ii). Both counterexamples were only briefly touched upon, to give a general idea of the reason why they can be used to highlight some difficulties faced by Field. They will be discussed in greater detail only in chapters 4 and 5, whereas my analysis of the role of mathematics in theory-construction covers chapters 2 and 3.

My task in the following chapters is twofold. On the one hand, I will fully establish the objections I formulated in this chapter against Field's account of applicability. On the other hand, I will develop an alternative account of applicability, which does not face the problems of Field's. My account includes uses of mathematics in applications which are neglected by Field, and also provides a unified setting to characterize both the applications based on representations and the ones which do not rely on them. In a sense, I exploit the shortcomings of Field's account of applicability in order to generate one which integrates and extends it, thus proving more comprehensive and realistic.

CHAPTER 2: MATHEMATIZATION IN PSYCHOPHYSICAL MEASUREMENT

1. Measurement, idealization, and mathematization

In part 1 I provided a motivation for the introduction of the idealizing hypotheses which characterize extensive measurement. In this and the next chapter I will illustrate why these hypotheses are scientifically significant: in other words, I intend to provide an explanation, at least in the restricted context of measurement, of the reason why it is sometimes scientifically important to reason on idealized models of certain phenomena rather than on more realistic characterizations of them

In axiomatic terms, this amounts to explaining why some problems of measurement are best studied using Hölder's theory rather than any of the weakened versions of that theory examined in part 1. The reason why it is so is that, with the idealizing hypotheses of Hölder's theory in place, it becomes possible to prove results about the measurability of an empirical domain which could not be obtained without resorting to idealization. These results are significant because they have made it possible to generalize the concept of measurement to experimental settings other than the ordinary physical ones. In particular, they have led to the construction of theories of measurement for psychophysical variables.

This outcome relies on a novel characterization of the concepts of measurement and measurement scale, which exploits the assimilability of the idealized models of Hölder's theory to continua, i.e. Dedekind complete, ordered structures without a least or greatest element. In other words, idealization makes it possible to structurally identify certain empirical settings with mathematical objects (e.g. continua) satisfying strong formal properties (e.g. Dedekind completeness). Once this is done, several mathematical concepts and techniques can be invoked to characterize the global properties of the idealized empirical settings, typically measurability. Thus, in presence of idealization, it is possible to develop a mathematical analysis of certain empirical settings and their properties, whose consequences are empirically important.

In this and the next chapter I will provide evidence for the last claim. In particular, I will do it by looking at psychophyhsical measurement as a case study. In this chapter I describe the interaction between idealizing hypotheses and mathematical concepts which leads to a generalization of extensive measurement. In the next chapter I describe a way of isolating an axiom system for psychophysical measurement, based on the above generalization. My aim is to make clear that the mathematical analysis of idealized extensive measurement is scientifically important, because it leads to the construction of new empirical theories which play a significant role in scientific practice.

This conclusion, as will become clear later, poses a problem for Field's discussion of applicability, because it reveals an essential use of mathematics that his eliminative approach to scientific theories cannot take into account. This point will become fully clear only in the next chapter. In this chapter I prepare the ground for it, introducing certain experimental forms of psychophysical measurement, describing the problems of empirical meaningfulness they pose and explaining how these problems can be tackled by developing an abstract analysis of measurability, based on mathematical concepts and idealization.

2. Direct psychophysical measurement

In this section I am going to illustrate certain experimental methods of psychophysical measurement, called direct psychophysical methods. The latter methods have been adopted in a systematic way by the American psychologist Stanley Smith Stevens and his collaborators between the 1930s and the 1970s and are still widely employed in psychophysical experimental studies.

In the present context, direct psychophysical measurement is particularly interesting because the methodological problems it poses can be solved by carrying out a suitable mathematical analysis of idealized extensive measurement. In particular, the experimental methods of psychophysics can be provided with an adequate theoretical basis by the combined use of idealization and mathematical concepts and properties. Before illustrating this point, let me explain what direct psychophysical measurement is. To this end, I need to spend a few words on psychophysics: the main task of this discipline consists in establishing the relationship between observable physical stimuli and their inner psychological representations, i.e. unobservable sensations. Closely related to this task is that of understanding the structure of sensory continua.

For example, psychophysicists are typically interested to know whether there is a law connecting the physical intensity of a sound to the intensity of the sensation it elicits in a human subject, usually called subjective loudness. Given the possibility of measuring the physical intensity of sounds on a numerical scale, the issue arises of whether it is possible to set up a numerical scale for sensation and find its relation to the physical one in the form of a numerical equation, the so called *psychophysical law*.

One way of constructing a scale of measurement for sensation, firstly applied to loudness, was presented in Stevens 1936 and provides an early example of direct psychophysical measurement[1]. Stevens gave numerical instructions to experimental subjects and used their responses directly to fix measures of sensation[2]. The scale construction procedure adopted is based on the following three steps:

a) Experimental subjects are presented with pure tones[3], of constant frequency (1000 cycles per second);
b) One fixed stimulus-intensity (40 decibels) is taken to correspond to the subjective loudness unit, called a Sone;
c) Each experimental subject is asked to halve the physical stimuli presented to him. For instance, he may be presented with the stimulus corresponding to the loudness unit and then he may be told to adjust a different sound until it appears to him half as loud as the original sound: in this case, the adjusted sound is assigned the number 1/2.

Iterating (c) one can find successive partitions of an interval of physical intensities which, if the experimental subjects' adjustments are to be trusted as indices of sensory variations, correspond to different and proportional subjective loudness levels. This method was conceived by Stevens as a way of obtaining a measurement scale for subjective loudness.

Moreover, if intensities on the physical scale are plotted against the numerical estimates of loudness, it is possible, through extrapolation, to fit the coordinates thus obtained[4] with a curve, whose equation may be taken as the psychophysical law for loudness. In Stevens' 1936 this is a power law of the form:

$$\text{(i) } s = ki^h$$

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[1] Later forms of direct psychophysical measurement (see e.g. Stevens 1975) are very similar to this one, so it suffices for present purposes to restrict attention to the early example only.

[2] A practical motivation for Stevens' methodology came from the need to estimate industrial noise in a way more sensitive to subjective impressions than the usual decibel measures: an informative discussion of this issue can be found in Churcher 1935.

[3] Intuitively, tones that are, within experimental limits, not generated by the superposition of interfering sound waves and do not therefore contain harmonics.

[4] E.g. *x*-coordinates may stand for physical intensities, while *y*-coordinates stand for subjective loudness intensities.

where k, h are two constants[5].

Thus, Stevens 1936 describes a method to use numbers as measures of subjective loudness and of finding, on the basis of numerical assignments, a law connecting sound intensity and loudness sensation. Analogous methods were used by Stevens and his collaborators long after the first construction of the Sone scale, and produced a wealth of experimental results consistent with its structure (these are presented in some detail in Stevens 1957a-b, 1975 and Stevens & Galanter 1957). These experimental techniques, based on direct subjective estimates of sensation, are collectively called 'direct psychophysical measurement' (cf. Shepard 1981 for this terminology). However, psychologists have long debated whether it is correct to talk about measurement in this case. The reason is that Stevens' methods [6] are based on highly problematic assumptions.

In particular, these methods take at face value certain subjective estimates of sensations made by experimental subjects and directed by numerical instructions. It is assumed that the numerical instructions are literally followed by the experimental subjects and that they know how to numerically partition an unobservable sensory continuum. The halving instruction described above, for example, is presupposed to be followed by an actual halving of sensation intensity by the experimental subject: however, there is no independent assurance that a subjective sensation can be so fractionated (a famous criticism of this possibility was made by William James in James 1950: 546).

Abilities on the part of observers to recognize a certain structure on their sensory continua and to competently represent numerically this structure are assumed, which cannot be experimentally controlled or tested. Stevens has considered this objection and has constructed scales based on non-numerical responses [7] (the techniques

[5] The same type of law was also found by Stevens to hold for several other sensory continua (see Stevens & Galanter 1957). It is worth mentioning that, in the case of loudness, Stevens and Davis (Stevens & Davis 1938: ch.13) observed that a power law also relates in guinea pigs the electrical activity of the cochlea to the physical intensity of auditory stimuli. This led Stevens to conjecture that the electrical cochlear response may be understood as a physiological correlate of the loudness sensation.

[6] A comprehensive description of the methodology can be found in Stevens 1958, 1959. Here I focus only on some aspects of it, which are relevant to the subsequent discussion.

[7] The method Stevens devised to this effect is called cross-modality matching. It consists in asking an experimental subject to match the subjective intensity of a physical stimulus with the subjective intensity of a physical stimulus on a different sensory modality. A typical example of cross-modality experiment is one where the subject is required to find a degree of brightness whose subjective intensity matches the subjective loudness of a sound. Following this kind of instruction, a subject associates systematically e.g. monochrome light intensities to sound intensities: since both physical

adopted and results obtained are described in Stevens 1975: ch.4). Even this move, though, leaves the issue of how a sensory continuum is structured unanswered. As a consequence, it is not clear whether Stevens' methods really measure, if anything, subjective sensations.

3. Additivity

A further problem arises when it is observed that the Sone scale is erected simply by fixing a unit of measure: this suggests some analogy with extensive measurement, where, once a unit is chosen, any other measure is fixed. It is against this analogy that some of the traditional objections to the possibility of psychological measurement in general, and psychophysical measurement in particular, have been directed. The physicist Norman Robert Campbell, who was the first to put forward these objections, wrote that:

> 'The only properties measurable directly [...] are those that are additive [...]. Non-additive properties are measured by an *indirect* process [...]' (Final Report 1940: 340). 'Sensations could be measured if they were additive and independent[8] [...] (Campbell & Jeffreys 1938: 136).

Campbell took direct measurement, relying on experimental operations and not on the preliminary availability of other measurement scales, to be possible only for what we would call extensive structures[9]. Non-additive properties like density[10], on the other hand, can be measured as soon as other scales, i.e. for volume and mass, are available.

Since Stevens' methods in the case of the Sone scale, as well as all the other sensory scales he later constructed employing similar methods, make use of a direct procedure, according to Campbell they are meaningless unless they can be based on

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intensities are measurable, it is possible to describe the numerical law describing them. This turns out to be a power law (see for instance Stevens 1975: 13–17). It is nonetheless problematic to relate the resulting numerical law to unobservable sensory information. For a detailed analysis of this difficulty see Shepard 1981 and for further objections to Stevens' methods and ideas on measurement see Rozeboom 1966 and Luce 1972. An axiomatic treatment of cross-modality matching can be found in Krantz 1972.

8 By 'independent' Campbell means that adding sensations should not produce an interference between them.

9 A full account of Campbell's view is to be found in Campbell 1928. For a discussion and critical assessment of his position see in particular Stevens 1946, 1951, Luce & Narens 1987 and Luce et al.1990b: 108–111.

10 Density is non-additive in the sense that there is no physical operation combining physical objects of different densities which yield an object whose density is the sum of their respective densities.

an explicitly given empirical additive operation[11]. But this is not the case and so Stevens' direct psychophysical measurement does not make empirical sense. Furthermore, Campbell observes:

> '[…] In physical measurement the further condition is imposed that the assignment must be unique to this extent, that, when a numeral has been assigned to one member of the group, the numeral to be assigned to any other member is or can be determined by facts within a limited range of 'experimental error' […] (Final Report 1940: 340).

Here Campbell is saying that physical quantities are characterized by the fact that they generate scales of measurement which are unique up to a choice of unit. If we take Campbell's additivity requirement to mean that the only quantities which can be measured directly are extensive, then it follows that all quantities measurable directly must generate scales which are unique up to a choice of unit (this fact can be proved using the axioms of extensive measurement, and I will formulate it as a theorem later in the chapter).

Thus, according to Campbell, Stevens' methods do not measure anything, being a direct procedure for which no additive operation can be identified, and so in particular they cannot produce a measurement scale which is unique up to a choice of unit, as claimed by Stevens (see e.g. Stevens 1958).

Whereas Campbell's objections to psychophysical measurement are never, at least explicitly, based on a reference to the theory of extensive measurement, a position recently advocated by the Australian psychologist Joel Michell makes use of extensive measurement to reach conclusions which are in many respects very close to Campbell's.

Michell (see in particular Michell 1994, 1999, 2003a and 2005) identifies quantitativeness with the presence of an additive operation (cf. Michell 1994: 401, Michell 2005: 291–292). He in particular refers to Hölder's axioms as a fundamental characterization of quantitativeness (e.g. in Michell 2005: 290), emphasizing that a variable is quantitative when it approximately satisfies these axioms. The problem of the meaningfulness of a measurement procedure reduces then to checking whether

[11] Campbell did not however believe in the absolute impossibility of measuring any kind of sensation. This has been recently emphasized in Masin 2006, which draws attention to the significance of two studies by Campbell (i.e. Campbell 1933 and Campbell & Marris 1935) on visual sensations and loudness respectively. Nonetheless, according to Campbell, Stevens' methods are ill-founded, not being based on any explicit additive operation.

they hold. Michell, however, allows for the possibility that attributes satisfying axioms different from Hölder's be quantitative, provided these axioms entail Hölder's theory[12].

Like Campbell, Michell keeps together the existence of an additive operation and a constraint on the degrees of freedom of the possible scales of measurement: if an attribute is quantitative in his sense, then it satisfies axioms that uniquely determine a scale of measurement, once a unit of measure is fixed. It is clear that Stevens' methods do not identify any quantitative attribute either in the sense of Campbell or in the sense of Michell.

4. Mathematics

Campbell's and Michell's objections can be answered and the legitimacy of Stevens' methods vindicated in two steps. Firstly, it is necessary to show that the property of being an extensive structure and that of giving rise to a scale of measurement which is uniquely determined by a choice of unit are independent. In particular, it should be proved that the latter property holds in some empirical structure which is not assumed to satisfy the axioms of extensive measurement and yet is representable on a numerical model. Secondly, it is necessary to provide a theory of psychophysical measurement for non-extensive structures which can be numerically represented by scales which are unique up to a choice of unit.

In this chapter I am going to show that numerical representability with extensive uniqueness is independent of the presence of an additive structure: in particular, I am going to illustrate an axiomatic characterization of this form of measurability. Only in the next chapter will I show that this axiomatic characterization can be used to build up a psychophysical theory of measurement.

For the moment, let me explain how the axioms leading to this theory are reached: this, as I already remarked, requires a generalization of the ordinary concept of measurability for extensive structures. Mathematics, in presence of idealization, plays an essential role in reaching this generalization. Essentiality is related to two factors. On the one hand, as the above quick remarks show, there is a clear way of formulating a possible answer to Campbell's and Michell's objections as a

[12] For instance, this happens if their primitives suffice to define a concatenation operation which satisfies Hölder's axioms of extensive measurement: Hölder 1901 already gave an example of this fact (cf. Hölder 1996b), while Michell's characteristic example is the measurement theory of Luce & Tukey 1964.

mathematical problem. A solution to this problem amounts to a proof of the independence between the presence of an extensive operation and the possibility of obtaining measurement scales which are uniquely fixed by the choice of one parameter.

On the other hand, the essentiality of mathematics is due to the fact that Campbell's and Michell's objections to Stevens' methods cannot be countered in a non-mathematical way. They could if it was possible to devise an experimental procedure yielding scales with the desired uniqueness property but not based on an additive operation. The problem is that such a procedure would have to assume that numbers can be used to meaningfully represent unobservable sensations: independently of any issue concerning additivity, the problem remains whether such a use of numbers can be considered at all legitimate or meaningful[13].

In order to solve this problem, a preliminary proof of the empirical meaningfulness of numerical assignments has to be given. Once again, it is necessary to set up a general mathematical framework from which the measurability of sensations can be derived, e.g. through a representation theorem. Moreover, the uniqueness properties of a scale may not be testable in a psychophysical context (for instance certain changes of unit may not be easily handled by subjects making numerical estimates) and so it is more acceptable to have a theoretical, formal justification for them. This requires using mathematics to obtain a proof of uniqueness.

For all the above reasons, an answer to Campbell's and Michell's objection has to rely on mathematics.

A comparison may at this point be made between this situation and Field's use of mathematics in the reformulation of gravitation theory. Field needs to isolate non-numerical axioms whose models have prescribed numerical representations and he also wants to prove uniqueness results concerning them. Mathematics is used at a meta-theoretical level to accomplish all this. The psychophysical case is similar because the same kinds of proofs are needed, but it differs from Field's treatment of gravitation theory in one important respect.

[13] At the same time it is desirable to try and vindicate the legitimacy of Stevens' procedures, for: 'The striking regularities of the data obtained by magnitude methods [Stevens' methods] [...] have impressed many of us and should challenge theoreticians to provide theories that encompass both these methods and the more traditional ones having to do with discrimination and detection' (Luce 1972: 98).

In psychophysics we need a more substantial use of mathematics because the empirical axioms cannot be reached directly, as in the case of gravitation theory. To see this point, note that Field can take for granted that that the numerical treatment of gravitation theory is acceptable and empirically meaningful. This is because it is clear what the numerical structures involved in this theory and their primitive or definable relations represent: e.g. it is clear that numerical coordinates describe positions in space or that certain numerical linear equations describe world lines in space-time. From this point of view, it is relatively easy to see which empirical facts underlie the numerical formulation of the theory and so empirical axioms for it can be isolated by looking at these facts.

On the contrary, in the case of psychophysics it cannot be taken for granted that the numerical formulation of a psychophysical law or the experimental construction of a numerical scale for sensations is meaningful, since it is not clear what the empirical interpretation of measures within a psychophysical experiment should be, unlike what happens in a setting like Field's. Because of these differences between psychophysics and the examples considered by Field, the former requires a more abstract mathematical treatment. As no obvious empirical interpretation is available for the numerical concepts entering Stevens' experiments, it becomes necessary to find abstract conditions for numerical representability rather than directly empirical axioms, as Field does. Only subsequently will these conditions be used to frame a psychophysical theory of measurement.

The way of identifying the relevant conditions which I will describe in the next sections consists in starting from a well-known and accepted form of measurement like extensive measurement and isolating its abstract features through mathematical analysis. These abstract features can be transferred to empirical settings other than extensive structures, in particular psychophysical ones.

The problem thus takes a decidedly mathematical shape: it amounts to studying the abstract features of the measurability and uniqueness of extensive scales, so that these properties can be described in a way independent of additivity.

I now turn to illustrating how this has been done in the mathematical literature concerned with psychological modelling.

5. From extensive uniqueness to ratio scalability

In this section I present a rigorous formulation of uniqueness for extensive measurement and give an abstract definition thereof: my discussion mainly relies on the general results presented in Cohen & Narens 1979 and Narens 1981a. The motivation for this technical discussion is that it provides a concrete example of a situation in which an applicative problem can be tackled only by increasing the level of abstraction and resorting to mathematical concepts.

In the present case, an abstract characterization of extensive measurability and uniqueness has to be reached: the deep properties which lie at their basis have to be identified, so that they can be exploited to construct a psychophysical theory of measurement.

The use of mathematics made to obtain these results cannot be reduced to the ones described by Field, i.e. to the deduction of the consequences of an empirical theory or to its mathematical formulation. Here mathematics is employed to obtain the *abstract generalization* of an empirically meaningful property. Mathematical concepts prove essential to this end, because (as will be seen later) they are used to articulate the generalization itself. This could not be reached if they were to be eliminated. Since Field's treatment of scientific theories pursues the elimination of mathematics from them, there is an opposition between the account of applicability this treatment generates and the actual use of mathematics made in scientific theories. In order to establish this claim and show that the opposition really exists, and thus that Field account of the applicative uses of mathematics is too restrictive, it is necessary to look in detail at how extensive measurability and uniqueness can be studied in an abstract fashion.

It is convenient to take, as a starting point, a property of the scales of measurement for idealized extensive structures, i.e. models of Hölder's theory. All of these structures satisfy the following:

Uniqueness theorem for extensive measurement

If $\mathbf{L}$ is a model of Hölder's theory and μ, ν two scales of measurement for $\mathbf{L}$ onto $\mathbf{R}$[14], then there is a positive real r such that:

$$r\mu = \nu.$$

In addition, for any positive real r, if μ is a measurement scale for $\mathbf{L}$ onto $\mathbf{R}$, $r\mu$ is as well.

[14] Recall that, as in chapter 1 of part 1, $\mathbf{R}$ is the set of positive reals endowed with order and addition.

To see the intuitive meaning of the uniqueness theorem, simply consider the fact that the usual extensive scales are related by positive conversion factors, corresponding to changes of unit. For example, if a numerical scale is based on the unit 'gram', then it suffices to multiply all of its values by the positive constant 10^{-3} in order to obtain a scale based on the unit 'kilogram'. The uniqueness theorem asserts that one might take a gram scale and determine all possible scales of mass on **R** simply by successively multiplying the values of the gram scale by all positive real constants. This is a formal generalization of what happens in practice when new scales are generated by applying constant conversion factors.

The objective now is to obtain a formulation of the uniqueness theorem which does not include any reference to extensive structures and thus can be generalized beyond extensive measurement. Mathematical concepts are important in this context because they are used to identify the general features of **L** which are responsible for the fact that the uniqueness theorem holds. By 'general features' I mean properties of **L** which can be spelled out without referring to its operation and which can be stated for arbitrary ordered structures (an ordering is usually assumed on measurable structures). The desired properties have an abstract character and mathematics provides a suitable abstract environment to describe and characterize them[15].

