Diamond-free Partial Orders

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Abstract

This thesis presents initial work in attempting to understand the class of 'diamond-free' 3-cs-transitive partial orders. The notion of diamond-freeness, proposed by Gray, says that for any $a \leq b$, the set of points between a and b is linearly ordered. A weak transitivity condition called '3-cs-transitivity' is taken from the corresponding notion for cycle-free partial orders, which in that case led to a complete classification [3] of the countable examples. This says that the automorphism group acts transitively on certain isomorphism classes of connected 3-element structures. Classification for diamond-free partial orders seems at present too ambitious, but the strategy is to seek classifications of natural subclasses, and to test conjectures suggested by motivating examples.

The body of the thesis is divided into three main inter-related chapters. The first of these, Chapter 3, adopts a topological approach, focussing on an analogue of topological covering maps. It is noted that the class of 'covering projections' between diamond-free partial orders can add symmetry or add cycles, and notions such as path connectedness transfer directly. The concept of the 'nerve' of a partial order makes this analogy concrete, and leads to useful observations about the fundamental group and the existence of an underlying cycle-free partial order called the universal cover.

In Chapter 4, the work of [1] is generalised to show how to decompose ranked diamond-free partial orders. As in the previous chapter, any diamond-free partial order is covered by a specific cycle-free partial order. The paper [1] constructs a diamond-free partial order with cycles of height 1 from a different cycle-free partial order through which the universal covering factors. This is extended to construct a sequence of diamond-free partial orders with cycles of finite height which are not only factors but have the chosen diamond-free partial order as a 'limit'. This leads to a better understanding of why structures with cycles only of height 1 are special, and the rest divide into structures with cycles of bounded height and a cycle-free backbone, and those for which the cycles have cofinal height. Even these can be expressed as limits of structures with cycles of

bounded height, though not directly.

A variety of constructions are presented in Chapter 5, based on an underlying cycle-free partial order, and an 'anomaly', which in the simplest case given in [5] is a 2-level Dedekind-MacNeille complete 3-cs-transitive partial order, but which here is allowed to be a partial order of greater complexity. A rich class of examples is found, which have very high degrees of homogeneity and help to answer a number of conjectures in the negative.

Contents

Chapter 1

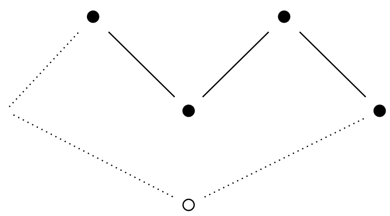
Introduction

Homogeneous structures as a field is a meeting point for algebra, combinatorics and model theory. This dates back perhaps sixty years to the classification of ultrahomogeneous structures by their families of finite substructures, which form amalgamation classes. A model-theoretic relational structure (here assumed to be countable) is said to be ultrahomogeneous if any isomorphism between finite substructures extends to an automorphism of the whole structure. Naturally this means that the automorphism group is extremely rich, may contain a lot of 'information' about the structure, and has a variety of interesting properties even as an abstract group. In some cases the original structure can even be recovered logically from the group in a reconstruction result. A survey of this is [8].

Ultrahomogeneity (also often referred to just as 'homogeneity' for short, for instance in [2]) is a large-scale property that often requires the diameter of the structure under any non-empty relation to be bounded. (Though this is not the case for an empty relation.) For instance, the random graph is the unique countable graph with the property that for any disjoint sets X and Y there is a point adjacent to all points of X and no points of Y. This immediately implies that all vertices of the random graph are either adjacent or that there is a point adjacent to both, putting them at distance 2. More generally, for instance in the case of connected partial orders, a pair of two points which are not directly adjacent necessarily has the same quantifier-free 2-type as any other pair of points which are not directly adjacent. But since we are assuming that the universe is connected there is a pair at some finite distance (generally 2), and other pairs with the same type must be connected in an analogous way. Thus in the case of ultrahomogeneous partial orders which are not trees, every pair (x, y) of points which are not directly comparable must have a point $z < x, z < y$ if any do, and similarly a point $t > x, t > y$ if any do. As the partial order is not a tree, this tells us that every pair of incomparable points is part of a diamond, namely a partial order consisting of two incomparable points lying in the same interval.

This means that if we wish to look at structures with a less trivial diameter it is necessary to restrict the class of substructures over which transitivity should hold. In particular, such substructures should be connected. In the context of partial orders, the class of 3-cs-transitive (a symmetry condition defined at 2.3) cycle-free partial orders has been classified in the work of Richard Warren. The motivation for this thesis is to look at diamond-free partial orders, which are a larger and more general class. In particular from [11] we know that a partial order is cycle-free iff its completion (as defined in 2.4) omits both diamonds and crowns. A crown is a finite, alternating path which is a closed cycle in that it starts and ends at the same point.

There are a number of reasons for looking particularly at suborders of size 3. It is known from [9] that a countable partial order is ultrahomogenous if and only if it is ≤ 4-transitive, that is n-transitive for $n \leq 4$. Connected substructures of size 3 are interesting since $3 > 2$ and distances between points do not 'collapse' to smaller values: distance complications arise when substructures of size 4 are considered. For instance, if we assume we are in the diamond-free case, the two incomparable points of a connected ramification-complete 3-element partial order must be at a distance of 2 and in only one way as there cannot be a diamond. However, the endpoints of a 4-element alternating path might be at a distance of 2 as in the following picture:



A central theme of the thesis is the interaction between digraphs and partial orders. Note that *digraph* is used in (at least) two slightly different senses. In [2], saying that (x, y) is a directed edge, is taken to imply that (y, x) is not also a directed edge so the relation is antisymmetric, whereas in [6] (x, y) and (y, x) may both be directed edges. Here, since digraphs will be constructed as irreflexive partial orders much of the time, ours will be of the former, antisymmetric type with no loops. Note that the two papers just cited classify classes of ultrahomogeneous digraphs. In the first case, this is all the countable antisymmetric ones, in the second it is all finite ones.

The above are two celebrated examples of classification. The original classification in this area [15] was by Lachlan and Woodrow and was of all the countable homogeneous undirected graphs. Cherlin's work in [2] was a major extension of this – for instance he gives $2^{\aleph_0}$ uncountably many structures whereas all other classifications mentioned here only have countably many structures up to isomorphism. The last classification relevant here is Schmerl's classification in [16] of the countable ultrahomogeneous partial orders. These fall into essentially three kinds: antichains of chains (including antichains as a special case), chains of antichains and the generic partial order embedding any finite partial order.

Now I discuss the relationship between partial orders and digraphs. Any antisymmetric irreflexive binary relation can be viewed as a digraph. However this comes about most naturally in the discrete case, and when for any two comparable points $x < y$, y is 'finitely far' above x, that is the interval contains a finite maximal chain. In particular, some of the finite or infinite chain cycle-free partial orders constructed by Warren are of this kind. For instance, an intuitive construction of a partial order of a kind we envisage (described precisely at 2.1) can be described thus. We start with a copy of $\mathbb{Z}$, at each point we would like there to be two incomparable ways to go up and down, so as the point is part of a chain giving us one way to go up and down, we add a point above and a point below not part of that chain and extend upwards and downwards so that all maximal chains are still isomorphic to $\mathbb{Z}$. Repeat this countably many times without identifying any vertices. As elaborated on later as part of the precise description, this corresponds to the construction of this partial order as a Cayley graph, and each point is introduced at the stage corresponding to the number of distinct consecutive blocks of the same letter in the corresponding word. This is a simple example of a cycle-free partial order with symmetry and nontrivial ramification, and has a clear existence as a digraph where (x, y) is an edge iff x is immediately above y. Indeed, this digraph is the unique digraph with in- and out-degree 2 at every vertex and a unique path between any two vertices. The partial order can be recovered as the transitive closure of the digraph relation.

Note that products and powers of linear orders are assumed in this thesis to have the lexicographic order. More complicated partial orders may have different order types of chains. Some are still discrete, such as $\mathbb{Z}^2$. This product order, however, now has infinite gaps, such as intervals of order type $\omega + \omega^*$, so the partial order contains strictly more information than the digraph as the digraph does not order points in different connected components, which may be comparable in the partial order. Some, such as $\mathbb{Q}$, are not discrete at all. A classification of countable 1-transitive linear orders is in [10].

Creed, Truss and Warren ([13], extended in [3] and [12]) give a classification of countable 3-cs-transitive partial orders (as in 2.3). These fall into several main types.

- Skeletal. Here maximal chains have length 2 and these pairs of points when embedded in the completion (from 2.4) become endpoints of infinite chains. In both this and the following case, in general, the alternating chain (ALT) embeds in the completion (and thus in the original).

- Sporadic. Here maximal chains are finite in the completion.
- Partial orders with infinite chains prior to completion.
- Partial orders where the completion does not embed ALT. ([12])

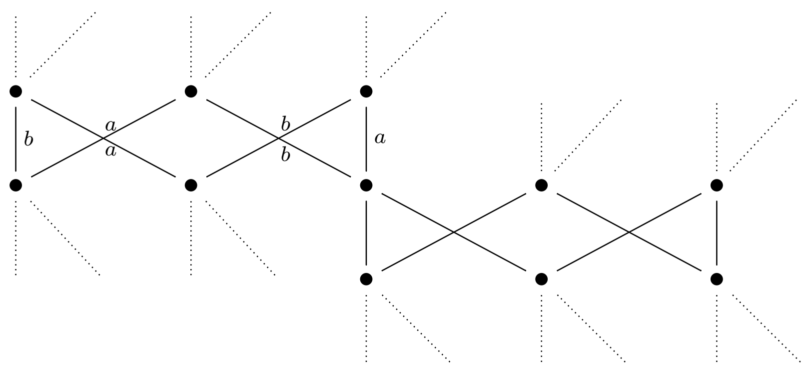
Skeletal ones are the most interesting ones for our present purposes, and provide a family of bipartite graphs which are transitive on 2-arcs if one adds a predicate picking out one half of the bipartition (locally 2-arc-transitive).

The paper by Gray and Truss, [5], is the immediate work preceding this thesis, and was initially motivated by a desire to find interesting examples of locally 2-arc-transitive bipartite graphs. A good number of these are given by skeletal or sporadic cycle-free partial orders, but these provide a limited range of possibilities. It was hard to construct non-cycle-free examples, until Gray proposed relaxing the 'cycle-free' constraint to 'diamond-free'. In view of the characterisation of cycle-free partial orders (CFPOs), this was quite a natural idea, and is equivalent to 'local linearity' - a requirement that intervals be chains. Some of the basic theory for CFPOs carries over to this situation, e.g. in 3-cs-transitive finite chain diamond-free partial orders, for $x < y$ elements of the original order, intervals of form (x, y) forming a maximal chain consist of two 1-transitive classes in the completion. These are the upward and downward ramification points, which may turn out to be the same collection of points.

It is worth emphasising that this apparently unpromising approach actually gives rise to many rich structures, a theme that will play a prominent role throughout the thesis. The situation is that such a bipartite graph forms a partial order, such that one part constitutes the upper points and the other the lower points, with the graph relation interpreted one way between them as an order relation. This is trivially a partial order as there is no triple to check for transitivity (there is no graph triangle). However, the completion may be highly non-trivial. The bipartite graph with upper part Y and lower part X is thought of as the 'skeleton' and the intermediate points Z of the completion form the 'flesh'.

To show that such complexity in the completion can actually arise, consider an order Z of the desired form (cycle-free initially, later diamond-free). As the points of Z arise as ramification points of the completion, Z must also be ramification complete and each point of Z must be a ramification point. That is, for each point at least one of its upward and downward ramification orders (as in 2.1) must be greater than one. One then adjoins suitable points for Y and X above and below maximal chains in Z, in such a way that every point z is bounded by some pair of points from X and Y. If there are two convex chains which have z as their greatest common point, then there are two points of Y which lie above some pair of such chains, and similarly if there are two convex chains which have z as their least common point. This means that each point of Z becomes a ramification point, that is the least upper bound or greatest lower bound in the completion, of two points in either X or Y. This is obviously possible as one could just adjoin the full set of chains quotiented by eventually being identical downwards for X and eventually being identical upwards for Y, but this will be a rather large set, generally being uncountable if Z is countable. We can do better if Z is countable. The adjoined sets can be chosen to be countable also, by constructively adjoining a pair of points in X or Y for points $z \in Z$ ramifying downwards or upwards respectively. Finally one can remove Z entirely to reach the bipartite structure, but it is not really absent as it can be recovered by completion. For good behaviour here we will generally work with choices of Z that are already complete in some way defined in 2.4.1.

A classic example of a diamond-free partial order which is not cycle-free is given by adjoining 6-crowns freely, one lower point of one 6-crown to one upper point of another. One of Rubin's motivating examples in [14] was the free group on 2 generators, whose Cayley graph corresponds to the digraph of the cycle-free partial order constructed earlier when we interpret the two generators as the two choices of successor for each point. It is natural to adapt that construction here, again taking the Cayley graph of a finitely presented group, in this case $\langle a, b|(ab^{-1})^3 = 1\rangle$. Here again there are two generators as each vertex has two successors and two predecessors, and the word relation corresponds to adding 6-crowns. Similar presentations of partial orders as presented groups are explored later in Chapter 3.



In [5] the principal constructions are carried out in the discrete case, and the interplay between partial orders and digraphs is exploited. To explain the easiest case, we would like to construct a locally 2-arc-transitive bipartite graph with the parts thought of as upper points Y, lower points X. This is trivially a partial order where $x < y$ iff $x \in X$ and $y \in Y$ are joined by an edge and so one may form its Dedekind-MacNeille completion(2.4.1). Any additional points lie in-between these two levels, and Z is the set of added completion points. If $X \cup Y$ is locally 2-arc-transitive then as a partial order it is 3-cs-transitive, and from this it follows that the maximal chains of Z are 1-transitive as linear orders.

In the initial case these maximal chains have order-type $\mathbb{Z}$, so the most straightforward way to violate cycle-freeness is to include 6-crowns having edges corresponding to consecutive pairs in the partial order Z. This construction is done in [5].

Some work was done in [5] on more general 2-transitive order-types. The order types which were discrete at the lowest level are most analogous to digraphs. These are $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ for $\alpha \geq 1$ but $\mathbb{Q}$, $\mathbb{Q}.2$ and $\mathbb{Q}_2$ (the rationals coloured in two interdense colours) were more problematical. Even in these cases, the crowns or other 'anomalies' were always adjoined on consecutive levels, that is their edges had order type 2. Note that Morel's list in [10] of all the countable 1-transitive linear orders comprises $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ for countable ordinals α. These ones correspond to where the chains are 1-transitive, which is typically where each point is both an upward and downward ramification point. The other two orders (thought of as 2-coloured orders) $\mathbb{Q}.2$ and $\mathbb{Q}_2$ are not 1-transitive as linear orders, but correspond to cases where the upward and downward ramifications points are distinct and when these are coloured in 2 colours are 1-transitive as coloured orders. In the $\mathbb{Q}.2$ case, the lower point of each pair ramifies upwards and the upper point ramifies downwards - otherwise the pair would be indistinguishable from a single point. Thus Warren's task in [13] was to deal with certain countable coloured 1-transitive linear orders, in addition to just monochromatic ones.

In generalising this, various challenges present themselves. The first is to provide constructions in which anomalies arise with 'legs' longer than 1. In the original case an n-crown which is included as an anomaly has its top points immediately above its bottom points, but we also wish to consider the possibility in which its top points may have distance 2 or more above the bottom points. We refer to these as 'extended' n-crowns, composed of 'legs' of order-type which may be greater than 1. It isn't immediately clear how to do this while retaining sufficient transitivity, one approach through explicit construction of a Cayley graph is discussed in 3.5.

A 'leg' of a cycle is a maximal chain. Another possibility is where dense 'legs' arise. One could consider for instance modifying the previous example to allow legs of order type $1 + \mathbb{Q} + 1$ (the order type of a closed interval in $\mathbb{Q}$), in a partial order with dense chains.

Another question arising is whether many different choices of substructure with cycles (referred to as 'anomalies') can be included. One might have a small family of such structures, which one would like to be joined together in specific ways. In Chapter 5 I describe a general construction based on a choice of anomaly or family of anomalies and a concept of ‘compatibility’ which answers most of these questions in the affirmative, providing 3-cs-transitive partial orders whose completions contain any compatible anomaly or family of anomalies. A common restriction made however is that the interior points of the constructed partial orders have infinite ramification order. To arrange finite ramification order seems harder.

The Reachability digraph, as defined in [1], is a way to recover information from a discrete partial order by exploring how far one can get from a single point by alternating chains. In structures where cycles have ‘legs’ of length 1 only, this is a smaller two-levelled partial order from which the original can be rebuilt. In Chapter 4 I try to extend the use of this information as far as possible to provide descriptions of diamond-free partial orders, so long as they are discrete and their connected components are recursively discrete (i.e. their interior maximal chains have order type $\mathbb{Z}^\alpha$). The main result of that chapter is that a 3-cs-transitive 2-level partial order whose completion has maximal chains of order type $\mathbb{Z}$ can be reconstructed from a ω-sequence of 2-level partial orders whose completions have finite height.

In Chapter 3 we define the concept of ‘covering projection’ which is a type of order homomorphism which comes up repeatedly in the succeeding chapters, and explore why it is a meaningful choice of homomorphism in this context. It emerges by examining the concept of a ‘nerve’ that these covering maps of ordered structures are analogous to covering maps in topology. I prove that diamond-free partial orders fall into families, each defined by a cycle-free partial order which is a universal cover for each partial order in the family. Groups arise in this context, most notably the fundamental group of the partial order which describes the cycles present. I also give some sufficient conditions in this chapter which allow the non-extremal points of a partial order to be constructed as a group presentation.

Chapter 2 provides definitions of the concepts which arise in this and later chapters. Among these is enough category theory to power constructions in later chapters, the model theoretic background to logical arguments, and proper definitions of the completion operations used throughout the thesis. I also prove some concrete results which illustrate some observations which can be made about the interior points by observing the extremal points.

Chapter 2

Introductory definitions

2.1 Discrete partial orders

A *digraph* G is a pair (V, E) where $E \subset V^2$ is an irreflexive antisymmetric relation. Here V is the set of vertices and E the set of edges. A digraph homomorphism is a map between vertices of digraphs which takes directed edges to directed edges, preserving direction. Such a map can (and here will) take non-adjacent points to two points connected by an edge in some cases, and in some others collapse them to the same point. An *embedding* is a rather stronger notion; it refers to an isomorphism between the embedded structure and its image.

In general throughout this thesis arrows will point from higher elements in a partial order to a lower one, and this will be the case for corresponding directed graphs as well.

Given x, y in a poset, the interval (x, y) consists of all elements z with $x < z$ and $z < y$, and will be empty in any case when $x < y$ is not the case. A poset is *discrete* if when (x, y) is nonempty there is an immediate successor of x in (x, y) and an immediate predecessor of y in (x, y). Note that this is true for 2-level partial orders in a trivial way, which is reasonable. Here by a *successor* of x we mean an element $z \in (x, y)$

such that (x, z) is empty and every $t \in (x, y)$ satisfies $z \leq t$. If the partial order is not diamond-free the set of successors of x in an interval (x, y) may not be a singleton - for instance in the diamond itself the top and bottom points form an interval in which there are two immediate successors to the bottom point. A point x may also have many sets of successors depending on the interval (x, y), and given an antichain $y_1, \ldots y_n$ of points above x the sets of successors $z_i \subseteq (x, y_i)$ for $1 \leq i \leq n$ need not in general be distinct. The term *predecessor* is defined dually, which means reversing the direction of the order relation. Each discrete poset $(P, <)$ gives rise to an adjacency digraph $(P, <')$ where $a <' b \iff a < b \wedge \forall c(a < c \leq b \implies c = b)$. If the digraph is connected, it is possible to recover the poset, so that when $<$ is used for the poset relation, $<'$ is the corresponding digraph relation and the two structures are identified.

An $n-$*arc* in a digraph (V, E) is a tuple of $n + 1$ points $(x_0, \ldots, x_n) \in V^{n+1}$ such that $(x_{i-1}, x_i) \in E$ for $i \in [1, n]$. For digraphs arising from posets, these correspond to finite chains which are maximal given their starting and ending points. A digraph is $n-$*arc-transitive* if the automorphism group acts transitively on the set of $n-$arcs. It is *highly-arc-transitive* if $n-$arc transitive for each n.

