REDUCTIVE PAIRS ARISING FROM REPRESENTATIONS

Oliver Frederick Goodbourn

PhD

University of York

Mathematics

September 2014

Abstract

We study the question of when a given rational representation of a reductive group G gives rise to a reductive pair $(GL(V), \rho(G))$, presenting complete classifications when ρ is the representation afforded by a simple module for the group $\mathrm{SL}_2(K)$, or a symmetric power of the natural module (the induced or dual Weyl modules for this group), where K is an algebraically closed field of any positive characteristic. We also present several classes of examples for the group $\mathrm{SL}_3(K)$ in some small characteristics, along with results allowing new examples to be generated.

Contents

Abstract **1**

Acknowledgements **3**

Author's Declaration **4**

1 Introduction **5**

1.1 Breakdown of Chapters 6

2 Preliminaries **9**

2.1 Affine varieties 9

2.2 Affine algebraic groups 13

2.3 Lie algebras of algebraic groups 16

2.4 Some important subgroups 17

2.5 Representations of algebraic groups 20

2.6 Weights and roots 21

2.7 Root systems and Weyl groups 22

2.8 Representation theory of semisimple algebraic groups 26

2.9 Motivation and definition of reductive pairs 32

2.10 First results and methods 39

3 SL_2 **45**

3.1 Basic facts 45

3.2 Simple modules 47

3.3 Symmetric Powers 58

4 SL_3 **and other simple groups** **69**

4.1 General statements for simple groups 69

4.2 SL_3 71

5 Conclusion **75**

A Table, Figures and GAP code **77**

A.1 Table 77

A.2 GAP code 78

A.3 Figures 83

Bibliography **87**

Acknowledgements

I am grateful to my supervisor Michael Bate for being generous with his time and very helpful thoughout the process; to Steve Donkin for providing useful advice and good ideas; to Steve Donkin (again) and Alison Parker for their scrutiny of the thesis and thoughtful suggestions; and not least to Ellie, to my family and friends for being very supportive over the course of the PhD:

Thanks.

Author's Declaration

With the exception of those results in the section on symmetric powers that are noted as due to Donkin, I declare that the work contained in this thesis is my own. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1

Introduction

The work in this thesis is an attempt to gain some insight into the situations in which we encounter a certain favourable type of embedding of algebraic groups, referred to as reductive pairs. We do this primarily via the methods of representation theory, as there is a large class of prospective examples of embeddings of the form $(\mathrm{GL}_n(K), \rho(G))$, where ρ is a rational representation of an algebraic group G, and when considering such examples we are equipped with a significant arsenal of established techniques and theory. Nevertheless, it is important to keep in mind that the problem is not purely representation theoretic: subtleties will arise from the nature and behaviour of G as an algebraic group, and these must be carefully dealt with before the full force of the representation theory may be brought to bear. Indeed, circumnavigating these issues will form a large part of the work within.

Introduced in Richardson's 1967 paper *Conjugacy classes in Lie algebras and algebraic groups*, the concept of a reductive pair of algebraic group has found much use (see for instance [25, 17, 2, 1, 3]). In a very loose sense, they are sometimes employed when seeking to prove results that attempt to salvage the good behaviour of groups over fields of characteristic 0 in the positive characteristic case. In many instances, the idea that for large enough characteristic, such behaviour is more likely to be correctly modelled is borne out by the relative likelihood of a particular pair of algebraic groups being a reductive pair in large characteristic. For instance, in chapter 2 section 2.9 we will see a result (found in [1, 3.3]) that when the characteristic is very large compared to the dimension of a G-module V, then $(\mathrm{GL}(V), \rho(G))$ is always a reductive pair: in the same paper, this fact is exploited to great effect. Nevertheless, the requirement on the characteristic is quite restrictive, and we will see that in specific cases more precise statements may be made.

1.1 Breakdown of Chapters

Chapter 2 contains sections describing the basics of the subject area, beginning with definitions of the basic objects such as affine varieties and algebraic groups. Next comes an overview of the representation theory of algebraic groups. The main purpose of these sections is to set out the notation and terminology to be used in the rest of the thesis. Towards the end of the chapter, focus turns to the main objects of study, namely reductive pairs of algebraic groups. Results are included from the literature that motivate the study of such pairs. The final section of the chapter contains results that will be applicable to each case discussed later, which are therefore set apart.

In chapter 3 we consider the group $\mathrm{SL}_2(K)$. There is first a reminder of the specifics of the representation theory of this group. In the next section we consider results specific to the simple $\mathrm{SL}_2(K)$-modules, including results of Doty and Henke [14] which provide a direct sum decomposition of the tensor product of two such simple modules into known indecomposable modules. In order that this result may be applied to the problem of whether or not a particular irreducible representation gives a reductive pair, we first need some results that tell us about the Lie algebra of the image of such a representation. With the combination of the above results, we finally present a complete classification of those simple modules for $\mathrm{SL}_2(K)$ that give reductive pairs, in any prime characteristic.

In the last part of chapter 3 we turn our attention to the symmetric powers of the natural module E for $\mathrm{SL}_2(K)$. These form another important class of modules in this case, the induced modules $\nabla(\lambda)$. After a series of character calculations (due to Donkin), we arrive at a complete classification of those induced modules that give reductive pairs, in any prime characteristic. The proof of this result relies upon the previous classification for simple modules mentioned earlier.

In chapter 4 we consider what can be said more generally for a simple algebraic group G. Although the first section contains results that are in theory applicable to any simple algebraic group, we quickly focus on the group $\mathrm{SL}_3(K)$. Since comparatively few explicit details of the representation theory of $\mathrm{SL}_3(K)$ are yet known (cf. chapter 3), the approach we present in this chapter is far less direct. Loosely speaking, the main result of the chapter tells us that, armed with knowledge of the composition factors of the module $V \otimes V^*$, we may in some circumstances conclude that V gives a reductive pair. Combining this with other results showing how to generate more examples of reductive pairs, we produce several infinite families of examples. This process involved computer calculations of composition factors, using Doty's Weyl Modules package [13] for GAP [16] .

Chapter 5 contains a summary of the work in the main text, along with some suggestions on new ideas to pursue.

Finally, chapter A is an appendix, including diagrams that illustrate certain concepts and results in the main text. Also included are some of the GAP code used to generate examples for the group $\mathrm{SL}_3(K)$, as well as some of the output of these processes.

Chapter 2

Preliminaries

Please note that many of the following definitions are presented out of their logical order: in forgoing a proper treatment of the material (which would require much more space), following the same structure as a standard book could add unnecessary complication. Standard references for this topic include *Linear Algebraic Groups*, by Humphreys [18], *Linear Algebraic Groups*, by Springer [28], and *Linear Algebraic Groups*, by Borel [6]; of these, the first two offer a gentler introduction to the neophyte.

2.1 Affine varieties

In this thesis, K denotes an algebraically closed field of positive characteristic p, unless otherwise stated.

Definition 2.1.1. An affine variety is a pair (V, A), where V is a set and $A \leq \mathrm{Map}(V, K)$ is a finitely generated subalgebra of the K-algebra of set maps from V to K, such that the the map sending each $x \in V$ to its evaluation map $\epsilon_x \in \mathrm{Hom}_{K-alg}(A, K)$ is a bijection.

We note that the algebra structure on $\mathrm{Map}(V, K)$ in the above definition is by pointwise operations. We will typically write $K[V]$ for the algebra A, and call this the coordinate algebra of V; in adopting this convention, it will be expedient to refer to "the affine variety V", leaving the coordinate algebra implicit where no confusion may arise.

Example 2.1.2. *We write $\mathbb{A}^n$ for the set K^n. Regarding elements of the polynomial algebra $K[T] := K[T_1, T_2, \ldots, T_n]$ as K-valued functions on $\mathbb{A}^n$, we have that $(\mathbb{A}^n, K[T])$ is an affine variety, which we will call* affine n-space.

Definition 2.1.3. Let $(V, K[V])$ be an affine variety. For any subset $S \subset K[V]$, define $\mathcal{V}(S) := \{x \in V \mid f(x) = 0 \ \forall f \in S\}$. The $\mathcal{V}(S)$ form the closed sets of a topology on V called the *Zariski topology.*

Unless otherwise stated, all mention to open or closed sets will be in reference to the relevant Zariski topology.

Example 2.1.4. *Let W be any closed subset of an affine variety V. Then W may be made into an affine variety with coordinate algebra $K[W] = \{f|_W \mid f \in K[V]\}$.*

Definition 2.1.5. Let V be an affine variety and $f \in K[V]$ with $f \neq 0$. A subset of V of the form $V_f := \{x \in V \mid f(x) \neq 0\}$ is called a *principal open set.*

Since the complement of V_f in V is $\mathcal{V}(\{f\})$, it is indeed an open set. The principal open sets form a basis for the Zariski topology. They may be regarded as affine varieties in their own right, with the coordinate algebra of V_f being the K-algebra $A_f := \{\frac{a}{f^r} \mid a \in A, r \geq 0\}$ (identified with a subalgebra of $\mathrm{Map}(V_f, K)$ in the obvious way). Arbitrary open subsets of affine varieties are not generally affine varieties themselves.

Example 2.1.6. *This allows us to view the set* $\mathrm{GL}_n(K)$ *as an affine variety: regarding* $\mathrm{Mat}_n(K)$ *as affine* n^2*-space,* $\mathrm{GL}_n(K)$ *is the principal open set* $\mathrm{Mat}_n(K)_d$ *defined by the nonvanishing of the determinant* d*. Its coordinate algebra is the polynomial algebra generated by the* n^2 *coordinate functions (restricted to* $GL_n(K)$*) and the rational function* $1/d$*.*

Recall that a topological space is called *irreducible* if any of the following equivalent conditions hold: every nonempty open set is dense; no two nonempty open sets are disjoint; or the whole space cannot be written as a union of two proper closed subsets. We remark that an affine variety is irreducible with respect to the Zariski topology if and only if its coordinate algebra is an integral domain. We remark at this point that irreducibility is a stronger condition than connectedness: a topological space is *connected* if it cannot be written as a disjoint union of two nonempty open subsets.

Definition 2.1.7. Suppose a topological space X contains a strictly increasing sequence of closed, irreducible subsets $X_0 \subset \cdots \subset X_n$ and no longer sequence of this sort. Then we say that n is the *dimension* of X, and write $\dim X = n$ to indicate this.

We note that for an irreducible topological space X and a proper, closed set Y, $\dim Y < \dim X$.

Definition 2.1.8. A topological space is called *Noetherian* if its closed (respectively, open) sets satisfy the descending (respectively, ascending) chain condition.

Every affine variety is Noetherian with respect to the Zariski topology. A Noetherian topological space has only a finite number of *irreducible components* (these are maximal irreducible subsets), which are uniquely determined up to order. In contrast, any nonempty topological space may be written as a disjoint union of one or more *connected components*, which are the maximal connected subsets.

Definition 2.1.9. Let V, W be affine varieties. A map $\phi : V \to W$ is called a *morphism of affine varieties* (or just a *morphism* when the context is clear) if for every map $f \in K[W]$ we have $f \circ \phi \in K[V]$; in this case, the map $\phi^\sharp : K[W] \to K[V]$ defined by $\phi^\sharp(f) = f \circ \phi$ is called the *comorphism* of ϕ.

Morphisms are continuous with respect to the Zariski topologies on the domain and codomain. An inverse morphism exists (and we call ϕ an isomorphism of affine varieties) precisely when the comorphism is an isomorphism of K-algebras. We note that given any affine variety V, there exists an isomorphism between V and some closed subset of affine n-space (depending on V). Since many of the examples we consider will most easily be realised as closed sets in an affine space, we will occasionally refer without further comment to the coordinate algebra of polynomial functions inherited from the full coordinate algebra of that affine space. In short, for an closed subset V of affine n-space, this coordinate algebra is $K[V] = K[T_1, \ldots, T_n]/\mathcal{I}(V)$, where $\mathcal{I}(V)$ is the collection of all polynomials in $K[T] = K[T_1, \ldots, T_n]$ vanishing on all of V.

Definition 2.1.10. Let V be an affine variety and $x \in V$. The *tangent space to V at the point x*, $T_x(V)$ is the K-vector space of linear maps $\alpha : K[V] \to K$ such that for all $f, g \in K[V]$, $\alpha(fg) = f(x)\alpha(g) + \alpha(f)g(x)$.

This notion of tangent spaces extends that of the familiar tangent space at a point to a curve or surface. A tangent space as defined above is finite dimensional. Letting m be the minimal dimension of the tangent space $T_x(V)$ for any $x \in V$, we call a point $y \in V$ *simple* if $\dim T_y(V) = m$. An important result is that the dimension of the tangent space at a simple point of an irreducible variety is equal to the (topological) dimension of that variety.

As in the study of manifolds, given a morphism between affine varieties, we may compute the differential of the morphism at a point, which is a linear map between tangent spaces.

Definition 2.1.11. Let $\phi : V \to W$ be a morphism of affine varieties and let $x \in V$. The map $d\phi_x : T_x(V) \to T_{\phi(x)}(W)$ defined by $d\phi_x(\alpha) = \alpha \circ \phi^\sharp$ is called the *differential of* ϕ *at* x.

Definition 2.1.12. Given affine varieties V and W, we identify $K[V] \otimes K[W]$ with a subalgebra of $\text{Map}(V \times W, K)$ by letting the pure tensor $f \otimes g$ map a pair $(x, y) \in V \times W$ to $f(x)g(y) \in K$ and extending linearly. With this identification, $(V \times W, K[V] \otimes K[W])$ is an affine variety, which we call the *product* of V and W.

We note that the product as defined here is a product in the categorical sense. That this set carries the structure of an affine variety is vital to the definition of the basic objects of study in this thesis, algebraic groups.

2.2 Affine algebraic groups

Definition 2.2.1. We define an *algebraic group* to be an affine variety G carrying the structure of a group, such that the group multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$ are morphisms of affine varieties.

We note that a more general definition is possible, but we will restrict our attention to affine algebraic groups. Not every subgroup of an algebraic group need be an algebraic group in its own right, however any closed subgroup is (where by *closed subgroup* we mean a subgroup whose set of elements is closed in the Zariski topology, thus inherits an affine variety structure as discussed above). In this thesis we will consider only algebraic subgroups of algebraic groups, unless otherwise stated.

Example 2.2.2. *The variety* $\mathrm{GL}_n(K)$ *defined above is an algebraic group when endowed with the standard group structure (matrix multiplication and inversion are defined in terms of polynomials in the coordinate functions and the determinant). Whereas affine* n*-space may be regarded as the prototypical example of an affine variety,* $\mathrm{GL}_n(K)$ *plays a similar role for algebraic groups: any algebraic group is isomorphic (as an algebraic group) to a closed subgroup of* $\mathrm{GL}_n(K)$ *for some* n*.*

Example 2.2.3. *The special linear group* $\mathrm{SL}_n(K)$ *is an algebraic subgroup of* $\mathrm{GL}_n(K)$*: it is a subgroup which is also closed, being the zero set of the function* $d - 1$*.*

Example 2.2.4. *The affine line* $\mathbb{A}^1 = K$ *has the structure of an Abelian group under addition; since addition and negation are polynomials in the coordinate function* X *on* $\mathbb{A}^1$*, we have an algebraic group, which we call the* additive group*,* $\mathbb{G}_a$*. On the other hand, we refer to the group of units of the field* K *as the* multiplicative group*, denoted* $\mathbb{G}_m$*. It, too, is clearly an algebraic group, isomorphic to* $\mathrm{GL}_1(K)$*.*

Example 2.2.5. *Any finite group* G *may be regarded as an algebraic group with coordinate algebra* $K[G] = \mathrm{Map}(G, K)$*.*

Definition 2.2.6. A *morphism of algebraic groups* (or *morphism* when the context is clear) is a morphism of the underlying affine varieties which is also a homomorphism with respect to the group structures. An *isomorphism of algebraic groups* is a morphism which is also an isomorphism of varieties.

Note: it is *not* true that a bijective morphism of algebraic groups is automatically an isomorphism of algebraic groups (although it is still an isomorphism of "abstract" groups). An important example of where this fails will be encountered repeatedly in this thesis in the form of the Frobenius morphism (defined shortly). When discussing algebraic groups, we will always use the word "isomorphic" to mean isomorphic *as algebraic groups*, unless otherwise stated.

Definition 2.2.7. Let G be a linear algebraic group (an algebraic group of matrices over K). We define the *Frobenius morphism* $F : G \to G$ by $F(a_{ij}) := (a_{ij}^p)$, that is, we raise the entries of the matrix to the p^{th} power. A more complicated definition can be made (see, for instance, [21, 3.1]), extending this notion to arbitrary algebraic groups.

As noted above, F is a bijective morphism of algebraic groups, but the inverse function is not a morphism (it involves taking p^{th} roots).

There is a rigorously defined notion of a quotient of an algebraic group G by a closed normal subgroup N. The result is an affine variety structure defined on the set of cosets G/N, which can be shown to satisfy the universal properties desired of a quotient.

Since an algebraic group is an affine variety, we may consider its irreducible components.

Definition 2.2.8. We write G° for the *identity component* of G, which is the (unique) irreducible component containing the identity element.

Proposition 2.2.9. *We have that G° is a normal subgroup of G of finite index, whose cosets are the irreducible components of G; these are therefore disjoint, whence they are also the connected components of G.*

Definition 2.2.10. We say that an algebraic group G is *connected* if $G = G^\circ$.

Note that this requirement that G be irreducible is a stronger condition than the usual topological notion of connectedness in topology, since every irreducible space is connected whilst the reverse is not true in general.

Example 2.2.11. *Without proof, we will state that the following algebraic groups are all connected:* $\mathbb{G}_a$, $\mathbb{G}_m$, $\mathrm{GL}_n(K)$, $\mathrm{SL}_n(K)$.

2.3 Lie algebras of algebraic groups

All points of a connected algebraic group G are simple: the map $f_{g,h} : G \to G$ defined by $f_{g,h}(x) = xg^{-1}h$ is an isomorphism of varieties whose differential at $g \in G$ is an isomorphism of vector spaces $T_g(G) \to T_h(G)$.

Since an algebraic group carries the structure of an affine variety, it is natural to consider the tangent spaces defined earlier as applied to this case. In fact, as with a Lie group, the tangent space at the identity element of an algebraic group carries the additional structure of a Lie algebra. Let V, W be affine varieties, $x \in V$, $y \in W$. Then the map $\Phi : T_x(V) \times T_y(W) \to T_{(x,y)}(V \times W)$ defined by $\Phi(\alpha, \beta)(f \otimes g) = \alpha(f)\beta(g)$ for $f \in K[V], g \in K[W]$ is a bijection; we will identify these sets.

Definition 2.3.1. Let G be an algebraic group. We will write $\mathrm{Lie}(G) := T_1(G)$, where 1 is the identity element of the group. We call this tangent space the *Lie algebra of* G, since it may be checked that it is a Lie algebra when equipped with a bracket defined as follows. Define a morphism $\phi : G \times G \to G$ by

$$\phi(x, y) = xyx^{-1}y^{-1}.$$

Using the identification of $\mathrm{Lie}(G \times G)$ with $\mathrm{Lie}(G) \times \mathrm{Lie}(G)$ via Φ as above, we consider the differential

$$d\phi_{(1,1)} : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G).$$

We set $[\alpha, \beta] = d\phi_{(1,1)}(\alpha, \beta)$.

Example 2.3.2. *It can be shown that* $\mathrm{Lie}(\mathrm{GL}_n(K))$ *is the Lie algebra of* $n \times n$ *matrices, with bracket the usual (commutator) Lie bracket for matrices.*

When discussing differentials of algebraic group morphisms, it will be convenient to drop the identity element from our notation, rendering $d\phi_1$ as simply $d\phi$. The differential of a morphism of algebraic groups is a homomorphism of Lie algebras.

