Hyperspherical Trigonometry, Related Elliptic Functions and Integrable Systems

Paul Richard Jennings

Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds

School of Mathematics

September 2013

For Lesley

Acknowledgements

I am extremely grateful to my supervisor Professor Frank Nijhoff for his continuing enthusiasm and support. He has provided my with inspiration and guidance through many discussions, which often veered wildly off topic, but were always interesting and thought provoking. I am also grateful to the UK Engineering and Physical Science Research Council (ESPRC) for funding my research. I would also like to thank my parents for their continuous encouragement and support. Finally, a very special thanks is owed to my dear wife, Lesley, who stood patiently by me as I pursued my PhD, even moving countries to be here with me (although Scotland is not a real country). Her love, support and encouragement have been essential.

This research has been carried out by myself and Professor Frank Nijhoff. My contribution to the work has been to carry out explicit calculations and provide proofs. The ideas and concepts were developed in discussion with Professor Nijhoff.

Abstract

The basic formulae of hyperspherical trigonometry in multi-dimensional Euclidean space are developed using multi-dimensional vector products, and their conversion to identities for elliptic functions is shown. The basic addition formulae for functions on the 3-sphere embedded in four-dimensional space are shown to lead to addition formulae for elliptic functions, associated with algebraic curves, which have two distinct moduli. Application of these formulae to the cases of a multi-dimensional Euler top and Double Elliptic Systems are given, providing a connection between the two.

A generalisation of the Lattice Potential Kadomtsev-Petviashvili (LPKP) equation is presented, using the method of Direct Linearisation based on an elliptic Cauchy kernel. This yields a $(3+1)$-dimensional lattice system with one of the lattice shifts singled out. The integrability of the lattice system is considered, presenting a Lax representation and soliton solutions. An associated continuous system is also derived, yielding a $(3+1)$-dimensional generalisation of the potential KP equation associated with an elliptic curve.

Contents

Dedication ii

Acknowledgements iii

Abstract iv

List of figures ix

1 Introduction **1**

1.1 Spherical Trigonometry 1

1.2 Elliptic Functions 10

1.3 Multi-dimensional Integrability 13

1.3.1 The d-Simplex Equations 13

1.3.2 Spherical Volumes 15

1.3.3 The Kadomtsev-Petviashvili Equation 19

1.3.4 Classification of Integrable Discrete Equations of Octahedron Type 23

1.4 From The Kadomtsev-Petviashvili Equation to the Calogero-Moser and Ruijsenaars-Schneider Models 24

1.5 Overview 26

1.6 Outline 27

2 Hyperspherical Trigonometry and Corresponding Elliptic Functions 29

2.1 Introduction 29

2.2 Multi-dimensional Vector Products 31

2.3 Hyperspherical Trigonometry 36

2.3.1 Spherical Trigonometry 36

2.3.2 Hyperspherical Trigonometry in Four-Dimensional Euclidean Space 41

2.3.3 m-Dimensional Hyperspherical Trigonometry 50

2.4 Hyperspherical Trigonometry and Elliptic Functions 55

2.4.1 Link between Spherical Trigonometry and the Jacobi Elliptic Functions 55

2.4.2 Link between Hyperspherical Trigonometry and Elliptic Functions 59

2.5 Hyperspherical Trigonometry and the Generalised Jacobi Functions . . . 61

2.5.1 Generalised Jacobi Functions 61

2.5.2 Hyperspherical Trigonometry and the Generalised Jacobi Functions 64

2.6 Four- and Five-Parts Formulae 65

2.6.1 Spherical Case 65

2.6.2 Four-dimensional Hyperspherical Case 67

2.7 Angle Addition Formulas 69

3 A Higher-Dimensional Euler Top and the DELL Model 71

3.1 Introduction 71

3.2 Nambu Mechanics 71

3.2.1 The Euler Top 74

3.3 Multidimensional Euler Top 76

3.3.1 Double Elliptic Systems 77

4 The Discrete Elliptic Kadomtsev-Petviashvili Equation 84

4.1 Introduction 84

4.2 Elliptic Matrix Structure 88

4.3 System Dynamics 93

4.4 Elliptic Lattice Structure 95

4.5 Lax Representation 103

4.6 Dimensional Reduction and Degeneration 108

4.6.1 Degeneration of the Elliptic Curve 108

4.6.2 Dimensional Reduction 108

4.7 Soliton Type Solutions 109

4.8 Conclusion 111

5 The Continuous Elliptic Kadomtsev-Petviashvili Equation 112

5.1 Introduction 112

5.2 Continuous Elliptic System 114

5.3 Lax Representation 126

5.4 Dimensional Reduction and Degeneration 134

5.4.1 Degeneration of the Elliptic Curve 134

5.4.2 Dimensional Reduction 134

5.5 Date, Jimbo and Miwa's elliptic KP system 135

6 Conclusion 138

6.1 Discussion 138

6.2 Future Work 140

Bibliography 142

List of figures

1.1 A great circle on a 2-sphere . . . 3
1.2 The spherical triangle $\triangle ABC$ . . . 4
1.3 Cosine Rule . . . 5
1.4 A sphere showing two lunes, ACBDA and ADBEA . . . 8
1.5 Girard's Theorem . . . 9
1.6 A Birectangular Tetrahedron . . . 16
1.7 The KP Lattice . . . 22
1.8 Overview of the Relevant Areas . . . 27

2.1 A spherical triangle . . . 37
2.2 A hyperspherical tetrahedron . . . 41

Chapter 1

Introduction

At first glance, hyperspherical trigonometry, elliptic functions and integrable systems may not appear to be all that closely connected. This chapter provides an introduction to the topics of study relevant for this thesis, giving an overview of previous results, and discusses how the various strands may be related. We begin with spherical trigonometry, and discuss how this relates to elliptic functions. We then move on to discuss integrability, in particular multi-dimensional integrable systems, and how they relate.

1.1 Spherical Trigonometry

Spherical trigonometry, a branch of geometry dealing with the goniometry, i.e. the measure of the angles, of triangles confined to a 2-sphere, is an area of mathematics that has existed since the ancient Greeks, with its foundations laid by Menelaus and Hipparchus. It is of vital importance for many calculations in astronomy, navigation and cartography. Further advances were made in the Islamic world, in order to help calculate their Holy Days based on the phases of the moon. These advances resulted in giving us the basis of spherical trigonometry in its modern form. However, it was not until the 17th century, in the western world, that spherical trigonometry was first considered as a separate mathematical discipline, independent of astronomy. One of the protagonists of this era, John Napier, in his work of 1614, treated spherical trigonometry alongside his work introducing logarithms [102]. Other protagonists include Delambre [42], Euler [48], Cagnoli [33] and l'Huillier [87]. A neat treatise of the subject was later given by Todhunter, and refined by Leatham [133].

Definition 1.1.1 (m-Sphere) *[57] An* **m-*sphere*** *is an m-dimensional hyper-surface, every point of which is equidistant from a fixed point, called the* ***centre*** *of the sphere. The straight line which connects any point on the surface to its centre is called a* ***radius***. *A 2-sphere is often just called a sphere, whilst for $m > 2$, an m-sphere may be called an $(m+1)$-dimensional hypersphere.*

Somewhat confusingly, this now universally accepted definition is at odds as to how a sphere was originally defined. Todhunter defines a sphere as 'a solid bounded by a surface every point of which is equally distant from a fixed point' [133], whereas the solid is now defined to be a 'ball'. Perhaps more confusingly, in geometry, the m in m-sphere refers to the number of co-ordinates in the underlying space, and not the dimension of the surface itself as in Definition 1.1.1 [39]. We, however, adopt the topological convention given in Definition 1.1.1.

For simplicity, in this section, we consider a 2-sphere of unit radius, that is with radius equal to 1, embedded in three-dimensional Euclidean space, $\mathbb{R}^3$, with its centre taken to be the origin.

Definition 1.1.2 (Great Circle) *[133] The section of surface of a sphere mapped by a plane passing through its centre is called a* ***great circle***. *(See figure 1.1).*

Definition 1.1.3 (Spherical Triangle) *[133] A* ***spherical triangle*** *is a triangle formed on a sphere bounded by three great circles, restricted such that each side is less than a*

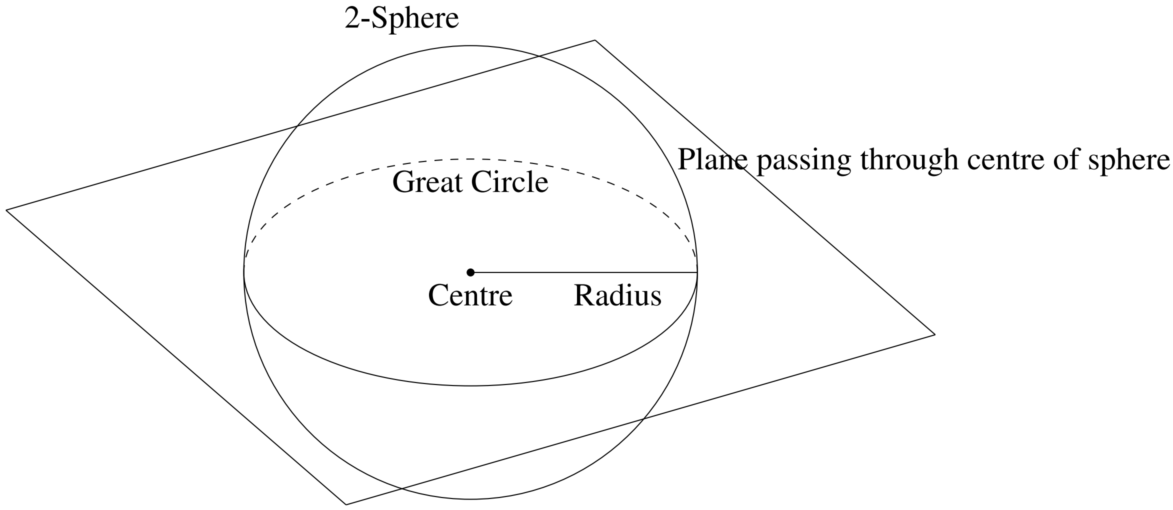


Figure 1.1: A great circle on a 2-sphere

semicircle, and hence, each angle is less than π. The three arcs of great circles that form the the spherical triangle are called the sides of the spherical triangle, and the angles between the intersection of these arcs, the spherical angles.

For simplicity, in this chapter, we denote the three vertices of a spherical triangle by A, B and C, also applying the same label to the spherical angles at each vertex. We label the sides of the spherical triangle by a, b and c, corresponding to the side opposite the vertex labeled by the same uppercase label. We label this spherical triangle $\triangle ABC$. Note that by restricting ourselves to the consideration of the unit sphere, we can identify the side lengths with the corresponding angles at the centre of the sphere. (See figure 1.2).

Proposition 1.1.4 (Cosine Rule) *[133] The cosine of one of the spherical angles of a spherical triangle in terms of the sines and cosines of the triangle's sides is given by:*

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \tag{1.1}$$

Proof

[133] First, we consider the case sides b and c are less than $\pi/2$. Consider the spherical

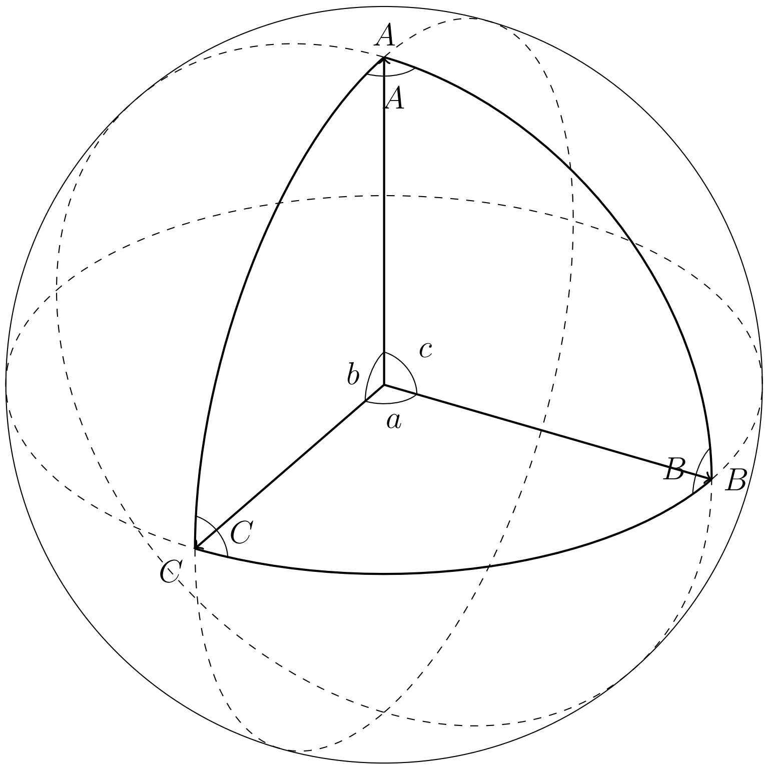


Figure 1.2: The spherical triangle $\triangle ABC$

triangle ABC, shown in 1.3. Let the tangent at A to the arc AC intersect the line in the direction OC at E, and similarly, let the tangent at A to the arc AB intersect the line in the direction of OB at D. Upon joining ED, the angle EAD is the angle of the spherical triangle at A, and the angle EOD measures the side a. Using the cosine rule for planar Euclidean triangles, for triangles ADE and ODE, we have:

$$DE^2 = AD^2 + AE^2 - 2AD \cdot AE \cos A, \tag{1.2a}$$

$$DE^2 = OD^2 + OE^2 - 2OD \cdot OE \cos a, \tag{1.2b}$$

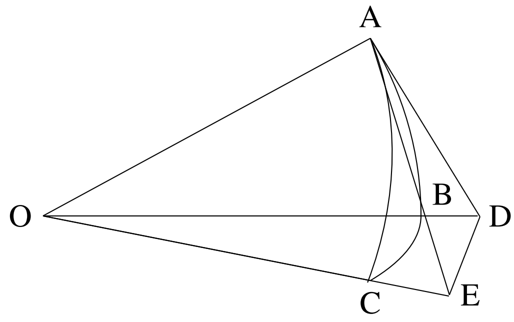


Figure 1.3: Cosine Rule

respectively. As for triangles OAD and OAE, as they are right-angled, we also have

$$OD^2 = OA^2 + AD^2, \tag{1.3a}$$

$$OE^2 = OA^2 + AE^2. \tag{1.3b}$$

$$\tag{1.3c}$$

Subtracting these formulae, we have

$$0 = 2OA^2 + 2AD \cdot AE \cos A - 2OD \cdot OE \cos a, \tag{1.4}$$

or equivalently,

$$\cos a = \frac{OA}{OE} \cdot \frac{OA}{OD} + \frac{AE}{OE} \cdot \frac{AD}{OD} \cos A, \tag{1.5}$$

from which the result follows. □

Proposition 1.1.5 (Sine Rule) *[133] This expresses the relationship between two sides of a spherical triangle and the angles opposite them:*

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \tag{1.6}$$

Proof

[133] Taking as a starting point,

$$\sin^2 A = 1 - \cos^2 A, \tag{1.7}$$

and substituting in the formula for the Cosine rule, (1.1), we have

$$\sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2, \tag{1.8a}$$

$$= \frac{\left(1-\cos^2 b\right)\left(1-\cos^2 c\right) - \left(\cos a - \cos b \cos c\right)^2}{\sin^2 b \sin^2 c}, \tag{1.8b}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c}, \tag{1.8c}$$

and hence,

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}}{\sin b \sin c}, \tag{1.9}$$

whereby we have taken the positive square root because $0 < A < \pi$. The Sine Rule then follows by symmetry. □

Definition 1.1.6 *[133] The* ***axis*** *of a great circle is the line through the centre of the sphere, perpendicular to the plane forming said great circle. The points where this axis meets the surface of the sphere are called its* ***poles****.*

Definition 1.1.7 (Polar Triangle) *[133] Let the points A, B and C form a spherical triangle, with the points A', B' and C', the poles of the arcs BC, CA and AB, respectively, which lie on the same sides of them as the opposite angles A, B and C. Then, the spherical triangle formed by the points A', B' and C' is the* ***polar triangle*** *to triangle ABC. Note that, although there are six poles, and in principle eight triangles formed by them, there is only one triangle in which the poles A', B' and C' lie toward the same parts with the corresponding angles A, B and C, and it is this which is the polar triangle.*

Proposition 1.1.8 (Polar Cosine Rule) *[133] This expresses the cosine of one of the sides of a spherical triangle in terms of the sines and cosines of the triangle's spherical angles:*

$$\cos a = \frac{\cos B \cos C + \cos A}{\sin B \sin C} \tag{1.10}$$

Proof

[133] From the cosine rule (1.1), we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \tag{1.11a}$$
$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}, \tag{1.11b}$$
$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}. \tag{1.11c}$$

Therefore, we have

$$\cos A + \cos B \cos C = \frac{\cos a - \cos b \cos c}{\sin b \sin c} + \left(\frac{\cos b - \cos a \cos c}{\sin a \sin c}\right)\left(\frac{\cos c - \cos a \cos b}{\sin a \sin b}\right), \tag{1.12}$$

$$= \frac{\cos a}{\sin^2 a \sin b \sin c}\left(1 - \cos^2 a - \cos^2 c - \cos^2 b + 2\cos a \cos b \cos c\right), \tag{1.13}$$

which, using the sine rule (1.6) reduces to

$$\cos A + \cos B \cos C = \cos a \sin B \sin C, \tag{1.14}$$

and so the result follows. □

Definition 1.1.9 *[133] A **lune** is a portion of a sphere contained between two great semicircles.*

Proposition 1.1.10 (Area of a Lune) *[133] The area of a lune of a unit sphere is given by*

$$\textit{Area of Lune} = 2A, \tag{1.15}$$

where the angle A is the lune's circular measure.

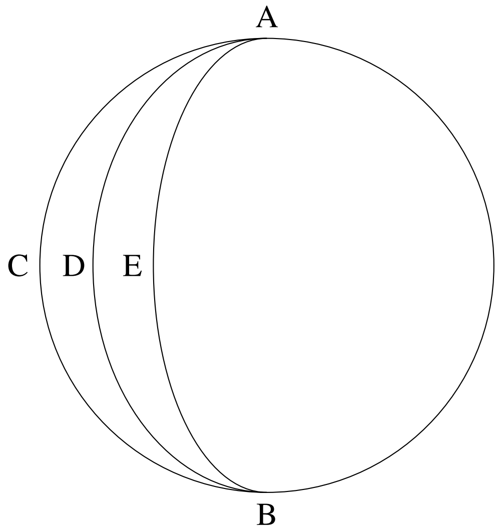


Figure 1.4: A sphere showing two lunes, ACBDA and ADBEA

Proof

[133] Let $ACBDA$ and $ADBEA$ be two lunes with equal angles at A, as in figure 1.4. These lunes may be placed atop each other, and so, have equal area. Therefore, the area of a lune is proportional to its angle,

$$\text{Area of Lune} = \kappa A, \tag{1.16}$$

for some constant of proportionality κ. As the entire sphere is a lune with area 4π, and angle $A = 2\pi$, it follows that

$$\text{Area of Lune} = 2A, \tag{1.17}$$

as required. □

Definition 1.1.11 (Spherical Excess) *[133] The expression*

$$A + B + C - \pi \tag{1.18}$$

is called the ***Spherical Excess*** *of a spherical triangle with spherical angles A, B and C.*

The formula to calculate the area of a spherical triangle was first shown by Girard in 1626.

Theorem 1.1.12 (Girard's Theorem) *[133] The area of a spherical triangle, with spherical angles A, B and C, is given by its Spherical Excess,*

$$\textit{Area of spherical triangle} = A + B + C - \pi. \tag{1.19}$$

Proof

[133] Let triangle ABC be a spherical triangle. Label the points opposite A, B and C,

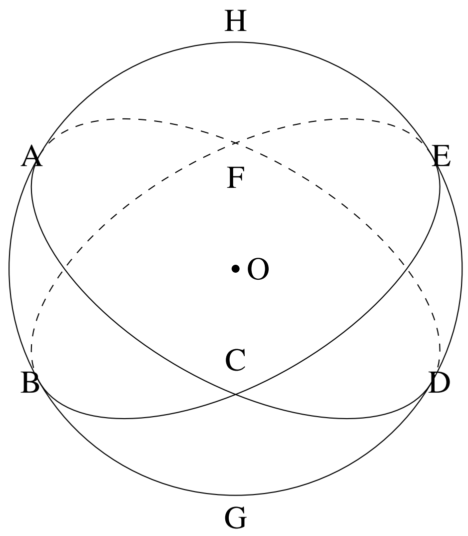


Figure 1.5: Girard's Theorem

D, E and F, respectively, such that the arcs AD, BE and CF form semicircles. (See figure 1.5).The triangle ABC now forms part of three lunes; $ABDCA$, $BCEAB$ and $CAFBC$.

Now, note that the triangles AFB and CDE have equal solid angle, and hence, side lengths. This means that their areas must in fact be equal, and so the area of the lune $CAFBC$ is equal to the sum of the areas of triangles ABC and CDE. We therefore have,

$$ABC + BGDC = \text{lune}\, ABDCA = 2A, \tag{1.20a}$$

$$ABC + AHEC = \text{lune}\, BCEAB = 2B, \tag{1.20b}$$

$$ABC + CDE = \text{lune}\, CAFBC = 2C, \tag{1.20c}$$

and so, summing these together gives

$$2ABC + \text{Area of a hemisphere} = 2(A + B + C). \tag{1.21}$$

As the area of a hemisphere of radius one is 2π, the result follows. □

1.2 Elliptic Functions

There has long existed a connection between spherical trigonometry and elliptic functions, dating back to almost the foundation of elliptic function theory itself. Elliptic functions were first devised by Legendre [86] as the inverse of elliptic integrals, with the theory built upon by Abel [1, 2, 3] who revealed their double periodicity. The theory was improved upon through the work of Jacobi [64] with the introduction of what came to be known as the Jacobi elliptic functions. A more complete study of the area was later undertaken by Weierstrass [135], who by the introduction of the Weierstrass $\wp$-function linked together the previous work. Other key protagonists include Liouville, Gauss, Euler and Frobenius. Complete treatises of the subject area are provided by Greenhill [54] and Cayley [36], as well as by Whittaker and Watson [136] and Akheizer [14]. The connection between spherical trigonometry and elliptic functions arises through the Jacobi elliptic functions and is owing to Legendre [86] and Lagrange [84]. A neat proof of this connection, which is looked at in more detail in Chapter 2, was provided by Irwin [62].

Definition 1.2.1 (Analytic Function) *[136] A function $f : \mathcal{D} \to \mathbb{C}$ defined on some open domain $\mathcal{D}$ in the complex plane is said to be* ***analytic*** *or holomorphic if the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \tag{1.22}$$

exists for all $z_0 \in \mathcal{D}$, i.e. f is complex differentiable in the domain $\mathcal{D}$.

Definition 1.2.2 (Pole) *[136] Let $\mathcal{D}$ be an open subset of the complex plane $\mathcal{C}$ with $z_0 \in \mathcal{D}$, and $f : \mathcal{D} \setminus \{z_0\} \to \mathbb{C}$ an analytic function over its domain. If there exists another analytic function $g : \mathcal{D} \to \mathbb{C}$ and a positive integer μ such that*

$$f(z) = \frac{g(z)}{(z - z_0)^{\mu}} \tag{1.23}$$

holds for all $z \in \mathcal{D} \setminus \{z_0\}$, then z_0 is a ***pole*** *of f. The smallest such μ is the order of the pole.*

Definition 1.2.3 (Meromorphic Function) *[136] A function $f : \mathcal{D} \to \mathbb{C}$ is* ***meromorphic*** *if it is analytic for all of $\mathcal{D}$ except a finite number of poles.*

Definition 1.2.4 (Doubly Periodic) *[14] A function $f : \mathcal{D} \to \mathbb{C}$ is said to be* ***doubly periodic*** *if for $2\omega, 2\omega' \in \mathbb{C}$ with $\omega/\omega' \notin \mathbb{R}$,*

$$f\left(z + 2m\omega + 2m'\omega'\right) = f\left(z\right), \tag{1.24}$$

for all $z \in \mathcal{D}$ and all $m, m' \in \mathbb{Z}$.

Definition 1.2.5 (Elliptic Function) *[14] An* ***elliptic function*** *is a doubly periodic function which is meromorphic on $\mathbb{C}$.*

The elliptic functions are closely connected to a family of complex algebraic curves called elliptic curves. These are curves which in appropriate coordinates for $w, z \in \mathbb{C}$ can be cast in the form

$$w^2 = \mathcal{R}(z), \tag{1.25}$$

where $\mathcal{R}$ is a polynomial of degree three or four in z [14].

Definition 1.2.6 (Weierstrass $\wp$-function) *[136] The Weierstrass $\wp$-function is given by the sum*

$$\wp(z;2\omega,2\omega') = \frac{1}{z^2} + \sum_{m^2+m'^2\neq 0}\left(\frac{1}{(z+2m\omega+2m'\omega')^2} - \frac{1}{(2m\omega+2m'\omega')^2}\right), \quad (1.26)$$

for all $m, m' \in \mathbb{Z}$ and $z \neq 2m\omega + 2m'\omega'$.

It may be shown that

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3, \quad (1.27)$$

with

$$g_2 = 60\sum_{m^2+m'^2\neq 0}\frac{1}{(2m\omega+2m'\omega')^4}, \quad (1.28)$$

and

$$g_3 = 140\sum_{m^2+m'^2\neq 0}\frac{1}{(2m\omega+2m'\omega')^6}, \quad (1.29)$$

and hence the pair $(\wp', \wp)$ parameterises an elliptic curve.

Definition 1.2.7 (Jacobi Elliptic functions) *[14] The Jacobi elliptic function sn$(u;k)$ is defined by the inversion of the elliptic integral*

$$u(x;k) = \int_0^{x=sn(u;k)}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad (1.30)$$

where k, the elliptic modulus, is constant. The functions cn$(u;k)$ and dn$(u;k)$ are then defined through the identities

$$sn^2(u;k) + cn^2(u;k) = 1, \quad (1.31a)$$

$$k^2sn^2(u;k) + dn^2(u;k) = 1, \quad (1.31b)$$

If we now let e, e' and e'' be the roots of the equation $4t^3 - g_2t - g_3 = 0$, then the Jacobi elliptic functions are related to the Weierstrass $\wp$-function [14] via

$$\wp(z) = e'' + \frac{e-e'}{\mathrm{sn}^2\left(\sqrt{e-e''}z;\sqrt{\frac{e'-e''}{e-e''}}\right)}. \quad (1.32)$$

There are a number of various other elliptic functions including, amongst others, the theta functions, as well as a number of other closely related functions, such as the Weierstrass ζ- and σ-functions [136].

There also exist connections between spherical trigonometry and integrable systems. A recent paper by Suris and Petrera [122] defined a mapping associated with the cosine rule for spherical triangles to define a mapping, which they showed to be integrable in the sense of multi-dimensional consistency [104, 103, 10, 11, 26]. However, it is an earlier connection that is of particular interest here, namely the connection between the tetrahedron equation and spherical trigonometry.

In chapter 2, we establish formulae for hyperspherical trigonometry, analogous to those of spherical trigonometry, with particular attention focused on the hyperspherical tetrahedral case. We also establish a link between these four-dimensional hyperspherical trigonometric formulae and elliptic functions.

1.3 Multi-dimensional Integrability

1.3.1 The d-Simplex Equations

It is well known that the Yang-Baxter (or triangle) equation [138, 19] and associated R-matrices form the basic algebraic structure underlying most integrable systems in two-dimensions [49]. The Yang-Baxter equation originally arose as a factorisability condition of a scattering matrix of three particles on a line, as well as a generalised star-triangle equation for Boltzmann weights in two-dimensional statistical mechanical models [141]. A brief history of the Yang-Baxter equation may be found in [117].

In their study of the tetrahedron equation, Bazhanov and Stroganov suggested the notion of d-simplex equations as the higher generalisations of the Yang-Baxter equation [25].

The d-simplex equations may be thought of as generalisations of the commutability conditions of transfer matrices in statistical mechanics, with the objects involved dependent on pairs of matrix indices [91]. The 3-simplex, or tetrahedron, equation was discovered by A. B. Zamolodchikov as the factorisability condition for the scattering of four straight strings into three-string amplitudes in a plane [142], building on work on the three-string case in [144]. It is also the condition for the commutativity of layer-to-layer transfer matrices in a three-dimensional lattice spin model [66, 91]. A first non-trivial solution was provided for the static case by Zamolodchikov [142]. Zamolodchikov later found a further solution as a feat of intuition [143], which is given in terms of spherical trigonometry. Zamolodchikov's solution was later verified by Baxter, with the solution dependent on the spherical excess [21, 20]. Further insights have been studied by Sergeev, Bazhanov, Korepanov, Kashaev, Mangazeev and Stroganov amongst others [126, 127, 22, 23, 24, 60, 61, 72, 73, 75, 76, 77, 92], but no complete solution has yet been found. Solutions are also given in [129, 71, 74] using advanced algebraic and categorical methods.

Nijhoff and Maillet showed that the hierarchy of simplex equations leads to the notion of classical integrability for discrete systems on multi-dimensional lattices [90, 89], with, in particular, the tetrahedron equation governing three-dimensional integrable lattice models.

As the Yang-Baxter equation may be solved in terms of the sum of planar angles or arc length, it is natural to conjecture that perhaps the tetrahedron equations, the three-dimensional analogues to the Yang-Baxter equation, may be solved in terms of the sum of the areas of spherical triangles. The extension of this would then be that perhaps that solutions to the d-simplex equations would be related to higher dimensional spherical volumes.

1.3.2 Spherical Volumes

Calculating the volume of spherical simplices is an old and difficult problem. The volume of Euclidean simplices on the other hand is relatively straightforward to calculate. For simplices in Euclidean space, the volume is obtained from the determinant of the vectors emanating from one vertex to the other vertices, with the result first discovered for Euclidean tetrahedra by Tartaglia in 1494. For the case of spherical simplices calculating the volume is not so simple. As part of his magnus opus "Theorie der Vielfachen Kontinuität" [125], Schläfli derived a differential formula for the volume of such simplices. This theorem may be proven either geometrically or by using Schläfli's Differential Volume Formula.

Theorem 1.3.1 (Schläfli's Differential Volume Formula) *[125] Let an $(m-1)$-dimensional spherical simplex S, for $m \geq 2$, have vertices $v_1, \ldots, v_m$, and dihedral angles $\varphi_{jk} = \angle(S_j, S_k)$, with $0 \leq j < k \leq m$, formed by the faces S_j and S_k of S, with apex $S_{jk} := S_j \cap S_k$. Then the differential of the volume function, V_m, on the set of all simplices in S^m is given by*

$$dV_m(S) = \frac{1}{m-1} \sum_{j,k=1:j<k}^{m+1} V_{m-2}(S_{jk}) d\varphi_{jk}, \quad V_0(S_{jk}) := 1. \tag{1.33}$$

A more combinatorial proof to Schläfli's was later provided by Peschl [118]. However, solving this differential relation is far from an easy matter, except in the $n = m$ case, where it provides an alternative proof to Girard's Theorem.

Proof
An alternative proof of Theorem 1.1.12

For spherical triangles, $m = 2$, and so, Schläfli's Differential Volume formula becomes

$$\mathrm{d}A = \sum_{i=1}^{3} \mathrm{d}\alpha_i, \tag{1.34}$$

where A is the area of the spherical triangle with spherical angles α_i, $i = 1, 2, 3$ Integrating this gives

$$A = \alpha_1 + \alpha_2 + \alpha_3 + c, \tag{1.35}$$

where $c = -\pi$ is the integration constant determined by considering, say, the area of a spherical triangle with planar angles $\alpha_1 = \alpha_2 = \alpha_3 = \frac{\pi}{2}$ and comparing this to the known area of $1/8$ of a sphere. $\square$

For the case of spherical tetrahedra, we take $m = 3$, leading Schläfli's Differential formula to become

$$\mathrm{d}V = \frac{1}{2} \sum_{j,k=1:j<k}^{4} \theta_{jk} \mathrm{d}\varphi_{jk}, \tag{1.36}$$

where θ_{jk} is the length of the side of the spherical tetrahedron opposite the corresponding dihedral angle φ_{jk}.

Schl ̈afli then used this differential relation to provide a formula for the simpler case of the volume of a spherical orthoscheme, tetrahedra with a particular property.

Definition 1.3.2 (Orthoscheme) *[134] An **Orthoscheme** is a m-dimensional simplex, defined by a series of m mutually orthogonal edges. This ensures that all of the faces of an orthoscheme are right angled triangles. A three-dimensional orthoscheme is called a **Birectangular Tetrahedron**.*

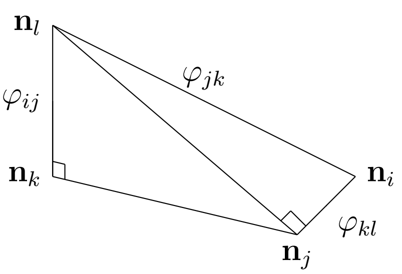


Figure 1.6: A Birectangular Tetrahedron

Proposition 1.3.3 *[5] The volume of a spherical birectangular tetrahedron, T, with dihedral angles φ_{ij}, φ_{jk} and φ_{kl}, as shown in figure 1.6 is given by*

$$\begin{aligned} V_3(T) = \frac{1}{4}\sum_{\mu=1}^{\infty}&\left(\frac{D\left(\varphi_{ij},\varphi_{jk},\varphi_{kl}\right)-\sin\left(\varphi_{ij}-\frac{\pi}{2}\right)\sin\left(\varphi_{kl}-\frac{\pi}{2}\right)}{D\left(\varphi_{ij},\varphi_{jk},\varphi_{kl}\right)+\sin\left(\varphi_{ij}-\frac{\pi}{2}\right)\sin\left(\varphi_{kl}-\frac{\pi}{2}\right)}\right)^{\mu} \\ &\cdot\cos\left(2\mu\left(\varphi_{ij}-\frac{\pi}{2}\right)\right)-\cos\left(2\mu\varphi_{jk}\right)+\cos\left(2\mu\left(\varphi_{kl}-\frac{\pi}{2}\right)\right)\mu^2, \end{aligned} \tag{1.37}$$

where

$$D\left(\varphi_{ij},\varphi_{jk},\varphi_{kl}\right)=\sqrt{\cos^2\left(\varphi_{ij}-\frac{\pi}{2}\right)\cos^2\left(\varphi_{kl}-\frac{\pi}{2}\right)-\cos^2\varphi_{jk}}. \tag{1.38}$$

As all tetrahedra may be expressed as the sum of orthoschemes, it is then possible to use this result to calculate the volume of all spherical tetrahedra. However, in practice determining the new dihedral angles of the orthoschemes is often difficult, and a closed form for the volume may not always be achievable.

