$\widetilde{\mathcal{J}}$- Simple restriction ω-semigroups

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Abstract

In 1985 Asibong-Ibe [1] considered the *-bisimple ample ω-semigroups and proved that they are isomorphic to certain generalised Bruck-Reilly extensions $BR(M, \theta)$ of a cancellative monoid M where θ is a morphism.

In 1993 he also proved in [2] that a similar structure theorem holds for $\mathcal{J}^*$-simple ample ω-semigroups. Recently this result has been generalised in [16] for $\widetilde{\ }$-bisimple restriction ω-semigroups.

Our objective is to study generalised Green's relations and restriction semigroups, together with their properties. Then we show that the analogue of a similar structure theorem for simple inverse ω-semigroups that was proved in [13] still holds if we replace simple inverse ω-semigroups by $\widetilde{\mathcal{J}}$-simple restriction ω-semigroups. The theory developed here closely parallels that in [13] and [2].

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Preface

In the first chapter we review some basic notions of semigroup theory, together with some results connected to Green's relations. We define a generalisation of Green's relations ($\mathcal{R}^*, \mathcal{L}^*, \mathcal{H}^*, \mathcal{D}^*, \mathcal{J}^*$) and their properties that enable us to study some non-regular classes of semigroups. In the second chapter we refine these definitions to further generalise Green's relations, and we introduce ample and restriction semigroups together with some basic results connecting them. Then we go on to introduce a generalisation of the Bruck-Reilly extension and prove that it is a $\widetilde{\mathcal{J}}$-simple restriction ω-semigroup. The third chapter contains our main result which is a structure theorem characterising $\widetilde{\mathcal{J}}$-simple restriction ω-semigroups isomorphic to these generalised Bruck-Reilly extensions.

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Author's declaration

The contents of Chapter 1 are mostly from [13], [14], and [19], consisting of various basic definitions and results, however Examples 1.10.8, 1.11.7, 1.11.8 and 1.11.9 are my own works. Chapter 2 consists mostly of some results which have been proven previously in [5], [9] and [10], however the $\widetilde{\mathcal{J}}$-simple restriction case is my own work. In Chapter 3 we prove some results which are analogous to those in [13], [1] and [2], however the main result is my own work.

Chapter 1

Preliminaries

1.1 Basic Theory

This chapter is devoted to the study of some elementary properties and examples of semigroups, and is mostly based on [13]. We assume that the reader is familiar with basic concepts of set theory and group theory. To fix notation, we start with some basic definitions regarding maps.

We say a *(partial) map* $\phi : A \to B$ is a subset of $A \times B$ such that for every $x \in A$, there exists exactly one (in the case of a *partial map*, at most one) element $y \in B$ such that $(x, y) \in \phi$. The *domain* of ϕ is

$$\text{dom } \phi = \{x \in A : \exists y \in B \text{ such that } (x, y) \in \phi\}$$

and the *image* of the map is

$$\text{im } \phi = \{y \in B : \exists x \in A \text{ such that } (x, y) \in \phi\}.$$

We denote the *kernel* of ϕ by ker ϕ and define it by

$$\ker \phi = \{(x, y) \in A : x\phi = y\phi\}.$$

It is worth reminding the reader that if $x \in \text{dom } \phi$, then $x\phi$ is called the image of x under ϕ.

Throughout this thesis, maps are written on the right and composed left to right. The composition of two maps is the usual composition, namely, let $\phi : A \to B$ and $\psi : B \to C$ be two maps, then we define a new map $\phi \circ \psi : A \to C$ by

$$x(\phi \circ \psi) = (x\phi)\psi.$$

This new map is called the *composition* of ϕ and ψ.

A map $\phi : A \to B$ is said to be *one-to-one (or injective)* if different elements in the domain A have distinct images. A map $\phi : A \to B$ is said to be an *onto (surjective)* map if every element of B is the image of some element in A, and we say that a map is *bijective* if it is both injective and surjective.

Let $\phi : A \to B$ be a map and $A' \subseteq A$ then $\phi|_{A'} : A' \to B$ is also a map, called *the restriction of* ϕ to A'.

1.2 Semigroups

A *binary operation* on a non-empty set S is a map $\cdot : S \times S \to S$. This operation is *associative* if for all $x, y, z \in S$

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

Semigroups play an important role in many areas of mathematics and computer science. The study of semigroups were first considered in the early 20th century. A *semigroup* is defined as an algebraic structure consisting of a set together with an associative binary operation. One can formulate the definition as follows:

Definition 1.2.1. A *semigroup* is a non-empty set S equipped with an associative binary operation '$\cdot$' and usually denoted by $(S, \cdot)$ or just S.

A *subsemigroup* of a semigroup S is a non-empty subset $T \subseteq S$ which is closed with respect to multiplication, that is, if $x, y \in T$ then $xy \in T$. We normally abbreviate this by $T \leq S$.

A *unary operation* is an operation with only one operand, that is, a transformation on S. We define a $(2,1)$-*semigroup* to be a set equipped with an associative binary operation and a unary operation, and $(2,1,1)$-*semigroup* is a set equipped with an associative binary operation and two unary operations. A $(2,1)$-*subsemigroup* or $(2,1,1)$-*subsemigroup* is a subsemigroup closed under the binary and unary operations.

Next we introduce two special elements in semigroup theory which are the *identity* element and the *zero* element. Let S be a semigroup. An element $e \in S$ is a *right (left) identity* of S, if for all $x \in S$

$$x \cdot e = x \ (e \cdot x = x).$$

If e is both a *right* and *left identity* of S, then e is called an *identity* of S. It is easy to prove that a semigroup S can have at most one identity.

We say an element z is a *right (left) zero* element in S if for all $x \in S$

$$x \cdot z = z \ (z \cdot x = z)$$

and z is a *two-sided zero* or just a *zero* if it is both a *right* and a *left zero*. It is also true that any semigroup contains at most one zero. Moreover, if every element of the semigroup is a left zero then the semigroup is called a *left zero semigroup*; a *right zero semigroup* is defined dually.

A *monoid* is a semigroup S with an identity element. It is worth mentioning here that, if S has no identity element, then it is very easy to adjoin an extra element 1 to S. We form a monoid

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise} \end{cases}$$

where the multiplication is defined in the obvious way

$$1 \cdot s = s \cdot 1 = s \ (\forall s \in S^1).$$

We refer to S^1 as the monoid obtained from S by *adjoining an identity if necessary.* As a consequence, we deduce that each semigroup can be extended to a monoid by adding at most one element.

Next we give some examples of semigroups and monoids below:

Example 1.2.2. The set $\mathbb{N} = \{1, 2, 3, ...\}$ with the addition operation forms a semigroup. The set $\mathbb{N}^0 = \{0, 1, 2, ...\}$ is a monoid under $+$ and $\times$.

Example 1.2.3. A ring is a semigroup under $\times$. If the ring has an identity then this semigroup is a monoid.

Example 1.2.4. The integers $\mathbb{Z}$ form a semigroup under both $+$, $\times$ operations. The semigroup $(\mathbb{Z}, +)$ is a monoid with identity 0, and $(\mathbb{Z}, \cdot)$ is a monoid with identity 1. In $(\mathbb{Z}, \cdot)$ the element 0 is a zero.

Example 1.2.5. Let I, J be non-empty sets and set $T = I \times J$ with the operation defined by

$$(i, j)(k, l) = (i, l) \ (\forall i, k \in I \text{ and } \ \forall j, l \in J).$$

Then $\cdot$ is associative and the semigroup $(T, \cdot)$ is called a *rectangular band.*

Example 1.2.6. Let $B = \mathbb{N}^0 \times \mathbb{N}^0$, and $(a, b), (c, d) \in B$. We define a binary operation on B by

$$(a, b)(c, d) = (a - b + t, d - c + t),$$

where $t = max\{b, c\}$. The given operation is associative and the resulting semigroup is called the *bicyclic monoid* (its identity is $(0, 0)$).

Example 1.2.7. [13] Let G be a group with identity element e, and let I, Λ be non-empty sets. Let $P = (p_{\lambda i})$ be an $I \times \Lambda$ matrix with entries in $G^0 (= G \cup \{0\}$ where 0 is a symbol not contained in G), and suppose that P is regular, in the sense that no row or column of P consists entirely of zeros. Formally,

$$(\forall i \in I)(\exists \lambda \in \Lambda)\ p_{\lambda i} \neq 0,$$
$$(\forall \lambda \in \Lambda)(\exists i \in I)\ p_{\lambda i} \neq 0.$$

Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define multiplication on S by

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

$$(i, a, \lambda)0 = 0(i, a, \lambda) = 00 = 0.$$

This multiplication is associative. The semigroup we thus defined above is denoted by $\mathcal{M}^0[G; I, \Lambda; P]$, and will be called the $I \times \Lambda$ *Rees matrix semigroup* over the group G with the regular sandwich matrix P.

Idempotents play a central role in semigroup theory. One reason is (as we will see in Section 1.7) that they locally connect semigroups to groups. We say an element e of S is an *idempotent* if

$$e.e = e \quad (e^2 = e).$$

The set of *idempotents* of S is denoted by

$$E(S) = \{e \in S : e^2 = e\}.$$

Surprisingly, $E(S)$ has a very strong influence on the whole structure of S. Notice that in some cases $E(S)$ may also equal S. In the rectangular band that we defined in Example 1.2.5, for any $(i, j) \in T$ we have

$$(i, j)^2 = (i, j)(i, j) = (i, j),$$

thus $E(T) = T$.

Let A, B be semigroups. Similarly to the corresponding notion in group theory we say that a map $\phi : A \to B$ is a *morphism (homomorphism)* if for all $x, y \in A$

$$(xy)\phi = (x\phi)(y\phi).$$

If A and B are monoids and ϕ also satisfies

$$1_A\phi = 1_B$$

(where 1_A and 1_B are identities of A and B respectively) then ϕ is called a *monoid morphism.* It is easy to see that im ϕ is a submonoid of B.

Let $(A, \cdot, ^+)$ and $(B, \cdot, ^+)$ be $(2,1)$-semigroups. A map $\phi : A \to B$ is a $(2,1)$-*morphism* if for all $x, y \in A$

$$(xy)\phi = x\phi \cdot y\phi \text{ and } (x\phi)^+ = x^+\phi.$$

We define a $(2,1,1)$-*morphism* similarly.

Similarly to morphisms, the image of a $(2,1)$- or a $(2,1,1)$-morphism is a $(2,1)$- or a $(2,1,1)$-subsemigroup respectively.

If $(A, \cdot)$ and $(B, \cdot)$ are monoids, and ϕ sends the identity of A to the identity of B, then we call it a monoid $(2,1)$-morphism (or monoid $(2,1,1)$-morphism) and in this case its image is a $(2,1)$-submonoid (or $(2,1,1)$-submonoid).

An *isomorphism* of semigroups is a map $\phi : A \to B$ which is a bijective morphism. Embeddings capture mathematically when one semigroup is contained in another. We say that a morphism $\phi : A \to B$ is an *embedding* if it is injective. We say a semigroup S is *embeddable* in another semigroup T, if there exists an *embedding* $\phi : S \to T$. If $\phi : S \to T$ is an embedding, then S is isomorphic to im ϕ, which is a subsemigroup of T, so we can think of S as being a subsemigroup of T.

Properties that reduce the gap between groups and semigroups are always important. Now we introduce one of these properties. The semigroup S is *left cancellative* if

$$(\forall a, b, c \in S)\ ca = cb \Rightarrow a = b,$$

right cancellative if

$$(\forall a, b, c \in S)\ ac = bc \Rightarrow a = b,$$

and *cancellative* if it is both *left* and *right cancellative.* It is clear that groups (and so also subsemigroups of groups) are cancellative, and on the other hand a finite cancellative semigroup is necessarily a group. Therefore cancellative semigroups are considered to be very close to groups. Most semigroups are not cancellative, for example the full matrix semigroup $\mathbb{Z}_{2\times 2}$, since

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}.$$

Note that a cancellative monoid has only one idempotent, namely its identity.

Another important property is *commutativity.* We say that a semigroup S is *commutative* if

$$ab = ba\ (\forall a, b \in S).$$

It is worth mentioning that a commutative semigroup can be embedded in a group if and only if it is cancellative. Evidently, the semigroups defined in Example 1.2.2 are all commutative semigroups, while a nontrivial rectangular band defined in Example 1.2.5 is not commutative.

1.3 Binary Relations

Binary relations are used in many branches of mathematics to model concepts like 'smaller than', 'equal to', and 'divides' in arithmetic, and so on. More generally, a binary relation is an arbitrary association between pairs of elements of a set. Intuitively the concept of a binary relation is as fundamental mathematically as the concept of a map. We now give the formal definition.

A *binary relation* ρ on a set X is a subset of $X \times X$. To simplify notation we write $x \; \rho \; y$ instead of $(x, y) \in \rho$, for any elements x and y in X. A special relation that is worth mentioning here is the *identity relation* on X,

$$id_X = \{(x, x) : x \in X\}$$

that is, two elements are related if and only if they are equal. Also we have the *universal relation* $X \times X$, in which everything is related to everything. We denote the set of all binary relations on X by $\mathcal{B}(X)$. An operation of composition is defined as:

$$\rho \circ \lambda = \{(x, y) \in X \times X : \exists z \text{ such that } (x, z) \in \rho \text{ and } (z, y) \in \lambda\}.$$

For each $\rho \in \mathcal{B}(X)$ the *converse* of ρ is defined by

$$\rho^{-1} = \{(y, x) : (x, y) \in \rho\}.$$

It is easy to see that the operation $\circ$ is associative. To see this, let $(x, y) \in X \times X$ then

$$\begin{aligned}
&(x, y) \in (\rho \circ \lambda) \circ \sigma \\
&\qquad \Leftrightarrow (\exists z \in X)(x, z) \in \rho \circ \lambda \text{ and } (z, y) \in \sigma, \\
&\qquad \Leftrightarrow (\exists z \in X)(\exists u \in X)(x, u) \in \rho, (u, z) \in \lambda \text{ and } (z, y) \in \sigma, \\
&\qquad \Leftrightarrow (\exists u \in X)(x, u) \in \rho \text{ and } (u, y) \in \lambda \circ \sigma, \\
&\qquad \Leftrightarrow (x, y) \in \rho \circ (\lambda \circ \sigma).
\end{aligned}$$

Thus we have proved that $(\mathcal{B}_X, \circ)$ is a semigroup.

One of the important binary relations that we are interested in is a *partial order*. We say that a binary relation ρ on a set X is a *partial order* if:

1. $(x, x) \in \rho$ for all x in X, that , ρ is *reflexive*;

2. $(\forall x, y \in X)(x, y) \in \rho$ and $(y, x) \in \rho \Rightarrow x = y$, that is, ρ is *antisymmetric*;

3. $(\forall x, y, z \in X)(x, y) \in \rho$ and $(y, z) \in \rho \Rightarrow (x, z) \in \rho$, that is, ρ is *transitive*.

   A partial order having the extra property

4. $(\forall x, y \in X)\ x \leq y$ or $y \leq x$, will be called a *total order*.

On the other hand, if we have *symmetric* property,

$$(\forall x, y \in X)(x, y) \in \rho \Rightarrow (y, x) \in \rho$$

instead of anti-symmetry, and ρ is also reflexive and transitive, then ρ is an *equivalence relation*.

If $\sim$ is an equivalence relation, then the *equivalence class* of an element a is denoted by $[a]$ and is defined as:

$$[a] = \{x \in X | a \sim x\}.$$

Given an equivalence relation $\sim$ on X, the set of all equivalence classes is denoted by $X/\sim$ and is called the *quotient set of X by $\sim$*. That is,

$$X/\sim = \{[a] : a \in X\}.$$

An important connection between maps and equivalences is given by:

Proposition 1.3.1. *[13] If $\phi : X \to Y$ is a map, then $\ker \phi$ is an equivalence.*

Let X be a non-empty set and let $(X, \leq)$ be a partially ordered set, then we say an element $x \in X$

is *minimal* if $(\forall y \in X)(y \leq x \Rightarrow y = x)$,

is *minimum* if $(\forall y \in X)(x \leq y)$,

is *maximal* if $(\forall y \in X)(x \leq y \Rightarrow y = x)$,

is *maximum* if $(\forall y \in X)(y \leq x)$.

Evidently a minimum element is also minimal, but the converse is not true.

Semilattices play an important role in this thesis. They can be obtained either as special semigroups or as special partially ordered sets. We include both approaches here.

Definition 1.3.2. Let Y be a non-empty subset of a partially ordered set $(X, \leq)$. We say that an element c of X is a *lower bound* of Y if $c \leq y$ for every $y \in Y$. If the set of lower bounds of Y is non-empty and has a *maximum element* d, we say that d is the *greatest lower bound*, or *meet*, of Y.

It is easy to see that the element d is unique if it exists, and we write

$$d = \bigwedge \{y : y \in Y\}.$$

If $Y = \{a, b\}$ then we write $d = a \wedge b$.

Definition 1.3.3. If $(X, \leq)$ is such that $a \wedge b$ exists for all $a, b \in X$, then we say that $(X, \leq)$ is a *lower semilattice*. Also if $\bigwedge\{y : y \in Y\}$ exists for every non-empty subset Y of X, then we say that $(X, \leq)$ is a *complete lower semilattice*. Analogous definitions are easily given for the *least upper bound*, or *join*

$$\bigvee \{y : y \in Y\},$$

for $a \vee b$, for an *upper semilattice* and for a *complete upper semilattice*.

It is easy to check that if $(X, \leq)$ is a lower semilattice then $(X, \wedge)$ is a commutative semigroup where all elements are idempotents.

To define *semilattices* as semigroups, we need to define a partial order on idempotents of semigroups. Let S be a semigroup. We define a relation $\leq$ on the set of idempotents $E(S)$ by

$$e \leq f \Leftrightarrow ef = fe = e.$$

We show now that $\leq$ is a partial order. Certainly it is clear that $e \leq e$ so it is reflexive, also, $f \leq e$ and $e \leq f$ together imply that $e = f$ so it is anti-symmetric. To show transitivity, notice that if $e \leq f$ and $f \leq g$, so that

$$ef = fe = e$$

and

$$fg = gf = f$$

then

$$eg = efg = ef = e \text{ and } ge = gfe = fe = e,$$

and so $e \leq g$.

Products of idempotents need not be idempotents in general, however, if idempotents commute (i.e. $ef = fe$ for all $e, f \in E(S)$), then $E(S)$ forms a subsemigroup. A semigroup is called a *semilattice*, if it is commutative and all elements are idempotents. The following lemma shows that a semilattice defined this way is a semilattice as a partially ordered set.

Lemma 1.3.4. *Let S be a semilattice. Then $ef \in S$ is the greatest lower bound of the elements e and f of S, that is, $(S, \leq)$ is a lower semilattice (as a partially ordered set).*

Proof. Since S is a semilattice, then $e \leq f$ if and only if $e = ef$ (as S is commutative) then we have that $ef = fe = ffe = eff$, which means that $ef \leq f$. Similarly, $ef \leq e$ holds. Thus ef is a lower bound of e and f. To show that ef is the greatest lower bound, let g be a lower bound of e and f, then $g = ge = gf$ and hence

$$g(ef) = (ge)f = gf = g,$$

thus $g \leq ef$. Therefore ef is the greatest lower bound of the elements e and f. □

The set of equivalences is partially ordered by set inclusion. In this partially ordered set, both $\rho \wedge \sigma$ and $\rho \vee \sigma$ exist. The former simply equals $\rho \cap \sigma$, and the latter is described by the following proposition.