The first step of the relevant mathematical analysis consists in drawing an analogy between the representation theorem for extensive measurement and the uniqueness theorem. Since the former theorem says that μ maps the structure of **L** onto that of **R**, one might ask whether uniqueness could be read in the same way. An affirmative answer can be given, by making explicit which features of **L** μ maps onto **R** as scale

[15] Incidentally note that, because mathematics is employed to isolate the general features of extensive uniqueness, it can be said to deliver an explanation of the reason why it holds. Experimental methods only show that uniqueness up to a choice of unit holds for an extensive structure in all the tested cases. Besides providing only inductive evidence, they do not point to a global feature of extensive structures which is responsible for uniqueness: they simply offer evidence for the fact that the physical operations performed on extensive structures constrain the resulting measurement scales in the familiar way. This experimentally based observation can be replaced by a global principle only when mathematics and idealization are introduced. In particular, as will be shown in a moment, one can say that extensive uniqueness depends on the strong symmetry properties of idealized extensive structures. The notion of symmetry is mathematical and the investigation of the role of symmetries in extensive structures is carried out in a mathematical fashion.

changes, i.e. the multiplications by a positive real constant mentioned in the uniqueness theorem.

To obtain the desired answer it is therefore necessary to look in some detail at the functions which are multiplications by a positive real constant. Consider any two *a*, *b* in the domain of **L**, whose measures are $\mu(a)$, $\mu(b)$ respectively: if these measures are multiplied by *r*, a positive real constant, the following properties hold:

(i) $r\mu(a) \leq r\mu(b)$ iff $\mu(a) \leq \mu(b)$;

(ii) $r\mu(a +_L b) = r\mu(a) + r\mu(b)$.

This means that a multiplication by *r* preserves the structure on **R**. It can also be seen that it is a bijective function[16] onto **R**. By definition, the multiplication by a positive constant *r* is therefore an isomorphism from R^+ onto R^+, i.e. a symmetry or, more technically, an *automorphism* of **R**.

As a consequence, all multiplications by a positive real constant are automorphisms of **R**: once the concept of automorphism is in place, mathematical theorems on automorphisms can be exploited to further the understanding of extensive uniqueness. In particular, it can be proved that that the multiplications by a positive real constant are *all* the automorphisms of **R** (the proof is immediate on the basis of a result in the theory of functional equations, given e.g. in Roberts 1979: 159–160). There are no other symmetries of **R**.

It follows from this (I omit a proof, since the result is intuitively clear) that what a scale of measurement μ maps onto the automorphisms of **R** are exactly the symmetries or automorphisms of **L**. The totality of the multiplications by a positive constant corresponds to the totality of the structure-preserving permutations of the domain of **L**. Thus, it is possible to reformulate the uniqueness theorem of extensive measurement as a statement saying that the automorphisms of any model of Hölder's theory are numerically representable onto the automorphisms of **R**.

We may forget about Hölder's theory and simply define an ordered structure **S** to be *ratio scalable* if it has a representation onto the ordered, positive reals which maps its automorphisms onto the multiplications by a positive real constant. The property of ratio scalability is abstract and does not make any reference to the availability of an additive operation (such a reference is implicit in the uniqueness theorem for

[16] Whenever $\mu(a)$ and $\mu(b)$ are different real numbers, so are $r\mu(a)$ and $r\mu(b)$, which means that a multiplication by *r* is an injective function. It is also surjective, because any positive real, which is $\mu(a)$ for some *a*, is obtained from $1/r(\mu(a))$, applying multiplication by *r*.

extensive measurement, as this is proved for models of Hölder's theory). On the other hand, it requires mathematical concepts to be formulated. This is also true of the axioms which an arbitrary ordered structure **S** must satisfy to be ratio scalable, as I will show in the next section. These axioms are conditions on an empirical structure **S** which spell out certain properties of its associated symmetries or automorphisms: for this reason they are mathematical in character.

6. Representability by automorphisms

The strategy pursued in the previous sections to define uniqueness avoiding any reference to extensive structures focuses on a shift from empirical structures to sets of transformations defined on them, i.e. their automorphisms. The introduction of mathematical concepts is necessary to avoid any references to the additive operation defined on extensive structures. This in turn serves the purpose of formulating certain uniqueness properties which hold for extensive structures in a general way.

The observations made in section 5 about the incompatibility of Field's eliminative treatment of scientific theories and the use of mathematics in scientific practice become now more concrete. The definition of ratio scalability is framed in mathematical terms, since it refers to automorphisms: dispensing with references to these structure-preserving functions would amount to getting rid of the generalization of extensive uniqueness it describes.

I am now going to strengthen this conclusion by showing that not only the concept of automorphism but certain mathematical properties of automorphisms are needed in order to characterize ratio scalability in an axiomatic fashion. The resulting axioms are very important because they preside over empirical theory-construction in psychophysics. With the preliminaries of section 5 in place, the axioms can be identified relatively quickly.

Suppose **S** is a ratio scalable, ordered structure. By definition of ratio scalability, **S** has a representation onto some real structure $\mathbf{R_S}$ and its automorphisms are mapped onto the multiplications by a positive constant. The latter functions correspond (*via* representation) to the symmetries of **S** and, in particular, their structural behaviour reflects the behaviour of the automorphisms of **S**. This means that it is possible to identify some important features of these automorphisms by looking at their numerical representations as multiplication by positive constants. In particular, since automorphisms are functions, the operation of functional composition is defined on them. But, relative to multiplications by a positive constant, functional composition is ordinary arithmetical multiplication. Thus, by looking at the properties of real multiplication, we can gather important information about the composition of the automorphisms of **S**.

Trivially, since multiplication is closed on the positive reals, associative, commutative, it has a neutral element (i.e. 1) and is invertible, it follows that functional composition on the automorphisms of **S** has the same properties. This is to say that the automorphisms of **S** form a *commutative or Abelian group* relative to functional composition. Furthermore, for any two positive real numbers r, s, it is possible to find a unique multiplication by a positive real x such that $xr = s$. This implies that, for any two objects a, b in the domain of S, there is a unique automorphism of **S** sending a into b. This property is called in the measurement theoretic literature *homogeneity* or 1-*point homogeneity* (cf. Narens 1981b: 254–255, Luce et al. 1990b: 115–117).

It thus follows that the automorphisms of **S** form a homogeneous Abelian group over a continuum (since **S** is ordered and is represented onto the continuum of the positive reals). A fundamental result proved in Narens 1981a says that *any continuum* **S** *whose associated group of automorphisms is Abelian and homogeneous is ratio scalable*[17].

In axiomatic terms this means that an ordered structure **S** is ratio scalable if it satisfies the following three conditions:

S1. **S** is a continuum relative to its ordering[18].

S2. The automorphism group of **S** is homogeneous.

S3. The automorphism group of **S** is commutative.

In the terminology of Narens 1981a, a model of the above axioms is called a *complete scalar structure*. Once S1 to S3 are in place, ratio scalability follows: since none of the axioms constrains the structure on **S** to include an additive operation, ratio scalability can in principle be obtained without an extensive structure in place (see the appendix for an explicit proof that this is actually the case). Note that S1 to

[17] This result can be seen as a generalization of the representation theorem for extensive measurement and it has actually been obtained by discussing the representability of certain structures with a binary operation generalizing the extensive one (not being assumed to be associative), as can be seen by surveying the results of Luce & Narens 1976, Cohen & Narens 1979 and their subsequent extension in Narens 1981a-b and Cohen 1988.

[18] This means that the ordering on **S** is Dedekind complete, dense, unbounded (i.e. there are no minimal or maximal elements) and that the domain of **S** contains a countable subset dense in it.

S3 may be taken to model an idealized empirical structure **S**, which is measurable on the positive reals in virtue of the axioms. As I will show in the next chapter, S1 to S3 guide the construction of a psychophysical theory of measurement. The important fact to emphasize is that these axioms have been obtained through an abstract analysis of extensive measurement, in particular by invoking certain mathematical concepts (group, automorphism) to identify the deep properties of extensive measurability and uniqueness. The axioms themselves state certain abstract properties arisen from this analysis.

This means that, as long as we want to reach a generalization of extensive uniqueness and employ it to construct a psychophysical theory, mathematical concepts and properties have to be relied upon. This scientific use of mathematics, as remarked in the previous chapter, cannot be taken into account by Field's eliminative treatment of scientific theories (this point will become fully clear only in the next chapter).

At this point I would like to clarify again that my concern here is not with providing an example of an indispensable use of mathematics and thus raise an objection against Field's claim that mathematics is dispensable from scientific theories.

In fact, even if mathematics, as I showed, is essential to reach the above concept of ratio scalability and to axiomatize it, this does not rule out the possibility of constructing measurement theories for psychophysics which do not rely on ratio scalability and the related mathematical notions. As long as this possibility is open, Field or an advocate of his position could say that the use of mathematics described in this chapter is not absolutely necessary in psychophysics because there may be psychophysical theories which do not rely upon it.

This is, I think, an acceptable objection, but it does not concern my discussion and my claims. The reason is that I am not looking at the tenability of Field's nominalism but at the tenability of his account of applicability. Thus, my primary objective is not to provide examples of allegedly indispensable uses of mathematics which might be adopted to raise problems for Field's nominalism. My distinct objective is to provide examples of uses of mathematics which play an important role in scientific practice but cannot be adequately characterized by the account of applicability generated by Field's treatment of scientific theories. What interests me is not whether the previous discussion of mathematics in psychophysical measurement, which relies essentially on mathematics, is the only way of solving the methodological problems posed by this form of measurement (in which case it would be an instance of a use of mathematics which is absolutely ineliminable).

What interests me is that mathematics is in fact applied in scientific practice in the essential way I have described: when I say ‘essential way’ I mean that there is no way of characterizing ratio scalability for arbitrary ordered structures without relying on mathematical concepts like group or automorphism (this however does not rule out the possibility of characterizing psychophysical measurement in a way which does not make use of ratio scalability and thus of the related mathematical concepts).

If mathematics is in fact used in an essential way in certain applications, an adequate account of applicability must be able to characterize this use of mathematics, precisely because it is a part of scientific practice. My objection to Field is then simply that his account does not characterize certain essential uses of mathematics which are actually present in science. I will get back to this point in the next chapter, section 5, once the importance of complete scalar structures in empirical theory-construction will become fully clear.

I now turn to making explicit the role played by idealization in the theoretical developments illustrated in this chapter. This concludes the account of idealization articulated in part 1 by providing an explanation of its importance in science.

7. Idealization

So far I have only cursorily mentioned that idealization is necessary to generalize extensive measurability and uniqueness to ratio scalability and, consequently, to isolate the axioms for complete scalar structures. The idealizations of extensive measurement are needed to this end because only when they are in place it is possible to have an isomorphic correspondence between an extensive structure and the additive, positive reals. Only when this happens the properties of the numerical automorphisms of **R** can be used to identify the properties of the automorphism of extensive structures. Conversely, only with the idealizations of extensive measurement in place extensive structures have automorphism groups which are rich enough to be assimilated to the group of automorphisms of **R**.

This correspondence between the two automorphism groups is responsible for the possibility of isolating S1 to S3, since all of them hold in the reals and can be transferred to an isomorphic structure. In fact it can be quickly shown that, if an extensive structure satisfies axioms S1 to S3, then it has to satisfy the idealizing hypotheses of extensive measurement, as discussed in part 1. For example, if S2 holds, then several infinitary idealizations must hold in the corresponding extensive structure. This is because homogeneity implies that its models cannot have maximal or minimal elements[19]; moreover, in case a binary concatenation operation is defined on these models, it has to be a closed operation (a formal proof of this fact can be found in Cohen & Narens 1979: 201). If S1 holds, the infinitary idealization of Dedekind completeness has to hold as well[20]. The transitivity of empirical equivalence is also implicitly at work, to deliver the uniqueness theorem for extensive measurement[21], which makes it possible to explicitly relate the automorphisms of **R** to those of an idealized extensive structure.

As a result, the above discussion of ratio scalability relies on the presence of all the idealizing hypotheses of extensive measurement to go through. This is enough to conclude that the benefits afforded by the abstract analysis of extensive uniqueness ultimately come from the use of idealizing hypotheses. It will be the task of the next chapter to illustrate these benefits: in short, they amount to the possibility of constructing an empirical theory of psychophysical measurement which vindicates Stevens' psychophysical methods, has a good empirical support and provides a unified framework in which many experimental results of psychology can be explained.

This example from psychophysics is instructive, not only because it shows that idealization is important, but also because it shows why it is important. When idealizations are in place, it is possible to deal with the empirical settings on which they are imposed in a mathematical fashion, because of the formal properties they satisfy. The resulting abstract investigation of the idealized empirical settings makes it possible to generalize some of their properties to new settings and leads to the construction of new empirical theories describing phenomena which are different from the ones initially investigated, on which the idealizations were imposed.

[19] This is because any automorphism would send a maximal (minimal) element into itself, against homogeneity (cf. the remarks in Niederée 1994: 535).
[20] It follows that, relative to a suitable order relation (defined in the appendix) and functional composition, the automorphisms of an extensive structure are Archimedean and so they satisfy this idealization of extensive measurement as well.
[21] A semiordered version of extensive measurement like that of Krantz 1967 would only yield an approximation of uniqueness, unless K5 were assumed.

This is what happens in the present context, where the starting point is the theory of extensive measurement but the analysis of measurability carried out on its models turns out to be relevant for non-extensive measurement, in particular psychophysical measurement.

Note that this does not mean that idealization is not important in the study of extensive measurement. The discussion of part 1 provided evidence for this conclusion by making it apparent that the ideal axioms of Hölder's theory give a unitary framework in which extensive structures can be studied. Furthermore, the assumption that quantities vary continuously is pervasive in science and it is clear that such an assumption, in presence of an additive operation, presupposes Hölder's theory (because its models are continua).

Nevertheless, I consider it important to stress that the idealizations of Hölder's theory are not *exclusively* relevant to extensive measurement and the physical sciences which make use of them. Indeed, they trigger developments of interest in other disciplines, like psychology.

These conclusions are important in themselves, but it also noteworthy that they highlight aspects of idealization which are usually not taken into due account in the philosophical literature. This observation parallels the contrast I drew in chapter 3 of part 1 between the traditional view of idealization as simplification or misdescription and my characterization of idealization as enrichment.

It is interesting to draw a similar parallel between my characterization of the importance of idealization and some recent ones, found in the philosophy of science. Some of these views diverge from the traditional one in their explanation of the reason why idealization is important (i.e. to make complex phenomena treatable), but they accept the traditional characterization of idealization as simplification. This can be clearly seen in Batterman 2002 and Strevens 2007[22]: they both accept the definition of idealization as a way of omitting details in the description of a physical system (they only consider physical examples) but they argue that this does not imply that it is always desirable or scientifically productive to add back detail to the idealization. Very briefly, Strevens maintains that idealizations are important because

[22] Liu 1999 and Liu 2004 also develop an account of the importance of idealization which departs from the traditional ones. I do not consider these works here because they characterize idealization in a way which differs from the traditional one, on which I wish to focus (Liu's characterization differs from mine, since it relates idealization to, roughly, the analysis of physical systems into their causal components).

they point out explicitly which details are not relevant to the description of a certain physical regularity [23], while Batterman observes that what often interests mathematical modellers in science is to capture patterns underlying many different phenomena, which an excess of detail in the description of the individual cases might overshadow[24].

In essence my account of the importance of idealization agrees with these conclusions, although from a different perspective. Batterman and Strevens emphasize the importance of idealized models in science because these models are able to capture fundamental patterns of physical behaviour. On my view of idealization as enrichment it is still true that enrichments are essential to capture fundamental empirical properties, like measurability with a prescribed degree of uniqueness: in this sense my idealizations work like the models of Batterman and Strevens. On the other hand, the reason why enrichments are fruitful is not that they omit negligible details, but rather that they introduce formal properties which allow the mathematical analysis of the enriched empirical settings. It is because idealization brings about the possibility of mathematization (and thus makes the conceptual resources of mathematics available) that it is important in science.

The main difference between my view and Batterman's or Strevens' is that I focus on the crucial connection between idealization (conceived as enrichment) and the possibility of mathematization, whereas they focus on the structure of physical phenomena without looking at the nature and motivation for the mathematics applied to them.

The importance of my account then consists in its drawing attention to the connection between idealization and mathematization. Idealization is important

[23] 'A good idealizing explanation is always better than its veridical counterpart, where by the veridical counterpart, I mean the explanation that corrects every one of the idealizing explanation's distortions [...] The best idealized models will be equivalent in one explanatorily central sense to the corresponding canonical models [i.e. fully satisfactory explanations of the phenomena]: when understood correctly, both models cite the same relevant factors and no irrelevant factors (Strevens 2007: 2, 25)

[24] '[...] it does seem that the mathematical modeller is interested primarily in accounting for repeatable patterns of behavior' (Batterman 2002: 26–27). Batterman then discusses methods to identify these patterns of behaviour, which study the development of phenomena 'for intermediate times and distances away from the boundaries such that the effects of random initial features or fine detail in the spatial structure of the boundaries have disappeared but the system is still far from its final equilibrium state (Barenblatt 1966: 94, quoted in Batterman 2002: 29). The idea is that the patterns at the basis of physical phenomena are studied for situations in which many details can be left aside. Batterman then provides a justification for the use and physical meaningfulness of models which neglect actual physical details, but his point is that it is these idealized models, rather than more detailed descriptions, which ultimately characterize physical behaviour in a law-like fashion.

because it makes it possible to prove facts of empirical relevance (e.g. the ratio scalability of complete scalar structures) by means of mathematical methods.

Appendix: a non-extensive instance of ratio scalability

My objective in this appendix is to show an example of a structure which is ratio scalable but does not possess any additive operation. My discussion follows Narens 1981a and Narens 1985: ch.2 and gives in a self-contained fashion a simple instance of a general result firstly proved in Narens 1981a: 20–21. The proof is interesting in itself because it illustrates a way, alternative to the type-approach discussed in chapter 2 of part 1, to describe a measurement scale as a reference frame generated by an empirical structure.

I assume axioms S1 to S3 and consider a an ordered structure **S** satisfying them, which has only one unary predicate defined on it. Thus **S** is **S** = $\langle S, <_S, P\rangle$ and it cannot be extensive, since it has no binary operations. Its ratio scalability can then be proved in two steps: the first consists in constructing a structure **A** based on the automorphisms of **S**, which is isomorphic to **S**. The second step consists in proving that **A** is ratio scalable and so that **S** must be as well. Note however that **S** has a representation on **A** and for this reason it makes sense to regard **A** as a system of coordinates or generalized measures for **S** (this perspective is put forward in Narens 2002a and Narens 2002b: 381–406, but the ideas at its basis have already been used in the coordinatization of geometry, for instance in Artin 1957 and Lipman 1961).

The domain of **A** is the set of automorphisms of **S**: since automorphisms form a group, a binary additive operation is defined on **A**. In addition, an ordering $<_A$ and a property P_A can be induced on it, by means of the following definitions, where f and g are arbitrary automorphisms of **S**:

(i) $f <_A g$ if there is x such that $f(x) <_S g(x)$.

(ii) $P_A(f)$ if and only if there is x such that $P(f(x))$[25].

Using (i) and S1 to S3 it can be checked that **A** is totally ordered by $<_A$, and, in fact, that it is a continuum. Thus, technically, **A** is an Abelian (i.e. commutative)

[25] It can be proved that the right hand side of both (i) and (ii) holds for all x if it holds for some x. I show this for (ii) by contradiction. Suppose there is y different from x and $P(f(x))$ but not $P(f(y))$. By S2 there is an automorphism h such that $h(x) = y$, and so we have not $P(f(h(x)))$ or, by S3, not $P(h(f(x)))$. Since h is an automorphism, it preserves P, and the last formula holds if and only if not $P(f(x))$, a contradiction.

Dedekind complete ordered group, with a unary predicate defined on it. Note that, because of S2, the function φ_x defined by:

$$\varphi_x(y) = \text{the (unique) automorphism } f \text{ which sends } x \text{ into } y$$

exists and it is an isomorphism from **S** onto **A**. The latter constitutes a reference frame for **S**, whose elements are precisely the unique functional correlations connecting a fixed unit x to any other object in the domain of **S**.

Exploiting **A**, a proof of the ratio scalability of **S** can finally be obtained. To see how, observe that, relative to its ordering and the operation of functional composition, **A** satisfies all of Hölder's axioms except for positivity. The failure of positivity depends on the interaction between automorphisms and order. Call *positive* an automorphism f if $f(x) >_A x$, for some (and thus all) x in the domain of **S**. An automorphism is *nonnegative* if it is positive or the identity.

Then it is easy to see that the inverse of positive f is negative[26]. Because S1 and S2 are assumed, there are bound to be automorphisms other than the identity in the domain of **A**. Thus their ordered group structure, implies that they can be distinguished according to their 'sign' into negative, positive and 'zero' (the identity being unsigned like zero in ordinary arithmetic). The composition of two negative automorphisms contradicts positivity.

Nevertheless, if we restrict attention to the positive part of **A**, the composition of two functions in it satisfies positivity and thus all of Hölder's axioms hold. It follows that the positive part of **A** is representable by some function μ on the positive reals with order and addition. This representation can be extended to the whole of **A** putting:

$$\nu(f) = \begin{cases} \mu(f) & \text{if } f \text{ is positive;} \\ 0 & \text{if } f \text{ is the identity;} \\ -\mu(f) & \text{if the inverse of } f \text{ is positive.} \end{cases}$$

As a result, ν is a representation of the ordered, Abelian group **A** on the additive group of the ordered reals. By taking exponentials on this representation[27] we obtain a new one, on the multiplicative, positive, ordered reals: call it $\mathbf{R_A}$. Since **S** is

[26] Otherwise, $f^{-1}(x) >_A x$. Since f is structure-preserving, applying it to both members we obtain: $x >_A f(x)$ and f, contradicting the positivity of f.

[27] This means that, for any ν(f), we convert it into $e^{\nu(f)}$: this transformation leaves the previous ordering on the ν(f) unaltered, because it is strictly monotonic, while it turns addition into multiplication because
$\nu(f) + \nu(g)$ is turned into $e^{\nu(f) + \nu(g)} = e^{\nu(f)} e^{\nu(g)}$. Since all values of the exponential are positive, we also have 'compressed' all measures of ν onto positive ones.

isomorphic to **A**, there is an isomorphism $g = \exp[\nu\varphi_x]$, mapping it onto $\mathbf{R_A}$. To conclude that **S** is ratio scalable it suffices to prove that its automorphisms are mapped by g onto the multiplications by a positive real. This is to say that α is an automorphism of $\mathbf{R_A}$ if and only if it is a multiplication by a positive real.