In a discrete poset corresponding to a digraph there are no cycles in the usual sense, since in a chain we cannot have $x = x_1 <' x_2 <' \ldots <' x_n = x$, so when the the term *cycle* is used, it refers to cycles in the graph which is the symmetric closure of the digraph.

Terminology for bipartite graphs is different, as they do not have non-trivial arcs of the above form. Instead arcs are taken in the graph sense, but the two parts of the graph are explicitly labelled as 'top' and 'bottom'. A 'locally 2-arc-transitive' bipartite graph Δ is one which is transitive on 2-arcs whose midpoints are in the same partition, so this condition implies that a V-shape, that is a 2-arc that starts and ends in the 'top' of the bipartition, may be mapped onto any other 2-arc of this form, but not in general onto an 2-arc that starts and ends in the 'bottom'. A similar condition applies to the other type of 2-arc, a Λ-shape. We may define powers of the digraph relation in the obvious way, namely $x <'^n y$ if for some n-arc $x_0 <' x_1 <' \ldots x_n$ we have $x = x_0$ and $y = x_n$. Observe that $x <'^1 y$ is equivalent to $x <' y$.

A *tree* in the traditional context of partial orders is a connected partial order P such that for every $x \in P$ the suborder $\{y \in P : y \leq x\}$ is linear. This implies that for every $x, y \in P$ there is some $z \in P$ such that $z \leq x$ and $z \leq y$. The tree P is rooted with root z if z is the least element. Upside down trees where the points above any point form a chain will also be referred to as trees. A suborder of a tree which is also a tree is a *subtree*.

Given a tree P and a point $x \in P$, an *upward cone* at x is a connected component of the suborder $\{y \in P : y > x\}$ and is a tree in its own right. If P is discrete then the cones at any point are rooted subtrees. The *branching order* of P at x is the number of distinct upward cones, if this is the same for any choice of x then we call it the branching order for P itself. In the more general notion of tree described now, the definition for upward cones is identical and one can define downward cones in the same way. This gives downward and upward branching orders for points of the tree. However it is also possible to have branching points not in the tree proper, though they will arise in the ramification completion defined in 2.4.1. The words *branch* and *ramify* are used interchangeably for trees (as are branching order and ramification order), but the latter will be preferred for partial orders with cycles.

More commonly in this thesis we will generalise a *tree* to be a cycle-free partial order (which will be properly defined later) T defined by a 1-transitive order type Z, an upward ramification order and a downward ramification order. If one additionally adds that the tree should only ramify at points of the tree, there is a unique 1-transitive cycle-free partial order with these characteristics, and can be constructed in a step-by-step fashion as a union of countably many approximations, starting with T_0 equal to a point. If the starting parameters are all countable then the result will be. At each stage, we need to ensure that the next approximation gives correct ramification for each point so far adjoined, and all the maximal chains have the correct order type.

To do this we define an upward ray to be the order type of $[x, \infty)$ for $x \in Z$ and a downward ray the order type of $(-\infty, x]$. This is independent of the choice of x because Z is 1-transitive. To go to the next approximation one takes all points $t \in T_i \setminus T_{i-1}$. These are newly added so only have upward and downward ramification order of 1 which is trivial. Then one amalgamates rays at each such point simultaneously equal to 1 less than the upward ramification order for upward rays and downward ramification order for downward rays. This process terminates in ω steps. This is equivalent, when upward and downward branching orders are equal to n and Z is the order type of an ordered abelian group, to the construction of the free product of n copies of the group Z. This is best thought of as amalgamating copies of Z which are infinite in both directions rather than rays, and in this case T_i corresponds to the addition of all words of length at most i. It is worth noting that [10] shows that each 1-transitive linear order is of the form of $\mathbb{Z}^\alpha$ or $\mathbb{Q}.\mathbb{Z}^\alpha$. These are all linearly orderable abelian groups. This construction has a great deal of symmetry: it is transitive on points, maximal chains, amalgams of maximal chains, or indeed any convex suborder, because one can from two copies of such a convex suborder conduct the amalgamation process to achive the same tree.

Note that a tree has maximal *branches* or maximal chains (the axiom of choice is assumed throughout this thesis without comment) and a countable tree has uncountably many such branches if it branches nontrivially, for instance if it has two incomparable points above or below any point in the tree.

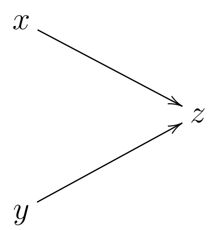
2.2 Category Theory

Some terminology from category theory will be used in this work. A standard text on the subject is [7]. In particular, all arrows and (co-)cones are taken in the categorical as opposed to the digraph sense.

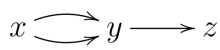
The notion of *duality* is the same in categories and orders. Intuitively one reverses the arrows or order in the definitions. For instance, the dual notion to a set unbounded below is a set unbounded above.

A *category* consists of classes P of points and A of arrows, here they will be proper sets, generally finite or countable. Each arrow has a starting point and an ending point. These are not necessarily distinct. The set of arrows A is closed under composition between arrows which link up correctly. There are always the identity arrows $1_p : p \to p$ for each point p with the property that whenever these are composed on the right or left they have no effect. A *subcategory*, which may also be called a *diagram* is a category which has point and arrows subclasses of the main category, it need not contain all the arrows between the points chosen.

A category is *filtered* when it satisfies two properties. The first is that for any two points x, y one can find a point and arrows satisfying the following, which is analogous to the joint embedding property in homogeneous structures.



The second is that for any pair of arrows $x \to y$ there is one arrow $y \to z$ such that the composites with the new arrow are equal. Note that in this condition x and y are not required to be distinct.

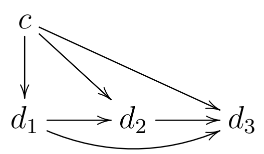


The consequence of all this is that for any finite collection of points and arrows (typically structures and embeddings respectively), there is a point (indeed, there are many) with exactly one arrow from each point of the collection to that point, such that the diagram commutes around the new point: any two ways to reach it from any other point are equal.

The dual notion is for the category to be *cofiltered*. In this case for any pair of points there is a pair of arrows from another point to both of them, and for any pair of arrows there is one arrow *into* the pair from the left, rather than out from the right, such that the composites are equal.

A point in a category is *initial* if for each point in the category there is a unique arrow from it to that point. Dually it is *terminal* if there is a unique arrow from each point to it. Existence of such points makes a category trivially cofiltered or filtered respectively.

A *cone* in a category C over a subcategory D consists of a point c and one arrow $c \to d$ for each $d \in D$ such that for any arrow $d_1 \to d_2$ in D, where d_1 and d_2 are not necessarily distinct, the composite of $c \to d_1$ with that arrow is equal to the arrow $c \to d_2$. The category of cones in C over D is the *slice category* C/D; there is an arrow between two cones c/D and c'/D if there is an arrow $t : c \to c'$ such that for every $d \in D$ the composite of t and the arrow $c' \to d$ is the arrow $c \to d$.



Dually a *cocone* in C under D consists of a point with arrows to it from each point in D, and the collection of cocones for a particular subcategory D form the *coslice category* $C\backslash D$.

A *limit* in C over D is, should it exist, a terminal object in the slice C/D. When C is a category of algebraic structures and suitable maps, it is equivalently an inverse limit in the algebraic sense, that is the unique structure with suitable maps to each structure in D commuting with the suitable maps in D such that any other structure with this property maps to it. The *colimit* under a subcategory is the initial object in the coslice category.

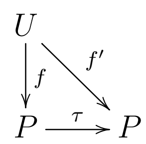
An *isomorphism* $a : x \to y$ is an invertible morphism. That is to say, if there exists an arrow a' such that a and a' compose one way to give 1_x and the other to give 1_y, then a is an isomorphism.

A *functor* $F : C \to D$ is a homomorphism of categories. Namely it is a map which takes points in C to points in D, arrows in C to arrows in D such that the start point of an arrow in the image is the image of the start point of an arrow in C and similarly for end points, it takes the identity arrow for each point in C to the identity arrow for the image of that point, and the composite of two arrows in C maps to the composite of the images of those arrows.

2.3 Homogeneous Structures

The language L used here is that of partial orders, either $(=, <)$ or $(=, \leq)$; the theory of partial orders is that of a transitive antisymmetric relation, which is respectively either irreflexive or reflexive - the two languages are interdefinable and will be used interchangeably as convenient. Automorphisms on a partial order P are bijections $P \to P$ which preserve the order relation. These form a group, Aut(P).

If U is a L-structure, any copy of U in P is exactly an embedding $f : U \to P$. If f and f' are copies of U in P, then there is an automorphism taking one to the other if one can find τ in Aut(P) such that the following diagram commutes.



As Aut$P$ is a group what we are looking at is its action on the set of embeddings of U into

P by right composition. Here we are saying that f' and f share an orbit of this action. If for any pair of copies of U there is an automorphism taking one to the other then we say P is *transitive* on copies of U. This corresponds to the group action being transitive.

There is an alternative notion: if $f, f' : U \to P$ are copies of U in P then there is one taking one to the other *setwise* if there are automorphisms $\sigma \in \mathrm{Aut}(U)$ and $\tau \in \mathrm{Aut}(P)$ such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\sigma} & U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ P & \xrightarrow{\tau} & P \end{array}$$

If for any pair of copies of U there is an automorphism taking one to the other setwise then we say P is *homogeneous* on copies of U. It is worth noting that 'transitivity' and 'homogeneity' are always used to mean different things, but may be interchanged depending on the author. Here transitivity in the case of finite embedded substructures treats them as ordered tuples, whereas homogeneity treats them as unordered subsets.

P is fully transitive if it is transitive on copies of every finite L-structure. Such ultrahomogeneous structures are ω-categorical if they are in a finite language L. Indeed, by the Ryll-Nardzewski theorem, a structure is ω-categorical iff there are only finitely many orbits of n-element substructures for each n. The finiteness of L ensures that there are only finitely many possible n-element non-isomorphic L-structures for each n, and each, if it exists, must correspond to a single orbit.

In this thesis we will not generally be looking at such structures. Instead the transitivity conditions will be on structures of size n - written *n-transitivity*. Even this may be too strong - 2-transitivity requires that any two antichains of size 2 be exchangeable, which tells us that any two points must be comparable or at the same distance (which might be one of several notions to be precisely defined later). Preferable to that is *n-cs-transitivity*, which is transitivity on *connected* structures of size n.

2.4 Dedekind-MacNeille Completions

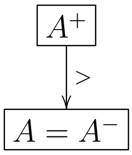
Recall the notion of the Dedekind completion of a linear order. There one takes the set of Dedekind cuts, that is the nonempty downward closed subsets which are bounded above. There is a bit of subtlety here - for instance in $\mathbb{Q}$ both $\{x : x \leq q\}$ and $\{x : x < q\}$ for $q \in \mathbb{Q}$ fit the description, and for consistency we choose $\{x : x \leq q\}$. These are totally ordered by inclusion and embed elements of the original order as the downsets generated by them.

A complete partial order is one where every cofiltered subset (convention here has arrows go down) has a supremum and every filtered subset has an infimum. Here because we have no use for the top and bottom elements of a traditional order completion we discard them by requiring that for a subset to have a supremum it must also be bounded above, and for it to have an infimum it must be bounded below.

The *Dedekind-MacNeille* completion of a poset P, written P^D, is defined to be the smallest order-complete suborder of the power set of P, $\mathcal{P}(P)$ (as a boolean algebra, so ordered by $A \leq B$ iff $A \subseteq B$) embedding principal ideals of the form $\{p \in P : p \leq q\}$ for $q \in P$.

Definition 2.4.1 *Given a partial order P and a subset $A \subset P$ define the set A^+ to be $\{p \in P : (\forall a \in A)a \leq p\}$ and similarly A^- to be $\{p \in P : (\forall a \in A)p \leq a\}$. Such a subset is an* ideal *if $A = (A^+)^-$ and both A and A^+ are nonempty. The Dedekind-MacNeille completion P^D is the set of ideals A. The set A^+ is the corresponding filter to A. The original partial order P embeds in P^D as the set of principal ideals $x^D = \{p \in P : p \leq x\}$, and the corresponding filters are called principal filters.*

An ideal is finitely generated if it is the smallest ideal containing some finite set $A \subset U$. Not all finite sets generate ideals as opposed to a pair of sets one of which is empty, and fewer still generate ideals which are not principal. The ramification completion U^+ of *a partial order U is the suborder of the Dedekind-MacNeille completion which contains only the finitely generated ideals or ideals corresponding to finitely generated filters.*



In the cycle-free and diamond-free cases, it is sufficient to consider just the ideals. Additionally in these cases the ramification points are just the points which are the supremum of a pair of points (or the dual notion) – this is immediate because the set of points at or below that ramification point must form a tree, so it is sufficient to take two points of P from downward cones of that tree which only meet at the root.

The tree can be partitioned into maximal downward cones meeting each other at the root and nowhere else; the number of such distinct cones is the downward *ramification order* of the point. There is a corresponding notion of upward ramification order which need not be equal to the downward ramification order. Both the full completion and the restriction to ramification points are closure operations. The use of such constructions originates in the paper of Warren [13].

It is worth noticing that in the theory of lattices and complete partial orders, which may be found described in [4], both empty and unbounded ideals (the latter corresponding to empty filters) are permitted, which adds to the completion a single maximum and a single minimum point. There is no further difference.

Remark 2.4.2 *Going from an ideal I to I^{+-} is a closure operation, so it does not remove any elements and $I^{+-+-} = I^{+-}$. By duality, J^{-+} is a closure operation on the filter, so I^{+-} retains all the bounds that it has previously.*

Proof

First, if $x \in I$, then $\forall y \in I^+ x \leq y$, so $x \in I^{+-}$. As $I \subseteq I^{+-}$, $I^{+-+} \subseteq I^+$, so $I^{+-+-} \subseteq I^{+-}$. As we already know $I^{+-} \subseteq I^{+-+-}$, the two must be equal. □

Remark 2.4.3 *Ideals have some recognisable properties: they are downward closed and, when an ideal I contained in a partial order P contains a subset S whose supremum s is in that partial order, I also contains s. However, it is not possible to tell directly from looking just at a suborder whether or not it is an ideal.*

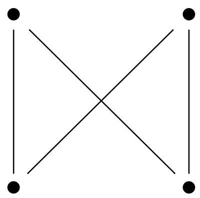
Proof

For any x contained in an ideal I, any $y < x$ is also a lower bound for anything for which x is a lower bound. For any S with supremum s, s is by definition a lower bound for anything for which S is a lower bound.

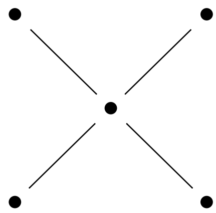
To get a counterexample for the last point, observe that the bottom 2 points of a $K_{2,2}$ form an ideal, but if a third point is added to the bottom also less than both top points, they no longer form an ideal although no relation involving them has changed. □

2.5 Cycle-free and Diamond-free partial orders

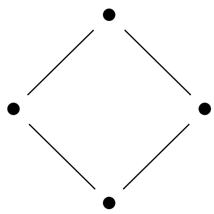
It is necessary to consider completions before defining the properties of cycle-freeness and diamond-freeness for a number of reasons. The classical example is that of the partial order $K_{2,2}$ (below left) which does not appear to be cycle-free (being a 4-crown) but has a completion where paths are unique (right), because prior to completion the infima and suprema that paths turned at were not present.



completes to



A partial order P is *diamond-free* if all intervals in the completion P^D as defined earlier are chains. Equivalently it does not embed a *diamond*, which is a specific 4-element partial order as shown.

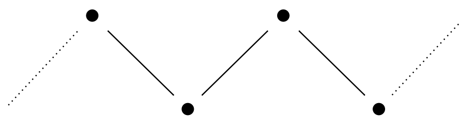


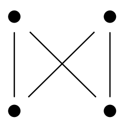
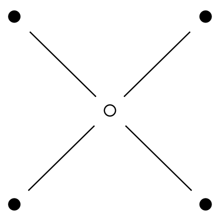
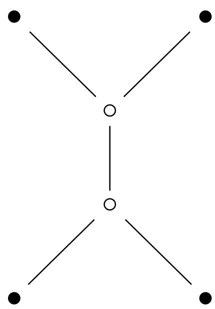
A *walk* of length n in a ramification complete diamond-free partial order is a sequence $x_0, \ldots, x_n$ of points such that x_i and x_{i+1} are comparable for $i \in [0, n)$ and the intervals generated by x_i, x_{i+1} and x_{i+1}, x_{i+2} for $i \in [0, n-2]$ meet only at x_{i+1}, so backtracking is prohibited. If the additional condition that the intervals generated by x_i, x_{i+1} and x_j, x_{j+1} only meet if $-1 \leq i - j \leq 1$ is imposed then the sequence is also a *path*. A walk starting and ending at the same point is a *crown*. A *cycle-free* partial order is a diamond-free partial order which has at most one walk between any two points in the Dedekind-MacNeille completion. An equivalent definition of a cycle-free partial order is that the Dedekind-MacNeille completion omits both diamonds and crowns.

Definition 2.5.1 *Let X be a linear order. A partial order P is X-*levelled *if there is a function $f : P \to X$ called a* level function *such that for any $x, y \in P$ with $x < y$ the restriction of f to $[x, y]$ is a bijection with the image of that interval. Level functions are often also assumed to be surjective. In particular in a 2-level partial order every point is* *either maximal or minimal. The word* rank *is used synonymously.*

It is important to note that ramification completion in general includes both finitely generated ideals and finitely generated filters - as there may be points which only ramify upwards or only downwards. An example of such a structure may be seen by taking a $\mathbb{Q}.2$-ranked cycle-free partial order which ramifies upward at the lower points in each pair and downward at the upper points only. Of course, vertex-transitivity suffices to give this but that is not implied by 3-cs-transitivity.

One important example of a 2-level cycle-free partial order is the alternating path Alt, namely the bipartite version of the graph of a line.



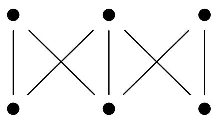
Lemma 2.5.2 *A partial order is ramification complete iff any embedding of $K_{2,2}$,*  *extends to an embedding of*  *or of* *, where in the first case the newly added point is infimum for the top two and supremum for the bottom two, and in the second case the newly added top point is infimum for the top two and the newly added bottom point is supremum for the bottom two.*

Proof

Suppose the partial order is not ramification complete, so there is a non-principal ideal I or filter F. If I is non-principal then so is $F = I^+$ and vice versa. Thus both actually arise. It is sufficient to pick an antichain of size 2 from the ideal and an antichain of size 2 from the filter to give the copy of $K_{2,2}$ whose completion does not embed in this way.

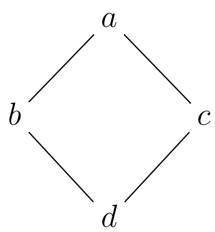
Suppose it embeds $K_{2,2}$, then the ideal generated by those lower elements has an upper bound set which is not a singleton, and so is non-principal. The filter generated by the upper elements is similarly non-principal. If they generate the same ideal/filter pair, then the resulting ramification point arises as in the first case, if not then in the second. $\square$

Lemma 2.5.3 *A* 2*-level partial order is diamond-free iff it does not contain the following induced suborder.*



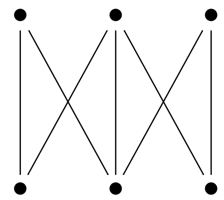
Proof

Consider a diamond:

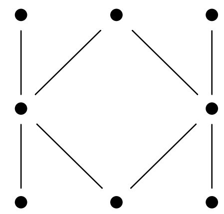


The intermediate vertices b and c of the diamond must be ramification points as they are neither maximal nor minimal. As such, there are points below and above each which are extremal. As c is not comparable to b there is a point above c which is not above b – if every extremal point above c were above b then $c \geq b$ in the ramification completion as its ideal would at least contain that corresponding to b. Repeating this argument gives three other extremal points in the diagram. The remaining two are simply obtained by taking an extremal point above a and one below d.

Conversely one can consider a couple of diagrams.



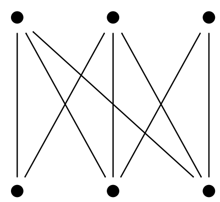
completes to



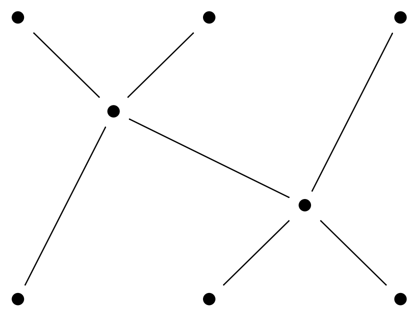
Diamonds arise here.