Definition 2.3.3. Let $\mathrm{Int}_x : G \to G$ be the inner automorphism defined by $x \in G$, so that $\mathrm{Int}_x(y) = xyx^{-1}$. We write Ad_x for the differential $d(\mathrm{Int}_x)$. It happens that Ad_x is an automorphism of the Lie algebra of G. The *adjoint representation* of G is the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$ sending each group element x to the Lie algebra homomorphism Ad_x.

2.4 Some important subgroups

We note that the notion of solvability of groups applies equally well to algebraic groups, since all the derived subgroups of an algebraic group G are themselves closed and normal (and connected if G is connected). Recall that subgroups and homomorphic images of solvable groups are solvable.

Definition 2.4.1. An algebraic group T is called a *torus* if it is isomorphic to a diagonal subgroup $D(n, K)$ of a general linear group. A group that is isomorphic to a subgroup of some $D(n, K)$ is called *diagonalisable.*

Note that $D(n, K)$ is isomorphic to a direct product of n copies of the multiplicative group $\mathbb{G}_m$ defined previously. We will usually speak of a torus T being *of a group* G, meaning that T is a subgroup of G that is a torus. For instance, $D(n, K)$ is a torus of $GL(n, K)$. Tori are solvable.

Definition 2.4.2. A *Borel subgroup* of G is a maximal closed, connected, solvable subgroup.

Definition 2.4.3. Let B be a Borel subgroup of an algebraic group G, and let P be any algebraic subgroup with $B \leq P \leq G$. Then P is called a *parabolic* subgroup of G.

The parabolic subgroups may equivalently be described as those closed subgroups of G such that the variety G/P is projective (assuming an adequate definition of homogeneous spaces and projective varieties has been given).

Theorem 2.4.4. *All Borel subgroups of* G *are conjugate, as are all maximal tori of* G*.*

A maximal torus is of course a torus properly contained in no other; since Borel subgroups are maximal amongst the connected, closed, solvable subgroups, each maximal torus is contained in a Borel. That the Borel subgroups are all conjugate is a consequence of Borel's fixed point theorem, and is false in general if we drop the assumption that K is algebraically closed. The common dimension of the maximal tori of G is called the *rank* of G.

Let V be a finite dimensional K-vector space. We recall the Jordan decomposition of an element $x \in \mathrm{GL}(V)$: there exist unique elements x_s and x_u in $\mathrm{GL}(V)$ with x_s semisimple, x_u unipotent and $x = x_s x_u = x_u x_s$. For an arbitrary algebraic group G there exists an analogous decomposition. We consider *right translation* *of functions by* x, written ρ_x, where $\rho_x(f)(y) = f(yx)$ for $f \in K[G]$, $x, y \in G$. Given $x \in G$ there exist unique elements s and u of G with $x = su = us$ such that ρ_s and ρ_u are semisimple and unipotent elements of $\mathrm{GL}(K[G])$, respectively[1]. We call s and u the *semisimple* and *unipotent parts* of x; if x is equal to its semisimple or unipotent part, we say that x is semisimple or unipotent, respectively. The subset G_u of all unipotent elements of G is closed in G. Any algebraic group all of whose elements are unipotent will itself be called a *unipotent group*. Unipotent groups are solvable.

Definition 2.4.5. Let G be an algebraic group. The *radical* of G, written $R(G)$ is the (uniquely determined) maximal connected normal solvable subgroup of G. If G is a non-trivial, connected group and $R(G)$ is trivial, we say that G is *semisimple.*

The *unipotent radical* of G, written $R_u(G)$ is the (again uniquely determined) maximal connected normal unipotent subgroup of G. It is the collection of unipotent elements of $R(G)$. If G is a non-trivial, connected group and $R_u(G)$ is trivial, we say that G is *reductive.*

Given a connected algebraic group G, the quotient $G/R(G)$ is semisimple, whereas $G/R_u(G)$ is reductive.

Example 2.4.6. *The group* $\mathrm{SL}_n(K)$ *is semisimple (thus also reductive), whilst the group* $\mathrm{GL}_n(K)$ *is reductive.*

Let G be a reductive group. Given a maximal torus T of G and a Borel subgroup B containing T, there exists a unique Borel subgroup B^- called the *opposite Borel* such that $B \cap B^- = T$. Let P be any parabolic subgroup of G.

[1]Since $K[G]$ is not finite dimensional, a certain amount of work is needed to make this statement rigorous. See for instance [18, 15.1]

Then P admits a *Levi decomposition* into a semi-direct product of a reductive subgroup $L \leq P$ and the unipotent radical $R_u(P)$. Such a subgroup L is called a *Levi subgroup* of P, and all such subgroups are conjugate by elements of $R_u(P)$.

2.5 Representations of algebraic groups

We will be concerned only with representations that preserve the essential structures of an algebraic group: the group structure and the variety structure.

Definition 2.5.1. A (finite dimensional) *rational representation* of an algebraic group G is a morphism of algebraic groups $\rho : G \to \mathrm{GL}_n(K)$ for some n. We say that a module for the group algebra KG is a *rational module* if and only a matrix representation afforded by the module is a rational representation. We will sometimes refer to a KG-module as a *G-module*.

We note that where V is an n-dimensional K-vector space, we may regard $\mathrm{GL}(V)$ as an algebraic group unambiguously by picking a basis for V. Hereafter and unless otherwise stated, in this thesis "module" will always mean rational KG-module. We may also consider infinite dimensional rational modules: in this case we require that the module V be locally finite dimensional (that is, every finite dimensional K-subspace of V is contained in a finite dimensional KG-submodule), and every finite dimensional submodule of V is rational in the sense already described. This allows us to consider the coordinate algebra $K[G]$ as a rational module; this is an important step in realising an embedding of an arbitrary algebraic group into some GL_n.

Lemma 2.5.2. *Direct sums, tensor products, linear duals and subquotients of finite dimensional rational modules are rational.*

These are easily proved by working with the coefficient functions defined by picking a basis. Using this result we can see that for a finite dimensional rational module V, we have in particular that the r^{th} symmetric power of V, S^rV is rational, since it is a quotient of $V^{\otimes m}$. We will frequently consider the (rational) natural module E for the group $\mathrm{GL}_n(K)$ or $\mathrm{SL}_n(K)$, consisting of column vectors of length n with the action being by matrix multiplication.

Definition 2.5.3. Let M be a G-module. We define M^{F^n}, the n^{th} *Frobenius twist of* M, as the G-module with the same underlying vector space as M, but for which the action of G follows n iterations of the Frobenius morphism $F : G \to G$. Thus, if ρ is the representation afforded by M, the representation afforded by M^{F^n} is $\rho \circ F \circ \cdots \circ F$ (where F occurs n times).

2.6 Weights and roots

Definition 2.6.1. A *character* of an algebraic group G is a morphism from G to the multiplicative group $\mathbb{G}_m$. If χ_1 and χ_2 are characters of G, then so is $\chi_1 + \chi_2$, defined by $(\chi_1 + \chi_2)(g) = \chi_1(g)\chi_2(g)$. This turns the set $X(G)$ of characters of G into an Abelian group.

A *cocharacter* is a morphism from $\mathbb{G}_m$ to G. For a commutative group G, the set of cocharacters, $Y(G)$ also forms an Abelian group.

Composing a character with a cocharacter yields a morphism $\mathbb{G}_m \to \mathbb{G}_m$. Since $X(\mathbb{G}_m) \cong \mathbb{Z}$, given a torus T we may associate to each pair $(\chi, \lambda) \in X(T) \times Y(T)$ an integer value which we will write $\langle \chi, \lambda \rangle$. We note also that (as a consequence of Dedekind's theorem), $X(T) \cong \mathbb{Z}^n$ and $Y(T) \cong \mathbb{Z}^n$ for any n-dimensional torus [28, 3.2.2].

Definition 2.6.2. If $G \leq \mathrm{GL}(V)$ is a closed subgroup, then for each $\chi \in X(G)$ we can define the *weight space* $V_\chi = \{v \in V \mid gv = \chi(g)v \text{ for all } g \in G\}$. Each V_χ is a KG-submodule of V, possibly the zero submodule. An nonzero element of V_χ is called a *semi-invariant* of *weight* χ. The *multiplicity* of a weight χ is the dimension of the weight space V_χ.

Suppose G is any algebraic group and we are given a rational representation $\rho : G \to \mathrm{GL}(W)$. There is an injection $X(\rho(G)) \to X(G)$ given by sending $\chi \in X(\rho(G))$ to $\chi \circ \rho$. Thus, where χ is a character of G induced by a character of $\rho(G)$, we may define V_χ in the obvious way.

Proposition 2.6.3. *Let $\rho : G \to GL(V)$ be a rational representation with V finite dimensional. Then the spaces V_χ for $\chi \in X(G)$ are linearly independent; in particular, only finitely many of them are non-zero. [18, 11.4]*

Definition 2.6.4. Let D be a diagonalisable subgroup of an algebraic group G (for instance a torus). Then $\mathrm{Ad}(D)$ is a diagonalisable subgroup of $\mathrm{GL}(\mathrm{Lie}(G))$. We may consider the weights of $\mathrm{Ad}(D)$ as defined above. The non-zero weights are the *roots of G relative to D*, and we write $\Phi(G, D)$ for the set of these.

2.7 Root systems and Weyl groups

We will now recall definitions pertaining to root systems, which will be of great importance in the later discussion of representations.

Definition 2.7.1. Let E be a finite dimensional Euclidean vector space. A *root system* in E is a subset of elements $\Phi \subset E$ (called *roots*) such that:

- Φ is a finite spanning set not containing the zero vector;
- for any $\alpha \in \Phi$, the only multiples of α contained in Φ are α and $-\alpha$;

- for any $\alpha \in \Phi$, the reflection σ_α in E in the hyperplane perpendicular to α leaves Φ stable; and

- for any $\alpha, \beta \in \Phi$, the vector $\sigma_\alpha(\beta) - \beta$ is an integral multiple of α.

Let Φ be a root system in E. It is always possible to choose a set of *positive roots*, $\Phi^+ \subset \Phi$ with the two properties:

- for each root $\alpha \in \Phi$, precisely one of the roots α and $-\alpha$ is in Φ; and

- if $\alpha \neq \beta$ are two roots in Φ^+, their sum $\alpha + \beta$ is also in Φ^+.

If Φ^+ is a set of positive roots, then the elements of the set $\Phi^- := -\Phi^+$ are called *negative roots*. A given choice of positive roots in turn determines a set Δ called a *base* of Φ, consisting of *simple roots*. These are those roots in Φ^+ which cannot be written as a sum of two elements in Φ^+. The base Δ forms a basis of E with the property that any root $\alpha \in \Phi$ can be written as a linear combination of roots in Δ with all coefficients either non-negative (the positive roots) or non-positive (the negative roots).

The (finite) group W generated by the reflections in hyperplanes perpendicular to the roots is called the *Weyl group* of the root system. If Δ is a choice of simple roots of Φ, then W is generated by the reflections $\{\sigma_\alpha \mid \alpha \in \Delta\}$. With these generators, Weyl group is an example of a finite Coxeter group. Thus it comes with a *longest element* w_0 determined by the choice of simple roots (this is the unique element for which the length of any expression as a reduced word in the simple reflections is maximal). The Weyl group permutes the set of bases of Φ simply transitively.

A root system Φ in E is called *irreducible* if it cannot be partitioned into two mutually orthogonal proper subsets.

Let the inner product on the Euclidean space E be denoted $(\ , \ )$. We write $\langle \alpha, \beta \rangle$ for the quantity defined by $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$, noting that with this notation, $\sigma_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$. For pairs of roots in a given base, these numbers are integers, called the *Cartan integers*.

Root systems Φ in E and Ψ in F are *isomorphic* if there is an isomorphism of vector spaces $E \to F$ preserving the Cartan integers. An isomorphism of root systems

A vector λ in E will be called an *abstract weight* if all the values $\langle \lambda, \alpha \rangle$ are integers. The set of abstract weights forms a lattice Λ in E; the lattice spanned by the roots is a subgroup of Λ of finite index. A base Δ of Φ, determines a corresponding basis B of Λ (necessarily of the same cardinality) with the property that for $\alpha_i \in \Delta$ and $\lambda_j \in B$, we have $\langle \lambda_j, \alpha_i \rangle = \delta_{ij}$. Elements of B are called *fundamental dominant weights*.

A non-zero $\mathbb{Z}$-linear combination of fundamental dominant weights is called *dominant* if all the coefficients are non-negative. We may define a partial order on E by declaring that $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a non-negative integer combination of simple roots (i.e. the difference is dominant). Thus $\Phi^+ = \{\alpha \in \Phi \mid \alpha > 0\}$.

Every root system may be decomposed as a disjoint union of uniquely determined irreducible root systems in subspaces of E. It is well known that the irreducible root systems are classified by *Dynkin diagrams*, which are certain graphs whose vertices and edges are determined by the simple roots and Cartan integers. For a detailed discussion, see [19]. The Dynkin diagrams are categorised as belonging to several families, and in this thesis we will mostly be concerned with those of type A. The appendix contains diagrams showing some dominant weights of a root system of type A_2, some of which are marked due to their relevance to a later calculation. In the diagrams, the numbers on the sides represent coordinates with respect to a choice of fundamental dominant weights, with the weights falling at the intersection points of the lines. Note that the weights with one coordinate negative are not dominant.

Let G be a connected algebraic group. It can be shown that for a torus S of G, the quotient of its normaliser by its centraliser is a *finite* group, $W(G,S) := N_G(S)/C_G(S)$.

Definition 2.7.2. Let T be a maximal torus of G. All such subgroups being conjugate in G, we refer to any group in the isomorphism class of $W(G,T)$ as the *Weyl group* W of G.

There is a natural action of W on the set of roots $\Phi(G,T)$ relative to a maximal torus of G. If $n \in N_G(T)$ represents an element σ of the Weyl group, then σ permutes the root spaces $\mathrm{Lie}(G)_\alpha$ as follows: $\mathrm{Ad}_n(\mathrm{Lie}(G)_\alpha) = \mathrm{Lie}(G)_{\sigma(\alpha)}$. [18, 24.1].

Through much work it can be shown that we have the following result, which is [18, 27.1].

Theorem 2.7.3. *Let G be a semisimple algebraic group, T a maximal torus of G and define $E := \mathbb{R} \otimes_{\mathbb{Z}} X(T)$. Then $\Phi := \Phi(G,T)$ is a root system in the space E in the sense defined above, with Weyl group isomorphic to $W(G,T)$.*

Having chosen a positive definite symmetric bilinear form f on E, we may define a new one by $(x,y) := \sum_{w \in W(G,T)} f(w \cdot x, w \cdot y)$. This new form is $W(G,T)$ invariant.

In fact, this result may be extended to deal with reductive (and not just semisimple) algebraic groups; for details, see [18, 27.1]. A choice of Borel subgroup containing T amounts to a choice of base for Φ.

Definition 2.7.4. Let G be a reductive algebraic group, T a maximal torus of G, and Δ the base of $\Phi(G,T)$ determined by a Borel subgroup B of G. Write each $\beta \in \Phi^+$ (uniquely) as $\sum_{\alpha\in\Delta} n_{\alpha\beta}\alpha$, where the $n_{\alpha\beta}$ are non-negative integers. We say that a prime p is *bad* for G if $p \mid n_{\alpha\beta}$ for any of the $n_{\alpha\beta}$; otherwise, we say that p is *good* for G. A semisimple algebraic group has finitely many minimal non-trivial connected closed normal subgroups, which we call the *simple components* of the group. We say that a good prime p is *very good* for G if $p \nmid l+1$ for any simple component of G whose Dynkin diagram is of type A_l.

For the remainder of the thesis we will adopt the convention that (when discussing a reductive group G) T is a (fixed) maximal torus of G, B is a (fixed) Borel subgroup containing T, and B^- is the opposite Borel group determined by B and T. We will write $U := R_u(B)$, noting that $B = TU$ is a Levi decomposition of B.

2.8 Representation theory of semisimple algebraic groups

Let G be a semisimple algebraic group. We are particularly interested in the irreducible KG-modules. For a rational representation ϕ of G, we will refer to the images in $X(T)$ of the weights of $\phi(T)$ as the *weights of* ϕ (or, if the associated G-module is V, the *weights of* V). It can be shown that the weights of a rational representation are abstract weights in the sense defined above [18, 31.1]. One might instead define the weights of V directly as the weights of the T-module V.

Definition 2.8.1. A KG-module V is *irreducible* (or *simple*) if it has no G-stable subspaces except 0 and V.

Definition 2.8.2. If $\phi : G \to \mathrm{GL}(V)$ $(V \neq 0)$ is a rational representation, then there exists a 1-dimensional subspace of V stable under $\phi(B)$. Any vector v

spanning such a subspace will be called a *maximal vector*. Such a vector belongs to a weight space V_λ for some weight λ.

The following proposition is found as [18, 31.2].

Proposition 2.8.3. *Let $V \neq 0$ be a rational KG-module, v a maximal vector in V of weight λ and V' the KG-submodule generated by v. Then the weights of V' are of the form $\lambda - \sum c_\alpha \alpha$, $\alpha \in \Phi^+$, $c_\alpha \in \mathbb{Z}^+$, and λ itself has multiplicity 1. Moreover, V' has a unique maximal submodule.*

The proof depends on knowing how certain subgroups of G, the *root groups* act on the weight spaces. If μ is any other weight of V', the proposition shows that $\mu < \lambda$ in the sense defined in the previous section.

Definition 2.8.4. Following the notation of the previous proposition, we call λ the *highest weight* of V'.

It can be shown that in a root system, every abstract weight is conjugate to precisely one dominant weight under the action of W (which one, of course, depends on the choice of base). A dominant weight λ is thus greater than any W-conjugate of λ. It can also be shown that W permutes the weights of any rational representation. Thus the highest weight is always a dominant weight.

If the KG-module V of proposition 2.8.3 is irreducible, it coincides with V'. In particular, we have the following result [18, 31.3]:

Proposition 2.8.5. *Let V be an irreducible KG-module. Then V contains a unique 1-dimensional B-stable subspace spanned by a maximal vector of some dominant weight λ, whose multiplicity is 1. All other weights of V take the form $\lambda - \sum c_\alpha \alpha$, $\alpha \in \Phi^+$, $c_\alpha \in \mathbb{Z}^+$.*

If V' is an irreducible KG-module with highest weight μ, then $V \cong V'$ (as KG-modules) if and only if $\lambda = \mu$.

Thus irreducible KG-modules, if they exist, are isomorphic precisely when they have the same highest weight. It is possible to construct an irreducible module with highest weight λ for λ any dominant weight [18, 31.4]:

Proposition 2.8.6. *Let λ be a dominant weight. Then there exists an irreducible KG-module of highest weight λ.*

Thus the isomorphism classes of KG-modules are in one-to-one correspondence with the dominant weights. We will write $X(T)_+$ for the set of dominant weights.

From here on, for any unexplained terminology in this section the reader is directed to Jantzen's book Representations of Algebraic Groups [21].

A character $\lambda \in X(G)$ of an algebraic group may be considered as a 1-dimensional representation by identifying $\mathbb{G}_m$ with $GL_1(K)$.

Definition 2.8.7. Let λ be a character of T (i.e. $\lambda \in X(T)$). Consider λ as a character of B by letting U act trivially. We write K_λ for the one dimensional B-module K with the action of T given by λ.

That we can define an action of the whole of B using a character of T follows from the Levi decomposition of B as TU. As a semidirect product of T with the normal subgroup $U \leq B$, there is a homomorphism $\phi : B \to T$ with kernel U which acts as the identity map on T. Thus $\lambda \circ \phi \in X(B)$ provides the desired action.

Definition 2.8.8. Let M be any B-module. We write $H^i(M)$ for $R^i \operatorname{ind}_B^G M$, where $R^i \operatorname{ind}_B^G$ is the i^{th} right derived functor of induction. We will abbreviate $H^i(K_\lambda)$ as $H^i(\lambda)$.