Definition 1.3.4 (Symmetric Tetrahedron) *[44] A **Symmetric Tetrahedron** is a tetrahedron with equal dihedral angles at opposite edges.*

Derevin, Mednykh and Pashkevich derived from Schläfli's formula a result for the volumes of symmetric spherical tetrahedra [44].

Proposition 1.3.5 *[44] The volume of a symmetric spherical tetrahedron, with dihedral angles φ_{ij}, φ_{ik} and φ_{jk}, is given by*

$$\begin{aligned} V_3(T) = -\int_{v}^{\infty}&\left(\operatorname{arcsinh}\frac{\cos\varphi_{ij}}{\sqrt{v^2-1}}+\operatorname{arcsinh}\frac{\cos\varphi_{ik}}{\sqrt{v^2-1}}+\operatorname{arcsinh}\frac{\cos\varphi_{jk}}{\sqrt{v^2-1}}\right. \\ &\left.-\operatorname{arcsinh}\frac{1}{\sqrt{v^2-1}}\right)\frac{dv}{v}, \end{aligned} \tag{1.39}$$

where v is defined by

$$v=\frac{1-\cos^2\varphi_{ij}-\cos^2\varphi_{ik}-\cos^2\varphi_{jk}-2\cos\varphi_{ij}\cos\varphi_{ik}\cos\varphi_{jk}}{\sqrt{f(\pi-\varphi_{ij},\varphi_{ik},\varphi_{jk})f(\varphi_{ij},\pi-\varphi_{ik},\varphi_{jk})f(\varphi_{ij},\varphi_{ik},\pi-\varphi_{jk})f(\pi-\varphi_{ij},\pi-\varphi_{ik},\pi-\varphi_{jk})}}, \tag{1.40}$$

with

$$f(A,B,C)=1+\cos A+\cos B+\cos C. \tag{1.41}$$

The general case was finally solved by Cho and Kim [37] in terms of the dilogarithm function, and later refined into a more symmetric form by Murakami with Yano in terms of the dihedral angles [100], and with Ushijima in terms of edge lengths [99].

Definition 1.3.6 (Dilogarithm) *[4] The **dilogarithm** function, $Li_2(z)$, is given by*

$$Li_2(z) = \begin{cases} -\int_0^z \frac{\log(1-t)}{t}dt, & z \in \mathbb{C} \setminus [1,\infty), \\ \sum_{t=1}^{\infty} \frac{z^t}{t^2}, & |z| < 1. \end{cases} \tag{1.42}$$

Proposition 1.3.7 *[100, 99] Let a spherical tetrahedron, T have dihedral angles φ_{ij}, and edge lengths θ_{ij}, $i, j = 1, 2, 3, 4$, $i < j$. Define*

$$a_{ij} = \mathit{exp}(\mathrm{i}\varphi_{ij}), \tag{1.43}$$

and similarly,

$$b_{ij} = \mathit{exp}(\mathrm{i}\theta_{ij}). \tag{1.44}$$

Then, the volume of T in terms of the dihedral angles is given by

$$\begin{aligned} V_3(T) = &\mathrm{Re}(L(a_{ij}, a_{ik}, a_{il}, a_{kl}, a_{jl}, a_{jk}; z_0)) + \pi \left(\mathit{arg}(-q_2) + \frac{1}{2} \sum_{i,j=1:i<j}^{4} \varphi_{ij} \right) \\ &- \frac{3}{2}\pi^2, \qquad \left(\mathit{mod}\ 2\pi^2\right), \end{aligned} \tag{1.45}$$

where

$$\begin{aligned} L(a_{ij}, a_{ik}, a_{il}, a_{kl}, a_{jl}, a_{jk}; z_0) =& \frac{1}{2} \Big(Li_2(z) + Li_2(a_{ij}^{-1}(a_{ik}^{-1}a_{kl}^{-1}a_{jl}^{-1}z) \\ &+ Li_2(a_{ij}^{-1}a_{ik}^{-1}a_{kl}^{-1}a_{jk}^{-1}z) - Li_2(a_{ik}^{-1}a_{ik}^{-1}a_{jl}^{-1}a_{jk}^{-1}z) \\ &- Li_2(-a_{ij}^{-1}a_{ik}^{-1}a_{ik}^{-1}z) - Li_2(-a_{ij}^{-1}a_{jl}^{-1}a_{jk}^{-1}z) \\ &- Li_2(-a_{ik}^{-1}a_{kl}^{-1}a_{jk}^{-1}z) - Li_2(-a_{ik}^{-1}a_{kl}^{-1}a_{jl}^{-1}z) \\ &+ \sum_{j=1}^{3} \log a_j \log a_{j+3} \Big), \end{aligned} \tag{1.46}$$

and

$$z_0 = \frac{-q_1 + \sqrt{q_1^2 - 4q_0q_2}}{2q_2}, \tag{1.47}$$

with

$$\begin{aligned} q_0 = a_{ij}a_{kl} + a_{ik}a_{jl} + a_{ik}a_{jk} + a_{ij}a_{ik}a_{jk} + a_{ij}a_{ik}a_{jl} \\ + a_{ik}a_{ik}a_{kl} + a_{jk}a_{jl}a_{jk} + a_{ij}a_{ik}a_{ik}a_{kl}a_{jl}a_{jk}, \end{aligned} \tag{1.48a}$$

$$\begin{aligned} q_1 = -\left(a_{ij} - a_{ij}^{-1}\right)\left(a_{kl} - a_{kl}^{-1}\right) - \left(a_{ik} - a_{ik}^{-1}\right)\left(a_{jl} - a_{jl}^{-1}\right) \\ - \left(a_{ik} - a_{ik}^{-1}\right)\left(a_{jk} - a_{jk}^{-1}\right), \end{aligned} \tag{1.48b}$$

$$\begin{aligned} q_2 = a_{ij}^{-1}a_{kl}^{-1} + a_{ik}^{-1}a_{jl}^{-1} + a_{ik}^{-1}a_{jk}^{-1} + a_{ij}^{-1}a_{ik}^{-1}a_{jk}^{-1} + a_{ij}^{-1}a_{ik}^{-1}a_{jl}^{-1} + a_{ik}^{-1}a_{ik}^{-1}a_{kl}^{-1} \\ + a_{kl}^{-1}a_{jl}^{-1}a_{jk}^{-1} + a_{ij}^{-1}a_{ik}^{-1}a_{ik}^{-1}a_{kl}^{-1}a_{jl}^{-1}a_{jk}^{-1}. \end{aligned} \tag{1.48c}$$

Similarly, in terms of edge lengths, the volume of T is given by

$$\begin{aligned} V_3(T) = \mathrm{Re}\left(\widetilde{L}\left(b_{kl}, b_{jl}, b_{jk}, b_{il}, b_{ik}, b_{ij}; \widetilde{z}_0\right)\right) - \pi\, arg(-\widetilde{q}_2) \\ - \sum_{j=1}^{6} l_j \frac{\partial\, \mathrm{Re}\left(\widetilde{L}\left(b_{kl}, b_{jl}, b_{jk}, b_{il}, b_{ik}, b_{ij}; \widetilde{z}\right)\right)}{\partial l_j}\Bigg|_{z=\widetilde{z}_0} - \frac{1}{2}\pi^2, \ \left(mod 2\pi^2\right), \end{aligned} \tag{1.49}$$

where $\widetilde{z}_0$ and $\widetilde{q}_2$ are obtained by substituting $-b_{ij}^{-1}$ for a_{ij}, $i, j = 1, 2, 3, 4$, $i < j$ in the definitions of z_0 and q_2 respectively.

Another interesting result was discovered by Zehrt [145] as part of his thesis, where he developed reduction formulae which reduce the problem of calculating volumes in even dimensional spherical and hyperbolic space, to the determination of odd dimensional volumes.

It is hoped that these volume formulae may prove useful in solving the d-simplex equations in future, although at the current time it is not clear as to whether the addition of volumes could be linear, like the addition of angles in a plane.

1.3.3 The Kadomtsev-Petviashvili Equation

The Kadomtsev-Petviashvili, or KP, equation is a partial differential equation, derived in 1970 to describe non-linear wave motion, as a $(2 + 1)$-dimensional generalisation of the

KdV equation [69]. For $u = u(x, y; t)$, it takes the potential form

$$\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u_x^2\right)_x = \frac{3}{4}u_{yy}, \tag{1.50}$$

with the KdV equation recovered under the dimensional reduction $u_y \to 0$, u becoming independent of y. It is believed that the KP hierarchy is connected to the tetrahedron equation in the same manner that the Yang-Baxter structure defines many of the algebraic properties of the KdV hierarchy.

The KP equation is integrable in that it possesses a Lax representation, namely a Lax pair.

Definition 1.3.8 (Lax Representation) *[105] A **Lax representation** is an underlying overdetermined system of linear partial differential in the continuous case, or difference in the discrete case, equations whose consistency condition leads to a non-linear partial differential, or difference, system respectively. The system arising from the consistency condition is then said to be **integrable**. A Lax representation of two linear equations is called a Lax pair.*

The Lax pair for the potential KP equation is given by

$$\varphi_y = \varphi_{xx} + 2u_x\varphi, \tag{1.51a}$$

$$\varphi_t = \varphi_{xy} + u_x\varphi_x + \frac{1}{2}\left(3u_y - u_{xx}\right)\varphi. \tag{1.51b}$$

In order to see how the potential KP equation follows from this Lax pair, consider the compatibility condition

$$\begin{aligned}\varphi_{yt} - \varphi_{ty} =& \varphi_{xxt} + 2u_{xt}\varphi + 2u_x\varphi_t - \varphi_{xyy} - u_{xy}\varphi_x - u_x\varphi_{xy} \\ & - \frac{3}{2}u_{yy}\varphi + \frac{1}{2}u_{xxy}\varphi - \frac{3}{2}u_y\varphi_y + \frac{1}{2}u_{xx}\varphi_y.\end{aligned} \tag{1.52}$$

Elimating all t and y derivatives of φ using the Lax pair, (1.51), this compatibility condition reduces to

$$\varphi_{yt} - \varphi_{ty} = 2\left(\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u_x^2\right)_x - \frac{3}{4}u_{yy}\right)\varphi = 0, \tag{1.53}$$

and the potential KP equation hence follows.

This Lax pair is a source of many of the system's integrability characteristics, and forms the starting point for many methods of finding solutions.

An analogous integrable lattice system, the Lattice Potential KP, or LPKP, equation, first produced by Nijhoff, Capel, Wiersma and Quispel [108], is given by

$$\begin{aligned}(p-\widetilde{u})\left(q-r+\widetilde{\dot{u}}-\widehat{\widetilde{u}}\right)+(q-\widehat{u})\left(r-p+\widehat{\widetilde{u}}-\widehat{\dot{u}}\right)&\\ +(r-\dot{u})\left(p-q+\widehat{\dot{u}}-\dot{\widetilde{u}}\right)&=0,\end{aligned} \tag{1.54}$$

whereby the system resides on a three-dimensional lattice, with shifts in $u=u(n,m,l)$ defined by

$$\widetilde{u}=u(n+1,m,l), \tag{1.55a}$$

$$\widehat{u}=u(n,m+1,l), \tag{1.55b}$$

$$\dot{u}=u(n,m,l+1), \tag{1.55c}$$

and corresponding lattice parameters p, q and r, respectively. Note that, the actual potential KP equation may be recovered through a direct series of continuum limits to (1.50). An alternative discretisation, following the same notation,

$$(q-r)\,\widetilde{u}\widehat{\dot{u}}+(r-p)\,\widehat{u}\dot{\widetilde{u}}+(p-q)\,\dot{u}\widetilde{\widehat{u}}=0, \tag{1.56}$$

was discovered by Hirota in [56]. Further features were discovered by Miwa [97], who provided a transformation linking the equation to the KP hierarchy, but its continuum limit does not lead directly to it.

Note that the equation (1.54) takes the form

$$f(\widetilde{u},\widehat{u},\dot{u},\widehat{\widetilde{u}},\widetilde{\dot{u}},\widehat{\dot{u}};p,q,r)=0, \tag{1.57}$$

and so, only involves six vertices of an elementary hexahedron, in a hexagonal configuration.

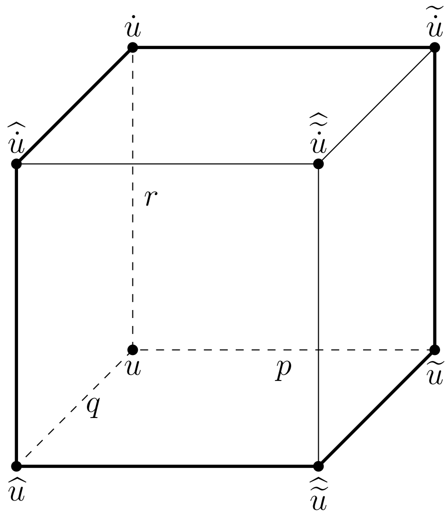


Figure 1.7: The KP Lattice

Integrability is understood in terms of multi-dimensional consistency: The system can be embedded in a consistent way in a multi-dimensional lattice with dimension greater than three, with the equation imposed on the sub-lattices of the three-dimensional facets of a four-dimensional hypercube. The system also possesses a Lax triplet [110], given by

$$\widetilde{\varphi} = (p - \widetilde{u})\,\varphi + \psi, \tag{1.58a}$$

$$\widehat{\varphi} = (q - \widehat{u})\,\varphi + \psi, \tag{1.58b}$$

$$\dot{\varphi} = (r - \dot{u})\,\varphi + \psi, \tag{1.58c}$$

where ψ is any function independent of the particular discrete direction. The LPKP system follows as a result of the compatibility conditions of these equations. For example, consider the compatibility between $\widehat{\varphi}$ and $\widetilde{\varphi}$,

$$\begin{aligned} \widehat{\widetilde{\varphi}} - \widetilde{\widehat{\varphi}} &= \left(p - \widehat{\widetilde{u}}\right)\widehat{\varphi} - \left(q - \widehat{\widetilde{u}}\right)\widetilde{\varphi}, \\ &= \left(\left(p - \widehat{\widetilde{u}}\right)(q - \widehat{u}) - \left(q - \widehat{\widetilde{u}}\right)(p - \widetilde{u})\right)\varphi + (p - q)\,\psi = 0. \end{aligned} \tag{1.59}$$

Similarly, from the other compatibility, we have

$$\dot{\widehat{\varphi}} - \widehat{\dot{\varphi}} = \left(\left(q - \widehat{\dot{u}}\right)(r - \dot{u}) - \left(r - \widehat{\dot{u}}\right)(q - \widehat{u})\right)\varphi + (q - r)\,\psi, \tag{1.60a}$$

$$\widetilde{\dot{\varphi}} - \dot{\widetilde{\varphi}} = \left(\left(r - \widehat{\dot{u}}\right)(p - \widetilde{u}) - \left(p - \widetilde{\dot{u}}\right)(r - \dot{u})\right)\varphi + (r - p)\,\psi. \tag{1.60b}$$

(1.60c)

Now, by considering the overall compatibility, through the combination

$$\left(\widehat{\widetilde{\varphi}} - \widetilde{\widehat{\varphi}}\right) + \left(\dot{\widehat{\varphi}} - \widehat{\dot{\varphi}}\right) + \left(\widetilde{\dot{\varphi}} - \dot{\widetilde{\varphi}}\right) = 0, \tag{1.61}$$

the LPKP equation follows.

We establish an elliptic extension of the KP equation, in the lattice setting in Chapter 4, and in the continuous setting in Chapter 5.

1.3.4 Classification of Integrable Discrete Equations of Octahedron Type

As a generalisation of their classification of discrete integrable quad equations [10, 11], Adler, Bobenko and Suris applied the property of multi-dimensional consistency to obtain a classification of integrable three-dimensional equations of Discrete Hirota KP or Octahedron type [9].

Definition 1.3.9 (Octahedron Type) *An equation is of* ***Octahedron Type*** *if using our earlier notation for lattice shifts, it can take the form*

$$F\left(\widehat{u}, \widetilde{u}, \dot{u}, \widehat{\widetilde{u}}, \widehat{\dot{u}}, \widetilde{\dot{u}}\right) = 0 \tag{1.62}$$

Definition 1.3.10 ($4D$ Consistency) *An equation, F of Octahedron type is $4D$* ***Consistent*** *if it may be imposed in a consistent way on all cubic faces of a four-dimensional hypercube.*

Under the assumptions:

- F is locally analytic in some domain, and $F = 0$ can be solved with respect to any variable,
- F is irreducible, i.e. the solution of $F = 0$ in terms of any variable depends on all other variables,

this classification results in the following equations:

$$\widehat{\widetilde{u}}\,\overline{\dot{u}} - \dot{\widetilde{u}}\,\widehat{\overline{u}} + \widehat{\overline{u}}\,\overline{\widetilde{u}} = 0, \tag{$\chi1$}$$

$$\frac{\left(\overline{\widetilde{u}} - \widehat{\widetilde{u}}\right)\left(\widehat{\overline{u}} - \widehat{\dot{u}}\right)\left(\overline{\dot{u}} - \dot{\widetilde{u}}\right)}{\left(\widehat{\widetilde{u}} - \widehat{\overline{u}}\right)\left(\widehat{\dot{u}} - \overline{\dot{u}}\right)\left(\dot{\widetilde{u}} - \overline{\widetilde{u}}\right)} = -1, \tag{$\chi2$}$$

$$\left(\dot{\widetilde{u}} - \widehat{\widetilde{u}}\right)\overline{\widetilde{u}} + \left(\widehat{\widetilde{u}} - \widehat{\dot{u}}\right)\widehat{\overline{u}} + \left(\widehat{\dot{u}} - \dot{\widetilde{u}}\right)\overline{\dot{u}} = 0, \tag{$\chi3$}$$

$$\frac{\dot{\widetilde{u}} - \widehat{\widetilde{u}}}{\overline{\widetilde{u}}} + \frac{\widehat{\widetilde{u}} - \widehat{\dot{u}}}{\widehat{\overline{u}}} + \frac{\widehat{\dot{u}} - \dot{\widetilde{u}}}{\overline{\dot{u}}} = 0, \tag{$\chi4$}$$

$$\frac{\dot{\widetilde{u}} - \widehat{\dot{u}}}{\overline{\dot{u}}} = \widehat{\widetilde{u}}\left(\frac{1}{\widehat{\overline{u}}} - \frac{1}{\overline{\widetilde{u}}}\right). \tag{$\chi5$}$$

All these equations had previously been derived through the process of direct linearisation [113, 35, 106, 107, 108].

1.4 From The Kadomtsev-Petviashvili Equation to the Calogero-Moser and Ruijsenaars-Schneider Models

The Calogero-Moser model [34, 98] is a one-dimensional many-body system with long range interactions. The equations of motion for the system in the elliptic case are given by

$$\ddot{x}_i = g \sum_{j=1:j\neq i}^{N} \wp'(x_i - x_j), \qquad i = 1, \ldots, N, \tag{1.64}$$

where $\wp(x)$ is the Weierstrass elliptic $\wp$-function, with half-periods ω and ω'. The system possess three degenerate cases:

- **The Rational Case** $\wp(x) \to x^{-2}$ as $2\omega \to \infty$, $2\omega' \to \mathrm{i}\infty$.
- **The Trigonometric Case** $\wp(x) \to (\sin x)^{-2} - 1/3$ as $2\omega = \pi/2$, $2\omega' \to \mathrm{i}\infty$.
- **The Hyperbolic Case** $\wp(x) \to (\sinh x)^{-2} + 1/3$ as $2\omega \to \infty$, $2\omega' \to \mathrm{i}\pi/2$.

A connection between the dynamics of the poles of special solutions of integrable nonlinear evolution equations and integrable systems of particles on the line was observed in [38] and [13]. This observation was then exploited by Krichever [78] in finding a connection between the pole solutions of the continuum KP equation and the Calogero-Moser system. Based upon this result, Nijhoff and Pang [109] showed that the pole solutions of a semi-discretised version of the KP equation is connected with a time-discretised version of the Calogero-Moser model, where the semi-discretised KP equation is given by

$$(p - q - \widehat{u} - \widetilde{u})\,\xi = (p - q + \widehat{u} - \widetilde{u})\left(u + \widehat{\widetilde{u}} - \widehat{u} - \widetilde{u}\right), \tag{1.65}$$

where $\widetilde{\cdot}$ and $\widehat{\cdot}$ are shifts in the time and space directions, respectively, associated with lattice parameters p and q, and ξ is a continuous variable. This identification gives the equations of motion for the elliptic discrete-time Calogero-Moser model to be

$$\zeta(x_i - \widetilde{x}_i) + \zeta(x_i - \underset{\sim}{x_i}) + \sum_{j=1:j\neq i}^{N} \left(\zeta(x_i - \widetilde{x}_j) + \zeta(x_i - \underset{\sim}{x_j}) - 2\zeta(x_i - x_j)\right), \qquad i \in \mathbb{N}, \tag{1.66}$$

where $\zeta(\cdot)$ is the Weierstrass zeta-function.

The Ruijsenaars-Schneider model is a relativistic variant of the Calogero-Moser model introduced in [123]. It was discovered by considering the Poincaré Poisson algebra associated with sine-Gordon solitons. The equations of motion are given by

$$\ddot{q}_i = \sum_{j\neq i} \dot{q}_i \dot{q}_j v(q_i - q_j), \qquad i \in \mathbb{N}, \tag{1.67}$$

with "potential"

$$v(x) = \frac{\wp'(x)}{\wp(\lambda) - \wp(x)}. \tag{1.68}$$

Nijhoff, Ragnisco and Kuznetsov [114] showed that the discrete-time Ruijsenaars-Schneider model follows from a reduction of the fully discrete KP equation, with three discrete variables, analogous to the reduction of the semi-discrete KP, used in the Calogero-Moser case. The equations of motion are then contained in the system

$$\frac{p}{\underset{\sim}{p}} \prod_{j=1:i\neq j}^{N} \frac{\sigma(q_i - q_j + \lambda)}{\sigma(q_i - q_j - \lambda)} = \prod_{j=1}^{N} \frac{\sigma(q_i - \widetilde{q}_j)\sigma(q_i - \underset{\sim}{q_j} + \lambda)}{\sigma(q_i - \underset{\sim}{q_j})\sigma(q_i - \widetilde{q}_j - \lambda)}, \qquad i \in \mathbb{N}, \tag{1.69}$$

where $\sigma(\cdot)$ is the Weierstrass sigma function.

More recently, the Lagrange formalisms of the Calogero-Moser and Ruijsenaars-Schneider models, in both continuous and discrete cases, have been established as examples of a Lagrange 1-form structure [139, 140] in the sense of the connection established between Lagrangian multi-forms and multi-dimensional consistency [88].

In the conclusion, Chapter 6, we speculate that similar reductions of the elliptic extension of the KP equation that we establish may result in the Double Elliptic (or DELL) model, a generalisation of the Calogero-Moser and Ruijsenaars-Schneider models, elliptic in both potential and momentum.

1.5 Overview

In this introduction, we have referred to numerous established, and some conjectured, connections between various areas. These connections are summarised in figure 1.8, with the established connections represented by solid arrows, and the conjectured ones by dashed lines.

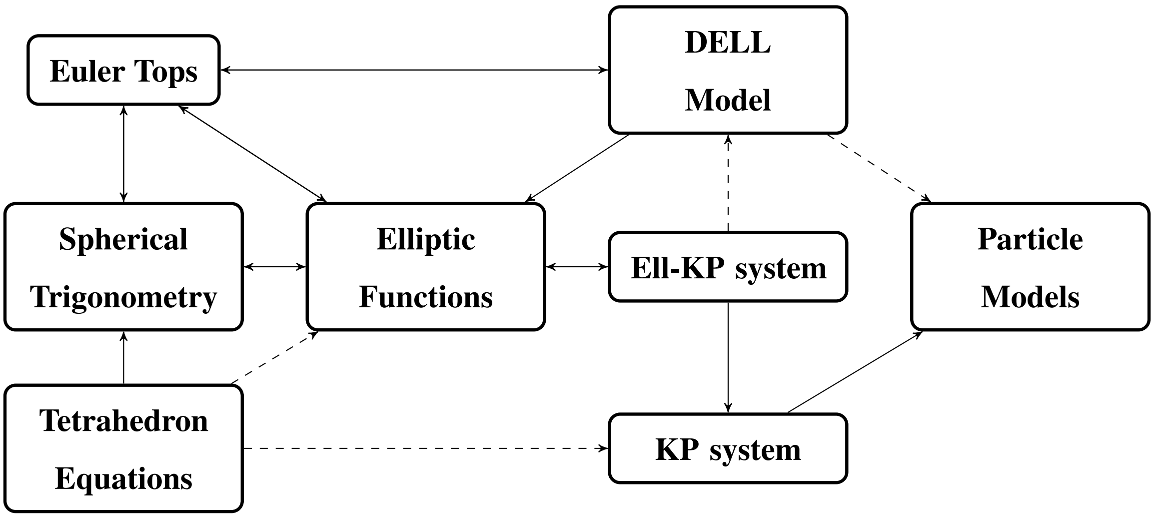


Figure 1.8: Overview of the Relevant Areas

1.6 Outline

The outline of this thesis is as follows.

In Chapter 2, we begin by reviewing multi-dimensional vector products as a higher dimensional analogue of the standard vector cross-products. We then review the basic formulae of spherical trigonometry given in this introductory chapter, re-deriving these formulae in terms of vector products, before deriving analogous formulae for the four-dimensional hyperspherical case by utilising the multi-dimensional vector products, following a similar method. We extend this method to produce analogous formulae for the general n-dimensional case. The derivation of the link between spherical trigonometry and the Jacobi elliptic functions is reviewed, and we generalise this to provide a link between four-dimensional hyperspherical trigonometry and elliptic functions. The chapter concludes with an introduction to the generalised Jacobi elliptic functions, as studied in depth by Pawellek [116], to which we derive a connection with the four-dimensional hyperspherical trigonometry.

Chapter 3 contains two examples of models where the connection between the generalised

Jacobi elliptic functions and four-dimensional hyperspherical trigonometry is exploited. The first example is that of a four-dimensional Euler top. We review a reformulation of the mechanics of the Euler top in terms of Nambu mechanics by Minic and Tze [94], and extend this to a four-dimensional generalisation, which we then solve in terms of the generalised Jacobi functions, making a connection with the hyperspherical trigonometry. The second example is the two-particle Double Elliptic (DELL) model, for which we show that the generalised Jacobi elliptic functions are the natural parameterisation for the Hamiltonian of the system.

Chapters 4 and 5 are concerned with an elliptic extension of the Kadomtsev-Petviashvili (KP) equation, the results of which have been published and are available in [68]. In chapter 4, we present a generalisation of the lattice potential Kadomtsev-Petviashvili (LPKP) equation using the method of direct linearisation based on an elliptic Cauchy kernel. This yields a $(3+1)$-dimensional system with one of the lattice shifts singled out. The integrability of the lattice system is considered, presenting a Lax representation and soliton solutions.

Chapter 5 contains the continuous analogue to the system derived in chapter 4, an elliptic generalisation of the continuous KP system, following the same direct linearisation method. The integrability of this system is also considered, with a Lax representation presented. The chapter concludes with a comparison between this continuous system and a similar elliptic generalisation of the KP equation found by Date, Jimbo and Miwa [41]. We also show that the Lax representation that they provide for their system does not correspond to the system that they give.

The thesis concludes with a recap of the new results, together with an outlook on the future.

Chapter 2

Hyperspherical Trigonometry and Corresponding Elliptic Functions

2.1 Introduction

Spherical trigonometry has a long history as a part of mathematics, as introduced in Chapter 1. However, for obvious reasons the trigonometry of hyperspheres in higher dimensions was not studied so intently. McMahon produced a number of formulae as generalisations of some of those from spherical trigonometry [93]. However, these formulae are not in a particularly accessible form, and so cannot be easily applied elsewhere. More recently, Sato considered the relationship between the dihedral angles of spherical simplices and those of their polars [124]. There also exists a significant amount of work looking into the 'Law of Sines', generalisations of the sine rule from the spherical case [46]. Various formulae have also arisen in the the work of Derevnin, Mednykh and Pashkevich in their work on spherical volumes [44, 43]. Recently, it has been shown by Petrera and Suris the the cosine rule for spherical triangles and tetrahedra define integrable systems [122]. Apart from McMahon's work and the work on the Law of Sines, it is only ever the four-dimensional hyperspherical case that is considered.

As discussed earlier, it is well known that elliptic functions are related to spherical trigonometry through their addition formulae [86, 84]. These functions are commonly defined through the inversion of integrals, and they consequently obey differential equations associated with certain algebraic curves. A great deal of research took place in this area in the 19th century, comprising works by many of the great mathematicians, including Euler [48], Jacobi [64], Legendre [86] and Frobenius [52]. As for higher-dimensional hyperspherical trigonometry no such connection is known. It may be expected that this link would be through higher genus elliptic curves, for example Abelian functions[18]. However, this is not in fact the case. We show that instead the link is through the 'Generalised Jacobi Functions' explicitly defined by Pawellek [116], and building upon Jacobi's work in this area [65], as an elliptic covering, dependent on two distinct moduli.

In this chapter, we develop a complete set of formulae for hyperspherical trigonometry and explore their link with elliptic functions. We establish a novel connection between the generalised Jacobi elliptic functions and the formulae of hyperspherical trigonometry in the four-dimensional case. We show that through this connection the basic addition formulae of hyperspherical trigonometry lead to addition formulae for these generalised Jacobi elliptic functions. We also derive corresponding angle addition formulae, as well as four- and five-part formulae, for the four-dimensional hyperspherical case.

The outline of this chapter is as follows. In section 2.2, we review multi-dimensional vector products as a higher-dimensional analogue of the standard cross-product of vectors, which we need to obtain the formulae of hyperspherical trigonometry. In section 2.3 we provide a summary of the well-known formulae of spherical trigonometry. We deduce analogous formulae for the four-dimensional hyperspherical case, and using the same principles, do the same for the general m-dimensional case. In section 2.4 we review the link between spherical trigonometry and the Jacobi elliptic functions, and generalise this to the link between four-dimensional hyperspherical trigonometry and elliptic functions.

Section 2.5 is a brief introduction to the generalised Jacobi functions [116], complete with a link between the formulae of hyperspherical trigonometry and these functions. We conclude the chapter with angle addition, and four- and five-part formulae in four-dimensional hyperspherical trigonometry.

2.2 Multi-dimensional Vector Products

The formulae involved in spherical trigonometry are dependent on the cross product between vectors. The vector product $\mathbf{a} \times \mathbf{b}$ in three-dimensional Euclidean space is a binary operation defined by

$$(\mathbf{a} \times \mathbf{b})_i = \det(\mathbf{a}, \mathbf{b}, \mathbf{e}_i), \qquad (i = 1, 2, 3), \tag{2.1}$$

with $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ the standard unit vectors in the orthogonal basis. This product has the properties:

- Anti-commutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- Vector Triple Product: $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} = -\begin{vmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix}$, from which it follows $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\mathbf{a} \times \mathbf{c}) \times \mathbf{b} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.
- Area of a Parallelogram: The modulus of the vector product, $|\mathbf{a} \times \mathbf{b}|$, is equivalent to the area of the parallelogram defined by these vectors, $|\mathbf{a} \times \mathbf{b}| = \sin\theta$, with θ the obtuse angle between them $(0 \leq \theta \leq \pi)$.

For higher-dimensional spherical trigonometry an m-ary operation between m vectors is required. A natural vector product in four dimensions will therefore be a ternary vector product, $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$, of three vectors in four-dimensional Euclidean space, defined in a similar manner, by

$$(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot \mathbf{d} = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}), \tag{2.2}$$

for all vectors $\mathbf{d} \in E_4$. More generally, this could be extended to an $(m-1)$-dimensional vector product as an m-ary operation of $m-1$ vectors in m-dimensional Euclidean space [53, 6], defined by the expression

$$(\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_{m-1}) \cdot \mathbf{a}_m = \det(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1}, \mathbf{a}_m). \tag{2.3}$$

It clearly follows from this definition that the multi-dimensional vector products are anti-symmetric with respect to the interchangement of their constituent vectors. Furthermore, these multi-dimensional vector products are perpendicular to any one of their constituent vectors.

In fact, this construction of the multi-dimensional vector product follows directly from the wedge product of the exterior algebra of a vector space V, denoted $\Lambda(V)$. This algebra is a direct sum of spaces $\Lambda^k(V)$ spanned by forms of degree k [137, 32].

Definition 2.2.1 *[67] The kth exterior power $\Lambda^k(V)$ of a finite-dimensional vector space V is the dual space of the vector space of alternating multi-linear forms of degree k on V. Elements of $\Lambda^k(V)$ are called k-forms.*

Given vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in V$ the exterior (or wedge) product $\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k \in \Lambda^k(V)$ is the linear map to a field F which on an alternating multi-linear form M takes the value

$$(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k)(M) = M(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k). \tag{2.4}$$

The wedge product between a k-form and an l-form gives a $(k+l)$-form [137]. Therefore, taking the wedge product between $(m-1)$ 1-forms, i.e. vectors, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{m-1} \in V$,

$$\boldsymbol{w} = \boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_{m-1}, \tag{2.5}$$

gives an $(m-1)$-form. The dimension of all such $\boldsymbol{w}$ is the dimension of $\Lambda^{m-1}(V)$,

$$\dim\Lambda^{m-1}(V) = m = \dim(V), \tag{2.6}$$

and as such, by duality we can identify $\boldsymbol{w}$ as a vector in V as they have the same dimension. This means that in fact there is a natural pairing, an inner product between $\boldsymbol{v} \in V$ and $w \in \Lambda^{m-1}(V)$, given again by the wedge product, $\boldsymbol{w} \wedge \boldsymbol{v} \in \Lambda^m(V)$. The result of this product has dimension one, i.e. is a number, and so this is in fact the dot product. If we let $\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be column vectors in $\mathbb{R}^m$, then

$$M(\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m) = \det(\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m) \tag{2.7}$$

is actually an alternating multi-linear form of degree m [67]. Therefore, we have

$$(\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k)(M) = \det(\boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m), \tag{2.8}$$

which is equivalent to (2.3). As such, the identities for the multi-dimensional vector product which follow may also be derived in terms of wedge products [137].