□

If an extra edge is added, however, there is no diamond.



completes to



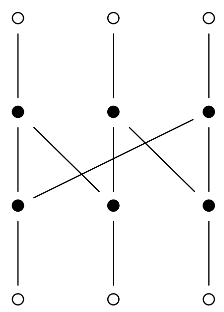
For that matter, $K_{3,3}$ simply completes by adding a single intermediate point

Lemma 2.5.4 *A diamond-free partial order P is cycle-free iff it does not contain any induced crowns.*

Proof

Recall that P is cycle-free by definition iff P^D is diamond-free and crown-free. The hypotheses give that P^D is diamond-free, so it suffices to prove that crowns in P^D somehow correspond to crowns in P.

Consider the completion P^D. If P contains crowns then so does P^D. Suppose that P^D is not cycle-free. Then it contains a cycle of minimal length, which is a crown. Each top element must have an upper bound which is an element of P, similarly each bottom element must have a lower bound which is an element of P. These form a crown, as in the following diagram.



To see that they form an induced crown suppose there is some additional relation. This immediately creates a shorter cycle, contradicting the assumption of minimality. The minimum length necessarily exists and is attained because any cycle has length which is a positive integer. □

The paper [11] provides more detailed proofs.

Chapter 3

Notes on covering projections

3.1 Introduction

The notion of *covering projection* between directed graphs arises in combinatorial contexts, some of which are mentioned in [17]. The concept of nerve originates in category theory, where one can take the nerve of a category to find an associated topological space. The usage of the term in this chapter is a specific case of that, as partial orders can be seen as categories with a unique arrow between any two comparable elements in the direction of the relation.

Here we define covering projections and describe the universal cover for a diamond-free partial order. This is the unique cycle-free partial order which contains a unique representation for each walk in the diamond-free partial order. Universal covers partition the class of diamond-free partial orders into families. We refer to the known classification of cycle-free partial orders and make a start on similarly describing diamond-free partial orders. We describe a small family of diamond-free partial orders which arise as Cayley graphs of groups.

3.2 Definitions

We will usually work with ramification-complete countable partial orders, but this will be apparent from the claims. This condition means that any finite non-empty set that is bounded above or respectively below has a least upper or lower bound. For any point u in a partial order U let the set of comparable points $c(u)$ be $\{x \in U : u \leq x \vee x \leq u\}$. A *covering projection* $f : A \to B$ is a surjective map such that for each $x \in A$ the restriction of f to $c(x)$ is an isomorphism with $c(f(x)) \subseteq B$. The identity map, for example, is always a covering projection. When there is such a function f we say A *covers* B.

Please note that in this paragraph when indices i, j are used they will range between 0 and $n-1$, including or excluding 0 as necessary so that all expressions refer to points listed in the sequence. A *walk* of length n in a ramification-complete partial order is a sequence $x_0, \ldots, x_n$ of points in that partial order such that x_i and x_{i+1} are comparable and x_i is the meet or join of x_{i-1} and x_{i+1}. Note that it is therefore alternating: if $x_i > x_{i-1}$ then $x_{i+1} < x_i$. The walk is said to be *between* x_0 and x_n. A *path*, or an *alternating path*, is a walk such that x_i is only comparable with x_{i-1} and x_{i+1}. This gives that points in two intervals $[x_i, x_{i+1}]$ and $[x_j, x_{j+1}]$ can only be comparable if the two intervals are adjacent, i.e. $j = i + 1$ or $i = j + 1$.

Lemma 3.2.1 *If any exist, every walk of minimum length between two points in a ramification-complete partial order is a path.*

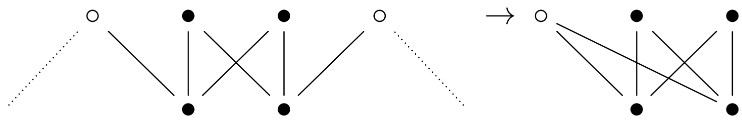
Proof

Suppose some walk $x_0, \ldots, x_n$ is not a path, then for some distinct i, j differing by more than 1, x_j is comparable with x_i. Without loss of generality $j > i$. Let i be minimal and j maximal. Then $x_0, \ldots, x_i, x_j, \ldots, x_n$ is a shorter path. Thus any walk which is not a path cannot be of minimum length. □

Remark 3.2.2 *If A covers B, then it is not true that A being Dedekind-MacNeille complete or ramification complete implies that B is. Nor is it true that an ideal in A maps to an ideal in B. However, it is true that if B is ramification complete then so is A, and if B is Dedekind-MacNeille complete then A is Dedekind-MacNeille complete.*

Proof

For a counterexample to the first point see that the alternating chain covers the 4-crown. In the second case, consider the following map.



In the fourth case, suppose there is some ideal/filter pair $I \subset A$, $J = I^+$, $I = J^-$. As f is a homomorphism, $f(I) \subseteq f(J)^-$, $f(J) \subseteq f(I)^+$, and by completeness of B there is some $x \in B$ such that $f(I) \subseteq \{x\}^-, f(J) \subseteq \{x\}^+$. As I is nonempty let x' be the preimage of x connected to some element of I. As f is a bijection restricted to elements comparable with x', we have x' greater than or equal to all elements of I and less than or equal to all elements of J, so the ideal and filter were in fact principal.

In the third case, suppose there is some ideal/filter pair as above, each finitely generated. Then one can pick a generating set and use the images of the generators to generate an ideal/filter pair such that the ideal contains $f(I)$ and the filter contains $f(J)$. The rest of the argument follows. □

An immediate corollary of this is that if B is Dedekind-MacNeille complete then A is as well, or if B is ramification complete then A is also ramification complete.

Result 3.2.3 *Ramification-complete diamond-free partial orders are exactly the partial orders covered by cycle-free partial orders.*

Proof

If there is a diamond in a partial order then anything covering it must also contain a diamond, as the diamond will be in the set of comparable points of one of its endpoints. So anything covered by a cycle-free partial order must be diamond-free. The proof that there exists a cycle-free partial order covering it is in Theorem 3.2.11. □

The following lemma shows how covering projections interact with Dedekind-MacNeille completion and justifies the choice of working most of the time with ramification-complete partial orders. First recall the construction of the Dedekind-MacNeille completion as consisting of ideals A with corresponding filters $A^+ = \{p \in P : (\forall a \in A) a \leq p\}$ obeying the condition that $(A^+)^- = \{p \in P : (\forall a \in A^+) p \leq a\}$ equals A.

Lemma 3.2.4 *Covering projections extend through Dedekind-MacNeille completion in some cases, i.e. if $f : U \to V$ is a covering projection taking ideals to ideals then there is a covering projection $f^D : U^D \to V^D$.*

Proof

Set f^D to be the pointwise application of f to the elements of U^D, which are ideals in the partial order U. First one must check that this map is surjective. Consider then an ideal $\mathcal{V}$ in V^D. This (and thus all its elements) is bounded above by some element $v \in V$. Let u be an element of U such that $f(u) = v$, so f is an isomorphism $f_{\leq u}$ when restricted to the elements directly below u. Set $\mathcal{U}$ to be the pointwise preimage in $f_{\leq u}$ of $\mathcal{V}$. Then $f^D(\mathcal{U}) = \mathcal{V}$.

It remains to check that $\mathcal{U}$ is an element of U^D. Set $\mathcal{U}^+ = \bigcap_{u' \in \mathcal{U}} \{u \in U : u \geq u'\}$. Then, as f is an isomorphism on any of the principal filters, it takes $\mathcal{U}^+$ isomorphically to

$\bigcap_{v' \in \mathcal{V}} \{v \in V : v \geq v'\} = \mathcal{V}^+$. By duality $\mathcal{U}$ itself is the set of lower bounds of $\mathcal{U}^+$, and thus an ideal.

It is clear that f^D is an order preserving map, so to show that it is a covering projection one just needs to check, without loss of generality, that for any $\mathcal{U} \in U^D$, the restriction of f^D to greater elements of U^D is an injection. It is sufficient to show that any element of V^D greater than $\mathcal{V} = f^D(\mathcal{U})$ has a unique preimage greater than $\mathcal{U}$. To see this, let $v \in \mathcal{V}$ and $\mathcal{W}$ be an arbitrary element of V^D greater than $\mathcal{V}$. Then every element of $\mathcal{W}^+$ is greater than v, so the corresponding filter of $\mathcal{W}$ has a unique pointwise preimage from which one can recover a unique preimage for $\mathcal{W}$. □

Corollary 3.2.5 *Restricting the above f^D to the ramification completion U^+ gives a covering projection f^+ to the ramification completion V^+.*

Proof

Observe that a finitely generated ideal contains its generators, which will have some common upper bound. f^D is an isomorphism on all ideals less than (the prime ideal for) that upper bound, and will take the generators and their supremum (and only this supremum) to their unique images and another supremum. □

The *path distance* is the length of the shortest path between two points in a connected partial order. In particular any point will have a path distance of 0 from itself, and comparable points will have a path distance of at most 1.

Lemma 3.2.6 *Let $f : U \to V$ be a covering projection. Then if $x, y \in V$ have a path distance $d(x, y)$, this is the least path distance of any preimage of the pair.*

Proof

No preimage can have a shorter path distance, as the image of this path would be at least a walk which could be contracted to a path of not greater length. Given a path realising the minimum distance, one may pull it back segment by segment to give a path in U. □

To illustrate this procedure in more detail, let $x = x_0, \ldots, x_n = y$ be a minimal path in V. If a preimage x'_i of x_i is fixed, there is a unique preimage of x_{i+1} which is comparable with x'_i, so there are unique preimages x'_j for all the x_j. If points on the walk with turns at x'_j were comparable, there would be a shorter walk which is not possible – its image would be a walk in V shorter than the path distance – so the walk must be a path.

The following corollary is immediate from reversing the process.

Corollary 3.2.7 *Let $f : U \to V$ be a covering projection. Then any path in U is the preimage of some walk in V.*

Corollary 3.2.8 *Let V be a diamond-free partial order, $f : U \to V$ be a covering projection, x a point of V and p a path containing x. Then given a preimage $x' \in U$ of x there is a unique preimage of the path p, and f is an isomorphism when restricted to that preimage.*

Proof

The preimage is found as in the previous lemma.

In particular the above is true for preimages of paths between points $x, y \in V$ given a choice of preimage for x. □

The following lemmata and corollary show the existence of a unique universal cover where they exist (in other words, for any diamond-free partial order).

Lemma 3.2.9 *Let $f : A \to B$ and $g : B \to C$ be covering projections. Then $gf : A \to C$ is a covering projection.*

Proof

The composite map is clearly a surjective order-preserving map. As for any $a \in A$ the function f is a bijection from $c(a)$ to $c(f(a))$ and g is a bijection from $c(f(a))$ to $c(gf(a))$ the composite is a bijection from $c(a)$ to $c(fg(a))$. □

Lemma 3.2.10 *Any covering projection between connected cycle-free partial orders is an isomorphism.*

Proof

Assume that the cycle-free partial orders in question are complete. It does not matter whether this means ramification complete or Dedekind-MacNeille complete. If not one may extend them to either completion by 3.2.4, and this extension can only be injective if the original map is.

Let $f : U \to V$ be such a map. It suffices to show that f is an injection. Let $a, b \in U$ be preimages of some $x \in V$ and p a path between them. A covering projection must take a path to a walk, and as V is cycle-free any walk between x and itself must be trivial, so $a = b$. □

Theorem 3.2.11 *For any ramification-complete connected diamond-free partial order V there are a connected cycle-free partial order U which is unique up to isomorphism and a covering map $f : U \to V$ (which is not unique), such that any covering projection from a connected space $g : W \to V$ gives rise to a unique covering projection $g' : U \to W$ such that $f = gg'$.*

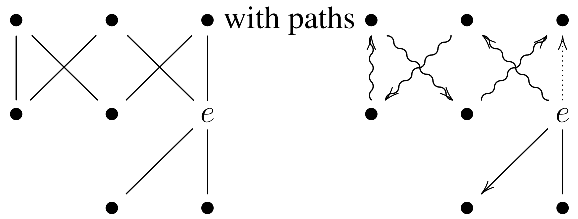
Proof

We start by constructing this cycle-free partial order U. Let e be an element of V. Let

U be the set of walks in V starting at e, with an order relation based on extension of walks, which will be formally described after the diagram. Intuitively, where two walks $x = (x_0, \dots x_n)$ and $y = (y_0, \dots y_m)$ are comparable if they have length differing by at most 1 and:

- If $(y_0, \dots y_{m-1}) = x$ then $y < x$ if $y_m < y_{m-1}$ and $y > x$ if $y_m > y_{m-1}$.
- If $y_0, \dots y_{m-1} = x_0, \dots x_{n-1}$ then $y < x$ if $y_m < x_n$ and $y > x$ if $y_m > x_n$.

This can be illustrated by a diagram. In the following diamond-free partial order, with the paths as illustrated $\bullet \rightsquigarrow \bullet$ is incomparable with the others, whereas $\bullet \dashrightarrow \bullet$ $>$ $\bullet \longrightarrow \bullet$.



This is in fact almost a transitive relation as if $(x_i) > (y_i)$ and $(y_i) > (z_i)$ then, if all three walks have the same length, then the last elements of each are comparable and $(x_i) > (z_i)$ by the second rule. If they are not then (y_i) cannot be longer than the other two as they would otherwise be identical. If (x_i) is shorter than the other two then (z_i) is an extension of (x_i) by an element less than that used to extend (x_i) to give (y_i), and so is less than (x_i). A similar argument applies if (z_i) is shorter than the other two. Because walks alternate, if (x_i) is longer than (y_i) then (z_i) must have the same length as (y_i) and a lower termination point, and similarly if (z_i) is longer than (y_i) then (x_i) must have the same length as (y_i) and a lower termination point. Adding the relation in these cases is sufficient to make it transitive, giving the following explicit form: $x = (x_0, \dots, x_n) \leq y = (y_0, \dots, y_m)$ iff $|n - m| \leq 1$ and

- $m = n + 1$, $(x_0, \ldots, x_{n-1}) = (y_0, \ldots, y_{m-2})$, $x_n \leq y_{m-1} \leq y_m$.
- $m = n$, $(x_0, \ldots, x_{n-1}) = (y_0, \ldots, y_{m-1})$, $x_n \leq y_m$.
- $m + 1 = n$, $(x_0, \ldots, x_{n-2}) = (y_0, \ldots, y_{m-1})$, $x_n \leq x_{n-1} \leq y_m$.

This is a cycle-free partial order: there is a unique path between two walks which is found by retracing them from the end points towards e until they meet.

Let $f : U \to V$ simply take walks to their endpoints in V. This map is surjective as V is connected. It is an order preserving map because whenever one walk walks in U is less than another their endpoints must be comparable and the endpoint of the first must be less than the endpoint of the second. To show that it is a bijection when restricted to points comparable with an element of U, let (x_i) be a walk in U, x its endpoint and x' its second last point or last turning point. If the walk has length 0 or 1 we may take the base point as x', and in the first case x as well. It suffices to show that there is exactly one element in U comparable with (x_i) corresponding with every element of V comparable with x.

Without loss of generality assume $x < x'$ and let $y \in V$ be comparable to x. If $y < x'$, in other words $y \in (x, x')$ or $y \leq x$ then (x_i) with x replaced by y is comparable with (x_i) and maps to y. If $y > x'$ then (x_i) truncated and with x' replaced by y is comparable with (x_i) and maps to y. Otherwise y is greater than x and not comparable with x', so the extension of (x_i) by y suffices.

The unique factoring of any other covering projection is a consequence of the universality property. Given the covering map $g : W \to V$, pick an element $e' \in W$ mapping under g to e. Having determined this point, every walk in V, that is element of U, has a unique pull-back to a walk in W. Let then g' be the map taking the element of U to the endpoint of the walk in W, giving the required property that $gg' = f$. This is a surjection because W is connected, so every element of W is reachable from e' by a walk, which is the unique pull-back of a walk in V.

Applying the previous lemma tells us that if we have another possible universal cover T, then there are unique covering maps $T \to U$ and vice versa such that both possible composites are isomorphisms. As the maps are surjective, T and U must be isomorphic. □

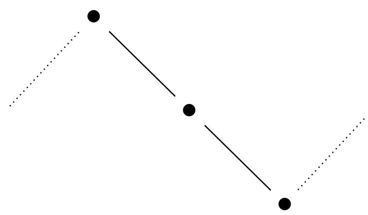
Remark 3.2.12 *If V is not complete but is connected and diamond-free then one may obtain the universal cover $f : U \to V$ by first completing V and obtaining the universal cover $f' : U' \to V^+$. Then we take $U = f'^{-1}(V)$ and f to be the restriction of f' to U, which is cycle-free because suborders of cycle-free partial orders are cycle-free. By the previous lemma and the universal property of the universal cover, this gives us the unique object with this property.*

In fact, one can see that, among diamond-free partial orders, the category of covering projections is divided into classes determined by the isomorphism type of the cycle-free partial order covering the class.

Example 3.2.13 *If the diamond-free partial order is vertex-transitive then so is the corresponding cycle-free partial order, as will be shown later. A connected ramification-complete vertex-transitive cycle-free partial order is determined by the order type of its maximal chains and the upward and downward ramification orders at each vertex. Since these are all fixed by any covering projection one can immediately see what the universal cover for a vertex-transitive diamond-free partial order is.*

Remark 3.2.14 *For an arbitrary connected partial order, one can take the partial order of walks, giving a unique cycle-free partial order. The resulting surjection is however not necessarily a covering projection. If the order is not diamond free intervals are not chains, so instead of describing a walk by just endpoints one must consider it as an image of a connected partial order. (Taking just endpoints gives us the diamond if we start with the diamond.)*

To be precise, a walk is now a map f from a connected cycle-free partial order K with designated endpoints (points whose connecting path is the entire partial order), which maps those endpoints to the endpoints of the walk and for an interval I from K the image $f(I)$ is convex. Two walks are comparable if they are (possibly trivial) upward or downward extensions of the same walk.

Example 3.2.15 *Starting with the diamond gives*  *, an extended version of the alternating path. Similarly, if we start with any ranked partial order such as a finite boolean algebra gives a cycle-free partial order with the same linear order of levels and each point has the same upward and downward ramification order as its preimages. In the case of the finite boolean algebra which has ramification orders determined by the level the point is on, the same function determines the ramification orders of points in the preimage by level.*

Define a *cycle* in a ramification-complete diamond-free partial order to be a walk $x_0, \ldots, x_n$ such that $x_0 = x_n$, $(x_0, \ldots, x_{n-1})$ and $(x_1, \ldots, x_n)$ are paths and x_n is the supremum or infimum of x_1 and x_{n-1}. Note that n is always even, as sequence elements are alternately suprema and infima of their immediate predecessors and successors. The length of such a cycle is n. The *cycle length* of a ramification-complete diamond-free partial order is the minimum length of a cycle in the order.

Lemma 3.2.16 *The cycle length of a ramification-complete diamond-free partial order is at least* 6.

Proof

A cycle of length 2 is nonsensical. A cycle of length 4 is also impossible as two points, if they have a supremum or infimum, have a unique one. They cannot have both a supremum and an infimum as that would constitute a diamond. So suppose $x_0, \dots, x_4$ were a cycle of length 4 then x_1 and x_3, as the meet or join of x_0 and x_2, would have to be equal. □

Corollary 3.2.17 *Let $f : U \to V$ be a covering projection between ramification-complete partial orders. Then the cycle length of U is at least that of V.*

Proof

The covering projection f takes cycles to sequences which may be truncated to give cycles. So, considering a cycle of minimum length in U, there is a cycle in V of length at most the cycle length of U. □

The path metric in a connected ramification-complete diamond-free partial order U gives the distance between two points to be the length of a path of minimal length between them. If X is a connected suborder of U, one can consider on this suborder both the path metric given by the order X and that induced from U, whose distances will be less or equal. The *diameter* of X in a specified metric is the greatest path-distance attained, if the distances are bounded. Note that while X is expected to be ramification-complete it is not expected to be convex as a suborder. One can also talk about the distance between subsets A, B of a partial order U, in which case it is the minimum over $a \in A$ and $b \in B$ of the distance between a and b. A significant part of the following essentially uses the fact that paths of length less than half the cycle length of the partial order are unique given the endpoints.