On the one hand, consider an arbitrary, fixed multiplication by a positive real, say r. This certainly preserves the order on $\mathbf{R_A}$ and so, to prove that it is an automorphism, it suffices to establish: $\mathbf{P}(x)$ if and only if $\mathbf{P}(rx)$, where **P** is the numerical representation of P on **S**. Since g represents the automorphisms of **S** *onto* positive reals in $\mathbf{R_A}$, then r is the image of some automorphism of **S**. In this case we have:

$$\mathbf{P}(x) \text{ iff } \mathrm{P}(g^{-1}(x)) \text{ iff } \mathrm{P}(g^{-1}(r)g^{-1}(x)) \text{ iff } g[\mathrm{P}(g^{-1}(r)g^{-1}(x))] \text{ iff } \mathbf{P}(rx).$$

On the other hand, if α is an automorphism of $\mathbf{R_A}$, it must be a multiplication by a positive constant. For, suppose $\alpha(1) = r$, where r is a positive real. Then the function which is a multiplication by r acts on 1 exactly like α: this means that there is x such that $\alpha(x) = r(x)$ and so this holds for any y[28]. This establishes the ratio scalability of **S** and the independence between this property and the presence of additive structure on **S**.

[28] Since $\mathbf{R_A}$ isomorphically represents a structure whose automorphisms commute, its automorphisms must commute, whatever they are. Now consider any y different from x: since y, x are positive reals there is a multiplication by a positive real which sends x into y, call it s. Thus $sx = y$ and multiplication by s is an automorphism of $\mathbf{R_A}$. We have $\alpha(y) = \alpha(s(x)) = s(\alpha(x)) = sr\mathrm{x} = rsx = ry$. Thus α and the multiplication by r coincide on any y.

CHAPTER 3: MATHEMATICAL THEORY CONSTRUCTION IN PSYCHOPHYSICS

1. Towards theory-construction

In the previous chapter I have examined a use of mathematical concepts and theorems to address a problem of applicative relevance for measurement in general and psychophysical measurement in particular. The problem was twofold: it concerned both the possibility of measuring empirical structures devoid of an additive operation and the possibility of doing so on scales of measurement satisfying uniqueness up to the choice of one parameter.

An abstract analysis of extensive measurement solved this problem by delivering a generalization of the properties of measurability and of extensive uniqueness, ultimately translated into the definition of ratio scalability and the axioms for complete scalar structures.

The role of mathematics in achieving this result is substantial. The problem of studying the precise interrelationship between being an extensive structure and being ratio scalable can only be stated and investigated in abstract terms, in order to become treatable. As remarked in the previous chapter, mathematics is needed because an appeal to experimental methods alone cannot settle the question of whether measurability can be generalized and made independent of the presence of an extensive structure. This is because either the experimental methods adopted are based on numerical estimates (like Stevens' methods) or they must lead to numerical measures. In the first case they cannot settle the question experimentally because it is not even clear whether the numerical estimates are empirically meaningful. In the second case, appeal to mathematics cannot be avoided because, in order to legitimize the use of numerical measures, one has to prove a representation theorem.

Thus, in order to show that, theoretically, there may be ratio scalable, non-extensive structures, it becomes necessary to turn to the mathematically informed strategy of looking at successful and acceptable examples of measurement in order to extract from them abstract, global properties, which are characteristic of measurability in general (with a prescribed degree of uniqueness). These properties have been isolated in the previous chapters, as axioms S1 to S3.

This chapter is devoted to showing the scientific importance of these axioms by illustrating how a psychophysical theory of measurement can be built up on their basis. Since axioms S1 to S3 are mathematical assumptions containing references to structure-preserving functions, groups and their properties, their employment in psychophysics exemplifies a use of mathematics in empirical theory-construction.

The main conclusion I will draw in this chapter is that Field's account of applicability cannot characterize this use of mathematics in applications nor the additional uses of mathematics which depend on it and which I will illustrate in later sections. This result will confirm the restrictiveness of Field's characterization of applicability and will also provide a guide to a more adequate account of applicability, which I am going to present in chapter 5.

Now, before looking at the role played by S1 to S3 in theory-construction, I wish to make some remarks on their status and the fact that Field cannot adequately account for their applicability.

2. A shortcoming of Field's account of applicability

Field, as I have shown in chapter 1, takes mathematics to be useful in science because it allows the mathematical formulation of empirical theories and makes it possible to prove the empirical consequences of empirical theories.

But the use of mathematics involved in studying measurability and ratio scalability which was described in the previous chapter differs from the ones Field considers and so his account of applicability fails to capture it. The use of mathematics involved in isolating axioms S1 to S3, which characterize ratio scalability, does not lead to the mathematical formulation of an empirical theory, but only to the algebraic axioms for complete scalar structures. In addition, mathematics is not used to prove any empirical consequence of an empirical theory in a sense compatible with Field's treatment of scientific theories.

To see why, some observations are necessary. When Field talks about using mathematics to make inferences about empirical structures, he has in mind inferences whose conclusions have a purely empirical interpretation.

Now, if the use made of mathematics to obtain axioms S1 to S3 really is one of those pointed out by Field, it has to consist in proving some empirical consequence of an empirical theory. In the present context, the empirical consequence in question is the ratio scalability of extensive structures, i.e. the fact that they satisfy S1 to S3. This property, though, can only be formulated as a property of the group of symmetries of an idealized extensive structure.

In other words, it can only be formulated by referring to the mathematical concepts of (homogeneous) group and (continuous) function. But Field's treatment of scientific theories is of an eliminative nature: for nominalistic reasons, he aims to avoid any reference to mathematical entities in stating the empirical properties of empirical structures. In particular, Field classifies references to functions among those to be eliminated[1] and it is safe to conjecture that he would take a similar stand with respect to references to groups. As a result, it is not possible for him to formulate the property of ratio scalability as an empirical one. So it is, *a fortiori*, not possible for him to treat it as a property of an empirical setting which has been deduced by means of mathematics.

This causes an *impasse* for the account of applicability which is generated by Field's nominalistically motivated, eliminative strategy: if Field adheres to his strategy, then he is forced to eliminate an empirically meaningful concept like ratio scalability. Alternatively, he has to revise his strategy, in which case he can accept the possibility that sometimes functions are used in an empirically meaningful way.

The next sections are devoted to showing why the first alternative is not viable and so Field would have to pursue the second if he wished to be able to give a realistic account of the applicability of mathematics[2]. Briefly, the reason is that, if an eliminative strategy is retained, then ratio scalability cannot be used as a property of certain empirical structures. If this is prohibited, then it is not possible to formulate a psychophysical theory which has been recently developed in the scientific literature, has proved theoretically fruitful and has also received good experimental confirmation. An account of applicability which neglects or eliminates a theory of this kind is clearly problematic.

The psychophysical theory I will discuss in the next sections to establish the problematicity of Field's account of applicability is a set of empirical axioms from which a representation theorem can be proved. Some of these axioms and their immediate consequences refer to continuous functions. If Field's eliminative strategy is to be pursued uniformly, then in this particular context it leads to the elimination of an empirical theory, because the latter contains references to functions (and, in its

[1] Note that this might already cause problems with the binary operation defined on extensive structures.

[2] Note that my objections, here, in the previous chapter and in the following ones, are not directed against Field's nominalism but against the adequacy of the account of applicability which Field articulates on the basis of his treatment of scientific theories, which is motivated by nominalism.

deductive developments, to groups as well). This fact implies that Field cannot pursue a uniform eliminative strategy. In the case of functions, for example, he needs to make a distinction between situations like his theory of space-time, where references to functions can be eliminated, and situations like the one I have just mentioned, in which references to functions cannot be eliminated and play an essential role in the formulation of an empirical theory[3].

In this context mathematical concepts and theorems are essential to reason about empirical problems, e.g. to obtain generalizations of empirical properties or to frame empirical theories. In other words, mathematical concepts are essential as a methodology for the investigation of empirical phenomena. It is this particular methodological use of mathematics which is not captured by Field's account.

This is not surprising, since he thinks that applied mathematics is introduced only through the mediation of a representation. One does not use directly mathematical concepts or theorems on empirical structures, but only applies them indirectly, i.e. because they can be mapped on an empirical structure by means of a representation. In a sense, the representation ensures that a certain piece of applied mathematics can be separated from the empirical structure to which it is applied. Applicability depends on a link between mathematics and an empirical setting but the mathematics involved can be kept completely distinct from the latter setting. The possibility of making this complete distinction ensures, among other things, that Field's eliminative strategy goes through.

My point is that certain applications of mathematics do not presuppose this kind of separation between mathematical concepts and empirical structures (I am not talking about mathematical entities, but only about formal, mathematical concepts, which may be introduced within an empirical context, at least if idealization is allowed). In the next sections I will show that this is the case for the psychophysical theory which can be built on axioms S1 to S3. These axioms contain mathematical concepts and the theory they determine is in essence based on the same concepts. We

[3] Note that this distinction does not necessarily disrupt nominalism because, even though certain references to functions cannot be eliminated, they are nonetheless given an empirical interpretation, which a nominalist may accept. What is disrupted is, on the other hand, Field's eliminative strategy. This strategy does not take into account the fact that some mathematical references, e.g. to functions, may become part of empirical theories, in which case it does not make sense to eliminate them.

Here I am not interested in suggesting how Field should modify his strategy to deal nominalistically with scientific theories in the light of this fact. I am just interested in emphasizing that this eliminative approach is harmful for the account of applicability Field has to offer.

have an instance of the integration of abstract concepts into an empirical (albeit idealized) setting.

This use of mathematics does not require the mediation of a representation. Field is not in a position to describe this use because he always seeks to reduce the application of mathematics to the mediation of a representation. I will now provide evidence for this conclusion by describing how the psychophysical theory I have alluded to above emerges from the axioms S1 to S3 for complete scalar structures.

3. Framing an empirical theory

As I observed in the previous section, axioms S1 to S3 may be taken as purely mathematical conditions: they are algebraic statements which spell out some properties of a family of continuous functions over an ordered set. For this reason, it may seem difficult to think that they can be used to isolate the axioms of an empirical theory, even in presence of idealization. What is difficult to understand is how abstract mathematical concepts like automorphism or properties like homogeneity (asserted by S2) can be relevant to the description of an empirical setting.

In the pyschophysical case an answer is relatively straightforward: these concepts and properties are empirically relevant because they can be interpreted upon an idealized empirical setting. More precisely, they can be used to so model psychological experiments as to ensure the measurability of the empirical variables involved in them (e.g. sensations, subjective responses to certain stimuli). The key conceptual move is to idealize psychophysical experiments like Stevens' in such a way that the notion of automorphism and the properties of automorphism groups assumed in S1 to S3 can be given an empirical interpretation on the idealized experiment.

Thus, despite their mathematical character, the axioms for complete scalar structures are useful in applications. Their usefulness really depends on the fact that they can be employed to shape an empirical setting, because they are interpretable on it. Once their interpretation is effected, the property of ratio scalability[4] is set within an empirical context and can be used to ensure measurability. Note that, as mentioned in section 2, this use of mathematics, i.e. the introduction of S1 to S3 into

[4] Ratio scalability follows from these axioms.

an empirical theory, does not rely on a representation theorem but only on providing an empirical interpretation for certain formal concepts and properties. Proving a representation theorem is a distinguishing feature of the strategy Field uses to describe applicability, but not of the application of mathematics I am going to describe.

Here axioms S1 to S3 are adopted to determine both the primitive concepts and the axioms of an empirical theory of psychophysical measurement. Once this theory is in place, it provides a sound foundation for Stevens' methods, conceived as a form of measurement.

Before giving the psychophysical axioms, let me briefly explain how the concepts entering the formulation of S1 to S3 can be set within an empirical context. This is a fundamental step to make, preliminary to the construction of an empirical theory on the basis of the axioms.

The idea is to use S1 to S3 to describe Stevens' experimental methods. The empirical interpretation of the axioms is therefore relative to these methods. To see how it can be determined, consider the procedure employed to set up the Sone scale (see the previous chapter, section 2). This procedure consists in presenting an observer with a reference physical stimulus t and asking him to produce a physical stimulus according to an instruction like: find x such that it is p times (subjectively) louder than t. Here the experimental subject is presented with a physical stimulus and produces another physical stimulus. This input-output process can be idealized into a function $f_{\mathbf{p}}$, indexed by a label describing a particular instruction. Psychophysical tasks involving different numerical instructions can then be modelled by different functions from physical stimuli to physical stimuli[5]. Functional composition makes empirical sense, because it corresponds to performing distinct tasks in succession. In other words, the concepts of function and functional composition become part of the description of an experimental setting.

Axioms S1 to S3, which talk about functions, can then be used to constrain this description in a desired way. It is at this stage that the algebraic axioms for complete scalar structures take an important part in the process of theory-construction. This

[5] Talking about functions is an idealization because an experimental subject may react to the same instruction on different occasions by producing different responses. Nevertheless, functions may be taken to describe the central tendencies (statistical means) of subjective responses. It should also be pointed out that this problem is not characteristic of psychophysics, since even in physical measurment repeated measurements of the same quantity usually generate different numbers.

can be concretely illustrated by looking at a psychophysical theory presented in Narens 1996, which is based on S1 to S3.

4. Empirical axiomatization

In view of the observations in the previous section, the responses of an experimental subject in Stevens' psychophysical experiments can be described (making some use of idealization) as functions from physical stimuli into physical stimuli. Formally, the stimuli may be modelled by a set X, in which case the relevant functions are correspondences defined on X. In this context an equality like:

$$f_{\mathbf{p}}(t) = x$$

means that, when presented with a stimulus t from X, an experimental subject produces a stimulus x in X when given instruction **p**. Narens 1996 codifies this state of affairs by an ordered triple like:

$$(x, \mathbf{p}, t).$$

The overall behaviour of an experimental subject can therefore be described by a set E of triples like the above. Note that $f_{\mathbf{p}}$ only relates elements of X to elements of X and is not assumed to have a numerical representation, even if the instruction **p** may be formulated in numerical terms. Stevens' methods, on the contrary, assume that $f_{\mathbf{p}}$ has a numerical interpretation induced by the numerical instruction given to the experimental subject (i.e. if the instruction is 'halve the subjective intensity of the stimulus', then the $f_{\mathbf{p}}$ is assumed to be interpretable on the multiplication by 1/2). The task of Narens 1996 is to drop this assumption and show that, in presence of suitable axioms on the behaviour of an experimental subject, functions like $f_{\mathbf{p}}$ can be legitimately given a numerical interpretation[6].

Narens does this by axiomatizing families of triples (x, **p**, t) in such a way that a family of functions over X can be defined from them, which is embeddable into a complete scalar structure, which of course satisfies axioms S1 to S3. The axioms on the triples (x, **p**, t) are indeed chosen with this result in view and so S1 to S3 determine their choice. To explain this last point it is best to start from a formulation of the axioms and comment on them once they have been listed.

A convenient way of stating the axioms consists in specifying beforehand the type of structure they axiomatize. In this case an ordered set of physical stimuli is

[6] In short, the existence of a numerical interpretation is no longer assumed but justified by means of a proof.

assumed, symbolized by $\langle X, \geq_X \rangle$: for example, the stimuli may be sounds ordered by intensity. Functions like $f_{\mathbf{p}}$ are also defined on X and they give rise to the structure:

$$\mathbf{X} = \langle X, \geq_X, \ldots, f_{\mathbf{p}}, \ldots \rangle.$$

The application of $f_{\mathbf{p}}$ to X correlates physical stimuli to physical stimuli. As already observed, the equality $f_{\mathbf{p}}(t) = x$ can be coded by the triple $(x, \mathbf{p}, t)$. Narens' axioms describe a structure like **X** by imposing conditions on ordered triples, from which it can in particular be proved that **X** is ratio scalable. In particular, **X** has a numerical representation in which any function $f_{\mathbf{p}}$ is interpreted on a multiplication by a positive real constant. This result shows that it is in principle possible to use subjective responses (the triples $(x, \mathbf{p}, t)$) based on numerical instructions (like **p**) to generate a psychological measurement scale exhibiting extensive uniqueness: to this extent, Narens 1996 vindicates Stevens' methods[7].

Now, with these observations in place, it can finally be shown how S1 to S3 are integrated into an empirical theory. First of all, S1 is simply assumed in the context of the new theory and it becomes:

N1: X is a continuum relative to the physical ordering of stimuli $\geq_X$.

Since the additional axioms serve the purpose of determining a complete scalar structure, they have to allow for the possibility of defining a group of automorphisms. To this effect, the following is assumed, where E is just the set of responses produced by an experimental subject:

N2:

(a) $(t, \mathbf{1}, t)$ is in E;

(b) For all t, x in X and p in N, there exist exactly one z in X and exactly one s in X such that: $(z, \mathbf{p}, t)$ is in E and $(x, \mathbf{p}, s)$ is in E;

(c) For all x, y, t, s in X , if $(x, \mathbf{p}, t)$ and $(y, \mathbf{p}, s)$ are in E, then $x \geq_X y$ iff $t \geq_X s$[8].

N2 is important for one main reason, i.e. because it makes it possible to introduce functions onto X. Once the following definition is given:

$$f_{\mathbf{p}}(t) = x \text{ if and only if } (x, \mathbf{p}, t) \text{ is in E},$$

[7] The axioms I will consider only describe the observable behaviour of a subject and do not relate it to sensations: so, strictly speaking, they are not psychophysical axioms but behavioural ones. I will however occasionally refer to them as psychophysical because they can be included into a wider theory in which they are assumed to provide information about subjective sensations (for the additional axioms of the wider theory see Narens 1996: 115–116).

[8] Narens 1996 also contains the assumptions that (i) E, the set of data describing the behaviour of an experimental subject, is included in the set $\{(x, \mathbf{p}, t): x, t \in X, p \in N\}$, where N is the set of positive integers; (ii)for all $(x, \mathbf{p}, t)$ in E, $x \geq_X t$.

condition (b) in N2 implies that any $f_{\mathbf{p}}$ is onto X, while (c) implies that it is one-to-one, because it preserves the order on X. As a result, any $f_{\mathbf{p}}$ is an order-preserving bijection onto X, i.e. an automorphism of $\langle X, \geq_X \rangle$. Condition (a) ensures the existence of the identity automorphism.

Now, in light of the above definition, the structure $\mathbf{X} = \langle X, \geq_X, \ldots, f_{\mathbf{p}}, \ldots \rangle$ can be identified. Since it is this structure that has to be ratio scalable, it is necessary to ensure that any $f_{\mathbf{p}}$ is an automorphism of **X**. But any $f_{\mathbf{p}}$ is already order-preserving, so it suffices to assume that it also preserves the other functions on X. This means that, for any $f_{\mathbf{p}}$ and $f_{\mathbf{q}}$: $f_{\mathbf{p}}(f_{\mathbf{q}}(x)) = f_{\mathbf{q}}(f_{\mathbf{p}}(x))$. Then the required axiom is:

N3: all psychological functions on X commute.

This commutativity property is fundamentally the translation of S3 in a psychological context. Once axioms N1 to N3 are supplemented with an Archimedean condition N4[9], the ratio scalability of **X** can be proved using them.

This is not surprising, because essentially S1 and S3 are two of the axioms. As for S2, stating the homogeneity of a group of automorphisms, it can be recovered from the given axioms, since they imply that the family of functions over **X** can be extended to a homogeneous group of functions, automorphisms of **X** (for a proof see Narens 1996: 121–125)

5. Mathematical theory-construction

A few observations are in order about the significance of the axiom system N1 to N4. The axioms and the primitives of the structure **X** are determined by the theory of ratio scalability, in the sense that the choice of axioms and primitives is directed by the need to recover S1 to S3.

In fact, these algebraic axioms have been integrated into the empirical theory of previous section: they have been directly (like S1 and S3) or indirectly (like S2) incorporated into it. We may talk about an instance of mathematically informed theory-construction.

I have already observed that the fact that mathematical notions like group or (continuous) function and mathematical properties like homogeneity are used in empirical theories causes a problem for Field's eliminative strategy. The problem, it

[9] N4 includes three assumptions: (i) For all $(x, \mathbf{p}, t)$ and $(y, \mathbf{q}, t)$ in E: $x >_X y$ iff $p > q$, where p, q are positive integers; (ii) For all x, t in X, if $x >_X t$ then there is y in X such that $y >_X x$ and $(y, \mathbf{p}, t) \in$ E; (iii) For all x, t in X, if $x >_X t$ there are y, z in X and p in N such that $x >_X z >_X t$ and $(y, \mathbf{p+1}, t) \in$ E and $(y, \mathbf{p}, z) \in$ E.

should be stressed, does not necessarily affect Field's nominalism, but the account of applicability generated by Field's treatment of scientific theories. In other words, if Field pursues an eliminative strategy and asks references to groups, functions and properties of functions to be dispensed with in the formulation of any empirical theory, he then has to dispense with the theory of Narens 1996.

If so, then he cannot account for the particular way in which mathematics is applied to obtain this theory (since he eliminates the theory in the first place) and, as a result, his account of applicability proves inadequate. At this point one may object, as noted in the previous chapter, that the fact that a psychophysical theory can be axiomatized through the essential use of mathematical references does not really pose a problem for Field. He might simply observe that there may be other psychophysical theories whose axioms do not involve any of the above mathematical references.