Proposition 1.3.5. *[13] Let ρ, σ be equivalences on a set S. Then $(a, b) \in \rho \vee \sigma$ if and only if, for some $n \in \mathbb{N}$, there exist elements $x_1, x_2, ..., x_{2n-1}$ in S such that*

$$(a, x_1) \in \rho, (x_1, x_2) \in \sigma, (x_2, x_3) \in \rho, ..., (x_{2n-1}, b) \in \sigma.$$

As a useful corollary to this we have:

Corollary 1.3.6. *[13] Let ρ, σ be equivalences on a set S such that $\rho \circ \sigma = \sigma \circ \rho$. Then $\rho \vee \sigma = \rho \circ \sigma$.*

1.4 Congruences

Another important relation that plays a significant role in semigroup theory is a *congruence*. Let S be a semigroup. A relation ρ on the set S is called *left compatible* with respect to the operation on S if

$$(\forall s, t, a \in S) \ \ (s, t) \in \rho \Rightarrow (as, at) \in \rho,$$

and *right compatible* if

$$(\forall s, t, a \in S) \ \ (s, t) \in \rho \Rightarrow (sa, ta) \in \rho.$$

It is called *compatible* if it is both left and right compatible.

A left (right) compatible equivalence is called a *left (right) congruence*. An equivalence relation ρ is called a *congruence* if for all $a, b, c, d \in S$

$$a \ \rho \ b \text{ and } c \ \rho \ d \Rightarrow ac \ \rho \ bd.$$

Then we have the following proposition:

Proposition 1.4.1. *A relation ρ on a semigroup S is a congruence if and only if it is both a left and a right congruence.*

Proof. Suppose that ρ is a congruence on S. By definition, ρ is an equivalence. If $a \ \rho \ b$ and $c \in S$, then as ρ is reflexive, we have that $c \ \rho \ c$. As ρ is a congruence we deduce that

$$ac \ \rho \ bc \text{ and } ca \ \rho \ cd,$$

so that ρ is right and left compatible and hence is a right congruence and a left congruence.

Conversely, suppose that ρ is both a right congruence and a left congruence on S. By definition, ρ is an equivalence. Suppose that $a \ \rho \ b$ and $c \ \rho \ d$. Then $ac \ \rho \ bc$ as ρ is right compatible, and $bc \ \rho \ bd$ as ρ is left compatible. But ρ is transitive, so that $ac \ \rho \ bd$ and we deduce that ρ is a congruence. □

If ρ is a congruence on a semigroup S then we can define a binary operation on the *quotient set* $S/\rho = \{[x] : x \in S\}$ in a natural way as follows:

$$[a][b] = [ab].$$

The following theorem shows how quotient semigroups and morphisms are connected:

Proposition 1.4.2. *[13] Let S be a semigroup, and let ρ be a congruence on S. Then S/ρ is a semigroup with respect to the operation defined above, and the map $\rho^\sharp$ from S onto S/ρ given by*

$$x\rho^\sharp = [x] \quad (x \in X)$$

is a morphism. Now let T be a semigroup and let $\phi : S \to T$ be a morphism. Then the relation ker ϕ is a congruence on S, and there is an injective morphism $\alpha : S/ker\ \phi \to T$ such that im α= im ϕ and $\rho^\sharp \circ \alpha = \phi$.

1.5 $\mathcal{T}_X$ and $\mathcal{PT}_X$

While Cayley's theorem enables us to view groups as groups of permutations of some set, the analogous result in semigroup theory represents semigroups as semigroups of maps from a set to itself.

Let X be a set, then a map from X to itself is called a *transformation* of X. The set of all maps (with identity id_X) $\alpha : X \to X$ forms a transformation monoid under the composition of maps. This semigroup is called the *full transformation* semigroup on X and is denoted by $\mathcal{T}_X$. Subsemigroups of $\mathcal{T}_X$ are called transformation semigroups. We will sometimes use the so-called double row notation for maps.

Let X be a set, then a map from A to B is called a *partial transformation* of X where $A, B \subseteq X$. We let

$$\mathcal{PT}_X = \{\phi : A \to B \mid A, B \subseteq X\}.$$

If $C \subseteq X$ and $\alpha \in \mathcal{PT}_X$ then let $C\alpha^{-1} = \{a \in \text{dom } \alpha : a\alpha \in C\}$ and using this notation we can define composition of partial transformations in the following way: if $\alpha, \beta \in \mathcal{PT}_X$, then

$$\text{dom } \alpha\beta = (\text{im } \alpha \cap \text{dom } \beta)\alpha^{-1},$$

$$\text{im } \alpha\beta = (\text{im } \alpha \cap \text{dom } \beta)\beta,$$

and

$$x(\alpha\beta) = (x\alpha)\beta$$

for all $x \in \text{dom } \alpha\beta$. This composition is associative, so $\mathcal{PT}_X$ is a semigroup, in fact, a monoid with identity id_X.

As an example of composition of partial transformations, let $\alpha,\ \beta \in \mathcal{PT}_5$ be defined as

$$\alpha = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix}.$$

That is, $\text{dom } \alpha = \{1, 3, 4, 5\}$, $\text{im } \alpha = \{1, 2, 3\}$. Then

$$\alpha\beta = \begin{pmatrix} 1 & 3 & 4 \\ 1 & 1 & 3 \end{pmatrix}.$$

As we know from group theory, the set of all bijections on X forms a group under composition, which is called the *symmetric group* on X and it is denoted by $\mathcal{S}_X$. It is clear from the definitions that

$$\mathcal{S}_X \subseteq \mathcal{T}_X \subseteq \mathcal{PT}_X.$$

Definition 1.5.1. Let S be a semigroup. An injective morphism $\phi : S \to \mathcal{T}_X$ is called a *faithful representation* of S. The image $S\phi$ of ϕ is a *transformation semigroup* isomorphic to S.

In group theory Cayley's theorem states that every group G is isomorphic to a subgroup of the symmetric group acting on G. The next theorem is the analogue of this in semigroup theory, that is, it shows that every semigroup is embeddable in $\mathcal{T}_X$ for some X.

Theorem 1.5.2. *If S is a semigroup and $X = S^1$ then there is a faithful representation $\phi : S \to \mathcal{T}_X$.*

Proof. For each $a \in S$, we define a map $\rho_a : S^1 \to S^1$ by

$$x\rho_a = xa \quad (x \in S^1).$$

Thus $\rho_a \in \mathcal{T}_X$, and so there is a map $\alpha : S \to \mathcal{T}_X$ given by

$$a\alpha = \rho_a \quad (a \in S).$$

To show that α is one-one, for all a, b in S, we have

$$a\alpha = b\alpha \Rightarrow \rho_a = \rho_b \text{ for all } x \in S^1$$

$$\Rightarrow 1a = 1b \Rightarrow a = b.$$

To show that α is a morphism, for all a, b in S and $x \in S^1$,

$$x(\rho_a\rho_b) = (x\rho_a)\rho_b = (xa)b = x(ab) = x\rho_{ab},$$

and so $(a\alpha)(b\alpha) = (ab)\alpha$. □

1.6 Ideals

Ideals of semigroups can be defined similarly to ideals of rings, however their roles are slightly different in semigroup theory. In ring theory all homomorphisms are determined by ideals, but this is not the case for semigroups. Still, ideals play an important role in semigroup theory. Before we define ideals, we are going to remind the reader about multiplication of subsets of a semigroup. If $A, B \subseteq S$ and $a \in S$ then we write

$$AB = \{xy : x \in A, y \in B\},$$

$$A^2 = AA = \{xy : x, y \in A\},$$

$$AaB = \{xay : x \in A, y \in B\}, \quad etc.$$

We say that a non-empty subset I of a semigroup S is a *left ideal* if $SI \subseteq I$ and a *right ideal* if $IS \subseteq I$. If I is both a left and a right ideal then we call it an *ideal*. In other words, a non-empty subset $I \subseteq S$ is a left ideal of S. If

$$\forall a \in I \text{ and } s \in S,\ sa \in I$$

and we say I is a right ideal if

$$\forall a \in I \text{ and } s \in S,\ as \in I$$

and an ideal if

$$\forall a \in I \text{ and } s \in S,\ sa, as \in I.$$

We notice from the definition that if I is a left (right) ideal of S, then I is a subsemigroup of S, since if $SI \subseteq I$ or $IS \subseteq I$, then certainly $II \subseteq I$. Of course not any subsemigroup is an ideal, for example in a non-trivial group G every subgroup different from G is a subsemigroup but is not an ideal.

Notice that if a is an element of a semigroup S without identity, then Sa need not contain a. The following facts will be used throughout this thesis:

$$S^1a = Sa \cup \{a\},$$

$$aS^1 = aS \cup \{a\},$$

$$S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}.$$

It is easy to see that S^1a is the smallest left ideal containing a and we shall call it the *principal left ideal* generated by a. The *principal right ideal* aS^1 is defined dually, and we shall call S^1aS^1 the *principal two-sided ideal* generated by a.

1.7 Green's relations

Green's relations were first introduced and studied by Green in 1951 [12]. They are the most important tools to understand a semigroup. There are five *Green's relations* $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$.

Let S be a semigroup. We define the following relations on S:

$$a \,\mathcal{L}\, b \text{ if and only if } S^1a = S^1b;$$

$$a \,\mathcal{R}\, b \text{ if and only if } aS^1 = bS^1;$$

$$a \,\mathcal{J}\, b \text{ if and only if } S^1aS^1 = S^1bS^1,$$

where $a, b \in S$. Thus $a \,\mathcal{L}\, b$ if and only if a and b generate the same principal left ideal. Similarly $a \,\mathcal{R}\, b$ if and only if a and b generate the same principal right ideal, and $a \,\mathcal{J}\, b$ if and only if a and b generate the same principal two-sided ideal.

If we need to emphasise on which semigroup these relations are defined, then we will use the notation $\mathcal{L}^S$, $\mathcal{R}^S$, etc.

It is easy to see that the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are all equivalence relations. The following proposition gives an equivalent definition of these relations.

Proposition 1.7.1. *Let a, b be elements of a semigroup S. Then*

$$a \,\mathcal{L}\, b \Leftrightarrow (\exists x, y \in S^1)(xa = b,\ yb = a),$$

$$a \,\mathcal{R}\, b \Leftrightarrow (\exists x, y \in S^1)(ax = b,\ by = a),$$

$$a \,\mathcal{J}\, b \Leftrightarrow (\exists x, y, u, v \in S^1)(xay = b,\ ubv = a).$$

An important property of $\mathcal{L}$ and $\mathcal{R}$ is given in the next proposition.

Proposition 1.7.2. *The relation $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.*

Proof. For any $a, b, c \in S$,

$$a \,\mathcal{L}\, b \;\Rightarrow\; S^1a = S^1b \;\Rightarrow\; S^1ac = S^1bc \;\Rightarrow\; ac \,\mathcal{L}\, bc,$$

and so $\mathcal{L}$ is a right congruence. Dual reasoning shows that $\mathcal{R}$ is a left congruence. □

An extremely useful result is:

Proposition 1.7.3. *The relations $\mathcal{L}$ and $\mathcal{R}$ commute.*

Proof. Let S be a semigroup, and suppose that $a, b \in S$, with $(a, b) \in \mathcal{L} \circ \mathcal{R}$. Then $\exists c \in S$ such that $a \,\mathcal{L}\, c \,\mathcal{R}\, b$. That is, $\exists s, t, u, v \in S^1$ such that

$$sa = c, \ tc = a,$$

$$cu = b, \ bv = c.$$

Let $d = tcu$, then we have

$$au = tcu = d, \quad dv = tcuv = tbv = tc = a;$$

thus, $a \,\mathcal{R}\, d$. Furthermore,

$$tb = tcu = d, \quad sd = stcu = sau = cu = b,$$

hence $d \,\mathcal{L}\, b$. We have proved that $(a, b) \in \mathcal{R} \circ \mathcal{L}$, so $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. The reverse inclusion follows in a similar way. □

As an immediate result of Corollary 1.3.6 and Proposition 1.7.3, we notice that

$$\mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R}.$$

The intersection of two equivalences is an equivalence. We define

$$\mathcal{H} = \mathcal{L} \wedge \mathcal{R}, \ \mathcal{D} = \mathcal{L} \vee \mathcal{R},$$

which is equivalent to saying:

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \ \mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

Namely, for any elements $a, b \in S$ we have:

$$a \,\mathcal{H}\, b \text{ if and only if } a \,\mathcal{L}\, b \text{ and } a \,\mathcal{R}\, b$$

and

$$a \,\mathcal{D}\, b \text{ if and only if } (\exists c \in S) \text{ such that } a \,\mathcal{L}\, c \text{ and } c \,\mathcal{R}\, b.$$

Clearly, $\mathcal{L}$ and $\mathcal{R}$ are contained in $\mathcal{J}$. Thus we can say $\mathcal{J}$ is an upper bound for the set $\{\mathcal{L}, \mathcal{R}\}$ and hence,

$$\mathcal{D} = \mathcal{L} \vee \mathcal{R} \subseteq \mathcal{J}$$

equivalently,

$$\mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}.$$

Notice that in any commutative semigroup Green's relations are all equal, that is,

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J},$$

and also in any group G we have

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = G \times G.$$

Since for any elementt $a \in G$ we have that $G^1 a = G$ and $aG^1 = G$.

In the following lemmas we characterise the Green's relations in the full transformation semigroups, Rees Matrix Semigroups and in the bicyclic semigroup.

Lemma 1.7.4. *[13] In $\mathcal{T}_X$*

1. $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $im\ \alpha = im\ \beta$;
2. $(\alpha, \beta) \in \mathcal{R}$ *if and only if* $ker\ \alpha = ker\ \beta$;
3. $(\alpha, \beta) \in \mathcal{D}$ *if and only if* $|im\ \alpha| = |im\ \beta|$;
4. $\mathcal{D} = \mathcal{J}$.

Lemma 1.7.5. *[13] Let $\mathcal{M}^0 = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees Matrix Semigroup over a group G. Then*

1. $(i, a, \lambda)\ \mathcal{L}\ (j, b, \mu)$ *if and only if* $\lambda = \mu$*;*
2. $(i, a, \lambda)\ \mathcal{R}\ (j, b, \mu)$ *if and only if* $i = j$*;*
3. $(i, a, \lambda)\ \mathcal{H}\ (j, b, \mu)$ *if and only if* $i = j$ *and* $\lambda = \mu$*;*
4. $\mathcal{D} = \mathcal{J}$ *and has two classes,* $\{0\}$ *and* $\mathcal{M}^0 \setminus \{0\}$.

Lemma 1.7.6. *In in the bicyclic semigroup B*

1. $(m, n)\ \mathcal{L}\ (p, q)$ *if and only if* $n = q$;
2. $(m, n)\ \mathcal{R}\ (p, q)$ *if and only if* $m = p$;
3. $(m, n)\ \mathcal{H}\ (p, q)$ *if and only if* $m = p$ *and* $n = q$.

Notation 1.7.7. Let S be a semigroup and let $a \in S$. The $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class, $\mathcal{J}$-class) containing the element a will be denoted by L_a (R_a, H_a, D_a, J_a).

1.8 Simple semigroups

A *(left, right) proper ideal* I of a semigroup S is an (left, respectively, right) ideal such that $I \neq S$. That is, such that $I \subseteq S$ and $I \neq S$. A semigroup S is called *right simple* if it contains no proper right ideals, dually a semigroup S is called *left simple* if it contains no proper left ideals, and a semigroup S is called *simple* if it has no proper two-sided ideals.

It is easy to see that a semigroup S is *right (left) simple* if and only if $\mathcal{R} = S \times S$ ($\mathcal{L} = S \times S$), and *simple* if and only if $\mathcal{J} = S \times S$. A semigroup S is called *bisimple* if $\mathcal{D} = S \times S$.

Since in a group G we have that $\mathcal{L} = \mathcal{R} = G \times G$, we conclude that groups are left and right simple. Thus G is simple.

The following example is crucial for our main result in Chapter 3.

Example 1.8.1. Let B be the bicyclic semigroup. Let $I \subseteq B$ be an ideal, and $(m, n) \in I$. Then we have $(0, n) = (0, m)(m, n) \in I$. Hence

$$(0, 0) = (0, n)(n, 0) \in I.$$

Take any arbitrary element $(a, b) \in B$. Then

$$(a, b) = (a, b)(0, 0) \in I$$

thus, $B \subseteq I$. Therefore $B = I$, so B is simple. In fact more is true: let $(m, n), (k, l) \in B$. Then

$$(m, n) \; \mathcal{R} \; (m, l) \; \mathcal{L} \; (k, l),$$

so that $(m, n) \; \mathcal{D} \; (k, l)$. Hence B is bisimple.

1.9 Regular Semigroups and Inverse Semigroups

The concept of regularity in a semigroup was adapted from an analogous condition for rings, already considered by J. von Neumann. An element a of a semigroup S is *regular* if there exists an element $x \in S$ such that $a = axa$, and a semigroup S is *regular* if each of its elements is regular.

For instance, all idempotents of a semigroup S are regular, since if

$$e \in E(S) \text{ then } e = eee.$$

Also all groups are regular since for every element $a \in G$ (where G is a group) we have $a = aa^{-1}a$, where a^{-1} is the inverse of a in the sense of group theory. A rectangular band T (Example 1.2.5) is regular, since for any elements $(i, j), (k, l) \in T$ we have

$$(i, j)(k, l)(i, j) = (i, j).$$

Definition 1.9.1. We say an element $a' \in S$ is an *inverse* of an element a of S if:

$$aa'a = a \text{ and } a'aa' = a'.$$

The set of all inverses of a is denoted by $V(a)$. It is clear that every regular element has an inverse, since if a is a regular element in S then there exists an element $x \in S$ such that $a = axa$, and it is easy to see that $xax \in V(a)$.

Notice that the inverse of an element need not be unique. For example, in a rectangular band T, for every $(i, j), (k, l) \in T$ we have

$$(i, j)(k, l)(i, j) = (i, j)$$

$$(k, l)(i, j)(k, l) = (k, l)$$

so every element is an inverse of every element.

Definition 1.9.2. We say that a semigroup S is an *inverse semigroup* if there exists a unary operation $x \to x^{-1}$ on S with the properties:

$$(x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad (\forall x \in S)$$

and

$$xx^{-1}yy^{-1} = yy^{-1}xx^{-1} \quad (\forall x, y \in S).$$

Then we have

Theorem 1.9.3. *[13] Let S be a semigroup. Then the following statements are equivalent:*

1. *S is an inverse semigroup;*
2. *S is regular, and its idempotents commute;*
3. *every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent;*
4. *every element of S has a unique inverse.*

Examples of inverse semigroups are bicyclic semigroups, semilattices, and groups. In the following definition we introduce another example of inverse semigroups which is analogous to a symmetric group.

Definition 1.9.4. Given a non-empty set X, we define $\mathcal{I}_X$ to be the set of all partial one-one maps of X. Then $\mathcal{I}_X$ is a semigroup under the composition of partial maps.