Lemma 3.2.18 *Let X and Y be connected ramification-complete diamond-free partial orders such that X embeds into Y as a suborder. Let n be the cycle length of Y. If X has* *diameter less than $n/2$ then the path distance in X between points $a, b \in X$ is equal to the path distance induced between those points as elements of Y.*

Proof

Suppose not. Then consider a counterexample (a, b) in X such that the distance between a and b is minimal among counterexamples. There is one path between them in X and another, shorter, path in Y. Reversing the latter path and concatenating it with the earlier one gives a walk of length less than n with the same start and end point. Where the paths were joined there may be some duplicated sections. Deleting these gives a cycle of length less than n, which contradicts the hypotheses. $\square$

Lemma 3.2.19 *Let $f : U \to V$ be a universal covering of the connected ramification-complete partial order V, $\sigma : V \to V$ be an automorphism of V and $u, u' \in U$ points such that $\sigma(f(u)) = f(u')$. Then there is a unique automorphism $\tau : U \to U$ taking u to u' such that $f\tau = \sigma f$, as in the following diagram.*

$$\begin{array}{ccc} U & \xrightarrow{\tau} & U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{\sigma} & V \end{array}$$

Proof

The map τ is obtained thus: any point in U is connected to u by some path and f maps this to a walk in V which is possibly moved by σ. This can be pulled back to give a walk in U. Reading off the endpoint of this walk gives the image of that point. As U is the universal cover and cycle-free, paths are unique and this is well-defined. The resulting map is an automorphism, as walks correspond uniquely to points in the universal cover and σ is a bijection on the set of walks in V.

Pick an arbitrary base point $v \in V$. By the properties so given, a point $u \in U$ corresponds to a unique walk in V from v to $f(u)$. The automorphism σ takes that to $\sigma(f(u))$, which is also the endpoint of $\tau(u)$ by definition. □

Definition 3.2.20 *Let $f : U \to V$ be a covering projection. The group $Aut(f)$ is the subgroup of elements g of $Aut(U)$ such that fg is an automorphism of V. Say f is a regular covering if, for every $v \in V$, $Aut(f)$ is transitive on all elements $u \in U$ such that $f(u) = v$.*

Remark 3.2.21 *In some cases there is an intermediate partial order W through which f factors as a covering projection such that automorphisms are images of automorphisms of W. If possible, the following diagram arises.*

$$\begin{array}{ccc} U & \xrightarrow{\tau} & U \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} \\ W & \xrightarrow{\rho} & W \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2} \\ V & \xrightarrow{\sigma} & V \end{array}$$

The following example illustrates this and the point that whereas a covering endomorphism to a cycle-free partial order, as shown earlier, is always an automorphism, this is not always the case for a diamond-free partial order. Consider the following partial orders. Let T be the Cayley graph of the free group F_2 on two generators and V that corresponding to the quotient of the free group by the word $z = (xy^{-1})^3$. Let U on the other hand be the quotient of T by the transitive closure of the equivalence relation that deems the words u and uz equivalent if the sum of the indices in u is positive. Then the map that takes a word u to xu is an automorphism both of V and T (it is simply an upward translation of the graph along a single edge) and an endomorphism on U (but not injective and so not an automorphism), and the map that takes u to $x^{-1}u$ is an automorphism of

V and T but ill-defined on U, so no map at all there (as it would attempt to unwind the cycles on the lowest level and this is not possible homomorphically).

In the event that there is an intermediate partial order W and any automorphism of V does in fact lift to an automorphism of W, call the covering $W \to V$ symmetry preserving.

Lemma 3.2.22 *Let $f : U \to V$ be a covering projection and n be the cycle length of V. Let X be a connected partial order with diameter less than n. Then f takes copies of X embedded in U to copies of X embedded in V.*

Proof

Let $Y \subset U$ be such an embedding. The claim fails only if points in the image of Y under f are comparable whereas they previously were not in U, but this would give a cycle shorter than n. $\Box$

Here is an easy application of this lemma.

Corollary 3.2.23 *Let $f : U \to V$ be a covering projection and n be the cycle length of V. Then f preserves path distances of at most $n/2$.*

Proof

Take the entire path (that is, the finite points of ramification) to be the connected partial order X in the previous lemma. This gives a path in V. If there is another path then there would be a cycle of length less than n. $\Box$

Theorem 3.2.24 *Let $f : U \to V$ be a regular, symmetry preserving covering and n be the cycle length of V. Let X be a connected partial order with diameter less than $n/2$. Then U is transitive over order embeddings of X if V is.*

Proof

Note that the bound on the diameter of X forces it to be cycle-free.

Given two embeddings a, b of X into U, the previous lemma states that they are mapped isomorphically to embeddings of X into V. As V is X-transitive, there is a map exchanging those images. As f is symmetry-preserving this corresponds to an automorphism of U exchanging embeddings c, d of X into U such that $fc = fa$ and $fd = fb$. As f is regular there are automorphisms of U exchanging a and c and b and d, and composition gives the required automorphism exchanging a and b. □

Remark 3.2.25 *It should not be expected that if V is a diamond-free partial order with cycle length n and X has diameter greater than $n/2$ that V can possibly be X-transitive. In general one may choose X to be a path of length greater than $n/2$. There will be embeddings into V such that the endpoints have distance less than the length of the path and embeddings without this property, and it is not possible to exchange them.*

Corollary 3.2.26 *If V is a ramification-complete 3-cs-transitive diamond-free partial order and U is its universal cover, then U is also 3-cs-transitive.*

Proof

The choice of three connected points gives a partial order with diameter at most 2. It is not possible to have a cycle of length 4 or less so the cycle length of a diamond-free partial order is at least 6. The result follows from the previous theorem. □

3.3 Categorical connections

Thanks to James Cranch for bringing the following to my attention.

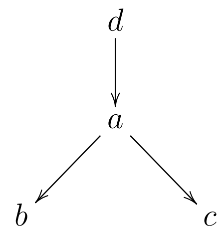
Definition 3.3.1 *The nerve of a partial order P, written $N(P)$ is the simplicial complex constructed thus:*

1. *The elements of the partial order are the points of the simplicial complex.*
2. *The line segments correspond to the pairs of the relation.*
3. *For every finite n, the n-simplices correspond to suborders of P which are $(n+1)$-element total orders.*

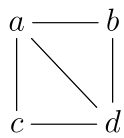
For $x \in P$ let the corresponding vertex in $N(P)$ be $N(x)$.

Remark 3.3.2 *If unit intervals are used for the line segments, this is a nicely metrisable space, with the restriction of the metric to the points bounded by the order metric if P is connected.*

The metric distance does not correspond exactly to the order distance. For instance the following order



has nerve



, where there are two triangles, abd and acd. It shares this nerve with the diamond and a distance between b and c of exactly $\sqrt{3}$, being twice the height of the equilateral triangle. If there were points above d this distance would be reduced, as for instance the height of the tetrahedron is less than the height of the triangle. Thus the least possible distance between two incomparable points is $\sqrt{2}$, twice the vertical height of the ω-simplex. This can be attained in this example by sticking countably many points between a and d.

Thus, for a point $u \in P$ the realisation in the nerve $N(P)$ of $c(u)$ consists exactly of the vertices in the 1-ball of u.

Result 3.3.3 *The height of a n-dimensional regular simplex is $\sqrt{\frac{n+1}{2n}}$, which is bounded below by $\frac{\sqrt{2}}{2}$. The circumradius of the same simplex is $\sqrt{\frac{n}{2(n+1)}}$, which is bounded above by $\frac{\sqrt{2}}{2}$.*

Proof

The proof is by induction on the dimension. By some elementary calculations, the ratio of the height to the circumradius of the regular n-simplex is $\frac{n+1}{n}$, giving the result about the circumradius. The circumcentre is the point furthest from the vertices. The circumcentre of the n-dimensional simplex is also the base of the vertical height of the $(n+1)$-dimensional simplex, so the height h of the $n+1$-dimensional simplex satisfies $h^2 + \frac{n}{2(n+1)} = 1$. This gives $h^2 = \frac{n+2}{2(n+1)}$ which is the induction step. $\square$

Remark 3.3.4 *Quite a lot of information is lost in going to the nerve from the order. For instance, all countable linear orders have the same nerve, the previously mentioned ω-simplex, and the isometries of this space correspond to the full symmetry group on the countably many points.*

This counterexample can be similarly generalised to partial orders. The example of the countable partial order with the following properties arises frequently in this thesis, it was used as an example in the introduction.

1. *Maximal chains have order type $\mathbb{Z}$.*

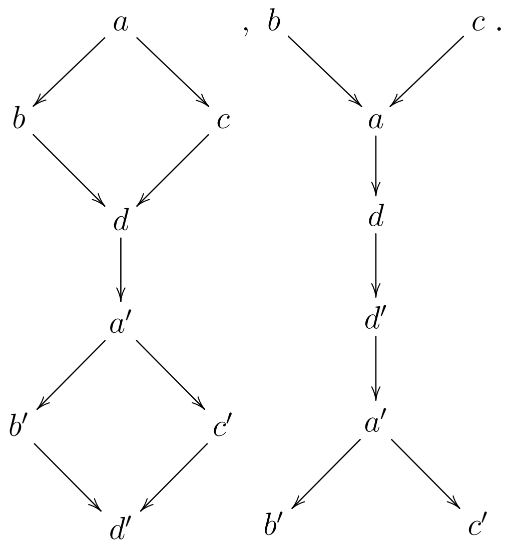
2. *Every vertex has upward and downward ramification orders 2.*

3. *Every vertex is the base point of a unique 6-crown.*

4. *Those 6-crowns are the only cycles which do not contain smaller cycles as subwalks.*

This partial order has a nerve which it shares with the other partial orders with the same properties, but with the order type of the maximal chains replaced with other 1*-transitive linear orders such as* $\mathbb{Z}^2$ *or even* $\mathbb{Q}$*, because those linear orders have the same nerve and all the partial orders so created have the same reduct with the comparability relation.*

Remark 3.3.5 *If one is not considering diamond-free partial orders then the loss of information means that some partial orders with cycles may have nerves which they share with cycle-free partial orders. The diamond is an example, a more complicated example is that the following pair of partial orders have the same nerve*



Here the labelling shows that we have simply moved points around without affecting the comparability relation on the partial order. In this case we are just permuting levels. One might hope that similar procedures will work for partial orders with cycle-free nerves in general, but this is not possible.

In fact, it is false that every cycle-free nerve of a partial order is the nerve of some diamond-free partial order. Consider the nerve of the analogue of the previous two-diamond partial order with a third diamond below the rest. The diamond-free partial order with the same nerve for the two-diamond example was unique up to permutation of the elements present in every chain (a, d, a', d') *and the two mutually exclusive pairs. This* *leaves no room to add a third mutually exclusive pair.*

Lemma 3.3.6 *Let P be a partial order and $N(P)$ its nerve. Then any path $f : [0,1] \to N(P)$ between two vertices of $N(P)$ is homotopic to the embedding in $N(P)$ of a walk in P between the vertices.*

Proof

The idea here is that where a path lies in one simplex it can be pushed out to one of the 1-dimensional simplices bordering that simplex. Fortunately, $[0,1]$ is compact. The balls of radius $\frac{\sqrt{2}}{2}$ around the vertices form an open cover of $N(P)$, so their preimages form an open cover of the interval, as do the connected components of their preimages. Take a finite subcover of this.

This gives a sequence of intervals in $[0,1]$ labelled by vertices such that any vertices labelling overlapping intervals are comparable. Shorten intervals as needed until any overlap is pairwise only. Construct the new map $g : [0,1] \to N(P)$ with image restricted to 1-dimensional simplices thus:

- If t is contained in one interval only labelled by the point x then set $g(t)$ equal to x.
- If t is contained in intervals labelled by points x and y which overlap on the interval (t_1, t_2) then set $g(t)$ equal to $\frac{(t-t_1)x+(t_2-t)y}{t_2-t_1}$.

If t is in the overlap of two intervals then $f(t)$ must be contained in a simplex which contains both the vertices. If t is in just one interval then $f(t)$ is contained in a simplex which contains that vertex. Thus, by convexity of individual simplices, the linear homotopy between f and g can be defined and the two maps are homotopic. □

Theorem 3.3.7 *If the nerve of a ramification-complete connected diamond-free partial order P is simply connected then P is cycle-free. Conversely, if P is connected and cycle-free then $N(P)$ is contractible and thus simply connected.*

Proof

In the previous proof the result that the interval has covering dimension 1 is subtly used to take a cover where each point is contained in at most two points. Suppose that P is not simply connected: thus it contains some manner of crown C. There is a path $N(C)$ corresponding to this crown, which we claim is a nontrivial loop. If it is homotopic to the identity, then there is a continuous map $f : [0,1]^2 \to N(P)$. Using the fact that the square has covering dimension 2 one constructs as above a homotopy to the 2-dimensional simplicial skeleton.

Such a homotopy is only able to rearrange the appearances of chains of length 3 of elements of P in the path, and cannot remove the cycles induced by crowns in P.

Conversely consider a point x in P. For any other point y in $N(P)$, set $d(y)$ to be 1 more than the least order-theoretic distance from x of a vertex of a simplex containing y, and $\pi(y)$ to be such a vertex with minimal order-theoretic distance. Let the unique path in P from x to $\pi(y)$ be $\psi(y)$, which is a path of length $d(y)$. The object is to find a homotopy from the map on $N(P)$ which is constant at $N(x)$ to the identity.

Let the homotopy be constructed in countably infinitely many steps starting from 1. At step n:

- Points y with $d(y) < n$ are already in the correct position so need not be moved.
- Points y with $d(y) = n$ start at a vertex of a simplex containing y and travel linearly to y as the simplex is convex.
- Points y with $d(y) > n$ start at the nth point of $\psi(y)$ and travel to the $(n+1)$th point of the sequence linearly along the 1-simplex.

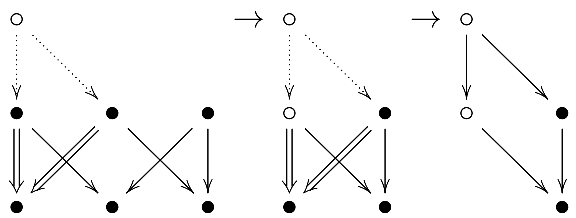
Squeezing all of those steps into the interval gives the continuous map needed. One explicit way to do this is to perform step n over the interval $[1 - 2^{1-n}, 1 - 2^n]$. In this case, each point y will be in its correct position at time $1 - 2^{-d(y)}$, which is a time before 1 as $d(y)$ is finite. At time 1, in the pointwise limit, each point will be in its proper location. □

Corollary 3.3.8 *The nerve of a ramification-complete connected partial order P is simply connected if P does not embed a crown.*

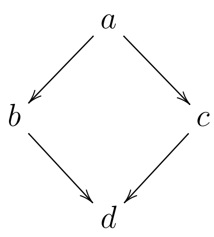
Proof

In a ramification-complete crown-free partial order paths are unique up to replacing an element by any other element comparable to the two adjacent points in the path, so the previous proof still works.

The converse is not true. A lattice, for example, probably contains many crowns, but every path tracing out such a crown can be turned into an insignificance by giving a homotopy to a constant path as in the example below.



To see what is happening here observe that when we have a diamond like this



then the path from b to c via d is homotopic to the path from b to c via a. To see this assume without loss of generality that we are talking about the path through the 1-simplices – an earlier lemma permits this. Then one can simply move d where it appears in that path to a linearly and do the same with the 1-simplices bd and dc, moving them at the same speed to ba and ac. Doing this once to the six-crown allows us to exchange the double-lined arrows with the dotted arrows, turning the structure into a slightly elongated four-crown. Forgetting the intermediate points gives us the four-crown, and a repetition of the process takes us to the diamond, whose nerve is contractible. Alternatively, if a lattice has a point greater than all other points one may simply contract the entire nerve linearly to that point, as every chain either contains it or may be extended to a chain containing it. □

Corollary 3.3.9 *Let P be a diamond-free partial order. Then any covering projection $f : Q \to P$ corresponds to a topological covering map $N(f) : N(Q) \to N(P)$. Furthermore, if f is universal so is $N(f)$.*

Proof

Let $x \in P$ and $y \in Q$ satisfy $f(y) = x$. Then the $\frac{\sqrt{2}}{2}$-ball around both $N(x)$ and $N(y)$ is determined by the simplices containing x and y respectively as vertices. These correspond to chains containing x and y. As f is an isomorphism between $c(x)$ and $c(y)$, $N(f)$ is a homeomorphism (indeed, an isometry) between $B_{\frac{\sqrt{2}}{2}}(x)$ and $B_{\frac{\sqrt{2}}{2}}(y)$. The balls around y for different choices of y cannot overlap as preimages of x cannot be comparable, so must be at distance at least $\sqrt{2}$. Indeed, they must be at a greater distance as two points at distance 2 in the partial order are in $c(z)$ for some z.

If f is universal then Q is cycle-free, which corresponds to $N(Q)$ being simply connected. This means that the topological projection is universal. □

Remark 3.3.10 *This is a functor from the category of diamond-free partial orders and covering maps to the category of topological spaces and covering maps. As distinct partial orders may have the same nerve, it is not injective on points. It is however bijective on arrows between two points.*

Formulating the order theory in terms of topology does more than justify the use of language such as 'covering projection'. It grants access to the rich theory of topological coverings.

First it is necessary to recall some definitions from algebraic topology.

Definition 3.3.11 *Let T be a topological space. A* path *is a continuous map $p : [0, 1] \to T$ and has endpoints $p(0)$ and $p(1)$. Two paths f and g may be concatenated if $f(1) = g(0)$. Then $f + g(t) = f(2t)$ if $t \leq 1/2$ and $g(2t - 1)$ if $t \geq 1/2$. If for any $a, b \in T$ there is a path from a to b then T is said to be* path-connected*. Assume it is so. Let $P(T)$ be the set of paths on T. This is a function space.*

A homotopy *between two paths $p, q \in P(T)$ is a map $h : [0, 1] \to P(T)$ such that $h(0) = p$, $h(1) = q$ and $h(.)(.)$ is a continuous map from $[0, 1]^2$ to T. Homotopy is an equivalence relation.*

Let $x \in T$ be a base point*. The loop space at x is the restriction of the space of paths to those starting from x and ending at x. Taking the quotient of this space by homotopy* *gives a topological group, the* fundamental group *at x, $\pi_1(T)$. If T is path-connected then this is independent of the choice of x. If the fundamental group of a space is trivial, that is every cycle is homotopic to the constant path, it is said to be* simply connected.

Let $f : U \to T$ be a continuous surjection between path-connected spaces. Then $x \in T$ is evenly covered *if there is an open set d satisfying $x \in d \in T$ such that $f^{-1}(d)$ is equal to $\sqcup_{i\in I}\{d_i \in U\}$ with the restriction of f to d_i a bijection to d for every $i \in I$. The function f is a* covering projection *and U is a* covering space *for T if every point in the image is evenly covered. Additionally if U is simply connected then it is the* universal cover *for T.*

The following result is a common example in category theory.

Result 3.3.12 *The Galois correspondence subgroups of the fundamental group of a path-connected space and its path-connected covers are in bijection, with embedded subgroups corresponding to supercovers. In particular, covering spaces of a space T correspond to subgroups of $\pi_1(T)$. Normal covers correspond to normal subgroups, and in the case of a normal cover $f : S \to T$ corresponding to $N \triangleleft \pi_1(T)$ the group of deck transformations is the quotient $\pi_1(T)/N$.*

Remark 3.3.13 *The deck transformation group of the universal cover of T is the same as $\pi_1(T)$, and that of any intermediate cover is an image of this group via the intermediate projection and thus a quotient.*

One use of this result is to look at the orbits of the deck transformation group of the universal cover, which are preimages of points in the image. Information about the cycle-free partial order and equivalence relation will give information about the quotient order being studied.