Let a M be a finite dimensional G-module, and recall that the *socle* of M, written $soc_G M$, is the sum of all the simple submodules of M; it is the largest semisimple G-submodule of M. The *head* of M, written $hd_G M$, is the quotient of M by its radical (the intersection of all maximal submodules); it is the largest semisimple homomorphic image of M [21, I,2.14; II,11.12].

Definition 2.8.9. Let $\lambda \in X(T)_+$. We will write $L(\lambda) := soc_G H^0(\lambda)$. It is a simple G-module of highest weight λ [21, II, 2.3-6].

In calculations and results about specific simple modules, we will typically refer to dominant weights by reference to their coordinates with respect to the fundamental dominant weights (see the earlier section on root systems). For example, for the group $SL_3(K)$, whose root system is of type A_2, we will refer to the simple module $L(a,b)$, where $(a,b) = a\lambda_\alpha + b\lambda_\beta$ (α and β being the two simple roots).

Definition 2.8.10. We will write $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha \in X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We define an action (the *dot* action) of W on $E = X(T) \otimes_Z \mathbb{R}$ as follows.

Definition 2.8.11. Let $\lambda \in E$, $w \in W$. Set $w \cdot \lambda := w(\lambda + \rho) - \rho$.

Definition 2.8.12. We set

$$X_r(T) := \{\lambda \in X(T) \mid 0 \leq \langle \lambda, \alpha \rangle < p^r \text{ for all } \alpha \in \Delta\} \subset X(T)_+.$$

In particular, we refer to the weights in $X_1(T)$ as *restricted*, and to $X_1(T)$ itself as the *restricted region*.

Definition 2.8.13. In the space $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, we define the affine reflection

$$\sigma_{\alpha,n}(\lambda) := \lambda - (\langle \lambda, \alpha \rangle - n)\,\alpha,$$

where $\alpha \in \Phi$, $\lambda \in E$ and $n \in \mathbb{N}$. We write W_p for the *affine Weyl group*, which is the group generated by the elements $\sigma_{\alpha,np}$.

We will consider the dot action of W_p on E, defined as above. In particular, we may regard $\sigma_{\alpha,np}$ as reflection with respect to the hyperplane

$$\{\lambda \in E \mid \langle \lambda + \rho, \alpha \rangle = np\}.$$

Definition 2.8.14. We define an *alcove* to be a subset of E of the form

$$A := \{\lambda \in E \mid (n_\alpha - 1)p < \langle \lambda + \rho, \alpha \rangle < n_\alpha p \text{ for all } \alpha \in \Phi^+\},$$

where each n_α is an integer depending on the positive root α.

In particular, we will write

$$A_0 := \{\lambda \in E \mid 0 < \langle \lambda + \rho, \alpha \rangle < p \text{ for all } \alpha \in \Phi^+\},$$

referring to this set as the *bottom alcove*.

For convenience, we will adopt the following notation.

Definition 2.8.15. Let $\lambda \in X(T)_+$. Then $\nabla(\lambda) := H^0(\lambda)$, and $\Delta(\lambda) := H^0(-w_0\lambda)^*$.

We note that $\nabla(\lambda)$ has simple socle $L(\lambda)$, whereas $\Delta(\lambda)$ has simple head $L(\lambda)$ – and this may help as a mnemonic to remember which means which. We may, on occasion, refer to the $\nabla(\lambda)$ as *induced modules* and the $\Delta(\lambda)$ as *Weyl modules*.

Definition 2.8.16. Let M be a finite dimensional G-module. We define the *formal character* of M as

$$\operatorname{ch} M := \sum_{\lambda \in X(T)} \dim M_\lambda e(\lambda) \in \mathbb{Z}[X(T)],$$

where $e(\lambda)$ is the canonical basis element associated to λ in the ring $\mathbb{Z}[X(T)]$. Since we write the group law in $X(T)$ additively, we have $e(\lambda)e(\mu) = e(\lambda+\mu)$.

The sum in the definition may be taken over finitely many weights, by 2.6.3.

Remark 2.8.17. *By [21, II,Remark 2.7], finite dimensional G-modules have the same composition factors (including multiplicities) if and only if they have the same formal character.*

Lemma 2.8.18. *We have* $\operatorname{ch}(M \otimes N) = \operatorname{ch} M \cdot \operatorname{ch} N$ *and* $\operatorname{ch}(M \oplus N) = \operatorname{ch} M + \operatorname{ch} N$ *for G-modules M and N.*

Definition 2.8.19. Given an element $\phi = \sum a_\lambda e(\lambda)$ of $\mathbb{Z}[X(T)]$, we will denote by ϕ^F the element $\sum a_\lambda e(\lambda)^p$.

Lemma 2.8.20. *For a G-module V, we have* $\operatorname{ch}(V^F) = (\operatorname{ch} V)^F$.

Definition 2.8.21. Let M be a G-module. An ascending filtration $0 = M_0 \leq M_1 \leq M_2 \leq \ldots$ of M such that each successive quotient module is either 0 or isomorphic to an induced module $\nabla(\lambda)$ for some $\lambda \in X(T)_+$ (depending on the quotient) is called a *good filtration*. If instead the quotients are each either 0 or isomorphic to some Weyl module $\Delta(\lambda)$ (with λ again depending on the quotient), we call the filtration a *Weyl filtration* of M.

Definition 2.8.22. A finite-dimensional G-module that has both a good filtration and a Weyl filtration will be called a *tilting module*.

We have the following important facts about tilting modules. For proofs, see [11].

Lemma 2.8.23. *Direct sums and tensor products of tilting modules are again tilting, as are direct summands of tilting modules.*

Lemma 2.8.24. *Tilting modules are isomorphic if and only if they have the same formal character.*

Lemma 2.8.25. *For each $\lambda \in X(T)_+$ there exists an indecomposable tilting module $T(\lambda)$ with unique highest weight λ. Furthermore, λ has multiplicity 1 as a weight of $T(\lambda)$. The $T(\lambda)$ form a complete set of inequivalent indecomposable tilting modules.*

Definition 2.8.26. Recall the definition of the Ext functors $\mathrm{Ext}_G(M,\cdot)$ as the right derived functors of the hom functor $\mathrm{Hom}_G(M,\cdot)$.

Definition 2.8.27. Let A and B be G-modules. An *extension of A by B* is a short exact sequence $0 \to B \to E \to A \to 0$ of G-modules; in this case we may also refer to E as an *extension of A by B.* Two extensions $0 \to B \to E \to A \to 0$ and $0 \to B \to E' \to A \to 0$ are considered *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow \wr & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

of G-modules and G-module homomorphisms.

There is a well-known bijection between equivalence classes of extensions of A by B and the Abelian group $\mathrm{Ext}^1_G(A,B)$.

2.9 Motivation and definition of reductive pairs

In the last sections of the preliminaries we define and discuss several ideas concerning the properties of particular embeddings of algebraic groups. That is, these are properties of a pair (G,H) consisting of a group G and a particular subgroup H; a different copy H' of H seen inside G may or may not give rise to a pair

(G, H') with the same property. The first such definition that we will look at is due to Serre, and generalises the important concept of complete reducibility from representation theory. We then give a definition of reductive pairs, and discuss several results pertaining to these.

Definition 2.9.1. A subgroup H of a connected, reductive algebraic group G is said to be *G-completely reducible* if whenever H is contained in a parabolic subgroup P of G, H is also contained in a Levi subgroup of that parabolic.

When $G = \mathrm{GL}_n(K)$, this condition can be shown to reduce to the usual notion of complete reducibility: that is, $H \leq \mathrm{GL}_n(K)$ is $\mathrm{GL}_n(K)$-completely reducible if and only if H acts completely reducibly on K^n. The notion turns out not to be very interesting in characteristic 0: in this case, a subgroup of an algebraic group G is G-completely reducible if and only if it is reductive [2].

Recall that the *centraliser* of a subgroup $H \leq G$ is

$$C_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H\}.$$

We define the *infinitesimal centraliser* of the closed subgroup H of an algebraic group G to be the set

$$\mathfrak{c}_{\operatorname{Lie} G}(H) = \{x \in \operatorname{Lie} G \mid \operatorname{Ad}_h(x) = x \text{ for all } h \in H\}.$$

In all cases, we have $\operatorname{Lie}(C_G(H)) \subset \mathfrak{c}_{\operatorname{Lie} G}(H)$.

Definition 2.9.2. A subgroup H of G is *separable in G* if the Lie algebra of the centraliser of H in G is equal to the centraliser of H in $\operatorname{Lie} G$, that is, $\operatorname{Lie} C_G(H) = \mathfrak{c}_{\operatorname{Lie} G}(H)$.

Equivalently, H is separable in G if and only if the spaces in the definition have the same dimension. Any closed subgroup H of a general linear group $GL(V)$ is separable in $GL(V)$ [2, 3.5]. The motivation for this definition comes from the following situation.

Remark 2.9.3. *The word "separable" will have several meanings in different contexts. First, we say that a field extension E/F is* separable *if either F has characteristic 0, or, if F has characteristic p and $x_1, x_2, \ldots, x_m \in E$ are linearly independent over F, then their* p^{th} *powers remain linearly independent over F.*

Let $\phi : X \to Y$ be a morphism of irreducible varieties. We call ϕ dominant *if the comorphism $\phi^\sharp : K[Y] \to K[X]$ is injective. When ϕ is dominant, the field of fractions of (the integral domain) $K[Y]$ may therefore be identified with a subfield of the field of fractions of $K[X]$. If this field extension is separable, then we call ϕ* separable*. Note that over fields of characteristic 0, all morphisms are separable (from the definitions).*

We say a closed subgroup H is topologically generated by $h_1, h_2 \ldots, h_n \in G$ *if H is the closure of the subgroup of G generated by these elements. We let G act on the variety G^n by simultaneous conjugation, that is*

$$g \cdot (g_1, \ldots, g_n) := (gg_1g^{-1}, \ldots, gg_ng^{-1})$$

for an n-tuple $(g_1, \ldots, g_n) \in G^n$. In this case, the orbit map $G \to G \cdot (h_1, \ldots, h_n)$ is a separable morphism if and only if H is separable in G in the sense of definition 2.9.2.

We now define reductive pairs, giving some of the results that motivated the work in this thesis. Reductive pairs were introduced by Richardson in [25]. In the paper appears the following result; the definition will follow. We write $C(g, G)$ for the G-conjugacy class of g.

Theorem 2.9.4 (Richardson)**.** *Let V be a finite-dimensional vector space over an algebraically closed field (of arbitrary characteristic); and let H be an algebraic subgroup of $\mathrm{GL}(V)$ such that $(\mathrm{GL}(V), H)$ is a reductive pair.*

1. *If $x \in \mathrm{Lie}\, H$, then $C(x, \mathrm{Lie}\,\mathrm{GL}(V)) \cap \mathrm{Lie}\, H$ is the union of a finite number of H-conjugacy classes.*

2. *If $h \in H$, then $C(h, \mathrm{GL}(V)) \cap H$ is the union of a finite number of H-conjugacy classes.*

If we let K have characteristic 0, then more can be said: in this case we may replace $\mathrm{GL}(V)$ in the statement with any connected reductive algebraic group G. The proof of theorem 2.9.4 uses the stipulation that $(\mathrm{GL}(V), H)$ be a reductive pair in an essential way, in an argument that has to do with tangent spaces to orbits of the conjugation action of G.

In the situation described in the statement of the theorem, every element of $C(h, \mathrm{GL}(V))$ has the same Jordan normal form. If we intersect this $\mathrm{GL}(V)$-conjugacy class with H, then the result is a union of H-conjugacy classes. The theorem tells us that we may write $C(h, \mathrm{GL}(V)) \cap H$ as a *finite* union of H-conjugacy classes. Hence there is only a finite number of H-conjugacy classes of h with a given Jordan normal form. In particular, there are only a finite number of H-conjugacy classes of unipotent elements, since all eigenvalues of such a matrix are 1. The use of reductive pairs also simplifies a result that was known prior to Richardson's paper: in good characteristic, all simple groups (and by extension all semisimple) algebraic groups have only finitely many conjugacy classes of unipotent elements. Richardson's proof involves finding an embedding of simple groups G of each type (except those of type A_n)[2] as a reductive pair

[2]That groups of type A_n have finitely many unipotent conjugacy classes follows from considering a correspondence between partitions of n and sizes of Jordan blocks.

$(\mathrm{GL}(V), G)$.

Definition 2.9.5. Let H be a closed, reductive subgroup of a reductive group G. We say that (G, H) is a *reductive pair* if $\operatorname{Lie} H$ is an H-module direct summand of $\operatorname{Lie} G$, where H acts via the adjoint representation of G.

Since their introduction, reductive pairs have found use in the work of others.

Theorem 2.9.6. *[3, 1.4] Let H be a reductive subgroup of a reductive group G such that (G, H) is a reductive pair. Let H' be a subgroup of H such that H' is a separable subgroup in G. Then H' is separable in H. Moreover, if H' is G-completely reducible, then it is also H-completely reducible.*

The same paper includes a result giving conditions under which reductive pairs of a certain form will not occur.

Theorem 2.9.7. *[3, Corollary 2.13] If $(\mathrm{GL}(V), G)$ is a reductive pair, then every subgroup of G is separable in G.*

In particular, G must then be separable in itself, and if this is not the case, then there cannot be any reductive pairs of the form $(\mathrm{GL}(V), G)$. That a group fail to be separable in itself does sometimes occur: in characteristic p, the group $\mathrm{SL}_p(K)$ has this property. To see this, consider that the infinitesimal centraliser of $\mathrm{SL}_p(K)$ contains the scalar matrices, since these all have trace 0 in characteristic p. Its dimension is thus positive. However, in characteristic p the requirement that the scalar matrices in $\mathrm{SL}_p(K)$ have determinant 1 is restrictive enough to ensure that the centre of $\mathrm{SL}_p(K)$ is trivial; its Lie algebra therefore has dimension 0.

Let G be simple, and define an integer $a(G)$ as the rank of G plus 1. For a reductive group G, define $a(G)$ as the maximum of the ranks of the simple components of G. The following theorem appears in a paper by Serre, and has a proof using several results, including a rather technical case-by-case analysis concerning groups of exceptional type [22]. Given a subgroup $H \leq G$, we will write $|G, H|$ for the index of H in G.

Theorem 2.9.8. *[27, 4.4] Suppose $p \geq a(G)$ and that $|H : H^\circ|$ is prime to p. Then H° is reductive if and only if H is G-completely reducible. (For much relevant discussion, see [3, 1.2]).*

In [1], a version of this result is presented with a less favourable bound; however, this time the proof does not rely on the case-by-case treatement found in the proof of 2.9.8, and is conceptually much simpler. We call a rational G-module *non-degenerate* if the identity component of the kernel of the representation afforded by V is a torus.

Theorem 2.9.9. *[1, 3.5] Suppose $p > 2\dim V - 2$ for a non-degenerate G-module V and that $|H : H^\circ|$ is prime to p. Then H° is reductive if and only if H is G-completely reducible.*

The uniformity of the proof of this result comes through its exploitation of a result in the same paper that guarantees the existence of a particular reductive pair. In particular, the following results are used.

Theorem 2.9.10. *[1, 3.3] Let H be a closed subgroup of G and V be a G-module.*

1. *Suppose that $p \geq \dim V$ and that $|H : H^\circ|$ is prime to p. If H is G-completely reducible, then V is a semisimple H-module.*

2. *Suppose that V is non-degenerate and $p > 2\dim V - 2$. If V is semisimple as an H-module, then H is G-completely reducible.*

Theorem 2.9.11. *[1, 3.1] Suppose* $p > 2\dim V - 2$*. Then* $(\mathrm{GL}(V), \rho(G))$ *is a reductive pair.*

Proof. Since we assume $p > 2\dim V - 2$, it is also true that $p \geq \dim V$, so that Jantzen's semisimplicity theorem [20, 2.1] tells us that V is semisimple. Again since $p > 2\dim V - 2$, Serre's theorem on the semisimplicity of tensor products [26, Thm 1] implies that $V \otimes V^*$ is also semisimple. This module is isomorphic to the Lie algebra $\mathrm{Lie}(\mathrm{GL}(V))$, so the submodule $\mathrm{Lie}(\rho(G))$ has a direct complement. □

Remark 2.9.12. *The assumption that* $(GL(V), \rho(G))$ *is a reductive pair is used in a crucial way in the proof of the second point of theorem 2.9.10; however, the method by which it is shown to be a reductive pair, by appeal to Jantzen and Serre's semisimplicty results, is not relevant to the proof. Although the bounds in both semisimplicity results are sharp (cf. [20, 26]), this is not to say that for a given* G *there might not be many examples where we get a reductive pair* $(\mathrm{GL}(V), \rho(G))$ *with* $p \leq 2\dim V - 2$*. In some respects, an investigation into this situation was the starting point of the present research project, commencing with the group* $\mathrm{SL}_2(K)$*.*

Since it will greatly improve the clarity and brevity of what follows, we make the following definition.

Definition 2.9.13. Let V be a G-module affording the representation $\rho : G \to \mathrm{GL}(V)$. We will say that V *gives a reductive pair* if and only if $(\mathrm{GL}(V), \rho(G))$ is a reductive pair.

2.10 First results and methods

In this section can be found discussion of results that will be applicable to each of the cases we study later. The general approach taken is as follows. Given a rational representation $\rho : G \to \mathrm{GL}(V)$ of an algebraic group G over an algebraically closed field K of positive characteristic, we want to know whether or not $(\mathrm{GL}(V), \rho(G))$ is a reductive pair. Considered as G-modules via the adjoint action of $\mathrm{GL}(V)$, we first identify the Lie algebra of $\mathrm{GL}(V)$ with $\mathrm{End}(V) \cong V \otimes V^*$. The main reason for doing this is to take advantage of results relating to the decomposition of tensor products. We then need to establish whether or not $\mathrm{Lie}\,\rho(G)$, or rather the image of this Lie algebra in $V \otimes V^*$, is a direct summand; in the following, if we speak of $\mathrm{Lie}\,\rho(G)$ as being a submodule of $V \otimes V^*$, we tacitly refer to this image. One of the main difficulties to overcome in this work is correctly identifying the submodule of $V \otimes V^*$ corresponding to the Lie algebra of $\rho(G)$. For the group $SL_2(K)$, we will work around this problem by showing that there can effectively be only one such submodule. In general, our approach will be to prove results about all modules isomorphic to $\mathrm{Lie}\,\rho(G)$.

We shall make much use of Steinberg's tensor product theorem for simply connected, semi-simple algebraic groups (below). A full proof can be found in Jantzen [21, 3.17]. The idea is to prove first that, given a weight $\lambda \in X_r(T)$ and a dominant weight μ, we have $L(\lambda + p^r\mu) \cong L(\lambda) \otimes L(\mu)^{F^r}$. The result as stated below then follows by induction. The proof in [21] capitalizes on the relationship between representations of the Frobenius kernels G_r with those of G, which is suggested in Cline, Parshall and Scott [10] (this paper contains a short proof of the result which does not require the methods of group schemes).

Theorem 2.10.1 (Steinberg's tensor product theorem)**.** *Let* $\lambda \in X(T)_+$ *with*

$\lambda = \sum_{i=0}^{m} p^i \lambda_i$, *where the* $\lambda_i \in X_1(T)$. *Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^F \otimes \cdots \otimes L(\lambda_m)^{F^m}.$$

Remark 2.10.2. *Every dominant weight may be written as such a sum. For example, in a root system of type* A_2 *in characteristic* $p = 5$, *consider the weight labelled* $(12, 70)$. *We may write* 12 *and* 70 *in base* 5 *as* $12 = 2 + (2 \times 5)$, $70 = 0 + (4 \times 5) + (2 \times 5^2)$. *Then* $(12, 70) = (2, 0) + 5(2, 4) + 5^2(0, 2)$. *This theorem frequently allows us to break down a complicated problem about an arbitrary simple module into a possibly easier problem about the (*finitely many*) restricted simple modules.*

The following result appears first in [5][3], stated in terms of modules for finite groups. The arguments used hold when considering modules for cocommutative Hopf algebras, so that the result is in fact applicable to a very wide range of classes of modules, including KG-modules for any group algebra KG. We will state the result only in the form in which we will use it.