We have the following theorem (the proof of which can be found in [13]):

Theorem 1.9.5. *The semigroup of all partial one-one maps $\mathcal{I}_X$ of some set X is an inverse semigroup.*

We call $\mathcal{I}_X$ the *symmetric inverse semigroup* on X. The unique inverse of an element $\alpha \in \mathcal{I}_X$ is the usual inverse of an injective partial map. Now we can state the Vagner-Preston Theorem which is the analogue of Cayley's Theorem for inverse semigroups (the proof can be found in [13]):

Theorem 1.9.6. *Let S be an inverse semigroup. Then there exists a symmetric inverse semigroup $\mathcal{I}_X$ on some set X, and an injective morphism ϕ from S into $\mathcal{I}_X$ given by*

$$a\phi = \rho_a$$

where $\rho_a : Saa^{-1} \to Sa^{-1}a,\ x \mapsto xa$.

1.10 Structure of $\mathcal{D}$-classes

Equivalence classes of $\mathcal{D}$ are called the $\mathcal{D}$-classes of a semigroup S. Evidently, $\mathcal{L}, \mathcal{R} \subseteq \mathcal{D}$ (since $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$), so every $\mathcal{D}$-class is the union of $\mathcal{R}$-classes (and also of $\mathcal{L}$-classes). There is an alternative way to guarantee that two elements are $\mathcal{D}$-related, which is

$$a \ \mathcal{D} \ b \Leftrightarrow R_a \cap L_b \neq \emptyset \Leftrightarrow L_a \cap R_b \neq \emptyset.$$

We can illustrate a $\mathcal{D}$-class by an *"eggbox"*, a notion first introduced by Clifford and Preston (1961). The *"eggbox"* is a rectangle where the rows represent the $\mathcal{R}$-classes and the columns represent the $\mathcal{L}$-classes and the intersections of $\mathcal{L}$-classes and $\mathcal{R}$-classes create the $\mathcal{H}$-classes. The following *eggbox* below has 3 $\mathcal{R}$-classes, 4 $\mathcal{L}$-classes, and 12 $\mathcal{H}$-classes:

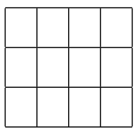

Figure 1.1: A $\mathcal{D}$-class

Let $(a, b) \in \mathcal{R}$ then there exist $x, y \in S^1$ with

$$ax = b, \ by = a.$$

The *right translation* $\rho_x : S \to S, \ s \mapsto sx$ maps a to b, in fact maps L_a into L_b. Similarly, ρ_y maps L_b into L_a. The composition map $\rho_x \rho_y : L_a \to L_a$ is just the identity map from L_a into L_a, also $\rho_y \rho_x : L_b \to L_b$ is just the identity map from L_b into L_b. As a consequence, the maps $\rho_x|_{L_a}$ and $\rho_y|_{L_b}$ are mutually inverse bijections from L_a into L_b and L_b into L_a, respectively. Dually if $x \in S$ then one can define the *left translation* $\lambda_x : S \to S, \ t \mapsto xt$.

Recall from [13] that if S is a semigroup, σ is a relation on S and ϕ is a partial transformation of S then we say that ϕ is *σ-class preserving* if for all $s \in \text{dom } \phi$ we have $s \ \sigma \ s\phi$. These observations mentioned above and their duals are formulated in Green's Lemmas:

Lemma 1.10.1. *Let a, b be $\mathcal{R}$-equivalent elements in a semigroup S, and let s, s' in S^1 be such that*

$$as = b, \ \ bs' = a.$$

Then the right translations $\rho_s|_{L_a}, \rho_{s'}|_{L_b}$ are mutually inverse $\mathcal{R}$-class-preserving bijections from L_a onto L_b and L_b onto L_a, respectively.

Lemma 1.10.2. *Let a, b be $\mathcal{L}$-equivalent elements in a semigroup S, and let t, t' in S^1 be such that*

$$ta = b, \quad t'b = a.$$

Then the left translations $\lambda_t|_{R_a}, \lambda_{t'}|_{R_b}$ are mutually inverse $\mathcal{L}$-class-preserving bijections from R_a onto R_b and R_b onto R_a, respectively.

Now we will present some results from [13] describing the structure of those $\mathcal{D}$-classes which contain regular elements.

Proposition 1.10.3. *If a is a regular element of a semigroup S, then every element of D_a is regular.*

A $\mathcal{D}$-class is called *regular* if all of its elements are regular. Regular $\mathcal{D}$-classes contain a lot of idempotents as we notice in the following proposition.

Proposition 1.10.4. *In a regular $\mathcal{D}$-class, every $\mathcal{L}$-class and every $\mathcal{R}$-class contains an idempotent.*

In a $\mathcal{D}$-class idempotents determine the position of inverses.

Theorem 1.10.5. *Let a be an element of a regular $\mathcal{D}$-class D in a semigroup S.*

1. *If $a' \in V(a)$, then $a' \in D$ and the two $\mathcal{H}$-classes $R_a \cap L_{a'}, L_a \cap R_{a'}$ contain the idempotents aa' and $a'a$ respectively.*
2. *If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain idempotents e, f, respectively, then H_b contains an inverse a^* of a such that*

$$aa^* = e, \ a^*a = f.$$

3. *No $\mathcal{H}$-class contains more than one inverse of a.*

As a result of this we have the next proposition:

Proposition 1.10.6. *Let e, f be idempotents in a semigroup S. Then $e \ \mathcal{D} \ f$ if and only if there exist an element a in S and an inverse a' of a such that $aa' = e, a'a = f$.*

Then we have an extremely useful theorem called Green's Theorem:

Theorem 1.10.7. *If H is an $\mathcal{H}$-class in a semigroup S then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S.*

In the following example we are going to determine Green's relations on a subsemigroup of $\mathcal{T}_4$.

Example 1.10.8. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{pmatrix}$, and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 2 \end{pmatrix}$.

The subsemigroup S generated by α and β consists of ten elements α, α^2, $\alpha^3, \beta, \beta^2, \beta^3, \alpha\beta, \alpha^2\beta, \alpha^3\beta$ and $\alpha\beta^2$.

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 3 \end{pmatrix},\ \alpha^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix},\ \beta^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix},$$

$$\beta^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 1 \end{pmatrix},\ \alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 3 & 3 \end{pmatrix},\ \alpha^2\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 1 \end{pmatrix},$$

$$\alpha^3\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix},\ \alpha\beta^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 1 \end{pmatrix}.$$

We use the Cayley table below to compute the Green's relations.

	α	α^2	α^3	β	β^2	β^3	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha\beta^2$
α	α^2	α^3	α^3	$\alpha\beta$	$\alpha\beta^2$	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
α^2	α^3	α^3	α^3	$\alpha^2\beta$	α^2	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
α^3	α^3	α^3	α^3	$\alpha^3\beta$	α^3	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
β	α^2	α^3	α^3	β^2	β^3	β^3	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^2
β^2	α^3	α^3	α^3	β^3	β^2	β^3	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
β^3	α^3	α^3	α^3	β^3	β^3	β^2	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
$\alpha\beta$	α^3	α^3	α^3	$\alpha\beta^2$	$\alpha\beta$	$\alpha\beta^2$	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
$\alpha^2\beta$	α^3	α^3	α^3	α^2	$\alpha^2\beta$	α^2	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
$\alpha^3\beta$	α^3	α^3	α^3	α^3	$\alpha^3\beta$	α^3	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3
$\alpha\beta^2$	α^3	α^3	α^3	$\alpha\beta$	$\alpha\beta^2$	$\alpha\beta$	$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^3\beta$	α^3

Then:
$\alpha S^1 = \{\alpha, \alpha^2, \alpha^3, \alpha\beta, \alpha\beta^2, \alpha^2\beta, \alpha^3\beta\}$;
$S^1\alpha = \{\alpha, \alpha^2, \alpha^3\}$;
$\alpha^2 S^1 = \{\alpha^2, \alpha^3, \alpha^2\beta, \alpha^3\beta\}$;
$S^1\alpha^2 = \{\alpha^2, \alpha^3\}$;
$\alpha^3 S^1 = \{\alpha^3, \alpha^3\beta\}$,
$S^1\alpha^3 = \{\alpha^3\}$;
$\beta S^1 = \{\beta, \alpha^2, \alpha^3, \beta^2, \beta^3, \alpha^2\beta, \alpha^3\beta\}$;
$S^1\beta = \{\beta, \beta^2, \beta^3, \alpha^2, \alpha^3, \alpha\beta, \alpha\beta^2, \alpha^2\beta, \alpha^3\beta\}$;
$\beta^2 S^1 = \{\beta^2, \beta^3, \alpha^3, \alpha^3\beta\}$;
$S^1\beta^2 = \{\beta^2, \beta^3, \alpha^2, \alpha^3, \alpha\beta, \alpha^2\beta, \alpha^3\beta\}$;
$\beta^3 S^1 = \{\beta^3, \beta^2, \alpha^3, \alpha^3\beta\}$;
$S^1\beta^3 = \{\beta^3, \beta^2, \alpha^2, \alpha^3, \alpha\beta, \alpha\beta^2, \alpha^3\beta, \alpha^2\beta\}$;
$\alpha\beta S^1 = \{\alpha\beta, \alpha^3, \alpha\beta^2, \alpha^3\beta\}$;
$S^1\alpha\beta = \{\alpha\beta, \alpha^2\beta, \alpha^3\beta\}$;

$\alpha^2\beta S^1 = \{\alpha^2\beta, \alpha^2, \alpha^3, \alpha^3\beta\}$;
$S^1\alpha^2\beta = \{\alpha^2\beta, \alpha^3\beta\}$;
$\alpha^3\beta S^1 = \{\alpha^3\beta, \alpha^3\}$;
$S^1\alpha^3\beta = \{\alpha^3\beta\}$;
$\alpha\beta^2 S^1 = \{\alpha\beta^2, \alpha^3, \alpha\beta, \alpha^3\beta\}$;
$S^1\alpha\beta^2 = \{\alpha\beta^2, \alpha^3\}$.

Hence we have $\alpha^2 \mathcal{R}\ \alpha^2\beta$, $\beta^2 \mathcal{R}\ \beta^3$, $\alpha^3 \mathcal{R}\ \alpha^3\beta$, and $\alpha\beta \mathcal{R}\ \alpha\beta^2$. Similarly one can see that $\mathcal{L}$ is the identity relation, so $\mathcal{D} = \mathcal{L} \vee \mathcal{R} = \mathcal{R}$.

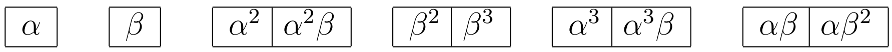

Figure 1.2: The $\mathcal{D}$-classes of a semigroup S

1.11 Green's *-relations

Green's *-relations were first introduced by Pastijn in [18] and were adopted by Fountain in [6]. They are useful in investigating non-regular semigroups. Let a, b be elements of a semigroup S, then we define *Green's *-relations* as follows

$$\mathcal{L}^* = \{(a,b) \in S \times S : (\forall x, y \in S^1)\ ax = ay \Leftrightarrow bx = by\},$$

$$\mathcal{R}^* = \{(a,b) \in S \times S : (\forall x, y \in S^1)\ xa = ya \Leftrightarrow xb = yb\},$$

$$\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*,$$

$$\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*.$$

Now we study various properties of these relations. Most of these were proved in [18] and [6], and some of them are folklore.

Proposition 1.11.1. *The relation $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence.*

Proof. The proof of the relation $\mathcal{L}^*$ to be an equivalence relation is straightforward, so we leave it for the reader to check it. Now let $z \in S$ then

$$\begin{aligned}(az)x = (az)y &\Leftrightarrow a(zx) = a(zy)\\ &\Leftrightarrow b(zx) = b(zy)\\ &\Leftrightarrow (bz)x = (bz)y.\end{aligned}$$

Therefore, $az\ \mathcal{L}^* bz$, which proves that $\mathcal{L}^*$ is a right congruence. Dually one can show that $\mathcal{R}^*$ is a left congruence. □

The next proposition shows the connection between $\mathcal{L}^*$ and $\mathcal{L}$.

Proposition 1.11.2. *In a semigroup S, $\mathcal{L} \subseteq \mathcal{L}^*$ and if $s, t \in S$ are regular and $s\ \mathcal{L}^* t$, then $s\ \mathcal{L}\ t$.*

Proof. Let $(a, b) \in \mathcal{L}$ so there exist $u, v \in S^1$ with

$$ua = b,\ vb = a.$$

Suppose that $ax = ay$ for some x, y in S^1, then we have

$$bx = uax = uay = by.$$

Conversely, if $bx = by$ then we have

$$ax = ay,$$

thus $(a, b) \in \mathcal{L}^*$.

Now suppose $s \ \mathcal{L}^* \ t$, and s, t are both regular. Then there exists $s' \in S$ such that $s = ss's$. Fix $x = 1$ (since $x \in S^1$). Then

$$s \cdot 1 = s \cdot s's \Rightarrow t \cdot 1 = t \cdot s's$$
$$\Rightarrow t = ts' \cdot s.$$

Similarly, if $t = tt't$ for some $t' \in S$ then

$$t \cdot 1 = t \cdot t't \Rightarrow s \cdot 1 = s \cdot t't$$
$$\Rightarrow s = st' \cdot t$$

thus $s \ \mathcal{L} \ t$. □

We have the dual result for $\mathcal{R}^*$:

Proposition 1.11.3. *In a semigroup S, $\mathcal{R} \subseteq \mathcal{R}^*$ and if $s, t \in S$ are regular and $s \ \mathcal{R}^* t$, then $s \ \mathcal{R} \ t$.*

Notice that the above results show that $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$ in a regular semigroup.

Lemma 1.11.4. *Let S and T be two semigroups such that $S \leq T$. Then for all $s, t \in S$*

$$s \ \mathcal{L}^T \ t \Rightarrow s \ \mathcal{L}^{*S} \ t$$

that is, if s and t are $\mathcal{L}$-related in T then s and t are $\mathcal{L}^*$-related in S.

Dually

$$s \ \mathcal{R}^T \ t \Rightarrow s \ \mathcal{R}^{*S} \ t.$$

Proof. Suppose that $s \ \mathcal{L}^T \ t$. Then $us = t, \ vt = s$, for some $u, v \in T^1$. Suppose that $sx = sy$, for some $x, y \in S^1$. Then

$$tx = usx = usy = ty.$$

Similarly, if $tx = ty$ then we have $sx = sy$. Hence $s \ \mathcal{L}^{*S} \ t$. Dually one can prove that $s \ \mathcal{R}^T \ t \Rightarrow s \ \mathcal{R}^{*S} \ t$. □

The next proposition explores another useful connection between $\mathcal{R}^*$ and $\mathcal{R}$.

Proposition 1.11.5. *If S is a semigroup then there is a semigroup T and an embedding $\alpha : S \to T$, in other words, T is an oversemigroup of S, such that for every $s, t \in S$ we have $s \;\mathcal{R}^*\; t$ if and only if $s\alpha \;\mathcal{R}\; t\alpha$.*

Proof. Let S be a semigroup and set $X = S^1$. We need to define a one-one morphism $S \to \mathcal{T}_X$. For $s \in S$, we define $\rho_s \in \mathcal{T}_X$ by

$$x\rho_s = xs \; (\forall x \in X).$$

Next we define the map $\alpha : S \to \mathcal{T}_X$ by $s\alpha = \rho_s$. By Theorem 1.5.2 the map $\alpha : S \to \mathcal{T}_X$ is an embedding. Now suppose that $s \;\mathcal{R}^* t$, that is, for any $x, y \in S^1$

$$xs = ys \Leftrightarrow xt = yt.$$

We have

$$\begin{aligned}(x, y) \in \text{ ker } \rho_s &\Leftrightarrow x\rho_s = y\rho_s \\ &\Leftrightarrow xs = ys \\ &\Leftrightarrow xt = yt \\ &\Leftrightarrow x\rho_t = y\rho_t \\ &\Leftrightarrow (x, y) \in \text{ker } \rho_t.\end{aligned}$$

Therefore, ker $\rho_s =$ ker ρ_t then ker $s\alpha =$ ker $t\alpha$, so by Lemma 1.7.4 we have $s\alpha \;\mathcal{R}\; t\alpha$. The converse part follows from Lemma 1.11.4. □

Note that the proof relies heavily on the fact that $\mathcal{R}$ on $\mathcal{T}_X$ is determined by the kernels, and $\mathcal{R}^*$ is clearly connected to these on $\mathcal{T}_X$. For convenience we now state the dual of Proposition 1.11.5 with the proof.

Proposition 1.11.6. *If S is a semigroup then there is a semigroup T and an embedding $\alpha : S \to T$ such that for every $s, t \in S$ we have $s \;\mathcal{L}^*\; t$ if and only if $s\alpha \;\mathcal{L}\; t\alpha$.*

Proof. Let $S^* = (S, \circ)$ be defined by

$$b \circ a = a \cdot b \; (\forall a, b \in S).$$

It is clear that S^* is also a semigroup and that $s \;\mathcal{L}^*\; t$ in S if and only if $s \;\mathcal{R}^*\; t$ in S^*. By Proposition 1.11.5, there exist a semigroup T and an embedding $\alpha : S^* \to T$ such that for every $s, t \in S^*$ we have $s \;\mathcal{R}^*\; t$ in S^* if and only if $s\alpha \;\mathcal{R}\; t\alpha$ in T. Note that α also maps the underlying set of the semigroup $(S, \cdot)$ to the semigroup $(T^*, \circ)$ (where T^* is defined similarly to S^*), so we only need to check that α is still a morphism from $(S, \cdot)$ to $(T^*, \circ)$. For this reason, let $a, b \in S$. Then

$$(a \cdot b)\alpha = (b \circ a)\alpha = b\alpha \ \cdot a\alpha = a\alpha \circ b\alpha.$$

Hence, S is embedded in T^*. Furthermore, if $s, t \in S$ are such that $s \ \mathcal{L}^* \ t$ in S then $s \ \mathcal{R}^* \ t$ in S^*, so $s\alpha \ \mathcal{R} \ t\alpha$ in T, which implies $s\alpha \ \mathcal{L} \ t\alpha$ in T^*. The converse part follows from Lemma 1.11.4. □

The following example shows that it is not always possible to achieve both of these aims with a single embedding.

Example 1.11.7. Let M be a cancellative monoid which is not embeddable in a group (such monoids exist, as was shown by Malcev in [17]), and let $a, b \in M$. For any $a, b \in M$ we have

$$ax = ay \ \Rightarrow \ x = y \ \Rightarrow \ bx = by.$$

Similarly,

$$bx = by \ \Rightarrow \ x = y \ \Rightarrow \ ax = ay,$$

so we deduce that all elements in M are $\mathcal{L}^*$-related, hence $\mathcal{L}^* = M \times M$. Similarly, we deduce that $\mathcal{R}^* = M \times M$. So $\mathcal{H}^* = M \times M$. Now suppose for contradiction that there exists an embedding $\alpha : M \to S$ (where S is a semigroup) such that

$$a \ \mathcal{L}^* b \text{ in } M \Leftrightarrow a\alpha \ \mathcal{L} \ b\alpha \text{ in } S,$$

$$a \ \mathcal{R}^* b \text{ in } M \Leftrightarrow a\alpha \ \mathcal{R} \ b\alpha \text{ in } S.$$

Since for any $a, b \in M$ we have that $a \ \mathcal{H}^* \ b$ in M, we have that $a\alpha \ \mathcal{H} \ b\alpha$ in S. In particular, let $x \in M$ be arbitrary. Then

$$x \ \mathcal{H}^* x^2 \Rightarrow x\alpha \ \mathcal{H} \ x^2\alpha = \ (x\alpha)^2,$$

so by Theorem 1.10.7, $H_{x\alpha}$ is a group. This means that the image of M lies in a single $\mathcal{H}$-class which is a group . This contradicts the fact that we can not embed M in a group.