Corollary 3.3.14 *Let $f : Q \to P$ be a universal covering of diamond-free partial orders. Then the group of deck transformations of f has an explicit isomorphism with the fundamental group of P.*

Proof

It is possible to prove this result elementarily. Given a base point $x \in P$, walks from x to itself are in bijection with the elements of the orbit $f^{-1}(x)$. If a canonical element x' of the orbit is chosen, the orbit consists of translates of x' via a free action. It suffices to check that composition is the same in each case, that is that the translation corresponding to the concatenation of two walks is the product of the corresponding translations. This is true by construction. □

Remark 3.3.15 *Let $f : Q \to P$ be a universal cover of a 3-cs-transitive diamond-free partial order. Then orbits of the deck transformation group $Aut(f)$ of Q (i.e. preimages of elements of P) satisfy:*

- *They are acted on transitively by (the lifts) of $Aut(P)$.*
- *Any two elements are at distance bounded by 2, and indeed by the cycle length of P.*
- *The deck transformation group acts freely on Q, thus consists only of translations.*
- *There is a subgroup of $Aut(Q)$ realising 3-cs-transitivity well-defined on the orbits.*

3.4 Constructions via Finitely Presented Groups

Here we are constructing partial orders as Cayley graphs of finitely presented groups. This is an obvious way in the context of group theory to generate connected partial orders, which will automatically be 1-transitive and connected, and the elements of the partial order will naturally be levelled. The challenge is that edges must be exchange-able. While initially this is used to create partial orders with maximal chains of order type $\mathbb{Z}$, the other 1-transitive countable linear orders are also ordered groups.

Here are a few preliminary definitions.

Recall that the *Cayley graph* of a group G with a set of generators X is a directed graph with labelled vertex set equal to the set of group elements G, each vertex labelled by its corresponding group element. The edge set of the graph is labelled by the elements of X, and there is a directed edge labelled x from the vertex a to b if and only if $ax = b$.

Given a set of generators X and a set R of words (*relators*) in those generators, the group presented by the generating set X with relations from R is the quotient of the free group on X by the normal closure in this free group of the set R. If both sets are finite this is a *finitely presented* group.

The *index sum* of a word or subword of a word in a group with a specified independent generating set is the sum of the powers of the generators. A word is *balanced* if its index sum is zero. A word is *reduced* if it does not contain any generator adjacent to its inverse. A word is *trivial* if it equals the identity in the free group.

Recall that a partial order is strongly arc transitive if it is transitive on unions of two maximal chains which overlap on a maximal ray (either upward or downward) starting from a point which are the same shape (in the infinite case just direction). If a partial order of finite height is strongly arc transitive is transitive on such structures, in this case it will depend on how long the ray is as well as whether it is upward or downward maximal.

Given an interval $[a, b]$ typically including zero the $[a, b]$th alternating closure of a point specified by a word w is the set of all words v with w as an initial segment such that any initial segment of v of which w is an initial segment has index sum contained in $[a, b]$. The $[a, b]$th alternating closure of the origin, represented by the empty word, is then the set of all words such that any initial segment has index sum in $[a, b]$. The $[k, n+k]$th alternating closure of a word w with index sum k will be referred to as the nth alternating closure for short.

Given a group presentation in which all relators are balanced there is a natural rank function to $\mathbb{Z}$ on words given by the index sum, this defines a discrete partial order whose Hasse diagram has the successors of any point (specified by a word w) be the words wg for g a generator. Edges on the nth level of this diagram are those between words of $n-1$th and nth rank. An *upward edge* is a non-inverted generator following an initial subword of rank $n-1$, and a *downward edge* is an inverted generator following an initial subword of rank n.

If the greatest index sum or *height* of any subword of a relator in the presentation is m then the *cycle structure* in the Cayley graph is the isomorphism class of the mth alternating closure of a point, which as the group is symmetric under translation (by left multiplication) does not depend on the point. The *word cycle* of a word is the suborder consisting of all the points given by initial segments of the word.

The following theorem makes it possible to break down Cayley graphs resulting from finitely presented groups into disconnected chunks, which means that a finite amount of symmetry is sufficient to ensure symmetry at arbitrary heights, namely strong arc-transitivity in a partial order with infinite chains.

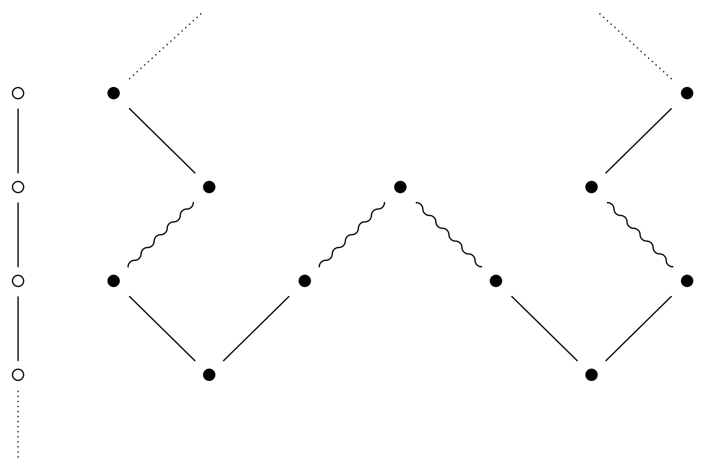
Theorem 3.4.1 *Let G be a finitely presented group with generating set X and one balanced relator r of height n. Then removing the nth alternating closure of a point is sufficient to disconnect the Cayley graph. In particular, any points with a nontrivial connecting path whose intersection with the removed section has on any level numbers x of upward edges and y of downward edges satisfying $x-y$ is not 0 are now not connected.*

Proof

Without loss of generality let the point be the origin. Walks in the Cayley graph correspond to not necessarily reduced words. As such, they are equivalent (in that they start and end at the same point) if the words are provably equivalent in the group through a finite number of applications of the relation. As the relation describes a cycle in the graph, an application of the relation corresponds to partially superimposing one copy of this cycle or a trivial word over part of the walk and taking the symmetric difference.

As the relator is balanced, its application at any level has an equal number of upward and downward edges. To verify this, consider the following cases. If a trivial word was applied, then the upward edge we are adding corresponds to an identical downward edge.

Otherwise, as the height of the relator is n, the levels taken up by the relator cannot extend both above and below the levels of the nth alternating closure, and in each component of the relator which lies in the alternating closure there must be as many upward edges at each level as downward edges. Consider the following diagram, where the vertical line at the left corresponds to the levels of the alternating closure, the diagram on the right indicates the component of the relator meeting the alternating closure, and the level in question is indicated by wavy lines.



If the relation is applied at the level such that its cycle is entirely included in the removed section, then if the walk intersected the removed section in a number of edges such that upward and downward edges have different counts then this is preserved by the application. □

This result can be immediately generalised with the same proof.

Corollary 3.4.2 *Let G be a finitely presented group with generating set X and a finite set R of balanced relators of height n. Then removing the nth alternating closure of a point is sufficient to disconnect the Cayley graph. In particular, any points with a nontrivial connecting path whose intersection with the removed section has on any level numbers x of upward edges and y of downward edges satisfying $x-y$ is not 0 are now not connected.*

We may want to apply the lemma to directly produce some examples of Cayley graphs with crowns with legs of uniform height greater than 1 given by relators in the presentation, obtaining high symmetry with finitely many conditions to check. Many more examples are possible with the corollary, however. This lemma provides some such examples. As these satisfy the conditions to be the ramification points of a two-level partial order, one may gain from this process a two-level locally 2-arc-transitive partial order. It may be the case, because of the symmetry of generators involved, that the two-level partial order may be explicitly given by considering words of the Cayley graph followed by $+\infty$ and $-\infty$ powers of the generators.

Lemma 3.4.3 *Let r be a balanced word of height n where every subword of locally maximal height attains the global maximum and every subword of locally minimal height attains the global minimum. Let G be a finitely presented group with generating set X and r as sole relator. Suppose also that the cycle structure of r in G is strongly arc-transitive. Finally, observe that $Sym(X)$ acts on balanced words of this kind at each level by changing only the edges of that level. Suppose that the stabilisers for r at each level are 1-transitive on X itself. Then the Cayley graph of G corresponding to this presentation is strongly arc-transitive.*

Proof

It is necessary to show that the Cayley graph is strongly arc transitive. Consider a

Y-structure, which is the union of two infinite maximal chains meeting in a maximal downward ray. Let there be two of these, J and J'. Since it is known Cayley graphs are vertex transitive, one may assume that both have their 'joining point' at the origin. The case for structures meeting in maximal upward rays is dual to this. Let n be the height of the word.

J is defined by three strings indexed by ω, one D of generators for the ray going downwards and two U_1 and U_2 for the rays going upwards. The other structure J' will have strings D', U_1' and U_2'. The plan is then, at each level, to choose a permutation of the generators such that the generator in D or U_1 and U_2 is mapped to the corresponding generator in J'. The nth alternating closure of the origin is strongly arc transitive, so can be dealt with separately. The remainder of each ray is in a separate component, so one can, in each component, apply 1-transitivity at each level.

This then defines a permutation of the Cayley graph taking J to J'. □

It can be quite difficult to find single words satisfying the conditions, so this corollary, whose proof is analogous, is more fruitful.

Corollary 3.4.4 *Let R be a set of balanced words of height n where every subword of locally maximal height attains the global maximum and every subword of locally minimal height attains the global minimum. Let G be a finitely presented group with generating set X and set of relations R. Suppose also that the height-n cycle structure in G is strongly arc-transitive. Finally, observe that $Sym(X)$ acts on balanced words of this kind at each level by changing only the edges of that level. Suppose that the stabilisers for r at each level are 1-transitive on X itself. Then the Cayley graph of G corresponding to this presentation is strongly arc-transitive.*

3.5 Examples and Generalisations

Let us find some appropriate words. Fix n to be the desired height of a word and q to be the number of generators, written $0, \ldots, q-1$. A V-shape consists of n downward edges (the inverses of generators) followed by n upward edges. The dual notion is that of the Λ-shape. A suitable word must embed every possible V-shape and Λ-shape of the appropriate height.

Consider the case of $q = 2$ and $n = 2$. Here $\overline{ab}$ is used to represent $a^{-1}b^{-1}$

Result 3.5.1 *No* perfect *word – that is one that embeds every V- and Λ-shape exactly once – is possible.*

Proof

There are 4 distinct V-shapes (we consider $\overline{00}11$ and $\overline{11}00$ the same V-shape. Exchanging generators on one level is an operation of order 2, so a perfect word would consist of four V-shapes, a total of 16 letters in a string. If this string is a line segment, it is necessary for S_2^2 to act faithfully on this, which must be by means of two reflections whose composition is a translation of order 2. However, there is a V-shape that goes $\overline{11}01$. Exchanging generators on the lower level takes this to $\overline{10}11$, which is a reflection of itself, and forces this action to be a reflection at a location $2 \mod 4$ along the word. By a similar argument the Λ-shape $11\overline{01}$ forces exchanging generators on the upper level to act on the string by a reflection at a location $0 \mod 4$ along the word. But the composition of such reflections is a translation by a length $4 \mod 8$, which cannot be of order 2 in a word of length 16. □

Less perfect words do exist. By a similar argument, if an imperfect word is made up of V-shapes then exchanging generators on the upper level should be a reflection and exchanging generators on the lower level the translation of order 2, which forces exchanging both to be the other reflection.

Example 3.5.2 *An example of such a word is* $\overline{10}10\overline{10}11|\overline{01}00\overline{11}00|\overline{11}00\overline{11}01|\overline{00}10\overline{10}10$. *This was formed by observing that there are two orbits of* V*-shapes under the exchanges, and including both of them in one of the sections, which ensures that they all arise. One of the orbits of* Λ*-shape is reflected over and the other is included in a section. If we conduct the construction with this word, we find a diamond-free partial order whose height-*1 *alternating closure is free (so* Alt*), and which embeds the* 16*-crown with legs of height* 2 *corresponding to the word itself. Because each* V*-shape occurs twice, there is not a unique* 16*-crown for each extended* V*-shape.*

This process, however, does not generalise very well to higher ramification orders or longer extended arcs, as neither S_2^3 nor C_3^2 act faithfully on the interval. Instead, one can use the earlier corollaries with finite sets of words.

Example 3.5.3 *One way to find such a set is to take some words (such as* $(\overline{00}11)^4$ *and* $(\overline{01}00\overline{10}11)^2$*) and extend to their orbits under a subgroup of symmetries of the generators at each level. In this case, the second word is mapped to itself when exchanging top or bottom generators, and the first word is mapped to* $(\overline{01}01)^4$. *We have a diamond-free partial order with no cycles of height* 1 *and a unique extended* 8*-crown with legs of height* 2 *for each extended* V*-shape of legs of height* 2.

Finally, we can generalise to give constructions with maximal chains of height greater than Z. This approach works for powers of $\mathbb{Z}$, but it is not clear how to do similar presentations for $\mathbb{Q}$, because the cycles must now interact densely and, even if rank is preserved, there are necessarily infinite many V-shapes which it must be possible to exchange. Once that is understood, results for order types $\mathbb{Q}.\mathbb{Z}^\alpha$ seem likely to be straightforward.

Remark 3.5.4 *It is possible to generalise these presentations to some other order types which are ordered groups. For instance, suppose that we have incomparable generators*

x_i and a discrete ordered group $G = \mathbb{Z}^\alpha$, with a generating set G'. Then we replace the x_i with incomparable copies of G', and each occurence of x_i in the presentation with its copy of the least generator of G'.

An example of this, starting with the presentation $\langle a, b : (ab^{-1})^3 = 1\rangle$ corresponding to a $\mathbb{Z}$-ranked partial order with 6-crowns is that if we use $\mathbb{Z}^2$, with two generators and replace the original generators with the smaller lexicographic generator, we get a $\mathbb{Z}^2$-ranked partial order with 6-crowns and no structure at the higher level.

The difficulties with finding more complex presentations of this form are connected with the problems in Chapter 5 with infinite ramification, but here it is possible to construct some examples. Suppose we have a connected partial order with finite ramification orders, expressed as the Cayley graph of some quotient of a finitely generated group. Then let us assume that there are n generators and we assume each V-shape of height k (of which there are $\frac{n^{2k-1}(n-1)}{2}$) is contained in a unique extended $2m$-cycle (and also the dual condition on Λ-shapes). This tells us that there are left and right bijections f, f' from V-shapes to Λ-shapes, depending on whether the image shares the left-hand side or the right-hand side of the V-shape. To have the $2m$-cycle, we require that $(f(f')^{-1})^m$ be the identity map. This pair of maps now contains enough information to write down a set of relations for the generators. In particular, there will be $\frac{n^{2k-1}(n-1)}{2m}$ words, so we observe that this fraction must be an integer.

Chapter 4

Finitary decomposition

4.1 Objectives

Here we apply the work of Cameron, Praeger and Wormald [1] on highly-arc-transitive connected digraphs to describe complete posets with maximal chains of order type $\mathbb{Z}$. These arise naturally as the poset of ramification points of the completions of two-level posets with appropriate transitivity properties. Covering projections, as defined earlier, are frequently used. The notion of $DL(\Delta)$ is also defined in that paper . The idea behind this structure is that by local 1-arc-transitivity of Δ, a bipartite graph, it is possible to create a unique digraph by freely amalgamating copies of Δ.

The standard construction is to take a tree where the upward branching order is equal to the size of the upper part of Δ and the downward one is equal to the size of the lower part of Δ. In this case by tree we mean a directed graph whose underlying graph is connected and contains no cycles. This tree should be regular with in-degree at each point equal to the size of the upper part of Δ and out-degree equal to the size of the lower part. The points of this tree will correspond to the copies of Δ, and each edge to a lower vertex of the copy of Δ corresponding to the source of the edge and an upper vertex of the copy corresponding to the end point of the edge. Edges into a point are then identified with the upper partition of the copy of Δ corresponding to that point, and edges out with the lower partition. This gives a natural subgraph of the line graph of that tree where the edges correspond to the edges in the copy of Δ for each point. This is $DL(\Delta)$, which will be called the *connected cover*.

4.2 The main theorem

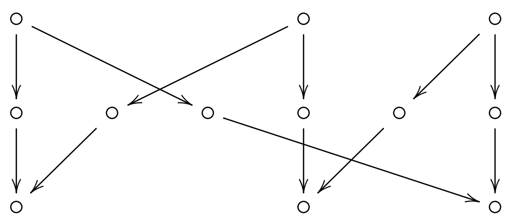
Consider digraphs and surjective homomorphisms. As the identity is a surjection and the composition of two surjections is surjective, these form a category. The $\mathbb{Z}$-ranked digraphs are those with an arrow to $\mathbb{Z}$. This corresponds to those digraphs with a rank function from the vertex set onto $\mathbb{Z}$ where if $x <' y$ the rank of x is exactly 1 less than that of y. This is exactly the "property Z" of [1].

As usual, we assume that these structures are connected, diamond-free and highly arc-transitive (which requirement gives regularity of the digraphs via vertex-transitivity). Note that with this constraint, any arrow, that is one of the surjective homomorphisms, between two digraphs with the same finite in- and out-degrees is in fact a covering projection, as the diamond-free condition forces all ancestor and descendant sets to be trees. For any locally 2-arc-transitive bipartite graph Δ which is complete as a 2-level partial order (in other words any two neighbourhoods of points meet in at most one point), $DL(\Delta)$ is such a digraph. Later the assumption of completeness is relaxed for more complex constructions. This does not describe all examples however, there are such digraphs which are not of the form $DL(\Delta)^D$ and it will be instructive to consider one.

Lemma 4.2.1 *There is a digraph Γ, still connected diamond-free and highly-arc-transitive which is not of the form $DL(\Delta)^D$ where Δ^D has finite height.*

Proof

A general example here is to take an arbitrary Dedekind-MacNeille complete diamond-free finitely (more than 2) ranked partial order and construct the countable free highly-arc-transitive digraph embedding that order. The idea is to take a starting partial order γ. Here we use a cycle with three maximal points, three minimal points and edges which are 2-arcs, namely an extended six-crown.



We will construct by amalgamation the highly arc-transitive partial order Γ satisfying the conditions that each point has countable upward and downward ramification orders and that each 2-arc is contained in countably many copies of γ.

The construction starts with P_0, a copy of γ, and enforces the conditions one step at a time, by adding countably many free rays above and below each point, by freely gluing countably many copies of γ to each 2-arc and by amalgamating to witness arc-transitivity.

To be precise, for each pair of arcs a and b (finitely many such are introduced at each step) one would like to be able to map within an approximation μ one glues $\mathbb{Z}$ copies of μ such that the b of each copy is glued to a of the next, giving a witness automorphism taking the first arc to the second.

At this point a new requirement is added demanding that any future additions be mirrored in each copy. As new points and arcs are introduced new instructions are added to the queue, by ordering these instructions correctly the whole process can be completed in countable time. A precise account of a generalisation of this process can be found in the next chapter. The outcome here is still ranked, as all amalgamations were free and will not interfere with a rank function.

The process is thus as follows:

1. Start with $P_0 = \gamma$.

2. At each stage designate a future time in ω for each point (including ones previously chosen for this purpose) such that rays will be freely amagalmated to that point.

3. At each stage designate a future time for each 2-arc (again including those previously chosen) to attach a copy of γ.

4. For each pair of 2-arcs a and b designate a future time t at which a partial isomorphism f between a and b will be chosen and used to amalgamate $\mathbb{Z}$ copies of P_t, giving an automorphism taking a to b corresponding to shifting down $\mathbb{Z}$.

5. When such an amalgamation is done add, immediately after every future stage a stage requiring that any changes made to one copy be mirrored so that this automorphism may be extended at that stage.

This is not of the form $DL(\Delta)^D$ for a Dedekind-MacNeille complete Δ, as $DL(\Delta)$ cannot have cycles ranging over more than 2 ranks of vertices. In fact this has cycles ranging between arbitrary ranks resulting from simple amalgamations of the original γ - consider a chain of arbitrary length of the γ, where an edge from a maximal point to a middle point in an arc of each (non-final) element of the chain is amalgamated with an edge from a minimal point to a middle point in an arc of the next element of the chain in a nontrivial manner. This prevents it from being of form $DL(\Delta)^D$ if Δ^D has bounded height, as that bound also bounds the height of possible cycles. □

Restricting this example Γ to finitely many ranks (n, say) and taking a connected component gives a structure that is transitive on arcs between the same ranks. Indeed, it is the free poset on those ranks transitive on arcs over the same ranks where each vertex not at a maximal rank has countably many vertices above it, each vertex not at a minimal rank has countably many vertices below it and each 2-arc is part of countably many copies of γ. This connected component is the completion of its suborder of elements of maximal and minimal rank, which is a two-level partial order, call this Γ_n.

One can therefore say that cycles between ranks $g(n)$ and $g(n)+n$ from Γ are all to be found in $DL(\Gamma_n)^D$, where $[g(n), g(n)+n]$ is the finite interval of ranks restricted to. If it were possible to construct a family of such Γ_n with a filtered diagram of embeddings such that every finite cycle (or every path from an arbitrarily fixed origin point) lay in some Γ_n for some n, the colimit of that family would have to be isomorphic to Γ itself, as every point and cycle would be in its proper place. This general result is outlined in the following theorem, whose proof is given in the six lemmata that follow.