Proposition 2.10.3 (Benson and Carlson)**.** *Let* M *and* N *be finite dimensional* KG*-modules. Then* K *is a summand of* $M \otimes N$ *if and only if the following two conditions are met:*

1. $M \cong N^*$
2. $p \nmid \dim N$

Moreover, if K *is a direct summand of* $N \otimes N^*$ *then it occurs with multiplicity one.*

[3]For a more detailed discussion the reader may consult Benson's book [4].

Remark 2.10.4. *In the case of a KG-module V, it is straight forward to see that, if $p \nmid \dim V$, there is a summand of $V \otimes V^*$ isomorphic to K. We have that $V \otimes V^* \cong \mathrm{End}(V) \cong \mathrm{Mat}_{\dim V}(K)$, the $\dim V \times \dim V$ matrices with entries in K. If $p \nmid \dim V$, then the set of scalar matrices is a direct summand of this module, with direct complement the collection of matrices with trace zero.*

The above result will itself be useful in a later section, but for now it is worth noting the following corollary.

Corollary 2.10.5 (Benson and Carlson)**.** *Suppose M is an indecomposable KG-module with $p \mid \dim M$. Then for any KG-module N and any indecomposable summand U of $M \otimes N$, we have $p \mid \dim U$.*

The proof is by contradiction, supposing first that $p \nmid \dim U$, then applying 2.10.3 twice, using the associativity of the tensor product.

Remark 2.10.6. *We will use this result as follows. Corollary 2.10.5 implies that for any indecomposable G-module M, if $p \mid \dim M$, then $M \otimes N$ can have no summands with dimension not divisible by p, for any G-module N. This will be of particular use in cases when $p \nmid \dim \mathrm{Lie}\, G$.*

By the reasoning described in the remark, we arrive at the following.

Proposition 2.10.7. *Let K have characteristic p, and suppose $p \nmid \dim \mathrm{Lie}\, G$. Let $V = V_1 \otimes \cdots \otimes V_r$ be a G-module such that one of the V_i is indecomposable and has dimension divisible by p. Then V does not give a reductive pair.*

Proof. Suppose the factor V_i is indecomposable and has dimension divisible by p. Write

$$V \otimes V^* \cong V_i \otimes (V_1 \otimes \cdots \otimes V_r \otimes (V_1 \otimes \cdots \otimes V_r)^*),$$

where we have rearranged the tensor product to bring V_i to the front. Now corollary 2.10.5 implies that all indecomposable summands of $V \otimes V^*$ have dimension divisible by p. In particular, $\operatorname{Lie} G$ (being indecomposable) cannot be a summand, hence the result. □

Lemma 2.10.8. *Let G be a reductive algebraic group over K and V a G-module. Then V^F gives a reductive pair if and only if V does.*

Proof. Recall that V^F and V are equal as K-vector spaces, so that $GL(V) = GL(V^F)$. Let the representation afforded by the module V be denoted by ρ. Since the Frobenius morphism is a bijection, the subgroups $\rho(G)$ and $\rho \circ F(G)$ of $GL(V)$ are equal, from which the result follows. □

Remark 2.10.9. *The above lemma is especially helpful when Steinberg's tensor product theorem is taken into account: if $\lambda = p\mu$, with λ, μ dominant weights, then $L(\lambda) = L(\mu)^F$.*

Lemma 2.10.10. *Let V, W be G-modules and let ρ be the representation afforded by V. Suppose that V gives a reductive pair and that the differential $d\rho$ is injective; suppose further that* $\operatorname{End}(W)$ *has a summand isomorphic to K. Then the module $V \otimes W^F$ gives a reductive pair.*

Proof. Let ρ and σ be the representations afforded by V and W respectively. If we consider the representation

$$\rho \otimes \sigma^F : G \to \mathrm{GL}(V \otimes W^F),$$

then the aim is to show that $\operatorname{Lie}(\rho \otimes \sigma^F(G))$ is a summand of $\operatorname{End}(V \otimes W^F)$.

First, we have the differential $d(\rho \otimes \sigma^F) : \operatorname{Lie} G \to \operatorname{End}(V \otimes W^F)$, and

$$\operatorname{Im}(d(\rho \otimes \sigma^F)) \subset \operatorname{Lie}(\rho \otimes \sigma^F(G)), \tag{2.1}$$

that is, the image of the differential is contained in the Lie algebra of the image of the representation.

By the properties of the differential ([6, 3.21]), $d(\rho \otimes \sigma^F) = d\rho \otimes 1_{W^F} + 1_V \otimes d\sigma^F$ (where here 1_{W^F} refers to the identity map on W^F); since the map σ^F is equal to $\sigma \circ F$ and the differential of the Frobenius morphism F is 0, we have that $d\sigma^F = 0$, whence $d(\rho \otimes \sigma^F) = d\rho \otimes 1_{W^F}$. Thus

$$\dim d(\rho \otimes \sigma^F)(\operatorname{Lie} G) = \dim(d\rho \otimes 1_{W^F})(\operatorname{Lie} G),$$

which, since $d\rho$ is an isomorphism onto its image, is equal to $\dim \operatorname{Lie} G$. However,

$$\dim \operatorname{Lie} G \geq \dim \operatorname{Lie}(\rho \otimes \sigma^F(G)),$$

since the Lie algebra on the right is that of an algebraic group morphic image of the group the Lie algebra of which is on the left. We therefore have equality in 2.1. Thus we may look for the image of the differential rather than the Lie algebra of the image when deciding if $V \otimes W^F$ gives a reductive pair.

We note that $\operatorname{End}(V \otimes W^F) \cong \operatorname{End}(V) \otimes \operatorname{End}(W^F)$ [7]. It will be convenient to identify these spaces via such an isomorphism. Thus

$$d(\rho \otimes \sigma^F) = d\rho \otimes 1_{W^F} : \operatorname{Lie} G \to \operatorname{End}(V) \otimes \operatorname{End}(W^F).$$

So we have $\operatorname{Lie}(\rho \otimes \sigma^F(G)) = (d\rho \otimes 1_{W^F})(\operatorname{Lie} G) = \{d\rho X \otimes 1_{W^F} \mid X \in \operatorname{Lie} G\} =$

$d\rho\, \mathrm{Lie}\, G \otimes K^F$. Finally, we note that since $\mathrm{Lie}\, G$ is a summand of $\mathrm{End}(V)$ and K^F is a summand of $\mathrm{End}(W^F)$, their tensor product $\mathrm{Lie}\, G \otimes K^F$ is a summand of $\mathrm{End}(V) \otimes \mathrm{End}(W^F) \cong \mathrm{End}(V \otimes W^F)$. □

Corollary 2.10.11. *Let G be a simple algebraic group over a field K of positive characteristic p such that p is very good for G, let λ be a restricted dominant weight such that the simple G-module $L(\lambda)$ gives a reductive pair, and let μ be a dominant weight such that $p \nmid \dim L(\mu)$. Then the module $L(\lambda + p^n\mu)$ gives a reductive pair for any integer $n \geq 1$.*

Proof. Since p is very good for G and G is simple, the Lie algebra of G is simple (as a Lie algebra, hence also as a module) [3]. Thus any homomorphism leaving $\mathrm{Lie}\, G$ is either the zero map or is injective; since λ is restricted, the differential is therefore injective. Since $L(\lambda)$ gives a reductive pair, we know that $\mathrm{Lie}\, G$ is a summand of $\mathrm{Lie}\, \mathrm{GL}(V)$. Since $p \nmid \dim L(\mu)$, proposition 2.10.3 tells us that this module has a summand isomorphic to K. We now apply lemma 2.10.10, noting that $L(\lambda) \otimes L(\mu)^{F^n} \cong L(\lambda + p^n\mu)$. □

Remark 2.10.12. *If we relax the conditions on λ and μ, more may be said. If λ, μ are required only to be dominant weights, then it may be that $L(\lambda) \otimes L(\mu)^{F^n}$ is not a simple module; however, the lemma still applies, so this module still gives a reductive pair. Weaker conditions can be found to ensure that the module is still simple, for instance requiring that $\lambda \in X_n(T)$ with non-zero restricted part (see discussion of Steinberg's tensor product theorem, 2.10.1).*

Chapter 3

SL_2

3.1 Basic facts

In this chapter we focus on the group $G = \mathrm{SL}_2(K)$. A great deal is known about this group and its representations, making it a logical choice for a first example.

Lemma 3.1.1. *(a) For $0 \leq u \leq p-1$ we have $T(u) = L(u) = \nabla(u) = \Delta(u)$.*

(b) For $p \leq u \leq 2p-2$ the module $T(u)$ is uniserial and its unique composition series has the form $[L(2p-2-u), L(u), L(2p-2-u)]$. Moreover, $T(u)$ is a non-split extension of $\Delta(2p-2-u)$ by $\Delta(u)$ (or, dually, of $\nabla(u)$ by $\nabla(2p-2-u)$). [14, 1.1]

It is well-known (see for instance [14]) that for $\mathrm{SL}_2(K)$ the module $\nabla(r) \cong S^r E$, the r^{th} symmetric power of the 2-dimensional natural module for $\mathrm{SL}_2(K)$. Thus the simple $\mathrm{SL}_2(K)$-modules are tensor products of Frobenius twists of such symmetric powers, by Steinberg's tensor product theorem. As in [14], we shall call the tilting modules $T(u)$ described in the previous result *fundamental*.

Remark 3.1.2. *As the characteristic gets larger, more of the dominant weights* *fall in the restricted region, so that more of the behaviour correctly models characteristic* 0 *theory.*

Lemma 3.1.3. *The simple* $\mathrm{SL}_2(K)$*-modules are self-dual. That is,* $L(\lambda)^* \cong L(\lambda)$.

Proof. By [21, II,2.5], the dual of $L(\lambda)$ is the module $L(-w_0\lambda)$. Since SL_2 is of type A_1 so that the Weyl group of SL_2 is of order 2, the longest element w_0 sends each weight to its negative, hence the result. □

Lemma 3.1.4. *Let* K *have characteristic* $p \geq 3$*, and let* $\rho : \mathrm{SL}_2(K) \to \mathrm{GL}(V)$ *be a rational representation. Then* $\mathrm{Lie}\,\rho(\mathrm{SL}_2(K))$ *is isomorphic to the Lie algebra* $\mathfrak{sl}_2(K)$ *of trace-zero* 2×2 *matrices with entries in* K.

Proof. The kernel of ρ is a closed normal subgroup of $\mathrm{SL}_2(K)$ and is therefore either trivial or is the centre of $\mathrm{SL}_2(K)$, which consists of those scalar matrices having determinant 1. Thus the image $\rho(\mathrm{SL}_2(K))$ is therefore isomorphic as an abstract group to $\mathrm{SL}_2(K)$, and as an algebraic group either to $\mathrm{SL}_2(K)$ or to $\mathrm{PGL}_2(K)$. Since $p \nmid 2$, the Lie algebras of both of these groups are isomorphic, and in particular are isomorphic to $\mathfrak{sl}_2(K)$. □

The following result is abridged from [23, 15.20], which lists the cases in which the adjoint representation of a simple algebraic group G is irreducible.

Theorem 3.1.5. *If* K *has characteristic* p *and* $p \nmid n$*, then* $\mathrm{Lie}\,\mathrm{SL}_n(K)$ *is an irreducible* $\mathrm{SL}_n(K)$*-module.*

Thus when deciding whether or not a representation gives a reductive pair, if the characteristic is greater than or equal to 3, we may be certain the Lie algebra we look for is isomorphic to $L(2)$, the 3-dimensional simple SL_2-module with highest weight 2. We will address the issue of characteristic 2 in the discussion of the main results of this chapter, as in each case the argument we use will be specific to the types of representation under consideration.

3.2 Simple modules

In [14], Doty and Henke provide a decomposition of an arbitrary tensor product of simple modules for SL_2 into a direct sum of tensor products of Frobenius twisted fundamental tilting modules. Since we are concerned with tensor products of simple modules with their duals, and since the simple SL_2 modules are self-dual, we will make use of this result in a simpler special case; we will nevertheless include the full statement (as theorem 3.2.5) for interest's sake.

Lemma 3.2.1. *Let $\lambda = a_0 + a_1 p + \cdots + a_r p^r$ be a non-negative integer. Then the dimension of the simple KSL_2-module $L(\lambda)$ is $\prod_{i=0}^{r}(a_i + 1)$.*

Proof. By Steinberg's tensor product theorem, we have that $L(\lambda) = L(a_0) \otimes L(a_1)^F \otimes \cdots \otimes L(a_r)^{F^R}$. Since the underlying vector space of a module M is the same as that of the module M^F, we have that the dimension of $L(\lambda)$ is the product of the dimensions of the $L(a_i)$. By lemma 3.1.1, the simple KSL_2-modules with restricted highest weight are symmetric powers of the 2-dimensional natural module E; it is well-known that the dimension of the i^{th} symmetric power $S^i E$ is the multiset coefficient[1]

$$\dim S^i E = \left(\!\!\binom{\dim E}{i}\!\!\right) = \binom{\dim E + i - 1}{i} = i + 1,$$

hence the result. □

Lemma 3.2.2. *Suppose $p > 3$, and let λ be a non-negative integer with base p*

[1] That is, choice with replacement.

expansion $\lambda = a_0 + a_1 p + \cdots + a_k p^k$*. Supppose at least one of the* $a_i = p-1$*. Then* $L(\lambda)$ *does not give a reductive pair.*

Proof. Apply proposition 2.10.7, noting that by theorem 2.10.1,

$$L(\lambda) \cong L(a_0) \otimes L(a_1)^F \otimes \cdots \otimes L(a_k)^{F^k},$$

and that by lemmas 3.2.1 and 2.10.8, the term $L(a_i)^{F^i}$ has dimension p. □

Remark 3.2.3. *This provides an initial constraint on the simple modules that can give rise to reductive pairs; we shall see that it agrees with the complete picture that will emerge for this case.*

Lemma 3.2.4 (Doty and Henke)**.** *Let L,*L' *be two simple modules with highest weights inclusively between 0 and p-1. Then* $L \otimes L'$ *is tilting, and isomorphic with the direct sum of* $T(u)$ *as* u *varies over a set* $W(L, L')$ *of weights which can be computed as follows. Let* r *(resp.,* s*) be the larger (resp., smaller) of the highest weights of* L, L'*. List the weights* $r+s, r+s-2, \ldots, r-s$*. For each* $u \geq p$ *on this list, strike out the number* $2p-2-u$ *from the list. What remains is the set* $W(L, L')$*. In other words, if* $S = \{r+s-2i\}_{i=0}^{s}$*, then*

$$W(L, L') = S - \{2p-2-u | u \in S, u \geq p\}.$$

In particular, $L \otimes L'$ *is indecomposable if and only if* $s = 0$ *or* $(r, s) = (p-1, 1)$*.*

Theorem 3.2.5 (Doty and Henke)**.** *Let* r, r' *be arbitrary non-negative integers. The tensor product* $L(r) \otimes L(r')$ *can be expressed as a direct sum of twisted tensor* *products of fundamental tilting modules. In fact, we have*

$$L(r) \otimes L(r') \cong \bigoplus_{\mathbf{u}} \left(\bigotimes T(u_i)^{F^i} \right),$$

where $\mathbf{u} = (u_0, \ldots, u_m)$ *ranges over all elements of the finite Cartesian product*

$$W = W(\delta_0(r), \delta_0(r')) \times W(\delta_1(r), \delta_1(r')) \times \cdots \times W(\delta_m(r), \delta_m(r'))$$

of the sets described in Lemma 3.2.4, and where m *is the* p*-adic length of the largest of* r, r'. *Given* $\mathbf{u}$ *as above, the corresponding indecomposable direct summand* $J(\mathbf{u}) = \otimes_{i=0}^{m} T(u_i)^{F^i}$ *is always contravariantly self-dual, with simple socle and head isomorphic with* $L(\sum_{i=0}^{m} \tilde{u}_i p^i)$, *where* $\tilde{u}_i$ *is defined by*

$$\tilde{u}_i = \begin{cases} u_i & \text{if } u_i \leq p-1, \\ 2p-2-u_i & \text{otherwise.} \end{cases} \tag{3.1}$$

We shall show that, given a non-negative integer λ, there is precisely one submodule of $L(\lambda) \otimes L(\lambda)^*$ isomorphic to $L(2)$. We shall do this in a series of results.

It is well known that tensor products of induced modules have filtrations by induced modules; in the case of $\mathrm{SL}_2(K)$, we may be very explicit about this.

Lemma 3.2.6. *There is a short exact sequence* $0 \to \nabla(\mu - 1) \otimes \nabla(\nu - 1) \to \nabla(\mu) \otimes \nabla(\nu) \to \nabla(\mu + \nu) \to 0$. *In particular, the module* $\nabla(\mu) \otimes \nabla(\nu)$ *has a filtration with sections isomorphic to* $\nabla(\mu + \nu), \nabla(\mu + \nu - 2), \ldots, \nabla(\mu - \nu)$ *(each with multiplicity one, with* $\nu \leq \mu$*).*

For the sake of brevity in the proof, we will define

$$u(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

Proof. Consider the map $\phi : \nabla(\mu) \otimes \nabla(\nu) \to \nabla(\mu + \nu)$ defined by sending a typical basis vector[2] $x^{\mu-a}y^a \otimes x^{\nu-b}y^b$ to $x^{\mu+\nu-a-b}y^{a+b}$ and extending linearly. Note that ϕ is clearly surjective; we shall show that it is an SL_2-module homomorphism.

We have that SL_2 is generated by the upper and lower unipotent subgroups U^+ and U^-; because of the symmetry of the roles of the symbols x and y in the symmetric powers, it is therefore enough to show that, for $g \in U^+ \subset SL_2$ and $X \in \nabla(\mu) \otimes \nabla(\nu)$, we have $g\phi(X) = \phi(gX)$. In particular, having shown the result for $g = u(c)$, $c \in K$, we may infer the same result for $u^-(c)$, swapping the symbols x and y in the argument. Now let $g = u(c)$, and let $X = x^{\mu-a}y^a \otimes x^{\nu-b}y^b$ be a basis element of $\nabla(\mu) \otimes \nabla(\nu)$. We have that $gX = x^{\mu-a}(cx + y)^a \otimes x^{\nu-b}(cx + y)^b$. Expanding these brackets using the binomial formula, we see

$$\begin{aligned} gX &= x^{\mu-a} \sum_{i=0}^{a} \binom{a}{i} c^{a-i} x^{a-i} y^i \otimes x^{\nu-b} \sum_{j=0}^{b} \binom{b}{j} c^{b-j} x^{b-j} y^j \\ &= \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} c^{a+b-i-j} x^{\mu-i} y^i \otimes x^{\nu-j} y^j, \end{aligned}$$

where we have used the bilinearity of the tensor product repeatedly. Since this is

[2] Here we shall abuse notation by neglecting to draw a distinction between x's and y's on different sides of the tensor product. Provided that we tacitly respect the distinction and do not attempt to "bring an x over the tensor product" (or similar), this will not cause a problem.

now expressed as a linear combination of basis vectors we can see that

$$\begin{aligned}\phi(gX) &= \sum_{i=0}^{a}\sum_{j=0}^{b}\binom{a}{i}\binom{b}{j}c^{a+b-i-j}x^{\mu+\nu-i-j}y^{i+j} \\ &= x^{\mu+\nu-a-b}\sum_{i=0}^{a}\sum_{j=0}^{b}\binom{a}{i}\binom{b}{j}c^{a+b-i-j}x^{a+b-i-j}y^{i+j} \\ &= x^{\mu+\nu-a-b}\sum_{\substack{0\le i\le a\\ 0\le j\le b\\ i+j=k}}\binom{a}{i}\binom{b}{j}c^{a+b-k}x^{a+b-k}y^{k},\end{aligned}$$

where we have combined the sums and brought out the powers of x in order to clarify an argument to follow. On the other hand, we have that

$$\begin{aligned}g\phi(X) &= gx^{\mu+\nu-a-b}y^{a+b} \\ &= x^{\mu+\nu-a-b}(cx+y)^{a+b} \\ &= x^{\mu+\nu-a-b}\sum_{k=0}^{a+b}\binom{a+b}{k}c^{a+b-k}x^{a+b-k}y^{k}.\end{aligned}$$

We now claim that $g\phi(X) = \phi(gX)$. To see this, consider the polynomial

$$(cx+y)^{a+b} = (cx+y)^{a}(cx+y)^{b}.$$

Once again using the binomial formula to expand each bracket, we compare coefficients of $x^{a+b-k}y^k$ on each side for a given $k \in \{0, 1, \ldots, a+b\}$. Doing so, we find that

$$\sum_{k=0}^{a+b}\binom{a+b}{k}c^{a+b-k} = \sum_{\substack{0\le i\le a\\ 0\le j\le b\\ i+j=k}}\binom{a}{i}\binom{b}{j}c^{a+b-k}.$$

Thus ϕ is a homomorphism of SL_2-modules.