Although this is a possible move, it only shows that Field could retain an eliminative approach to scientific theories by neglecting some (for him) problematic cases. The point is that this option is not available, if the problem of giving a realistic account of applicability is taken as the central one, as I am doing here. The reason is that the theory of Narens 1996 has been presented to deal with certain empirical phenomena, its axioms have a clear empirical meaning and they make successful empirical predictions. Moreover, some axioms have been tested and experimentally proved to hold (evidence for this and the previous remarks will be offered in a moment). In other words, we are in presence of an empirical theory which is not purely speculative, but is applied with some degree of success in scientific practice. In other words, the application of mathematical concepts and theorems which has led to this theory is a fact of relevance, which has some influence on experimental researches in psychology.

A satisfactory account of applicability simply cannot ignore this fact, because it provides an important instance of the way mathematics is successfully applied. For this reason, the possible move of ignoring the theory of Narens 1996 in order to adhere to a purely eliminative strategy won't work, if applicability is taken seriously. Such a move would impoverish the resulting analysis of applicability because it corresponds to a refusal to study certain theories which are nonetheless of importance in scientific practice.

The main consequence of all this is the same I pointed out in section 2. Either Field's strategy is uniformly applied, in which case it does not generate a satisfactory account of applicability for the reasons just stated, or it has to be revised. If the objective is to reach a satisfactory account of applicability, then the strategy has to be revised.

The revision would amount to making a distinction between situations in which mathematical references can be eliminated and situations in which they are integrated into empirical theories. Here, however, I am not primarily interested in describing a possible revision of Field's approach to scientific theories, but only in improving his account of applicability. So my conclusion is simply that mathematical concepts and theorems can either be applied as Field says, by means of representations, or they can be applied by means of integration within an empirical theory.

The second form of application identifies a use of mathematics in science which does not correspond to any of those described by Field. In fact, Field's account does not capture this use of mathematics or the other uses induced by it. Two of them are of particular relevance and follow from the fact that mathematical concepts and properties can be incorporated into an empirical theory: these very concepts (i) provide a unitary conceptual framework for the empirical phenomena described by the theory and also (ii) guide experimental researches.

My task in the remainder of this chapter is to provide evidence for (i) and (ii), i.e. for the fact that the algebraic concepts and properties involved in S1 to S3 work[10] as explanatory principles of psychological phenomena and as guides to experimental tests. While doing so, I will also be able to substantiate my previous observations about the good empirical confirmation of Narens' theory and its importance for experimental investigations in psychology.

6. Abstract principles and experimental results

In have already stressed that any satisfactory account of applicability has to take into account Narens' theory and the way mathematics is used in it. This depends on the fact that this theory is not a purely abstract construct but plays an important role in actual psychological investigations. I am now going to provide grounds for this statement and show why mathematics is important in directing these investigations.

[10] Once introduced into the theory of Narens 1996.

By the remarks in sections 4 and 5 it should be clear that the axioms of complete scalar structures have guided the choice of axioms in Narens 1996. It can be shown that they also guide the understanding of the phenomena described by the axioms. This observation can be illustrated by an example. One of the axioms of complete scalar structures (i.e. S2) states the homogeneity of a group of automorphisms. Axioms N1 to N4 imply that, given a structure:

$$\mathbf{X} = \langle X, \geq_X, \ldots f_p \ldots \rangle,$$

there is a homogeneous group of automorphisms over it. This amounts to integrating the formal property of homogeneity within an empirical context. Homogeneity is particularly interesting because it makes it possible to derive several empirical results concerning **X**. Thus, these results can be unified within a single conceptual framework based on an abstract property. This is a fact of great importance: the reason is that psychological experiments like Stevens' have often revealed certain regularities without providing for them any broad theoretical justification. In the absence of a theory which implies them as consequences, it is difficult to evaluate how relevant they are and what kind of information they provide about sensory processes or behavioural responses.

The presence of formal properties like homogeneity within an empirical theory makes it possible to solve this kind of problem precisely because it provides a theoretical basis and an explanatory principle for observed empirical regularities. In this section I will elucidate this point by means of an example, while in the next section I will strengthen the conclusions reached here.

My example of a psychological regularity for which homogeneity works as theoretical basis and explanatory principle requires some preliminary observations on **X**. As I earlier pointed out, functions like f_p on X are indexed by labels which correspond to the experimental instructions on which the functions are based. So far I have only considered instructions of the form 'find a stimulus x which is p times as loud as t', but other types of instructions can equally well be associated to the functions on X.

Here I want to focus on the instructions used in equal-difference and equal-ratio estimation experiments. In order to understand what these are, take the case of loudness to fix ideas. Then, an experiment involving equal-difference estimations works as follows: the experimental subject is presented with two stimuli x, y, e.g. two pure tones, and asked to find a stimulus z such that he perceives the same subjective difference in loudness between the couples x, y and y, z.

The responses of the experimental subject may thus be described by a *difference function* indexed by the reference points x, y and defined as follows:

$$\text{(i) } D_{xy}(y) = z.$$

An experiment involving equal ratios is entirely analogous to the one just described, with the difference that the instructions given to the experimental subject require him to find a stimulus c such that, given a, b, the subjective loudness ratio a bears to b equals the one b bears to c[11]. Then a ratio function may be defined by:

$$\text{(ii) } R_{ab}(a) = c.$$

Now, experimental results described in Garner 1954, discussed in Torgerson 1961 and surveyed in Birnbaum 1982 suggest that difference functions and ratio functions describe the same psychological correlation between stimuli, i.e. that equal subjective intervals are also equal subjective ratios. In other words, any ratio function is a certain difference function and viceversa. This suggestion has been called 'Torgerson's conjecture' in the psychological literature (see e.g. Narens 2006: 293–294). Then Torgerson's conjecture is a hypothesis based on observed regularities, but not on theoretical principles.

On the other hand, it becomes possible to prove it as a theorem from a theory which slightly strengthens that of Narens 1996 and contains homogeneity as an assumption[12]. In this case, the question 'under what kinds of general conditions about subjective processing of stimuli should we expect a result like this [Torgerson's conjecture]?' (Narens 1997: 190) can be answered by a theorem based on homogeneity. It becomes possible to identify theoretical conditions underlying an

[11] Here 'ratio' and 'difference' are taken as words occurring in the relevant experimental instructions and not necessarily as words referring to relations between sensations which resemble numerical relations.

[12] In particular, Narens assumes **X** to be a continuum relative to its ordering, he takes ratio and difference functions to be order-preserving and defined on X and requires that ratio functions be homogeneous, commutative and closed under composition. If ratio functions and difference functions commute (a result which has received experimental confirmation, as will be seen in the next section), then ratio functions alone constitute a homogeneous group of automorphisms on **X**. As a result, **X** is ratio scalable and it follows from a theorem of Narens (Narens 2006: 295) that difference functions are represented on multiplications by positive constant, exactly as ratio functions. Note that, since ratio functions are represented onto the multiplications by a positive constant, this fact entails that there exists a simultaneous representation for ratio and difference functions on the same numerical functions. Exploiting this fact it can be shown that: for all x, y in X (the domain of **X**) there are a, b in X such that $D_{xy} = R_{ab}$. In presence of further axioms it can also be proved that, for all a, b in X there are x, y in X such that $R_{ab} = D_{xy}$ (cf. Narens 2006: 294). This establishes the coincidence of ratio and difference functions.

empirical regularity. One can say that Torgerson's conjecture holds because psychological structures have strong symmetry properties, in particular homogeneity. In this sense, homogeneity becomes an explanatory principle. The interesting fact is that its adequacy as explanatory principle can be tested, since it implies several empirical consequences which can be checked experimentally. For this reason, homogeneity can also be seen as an explanatory principle which unifies different results, as I am going to show in the next section.

7. Unification

The foregoing discussion has shown that an algebraic principle like homogeneity can be introduced into an empirical theory and works in this context as the theoretical basis and explanatory principle of certain empirical regularities. This role of homogeneity exemplifies a use of mathematics as methodology to articulate a conceptual framework within which empirical phenomena can be set. Such a use of mathematics depends on its direct integration into an empirical theory, which is not accounted for by Field's eliminative strategy (for the reasons I have made explicit in the previous sections). For this reason Field's characterization of applicability fails to identify it.

This conclusion is made more relevant by the fact that homogeneity is employed not only to prove Torgerson's conjecture but also to derive other important psychological regularities which experimental researches have identified.

One of these regularities concerns the correlation between stimuli and responses in Stevens' experiments. Stevens 1936 plotted sound intensities measured on a physical scale against the numerical estimates of loudness obtained from his methods, finding the resulting coordinates to be distributed along a curve described by a power law like:

$$r = ks^h$$

with k, h constants, s the measure of a physical stimulus and r the psychological measure of a response.

This power law was taken by Stevens to be a psychophysical law. Several experiments on loudness and other sensory modalities (surveyed e.g. in Stevens & Galanter 1957, Stevens 1975) led to a law of the same form. In other words Stevens' experimental findings revealed a certain consistency in the way the responses arising from his experiments are related to the physical intensities of the stimuli involved: this consistency was translated by Stevens into numerical terms as a particular law.

As in the previous section, we are in presence of an empirical regularity lacking theoretical justification. Again, Narens' framework, in particular the assumption of homogeneity, provides the missing justification in the form of a theorem.

To see how, consider a structure **X** satisfying axioms N1 to N4 above. Since X is a set of physical stimuli ordered by physical intensity, it is possible to obtain from it an extensive structure ***X**, whose primitives are the order relation $\geq_X$ and a concatenation operation $+_X$ defined on the physical magnitudes in X. Being extensive and based on a continuum, ***X** is ratio scalable. At the same time, the psychological structure **X** is ratio scalable. Now a numerical function relating the representation for **X** (as dependent variable) to the representation for ***X** (as independent variable) can be proved to be a power function like Stevens', exploiting a result from the theory of functional equations[13]. The proof relies on homogeneity, because it delivers ratio scalability.

This means that, if a suitable psychological theory assuming homogeneity holds, then the experimental results obtained by Stevens can be based on an explicit theoretical principle, which is the same grounding Torgerson's conjecture. The content of this conjecture and Stevens' researches on the form of psychophysical laws have been pursued independently of each other, thus it is noteworthy that a principle can be found which unifies them.

The same principle of homogeneity lies also at the basis of the measurability of psychological structures and implies, as a further testable consequence, the non-contextuality of psychological variables (very roughly, this means that there are no absolute reference stimuli from which psychophysical scales must be constructed[14]).

The main consequence of all this is that one fundamental abstract principle, intuitively stating that psychological structures are highly symmetrical (they have many symmetries, i.e. automorphisms), provides a unitary theoretical ground for

[13] The relevant hypotheses include the continuity and positivity of the function, which are satisfied here. More importantly, they require that a change of scale on ***X** should induce a corresponding change of scale on **X**. These conditions are spelled out rigorously in Luce 1959: 85–87 and generalized to multivariate functions in Luce 1964.

[14] Formally, contextuality means that the primitives of **X** include at least one particular, fixed stimulus *a* of X. If this happens, than all the automorphisms of **X** must preserve *a*, in which case there is no automorphism sending *a* into any other element of X. This contradicts homogeneity. Contraposing, homogeneity implies non-contextuality.

several psychological facts, some of which have actually been confirmed by experiment.

As a result, the theory of Narens 1996 proves important in scientific investigations and it offers an example of the role abstract, mathematical properties may assume in providing a broad theoretical background for many experimentally identified regularities. Note that this is not achieved through the introduction of a representation but simply by using certain mathematical properties to build up an empirical theory.

8. Mathematical properties and experimental research

The last two sections have shown how mathematical properties can shape an empirical theory and work as general principles which provide a unified theoretical justification for several empirical facts.

This particular use of mathematics does not depend on the presence of a representation, because the relevant mathematical concepts are directly integrated into an empirical theory. As a consequence, the resulting empirical theory contains mathematical references (e.g. to functions and groups) which it would not make sense to eliminate, because this would entail the elimination of the empirical theory itself.

Since Field's treatment of scientific theories implies the elimination of mathematical references and makes the applicability of mathematics depend upon the presence of a representation, it cannot deal, and thus account for, the uses of mathematics I have illustrated so far with reference to the psychological theory of Narens 1996.

This is also true of a final use of mathematics I wish to mention here, which consists in orienting experimental researches. As I have already stressed, the axioms of Narens 1996 give conditions on a set of subjective responses of the form (x, $\mathbf{p}$, t) which are designed to recover the axioms of complete scalar structures. These axioms, which are algebraic, play an important role when set in the context of Narens' empirical theory. One of them, namely homogeneity, has an explanatory relevance.

Another axiom, namely commutativity, although not equally relevant, is testable. Since commutativity is a key combinatorial property needed in order to prove the ratio scalability of $\mathbf{X}$, testing it is of considerable importance. If confirmed, commutativity provides evidence for the correctness of the theory of Narens 1996 and the possibility of measuring subjective responses or sensations. Note however that the relevance of commutativity is not revealed by experiment, but established by the mathematical theory of ratio scalability.

Commutativity and its testability are important because it is an axiom of the theory of complete scalar structures, i.e. a condition needed to prove the measurability of an empirical structure. Thus, commutativity is isolated as a relevant property at a theoretical, in fact mathematical, level. Its theoretical relevance motivates its testing and so it is mathematical analysis which ultimately identifies the empirical properties it is important to check: in this sense it orients experimental researches.

It is noteworthy that attention to a property like the commutativity of psychological estimates has only arisen after Narens 1996 was published, i.e. after a formal theory of measurement for psychophysical experiment was in place. This shows that in some cases it is not really by experimental investigations but rather through theoretical reflection that the empirical regularities crucial to ensure the acceptability of certain numerical methods are identified.

The case of commutativity is an interesting one, because it shows a nice harmony between theoretical assumptions and experimental confirmation. The property of commutativity spelled out in N3 says, in empirical terms, that the performance of a finite number of tasks on a physical continuum generates a response which is independent of the order in which the tasks are performed.

In the context of difference and ratio functions, illustrated in section 6, commutativity takes two different forms: it either says that ratio functions commute, or that ratio and difference functions commute. The second property is called generalized commutativity. It is clear that, for short sequences of two or three estimation tasks, these properties can be tested.

Recent studies, particularly Ellermeier & Faulhammer 2000 and Ellermeier et al. 2003 (see also Zimmer 2005), have tested commutativity and generalized commutativity, finding both to hold. The basic combinatorial assumption of Narens 1996 is thus confirmed.

9. Extending Field's account

The discussion of psychophysics just carried out has provided a concrete example of the manifold uses of mathematics in science. In particular, it has become clear that many of them are neglected by Field's account, which does not pay sufficient attention to the crucial importance of mathematics in directing empirical theory-construction, providing explanatory principles for empirical phenomena and orienting experimental researches.

These uses of mathematics are characteristic of a theory like that of Narens 1996. The discussion of the last two chapters has made clear that applicability cannot be adequately characterized for this theory if Field's eliminative strategy is retained. This is because the mathematical references occurring in the theory of complete scalar structures are employed in an essential way in the theory of Narens 1996. In particular, axioms N1 to N4 are used to define functions which satisfy the axioms for complete scalar structures. If functions and properties of functions were always to be eliminated when they occur within a scientific theory, as Field 1980 suggests, then the theory of Narens 1996 would become useless, as it is impossible to develop it without talking about functions.

As I have pointed out, this fact does not imply that Field's approach to scientific theories should be simply rejected, but it requires its revision, in particular it forbids it to be a purely eliminative approach, if it is to deliver an acceptable account of applicability.

Now, the consequence of this result which is of interest to me is that Field's approach to scientific theories, without revision, generates an inadequate account of applicability. The question then arises whether it is possible to recover the important insights provided by Field on applicability and embed them within a wider, more comprehensive and realistic account. I will reach a positive answer to this question only in chapter 5. This chapter has, among other things, served the purpose of clarifying that an account improving on Field has to include many different uses of mathematics in applications, not considered by him.

The next chapter will serve the purpose of showing in which direction a new and better characterization of applicability has to extend Field's. Observations from previous sections of this chapter already provide information on this point. To see this, recall that the algebraic theory of complete scalar structures has been applied to psychophysics in a way which did not require the mediation of a representation.

Yet applicability arises for Field when a mathematical model can be found, which reflects the structural properties of an empirical domain. The empirically meaningful use of mathematical properties depends on their being abstract counterparts of empirical properties, determined by a representation. This is not what happens with Narens' theory, since mathematical concepts and properties are integrated into it and they do not arise from a mathematical representation of suitable empirical concepts and properties.

Thus, the fact that Field explains applicability on the basis of representation is a problem for him, since he cannot take into account applications of mathematics which can be carried out without a representation in place or a representation theorem being proved.

This is true of the psychophysical example discussed here, but the latter is by no means an isolated case. There are many other forms of applications which do not rely on representations. In the next two chapters I will provide evidence for this observation and consider several cases in which mathematics is applied without relying on representations.

This conclusion will show that a comprehensive characterization of applicability which retains the benefits of Field's account and overcomes its drawbacks must generalize that account to encompass both the applications of mathematics which are based on representations and the applications which are not. I reach this generalization in chapter 5 and prepare the ground for its formulation in the next chapter

CHAPTER 4: APPLICABILITY WITHOUT REPRESENTATION

1. Applicability and structure-preservation

Almost all of the applications I have discussed so far rely on representation theorems, i.e. on the relation of structure-preservation, which holds between an empirical setting (modelling suitable empirical axioms) and a numerical setting[1]. The same is true of the applications discussed in Field 1980. More precisely, since Field's treatment of scientific theories is based on proving representation theorems, the account of applicability he gives depends on structure-preservation[2]. The latter ensures that the structural features of an empirical setting can be translated into those of a mathematical model: the possibility of using mathematics to study the empirical world rests, according to Field, on this fact.

It is clear that an account of applicability based on mappings and structure-preservation is quite appealing, since it provides a clear and formally precise account of the relationship between empirical structures and the mathematical models used to investigate them. This is probably one of the reasons why structure-preservation and mappings are often taken to be the basic ingredients of a satisfactory explanation of applicability. At the same time, they constitute a natural and plausible way of characterizing it for several versions of structuralism in the philosophy of mathematics and the philosophy of science. This is probably one reason why applicability is more or less explicitly coupled with structure-preservation in a wide variety of philosophical works.

For example, this coupling is entirely consonant with Shapiro's and Resnik's distinct versions of mathematical structuralism (expounded respectively in Resnik 1997 and Shapiro 1997) as confirmed, most notably, by the resemblance of the framework of Field 1980 to a structuralist framework, pointed out in Resnik 1985a-b and Shapiro 2000: 237. Furthermore, Pincock 2004a: 71 observes that a mapping-based approach to applicability is well-suited to other forms of structuralism like

[1] In the psychophysical theory examined in the previous chapter, the mathematical concepts and properties of the algebraic axioms S1 to S3 are applied to an empirical theory without making use of a representation, as I already pointed out (this fact is distinct from the numerical representabiilty of the models of the psychophysical theory). Nevertheless, the applicability of these concepts depends on their ability to capture the features of some empirical structure. Thus, mathematics is applied in virtue of its ability to characterize the structure of an empirical setting, as happens when representations are employed. The applications of mathematics I discuss here and in the next chapter do not rely on the fact that the mathematics employed describes the structural features of an empirical system.

[2] Since a representation theorem in essence states the existence of a structure-preserving function.

Hellman's (see Hellman 1989 and the concise survey in Hellman 2005), which are based, unlike Resnik's and Shapiro's positions, on an anti-realist view on the existence of mathematical entities[3]. In fact, the view that the applicability of mathematics rests on mappings seems relatively popular among philosophers of mathematics, as shown e.g. by Burgess & Rosen 1997: 69–70, Baker 2003 and Pincock 2004a-b.

Similar positions are also characteristic of the structuralist tradition in the philosophy of science. It is fair to say that this has originated from Patrick Suppes' very influential idea of characterizing empirical settings by means of set-theoretical predicates[4]. This idea has stimulated a vast amount of work characterizing scientific theories in model-theoretical terms[5] and the application of mathematics in terms of representation theorems for the relevant empirical structures. Clearly, the concepts of mapping and structure-preservation are essential in this context.

These quick references show that the idea, so important to Field, that applications are based on mappings has become pervasive in philosophy. This idea is certainly correct for many important applications. What I want to show in this chapter, though, is that it should not be extended to the totality of applications. There are circumstances in which representation theorems do not play any role in making mathematics applicable: in this case I talk about *non-representational* applications. In the next sections I will provide evidence for this claim by looking at one example of non-representational application in detail. My objective is twofold: on the one hand, I want to show with this example that there are non-representational applications. As a consequence, Field's account of applicability (as any other account exclusively focusing on structure-preservation) is too restrictive: being based on representations, it cannot describe non-representational applications.

On the other hand, I am going to extract from the instance of non-representational application I will discuss a characterization of applicability which is alternative to the one based on structure-preservation. I will give this characterization in outline at the end of this chapter. In the next chapter I am going to make it more precise and show

[3] Thus, the idea that applications are based on mappings may be considered more stable than the ontological tendencies exhibited by different versions of mathematical structuralism.

[4] A precise formalization of this concept is given in Da Costa & Chuaqui 1988, while a comprehensive survey of Suppes' own work based on this perspective is presented in Suppes 2002.

[5] Two famous and important books doing this are Sneed 1971 and Stegmüller 1979. The idea of using set-theoretic predicates to characterize set theory is also at the core of the mathematical framework set up in Krantz et al. 1971, Luce et al. 1990a-b and many other works on measurement.

that it can be used as the basis for an account of applicability which improves on Field's, because it encompasses both representational and non-representational applications.

2. An archaeological problem

My example of a non-representational application comes from archaeology. It has to do with determining chronologies for families of deposits in a site (e.g. graves in a field). Often, even if the time-span of the deposits is known, their relative chronology is ignored and has to be reconstructed (i.e. the temporal succession of the deposits has to be determined). In this context, the relevant empirical structure is given by the deposits with a relation of chronological order defined on them. The problem consists in recovering this order from archaeological data coming from the deposits, e.g. concerning the types of artefacts contained in each of them.