The following nested product identity involving five vectors holds for the triple vector product in $\mathbb{R}^4$,

$$(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \times \mathbf{d} \times \mathbf{e} = -\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ (\mathbf{a} \cdot \mathbf{d}) & (\mathbf{b} \cdot \mathbf{d}) & (\mathbf{c} \cdot \mathbf{d}) \\ (\mathbf{a} \cdot \mathbf{e}) & (\mathbf{b} \cdot \mathbf{e}) & (\mathbf{c} \cdot \mathbf{e}) \end{vmatrix}. \tag{2.9}$$

This follows from the more general higher-dimensional analogue.

Proposition 2.2.2 (Nested Vector Product Identity) *[6] For $(2m-1)$ vectors, $\mathbf{a}_i \in \mathbb{R}^{m+1}$, $i = 1, \ldots, 2m-1$,*

$$(\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_m) \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-1} =$$
$$-\begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_m \\ (\mathbf{a}_1 \cdot \mathbf{a}_{m+1}) & (\mathbf{a}_2 \cdot \mathbf{a}_{m+1}) & \ldots & (\mathbf{a}_m \cdot \mathbf{a}_{m+1}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mathbf{a}_1 \cdot \mathbf{a}_{2m-1}) & (\mathbf{a}_2 \cdot \mathbf{a}_{2m-1}) & \ldots & (\mathbf{a}_m \cdot \mathbf{a}_{2m-1}) \end{vmatrix}. \tag{2.10}$$

Proof

Consider the determinantal expression

$$\det(\mathbf{a}_1, \dots, \mathbf{a}_{m+1}) = \sum_{i_1} \cdots \sum_{i_{m+1}} \varepsilon_{i_1 \dots i_{m+1}} (\mathbf{a}_1)_{i_1} \dots (\mathbf{a}_{m+1})_{i_{m+1}}, \tag{2.11}$$

where $\varepsilon_{i_1 \dots i_{m+1}}$ is the $(m+1)$-dimensional Levi-Civita symbol. From this it follows that

$$\begin{aligned} &[(\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_m) \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-1}]_{j_m} \\ &\quad = \sum_{i_1} \cdots \sum_{i_{m+1}} \sum_{j_1} \cdots \sum_{j_m} \epsilon_{j_1 i_1 \dots i_m} \epsilon_{j_1 \dots j_{m+1}} \\ &\qquad\qquad \cdot (\mathbf{a}_1)_{i_1} \dots (\mathbf{a}_m)_{i_m} (\mathbf{a}_{m+1})_{j_2} \dots (\mathbf{a}_{2m-1})_{j_m}. \end{aligned} \tag{2.12}$$

Noting that the Levi-Civita symbol satisfies the following product rule

$$\sum_{j_1} \epsilon_{j_1 i_1 \dots i_m} \epsilon_{j_1 \dots j_{m+1}} = \begin{vmatrix} \delta_{i_1 j_2} & \dots & \delta_{i_1 j_{m+1}} \\ \vdots & & \vdots \\ \delta_{i_m j_2} & \dots & \delta_{i_m j_{m+1}} \end{vmatrix}, \tag{2.13}$$

the result follows. □

Vectorial addition identities for these multi-dimensional vector products follow from the Plücker relations in projective geometry. The Plücker relations are identities involving minors of non-square matrices which are the Plücker coordinates of corresponding Grassmannians.

Proposition 2.2.3 (Plücker Relations) *[130] For $(2m-2)$ vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{2m-2} \in \mathbb{R}^{m-1}$,*

$$\begin{aligned} &(\mathbf{a}_1, \mathbf{a}_{m+1}, \dots, \mathbf{a}_{2m-2})(\mathbf{a}_2, \dots, \mathbf{a}_m) - (\mathbf{a}_2, \mathbf{a}_{m+1}, \dots, \mathbf{a}_{2m-2})(\mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_m) \\ &\qquad + \cdots + (-1)^{m-1} (\mathbf{a}_m, \mathbf{a}_{m+1}, \dots, \mathbf{a}_{2m-2})(\mathbf{a}_1, \dots, \mathbf{a}_{m-1}) = 0, \end{aligned} \tag{2.14}$$

where $(\boldsymbol{a}_1, \dots, \boldsymbol{a}_m)$ represents the determinant of the matrix whose ith column is the vector $\boldsymbol{a}_i$.

Proof

[130] Consider m vectors, $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$ in $\mathbb{R}^{m-1}$. As these m vectors are in $(m-1)$-dimensional space they must be linearly dependent, and hence,

$$\begin{vmatrix} (\mathbf{a}_1) & (\mathbf{a}_2) & \cdots & (\mathbf{a}_m) \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{vmatrix} = 0, \tag{2.15}$$

where $(\mathbf{a}_i)$ represents the column vector of the components of $\mathbf{a}_i$, $i \in \{1, 2, \dots, m\}$. This implies

$$\mathbf{a}_1(\mathbf{a}_2, \dots, \mathbf{a}_m) - \mathbf{a_2}(\mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_m) + \cdots + (-1)^{m-1}\mathbf{a}_m(\mathbf{a}_1, \dots, \mathbf{a}_{m-1}) = 0, \tag{2.16}$$

which in turn gives

$$\begin{aligned}(\mathbf{a}_1, \mathbf{a}_{m+1}, \dots, \mathbf{a}_{2m-2})(\mathbf{a}_2, \dots, \mathbf{a}_m) - (\mathbf{a}_2, \mathbf{a}_{m+1}, \dots, \mathbf{a}_{2m-2})(\mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_m)& \\ + \cdots + (-1)^{m-1}(\mathbf{a}_m, \mathbf{a}_{m+1}, \dots, \mathbf{a}_{2m-2})(\mathbf{a}_1, \dots, \mathbf{a}_{m-1}) = 0,&\end{aligned} \tag{2.17}$$

for some arbitrary $\mathbf{a}_{m+1}, \dots, \mathbf{a}_{2m-2} \in \mathbb{R}^{m-1}$, as required. $\square$

Corollary 2.2.4 *For vectors* $\mathbf{a}_1, \dots, \mathbf{a}_{2m-2} \in \mathbb{R}^{m-1}$,

$$\begin{aligned}&(\mathbf{a}_1 \times \cdots \times \mathbf{a}_m) \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-1} \\ &\quad + \mathbf{a}_m \times (\mathbf{a}_1 \times \cdots \times \mathbf{a}_{m-1} \times \mathbf{a}_{m+1}) \times \mathbf{a}_{m+2} \times \cdots \times \mathbf{a}_{2m-1} + \dots \\ &\quad + \mathbf{a}_m \times \cdots \times \mathbf{a}_{2m-2} \times (\mathbf{a}_1 \times \cdots \times \mathbf{a}_{m-1} \times \mathbf{a}_{2m-1}) \\ &\qquad = \mathbf{a}_1 \times \cdots \times \mathbf{a}_{m-1} \times (\mathbf{a}_m \times \cdots \times \mathbf{a}_{2m-1}).\end{aligned} \tag{2.18}$$

Proof

From the Plücker relations, (2.14), set $\mathbf{a}_{2m-2} = \mathbf{a}_m = e_i$ and sum over i. This implies

$$\begin{aligned}&(\mathbf{a}_1 \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-3}) \cdot (\mathbf{a}_2 \times \cdots \times \mathbf{a}_{m-1}) \\ &\quad - (\mathbf{a}_2 \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-3}) \cdot (\mathbf{a}_1 \times \mathbf{a}_3 \cdots \times \mathbf{a}_{m-1}) \\ &\quad + \cdots + (-1)^m(\mathbf{a}_{m-1} \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-3}) \cdot (\mathbf{a}_1 \times \cdots \times \mathbf{a}_{m-2}) = 0,\end{aligned} \tag{2.19}$$

which, using $(\mathbf{a}_1 \times \mathbf{a}_2 \times \ldots \mathbf{a}_{m-2}) \cdot \mathbf{a}_{m-1} = -(\mathbf{a}_2 \times \cdots \times \mathbf{a}_{m-1}) \cdot \mathbf{a}_1$, implies

$$\begin{aligned} &\mathbf{a}_1 \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-4} \times (\mathbf{a_2} \times \cdots \times \mathbf{a}_{m-1}) \\ &\qquad - \mathbf{a}_2 \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-4} \times (\mathbf{a_1} \times \mathbf{a}_3 \times \cdots \times \mathbf{a}_{m-1}) \\ &\qquad + \cdots + (-1)^m \mathbf{a}_{m-1} \times \mathbf{a}_{m+1} \times \cdots \times \mathbf{a}_{2m-4} \times (\mathbf{a_1} \times \cdots \times \mathbf{a}_{m-2}) = 0. \end{aligned} \tag{2.20}$$

Summing over $(m-2)$ copies, the result follows. □

Corollary 2.2.5 *More specifically, for* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathbb{R}^4$,

$$(\mathbf{a}\times\mathbf{b}\times\mathbf{c})\times\mathbf{d}\times\mathbf{e}+(\mathbf{a}\times\mathbf{b}\times\mathbf{d})\times\mathbf{e}\times\mathbf{c}+(\mathbf{a}\times\mathbf{b}\times\mathbf{e})\times\mathbf{c}\times\mathbf{d} = \mathbf{a}\times\mathbf{b}\times(\mathbf{c}\times\mathbf{d}\times\mathbf{e}). \tag{2.21}$$

This identity will be particularly important in proving the various hyperspherical identities in the four-dimensional case.

2.3 Hyperspherical Trigonometry

In this section we review the formulae for spherical trigonometry as given in chapter 1, as a preparation for developing similar formulae in higher dimensions. We take a novel approach, exploiting the higher-dimensional vector product introduced in section 2.2. We first do this for the four-dimensional hyperspherical case, before extending this to the general m-dimensional case, providing a complete new set of formulae.

2.3.1 Spherical Trigonometry

We review the derivation of the basic formulae of spherical trigonometry, re-deriving the formulae in terms of vector products. Consider again a big spherical triangle on the surface of a 2-sphere of unit radius embedded in three-dimensional Euclidean space,with the centre of the sphere to be the origin in $\mathbb{R}^3$ and denote the position vectors of the three vertices of the spherical triangle by $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$, with the angles, θ_{ij}, between them, corresponding to the edges, being defined by

$$\mathbf{n}_i \cdot \mathbf{n}_j \equiv \cos\theta_{ij}, \tag{2.22}$$

for $i, j = 1, 2, 3$. Introduce the vectors

$$\mathbf{u}_{ij} \equiv \frac{\mathbf{n}_i \times \mathbf{n}_j}{|\mathbf{n}_i \times \mathbf{n}_j|}, \tag{2.23}$$

and define the spherical angles, α_j, between them by

$$\mathbf{u}_{ij} \cdot \mathbf{u}_{jk} \equiv -\cos\alpha_j. \tag{2.24}$$

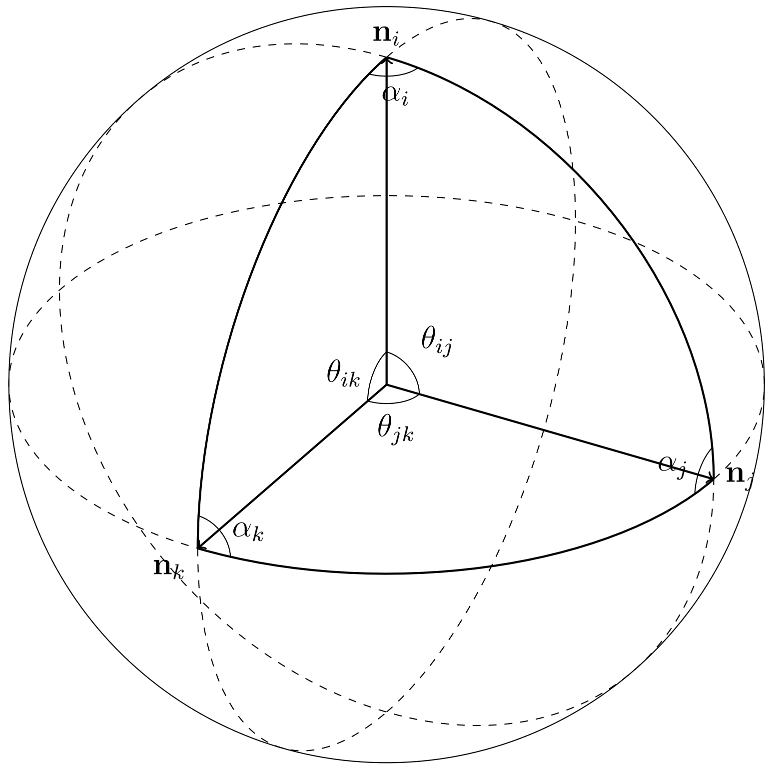


Figure 2.1: A spherical triangle

We may now define a polar triangle in terms of vectors.

Definition 2.3.1 (Polar Triangle) *The spherical triangle defined by the vectors* $\mathbf{u}_{ij}$, $\mathbf{u}_{ki}$, $\mathbf{u}_{jk}$ *is called the* $\mathbf{polar}$ *of the spherical triangle defined by the vectors* $\mathbf{n}_i$, $\mathbf{n}_j$, $\mathbf{n}_k$.

By considering various relations for the scalar and vector products between the polar vectors $\mathbf{u}_{ij}$, we can define a number of important relations between the angles of a spherical triangle.

Proposition 2.3.2 (Cosine Rule)

$$\cos\alpha_j = \frac{\cos\theta_{ik} - \cos\theta_{ij}\cos\theta_{jk}}{\sin\theta_{ij}\sin\theta_{jk}}, \tag{2.25}$$

for all $i, j, k = 1, 2, 3$.

Proof

Consider the scalar product,

$$\begin{aligned}
\mathbf{u}_{ij}\cdot\mathbf{u}_{jk} =& \frac{(\mathbf{n}_i\times\mathbf{n}_j)\cdot(\mathbf{n}_j\times\mathbf{n}_k)}{|\mathbf{n}_i\times\mathbf{n}_j|\,|\mathbf{n}_j\times\mathbf{n}_k|},\\
=& \frac{(\mathbf{n}_i\cdot\mathbf{n}_j)(\mathbf{n}_j\cdot\mathbf{n}_k)-(\mathbf{n}_i\cdot\mathbf{n}_k)(\mathbf{n}_j\cdot\mathbf{n}_j)}{\sin\theta_{ij}\sin\theta_{jk}},\\
=& \frac{\cos\theta_{ij}\cos\theta_{jk}-\cos\theta_{ik}}{\sin\theta_{ij}\sin\theta_{jk}}.
\end{aligned} \tag{2.26}$$

Hence, comparing with (2.24), the cosine rule follows. □

Definition 2.3.3 *The* ***generalised sine function of three variables****,* $\sin(\theta_{ij},\theta_{jk},\theta_{ik})$*,is defined by*

$$\begin{aligned}
\sin(\theta_{ij},\theta_{jk},\theta_{ik}) &= \sqrt{1-\cos^2\theta_{ij}-\cos^2\theta_{jk}-\cos^2\theta_{ik}+2\cos\theta_{jk}\cos\theta_{ij}\cos\theta_{ik}},\\
&= \begin{vmatrix} 1 & \cos\theta_{ij} & \cos\theta_{ik}\\ \cos\theta_{ij} & 1 & \cos\theta_{jk}\\ \cos\theta_{ik} & \cos\theta_{jk} & 1\end{vmatrix}^{\frac{1}{2}},
\end{aligned} \tag{2.27}$$

with $(0 \leq \theta_{ij}, \theta_{ik}, \theta_{jk} \leq \pi)$. *Note that the restrictions for a spherical triangle, that each side is less than a semi-circle and each angle less than* π, *ensure that the generalised sine function is always real and positive.*

This sine function is equivalent to the volume of a parallelepiped with edges $\mathbf{n}_i$, $\mathbf{n}_j$ and $\mathbf{n}_k$, given by

$$\text{Volume} = |\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k)| = \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}). \tag{2.28}$$

Proposition 2.3.4 (Sine Rule)

$$\frac{\sin \alpha_i}{\sin \theta_{jk}} = \frac{\sin \alpha_j}{\sin \theta_{ik}} = \frac{\sin \alpha_k}{\sin \theta_{ij}} = \frac{\sin(\theta_{ij}, \theta_{jk}, \theta_{ik})}{\sin \theta_{ij} \sin \theta_{jk} \sin \theta_{ik}} = k, \tag{2.29}$$

where k *is a constant.*

Proof

Consider the ratio

$$\frac{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}|}{|\mathbf{n}_i \times \mathbf{n}_k|} = \frac{\sin \alpha_j}{\sin \theta_{ik}}. \tag{2.30}$$

Now, note the vector product

$$\mathbf{u}_{ij} \times \mathbf{u}_{jk} = \frac{(\mathbf{n}_i \times \mathbf{n}_j) \times (\mathbf{n}_j \times \mathbf{n}_k)}{|\mathbf{n}_i \times \mathbf{n}_j||\mathbf{n}_j \times \mathbf{n}_k|} = \frac{(\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k))\, \mathbf{n}_j}{\sin \theta_{ij} \sin \theta_{jk}}. \tag{2.31}$$

This implies

$$|\mathbf{u}_{ij} \times \mathbf{u}_{jk}| = \left| \frac{\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k)}{\sin \theta_{ij} \sin \theta_{jk}} \right| = \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})}{\sin \theta_{ij} \sin \theta_{jk}}, \tag{2.32}$$

and hence, the sine rule follows. $\square$

Proposition 2.3.5 (Polar Cosine Rule)

$$\cos \theta_{jk} = \frac{\cos \alpha_j \cos \alpha_k + \cos \alpha_i}{\sin \alpha_j \sin \alpha_k}, \tag{2.33}$$

for all $i, j, k = 1, 2, 3$.

Proof

Consider the product

$$\frac{(\mathbf{u}_{ij} \times \mathbf{u}_{jk}) \cdot (\mathbf{u}_{jk} \times \mathbf{u}_{ik})}{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}||\mathbf{u}_{jk} \times \mathbf{u}_{ik}|}. \tag{2.34}$$

This product can be calculated in two ways. First,

$$\begin{aligned}\frac{(\mathbf{u}_{ij} \times \mathbf{u}_{jk}) \cdot (\mathbf{u}_{jk} \times \mathbf{u}_{ik})}{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}||\mathbf{u}_{jk} \times \mathbf{u}_{ik}|} &= \frac{(\mathbf{u}_{ij} \cdot \mathbf{u}_{jk})(\mathbf{u}_{jk} \cdot \mathbf{u}_{ik}) - (\mathbf{u}_{jk} \cdot \mathbf{u}_{jk})(\mathbf{u}_{ij} \cdot \mathbf{u}_{ik})}{\sin\alpha_j \sin\alpha_k}, \\ &= \frac{-\cos\alpha_j \cos\alpha_k - \cos\alpha_i}{\sin\alpha_j \sin\alpha_k},\end{aligned} \tag{2.35}$$

and second, using (2.31),

$$\begin{aligned}&\frac{(\mathbf{u}_{ij} \times \mathbf{u}_{jk}) \cdot (\mathbf{u}_{jk} \times \mathbf{u}_{ik})}{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}||\mathbf{u}_{jk} \times \mathbf{u}_{ik}|} \\ &\qquad = \frac{1}{\sin\alpha_j \sin\alpha_k} \left(\frac{\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k)}{\sin\theta_{ij} \sin\theta_{jk}} \mathbf{n}_j \right) \cdot \left(\frac{\mathbf{n}_i \cdot (\mathbf{n}_k \times \mathbf{n}_j)}{\sin\theta_{ik} \sin\theta_{jk}} \mathbf{n}_k \right), \\ &\qquad = -\frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{jk})}{\sin\alpha_j \sin\alpha_k \sin\theta_{ij} \sin^2\theta_{jk} \sin\theta_{ik}} \cos\theta_{jk}.\end{aligned} \tag{2.36}$$

Reducing this using the sine rule and equating with the previous result gives the polar cosine rule. □

Note that, by now considering

$$k^2 = \frac{\sin^2\alpha_i}{\sin^2\theta_{jk}} = \frac{\sin^2\alpha_i}{1 - \cos^2\theta_{jk}}, \tag{2.37}$$

and substituting in the polar cosine rule,

$$\begin{aligned}k^2 &= \frac{\sin^2\alpha_i}{1 - \left(\frac{\cos\alpha_i + \cos\alpha_j \cos\alpha_k}{\sin\alpha_j \sin\alpha_k} \right)^2}, \\ &= \frac{\sin^2\alpha_i \sin^2\alpha_j \sin^2\alpha_k}{1 - \cos^2\alpha_i - \cos^2\alpha_j - \cos^2\alpha_k - 2\cos\alpha_i \cos\alpha_j \cos\alpha_k},\end{aligned} \tag{2.38}$$

the constant k may also be written in terms of the spherical angles, so that the sine rule now becomes

$$\begin{aligned}\frac{\sin\alpha_i}{\sin\theta_{jk}} &= \frac{\sin\alpha_j}{\sin\theta_{ik}} = \frac{\sin\alpha_k}{\sin\theta_{ij}} \\ &= \frac{\sin\alpha_i \sin\alpha_j \sin\alpha_k}{\sqrt{1 - \cos^2\alpha_i - \cos^2\alpha_j - \cos^2\alpha_k - 2\cos\alpha_i \cos\alpha_j \cos\alpha_k}}.\end{aligned} \tag{2.39}$$

We will use both forms of the sine rule later when discussing the link between spherical trigonometry and the Jacobi elliptic functions.

2.3.2 Hyperspherical Trigonometry in Four-Dimensional Euclidean Space

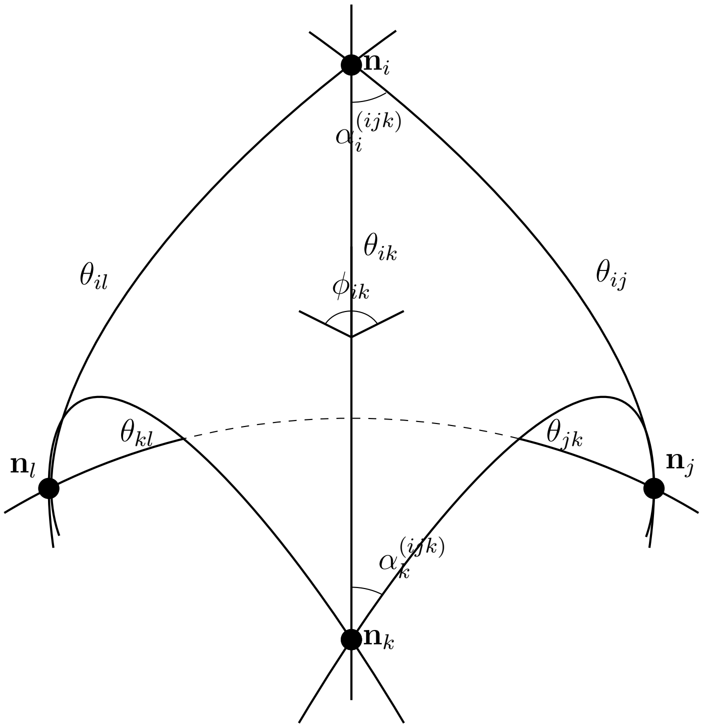


Figure 2.2: A hyperspherical tetrahedron

We now extend these principles to the four-dimensional hyperspherical case. Consider a 3-sphere embedded in four-dimensional Euclidean space, $\mathbb{R}^4$. In this case, we have four unit vectors, $\mathbf{n}_i, i = 1, \ldots, 4$ pointing to the four vertices of a hyperspherical tetrahedron.

The angles between these unit vectors, θ_{ij}, are defined, in the same way as for the spherical case, by

$$\mathbf{n}_i \cdot \mathbf{n}_j \equiv \cos\theta_{ij}, \tag{2.40}$$

whereas now we must also define orthogonal vectors $\mathbf{u}_{ijk}$ to each hyperplane $[i,j,k]$ (defined by $\mathbf{n}_i$, $\mathbf{n}_j$, $\mathbf{n}_k$). This is done using the ternary cross product,

$$\mathbf{u}_{ijk} \equiv \frac{\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k}{|\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k|}. \tag{2.41}$$

The four vectors $\mathbf{u}_{123}$, $\mathbf{u}_{124}$, $\mathbf{u}_{134}$, $\mathbf{u}_{234}$ define the polar of the hyperspherical tetrahedron, between which we define dihedral angles ϕ_{jk} by

$$\mathbf{u}_{ijk} \cdot \mathbf{u}_{jkl} \equiv -\cos\phi_{jk}. \tag{2.42}$$

We also have

$$|\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k| = \sin(\theta_{ij}, \theta_{jk}, \theta_{ik}), \tag{2.43}$$

together with

$$|\mathbf{u}_{ijk} \times \mathbf{u}_{jkl} \times \mathbf{u}_{kli}| = \sin(\phi_{jk}, \phi_{ik}, \phi_{kl}). \tag{2.44}$$

The four faces of the hyperspherical tetrahedron are spherical triangles, with the various sine and cosine rules for the spherical case still holding true for these. For ease of notation, we now label the angles in the faces of the spherical triangles $\alpha_i^{(ijk)}$ to indicate which triangle is being considered.

In addition to the previous spherical relations, there are now various relations involving the dihedral angles, ϕ_{ij}, which connect together the various faces of the hyperspherical tetrahedron. By taking various relations between the scalar and vector products for the vectors $\mathbf{u}_{ijk}$, we can again derive a number of relations.

Proposition 2.3.6 (Cosine Rule)

$$\cos\phi_{jk} = \frac{-\begin{vmatrix} \cos\theta_{ij} & \cos\theta_{ik} & \cos\theta_{il} \\ 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{jk} & 1 & \cos\theta_{kl} \end{vmatrix}}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{jk},\theta_{jl},\theta_{kl})}. \tag{2.45}$$

for all $i, j, k, l = 1, 2, 3, 4$.

Proof

From the computation

$$\begin{aligned}(\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k)\cdot(\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l) &= -(\mathbf{n}_j \times \mathbf{n}_k \times (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l))\cdot \mathbf{n}_i, \\ &= \begin{vmatrix} \mathbf{n}_i\cdot\mathbf{n}_j & \mathbf{n}_i\cdot\mathbf{n}_k & \mathbf{n}_i\cdot\mathbf{n}_l \\ \mathbf{n}_j\cdot\mathbf{n}_j & \mathbf{n}_j\cdot\mathbf{n}_k & \mathbf{n}_j\cdot\mathbf{n}_l \\ \mathbf{n}_k\cdot\mathbf{n}_j & \mathbf{n}_k\cdot\mathbf{n}_k & \mathbf{n}_k\cdot\mathbf{n}_l \end{vmatrix},\end{aligned} \tag{2.46}$$

we find that for the following inner product (for all $i, j, k, l = 1, 2, 3, 4$) we have

$$\begin{aligned}\mathbf{u}_{ijk}\cdot\mathbf{u}_{jkl} &= \frac{(\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k)\cdot(\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l)}{|\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k|\,|\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l|}, \\ &= \frac{\begin{vmatrix} \mathbf{n}_i\cdot\mathbf{n}_j & \mathbf{n}_i\cdot\mathbf{n}_k & \mathbf{n}_i\cdot\mathbf{n}_l \\ \mathbf{n}_j\cdot\mathbf{n}_j & \mathbf{n}_j\cdot\mathbf{n}_k & \mathbf{n}_j\cdot\mathbf{n}_l \\ \mathbf{n}_k\cdot\mathbf{n}_j & \mathbf{n}_k\cdot\mathbf{n}_k & \mathbf{n}_k\cdot\mathbf{n}_l \end{vmatrix}}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{jk},\theta_{jl},\theta_{kl})}.\end{aligned} \tag{2.47}$$

Hence, the cosine rule follows. □

We now need to extend the definition of the triple sine function to the four-dimensional case, to give a further generalisation of the sine function dependent on six variables.

Definition 2.3.7 *The generalised sine function of six variables,* $\sin(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl})$*, is the four-dimensional analogue of the triple sine function,* *and is given by*

$$\sin(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl}) = \begin{vmatrix} 1 & \cos\theta_{ij} & \cos\theta_{ik} & \cos\theta_{il} \\ \cos\theta_{ij} & 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{ik} & \cos\theta_{jk} & 1 & \cos\theta_{kl} \\ \cos\theta_{il} & \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix}^{\frac{1}{2}}, \tag{2.48}$$

with $0 \le \theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl} \le \pi$. *Note that the restrictions on being a spherical tetrahedron ensure that the generalised sine function is always real and positive.*

This sine function is equivalent to the volume of a four-dimensional parallelotope with edges $\mathbf{n}_i$, $\mathbf{n}_j$, $\mathbf{n}_k$ and $\mathbf{n}_l$, given by

$$\text{Volume} = |\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l)| = \sin(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl}). \tag{2.49}$$

Proposition 2.3.8 (Sine Rule)

$$\begin{aligned} \frac{\sin(\phi_{ij},\phi_{ik},\phi_{il})}{\sin(\theta_{jk},\theta_{jl},\theta_{kl})} = \frac{\sin(\phi_{ij},\phi_{jk},\phi_{jl})}{\sin(\theta_{ik},\theta_{il},\theta_{kl})} = \frac{\sin(\phi_{ik},\phi_{jk},\phi_{kl})}{\sin(\theta_{ij},\theta_{il},\theta_{jl})} = \frac{\sin(\phi_{il},\phi_{jl},\phi_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})}, \\ = \frac{\sin^2(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{ij},\theta_{il},\theta_{jl})\sin(\theta_{ik},\theta_{il},\theta_{kl})\sin(\theta_{jk},\theta_{jl},\theta_{kl})} = k_H, \end{aligned} \tag{2.50}$$

where k_H *is constant.*

Proof

Consider the triple product

$$\begin{aligned} \mathbf{u}_{ijk} \times \mathbf{u}_{jkl} \times \mathbf{u}_{kli} &= \frac{(\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k) \times (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l) \times (\mathbf{n}_k \times \mathbf{n}_l \times \mathbf{n}_i)}{|\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k||\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l||\mathbf{n}_k \times \mathbf{n}_l \times \mathbf{n}_i|}, \\ &= \frac{(\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l))^2}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{jk},\theta_{jl},\theta_{kl})\sin(\theta_{kl},\theta_{ki},\theta_{li})}\,\mathbf{n}_k. \end{aligned} \tag{2.51}$$

This implies its modulus is

$$\begin{aligned} |\mathbf{u}_{ijk} \times \mathbf{u}_{jkl} \times \mathbf{u}_{kli}| &= \frac{(\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l))^2}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{jk},\theta_{jl},\theta_{kl})\sin(\theta_{kl},\theta_{ki},\theta_{li})}, \\ &= \frac{\sin^2(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{jk},\theta_{jl},\theta_{kl})\sin(\theta_{kl},\theta_{ki},\theta_{li})}. \end{aligned} \tag{2.52}$$

From this we have that the ratio

$$\frac{|\mathbf{u}_{ijk} \times \mathbf{u}_{jkl} \times \mathbf{u}_{kli}|}{|\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_l|} = \frac{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})} \tag{2.53}$$

is symmetric under the interchange of the labels i, j, k, l. Thus, it follows that

$$\begin{aligned} &\frac{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})} \\ &\qquad = \frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{kl}, \theta_{ki}, \theta_{li}) \sin(\theta_{ij}, \theta_{il}, \theta_{jl})}, \end{aligned} \tag{2.54}$$

and hence, the hyperspherical sine rule follows. □

There also exists a simpler sine relationship between the standard sine functions of the central angles and the sine of the dihedral angles.

Proposition 2.3.9

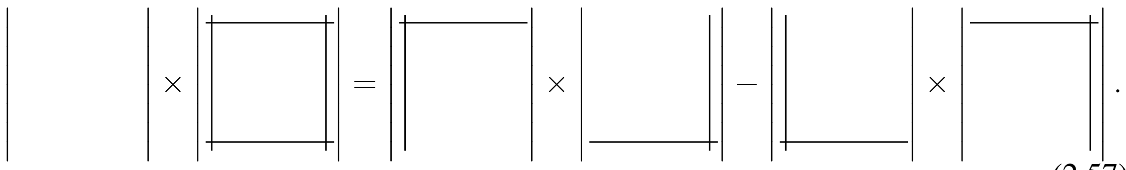
$$\sin \phi_{kl} = \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ik}, \theta_{il}, \theta_{kl}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl})} \sin \theta_{kl}. \tag{2.55}$$

In order to prove this proposition, we require a determinantal identity, attributed to Desnanot for the $n \leq 6$ case [45], and to Jacobi in the general case [64]. The Desnanot-Jacobi identity is also sometimes known as the Lewis Carroll formula of Dodgson Condensation [31].