Next, consider the map $\psi : \nabla(\mu-1)\otimes\nabla(\nu-1) \to \nabla(\mu)\otimes\nabla(\nu)$ defined by the following.

$$x^{\mu-1-a}y^a \otimes x^{\nu-1-b}y^b \mapsto x^{\mu-a}y^a \otimes x^{\nu-1-b}y^{b+1} - x^{\mu-1-a}y^{a+1} \otimes x^{\nu-b}y^b.$$

That is, we increase the power of x on the left, then balance the effect this has on the weight by increasing the power of y on the right; to ensure that the end result is an element of the kernel of ϕ, we then subtract from this the vector we get by applying the same rule reversed. Now, ψ is clearly injective, and we shall now see that it is an SL_2-module homomorphism. As before, we check this for an element of U^+.

Let $g = u(c)$, $X = x^{\mu-1-a}y^a \otimes x^{\nu-1-b}y^b$. Then

$$\psi(gX) = \sum_{i=0}^{a}\sum_{j=0}^{b}\binom{a}{i}\binom{b}{j}c^{a+b-i-j}\left(x^{\mu-i}y^i \otimes x^{\nu-1-j}y^{j+1} - x^{\mu-1-i}y^{i+1} \otimes x^{\nu-j}y^j\right),$$

where the omitted steps in the calculation are similar to those performed for the map ϕ. Next,

$$\begin{aligned} g\psi(X) &= x^{\mu-a}(cx+y)^a \otimes x^{\nu-1-b}(cx+y)^{b+1} - x^{\mu-1-a}(cx+y)^{a+1} \otimes x^{\nu-b}(cx+y)^b \\ &= x^{\mu-a}(cx+y)^a \otimes x^{\nu-1-b}(cx+y)^b(cx+y) - x^{\mu-1-a}(cx+y)^a(cx+y) \\ &\qquad \otimes x^{\nu-b}(cx+y)^b \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{a}\binom{a}{k}c^{a-k}x^{\mu-k}y^k \otimes \sum_{l=0}^{b}\left(\binom{b}{l}c^{b-l}x^{\nu-1-l}y^l(cx+y)\right) \\ &\quad - \sum_{m=0}^{a}\left(\binom{a}{m}c^{a-m}x^{\mu-1-m}y^m(cx+y)\right) \otimes \sum_{n=0}^{b}\binom{b}{n}c^{b-n}x^{\nu-n}y^n \end{aligned}$$

$$= \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} c^{a+b-i-j} \big(x^{\mu-i} y^{i} \otimes x^{\nu-1-j} y^{j} (cx+y) - x^{\mu-1-i} y^{i} (cx+y) \otimes x^{\nu-j} y^{j} \big),$$

and note that the term in brackets is equal to

$$x^{\mu-i}y^{i} \otimes cx^{\nu-j}y^{j} + x^{\mu-i}y^{i} \otimes x^{\nu-1-j}y^{j+1} - cx^{\mu-i}y^{i} \otimes x^{\nu-j}y^{j} - x^{\mu-1-i}y^{i+i} \otimes x^{\nu-j}y^{j},$$

the first and third terms of which cancel, leaving

$$x^{\mu-i}y^{i} \otimes x^{\nu-1-j}y^{j+1} - x^{\mu-1-i}y^{i+1} \otimes x^{\nu-j}y^{j}.$$

Thus we see that $\psi(gX) = g\psi(X)$, as required.

From the above, it is clear that for a given pair of dominant weights (μ, ν) with $\mu \geq \nu$ we have an exact sequence of SL_2-modules $\nabla(\mu-1) \otimes \nabla(\nu-1) \hookrightarrow \nabla(\mu) \otimes \nabla(\nu) \twoheadrightarrow \nabla(\mu+\nu)$. In fact we see

$$\begin{array}{ccccccc}
\nabla(\mu-\nu) \otimes \nabla(0) & \rightarrow & \cdots & \rightarrow & \nabla(\mu-1) \otimes \nabla(\nu-1) & \rightarrow & \nabla(\mu) \otimes \nabla(\nu) \\
\downarrow & & & & \downarrow & & \downarrow \\
\nabla(\mu-\nu) & & & & \nabla(\mu+\nu-2) & & \nabla(\mu+\nu),
\end{array}$$

where the horizontal maps are injective, and the vertical maps are surjective. In particular, we see that $\nabla(\mu) \otimes \nabla(\nu)$ has a filtration with sections $\nabla(\mu+\nu), \nabla(\mu+\nu-2), \ldots, \nabla(\mu-\nu)$. □

Lemma 3.2.7. $\dim \mathrm{Hom}_{SL_2}(\Delta(\lambda), \nabla(\mu) \otimes \nabla(\nu)) \leq 1$.

Proof. By the zero case of [21, Prop. 4.13], we know that

$$\dim \mathrm{Hom}_G(\Delta(\alpha), \nabla(\beta)) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{3.2}$$

We also know that for a short exact sequence $0 \to Y_1 \to Y \to Y_2 \to 0$ of KG-modules, $\dim \mathrm{Hom}_G(X, Y) \leq \dim \mathrm{Hom}_G(X, Y_1) + \dim \mathrm{Hom}_G(X, Y_2)$ for a given module X. We apply this result inductively to the filtration of lemma 3.2.6, giving

$$\dim \mathrm{Hom}_G(\Delta(\lambda), \nabla(\mu) \otimes \nabla(\nu)) \leq \sum_{i=0}^{\nu} \dim \mathrm{Hom}_G(\Delta(\lambda), \nabla(\mu + \nu - 2i)). \tag{3.3}$$

Since at most one of the integers $\mu + \nu - 2i$ is equal to λ, formula 3.2 implies that the right hand side is at most 1. □

In fact, 3.3 may be shown to be an equality, using [12, Prop. A2.2]

Corollary 3.2.8. $\dim \mathrm{Hom}_{SL_2}(L(\lambda), L(\mu) \otimes L(\nu)) \leq 1$.

Proof. Note that $\nabla(\lambda)$ has socle isomorphic to $L(\lambda)$, and $\Delta(\lambda)$ has head isomorphic to $L(\lambda)$. Thus $\mathrm{Hom}_{SL_2}(L(\lambda), L(\mu) \otimes L(\nu))$ embeds in $\mathrm{Hom}_{SL_2}(\Delta(\lambda), L(\mu) \otimes L(\nu))$. Next, $\mathrm{Hom}_{SL_2}(\Delta(\lambda), L(\mu) \otimes L(\nu)) \leq \mathrm{Hom}_{SL_2}(\Delta(\lambda), \nabla(\mu) \otimes \nabla(\nu))$. Thus it is enough to prove that this latter space has dimension ≤ 1, which is lemma 3.2.7.

□

Remark 3.2.9. *Recall that in characteristic $p \neq 2$, the Lie algebra of the image of SL_2 under a representation is isomorphic to the simple module $L(2)$ (cf. 3.1.4). The argument above shows in particular that there is at most (hence exactly) one submodule of any tensor product $L(\lambda) \otimes L(\lambda)^* \cong L(\lambda) \otimes L(\lambda)$ that is isomorphic* *to $L(2)$, hence to the Lie algebra of the image of SL_2 in the representation $L(\lambda)$. Thus, if we find such a submodule (whether or not it appears as a summand) we may be certain that it is in fact the Lie algebra of the image; this is a great aid in deciding whether an SL_2-module gives a reductive pair.*

We are now ready to state the main result of this section, which classifies the simple SL_2-modules giving reductive pairs. The majority of the work has been done already, and the proof will therefore take the form of a few observations, minor calculations and checks.

Theorem 3.2.10. *Let $\lambda \neq 0$ be a non-negative integer with base p expansion $\lambda = a_0 + a_1 p + \cdots + a_k p^k$ and $\rho : SL_2(K) \to GL(L(\lambda))$ the representation afforded by the simple module $L(\lambda)$. Let l be the smallest integer for which $a_l \neq 0$. Then, when $p > 3$, $(GL(L(\lambda)), \rho(SL_2))$ is a reductive pair if and only if*

1. *$a_i \leq p - 2$ for all i, and*

2. *$a_l \leq p - 3$.*

If $p = 3$, then $(GL(L(\lambda)), \rho(SL_2))$ is a reductive pair if and only if all $a_i \leq 1$ except for *a_l, which can be* 1 or 2*.*

If $p = 2$, $(GL(L(\lambda)), \rho(SL_2))$ is never a reductive pair.

Proof. First, let μ be the integer defined by factoring λ by the highest power of p dividing it: that is, setting $\mu := \sum_{j=0}^{k-l} b_j p^j$, where $b_j := a_{j+l}$. By lemma 2.10.8, we know that $L(\lambda)$ gives a reductive pair if and only if $L(\mu)$ does. Thus we may assume in the proof that $l = 0$, so that λ is not a multiple of p.

It is now enough to check when one of the summands in Doty and Henke's decomposition of the tensor product is isomorphic to $L(2)$: by corollary 3.2.8 we know that there is precisely one submodule of $L(\lambda) \otimes L(\lambda)$ isomorphic to $L(2)$, so if one of the summands is isomorphic to $L(2)$, it must be the Lie algebra (rather, its image under the various implicit isomorphisms); if none of the summands are isomorphic to $L(2)$, then we know the submodule we care about cannot split off, by Krull-Schmidt.

The term $J(\mathbf{u})$ in [14, theorem 2.1] is isomorphic to $L(2)$ precisely when $\mathbf{u} = (2, 0, \ldots, 0)$: certainly $T(2) \otimes T(0)^F \otimes \cdots \otimes T(0)^{F^k} \cong L(2)$, whereas $L(2) \not\cong M^F$ for any KG-module M, or indeed to any tensor product of more than one non-trivial Frobenius twisted fundamental tilting module.

Now suppose $p > 3$. Using the terminology of Doty and Henke's results, we must ensure that the set

$$W = W(a_l, a_l) \times W(a_{l+1}, a_{l+1}) \times \cdots \times W(a_k, a_k)$$

contains the element $(2, 0, \ldots, 0)$. Thus $W(a_l, a_l)$ must contain 2, and the rest of the sets $W(a_i, a_i)$ must all contain 0. Recall that

$$W(a, a) := S \setminus \{2p - 2 - u \mid u \in S, u \geq p\},$$

where $S := \{2a, 2a - 2, \ldots, 0\}$. The higher the number a is, the more elements of S will be removed to form W, working downwards in twos from $p - 3$, which is removed whenever S contains the element $p + 1$ (that is, when $a \geq \frac{p+1}{2}$). When $a = p - 2$, we must remove $2p - 2 - (2p - 4) = 2$, and when $a = p - 1$ we remove 0, giving us the stated conditions on the coefficients of λ.

For the case where $p = 3$, we note that in this case,

$$W(0,0) = \{0\},$$
$$W(1,1) = \{2,0\}, \text{ and}$$
$$W(2,2) = \{4,2\} \ (= \{4,2,0\} \setminus \{6-2-4=0\}),$$

hence the Cartesian product the set W contains the element $(2,0,\dots,0)$ precisely under the stated conditions on the coefficients.

When $p = 2$, we see by [14, 2.3] that $L(r) \otimes L(r')$ is indecomposable for any non-negative integers r, r'. Thus for the image in $L(\lambda) \otimes L(\lambda)$ of $\mathrm{Lie}\,\rho\,\mathrm{SL}_2(K)$ to be a summand of that space would require that these spaces be isomorphic. This is not so, since the image is 3-dimensional (albeit no longer simple), and 3 is not a square number, which the dimension of $L(\lambda) \otimes L(\lambda)^*$ always is.

$\square$

Example 3.2.11. *Suppose K has characteristic 5. Then the simple module with highest weight 450 does not give a reductive pair, since $450 = 0.1 + 0.5 + 3.5^2 + 3.5^3$. By lemma 3.2.1, $L(450)$ has dimension $(3+1)(3+1) = 16$; since $2 \times 16 - 2 = 30$ is greater than $p = 5$, we are within the bound established in theorem 2.9.9. We note that $L(451)$ does give a reductive pair, since the lowest non-zero coefficient of the base 5 expansion of 451 is no longer greater than $p - 3 = 2$. Note further that in this case $2 < p < 2 \dim L(451) - 2 = 62$. Thus we have examples within the bound which do and do not give reductive pairs.*

Example 3.2.12. *As another example, let $p > 3$ and consider the simple module $L(1 + \frac{p(p-1)}{2}) = L(1) \otimes L(\frac{p-1}{2})^F$ by theorem 2.10.1 . By lemma 3.2.1, this module has dimension $p + 1$, again within the bound. This module will be of interest in* *the next section of this chapter, where we will compare its behaviour with that of a symmetric power of the natural module.*

3.3 Symmetric Powers

The character calculations in this section and the proof of lemma 3.3.1 are due to Donkin.

Recall the definition of tilting modules from the preliminaries chapter. The following lemma is a special case of [24, Lemma 3.3], which is proved in a similar way, the argument being due to Donkin.

Lemma 3.3.1. *If* $r, s \geq 0$ *with* $|r - s| \leq 1$*, then* $\nabla(r) \otimes \Delta(s)$ *is a tilting module.*

Proof. Let B be the Borel subgroup

$$B := \left\{ \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix} \middle| \, t \in K^{\times}, x \in K \right\}.$$

Then $\nabla(r) \otimes \Delta(s) = \operatorname{ind}_B^G K_r \otimes \Delta(s) = \operatorname{ind}_B^G (K_r \otimes \Delta(s))$, with the first equality being by the definition of $\nabla(r)$, and the second by the tensor identity ([21] I,3.6), where first we regard $\Delta(s)$ as a G-module, then as a B-module, as appropriate (the equalities themselves are of G-modules). Now, $K_r \otimes \Delta(s)$ has a B-module filtration with sections K_m, where m ranges over $\{r + s - 2i \mid 0 \leq i \leq s\}$. By Kempf's vanishing theorem ([21], II,4.5), $R^i \operatorname{ind}_B^G K_m = 0$ for all $i > 0$; using this and considering the long exact sequence of induction, we see that $\nabla(r) \otimes \Delta(s)$ has a G-module filtration with sections $\operatorname{ind}_B^G K_m$, that is, a filtration by $\nabla(m)$'s. Since $(\nabla(r) \otimes \Delta(s))^* = \Delta(r) \otimes \nabla(s) \cong \nabla(s) \otimes \Delta(r)$ also has such a filtration, we see by the same argument (with r and s reversed) that $\nabla(r) \otimes \Delta(s)$ is tilting.

□

Define

$$Y(r) := \begin{cases} \nabla(m) \otimes \Delta(m) & \text{if } r = 2m \text{ is even,} \\ \nabla(m+1) \otimes \Delta(m) & \text{if } r = 2m+1 \text{ is odd.} \end{cases}$$

Note that for even r, $Y(r) \cong \nabla(m) \otimes (\nabla(m))^*$. Let $\chi(m) := \operatorname{ch} \nabla(m)$ and $\psi_r := \operatorname{ch} Y(r)$. Noting that the weights of the dual V^* of a module are in this case the negatives of the weights of V, with multiplicities, we have $\chi(m) = \operatorname{ch} \Delta(m)$. Alternatively, see [21, II,5.7-11] for more a general discussion of χ. One can calculate directly (by working out the weight spaces) that

$$\chi(m) = x^m + x^{m-2} + \cdots + x^{-m}.$$

We will use the following well-known formula.

Lemma 3.3.2 (Clebsch-Gordan formula)**.** *Let $\lambda \geq \mu$ be non-negative integers. Then we have*

$$\operatorname{ch}(\nabla(\lambda) \otimes \Delta(\mu)) = \operatorname{ch} \nabla(\lambda + \mu) + \operatorname{ch} \nabla(\lambda + \mu - 2) + \cdots + \operatorname{ch} \nabla(\lambda - \mu).$$

Next, using lemma 2.8.18 to calculate it directly, or using the Clebsch-Gordan formula above, we may see that

$$\psi_r = \begin{cases} \chi(r) + \chi(r-2) + \cdots + \chi(2) + \chi(0) & r \text{ even,} \\ \chi(r) + \chi(r-2) + \cdots + \chi(3) + \chi(1) & r \text{ odd.} \end{cases}$$

We will show that for all non-negative, even integers r, ψ_r is a sum of characters of certain tilting modules. By 2.8.24, this implies an isomorphism of tilting modules between $Y(r)$ and the direct sum of the tilting modules whose characters appear. Doing this will allow us to reduce the problem of determining (for each non-negative integer m) whether $\nabla(m)$ gives a reductive pair for SL_2 to the already covered case of simple modules. We will proceed via several calculations, to accommodate different values of r. It is worth noting that Donkin's calculations cover a wider class of examples than is included here.

Suppose $p > 2$. First, we let a be an even integer with $0 \leq a \leq p-3$, and write

$$\psi_{2pm+a} = \chi(2pm+a) + \chi(2pm+a-2) + \cdots + \chi(2) + \chi(0). \tag{3.4}$$

Next, we factor the whole expression by $\chi(p-1)$:

$$\begin{aligned}\psi_{2pm+a} = \chi(p-1)\Big[&\chi((2m-1)p+a+1) + \chi((2m-3)p+a+1) + \cdots \\ &+ \chi(p+a+1)\Big] + \chi(a) + \chi(a-2) + \cdots + \chi(0).\end{aligned} \tag{3.5}$$

To see this equality, note that using the Clebsch-Gordan formula 3.3.2, we have that the "top" term of $\chi(p-1)\chi((2m-1)p+a+1)$ is $\chi(2pm+a)$, whilst the "bottom" term is $\chi((2m-2)p+a+2)$ (where by top and bottom we refer to the usual order of the integers on the arguments). The bottom term is therefore the one "2 greater than" (in the same sense) the top term of $\chi(p-1)\chi((2m-3)p+a+1)$. In fact, the terms we obtain by multiplying out the square bracket continue to "stack" in this fashion; after all terms have been expanded, we are left with

$$\begin{aligned}&\chi(p-1)\Big[\chi((2m-1)p+a+1) + \cdots + \chi(p+a+1)\Big] \\ &\qquad = \chi(2pm+a) + \chi(2pm+a-2) + \cdots + \chi(a+2),\end{aligned}$$

and for this to equal ψ_{2pm+a} we must add on $\psi_a = \chi(a) + \chi(a-2) + \cdots + \chi(0)$ (the rest of the terms).

We will make use of the following lemma, which is [29, 6.1.1].