Given these preliminaries, it is already possible to give an impression of the reason why this problem requires a non-representational application of mathematics. A representational application is based on the fact that a mathematical structure is introduced to study an empirical setting by means of a suitable mapping. The applicability of mathematics depends on the link between the mathematical structure and the empirical setting. The mathematics which is employed in the present context, however, is applicable without being introduced by means of a structure-preserving function between a suitable mathematical model and a mathematical structure.

To see why, two things must be observed. Firstly, the empirical structure of relevance in the archaeological problem is an ordering of deposits. Secondly, if mathematics were applied to it through a representation, this representation would have to link the ordering of deposits to some other ordering, e.g. a numerical one.

Now, the problem at hand is that of determining the particular ordering of a family of deposits. The above representation can be used to introduce mathematics on this problem in two ways, namely in an existential or in a 'constructive' way. In the case of the existential way, only the representability relation is used, irrespectively of whether a representation is actually constructed. Representability can be established here: since the deposits determine a finite linear ordering relative to their chronology, it follows that they are homomorphic to a mathematical ordered structure, e.g. a numerical one. Thus, one might use this fact to introduce mathematics in the archaeological problem. As for the constructive way of introducing a representation, this consists in providing an explicit link between the deposits to be ordered chronologically and their mathematical model, e.g. ordered numbers.

The point is that neither the existential nor the constructive use of a representation to introduce mathematics leads to the solution of the archaeological problem. Clearly, the existential use is of no relevance here, because it does not provide any information about the *particular* ordering of deposits being sought.

The constructive use is not relevant either: if we knew how to represent an empirical ordering, say numerically (i.e. if we knew which numbers are assigned to which deposits), this would depend on the fact that we already know how the empirical ordering is structured, which is precisely what is not known and has to be discovered here.

In short, the only representation which is relevant to the archaeological problem, i.e. the one joining ordered deposits to a suitable mathematical structure, cannot be used to solve the problem through the introduction of a mathematical structure. For this reason, the mathematics which is applied to determine the sought chronology is not applicable in virtue of a representation: it does not rely on a relation of structure-preservation between an empirical and a mathematical model. We have an example of non-representational application.

The following sections are devoted to explaining how mathematics is applied in this context: it is nonetheless convenient to briefly illustrate it here. The starting point for the introduction of mathematics to determine an archaeological chronology is a set of data describing the archaeological composition of a family of deposits (saying which artefacts occur in which deposits, as I noted above). The data is presented in a particular form, i.e. as a matrix. The rows of this matrix can be associated to the deposits, so that the chronological order of the deposits corresponds to a particular ordering of the rows[6]. Now, under a suitable empirical hypothesis, it is possible to show that, when the rows of the matrix are in the right order (i.e. the one reflecting the chronological ordering of the deposits), the entries of the matrix assume a particular configuration. To apply mathematics here amounts to finding a

[6] This is a structural correspondence, but the initial ordering of the matrix's rows does not in general reflect the chronology (otherwise no problems would arise), so this structural correspondence has to be made explicit and it is to this effect that mathematics is introduced (note that it is not introduced on the basis of the correspondence, but to identify it).

method to bring the matrix into the desired configuration, from which a chronology can be read off.

Thus, mathematics is applied to the analysis of a set of data (presented in matrix form): it is in particular used to develop an algorithm which decides an empirically relevant property of the data set[7]. The same approach, based on data analysis and the construction of an algorithm, can be found in many other applications: for example, the mathematical techniques developed to solve the archaeological problem I have described are also adopted in a variety of other contexts, e.g. psychology, genetics and management[8] (cf. the remarks in Shuchat 1984: 3). Thus, even if I focus only on one example, my discussion can easily be generalized to a variety of different applications and empirical problems.

3. Seriation

The archaeological problem of determining a chronological ordering described in section 2 is called the problem of *seriation*. There are cases in which a seriation cannot be obtained exploiting chemical or physical techniques (e.g. carbon-dating) and informative documents like inscriptions are unavailable. It is because these cases arise that a mathematical solution to the seriation problem has been looked for.

Its starting point is a particular presentation of the information concerning the composition of the deposits to be chronologically ordered. Suppose there are m of them, containing n different types of artefacts overall: in this case it is possible to write down an *incidence matrix* M, saying which artefacts are present in which graves. The matrix is defined by the following two conditions, determining all of its entries:

i) $m_{ij} = 0$ if the j-th artefact does not occur in the i-th grave;

ii) $m_{ij} = 1$ if the j-th artefact occurs in the i-th grave.

It follows that M has 0 or 1 alone as possible entries. This matrix may be called a *data space*. Mathematics is applied to seriation as an analysis of the combinatorial properties of the data space (this remark will be clarified in the next sections). In order for this use of mathematics to yield the desired chronological information, it is necessary to establish an explicit connection between the given data space and the

[7] This is not a structural property of the deposits, as will be seen, but only a property of the data in matrix form.

[8] An application to genetics is discussed in the appendix.

problem to be solved. As I will show in a moment, once this link is established, the problem of seriation can be formulated in purely mathematical terms and its solution corresponds to the solution of a mathematical problem concerning the properties of M. The link between the data space and the seriation problem is provided by the following empirical assumption:

> (s) any artefact present in the graves belonging to a given period is also present in all graves belonging to any later period (within the time-span considered)[9].

Assumption (s) is equivalent to a particular property of the incidence matrix M[10]. To see this, suppose a chronology for the deposits is known: then it can be used to order the rows of M. This is formally accomplished by applying a permutation matrix P to M (which may be the identity matrix of order m[11]). The effect of this is merely to change the order of the rows in M, so one can think of the application of P to M, denoted PM, simply as a concrete operation which consists in rewriting the rows of M in a different order (or the same, in case the identity permutation is applied). Because of assumption (s), the permuted matrix PM must look a particular way, as I will now demonstrate.

Suppose that the first row in PM corresponds to the earliest deposit, and the subsequent rows follow in chronological order. Suppose, in addition, that the *ij*-th entry of PM is 1: this means that the *j*-th artefact occurs in the *i*-th deposit. The empirical assumption (s) implies that the *j*-th artefact has also to be in all deposits chronologically posterior to the *i*-th, i.e. in all deposits corresponding to the rows of PM from $i + 1$ to m: in other words, if we look at the *j*-th column, it must contain an uninterrupted sequence of 1's from i to m.

The remaining entries of the *j*-th column, lying above the *i*-th, are also constrained by (s). In particular, we cannot have the following configuration in the *j*-thcolumn:

$$\begin{array}{cc} 1 & i-2 \\ 0 & i-1 \\ 1 & i \end{array}$$

[9] This hypothesis is an idealization, especially if the graves span a very large epoch, in which case old-fashioned artefacts may disappear when new stylistic trends emerge and impose themselves (think e.g. of pottery): however, for shorter time spans the assumption appears more realistic. A discussion of idealization (s) is given in section 7.

[10] A seriation method which does not assume (s) but uses it only as a test hypothesis is described in section 10.

[11] That is, the matrix with m rows and m columns, whose entries are 1 along the diagonal and 0 everywhere else.

where the indices on the right mark the relevant rows of the j-th column. The reason is that (s) entails that, if 1 occurs at some point along a column, then it has to occur at all points below it: this requirement is violated in the above arrangement. These remarks suffice to show that PM must be such that, in any of its columns, it is never the case that 0-entries interrupt or are interposed between 1-entries. All 1-entries must be bunched together: they must occur consecutively in each column. A matrix satisfying this property is called a *Petrie* matrix. Any incidence matrix M which can be turned by a row-permutation into a Petrie matrix is called *pre-Petrie*. In this case M is pre-Petrie and PM is Petrie. The assumption (s) is equivalent to the fact that the incidence matrix M is pre-Petrie.

But, whereas (s) is an empirical statement, the property 'pre-Petrie' is a combinatorial property of incidence matrices, so it is possible to study it in a purely mathematical fashion to obtain empirically meaningful results. Given this background, the solution of two mathematical problems provides a solution to the archaeological problem. The mathematical problems are the following:

i) Find necessary and sufficient conditions under which an incidence matrix M is a pre-Petrie matrix.
ii) Find an algorithm to decide whether an incidence matrix M is pre-Petrie and, in case it is, to bring it into Petrie form.

If (i) is solved, its solution can be exploited to identify the properties which have to be checked by the algorithm which solves problem (ii). When this algorithm is in place, the seriation problem can be solved by applying the algorithm to an incidence matrix constructed from archaeological data. I will discuss the way (i) and (ii) are mathematically dealt with in sections 5 and 6. Before doing it, I wish to further clarify the reason why the use of mathematics in seriation is non-representational.

4. The non-representational character of the seriation problem

In this section I intend to articulate my previous explanation (cf. section 2) of the reason why mathematics is applied to seriation in a non-representational fashion. I have already pointed out that it be applied to seriation by means of a representation. If such a representation were available, it would map the chronological ordering of a family of deposits into some numerical ordering. But the availability of such a representation, in an existential or constructive sense, is not relevant to solving the given problem (since either it doesn't solve it or it implies that it has already been solved).

Even if this point is granted, something may be objected against the idea that the mathematics of seriation is not introduced in virtue of some structural correspondence established by a representation. It may be insisted that there is in fact a representational component in the way seriation is formulated. To see why, observe that the incidence matrix M defined in the previous section can be conceived of as a representation of the empirical relation of incidence which exists between deposits and artefacts. The corresponding representation theorem states that the j-th artefact lies in the i-th deposit if and only if the entry of M corresponding to the intersection of its i-th row and j-th column is 1.

Although this is correct, it does not contradict the non-representational character of the seriation problem. The representation just described does not map an empirical structure into a mathematical one, but rather joins two empirical structures. To see this point, consider how M is actually constructed: one may start with a table listing the artefacts occurring in the deposits being investigated. Then, by looking at each deposit, one can write '1' next to the artefacts in the list which occur in the deposit and '0' next to the artefacts not occurring in it: for that matter, any labels a, b are equally good[12]. As a result, a finite array of 0 and 1 (or a and b) is produced. This is an object given for syntactic manipulation and mathematics is used to guide this kind of manipulation[13].

There is a difference between this use of mathematics and its use in presence of a representation relating a mathematical model to an empirical setting. In the latter case, the mathematical model's properties are exploited to make inferences whose conclusions can later be interpreted into the empirical setting under investigation. This does not change the structural configuration of the mathematical model.

On the contrary, in the archaeological case the use of mathematics over M changes its configurations. In addition it is unnecessary, in order to find a seriation, to keep track of how the incidence relations between artefacts and deposits[14]change following changes in M's configuration (this point will be explained in a moment). In other words, mathematics is not applied because it preserves the empirical

[12] The same holds for all the subsequent uses of incidence matrix I will describe.
[13] Most importantly, it is used to prove the conditions under which suitable manipulations lead to a particular configuration.
[14] These are represented, as already observed, by the row-column intersections in M.

structure represented by M and it is also used to recombine its elements: none of these things ever happen when mathematics is introduced by means of a representation.

It then follows that we are not in presence of a representational application: mathematics is not introduced because it reflects the properties of an empirical setting, possibly represented by M[15], but only because it provides an effective methodology to deal with the syntactically presented information of M. So, even if M is conceived as a representation, it does not link an empirical structure to a mathematical one, but only an empirical structure to another empirical structure M, understood as a syntactic object. Mathematics is then applicable to M because M can be studied combinatorially as a discrete array of parts and because it encodes archaeological information.

In fact, we don't even need to talk about M being a representation, because it is not necessary to have identified an empirical structure on deposits and artefacts to construct M *via* some mapping: it is only required to list the artefacts occurring in the several deposits in a suitable way. The use of a representation to describe M is dispensable and, if allowed, only determines a link between empirical structures.

Moreover, given M, no use of representation is involved in the application of mathematics to seriation. With M in place, mathematics is introduced to bring it into a Petrie form. Such a use of mathematics does not rest on M being a representation or on the presence of any other representation. It only relies on the assumption (s) of the previous section and the fact that a row-permutation of M is a Petrie matrix. In principle, one could even dispense with the assumption that M describes empirical data. One could simply start with a given incidence matrix M, turn it by a mathematical algorithm into a Petrie matrix PM (if possible), and then check whether its rows describe the distribution of certain artefacts in a family of deposits[16]. If so, then the rows of PM give a relative chronology for the deposits. Nowhere in this mathematical treatment of seriation does the concept of representation play an essential role to allow the introduction of mathematics. In fact, a representation can be constructed only *after* mathematics has been applied. Knowing the chronology of a family of deposits, one can proceed to represent it, e.g. numerically.

[15] In addition, mathematics is not applied to study the incidence relation between artefacts and deposits, but another empirical structure, i.e. the chronological ordering of deposits.
[16] The checking does not require any structure-preserving mapping to be carried out.

With these observations I hope to have made it sufficiently clear why it makes sense to regard the application of mathematics to seriation as an example of non-representational application.

5. Non-representational applicability and reconfiguration

Having clarified the non-representational character of applicability in seriation, I now turn to isolating the distinctive features of the way mathematics is applied to it. The characterization I propose can be extended to other forms of non-representational applications and, as will be seen in the next chapter, ultimately generalized to representational ones.

To fix ideas, however, my focus remains on seriation and my attempt consists in describing how mathematics is applied in this context. It was already clear in section 3 that mathematics acts on the 'data space' M. The latter is the starting point for its application. Due to a suitable empirical hypothesis ((s) of section 3), it is possible to correlate a particular configurational property of the data space to the solution of seriation. More concretely, when M is turned into Petrie form, the order of its rows expresses a relative chronology of deposits. For this reason, once the data space is given, mathematics can be applied to it as a finite object which can be subjected to certain combinatorial transformations, i.e. row-permutations.

This application consists, roughly, in an 'action' of mathematical concepts and theorems over the data space. The problem now is to qualify in more precise terms this action, so that the non-representational use of mathematics in seriation can be characterized as a particular treatment of a data space.

In order to achieve this result, it is necessary to ask again what mathematics has to do in order to provide a solution to the seriation problem. The answer is clear: it has to provide an algorithm which turns an incidence matrix describing archaeological data into Petrie form. The algorithm must provide a uniform and general method to deal with incidence matrices. For this reason, it must work as follows: firstly, it is given as input an incidence matrix M; then it checks whether assumption (s) holds by checking whether M is pre-Petrie; finally, if M is indeed pre-Petrie, the algorithm permutes its rows and turns it into a Petrie matrix.

The first step of the algorithm requires the availability of the data space M. The second step requires a characterization of the property of being pre-Petrie, i.e. the isolation of necessary and sufficient conditions for it, which can be checked by the algorithm. The final step requires a combinatorial method of computation turning M into Petrie form. Note that the last two requirements are problems (i) and (ii) of section 3. Mathematics is applied to the data space M to solve these two problems. To be precise, mathematical concepts and theorems are applied to M in order to solve them: in this sense mathematics works as a methodology applied to carry out a form of data analysis.

In order to clarify the last remarks, it is convenient to have a look in some detail at the way problems (i) and (ii) are actually solved. This requires some preliminaries, which I provide in this section, while (i) and (ii) will be dealt with in section 6. In particular, (ii) can be solved when (i) is, and so the preliminaries I need to cover concern the strategy adopted to reach a solution of (i), i.e. to determine necessary and sufficient conditions for the property of being pre-Petrie.

The way in which these conditions are identified is particularly interesting, because it exemplifies an important aspect of the use of mathematics in applications, which I will later take as one of the main features of my characterization of non-representational applicability.

Necessary and sufficient conditions for the property of being pre-Petrie can be identified by effecting a *reconfiguration* of the data space M. Generally speaking, this means that a certain structure is defined on M, and so additional mathematical concepts and theorems can be applied to study it. In the present case, M is endowed with a graph-theoretical structure. Its introduction corresponds to an expansion of the conceptual resources available to deal with the seriation problem. Finding a solution to this problem then becomes easier, because it is possible to rely not only on the concept of incidence matrix, but also on that of graph and the relevant graph-theoretical theorems, which become applicable to M.

It remains to be explained how this application of graphs is obtained by a reconfiguration of the data space M. To this end, a technical digression is necessary. Consider the columns of M: each of them is a finite, ordered sequence of 0 and 1. For example, suppose two of M's columns are:

$$m_1 = (1, 0, 1) \text{ and } m_2 = (1, 1, 0).$$

These are two ordered sequences which have a '1' entry in the first projection, i.e. their first elements are equal and 1. Their second and third projections, on the other hand, are different. If, for any two columns of an incidence matrix M, some but not all of their corresponding projections are equal and 1, these columns are said to *overlap*. Thus, m_1 and m_2 overlap.

With these preliminary definitions in place, it is possible to define a graph, called the *overlap graph*, on an incidence matrix M. A graph (without multiple edges or loops) can be set-theoretically defined as an ordered couple $\mathbf{G} = \langle V, E \rangle$, where V is a nonempty set of 'vertices', on which a binary relation 'edge', symbolized by E, is defined: this relation is irreflexive and symmetric[17]. Finite graphs may be rendered pictorially representing their vertices by dots and joining any two dots by a straight line whenever E holds on the corresponding vertices.

Now, in order to define a graph on an incidence matrix M, it suffices to state which parts of the matrix will be taken as vertices and which relation between these parts will be taken as the 'edge' relation. The vertices of the sought graph are just the columns of M and the 'edge' relation is just the 'overlap' relation defined above. Thus we can define an overlap graph on M: by doing so, we no longer focus on the arrangement of 0 and 1 in the single entries of the matrix but only on columns and a particular relation between columns. We look at M from a different perspective and identify on it a new structure: for this reason it makes sense to talk about a reconfiguration of the data space M. Thanks to this, necessary and sufficient conditions for the property of being pre-Petrie can be spelled out.

6. Deductive analysis and algorithm construction

The discussion in the previous section has shown that an important way in which mathematics is applied to the seriation problem is by effecting a reconfiguration of a data space M. This reconfiguration makes new concepts and theorems available to study M.

In particular, the overlap graph on M determines a partition of the columns of M. This is because, in general, some columns will be non-overlapping, in which case they have no edges between them when conceived as vertices of the overlap graph. Then it is possible to divide the columns of M into groups of overlapping columns, which are *submatrices* of M. In short, the columns of different submatrices do not overlap, while all columns within the same submatrix overlap. The submatrices into which M is partitioned are called its *overlap graph components*. Exploiting this fact

[17] This means that, for any v, w in V, we have: (i) not(vEv) and (ii) if (vEw) then (wEv). For this definition of graph cf. Hedman 2004: 66.

and suitable graph-theoretical theorems (cf. Fulkerson & Gross 1965: 842–843), it is possible to prove that:

Theorem (Fulkerson & Gross 1965: 844–845):

An incidence matrix M is pre-Petrie if and only if each of its overlap graph components is.

This theorem gives necessary and sufficient conditions for the property of being pre-Petrie (thus it solves problem (i) of section 3). The way it is obtained passes through three stages: the first consists in the construction of an incidence matrix M; the second stage consists in the reconfiguration of M, which leads to the introduction of graph-theoretical concepts; the third step consists in the deductive analysis of this reconfiguration, i.e. the use of graph-theoretical theorems to study the reconfiguration of M.

From this deductive analysis the above theorem issues. Its importance lies in the fact that it reduces the problem of checking whether a matrix M is pre-Petrie to the problem of checking whether certain submatrices of M are. This theorem is important for the construction of an algorithm to turn M into a Petrie form because, intuitively, it shows that this algorithm can be obtained by reducing certain submatrices of M to Petrie form and then 'pasting together' in a suitable way the results (for the technical details see Fulkerson & Gross 1965: 845–849). Thus problem (ii) of section 3, i.e. the construction of an algorithm which decides the seriation problem, can also be solved.

The observations made so far make it possible to provide a characterization of the applicability of mathematics to seriation, which can be formulated in general terms. Applicability in this context can be understood as the deductive analysis of a data space, carried out by effecting reconfigurations of the data space itself and leading to the construction of algorithms.

Note that the concept of representation is not needed to specify any of the features of applicability as just characterized[18]. This point can be explained by looking again at seriation. In this case the reconfiguration of the data space M does not require structure-preservation. In particular, the introduction of graph-theoretical concepts does not depend on any previously given empirical structure but only on the 'logic'

[18] This does not rule out the possibility that structure-preservation may be compatible with the characterization: e.g. some data spaces may be obtained by representation, even if this is not always the case. I will say more on this point in the next chapter.

of data analysis: it is not representational because the overlap graph is not used to represent a structural aspect of an empirical setting[19] but to reduce a combinatorial property of M (being pre-Petrie) to a combinatorial property of its submatrices (as established by the theorem above).

For analogous reasons, the construction of an algorithm which decides the property of being pre-Petrie and turns M into Petrie form (if possible) is itself non-representational: it is entirely based on the manipulation of the data space M. The reason why this manipulation ends up with the solution to a concrete problem is that the algorithm leaves undisturbed all the empirical information carried by M. What it does is simply to rearrange its rows in a manner making apparent the relative chronological position of archaeological sites or deposits. The important applicative role played by mathematics is non-representational and it consists in identifying the key properties which govern the possibility of a suitable rearrangement of M.

These observations consolidate my previous remarks about the non-representational character of certain applications of mathematics. Because these applications exist, no representational account of applicability can be exhaustive: there are applications it is bound to leave out. This is in particular true for Field's account of applicability.

If a representational account of applicability is too restrictive, then it is natural to ask whether an alternative account can be given, which does not suffer of the same limitation. My discussion of seriation has already led to a characterization of applicability which is not subordinated to the concept of representation. I will show in the next chapter that this characterization can be generalized to other non-representational applications but also to representational ones, in particular the scientific theories discussed in Field 1980. The consequence of all this is that a characterization of applicability along the lines I have indicated in this section (i.e. as the deductive analysis of a data space) improves on Field's. The next chapter is devoted to establishing this result. Before doing it, however, I wish to conclude this chapter by discussing the status of the empirical assumption on which all the previous discussion of seriation has been based, namely:

[19] Empirically, the edge relation on overlap graph says that certain types of artefacts occur simultaneously in some, but not all, deposits. However, this piece of empirical information arises 'accidentally' (i.e. just because the entries of M describe occurrences of artefacts in deposits) and is not used to reach a solution of the seriation problem.