Theorem 2.3.10 (Desnanot-Jacobi Identity) *[115] Let M be a $n \times n$ square matrix, and denote by M_i^p the matrix obtained by removing both the i-th row and p-th column. Similarly, let $M_{i,j}^{p,q}$ denote the matrix obtained by deleting the i-th and j-th rows, and the p-th and q-th columns, respectively, with $1 \leq i, j, p, q \leq n$. Then,*

$$\det(M) \det\left(M_{1,n}^{1,n}\right) = \det\left(M_1^1\right) \det\left(M_n^n\right) - \det\left(M_1^n\right) \det\left(M_n^1\right). \tag{2.56}$$

Diagrammatically, this is represented as



(2.57)

Proof

Proof of Proposition 2.3.9

Apply the Desnanot-Jacobi determinant identity to $\sin^4(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})$,

$$
\begin{aligned}
&\begin{vmatrix} 1 & \cos\theta_{ij} & \cos\theta_{ik} & \cos\theta_{jk} \\ \cos\theta_{ij} & 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{ik} & \cos\theta_{jk} & 1 & \cos\theta_{kl} \\ \cos\theta_{il} & \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix} \begin{vmatrix} 1 & \cos\theta_{kl} \\ \cos\theta_{kl} & 1 \end{vmatrix} \\
&\quad = \begin{vmatrix} 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{jk} & 1 & \cos\theta_{kl} \\ \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix} \begin{vmatrix} 1 & \cos\theta_{ik} & \cos\theta_{jk} \\ \cos\theta_{ik} & 1 & \cos\theta_{kl} \\ \cos\theta_{il} & \cos\theta_{kl} & 1 \end{vmatrix} \\
&\qquad - \begin{vmatrix} \cos\theta_{ij} & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{ik} & 1 & \cos\theta_{kl} \\ \cos\theta_{il} & \cos\theta_{kl} & 1 \end{vmatrix} \begin{vmatrix} \cos\theta_{ij} & \cos\theta_{ik} & \cos\theta_{jk} \\ \cos\theta_{jk} & 1 & \cos\theta_{kl} \\ \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix}.
\end{aligned} \tag{2.58}
$$

The result follows directly. □

From this proposition, it hence follows that the hyperspherical sine rule may be rewritten as

$$
\frac{\sin\phi_{ij}}{\sin\theta_{ij}}\frac{\sin\phi_{kl}}{\sin\theta_{kl}} = \frac{\sin\phi_{ik}}{\sin\theta_{ik}}\frac{\sin\phi_{jl}}{\sin\theta_{jl}} = \frac{\sin\phi_{il}}{\sin\theta_{il}}\frac{\sin\phi_{jk}}{\sin\theta_{jk}} = k_H. \tag{2.59}
$$

The constant k_H may also be neatly expressed in terms of cosines [44]. For this we need the polar cosine rule.

Proposition 2.3.11 (Polar Cosine Rule)

$$\cos\theta_{kl} = \frac{\begin{vmatrix} -\cos\phi_{jk} & \cos\phi_{ik} & -\cos\phi_{ij} \\ 1 & -\cos\phi_{kl} & \cos\phi_{jl} \\ -\cos\phi_{kl} & 1 & -\cos\phi_{il} \end{vmatrix}}{\sin(\phi_{ik},\phi_{jk},\phi_{kl})\sin(\phi_{il},\phi_{jl},\phi_{kl})}. \tag{2.60}$$

for all $i, j, k, l = 1, 2, 3, 4$.

Proof

Consider the product

$$\begin{aligned} &\frac{(\mathbf{u}_{ijk}\times\mathbf{u}_{jkl}\times\mathbf{u}_{kli})\cdot(\mathbf{u}_{jkl}\times\mathbf{u}_{kli}\times\mathbf{u}_{lij})}{|\mathbf{u}_{ijk}\times\mathbf{u}_{jkl}\times\mathbf{u}_{kli}||\mathbf{u}_{jkl}\times\mathbf{u}_{kli}\times\mathbf{u}_{lij}|} \\ &\qquad = \frac{\begin{vmatrix} \mathbf{u}_{ijk}\cdot\mathbf{u}_{jkl} & \mathbf{u}_{ijk}\cdot\mathbf{u}_{kli} & \mathbf{u}_{ijk}\cdot\mathbf{u}_{lij} \\ \mathbf{u}_{jkl}\cdot\mathbf{u}_{jkl} & \mathbf{u}_{jkl}\cdot\mathbf{u}_{kli} & \mathbf{u}_{jkl}\cdot\mathbf{u}_{lij} \\ \mathbf{u}_{kli}\cdot\mathbf{u}_{jkl} & \mathbf{u}_{kli}\cdot\mathbf{u}_{kli} & \mathbf{u}_{kli}\cdot\mathbf{u}_{lij} \end{vmatrix}}{\sin(\phi_{ik},\phi_{jk},\phi_{kl})\sin(\phi_{il},\phi_{jl},\phi_{kl})}, \\ &\qquad = \frac{\begin{vmatrix} -\cos\phi_{jk} & \cos\phi_{ik} & -\cos\phi_{ij} \\ 1 & -\cos\phi_{kl} & \cos\phi_{jl} \\ -\cos\phi_{kl} & 1 & -\cos\phi_{il} \end{vmatrix}}{\sin(\phi_{ik},\phi_{jk},\phi_{kl})\sin(\phi_{il},\phi_{jl},\phi_{kl})}. \end{aligned} \tag{2.61}$$

On the other hand, using (2.51), we have

$$\mathbf{u}_{ijk}\times\mathbf{u}_{jkl}\times\mathbf{u}_{kli} = \frac{(\mathbf{n}_i\cdot(\mathbf{n}_j\times\mathbf{n}_k\times\mathbf{n}_l))^2}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{jk},\theta_{jl},\theta_{kl})\sin(\theta_{kl},\theta_{ki},\theta_{li})}\,\mathbf{n}_k. \tag{2.62}$$

Reducing this using the sine rule and equating the two results gives the hyperspherical polar cosine rule. □

Proposition 2.3.12

$$\begin{aligned}\frac{\cos\phi_{ij}\cos\phi_{kl}-\cos\phi_{ik}\cos\phi_{jl}}{\cos\theta_{ij}\cos\theta_{kl}-\cos\theta_{ik}\cos\theta_{jl}} &= \frac{\cos\phi_{ik}\cos\phi_{jl}-\cos\phi_{il}\cos\phi_{jk}}{\cos\theta_{ik}\cos\theta_{jl}-\cos\theta_{il}\cos\theta_{jk}},\\ &= \frac{\cos\phi_{il}\cos\phi_{jk}-\cos\phi_{ij}\cos\phi_{kl}}{\cos\theta_{il}\cos\theta_{jk}-\cos\theta_{ij}\cos\theta_{kl}},\\ &= k_H.\end{aligned} \tag{2.63}$$

Proof

Consider the numerator

$$\cos\phi_{ij}\cos\phi_{kl}-\cos\phi_{ik}\cos\phi_{jl}, \tag{2.64}$$

and rewrite this in terms of the central angles using the cosine rule, (2.45), to obtain

$$\begin{aligned}&\cos\phi_{ij}\cos\phi_{kl}-\cos\phi_{ik}\cos\phi_{jl} = \\ &\frac{\begin{vmatrix}\cos\theta_{ik} & \cos\theta_{jk} & \cos\theta_{kl}\\ 1 & \cos\theta_{ij} & \cos\theta_{il}\\ \cos\theta_{ij} & 1 & \cos\theta_{jl}\end{vmatrix}}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{ij},\theta_{il},\theta_{jl})}\frac{\begin{vmatrix}\cos\theta_{ik} & \cos\theta_{il} & \cos\theta_{ij}\\ 1 & \cos\theta_{kl} & \cos\theta_{jk}\\ \cos\theta_{kl} & 1 & \cos\theta_{jl}\end{vmatrix}}{\sin(\theta_{ik},\theta_{il},\theta_{kl})\sin(\theta_{jk},\theta_{jl},\theta_{kl})}\\ &-\frac{\begin{vmatrix}\cos\theta_{ij} & \cos\theta_{jk} & \cos\theta_{jl}\\ 1 & \cos\theta_{ik} & \cos\theta_{il}\\ \cos\theta_{ik} & 1 & \cos\theta_{kl}\end{vmatrix}}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin(\theta_{ik},\theta_{il},\theta_{jk})}\frac{\begin{vmatrix}\cos\theta_{ij} & \cos\theta_{il} & \cos\theta_{ik}\\ 1 & \cos\theta_{jl} & \cos\theta_{jk}\\ \cos\theta_{jl} & 1 & \cos\theta_{kl}\end{vmatrix}}{\sin(\theta_{ij},\theta_{il},\theta_{jl})\sin(\theta_{jk},\theta_{jl},\theta_{kl})}.\end{aligned} \tag{2.65}$$

Multiplying this out and factorising, this reduces to

$$\cos\phi_{ij}\cos\phi_{kl}-\cos\phi_{ik}\cos\phi_{jl} = (\cos\theta_{ij}\cos\theta_{kl}-\cos\theta_{ik}\cos\theta_{jl})k_H, \tag{2.66}$$

from which the result follows. □

Furthermore, in four dimensions there are also relations involving the spherical angles $\alpha_j^{(j\cdot\cdot)}$ at a common vertex, $\mathbf{n}_j$, of the tetrahedron involving the different spherical triangles.

Proposition 2.3.13 (Vertex Cosine Rule)

$$\cos\phi_{jk} = \frac{\cos\alpha_j^{(ijk)}\cos\alpha_j^{(jkl)}-\cos\alpha_j^{(ijl)}}{\sin\alpha_j^{(ijk)}\sin\alpha_j^{(jkl)}}. \tag{2.67}$$

Proof

Expanding out the cosine rule (2.45) and applying the sine rule (2.50) to the denominator gives

$$\begin{aligned}\cos\phi_{jk} =&(\cos\theta_{ij}\cos\theta_{jl}+\cos\theta_{ik}\cos\theta_{kl}-\cos\theta_{ij}\cos\theta_{jk}\cos\theta_{kl}\\ &-\cos\theta_{ik}\cos\theta_{jk}\cos\theta_{jl}-\cos\theta_{il}+\cos\theta_{il}\cos^2\theta_{jk})\\ &/\sin\alpha_j^{(ijk)}\sin\alpha_j^{(jkl)}\sin\theta_{ij}\sin^2\theta_{jk}\sin\theta_{jl}.\end{aligned} \tag{2.68}$$

This can be factorised as

$$\cos\phi_{jk}=\frac{\left(\frac{\cos\theta_{ik}-\cos\theta_{ij}\cos\theta_{jk}}{\sin\theta_{ij}\sin\theta_{jk}}\right)\left(\frac{\cos\theta_{kl}-\cos\theta_{jk}\cos\theta_{jl}}{\sin\theta_{jk}\sin\theta_{jl}}\right)-\left(\frac{\cos\theta_{il}-\cos\theta_{ij}\cos\theta_{jl}}{\sin\theta_{ij}\sin\theta_{jl}}\right)}{\sin\alpha_j^{(ijk)}\sin\alpha_j^{(jkl)}}. \tag{2.69}$$

Applying (2.25) the result follows. $\square$

This set of relations implies, in turn, corresponding vertex polar cosine relations, expressing the spherical angles at a vertex in terms of the dihedral angles as follows:

$$\cos\alpha_j^{(ijl)}=\frac{\cos\phi_{jk}+\cos\phi_{ij}\cos\phi_{jl}}{\sin\phi_{ij}\sin\phi_{jl}}, \tag{2.70}$$

together with a corresponding sine rule of the form

$$\frac{\sin\phi_{jk}}{\sin\alpha_j^{(ijl)}}=\frac{\sin\phi_{jl}}{\sin\alpha_j^{(ijk)}}=\frac{\sin\phi_{ij}}{\sin\alpha_j^{(jkl)}}=\frac{\sin(\alpha_j^{(ijk)},\alpha_j^{(ijl)},\alpha_j^{(jkl)})}{\sin\alpha_j^{(ijk)}\sin\alpha_j^{(ijl)}\sin\alpha_j^{(jkl)}}, \tag{2.71}$$

for distinct $i,j,k,l\in\{1,2,3,4\}$. Note the similarities between these formulae and those of the spherical case.

We also have another new formula which again follows from the Desnanot-Jacobi identity, which proves particularly useful when providing the link between hyperspherical trigonometry and elliptic functions given later.

Proposition 2.3.14

$$\begin{aligned}&\frac{\sin^2(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl})\cos\alpha_l^{(jkl)}\sin\theta_{jl}\sin\theta_{kl}}{\sin^2(\theta_{jk},\theta_{jl},\theta_{kl})}\\ &\qquad=\sin(\theta_{ij},\theta_{il},\theta_{jl})\sin(\theta_{ik},\theta_{il},\theta_{kl})\times\left(\cos\phi_{jl}\cos\phi_{kl}-\cos\phi_{il}\right).\end{aligned} \tag{2.72}$$

Proof

Consider the Desnanot-Jacobi determinantal identity applied to $\sin^4(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})$. This yields

$$\begin{vmatrix} \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{kl} & 1 \end{vmatrix} \begin{vmatrix} 1 & \cos\theta_{ij} & \cos\theta_{ik} & \cos\theta_{il} \\ \cos\theta_{ij} & 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{ik} & \cos\theta_{jk} & 1 & \cos\theta_{kl} \\ \cos\theta_{il} & \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{jk} & 1 & \cos\theta_{kl} \\ \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix} \begin{vmatrix} 1 & \cos\theta_{ik} & \cos\theta_{il} \\ \cos\theta_{ij} & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{il} & \cos\theta_{kl} & 1 \end{vmatrix} \tag{2.73}$$

$$- \begin{vmatrix} \cos\theta_{ij} & \cos\theta_{ik} & \cos\theta_{il} \\ 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix} \begin{vmatrix} \cos\theta_{ij} & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{ik} & 1 & \cos\theta_{kl} \\ \cos\theta_{il} & \cos\theta_{kl} & 1 \end{vmatrix},$$

from which, the result follows. □

2.3.3 m-Dimensional Hyperspherical Trigonometry

We now generalise this work to provide a both new and complete set of formulae for m-dimensional hyperspherical trigonometry. Consider an $(m-1)$-dimensional hyperspherical simplex on the surface of an m-dimensional hypersphere, an $(m-1)$-sphere. Let the m vectors, $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_m$ be the position vectors of the m vertices of this simplex, such that the edges of this simplex are given by $\theta^{[1]}_{i_j i_k}$, with

$$\mathbf{n}_{i_j} \cdot \mathbf{n}_{i_k} \equiv \cos\theta^{[1]}_{i_j i_k}, \tag{2.74}$$

for $i_j, i_k \in \{1, 2, \ldots, m\}$. Now define orthogonal vectors $\mathbf{u}_{i_1 i_2 \ldots i_{m-1}}$ using the $(m-1)$-ary vector product,

$$\mathbf{u}_{i_1 i_2 \ldots i_{m-1}} \equiv \frac{\mathbf{n}_{i_1} \times \mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_{m-1}}}{|\mathbf{n}_{i_1} \times \mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_{m-1}}|}. \tag{2.75}$$

Define the angles between these orthogonal vectors by

$$\mathbf{u}_{i_1 i_2 \ldots i_{m-1}} \cdot \mathbf{u}_{i_2 i_3 \ldots i_m} \equiv -\cos\theta^{[m-1](i_1 \ldots i_m)}_{i_2 \ldots i_{m-1}}. \tag{2.76}$$

Note that the superscript $(i_1 \ldots i_m)$ denotes the simplex that is being considered. In the m-dimensional case this superscript may be omitted. We also have

$$|\mathbf{n}_{i_1} \times \mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_{m-1}}| = \sin\left(\Theta^{[1]}_{i_1 \cdots_{m-1}}\right), \tag{2.77}$$

where $\sin\left(\Theta^{[1]}_{i_1 \cdots_{m-1}}\right)$ is the generalised sine function of $m(m-1)/2$ variables,

$$\sin\left(\Theta^{[1]}_{i_1 \cdots_{m-1}}\right) = \sin\left(\left\{\theta^{[1]}_{i_j i_k} \middle| i_j < i_k,\ i_j, i_k = 1, \ldots, m-1\right\}\right), \tag{2.78}$$

together with

$$|\mathbf{u}_{i_1 \ldots i_{m-1}} \times \mathbf{u}_{i_2 \ldots i_m} \times \mathbf{u}_{i_3 \ldots i_m i_1} \times \cdots \times \mathbf{u}_{i_{m-1} i_m i_1 \ldots i_{m-3}}| = \sin\left(\Theta^{[m-1]}_{i_1 \ldots i_m}\right), \tag{2.79}$$

where similarly, $\sin\left(\Theta^{[m-1]}_{i_1 \ldots i_m}\right)$ is the multiple sine of all of the angles between each pair of orthogonal vectors. There follow a number of identities, similar to those for the lower-dimensional cases.

Proposition 2.3.15 (Cosine Rule)

$$\cos\theta^{[m-1]}_{i_2 \ldots i_{m-1}} = -\frac{\begin{vmatrix} \cos\theta^{[1]}_{i_1 i_2} & \ldots & \cos\theta^{[1]}_{i_1 i_m} \\ \vdots & \ddots & \vdots \\ \cos\theta^{[1]}_{i_{m-1} i_2} & \ldots & \cos\theta^{[1]}_{i_{m-1} i_m} \end{vmatrix}}{\sin\left(\Theta^{[1]}_{i_1 \ldots i_{m-1}}\right)\sin\left(\Theta^{[1]}_{i_2 \ldots i_m}\right)}. \tag{2.80}$$

Proof

Consider the scalar product

$$\begin{aligned} \mathbf{u}_{i_1 \ldots i_{m-1}} \cdot \mathbf{u}_{i_2 \ldots i_m} &= \frac{(\mathbf{n}_{i_1} \times \cdots \times \mathbf{n}_{i_{m-1}}) \cdot (\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m})}{|\mathbf{n}_{i_1} \times \cdots \times \mathbf{n}_{i_{m-1}}||\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m}|}, \\ &= \frac{\begin{vmatrix} \cos\theta^{[1]}_{i_1 i_2} & \ldots & \cos\theta^{[1]}_{i_1 i_m} \\ \vdots & \ddots & \vdots \\ \cos\theta^{[1]}_{i_{m-1} i_2} & \ldots & \cos\theta^{[1]}_{i_{m-1} i_m} \end{vmatrix}}{\sin\left(\Theta^{[1]}_{i_1 \ldots i_{m-1}}\right)\sin\left(\Theta^{[1]}_{i_2 \ldots i_m}\right)}. \end{aligned} \tag{2.81}$$

Hence the cosine rule follows. □

Proposition 2.3.16 (Sine Rule)

$$\frac{\sin\left(\Theta^{[1](i_{m-1})}_{i_1\ldots i_m}\right)}{\sin\left(\Theta^{[m-1]}_{i_m i_1\ldots i_{m-2}}\right)} == \frac{\sin^{m-2}\left(\Theta^{[1]}_{i_1\ldots i_m}\right)}{\sin\left(\Theta^{[m-1]}_{i_1\ldots i_{m-1}}\right)\ldots\sin\left(\Theta^{[m-1]}_{i_{m-1}i_m i_1\ldots i_{m-3}}\right)} = k, \tag{2.82}$$

where k is constant, and

$$\sin\left(\Theta^{[1](i_{m-1})}_{i_1\ldots i_m}\right) = \sin\left(\left\{\theta^{[1]}_{j_1\ldots j_{m-1}}\middle| i_{m-1} = j_k \,\text{for some}\, k \in \{1,\ldots,m-1\}\right\}\right). \tag{2.83}$$

Proof

Consider the ratio

$$\frac{|\mathbf{u}_{i_1\ldots i_{m-1}} \times \mathbf{u}_{i_2\ldots i_m} \times \cdots \times \mathbf{u}_{i_{m-1}i_m i_1\ldots i_{m-3}}|}{|\mathbf{u}_{i_m i_1\ldots i_{m-2}}|} = \frac{\sin\left(\Theta^{[1](i_{m-1})}_{i_1\ldots i_m}\right)}{\sin\left(\Theta^{[m-1]}_{i_m i_1\ldots i_{m-2}}\right)}. \tag{2.84}$$

However, using the nested multi-dimensional vector product identity, (2.10),

$$\begin{aligned}(\mathbf{u}_{i_1\ldots i_{m-1}} &\times \mathbf{u}_{i_2\ldots i_m} \times \cdots \times \mathbf{u}_{i_{m-1}i_m i_1\ldots i_{m-3}}) \\ &= \frac{(\mathbf{n}_{i_1} \times \cdots \times \mathbf{n}_{i_{m-1}}) \times \cdots \times (\mathbf{n}_{i_{m-1}} \times \mathbf{n}_{i_m} \times \mathbf{n}_{i_{m-1}} \cdots \times \mathbf{n}_{i_{m-3}})}{|\mathbf{n}_{i_1} \times \cdots \times \mathbf{n}_{i_{m-1}}|\ldots|\mathbf{n}_{i_{m-1}} \times \mathbf{n}_{i_m} \times \mathbf{n}_{i_{m-1}} \cdots \times \mathbf{n}_{i_{m-3}}|}, \\ &= \frac{(\mathbf{n}_{i_1}\cdot(\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m}))^{m-2}\mathbf{n}_{i_{m-1}}}{\sin\left(\Theta^{[m-1]}_{i_1\ldots i_{m-1}}\right)\ldots\sin\left(\Theta^{[m-1]}_{i_{m-1}i_m i_1\ldots i_{m-3}}\right)}.\end{aligned} \tag{2.85}$$

Hence,

$$\begin{aligned}|\mathbf{u}_{i_1\ldots i_{m-1}} &\times \mathbf{u}_{i_2\ldots i_m} \times \cdots \times \mathbf{u}_{i_{m-1}i_m i_1\ldots i_{m-3}}| \\ &= \frac{(\mathbf{n}_{i_1}\cdot(\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m}))^{m-2}}{\sin(\Theta^{[m-1]}_{i_1\ldots i_{m-1}})\ldots\sin\left(\Theta^{[m-1]}_{i_{m-1}i_m i_1\ldots i_{m-3}}\right)}, \\ &= \frac{\sin^{m-2}\left(\Theta^{[1]}_{i_1\ldots i_m}\right)}{\sin\left(\Theta^{[m-1]}_{i_1\ldots i_{m-1}}\right)\ldots\sin\left(\Theta^{[m-1]}_{i_{m-1}i_m i_1\ldots i_{m-3}}\right)},\end{aligned} \tag{2.86}$$

and so the sine rule follows. □

Proposition 2.3.17 (Polar Cosine Rule)

$$\cos\theta^{[1]}_{i_{m-1}i_m} = \det(X), \tag{2.87}$$

where

$$(X)_{jk} = \begin{cases} -1; & j = k-1, \\ -\cos\theta^{[m-1]}_{i_1\ldots i_{j-1}i_{j+1}\ldots i_{k-2}i_k\ldots i_m}; & j \neq k-1,\ m \text{ odd}, \\ (-1)^{j+k+1}\cos\theta^{[m-1]}_{i_1\ldots i_{j-1}i_{j+1}\ldots i_{k-2}i_k\ldots i_m}; & j \neq k-1,\ m \text{ even}. \end{cases} \tag{2.88}$$

Proof

Consider the product

$$(\mathbf{u}_{i_1\ldots i_{m-1}} \times \mathbf{u}_{i_2\ldots i_m} \times \mathbf{u}_{i_3\ldots i_m i_1} \times \cdots \times \mathbf{u}_{i_{m-1}i_m i_1\ldots i_{m-3}}) \cdot (\mathbf{u}_{i_2\ldots i_m} \times \cdots \times \mathbf{u}_{i_m i_1\ldots i_{m-2}}) \\ = \det(X), \tag{2.89}$$

where X is as defined previously. Alternatively, recalling (2.85) this product may be written as

$$\begin{aligned} &(\mathbf{u}_{i_1\ldots i_{m-1}} \times \mathbf{u}_{i_2\ldots i_m} \times \mathbf{u}_{i_3\ldots i_m i_1} \times \cdots \times \mathbf{u}_{i_{m-1}i_m i_1\ldots i_{m-3}}) \cdot (\mathbf{u}_{i_2\ldots i_m} \times \cdots \times \mathbf{u}_{i_m i_1\ldots i_{m-2}}), \\ &= \left(\frac{(\mathbf{n}_{i_1}\cdot(\mathbf{n}_{i_2}\times\cdots\times\mathbf{n}_{i_m}))^{m-2}\mathbf{n}_{i_{m-1}}}{\sin\left(\Theta^{[m-1]}_{i_1\ldots i_{m-1}}\right)\ldots\sin\left(\Theta^{[m-1]}_{i_{m-1}i_m i_1\ldots i_{m-3}}\right)}\right) \\ &\qquad\qquad \cdot\left(\frac{(\mathbf{n}_{i_2}\cdot(\mathbf{n}_{i_3}\times\cdots\times\mathbf{n}_{i_m}\times\mathbf{n}_{i_{m+1}}))^{m-2}\mathbf{n}_{i_m}}{\sin\left(\Theta^{[m-1]}_{i_2\ldots i_m}\right)\ldots\sin\left(\Theta^{[m-1]}_{i_m i_1\ldots i_{m-2}}\right)}\right), \\ &= \frac{\sin^{2m-4}(\Theta^{[1]}_{i_1\ldots i_m})\cos\theta^{[1]}_{i_{m-1}i_m}}{\sin(\Theta^{[m-1]}_{i_1\ldots i_{m-1}})\sin^2\left(\Theta^{[m-1]}_{i_2\ldots i_m}\right)\ldots\sin^2\left(\Theta^{[m-1]}_{i_{m-1}i_m i_1\ldots i_{m-3}}\right)\sin\left(\Theta^{[m-1]}_{i_m i_1\ldots i_{m-2}}\right)} \end{aligned} \tag{2.90}$$

from which, simplifying using the sine rule, the polar cosine rule then follows. □

Note that the facets of the $(m-1)$-dimensional simplex are $(m-2)$-dimensional simplices with the various cosine and sine rules still holding true in these facets. The same applies to these $(m-2)$-dimensional facets, and so on, forming a complete hierarchy of cosine and sine rules between the angles for the j-dimensional facets, and those for the $(j-1)$-dimensional facets. So, in terms of the cosine rule we have

$$\cos\theta^{[j]}_{i_2\ldots i_{j-1}} = \frac{-\begin{vmatrix} \cos\theta^{[j-1](i_1\ldots i_{j-1})}_{i_2\ldots i_{j-1}} & \cos\theta^{[j-1](i_1\ldots i_{j-2}i_j)}_{i_2\ldots i_{j-1}} \\ 1 & \cos\theta^{[j-1](i_2\ldots i_j)}_{i_2\ldots i_{j-1}} \end{vmatrix}}{\sin\theta^{[j-1](i_1\ldots i_{j-1})}_{i_2\ldots i_{j-1}}\sin\theta^{[j-1](i_2\ldots i_j)}_{i_2\ldots i_{j-1}}}. \tag{2.91}$$

This can be extended so that the angles of any facet can be expressed in terms of those for facets of any other dimensional to give a cosine rule of the form

$$\begin{aligned}&\cos\theta^{[j]}_{i_2\ldots i_{j-1}}\\ &= \frac{-\begin{vmatrix} \cos\theta^{[k](i_1\ldots i_k i_{k+1})}_{i_2\ldots i_k} & \cos\theta^{[k](i_1\ldots i_k i_{k+2})}_{i_2\ldots i_k} & \cdots & \cos\theta^{[k](i_1\ldots i_k i_j)}_{i_2\ldots i_k} \\ \cos\theta^{[k](i_2\ldots i_k i_{k+1})}_{i_2\ldots i_{k+1} i_{k+1}} & \cos\theta^{[k](i_2\ldots i_{k+2})}_{i_2\ldots i_k} & \cdots & \cos\theta^{[k](i_2\ldots i_{k+1} i_j)}_{i_2\ldots i_k} \\ \vdots & \vdots & \ddots & \vdots \\ \cos\theta^{[k](i_{j-1}i_2\ldots i_{k+1})}_{i_2\ldots i_k} & \cos\theta^{[k](i_{j-1}i_2\ldots i_k i_{k+2})}_{i_2\ldots i_k} & \cdots & \cos\theta^{[k](i_{j-1}i_2\ldots i_k i_j)}_{i_2\ldots i_k} \end{vmatrix}}{\sin\left(\Theta^{[k](i_1\ldots i_k)}_{i_2\ldots i_k}\right)\sin\left(\Theta^{[k](i_2\ldots i_{k+1})}_{i_2\ldots i_k}\right)}.\end{aligned} \tag{2.92}$$

From these cosine rules, there arise corresponding sine rules. Specifically, note that

$$\sin\theta^{[j]}_{i_2\ldots i_j} = k_{j-1}\sin\theta^{[j-1]}_{i_2\ldots i_{j-1}}, \tag{2.93}$$

where

$$k_{j-1} = \frac{\sin\left(\theta^{[j-1](i_1\ldots i_j)}_{i_2\ldots i_{j-1}}, \theta^{[j-1](i_2\ldots i_{j+1})}_{i_2\ldots i_{j-1}}, \theta^{[j-1](i_1\ldots i_{j-1}i_{j+1})}_{i_2\ldots i_{j-1}}\right)}{\sin\left(\theta^{[j-1](i_1\ldots i_j)}_{i_2\ldots i_{j-1}}\right)\sin\left(\theta^{[j-1](i_2\ldots i_{j+1})}_{i_2\ldots i_{j-1}}\right)\sin\left(\theta^{[j-1](i_1\ldots i_{j-1}i_{j+1})}_{i_2\ldots i_{j-1}}\right)}, \tag{2.94}$$

with k_{j-1} being constant. Hence, from this, it follows that a full hierarchy of intertwined sine rules exists between the angles of any two facets pf different dimensions.

2.4 Hyperspherical Trigonometry and Elliptic Functions

2.4.1 Link between Spherical Trigonometry and the Jacobi Elliptic Functions

There exists a well-known link between the formulae of spherical trigonometry and the Jacobi elliptic functions through their addition formulae [86, 84]. Here, we correct a derivation given by Irwin [62], taking as a starting point the sine rule for spherical trigonometry,

$$\frac{\sin\alpha_i}{\sin\theta_{jk}} = \frac{\sin\alpha_j}{\sin\theta_{ik}} = \frac{\sin\alpha_k}{\sin\theta_{ij}} = k, \tag{2.95}$$

where

$$k = \frac{\sin(\theta_{ij},\theta_{ik},\theta_{jk})}{\sin\theta_{ij}\sin\theta_{ik}\sin\theta_{jk}}, \tag{2.96}$$

and considering the expression

$$W = k^2\sin^2\theta_{ij}\sin^2\theta_{ik}\sin^2\theta_{jk} - \sin^2(\theta_{ij},\theta_{ik},\theta_{jk}). \tag{2.97}$$

It follows that the derivative of W with respect to one of the angles, θ_{ij}, is

$$\begin{aligned}\frac{\partial W}{\partial\theta_{ij}} &= -2\sin\theta_{ij}(\cos\theta_{ij} - \cos\theta_{ik}\cos\theta_{jk} - k^2\cos\theta_{ij}\sin^2\theta_{ik}\sin^2\theta_{jk}),\\ &= 2\sin\theta_{ij}\sin\theta_{ik}\sin\theta_{jk}\cos\alpha_i\cos\alpha_j.\end{aligned} \tag{2.98}$$

If the radial angles θ_{ij}, θ_{ik} and θ_{jk} are all varied in such a way as to keep k constant, then since this is equivalent to the condition $W = 0$, this implies

$$\cos\alpha_i\cos\alpha_j\mathrm{d}\theta_{ij} + \cos\alpha_i\cos\alpha_k\mathrm{d}\theta_{ik} + \cos\alpha_j\cos\alpha_k\mathrm{d}\theta_{jk} = 0, \tag{2.99}$$

which is of course equivalent to

$$\frac{\mathrm{d}\theta_{\mathrm{ij}}}{\cos\alpha_k} + \frac{\mathrm{d}\theta_{\mathrm{jk}}}{\cos\alpha_i} + \frac{\mathrm{d}\theta_{\mathrm{ik}}}{\cos\alpha_j} = 0. \tag{2.100}$$

However, the sine rule gives

$$\cos\alpha_i = \pm\sqrt{1 - k^2\sin^2\theta_{jk}},\ \text{etc.}, \tag{2.101}$$

and so, as the signs of θ_{ij}, θ_{ik} and θ_{jk} can be chosen arbitrarily, without loss of generality they can all be chosen to be positive, giving

$$\frac{\mathrm{d}\theta_{\mathrm{ij}}}{\sqrt{1-k^2\sin^2\theta_{ij}}}+\frac{\mathrm{d}\theta_{\mathrm{ik}}}{\sqrt{1-k^2\sin^2\theta_{ik}}}+\frac{\mathrm{d}\theta_{\mathrm{jk}}}{\sqrt{1-k^2\sin^2\theta_{jk}}}=0 \tag{2.102}$$

Writing $\theta_{ij}=\mathrm{am}(a_k)$, with

$$a_k=\int_0^{\mathrm{am}(a_k)}\frac{dt}{\sqrt{1-k^2\sin^2 t}}, \tag{2.103}$$

and similarly, $\theta_{ik}=\mathrm{am}(a_j)$ and $\theta_{jk}=\mathrm{am}(a_i)$, the integral of the differential relation, (2.102), implies that

$$a_i+a_j+a_k=\gamma, \tag{2.104}$$

with γ fixed by the triangle considered, together with

$$\begin{aligned}\sin\theta_{ij}&=\sin(\mathrm{am}(a_k))=\mathrm{sn}(a_k),\\ \sin\theta_{ik}&=\sin(\mathrm{am}(a_j))=\mathrm{sn}(a_j),\\ \sin\theta_{jk}&=\sin(\mathrm{am}(a_i))=\mathrm{sn}(a_i).\end{aligned} \tag{2.105}$$