Lemma 3.3.3. *1. Let $i, j \in \{0, \ldots, p-2\}$ with $i + j = p - 2$. Then for any $n \in \mathbb{N}$, there exists a short exact sequence of $\mathrm{SL}_2(K)$-modules:*

$$0 \to \nabla(i) \otimes \nabla(n)^F \to \nabla(pn - i) \to \nabla(j) \otimes \nabla(n-1)^F \to 0.$$

2. $\nabla(np-1) \cong \nabla(p-1) \otimes \nabla(n-1)^F$ is an isomorphism of $\mathrm{SL}_2(K)$-modules.

Recalling that $a \leq p - 3$, we apply this lemma to the all of the characters within the square bracket in equation 3.5 to see

$$\begin{aligned} \psi_{2pm+a} = \chi(p-1)\Big[&\big(\chi(a+1)\chi(2m-1)^F + \chi(p-a-3)\chi(2m-2)^F\big) \\ &+ \big(\chi(a+1)\chi(2m-3)^F + \chi(p-a-3)\chi(2m-4)^F\big) + \cdots \\ &+ \chi(a+1)\chi(1)^F + \chi(p-a-3)\Big] + \chi(a) + \chi(a-2) + \cdots + \chi(0). \end{aligned} \tag{3.6}$$

We may now collect the terms in the bracket into two sets, rewriting the right-hand side of equation 3.6 as follows.

$$\psi_{2pm+a} = \chi(p-1)\chi(a+1)\psi^F_{2m-1} + \chi(p-1)\chi(p-a-3)\psi^F_{2m-2} + \psi_a. \tag{3.7}$$

Now, $\nabla(\lambda)$ is tilting for $0 \leq \lambda \leq p-1$ [14, Lemma 1.1], and although $Y(r)^F$ is not tilting, $\nabla(p-1) \otimes Y(r)^F$ is. To see this, we note that if we apply a Frobenius twist to $Y(r)$ and to a good filtration of $Y(r)$, the result is no longer good: we get sections $\nabla(\mu)^F$ (where $Y(r)$ had sections $\nabla(\mu)$, say). However $\nabla(p-1) \otimes \nabla(\mu)^F \cong \nabla((\mu+1)p-1))$ as a KSL_2-module, by lemma 3.3.3. We also use the following isomorphism of modules: $(A\otimes C)/(B\otimes C) \cong (A/B)\otimes C$, showing that $\nabla(p-1)\otimes Y(r)^F$ has a good filtration. Ultimately, $\nabla(b)\otimes\Big(\nabla(p-1)\otimes Y(m)^F\Big)$ is tilting ($0 \le b \le p-1$), being a tensor product of two tilting modules. Since all the terms in 3.7 are characters of tilting modules, and since we have had equality at all stages of the calculation, we must have

$$\begin{aligned} Y(2pm+a) &\cong \nabla(p-1)\otimes\nabla(a+1)\otimes Y(2m-1)^F \\ &\qquad \oplus \nabla(p-1)\otimes\nabla(p-a-3)\otimes Y(2m-2)^F \oplus Y(a). \end{aligned} \tag{3.8}$$

Next, we consider the case where $a = p-1$. Following the same steps as in equations 3.5, 3.6, 3.7, but using the other case of lemma 3.3.3, we arrive at the following for this case.

$$\begin{aligned} Y(2pm+p-1) &\cong \nabla(p-1)\otimes Y(2m)^F \\ &\qquad \oplus \nabla(p-1)\otimes\nabla(p-2)\otimes Y(2m-1)^F \oplus Y(p-3). \end{aligned} \tag{3.9}$$

We next consider $Y(p-1+2pm+a)$ with $0 \le a \le p-3$, a even. Again, by similar manipulation we see

$$\begin{aligned} \psi_{p-1+2pm+a} &= \chi(p-1)\Big[\chi(2pm+a)+\chi(2p(m-1)+a)+\cdots \\ &\quad + \chi(2p+a)+\chi(a)\Big] + \chi(p-1-a-2)+\chi(p-1-a-4)+\cdots+\chi(0). \end{aligned}$$

Next,

$$\begin{aligned} \psi_{p-1+2pm+a} &= \chi(p-1)\Big[\chi(a)\chi(2m)^F + \chi(p-2-a)\chi(2m-1)^F \\ &\quad + \cdots + \chi(a)\chi(2)^F + \chi(p-2-a)\chi(1)^F + \chi(a)\Big] + \psi_{p-a-3}. \end{aligned}$$

Finally, we collect the terms:

$$\psi_{p-1+2pm+a} = \chi(p-1)\chi(a)\psi^F_{2m} + \chi(p-1)\chi(p-2-a)\psi^F_{2m-1} + \psi_{p-a-3},$$

whence (for $0 \le a \le p-3$)

$$\begin{aligned} Y(p-1+2mp+a) &\cong \nabla(p-1)\otimes\nabla(a)\otimes Y(2m)^F \\ &\quad \oplus \nabla(p-1)\otimes\nabla(p-2-a)\otimes Y(2m-1)^F \oplus Y(p-a-3). \end{aligned} \tag{3.10}$$

Another, simpler calculation yields

$$Y(2pm+2p-2) = \nabla(p-1)\otimes\nabla(p-1)\otimes Y(2m)^F. \tag{3.11}$$

Considering 3.8, 3.9, 3.10 and 3.11, we have expressions for $Y(r)$ for each even, non-negative r. Thus we have expressions for $\nabla(m)\otimes\Delta(m)$ with $r = 2m$ – that is, all non-negative integers m.

Theorem 3.3.4. *Let K be an algebraically closed field of characteristic $p > 0$, let n be a non-negative integer and let $\rho : SL_2(K) \to GL(\nabla(n))$ be the representation afforded by $\nabla(n)$, the n^{th} symmetric power of the natural module.*

1. *If K has characteristic $p > 3$, then $(GL(\nabla(n)), \rho(SL_2))$ is a reductive pair if and only if $n \not\equiv p, p-1$ or $p-2$ (mod p).*
2. *If K has characteristic 3, then $(\mathrm{GL}(\nabla(n)), \rho(SL_2))$ is a reductive pair if and only if $n \equiv 1, 2, \ldots 6$ (mod 9).*
3. *If K has characteristic 2, then $(\mathrm{GL}(\nabla(n)), \rho(SL_2))$ is never a reductive pair.*

Proof. First suppose K has characteristic $p > 3$. Although in proving this result we will deal with several cases, they all resolve to the same issue: what ultimately matters is the residue class of n modulo p. As in the calculations above, we write $r = 2n$ and $\nabla(n) \otimes \nabla(n)^* \cong \nabla(n) \otimes \Delta(n) = Y(r)$ as a direct sum of tilting modules. Recall that by 3.2.8, if we exhibit a summand of $\nabla(n) \otimes \Delta(n)$ that is isomorphic to $L(2)$, then we know $\nabla(n)$ does give a reductive pair. On the other hand, if we can show that in a given decomposition of $\nabla(n) \otimes \Delta(n)$ into (not necessarily indecomposable) direct summands M_i, none of the M_i has a summand isomorphic to $L(2)$, then $\nabla(n)$ cannot give a reductive pair, by the Krull-Schmidt theorem.

By 3.8, we see that when $n \equiv 0, 1, \ldots, \frac{p-3}{2} \pmod{p}$, $\nabla(n)$ gives a reductive pair if and only if $L(2\overline{n})$ gives a reductive pair, where $\overline{n}$ is the least residue of n modulo p. When $\overline{n} = 0$ (that is, when $n = pm$ for some m), $\nabla(n)$ does not give a reductive pair: by corollary 2.10.5, we know that no module isomorphic to $L(2)$ can be a summand of any module of the form $\nabla(p-1) \otimes N$ (where N is a G-module), since $\nabla(p-1)$ is p-dimensional and indecomposable (it is irreducible). We combine this this with the last observation in the previous paragraph, noting that none of the summands in 3.8 has a summand isomorphic to $L(2)$. In the rest of the cases we have $0 < 2\overline{n} \leq p-3$, so that $\nabla(n)$ does give a reductive pair.

By 3.9 and 3.10, we see that if $n \equiv \frac{p-1}{2}, \frac{p-1}{2}+1, \ldots, p-3 \pmod{p}$, then $\nabla(n)$ does give a reductive pair, while for $n \equiv p-2$ it does not. This is because if $n \equiv \frac{p-1}{2} + \frac{a}{2} \pmod{p}$ where $0 \leq a \leq p-3$ is even, then $\nabla(n)$ gives a reductive pair if and only if $L(\frac{p-a-3}{2})$ gives a reductive pair, which is true for $0 \leq a \leq p-5$ but false for $a = p-3$.

Finally we look at 3.11. Thus if $n \equiv p-1 \pmod{p}$, then $\nabla(n)$ does not give a reductive pair, as corollary 2.10.5 shows $\nabla(n) \otimes \Delta(n)$ does not have a summand isomorphic to $L(2)$ (as before). Combining this and the observations about other residue classes above, the proof for $p > 3$ is complete.

Now suppose $p = 3$. In this case, letting $a = 0$ in equation 3.8, we have

$$Y(6m) \cong \nabla(2) \otimes \nabla(1) \otimes Y(2m-1)^F \oplus \nabla(2) \otimes \nabla(0) \otimes Y(2m-2)^F \oplus Y(0). \quad (3.12)$$

Equation 3.9 becomes

$$Y(6m+2) \cong \nabla(2) \otimes Y(2m)^F \oplus \nabla(2) \otimes \nabla(1) \otimes Y(2m-1)^F \oplus Y(0). \quad (3.13)$$

Finally, equation 3.11 becomes

$$Y(6m+4) = \nabla(2) \otimes \nabla(2) \otimes Y(2m)^F. \quad (3.14)$$

From 3.12, we see that if $p \nmid m$, then $Y(6m)$ has a summand isomorphic to $L(2)$. This is because the term $\left(\nabla(2) \otimes \nabla(0) \otimes Y(2m-2)^F\right)$ itself has a summand $L(2) \otimes L(0) \otimes L(0)^F$, using proposition 2.10.3, noting that $Y(2m-2)^F = (\nabla(m-1) \otimes \Delta(m-1))^F$. Since $Y(6m) = \nabla(3m) \otimes \Delta(3m)$, we have therefore have that $\nabla(3m)$ gives a reductive pair for each m coprime to 3, namely $\nabla(3 \times 1), \nabla(3 \times 4), \nabla(3 \times 7)$, or generally those of the form $\nabla(3 + 9k)$; and also $\nabla(3 \times 2), \nabla(3 \times 5)$, or in other words those of the form $\nabla(6+9k)$. On the other hand, again using lemma 3.2.4, $L(2) \otimes L(1) \cong T(3)$ in characteristic 3. Thus $L(2)$ cannot be a summand of either of the other terms in equation 3.12, for reasons we now explain. The composition factors of $T(3)$ in characteristic 3 are $L(1), L(3), L(1)$, by lemma 3.1.1. Hence the composition factors of $T(3) \otimes Y(2m-1)^F$ are the composition factors of $L(1) \otimes Y(2m-1)^F$ (twice each) and the composition factors of $L(3) \otimes Y(2m-1)^F$. If $T(3) \otimes Y(2m-1)^F$ has a summand isomorphic to $L(2)$, then this summand is in particular a submodule, and, being simple, is therefore a composition factor. Thus it is enough to know that $L(2)$ cannot be a composition factor of either $L(1) \otimes Y(2m-1)^F$ or $L(3) \otimes Y(2m-1)^F \cong (L(1) \otimes Y(2m-1))^F$; considering the weights of these modules, we see that no composition factor of either may have highest weight congruent to 2 modulo 3. Finally, $Y(0)$ is 1-dimensional, so $L(2)$ cannot be a submodule.

From 3.13, using the same reasoning, $Y(6m+2)$ does have a summand isomorphic to $L(2)$ if $p \nmid m+1$, noting that $Y(2m)^F = (\nabla(m) \otimes \Delta(m))^F$. Then, following the same process as for the previous case, we see that we get a reductive pair from each $\nabla(1+9k)$ and each $\nabla(4+9k)$. Since the other terms in equation 3.13 are the same as in the previous case, the same reasoning shows that $L(2)$ is not a summand of either of the those terms.

From 3.14, we see that $Y(6m+4)$ has a summand isomorphic to $L(2)$ if $p \nmid m+1$. To see this, note that by lemma 3.2.4 we have $L(2) \otimes L(2) \cong T(4) \oplus L(2)$ then apply the same reasoning as before. In this case, we see that we get reductive pairs from $\nabla(2+9k)$ and $\nabla(5+9k)$. Again, these are the only ways to get a summand isomorphic to $L(2)$.

To summarise: from the cases above, we see that when $p = 3$, $\nabla(n)$ gives a reductive pair if and only if $n \equiv 1, 2, \ldots 6 \pmod 9$.

Finally, suppose $p = 2$. First recall that direct summands of tilting modules are tilting modules, by 2.8.23. Thus, if the image of $\operatorname{Lie} \rho(\mathrm{SL}_2(K))$ in $\nabla(n) \otimes \Delta(n)$ is not a tilting module, it cannot be a summand. With this in mind, consider that the differential $d\rho : \mathfrak{sl}_2(K) \to \operatorname{Lie}(\rho\, \mathrm{SL}_2(K))$ is injective unless n is even, in which case it has as its kernel the scalar matrices. If n is odd, $\operatorname{Lie} \rho(\mathrm{SL}_2)$ is therefore the $\mathrm{SL}_2(K)$-module $\nabla(2)$, which is not a tilting module. Thus $\nabla(n)$

does not give a reductive pair when n is odd in characteristic 2. If n is even, then $\operatorname{Lie}\rho(\mathrm{SL}_2)$ has a 2-dimensional simple submodule coming from the image $d\rho\left(\mathfrak{sl}_2(K)\right)$. Since $\mathfrak{sl}_2(K)$ is the $\mathrm{SL}_2(K)$-module $\nabla(2)$, this simple module must be $L(2)$ (which in characteristic 2 is $L(1)^F$, which is 2-dimensional). There are then two possibilities: either $\operatorname{Lie}\rho(\mathrm{SL}_2)$ is indecomposable, in which case it is the module $\Delta(2)$; or else it has a decomposition as $L(2) \oplus L(0)$. Since $\Delta(2)$ and $L(2)$ are not tilting, in either of these cases this is enough information to conclude that $\nabla(n)$ does not give a reductive pair for even n in characteristic 2.

□

Example 3.3.5. *Let k have characteristic $p > 3$. By 3.3.4, we see that $\nabla(p)$ does not give rise to a reductive pair; by 3.2.10, we see that the simple module $L(1+\frac{p(p-1)}{2}) = L(1) \otimes L(\frac{p-1}{2})^F$ does. Both of these modules have dimension $p+1$, and we note that $2 < p < 2(p+1) - 2 = 2p$. Thus, if the characteristic is not 3, we have examples of both sorts of behaviour between the bounds in [3, 3.1].*

Example 3.3.6. *We can use lemma 2.10.10 to generate many more examples of $\mathrm{SL}_2(K)$-modules giving reductive pairs. For instance, suppose K has characteristic $p > 2$. Let ρ_i be the representation afforded by $L(i)$. Since $\operatorname{Lie}(\mathrm{SL}_2(K))$ is simple, $d\rho_i$ is injective for each i. The dimension of $\nabla(r)$ being $r+1$, we see that $\nabla(r) \otimes \nabla(r)^*$ has the trivial module K as a summand if and only if $p \nmid r+1$ (using proposition 2.10.3). Therefore for each $n \geq 1$ the following modules all give reductive pairs.*

$L(1) \otimes \nabla(1)^{F^n}$

$L(1) \otimes \nabla(2)^{F^n}$

$L(1) \otimes \nabla(3)^{F^n}$

$\dots$

$L(1) \otimes \nabla(p-1)^{F^n}$.

This could have been predicted by theorem 3.2.10, since each of these modules is simple. However, we also get the following for each λ with non-zero restricted part[3] *such that $L(\lambda)$ gives a reductive pair, and each $k \in \mathbb{N}$.*

$L(\lambda) \otimes \nabla(1+kp)^{F^n}$

$L(\lambda) \otimes \nabla(2+kp)^{F^n}$

$L(\lambda) \otimes \nabla(3+kp)^{F^n}$

$\dots$

$L(\lambda) \otimes \nabla(p-1+kp)^{F^n}$.

[3]That is, where $\lambda = \lambda_0 + p\lambda'$, with $0 \neq \lambda_0 \in X_1(T)$ and λ' dominant.

Chapter 4

SL_3 and other simple groups

In this chapter we consider some results that can sometimes be applied to arbitrary simple algebraic groups to decide that a particular module does give a reductive pair. Unfortunately, the results tend to require more information than is readily available in order to draw useful conclusions. We thus quickly turn our attention to $\mathrm{SL}_3(K)$, as for this group a certain amount has already been worked out in detail [30].

4.1 General statements for simple groups

The lemma below is stated in the form most applicable to our use in this chapter. However, in chapter 5 we discuss other potential applications.

Lemma 4.1.1. *Let R be a ring and let M be an R-module with a filtration*

$$0 \leq M_1 \leq M_2 \leq \ldots \leq M_n = M.$$

Suppose every quotient $L_i := M_i/M_{i-1}$ is such that $\mathrm{Ext}^1_R(L_i, M_1) = 0$. (Equiva-

lently, every short exact sequence $0 \to M_1 \to E \to L_i \to 0$ *splits). Then* M_1 *is an* R*-module direct summand of* M*:* $M = M_1 \oplus W$ *for some* $W \leq M$*.*

Proof. Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ of R-modules and an R-module B, we may consider the long exact sequence of the Ext functor in the first variable,

$$\cdots \to \operatorname{Ext}^n_R(A'', B) \to \operatorname{Ext}^n_R(A, B) \to \operatorname{Ext}^n_R(A', B) \to \operatorname{Ext}^{n+1}_R(A'', B) \to \cdots .$$

In the notation of the statement, we are given exact sequences $0 \to M_{i-1} \to M_i \to L_i \to 0$, and we therefore have exact sequences

$$\operatorname{Ext}^1_R(L_i, M_1) \to \operatorname{Ext}^1_R(M_i, M_1) \to \operatorname{Ext}^1_R(M_{i-1}, M_1),$$

in which the first term is 0 by hypothesis, and the last term is 0 by induction (the base case being clear). Thus M_1 is a summand of each M_i, in particular $M_n = M$. □

Remark 4.1.2. *Provided that we can work out the composition factors of* $V \otimes V^*$ *for a* G*-module* V*, we may sometimes use lemma 4.1.1 to show that* V *gives a reductive pair (this method will not tell us that a given module* does not *give a reductive pair). We note that (the image of) the Lie algebra of the image of* G *is a simple submodule of* $V \otimes V^*$ *(subject, potentially, to minor constraints on the characteristic), so that* $0 \leq \operatorname{Lie} G \leq V \otimes V^*$*; this may be refined into a composition series for* $V \otimes V^*$ *with* $\operatorname{Lie} G$ *at the bottom. Thus, if we know a posteriori that all extensions of the Lie algebra by the other composition factors must be split, lemma 4.1.1 tells us that the Lie algebra is a direct summand of* $V \otimes V^*$ *(whence the module* V *gives a reductive pair).*

We also note at this point that lemma 4.1.1 implies equally that all the submodules of $V \otimes V^*$ in the same isomorphism class as $\mathrm{Lie}\, G$ *are summands under the same hypotheses.*

It will be useful to have some results that tell us about extensions of simple modules. First of all, we recall the linkage principle (this may be found as [21, Corollary 6.17]). The proposition is proved in Jantzen as a corollary of the strong linkage principle, which, roughly speaking, tells us that for a simple module with highest weight μ to be a composition factor of a cohomology module $H^i(w.\lambda)$ with w an element of the Weyl group, we must have that $\mu \uparrow \lambda$. That is, there must be a finite sequence of affine reflections σ_i, applied one after another ultimately taking μ to λ, with $\mu \leq \sigma_1 \cdot \mu \leq \sigma_2 \cdot (\sigma_1 \cdot \mu) \leq \cdots \leq \sigma_k \cdot (\sigma_{k-1} \cdot (\ldots)) = \lambda$.