(s) any artefact present in the graves belonging to a given period is present in all graves belonging to any later period (within the time-span considered).

As I already remarked, this assumption can be taken as an idealization (I will explain why in a moment). For this reason, a discussion of its status makes it possible to check whether the account of idealization I have provided in part 1, which only involved representational applications, can be extended to non-representational applications as well.

7. Idealization and enrichment

The reason why assumption (s) is an idealization is in essence that, given an incidence matrix M constructed from archaeological data, 'one can be virtually sure that no re-arrangement of the rows will create a perfect Petrie pattern' (Kendall 1969: 566). This means that the incidence matrices encountered in archaeology are not, as a rule, pre-Petrie matrices. In other words, archaeological matrices are in general less regular then pre-Petrie matrices: the latter possess a combinatorial property which the former lack. But the property of being pre-Petrie for an incidence matrix is equivalent to the fact that it is constructed from a family of deposits satisfying (s). The assumption (s) imposes on archaeological incidence matrices a property (being pre-Petrie) which they usually lack in practice.

For this reason it is quite natural to consider (s) as a regularity idealization, not unlike the transitivity of empirical equivalence in extensive measurement (cf. ch.4 of part 1). What (s) does is to assume that archaeological incidence matrices will behave more regularly than they actually do[20]. In the same way, the assumption of the transitivity of empirical equivalence in extensive measurement assumed extensive structures to behave more regularly than they do in actual experiments.

In the latter case, the regularity idealization can be justified by showing that actual extensive structures converge to structures with a transitive empirical equivalence as the precision of measurement increases (as proved in section 10 of ch.4, part 1). Convergence shows that it is acceptable to assume the transitivity of empirical equivalence on extensive structures to be taken as enrichments of the actual experimental settings, which only approximate them.

[20] This means that they will exhibit more regular configuration patterns of their entries, at least under suitable row-permutations, than they actually do.

The problem here is similar: it consists in finding a justification for (s) and showing that it is acceptable to use (s) to describe an idealized situation which is only approximated by actual archaeological incidence matrices. Because (s) is equivalent to the fact that an incidence matrix M is pre-Petrie, it is enough to reach this conclusion for the property of being pre-Petrie.

To this effect, it suffices to observe that, despite being an idealization, not satisfied by archaeological incidence matrices, the property of being pre-Petrie plays a crucial role in tackling the seriation problem as a 'limiting' case, to which non pre-Petrie matrices can get more or less close. The idea is that, even if the property of being pre-Petrie fails for it, an incidence matrix M can still be used to indicate a chronology, provided it is turned into a form which is the closest possible to the Petrie form. It suffices to determine how much an incidence matrix differs from a pre-Petrie one or, simply, how much its 'optimal' rearrangement differs from a Petrie form.

The notion of difference here implies the possibility of comparing an idealized situation with actual one: so idealization is crucial and cannot be eliminated, because it is needed to obtain chronological information from a non-idealized situation. In this sense the use of the idealization is justified.

Several evaluations of how much an incidence matrix differs from a Petrie form when suitably rearranged are possible: a survey can be found in Shuchat 1984. Here I will not go into the mathematical details but only point out that evaluating how well a Petrie form can be approximated by an incidence matrix requires a measure of the approximation. This calls for an enrichment of the mathematical techniques and concepts adopted to deal with seriation. The ones explored in the previous sections were only based on incidence relations, whereas, as soon as approximations have to be evaluated, additional concepts and theorems are needed (see Shuchat 1984: 4–12).

This suggests that the notion of idealization as enrichment developed in part 1 for extensive measurement is in some respects similar to that occurring in non-representational applications, provided a different qualification of 'enrichment' is adopted. In the extensive and, more generally, representational case idealization is responsible for the embedding of an empirical setting within a richer framework satisfying certain strong mathematical properties.

In the archaeological, non-representational case, the use of idealizing hypotheses is responsible for an enrichment of the space of mathematical concepts and theorems adopted to deal with actual empirical problems. I will show in the next chapter that this notion of idealization applies to other non-representational applications as well.

Appendix: matrices, graphs, and genetic mapping

As I already observed, incidence matrices arise in a variety of applicative problems, not only in archaeology: for this reason, the mathematical techniques used to deal with them are important for many other disciplines. These techniques have a non-representational character and the empirical problems which are solved by their aid are examples of non-representational applications.

Here I wish to illustrate one of them, in order to give a concrete idea of the fact that my discussion of seriation can be readily generalized to other quite different applications. The example I have chosen has to do with genetic mapping, i.e. the problem of determining the geometrical arrangement of the hereditary material. Pioneering studies in this areas have been conducted by T.H. Morgan and A. H. Sturtevant (see in particular Morgan 1911 and Sturtevant 1913), who investigated the structure of sex-related chromosomes in *Drosophila Melanogaster*, more precisely the arrangement of genes on them, by looking at the frequency of crossing-overs occurring within populations of this insect (such frequency being understood as an index of the distance between two genes on a chromosome). Their experimental conclusions, a part of current genetic theory (see for instance Lewins 2000: 3), have led to a linear model of the chromosomes, understood as sequences of linearly ordered genes.

The related question of whether individual genes follow the same pattern and thus can be regarded as linearly ordered portions of the hereditary material, was firstly addressed in Benzer 1959. Benzer's study was carried out on a particular virus, phage T4, infecting two different bacterial strains, called B and K. His experiments were based on the following facts:

> '[…] T4 phage of the "standard" form can multiply normally in either of two bacterial host strains, B or K. From the standard form of T4, rII mutants occasionally arise […] that are defective in growth on K. […] Thus, the standard type is able to perform some necessary function which the mutant cannot. Our interest is in the genetic structure that controls this particular function of the phage' (Benzer 1959: 1608).

The problem for Benzer was to decide whether the genetic structure responsible for the phage's ability to grow in K, called the rII region, is a linear array of parts. One way of checking it is by putting different mutants in K and seeing whether they grow or not: if they are different, they may, *taken together*, possess the full hereditary material of the rII region, which is needed for growth, even though, taken separately, they do not. In this case it may be supposed that disjoint parts of the rII region of each mutant are blemished, making impossible for them to grow individually in K, but possible for them to grow there together. If so, they may eventually produce standard phages, which will individually grow in K. In other words, by testing whether the progeny of two mutants can or cannot grow in K, it is possible to decide whether the mutants' blemishes are disjoint or they overlap. If they are disjoint, there is the possibility that they 'recombine' to produce standard phages, while, if they overlap, this possibility cannot arise. As a consequence, it is possible to give an incidence matrix N describing the behaviour of T4 mutants as follows:

i) $n_{ij} = 1$ if the i-th and j-th mutant produce standard type phages in K;

ii) $n_{ij} = 0$ if the i-th and j-th mutant do not produce standard type phages in K.

Benzer used an incidence matrix like N, together with information about the altered elements of the phages, to produce strong evidence in favour of a linear model of the gene[21]. However, linearity can be checked on the basis of N alone, i.e. with information coming exclusively from the results of the recombinations of the genetic material of mutant phages.

This is because the columns of N can be associated to certain alterations of the rII region: it is assumed that such alterations are connected, i.e. not interrupted by unaltered portions of the hereditary material. In this case N provides information about the way the blemished portions of rII overlap. Because of this, it is possible to use N to describe the overlap relations encoded in it.

These relations, in turn, can be used to determine a graph and, in case this is an interval graph[22], it can be proved that it describes a linear array. For these reasons, the problem of deciding whether the fine structure of genes is linear can be reduced

[21] He also supposed that there are no unaltered portions of the rII region between altered ones (this hypothesis does not force a linear model, because it is for instance compatible with the presence of branches, as remarked in Benzer 1959: 1619).

[22] An interval graph is a graph whose vertices can be associated to intervals on the real line, and whose edges represent nonempty intersections between these intervals.

to the mathematical problem of deciding *when a certain incidence matrix determines an interval graph*. It turns out that the property of being pre-Petrie can be used to isolate necessary and sufficient conditions to solve this problem (see Fulkerson & Gross 1965: 853 for details).

Mathematics is used here in a way which very closely resembles the case of seriation. The starting point for the application of mathematics is an incidence matrix, i.e. the same type of data space encountered in archaeology. This data space is subjected to reconfiguration, in such a way that graph-theoretical concepts can be applied to it. Once this is done, graph-theoretical theorems can be used to decide whether the reconfiguration is an interval graph, by reducing this problem to deciding whether a certain incidence matrix is pre-Petrie, which can be done by a suitable algorithm.

We thus have another example of non-representational application, with the same structure as the archaeological example. This already suggests the possibility of extending the characterization of applicability I have given in this chapter[23] to an entire family of empirical problems, namely all those for which techniques similar to those used for seriation work. In the next chapter I will show that the same characterization of applicability extends considerably further.

[23] As the deductive analysis of a data space.

CHAPTER 5: DATA SPACES AND APPLICABILITY

1. Extending the account

The previous chapter contained a detailed analysis of a particular non-representational application of mathematics, taken from archaeology. Mathematics could be used to solve an empirical problem (seriation) because it worked on a data space carrying empirical information (the composition of archaeological deposits). In this chapter I introduce a general characterization of the applicability of mathematics which is based on the analysis of the archaeological example developed in the previous chapter.

Three concepts have emerged from that analysis, namely those of *data space*, *reconfiguration of a data space* and *deductive analysis*. In the archaeological case the data space was an incidence matrix. By suitable definitions a graph could be defined on that matrix (more precisely on its columns): this particular isolation of a new structure from the given data space was called a reconfiguration. Finally, the relevant deductive analysis consisted in the application of theorems from graph theory (which I did not discuss) in order to determine necessary and sufficient conditions for the property 'pre-Petrie', which led to the construction of an algorithm and, thus, to the solution of the seriation problem.

In general terms, we may think of a data space as a particular symbolic presentation of empirical information: this information does not necessarily have to specify the domain of an empirical setting and describe the structural properties of its possible relations and operations. An incidence matrix is a case in point, since it can be understood just as a system of labels indicating the presence or absence of something (e.g. an artefact in a deposit). Moreover, as will be seen in the following sections, the data space doesn't have to be finite and effectively given: it may be idealized, e.g. infinitely large.

Once a possibly idealized data space is given, and a given empirical problem is specified, mathematics can be applied to work on the data space and solve the problem. In other words mathematics has an essentially conceptual role: it articulates a methodology to process empirical information in a suitable way and reach a solution (if one exists) to empirical problems. In general, the processing of the information relies on an expansion of the available conceptual resources, which is obtained by defining structures on the data space (graphs, in the archaeological case but also geometrical objects or coordinate systems in other cases to be discussed later). This isolation of structures from the data space may be called data space reconfiguration.

Once the original data space and its related structures (the reconfigurations) are in place, mathematical theorems concerning these structures can be applied to study them and, ultimately, to reach a solution to the given empirical problem. The systematic application of theorems to data spaces and their reconfigurations is what I call deductive analysis[1].

The characterization of the applicability of mathematics I have just illustrated looks quite general. However, as things stand, it may be objected that it has been constructed on the basis of really just one example and so it is difficult to decide whether it really captures the essential features of other situations. The task of this chapter is to show by examples and some general arguments that this is the case. In fact, the concept of applicability as the deductive analysis of a data space and its reconfigurations captures the salient features of all the forms of applications examined so far, i.e. the application of mathematics to extensive measurement, the theories of measurement which reduce to it, the theories discussed in Field 1980, the generalization of extensive measurement employed in psychophysics, archaeology, genetics and also economics, ecology and several other disciplines, as I will show in the following sections.

The generality of my account of applicability depends on two facts: (i) on the one hand, it is adequate to describe several different mathematical techniques used in applications; (ii) on the other hand, it captures the salient features of particular mathematical techniques which find applications in a very wide variety of situations.

Some evidence for (ii) has already been provided by pointing out that the mathematics used in archaeology to find seriations has applications in rather different fields, e.g. genetic mapping. In this chapter I strengthen this conclusion by showing that my account of applicability includes an important family of optimization problems, which can be treated in a mathematically unitary way but arise in a wide variety of different areas, from farming to cosmology (cf. the discussion in Franklin 1980: 1–9). I look at the example from optimization in detail because, besides being

[1] Deductive analysis relies on theorems and may lead to the construction of algorithms but also to other proofs solving the problem at hand, e.g. representation proofs (again, this point will be explained later).

amenable to my account based on data spaces, it provides a significant instance of non-representational application. In particular, it shows that a wide family of applicative problems is solved through a non-representational use of mathematics. This suggests that non-representational applications are not less pervasive than representational ones and thus reinforces the impression that an adequate account of applicability should take them into account.

After discussing optimization, I intend to turn to (i) and give explicit evidence for the generality of my account of applicability by showing how to reduce to it all the representational applications of mathematics examined in previous chapters. This conclusion is significant because it implies that, when applicability is qualified in terms of data spaces, the resulting account properly includes that of Field 1980, in the sense that it can deal with the scientific theories discussed there and many additional ones, which Field cannot take into account in view of their non-representational character.

Before proceeding to illustrate this and the previous points, let me take the next section to identify a particular difference between representational and non-representational applications, which will be important both to clarify why an account of applicability which encompasses non-representational and representational cases is desirable and to elucidate the particular features of the non-representational applications I am going to discuss in sections 4 to 6 below.

2. Purpose of representation

The distinction between representational and non-representational applications can be based on the purposes for which mathematics is applied in each case. Certain applicative purposes, or the nature of certain empirical problems, will require the introduction of mathematics through a representation, while other problems and applicative purposes will not. This point is important because it explains why not all applications can be representational. As long as it is possible to identify the problems which call for a representational use of mathematics, and to show, at the same time, that other problems emerging in scientific investigations require essentially different ways of using or introducing mathematics, it becomes clear that a representational characterization of applicability cannot be exhaustive.

Such a characterization is in fact restricted to a certain type of applicative problem and cannot take into account problems of a different nature, which nonetheless frequently occur in science.

In order to put some flesh on these remarks, in this section I give a general description of representational applications and identify their applicative purpose. It will then be easy to contrast them with non-representational applications in the next sections.

Representational applications are based on a (representation) theorem which is proved using a suitable system of empirical axioms: this theorem asserts that the models of the axioms can be described mathematically in a specified way. Sometimes all models prove embeddable into a unique mathematical structure, as happens with extensive structures relative to the positive, ordered reals with addition. In this case, the following remark applies:

> 'When a scientific theory quantifies over real numbers, for example, it is exploiting the structure of the real number line in order to make assertions and predictions about the physical world' (Baker 2003: 54).

The fact that a representation connects the real continuum (its positive part in extensive measurement) to a physical system makes it possible to study this system in mathematical terms. As noted in the above quotation, only the structural properties of the mathematics are considered, because these are precisely the properties which are preserved by a representation in the passage from an empirical to a mathematical structure. For this reason, certain properties of the mathematical model correspond, *via* representation, to true physical properties (e.g. in extensive measurement the commutativity of arithmetical addition corresponds to the commutativity of physical concatenation). In other words:

> 'According to the mapping account of applications, the truth of a statement of applied mathematics [...] depends on the existence of a mapping of a certain kind from a physical situation to a mathematical domain. [...] applications will involve [...] mappings such as homomorphisms that respect certain features of the physical situations [...]' (Pincock 2004a: 69).

The description Pincock gives of what he calls the mapping account of applicability, as well as the remarks by Baker, are of course not new. In particular, they have been long codified in the context of psychology and measurement theory, with the purpose of providing an explanation of what it means to apply a mathematical model. The canonical answer is the following:

> 'With some segment of the real world as his starting point, the scientist [...] maps his object system into one of the mathematical systems or models. By mathematical argument [...] certain mathematical conclusions are arrived at as necessary logical consequences of the postulates of the system. The mathematical conclusions are then converted into physical conclusions by a process we shall call interpretation [...]' (Coombs et al. 1954: 132–133) [2].

Representational applications can be understood, in light of all the above quotations, as based on mappings which allow the physical interpretation of results obtained by mathematical inferences on a mathematical model. When a representation exists, it opens the possibility of studying an empirical system by means of the properties of a mathematical structure. The representation 'projects' the empirical system into a mathematical model which can be used to recover information about the empirical system itself.

The reason why representations are important is that they allow the introduction and applicability of mathematical models which reflect some empirical structure. In this case:

> '[...], the usefulness of mathematics is much like the usefulness of a city street map. Once the structural similarity between the map and the city's streets is established, the map can stand in for the world' (Bueno & Colyvan 2008: 2).

The point is that representational applications are important when one wants to have a 'map' of a certain empirical setting, i.e. a system of reference in which a space of 'points' related in suitable ways stands for an empirical domain in which objects interact in structurally corresponding ways. What is important about the 'map' is that it only charts the empirical relations of relevance, leaving out any other detail of the empirical system (for instance rigid rods in an extensive structure are simply described as points along a linear order). It is therefore possible to exclusively focus on the charted empirical relations and study their properties: the latter are

[2] More recent references on the subject essentially provide the same account. For instance Luce and Suppes write that 'Representational measurement is [...] an attempt to understand the nature of empirical observations that can be usefully recoded [...] in tems of familiar mathematical structures'. (Luce & Suppes 2002: 1). This means that in representational measurement one looks for empirical axioms which allow the introduction of a mathematical model by a suitable mapping.

informative about the actual world because they have an empirical interpretation, given a representation.

For this reason, the purpose of representational applications is (r) to gather information about an empirical setting by making inferences on a mathematical reference frame which preserves the structure of that setting. The applicative purpose (r) requires a reference frame, proved to exist by means of a representation theorem, in order for mathematics to be applicable. With this discussion in place it becomes easy to identify non-representational applications by showing that they correspond to applicative purposes other than (r). As will be shown later, the account of applicability based on data spaces is general because, unlike a purely representational one, it can successfully describe uses of mathematics corresponding to many different applicative purposes.

3. Non-representational purposes

A non-representational applicative purpose is one which cannot be reduced to (r): this means in general that a non-representational application will make use of mathematics even if (i) mathematical properties are not introduced by means of a reference frame for an empirical structure and (ii) mathematics is not used to deduce the properties of an empirical structure.

An example of (ii) will be discussed later. On the other hand, to see why (i) is the case in certain situations, just recall the examples from archaeology and genetics I have given in the previous chapter. In the case of seriation one starts without knowing the chronological ordering of a finite set of deposits. Information about that structure has to be gathered, but this is not done by means of a mapping joining it to a reference frame (e.g. a numerical linear order). If such a mapping could be explicitly given, then the applicative problem of seriation would not even arise. On the other hand, the mere fact that the mapping exists is of no help in solving the problem of determining a relative chronology.

In the case of genetic mapping (described in the appendix to chapter 4), the situation is in one respect more radical: whereas the structural properties of a linear ordering could be assumed for the deposits in archaeology, the genetics example shows that there may be situations in which even the structural properties of an empirical system to be investigated are unknown. The problem is to find them, e.g. to decide whether a gene is a linear order of parts or not. We cannot reason in representational terms because there is no way of specifying a mathematical model for the gene unless we have information about its structure. Mathematics may be applied to extract this information from suitable data, but it is not applied in a representational fashion, because a reference frame for the geometrical structure of the gene is not available.

Applicability in both seriation and genetic mapping, i.e. the fact that mathematics can be successfully applied to solve these problems, does not depend on the presence of a representation but on suitable empirical hypotheses and the fact that the mathematics involved works on empirically meaningful data. The application of mathematics involved here is non-representational, because the mathematics introduced to obtain a seriation or to map the genes does not correspond to the formal properties of a model which preserves the structure of some empirical setting, as happens according to (r). When algorithms are introduced to deal with incidence matrices and graph-theoretical concepts are used to study their partitions into submatrices, these techniques are a methodology applied to an effectively given data space. In short, the mathematics involved does not arise from a mathematical model understood in the representational sense.

So far I have only exploited examples which have already been discussed in detail to show that they describe applications of mathematics which do not conform to the representational paradigm, because they are motivated by applicative purposes which differ from (r). In the next three sections I intend to strengthen this conclusion by introducing a new example of non-representational application, whose underlying purpose differs from (r) in a more radical way than the purposes arising in archaeology and genetics.

To see why this is the case, note that often mathematics is used to make rational or optimal choices or to study the outcomes of different courses of actions (as in game theory). In these circumstances it is not really applied to illustrate the features of something given, like a physical system, an archaeological site or a fragment of genetic material. Rather, it is applied to make decisions about future actions or policies. The reason why this does not amount to studying an empirical structure can be briefly mentioned, but will be explained more clearly in the next section. Consider an application of mathematics which solves the problem of making an optimal choice. One may think that an empirical structure is involved here: in particular, one may have a domain of empirical choices ordered according to how much advantageous they are. Optimality may then be defined in terms of the given order. This is usually not the case: in general one has a domain of choices which is given as a data space, but on which no order relevant to defining optimality exists. The optimality condition does not describe a property of an empirical structure of choices: it only formulates a constraint on a search for a point (the optimal point) within a data space of choices. Finding an optimal point does not correspond to studying a property of an empirical structure. As a consequence, this kind of applicative problem cannot be solved in a representational way, because the purpose for which mathematics has to be applied to it differs from (r). I am now going to make sense of this remark by looking at a family of optimization problems and the way mathematics is applied to them.

I will spend some time discussing this example because, apart from its significance, it gives me the opportunity of concretely showing how to apply my account of applicability, as described in section 1, to a situation which considerably departs from the examples which have led me to isolate the notions of data space, reconfiguration and deductive analysis.