Therefore, having introduced uniformising variables a_i, $i=1,2,3$, associated with the three spherical angles θ_{jk}, the various spherical trigonometric functions can be identified with the Jacobi elliptic functions by using the identifications

$$\sin(\theta_{jk})=\mathrm{sn}(a_i;k) \iff \sin\alpha_i\equiv k\,\mathrm{sn}(a_i;k), \tag{2.106}$$

in which k is the modulus of the elliptic function given by (2.96). These identifications, through the usual relations between the three Jacobi elliptic functions sn, cn and dn,

$$\mathrm{cn}^2(u;k)+\mathrm{sn}^2(u;k)=1, \qquad \mathrm{dn}^2(u;k)+k^2\mathrm{sn}^2(u;k)=1, \tag{2.107}$$

lead to

$$\cos\theta_{jk}=\mathrm{cn}(a_i;k), \text{ and } \cos\alpha_i=\mathrm{dn}(a_i;k), \tag{2.108}$$

for i, j, k cyclic. Addition formulae follow readily from the various spherical trigonometric relations. In particular, the cosine and polar cosine rules yield the relations

$$\begin{aligned} \mathrm{cn}(a_i) &= \mathrm{cn}(a_j)\mathrm{cn}(a_k) + \mathrm{sn}(a_j)\mathrm{sn}(a_k)\mathrm{dn}(a_i), \\ \mathrm{dn}(a_i) &= -\mathrm{dn}(a_j)\mathrm{dn}(a_k) + k^2\mathrm{sn}(a_j)\mathrm{sn}(a_k)\mathrm{cn}(a_i), \end{aligned} \tag{2.109}$$

respectively. Solving these relations as functions of a_i gives

$$\mathrm{cn}(a_i) = \frac{\mathrm{dn}(a_j)\mathrm{dn}(a_k)\mathrm{sn}(a_j)\mathrm{sn}(a_k) - \mathrm{cn}(a_j)\mathrm{cn}(a_k)}{1 - k^2\mathrm{sn}^2(a_j)\mathrm{sn}^2(a_k)}, \tag{2.110}$$

$$\mathrm{dn}(a_i) = \frac{\mathrm{dn}(a_j)\mathrm{dn}(a_k) - k^2\mathrm{cn}(a_j)\mathrm{cn}(a_k)\mathrm{sn}(a_j)\mathrm{sn}(a_k)}{1 - k^2\mathrm{sn}^2(a_j)\mathrm{sn}^2(a_k)}, \tag{2.111}$$

the addition formulae for the Jacobi elliptic functions with

$$\begin{aligned} \mathrm{cn}(a_i) &= -\mathrm{cn}(a_j + a_k), \\ \mathrm{dn}(a_i) &= \mathrm{dn}(a_j + a_k). \end{aligned} \tag{2.112}$$

The Jacobi elliptic functions are periodic with half periods $K(k)$ and $K'(k)$, meaning that

$$\begin{aligned} \mathrm{cn}(a_j + 2\mu K + 2\nu i K'; k) &= (-1)^{\mu+\nu}\mathrm{cn}(a_j; k), \\ \mathrm{dn}(a_j + 2\mu K + 2\nu i K'; k) &= (-1)^{\nu}\mathrm{dn}(a_j; k), \end{aligned} \tag{2.113}$$

for all a_j,and all integers μ, ν, and so we must restrict γ such that $a_i + a_j + a_k = 2K$. Irwin incorrectly sets this equal to zero. Note the similarity here between this and the spherical excess. This condition is in fact that the fixing of k also fixes the area of the spherical triangle, under changes of the angles. The addition formula for sn follows as a consequence. From these addition formulae a number of intertwined addition relations follow

$$\mathrm{cn}(a_j)\mathrm{sn}(a_i + a_j) = \mathrm{sn}(a_i)\mathrm{dn}(a_j) + \mathrm{dn}(a_i)\mathrm{sn}(a_j)\mathrm{cn}(a_i + a_j), \tag{2.114}$$

$$\mathrm{dn}(a_i)\mathrm{sn}(a_i + a_j) = \mathrm{cn}(a_i)\mathrm{sn}(a_j) + \mathrm{sn}(a_i)\mathrm{cn}(a_j)\mathrm{dn}(a_i + a_j), \tag{2.115}$$

$$\mathrm{sn}(a_i)\mathrm{cn}(a_i + a_j) + \mathrm{sn}(a_j)\mathrm{dn}(a_i + a_j) = \mathrm{cn}(a_i)\mathrm{dn}(a_j)\mathrm{sn}(a_i + a_j). \tag{2.116}$$

Denoting

$$w_1(a_j) = \frac{\rho}{\mathrm{sn}(a_j)},\ w_2(a_j) = \frac{\rho\,\mathrm{cn}(a_j)}{\mathrm{sn}(a_j)},\ w_3(a_j) = \frac{\rho\,\mathrm{dn}(a_j)}{\mathrm{sn}(a_j)}, \tag{2.117}$$

these addition formulae may be rewritten as

$$w_i(a_j)w_j(a_i) + w_j(a_k)w_k(a_j) + w_k(a_i)w_i(a_k) = 0,\ \ i,j,k = 1,2,3, \tag{2.118}$$

with $a_1+a_2+a_3 = 2K$. Note that this relation is the functional Yang-Baxter relation[21]. By identifying uniformising variables with the facial angles as opposed to the radial ones, we can derive a similar relationship. This follows from (2.39) by letting

$$\begin{aligned}\overline{W} = {} & \frac{1}{k^2}\sin^2\alpha_i \sin^2\alpha_j \sin^2\alpha_k \\ & - (1 - \cos^2\alpha_i - \cos^2\alpha_j - \cos^2\alpha_k - 2\cos\alpha_i\cos\alpha_j\cos\alpha_k).\end{aligned} \tag{2.119}$$

Applying Irwin's method and varying α_i, α_j and α_k as to keep k constant, it follows that

$$\frac{\mathrm{d}\alpha_i}{\sqrt{1-\frac{1}{k^2}\sin^2\alpha_i}} + \frac{\mathrm{d}\alpha_j}{\sqrt{1-\frac{1}{k^2}\sin^2\alpha_i}} + \frac{\mathrm{d}\alpha_k}{\sqrt{1-\frac{1}{k^2}\sin^2\alpha_i}} = 0. \tag{2.120}$$

Writing $\alpha_i = \mathrm{am}(b_i)$, with

$$b_i = \int_0^{am(b_i)} \frac{\mathrm{d}t}{\sqrt{1-\frac{1}{k^2}\sin^2 t}}, \tag{2.121}$$

and similarly, $\alpha_j = \mathrm{am}(b_j)$ and $\alpha_k = \mathrm{am}(b_k)$, the integral of (2.120) implies

$$b_i + b_j + b_k = \gamma. \tag{2.122}$$

Therefore, having introduced spherical angles b_i, $i = 1,2,3$, this time associated with the three spherical angles α_i, the various spherical trigonometric functions can be identified with the Jacobi elliptic functions via the identifications

$$\sin\alpha_i = \mathrm{sn}\left(b_i;\frac{1}{k}\right) \iff \sin\theta_{jk} = \frac{1}{k}\mathrm{sn}\left(b_i;\frac{1}{k}\right), \tag{2.123}$$

in which $1/k$ is the modulus of the elliptic function. These identifications lead to

$$\cos\alpha_i = \mathrm{cn}\left(b_i;\frac{1}{k}\right),\ \text{and}\ \cos\theta_{jk} = \mathrm{dn}\left(b_i;\frac{1}{k}\right). \tag{2.124}$$

In this case, the Jacobi elliptic function addition formulae are satisfied providing we have

$$\begin{aligned} \mathrm{cn}(b_i) &= \mathrm{cn}(b_j + b_k), \\ \mathrm{dn}(b_i) &= -\mathrm{dn}(b_j + b_k), \end{aligned} \tag{2.125}$$

and therefore, this time the periodicity ensures $\gamma = 2K + 2iK'$,

$$b_i + b_j + b_k = 2K + 2iK'. \tag{2.126}$$

Note that making the identification in this way results in a slightly different restriction, dependent on both K and also K'.

2.4.2 Link between Hyperspherical Trigonometry and Elliptic Functions

We now look to provide a novel link between the four-dimensional hyperspherical case and elliptic functions by producing a generalisation of Irwin's procedure as before. We first consider a general unrestricted hyperspherical tetrahedron before focusing on the specialised restricted case of a symmetric hyperspherical tetrahedron in order to provide clarity.

General Hyperspherical Tetrahedron

We take as a starting point the hyperspherical sine rule relation,

$$\begin{aligned} &\frac{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})} \\ &\qquad = \frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{ik}, \theta_{il}, \theta_{kl}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl})} = k, \end{aligned} \tag{2.127}$$

from which we introduce

$$\begin{aligned} W = &\sin^4(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \\ &- k^2 \sin^2(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin^2(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin^2(\theta_{ik}, \theta_{il}, \theta_{kl}) \sin^2(\theta_{jk}, \theta_{jl}, \theta_{kl}). \end{aligned} \tag{2.128}$$

Taking W's derivative with respect to θ_{ij} gives

$$\begin{aligned}\frac{\partial W}{\partial \theta_{ij}} = -2\sin(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl})\sin(\theta_{ik},\theta_{il},\theta_{kl})\sin(\theta_{jk},\theta_{jl},\theta_{kl})\\ \times\sin(\theta_{ij},\theta_{il},\theta_{jl})\sin(\theta_{ij},\theta_{ik},\theta_{jk})\sin\phi_{ij}\\ \times\left(\cos\phi_{il}\cos\phi_{jl}+\cos\phi_{ik}\cos\phi_{jk}\right),\end{aligned} \tag{2.129}$$

which we have simplified using the various spherical and hyperspherical identities from earlier. If we vary the length of the six sides such that k remains constant, then since this is equivalent to $W = 0$, setting $\mathrm{d}W = 0$ implies

$$\sum_{perm}\sin\phi_{ij}\left(\cos\phi_{il}\cos\phi_{jl}+\cos\phi_{ik}\cos\phi_{jk}\right)\mathrm{d}\theta_{ij} = 0, \tag{2.130}$$

where $perm$ denotes all six index pairs, ij, ik, il, jk, jl and kl. Note the similarity between this and the spherical case, again perhaps suggesting some sort of connection with elliptic functions. This, however, cannot be simplified along the same lines as the spherical case, and so we now focus on the specialised symmetric hyperspherical case in order to provide clarity.

Symmetric Hyperspherical Tetrahedron

We now consider this relationship in terms of the more specific case of a symmetric tetrahedron to, for the first time, produce a clear link between the functions of hyperspherical trigonometry and elliptic functions. Recall from Definition 1.3.4, that a Symmetric Tetrahedron is a tetrahedron in which the opposite dihedral angles are equal, i.e.

$$\phi_{ij} = \phi_{kl}, \quad \phi_{ik} = \phi_{jl}, \quad \phi_{il} = \phi_{jk}. \tag{2.131}$$

Restricting the spherical tetrahedron to be a symmetric one, the differential relationship (2.130) reduces to

$$\sum_{perm}\sin\phi_{ij}\cos\phi_{ik}\cos\phi_{jk}\mathrm{d}\theta_{ij} = 0, \tag{2.132}$$

or equivalently,

$$\sum_{perm} \frac{\sin \phi_{ij}}{\cos \phi_{ij}} \mathrm{d}\theta_{ij} = 0. \tag{2.133}$$

The restrictions on the tetrahedron being symmetric also simplify the hyperspherical sine rule, (2.50), reducing it to

$$\sin^2 \phi_{ij} = k_H \sin^2 \theta_{ij}. \tag{2.134}$$

Substituting this into (2.133), this becomes

$$\sum_{perm} \frac{\sqrt{k_H} \sin \theta_{ij}}{\sqrt{1 - k_H \sin^2 \theta_{ij}}} \mathrm{d}\theta_{ij} = 0, \tag{2.135}$$

a sum of elliptic integrals of the form

$$\int \frac{\sqrt{k_H} u}{\sqrt{(1 - k_H u^2)(1 - u^2)}} \mathrm{d}u. \tag{2.136}$$

Hence, we have shown that for the case of a symmetric hyperspherical tetrahedron, the hyperspherical trigonometric functions may be associated with elliptic functions. We now investigate this association further for the general case.

2.5 Hyperspherical Trigonometry and the Generalised Jacobi Functions

In this section we review the generalised Jacobi functions as introduced by Pawellek[116], and provide a novel link between these functions and the formulae of hyperspherical trigonometry.

2.5.1 Generalised Jacobi Functions

In [116], Pawellek introduced the generalised Jacobi functions $s(u; k_1, k_2)$, $c(u; k_1, k_2)$, $d_1(u; k_1, k_2)$ and $d_2(u; k_1, k_2)$. These functions are defined on algebraic curves with two distinct moduli, and are based upon Jacobi's elliptic functions [65]. After assuming without loss of generality that $1 > k_1 > k_2 > 0$ as moduli parameters, they are defined as the inversion of the hyperelliptic integrals

$$u(x; k_1, k_2) = \int_0^{x=s(u)} \frac{dt}{\sqrt{(1-t^2)(1-k_1^2t^2)(1-k_2^2t^2)}}, \tag{2.137}$$

$$u(x; k_1, k_2) = \int_{x=c(u)}^{1} \frac{dt}{\sqrt{(1-t^2)(k_1'^2+k_1^2t^2)(k_2'^2+k_2^2t^2)}}, \tag{2.138}$$

$$u(x; k_1, k_2) = k_1 \int_{x=d_1(u)}^{1} \frac{dt}{\sqrt{(1-t^2)(t^2-k_1'^2)(k_1^2-k_2^2+k_2^2t^2)}}, \tag{2.139}$$

$$u(x; k_1, k_2) = k_2 \int_{x=d_2(u)}^{1} \frac{dt}{\sqrt{(1-t^2)(t^2-k_2'^2)(k_2^2-k_1^2+k_1^2t^2)}}, \tag{2.140}$$

respectively, with $k_i' = \sqrt{1-k_i^2}$. These generalised Jacobi functions are associated with an algebraic curve of the form

$$\mathcal{C} : y^2 = (1-x^2)(1-k_1^2x^2)(1-k_2^2x^2), \tag{2.141}$$

which can be modelled as a Riemann surface of genus 2, but is, in fact, a double cover of an elliptic curve $\mathcal{E}$,

$$\mathcal{E} : w^2 = z(1-z)(1-k_1^2z)(1-k_2^2z), \tag{2.142}$$

via the cover map $\mathcal{C} \xrightarrow{\pi} \mathcal{E}$,

$$(w, z) = \pi(y, x) = (xy, x^2). \tag{2.143}$$

The functions satisfy a number of identities,

$$\begin{aligned} &c^2(u) = 1 - s^2(u), \qquad d_1^2(u) = 1 - k_1^2s^2(u), \qquad d_2^2(u) = 1 - k_2^2s^2(u), \\ &d_i^2(u) - k_i^2c^2(u) = 1 - k_i^2, \quad i = 1, 2; \qquad k_1^2d_2^2(u) - k_2^2d_1^2(u) = k_1^2 - k_2^2, \end{aligned} \tag{2.144}$$

and are related to the Jacobi elliptic functions by

$$s(u; k_1, k_2) = \frac{\mathrm{sn}(k_2'u; \kappa)}{\sqrt{k_2'^2 + k_2^2 sn^2(k_2'u; \kappa)}}, \quad c(u; k_1, k_2) = \frac{k_2'\mathrm{cn}(k_2'u; \kappa)}{\sqrt{1 - k_2^2 cn^2(k_2'u; \kappa)}}, \tag{2.145}$$

$$d_1(u; k_1, k_2) = \frac{\sqrt{k_1^2 - k_2^2}\mathrm{dn}(k_2'u; \kappa)}{\sqrt{k_1^2 - k_2^2\mathrm{dn}^2(k_2'u; \kappa)}}, \quad d_2(u; k_1, k_2) = \frac{\sqrt{k_1^2 - k_2^2}}{\sqrt{k_1^2 - k_2^2\mathrm{dn}^2(k_2'u; \kappa)}}, \tag{2.146}$$

with

$$\kappa = \frac{k_1^2 - k_2^2}{1 - k_2^2}. \tag{2.147}$$

Note that although these generalised Jacobi functions may be written in terms of the Jacobi elliptic functions, they are not in fact elliptic functions themselves. Rather, they are two-valued functions on the complex plane, with square-root branch points. The first derivatives of these functions are given by

$$s'(u) = c(u)d_1(u)d_2(u), \qquad c'(u) = -s(u)d_1(u)d_2(u), \tag{2.148}$$

$$d_1'(u) = -k_1^2 s(u)c(u)d_2(u), \qquad d_2'(u) = -k_2^2 s(u)c(u)d_1(u). \tag{2.149}$$

These generalised Jacobi functions with moduli k_1 and k_2 satisfy the following addition formulae:

$$s(u \pm v) = \frac{s(u)d_2(u)c(v)d_1(v) \pm s(v)d_2(v)c(u)d_1(u)}{\sqrt{\Delta^\pm}}, \tag{2.150a}$$

$$c(u \pm v) = \frac{c(u)d_2(u)c(v)d_2(v) \mp k_2'^2 s(u)d_1(u)s(v)d_1(v)}{\sqrt{\Delta^\pm}}, \tag{2.150b}$$

$$d_1(u \pm v) = \frac{d_1(u)d_2(u)d_1(v)d_2(v) \mp \kappa^2 k_2'^2 s(u)c(u)s(v)c(v)}{\sqrt{\Delta^\pm}}, \tag{2.150c}$$

$$d_2(u \pm v) = \frac{d_2^2(u)d_2^2(v) - \kappa^2 k_2'^4 s^2(u)s^2(v)}{\sqrt{\Delta^\pm}}, \tag{2.150d}$$

$$\tag{2.150e}$$

where

$$\Delta^\pm = \left[d_2^2(u)d_2^2(v) - \kappa^2 k_2'^4 s^2(u)s^2(v)\right]^2 + k_2^2\left[s(u)d_2(u)c(v)d_1(v) \pm s(v)d_2(v)c(u)d_1(u)\right]. \tag{2.151}$$

These formulae follow from the addition formula for the standard Jacobi elliptic functions [116].

2.5.2 Hyperspherical Trigonometry and the Generalised Jacobi Functions

We are also able to show a new connection between the generalised Jacobi functions and hyperspherical trigonometry. As the generalised Jacobi functions are dependent on two moduli, then so must the hyperspherical trigonometry be. It is the interplay of these moduli that govern the connection. Recall that for a hyperspherical tetrahedron we have

$$\sin\alpha_i^{(ijk)} = k_1 \sin\theta_{jk}, \tag{2.152}$$

and

$$\sin\phi_{il} = k_2 \sin\alpha_i^{(ijk)}, \tag{2.153}$$

for distinct i, j, k, l, where

$$k_1 = \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})}{\sin\theta_{ij}\sin\theta_{ik}\sin\theta_{jk}}, \tag{2.154}$$

and

$$k_2 = \frac{\sin(\alpha_i^{(ijk)}, \alpha_i^{(ijl)}, \alpha_i^{(ikl)})}{\sin\alpha_i^{(ijk)}\sin\alpha_i^{(ijl)}\sin\alpha_i^{(ikl)}}, \tag{2.155}$$

respectively. Separately, these each have the same link to the Jacobi elliptic functions as the functions of a spherical triangle. They imply that

$$\sin\phi_{il} = k_1 k_2 \sin\theta_{jk}. \tag{2.156}$$

Introducing uniformising variables a_{jk}, $j, k = 1, 2, 3, 4$ with $k > j$, associated with the six θ_{jk}, such that

$$a_{jk} = \int_0^{\theta_{jk}} \frac{\mathrm{d}t}{\sqrt{(1 - k_1^2\sin^2 t)(1 - k_1^2 k_2^2 \sin^2 t)}}, \tag{2.157}$$

with $\theta_{jk} = \mathrm{am}(a_{jk})$, the various hyperspherical trigonometric functions can be identified with Pawellek's generalised Jacobi functions via the identifications

$$s(a_{jk}) \equiv \sin\theta_{jk} \iff k_1 s(a_{jk}) = \sin\alpha_i^{(ijk)} \iff k_1 k_2 s(a_{jk}) = \sin\phi_{il}, \tag{2.158}$$

in which, k_1 and $k_1 k_2$ are the two moduli of the functions, and are as given earlier. From these and the identities listed previously it follows that

$$c(a_{jk}) = \cos\theta_{jk},\ d_1(a_{jk}) = \cos\alpha_i^{(ijk)},\ d_2(a_{jk}) = \cos\phi_{il}. \tag{2.159}$$

Note that under these identifications the identities (2.144) for the generalised Jacobi functions are still obeyed.

The modulus k_1 governs the relations between the spherical trigonometry of each of the faces, and the elliptic functions. The second modulus, $k_1 k_2$, acts as an overall modulus for the spherical tetrahedron. Note its dependence on the first modulus, k_1. These moduli remain constant, so that if one of the vertices of the spherical tetrahedron were moved, the others must be adjusted to compensate. These movements result in a change to the faces of the tetrahedron, but not to k_1, the facial modulus.

2.6 Four- and Five-Parts Formulae

In this section, we consider the expansion of one of the radial vectors, $\mathbf{n}_i$, in terms of the other radial vectors. We compare this with the expansion in terms of an orthogonal frame, obtained by Gram-Schmidt orthonormalisation to obtain for the first time the four-parts formula for the hyperspherical case. We follow a similar procedure for the orthogonal vectors, $\mathbf{u}_{ijk}$, to obtain the novel five-parts formula. We believe these expansions may be of particular use in considering Sergeev's model for the tetrahedron equation[126]. This will require further investigation in future.

2.6.1 Spherical Case

We write a general position vector on the surface of a sphere in terms of its basis vectors, and derive the four and five parts formulae.

Given two vectors in three dimensions, we can express any other vector on the sphere in terms of these by

$$\mathbf{n}_i = A\mathbf{n}_j + B\mathbf{n}_k + C\mathbf{n}_j \times \mathbf{n}_k, \tag{2.160}$$

with

$$\begin{aligned} \begin{pmatrix} A \\ B \end{pmatrix} &= \begin{pmatrix} 1 & \cos\theta_{jk} \\ \cos\theta_{jk} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta_{ij} \\ \cos\theta_{ik} \end{pmatrix} \\ &= \frac{1}{\sin^2\theta_{jk}} \begin{pmatrix} \sin\theta_{ik}\sin\theta_{jk}\cos\alpha_k \\ \sin\theta_{ij}\sin\theta_{jk}\cos\alpha_j \end{pmatrix}, \end{aligned} \tag{2.161}$$

and

$$C = \frac{\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k)}{\sin^2\theta_{jk}} = \frac{\sin\alpha_i \sin\theta_{ik}\sin\theta_{ij}}{\sin^2\theta_{jk}}, \tag{2.162}$$

giving

$$\mathbf{n}_i = \frac{\sin\theta_{ik}\cos\alpha_k}{\sin\theta_{jk}}\mathbf{n}_j + \frac{\sin\theta_{ij}\cos\alpha_j}{\sin\theta_{jk}}\mathbf{n}_k + \frac{\sin\alpha_i\sin\theta_{ik}\sin\theta_{ij}}{\sin^2\theta_{jk}}\mathbf{n}_j \times \mathbf{n}_k. \tag{2.163}$$

Using the spherical sine rule this reduces to

$$\mathbf{n}_i = \frac{\sin\alpha_j\cos\alpha_k}{\sin\alpha_i}\mathbf{n}_j + \frac{\sin\alpha_k\cos\alpha_j}{\sin\alpha_i}\mathbf{n}_k + \frac{\sin\alpha_i\sin\sin\alpha_k}{\sin\alpha_i}\mathbf{n}_j \times \mathbf{n}_k. \tag{2.164}$$

Using the Gram-Schmidt Process $\mathbf{n}_i$ may be written in terms of an orthonormal basis,

$$\mathbf{n}_i = A'\mathbf{n}_j + B'\mathbf{n}'_k + C'\mathbf{n}_j \times \mathbf{n}'_k, \tag{2.165}$$

where $\mathbf{n}_j$ is as is before, and $\mathbf{n}'_k$ is given by

$$\mathbf{n}'_k = \frac{\mathbf{n}_k - (\mathbf{n}_j \cdot \mathbf{n}_k)\mathbf{n}_j}{|\mathbf{n}_k - (\mathbf{n}_j \cdot \mathbf{n}_k)\mathbf{n}_j|} = \frac{\mathbf{n}_k - (\mathbf{n}_j \cdot \mathbf{n}_k)\mathbf{n}_j}{\sin\theta_{jk}}. \tag{2.166}$$

This implies

$$C' = (\sin\theta_{jk})C, \qquad B' = (\sin\theta_{jk})B, \qquad A' = A + \cot\theta_{jk}B', \tag{2.167}$$

from which it follows that

$$\begin{aligned} \mathbf{n}_i = \frac{\sin\theta_{ik}\cos\alpha_k + \cos\theta_{jk}\sin\theta_{ij}\cos\alpha_j}{\sin\theta_{jk}}&\mathbf{n}_j \\ &+ \sin\theta_{ij}\cos\alpha_j\mathbf{n}'_k + \sin\theta_{ij}\sin\alpha_j\mathbf{n}_j \times \mathbf{n}'_k. \end{aligned} \tag{2.168}$$

Equating the two expression we have for $\mathbf{n_i}$ gives the four-parts formula

$$\cot\theta_{ij}\sin\theta_{jk} = \cot\alpha_k\sin\alpha_j + \cos\theta_{jk}\cos\alpha_j. \tag{2.169}$$

Similarly, for

$$\mathbf{u}_{ij} = P\mathbf{u}_{ik} + Q\mathbf{u}_{kj} + R\mathbf{u}_{ik}\times\mathbf{u}_{kj} \tag{2.170}$$

it follows that

$$\mathbf{u}_{ij} = \frac{\sin\alpha_j}{\sin\alpha_k}\cos\theta_{jk}\mathbf{u}_{ik} + \frac{\sin\alpha_i}{\sin\alpha_k}\cos\theta_{ik}\mathbf{u}_{kj} - \frac{\sin\theta_{ik}\sin\theta_{kj}}{\sin\theta_{ij}}\mathbf{u}_{ik}\times\mathbf{u}_{kj}, \tag{2.171}$$

or alternatively, using the spherical sine rule,

$$\mathbf{u}_{ij} = \frac{\sin\theta_{ik}\cos\theta_{jk}}{\sin\theta_{ij}}\mathbf{u}_{ik} + \frac{\sin\theta_{jk}\cos\theta_{ik}}{\sin\theta_{ij}}\mathbf{u}_{kj} - \frac{\sin\theta_{ik}\sin\theta_{kj}}{\sin\theta_{ij}}\mathbf{u}_{ik}\times\mathbf{u}_{kj}. \tag{2.172}$$

Similarly, using the Gram-Schmidt Process,

$$\begin{aligned}\mathbf{u}_{ij} = {}& \frac{\sin\alpha_j\cos\theta_{jk} - \cos\alpha_k\sin\alpha_i\cos\theta_{ik}}{\sin\alpha_k}\mathbf{u}_{ik} \\ &+ \sin\alpha_i\cos\theta_{ik}\mathbf{u}'_{kj} + \sin\theta_{kj}\sin\alpha_j\mathbf{u}_{ij}\times\mathbf{u}'_{kj}.\end{aligned} \tag{2.173}$$

Equating the two expression we have for $\mathbf{u}_{ij}$ gives the five-parts formula

$$\cot\theta_{jk}\sin\alpha_j = \cos\alpha_i\sin\alpha_k + \cos\theta_{ik}\sin\alpha_i\cos\alpha_k. \tag{2.174}$$

2.6.2 Four-dimensional Hyperspherical Case

Following a similar process to the spherical case, in four-dimensional Euclidean space, for the vectors $\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k, \mathbf{n}_l \in E_4$ of a hyperspherical tetrahedron we have the expansions

$$\begin{aligned}\mathbf{n}_i = {}& \frac{\sin(\theta_{ik},\theta_{il},\theta_{kl})\ \cos\phi_{kl}}{\sin(\theta_{jk},\theta_{jl},\theta_{kl})}\mathbf{n}_j + \frac{\sin(\theta_{ij},\theta_{il},\theta_{jl})\ \cos\phi_{jl}}{\sin(\theta_{jk},\theta_{jl},\theta_{kl})}\mathbf{n}_k \\ & \frac{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\ \cos\phi_{jk}}{\sin(\theta_{jk},\theta_{jl},\theta_{kl})}\mathbf{n}_l + \\ & + \frac{\sqrt{\sin(\theta_{ij},\theta_{ik},\theta_{jk})\ \sin(\theta_{ij},\theta_{il},\theta_{jl})\ \sin(\theta_{ik},\theta_{il},\theta_{kl})\ \sin(\phi_{ij},\phi_{ik},\phi_{il})}}{\left(\sin(\theta_{jk},\theta_{jl},\theta_{kl})\right)^2}\,\mathbf{n}_j\times\mathbf{n}_k\times\mathbf{n}_l,\end{aligned} \tag{2.175}$$

as well as

$$\begin{aligned}\mathbf{n}_i = \cos\theta_{ij}\,\mathbf{n}_j + \sin\theta_{ij}\cos\alpha_j^{(ijk)}\,\mathbf{n}_k' + \sin\theta_{ij}\sin\alpha_j^{(ijk)}\cos\phi_{jk}\,\mathbf{n}_l' \\ + \sin\theta_{ij}\sin\alpha_j^{(ijk)}\sin\phi_{jk}\,\mathbf{n}_j\times\mathbf{n}_k'\times\mathbf{n}_l'\ ,\end{aligned} \tag{2.176}$$

in a (Gram-Schmidt) orthonormal basis $\mathbf{n}_j, \mathbf{n}_k', \mathbf{n}_l'$. Comparing the results provides the following set of equations:

$$\begin{aligned}&\frac{\sin(\theta_{jk},\theta_{jl},\theta_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})}\cos\phi_{jk} \\ &\quad = \cos\theta_{il}\left(1 - \tan\theta_{il}\cot\theta_{ik}\cos\phi_{ij}\frac{\sin\alpha_i^{(ijl)}}{\sin\alpha_i^{(ijk)}}\right. \\ &\qquad\qquad \left. - \tan\theta_{il}\cot\theta_{ij}\cos\alpha_i^{(ijl)}\left(1 - \tan\alpha_i^{(ijl)}\cot\alpha_i^{(ijk)}\cos\phi_{ij}\right)\right)\ ,\end{aligned} \tag{2.177a}$$

$$\frac{\sin(\theta_{ik},\theta_{il},\theta_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})}\cos\phi_{ik} = \frac{\sin\theta_{il}\cos\alpha_i^{(ijl)}}{\sin\theta_{ij}}\left(1 - \tan\alpha_i^{(ijl)}\cot\alpha_i^{(ijk)}\cos\phi_{ij}\right)\ , \tag{2.177b}$$

which form the hyperspherical analogue of the four-parts formula.

Similarly, for the expansions of the polar vectors $\mathbf{u}_{ijk}, \mathbf{u}_{ijl}, \mathbf{u}_{ikl}, \mathbf{u}_{jkl}$ we have

$$\begin{aligned}\mathbf{u}_{ijk} =& \frac{\sin(\theta_{ij},\theta_{il},\theta_{jl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})}\cos\theta_{kl}\mathbf{u}_{ijl} - \frac{\sin(\theta_{jk},\theta_{jl},\theta_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})}\cos\theta_{il}\mathbf{u}_{jlk} \\ &+ \frac{\sin(\theta_{ik},\theta_{il},\theta_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})}\cos\theta_{jl}\mathbf{u}_{lki} \\ &+ \frac{\sin(\theta_{ij},\theta_{il},\theta_{jl})\sin(\theta_{jk},\theta_{jl},\theta_{kl})\sin(\theta_{ik},\theta_{il},\theta_{kl})}{\sin(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl})}\mathbf{u}_{ijl}\times\mathbf{u}_{jlk}\times\mathbf{u}_{kli},\end{aligned} \tag{2.178}$$

and in terms of an orthonormal basis,

$$\begin{aligned}\mathbf{u}_{ijk} =& \left(\frac{\sin(\theta_{ij},\theta_{il},\theta_{jl})\cos\theta_{kl}}{\sin(\theta_{ij},\theta_{ik},\theta_{jk})} - \left(\frac{\cos\phi_{jk} - \cos\phi_{jl}\cos\phi_{ij}}{\sin\phi_{jl}}\right)\cos\phi_{jl}\right. \\ &\qquad \left. + \frac{\cos\theta_j l\cos\phi_{il}\sin\phi_{jl}\sin(\phi_{ij},\phi_{jk},\phi_{jl})}{\sin(\phi_{il},\phi_{jl},\phi_{kl})}\right)\mathbf{u}_{ijl} \\ &+ \frac{\cos\phi_{jk} - \cos\phi_{ij}\cos\phi_{jl}}{\sin\phi_{jl}}\mathbf{u}_{jlk}' + \frac{\cos\theta_{jl}\sin(\phi_{ij},\phi_{jk},\phi_{jl})}{\sin\phi_{ij}}\mathbf{u}_{lki}' \\ &+ \sin(\theta_{ij},\theta_{ik},\theta_{il},\theta_{jk},\theta_{jl},\theta_{kl})\mathbf{u}_{ijl}\times\mathbf{u}_{jlk}'\times\mathbf{u}_{kli}'.\end{aligned} \tag{2.179}$$

Comparing the results provides

$$\begin{aligned}\cos\phi_{jk}\cos\phi_{jl}\sin(\theta_{ij},\theta_{ik},\theta_{jk}) = &\cos\phi_{ij}\sin\phi_{jl}\sin(\theta_{ij},\theta_{ik},\theta_{jk})\\ &+\cos\theta_{kl}\sin\phi_{jl}\sin(\theta_{ij},\theta_{il},\theta_{jl})\\ &+\cos^2\phi_{jl}\cos\phi_{ij}\sin(\theta_{ij},\theta_{ik},\theta_{jk})\\ &+\cos\theta_{jl}\cos\phi_{il}\sin^2\phi_{jl}\sin(\theta_{ik},\theta_{il},\theta_{kl}),\end{aligned} \tag{2.180}$$

the hyperspherical analogue of the five-parts formula.