Proposition 4.1.3. *Let $\lambda, \mu \in X(T)_+$. If $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$, then $\lambda \in W_p \cdot \mu$.*

It will be useful for us to restate this as saying that if $\lambda \notin W_p \cdot \mu$, then any extension of $L(\lambda)$ by $L(\mu)$ must be split.

The following lemma is found in [21, Section 2.12].

Lemma 4.1.4. *We have that $\mathrm{Ext}^1_G(L(\lambda), L(\lambda)) = 0$ for all $\lambda \in X(T)_+$.*

The idea of the proof is to show that an exact sequence $0 \to L(\lambda) \to M \to L(\lambda) \to 0$ splits, by carefully picking an element of the λ weight space of $L(\lambda)$ and considering the action of the algebra of distributions $\mathrm{Dist}(G)$ (for a definition, see [21]).

4.2 SL_3

Now and for the remainder of this chapter let $G = \mathrm{SL}_3(K)$, K algebraically closed and of characteristic $p \neq 0$. Consider a rational representation $\rho : \mathrm{SL}_3(K) \to$

GL(V). If the representation is non-trivial, the kernel of ρ is either trivial or the centre of $\mathrm{SL}_3(K)$, and is in either case finite. When $p \neq 3$, the adjoint representation is irreducible and has highest weight $(1,1)$. Less is known about the representation theory of SL_3 than that of SL_2, and although some progress has been made (e.g. [8, 9]), no complete tensor product decomposition (as in [14]) seems to have appeared at the time of writing.

In Yehia's PhD thesis [30], $\mathrm{Ext}^1_G(L(\mu), L(\lambda))$ is shown to be at most one-dimensional for G of type A_2. Furthermore, for each dominant weight λ the set of weights

$$A(\lambda) := \{\mu \in X(T)_+ \mid \mathrm{Ext}^1_G(L(\mu), L(\lambda)) \neq 0\}$$

is determined explicitly. For our purposes it will be enough to consider the following result, which is an abridgement of [30, Proposition 4.1.1].

Proposition 4.2.1. *Suppose $\lambda, \mu \in X(T)_+$, $\lambda = \lambda_0 + p\lambda'$, $\mu = \mu_0 + p\mu'$, where $\lambda_0, \mu_0 \in X_1$ (the restricted region) and $\lambda', \mu' \in X(T)_+$. Moreover suppose $\lambda_0 \neq \mu_0$. Then*

1. *A necessary condition for $\mathrm{Ext}^1_G(L(\mu), L(\lambda))$ to be non-zero is that μ_0 is in the W_p orbit of λ_0.*

2. *If $\lambda_0 = (r, s) \in A_0$ (the bottom alcove) and μ_0 is* not *one of $(p-s-2, p-r-2), (r+s+1, p-s-2)$ or $(p-r-2, r+s+1)$, we have $\mathrm{Ext}^1_G(L(\mu), L(\lambda)) = 0$.*

Hence for $\mathrm{SL}_3(K)$ we may do significantly better than using linkage alone. The full result in [30] considers in point (2) any λ_0 in the restricted region.

Stephen Doty has written a package [13] for the computer algebra software GAP [16] which can perform calculations pertaining to Weyl modules. We used this software to calculate the composition factors of tensor products of simple modules with small highest weights in type A_2 in a variety of small characteristics. It quickly began to take too long for the computer to complete the calculations as the characteristic or weights increased. Some of the results themselves may be seen in the appendix.

We may (somewhat crudely) use proposition 4.2.1 in combination with the linkage principle 4.1.3 in order to create a "mask" of those dominant weights for which extensions of the simple module $L(1,1)$ by simple modules with this highest weight are all split. The result is a reduced selection of weights to look for amongst the composition factors of determined by the computer calculations. Please see the figures in the appendix illustrating this process. If none of the designated weights appear, we may be certain (by 4.1.2) that the module in question does give a reductive pair. We note at this point that the same process could in theory be applied to any of the simple groups, albeit without the aid of 4.2.1, which is specific to the case of type A_2: we only require that the Lie algebra be a simple module (which as noted previously is true when the characteristic is very good for the group).

Example 4.2.2. *Let $p = 5$. By considering characters, the composition factors of $L(5,1) \otimes L(5,1)^*$ are calculated to have highest weights (with multiplicities following)*

$$(6,6), 1, (5,5), 1, (1,1), 1, (0,0)1.$$

The weight $(6,6)$ is in the W_p orbit of $(1,1)$, so the linkage principle alone does not allow us to rule out this weight as possibly contributing a non-split extension. However, $(6,6)$ is not *one of the weights specified by 4.2.1. We may therefore infer that $L(5,1)$ gives a reductive pair in this case.*

The following is an easy application of one of the preliminary results.

Proposition 4.2.3. *Let K have characteristic 7 and let $n \geq 1$ be an integer. Let $\lambda \in A = \{(1,0),(0,1),(1,1),(2,0),(0,2),(2,1),(1,2),(3,0),(0,3)\}$ and $\mu \in B = \{(1,0),(0,1),(1,1),(2,0),(0,2),(2,1),(1,2),(2,2),(3,0),(0,3),(3,1),(1,3)\}$. Then the module $L(\lambda + p^n\mu)$ gives a reductive pair.*

Proof. The simple modules with highest weight in the set A of the statement all give reductive pairs (cf. table of results in the appendix), and their highest weights are restricted. Since the weights in B are all in the bottom alcove, [21, II,5.6] implies that each simple module with highest weight in B is equal to the induced module with the same highest weight. Then note that $p = 7$ does not divide the dimensions (calculated using Weyl's dimension formula [15, Cor.24.6]) of the these simple modules. Thus the conditions of corollary 2.10.11 are satisfied, and for $\lambda \in A, \mu \in B$ we have that $L(\lambda) \otimes L(\mu)^F \cong L(\lambda + p^n\mu)$ gives a reductive pair.

□

Remark 4.2.4. *The list above is not claimed by any means to be complete. By remark 2.10.12, many more classes of examples of modules (simple or otherwise) giving reductive pairs could readily be determined by this method.Note also that the only significance of the prime being 7 was the ready availability of the calculated results in the appendix.*

Chapter 5

Conclusion

In the above we have seen full treatments of the cases of simple modules and induced modules for the group $\mathrm{SL}_2(K)$ and several examples for the group $\mathrm{SL}_3(K)$. There are numerous directions that the project may now take, some of which we discuss now.

The work on $\mathrm{SL}_2(K)$ was left at its current state due to a desire to consider a wider class of examples. It should be easy to continue to explore other classes of examples for this group, due to its relative simplicity.

The difficulties encountered by the authors of [8, 9] could possibly be reduced by considering the smaller class of examples relevant to our problem: that is, it may be more tractable to restrict attention to tensor products $L(\lambda) \otimes L(\lambda)^*$ instead of arbitrary $L(\mu) \otimes L(\nu)$. If so, it may be possible to say a great deal more about the problem for $\mathrm{SL}_3(K)$.

Some thought has been devoted to possible applications of (the proof of) lemma 4.1.1 and the remark following it. For example, there is no need to require that the sequence of modules in the lemma be a filtration: it would be enough to have

$$M_1 \oplus M' \leq M_2 \leq \ldots \leq M_n = M,$$

making the necessary changes to the proof. Moreover, the same lemma could potentially be applied to more than just simple modules: the author began to investigate such an application, but requiring greater knowledge of the specifics of (in this case) tilting modules for SL_3, this did not get very far.

There seems to be no reason that the methods of section 4.1 could not be applied to other simple groups. An investigation into determining (or restricting) the composition factors of $L(\lambda) \otimes L(\lambda)^*$ may in some cases eliminate the need for computers. The composition factors of $L(n) \otimes L(n)$ for $\mathrm{SL}_2(K)$ seem easy to predict: if $r \neq 0$ and we write it in base p with coefficients a_i, then the composition factors that appear are the numbers of the form

$$2 \times \Big((a_0 - b_0) + (a_1 - b_1)p + \cdots + (a_k - b_k)p^k\Big),$$

where $0 \leq b_i \leq a_i$ for all i; we think this could be proved by working with characters. For $\mathrm{SL}_3(K)$, those of $L(a,b) \otimes L(b,a)$ seem a little more difficult. Example A.2.1 in the appendix shows composition factors of some tensor products $L(a,b) \otimes L(b,a)$ for $\mathrm{SL}_3(K)$ when K has characteristic 7. There are some clear patterns, aside from the obvious limits imposed by dimension et cetera. For instance, the weights appearing seem to form chains of the form $\ldots, (x-4, y+2), (x-2, y+1), (x, y), (x+1, y-2), (x+2, y-4), \ldots$, without gaps.

Finally, in all cases, it seems desirable to have more methods of combining existing examples. Some work could be done on seeing how far this may be taken, and whether any interesting modules may be treated by a combination of easier examples.

Appendix A

Table, Figures and GAP code

A.1 Table

The following table shows a sample of results generated using GAP. The rows and columns are labelled by dominant weights, and a given cell (a, b) contains a "Y" if the method described in the main text tells us that $L(a, b)$ gives a reductive pair, or is left blank if the results are inconclusive (cf. 4.1.2). The cell $(0, 0)$ contains an "N" because this module does not give a reductive pair, its dimension being too small. The pattern shown in the table appears to tile (e.g. cells $(14, 0)$ — $(14, 3)$ contain "Y"s, as do $(15, 0)$—$(15, 2)$, $(16, 0), (16, 1)$ and $(17, 0)$; the same "triangle" appears starting at $(14, 7)$, etc), but the calculation becomes too lengthy to continue far with this. Every weight (a, b) in this table with an inconclusive result is such that $L(a, b) \otimes L(a, b)^*$ has $L(4, 4)$ as a composition factor. A similar investigation of the results for $p = 5$ (not included, for brevity) shows that, in each case where the test is inconclusive, the module in question has $L(2, 2)$ as a composition factor. When $p = 7$, $(4, 4)$ is the reflection in the line joining $(-1, p - 1)$ and $(p - 1, 1)$ of the highest weight of the Lie algebra; for $p = 5$, the same is true of $(2,2)$.

SL_3, $p=7$, does $L(a,b)$ a reductive pair?

Y=yes, N=no; otherwise no conclusion drawn

wt	0	1	2	3	4	5	6	7	8	9	10	11	12
0	N	Y	Y	Y				Y	Y	Y	Y		
1	Y	Y	Y					Y	Y	Y			
2	Y	Y						Y	Y				
3	Y							Y					
4													
5													
6													
7	Y	Y	Y	Y				Y	Y	Y	Y		
8	Y	Y	Y					Y	Y	Y			
9	Y	Y						Y	Y				
10	Y							Y					
11													
12													

A.2 GAP code

Here is the GAP code that was used to generate examples. As mentioned in the text, it relies on Doty's package Weyl Modules for GAP. The first routine takes a prime p and a weight ab and determines then prints a list of composition factors of the tensor product of the simple module $L(ab) \otimes L(ab)^*$ for SL_3 in characteristic p.

```
compfacs:=function(p,ab)
local ch1, flip, ch2, prodch, thelist, runningtotal, place;
flip:=[ab[2],ab[1]];
ch1:=SimpleCharacter(p,ab,"A",2);
```

```
ch2:=SimpleCharacter(p,flip,"A",2);
prodch:=ProductCharacter(ch1,ch2);
thelist:=DecomposeCharacter(prodch,p,"A",2);
place:=0;
runningtotal:=0;
for place in [1..Length(thelist)/2] do;
runningtotal:=runningtotal + thelist[2*place];
od;
Print("p = ", p, ", wt (", ab[1], ",", ab[2], ").  ", thelist, ".
Total composition length: ", runningtotal, "\n");
end;
```

The next routine iterates the first to create a list of results, which it then prints. In the code below, weights up to $(3p, 3p)$ are being examined.

```
compfacslist:=function(p)
local i, j;
for i in [0..3*p] do;
j:=0;
while j <= i do;
compfacs(p,[i,j]);
j:= j+1;
od;
od;
end;
```

A sample of output follows, showing a section of the results used to create the table above. The lines in the list show the composition factors $[m, n]$ (with multiplicities following) of the simple module of weight (a, b).

Example A.2.1. *compfacslist(7); p = 7, wt (0,0). [[0, 0], 1]. Total composition length: 1*

p = 7, wt (1,0). [[1, 1], 1, [0, 0], 1]. Total composition length: 2

p = 7, wt (1,1). [[2, 2], 1, [0, 3], 1, [3, 0], 1, [1, 1], 2, [0, 0], 1]. Total composition length: 6

p = 7, wt (2,0). [[2, 2], 1, [1, 1], 1, [0, 0], 1]. Total composition length: 3

p = 7, wt (2,1). [[3, 3], 1, [1, 4], 1, [4, 1], 1, [2, 2], 3, [0, 3], 1, [3, 0], 1, [1, 1], 2, [0, 0], 1]. Total composition length: 11

p = 7, wt (2,2). [[4, 4], 1, [2, 5], 1, [5, 2], 1, [3, 3], 2, [0, 6], 1, [6, 0], 1, [1, 4], 2, [4, 1], 2, [2, 2], 5, [0, 3], 2, [3, 0], 2, [1, 1], 3, [0, 0], 1]. Total composition length: 24

p = 7, wt (3,0). [[3, 3], 1, [2, 2], 2, [1, 1], 1, [0, 0], 1]. Total composition length: 5

p = 7, wt (3,1). [[4, 4], 1, [2, 5], 1, [5, 2], 1, [3, 3], 2, [1, 4], 1, [4, 1], 1, [2, 2], 4, [0, 3], 2, [3, 0], 2, [1, 1], 3, [0, 0], 1]. Total composition length: 19

p = 7, wt (3,2). [[5, 5], 1, [3, 6], 1, [6, 3], 1, [4, 4], 2, [1, 7], 1, [7, 1], 1, [2, 5], 3, [5, 2], 3, [3, 3], 3, [0, 6], 1, [6, 0], 1, [1, 4], 2, [4, 1], 2, [2, 2], 6, [0, 3], 3, [3, 0], 3, [1, 1], 4, [0, 0], 2]. Total composition length: 40

p = 7, wt (3,3). [[6, 6], 1, [4, 7], 1, [7, 4], 1, [5, 5], 2, [2, 8], 1, [8, 2], 1, [0, 9], 1, [9, 0], 1, [4, 4], 2, [3, 3], 2, [2, 2], 2, [1, 1], 2, [0, 0], 2]. Total composition length: 19

p = 7, wt (4,0). [[4, 4], 1, [3, 3], 1, [2, 2], 2, [1, 1], 2, [0, 0], 1]. Total composition length: 7

p = 7, wt (4,1). [[5, 5], 1, [3, 6], 1, [6, 3], 1, [4, 4], 2, [2, 5], 1, [5, 2], 1, [3, 3], 2, [1, 4], 1, [4, 1], 1, [2, 2], 4, [0, 3], 2, [3, 0], 2, [1, 1], 4, [0, 0], 2]. Total composition length: 25

p = 7, wt (4,2). [[6, 6], 1, [4, 7], 1, [7, 4], 1, [5, 5], 2, [2, 8], 1, [8, 2], 1, [4, 4], 2, [3, 3], 1, [2, 2], 2, [1, 1], 2, [0, 0], 2]. Total composition length: 16

p = 7, wt (4,3). [[7, 7], 1, [5, 8], 2, [8, 5], 2, [6, 6], 2, [3, 9], 1, [9, 3], 1, [4, 7], 5, [7, 4], 5, [1, 10], 1, [10, 1], 1, [5, 5], 8, [2, 8], 2, [8, 2], 2, [3, 6], 2, [6, 3], 2, [0, 9], 1, [9, 0], 1, [4, 4], 4, [1, 7], 1, [7, 1], 1, [2, 5], 2, [5, 2], 2, [3, 3], 3, [1, 4], 1, [4, 1], 1, [2, 2], 4, [0, 3], 2, [3, 0], 2, [1, 1], 4, [0, 0], 6]. Total composition length: 72

p = 7, wt (4,4). [[8, 8], 1, [6, 9], 1, [9, 6], 1, [7, 7], 2, [4, 10], 2, [10, 4], 2, [5, 8], 4, [8, 5], 4, [2, 11], 1, [11, 2], 1, [6, 6], 3, [3, 9], 3, [9, 3], 3, [0, 12], 1, [12, 0], 1, [4, 7], 9, [7, 4], 9, [1, 10], 3, [10, 1], 3, [5, 5], 14, [2, 8], 6, [8, 2], 6, [3, 6], 5, [6, 3], 5, [0, 9], 1, [9, 0], 1, [4, 4], 10, [1, 7], 3, [7, 1], 3, [2, 5], 6, [5, 2], 6, [3, 3], 4, [0, 6], 2, [6, 0], 2, [1, 4], 2, [4, 1], 2, [2, 2], 6, [0, 3], 4, [3, 0], 4, [1, 1], 8, [0, 0], 10]. Total composition length: 164

p = 7, wt (5,0). [[5, 5], 1, [4, 4], 1, [3, 3], 1, [2, 2], 2, [1, 1], 2, [0, 0], 2]. Total composition length: 9

p = 7, wt (5,1). [[6, 6], 1, [4, 7], 1, [7, 4], 1, [5, 5], 2, [4, 4], 1, [3, 3], 1, [2, 2], 2, [1, 1], 2, [0, 0], 2]. Total composition length: 13

p = 7, wt (5,2). [[7, 7], 1, [5, 8], 2, [8, 5], 2, [6, 6], 2, [3, 9], 1, [9, 3], 1, [4, 7], 5, [7, 4], 5, [5, 5], 8, [2, 8], 1, [8, 2], 1, [3, 6], 2, [6, 3], 2, [4, 4], 3, [2, 5], 1, [5, 2], 1, [3, 3], 2, [1, 4], 1, [4, 1], 1, [2, 2], 4, [0, 3], 2, [3, 0], 2, [1, 1], 4, [0, 0], 6]. Total composition length: 60

p = 7, wt (5,3). [[8, 8], 1, [6, 9], 1, [9, 6], 1, [7, 7], 2, [4, 10], 2, [10, 4], 2, [5, 8], 4, [8, 5], 4, [2, 11], 1, [11, 2], 1, [6, 6], 3, [3, 9], 3, [9, 3], 3, [4, 7], 9, [7, 4], 9, [1, 10], 2, [10, 1], 2, [5, 5], 14, [2, 8], 5, [8, 2], 5, [3, 6], 5, [6, 3], 5, [0, 9], 1, [9, 0], 1, [4, 4], 9, [1, 7], 2, [7, 1], 2, [2, 5], 5, [5, 2], 5, [3, 3], 4, [0, 6], 1, [6, 0], 1, [1, 4], 2, [4, 1], 2, [2, 2], 6, [0, 3], 4, [3, 0], 4, [1, 1], 8, [0, 0], 10]. Total composition length: 151

p = 7, wt (5,4). [[9, 9], 1, [7, 10], 1, [10, 7], 1, [8, 8], 2, [5, 11], 2, [11, 5], 2, [6, 9], 2, [9, 6], 2, [3, 12], 2, [12, 3], 2, [7, 7], 3, [4, 10], 4, [10, 4], 4, [1, 13], 1, [13, 1], 1, [5, 8], 6, [8, 5], 6, [2, 11], 3, [11, 2], 3, [6, 6], 4, [3, 9], 5, [9, 3], 5, [0, 12], 1, [12, 0], 1, [4, 7], 13, [7, 4], 13, [1, 10], 6, [10, 1], 6, [5, 5], 20, [2, 8], 9, [8, 2], 9, [3, 6], 8, [6, 3], 8, [0, 9], 4, [9, 0], 4, [4, 4], 15, [1, 7], 6, [7, 1], 6, [2, 5], 11, [5, 2], 11, [3, 3], 9, [0, 6], 2, [6, 0], 2, [1, 4], 3, [4, 1], 3, [2, 2], 10, [0, 3], 8, [3, 0], 8, [1, 1], 12, [0, 0], 14]. Total composition length: 284