Unlike the relations $\mathcal{L}$ and $\mathcal{R}$, the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ need not commute. The next example shows that in general $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$.

Example 1.11.8. We are going to construct a subsemigroup of a *Brandt semigroup* for which $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$. *Brandt semigroups* are special Rees matrix semigroups (see Example 1.2.7), where the non-empty sets I

and Λ are equal and the sandwich matrix is the identity matrix. That is, $B(G, I) = \mathcal{M}^0[G; I, I; P]$, where P is defined by

$$p_{ij} = \begin{cases} e & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $B(\{e\}, 5)$ be the Brandt semigroup over the trivial group $G = \{e\}$, where $I = \{1, 2, 3, 4, 5\}$. Let

$$T = \{(i, e, j) | i \leq j\} \cup \{0\}$$

be a subset of $B(\{e\}, 5)$. First, we check that T is indeed a subsemigroup of $B(\{e\}, 5)$. We need to prove that

$$a, b \in T \Rightarrow ab \in T.$$

Let (i, e, j) and (k, e, l) be arbitrary non-zero elements of T. Then

$$(i, e, j)(k, e, l) = \begin{cases} (i, e, l) & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

if $(i, e, j)(k, e, l) = 0$, then we are done, on the other hand if

$$(i, e, j)(k, e, l) = (i, e, l),$$

then $j = k$, and so

$$i \leq j = k \leq l \Rightarrow i \leq l,$$

hence $(i, e, l) \in T$. Thus T is a subsemigroup of $B(\{e\}, 5)$. Secondly, we need to characterise $\mathcal{R}^*$ and $\mathcal{L}^*$ in T. It is easy to see that $\{0\}$ is always an $\mathcal{R}^*$-class and an $\mathcal{L}^*$-class. For any non-zero $(i, e, \lambda), (j, e, \mu)$ in T we claim that

$$(i, e, \lambda) \ \mathcal{R}^* \ (j, e, \mu) \text{ if and only if } \ i = j;$$

$$(i, e, \lambda) \ \mathcal{L}^* \ (j, e, \mu) \text{ if and only if } \ \lambda = \mu.$$

Proof. Suppose that $(i, e, \lambda) \ \mathcal{R}^* \ (j, e, \mu)$ in T. If

$$(i, e, i)(i, e, \lambda) = 1 \cdot (i, e, \lambda)$$

then

$$(i, e, i)(j, e, \mu) = 1 \cdot (j, e, \mu).$$

Then $i = j$, since $1 \cdot (j, e, \mu) \neq 0$.

Conversely, suppose that $i = j$, then by Lemma 1.7.5, $(i, e, \lambda) \ \mathcal{R} \ (j, e, \mu)$ in $B(\{e\}, 5)$, then by Lemma 1.11.4, $(i, e, \lambda) \ \mathcal{R}^* \ (j, e, \mu)$ in T. The proof for $\mathcal{L}^*$ is dual. $\square$

Finally, let $a = (1, e, 1), b = (5, e, 5) \in T$, we have

$$(a, b) \in \mathcal{R}^* \circ \mathcal{L}^*,$$

since for $c = (1, e, 5) \in T$ we have that

$$a \; \mathcal{R}^* c \; \mathcal{L}^* b.$$

On the other hand, $(a, b) \notin \mathcal{L}^* \circ \mathcal{R}^*$, since there is no such $c' \in T$ with

$$a \; \mathcal{L}^* c' \; \mathcal{R}^* b$$

(clearly, c' must be $(5, e, 1) \notin T$). Hence, $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$ in T.

In section 1.7 we characterised the Green's relations of $\mathcal{T}_X$. In the following example we are going to characterise the Green's relations and the Green's *- relations in a subsemigroup of $\mathcal{T}_X$.

Example 1.11.9. Let X be a set and $A \subset X$ (that is, $A \neq X$) such that $|A| \gneq 1$. Define

$$S = \{\alpha \in \mathcal{T}_X : \text{ im } \alpha \subseteq A\}.$$

We are going to determine $\mathcal{L}, \mathcal{R}, \mathcal{L}^*$, and $\mathcal{R}^*$ in S. To determine $\mathcal{L}$ in S, we know that in $\mathcal{T}_X$ any two elements are $\mathcal{L}$-related if and only if they have the same images, however, this is not a sufficient condition in S, so we have to find extra conditions.

Precisely, for all α, β in S, $\alpha \; \mathcal{L} \; \beta$ if and only if $\alpha = \beta$ or:

1. im $\alpha =$ im β and
2. $\forall a \in \text{im } \alpha = \text{im } \beta, \exists b, c \in A$ such that $b\alpha = c\beta = a$.

Proof. Suppose that $\alpha \; \mathcal{L} \; \beta$ in S and $\alpha \neq \beta$, then $\alpha \; \mathcal{L} \; \beta$ in $\mathcal{T}_X$. Therefore by Lemma 1.7.4, im $\alpha =$ im β. Let $\gamma \in S$ be such that $\beta = \gamma\alpha$.

To show that the second condition holds, let $a \in \text{im } \alpha = \text{im } \beta$. Then $a = x\beta = x\gamma\alpha$ for some $x \in X$, so if we let $b = x\gamma \in A$, then $b\alpha = x\beta = a$.

Conversely, suppose that Conditions 1 and 2 hold. For each $x \in X$ we have $x\beta \in \text{im } \beta = \text{im } \alpha$, so by Condition 2, there exists $b_x \in A$ such that $x\beta = b_x\alpha$. Then we define $\gamma : X \to A$ by

$$x\gamma = b_x,$$

for all $x \in X$. Then $\gamma\alpha = \beta$. Similarly, we can define $\pi \in S$ such that $\pi\beta = \alpha$. Hence $\alpha \; \mathcal{L} \; \beta$ in S. □

To determine $\mathcal{R}$ in S, we have the same condition that holds for two elements to be $\mathcal{R}$-related in $\mathcal{T}_X$, that is, for any α, β in S

$$\alpha \ \mathcal{R} \ \beta \Leftrightarrow \text{ ker } \alpha = \text{ker } \beta.$$

Proof. Suppose that $\alpha \ \mathcal{R} \ \beta$ in S then $\alpha \ \mathcal{R} \ \beta$ in $\mathcal{T}_X$. Therefore by Lemma 1.7.4, ker $\alpha =$ ker β.

Conversely, suppose that ker $\alpha =$ ker β. Let $x \in \text{im } \alpha$, and for all $y \in X$ such that $y\alpha = x$, define $x\gamma = y\beta$. For each $x \notin \text{im } \alpha$, let $x\gamma \in A$ be arbitrary. Then

$$y\alpha\gamma = x\gamma = y\beta \quad (\forall y \in X)$$

It is clear that $x\gamma$ is well-defined since, if $y\alpha = y'\alpha$, then

$$(y, y') \in \text{ker } \alpha \subseteq \text{ker } \beta \text{ so } y\beta = y'\beta.$$

Hence $\alpha\gamma = \beta$. Similarly, we can define $\pi \in S$ such that $\beta\pi = \alpha$. Hence $\alpha \ \mathcal{R} \ \beta$. □

To determine $\mathcal{L}^*$ in S, we claim that

$$\alpha \ \mathcal{L}^*\beta \text{ in } S \Leftrightarrow \text{im } \alpha = \text{im } \beta.$$

Proof. Suppose that im $\alpha =$ im β. Then $\alpha \ \mathcal{L} \ \beta$ in $\mathcal{T}_X$, then by Lemma 1.11.4, $\alpha \ \mathcal{L}^*\beta$ in S.

Conversely, suppose that $\alpha \ \mathcal{L}^*\beta$. Let $\gamma : X \to A$ be defined by

$$x\gamma = \begin{cases} x & \text{if } x \in \text{im } \alpha, \\ a_0 & \text{if } x \notin \text{im } \alpha. \end{cases}$$

where $a_0 \in A$ is fixed. Then $\alpha\gamma = \alpha \cdot 1$ and so as $\alpha \ \mathcal{L}^* \ \beta$ we have $\beta\gamma = \beta \cdot 1$, and thus

$$\text{im } \beta = \text{im } (\beta\gamma) \subseteq \text{im } \gamma = \text{im } \alpha.$$

Similarly one can show that im $\alpha \subseteq$ im β, so that im $\alpha =$ im β. □

Notice that in the previous example, we have $\mathcal{L} \neq \mathcal{L}^*$ in S. To show this let $A = \{1, 2, 3\}$ be a subset of $X = \{1, 2, 3, 4, 5\}$. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 2 & 3 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 1 & 2 \end{pmatrix}.$$

Obviously im $\alpha =$ im β, so $\alpha \ \mathcal{L}^*\beta$. However α and β are not $\mathcal{L}$-related, since there is no such element γ in S with $\gamma\alpha = \beta$, because $\nexists c \in A$ such that $c\alpha = 3$. So Condition 2 is not satisfied.

Chapter 2

Ample and restriction semigroups

2.1 Ample and restriction semigroups

Our plan is to generalise certain results of inverse semigroup theory to a wider class of semigroups, namely to restriction semigroups, which were first introduced by Gould (see [10]). In the structure theory of inverse semigroups, there are two approaches to build up inverse semigroups. That is, inverse semigroups can be defined both in structural terms (e.g. semigroups having unique idempotents in $\mathcal{R}$- and $\mathcal{L}$-classes) and in algebraic terms (e.g. semigroups having an additional unary operation, and satisfying certain identities). Similarly, there are two possible ways to define ample and restriction semigroups. Here we include both ways and indicate how they are related to each other. We shall provide some more relations to study non-regular semigroups. Before defining ample semigroups, we need to say what is meant by an abundant semigroup. The latter two classes were introduced by Fountain in [6]. Most of the results in this chapter are folklore.

Definition 2.1.1. A semigroup S is *left abundant* if every $\mathcal{R}^*$-class contains an idempotent. A semigroup S is *left adequate* if it is *left abundant* and idempotents commute. In a *left adequate* semigroup every $\mathcal{R}^*$-class contains a unique idempotent, and we denote by a^+ the unique idempotent in the $\mathcal{R}^*$-class of a.

We define *right abundant* and *right adequate* semigroups dually and for an element a in a right adequate semigroups we denote the unique idempotent in the $\mathcal{L}^*$-class of a by a^*. Also we say that a semigroup is *abundant (adequate)* if it is both *right* and *left abundant (adequate)*. We summarise some properties of adequate semigroups in the following lemma.

Lemma 2.1.2. *For elements a, b of an adequate semigroup S, we have*

(i) $a\ \mathcal{R}^*b$ *if and only if* $a^+ = b^+$ ($a\ \mathcal{L}^*b$ *if and only if* $a^* = b^*$);

(ii) $(ab)^* = (a^*b)^*$; $(ab)^+ = (ab^+)^+$;

(iii) $aa^* = a = a^+a$.

Now we are going to give the structural definition of ample semigroups:

Definition 2.1.3. A left adequate semigroup is *left ample* if it satisfies the left ample condition. That is, $ae = (ae)^+a$ for every $a \in S$ and $e \in E(S)$.

Dually we say S is a *right ample* semigroup if the *right ample condition* $ea = a(ea)^*$ is satisfied.

We say a semigroup is an *ample* semigroup if it is both a left and a right ample semigroup.

In the following Lemmas we give an alternative definition of (left/right) ample semigroups:

Lemma 2.1.4. *[10] Let S be a $(2,1)$-semigroup satisfying the following identities: for all $a, b, c \in S$ and $E = \{a^+ : a \in S\}$ we have*

$$a^+b^+ = b^+a^+,\ a^+a = a,$$

$$(a^+)^+ = a^+,\ (ab)^+ = (ab^+)^+,\ ab^+ = (ab^+)^+a,$$

and

$$a^2 = a \Rightarrow a = a^+,$$

$$ac = bc \Rightarrow ac^+ = bc^+.$$

Then S is a left ample semigroup.

Conversely, if S is a left ample semigroup. Then for any $a, b, c \in S$, and $E = \{a^+ : a \in S\}$, S satisfies the following identities

$$a^+b^+ = b^+a^+,\ a^+a = a,$$

$$(a^+)^+ = a^+,\ (ab)^+ = (ab^+)^+,\ ab^+ = (ab^+)^+a,$$

and

$$a^2 = a \Rightarrow a = a^+,$$

$$ac = bc \Rightarrow ac^+ = bc^+.$$

Proof. Suppose that the identities hold. Then by making use of the implication $ac = bc \Rightarrow ac^+ = bc^+$, one can show that for all $a, b \in S$ we have

$$a \,\mathcal{R}^* b \text{ if and only if } a^+ = b^+.$$

Also it is easy to check that the given identities imply that $E(S)$ is a semilattice. To see that S satisfies the left congruence condition, suppose that $(ab)^+ = (ab^+)^+$ and $a \,\mathcal{R}^*\, b$, by Lemma 2.1.2, $a^+ = b^+$ and by the given identities we have (for some $c \in S$)

$$(ca)^+ = (ca^+)^+ = (cb^+)^+ = (cb)^+,$$

thus $ca \,\mathcal{R}^*\, cb$. The result follows.

Conversely, the proof is straightforward as one can easily deduce the identities from the fact that S is left ample. However, we prove the most complicated one. Let $a, b \in S$. As $b \,\mathcal{R}^*\, b^+$, and S satisfies the left congruence condition we have that $ab \,\mathcal{R}^*\, ab^+$. Then by Lemma 2.1.2, $(ab)^+ = (ab^+)^+$. □

For convenience we now state the dual of Lemma 2.1.4.

Lemma 2.1.5. *[10] Let S be a $(2, 1)$-semigroup satisfying the following identities: for all $a, b, c \in S$ and $E = \{a^* : a \in S\}$ we have*

$$a^*b^* = b^*a^*, \ aa^* = a,$$

$$(a^*)^* = a^*, \ (ab)^* = (a^*b)^*, \ b^*a = a(b^*a)^*,$$

and

$$a^2 = a \Rightarrow a = a^*,$$

$$ca = cb \Rightarrow c^*a = c^*b.$$

Then S is a right ample semigroup.

Conversely, if S a is right ample semigroup. Then for any $a, b, c \in S$, and $E = \{a^* : a \in S\}$, S satisfies the following identities

$$a^*b^* = b^*a^*, \ aa^* = a,$$

$$(a^*)^* = a^*, \ (ab)^* = (a^*b)^*, \ b^*a = a(b^*a)^*,$$

and

$$a^2 = a \Rightarrow a = a^*,$$

$$ca = cb \Rightarrow c^*a = c^*b.$$

As a consequence of Lemma 2.1.4 and Lemma 2.1.5 one can say that the $(2,1,1)$-semigroup $(S,\cdot,^+,^*)$ is ample if $(S,\cdot,^+)$ is left ample and $(S,\cdot,^*)$ is right ample.

We now state a useful corollary of Lemma 2.1.4.

Corollary 2.1.6. *Let S be a $(2,1)$-subsemigroup of a left ample semigroup. Then S is also left ample.*

Proof. Clearly a $(2,1)$-subsemigroup also satisfies the conditions appearing in Lemma 2.1.4. □

The following proposition shows that the notion of an ample semigroup generalises that of an inverse semigroup.

Proposition 2.1.7. *Inverse semigroups are ample semigroups.*

Proof. We only prove that inverse semigroups are left ample semigroups, as the proof that they are right ample is dual.

Suppose that S is an inverse semigroup, then S is regular so $\mathcal{R} = \mathcal{R}^*$ and every $\mathcal{R}^*$-class contains an idempotent. Furthermore, idempotents commute, so S is left adequate. It remains to show that S satisfies the left ample condition, that is $ae = (ae)^+a$ for all $e \in E(S)$ and $a \in S$. It is clear that $a^+ = aa^{-1}$ is the unique idempotent in R^*_a, and since S is an inverse semigroup we have $e = e^{-1}$ and $a = aa^{-1}a$. Then

$$\begin{aligned} ae = aa^{-1}ae &= a(a^{-1}a)e \\ &= ae(a^{-1}a) \\ &= (aea^{-1})a \\ &= (ae)^+a, \end{aligned}$$

that is, the left ample condition holds. □

We know that $\mathcal{I}_X$ is an inverse semigroup (Theorem 1.9.5). One can see in the following proposition that every left ample semigroup can be embedded in $\mathcal{I}_X$ for some set X. The proof is similar to the proof of Theorem 6.2 in [10].

Proposition 2.1.8. *Left ample semigroups are exactly the (2,1)-semigroups which are embeddable in symmetric inverse semigroups. That is, S is left ample if and only if there exists an injective (2,1)-morphism $\theta : S \to \mathcal{I}_X$ for some set X (on $\mathcal{I}_X$, $\alpha^+ = \alpha\alpha^{-1} = \mathrm{id}_{\mathrm{dom}\ \alpha}$).*

Proof. Suppose that S is a left ample semigroup. Take $X = S$, then we define

$$\theta : S \to \mathcal{I}_S$$

by

$$s\theta = \rho_s \text{ for } s \in S$$

where

$$\text{dom } \rho_s = Ss^+, \; x\rho_s = xs \text{ for all } x \in \text{dom } \rho_s.$$

To see that $\rho_s \in \mathcal{I}_S$, let $x, y \in$ dom ρ_s. If $x\rho_s = y\rho_s$ then $xs = ys$ so $xs^+ = ys^+$ as $s \; \mathcal{R}^* s^+$

$$x = xs^+ = ys^+ = y.$$

So ρ_s is one-one. Also we can see the image is

$$\text{im } \rho_s = (Ss^+)\rho_s = Ss^+s = Ss.$$

We need to show that θ is a one-one (2,1)-morphism.

$$(s\theta)^+ = (\rho_s)^+ = id_{\text{dom } \rho_s} = id_{Ss^+} = s^+\theta,$$

so θ is a $^+$-morphism. To show that θ is a semigroup morphism, that is, for all $s, t \in S$

$$(st)\theta = s\theta t\theta,$$

it is enough to show that dom ρ_{st} = dom $\rho_s\rho_t$. Since

$$\rho_{st} : \text{dom } \rho_{st} \to \text{im } \rho_{st}, \; x \mapsto x(st)$$

on the other hand we have

$$\rho_s\rho_t : \text{dom } \rho_s\rho_t \to \text{im } \rho_s\rho_t, \; x \mapsto x(st).$$

Let $x \in$ dom ρ_{st}. Then $x(st)^+ = x$ so that $xs^+ = x(st)^+s^+ = x(st)^+ = x$ and $x \in$ dom ρ_s. Now $x\rho_s = xs = x(st)^+s = x(st^+)^+s = xst^+ \in$ dom ρ_t so that $x \in$ dom $\rho_s\rho_t$.