2.7 Angle Addition Formulas

We derive the cosine addition formulae by collapsing a spherical triangle. We then extend this to see what happens when we collapse a hyperspherical tetrahedron. Recall that the generalised sine function of three variables gives the volume of a three-dimensional parallelepiped defined by vectors $\mathbf{n}_i, \mathbf{n}_j$ and $\mathbf{n}_k$ embedded in four-dimensional Euclidean space,

$$\sin(\theta_{ij},\theta_{ik},\theta_{jk}) = \begin{vmatrix} 1 & \cos\theta_{ij} & \cos\theta_{ik}\\ \cos\theta_{ij} & 1 & \cos\theta_{jk}\\ \cos\theta_{ik} & \cos\theta_{jk} & 1\end{vmatrix}^{\frac{1}{2}}. \tag{2.181}$$

When the three vectors $\mathbf{n}_i, \mathbf{n}_j$ and $\mathbf{n}_k$ become coplanar, the volume of the parallelepiped collapses to zero, and hence, so does the generalised sine function. When this occurs

$$\sin^2(\theta_{ij},\theta_{ik},\theta_{jk}) = 0. \tag{2.182}$$

By expanding this out and completing the square in terms of $\cos\theta_{ij}$, this reduces to

$$(\cos\theta_{ij} - \cos\theta_{ik}\cos\theta_{jk})^2 = \sin^2\theta_{ik}\sin^2\theta_{jk}. \tag{2.183}$$

Solving this gives

$$\cos\theta_{ij} = \cos\theta_{ik}\cos\theta_{jk} \pm \sin\theta_{ik}\sin\theta_{jk}. \tag{2.184}$$

This occurs when $\theta_{ij} = \theta ik \mp \theta jk$, hence giving us the standard addition formula for cosine,

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B. \tag{2.185}$$

Similarly, the generalised sine function of six variables gives the volume of a four-dimensional parallelotope defined by vectors $\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k$ and $\mathbf{n}_l$ embedded in five-dimensional Euclidean space,

$$\sin(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) = \begin{vmatrix} 1 & \cos\theta_{ij} & \cos\theta_{ik} & \cos\theta_{il} \\ \cos\theta_{ij} & 1 & \cos\theta_{jk} & \cos\theta_{jl} \\ \cos\theta_{ik} & \cos\theta_{jk} & 1 & \cos\theta_{kl} \\ \cos\theta_{il} & \cos\theta_{jl} & \cos\theta_{kl} & 1 \end{vmatrix}^{\frac{1}{2}}. \tag{2.186}$$

When the four vectors $\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k$ and $\mathbf{n}_l$ become linearly dependent, the volume of the 4-parallelepiped collapses to zero, and hence, so does the generalised sine function. When this occurs, recalling the hyperspherical sine rule,

$$\sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) = 0 \iff k_H = 0. \tag{2.187}$$

This gives

$$\begin{aligned} \sin(\phi_{ij}, \phi_{ik}, \phi_{il}) &= 0 \\ \sin(\phi_{ij}, \phi_{jk}, \phi_{jl}) &= 0 \\ \sin(\phi_{ik}, \phi_{jk}, \phi_{kl}) &= 0 \\ \sin(\phi_{il}, \phi_{jl}, \phi_{kl}) &= 0 \end{aligned}, \tag{2.188}$$

which, in turn from the spherical case implies that the three dihedral angles around any vertex obey the cosine rule.

Both spherical trigonometry and elliptic functions already have many established real world applications. In the next chapter, we will present two applications where hyperspherical trigonometry and its connection with the generalised Jacobi functions may be exploited. These examples in fact turn out to be closely related.

Chapter 3

A Higher-Dimensional Euler Top and the DELL Model

3.1 Introduction

As an example of where the hyperspherical formulae and their connection with the generalised Jacobi functions, derived in the previous chapter, may be used, we consider a multi-dimensional generalisation of the Euler top in relation to Nambu mechanics. We provide a link between this example and the double elliptic model.

3.2 Nambu Mechanics

In 1973 Nambu introduced a generalisation of Hamiltonian mechanics involving multiple Hamiltonians [101]. In his formulation, an m-tuple of "canonical" variables replaces a pair of canonically conjugated co-ordinates, and an m-ary operation, the Nambu bracket, replaces the binary Poisson bracket. This formulation has come to be known as Nambu mechanics. Takhtajan later axiomised Nambu mechanics [132].

Definition 3.2.1 (Nambu-Poisson Manifold) *[132] A smooth manifold X is called a* ***Nambu-Poisson manifold*** *of order m on its function ring $A = \mathbb{C}^{\infty}(X)$ if there exists a map $\{,\ldots,\} : A^{\otimes m} \mapsto A$, a generalised Nambu bracket of order m, with the following properties:*

- *"Skew"-symmetry*

$$\{A_1, A_2, \ldots, A_m\} = (-1)^{\epsilon(p)}\{A_{p(1)}, A_{p(2)}, \ldots, A_{p(m)}\}, \tag{3.1}$$

  where $p(i)$ is the permutation of indices and $\epsilon(p)$ is the parity of the permutation.

- *Derivation (the Leibniz rule)*

$$\{A_1A_2, A_3, \ldots, A_{m+1}\} = A_1\{A_2, A_3, \ldots, A_{m+1}\} + \{A_1, A_3, \ldots, A_{m+1}\}A_2. \tag{3.2}$$

- *Fundamental identity (analogue of Jacobi identity)*

$$\begin{aligned} &\{\{A_1, \ldots, A_m\}, A_{m+1}, \ldots, A_{2m-1}\} \\ &\quad + \{A_m, \{A_1, \ldots, A_{m-1}, A_{m+1}\}, A_{m+2}, \ldots, A_{2m-1}\} \\ &\qquad + \cdots + \{A_m, \ldots, A_{2m-2}, \{A_1, \ldots, A_{m-1}, A_{2m-1}\}\} \\ &\qquad\quad = \{A_1, \ldots, A_{m-1}, \{A_m, \ldots, A_{2m-1}\}\}. \end{aligned} \tag{3.3}$$

Explicitly, note that for m equals two, the Poisson bracket is recovered, and for Nambu brackets of order three, this is a ternary relationship, defined by

- Skew-symmetry

$$\{A_1, A_2, A_3\} = (-1)^{\epsilon(p)}\{A_{p(1)}, A_{p(2)}, A_{p(3)}\}, \tag{3.4}$$

  where $p(i)$ is the permutation of indices and $\epsilon(p)$ is the parity of the permutation.

- Derivation (the Leibniz rule)

$$\{A_1A_2, A_3, A_4\} = A_1\{A_2, A_3, A_4\} + \{A_1, A_3, A_4\}A_2. \tag{3.5}$$

- Fundamental identity (analogue of Jacobi identity)

$$\begin{aligned}&\{\{A_1, A_2, A_3\}, A_4, A_5\} + \{A_3, \{A_1, A_2, A_4\}, A_5\} + \{A_3, A_4, \{A_1, A_2, A_5\}\}\\&= \{A_1, A_2, \{A_3, A_4, A_5\}\}.\end{aligned} \tag{3.6}$$

The dynamics on a Nambu-Poisson manifold are determined by $m-1$ Hamiltonians $H_1, \ldots, H_{m-1}$ and are described by the generalised Nambu-Hamiltonian equations of motion

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H_1, \ldots, H_{m-1}, f\}, \qquad f \in A. \tag{3.7}$$

We restrict ourselves to Nambu's original example of phase space $X = \mathbb{R}^m$ with coordinates $x_1, \ldots, x_m$ and "canonical" Nambu bracket given by

$$\{H_1, \ldots, H_{m-1}, f\} = \sum_{i_2, \ldots, i_m} \epsilon_{i_1 \ldots i_m} \partial_{x_{i_1}} H_1 \ldots \partial_{x_{i_{m-1}}} H_1 \partial_{x_{i_m}} f, \tag{3.8}$$

where $\epsilon_{i_1 \ldots i_m}$ is the m-dimensional Levi Civita symbol [101].

Takhtajan [132] also extended the canonical formalism of Hamiltonian mechanics based on the Poincaré-Cartan integral invariant and the principle of least action [16] to Nambu mechanics. Let $\widetilde{X} = \mathbb{R}^{m+1}$ with coordinates $x_1, \ldots, x_{m+1}$ be the extended phase space to a Nambu-Poisson manifold $X = \mathbb{R}^{>}$.

Definition 3.2.2 *[132] The following $(m-1)$-form $\omega^{(m-1)}$ on $\widetilde{X}$,*

$$\omega^{(m-1)} = x_1 \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_m - H_1 \mathrm{d}H_2 \wedge \cdots \wedge \mathrm{d}H_{m-1} \wedge \mathrm{d}x_{m+1}, \tag{3.9}$$

*is called the **generalised Poincaré-Cartan form for Nambu mechanics**.*

Definition 3.2.3 *The integral of the generalised Poincaré-Cartan form over $(m-1)$-chains in the extended phase space $\widetilde{X}$,*

$$S(C_{m-1}) = \int_{C_{m-1}} \omega^{(m-1)}, \tag{3.10}$$

*is called the **actional function for Nambu mechanics**. Admissible variations are those which do not change projections of the boundary ∂C_{m-1} on the $x_2 x_3 \ldots x_m$-hyperplane.*

The equations of motion in terms of the canonical Nambu bracket then follow from the principle of least action [132], $\delta S = 0$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{H_1, \ldots, H_{m-1}, f\}, \quad f \in A. \tag{3.11}$$

We will now consider the Euler top in terms of this formulation as considered in [94].

3.2.1 The Euler Top

The Euler top describes a solid body, attached to a fixed point, rotating in the absence of any external torque [17, 40]. It was first considered by Euler in 1765 [47], when he considered the object's equations of motion in the frame rotating with the body. Full details on the tops variational formulation and Hamiltonian description in terms of the $so(3)$ Lie-Poisson bracket may be found in [58]. However, it is an alternative formulation that interests us here. In their paper on Nambu quantum mechanics [94], Minic and Tze reformulated the mechanics of this top in terms of a Nambu system of order three, which they then solved in terms of the Jacobi elliptic functions.

The Euler top admits two conserved quantities, the total energy,

$$H_1 = \frac{1}{2}\sum_{i=1}^{3}\frac{1}{I_i}M_i^2, \tag{3.12}$$

and the square of the angular momentum,

$$H_2 = \frac{1}{2}\sum_{i=1}^{3}M_i^2, \tag{3.13}$$

where I_i denote the principal moments of inertia, and M_i the components of the angular momentum. Minic and Tze take both of these conserved quantities to be Hamiltonians for the the system, with the action of the top given by the actional function for Nambu mechanics

$$S(C_2) = \int_{C_2} M_1 \mathrm{d}M_2 \wedge \mathrm{d}M_3 - H_1 \mathrm{d}H_2 \wedge \mathrm{d}t, \tag{3.14}$$

the integral of the generalised Poincaré-Cartan form for Nambu mechanics over permissible two-chains. The equations of motion,

$$\frac{\mathrm{d}M_i}{\mathrm{d}t} = \{H_1, H_2, M_i\}, \qquad i = 1, 2, 3, \tag{3.15}$$

then follow from $\delta S = 0$, where we take $\{H_1, H_2, M_i\}$ to be the canonical Nambu 3-bracket, a ternary bracket given by

$$\{H_1, H_2, M_i\} = \epsilon^{jkl}\partial_{M_j} H_1 \partial_{M_k} H_2 \partial_{M_l} M_i. \tag{3.16}$$

Without loss of generality, taking $I_3 > I_2 > I_1$, this gives the equations of motions for the Euler top to be

$$\frac{\mathrm{d}M_1}{\mathrm{d}t} = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) M_2 M_3, \tag{3.17a}$$

$$\frac{\mathrm{d}M_2}{\mathrm{d}t} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) M_1 M_3, \tag{3.17b}$$

$$\frac{\mathrm{d}M_3}{\mathrm{d}t} = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) M_1 M_2, \tag{3.17c}$$

which for initial time t_0 have solutions of the form

$$M_1(t) = A_1 \mathrm{sn}(K(t - t_0); k), \tag{3.18a}$$

$$M_2(t) = A_2 \mathrm{cn}(K(t - t_0); k), \tag{3.18b}$$

$$M_3(t) = A_3 \mathrm{dn}(K(t - t_0); k), \tag{3.18c}$$

with K a constant, and with

$$A_1^2 = \frac{-K^2 k^2 I_1^2 I_2 I_3}{(I_2 - I_1)(I_3 - I_1)}, \tag{3.19a}$$

$$A_2^2 = \frac{K^2 k^2 I_1 I_2^2 I_3}{(I_3 - I_2)(I_2 - I_1)}, \tag{3.19b}$$

$$A_3^2 = \frac{K^2 I_1 I_2 I_3^3}{(I_3 - I_2)(I_3 - I_1)}, \tag{3.19c}$$

and where the modulus k of the Jacobi elliptic functions is given by

$$k = \frac{I_2 - I_1}{I_3 - I_2} \frac{2H_1 I_3 - H_2}{H_2 - 2I_1 H_1}. \tag{3.20}$$

Using Irwin's connection [62], these solutions may be reparameterised in terms of spherical trigonometry. Note that in this case $\theta_{ij} = \alpha_k$ for all $i, j, k = 1, 2, 3$.

3.3 Multidimensional Euler Top

We consider a higher dimensional analogue of the Euler top, formulating the top's mechanics in terms of a Nambu system of order four, and then solving this system in terms of the generalised Jacobi functions. Consider a four-dimensional spinning top given by the actional function for Nambu mechanics as defined in Definition 3.2.3,

$$S(C_3) = \int_{C_3} M_1 \mathrm{d}M_2 \wedge \mathrm{d}M_3 \wedge \mathrm{d}M_4 - H_1 \mathrm{d}H_2 \wedge \mathrm{d}H_3 \wedge \mathrm{d}t, \tag{3.21}$$

where H_1, H_2 and H_3 are Hamiltonians, given by

$$H_1 = \frac{1}{2}\sum_{i=1}^{4} M_i^2, \tag{3.22}$$

$$H_2 = \frac{1}{2}\sum_{i=1}^{4} \alpha_i M_i^2, \tag{3.23}$$

and

$$H_3 = \frac{1}{2}\sum_{i=1}^{4} \beta_i M_i^2, \tag{3.24}$$

with α_i and β_i constants, and M_i the co-ordinates in the phase space $X = \mathbb{R}^4$. The top's equations of motion are given by the Nambu-Poisson brackets

$$\frac{dM_i}{dt} = \{H_1, H_2, H_3, M_i\}, \tag{3.25}$$

whereby we use the canonical Nambu-Poisson bracket of order four,

$$\{H_1, H_2, H_3, M_i\} = \epsilon^{jklm} \partial_{M_j} H_1 \partial_{M_k} H_2 \partial_{M_l} H_3 \partial_{M_m} M_i. \tag{3.26}$$

This gives the intertwined differential system of four variables

$$\dot{M}_1 = (\alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_3\beta_4 - \alpha_4\beta_3 + \alpha_4\beta_2 - \alpha_2\beta_4)\, M_2 M_3 M_4, \tag{3.27a}$$

$$\dot{M}_2 = (\alpha_1\beta_4 - \alpha_4\beta_1 + \alpha_3\beta_1 - \alpha_1\beta_3 + \alpha_4\beta_3 - \alpha_3\beta_4)\, M_1 M_3 M_4, \tag{3.27b}$$

$$\dot{M}_3 = (\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2\beta_4 - \alpha_4\beta_2 + \alpha_4\beta_1 - \alpha_1\beta_4)\, M_1 M_2 M_4, \tag{3.27c}$$

$$\dot{M}_4 = (\alpha_1\beta_3 - \alpha_3\beta_1 + \alpha_2\beta_1 - \alpha_1\beta_2 + \alpha_3\beta_2 - \alpha_2\beta_3)\, M_1 M_2 M_3. \tag{3.27d}$$

We now see that this system is in fact the four-dimensional case of Fairlie's system in [50], which Ivanov showed to be an $so(4)$ Nahm top [63]. The system was later shown to be related to the generalised Kovalelskaya system [27] by Petrera and Suris [121].

These equations of motion can be integrated in closed form in terms of the generalised Jacobi elliptic functions s, c, d_1 and d_2, giving

$$M_1(t) = A_1 s(K(t-t_0); k_1, k_2), \tag{3.28a}$$

$$M_2(t) = A_2 c(K(t-t_0); k_1, k_2), \tag{3.28b}$$

$$M_3(t) = A_3 d_1(K(t-t_0); k_1, k_2), \tag{3.28c}$$

$$M_4(t) = A_4 d_2(K(t-t_0); k_1, k_2), \tag{3.28d}$$

with t_0 the initial time, and A_i, K constants satisfying

$$\frac{A_1 K}{A_2 A_3 A_4} = \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_3\beta_4 - \alpha_4\beta_3 + \alpha_4\beta_2 - \alpha_2\beta_4, \tag{3.29a}$$

$$-\frac{A_2 K}{A_1 A_3 A_4} = \alpha_1\beta_4 - \alpha_4\beta_1 + \alpha_3\beta_1 - \alpha_1\beta_3 + \alpha_4\beta_3 - \alpha_3\beta_4, \tag{3.29b}$$

$$-\frac{k_1^2 A_3 K}{A_1 A_2 A_4} = \alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_2\beta_4 - \alpha_4\beta_2 + \alpha_4\beta_1 - \alpha_1\beta_4, \tag{3.29c}$$

$$-\frac{k_2^2 A_4 K}{A_1 A_2 A_3} = \alpha_1\beta_3 - \alpha_3\beta_1 + \alpha_2\beta_1 - \alpha_1\beta_2 + \alpha_3\beta_2 - \alpha_2\beta_3. \tag{3.29d}$$

Hence, the motion of the top may be entirely parameterised in terms of the generalised Jacobi elliptic functions, and therefore, through the link with hyperspherical trigonometry derived in section 2.5.2, the angles of hyperspherical tetrahedra. Note again that this identification ensures that $\theta_{ij} = \alpha_k = \phi_{kl}$, for all $i, j, k, l = 1, 2, 3, 4$.

We will now consider another example that utilises the connection between hyperspherical trigonometry and the generalised Jacobi elliptic functions.

3.3.1 Double Elliptic Systems

The so-called DELL, or double-elliptic, model is a conjectured generalisation of the Calogero-Moser and Ruijsenaars-Schneider models (integrable many-body systems), which is elliptic in both the momentum and position variables. So far, only the two-particle model (reducing to one degree of freedom) has been explicitly constructed [28], derived by considering the dual elliptic Calogero-Moser model, and imposing an elliptic position dependence. A possible Hamiltonian for the three-particle model was later suggested in terms of Riemann theta functions [30, 15], supported by numerical evidence in [95], although this has yet to be proven to define the DELL model.

This two-particle model has one co-ordinate variable q, and one momentum variable p. In general, the notion of duality between two arbitrary Hamiltonians, $h(p, q)$ and $H(P, Q)$, may be described by the relationship

$$h(p, q) = f(Q), \tag{3.30a}$$

$$H(P, Q) = F(q), \tag{3.30b}$$

such that there exists an anti-symplectic map

$$\mathrm{d}P \wedge \mathrm{d}Q = -\mathrm{d}p \wedge \mathrm{d}q. \tag{3.31}$$

Assuming the two Hamiltonians can be written in the form

$$h(p, q) = h_0(p) + g^2 h_1(p, q), \tag{3.32a}$$

$$H(P, Q) = H_0(P) + g^2 H_1(P, Q), \tag{3.32b}$$

then the duality conditions, (3.30), may be expressed

$$h_0(Q) = h(p, q), \tag{3.33a}$$

$$H_0(Q) = H(P, Q), \tag{3.33b}$$

$$\frac{\partial h(p, q)}{\partial p} H_0'(q) = h_0'(Q) \frac{\partial H(P, Q)}{\partial P}. \tag{3.33c}$$

In the elliptic Calogero-Moser case, the Hamiltonian is given by

$$h(p, q) = \frac{p^2}{2} + \frac{g^2}{\mathrm{sn}(q; k)}, \tag{3.34}$$

and hence, in this case,

$$h_0(p) = \frac{p^2}{2}. \tag{3.35}$$

A Hamiltonian $H(P,Q) = H_0(q)$ dual in the momentum, such that $H_0(q) = \text{cn}(q;k)$ is chosen, and as a result the duality conditions, (3.30), now become

$$\frac{Q^2}{2} = \frac{p^2}{2} + \frac{g^2}{\text{sn}(q;k)}, \tag{3.36a}$$

$$\text{cn}(q;k) = H(P,Q), \tag{3.36b}$$

$$p\text{cn}'(q;k) = Q\frac{\partial H(P,Q)}{\partial P}. \tag{3.36c}$$

The pair (p,q) can then be eliminated from the triplet, yielding the differential equation

$$\left(\frac{\partial H(P,Q)}{\partial P}\right)^2 = \left(1 - \frac{2g^2}{Q^2} - H^2\right)\left(k'^2 + k^2H^2\right). \tag{3.37}$$

This differential equation may then be solved in terms of the Jacobi elliptic functions, yielding the solution for the Hamiltonian of the dual elliptic Calogero-Moser system to be

$$H(P,Q) = \text{cn}(q;k) = \alpha(Q)\text{cn}\left(P\sqrt{k'^2 + k^2\alpha^2(Q)}; \frac{k\alpha(Q)}{\sqrt{k'^2 + k^2\alpha^2(Q)}}\right), \tag{3.38}$$

with

$$\alpha^2(Q) = \alpha^2_{rat}(Q) = 1 - \frac{2g^2}{Q^2}. \tag{3.39}$$

The DELL Hamiltonian then follows by enforcing an elliptic positional dependence,

$$\alpha^2(Q) = \alpha^2_{ell}(Q;\widetilde{k}) = 1 - \frac{2g^2}{\text{sn}^2(Q;\widetilde{k})}. \tag{3.40}$$

Note that the elliptic curves for q and Q may not necessarily be the same, and so $\widetilde{k} \neq k$ in general. Therefore, the DELL Hamiltonian is given explicitly by

$$H(P,Q) = \text{cn}(q;k) = \alpha_{ell}(Q;\widetilde{k})\text{cn}\left(P\sqrt{k'^2 + k^2\alpha^2_{ell}(Q;\widetilde{k})}; \frac{k\alpha_{ell}(Q;\widetilde{k})}{\sqrt{k'^2 + k^2\alpha^2_{ell}(Q;\widetilde{k})}}\right). \tag{3.41}$$

We observe that the generalised Jacobi elliptic functions in fact provide a more natural parameterisation for this Hamiltonian. Using the identity

$$\mathrm{cn}(k_2'u;\kappa) = \frac{c(u;k_1,k_2)}{d_2(u;k_1,k_2)}, \qquad \kappa^2 = \frac{k_1^2-k_2^2}{k_2'^2}, \tag{3.42}$$

given by Pawellek, it follows that the Hamiltonian takes the much neater form

$$H(P,Q) = \alpha_{ell}(Q;\widetilde{k})\frac{c(P;k_1,k_2)}{d_2(P;k_1,k_2)}, \tag{3.43}$$

with

$$k_1 = k, \tag{3.44a}$$

$$k_2 = k\sqrt{\frac{2g^2}{\mathrm{sn}(Q;\widetilde{k})}}. \tag{3.44b}$$

This system's equations of motion then follow from Hamilton's equations

$$\dot{P} = -\frac{\partial H}{\partial Q} = -\alpha_{ell}'\left(Q;\widetilde{k}\right)\frac{c\,(P;k_1,k_2)}{d_2\,(P;k_1,k_2)}, \tag{3.45a}$$

$$\dot{Q} = \frac{\partial H}{\partial P} = -k_2'^2\alpha_{ell}\left(Q;\widetilde{k}\right)\frac{s\,(P;k_1,k_2)\,d_1\,(P;k_1,k_2)}{d_2^2\,(P;k_1,k_2)}, \tag{3.45b}$$

where

$$\alpha_{ell}'\left(Q;\widetilde{k}\right) = \frac{\mathrm{d}\alpha_{ell}\left(Q;\widetilde{k}\right)}{\mathrm{d}Q} = 4g^2\mathrm{sn}\left(Q;\widetilde{k}\right)\mathrm{cn}\left(Q;\widetilde{k}\right)\mathrm{dn}\left(Q;\widetilde{k}\right). \tag{3.46}$$

Therefore, following the reparameterisation, it makes sense to take a closer look at the model in terms of the generalised Jacobi elliptic functions.

In a later paper [29], the following system of four quadrics in $\mathbb{C}^6$ was chosen to provide a phase space for this two-body double elliptic system:

$$\begin{aligned} Q_1: &\quad x_1^2 - x_2^2 = 1,\\ Q_2: &\quad x_1^3 - x_3^2 = \widetilde{k}^2,\\ Q_3: &\quad 2g^2x_1^2 + x_4^2 + x_5^2 = 1,\\ Q_4: &\quad 2g^2x_1^2 + x_4^2 + k^{-2}x_6^2 = k^{-2}, \end{aligned} \tag{3.47}$$

where g is a coupling constant, and x_i, $i = 1, 2, 3, 4$ are affine coordinates. The first pair of equations provides the embedding of an elliptic curve, while the second pair provides a second elliptic curve which is locally fibred over the first. Note that when $g = 0$, the system simply becomes a pair of elliptic curves embedded in $\mathbb{C}^3 \times \mathbb{C}^3$. The Poisson brackets are given by

$$\{x_i, x_j\} = \epsilon_{ijk_1\ldots k_4} \frac{\partial Q_1}{\partial x_{k_1}} \frac{\partial Q_2}{\partial x_{k_2}} \frac{\partial Q_3}{\partial x_{k_3}} \frac{\partial Q_4}{\partial x_{k_4}}, \tag{3.48}$$

with the polynomials Q_i themselves yielding the Casamirs of the algebra. The relevant Poisson brackets [29] for this system of quadrics are then

$$\{x_1, x_2\} = \{x_1, x_3\} = \{x_2, x_3\} = 0, \tag{3.49a}$$

$$\dot{x}_1 = \{x_1, x_5\} = -x_2x_3x_4x_6, \tag{3.49b}$$

$$\dot{x}_2 = \{x_2, x_5\} = -x_1x_3x_4x_6, \tag{3.49c}$$

$$\dot{x}_3 = \{x_3, x_5\} = -x_1x_2x_4x_6, \tag{3.49d}$$

$$\dot{x}_4 = \{x_4, x_5\} = -g^2x_1x_2x_3x_6, \tag{3.49e}$$

$$\dot{x}_5 = \{x_5, x_5\} = 0, \tag{3.49f}$$

$$\dot{x}_6 = \{x_6, x_5\} = 0. \tag{3.49g}$$

Again, the authors present the solutions to this system in terms of the Jacobi elliptic functions, as

$$x_1 = \frac{1}{\mathrm{sn}(Q;\widetilde{k})}, \tag{3.50a}$$

$$x_2 = \frac{\mathrm{cn}(Q;\widetilde{k})}{\mathrm{sn}(Q;\widetilde{k})}, \tag{3.50b}$$

$$x_3 = \frac{\mathrm{dn}(Q;\widetilde{k})}{\mathrm{sn}(Q;\widetilde{k})}, \tag{3.50c}$$

$$x_4 = \alpha_{ell}(Q;\widetilde{k})\mathrm{sn}\left(P\sqrt{k'^2 + k^2\alpha_{ell}^2(Q;\widetilde{k})}; \frac{k\alpha_{ell}(Q;\widetilde{k})}{\sqrt{k'^2 + k^2\alpha_{ell}^2(Q;\widetilde{k})}}\right), \tag{3.50d}$$

$$x_5 = \alpha_{ell}(Q;\widetilde{k})\mathrm{cn}\left(P\sqrt{k'^2 + k^2\alpha_{ell}^2(Q;\widetilde{k})}; \frac{k\alpha_{ell}(Q;\widetilde{k})}{\sqrt{k'^2 + k^2\alpha_{ell}^2(Q;\widetilde{k})}}\right), \tag{3.50e}$$

$$x_6 = \sqrt{k'^2 + k^2\alpha_{ell}^2(Q;\widetilde{k})}\mathrm{sn}\left(P\sqrt{k'^2 + k^2\alpha_{ell}^2(Q;\widetilde{k})}; \frac{k\alpha_{ell}(Q;\widetilde{k})}{\sqrt{k'^2 + k^2\alpha_{ell}^2(Q;\widetilde{k})}}\right), \tag{3.50f}$$

However, these equations of motion are more naturally solved in terms of the generalised Jacobi elliptic functions, with solutions

$$x_1 = A_1 s(K(t - t_0)|k_1, k_2), \tag{3.51a}$$

$$x_2 = A_2 c(K(t - t_0)|k_1, k_2), \tag{3.51b}$$

$$x_3 = A_3 d_1(K(t - t_0)|k_1, k_2), \tag{3.51c}$$

$$x_4 = A_4 d_2(K(t - t_0)|k_1, k_2), \tag{3.51d}$$

$$x_5 = E, \text{ the energy}, \tag{3.51e}$$

$$x_6 = K, \text{ constant}, \tag{3.51f}$$

with

$$k_1 = \frac{1}{\widetilde{k}}, \tag{3.52a}$$

$$k_2 = \widetilde{k}\sqrt{-2g^2}, \tag{3.52b}$$

$$A_1^2 = \frac{k_1 k_2}{\sqrt{-2g^2}}, \tag{3.52c}$$

$$A_2^2 = \frac{-k_1 k_2}{\sqrt{-2g^2}}, \tag{3.52d}$$

$$A_3^2 = \frac{-k_2}{k_1\sqrt{-2g^2}}, \tag{3.52e}$$

$$A_4^2 = \frac{-k_1\sqrt{-2g^2}}{k_2}, \tag{3.52f}$$

respectively. Note that K is related to the energy, E, through the final two quadrics

$$k^2(1 - E^2) = 1 - K^2. \tag{3.53}$$

This connection through the quadrics supports the assertion that the generalised Jacobi elliptic functions are the natural parameterisation for the two-particle DELL system. Note also, the similarity between the equations of motion derived from the quadrics and those for the earlier four-dimensional Euler top example (3.27). By equating M_j for the four-dimensional Euler top with x_j, for $j = 1, 2, 3, 4$, these models are in fact the same up to scaling.

In the next chapter, we will derive another elliptic integrable system, an elliptic extension of the KP equation. As discussed in Chapter 1, the Calogero-Moser and Ruijsenaars-Schneider systems can be derived through reductions of the fully-discrete and semi-discrete KP equations, respectively. We speculate in the conclusion, Chapter 6, that perhaps the DELL model may be related to a reduction of this elliptic extension of the KP equation.

Chapter 4

The Discrete Elliptic Kadomtsev-Petviashvili Equation

4.1 Introduction

To our knowledge, to date, there exist only four truly elliptic integrable lattice systems of partial difference equations, in the sense that they are naturally associated with an elliptic curve. These are:

- The lattice Landau-Lifshitz equations [111], resulting from a discretisation of the Sklyanin Lax pair [128].
- Adler's lattice Krichever-Novikov system [7], resulting from the permutability condition of the Bäcklund transformations of the Krichever-Novikov equation [80].
- Adler and Yamilov's system, arising from the consideration of Darboux chains [12].
- The elliptic KdV system, resulting from a discrete linearisation scheme with an elliptic Cauchy kernel [112].

Apart from these, there is also a 5-point scalar equation arising from the permutability of the Bäcklund transformations for the Landau-Lifshitz equation [8], although it is not clear as to whether this constitutes a true discretisation of the Landau-Lifshitz equation by itself.

In this chapter, we present an elliptic extension of the lattice potential Kadomtsev-Petviashvili (LPKP) equation (1.54). This is, to our knowledge, the first elliptic lattice system to reside in higher dimensions. We will show that the system is naturally a $(3+1)$-dimensional system, as opposed to the $(2+1)$-dimensional system that may be expected. This is contrary to expectation as there is nothing to suggest generalising to the elliptic case would entail a move to higher dimensions.

The elliptic lattice KP system, which we will refer to as Ell-dKP, is derived through a direct linearisation scheme following a similar method to that used in [112] , employing an infinite matrix structure based on an elliptic Cauchy kernel. This systematic approach for the derivation of integrable lattice equations arose in the series of papers [113, 35, 106]. This scheme proves to be a powerful tool with the ability to provide, amongst others, a Lax representation, Bäcklund and Miura transformations and hierarchies of commuting flows. It relies upon two key objects:

- Linear dynamics (either discrete or continuous) residing in the plane wave factors, denoted ρ_k.
- A Cauchy kernel, $\Omega(k, k')$, through which the connection with the non-linear equations arises.

The resulting integrable lattice system comprises a set of simultaneous equations involving several components. As a result of the multicomponent form, there are various ways of expressing the system. One way is given by the following system of discrete equations:

$$
\begin{aligned}
&(p-\widetilde{u})\left(q-r+\dot{\widetilde{u}}-\widetilde{\widehat{u}}\right)+(q-\widehat{u})\left(r-p+\widehat{\widetilde{u}}-\widehat{\dot{u}}\right)\\
&\qquad+(r-\dot{u})\left(p-q+\dot{\widehat{u}}-\dot{\widetilde{u}}\right)=g\left(\widetilde{\widehat{s}}'\left(\widetilde{s}-\widehat{s}\right)+\widehat{\dot{s}}'\left(\widehat{s}-\dot{s}\right)+\dot{\widetilde{s}}'\left(\dot{s}-\widetilde{s}\right)\right), \qquad (4.1a)\\
&\frac{(p+\dot{u})\,\dot{\widetilde{s}}-(q+\dot{u})\,\widehat{\dot{s}}+\widehat{\dot{w}}-\widetilde{\dot{w}}}{\dot{s}}+\frac{(q+\widetilde{u})\,\widehat{\widetilde{s}}-(r+\widetilde{u})\,\widetilde{\dot{s}}+\widetilde{\dot{w}}-\widetilde{\widehat{w}}}{\widetilde{s}}\\
&\qquad+\frac{(r+\widehat{u})\,\dot{\widehat{s}}-(p+\widehat{u})\,\widehat{\widetilde{s}}+\widehat{\widetilde{w}}-\widehat{\dot{w}}}{\widehat{s}}=0, \qquad (4.1b)\\
&\frac{\left(p-\dot{\widetilde{u}}\right)\dot{s}'-\left(r-\dot{\widetilde{u}}\right)\widetilde{s}'+\dot{w}'-\widetilde{w}'}{\dot{\widetilde{s}}'}+\frac{\left(r-\widehat{\dot{u}}\right)\widehat{s}'-\left(q-\widehat{\dot{u}}\right)\dot{s}'+\widehat{w}'-\dot{w}'}{\widehat{\dot{s}}'}\\
&\qquad+\frac{\left(q-\widetilde{\widehat{u}}\right)\widetilde{s}'-\left(p-\widehat{\widetilde{u}}\right)\widehat{s}'+\widetilde{w}'-\widehat{w}'}{\widetilde{\widehat{s}}'}=0, \qquad (4.1c)\\
&\left(p+u-\frac{\widetilde{w}}{\widetilde{s}}\right)\left(p-\widetilde{u}+\frac{w}{s}\right)=p^2+\left(\widetilde{\underline{U}}_{1,0}-\widetilde{U}_{0,1}\right)\\
&\qquad-\left(U_{1,0}-U_{0,1}\right)+\frac{\widetilde{w}}{\widetilde{s}}\left(\widetilde{u}-\underline{\widetilde{u}}\right)-\left(\frac{1}{\widetilde{s}\underline{s}'}+3e+g\widetilde{s}'s\right), \qquad (4.1d)\\
&\left(q+u-\frac{\widehat{w}}{\widehat{s}}\right)\left(q-\widehat{u}+\frac{w}{s}\right)=q^2+\left(\widehat{\underline{U}}_{1,0}-\widehat{U}_{0,1}\right)\\
&\qquad-\left(U_{1,0}-U_{0,1}\right)+\frac{\widehat{w}}{\widehat{s}}\left(\widehat{u}-\underline{\widehat{u}}\right)-\left(\frac{1}{\widehat{s}\underline{s}'}+3e+g\widehat{s}'s\right), \qquad (4.1e)\\
&\left(r+u-\frac{\dot{w}}{\dot{s}}\right)\left(r-\dot{u}+\frac{w}{s}\right)=r^2+\left(\dot{\underline{U}}_{1,0}-\dot{U}_{0,1}\right)\\
&\qquad-\left(U_{1,0}-U_{0,1}\right)+\frac{\dot{w}}{\dot{s}}\left(\dot{u}-\underline{\dot{u}}\right)-\left(\frac{1}{\dot{s}\underline{s}'}+3e+g\dot{s}'s\right), \qquad (4.1f)\\
&s'\overline{w}=w'\overline{s}. \qquad (4.1g)
\end{aligned}
$$

Here, e and g are fixed parameters, effectively the moduli of an elliptic curve

$$\Gamma : y^2 = 1/x + 3e + gx. \qquad (4.2)$$

In terms of notation to describe the lattice system, for simplicity we let $u = u(n,m,l,N)$ denote the dependent variable for lattice points labeled by the quadruplet $(n,m,l,N) \in \mathbb{Z}^4$. The variables p, q and r are the continuous lattice parameters, associated with the grid size in the lattice directions, given by the independent variables n, m and l, respectively.