This will provide evidence for the fact that these concepts are adequate to spell out the salient features of widely different forms of applications.

4. Linear programs

In order to present the type of optimization problem I am interested in, let me recall here the main features of the optimization problem from finance I briefly described in chapter 1. The issue was to get a maximal return of interest from a capital C, to be partly invested into loans and partly into securities. These were the unknowns of the problem, which is convenient here to symbolize by x_1 and x_2. On these unknowns constraints were imposed, in the form of equalities and inequalities. For example, we had:

$$\text{(i) } x_1 + x_2 = \mathrm{C},$$

and additional conditions like:

$$\text{(ii) } a_1x_1 + a_2x_2 \leq b_1$$

with a_1, a_2 and b_1 real constants. Furthermore, the sign of all the unknowns was required to be nonnegative. If the interest rate of x_i is r_i, then the problem of maximizing the return of interest becomes that of solving a system of linear equalities like (i) and (ii), subject to the nonnegativity of the unknowns occurring in them and the further condition:

$$r_1x_1 + r_2x_2 = \max.$$

which is arithmetically equivalent to:

$$\text{(iii)} \quad -(r_1x_1 + r_2x_2) = \min.$$

Note that this problem reduces to that of finding two nonnegative numbers x_1, x_2 solving the system (i), (ii), (iii). Once these numbers are found, the problem is solved and one knows what investment will be optimal. It is clear that the hypothetical presence of some 'empirical structure of investments' is not relevant to the position or the solution of this problem. Numerical values can be directly specified to quantify investments, without the need of previously laying down e.g. empirical axioms to describe their possible non-numerical properties (it is also doubtful whether any such description would be more than a rephrasing of the numerical properties which are used in elementary financial computations involving order and addition).

However, one may insist, in light of the suggestion I made at the end of section 3, that an empirical structure of choices, numerically represented by couples (x_1, x_2), is in fact present. In this case one may conjecture that finding an optimal choice of investment means to describe a property of this structure and thus to work within the representational paradigm. The reason why this conclusion cannot be drawn is that optimality is not formulated in terms of the property of a hypothetical empirical structure of choices. This is because the two parameters r_1 and r_2 occurring in the maximality condition are interest rates. They are exogenous parameters, which do not have an empirical interpretation on a hypothetical empirical structure of choices. As a consequence, the optimality condition has no such empirical interpretation. Then finding an optimal choice of investments in the previous problem is not equivalent to studying the property of a given empirical structure. In the light of these observations, it can be concluded that we are in presence of a non-representational application of mathematics.

The example from finance I have just described is only a special case of a type of optimization problem called a linear program. The observations I have made so far generalize to linear programs, because mathematics is applied to them as it is applied to the example I have just given and because linear programs include an optimality condition which is not reducible to the property of an empirical structure of choices.

Given the example from finance, it is not difficult to define a linear program in fully general terms. The only thing needed is a generalization of the structure of (i) and (ii) above plus nonnegativity conditions to an arbitrary fixed number n of unknowns occurring within an arbitrary fixed number m of constraints (linear equalities or inequalities like (ii)). This simply means that the i-th constraint ($i = 1, ..., m$) will have the form:

$$\text{(iv)}\ a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n \geq b_i$$

where a_{ij} and b_i are real numbers and x_k ($k = 1, ..., n$) are unknowns[3]. The nonnegativity conditions are:

$$\text{(v)}\ x_k \geq 0$$

and the optimality constraint, which can always be described as a minimization condition, is:

$$\text{(vi)}\ c_1x_1 + c_2x_2 ... + c_nx_n = min.$$

where c_k is a real constant for any k (the c_k are particular, fixed parameters which specify the optimality condition involved in the problem). A linear program is just a set of conditions like (iv) to (vi). These can be formulated in a more compact form using matrix notation, in which case they become respectively:

$$(1)\ \mathbf{Ax} \geq \mathbf{b}^4;\ (2)\ \mathbf{x} \geq \mathbf{0}^5;\ (3)\ \mathbf{c}\cdot\mathbf{x} = min.^6$$

A solution to a linear program is a vector $\mathbf{x} = (x_1, x_2, ..., x_n)$ satisfying (1), (2) and (3). Any vector satisfying (1) and (2) is called a *feasible solution* for the given problem: a feasible solution satisfying (3) is called an *optimal solution*, and it is the one looked for. Linear programming studies, at an abstract level, the conditions under which optimal solutions exist and are unique and also provides algorithms to compute them, in case they exist.

As I already remarked, linear programs give rise to non-representational applications of mathematics. This is suggested in general by the format of these problems, which is not based on an axiomatic theory from which a representation theorem can be proved or on the identification of a particular type of empirical structure to be modelled mathematically. The constraints of a linear program can simply be understood as providing a system of data relative to a concrete problem, e.g. conditions on investments.

[3] Conditions (i) and (ii) above can always be reduced to inequalities like (iv). For an equality like (i), it suffices to split it into two inequalities and multiply both members of one inequality by – 1. For an equality like (ii), it suffices to multiply its members by – 1.
[4] Here A is the $m \times n$ matrix (m rows for m equalities, n columns for n unknowns) whose coefficients are a_{ij} and **x**, **b** are, respectively, an n-component and an m-component column vector.
[5] **0** is a vector having all n components equal to zero.
[6] **c** is a vector having components c_k and '·' denotes the dot product on vectors.

Now the issue is to decide whether my characterization of the application of mathematics as the deductive analysis of a data space and its reconfigurations (as presented in section 1) applies here as it did to archaeology and genetics. To this end it is necessary to identify data spaces for linear programs and to show that mathematics works on reconfigurations of these spaces, making use of suitable concepts and theorems, in order to determine a solution to the linear programs. This is the subject of the next two sections.

5. Data spaces for linear programs

It is plausible to identify the data space for a linear program with the set of nonnegative, real-valued, n-component vectors which includes the set of its feasible solutions (and thus the optimal solution, if there is one)[7]. In this case we have an infinitely large data space, made of n-component, real valued vectors, within which a certain feasibility region is 'delimited' by the constraints in (1) and (2) of the previous section. In more technical terms, since feasible solutions are only a portion of the space of all n-component real-valued vectors, one might say that linear programs codify restricted searches (for optimality) within vector spaces. To each vector in the feasibility region corresponds an empirical choice, e.g. an investment.

The applicative problem which mathematics is applied to solve here consists in deciding whether, for a given linear program, its solution exists and, in this case, whether it is unique. Existence of a solution corresponds to the possibility of making an optimal choice, whereas uniqueness corresponds to the necessity to make that choice, if optimality is sought under the given constraints. Analogies with the examples of non-representational applications examined in the previous chapter can be easily discerned: we start in both cases from a data space and we need to answer a 'yes-or-no' problem ('is the incidence matrix pre-Petrie?' in archaeology and 'is there an optimal solution?' in linear programming, which becomes 'is there an optimal investment?' in the finance example discussed above). Because the whole problem revolves around finding an answer, it calls for the construction of an (efficient) algorithm.

[7] Here I take the data space to include vectors which are not feasible solutions. These may be given an empirical interpretation as the potential choices which are excluded by the constraints of a linear program.

I will not describe, even in outline, algorithms for linear programs, but only point to their conceptual role. I will only discuss the proof of existence of an optimal solution and the deductive analysis it generates over the data space of a linear program. This analysis lies at the basis of the construction of an algorithm to solve linear programs, exactly as happened in the seriation example of the previous chapter, where the analysis of the property 'pre-Petrie' led to an algorithm deciding it. A further, important similarity between seriation and linear programming lies in their crucial use of mathematical concepts and theorems applied to reconfigurations of the data space. This involved the introduction of graphs and graph-theoretical results in the archaeological case and it involves the introduction of geometrical objects and convex analysis results here.

Reconfiguration plays a major role in establishing the necessity of a certain condition for the existence of an optimal solution to a linear problem, whereas sufficiency does not require it. I will nonetheless give the proof of sufficiency because it is necessary to understand that of necessity and can be very briefly illustrated. This proof is based on the preliminary definition of a *dual* linear program, associated to the initially given one or *primal*. If the primal is determined by conditions:

$$\mathrm{A}\mathbf{x} \geq \mathbf{b}; \quad \mathbf{x} \geq \mathbf{0}; \quad \mathbf{c}\cdot\mathbf{x} = min.$$

then its dual is given by:

$$\mathrm{A}'\mathbf{y} \leq \mathbf{c}; \quad \mathbf{y} \geq \mathbf{0}; \quad \mathbf{b}\cdot\mathbf{y} = max.$$

where A′ is the transpose[8] of A and **y** has m components, if A is an $m \times n$ matrix[9]. If **y** is a feasible solution for the dual and **x** is feasible solution for the primal, then:

[8] That is, A′ is obtained by interchanging the rows and columns of A.

[9] The motivation for this approach is to be found in the fact that there naturally arise in applications cases in which the solution of a minimization problem is correlated to the solution of a maximization problem: for instance, there are situations in which a utility maximization problem and expenditure minimization problem have the same solution. This is something which happens in economics, typically when utility is to be maximized, subject to a budget constraint (which may be an equality or an inequality): under suitable assumptions on the utility function (see Varian 1992: 94–97), this problem is equivalent to minimizing expenditure subject to the attainment of a fixed level of utility. Intuitively, the highest utility one can get with a fixed amount of money to spend is also attained by spending the least possible, subject to attaining that level of utility. In this case the minimization problem is called the *dual* of the maximization problem. Under suitable conditions, it can be proved that a choice of a bundle of goods solving the maximization problem also solves its dual, and viceversa (for a proof of this fact see Varian 1992: 113 and, in a more general context, Lambert 1985:136–138).
It turns out that the solution to the dual of a given linear program has an important, general empirical interpretation. The dual vector **y** may provide information about the variation of the primal problem when its requirement parameters vary. It turns out that the dual vector is crucial precisely to study how

$$\mathbf{by} \leq (\mathrm{A}\mathbf{x})\mathbf{y} = \mathbf{x}'\mathrm{A}'\mathbf{y} \leq \mathbf{x}'\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{x}^{10} \text{ and thus } \mathbf{by} \leq \mathbf{c}\cdot\mathbf{x}.$$

As a consequence we have:

(S) if the equality $\mathbf{c}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{y}$ obtains, $\mathbf{x}$, $\mathbf{y}$ are optimal for the corresponding problems[11].

The equality in (S) yields a sufficient condition for the existence of an optimal solution to the primal linear program. It can be rewritten as: $\mathbf{c}\cdot\mathbf{x} - \mathbf{b}\cdot\mathbf{y} = 0$. The converse of (S) also holds, i.e. the only case in which the primal has an optimal solution is the one for which $\mathbf{c}\cdot\mathbf{x} - \mathbf{b}\cdot\mathbf{y} = 0$. Then (S) is also a necessary condition for optimality: as already observed, the proof of this fact rests on a significant reconfiguration of the data space for a linear program[12].

6. Reconfiguration

Proving the necessity of (S) means to prove that it is impossible that an optimal solution to a primal problem exists while an optimal solution to its dual does not. In other words, either no optimal solutions exist for any of them or both have simultaneous optimal solutions. Roughly, this exclusive alternative between two distinct situations is the content of the *duality theorem* of linear programming, which establishes the necessity of (S) (for details see Franklin 1980: 62–65). Since this theorem expresses an alternative, its proof can be reached by writing the primal and dual problem as a unique linear program, for which an exclusive alternative can be established, leading to duality.

In particular, the primal can be compactly formulated as the following linear system:

(i) $\mathrm{A}\mathbf{x} \leq \mathbf{b}$[13].

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optimality changes under perturbations of $\mathbf{b}$ (see in this connection Franklin 1980: 68–69), i.e. under a particular shifting of the data space relative to the primal problem.

[10] The proof goes through using the associativity of matrix multiplication and properties of transposes. In particular, if A is an m×n matrix and $\mathbf{x}$ is a column vector with n components, the matrix multiplication A$\mathbf{x}$ is defined, while $\mathbf{x}$A is not. However $\mathbf{x}'\mathrm{A}'$, i.e. the transpose of A$\mathbf{x}$, is defined.

[11] We have $\mathbf{b}\cdot\mathbf{y} \leq \mathbf{c}\cdot\mathbf{x}'$ for any feasible $\mathbf{x}'$ so, if $\mathbf{x}$ is such that $\mathbf{b}\cdot\mathbf{y} = \mathbf{c}\cdot\mathbf{x}$, then $\mathbf{c}\cdot\mathbf{x} \leq \mathbf{c}\cdot\mathbf{x}'$ for any feasible $\mathbf{x}'$, and $\mathbf{x}$ is optimal. An analogous argument proves the optimality of $\mathbf{y}$.

[12] As will be seen in the next section, the primal and dual program can be written compactly as a single linear program, so the introduction of the dual does not really make the structure of the original problem essentially more complicated.

[13] This can be done as follows: first of all, note that $\mathbf{c}\cdot\mathbf{x} - \mathbf{b}\cdot\mathbf{y} = 0$ is equivalent to $\mathbf{c}\cdot\mathbf{x} - \mathbf{b}\cdot\mathbf{y} \leq 0$, because $\mathbf{b}\cdot\mathbf{y}$ is a lower bound for $\mathbf{c}\cdot\mathbf{x}$, as shown above. Using this inequality and writing (p) and (d) in a composite matrix, the resulting system of takes the form:

$$\mathrm{A}^*\mathbf{x}^* \leq \mathbf{b}^*$$

where:

Then the duality theorem rests on the proof of the exclusive alternative:

(L) either $\mathbf{Ax} \leq \mathbf{b}$ has a nonnegative solution or the system ($\mathrm{A}'\mathbf{y} \geq \mathbf{0}$; $\mathbf{b}\cdot\mathbf{y} < 0$) has.

The whole issue reduces to proving (L) and it is at this point that a reconfiguration of the data space plays a crucial role. In short, the alternative (L) can be proved from a theorem called *Farkas Alternative* and stating that:

(F) either $\mathbf{Ax} = \mathbf{b}$ has a nonnegative solution or the system ($\mathrm{A}'\mathbf{y} \geq \mathbf{0}$; $\mathbf{b}\cdot\mathbf{y} < 0$) has a solution (not necessarily nonnegative)[14].

The data space of (F) (and (L)) is of the same kind as the data space of the relevant linear programs, i.e. an infinitely large family of vectors (in fact, because of (i), the same data spaces are involved in these statements). Thus the isolation of necessary conditions for optimality can ultimately be made to rest upon (F).

Two steps are required in order to prove (F): in the first step, a geometrical interpretation is found for the algebraic solvability of the system $\mathbf{Ax} = \mathbf{b}$. In particular one considers the columns of matrix A as vectors and looks at the space generated by summing together positive multiples of them, called the *finite cone* generated by A. From this point of view, the issue of whether $\mathbf{Ax} = \mathbf{b}$ has a solution can be reduced to the geometrical problem of establishing whether $\mathbf{b}$ lies in the finite cone generated by A. Algebraic solvability becomes equivalent to a geometrical incidence relation.

Now, due to certain geometrical properties of finite cones (namely, convexity and closedness), it is possible to establish (F) by showing that, if the first alternative in it does not hold, then the geometrical behaviour of the finite cone generated by A and a vector $\mathbf{x}$ not solving $\mathbf{Ax} = \mathbf{b}$ determines the second alternative (this is done by invoking the *separating hyperplane theorem*: details of the proof are in Franklin 1980: 54–56).

A reconfiguration of a data space is effected here by isolating geometrical objects (finite cones) and geometrical relations (incidence) within a space of vectors[15].

$$\mathrm{A}^* = \begin{matrix} A & 0 \\ 0 & A' \\ c' & -b \end{matrix}\ ; \quad \mathrm{x}^* = \begin{matrix} x \\ y \end{matrix} \ \text{and} \quad \mathrm{b}^* = \begin{matrix} -b \\ c \\ 0 \end{matrix}$$

[14] Briefly, the proof that (L) entails (F) runs as follows: starting from $\mathbf{Ax} \leq \mathbf{b}$, add nonnegative slack variables to obtain an m-component vector $\mathbf{z}$. Then it follows that: $\mathbf{Ax} + \mathrm{I}\mathbf{z} = \mathbf{b}$ where I is the identity matrix of order m (i.e. with *m* columns and *m* rows, whose entries are everywhere 0 except on the diagonal, where they are 1). Given the last equality, (F) implies that either it has a nonnegative solution $\mathbf{x}$, $\mathbf{z}$ or the system ($\mathrm{A}'\mathbf{y} + \mathrm{I}\mathbf{y} \geq \mathbf{0}$; $\mathbf{b}\cdot\mathbf{y} < 0$) has a solution. Since the first inequality can be described by the matrix product $[\mathrm{A}, \mathrm{I}]'\mathbf{y} = [\mathrm{A}'\mathbf{y}, \mathrm{I}\mathbf{y}]$, it follows that $\mathrm{I}\mathbf{y} \geq \mathbf{0}$ i.e. that each component of $\mathbf{y}$ must be nonnegative. This establishes the result.

Deductive analysis is applied to the reconfiguration thus obtained: in particular, theorems holding for the geometrical objects and relations identified on the data space (i.e. the separating hyperplane theorem) are used. By their aid a fundamental theorem of linear programming can be proved, yielding necessary conditions for the solvability of any linear program.

The last observations show that the way in which mathematics is applied in linear programming is in essence the same found in archaeology or genetic mapping. The characterization of applicability illustrated in chapter 4 works here as well. One starts from the presentation of a problem in numerical terms, as a system of constraints. These constraints organize the search of a particular piece of information, i.e. an optimal solution to the given problem. Thus, the constraints do not describe an empirical structure but restrict the space in which an optimal solution has to be sought. In the example on investment management of section 4, for instance, the constraints of the linear program say that investments have to be nonnegative, that at least a certain amount of money has to be invested in loans and so on. They specify empirically meaningful conditions, in the sense that they determine an actual investment choice, but they do not really describe an empirical structure: they only impose restrictions on the degrees of freedom of the choice.

Once the constraints of a linear program are given, they determine a data space. Then mathematics is used to direct the search for optimality within this space. Crucial in this process is the appeal to reconfigurations of the data space, which makes additional mathematical resources available to find conditions under which an optimal solution exists and, ultimately, to construct an algorithm which computes it.

In this connection it is particularly interesting to observe that there is an algorithm for linear programs, called the *lexicographic simplex tableaux*, which is by itself sufficient to deliver a proof of the duality theorem. I won't enter into details but it is sufficient to observe that an algorithm for linear programs must be able to tell whether they have or not a solution. Thus, in light of the previous discussion, an algorithm gives a positive answer exactly when the equality $\mathbf{c}\cdot\mathbf{x} - \mathbf{b}\cdot\mathbf{y} = 0$ holds. In other words, the algorithm can be directly used to isolate the necessary and sufficient conditions for the solvability of a linear program. This means that the algorithm alone may be used to provide a proof of the duality theorem and its converse.

[15] In fact, two linear programs: a primal and its dual. But, again, they can be written as a single linear program.

The interest of this fact depends on the syntactic, finitary nature of algorithms: this shows that the use of an infinitely large data space of vectors and of its topological properties (e.g. the closedness of finite cones) in order to study the solvability conditions of a linear program has an essentially methodological role. Since usually the data space for a linear program is idealized (in the sense that one takes certain quantities, e.g. loan investments, to vary continuously), it can be seen that the idealization involved is not ontologically significant, as it can be replaced by the application of a finitary method.

Nevertheless, the analysis of a linear program in terms of idealized data spaces is fruitful because it provides a guide to the formulation of an algorithm, e.g. by identifying the conditions the algorithm has to check in order to decide whether optimality can be reached. For this reason there remains an essential similarity between concretely given data spaces and infinitely large ones, namely their conceptual relevance to studying the solvability of a type of applicative problem. The application of mathematics in both situations can be characterized in the same way.

7. Data spaces and representational applications

So far I have shown that my account of the applicability of mathematics in terms of data spaces extends beyond archaeology, to the family of problems which can be dealt with by means of linear programs. Similar observations immediately extend to applicative problems which can be reduced to linear programs. For example, the search for equilibria in two-persons, zero-sum games can be solved constructively by means of linear programming algorithms.

My account also applies without particular troubles to non-linear programming[16]. At the same time, it covers applications which are quite different from the ones explored so far. A good example is modelling with differential equations, as it is used in e.g. epidemics or population dynamics: in particular, one may take ordinary first-order differential equations and identify their direction fields with their data space, while taking the deductive analysis of this data space to proceed either by the techniques of the calculus or by means of algorithms, i.e. Euler's method to approximate the solutions of first-order equations.

[16] Which deals with the same kind of problems encountered linear programming, with the sole exception that the algebraic form of the constraints is different.

These brief remarks, together with the examples discussed in the last two chapters, already give an idea of how comprehensive the characterization of applicability I have proposed is. More can be done to consolidate this impression: in particular it can be shown that all the applications I have examined in previous chapters can be dealt with in terms of data spaces and deductive analysis. This is what I am going to do in the next two sections. My account of applicability has the conceptual resources to encompass a representational one. In particular, it includes the applications discussed in Field 1980.

It is not too difficult to show how representational applications can be characterized in terms of data spaces. This, in fact, has already been done in chapter 2 of part 1 for extensive structures and that approach can be generalized considerably. In order to understand this remark, it is useful to look at a simple example of representational application, provided by the measurement of length-order on finite collections of rigid rods (cf. part 1, chapter 1, section 6). Here we have an empirical structure whose domain is a finite collection of rods. A binary relation of length-order, denoted by $\leq_L$, is defined on them and it obeys the following axioms:

1. Reflexivity: $\forall x\ (x \leq_L x)$;
2. Antisymmetry: $\forall x \forall y\ (x \leq_L y \wedge y \leq_L x \rightarrow x =_L y)$;
3. Transitivity: $\forall x \forall y \forall z\ (x \leq_L y \wedge y \leq_L z \rightarrow x \leq_L z)$;
4. Trichotomy: $\forall x \forall y\ (x <_L y \vee\ y <_L x \vee x =_L y)$[17].