Conversely, let $x \in$ dom $\rho_s\rho_t$. Then $x \in$ dom ρ_s and $x\rho_s = xs \in$ dom ρ_t. Hence $x \in Ss^+$ and $xs \in St^+$. Then by using the ample condition we have

$$x(st)^+ = (x(st)^+)^+x = (xst)^+x = (xst^+)^+x$$

$$= (xs)^+x = (xs^+)^+x = x^+x = x.$$

Hence $x \in S(st)^+ = \text{dom } \rho_{st}$. Therefore dom $\rho_{st} = \text{dom } \rho_s\rho_t$. Finally, to show that θ is one-one, suppose that $s\theta = t\theta$ for some s, t in S. Then

$$\begin{aligned} \text{dom } \rho_s = \text{dom } \rho_t &\Rightarrow Ss^+ = St^+ \\ &\Rightarrow s^+ = t^+ \end{aligned}$$

since $s^+ \mathcal{L}\ t^+$ and idempotents commute. Hence

$$\begin{aligned} s^+ = t^+ \in \text{dom } \rho_s = \text{dom } \rho_t &\Rightarrow s^+\rho_s = t^+\rho_t \\ &\Rightarrow s^+s = t^+t \\ &\Rightarrow s = t. \end{aligned}$$

Conversely, suppose that there exists an injective (2,1)-morphism $\theta : S \to \mathcal{I}_S$. That is, im $\theta \subseteq \mathcal{I}_S$, and im θ is a (2,1)-subsemigroup of $\mathcal{I}_S$, and $S \cong$ im θ as a $(2,1)$-semigroup. By Lemma 2.1.6 a $(2,1)$-subsemigroup of $\mathcal{I}_X$ is also a left ample semigroup, so S is left ample. □

Notice that one can easily see that the cancellative monoid M, introduced in Example 1.11.7 is a special ample semigroup with one idempotent. So that similarly to Proposition 1.11.6 and Example 1.11.7 one can show that it is not possible to achieve both of the aims with a single embedding. In general ample semigroups cannot be embedded in $\mathcal{I}_X$ as $(2,1,1)$-subsemigroups.

2.2 Relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$

There are two ways to introduce restriction semigroups. We are going to explain both definitions in this section. Here we shall provide and give a careful definition of the relations $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ that we are going to use instead of $\mathcal{L}^*$ and $\mathcal{R}^*$. Let S be a semigroup and let $E \subseteq E(S)$. Define the relation

$$\widetilde{\mathcal{L}}_E = \{(a, b) \in S \times S : ae = a \Leftrightarrow be = b \text{ for all } e \in E\}.$$

The relation $\widetilde{\mathcal{R}}_E$ is defined dually. We define

$$\widetilde{\mathcal{H}}_E = \widetilde{\mathcal{L}}_E \cap \widetilde{\mathcal{R}}_E \text{ and } \widetilde{\mathcal{D}}_E = \widetilde{\mathcal{L}}_E \vee \widetilde{\mathcal{R}}_E.$$

For convenience we denote the $\widetilde{\mathcal{R}}_E$-class ($\widetilde{\mathcal{L}}_E$-class, $\widetilde{\mathcal{H}}_E$-class, $\widetilde{\mathcal{D}}_E$-class) of any $a \in S$ by $\widetilde{R}^a_E$ ($\widetilde{L}^a_E, \widetilde{H}^a_E, \widetilde{D}^a_E$).

Proposition 2.2.1. *Let S be a semigroup and $E \subseteq E(S)$, and let $a \in S$, $e \in E$ then the following statements are equivalent:*

(i) $e \, \widetilde{\mathcal{L}}_E \, a$;

(ii) $ae = a$ *and for all* $f \in E$, $af = a \Rightarrow ef = e$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $e \, \widetilde{\mathcal{L}}_E \, a$. We have for all $f \in E$,

$$af = a \Rightarrow ef = e.$$

Furthermore, $e \in E$, so

$$ee = e \Rightarrow ae = a.$$

(ii) $\Rightarrow$ (i) Suppose that $ae = a$ and $\forall f \in E$,

$$af = a \Rightarrow ef = e.$$

Suppose $ef = e$. Then

$$af = aef = ae = a$$

and so $e \, \widetilde{\mathcal{L}}_E \, a$ as required. □

The dual statement is true for $\widetilde{\mathcal{R}}_E$.

Proposition 2.2.2. *Let S be a semigroup, $E \subseteq E(S)$ and let $a \in S$, $e \in E$. Then the following statements are equivalent:*

(i) $a \, \widetilde{\mathcal{R}}_E \, e$;

(ii) $ea = a$ *and for all* $f \in E$, $fa = a \Rightarrow fe = e$.

As a result we have the following Corollary:

Corollary 2.2.3. *Let* $a \in S$*, where* S *is a semigroup and let* $E \subseteq E(S)$ *be a semilattice. Then* a *is* $\widetilde{\mathcal{L}}_E$*-related to at most one idempotent in* E*.*

Proof. Suppose that we have $e, f \in E$ such that $e\ \widetilde{\mathcal{L}}_E\ a$ and $f\ \widetilde{\mathcal{L}}_E\ a$, so $e\ \widetilde{\mathcal{L}}_E\ f$. Then by Proposition 2.2.1, $ef = e$ and $fe = f$ and so

$$e = ef = fe = f.$$

□

In a similar way to the *-relations, the $^\sim$-relations are also related to Green's relations as follows:

Proposition 2.2.4. *In any semigroup* S *we have* $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$*. If* S *is regular, and* $E = E(S)$ *then* $\widetilde{\mathcal{R}}_E \subseteq \mathcal{R}$ *and so* $\widetilde{\mathcal{R}}_E \subseteq \mathcal{R}^*$*.*

Dually we have $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_E$*, and if* S *is regular, and* $E = E(S)$ *then* $\widetilde{\mathcal{L}}_E \subseteq \mathcal{L}$ *and so* $\widetilde{\mathcal{L}}_E \subseteq \mathcal{L}^*$*.*

Proof. Suppose that $(a, b) \in \mathcal{R}$, then $\exists u, v \in S$ with $au = b,\ bv = a$ (for any $e \in E$) we have

$$ea = a \Rightarrow eb = eau = au = b.$$

Similarly,

$$eb = b \Rightarrow ea = ebv = bv = a.$$

Therefore $(a, b) \in \widetilde{\mathcal{R}}_E$. That is, $\mathcal{R} \subseteq \widetilde{\mathcal{R}}_E$.

Suppose that $a\ \mathcal{R}^*\ b$ and $e \in E$. Then we have $ua = va$ if and only if $ub = vb$. Then by letting $u = e$ and $v = 1$ in the definition of $\mathcal{R}^*$, we see that $a\ \widetilde{\mathcal{R}}_E\ b$. Then $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_E$.

Now suppose that S is regular and that $E = E(S)$. Let $a\ \widetilde{\mathcal{R}}_E\ b$, then since a and b are regular, there exist $x, y \in S$ such that $a = axa$, and $b = byb$. It is clear that ax and by are both idempotents in S. Then, since $a\ \widetilde{\mathcal{R}}_E\ b$,

$$a = ax \cdot a \Rightarrow b = ax \cdot b = a \cdot xb$$

and similarly,

$$b = by \cdot b \Rightarrow a = by \cdot a = b \cdot ya.$$

Hence $a\ \mathcal{R}\ b$. Thus, $\mathcal{R} = \widetilde{\mathcal{R}}_E$. By Proposition 1.11.3 in a regular semigroup $\mathcal{R}^* = \mathcal{R}$. Thus $\widetilde{\mathcal{R}}_E = \mathcal{R}^*$ as required.

The proof for $\mathcal{L}$-relation is dual. □

Note that in the previous proof the fact that $E = E(S)$ was crucial. In general, the choice of E strongly influences $\widetilde{\mathcal{L}}_E$, and $\widetilde{\mathcal{R}}_E$, for example if M is a monoid with identity e , then $\widetilde{\mathcal{L}}_{\{e\}} = \widetilde{\mathcal{R}}_{\{e\}} = M \times M$.

Proposition 2.2.5. *Let S be a semigroup and let $E \subseteq E(S)$. If $e, f \in E$ then*

$$e \; \mathcal{R} \; f \Leftrightarrow e \; \widetilde{\mathcal{R}}_E \; f.$$

Dually,

$$e \; \mathcal{L} \; f \Leftrightarrow e \; \widetilde{\mathcal{L}}_E \; f.$$

Proof. Suppose that $e \; \widetilde{\mathcal{R}}_E \; f$, then

$$e \cdot e = e \Rightarrow e \cdot f = f,$$

$$f \cdot f = f \Rightarrow f \cdot e = e.$$

Hence $e \; \mathcal{R} \; f$. The converse follows from Proposition 2.2.4. The proof for $\mathcal{L}$ is dual. □

The relations $\mathcal{R}^*$ and $\widetilde{\mathcal{R}}_E$ also turn out to be equal on a left adequate semigroup.

Proposition 2.2.6. *[3] If S is left adequate, then*

$$\mathcal{R}^* = \widetilde{\mathcal{R}}_{E(S)}.$$

Definition 2.2.7. Let S be a semigroup and E be a set of idempotents of S. Then S satisfies the *left congruence condition* with respect to E if $\widetilde{\mathcal{R}}_E$ is a left congruence. Dually one can define the *right congruence condition.* We say a semigroup S satisfies the *congruence condition* if it satisfies the left and right congruence conditions.

Similarly to the ample case, we have two ways to introduce restriction semigroups. Here is the structural way to define restriction semigroups:

Definition 2.2.8. A semigroup S with a distinguished set of idempotents $E \subseteq E(S)$ is *left restriction* if

(i) E is a semilattice;

(ii) $\widetilde{\mathcal{R}}_E$ is a left congruence;

(iii) every $\widetilde{\mathcal{R}}_E$-class contains exactly one idempotent (if $a \in S$ we denote the unique idempotent in $\widetilde{R}_a$ by a^+, where $\widetilde{R}_a$ is the $\widetilde{\mathcal{R}}_E$-class containing the element a);

(iv) the left ample condition is satisfied, that is, $ae = (ae)^+a$ for every $a \in S$ and $e \in E$. This is equivalent to $ab^+ = (ab^+)^+a$ for every $a, b \in S$.

Dually we can define *right restriction* semigroups: we denote the unique idempotent in $\widetilde{L}_a$ by a^*. We say that S is a *restriction* semigroup if it is both a left and a right restriction semigroup with respect to the same semilattice $E \subseteq E(S)$.

Throughout this thesis we also use an equivalent definition for a restriction semigroup as it is given in the following lemmas:

Lemma 2.2.9. *[10] A (2,1)-semigroup $(S, \cdot, ^+)$ is a left restriction semigroup if and only if S satisfies the identities*

$$a^+a = a,\ a^+b^+ = b^+a^+,\ (a^+b)^+ = a^+b^+,\ ab^+ = (ab)^+a.$$

Lemma 2.2.10. *[10] A $(2,1)$-semigroup $(S, \cdot, ^*)$ is a right restriction semigroup if and only if S satisfies the identities*

$$aa^* = a,\ a^*b^* = b^*a^*,\ (ab^*)^* = a^*b^*,\ a^*b = a(ab)^*.$$

We say that S is a restriction semigroup if it is both a left and a right restriction semigroup, and also satisfies the identities

$$(a^+)^* = a^+, \quad \text{and} \quad (a^*)^+ = a^*.$$

As a result we can see that

$$E = \{a^+ : a \in S\} = \{a^* : a \in S\}.$$

Note the connection between the two approaches (Lemmas 2.2.9, Lemma 2.2.10 and Definition 2.2.8): if S is restriction in the first sense (Definition 2.2.8) then it is easy to see that the identities hold. On the other hand, if S is a $(2, 1, 1)$-semigroup satisfying these identities in Lemmas 2.2.9 and Lemma 2.2.10 then it is easy to see that if we set

$$E = \{a^+ : a \in S\} = \{a^* : a \in S\},$$

then E is a semilattice, and for all $a, b \in S$ we have

$$a\ \widetilde{\mathcal{R}}_E\ b \Leftrightarrow a^+ = b^+$$

and that a^+ is the unique element of E that is $\widetilde{\mathcal{R}}_E$-related to a, and dually for $\widetilde{\mathcal{L}}_E$.

Corollary 2.2.11. *Let S be a $(2,1)$-subsemigroup of a left restriction semigroup. Then S is also left restriction.*

From the definition of restriction semigroups one can formulate the following proposition.

Proposition 2.2.12. *Let S be an inverse semigroup. Then S is a restriction semigroup with respect to the semilattice $E(S)$.*

Proof. Let S be an inverse semigroup and let $E = E(S)$ be its semilattice. Let $a \in S$. We use the fact that in inverse semigroup $\widetilde{\mathcal{R}}_E = \mathcal{R}$. Thus $a \, \widetilde{\mathcal{R}}_E \, aa^{-1}$. Since E is a semilattice, $aa^{-1}(= a^+)$ is the unique idempotent in $\widetilde{R}_a$. Also by Proposition 1.11.1 $\widetilde{\mathcal{R}}_E$ is a left congruence.

Dually, one can use the fact that $\widetilde{\mathcal{L}}_E = \mathcal{L}$ to deduce the requested conditions. Also in a similar way to the proof of Proposition 2.1.7, one can show that the ample conditions hold. Then the result follows. □

Partial transformation monoids play a similar role in the theory of restriction semigroups as the symmetric inverse monoids do in the theory of inverse (ample) semigroups. To show this, first we need to characterise the $\widetilde{\ }$-relation on them. We claim that for $\alpha, \beta \in \mathcal{PT}_X$ and $E = \{\mathrm{id}_Y : Y \subseteq X\}$,

(i) $\alpha \, \widetilde{\mathcal{R}}_E \, \beta \Leftrightarrow \mathrm{dom}\ \alpha = \mathrm{dom}\ \beta$;

(ii) $\alpha \, \widetilde{\mathcal{L}}_E \, \beta \Leftrightarrow \mathrm{im}\ \alpha = \mathrm{im}\ \beta$.

Proof. (i) Suppose that $\alpha \, \widetilde{\mathcal{R}}_E \, \beta$. Let $Y = \mathrm{dom}\ \alpha$. Then $\mathrm{id}_Y \cdot \alpha = \alpha$ and so as $\alpha \, \widetilde{\mathcal{R}}_E \, \beta$, $\mathrm{id}_Y \cdot \beta = \beta$ which gives $\mathrm{dom}\ \beta \subseteq Y = \mathrm{dom}\ \alpha$.

Similarly, one can show that $\mathrm{dom}\ \alpha \subseteq \mathrm{dom}\ \beta$. Thus

$$\mathrm{dom}\ \alpha = \mathrm{dom}\ \beta.$$

Conversely, suppose that $\mathrm{dom}\ \alpha = \mathrm{dom}\ \beta = Y$, then we have

$$\begin{aligned} \mathrm{id}_Y \cdot \alpha = \alpha &\Leftrightarrow \mathrm{dom}(\mathrm{id}_Y \cdot \alpha) = \mathrm{dom}\ \alpha \\ &\Leftrightarrow \mathrm{im}\ \mathrm{id}_Y \cap \mathrm{dom}\ \alpha = \mathrm{dom}\ \alpha \\ &\Leftrightarrow \mathrm{im}\ \mathrm{id}_Y \cap \mathrm{dom}\ \beta = \mathrm{dom}\ \beta \\ &\Leftrightarrow \mathrm{dom}(\mathrm{id}_Y \cdot \beta) = \mathrm{dom}\ \beta \\ &\Leftrightarrow \mathrm{id}_Y \cdot \beta = \beta. \end{aligned}$$

(ii) Suppose that $\alpha \widetilde{\mathcal{L}}_E \beta$. Let $Y = \text{im } \alpha$. Then $\alpha \cdot \text{id}_Y = \alpha$ and so as $\alpha \widetilde{\mathcal{L}}_E \beta$, $\beta \cdot \text{id}_Y = \beta$ which gives $\text{im } \beta \subseteq Y = \text{im } \alpha$.

Similarly, one can show that $\text{im } \alpha \subseteq \text{im } \beta$. Thus

$$\text{im } \alpha = \text{im } \beta.$$

Conversely, suppose that $\text{im } \alpha = \text{im } \beta = Y \subseteq Y$, then we have

$$\begin{aligned} \alpha \cdot \text{id}_Y = \alpha &\Leftrightarrow \text{im}(\alpha \cdot \text{id}_Y) = \text{im } \alpha \\ &\Leftrightarrow \text{im } \alpha \cap \text{dom id}_Y = \text{im } \alpha \\ &\Leftrightarrow \text{im } \beta \cap \text{dom id}_Y = \text{im } \beta \\ &\Leftrightarrow \text{im}(\beta \cdot \text{id}_Y) = \text{im } \beta \\ &\Leftrightarrow \beta \cdot \text{id}_Y = \beta. \end{aligned}$$

□

Now we can state the following useful result (note that we cannot apply Proposition 2.2.4, because $E(\mathcal{PT}_X) \neq \{\text{id}_Y : Y \subseteq X\}$).

Proposition 2.2.13. *The partial transformation monoids $\mathcal{PT}_X$ are left restriction semigroups where*

$$E = \{\text{id}_Y : Y \subseteq X\}.$$

Proof. Let $\alpha \in \mathcal{PT}_X$, then obviously $\text{id}_{\text{dom } \alpha}$ is the unique element of E $\widetilde{\mathcal{R}}_E$-related to α, so $\alpha^+ = \text{id}_{\text{dom } \alpha}$. Now let $\alpha, \beta, \gamma \in \mathcal{PT}_X$, and suppose $\alpha \widetilde{\mathcal{R}}_E \beta$. Then

$$\begin{aligned} \text{dom } \gamma\alpha &= (\text{im } \gamma \cap \text{dom } \alpha)\gamma^{-1} \\ &= (\text{im } \gamma \cap \text{dom } \beta)\gamma^{-1} \\ &= \text{dom } \gamma\beta \end{aligned}$$

since $\text{dom } \alpha = \text{dom } \beta$. Therefore $\gamma\alpha \widetilde{\mathcal{R}}_E \gamma\beta$, thus $\widetilde{\mathcal{R}}_E$ is a left congruence. Finally, let $\alpha \in \mathcal{PT}_X$ and $\text{id}_Y \in E$. Since $(\alpha \cdot \text{id}_Y)^+$ and id_Y are both identity maps, it is easy to see that $(\alpha \cdot \text{id}_Y)^+\alpha = \alpha \cdot \text{id}_Y$ if and only if $\text{dom } (\alpha \cdot \text{id}_Y)^+\alpha = \text{dom } \alpha \cdot \text{id}_Y$. Then

$$\begin{aligned} \text{dom}(\alpha \cdot \text{id}_Y)^+\alpha &= (\text{im } (\alpha \cdot \text{id}_Y)^+ \cap \text{dom } \alpha)((\alpha \cdot \text{id}_Y)^+)^{-1} \\ &= \text{dom } \alpha \cdot \text{id}_Y \cap \text{dom } \alpha \\ &= \text{dom } \alpha \cdot \text{id}_Y. \ [\text{since dom } \alpha \cdot \text{id}_Y \subseteq \text{dom } \alpha] \end{aligned}$$

Thus the left ample condition holds. Therefore $\mathcal{PT}_X$ is a left restriction semigroup. □

The following example shows that the partial transformation monoids $\mathcal{PT}_X$ are not right restriction if $|X| > 1$ and $E = \{\mathrm{id}_Y : Y \subseteq X\}$.