Theorem 4.2.2 *Any connected Dedekind-MacNeille complete $\mathbb{Z}$-ranked diamond-free highly-arc-transitive digraph with nontrivial upward and downward ramification arises as the colimit of a filtered diagram, specifically of shape corresponding to the square-free natural numbers, with internal arrows corresponding to the "divides" relation, which are the DM-completions of structures of the form $DL(\Lambda)$ for some bipartite Λ.*

The proof proceeds as follows: given such a structure $Z = (Z, <')$ (with corresponding partial order $(Z, <)$ the transitive closure of the digraph relation), first construct the sequence X_i of covering structures and the internal covering maps. Then it is necessary to add the internal maps between the X_i and check their well-behaviour. Then it is sufficient to prove that this cocone is initial.

To form X_i the following result is needed:

Lemma 4.2.3 *For $n > 0$ there are $Z_n \subseteq (Z, \leq'^n)$, which are connected $\mathbb{Z}$-ranked highly arc-transitive digraphs, each of which has Z as its Dedekind-MacNeille completion. Each Z_n will be taken to be one connected component of $(Z, \leq'^n)$.*

Proof

First claim that there are n isomorphic components of $(Z, \leq'^n)$ of the form Z_n corresponding to the pre-images of the residues mod n in the rank function.

This requires that such a pre-image be connected. Between two points of the same level mod n take any path $x_0, \ldots, x_m$ in $(Z, <')$ by connectedness of $(Z, <')$. Construct a path y_i in $(Z, <'^n)$ thus: $y_i = x_i$ iff x_i is also in the same level mod n as x_0. If $x_i > x_{i-1}$ and $x_{i+1} < x_i$, so x_i is a local maximum, take y_i to be a point above x_i of minimal rank of correct residue mod n. If x_i is a local minimum take y_i to be a point below x_i of minimal appropriate rank. Otherwise omit y_i. Collapsing gives a path in $(Z, <'^n)$.

It is clear that there is no path in $(Z, <'^n)$ between any points of different levels mod n. The digraph Z is vertex-transitive, so it has level-shifting automorphisms which shift points to levels $1 \ldots n-1$ above themselves, which gives that the components are isomorphic.

Any arc of finite length s in $(Z, <'^n)$ has an extension to an arc of length ns in $(Z, <')$ by filling in the gaps (this is unique by the diamond-free assumption), and the automorphism in $(Z, <')$ gives one in $(Z, <'^n)$.

The fact that ${Z_n}^D = Z$ follows from the Dedekind-MacNeille closure property of Z itself. Since Z is DM-complete, it suffices to show that each element of Z arises as a ramification point of points in Z_n. The number of points of Z_n directly above any given point of Z not in Z_n is at least equal to the upper ramification order (indeed, the mth power, where m is the number of levels the next level of Z_n is above it, as ancestor sets are trees) and similarly for the number of points of Z_n directly below it, where both ramification orders are greater than one. It therefore arises as the ideal generated by the points below it and, if ramification orders are finite, finitely generated by the set of points directly below it, which is paired with the filter generated by the points above it.

□

Lemma 4.2.4 *There exist bipartite Λ_n such that $DL(\Lambda_n) \to Z_n$ as a covering projection.*

This is true by the work in [1]. Note that Λ_n is constructed thus: take any 1-arc and close by taking all 1-arcs which are part of alternating walks starting with the 1-arc in question. The presence of the rank function means this gives a bipartite structure, such that any connected bipartite structure embedded in Z_n is a substructure of an embedding of Λ_n. This structure is called the *alternating closure*.

The desired structures X_n will in fact take the form $DL(\Lambda_n)^D$. It is possible to imagine an X_0 as the tree with the same ramification orders as Z, but it will not be considered here as it adds little to the sequence. It is necessary to show it maps to Z in the appropriate way.

Lemma 4.2.5 *There is a covering projection $DL(\Lambda_n)^D \to Z$ extending the above map.*

Proof

It is obviously not generally true that if there is a covering projection between two posets then it extends to a covering projection between the order completions: consider a tree covering a non-complete structure, the tree will only complete to itself. Thus some understanding of the structure of $DL(\Lambda_n)$ will be required. Specifically of use is the property that there are no cycles without repeated points containing directed s-arcs where $s > 1$, in other words that the only cycles are of height 1.

Consider this characterisation for the nonprincipal ideals of $DL(\Lambda_n)$: each is generated by some subset of the bottom elements of some copy of Λ_n.

To prove this, let p be a nonprincipal ideal in $DL(\Lambda_n)$ and p^+ the corresponding filter. Let p' be the set of maximal elements of p. This is nonempty, as the ranks of elements of p

must be bounded above, for instance by the rank of some element of p^+. Similar let $p^{+'}$ be the set of minimal elements of p^+.

Now prove that the elements of p' and $p^{+'}$ are all of fixed and adjacent ranks. Suppose not. Then it is possible to find $u \in p'$ and $u^+ \in p^{+'}$ not of adjacent rank. So there exists an element v of $DL(\Lambda_n)$ such that $u < v < u^+$, u' an element of p' such that $u' \not\leq v$ (this exists as $v \notin p^+$) and $u^{+'}$ an element of $p^{+'}$ such that $v \not\leq u^{+'}$ (existing as $v \notin p$). Then a walk from u to u^+ via v, then to u', then to $u^{+'}$ and back to u collapses to a cycle containing an irreducible arc of length > 2, namely that between v and its immediate neighbours in the $u - v - u^+$ arc, giving a contradiction.

Thus $DL(\Lambda_n)^D$ in fact looks like $DL(\Lambda_n)$ locally completed over each copy of Λ_n. Z arises from a similar local completion, which allows for a concrete characterisation of the desired map.

Let τ_n be a covering projection $DL(\Lambda_n) \to Z_n$. The extension υ_n can be defined: it must map ideals pointwise. Thus for $I \subset DL(\Lambda_n)$ an ideal, take $\upsilon_n(I) = \{\tau_n(x) : x \in I\}$. It is necessary to check both that this is downward closed and that $\upsilon_n(I)$ is still an ideal. Downward closure is immediate: suppose $a = \tau_n(b) \in \upsilon_n(I), c < a$. Then c is in the ancestor set of a, but τ_n is a bijection between the ancestor set of b and that of a, so c has a pre-image in I. To verify that it is an ideal, set $J = \{\tau_n(x) : x \in I^+\}$ where I^+ is the set of upper bounds to I, the filter corresponding to it. By the same argument J is also upward closed. As J is the pointwise image of the intersection of the descendant sets of points of I, it is also the intersection of the descendant sets of $\upsilon_n(I)$, and so it is the set of upper bounds of $\upsilon_n(I)$. Similarly $\upsilon_n(I)$ is the set of lower bounds of J, thus an ideal.

More concretely, the extension agrees with the original map on principal ideals and takes an ideal generated by a set p to that generated by its image - and it has been shown that such generating sets live inside single copies of Λ_n. Thus, when one restricts τ_n to a single copy of Λ_n inside $DL(\Lambda_n)$ so that it is a bijection (as shown in [1]), υ_n must be a bijection between the closure of that copy and its image, which suffices to show that it is a surjective homomorphism, as every element and arc of Z arises in some such image. Specifically each such element arises from an ideal-filter pair whose extremal points form a bipartite structure (as constructed in Lemma 4.2.3) and is thus part of a copy of a Λ_n.

The same argument shows υ_n to be a covering projection as it is bijective on neighbourhoods of points: for any point its in-neighbourhood lives inside a single copy of ${\Lambda_n}^D$, as does its out-neighbourhood. $\square$

Recall that Z_n is any connected component of $(Z, <'^n)$, and that there are n isomorphic such components, which give n disjoint embeddings of Z_n into Z. This gives us a choice of embeddings $Z_n \subset Z$ and thus isomorphisms $Z_n^D \cong Z$. Choose appropriate projections $\pi_n : DL(\Lambda_n)^D \to Z_n^D$ for each n by extending the τ_n, though with certain constraints. Specifically ensure that the copy of Z_n in the image of $DL(\Lambda_n)$ takes ranks $g(n) \mod n$, where $g(n)$ is a function to be defined in the next lemma. Please note that these $X_n = DL(\Lambda_n)^D$ are also $\mathbb{Z}$ ranked and the rank function is chosen in such a way that π_n preserves the rank.

Lemma 4.2.6 *When S is the set of square-free natural numbers, there exists $g : S \to \mathbb{Z}$ with the following properties.*

1. *For each $x \in \mathbb{Z}$ and $y, z \in \mathbb{N}$ defining an interval $[x - z, x + y]$ around it, we need $p \in \mathbb{N}$ not necessarily prime and $n \in S$ such that distinct elements of the interval $[x - z - g(n), x + y - g(n)]$ have distinct residues $\mod p$, and 0 is not one of the residues. It is necessary to translate, for instance if 0 is part of the original interval.*

2. *For each $a \in S$ and $b \in S$ such that b divides a, g satisfies $g(b) \equiv g(a) \mod b$.*

The motivation for the first condition is to ensure (as in Lemma 4.2.8) that all cycle structures are eventually captured, and the second makes Lemma 4.2.7 true for sufficiently many pairs in S^2 for the diagram to be filtered.

Proof

Construct g satisfying these properties, first constructing it on primes to satisfy the first condition, then extending it to composite numbers while ensuring the second condition is fulfilled.

Since the first condition places countably many constraints and there is room to make countably many choices, a just-do-it approach is effective.

There are countably many criteria of form 1 for specific x, y, z. Enumerate these with order type ω, and for each constraint that comes up in order take the least prime p greater than $y+z+2$ for which $g(p)$ has not been determined, and set $g(p)$ to be $x-z-1$. Then $[x-z, x+y]$ takes residues a subset of $[1, p-1] \mod p$, witnessing the constraint of form 1 in question. Set g arbitrarily on primes for which it has not been defined.

Now to satisfy the second condition invoke the full form of the Chinese Remainder Theorem: given pairwise coprime natural numbers $n_1, \ldots, n_k$, integers $a_1, \ldots, a_k$ there exists x satisfying $x \equiv a_i \mod n_i$ for each i, and x is unique modulo Πn_i.

All numbers n in S are products of distinct primes p_i. Condition 2 states that $g(n) \equiv g(p_i)$ for each such i. Application of the Chinese Remainder Theorem where the n_i correspond to the p_i and the a_i the $g(p_i)$ gives a unique value for $g(n)$, as the p_i are coprime and the result is up to equivalence modulo $n = \Pi p_i$.

Now check that this satisfies condition 2 when b divides a, that is $g(b) \equiv g(a) \mod b$. In that case $g(a)$ has the same residue as $g(b)$ modulo every prime factor of b, which compels it to have the same residue modulo b. So $g(a) \equiv g(b) \mod b$ as required. □

Lemma 4.2.7 *There exist $f_{ij} : X_i \to X_j$ for i dividing j in S such that $f_{jk} f_{ij} = f_{ik}$ and $\pi_i = \pi_j f_{ij}$. In other words there is a cocone from a filtered diagram when restricting to X_i for $i \in S$.*

Proof

Use the familiar notion of path-lifting in the digraph context. Start by fixing base points: pick t_1 in X_1, then for other n pick t_n in X_n such that $\pi_n(t_n) = \pi_1(t_1)$.

Now, seek to construct the map f_{mn}. The idea is as follows: identify points of X_m with paths in X_m from the base point, quotiented by cycles in X_m. These map by π_m to paths in Z, and lift to paths in X_n via π_n, which correspond to points in X_n.

The uniqueness of this path-lifting for a particular path is obvious by induction on the length or number of alternations of the path (at each stage there is a bijection from the covering projections restricted to the ancestor/descendant tree of the last point on the path).

The problem is to ensure that it is well-defined, that is paths equivalent up to cycles in X_m correspond to paths equivalent up to cycles in X_n. Consider what cycles from Z are represented in X_t: these are exactly the cycles that do not cross a rank of the form $g(t) \mod t$, as by the construction of X_t as $DL(\Lambda_t)^D$ there is a unique path between two different copies of Λ_t in $DL(\Lambda_t)$. There are no cycles in X_t not in Z, as X_t covers Z. It suffices then to check that the cycles represented in X_m are all represented in X_n.

Fortunately, this is the case: any cycle in X_m lies in a copy of ${\Lambda_m}^D$, and is thus constrained to at most m ranks between $g(m) + am$ and $g(m) + (a + 1)m$ for some $a \in \mathbb{Z}$. But by the condition on $g(n)$ from Lemma 4.2.6 that $g(m) \equiv g(n) \mod m$ this interval $[g(m) + am, g(m) + (a + 1)m]$ is wholly contained in some interval of the form $[g(n) + bn, g(n) + (b + 1)n]$ for some $b \in \mathbb{Z}$. Therefore the copy of ${\Lambda_m}^D$ lies in a single copy of ${\Lambda_n}^D$, and thus the cycle is also contained in X_n.

This map is a homomorphism because if $a <' b$ in X_m then b can be represented by a path to a extended by one arc to b, which will map to a path to $f_{mn}(a)$ extended by one arc to $f_{mn}(b)$. It is surjective because points in X_n correspond to paths in X_n from the base point, which go to paths in Z and can be lifted (non-uniquely) to paths in X_m, and points corresponding to these paths will map to the desired point.

These internal maps commute under composition because they are unique given the fixed choices of base point. □

Lemma 4.2.8 *This cocone is initial: given a Z' with $\pi'_n : X_n \to Z'$ such that $\pi'_i = \pi'_j f_{ij}$ where all indices lie in S, there exists a unique $f : Z \to Z'$.*

Proof

Let Z' be the base of another cocone, with projections ${\pi_n}'$. It is necessary to find a unique $f : Z \to Z'$. For each point $x \in Z$, consider the sequence of subsets U_n of Z' given by $\{{\pi_n}'(u) : \pi_n(u) = x\}$. They are obviously all nonempty. First one needs to show that these sets are all equal for $n \in S$. Suppose $a \in X_n$, $\pi_n(a) = x$, ${\pi_n}'(a) = b \in U_n$ then for m a multiple of n it is true that $f_{nm}(a) \in X_m$, $\pi_m f_{nm}(a) = \pi_n(a) = x$, ${\pi_m}' f_{nm}(a) = {\pi_n}'(a) = b \in U_m$ and when m divides n then $a' : f_{mn}(a') = a$ by surjectivity of f_{mn}, with $\pi_m(a') = x$ and ${\pi_m}'(a') = b$. Transitivity does the rest, as the structure is filtered. Call that one set U.

Claim that this is a fixed singleton $\{y\}$. Observe that because it is the image of a surjective homomorphism from a connected structure Z' must be connected. Suppose then that there are distinct elements y and y' in U. There is some path between them. Let the points on that path span ranks of an interval of width n, such as $[0, n]$. This path is the homomorphic image of a path of the same length which spans ranks in an interval of the same width in each X_i. Trap any such path in X_p, where p is sufficiently large, in a copy of ${\Lambda_p}^D$, so the restriction to that copy of π_p is a bijection, which means that there can only be at most one pre-image of x, giving a contradiction.

To ensure that this is possible take the π_n so that any rank is arbitrarily far away from the closest elements of the corresponding Z_n. For instance, having the Z_n take ranks 0

$\mod 1, 0 \mod 2, 0 \mod 3$ and so on would make this not the case for rank 0 (or indeed any other rank, since one could not ensure arbitrary distance below any positive rank, or above any negative rank). The criteria are "rank x needs y space above it and z space below it", and are satisfied by Lemma 4.2.6.

This ensures that for each $x \in Z$ there is a distinguished $y \in Z'$. This correspondence is the desired f. By the surjective nature of the π', f is surjective, and it is a homomorphism as if $x \leq' x'$ then this is preserved by each π and each π'. This f is forced by the choice of the π' and the commutativity requirement, so it is unique. □

This completes the proof of Theorem 4.2.2.

4.3 Remarks

1. At first glance this result is limited to structures with chain type $\mathbb{Z}$, as all other vertex-transitive countable linear orders are of the form $\mathbb{Z}^\alpha$ for $\alpha \geq 2$ or $\mathbb{Q}.\mathbb{Z}^\alpha$ for $\alpha \geq 0$, which are not DM-complete.

2. Lemma 4.2.6 can be modified so that the diagram uses all the X_n: simply define g on primes as required, then define it on squares of primes by choosing values with correct residues modulo the prime. Similarly inductively define it on all prime powers to obey condition 2, and the extension to composite numbers is forced. In this case all the X_i are used, indexed by the positive integers and with arrows corresponding to divisibility. Note however that no new information is added by the X_n for $n \notin S$.

3. An alternative diagram structure (which is totally-ordered) for the X_i satisfying the conditions of 4.2.6 is suggested by Nathan Bowler: take S to be the set of powers of 3 instead of the square-free integers and $g(3^i) = \frac{3^i-1}{2}$, which determines

the maps between the $X_i : i \in S$. This satisfies both criteria: if $i < j$ then $g(3^j) - g(3^i) = (3^i)\frac{3^j-1}{2} \equiv 0 \mod 3^i$. Given an interval $[x, y]$ take X_{3^i} such that $2.3^i > \max(|x|, |y|)$, or indeed any later X_j. The same works for any other odd prime.

4. It is possible that Z only contains cycles of height 1, that is spanning two adjacent ranks, and then $Z = X_1$, as seen in [5]. However $Z = X_n$ would mean that not only cycles were bounded to have height at most n, but that they would not cross the boundaries which arise every n ranks. This is not possible, as if $Z \neq X_1$ then none of the X_n are actually vertex-transitive: points embedded in X_n originally from the Z_n used will be contained in no cycles which both go above and below such points in rank, whereas there will be other points which are contained in such cycles. In other words, if the sequence X_i is not constant it is not eventually so.

5. The following result ensures that given a diagram of the form used it is meaningful to discuss the colimit of that diagram without necessarily explicitly identifying its colimit.

Lemma 4.3.1 *The category (Digraphs, homomorphic surjections) has all nonempty connected colimits.*

Proof

The proof is due to Nathan Bowler and reproduced here. Here is a list of useful facts.

(a) All the legs of any colimiting cocone for a connected diagram of epimorphisms are again epic.

(b) The category of sets has all small colimits.

(c) Let D be the category with 2 objects and 2 maps in the same direction between them. The category of digraphs is the category of functors from D to the category of sets.

(d) Colimits in such functor categories are computed pointwise.

(e) A map of digraphs is epic iff it is a surjection.

If one specifies a diagram of the appropriate form in the category, then by (e) this corresponds to a connected diagram of epimorphisms of that shape in the category of digraphs. By (b)-(d) above this has a colimit (indeed, (Digraphs, homomorphisms) is a Grothendieck topos), and by (a) the legs of the colimiting cone are epic and so surjections (by (e) again). It is necessary to check that the unique map thus resulting between two epic cocones is epic: this is the case because if two maps are equalised by the leg from the second cocone they are equalised by its composite with the leg from the colimiting cocone, and that leg is epic. □

It is still necessary to check the transitivity conditions in the specific case.

6. A Y-structure consists of a point, two maximal upward rays from it and one maximal downward ray from it, in a way analogous to a maximal chain which can be thought of as consisting of any point in it with one maximal upward and one maximal downward ray from it. A $\bar{Y}$-structure is the reversed notion with two downward and one upward ray. Consideration of these is motivated by thinking of partial orders as the interiors of the completions of two-level partial orders of extreme points; whereas choosing two comparable points gives a maximal chain, choosing three points two of which are comparable to the other one gives one of these structures.

   While the Z_i are highly arc-transitive as shown in Lemma 4.2.3, it is not generally true, even if Z is Y- and $\bar{Y}$-structure transitive, that they are Y-structure transitive, as Y-structures in Z_i are different depending on whether the point of amalgamation of the rays in Z is in the Z_i or not, and, if not, on the height mod i at which it occurs. In particular, the Λ_i are not locally 2-arc-transitive. Otherwise the following lemma could be used along with the converse of Theorem 3.13 of [5] to show that the X_i are vertex-transitive, which is false.

Lemma 4.3.2 *Let D be a connected highly arc-transitive digraph which has the following properties*

(a) *D is diamond-free.*

(b) *D has nontrivial upward and downward ramification.*

(c) *D is transitive on Y and $\bar{Y}$-structures.*

Then there is a connected locally 2-arc-transitive bipartite graph corresponding to a partial order M such that $M^D = M \sqcup D^D$, and further $|M| = |D|$.