p = 7, wt (5,5). [[10, 10], 1, [8, 11], 1, [11, 8], 1, [9, 9], 3, [6, 12], 1, [12, 6], 1, [7, 10], 2, [10, 7], 2, [4, 13], 2, [13, 4], 2, [8, 8], 3, [5, 11], 4, [11, 5], 4, [2, 14], 2, [14, 2], 2, [6, 9], 3, [9, 6], 3, [3, 12], 6, [12, 3], 6, [0, 15], 1, [15, 0], 1, [7, 7], 4, [4, 10], 6, [10, 4], 6, [1, 13], 3, [13, 1], 3, [5, 8], 8, [8, 5], 8, [2, 11], 6, [11, 2], 6, [6, 6], 5, [3, 9], 7, [9, 3], 7, [0, 12], 2, [12, 0], 2, [4, 7], 17, [7, 4], 17, [1, 10], 9, [10, 1], 9, [5, 5], 26, [2, 8], 12, [8, 2], 12, [3, 6], 11, [6, 3], 11, [0, 9], 8, [9, 0], 8, [4, 4], 20, [1, 7], 9, [7, 1], 9, [2, 5], 14, [5, 2], 14, [3, 3], 12, [0, 6], 3, [6, 0], 3, [1, 4], 6, [4, 1], 6, [2, 2], 16, [0, 3], 12, [3, 0], 12, [1, 1], 16, [0, 0], 18]. Total composition length: 434

p = 7, wt (6,0). [[6, 6], 1, [5, 5], 1, [4, 4], 1, [3, 3], 1, [2, 2], 2, [1, 1], 2, [0, 0], 2]. Total composition length: 10

p = 7, wt (6,1). [[7, 7], 1, [5, 8], 2, [8, 5], 2, [6, 6], 2, [4, 7], 4, [7, 4], 4, [5, 5], 7, [3, 6], 1, [6, 3], 1, [4, 4], 2, [2, 5], 1, [5, 2], 1, [3, 3], 2, [1, 4], 1, [4, 1], 1, [2, 2], 4, [0, 3], 2, [3, 0], 2, [1, 1], 4, [0, 0], 6]. Total composition length: 50

p = 7, wt (6,2). [[8, 8], 1, [6, 9], 1, [9, 6], 1, [7, 7], 2, [4, 10], 2, [10, 4], 2, [5, 8], 4, [8, 5], 4, [6, 6], 3, [3, 9], 2, [9, 3], 2, [4, 7], 8, [7, 4], 8, [5, 5], 13, [2, 8], 4, [8, 2], 4, [3, 6], 4, [6, 3], 4, [4, 4], 8, [1, 7], 1, [7, 1], 1, [2, 5], 3, [5, 2], 3, [3, 3], 3, [0, 6], 1, [6, 0], 1, [1, 4], 2, [4, 1], 2, [2, 2], 6, [0, 3], 3, [3, 0], 3, [1, 1], 8, [0, 0], 10]. Total composition length: 124

p = 7, wt (6,3). [[9, 9], 1, [7, 10], 1, [10, 7], 1, [8, 8], 2, [5, 11], 2, [11, 5], 2, [6, 9], *2, [9, 6], 2, [3, 12], 2, [12, 3], 2, [7, 7], 3, [4, 10], 4, [10, 4], 4, [5, 8], 6, [8, 5], 6, [2, 11], 2, [11, 2], 2, [6, 6], 4, [3, 9], 4, [9, 3], 4, [4, 7], 12, [7, 4], 12, [1, 10], 4, [10, 1], 4, [5, 5], 19, [2, 8], 8, [8, 2], 8, [3, 6], 7, [6, 3], 7, [0, 9], 4, [9, 0], 4, [4, 4], 14, [1, 7], 5, [7, 1], 5, [2, 5], 9, [5, 2], 9, [3, 3], 9, [0, 6], 1, [6, 0], 1, [1, 4], 2, [4, 1], 2, [2, 2], 10, [0, 3], 7, [3, 0], 7, [1, 1], 12, [0, 0], 14]. Total composition length: 252*

p = 7, wt (6,4). [[10, 10], 1, [8, 11], 1, [11, 8], 1, [9, 9], 3, [6, 12], 1, [12, 6], 1, [7, 10], 2, [10, 7], 2, [4, 13], 2, [13, 4], 2, [8, 8], 3, [5, 11], 4, [11, 5], 4, [2, 14], 2, [14, 2], 2, [6, 9], 3, [9, 6], 3, [3, 12], 6, [12, 3], 6, [7, 7], 4, [4, 10], 6, [10, 4], 6, [1, 13], 2, [13, 1], 2, [5, 8], 8, [8, 5], 8, [2, 11], 4, [11, 2], 4, [6, 6], 5, [3, 9], 6, [9, 3], 6, [0, 12], 2, [12, 0], 2, [4, 7], 16, [7, 4], 16, [1, 10], 8, [10, 1], 8, [5, 5], 25, [2, 8], 12, [8, 2], 12, [3, 6], 10, [6, 3], 10, [0, 9], 8, [9, 0], 8, [4, 4], 20, [1, 7], 8, [7, 1], 8, [2, 5], 13, [5, 2], 13, [3, 3], 12, [0, 6], 3, [6, 0], 3, [1, 4], 5, [4, 1], 5, [2, 2], 16, [0, 3], 10, [3, 0], 10, [1, 1], 16, [0, 0], 18]. Total composition length: 407

p = 7, wt (6,5). [[11, 11], 1, [9, 12], 1, [12, 9], 1, [10, 10], 2, [7, 13], 1, [13, 7], 1, [8, 11], 2, [11, 8], 2, [5, 14], 2, [14, 5], 2, [9, 9], 5, [6, 12], 4, [12, 6], 4, [3, 15], 2, [15, 3], 2, [7, 10], 4, [10, 7], 4, [4, 13], 4, [13, 4], 4, [1, 16], 2, [16, 1], 2, [8, 8], 5, [5, 11], 8, [11, 5], 8, [2, 14], 4, [14, 2], 4, [6, 9], 4, [9, 6], 4, [3, 12], 10, [12, 3], 10, [0, 15], 2, [15, 0], 2, [7, 7], 5, [4, 10], 10, [10, 4], 10, [1, 13], 4, [13, 1], 4, [5, 8], 10, [8, 5], 10, [2, 11], 8, [11, 2], 8, [6, 6], 6, [3, 9], 8, [9, 3], 8, [0, 12], 4, [12, 0], 4, [4, 7], 20, [7, 4], 20, [1, 10], 12, [10, 1], 12, [5, 5], 31, [2, 8], 16, [8, 2], 16, [3, 6], 12, [6, 3], 12, [0, 9], 12, [9, 0], 12, [4, 4], 23, [1, 7], 12, [7, 1], 12, [2, 5], 16, [5, 2], 16, [3, 3], 15, [0, 6], 4, [6, 0], 4, [1, 4], 8, [4, 1], 8, [2, 2], 22, [0, 3], 16, [3, 0], 16, [1, 1], 22, [0, 0], 22]. Total composition length: 603

p = 7, wt (6,6). [[12, 12], 1, [10, 13], 1, [13, 10], 1, [11, 11], 2, [8, 14], 1, [14, 8], 1, [9, 12], 3, [12, 9], 3, [6, 15], 1, [15, 6], 1, [10, 10], 3, [7, 13], 2, [13, 7], 2, [4, 16], 2, [16, 4], 2, [8, 11], 4, [11, 8], 4, [5, 14], 4, [14, 5], 4, [2, 17], 2, [17, 2], 2, [9, 9], 7, [6, 12], 7, [12, 6], 7, [3, 15], 6, [15, 3], 6, [0, 18], 2, [18, 0], 2, [7, 10], 8, [10, 7], 8, [4, 13], 7, [13, 4], 7, [1, 16], 4, [16, 1], 4, [8, 8], 7, [5, 11], 14, [11, 5], 14, [2, 14], 6, [14, 2], 6, [6, 9], 7, [9, 6], 7, [3, 12], 14, [12, 3], 14, [0, 15], 3, [15, 0], 3, [7, 7], 7, [4, 10], 14, [10, 4], 14, [1, 13], 5, [13, 1], 5, [5, 8], 14, [8, 5], 14, [2, 11], 10, [11, 2], 10, [6, 6], 7, [3, 9], 10, [9, 3], 10, [0, 12], 6, [12, 0], 6, [4, 7], 24, [7, 4], 24, [1, 10], 16, [10, 1], 16, [5, 5], 34, [2, 8], 20, [8, 2], 20, [3, 6], 13, [6, 3], 13, [0, 9], 16, [9, 0], 16, [4, 4], 26, [1, 7], 17, [7, 1], 17, [2, 5], 18, [5, 2], 18, [3, 3], 18, [0, 6], 5, [6, 0], 5, [1, 4], 10, [4, 1], 10, [2, 2], 28, [0, 3], 20, [3, 0], 20, [1, 1], 28, [0, 0], 28]. Total composition length: 828

p = 7, wt (7,0). [[7, 7], 1, [0, 0], 1]. Total composition length: 2

p = 7, wt (7,1). [[8, 8], 1, [7, 7], 1, [1, 1], 1, [0, 0], 1]. Total composition length: 4

p = 7, wt (7,2). [[9, 9], 1, [8, 8], 1, [7, 7], 1, [2, 2], 1, [1, 1], 1, [0, 0], 1]. Total composition length: 6

A.3 Figures

Figures A.1 and A.2 are diagrams showing a section of the weight lattice of $SL_2(K)$. In the pictures, a weight is marked with the symbol " ○ " if it is linked to the weight $(1,1)$, which is marked with a star. A weight is marked with the symbol "□" if it is a translate by some $m(p,0)+n(0,p)$ of one of the weights $(p-3,p-3)$, $(3,p-3)$ or $(p-3,3)$. Thus the set of weights so marked (properly) contains the set of those weights (a,b) which proposition 4.2.1 does not claim give non-zero $\mathrm{Ext}^1_G(L(a,b),L(1,1))$ (cf. 4.2.1). Where a weight is marked with both symbols, this is a weight to look for amongst the composition factors (cf. 4.1.2).

Figure A.3 shows some results calculated for SL_2 using the theorem in the main text.

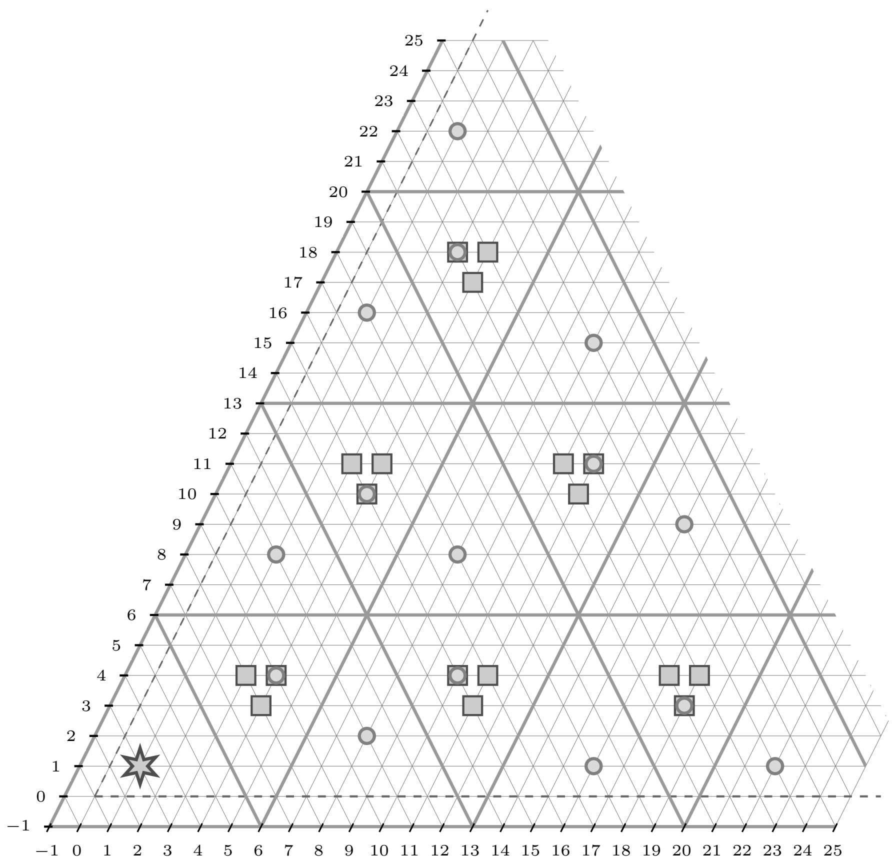

Figure A.1: $\mathrm{SL}_3(K)$, characteristic 7

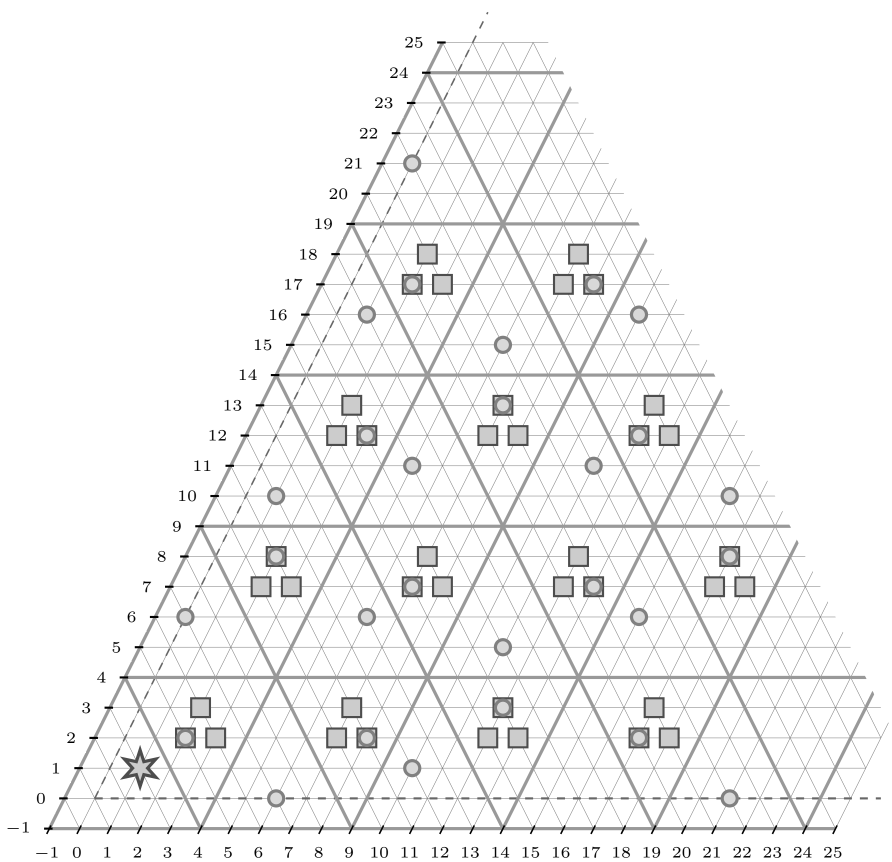

Figure A.2: $\mathrm{SL}_3(K)$, characteristic 5

	p=3	p=5	p=7
0	○	○	○
1	●	●	●
2	●	●	●
3	●	○	●
4	●	○	●
5	●	●	○
6	●	●	○
7	○	●	●
8	○	○	●
9	●	○	●
10	●	●	●
11	●	●	●
12	●	●	○
13	●	○	○
14	●	○	●
15	●	○	●
16	○	●	●
17	○	●	●
18	●	○	●
19	○	○	○
20	○	○	○
21	○	○	●
22	○	○	●
23	○	○	●
24	○	○	●
25	○	●	●
26	○	●	○
27	●	●	○
28	●	○	●
29	●	○	●
30	●	●	●
31		●	●
32		●	●
33		○	○
34		○	○
35		●	○
36		●	●
37		●	●
38		○	●
39		○	●
40		○	○
41			○
42			○
43			○
44			○
45			○
46			○
47			○
48			○
49			●
50			●

Black (resp. white) dots indicate that the simple module with this highest weight gives (resp. does not give) a reductive pair in the given characteristic

Figure A.3: Examples for SL_2.

Bibliography

[1] M. Bate, S. Herpel, B. Martin, and G. Röhrle. G-complete reducibility and semisimple modules. *Bull. Lond. Math. Soc.*, 43(6):1069–1078, 2011.

[2] M. Bate, B. Martin, and G. Röhrle. A geometric approach to complete reducibility. *Invent. Math.*, 161(1):177–218, 2005.

[3] M. Bate, B. Martin, G. Röhrle, and R. Tange. Complete reducibility and separability. *Trans. Amer. Math. Soc.*, 362(8):4283–4311, 2010.

[4] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Basic representation theory of finite groups and associative algebras.

[5] D. J. Benson and J. F. Carlson. Nilpotent elements in the Green ring. *J. Algebra*, 104(2):329–350, 1986.

[6] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[7] N. Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)].

[8] C. Bowman, S. R. Doty, and S. Martin. Decomposition of tensor products of modular irreducible representations for SL_3. *Int. Electron. J. Algebra*, 9:177–219, 2011. With an appendix by C. M. Ringel.

[9] C. Bowman, S. R. Doty, and S. Martin. Decomposition of tensor products of modular irreducible representations for SL_3: the $p \geq 5$ case. 2011. arXiv:1111.5811.

[10] E. Cline, B. Parshall, and L. Scott. On the tensor product theorem for algebraic groups. *J. Algebra*, 63(1):264–267, 1980.

[11] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[12] S. Donkin. *The q-Schur algebra*, volume 253 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[13] S. Doty. Weyl modules gap package. `http://doty.math.luc.edu/weylmodules`. Accessed: 15/09/2014.

[14] S. Doty and A. Henke. Decomposition of tensor products of modular irreducibles for SL_2. *Q. J. Math.*, 56(2):189–207, 2005.

[15] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[16] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, 2014.

[17] S. Herpel. On the smoothness of centralizers in reductive groups. *Trans. Amer. Math. Soc.*, 365(7):3753–3774, 2013.

[18] J. E. Humphreys. *Linear algebraic groups.* Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.

[19] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[20] J. C. Jantzen. Low-dimensional representations of reductive groups are semisimple. In *Algebraic groups and Lie groups*, volume 9 of *Austral. Math. Soc. Lect. Ser.*, pages 255–266. Cambridge Univ. Press, Cambridge, 1997.

[21] J. C. Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2003.

[22] M. W. Liebeck and G. M. Seitz. Reductive subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 121(580):vi+111, 1996.

[23] G. Malle and D. Testerman. *Linear algebraic groups and finite groups of Lie type*, volume 133 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2011.

[24] A. E. Parker. Higher extensions between modules for SL_2. *Adv. Math.*, 209(1):381–405, 2007.

[25] R. W. Richardson, Jr. Conjugacy classes in Lie algebras and algebraic groups. *Ann. of Math. (2)*, 86:1–15, 1967.

[26] J.-P. Serre. Sur la semi-simplicité des produits tensoriels de représentations de groupes. *Invent. Math.*, 116(1-3):513–530, 1994.

[27] J.-P. Serre. Complète réductibilité. *Astérisque*, (299):Exp. No. 932, viii, 195–217, 2005. Séminaire Bourbaki. Vol. 2003/2004.

[28] T. A. Springer. *Linear algebraic groups.* Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.

[29] S Xanthopoulos. *On a question of Verma about indecomposable representations of algebraic groups and of their Lie algebras.* PhD thesis, Queen Mary and Westfield College, University of London, 1992.

[30] S. E.-B Yehia. *Extensions of Simple Modules for the Universal Chevalley Groups and its Parabolic Subgroups.* PhD thesis, Mathematics Institute, Warwick, 1982.