Example 2.2.14. Let $X = \{1, 2\}$ and $A = \{1\}$ be a subset of X. Let $\alpha, \beta \in \mathcal{PT}_X$ then

$$\alpha \ \widetilde{\mathcal{L}}_E \ \beta \Leftrightarrow \mathrm{im}\ \alpha = \mathrm{im}\ \beta,$$

so $\alpha^* = \mathrm{id}_{\mathrm{im}\ \alpha}$. Now let

$$\alpha = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \text{ and } \mathrm{id}_Y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then $\mathrm{id}_Y \cdot \alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $(\mathrm{id}_Y \cdot \alpha)^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
however, $\alpha \cdot (\mathrm{id}_Y \cdot \alpha)^* = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \neq \mathrm{id}_Y \cdot \alpha$. Hence the right ample condition does not hold.

Now we state a similar version of Proposition 2.1.8 for left restriction semigroups. The proof is similar to the proof of Theorem 6.2 in [10].

Proposition 2.2.15. *A (2,1)-semigroup S is left restriction if and only if it is embeddable into a partial transformation monoid, that is, if and only if there exists an injective (2,1)-morphism $\theta : S \to \mathcal{PT}_X$ for some X, where*

$$E = \{\mathrm{id}_Y : Y \subseteq X\}.$$

Proof. Suppose that S is a left restriction semigroup. Fix $S = X$, and for all $s \in S$ we define

$$\theta : S \to \mathcal{PT}_S$$

by

$$s\theta = \rho_s$$

where dom $\rho_s = Ss^+,\ x\rho_s = xs$ for all $x \in \mathrm{dom}\rho_s$.

We have

$$s^+\theta = \mathrm{id}_{Ss^+} = \mathrm{id}_{\mathrm{dom}\ \rho_s} = (\rho_s)^+ = (s\theta)^+$$

hence θ is a $^+$-morphism. Similarly to the proof of Proposition 2.1.8, one can show that θ is a morphism.

If $s\theta = t\theta$, then $Ss^+ = St^+$ so that $s^+ \ \mathcal{L} \ t^+$ and as E is commutative, then $s^+ = t^+$. Further, $\rho_s = \rho_t$ gives

$$s^+\rho_s = t^+\rho_t$$

so that $s^+s = t^+t$ and $s = t$. Thus θ is one-one. Therefore S is embeddable in $\mathcal{PT}_S$.

Conversely, $S \cong \text{im } \theta$ is a $(2,1)$-subsemigroup of $\mathcal{PT}_X$, so by Corollary 2.2.11, S is left restriction semigroup. $\square$

2.3 Generalised Bruck-Reilly semigroups

The main result of this thesis is the characterisation of certain $\widetilde{\mathcal{J}}$-simple restriction semigroups as generalised Bruck-Reilly semigroups. In this section we prove the easier part of this characterisation. First we define the relation $\widetilde{\mathcal{J}}_E$ and Bruck-Reilly semigroups.

Definition 2.3.1. Let S be a restriction semigroup (so it has two unary operations * and $^+$). An ideal I of S is said to be a$\,\widetilde{}\,$*-ideal* if it is the union of $\widetilde{\mathcal{R}}_E$- and $\widetilde{\mathcal{L}}_E$-classes, that is, if $a \in I$ then $\widetilde{R}_a,\ \widetilde{L}_a \subseteq I$. The smallest $\widetilde{}\,$*-ideal* containing a which is the union of $\widetilde{\mathcal{D}}_E$-classes is denoted by $\widetilde{J}(a)$. We define the relation $\widetilde{\mathcal{J}}_E$ on S by

$$a\ \widetilde{\mathcal{J}}_E\ b \Leftrightarrow \widetilde{J}(a) = \widetilde{J}(b).$$

The following useful lemma is essentially derived from Lemma 1.7 in [6].

Lemma 2.3.2. *Let S be a semigroup and $a, b \in S$. Then $b \in \widetilde{J}(a)$ if and only if there are elements $a_0, a_1, ..., a_n \in S,\ x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S^1$ such that $a = a_0$, $b = a_n$ and $a_i\ \widetilde{\mathcal{D}}_E\ x_i a_{i-1} y_i$, for $i = 1, 2, ..., n$.*

Proof. Let I be the set of all elements $b \in S$ which satisfy the given condition. If $a_{i-1} \in \widetilde{J}(a)$, then $x_i a_{i-1} y_i \in \widetilde{J}(a)$, since $\widetilde{J}(a)$ is an ideal, and hence $a_i \in \widetilde{J}(a)$, since $\widetilde{J}(a)$ is an$\,\widetilde{}\,$-ideal. Since $a_0 = a \in \widetilde{J}(a)$, we see that $a_i \in \widetilde{J}(a)$ for $i = 1, 2, ..., n$. In particular, $b \in \widetilde{J}(a)$ and so $I \subseteq \widetilde{J}(a)$.

Now if $b \in I$, it is clear that $sbt \in I$, for all $s, t \in S$ and that $\widetilde{D}_b \subseteq I$. Hence I is a$\,\widetilde{}\,$-ideal and since $a \in I$, we have $\widetilde{J}(a) = I$. □

For later use we need the following technical observation.

Corollary 2.3.3. *If $\mathcal{D} = \widetilde{\mathcal{D}}_E$ then $\widetilde{J}(a) = S^1 a S^1$.*

Proof. □

Definition 2.3.4. Let T be a monoid and let $\theta : T \to T$ be a monoid morphism. We define a multiplication on the set $\mathbb{N}^0 \times T \times \mathbb{N}^0$ by

$$(m, a, n)(p, b, q) = (m - n + t, a\theta^{t-n} b\theta^{t-p}, q - p + t)$$

where $t = \max\{n, p\}$ and we define $x\theta^0 = x$ for all $x \in T$. This multiplication is associative (see [13]) and has an identity, namely $(0, e, 0)$ where e is the identity of T. We call the resulting monoid the *Bruck-Reilly monoid* and we denote it by $BR(T, \theta)$.

We are going to investigate three special Bruck-Reilly semigroups "the ample case, the $\widetilde{\mathcal{D}}$-simple restriction case, and the $\widetilde{\mathcal{J}}$-simple restriction case".

2.3.1 The ample case

We are going to study some properties of the *-relations in $BR(T,\theta)$ and show that a certain Bruck-Reilly semigroup is ample.

Proposition 2.3.5. *[1] Let T be a cancellative monoid with identity e and $\theta : T \to T$ a monoid morphism. Let $BR(T,\theta)$ is a monoid with identity $(0,e,0)$. Then for all (m,a,n) and $(p,b,q) \in BR(T,\theta)$ we have*

1. *$(m,a,n)\ \mathcal{R}^*\ (p,b,q) \Leftrightarrow m=p$;*
2. *$(m,a,n)\ \mathcal{L}^*\ (p,b,q) \Leftrightarrow n=q$;*
3. *$(m,a,n)\ \mathcal{H}^*\ (p,b,q) \Leftrightarrow m=p$ and $n=q$;*
4. *$(m,a,n)\ \mathcal{D}^*\ (p,b,q)$. That is, $\mathcal{D}^*$ is the universal relation.*

Proof. 1. Suppose that $(m,a,n)\,\mathcal{R}^*(p,b,q)$. For $(0,e,0),(m,e,m) \in BR(T,\theta)$ we have

$$(0,e,0)(m,a,n) = (m,e,m)(m,a,n)$$

then

$$(0,e,0)(p,b,q) = (m,e,m)(p,b,q)$$

so we have

$$(p,b,q) = (t, e\theta^{t-m}b\theta^{t-p}, q-p+t)$$

where $t = \max\{p,m\} = p$, so $m \le p$. Similarly one can prove that $p \le m$. Therefore $p=m$ as required.

For the converse direction, suppose that $m=p$. For any arbitrary elements $(i,c,j),(l,d,k) \in BR(T,\theta)$ we have

$$(i,c,j)(m,a,n) = (l,d,k)(m,a,n)$$

if and only if

$$(i-j+t, c\theta^{t-j}a\theta^{t-m}, n-m+t) = (l-k+t', d\theta^{t'-k}a\theta^{t'-m}, n-m+t')$$

where $t=\max\{j,m\}$ and $t'=\max\{k,m\}$. From the third components we get that $t=t'$, and the second component gives

$$c\theta^{t-j}a\theta^{t-m} = d\theta^{t'-k}a\theta^{t'-m} \Rightarrow c\theta^{t-j} = d\theta^{t'-k}$$

then we multiply both sides from the right by $b\theta^{t-p}$, and we get

$$c\theta^{t-j}b\theta^{t-p} = d\theta^{t'-k}b\theta^{t'-p}.$$

Thus

$$(i-j+t, c\theta^{t-j}b\theta^{t-p}, q-p+t) = (l-k+t', d\theta^{t'-k}b\theta^{t'-p}, q-p+t')$$

that is,

$$(i,c,j)(p,b,q) = (l,d,k)(p,b,q).$$

Similarly, we can prove the converse implication. Then

$$(m,a,n)\ \mathcal{R}^*(p,b,q).$$

2. This is dual to (1).

3. This is a consequence of (1) and (2).

4. $(m,a,n)\ \mathcal{D}^*\ (p,b,q)$, since we always have $(m,a,q) \in BR(T,\theta)$ with

$$(m,a,n)\ \mathcal{R}^*(m,a,q)\ \mathcal{L}^*(p,b,q).$$

□

Proposition 2.3.6. *Let T be a cancellative monoid with identity e, and let $\theta : T \to T$ be a morphism. Then $BR(T,\theta)$ is an ample semigroup.*

Proof. Note that $E(BR(T,\theta)) = \{(m,e,m) : m \in \mathbb{N}^0\}$. Let (m,e,m), $(n,e,n) \in E(BR(T,\theta))$. We know that

$$\begin{aligned}(m,e,m)(n,e,n) &= (t, e\theta^{t-m}e\theta^{t-n}, t), \text{ where } t = \max\{m,n\} \\ &= (t,e,t) \\ &= (n,e,n)(m,e,m).\end{aligned}$$

Thus the idempotents of $BR(T,\theta)$ commute. So every $\mathcal{R}^*$-class contains at most one idempotent and since we have $(m,a,n)\mathcal{R}^*(m,e,m)$, that is, every element in $BR(T,\theta)$ is $\mathcal{R}^*$-related to an idempotent. Similarly one can show that every $\mathcal{L}^*$-class contains exactly one idempotent and every element in $BR(T,\theta)$ is $\mathcal{L}^*$-related to an idempotent. Thus $BR(T,\theta)$ is an adequate semigroup. It remains to show that the ample conditions are also satisfied.

It is clear that $(m,a,n)^+ = (m,e,m)$ and $(m,a,n)^* = (n,e,n)$. Let $(m,a,n) \in BR(T,\theta)$, and $(l,e,l) \in E(BR(T,\theta))$,

$$(l,e,l)(m,a,n) = (t, a\theta^{t-m}, n-m+t) \quad (\text{where } t = \max\{l,m\})$$

then we have

$$(t, a\theta^{t-m}, n-m+t)^* = (n-m+t, e, n-m+t).$$

Now

$$
\begin{aligned}
(m,a,n)((l,e,l)(m,a,n))^* &= (m,a,n)(n-m+t,e,n-m+t)\\
&= (m-n+s, a\theta^{s-n}, s)
\end{aligned}
$$

(where $s = \max\{n, n-m+t\} = n-m+t$, because $t \geq m$.)

$$
\begin{aligned}
&= (t, a\theta^{t-m}, n-m+t)\\
&= (l,e,l)(m,a,n).
\end{aligned}
$$

So the right ample condition is satisfied. By the same method one can prove that the left ample condition also holds, that is,

$$(m,a,n)(l,e,l) = ((m,a,n)(l,e,l))^+(m,a,n).$$

□

2.3.2 The $\widetilde{\mathcal{D}}$-simple restriction case

We introduce the restriction semigroup version of the Bruck-Reilly semigroups. A weaker version can be found in [3].

Proposition 2.3.7. *Let T be an arbitrary monoid with identity e and let $\theta : T \to T$ be a monoid morphism. Let $E = \{(m, e, m) : m \in \mathbb{N}^0\}$ so that E is a subset of idempotents of $BR(T, \theta)$. Then for any arbitrary elements (m, a, n) and (p, b, q) in $BR(T, \theta)$ we have*

1. $(m, a, n)\ \widetilde{\mathcal{R}}_E\ (p, b, q) \Leftrightarrow m = p$,
2. $(m, a, n)\ \widetilde{\mathcal{L}}_E\ (p, b, q) \Leftrightarrow n = q$,
3. $(m, a, n)\ \widetilde{\mathcal{H}}_E\ (p, b, q) \Leftrightarrow m = p$ *and* $n = q$,
4. $(m, a, n)\ \widetilde{\mathcal{D}}_E\ (p, b, q)$. *That is, $\widetilde{\mathcal{D}}_E$ is the universal relation.*

Proof. 1. Suppose that $(m, a, n)\ \widetilde{\mathcal{R}}_E\ (p, b, q)$. We have

$$(m, e, m)(m, a, n) = (m, a, n)$$

so that

$$\begin{aligned}(m, e, m)(p, b, q) &= (t, e\theta^{t-m}b\theta^{t-p}, q - p + t)\\ &= (p, b, q)\end{aligned}$$

where $t = \max\{m, p\} = p$, so $m \leq p$. In the same method one can prove that $p \leq m$. Then we deduce that $m = p$ as required.

Conversely, if $m = p$, let $(l, e, l) \in E$ be such that

$$(l, e, l)(m, a, n) = (m, a, n).$$

Then necessarily $l \leq m$, and

$$(l, e, l)(p, b, q) = (t, e\theta^{t-l}b\theta^{t-p}, q - p + t)$$

where $t = \max\{l, p\} = p$ (since $m = p$). Therefore

$$(l, e, l)(p, b, q) = (p, b, q).$$

One can use the same trick to show

$$(l, e, l)(p, b, q) = (p, b, q) \Rightarrow (l, e, l)(m, a, n) = (m, a, n).$$

Thus

$$(m, a, n)\ \widetilde{\mathcal{R}}_E\ (p, b, q).$$

2. This is the dual of (1).

3. This is a consequence of (1) and (2).

4. We always have $(m, a, q) \in BR(T, \theta)$ with

$$(m, a, n) \ \widetilde{\mathcal{R}}_E \ (m, a, q) \ \widetilde{\mathcal{L}}_E \ (p, b, q).$$

Thus $(m, a, n) \ \widetilde{\mathcal{D}}_E \ (p, b, q)$.

□

Proposition 2.3.8. *Let T be an arbitrary monoid and let $\theta : T \to T$ be a monoid morphism. Let $E = \{(m, e, m) : m \in \mathbb{N}^0\}$, so that E is a subset of idempotents of $BR(T, \theta)$. Then $BR(T, \theta)$ is a restriction semigroup (with respect to E).*

Proof. To show that $BR(T, \theta)$ is a restriction semigroup we shall check that the conditions of Definition 2.2.8 hold.

By a similar argument in Proposition 2.3.6 we see that idempotents of $BR(T, \theta)$ commute. Clearly then (m, e, m) is the unique idempotent of E which is $\widetilde{\mathcal{R}}_E$-related to any (m, a, n). We therefore put

$$(m, a, n)^+ = (m, e, m).$$

Dually, one can show that (n, e, n) is the unique idempotent of E that is $\widetilde{\mathcal{L}}_E$-related to (m, a, n), that is, $(m, a, n)^* = (n, e, n)$.

Let $(m, a, n) \ \widetilde{\mathcal{L}}_E \ (p, b, n)$, and let (k, c, l) be any element in $BR(T, \theta)$. Now

$$(m, a, n)(k, c, l) = (m - n + t, a\theta^{t-n}c\theta^{t-k}, l - k + t)$$

where $t = max\{n, k\}$. We also have

$$(p, b, n)(k, c, l) = (p - n + t, b\theta^{t-n}c\theta^{t-k}, l - k + t).$$

Hence $(m, a, n)(k, c, l) \ \widetilde{\mathcal{L}}_E \ (p, b, n)(k, c, l)$, so that $\widetilde{\mathcal{L}}_E$ is a right congruence. Dually, one can prove that $\widetilde{\mathcal{R}}_E$ is a left congruence.

Again a similar argument in Proposition 2.3.6 can be used here in order to show that the ample conditions are satisfied, Hence $BR(T, \theta)$ is a restriction semigroup. □

2.3.3 The $\widetilde{\mathcal{J}}$-simple restriction case

The Bruck-Reilly semigroups introduced in Proposition 2.3.8 are $\widetilde{\mathcal{D}}_E$-simple, that is, $\widetilde{\mathcal{D}}_E$ is the universal relation on them. In order to characterise $\widetilde{\mathcal{J}}_E$-simple restriction ω-semigroups, we need a slightly more complicated generalisation. Before we look at our next proposition, we need to define a strong semilattice of monoids [3].

Definition 2.3.9. Let T be a semigroup which is a disjoint union of monoids M_i where the indices i form a semilattice Y.

Suppose that for all $i, j \in Y$, $M_iM_j \subseteq M_{ij}$. Then T is called a *semilattice* Y *of monoids* M_i where $i \in Y$. Furthermore, if for any $i, j \in Y$ where $i \geq j$, there exist a monoid morphism $\phi_{i,j} : M_i \to M_j$ such that:

(i) $\phi_{i,i} = \mathrm{id}_{M_i}$ for all $i \in Y$;

(ii) for $i, j, k \in Y$, where $i \geq j \geq k$, $\phi_{i,j}\phi_{j,k} = \phi_{i,k}$.

Then $\phi_{i,j}$ is called a *connecting morphism.*

Furthermore, if for all $a, b \in T$ where $a \in M_i$ and $b \in M_j$ we have that

$$ab = (a\phi_{i,ij})(b\phi_{j,ij}),$$

then $T = [Y; M_i; \phi_{i,j}]$ is called a *strong semilattice* Y of monoids M_i with connecting morphisms $\phi_{i,j}$.

It is worth mentioning that the operation $\cdot$ is associative and e_0 is the identity of T and the multiplication in T extends the multiplication in each M_i.

We are going to characterise the relations $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E, \widetilde{\mathcal{H}}_E, \widetilde{\mathcal{D}}_E$, and $\widetilde{\mathcal{J}}_E$ on a certain Bruck-Reilly semigroup, which will be $\widetilde{\mathcal{J}}$-simple with finitely many $\widetilde{\mathcal{D}}_E$ -classes.