When the above axioms hold, we can order the rods satisfying them from shortest to longest and in particular assign to them, assuming there are *n* rods, the first *n* positive integers (with the possible exception that rods having the same length are assigned the same positive integer). If μ is the function which assigns numbers to rods, then a representation theorem for them can be proved, to the effect that:

$$a \geq_L b \text{ if and only if } \mu(a) \geq \mu(b)$$

where $\mu(a)$ and $\mu(b)$ are positive integers. The representation theorem introduces a measurement scale for rigid rods. The question is how this result can be expressed in terms of data spaces. The answer depends on the characterization of representational applications offered in section 2 of this chapter. Since representations are used to determine reference frames for empirical structures, representational applications

[17] That the disjunction is exclusive follows from the other axioms.

will have to correspond to reconfigurations of particular data spaces leading to the construction of reference frames.

This general idea can be made precise by means of the strategy illustrated in chapter 2 of part 1 to obtain a type-based characterization of the representation theorem for extensive measurement. Consider the first-order language for the theory of length-order described above (axioms 1 to 4). In addition, suppose the language contains one constant for each rod to be measured. Then the data space for length-order can be described as the set S of all atomic formulas of the form[18] $\underline{x \trianglerighteq_{L} \mathrm{k}}$, where $\underline{\mathrm{k}}$ is a metavariable ranging over the constants naming the rods and x is a free variable. By assigning a rod a to x, it is possible to isolate a subset S_a of S, determined by all the atomic formulas in S which are satisfied under the given assignment. For instance, if a is assigned to x and a is longer than b but strictly shorter than c, then $\underline{x \trianglerighteq_{L} b}$ is in S_a, while $\underline{x \trianglerighteq_{L} c}$ is not. It can be checked, using the axioms, that, if $\underline{x \trianglerighteq_{L} b}$ but not viceversa, then S_a strictly includes S_b, strictness coming from the fact that at least the formula $\underline{x \geq_{L} a}$ is in the former set but not in the latter.

As a result, it is possible to order the subsets S_a of the data space S and take them as measures of rods relative to length-order. The introduction of an ordered structure on S, whose elements are the S_a, corresponds to a reconfiguration of S effected on the basis of the axioms for length-order. Once the reconfiguration is carried out, it is possible to work deductively on a reference frame for length-order and use it make deductions describing the physical behaviour of rigid rods. It is therefore apparent that the measurement of length-order can be described in terms of data spaces, deductive analysis and reconfigurations.

In the more complex case of extensive measurement, the same treatment is available, since now the infinite data space can be identified with the set of atomic formulas of the form:

$$\underline{Mx <_{L} Ny}$$

which takes into account the presence of a concatenation operation. The discussion of T-types in chapter 2 of part 1 can then be read as an account of extensive representation in terms of data spaces. The fact that T-types can be used in all the theories of extensive measurement I have considered in part 1 shows that data spaces do not only characterize classical extensive measurement, but also several of its

[18] I underline formulas to distinguish them from the relations they denote.

weakened forms, besides the theories of measurement which can be reduced to extensive measurement, e.g. all those axiomatized in chapters 4 and 6 of Krantz et al. 1971.

8. Further extensions

My previous remarks on the possibility of characterizing representational applications in terms of data spaces rely on the type-approach to measurement developed in Niederée 1987, 1992. Since the main examples of representation in Field 1980 are based on the presence of some additive structure and so can be reduced to extensive measurement, it can already be concluded that the representational applications Field considers are amenable to a treatment in terms of data spaces. It follows that the concept of data space is adequate to deal with representational and non-representational applications and it has the advantage of generating a characterization of applicability which is more general than Field's.

It is however difficult to see exactly how general the data space characterization is and indicate exact boundaries for it. My task so far has been to show that it is sufficiently general, in the sense that the salient features of many forms of applications can be described by means of it. I have taken some care to discuss a few selected examples (archaeology, genetics, linear programming) mainly to show that, even though the concepts of data space, reconfiguration and deductive analysis are not formal concepts which can be treated in a strictly mathematical fashion, as the concepts of representation and structure-preservation, they nonetheless provide a clear analysis of the way mathematics is applied (while they are also related to mathematical techniques like the type-approach). Here I have not the space to concretely show that there are many other forms of applications which can be successfully analysed using my account of applicability (I have just mentioned a few in the previous sections, i.e. non-linear programming, elementary differential equations and game theory).

I can however provide an argument to support the generality of my approach and one more (generalizable) example. The argument revolves around the fact that the type-approach of Niederée generates data space accounts of applications. Theoretically, this kind of approach is very general since it goes through as soon as a set of formulas can be identified, which suffices to discriminate objects or classes of objects within an empirical structure: in principle, as soon as separability holds for that structure relative to a suitable set of formulas, the construction of a reference frame is in principle possible, since it suffices to induce a given empirical structure over the subsets of formulas which suffice to distinguish the several objects within that structure.

In a sense, if one can explicitly give a representation for an empirical structure, this means that enough structural information is available to this end and thus that it separates the objects within the empirical structure. Then the type approach is applicable. The only problematic cases are those in which only a purely existential proof of representation is available (this happens with the representation theorem for arbitrary, infinitely large ordered structures): however one can read this proof as an existential statement saying that there is a reconfiguration of the relevant data space (e.g. formulas describing the ordered structure to be measured) which is a reference frame.

If this move is accepted, then ordinary representational applications can all be included in my account of applicability and this provides enough evidence in favour of its sufficient generality. In particular, my account locally[19] generates a unified characterization of representational applications by identifying their data spaces as sets of formulas which ensure the separability of an empirical domain. The relevant sets of formulas are structured as reference frames by means of suitable reconfigurations and these reconfigurations are used to study the original empirical structures, in the sense of (r) of section 2 (in this case (r) corresponds to the deductive analysis of the reconfiguration of a data space).

Further proof of the generality of my account comes from the fact that certain axiom systems designed to establish representation theorems can be directly seen as axiomatizations of data spaces. This is in fact the case in psychophysical measurement. Narens 1996 provides an axiom system which constrains the properties of behavioural responses, i.e. data described by triples of the form:

$$(x, \mathbf{p}, t).$$

If the set of these triples is taken as the initial data space, the proof of representation adopted by Narens relies essentially on a reconfiguration of this data space as a space of functions, followed by its extension to a richer data space, identified with the homogeneous set of automorphisms of an empirical structure. Since it is possible to

[19] That is, once it is restricted to representational applications.

construct a measuring structure for a model of Narens' axioms which is entirely based on automorphisms (as explained in the appendix to chapter 2 of this part), this amounts to saying that a suitable reconfiguration of the data space of responses (as functions) makes it possible to use them in order to set up a reference frame for sensations. This frame can then be used to make deductions about the initial empirical structure. At the same time, it is possible to take the application of the theory of complete scalar structures to psychophysics (something which could not be done on the basis of Field's eliminative account, as explained in chapters 2 and 3 of this part) as a deductive use of mathematics employed to carry out the reconfiguration of the data space of triples (x, $\mathbf{p}$, t). The fact that the properties of complete scalar structures are also used as explanatory principles in psychology and as guides to experimental researches[20] then depends on the fact that they shape a space of empirical data in a particular way.

In sum, both the fact just observed that certain representational applications are based directly on the axiomatization of a data space and the generality inherent in the type-approach developed by Niederée suggest that a wide variety of applications, including all ordinary representational ones, can be described in terms of data spaces: in particular, the ones Field considers can.

As a consequence, the account of applicability in terms of data spaces I have articulated proves more comprehensive and more realistic than his. I have not so far discussed what the consequences of replacing Field's account with mine may be, as far as Field's nominalism is concerned. Although this is an interesting and important problem, it is not my primary concern here, while much more space would be required to adequately discuss it. I only wish to remark that my account of applicability is not incompatible with nominalism: in particular, because it stresses the deductive (and thus syntactic) use of mathematics on data spaces, which are symbolic presentations of empirical information, my account does not necessarily require a distinctive mathematical ontology to be available for applications, both representational and non-representational, to be possible[21].

[20] These uses of mathematics, as observed in chapter 3, cannot be adequately accounted for by Field.

[21] This conclusion can be reached allowing for idealization, which is after all freely used by Field at the same time, it is possible at least under some circumstances, to show that idealization itself has a purely methodological role and does not lead to ontological commitments to idealized empirical settings or data spaces. This is what happens with linear programs.

As a consequence, my account of applicability does not by itself generate troubles for nominalism, although it calls for a revision of Field's eliminative strategy. By observations presented in chapters 2 and 3 above, there are applications of mathematics which end with the integration of mathematical concepts and properties into idealized empirical theories[22]. These theories then contain references which Field counts among those he wants to eliminate, but of course in these cases they cannot be eliminated without getting rid of the empirical theories founded upon them. Since some of these theories are important in scientific practice, a nominalist wishing to take the problem of applicability seriously should not pursue a uniformly eliminative strategy.

9. Concluding remarks

The previous chapter and this one have served the purpose of developing an account of applicability which extends Field's and highlights the limitations of the perspective, implicit in much philosophy of science and philosophy of mathematics, that the applicability of a piece of mathematics is based on its representing some empirical structure.

My account of applicability can dispense with this idea and describes many representational and non-representational applications within a unified conceptual framework, based on the key ideas of data space, deductive analysis and reconfiguration. It is easy to see, once this framework is in place, that different applicative problems and purposes require different mathematical techniques: only when these problems and purposes are of a specified kind do representational applications arise. Under different circumstances they do not.

My account, however, does not have an advantage which is peculiar to Field's, namely the ability to specify in exact mathematical terms what applicability amounts to (i.e. by proving a representation theorem). Nevertheless, as I have pointed out, this disadvantage of my account is compensated by the fact that it is more general than Field's account and proves capable to provide an informative analysis of

[22] Pincock 2007: 257–260 describes this use of mathematics introducing the concept of 'abstract explanation', but he is willing, at least as far as his example (the graph-theoretical solution to the Königsberg's bridges problem) is concerned, to concede that abstract explanations may be formulated in representational terms. This conclusion can be avoided if idealized empirical settings are allowed, in which case mathematical concepts may directly be applied to their description, i.e. without the mediation of a representation. Presumably the same can be said for many non-idealized settings whose macroscopic features may be described as discrete structures to which mathematical concepts naturally apply.

applicability, as I have tried to show by looking in detail at the structure of a few non-representational applications and characterizing them in terms of data spaces. At the same time, even if I don't propose a mathematical approach to applicability, mathematical techniques can be used to articulate my account, particularly the type-approach of Niederée and, in fact, also Narens' techniques leading to the construction of measurement structures from automorphism groups.

Thus, I can conclude that overall my account of applicability has definite advantages over Field's and proves adequately general in the sense that it can deal with many very different types of applications.

On the other hand, I don't intend to claim that all possible forms of applications can be reduced to the data space account. I am satisfied to have shown that this account can easily be extended to include ordinary measurement-theoretic and geometric representations, some of their generalizations (e.g. psychophysical theories of measurement), combinatorial applications and programming.

The reason why I haven't looked for an even wider characterization of applicability is that there is a trade-off between the generality of an account of applicability and the informativity of the concepts it relies upon.

Too much exactness may become a hindrance to generality by introducing unnecessary restrictions: this is exemplified by the limitations inherent in the representational approach. On this approach, we have a mathematically precise characterization of applicability, yet this is achieved at the cost of overlooking many kinds of applications.

On the other hand, there may be very general accounts of applicability which are not really informative and end up being too vague. The data space account is supposed to be a good compromise between generality and exactness. The loss in mathematical exactness it allows, if compared with a representational approach, is compensated by the availability of a unified conceptual framework characterizing many applications. The characterization is acceptable because it generates a detailed analysis of applicability when used to study specific examples.

One final question arising in connection with my proposal of explaining applicability in terms of data spaces has to do with idealization. I already gave a characterization of idealization as enrichment in part 1, which works for several theories of measurement and in principle extends to the scientific theories studied in Field 1980. All these theories, however, provide examples of representational applications, so the problem is whether a similar notion of idealization can be found in non-representational ones as well. At present I do not have a conclusive answer to this problem, but I can point to a plausible answer, solely based on the examples I have examined above. In chapter 4, idealization was described as a kind of enrichment, involving the properties of a data space. Even if archaeological data do not lead to a pre-Petrie matrix, the property 'pre-Petrie', which is suggested by an idealizing hypothesis, is nonetheless useful in chronological investigations and can actually be employed to determine probable chronologies in terms of divergence of an incidence matrix from a Petrie form.

In the case of linear programming, idealization is implicit in taking the space of feasible solutions to be an infinite family of vectors. Even though actual algorithmic methods to solve linear programs only have to check finitely many vectors, the use of idealization is important because it makes it possible to study the conditions under which linear programs are solvable and thus also to indicate which conditions an algorithm for these problems has to check. In this sense idealization corresponds to the assumption of a rich data space, which has a methodological importance.

These cursory observations suggest that non-representational applications make use of idealizations on their data spaces, pretty much as representational applications make use of idealizations on the empirical settings to be represented. In the latter case, once an idealization is in place, it is possible to derive global properties of empirical settings which could not be obtained without idealization. In non-representational applications idealized data spaces make it possible the introduction of new methodologies which would not have been possible without them.

On the basis of these conclusions it seems plausible to think that a relatively unitary concept of idealization can be developed for representational and non-representational applications.

CONCLUSIONS

1. Applicability

In the previous chapters I have developed an account of idealization in extensive measurement and related theories, together with a characterization of the applicability of mathematics to a large family of empirical settings and problems, both representational and non-representational.

The motivation for both these projects came from Field's account of applicability: my objective has been to overcome its drawbacks without forsaking the insights provided by his analysis of the way mathematics is used in science (in particular, in representational applications).

What I have done was to integrate these insights into a wider framework which yields a more comprehensive and realistic account of applicability. Let me now summarize the main features of my account, i.e. its treatment of idealization and of the role played by mathematics in applications.

2. Idealization

As I pointed out in chapter 1 of part 1, Field 1980 can be read as a study of the applicability of mathematics to scientific theories. The theories Field considers are idealized ones, i.e. they presuppose untestable physical properties[1] or ones which are contradicted by observation[2]. For this reason, whenever he talks about applicability, he in fact refers to the applicability of mathematics to idealized theories. This is acceptable in the context of Field's nominalistic project, since its aim is simply to avoid mathematical references in the formulation of scientific theories and not to explain idealization. On the other hand, this explanation becomes crucial if the question is to provide an analysis of the applicability of mathematics to the actual world, as opposed to an idealized counterpart thereof.

In particular, it is necessary (i) to describe the relation of idealizations to actual empirical settings and (ii) to clarify why they are used in the place of weaker or more realistic descriptions of those settings.

[1] E.g. that certain quantities like the gravitational potential vary continuously.

[2] E.g. the transitivity of empirical equivalence in length-measurement, which is implicit in the measurement of spatial distances possible on the basis of Field's geometrical axioms.

As for point (i), I have qualified idealization as a form of enrichment and generalization of the experimentally observed features of empirical settings, focusing on extensive measurement and the theories which can be reduced to it (these include Field's examples).

This result was reached by looking for a motivation to introduce idealizing hypothesis and to extend weaker theories of extensive measurement to a strongly idealized one (Hölder's theory). By looking at the distinct ways in which different idealizing hypotheses can be justified, I have been able to classify these hypotheses into different categories. There are infinitary idealizations, which simply generalize formal properties already satisfied by an empirical setting (e.g. closure in extensive measurement), regularity idealization, which introduce formal properties arising from the asymptotic behaviour of an empirical domain (e.g. the transitivity of empirical equivalence) and finally there are structural idealizations, which constrain the conceptualization of a certain empirical setting (e.g. Archimedes' axiom).

For this reason it is possible to discern several stages in the introduction of idealizations (cf. the discussion in part 1, chapter 5, section 5): one starts from the realistic description of an empirical setting, then extends it to a richer one through the generalization of some of its properties; subsequently, the generalized setting is endowed with additional properties characteristic of its asymptotic behaviour. In the background, suitable structural idealizations guide this process of enrichment and lead it to a characteristic form: this is what happens by assuming an Archimedean axiom in extensive measurement, which ensures the extensibility of the settings satisfying it (in presence of the other axioms) to a structure isomorphic to a numerical continuum.

This description of idealization as a process of enrichment performed in stages and guided by certain assumptions differs from the traditional characterization found in the philosophy of science, according to which idealization consists in a selective description of the features of complex phenomena for simplificatory purposes[3].

The idealizations occurring in extensive measurement are not used as provisional simplified characterizations of phenomena which will later have to be described in

[3] Often this involves the assumption of hypotheses which are contradicted by experiment, like the transitivity of empirical equivalence in extensive measurement, or untestable ones (e.g. the assumption that mean velocities converge to instantaneous velocity), like the infinitary hypotheses of extensive measurement.

greater detail. Rather, idealization provides here a unified conceptual framework in which several different situations can be embedded.

Having one such framework in place is scientifically important because of its degree of generality. In its presence one can identify certain deep properties of empirical structures, e.g. extensive measurability. Thus idealization, conceived as enrichment, is fundamentally related to the formulability of global empirical principles, which in the case of measurement describe quantitative properties of empirical settings. The particular reason why reaching global principles is important is that they can be used as the basis for new applications. I have shown this for psychophysical measurement. The possibility of extending extensive measurability to a psychophysical context depends essentially on the use of idealization and the introduction of mathematical concepts it makes possible. This provides a strong reason why idealization is scientifically productive: it plays a fundamental role in generalizing empirical concepts and opening the way for new applications of mathematics.

3. Mathematization

Adding an account of idealization to Field's account of applicability just supplies something omitted from his treatment of scientific theories. This does not modify Field's views but rather supplements them. On the other hand, a modification becomes necessary when one observes that his exclusive focus on representation theorems makes his picture of applicability unnecessarily restrictive.

In short, Field thinks mathematics is applicable to an empirical setting because mathematical entities can be used as abstract counterparts of the objects and properties within that setting: the 'counterpart' relation is established by means of a representation theorem. For instance real arithmetic is applicable to extensive structures because numerical measures can be used (through representation) as abstract counterparts of objects belonging to an extensive domain. Similarly, the empirical operations which can be performed within an extensive structure can be studied mathematically because arithmetical addition provides an abstract counterpart for it.

It is possible to regard the abstract counterparts of objects within an empirical setting as points of a reference frame, whose abstract structure specifies the empirical correlations of the empirical objects themselves. For example, an extensive structure can be represented on a one-dimensional reference frame, whose points (the numerical measures) correspond to objects in the extensive structure: the relative position relations of the points along the one-dimensional frame gives the relative magnitude relations of the objects within the extensive structure.

The characterization of applicability in representational terms or, equivalently, in terms of reference frames or abstract counterparts, is a useful and insightful one, because it makes explicit the empirical interpretation of the mathematics which is applied to the description of an empirical setting. Nevertheless, equating applicability and representability is problematic.

The reason why representations and reference frames are not suited to describe all sorts of applications of mathematics has been discussed in chapters 4 and 5 of part 2. There are situations in which an empirical structure to be represented is simply not given: only a set of data is available, which may be used to reconstruct the internal relations of the empirical structure. In order to deal with these situations a framework which is alternative to the representational one has to be introduced.

The one I have developed to this end is based on the idea that applied mathematics is in essence a methodology to process empirically meaningful data.

One starts with a certain amount of empirical information, which I have called a data space. The data space is to be exploited to solve a particular applicative problem[4] and this is done in two ways. By identifying certain mathematically describable patterns on the data space, which I called reconfigurations, and by using mathematical theorems to study the properties of these patterns, a process I have called deductive analysis (cf. in particular part 2, chapter 5, section 1). Although these concepts have been used to deal with non-representational applications, they are also suitable to characterize representational ones.

The reason is that in this case we start from an empirical structure and we want to prove the existence of a reference frame (or representing structure) for it. In order to do so we gather data about the empirical structure and use them to prove a representation theorem. The data provide information about the formal properties of the empirical structure's relations and operations. For this reason they can be described by a suitable family F of formulas. Using techniques described in chapter 2

[4] For example, to decide whether an empirical structure is linearly ordered, to construct a reference frame, and so on.

of part 1, it is possible to isolate from this family F certain subfamilies, which can be assigned to the objects in the empirical structure. Finally, the latter structure can be used to induce its relations and operations on the subfamilies of F. The result is that these subfamilies determine a representing structure for the original empirical structure. In this case F is the relevant data space, the representing structure carved out of it is the relevant reconfiguration and the use of mathematics to prove that this reconfiguration is structurally similar to a corresponding empirical setting is the relevant deductive analysis (this deductive analysis amounts to showing that inferences on a representing structure provide empirically meaningful information).

In general, representational applications are those whose data spaces are used to erect reference frames based on the description of empirical structures, whereas non-representational applications are those whose mathematics is directed toward other objectives, e.g. finding optimal choices (as shown in the discussion of linear programs in chapter 5 of part 2) within a given data space of possible choices or determining the features of an empirical structure on the basis of a data space containing information relevant to their description (as shown in the discussion of seriation in chapter 4 of part 2) and so on. Each of these situations is characterized by a particular mathematical treatment, depending on the type of applicative problem to be solved. It may be said that different kinds of applicative problems are subjected to different forms of mathematization. This view is quite general and it properly includes Field's own view. It also modifies it in the sense that it waives its restriction to representational applications only.

4. Summary

In this thesis I have contributed to clarifying some basic issues concerning the applicability of mathematics, through a critical analysis of Field's account. The main results I have reached are a novel characterization of idealization and a unified characterization of a large family of applications, based on the concept of data space. These results are relevant both in themselves and as integrations and extensions of Field's own account of applicability, which lead to a more comprehensive and realistic alternative.

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