Proof

The proof of this lemma is a modification of Theorem 3.13 of [5], which further requires that D be DM-complete and have countable ramification (and thus be countable). When these conditions are relaxed, the following changes are necessary to the proof.

(a) M_0 has the cardinality of D, and as each extension preserves cardinality so does M.

(b) Abandoning the intersection property is equivalent to losing DM-completeness, with the immediate consequence that all completions contain D^D as opposed to D.

□

Suppose then that the Λ_i are locally 2-arc-transitive. The proof of Theorem 3.14 of [5] can be applied to give that $DL(\Lambda_i)$ is transitive on Y and $\bar{Y}$ structures. The previous lemma says that $X_i = DL(\Lambda_i)^D$ is the completion of a locally 2-arc-transitive bipartite graph. As X_i is diamond-free and $\mathbb{Z}$-ranked, any element is the supremum of two points in that graph, and so X_i is vertex-transitive, which is a contradiction.

4.4 Generalisation

Here are some attempts to use the same techniques for partial orders with other chain types. As these are not connected as digraphs, existing definitions must be generalised. An arrow, or covering projection between partial orders U and V, is a surjective order homomorphism $f : U \to V$ such that, for $x \in U$, f is a bijection between the suborder of U of points comparable with x and the suborder of V of points comparable with $f(x)$. Orders remain diamond-free, so these sets are trees.

As previous results were about digraphs, a lemma about colimits in this new environment is needed.

Lemma 4.4.1 *The category (Posets, homomorphisms) is cocomplete and for a filtered diagram of epimorphisms the legs of the colimiting cocone are epic.*

Proof
Use the well-known theorem that a category is (co-)complete if it has (co-)products and (co-)equalisers. The limit case of this result is found on page 113 of [7]. It is not easy to work in Posets itself, so instead consider the supercategory (Quasiorders, homomorphisms). A quasiorder is a transitive relation or alternatively a partial order of equivalence classes, where any two elements of a single equivalence class are related by the order relation symmetrically. This does have coproducts, which are disjoint unions. It also has coequalisers, namely the latter object with the quotients forced by the parallel pair. Thus (Quasiorders, homomorphisms) is cocomplete.

Suppose then that one wants the colimit for a diagram in Posets. This is also a diagram in Quasiorders. Consider the coslice category over this diagram in Quasiorders, that is, the category of cocones. This is a special case of a comma category. It is known that this has an initial object, the quasiorder colimit. Every map from this initial object to a poset must map each isomorphism class in that quasiorder to a single point, as a homomorphism would preserve the symmetric order relation, which is unacceptable in a poset. Thus such maps factor uniquely through the poset that is the quasiorder colimit with isomorphism classes collapsed to points, and this is initial in the subcategory that is the coslice category in Posets (which, by this observation, is also nonempty). Thus Posets is cocomplete.

These colimits can be described explicitly. The colimit in Quasiorders is the disjoint union of all quasiorders in the diagram, quotiented by all equivalences forced by morphisms in the diagram. The colimit in Posets is therefore this quotiented disjoint union with all isomorphism classes collapsed to points.

Thus, given a filtered diagram of surjective homomorphisms, an object a in that diagram and a point in the colimit that point originally comes from another object in the diagram, but those two objects have surjective arrows to a third object, so the point also corresponds to a point in the third object, and indeed one in a. As this is true for all points in the colimit and objects all the legs are surjective homomorphisms also. □

An alternative proof of this result due to Nathan Bowler is as follows: there is an obvious functor G from Posets to Digraphs, namely interpreting the order relation as a digraph relation. This is clearly full and faithful because the definitions of homomorphism are identical. This functor has a left adjoint F defined thus: given a digraph D, there is a reachability quasiorder on D. This is given by $x \leq y$ iff there is a path in D from x to y. Set FD to be the partial order of equivalence classes on that quasiorder. In this way the category of posets can be identified as a reflective subcategory of the category of digraphs, so Posets is cocomplete. It is a general categorical fact that the legs of colimiting cocones for connected diagrams of epimorphisms are again epic.

In particular, when one can discuss either connected digraphs or posets, the notions are equivalent, as the explicit constructions line up. This of course fails when the digraph is not connected and thus does not contain enough information.

A note on order types: $\mathbb{Z}^\alpha$ here refers to the *colexicographic power* of functions $\alpha \to \mathbb{Z}$ with finite support, as in [10]. Such functions are ordered "from the right": given f, g have $f < g$ iff there is $\beta \in \alpha$ with $f(\beta) < g(\beta)$ and $(\forall \gamma > \beta) f(\gamma) = g(\gamma)$. These are considered because they arise in Morel's classification of countable transitive linear orders in [10]. Of course any chain of the form (x, y) in the set of ramification points of $X \sqcup Y$ must be transitive if that bipartite graph is locally 2-arc-transitive.

It is possible to widen the scope of Lemma 4.2.5, which extends a covering projection between posets to a covering projection between their completions, by exploiting the fact that the structure of the completion is precisely known. In that case the completion was formed by filling in copies of Λ_n in an amalgam of copies Λ_n to give the corresponding amalgam of copies of $(\Lambda_n)^D$. The following lemma extends the previous result with two main applications. One is that it allows one to compress a $\mathbb{Z}^{\beta+1}$-height structure into a $\mathbb{Z}$-height structure by treating the $\mathbb{Z}^\beta$ blocks as 2-level posets, and be confident of later recovering the full structure. It also allows the reconstruction of the $\mathbb{Z}$ part of the $\mathbb{Z}.\mathbb{Z}^\beta$ structure once the $\mathbb{Z}^\beta$ blocks have been compressed through the method described earlier in the chapter, but without direct appeal to Dedekind-MacNeille completion, as that would immediately expand the structure to something of height $\mathbb{Z}^{\beta+1}$ and make it difficult to discuss connectedness or the like.

Here a $\mathbb{Z}$-ranked poset is a discrete poset for which the corresponding directed graph is connected and has a rank function to $\mathbb{Z}$. Requiring that it be arc-transitive ensures that it can be moved up and down ranks.

If one partial order is embedded as the extremal elements of another in the way $f : \kappa \leq \mu$, then suppose an order X is a gluing of copies of κ, that is X is the transitive closure of $\frac{Y\times\kappa}{\cong}$ for some equivalence relation $\cong$, with the partial order of κ preserved, one can embed X canonically within $\frac{Y\times\mu}{\cong}$ by simply locally applying f to each copy of κ. The larger partial order is considered a gluing of copies of μ. In particular, if as in the following lemma we have $\Lambda \leq \Gamma$ then $DL(\Gamma)$ is defined to be the outcome of gluing copies of Γ in place of the copies of Λ in $DL(\Lambda)$.

Lemma 4.4.2 *Let there be given a $\mathbb{Z}$-ranked 1-arc-transitive poset Z' and a connected 2-level-poset Λ divided into two levels as $\Lambda^+ \sqcup \Lambda^-$ which is the alternating closure of Z'. Let there be a poset Γ and an embedding $u : \Lambda \leq \Gamma$ such that the elements of Λ are exactly the extremal elements of Γ – elements of Λ^+ are maximal in Γ, elements of Λ^- are minimal in Γ and every element of Γ is bounded by elements of Λ. Let σ be a covering projection $\sigma : DL(\Lambda) \to Z'$ as defined in Lemma 4.2.4, existing as Λ is the alternating closure. Then there is a covering projection $\sigma^{(*)}$ extending σ as follows, where Z'' is a gluing of copies of Γ replacing Z':*

$$\begin{array}{ccc} DL(\Lambda) & \xrightarrow{\sigma} & Z' \\ \downarrow & & \downarrow \\ DL(\Gamma) & \xrightarrow{\sigma^{(*)}} & Z'' \end{array}$$

The vertical arrows in the diagram are both extensions, i.e. the restriction to any copy of Λ is a copy of u.

Before proving this result it is necessary to define the sense in which "glued" is used here. The motivation comes from looking at $DL(\Lambda)$, from which the original tree used to construct $DL(\Lambda)$ (in which each point is later replaced by a copy of Λ) can be recovered by identifying each such copy of Λ with its set of lower points. First define an equivalence relation $\sim$ on Z' as the transitive closure of the relation $x \sim y$ iff $(\exists z)x, y <' z$ – so $[x]$ is the set of all points reachable by an alternating path of even length from x, which paths must start by going upwards. The set of all points reachable by an alternating path of odd length from x is $[x]^+$, alternatively this is the set of points y such that $(\exists z \sim x) z <' y$.

Then Z' is a gluing of copies of Λ if there are maps f^+ and f^- from Z' to Λ^+ and Λ^- respectively, such that for each $x \in Z'$ the maps $f^-|_{[x]}$ and $f^+|_{[x]^+}$ are bijections, and $f = f^-|_{[x]} \cup f^+|_{[x]^+}$ is an order isomorphism.

Proof

The idea here is to construct the glued copies of Γ by simply taking a copy of Γ for each copy of Λ and gluing them at the points in Λ. So first consider $(Z'/\sim) \times \Gamma$ with the order induced on each copy by the order on Γ. Define the relation $\equiv$ thus: if $a \in [b]^+$ then $([a], f^-(a)) \equiv ([b], f^+(a))$. This is an equivalence relation: the classes are of size 2 and correspond to identifying each point as part of the copy of Λ for which it is on the bottom and as part of that for which it is on the top.

This behaviour is evident as the classes can be stated explicitly.

1. $([x], u)$ is in a class of its own if $u \in \Gamma \setminus \Lambda$.
2. $([x], u)$, if $u \in \Lambda^+$, is in a class with $([v], f^-(v))$, where v is the preimage of u in $f^+|_{[x]^+}$.
3. $([x], u)$, if $u \in \Lambda^-$, is in a class with $([w], f^+(v))$, where v is the preimage of u in $f^-|_{[x]}$ and $w <' v$.

Then $Z'^{(*)} = ((Z'/\sim) \times \Gamma)/\equiv$, with the transitive closure of the order. This is a poset as if $x \not\sim y$ and $([x], u) \leq ([y], v)$ then x has rank lower than y. If $\Lambda = \Gamma$ then Z' is recovered: x becomes $([x], f^-(x)) \equiv ([w], f^+(x))$, where $w <' x$. All such w are equivalent under $\sim$ as wxw' is an even alternating path as required. Indeed, even when $\Lambda \neq \Gamma$ one can see that Z' embeds into $Z'^{(*)}$ in the same way.

The partial order $DL(\Lambda)$ is $\mathbb{Z}$-ranked as the connected tree (and thus its line graph) is ranked. It has functions g^+ and g^- corresponding to f^+ and f^- which must be the composites of those functions with σ. By the construction of σ in [1], this takes $\sim$-equivalence classes to $\sim$-equivalence classes bijectively, which is to say it bijects between copies of Λ, specifically those copies which are full alternating closures of arcs. Thus $DL(\Lambda)$, along with the maps g^+ and g^-, satisfies the conditions necessary to define $DL(\Lambda)^{(*)}$.

The extension $\sigma^{(*)}$ is obvious – $([x], u)$ must go to $([\sigma(x)], u)$ – and is a bijection on copies of Γ. It is obviously well defined by the above observation that it bijects between copies of Λ. The covering projection property is inherited from σ: any point comparable with $x \in DL(\Lambda)^{(*)}$ is either a copy of Γ with it, in which case it is in the copy of Γ to which that copy was mapped, or in an interval $[y, z]$ where $y <' z \in DL(\Lambda)$ and as such intervals also lie in a single copy of Γ they are mapped bijectively. $\square$

This can be applied as follows, in results which approximate partial orders by ones of bounded cycle height. The height of a cycle in a ranked partial order is the order type of the subset of ranks spanned by the cycle. Where infinite, this is equivalent to the order type of the subset of ranks spanned by some maximal chain contained in the cycle, as a cycle is a finite union of such chains. Say the *cycle height* of a partial order is bounded by some linear order when all cycles contained in the partial order have heights which are suborders of the bound.

Lemma 4.4.3 *If $X \sqcup Y$ is a locally 2-arc-transitive two-level partial order whose set of ramification points $X \sqcup Y \sqcup N$ has ranks in $1 + \mathbb{Z}^2 + 1$, then N arises as the colimit of an ω-sequence of partial orders with maximal chain type $\mathbb{Z}^2$ and bounded (infinite) cycle height, in this case bounded by $n.\mathbb{Z}$, where n depends on the element of the sequence.*

Proof

The method here is to derive a $\mathbb{Z}$-ranked poset from N, such posets we know arise as a colimit. The shape of the colimit comes from remark 3. The previous lemma lets us recover the full poset from the $\mathbb{Z}$-ranked abstraction.

The idea is to form the $\mathbb{Z}$-ranked poset by considering the set of points which can be added and considered to lie at the top of some maximal connected (in the digraph sense, so of length at most $\mathbb{Z}$) chain and at the bottom of some other chain. Take the set of infinite connected (in the digraph sense) chains in N with no maximal element and an upper bound in N (equivalently in $X \sqcup Y$) and quotient by the equivalence relation that two chains are equivalent if they are eventually equal (or share the same set of upper bounds) to give the set Z. This is ordered by $u < v$ iff all points in v are upper bounds for u. This corresponds to the set of minimal upper bounds of such chains in $(X \sqcup Y)^D$, with the order of the completion. Each point in Z can be identified with a pair of points in N bounding it, and so Z is countable if N is. As those points in N have elements of $X \sqcup Y$ above and below them, so do the points of Z.

The partial order Z is connected in the digraph sense if N is connected as a poset: as the maximal chains in N are of type $\mathbb{Z}^2$ the length of any interval is bounded by a finite multiple of $\mathbb{Z}$, and so a finite path in Z can be found.

It is true that Z is highly-arc-transitive: given an arc $z_1 \ldots z_n$ in Z, and a point $z_0 <' z_1$, this can be distinguished by taking $u \in N$ such that $z_0 < u < z_1$ in $(X \sqcup Y)^D$, and taking a 2-arc in $X \sqcup Y$ given by a point above z_n, a point below u and a point above u not above z_1. An automorphism switching such an arc will also move the corresponding arc (indeed, ray) in Z.

This allows the construction of the nth powers of the digraph relation on Z as 1-arc-transitive digraphs with covering projections as in Lemma 4.2.4, and the previous lemma replaces Lemma 4.2.5, giving a sequence of covering projections with Z as the colimit.

The connected components of N are isomorphic, and are partial orders bounded by elements of Z. Adjoining these elements gives a Γ. This allows the translation of the previous sequence into a sequence of covering projections with $N \sqcup Z$ as the colimit.

Removing the elements of Z from each poset in the digraph removes Z from the disjoint union which is the colimit, leaving N. □

This generalises to give the following result.

Theorem 4.4.4 *If $X \sqcup Y$ is a locally 2-arc-transitive two-level partial order whose set of ramification points $X \sqcup Y \sqcup N$ is connected and has ranks in $1 + \mathbb{Z}^\alpha + 1$, N arises as the colimit of a $\max(\mathrm{cf}(\alpha), \omega)$-sequence of partial orders with maximal chain type $\mathbb{Z}^\alpha$ and cycle height strictly less than $\mathbb{Z}^\alpha$ and cardinality at most $|X \sqcup Y|$.*

Proof

The proof of this result comes in two parts. First consider the case where $\alpha = \beta + 1$ is a successor ordinal. There are canonical embeddings $\mathbb{Z}^\beta \to \mathbb{Z}^{\beta+1}$, indexed by $\mathbb{Z}$, which simply add a final coordinate equal to the index. Set Z, as in the previous proof, to be the set of infinite chains taking ranks exactly the image of such an embedding with an upper bound ranked outside that embedding. The rest of the argument proceeds identically. The cardinality bound arises because $|Z| \leq |N|$, and $|N| = |X \sqcup Y|$ because ramification points correspond to finitely generated ideals.

Now let α be a limit. Let P be a cofinal subset of α of order type $\mathrm{cf}(\alpha)$. Pick an element x of rank 0 in N, and for each β in P set U_β to be the connected component of α of elements with support subset of β containing x. Then each U_β is included in $U_{\beta'}$ for $\beta \leq \beta'$, and their union is equal to N: as N is connected, for each y in N there is a finite path from x to y in the poset. Let γ be the greatest coordinate in the supports of the ranks of the path elements. P is cofinal, so contains elements greater than γ, and so y appears in some U_β. □

This result is not in itself very enlightening, as the U_β are not well described. To gain a better understanding of the situation in certain cases, in particular where the cardinality of the structure results from its height, consider one motivation for this study.

Lemma 4.4.5 *If $X \sqcup Y$ is a locally 2-arc-transitive two-level partial order whose set of ramification points $X \sqcup Y \sqcup N$ is connected and has ranks in $1 + \mathbb{Z}^\alpha + 1$, where α is a limit ordinal, and additionally $|X \sqcup Y| \leq |\alpha|$, then $X \sqcup Y$ arises as the colimit of an α-sequence of connected locally 2-arc-transitive two-level partial orders whose sets of ramification points are of rank strictly less than $\mathbb{Z}^\alpha$.*

Proof

Define U_β as above, with $P = \alpha$ (actually, the greater of the initial ordinal of $X \sqcup Y$ and $\mathrm{cf}(\alpha)$ will suffice). The two-level partial orders will take the form $X_\beta \sqcup Y_\beta$, and will correspond to maximal chains in U_β with external bounds in $X \sqcup Y$.

Each such structure is connected as U_β is: given a and b in $X_\beta \sqcup Y_\beta$, supposing without loss of generality that a is a maximal element and b is minimal, pick a' and b' comparable with a and b respectively, then take a path (x_i) in U_β with $x_1 = a'$ and $x_n = b'$. Write (x'_i) to equal (x_i) with added points $x'_0 = a$ and $x'_{n+1} = b$. Define the path (y_i) in $X_\beta \sqcup Y_\beta$ thus: if $x_{i+1} > x_i$ then set y_{i+1} to be an element in $X_\beta \sqcup Y_\beta$ above x_{i+1}, and similarly if $x_{i+1} < x_i$ then set y_{i+1} to be an element in $X_\beta \sqcup Y_\beta$ below x_{i+1}.

Local 2-arc-transitivity comes from that of $X \sqcup Y$. For instance, take a, b maximal and c minimal in $X_\beta \sqcup Y_\beta$, with $a, b > c$. Take a', b', c' in $X \sqcup Y$ with $a' > a$, $b' > b$, $c' < c$. The triple a, b, c can be recovered from these three points by taking the unique greatest bounded by a', b', c' and taking the component of rank within $\mathbb{Z}^\beta$ around it.

It is necessary to define the inclusions $X_\beta \sqcup Y_\beta \to X_{\beta'} \sqcup Y_{\beta'}$ for $\beta < \beta'$. To do this, for each z in $X_\beta \sqcup Y_\beta$ that is not comparable with $\bigcup_{\gamma<\beta} X_\gamma \sqcup Y_\gamma$, if z is minimal in $X_\beta \sqcup Y_\beta$ pick a point of $X \sqcup Y$ below z, and for each $\gamma > \beta$ include z as the point of $X_\gamma \sqcup Y_\gamma$ meeting that ray. Maximal points are dealt with similarly.

The union of the $X_\beta \sqcup Y_\beta$ ramifies to give the ramification points of $X \sqcup Y$: it is not necessarily isomorphic because it may not be locally 2-arc-transitive. This does however ensure that it has the correct cardinality, as in each case the set of ramification points has the same cardinality as the starting bipartite graph. It is not necessarily $X \sqcup Y$ by the natural correspondence: it may be a proper subset. To ensure every point in $X \sqcup Y$ appears in the union, well-order $X \sqcup Y$ by its initial ordinal and at each point when choosing a ray choose that given by the first point in $X \sqcup Y$ that will work. Because it is possible to make α many choices and this is at least that initial ordinal, each point has been chosen at some stage in the process. The colimit is then exactly $X \sqcup Y$ as desired.
□

The cardinality condition in this lemma is satisfied when the size of the partial order is a consequence of its height rather than its ramification degree and the two-level partial order to be described is of minimal cardinality, in particular when the partial order is countable. In cases where the two-level partial order is not of minimal cardinality the statement is generally not true.

Chapter 5

Free constructions

5.1 Background

In this chapter we construct partial orders with a range of homogeneity properties. In general the partial orders under discussion will be subsets of the set of ramification points of the Dedekind-MacNeille completion of two-level locally 2-arc-transitive partial orders. The completions will add points in between the points of the original partial order, which is considered the set of *extremal* points, divided into minimal and maximal points. It will be additionally assumed that these are diamond free, with the consequence that intervals are chains. Here this makes it clearer what is happening when I amalgamate over them.