The fourth variable N is singled out in that it has an associated lattice parameter, P, which, in order to simplify closing the system, we will later set to $P = 0$ effectively resulting in a (4.1) being a $3+1$-dimensional system. For shifts in these lattice directions, we use the notation

$$\widetilde{u} = u(n+1,m,l,N), \quad \widehat{u} = u(n,m+1,l,N), \quad \dot{u} = u(n,m,l+1,N), \tag{4.3}$$

together with

$$\breve{u} = u(n,m,l,N+1). \tag{4.4}$$

Note that, combined shifts may be represented by multiple diacritics and therefore we have the following

$$\begin{aligned}
\widehat{\widetilde{u}} &= u(n+1,m+1,l,N), & \dot{\widetilde{u}} &= u(n+1,m,l+1,N),\\
\dot{\widehat{u}} &= u(n,m+1,l+1,N), & \widehat{\breve{u}} &= u(n,m+1,l,N+1),\\
\breve{\widetilde{u}} &= u(n+1,m,l,N+1), & \dot{\breve{u}} &= u(n,m,l+1,N+1),\\
\widehat{\dot{\widetilde{u}}} &= u(n+1,m+1,l+1,N), & \widehat{\dot{\breve{u}}} &= u(n,m+1,l+1,N+1),\\
\widehat{\breve{\widetilde{u}}} &= u(n+1,m+1,l,N+1), & \breve{\dot{\widetilde{u}}} &= u(n+1,m,l+1,N+1),
\end{aligned} \tag{4.5}$$

completing the notation for the vertices of an elementary lattice octachoron. Backward shifts are represented by undershifts, i.e.,

$$\underline{u} = u(n,m,l,N-1), \tag{4.6}$$

with repeated undershifts representing repeated backward shifts in the corresponding lattice direction. Upon setting $P = 0$, for clarity we use use the notation $\bar{\cdot}$ for the lattice shift, replacing $\breve{\cdot}$.

Integrability of the system is understood in terms of the existence of a Lax representation and soliton solutions, both of which we derive later.

The equations (4.1) are a system of seven interconnected equations for dependent variables u, s, s', w, w', $U_{0,1}$ and $U_{1,0}$, but the latter two are understood to be eliminated by pairwise combinations of (4.1d, 4.1e, 4.1f). (We prefer to leave $U_{0,1}$ and $U_{1,0}$ in the system to make clear the dependence on the modulus e, although we do include another presentation later in (4.71).) This system is a natural higher-dimensional extension of the lattice potential KP equation [51]. In fact, by setting $g = 0$ in equation (4.1a), the elliptic curve degenerates into a rational curve, and equation (1.54) is recovered. The elliptic KdV equation [112] may also be recovered through a dimensional reduction. We look in more detail at these reductions in section 4.6.

In this chapter, we will provide a derivation of the lattice elliptic KP system, and establish some of its properties.

4.2 Elliptic Matrix Structure

The starting point for the elliptic analogue to the lattice potential KP equation is the algebra of centred infinite elliptic matrices, $\mathcal{A}$. This is an associative algebra, with identity $\mathbf{1}$, which is quasigraded, with grading given by two types of raising operator, $\boldsymbol{\Lambda}$ of degree one, and $\boldsymbol{L}$ of degree two, such that

$$\boldsymbol{\Lambda}^2 = \boldsymbol{L} + 3e\mathbf{1} + g\boldsymbol{L}^{-1}, \quad \boldsymbol{L}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{L}, \tag{4.7}$$

where $e, g \in \mathbb{C}$ are the moduli of the elliptic curve given by this relation [112]. This algebra of centred elliptic matrices was first introduced in [112], with some of the constituents reminiscent of universal Grassman manifolds [131]. We use the term quasigraded as the structure of this algebra is reminiscent of quasigraded Lie algebras, see for example [59] and Krichever-Novikov algebras [81, 82, 83]. Non-elliptic infinite matrices are built from the action of one index raising operator $\boldsymbol{\Lambda}$ and its conjugate, together with the projector $\boldsymbol{O}$. The projector $\boldsymbol{O}$ is the projection matrix on the central entry, with $(\boldsymbol{O} \cdot \boldsymbol{A})_{i,j} = \delta_{i,0}\boldsymbol{A}_{0,j}$ and $(\boldsymbol{A} \cdot \boldsymbol{O})_{i,j} = \delta_{0,j}\boldsymbol{A}_{i,0}$ for any $\boldsymbol{A} \in \mathcal{A}$. In the non-elliptic case, matrices may be given by

$$\boldsymbol{A} = \sum_{i,j} a_{ij}\boldsymbol{\Lambda}^i\boldsymbol{O}\ {}^t\boldsymbol{\Lambda}^j. \tag{4.8}$$

The infinite elliptic matrices are built from the action of two index raising operators $\boldsymbol{\Lambda}$ and $\boldsymbol{L}$ and their conjugates and projector $\boldsymbol{O}$. We adopt the following convention of labeling the entries of these elliptic matrices, $\boldsymbol{A} \in \mathcal{A}$, with $\boldsymbol{A} = (A_{i,j})$ for $i, j \in \mathbb{Z}$, and central entry $A_{0,0}$ as follows:

$$A_{2i,2j} = \left(\boldsymbol{L}^i \boldsymbol{A}\ \ {}^t\boldsymbol{L}^j\right)_{0,0}, \tag{4.9a}$$

$$A_{2i+1,2j} = \left(\boldsymbol{L}^i \boldsymbol{\Lambda} \boldsymbol{A}\ \ {}^t\boldsymbol{L}^j\right)_{0,0}, \tag{4.9b}$$

$$A_{2i,2j+1} = \left(\boldsymbol{L}^i \boldsymbol{A}\ \ {}^t\boldsymbol{\Lambda}\ \ {}^t\boldsymbol{L}^j\right)_{0,0}, \tag{4.9c}$$

$$A_{2i+1,2j+1} = \left(\boldsymbol{L}^i \boldsymbol{\Lambda} \boldsymbol{A}\ \ {}^t\boldsymbol{\Lambda}\ \ {}^t\boldsymbol{L}^j\right)_{0,0}. \tag{4.9d}$$

As such, these infinite elliptic matrices may be given by

$$\begin{aligned}\boldsymbol{A} = \sum_{i,j} & a_{2i,2j} \boldsymbol{L}^i \boldsymbol{O}\ \ {}^t\boldsymbol{L}^j + a_{2i+1,2j} \boldsymbol{L}^i \boldsymbol{\Lambda} \boldsymbol{O}\ \ {}^t\boldsymbol{L}^j \\ & + a_{2i,2j+1} \boldsymbol{L}^i \boldsymbol{O}\ \ {}^t\boldsymbol{\Lambda}\ \ {}^t\boldsymbol{L}^j + a_{2i+1,2j+1} \boldsymbol{L}^i \boldsymbol{\Lambda} \boldsymbol{O}\ \ {}^t\boldsymbol{\Lambda}\ \ {}^t\boldsymbol{L}^j,\end{aligned} \tag{4.10}$$

Subject to (4.7). In order to be able to multiply these infinite elliptic matrices, we impose

$$\boldsymbol{O}\ \ {}^t\boldsymbol{L}^i \boldsymbol{L}^j \boldsymbol{O} = \delta_{ij} \boldsymbol{O}, \tag{4.11a}$$

$$\boldsymbol{O}\ \ {}^t\boldsymbol{\Lambda}\ \ {}^t\boldsymbol{L}^i \boldsymbol{L}^j \boldsymbol{\Lambda} \boldsymbol{O} = \delta_{ij} \boldsymbol{O}, \tag{4.11b}$$

$$\boldsymbol{O}\ \ {}^t\boldsymbol{\Lambda}\ \ {}^t\boldsymbol{L}^i \boldsymbol{L}^j \boldsymbol{O} = 0, \tag{4.11c}$$

$$\boldsymbol{O}\ \ {}^t\boldsymbol{L}^i \boldsymbol{L}^j \boldsymbol{\Lambda} \boldsymbol{O} = 0. \tag{4.11d}$$

In order to make these infinite elliptic matrices more explicit, we consider their action on infinite column vectors in a basis of monomials whose entries are given by

$$(\boldsymbol{c}_\kappa)_\alpha = \begin{cases} K^{\alpha/2}, & \alpha \text{ is even}, \\ kK^{(\alpha+1)/2}, & \alpha \text{ is odd}, \end{cases} \tag{4.12}$$

whereby k, K are the eigenvalues of $\boldsymbol{\Lambda}$ and $\boldsymbol{L}$ respectively

$$\boldsymbol{\Lambda} \boldsymbol{c}_\kappa = k \boldsymbol{c}_\kappa, \quad \boldsymbol{L} \mathbf{c}_\kappa = K \mathbf{c}_\kappa \tag{4.13}$$

and we have introduced the variable κ as a uniformising variable for the curve Γ, (4.2). The corresponding row vector is given by the transpose

$$\left({}^{t}\boldsymbol{c}_{\kappa'} \right)_{\alpha} = \begin{cases} \left(K'\right)^{\alpha/2}, & \alpha \text{ is even}, \\ k' \left(K'\right)^{(\alpha+1)/2}, & \alpha \text{ is odd}, \end{cases} \tag{4.14}$$

with k' and K' the eigenvalues of ${}^{t}\boldsymbol{\Lambda}$ and ${}^{t}\boldsymbol{\Lambda}$ respectively,

$${}^{t}\boldsymbol{c}_{\kappa'} \; {}^{t}\boldsymbol{\Lambda} = k' \boldsymbol{c}_{\kappa'}, \quad {}^{t}\mathbf{c}_{\kappa'} \; {}^{t}\boldsymbol{L} = K' \mathbf{c}_{\kappa'}, \tag{4.15}$$

If we now introduce the Weierstrass $\wp$-function with half-periods ω and ω',

$$\wp\left(\kappa\right) = \wp\left(\kappa; 2\omega, 2\omega'\right), \tag{4.16}$$

this now allows the raising operators to be realised in terms of the Weierstrass $\wp$-function in the form

$$\boldsymbol{L} \leftrightarrow \wp\left(\kappa\right) - e, \quad 2\boldsymbol{\Lambda}\boldsymbol{L} \leftrightarrow \wp'\left(\kappa\right), \tag{4.17}$$

whereby $e = \wp\left(\omega\right)$. A realisation of the left, and right, index raising operators, respectively, then follows:

$$\boldsymbol{\Lambda} \leftrightarrow \zeta\left(\kappa + \omega\right) - \zeta\left(\kappa\right) - \zeta\left(\omega\right) = \frac{1}{2}\frac{\wp'\left(\kappa\right)}{\wp\left(\kappa\right) - e}, \qquad \boldsymbol{L} \leftrightarrow \wp\left(\kappa\right) - e, \tag{4.18a}$$

$${}^{t}\boldsymbol{\Lambda} \leftrightarrow \zeta\left(\kappa' + \omega\right) - \zeta\left(\kappa'\right) - \zeta\left(\omega\right) = \frac{1}{2}\frac{\wp'\left(\kappa'\right)}{\wp\left(\kappa'\right) - e}, \qquad {}^{t}\boldsymbol{L} \leftrightarrow \wp\left(\kappa'\right) - e, \tag{4.18b}$$

where ζ is the Weierstrass ζ-function. Defining $-w''$ as the sum of the half-periods

$$\omega'' := -\omega - \omega', \tag{4.19}$$

together with

$$e := \wp\left(\omega\right), \quad e' := \wp\left(\omega'\right), \quad e'' := \wp\left(\omega''\right), \tag{4.20}$$

ensures that e, e' and $e''$are the branch points for the Weierstrass curve

$$y^2 = 4\left(x - e\right)\left(x - e'\right)\left(x - e''\right), \tag{4.21}$$

which can be rewritten in terms of the pairs of eigenvalues (k, K) and (k', K') using the parameterisation

$$K = \wp(\kappa) - e, \qquad 2kK = \wp'(\kappa), \tag{4.22a}$$

$$K' = \wp(\kappa') - e, \qquad 2k'K' = \wp'(\kappa'), \tag{4.22b}$$

as

$$k^2 = K + 3e + \frac{g}{K}, \qquad k'^2 = K' + 3e + \frac{g}{K'} \tag{4.23}$$

with $g = (e - e')(e - e'')$. For the sake of the construction of the integrable system, we introduce the Cauchy kernel type object

$$\boldsymbol{\Omega} \leftrightarrow \Omega(k, k') = \frac{k - k'}{K - K'} = \frac{1 - g/(KK')}{k + k'} = {}^t\boldsymbol{c}_{\kappa'}\boldsymbol{\Omega}\boldsymbol{c}_\kappa. \tag{4.24}$$

with

$$k = \frac{1}{2}\frac{\wp(\kappa)}{\wp(\kappa) - e}, \qquad K = \wp(\kappa) - e, \tag{4.25a}$$

$$k' = \frac{1}{2}\frac{\wp(\kappa')}{\wp(\kappa') - e}, \qquad K' = \wp(\kappa') - e. \tag{4.25b}$$

Results for the Cauchy kernel (4.24) follow from the following addition formulae for the Weierstrass functions [136]:

$$(\zeta(\kappa + \omega) - \zeta(\kappa) - \zeta(\omega))^2 = \wp(\kappa + \omega) + \wp(\kappa) + e, \tag{4.26a}$$

$$(\wp(\kappa + \omega) - e)(\wp(\kappa) - e) = (e - e')(e - e'') = g. \tag{4.26b}$$

Definition 4.2.1 $\boldsymbol{\Omega}$ *is the formal Cauchy Kernel for the system if it obeys the following identities:*

$$\boldsymbol{\Omega\Lambda} + {}^t\boldsymbol{\Lambda\Omega} = \boldsymbol{O} - g\ {}^t\boldsymbol{L}^{-1}\boldsymbol{O}\boldsymbol{L}^{-1} =: \widehat{\boldsymbol{O}}_1, \tag{4.27a}$$

$$\boldsymbol{\Omega L} - {}^t\boldsymbol{L\Omega} = \boldsymbol{O\Lambda} - {}^t\boldsymbol{\Lambda O}, \tag{4.27b}$$

where $\boldsymbol{O}$ *is the projection matrix on the central element defined earlier.*

Proposition 4.2.2 *Definition 4.2.1 is consistent with the representation given in (4.24).*

Proof

Consider the operators applied to the infinite component vectors $\mathbf{c}_\kappa$ and ${}^t\mathbf{c}_{\kappa'}$ in the following manner:

- First,

$$ {}^t\mathbf{c}_{\kappa'}\left(\mathbf{\Omega\Lambda} + {}^t\mathbf{\Lambda\Omega}\right)\mathbf{c}_\kappa = {}^t\mathbf{c}_{\kappa'}\mathbf{\Omega}\left(\mathbf{\Lambda c}_\kappa\right) + \left({}^t\mathbf{c}_{\kappa'}\ {}^t\mathbf{\Lambda}\right)\mathbf{\Omega c}_\kappa, \tag{4.28} $$

  which using (4.13), becomes

$$ {}^t\mathbf{c}_{\kappa'}\left(\mathbf{\Omega\Lambda} + {}^t\mathbf{\Lambda\Omega}\right)\mathbf{c}_\kappa = {}^t\mathbf{c}_{\kappa'}\left(k+k'\right)\mathbf{\Omega c}_\kappa. \tag{4.29} $$

  By using the definition of the Cauchy Kernel (4.27), and (4.13), the result follows from

$$ {}^t\mathbf{c}_{\kappa'}\left(\mathbf{\Omega\Lambda} + {}^t\mathbf{\Lambda\Omega}\right)\mathbf{c}_\kappa = {}^t\mathbf{c}_{\kappa'}\left(\boldsymbol{O} - g\frac{1}{K'}\boldsymbol{O}\frac{1}{K}\right)\mathbf{c}_\kappa. \tag{4.30} $$

- Second, applying (4.13) and again using the definition of the Cauchy Kernel (4.27), consider

$$ \begin{aligned} {}^t\mathbf{c}_{\kappa'}\left(\mathbf{\Omega}\boldsymbol{L} - {}^t\boldsymbol{L}\mathbf{\Omega}\right)\mathbf{c}_\kappa &= {}^t\mathbf{c}_{\kappa'}\left(K-K'\right)\mathbf{c}_\kappa, \\ &= {}^t\mathbf{c}_{\kappa'}\left(k-k'\right)\mathbf{c}_\kappa. \end{aligned} \tag{4.31} $$

  By now, applying (4.13),

$$ {}^t\mathbf{c}_{\kappa'}\left(\mathbf{\Omega}\boldsymbol{L} + {}^t\mathbf{\Lambda}\boldsymbol{O}\right)\mathbf{c}_\kappa = {}^t\mathbf{c}_{\kappa'}\left(\boldsymbol{O}\mathbf{\Lambda} - {}^t\mathbf{\Lambda}\boldsymbol{O}\right)\mathbf{c}_\kappa, \tag{4.32} $$

  the result follows.

□

4.3 System Dynamics

The dynamics of the system are encoded by $\boldsymbol{C} \in \mathcal{A}$, given by the formal integral

$$\boldsymbol{C} = \iint_D \mathrm{d}\mu\left(\lambda, \lambda'\right)\, \rho_\lambda \mathbf{c}_\lambda \;\; {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'}, \tag{4.33}$$

over an arbitrary domain , D, on the space of variables λ, λ'. ρ_κ and σ'_κ, the plane wave factors, are discrete exponential functions, initially given by

$$\rho_\kappa(n, N) = (p + k)^n (P - K)^N \rho_\kappa(0, 0), \tag{4.34a}$$

$$\sigma_{\kappa'}(n, N) = (p - k')^{-n} (P - K')^{-N} \sigma_{\kappa'}(0, 0), \tag{4.34b}$$

respectively. We will later redefine the plane wave factors to incorporate the other two lattice directions. However, for now we use the simplified versions, (4.34). The factors $\mathbf{c}_\kappa$ and ${}^t\mathbf{c}_{\kappa'}$ are the infinite component vectors defined earlier, in (4.12). The integration measure, $\mathrm{d}\mu\left(\lambda, \lambda'\right)$, is in principle arbitrary, but we assume that basic operations, such as differentiation, and shifts, with respect to the parameters, commute with the integrations. Lattice shifts of these plane wave factors result in

$$\widetilde{\rho}_\kappa = \rho_\kappa(n + 1, N) = (p + k)\rho_\kappa, \quad \widetilde{\sigma}_{\kappa'} = \sigma_{\kappa'}(n + 1, N) = (p - k')^{-1}\sigma_{\kappa'}, \tag{4.35a}$$

$$\breve{\rho}_\kappa = \rho_\kappa(n, N + 1) = (P - K)\rho_\kappa, \quad \breve{\sigma}_{\kappa'} = \sigma_{\kappa'}(n, N + 1) = (P - K')^{-1}\sigma_{\kappa'}, \tag{4.35b}$$

which, for $\boldsymbol{C}$ imply

$$\widetilde{\boldsymbol{C}}\left(p - \;{}^t\boldsymbol{\Lambda}\right) = \left(p + \boldsymbol{\Lambda}\right)\boldsymbol{C}, \tag{4.36a}$$

$$\breve{\boldsymbol{C}}\left(P - \;{}^t\boldsymbol{L}\right) = \left(P - \boldsymbol{L}\right)\boldsymbol{C}. \tag{4.36b}$$

The main object from which the nonlinear equations are obtained is the infinite matrix $\boldsymbol{U} \in \mathcal{A}$, defined by

$$\boldsymbol{U} \equiv \left(\mathbf{1} - \boldsymbol{U}\boldsymbol{\Omega}\right)\boldsymbol{C}. \tag{4.37}$$

with components $U_{i,j}$, together with infinite component vectors, $\boldsymbol{u}_\kappa$, defined through the equations,

$$\boldsymbol{u}_\kappa + \rho_\kappa \boldsymbol{U\Omega c}_\kappa = \rho_\kappa \boldsymbol{c}_\kappa, \tag{4.38a}$$

$${}^t\boldsymbol{u}_{\kappa'} + {}^t\boldsymbol{c}_{\kappa'}\boldsymbol{\Omega}\ {}^t\boldsymbol{U}\sigma_{\kappa'} = {}^t\boldsymbol{c}_{\kappa'}\sigma_{\kappa'}. \tag{4.38b}$$

Proposition 4.3.1 *The infinite matrix $\boldsymbol{U}$ may then be expressed as*

$$\boldsymbol{U} = \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right)\mathbf{u}_\lambda\ {}^t\mathbf{c}_{\lambda'}\sigma_{\lambda'}. \tag{4.39}$$

Proof

Consider $\boldsymbol{U} - \boldsymbol{C}$,

$$\boldsymbol{U} - \boldsymbol{C} = \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right)\left(\mathbf{u}_\lambda\ {}^t\mathbf{c}_{\lambda'}\sigma_{\lambda'} - \rho_\lambda\mathbf{c}_\lambda\ {}^t\mathbf{c}_{\lambda'}\sigma_{\lambda'}\right), \tag{4.40a}$$

$$= \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right)\left(\mathbf{u}_\lambda - \rho_\lambda\mathbf{c}_\lambda\right)\left({}^t\mathbf{c}_{\lambda'}\sigma_{\lambda'}\right). \tag{4.40b}$$

Now, from (4.38a), we have

$$\mathbf{u}_\kappa - \rho_\kappa\mathbf{c}_\kappa = -\rho_\kappa\boldsymbol{U\Omega}\mathbf{c}_\kappa, \tag{4.41}$$

and so $\boldsymbol{U} - \boldsymbol{C}$ becomes

$$\boldsymbol{U} - \boldsymbol{C} = -\boldsymbol{U\Omega}\iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right)\rho_\lambda\mathbf{c}_\lambda\ {}^t\mathbf{c}_{\lambda'}\sigma_{\lambda'}, \tag{4.42a}$$

$$= -\boldsymbol{U\Omega C} \tag{4.42b}$$

as required. □

For this infinite matrix $\boldsymbol{U}$, discrete Riccati type shift relations follow from (4.36) for $\boldsymbol{C}$, giving

$$\widetilde{\boldsymbol{U}}\left(p - {}^t\boldsymbol{\Lambda}\right) = (p + \boldsymbol{\Lambda})\boldsymbol{U} - \widetilde{\boldsymbol{U}}\left(\boldsymbol{O} - g\ {}^t\boldsymbol{L}^{-1}\boldsymbol{O}\boldsymbol{L}^{-1}\right)\boldsymbol{U}, \tag{4.43a}$$

$$\widehat{\boldsymbol{U}}\left(q - {}^t\boldsymbol{\Lambda}\right) = (q + \boldsymbol{\Lambda})\boldsymbol{U} - \widehat{\boldsymbol{U}}\left(\boldsymbol{O} - g\ {}^t\boldsymbol{L}^{-1}\boldsymbol{O}\boldsymbol{L}^{-1}\right)\boldsymbol{U}, \tag{4.43b}$$

$$\dot{\boldsymbol{U}}\left(r - {}^t\boldsymbol{\Lambda}\right) = (r + \boldsymbol{\Lambda})\boldsymbol{U} - \dot{\boldsymbol{U}}\left(\boldsymbol{O} - g\ {}^t\boldsymbol{L}^{-1}\boldsymbol{O}\boldsymbol{L}^{-1}\right)\boldsymbol{U}, \tag{4.43c}$$

$$\breve{\boldsymbol{U}}\left(P - {}^t\boldsymbol{L}\right) = (P - \boldsymbol{L})\boldsymbol{U} + \breve{\boldsymbol{U}}\left(\boldsymbol{O}\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{U}, \tag{4.43d}$$

respectively. In particular, from (4.43d), it follows that

$$\boldsymbol{U}\left(P - {}^t\boldsymbol{L}\right)^{-1} = (P - \boldsymbol{L})^{-1}\breve{\boldsymbol{U}} - \boldsymbol{U}\left(P - {}^t\boldsymbol{L}\right)^{-1}\left(\boldsymbol{O}\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)(P - \boldsymbol{L})^{-1}\breve{\boldsymbol{U}}. \tag{4.44}$$

For what follows in this and the next chapter, we will use only the case $P = 0$, for which we use the notation $\breve{\boldsymbol{U}} = \overline{\boldsymbol{U}}$, and hence, we obtain the relations

$$\boldsymbol{U}\ {}^t\boldsymbol{L}^{-1} = \boldsymbol{L}^{-1}\overline{\boldsymbol{U}} + \boldsymbol{U}\ {}^t\boldsymbol{L}^{-1}\left(\boldsymbol{O}\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{L}^{-1}\overline{\boldsymbol{U}}, \tag{4.45}$$

if and only if

$$\overline{\boldsymbol{U}}\ {}^t\boldsymbol{L} = \boldsymbol{L}\boldsymbol{U} - \overline{\boldsymbol{U}}\left(\boldsymbol{O}\boldsymbol{\Lambda} - {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{U}. \tag{4.46}$$

Using these shift relations we are now able to derive a system of relations in terms of $\boldsymbol{U}$'s matrix entries. We will then close this system. Without setting $P = 0$ it would become difficult to achieve this closure.

4.4 Elliptic Lattice Structure

Having obtained the basic relations in the previous section (4.45, 4.43) in terms of the elliptic matrix $\boldsymbol{U}$, closed-form equations can now be derived in terms of a well-chosen set of entries. To do this we single out the following entries:

$$\begin{aligned} u &= U_{0,0}, & s &= U_{-2,0}, & s' &= U_{0,-2}, \\ h &= U_{-2,-2}, & v &= 1 - U_{-1,0}, & v' &= 1 - U_{0,-1}, \\ & & w &= 1 + U_{-2,1}, & w' &= 1 + U_{1,-2}, \end{aligned} \tag{4.47}$$

Firstly, we can derive a number of relations involving bar-shifted variables. Starting with (4.45), which involves only the $\bar{\cdot}$ shifts we can simply read off

$$s'\overline{v} = \overline{s}v'. \tag{4.48}$$

Other bar-shifted relations may be arrived at by applying powers, $a, b, c, d \in \{0, 1\}$, of the operators $\boldsymbol{\Lambda}$ and $\boldsymbol{L}$ to the left hand side of (4.45), and of their transposes to the right,

$$\begin{aligned} &\boldsymbol{L}^a\boldsymbol{\Lambda}^b\left(\boldsymbol{U}\ \ {}^t\boldsymbol{L}^{-1}\right)\ \ {}^t\boldsymbol{L}^c\ \ {}^t\boldsymbol{\Lambda}^d \\ &\quad = \boldsymbol{L}^a\boldsymbol{\Lambda}^b\left(\boldsymbol{L}^{-1}\overline{\boldsymbol{U}} + \boldsymbol{U}\ \ {}^t\boldsymbol{L}^{-1}\left(\boldsymbol{O}\boldsymbol{\Lambda} -\ \ {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{L}^{-1}\overline{\boldsymbol{U}}\right)\ \ {}^t\boldsymbol{L}^c\ \ {}^t\boldsymbol{\Lambda}^d. \end{aligned} \tag{4.49}$$

Firstly, applying $\boldsymbol{\Lambda}$ to the left implies

$$\begin{aligned} &\boldsymbol{\Lambda}\boldsymbol{U}\ \ {}^t\boldsymbol{L}^{-1} = \boldsymbol{\Lambda}\boldsymbol{L}^{-1}\overline{\boldsymbol{U}} + \boldsymbol{\Lambda}\boldsymbol{U}\ \ {}^t\boldsymbol{L}^{-1}\left(\boldsymbol{O}\boldsymbol{\Lambda} -\ \ {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{L}^{-1}\overline{U}, \\ &\Rightarrow w' - 1 = 1 - \overline{v} + (w' - 1)(1 - \overline{v}) - U_{1,-1}\overline{s}, \\ &\quad \Rightarrow U_{1,-1} = \frac{1 - \overline{v}w'}{\overline{s}}. \end{aligned} \tag{4.50}$$

Similarly, applying ${}^t\boldsymbol{\Lambda}$ to the right implies

$$\overline{U}_{-1,1} = \frac{1 - \overline{v}w'}{s'}. \tag{4.51}$$

By applying $\boldsymbol{L}$ to the left we get,

$$\overline{U}_{-4,0} = \overline{s}U_{-2,-1} + \overline{v}h, \tag{4.52}$$

and similarly, applying ${}^t\boldsymbol{L}$ to the right,

$$U_{0,-4} = s'\overline{U}_{-1,-2} + v'\overline{h}, \tag{4.53}$$

respectively. A number of other relations follow from (4.46) in a similar manner. Applying firstly $\boldsymbol{L}$ to the left gives

$$\overline{U}_{-2,2} = u\overline{w} - \overline{s}u_{1,0}, \tag{4.54}$$

and secondly, ${}^t\boldsymbol{L}$ to the right

$$U_{2,-2} = \overline{u}w' - s'\overline{u}_{0,1}, \tag{4.55}$$

respectively, whilst applying both together yields

$$\overline{s}w' = s'\overline{w}. \tag{4.56}$$

Therefore, from equations (4.48, 4.51, 4.50, 4.56), it follows that

$$\frac{\overline{s}}{s'} = \frac{\overline{v}}{v'} = \frac{\overline{w}}{w'} = \frac{\overline{U}_{-1,1}}{U_{1,-1}}. \tag{4.57}$$

From (4.46) we also have

$$\overline{U}_{0,2} = U_{2,0} - \overline{u}U_{1,0} + \overline{U}_{0,1}u. \tag{4.58}$$

For the other lattice shift directions, we take as a starting point equation (4.43a), from which the following equations are derived:

$$p\left(\widetilde{u} - u\right) + \widetilde{u}u = \widetilde{U}_{0,1} + U_{1,0} + g\widetilde{s'}s, \tag{4.59a}$$

$$p - gh = \frac{\left(p - \widetilde{u}\right)s' + w' - \widetilde{v'}}{\widetilde{s'}}, \tag{4.59b}$$

$$p + g\widetilde{h} = \frac{\left(p + u\right)\widetilde{s} + v - \widetilde{w}}{s}, \tag{4.59c}$$

$$U_{-1,-2} + \widetilde{U}_{-2,-1} = p\left(\widetilde{h} - h\right) - g\widetilde{h}h + \widetilde{\widetilde{s}}s', \tag{4.59d}$$

$$p\left(v - \widetilde{v}\right) = u\widetilde{v} + \widetilde{U}_{-1,1} + gs\left(\widetilde{U}_{-1,-2} + \underline{U}_{-2,-1}\right) + 3es + gv\underline{h}, \tag{4.59e}$$

$$p\left(v' - \widetilde{v'}\right) = \widetilde{u}v' + U_{1,-1} + g\widetilde{s'}\left(U_{-2,-1} + \overline{\widetilde{\overline{U}}}_{-1,-2}\right) + 3e\widetilde{s'} + gv'\overline{h}, \tag{4.59f}$$

$$p\left(\widetilde{w} - w\right) = \underline{\widetilde{u}}\widetilde{w} - \widetilde{s}\left(\underline{\widetilde{U}}_{1,0} + U_{0,1}\right) + U_{-1,1} + 3e\widetilde{s} + g\widetilde{h}w, \tag{4.59g}$$

$$p\left(\widetilde{w'} - w'\right) = \overline{u}w' - s'\left(\widetilde{U}_{1,0} + \overline{U}_{0,1}\right) + \widetilde{U}_{1,-1} + 3es' + gh\widetilde{w'}. \tag{4.59h}$$

These involve only the $\widetilde{\cdot}$ shifts, but similar relations are obtained in an obvious way by replacing p with q and r, and the $\widetilde{\cdot}$ shifts with $\widehat{\cdot}$ and $\dot{\cdot}$ shifts, respectively. By combining the various relations, and eliminating the variables h, v, v', $U_{-1,-2}$ and $U_{-2,-1}$, the closed-form system of partial difference equations given earlier in (4.1) can be found.