Proposition 2.3.10. *Let* $T = \bigcup_{i=0}^{d-1} M_i$ *be a strong semilattice of the monoids* M_i *where* $d \in \mathbb{N}^0$, *the indices* i *form a chain* $0 > 1 > \cdots > d-1$ *and the connecting morphisms are all monoid morphisms. Let* $\theta : T \to M_0$ *be a monoid morphism. Further, let*

$$E = \{(m, e_i, m) : m \in \mathbb{N}^0, 0 \leq i \leq d-1\}$$

where e_i *is the identity of* M_i. *Then* $\widetilde{\ }$*-relations on* $BR(T, \theta)$ *are characterised as follows*

1. $(m, a, n)\ \widetilde{\mathcal{R}}_E\ (p, b, q) \Leftrightarrow m = p,$ *and* $a, b \in M_i$ *(for some* i*);*

2. $(m,a,n)\ \widetilde{\mathcal{L}}_E\ (p,b,q) \Leftrightarrow n = q,$ *and* $a, b \in M_i$ *(for some* i*);*

3. $(m,a,n)\ \widetilde{\mathcal{H}}_E\ (p,b,q) \Leftrightarrow m = p$ *and* $n = q,$ *and* $a, b \in M_i$ *(for some* i*);*

4. $(m,a,n)\ \widetilde{\mathcal{D}}_E\ (p,b,q) \Leftrightarrow\ a, b \in M_i$ *(for some* i*), so* $BR(T,\theta)$ *has* $d\ \widetilde{\mathcal{D}}_E$*-classes;*

5. $(m,a,n)\ \widetilde{\mathcal{J}}_E\ (p,b,q)$. *That is,* $\widetilde{\mathcal{J}}_E$ *is the universal relation.*

Proof.

1. Suppose that $(m,a,n)\ \widetilde{\mathcal{R}}_E\ (p,b,q)$ where $a \in M_i$ and $b \in M_j$ (for some i and j). Then for $(m, e_i, m) \in E$,

$$(m, e_i, m)(m, a, n) = (m, a, n)$$

so that

$$(m, e_i, m)(p, b, q) = (p, b, q).$$

Thus

$$(t, e_i\theta^{t-m}b\theta^{t-p}, q - p + t) = (p, b, q)$$

where $t = \max\{m, p\} = p$, saying

$$m \leq p. \tag{2.1}$$

Similarly one can prove that

$$p \leq m. \tag{2.2}$$

So $m = p$, which implies that $e_i\theta^{t-m} = e_i$. We know that $e_i \in M_i$, $b \in M_j$, so $e_ib \in M_{\max\{i,j\}}$, which means that $\max\{i, j\} = j$, that is, $i \leq j$. Similarly $e_ja = a$ implies that $j \leq i$. As a result $m = p$ and $i = j$.

Conversely, suppose that $p = m$ and $a, b \in M_i$, and let $(k, e_j, k) \in E$ be such that

$$(k, e_j, k)(m, a, n) = (m, a, n).$$

Then $k \leq m$, and $j \leq i$. So

$$\begin{aligned}(k, e_j, k)(m, b, q) &= (m, e_j\theta^{m-k}b, q)\\ &= (m, b, q).\end{aligned}$$

In a similar way one can show that, if for $(l, e_j, l) \in E$ we have

$$(l, e_j, l)(p, b, q) = (p, b, q),$$

then

$$(l, e_j, l)(p, a, n) = (p, a, n).$$

So that

$$(m, a, n)\ \widetilde{\mathcal{R}}_E\ (p, b, q).$$

2. The proof is a dual of (1).

3. The proof is a consequence of (1) and (2).

4. Suppose that $(m, a, n)\ \widetilde{\mathcal{D}}_E\ (p, b, q)$. Then there exists an element $(m, c, q) \in BR(T, \theta)$ with

$$(m, a, n)\ \widetilde{\mathcal{R}}_E\ (m, c, q)\ \widetilde{\mathcal{L}}_E\ (p, b, q)$$

obviously, we deduce that $a, b, c \in M_i$ for some i.

Conversely, suppose that $a, b, \in M_i$, then clearly we have

$$(m, a, n)\ \widetilde{\mathcal{R}}_E\ (m, a, q)\ \widetilde{\mathcal{L}}_E\ (p, b, q)$$

therefore

$$(m, a, n)\ \widetilde{\mathcal{D}}_E\ (p, b, q).$$

5. Let $(m, a, n), (p, b, q) \in BR(T, \theta)$ where $a \in M_i$ and $b \in M_j$. Then we have

$$(p, e_j, m+1)(m, a, n) = (p-(m+1)+t, e_j\theta^{t-m-1}a\theta^{t-m}, n-m+t)$$
$$= (p, e_j(a\theta), n+1)$$

where $t = \max\{m+1, m\} = m+1$. Clearly $e_j(a\theta) \in M_j$. Then

$$(p, e_j(a\theta), n+1)\ \widetilde{\mathcal{D}}_E\ (p, b, q)$$

Similarly $(m, e_i, p+1)(p, b, q)\ \widetilde{\mathcal{D}}_E\ (m, a, n)$, thus

$$(m, a, n)\ \widetilde{\mathcal{J}}_E\ (p, b, q).$$

Then by Lemma 2.3.2 we conclude that $BR(T, \theta)$ is $\widetilde{\mathcal{J}}_E$-simple.

□

Let $E = \{f_i : i \in \mathbb{N}^0\}$ be a set of idempotents of a semigroup S, where $f_i \leq f_j$ if and only if $i \geq j$ $(\forall i, j \in \mathbb{N}^0)$. Then E is called C_ω. That is, C_ω is a descending chain

$$f_0 > f_1 > f_2 > \dots\ .$$

Definition 2.3.11. A restriction semigroup S with semilattice of distinguished idempotents E is an *ω-semigroup* if E is isomorphic to C_ω.

We show now that $BR(T,\theta)$ is an ω-semigroup where $T = \bigcup_{i=0}^{d-1} M_i$. Let $(m, e_i, m), (n, e_j, n) \in E$ where $m > n$. Then

$$(m, e_i, m)(n, e_j, n) = (m, e_i(e_j\theta^{m-n}), m) = (m, e_i, m),$$

because $(e_j\theta^{m-n})$ is the identity of T, so that $(m, e_i, m) < (n, e_j, n)$. On the other hand if $m = n$, and $i \geq j$, then

$$(m, e_i, m)(m, e_j, m) = (m, e_ie_j, m) = (m, e_i, m).$$

Altogether we have that $(m, e_i, m) \geq (n, e_j, n)$ if and only if $m < n$, or if $m = n$ and $i \leq j$. So E is the chain

$$\begin{aligned} &(0, e_0, 0) > (0, e_1, 0) > \cdots > (0, e_{d-1}, 0) \\ &> (1, e_0, 1) > (1, e_1, 1) > \cdots > (1, e_{d-1}, 1) \\ &> (2, e_0, 2) > (2, e_1, 2) > \cdots > (2, e_{d-1}, 2) \\ &> \cdots \end{aligned}$$

Hence $BR(T,\theta)$ is a $\widetilde{\mathcal{J}}_E$-simple restriction ω-semigroup.

Now we shall consider some ideas from [11] which generalise important results on Green's relations from [13] to Green's$^\sim$-relations. In [11] these were stated in a more general form but we are going to rephrase them to suit our objective.

Definition 2.3.12. Let S be a semigroup and E be a set of idempotents. An element $a \in S$ is *E-regular* if a has an inverse a° such that $aa^\circ, a^\circ a \in E$.

The analogues of Green's Lemmas (Lemma 1.10.1, and Lemma 1.10.2) hold if we replace $\mathcal{R}, \mathcal{L}$ by $\widetilde{\mathcal{R}}_E, \widetilde{\mathcal{L}}_E$, respectively, if there exists a suitable E-regular element.

Lemma 2.3.13. *[11] Suppose that $\widetilde{\mathcal{L}}_E$ is a right congruence and S has an E-regular element a such that $e = aa^\circ$ and $f = a^\circ a$. Then the right translations*

$$\rho_a : \widetilde{L}_E^e \to \widetilde{L}_E^f \quad \textit{and} \quad \rho_{a^\circ} : \widetilde{L}_E^f \to \widetilde{L}_E^e$$

are mutually inverse $\widetilde{\mathcal{R}}_E$-class preserving bijections.

Dually we state the following lemma:

Lemma 2.3.14. *[11] Suppose that $\widetilde{\mathcal{R}}_E$ is a right congruence and S has an E-regular element a such that $e = aa^\circ$ and $f = a^\circ a$. Then the left translations*

$$\lambda_{a^\circ} : \widetilde{R}_E^e \to \widetilde{R}_E^f \quad \text{and} \quad \lambda_a : \widetilde{R}_E^f \to \widetilde{R}_E^e$$

are mutually inverse $\widetilde{\mathcal{L}}_E$-class preserving bijections.

A useful corollary combining the two lemmas is the following:

Corollary 2.3.15. *[11] Let S be a restriction semigroup. Let a be an E-regular element of S such that $e = aa^\circ$ and $f = a^\circ a$. Then $|\widetilde{H}_E^e| = |\widetilde{H}_E^f|$.*

In the next lemma we give a condition ensuring that the relations $\widetilde{\mathcal{R}}_E$, $\widetilde{\mathcal{L}}_E$ commute:

Lemma 2.3.16. *[11] If S is a restriction semigroup such that every $\widetilde{\mathcal{H}}_E$-class contains an E-regular element, then $\widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E = \widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E$ (so that $\widetilde{\mathcal{D}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E$) and if $a, b \in S$ with $a \; \widetilde{\mathcal{D}}_E \; b$, then $|\widetilde{H}_E^a| = |\widetilde{H}_E^b|$.*

Definition 2.3.17. Let $V \subseteq S$. We say that V is an $\widetilde{\mathcal{H}}_E$-transversal of S if

$$|V \cap \widetilde{H}_E^a| = 1 \quad \text{for all} \quad a \in S.$$

Definition 2.3.18. Let U be an inverse subsemigroup of S consisting of E-regular elements such that $E \subseteq U$. If U is an $\widetilde{\mathcal{H}}_E$-transversal of S, then U is called an *inverse skeleton* of S.

To apply this definition to the semigroup $BR(T, \theta)$, one can easily see that

$$U = \{(m, e_i, n) : m, n \in \mathbb{N}^0, 0 \leq i \leq d - 1\}$$

is an inverse skeleton of $BR(T, \theta)$, since U is an inverse subsemigroup of $BR(T, \theta)$ consisting of E-regular elements, that is, for any $(m, e_i, n) \in U$

$$(m, e_i, n)^\circ = (n, e_i, m)$$

and clearly

$$(m, e_i, n)(m, e_i, n)^\circ, (m, e_i, n)^\circ(m, e_i, n) \in E,$$

furthermore U intersects every $\widetilde{\mathcal{H}}_E$-class exactly once.

To summarise this section, we have shown that $BR(T, \theta)$ is a $\widetilde{\mathcal{J}}_E$-simple restriction ω-semigroup having an inverse skeleton U(therefore satisfying $\widetilde{\mathcal{L}}_E \circ \widetilde{\mathcal{R}}_E = \widetilde{\mathcal{R}}_E \circ \widetilde{\mathcal{L}}_E$).

Chapter 3

The main result

3.1 A structure theorem

This chapter is devoted to the proof of the converse part of our main result, which is a structure theorem for certain $\widetilde{\mathcal{J}}$-simple restriction ω-semigroups. The analogous result was proved in [13] for simple inverse ω-semigroups and in [2] for $\mathcal{J}^*$-simple ample semigroups. Our objective is to generalise it for some $\widetilde{\mathcal{J}}$-simple restriction ω-semigroups . Now we need to introduce some concepts before we begin to prove our main result.

Definition 3.1.1. Let E be a semilattice. For each e in E, we say $\langle e \rangle = Ee = \{i \in E : i \leq e\}$ is a principal ideal (and subsemilattice) of E, and the *uniformity* relation $\mathcal{U}$ on E is given by

$$\mathcal{U} = \{(e, f) \in E \times E : Ee \simeq Ef\}.$$

For each (e, f) in $\mathcal{U}$ we define $T_{e,f}$ to be the set of all isomorphisms from Ee onto Ef. Let

$$T_E = \bigcup \{T_{e,f} : (e, f) \in \mathcal{U}\},$$

then T_E is a semigroup under composition of partial mappings. We call T_E the *Munn semigroup* of the semilattice E.

The following example from [13] shows that the Munn semigroup of the semilattice C_ω is isomorphic to the bicyclic semigroup.

Example 3.1.2. Let $E = C_\omega = \{e_0, e_1, e_2, ...\}$, with

$$e_0 > e_1 > e_2 > ... .$$

Then

$$E_{e_n} = \{e_n, e_{n+1}, e_{n+2}, ...\}.$$

For every $m, n \in \mathbb{N}^0$ we have $Ee_m \simeq Ee_n$. Thus the uniformity relation $\mathcal{U}$ is universal, that is, $\mathcal{U} = E \times E$. The only isomorphism from Ee_m onto Ee_n is $\alpha_{m,n}$ given by

$$e_k\alpha_{m,n} = e_{k-m+n} \quad (k \geq m).$$

Thus $\alpha_{n,m} : Ee_n \to Ee_m$ is defined by

$$e_l\alpha_{n,m} = e_{l-n+m} \quad (l \geq n) \text{ is the inverse of } \ \alpha_{m,n}.$$

If we take any two elements $\alpha_{m,n}$ and $\alpha_{p,q}$ of T_E, then

$$\begin{aligned}\mathrm{dom}(\alpha_{m,n}\alpha_{p,q}) &= (\mathrm{im}\ \alpha_{m,n} \cap \mathrm{dom}\ \alpha_{p,q})(\alpha_{m,n})^{-1} \\ &= (Ee_n \cap Ee_p)\alpha_{n,m} \\ &= E_{e_{t-n+m}} \quad \text{where } t = \max\{n, p\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{im}(\alpha_{m,n}\alpha_{p,q}) &= (\mathrm{im}\ \alpha_{m,n} \cap \mathrm{dom}\ \alpha_{p,q})\alpha_{p,q} \\ &= (Ee_n \cap Ee_p)(\alpha_{p,q}) \\ &= E_{e_{t-p+q}}.\end{aligned}$$

Then the product $\alpha_{m,n}\alpha_{p,q} : Ee_{t+m-n} \to Ee_{t+q-p}$ is given by

$$\alpha_{m,n}\alpha_{p,q} = \alpha_{m-n+t,q-p+t} \qquad (\text{where } t = \max\{n, p\}).$$

Thus we can identify the Munn semigroup of the semilattice C_ω with the bicyclic semigroup that was introduced in Example 1.2.6.

We conclude this discussion by giving the definition of the congruence μ studied explicitly in [5]. We shall first consider the following definition:

Definition 3.1.3. For each element s of a restriction semigroup S, define the mapping $\alpha_s : \langle s^+\rangle \to \langle s^*\rangle$ given by

$$x\alpha_s = (xs)^* \quad (x \in \langle s^+\rangle),$$

and the mapping $\beta_s : \langle s^*\rangle \to \langle s^+\rangle$ given by

$$y\beta_s = (sy)^+ \quad (y \in \langle s^*\rangle).$$

Notice that it is easy to see that β_s is the inverse map of α_s, since for any $x \in \langle s^+\rangle$, by using the ample identities we have

$$x\alpha_s\beta_s = (s(xs)^*)^+ = (xs)^+ = xs^+ = x,$$

as $x \leq s^+$, so $\alpha_s\beta_s$ is the identity map on $\langle s^+\rangle$, and similarly, $\beta_s\alpha_s$ is the identity map on $\langle s^*\rangle$.

Then we have the following lemma:

Lemma 3.1.4. *The maps α_s and β_s are isomorphisms, so $\alpha_s, \beta_s \in T_E$.*

Proof. We have already shown that β_s is the inverse of α_s, so they are bijective.

To show that α_s is a morphism, let $e, f \in \langle s^+ \rangle$, then we have

$$(e\alpha_s)(f\alpha_s) = (es)^*(fs)^* = ((es)(fs)^*)^* = ((ef)ss^*)^* = ((ef)s)^* = (ef)\alpha_s.$$

Thus the map α_s is a morphism. In a similar method one can prove that β_s is also a morphism. $\square$

For our structure theorem we need to determine the maximum congruence μ contained in $\widetilde{\mathcal{H}}_E$, by using the approach in [7] and [5]. We say

$$\begin{aligned}\mu = \{(s,t) \in S \times S : s^+ = t^+,\ s^* = t^* \text{ and } (xs)^* = (xt)^*,\ (sy)^+ = (ty)^+ \\ \text{for all } x \in \langle s^+ \rangle \text{ and for all } y \in \langle s^* \rangle\} \\ = \{(s,t) \in S \times S : \alpha_s = \alpha_t\} = \{(s,t) \in S \times S : \beta_s = \beta_t\}.\end{aligned}$$

Let $(s,t) \in \mu$. Then

$$s^+ = t^+,\ s^* = t^*.$$

That is,

$$s\ \widetilde{\mathcal{R}}_E\ t, \text{ and } s\ \widetilde{\mathcal{L}}_E\ t.$$

Thus, $s\ \widetilde{\mathcal{H}}_E\ t$, so $\mu \subseteq \widetilde{\mathcal{H}}_E$.

The following proposition shows (first proved by Fountain in [7] for adequate and ample semigroups) μ is the kernel of an important morphism.

Theorem 3.1.5. *For every restriction semigroup S with semilattice of idempotents E, the map $\phi : S \to T_E,\ s \mapsto \alpha_s$ is a $(2,1)$-morphism whose kernel is μ such that $\phi|_E : E \to T_E$ is bijective with $\mathrm{im}\phi|_E = E(T_E)$.*

Proof. Let $s, t \in S$. We omit the proof that ϕ is a $(2,1)$-morphism as it is quite similar to the proof of Proposition 2.1.8. For the second part of the proof clearly we have

$$\ker \phi = \{(s,t) \in S \times S : s\phi = t\phi\} = \{(s,t) \in S \times S : \alpha_s = \alpha_t\} = \mu.$$

Now to show $\phi|_E$ is injective, let $e, f \in E$. If $e\phi|_E = f\phi|_E$ then $e\alpha_e = f\alpha_f$ that is, $\langle e \rangle = \langle f \rangle$, so that $e = f$. We also have that $e\phi = \alpha_e$ for all $e \in E$, and since $E(T_E) = \{\alpha_e : e \in E\}$, we conclude that $E\phi = E(T_E)$. $\square$

We deduce from the previous theorem the following crucial property of $\widetilde{\mathcal{H}}_E$.

Lemma 3.1.6. *Suppose that S is a restriction ω-semigroup. Then $\widetilde{\mathcal{H}}_E = \mu$, that is, $\widetilde{\mathcal{H}}_E$ is a congruence on S.*

Proof. We have already noticed that $\mu \subseteq \widetilde{\mathcal{H}}_E$.

Conversely, let $(s,t) \in \widetilde{\mathcal{H}}_E$, then there exist $e_m, e_n \in E = \{e_0, e_1, ...\}$ such that

$$s^+ = t^+ = e_m,\ s^* = t^* = e_n,$$

then

$$\langle s^+ \rangle = \langle t^+ \rangle = \langle e_m \rangle \text{ and } \langle s^* \rangle = \langle t^* \rangle = \langle e_n \rangle.$$

We know $\alpha_s, \alpha_t \in T_E$, so there is a unique isomorphism from $\langle e_m \rangle$ onto $\langle e_n \rangle$ which is $\alpha_{m,n} \in T_E$, that is

$$\alpha_s = \alpha_t = \alpha_{m,n}.$$

This implies that $(s,t) \in$ker $\phi = \mu$. □

Now we recall some definitions from [13] to support our objective.