This builds on previous work by John Truss and Robert Gray discussing such partial orders and giving examples embedding cycles 'of height 1', namely ones which consist of upper and lower points without intermediates, and which admit level functions, which are order preserving maps to the linear order of the example in question. Here a fairly general construction is given which allows one to produce a wide range of examples demonstrating certain kinds of bad behaviour.

5.2 Construction

The idea behind the construction is that one takes some suitably compatible suborder δ, called an *anomaly*. Having specified the order type of the maximal chains which our resulting structure will have, these two pieces of information will determine a partial order embedding δ with suitable symmetries.

It is effective to take a just-do-it approach, as the following result in the case of order type $\mathbb{Z}$ shows. This means that we enforce the desired properties one at a time, and remind ourselves to keep them enforced.

Result 5.2.1 *Given a countable partial order δ which is connected and compatible (to be defined later in this section) with a connected partial order with maximal chains of type $\mathbb{Z}$ - which will be defined later - and diamond-free, we construct a connected partial order $P(\delta) = X \sqcup Z \sqcup Y$ such that $X \sqcup Y$ is a 3-cs-homogeneous 2-level countable partial order, $P(\delta)$ is ramification complete and countable and Z is a discrete partial order without endpoints satisfying*

1. $P = (X \sqcup Y)^D$
2. *For $x \in X, y \in Y$ the interval $[x, y]$ has order type $1 + \mathbb{Z} + 1$*
3. *Any 2-arc in Z is contained in a copy of δ.*
4. *Any point in Z has infinite upward and downward ramification order.*

Proof

We take the 'just do it' approach to constructing our structure, taking it as a union of approximations with level functions onto $1 + \mathbb{Z} + 1$. An approximation P_n will be a finite union of rays with endpoints and copies of δ. The first approximation P_1 will be an interval of order type $1 + \mathbb{Z} + 1$ (itself P_0 for sake of argument) with a copy of δ amalgamated at a point. We have the following tasks.

1. For a point in Z add countably many upward rays terminating in points in X and downward rays terminating in points in Y.

2. For any pair (u_1, u_2) of isomorphic connected 3-subsets of $X \sqcup Y$, we take $\mathbb{Z}$ copies of the approximation P_i and choose a map f from the convex closure of u_1 to that of u_2 in P_i (these convex closures look like Y-structures or upside-down Y-structures). This allows us to define an amalgamation of $\mathbb{Z} \times P_i$ over those structures and an automorphism τ corresponding to incrementing the integer index. For any subsequent additions to later approximations P_j we repeat those changes so that τ may be extended to a more detailed automorphism.

This gives rise to a countable sequence of tasks which thus gives a countable sequence of approximations, whose union satisfies the conditions. We need to check that at each step the approximation is diamond free and ramification complete. For the former, consider the steps in the construction. We start with the union of an interval and δ, which is diamond-free because δ is. We add rays, which is safe. When amalgamating a Y-structure, an interval in the amalgam is unchanged from the interval in the starting structure if not both endpoints are on the path. If both endpoints are on the path then the entire interval is contained in the path, and the same holds true for the interval in the amalgam. For the latter, amalgamation of rays preserves ramification completeness. When we amalgamate over a Y-structure, a pair of points from distinct copies of the previous approximation will have a common upper bound but not be comparable only if they are both below some point of the amalgamated Y-structure. The least such point will be their supremum. □

It is not clear at this stage that this process will give us a unique resulting object, what we get might vary depending on the order in which we choose to amalgamate approximations or freely add rays. With the procedure given all that can be easily proved is that the process behaves correctly on a finitary level.

Result 5.2.2 *Any finite union of intervals contained in some structure A generated by the above procedure (depending on the choice of order of the operations) is contained in any other such structure B.*

Proof

This is a consequence of two facts, first that any such finite union arises in some approximation a arising during the construction of A, and this embeds into some approximation arising during the construction of B, namely when all the tasks involved in the creation of a have been fulfilled. To prove the latter, let $a_0 \subset \ldots \subset a_n = a$ be an initial segment of the sequence of approximations whose limit is A and $b_0 \subset \ldots$ be the sequence of approximations whose limit is B.

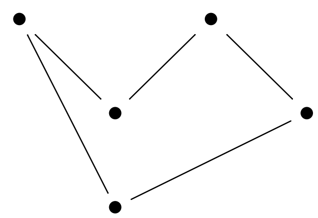
We induce over the a_i, and for each a_i will find a b_j and an embedding extending previous embeddings. $a_0 = b_0$. Given an embedding for a_k into b_l, a_{k+1} is constructed from a_k by adding a ray or through some free amalgamation, and this task is executed at some b_m with $m > l$, so a_{k+1} embeds into b_m. $\square$

A better way to understand what is happening here is to take a much more formal presentation. Before going into that, note that while the above construction used the order type of $1 + \mathbb{Z} + 1$ for maximal chains, it did not use the property that the countable 1-transitive linear order in that particular case was $\mathbb{Z}$. We simultaneously generalise this to the wider options including $\mathbb{Z}^\alpha$ and $\mathbb{Q}.\mathbb{Z}^\alpha$ and give a more precise interpretation of compatibility.

Definition 5.2.3 *Let T_Z be the free countable tree with maximal chains of order type $1 + Z + 1$, where Z is a 1-transitive linear order. This is a clearly understood partial order and T'_Z be the subset of points which are not extremal in that tree. Recall that a homomorphism $u : V \to W$ between two discrete connected partial orders is a covering projection if for each $x \in V$ u is an isomorphism between the immediate successors of x*

and those of $f(x)$, and similarly with predecessors. In the general case, let us just say that u is an isomorphism between the set of points comparable with x and those comparable with $f(x)$. An anomaly δ is compatible *with Z if there is a surjective covering projection f from a convex subset $U \subset T_Z$ to δ such that the image of $U \cap T'_Z$ is a connected poset. This will ensure that there is a full covering projection from T_Z to the outcome of the construction.*

For example, any Z-ranked partial order is immediately compatible with Z, but anything with a diamond is automatically incompatible. Compatibility is much weaker than being rankable, however. For instance, the following partial order is compatible with $\mathbb{Z}$:



This is because its intervals look like intervals in $\mathbb{Z}$. It is also compatible with $\mathbb{Z}^2$, but not $\mathbb{Q}$. Using a variant on it with a 6-crown with $\mathbb{Z}$ will give an example of a connected partial order with maximal chains of order type $\mathbb{Z}$ not admitting a rank function. One can be significantly more exotic. For instance, one can instead of unbalancing with a leg of length 2 use a leg of order type $\omega + \omega^*$.

Result 5.2.4 *Given a connected diamond-free anomaly δ and a countable 1-transitive linear order type Z, if there is a maximally free partial order formed by the construction with maximal chains of order type $1 + Z + 1$ and embedding δ, then this is unique.*

Proof

δ can be extended to ramify infinitely and freely at each point, which is done formally by considering a choice of f as above such that for any point in the domain, arbitrarily many open intervals directly above and below it are not in the domain, then taking T_Z/f, that is the quotient of T_Z by identifying points with the same image under f. Call this a "page" in the construction, represented by $T_Z(\delta)$.

Now one can pick a distinguished 3-arc of extremal points in each direction, that is one with two maximal points and one with two minimal points. Let Q be a set of the 3-arcs of extremal points which may or may not include the distinguished arcs. We need to ensure that it is possible to map any element of Q to the distinguished 3-arc in the correct direction, so let R be a set of bijections from distinguished arcs to elements of Q, such that there is a map corresponding to each element of Q. When $Z = \mathbb{Z}$ and the resulting orders are connected in a graph theoretic sense, there can only be one map between two 3-arcs, but for any other order type there is greater flexibility. For example, if $T_Z(\delta)$ possesses a level function mapping to Z, then one may have different results depending on whether one picks maps which preserve this or not.

For any specific map from R we wish to extend to an automorphism, we will amalagamate a copy of the existing approximation onto the chosen arc using that map. To do this all at once, let $\mathbb{F}_R$ be the free group with the elements of R as generators. The final construction will be a quotient of $T_Z(\delta) \times \mathbb{F}_R$, using the symmetric and transitive closure of the relation $(a, w) \cong (b, v)$, where w and v are words in the free group, v is an extension of w by a character r, and r is a map taking a to b. That is to say, if a is an element of a distinguished arc and b is an element of a distinguished arc q of the same orientation, then $f(a) = b \implies (a, w) \cong (b, wf)$. □

Result 5.2.5 *The construction described above gives a diamond-free partial order $P = X \sqcup M \sqcup Y$ such that $X \sqcup Y$ is a 3-cs-homogeneous 2-level countable partial order and*

P is ramification complete and countable, indeed the same as the just-do-it approach described above.

Proof

Observe that free amalgamations cannot create diamonds, as they cannot create any new cycles at all. The original approach adds rays with endpoints and free amalgamations, but as any ray added after an amalgamation must be replicated to the original "page" containing the starting anomaly, one might simply have chosen to add the ray there before beginning the amalgamation. Thus one may move all free additions of rays to before any amalgamation, and that is what has been done here. It remains to observe that the cyclic subgroups of the free group generated by the generators given correspond to the amalgamations required to extend the maps corresponding to those generators to automorphisms, and the rest is forced as those maps are extended. Essentially, any extremal 3-arc on one page is identified with the designated arc on a page connected to it. □

Instead of using the free group itself on those generators it is possible to use quotients of the free group such as the free abelian group on the generating set, but this will introduce additional cycles in the constructed structure. The resulting construction from using a quotient of the free group is a surjective image of the construction using the free group, via the projection map.

This construction gives plenty of examples of partial orders embedding cycles where the arcs have lengths greater than one. Indeed, if one takes an unbalanced cycle, such as one where all arcs have lengths of two except one which has a length of three (and using $Z = \mathbb{Z}$), one has an example of such an order with maximal chains of order type $1+\mathbb{Z}+1$ with no level function to that order type.

We get a family of distinct constructions if we consider an invariant. An example of such an invariant is the minimum alternation number of any cycle in the partial order. This is the minimum number of vertices needed to identify the cycle, and the amalgamations used in the constructions cannot decrease this number, so two constructions starting from anomalies with different minimum alternation numbers will necessarily be non-isomorphic. Indeed, any properly new cycles formed must actually have a higher number, so two constructions can be distinguished by their cycles with the least alternations.

Note that it is now straightforward to show that the construction is well-defined if we have chosen the automorphisms between Y-structures beforehand.

For the construction to be at all useful for classification purposes it needs to behave in a sensible fashion. The following result shows that the construction is in some cases idempotent; that is if we use the output of the construction as the anomaly the new constructed object is isomorphic to the original object.

Result 5.2.6 *Suppose that for each approximation in our class of approximations and each pair of Y-structures we wish to amalgamate over there is a unique approximation witnessing this amalgamation which is embedded in each other approximation witnessing the amalgamation. For instance there may be a level function and a requirement that maps be level-preserving. Then given an anomaly δ, let $R = P(\delta)$ be the outcome of the construction starting from δ and $P(R)$ the outcome of the construction starting from R. $P(R)$ and R are isomorphic.*

Proof

The basic idea here is that when the construction is done again a “page” of form $T(R)$ is made by freely extending R, but these free additions can be shifted back to the creation of the page $T(\delta)$ made when first constructing R. Recall that $P(\delta)$ is constructed from δ with some set of maps U.

Suppose we freely amalgamate a tree of the correct form to a point of $T(\delta)$ to give $T'(\delta)$ and apply the construction with the same set of maps. This gives R with some (up to countably many) trees amalgamated freely at each point, which, by a back and forth argument, is isomorphic to T(R). To demonstrate this identify the copies of R and since all other points are attached freely given a point in either object it is simply identified by the point of R to which it is attached and whether it is part of a tree from that point with any previously identified points, if it is the trees are free and infinitely ramifying so it is possible to find a corresponding point, and if not there are infinitely many trees so one may pick a fresh one.

Note also that $T'(\delta)$ and $T(\delta)$ are isomorphic. $P(R)$ is simply the object constructed from $T'(\delta)$, which is the same as R, the object constructed from $T(\delta)$. $\square$

5.3 Difficulties with Classification

We would like to show that the construction is in some way universal, at least in a finitary case. There are a few problems with this. One is that the age of an outcome of the construction is insufficient to distinguish it up to isomorphism. This is unfortunate, but at least it is straightforward to give a characterisation of the finite substructures which arise.

Result 5.3.1 *For a given anomaly A, the finite substructures of $P(A)$ are exactly substructures of amalgams of copies of A over Y-shapes and substructures of Y-shapes (such as points).*

Proof

Any finite substructure emerges and is covered by copies of A at a finite step during the construction (that is, after the application of finitely many maps). The free group on finitely many maps is infinite but one can find a finite connected component giving every one of the finitely many relations as a translate of a relation from A. $\square$

Result 5.3.2 *If A is a free amalgam of connected finite substructures of B including B itself, $P(A)$ and $P(B)$ contain the same finite substructures.*

Proof

Any amalgam of copies of A is a substructure of an amalgam of copies of B, and the rest is immediate from the previous result. □

It is fairly straightforward to choose A and B such that $P(A)$ and $P(B)$ are not isomorphic. Recall that the 8-crown is an 8-cycle of alternating upward and downward edges, pick two of the maximal points which are distance 4 from each other. Adding an antenna of length n means to add a path of length $2n$ starting from one maximal point and going to the other, which alternate up and down (so these paths lie above the original 8-crown). Then let B be a 8-crown with an antenna of, say, length 2, and A a free amalgam of B with an unadorned 8-crown. Then $P(A)$ and $P(B)$ have the same finite substructures, but while every 8-crown in $P(B)$ will have an antenna this is not the case for every 8-crown in $P(A)$, so the two are not isomorphic.

This allows the identification of an uncountable class of non-isomorphic structures with the same finite substructures corresponding to proper subsets of $\mathbb{N}$. Given such a subset u one can construct an 8-crown $C(u)$ with antennae freely added of lengths corresponding to elements of u. By $D(u)$ call the free amalgam of $C(u)$ and $C(\mathbb{N})$, then for any choice of u the construction $P(D(u))$ is distinct but has the same finite substructures as just $P(C(\mathbb{N}))$.

The implication of this is that it is necessary not only to know the class of substructures of a partial order to classify it but also to know how they are embedded. It is difficult to describe how much data this requires; it is possible, for some finite suborder u, that it embed into every v in a unique way, that it embed into every v unless it embeds into some v' or other, or some other rule. Examples arise quite naturally by choosing subsets of $P(\mathbb{N})$. This is all visible in the elementary theory of the structure but that may itself not be enough to determine it – $\mathbb{Z}^2$ and $\mathbb{Z}^3$ are elementarily equivalent, for instance.

5.4 Amalgamation closure

Remark 5.4.1 *A n-levelled partial order W can be given an order-preserving homomorphism to $[0, n]$. It is locally amalgamation closed if given two Y-structures with ramification point at the same level amalgamation of two copies of W over those structures gives W itself. It is transitive on local Y-structures if given two Y-structures with ramification point at the same level there is an automorphism of W moving one to the other.*

Result 5.4.2 *If Q is a construction by repeated amalgamation as described in this chapter and $Q(n)$ is its alternating closure of height n then $Q(n)$ is locally amalgamation closed and transitive on local Y-structures.*

Proof

Consider two Y-structures in $Q(n)$. These can be extended to Y-structures in Q itself, which must be exchanged by an automorphism of Q, whose restriction to $Q(n)$ will be an automorphism of that. This gives local transitivity. Similarly, Q will have been repeatedly amalgamated with itself over those Y-structures and this amalgamation ensured to preserve Q, so the same will hold for $Q(n)$. □

Remark 5.4.3 *If δ is a finite height partial order which is locally amalgamation closed and transitive on local Y-structures then a construction arising from δ may not have just δ as its alternating closure of the appropriate height.*

That alternating closure will of course contain δ. To see a counterexample to the case that it is δ consider the case where δ has two non-isomorphic cycles of minimal alternation number, one restricted to lower levels of δ and the other to upper levels. In the construction those cycles will necessarily arise on all levels of the alternating closure of the height of δ, but it is possible for δ to satisfy the hypotheses and not contain the cycles on all levels.

5.5 Many-levelled partial orders

Recall the standard classification of symmetric many-levelled partial orders . First colour the points of the (ramification-complete) partial order two colours. The blue points should be the points of the pre-completion partial order satisfying the appropriate symmetry conditions; these were previously the extremal points. The red points are the added ramification points. Consider a sufficiently long (in particular, it should not be monochromatic) maximal chain in this partial order. When the partial order is 3-cs-transitive, it is transitive on coloured sets of size 2.

The main difficulty added here is that 3-cs-transitivity on blue points will require considering connected sets that look like chains of length 3, which require for instance the blue points to form a dense partial order. If we do not require transitivity on chains of this sort we have the condition of being transitive solely on V- and Λ-shapes. Let us call that V-transitivity.

We need to expand the definition of *compatible* for the following result. In this case T is a cycle-free partial order so compatibility of an anomaly δ just requires that it be the image of a surjective cover of a connected suborder of T.

Theorem 5.5.1 *Let T be a 3-blue-cs-transitive 2-coloured cycle-free partial order. Let δ be a diamond-free partial order compatible with T. Suppose T has infinite upward* *and downward ramification at each red point and that each red point is bounded above and below by blue points. Each blue point should also bound infinitely many totally incomparable chains both above and below. Further suppose that R is a generating set of partial automorphisms of convex closures of 3-blue-connected sets such that T is preserved upon amalgamation with itself over such maps, starting with a distinguished 3-arc for each type of 3-blue-connected set. Then we may construct a 3-blue-cs-transitive 2-coloured cycle-free partial order embedding δ which is. This will be Dedekind-MacNeille complete if T is and countable if T and R are.*

Proof

The proof is as before, one takes a partial map $f : T \to \delta$ witnessing the compatibility of δ, whose domain is distinct from the distinguished connected sets of R and constructs the page $T(\delta)$ given by taking the quotient of T by preimages in f, i.e. $T/\equiv$ where $x \equiv y$ iff $x = y$ or $x, y \in f^{-1}(d)$ for some $d \in \delta$. This is now a projective image of T embedding δ.

The elements of R then give bijections between the distinguished arcs and all other arcs, and we conduct the amalgamation exactly as previously, taking a quotient of $T(\delta) \times \mathbb{F}_R$. If both T and R are countable then so are both sides of this direct product, so the resulting partial order must be. □

The demand for infinite ramification everywhere is rather restricting to be honest. It is a consequence of this technique of creating automorphisms via countable amalgamation everywhere. While, with ramification orders of 2, amalgamating over a Y-structure might not change the upward ramification order at the point in the middle, it will add an additional downward edge which is not identified with another, and the same will occur at all other points.

The sort of global identification necessary to enable ramification orders to be constrained mean that we must start with a partial order with finite ramification, and that the functions over which we amalgamate must be total covering projections, meaning that instead of creating a greater structure at each stage we are instead simply taking quotients over some choice of identifications. This can be done in some cases. As seen with the group presentations, sometimes it can be done in a very clear fashion. It is not known at present how this should be done in general, as an identifying map must be found for each pair of Y-shapes at some intermediate approximation stage.

Chapter 6

Conclusions

This thesis explores a few approaches to understanding and categorising diamond-free 3-cs-transitive partial orders. Much work remains to be done on each topic.

Chapter 3 establishes categorical relationships between covering projections of partial orders and those of related topological spaces, showing that the properties of each cycle-free partial order are invariants for a family of diamond-free partial orders. We observe that there are many closely related groups in play here: the fundamental group of loops, symmetry groups witnessing transitivity, and any concrete groups which can actually be ordered by the partial order. It would be fruitful to investigate how these are connected.

Cayley graphs of discrete groups have been found, giving explicit examples of partial orders with cycles with legs of extended height. These constitute one motivation for the decompositions of height greater than 1 in the succeeding chapter. These currently are only known for discrete orders. Understanding how to get partial orders embedding $\mathbb{Q}$ in this way should enable examples to be found for all countable order-types. Chapter 5 provides a class of constructions with high degrees of homogeneity. Even among the class of partial orders with this degree of homogeneity, it is not presently clear what data is sufficient to classify any individual example. It is known that many plausible candidates, such as the age, are insufficient. Identifying the data in this case should shed light on the wider classification problem.

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