Definition 3.1.7. A subsemigroup T of a restriction semigroup S is called *full* if it contains all the distinguished idempotents of S.

Definition 3.1.8. For $d = 1, 2, 3, ...$ we define the set

$$B_d = \{(m,n) \in \mathbb{N}^0 \times \mathbb{N}^0 : m \equiv n (\text{mod } d)\}.$$

Then it is easy to see that B_d is a subsemigroup of the bicyclic semigroup $B = \mathbb{N}^0 \times \mathbb{N}^0$.

Lemma 3.1.9. *[13] The subsemigroups B_d are exactly the simple, full inverse subsemigroups of B.*

From now on we fix a $\widetilde{\mathcal{J}}$-simple restriction ω-semigroup S with distinguished semilattice of idempotents

$$e_0 > e_1 > \cdots > e_{d-1} > \cdots$$

with inverse skeleton I. Then by Lemma 2.3.16, $\widetilde{\mathcal{L}}_E$ and $\widetilde{\mathcal{R}}_E$ commute. For $m, n \in \mathbb{N}^0$ we define

$$\widetilde{R}_m = \{a \in S : a\ \widetilde{\mathcal{R}}_E\ e_m\} \text{ and } \widetilde{L}_n = \{a \in S : a\ \widetilde{\mathcal{L}}_E\ e_n\}$$

and

$$\widetilde{H}_{m,n} = \widetilde{R}_m \cap \widetilde{L}_n = \{a \in S : a\ \widetilde{\mathcal{R}}_E\ e_m,\ a\ \widetilde{\mathcal{L}}_E\ e_n\}.$$

Lemma 3.1.10. *Let $\phi : S \to T_E$ be defined as in Theorem 3.1.5. By Example 3.1.2 we identify T_E with B via the isomorphism $\alpha_{m,n} \mapsto (m,n)$ for all $\alpha_{m,n} \in T_E$. Then im $\phi \cong B_d$ where d is the number of $\widetilde{\mathcal{D}}$-classes of S (in particular, S has finitely many $\widetilde{\mathcal{D}}$-classes). As a consequence*

$$\widetilde{H}_{m,n} \neq \emptyset \Leftrightarrow m \equiv n \pmod{d}$$

and

$$\widetilde{H}_{m,n}\widetilde{H}_{p,q} \subseteq \widetilde{H}_{m-n+t,q-p+t}$$

where $t = max\{n.p\}$.

Proof. Define $\iota : I \to S/\widetilde{\mathcal{H}}_E,\ i \mapsto [i]$. It is easy to see that ι is an isomorphism, which implies that im ϕ is an inverse subsemigroup of T_E, because we have im $\phi \cong S/\widetilde{\mathcal{H}}_E \cong I$ and by Example 3.1.2 we know that T_E is isomorphic to B. By Theorem 3.1.5, im ϕ is a full subsemigroup of $T_E = B$. Furthermore, we want to show that im ϕ is $\widetilde{\mathcal{J}}$-simple, that is, to show that all elements in im ϕ are $\widetilde{\mathcal{J}}_E$-related. To see that, we claim that for any $s,t \in S$ we have

$$s\ \widetilde{\mathcal{R}}_E\ t \Rightarrow s\phi\ \widetilde{\mathcal{R}}_{E\phi}\ t\phi.$$

Suppose that $s\ \widetilde{\mathcal{R}}_E\ t$, so $s^+ = t^+$. Then we have that

$$(s\phi)^+ = s^+\phi = t^+\phi = (t\phi)^+$$

thus $s\phi\ \widetilde{\mathcal{R}}_{E\phi}\ t\phi$. Dually one can show that

$$s\ \widetilde{\mathcal{L}}_E\ t \Rightarrow s\phi\ \widetilde{\mathcal{L}}_{E\phi}\ t\phi.$$

As a consequence

$$s\ \widetilde{\mathcal{H}}_E\ t \Rightarrow s\phi\ \widetilde{\mathcal{H}}_{E\phi}\ t\phi \text{ and } s\ \widetilde{\mathcal{D}}_E\ t \Rightarrow s\phi\ \widetilde{\mathcal{D}}_{E\phi}\ t\phi.$$

Let $s,t \in S$ such that $s\ \widetilde{\mathcal{J}}_E\ t$, using Lemma 2.3.2, there are elements $s_0, s_1, ..., s_n \in S, x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S^1$ such that $s = s_0, t = s_n$ then we have that

$$x_1 s_0 y_1\ \widetilde{\mathcal{D}}_E\ s_1,\ ...\ x_n s_{n-1} y_n\ \widetilde{\mathcal{D}}_E\ s_n = t$$

implies that

$$x_1\phi s_0\phi y_1\phi\ \widetilde{\mathcal{D}}_{E\phi}\ s_1\phi,\ ...\ x_n\phi s_{n-1}\phi y_n\phi\ \widetilde{\mathcal{D}}_{E\phi}\ s_n\phi = t\phi$$

and similarly $t\phi$ can be connected to $s\phi$, showing that $s\phi\ \widetilde{\mathcal{J}}_{E\phi}\ t\phi$ in im ϕ. Again by Theorem 3.1.5 we know that $E\phi = E(T_E)$. Since im ϕ is an inverse semigroup, this shows that $\widetilde{\mathcal{J}}_{E\phi} = \mathcal{J}$. Thus im ϕ is a simple full inverse subsemigroup of $T_E = B$. As a consequence (by Lemma 3.1.9), we conclude that S has finitely many $\widetilde{\mathcal{D}}_E$-classes, and that im $\phi = B_d$ where d is the number of $\widetilde{\mathcal{D}}_E$-classes. ∎

From now on $S = \bigcup_{(m,n)\in B_d} \widetilde{H}_{m,n}$, where d is the number of $\widetilde{\mathcal{D}}$-classes of S . We define $T = \bigcup_{i=0}^{d-1} \widetilde{H}_{i,i}$. Let $0 \leq i, j \leq d-1$. Then we have

$$\widetilde{H}_{i,i}\widetilde{H}_{j,j},\ \widetilde{H}_{j,j}\widetilde{H}_{i,i} \subseteq \widetilde{H}_{t,t}$$

where $t = max\{i, j\}$. Thus T is a subsemigroup of S, and $\widetilde{H}_{i,i}$ is a monoid with identity e_i.

We define $\varphi_{i,j} : \widetilde{H}_{i,i} \to \widetilde{H}_{j,j}$ by $m\varphi_{i,j} = e_j m$ where $i \leq j$. For all $m \in \widetilde{H}_{i,i}$ and $e_j \in \widetilde{H}_{j,j}$ we have that $me_j, e_j m \in \widetilde{H}_{j,j}$, so that

$$me_j = e_t(me_j)e_t = e_t m e_t = e_t(e_j m)e_t = e_j m.$$

Moreover one can show that the maps are morphisms, that is,

$$mn = m\varphi_{i,t} \cdot n\varphi_{i,t}$$

where $t = max\{i, j\}$, and they satisfy:

(i) $\varphi_{i,i}$ is the identity map ;

(ii) $\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k}$ for $k \leq\ j \leq i$.

Thus T is a strong semilattice of the monoids $\widetilde{H}_{i,i}$ whose semilattice is isomorphic to the chain

$$e_0 > e_1 > \cdots > e_{d-1}$$

with the connecting morphisms $\varphi_{i,j}$.

We fix $a \in \widetilde{H}_{0,d} \cap I$ for the rest of this chapter. The following lemma establishes some important equations.

Lemma 3.1.11. *Take* $a \in \widetilde{H}_{0,d} \cap I$ *then* $a^{-1} \in \widetilde{H}_{d,0}$, *and* $a^k a^{-k} = e_0$, $a^{-k}a^k = e_{kd}$.

Proof. Clearly since $(0, d)^{-1} = (d, 0)$ in $B_d \cong I$, we have that $a^{-1} \in \widetilde{H}_{d,0}$. Also we have

$$a^2 = a.a \in \widetilde{H}_{0,d}\widetilde{H}_{0,d} \subseteq \widetilde{H}_{0,2d}\ ,$$

and more generally by induction we find that

$$a^k \in \widetilde{H}_{0,kd}\ , \quad a^{-k} \in \widetilde{H}_{kd,0} \quad (n \in \mathbb{N}^0).$$

Now

$$a^k a^{-k} \in \widetilde{H}_{0,kd}\widetilde{H}_{kd,0} \subseteq \widetilde{H}_{0,0}\ \Rightarrow a^k a^{-k} = e_0,$$

since $a^k a^{-k} \in I$ and I intersects each $\widetilde{\mathcal{H}}$-class exactly once and $E \subseteq I$. Similarly one can prove that

$$a^{-k}a^k \in \widetilde{H}_{kd,0}\widetilde{H}_{0,kd} \subseteq \widetilde{H}_{kd,kd} \ \Rightarrow a^{-k}a^k = e_{kd}.$$

□

The following lemma shows that the elements of S can be uniquely expressed as certain products.

Lemma 3.1.12. *Every element s of S can be uniquely written in the form $s = a^{-m}t_i a^n$ where $m, n \in \mathbb{N}^0,\ t_i \in T\ (t_i \in \widetilde{H}_{i,i})$.*

Proof. Let $s \in S$. First we show that m, n, i are determined by the $\widetilde{\mathcal{H}}$-class of s. By Lemma 3.1.11, we have that for any $m \in \mathbb{N}^0$

$$a^m \in \widetilde{H}_{0,md},\ a^{-m} \in \widetilde{H}_{md,0},\ a^m a^{-m} \in \widetilde{H}_{md,md}.$$

Let $m, n \in \mathbb{N}^0,\ 0 \le i \le d-1$, and $t_i \in \widetilde{H}_{i,i}$, then we have

$$a^{-m}t_i a^n \in \widetilde{H}_{md,0}\ \widetilde{H}_{i,i}\ \widetilde{H}_{0,nd} \subseteq \widetilde{H}_{md+i,nd+i},$$

so if $s \in \widetilde{H}_{k,l}$ and $s = a^{-m}t_i a^n$ then $i \equiv k \equiv l\ (mod\ d)$ and $k = i + md$, $l = i + nd$. So $m = (k-i)/d$ and $n = (l-i)/d$. Hence we can define a map $\phi : \widetilde{H}_{i,i} \to \widetilde{H}_{md+i,nd+i}$ given by

$$t_i\phi = a^{-m}t_i a^n.$$

If $t_i\phi = s_i\phi$, then

$$\begin{aligned} a^{-m}t_i\ a^n = a^{-m}s_i\ a^n\ &\Rightarrow\ a^m a^{-m}t_i\ a^n a^{-n} = a^m a^{-m}s_i\ a^n a^{-n} \\ &\Rightarrow e_0 t_i e_0 = e_0 s_i e_0 \\ &\Rightarrow t_i = s_i. \end{aligned}$$

hence the map ϕ is injective.
The map ϕ is surjective, since for any $x \in \widetilde{H}_{md+i,nd+i}$, we have

$$t_i = a^m x a^{-n} \in \widetilde{H}_{0,md}\ \widetilde{H}_{md+i,nd+i}\ \widetilde{H}_{nd,0} \subseteq \widetilde{H}_{i,i},$$

then

$$\begin{aligned} a^{-m}t_i a^n &= a^{-m}(a^m x a^{-n})a^n \\ &= e_{md} x e_{nd} \end{aligned}$$

$$= e_{md} e_{md+i} x e_{nd+i} e_{nd}$$
$$= e_{md+i} x e_{nd+i} = x.$$

Hence ϕ is a bijection from $\widetilde{H}_{i,i}$ to $\widetilde{H}_{md+i,nd+i}$, with the inverse map given by

$$y \mapsto a^m y a^{-n} \quad (y \in \widetilde{H}_{md+i,nd+i}),$$

showing that s can be uniquely written as $a^{-m} t_i a^n$ where $t_i \in \widetilde{H}_{i,i}$. □

Then we have the following important results:

Lemma 3.1.13. *For any $t \in T$ there exists a unique $t' \in \widetilde{H}_{0,0}$ such that $at = t'a$. We also have $ta^{-1} = a^{-1}t'$. Let θ be the map $\theta : T \to \widetilde{H}_{0,0}$, $t \mapsto at'$. Then $a^k t = (t\theta^k)a^k$ and $ta^{-k} = a^{-k}(t\theta^k)$. Then the map θ is a monoid morphism.*

Proof. To see the uniqueness of t', let $at = t'a = t''a$. Then

$$t'aa^{-1} = t''aa^{-1} \Rightarrow t'e_0 = t''e_0$$
$$\Rightarrow t' = t''.$$

Let $t' = ata^{-1}$. Then

$$t' = ata^{-1} \in \widetilde{H}_{0,d}\widetilde{H}_{i,i}\widetilde{H}_{d,0} \subseteq \widetilde{H}_{0,0}.$$

Since $at \in \widetilde{H}_{0,d}$ and $ta^{-1} \in \widetilde{H}_{d,0}$, we have

$$t'a = ata^{-1}a = ate_d = at, \ a^{-1}t' = a^{-1}ata^{-1} = e_d ta^{-1} = ta^{-1}.$$

Now let us define $\theta : T \to \widetilde{H}_{0,0}$, $t \mapsto at'$. Then for $t, s \in T$ we have that

$$(ts)\theta = a(ts)a^{-1} = at \ e_d \ sa^{-1} = ata^{-1}asa^{-1} = (t\theta)(s\theta)$$

and

$$e_0\theta = ae_0a^{-1} = e_0$$

since I is an inverse subsemigroup and

$$ae_0a^{-1} \in \widetilde{H}_{0,d}\widetilde{H}_{d,d}\widetilde{H}_{d,0} \subseteq \widetilde{H}_{0,0}.$$

This shows that θ is a monoid morphism. Furthermore, we have that

$$(t\theta^k)a^k = (t\theta^{k-1})\theta aa^{k-1} = a(t\theta^{k-1})a^{k-1} = a(a^{k-1}t) = a^k t,$$

and similarly one can show that

$$a^{-k}(t\theta^k) = ta^{-k}.$$

□

The following theorem is the analogue of Theorem 5.7.6 in [13].

Theorem 3.1.14. *Let* $T = \bigcup_{i=0}^{d-1} M_i$ *be a strong semilattice of monoids of length* d (≥ 1)*. If* θ *is a monoid morphism from* T *into* M_0*, then the Bruck-Reilly extension* $S = BR(T, \theta)$ *of* T *determined by* θ *is a* $\widetilde{\mathcal{J}}$*-simple restriction* ω*-semigroup with inverse skeleton*

$$U = \{(m, e_i, n) : m, n \in \mathbb{N}^0,\ 0 \leq i \leq d-1\}.$$

Conversely, every $\widetilde{\mathcal{J}}$*-simple restriction* ω*-semigroup* S *with an inverse skeleton* I *is isomorphic to some* $BR(T, \theta)$ *constructed this way.*

Proof. In Chapter 2 we proved the direct part. For the converse part, let S be a $\widetilde{\mathcal{J}}$ -simple restriction ω-semigroup with inverse skeleton I. By Lemma 3.1.12 any element s of S has a unique expression in the form $a^{-m}t_ia^n$ for $t_i \in \widetilde{H}_{i,i}$ where a is the unique element in $\widetilde{H}_{0,d} \cap I$ and $T = \bigcup_{i=0}^{d-1} \widetilde{H}_{i,i}$. Thus we can define a bijection

$$\psi : S \to \mathbb{N}^0 \times T \times \mathbb{N}^0$$

by

$$(a^{-m}t_ia^n)\psi = (m, t_i, n)$$

Also by Lemma 3.1.13, for any $t_i \in T$, there exists a unique $t_i' \in \widetilde{H}_{0,0}$ such that

$$at_i = t_i'a,\ t_ia^{-1} = a^{-1}t_i'.$$

We define $\theta : T \to \widetilde{H}_{0,0}$, $t \mapsto at'$. Then ϕ is a monoid morphism, and for all $k \in \mathbb{N}$ we have

$$a^kt_i = t_i\theta^ka^k,\ t_ia^{-k} = a^{-k}(t_i\theta^k)$$

Let $a^{-m}t_ia^n, a^{-p}s_ja^q \in S$ where $t_i, s_j \in T$. So that we have two cases to study, that is, if $n \leq p$, then

$$\begin{aligned}(a^{-m}t_ia^n)(a^{-p}s_ja^q) &= a^{-m}t_ia^{-(p-n)}s_ja^q \\ &= a^{-m}a^{-(p-n)}(t_i\theta^{p-n})s_ja^q \\ &= a^{-(m-n+p)}(t_i\theta^{p-n})s_ja^q\end{aligned}$$

and if $n \geq p$, then

$$\begin{aligned}(a^{-m}t_ia^n)(a^{-p}s_ja^q) &= a^{-m}t_ia^{n-p}s_ja^q \\ &= a^{-m}t_i(s_j\theta^{n-p})a^{n-p}a^q \\ &= a^{-m}t_i(s_j\theta^{n-p})a^{q-p+n}.\end{aligned}$$

Thus

$$(a^{-m}t_ia^n)(a^{-p}s_ja^q) = a^{-(m-n+t)}(t_i\theta^{t-n})(s_j\theta^{t-p})a^{q-p+t},$$

where $t = \max\{n, p\}$. This shows that ψ is a semigroup morphism.

It remains to show that ψ is a $^+$-morphism and a *-morphism. Note that

$$e_{md+i} = a^{-m}e_ia^m,$$

because

$$a^{-m}e_ia^n \in \widetilde{H}_{md+i,nd+i} \cap I.$$

So $e_{md+i}\psi = (m, e_i, m)$. Then

$$(a^{-m}t_ia^n)^+\psi = e_{md+i}\psi = (m, e_i, m) = (m, t_i, n)^+ = ((a^{-m}t_ia^n)\psi)^+$$

and dually we have

$$(a^{-m}t_ia^n)^*\psi = e_{nd+i}\psi = (n, e_i, n) = (m, t_i, n)^* = ((a^{-m}t_ia^n)\psi)^*$$

hence ψ is a $(2, 1, 1)$-morphism.
That is, we have shown that $\psi : S \to \mathbb{N}^0 \times T \times \mathbb{N}^0$ is an isomorphism from S onto $BR(T, \theta)$. □

Corollary 3.1.15. *[16] Let S be a bisimple restriction ω-semigroup which satisfies the condition that every $\widetilde{\mathcal{H}}$-class contains an element of the set $Reg_E(S) = \{a \in S : (\exists e, f \in E) e \mathrel{\mathcal{L}} a \mathrel{\mathcal{R}} f\}$. Then $S \cong BR(T, \theta)$ where T is a monoid with identity e and $E = \{(i, e, i) : i \in \mathbb{N}^0\}$.*

Proof. It is easy to check that if $a \in Reg_E(S) \cap \widetilde{H}_{0,1}$ then $I = \{a^{-m}a^n : m, n \in \mathbb{N}^0\}$ is an inverse skeleton of S (isomorphic to B). Furthermore, if S is bisimple, then d in Lemma 3.1.9 equals 1, so T is just a monoid with identity $e_0 = e$, and then by Theorem 3.1.5, the result follows. □

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