In particular, by considering (4.59a) with its analogous equations in the other lattice directions

$$p\,(\widetilde{u} - u) + \widetilde{u}u = \widetilde{U}_{0,1} + U_{1,0} + g\widetilde{s}'s, \tag{4.60a}$$

$$q\,(\widehat{u} - u) + \widehat{u}u = \widehat{U}_{0,1} + U_{1,0} + g\widehat{s}'s, \tag{4.60b}$$

$$r\,(\dot{u} - u) + \dot{u}u = \widetilde{U}_{0,1} + U_{1,0} + g\dot{s}'s, \tag{4.60c}$$

we can combine them to eliminate both $U_{0,1}$ and $U_{1,0}$, by considering the combination $\widehat{(4.60a)} - \dot{(4.60a)} + \dot{(4.60b)} - \widetilde{(4.60b)} + \widetilde{(4.60c)} - \widehat{(4.60c)}$, giving (4.1a),

$$\begin{aligned}(p - \widetilde{u})\left(q - r + \dot{\widetilde{u}} - \widehat{\widetilde{u}}\right) + (q - \widehat{u})\left(r - p + \widetilde{\widehat{u}} - \dot{\widehat{u}}\right)\\ + (r - \dot{u})\left(p - q + \widehat{\dot{u}} - \widetilde{\dot{u}}\right) = g\left(\widetilde{\widehat{s}}'\,(\widetilde{s} - \widehat{s}) + \widehat{\dot{s}}'\,(\widehat{s} - \dot{s}) + \dot{\widetilde{s}}'\,(\dot{s} - \widetilde{s})\right).\end{aligned} \tag{4.61}$$

For the next equations in the system, from (4.59b) and (4.59c), we take as a starting point the equations

$$p - gh = \frac{(p - \widetilde{u})\,s' + w' - \widetilde{v}'}{\widetilde{s}'}, \tag{4.62a}$$

$$q - gh = \frac{(q - \widehat{u})\,s' + w' - \widehat{v}'}{\widehat{s}'}, \tag{4.62b}$$

$$r - gh = \frac{(r - \dot{u})\,s' + w' - \dot{v}'}{\dot{s}'}, \tag{4.62c}$$

$$p + g\widetilde{h} = \frac{(p + u)\,\widetilde{s} + v - \widetilde{w}}{s}, \tag{4.62d}$$

$$q + g\widehat{h} = \frac{(q + u)\,\widehat{s} + v - \widehat{w}}{s}, \tag{4.62e}$$

$$r + g\dot{h} = \frac{(r + u)\,\dot{s} + v - \dot{w}}{s}, \tag{4.62f}$$

and eliminate h, v and v' variables. We can do this in two different ways. First, we consider the combinations $\widehat{(4.62a)} - \widetilde{(4.62b)}$ and $(4.62d) - (4.62e)$, to eliminate v,

$$p - q + g\left(\widetilde{h} - \widehat{h}\right) = \frac{\left(p - \widehat{\widetilde{u}}\right)\widehat{s}' + \widehat{w}' - \left(q - \widehat{\widetilde{u}}\right)\widetilde{s}' - \widetilde{w}'}{\widehat{\widetilde{s}}}, \tag{4.63a}$$

$$p - q + g\left(\widetilde{h} - \widehat{h}\right) = \frac{(p+u)\,\widetilde{s} - \widetilde{w} - (q+u)\,\widehat{s} + \widehat{w}}{s}, \tag{4.63b}$$

which we then equate in order to eliminate h, yielding

$$\begin{aligned} \left((p+u)\,\widetilde{s} - (q+u)\,\widehat{s}\right)\widehat{\widetilde{s}}' - \left(\left(p - \widehat{\widetilde{u}}\right)\widehat{s}' - \left(q - \widehat{\widetilde{u}}\right)\widetilde{s}'\right)s \\ = \left(\widetilde{w}' - \widehat{w}'\right)s - \left(\widetilde{w} - \widehat{w}\right)\widehat{\widetilde{s}}'. \end{aligned} \tag{4.64}$$

Repeating this elimination process for the other pairs of equations gives

$$\begin{aligned} \left((q+u)\,\widehat{s} - (r+u)\,\dot{s}\right)\widehat{\dot{s}}' - \left(\left(q - \widehat{\dot{u}}\right)\dot{s}' - \left(r - \widehat{\dot{u}}\right)\widehat{s}'\right)s \\ = \left(\widehat{w}' - \dot{w}'\right)s - \left(\widehat{w} - \dot{w}\right)\widehat{\dot{s}}', \end{aligned} \tag{4.65a}$$

$$\begin{aligned} \left((r+u)\,\dot{s} - (p+u)\,\widetilde{s}\right)\widetilde{\dot{s}}' - \left(\left(r - \widetilde{\dot{u}}\right)\widetilde{s}' - \left(p - \widetilde{\dot{u}}\right)\dot{s}'\right)s \\ = \left(\dot{w}' - \widetilde{w}'\right)s - \left(\dot{w} - \widetilde{w}\right)\widetilde{\dot{s}}'. \end{aligned} \tag{4.65b}$$

Similarly, considering the combination $(4.62a) - (4.62b) - \overline{\widehat{(4.62d)}} + \overline{\widetilde{(4.62e)}}$ gives

$$\begin{aligned} &\left((p - \widetilde{u})\,\frac{s'\widetilde{\overline{w}}}{\widetilde{w}'} + \left(q + \widetilde{\overline{u}}\right)\widehat{\widetilde{\overline{s}}}\right)\widehat{\overline{s}} - \left(\left(p + \widehat{\overline{u}}\right)\widehat{\widetilde{\overline{s}}} + (q - \widehat{u})\,\frac{s'\widehat{\overline{w}}}{\widehat{w}'}\right)\widetilde{\overline{s}} \\ &= w'\left(\frac{\widehat{\overline{w}}}{\widehat{w}'}\widetilde{\overline{s}} - \frac{\widetilde{\overline{w}}}{\widetilde{w}'}\widehat{\overline{s}}\right) + \widehat{\widetilde{\overline{w}}}\left(\widetilde{\overline{s}} - \widehat{\overline{s}}\right). \end{aligned} \tag{4.66}$$

Versions with the other pairs of lattice parameters and lattice shifts also follow in a similar manner. Second, the v and h variables may be eliminated by considering using only the triplet of equations (4.62a, 4.62b, 4.62c). This elimination is achieved by considering the combination $\widehat{(4.62a)} - \dot{(4.62a)} + \dot{(4.62b)} - \widetilde{(4.62b)} + \widetilde{(4.62c)} - \widehat{(4.62c)}$,

$$\begin{aligned} &\frac{\left(p - \widehat{\widetilde{u}}\right)\widehat{s}' - \left(q - \widehat{\widetilde{u}}\right)\widetilde{s}' + \widehat{w}' - \widetilde{w}'}{\widehat{\widetilde{s}}'} - \frac{\left(p - \dot{\widetilde{u}}\right)\dot{s}' - \left(r - \widetilde{\dot{u}}\right)\widetilde{s}' + \dot{w}' - \widetilde{w}'}{\widetilde{\dot{s}}'} \\ &\qquad + \frac{\left(q - \dot{\widehat{u}}\right)\dot{s}' - \left(r - \widehat{\dot{u}}\right)\widehat{s}' + \dot{w}' - \widehat{w}'}{\widehat{\dot{s}}'} = 0 \end{aligned} \tag{4.67}$$

Similarly, from the the triplet of equations (4.62d, 4.62e, 4.62f), we get

$$
\begin{aligned}
\frac{(p+\widehat{u})\,\widehat{\widetilde{s}} - (r+\widehat{u})\,\widehat{\dot{s}} + \widehat{\dot{w}} - \widehat{\widetilde{w}}}{\widehat{s}} &- \frac{(q+\widetilde{u})\,\widetilde{\widehat{s}} - (r+\widetilde{u})\,\widetilde{\dot{s}} + \widetilde{\dot{w}} - \widetilde{\widehat{w}}}{\widetilde{s}} \\
&+ \frac{(q+\dot{u})\,\dot{\widehat{s}} - (p+\dot{u})\,\dot{\widetilde{s}} + \dot{\widetilde{s}} - \dot{\widehat{s}}}{\dot{s}} = 0, \quad (4.68)
\end{aligned}
$$

A relationship involving the elliptic curve is also needed. This relationship may be derived by considering the product

$$
\left(p + u - \frac{\widetilde{w}}{\widetilde{s}}\right)\left(p - \widetilde{u} - \frac{w}{s}\right), \quad (4.69)
$$

and substituting in (4.59a, 4.59g), leading to (4.1d),

$$
\begin{aligned}
&\left(p + u - \frac{\widetilde{w}}{\widetilde{s}}\right)\left(p - \widetilde{u} - \frac{w}{s}\right) \\
&\quad = p^2 - \left(\frac{1}{\widetilde{s}\underline{s}'} + 3e + g\widetilde{s}'s\right) + \frac{\widetilde{w}}{\widetilde{s}}\left(\widetilde{u} - \underline{u}\right) + \left(\widetilde{\underline{U}}_{1,0} - \widetilde{U}_{0,1}\right) - \left(U_{1,0} - U_{0,1}\right), \quad (4.70)
\end{aligned}
$$

with similar relations existing for the $\widehat{\cdot}$ and $\dot{\cdot}$ lattice directions. As referred to earlier, this representation of the system was chosen, despite containing the extra variables $U_{0,1}$ and $U_{1,0}$, as it clearly demonstrates the dependence on the elliptic moduli. However, both of these may both be eliminated, giving

$$
\begin{aligned}
&\left(p + \widehat{u} - \frac{\widehat{\widetilde{w}}}{\widehat{\widetilde{s}}}\right)\left(p - \widehat{\widetilde{u}} - \frac{\widehat{w}}{\widehat{s}}\right) - \left(p + \dot{u} - \frac{\dot{\widetilde{w}}}{\dot{\widetilde{s}}}\right)\left(p - \dot{\widetilde{u}} - \frac{\dot{w}}{\dot{s}}\right) \\
&\quad + \left(q + \dot{u} - \frac{\dot{\widehat{w}}}{\dot{\widehat{s}}}\right)\left(q - \dot{\widehat{u}} - \frac{\dot{w}}{\dot{s}}\right) - \left(q + \widetilde{u} - \frac{\widetilde{\widehat{w}}}{\widetilde{\widehat{s}}}\right)\left(q - \widetilde{\widehat{u}} - \frac{\widetilde{w}}{\widetilde{s}}\right) \\
&\quad + \left(r + \widetilde{u} - \frac{\widetilde{\dot{w}}}{\widetilde{\dot{s}}}\right)\left(r - \widetilde{\dot{u}} - \frac{\widetilde{w}}{\widetilde{s}}\right) - \left(r + \widehat{u} - \frac{\widehat{\dot{w}}}{\widehat{\dot{s}}}\right)\left(p - \widehat{\dot{u}} - \frac{\widehat{w}}{\widehat{s}}\right) \\
&= \left(\frac{1}{\dot{\widetilde{s}}\underline{\dot{s}}'} + 3e + g\dot{\widetilde{s}}'\dot{s}\right) - \left(\frac{1}{\widehat{\widetilde{s}}\underline{\widetilde{\widehat{s}}}'} + 3e + g\widehat{\widetilde{s}}'\widehat{s}\right) + \left(\frac{1}{\widetilde{\widehat{s}}\underline{\widetilde{\widehat{s}}}'} + 3e + g\widetilde{\widehat{s}}'\widetilde{s}\right) \\
&\quad - \left(\frac{1}{\dot{\widehat{s}}\underline{\dot{s}}'} + 3e + g\dot{\widehat{s}}'\dot{s}\right) + \left(\frac{1}{\widehat{\dot{s}}\underline{\widehat{s}}'} + 3e + g\widehat{\dot{s}}'\widehat{s}\right) - \left(\frac{1}{\widetilde{\dot{s}}\underline{\widetilde{s}}'} + 3e + g\widetilde{\dot{s}}'\widetilde{s}\right) \\
&\quad + \frac{\widehat{w}}{\widehat{s}}\left(\widehat{\widetilde{u}} - \underline{\widehat{\widetilde{u}}}\right) - \frac{\dot{w}}{\dot{s}}\left(\dot{\widetilde{u}} - \underline{\dot{\widetilde{u}}}\right) + \frac{\dot{w}}{\dot{s}}\left(\dot{\widehat{u}} - \underline{\dot{\widehat{u}}}\right) \\
&\quad - \frac{\widetilde{w}}{\widetilde{s}}\left(\widetilde{\widehat{u}} - \underline{\widetilde{\widehat{u}}}\right) + \frac{\widetilde{w}}{\widetilde{s}}\left(\widetilde{\dot{u}} - \underline{\widetilde{\dot{u}}}\right) - \frac{\widehat{w}}{\widehat{s}}\left(\widehat{\dot{u}} - \underline{\widehat{\dot{u}}}\right). \quad (4.71)
\end{aligned}
$$

A similar 'primed' version of this equation also exists,

$$
\begin{aligned}
&\left(p+u-\frac{\widetilde{w}'}{\widetilde{s}'}\right)\left(p-\widetilde{u}-\frac{w'}{s'}\right) \\
&= p^2-\left(\frac{1}{\overline{\widetilde{s}}s'}+3e+g\widetilde{s}'s\right)+\frac{w'}{s'}\left(\widetilde{u}-\overline{\widetilde{u}}\right)+\left(\widetilde{U}_{1,0}-\widetilde{U}_{0,1}\right)-\left(U_{1,0}-\overline{U}_{0,1}\right), \qquad (4.72)
\end{aligned}
$$

which, upon eliminating $U_{0,1}$ and $U_{1,0}$, leads to

$$
\begin{aligned}
&\left(p+\widehat{u}-\frac{\widehat{\widetilde{w}}'}{\widehat{\widetilde{s}}'}\right)\left(p-\widehat{\widetilde{u}}-\frac{\widehat{w}'}{\widehat{s}'}\right)-\left(p+\dot{u}-\frac{\dot{\widetilde{w}}'}{\dot{\widetilde{s}}'}\right)\left(p-\dot{\widetilde{u}}-\frac{\dot{w}'}{\dot{s}'}\right) \\
&\quad+\left(q+\dot{u}-\frac{\dot{\widehat{w}}'}{\dot{\widehat{s}}'}\right)\left(q-\dot{\widehat{u}}-\frac{\dot{w}'}{\dot{s}'}\right)-\left(q+\widetilde{u}-\frac{\widetilde{\widehat{w}}'}{\widetilde{\widehat{s}}'}\right)\left(q-\widetilde{\widehat{u}}-\frac{\widetilde{w}'}{\widetilde{s}'}\right) \\
&\quad+\left(r+\widetilde{u}-\frac{\widetilde{\dot{w}}'}{\widetilde{\dot{s}}'}\right)\left(r-\widetilde{\dot{u}}-\frac{\widetilde{w}'}{\widetilde{s}'}\right)-\left(r+\widehat{u}-\frac{\widehat{\dot{w}}'}{\widehat{\dot{s}}'}\right)\left(p-\widehat{\dot{u}}-\frac{\widehat{w}'}{\widehat{s}'}\right) \\
&\quad=\left(\frac{1}{\dot{\overline{\widetilde{s}}}\dot{s}'}+3e+g\dot{\widetilde{s}}'\dot{s}\right)-\left(\frac{1}{\widehat{\overline{\widetilde{s}}}\widehat{s}'}+3e+g\widehat{\widetilde{s}}'\widehat{s}\right)+\left(\frac{1}{\widetilde{\overline{\widehat{s}}}\widetilde{s}'}+3e+g\widetilde{\widehat{s}}'\widetilde{s}\right) \\
&\quad-\left(\frac{1}{\dot{\overline{\widehat{s}}}\dot{s}'}+3e+g\dot{\widehat{s}}'\dot{s}\right)+\left(\frac{1}{\widehat{\overline{\dot{s}}}\widehat{s}'}+3e+g\widehat{\dot{s}}'\widehat{s}\right)-\left(\frac{1}{\widetilde{\overline{\dot{s}}}\widetilde{s}'}+3e+g\widetilde{\dot{s}}'\widetilde{s}\right) \\
&\qquad+\frac{\widehat{w}'}{\widehat{s}'}\left(\widehat{\widetilde{u}}-\overline{\widehat{\widetilde{u}}}\right)-\frac{\dot{w}'}{\dot{s}'}\left(\dot{\widetilde{u}}-\overline{\dot{\widetilde{u}}}\right)+\frac{\dot{w}'}{\dot{s}'}\left(\dot{\widehat{u}}-\overline{\dot{\widehat{u}}}\right) \\
&\qquad-\frac{\widetilde{w}'}{\widetilde{s}'}\left(\widetilde{\widehat{u}}-\overline{\widetilde{\widehat{u}}}\right)+\frac{\widetilde{w}'}{\widetilde{s}'}\left(\widetilde{\dot{u}}-\overline{\widetilde{\dot{u}}}\right)-\frac{\widehat{w}'}{\widehat{s}'}\left(\widehat{\dot{u}}-\overline{\widehat{\dot{u}}}\right). \qquad (4.73)
\end{aligned}
$$

The system is completed by an equation relating the primed versions of s and w, with their bar-shifted versions

$$s'\overline{w}=w'\overline{s}, \qquad (4.74)$$

derived earlier. This results in a closed-form elliptic lattice system that can be expressed only in terms of the variables u, s, s' and w.

A dual system, similar to (4.1), in terms of the variables h, v, s and s' can also be derived, shadowing the system. This system is given by the following set of equations:

$$
\begin{aligned}
&\left(p - g\widetilde{h}\right)\left(r - q + g\widetilde{\dot{h}} - g\widetilde{\widehat{h}}\right) + \left(q - g\widehat{h}\right)\left(p - r + g\widehat{\widetilde{h}} - g\widehat{\dot{h}}\right) \\
&\qquad + \left(r - g\dot{h}\right)\left(q - p + g\dot{\widehat{h}} - g\dot{\widetilde{h}}\right) = g\left(\widetilde{\widehat{s}}\,(\widetilde{s}' - \widehat{s}') + \widehat{\dot{s}}\,(\widehat{s}' - \dot{s}') + \dot{\widetilde{s}}\,(\dot{s}' - \widetilde{s}')\right),
\end{aligned} \tag{4.75a}
$$

$$
\begin{aligned}
&\frac{\left(p - g\dot{h}\right)\dot{\widetilde{s}}' - \left(q - g\dot{h}\right)\widehat{\dot{s}}' - \widehat{\dot{v}}' + \widetilde{\dot{v}}'}{\dot{s}'} + \frac{\left(q - g\widetilde{h}\right)\widetilde{\widehat{s}}' - \left(r + g\widetilde{h}\right)\widetilde{\dot{s}}' - \widetilde{\dot{v}}' + \widetilde{\widehat{v}}'}{\widetilde{s}'} \\
&\qquad + \frac{\left(r - g\widehat{h}\right)\dot{\widehat{s}}' - \left(p - g\widehat{h}\right)\widetilde{\widehat{s}}' - \widehat{\widetilde{w}}' + \widehat{\dot{w}}'}{\widehat{s}'} = 0,
\end{aligned} \tag{4.75b}
$$

$$
\begin{aligned}
&\frac{\left(p + g\dot{\widetilde{h}}\right)\dot{s} - \left(r + g\dot{\widetilde{h}}\right)\widetilde{s} - \dot{v} + \widetilde{v}}{\dot{\widetilde{s}}} + \frac{\left(r + g\widehat{\dot{h}}\right)\widehat{s} - \left(q + g\widehat{\dot{h}}\right)\dot{s} - \widehat{v} + \dot{v}}{\widehat{\dot{s}}} \\
&\qquad + \frac{\left(q + g\widetilde{\widehat{h}}\right)\widetilde{s} - \left(p + g\widehat{\widetilde{h}}\right)\widehat{s} - \widetilde{v} + \widehat{v}}{\widehat{\widetilde{s}}} = 0,
\end{aligned} \tag{4.75c}
$$

$$
\begin{aligned}
&\left(p - gh + \frac{\widetilde{v}}{\widetilde{s}}\right)\left(p + g\widetilde{h} - \frac{v}{s}\right) = p^2 + g\left(\widetilde{U}_{-2,-1} - \widetilde{U}_{-1,-2}\right) \\
&\qquad - g\left(\underline{U}_{-2,-1} - U_{-1,-2}\right) + \frac{gv}{s}\left(h - \underline{h}\right) - \left(\frac{1}{\underline{\widetilde{s}}' s} + 3e + g\widetilde{s}s'\right),
\end{aligned} \tag{4.75d}
$$

$$
\begin{aligned}
&\left(q - gh + \frac{\widehat{v}}{\widehat{s}}\right)\left(q + g\widehat{h} + \frac{v}{s}\right) = q^2 + g\left(\widehat{U}_{-2,-1} - \widehat{U}_{-1,-2}\right) \\
&\qquad - g\left(\underline{U}_{-2,-1} - U_{-1,-2}\right) + \frac{gv}{s}\left(h - \underline{h}\right) - \left(\frac{1}{\underline{\widehat{s}}' s} + 3e + g\widehat{s}s'\right),
\end{aligned} \tag{4.75e}
$$

$$
\begin{aligned}
&\left(r - gh + \frac{\dot{v}}{\dot{s}}\right)\left(r + g\dot{h} - \frac{v}{s}\right) = r^2 + g\left(\dot{U}_{-2,-1} - \dot{U}_{-1,-2}\right) \\
&\qquad - g\left(\underline{U}_{-2,-1} - U_{-1,-2}\right) + \frac{gv}{s}\left(h - \underline{h}\right) - \left(\frac{1}{\underline{\dot{s}}' s} + 3e + g\dot{s}s'\right),
\end{aligned} \tag{4.75f}
$$

$$
s'\overline{v} = v'\overline{s}, \tag{4.75g}
$$

This system can be seen as the result of applying an automorphism of the curve. In fact, consider a shift in the parameter κ by a half period ω,

$$
\kappa \mapsto \kappa + \omega. \tag{4.76}
$$

As a result, from (4.25), using the periodicity of $\wp(\cdot)$, k maps to

$$k \mapsto \frac{1}{2}\frac{\wp'(\kappa+\omega)}{\wp(\kappa+\omega)-e} = \zeta(\kappa) - \zeta(\kappa+\omega) + \zeta(\omega). \tag{4.77}$$

Using the identity

$$\wp(a+b) = \frac{1}{4}\left(\frac{\wp'(a)-\wp'(b)}{\wp(a)-\wp(b)}\right)^2 - \wp(a) - \wp(b), \tag{4.78}$$

together with (4.26a), this returns

$$k \mapsto -k \tag{4.79}$$

Similarly, again using (4.78),

$$K \mapsto \wp(\kappa+\omega) - e, \tag{4.80a}$$

$$\mapsto k^2 - K - 2e, \tag{4.80b}$$

which using the elliptic curve Γ, (4.2), may be rewritten as

$$K \mapsto \frac{g}{K}. \tag{4.81}$$

Similarly, the pair

$$(k', K') \mapsto \left(-k', \frac{g}{K'}\right). \tag{4.82}$$

Applying these mappings to the Cauchy kernel, (4.24), hence gives

$$\Omega \mapsto \frac{KK'}{g}\Omega \leftrightarrow \frac{1}{g} \quad {}^t\boldsymbol{L}\Omega\boldsymbol{L}, \tag{4.83}$$

and so from this automorphism the system (4.75) follows.

We will now consider some of the properties of the system (4.1).

4.5 Lax Representation

To obtain a Lax representation, we make use of the infinite component vectors, $\boldsymbol{u}_\kappa$, introduced in (4.38a), which can be rewritten as

$$\boldsymbol{u}_\kappa \equiv (\boldsymbol{1} - \boldsymbol{U\Omega})\,\rho_\kappa \boldsymbol{c}_\kappa. \tag{4.84}$$

For these vectors the following set of shift relations can be derived:

Proposition 4.5.1

$$\widetilde{\boldsymbol{u}}_\kappa = (p+\boldsymbol{\Lambda})\,\boldsymbol{u}_\kappa - \widetilde{\boldsymbol{U}}\left(\boldsymbol{O} - g\ \ {}^t\boldsymbol{L}^{-1}\boldsymbol{O}\boldsymbol{L}^{-1}\right)\boldsymbol{u}_\kappa, \tag{4.85a}$$

$$\widehat{\boldsymbol{u}}_\kappa = (q+\boldsymbol{\Lambda})\,\boldsymbol{u}_\kappa - \widehat{\boldsymbol{U}}\left(\boldsymbol{O} - g\ \ {}^t\boldsymbol{L}^{-1}\boldsymbol{O}\boldsymbol{L}^{-1}\right)\boldsymbol{u}_\kappa, \tag{4.85b}$$

$$\dot{\boldsymbol{u}}_\kappa = (r+\boldsymbol{\Lambda})\,\boldsymbol{u}_\kappa - \dot{\boldsymbol{U}}\left(\boldsymbol{O} - g\ \ {}^t\boldsymbol{L}^{-1}\boldsymbol{O}\boldsymbol{L}^{-1}\right)\boldsymbol{u}_\kappa, \tag{4.85c}$$

$$\overline{\boldsymbol{u}}_\kappa = -\boldsymbol{L}\boldsymbol{u}_\kappa + \overline{\boldsymbol{U}}\left(\boldsymbol{O}\boldsymbol{\Lambda} - \ \ {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{u}_\kappa, \tag{4.85d}$$

$$\boldsymbol{u}_\kappa = -\boldsymbol{L}^{-1}\overline{\boldsymbol{u}}_\kappa + \boldsymbol{U}\ \ {}^t\boldsymbol{L}^{-1}\left(\boldsymbol{O}\boldsymbol{\Lambda} - \ \ {}^t\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{L}^{-1}\overline{\boldsymbol{u}}_\kappa. \tag{4.85e}$$

Proof

Start with the definition of $\boldsymbol{u}_\kappa$, (4.84), and consider $\widetilde{\boldsymbol{u}}_\kappa$,

$$\widetilde{\boldsymbol{u}}_\kappa = \left(\boldsymbol{1} - \widetilde{\boldsymbol{U}}\boldsymbol{\Omega}\right)\widetilde{\rho}_\kappa \boldsymbol{c}_\kappa. \tag{4.86}$$

Recalling, from (4.35a, 4.13),

$$\widetilde{\rho}_\kappa = (p+k)\,\rho_\kappa, \tag{4.87a}$$

$$\boldsymbol{\Lambda}\boldsymbol{c}_\kappa = k\boldsymbol{c}_\kappa, \tag{4.87b}$$

this implies

$$\begin{aligned}\widetilde{\boldsymbol{u}}_\kappa &= \left(\boldsymbol{1} - \widetilde{\boldsymbol{U}}\boldsymbol{\Omega}\right)(p+k)\,\rho_\kappa\boldsymbol{c}_\kappa, &(4.88a)\\ &= \left(\boldsymbol{1} - \widetilde{\boldsymbol{U}}\boldsymbol{\Omega}\right)(p+\boldsymbol{\Lambda})\,\rho_\kappa\boldsymbol{c}_\kappa, &(4.88b)\\ &= (p+\boldsymbol{\Lambda})\,\boldsymbol{u}_\kappa + \Big((p+\boldsymbol{\Lambda})\,\boldsymbol{U}\boldsymbol{\Omega} - \widetilde{U}\boldsymbol{\Omega}\,(p+\boldsymbol{\Lambda})\Big). &(4.88c)\end{aligned}$$

Using the relation (4.43a),

$$(p+\boldsymbol{\Lambda})\,\boldsymbol{U} = \widetilde{U}\left(p - \ \ {}^t\boldsymbol{\Lambda}\right) + \widetilde{\boldsymbol{U}}\widehat{\boldsymbol{O}}_1\boldsymbol{U} \tag{4.89}$$

this reduces to

$$\widetilde{\boldsymbol{u}}_\kappa = (p+\boldsymbol{\Lambda})\,\boldsymbol{u}_\kappa - \widetilde{\boldsymbol{U}}\left(\boldsymbol{\Omega}\boldsymbol{\Lambda} + \ \ {}^t\boldsymbol{\Lambda}\boldsymbol{\Omega} - \widehat{\boldsymbol{O}}_1\boldsymbol{U}\boldsymbol{\Omega}\right)\rho_\kappa\boldsymbol{c}_\kappa, \tag{4.90}$$

whereby the result follows, recalling (4.27a),

$$\widehat{\boldsymbol{O}}_1 = \boldsymbol{\Omega}\boldsymbol{\Lambda} + \quad {}^{t}\boldsymbol{\Lambda}\boldsymbol{\Omega}. \tag{4.91}$$

The proof for the $\widehat{\cdot}$ and $\dot{\,}$ shift directions is almost identical, except with the shift directions replaced and the lattice parameters by q and r respectively. For $\overline{\boldsymbol{u}}_\kappa$, we use

$$\overline{\rho}_\kappa = -K\rho_\kappa, \tag{4.92a}$$

$$\boldsymbol{L}\boldsymbol{c}_\kappa = K\boldsymbol{c}_\kappa. \tag{4.92b}$$

This gives

$$\begin{aligned}
\overline{\boldsymbol{u}}_\kappa &= \left(\boldsymbol{1} - \overline{\boldsymbol{U}}\boldsymbol{\Omega}\right)\overline{\rho}_\kappa \boldsymbol{c}_\kappa, && (4.93\text{a})\\
&= \left(\boldsymbol{1} - \overline{\boldsymbol{U}}\boldsymbol{\Omega}\right)(-K)\,\rho_\kappa \boldsymbol{c}_\kappa, && (4.93\text{b})\\
&= -\boldsymbol{L}\boldsymbol{u}_\kappa + \left(\boldsymbol{L}\boldsymbol{U}\boldsymbol{\Omega} - \overline{\boldsymbol{U}}\boldsymbol{\Omega}\boldsymbol{L}\right)\rho_\kappa \boldsymbol{c}_\kappa, && (4.93\text{c})\\
& && (4.93\text{d})
\end{aligned}$$

Recalling the relation (4.46),

$$\boldsymbol{L}\boldsymbol{U} = \overline{\boldsymbol{U}} \quad {}^{t}\boldsymbol{L} + \overline{\boldsymbol{U}}\left(\boldsymbol{O}\boldsymbol{\Lambda} - \quad {}^{t}\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{U} \tag{4.94}$$

this reduces to

$$\overline{\boldsymbol{u}}_\kappa = -\boldsymbol{L}\boldsymbol{u}_\kappa - \overline{\boldsymbol{U}}\left(\boldsymbol{\Omega}\boldsymbol{L} - \quad {}^{t}\boldsymbol{L}\boldsymbol{\Omega} - \left(\boldsymbol{O}\boldsymbol{\Lambda} - \quad {}^{t}\boldsymbol{\Lambda}\boldsymbol{O}\right)\boldsymbol{U}\boldsymbol{\Omega}\right)\rho_\kappa \boldsymbol{c}_\kappa. \tag{4.95}$$

Using the definition (4.27b), the result is obtained. □

Setting $(\boldsymbol{u}_\kappa)_i = \varphi_i$, and introducing the 2-component vector $\boldsymbol{\varphi} = (\varphi_0, \varphi_1)^T$, we can derive the following Lax triplet:

$$\widetilde{\boldsymbol{\varphi}} = A_0\boldsymbol{\varphi} + A_1\underline{\boldsymbol{\varphi}} + J\overline{\boldsymbol{\varphi}}, \tag{4.96a}$$

$$\widehat{\boldsymbol{\varphi}} = B_0\boldsymbol{\varphi} + B_1\underline{\boldsymbol{\varphi}} + J\overline{\boldsymbol{\varphi}}, \tag{4.96b}$$

$$\dot{\boldsymbol{\varphi}} = C_0\boldsymbol{\varphi} + C_1\underline{\boldsymbol{\varphi}} + J\overline{\boldsymbol{\varphi}}, \tag{4.96c}$$

where

$$A_0 = \begin{pmatrix} p - \widetilde{u} & 1 \\ 3e - \overline{U}_{0,1} - \widetilde{U}_{1,0} & p + \overline{u} \end{pmatrix}, \tag{4.97a}$$

$$A_1 = g \begin{pmatrix} -\widetilde{s}'w & \widetilde{s}'s \\ -\widetilde{w}'w & \widetilde{w}'s \end{pmatrix}, \tag{4.97b}$$

$$J = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{4.97c}$$

with B_i and C_i, $i = 0, 1$, equivalent to A_i, but with p and $\widetilde{\cdot}$ shifts, replaced by q and r and $\widehat{\cdot}$ and $\dot{\cdot}$ shifts, respectively. Note that the system possesses a Lax triplet rather than a Lax pair. This is analogous to the Lax representation for the LPKP system in [108], which can then be reduced to a Lax pair. This reduction is not possible in this case. The system (4.96) is subject to a number of pairwise compatibility relations, for example, between the $\widetilde{\cdot}$ and $\widehat{\cdot}$ directions resulting in the condition

$$\begin{aligned}\left(\widehat{A}_0 J + J\overline{B}_0 - \widetilde{B}_0 J - J\overline{A}_0\right)\overline{\boldsymbol{\varphi}} + \left(\widehat{A}_0 B_0 + \widehat{A}_1 J + J\overline{B}_1 - \widetilde{B}_0 A_0 - \widetilde{B}_1 J - J\overline{A}_1\right)\boldsymbol{\varphi} \\ + \left(\widehat{A}_0 B_1 + \widehat{A}_1 \underline{B}_0 - \widetilde{B}_0 A_1 - \widetilde{B}_1 \underline{A}_0\right)\underline{\boldsymbol{\varphi}} + \left(\widehat{A}_1 \underline{B}_1 - \widetilde{B}_1 \underline{A}_1\right)\underline{\underline{\boldsymbol{\varphi}}} = 0.\end{aligned} \tag{4.98}$$

This compatibility condition results in the following system of eight equations:

$$p\left(\widehat{\widetilde{u}}-\widehat{u}\right)-q\left(\widehat{\widetilde{u}}-\widetilde{u}\right)+\widetilde{U}_{1,0}-\widehat{U}_{1,0}+g\widehat{\widetilde{s}}'\left(\widetilde{s}-\widehat{s}\right)+\widehat{\widetilde{u}}\left(\widehat{u}-\widetilde{u}\right)=0, \tag{4.99a}$$

$$\begin{aligned}&(q-\widehat{u})\left(3e-\widehat{\overline{U}}_{0,1}-\widehat{\widetilde{U}}_{1,0}\right)+\left(p+\widehat{\overline{u}}\right)\left(3e-\overline{U}_{0,1}-\widehat{U}_{1,0}\right)\\ &\qquad-(p-\widetilde{u})\left(3e-\widetilde{\overline{U}}_{0,1}-\widetilde{\widehat{U}}_{1,0}\right)-\left(q+\widetilde{\overline{u}}\right)\left(3e-\overline{U}_{0,1}-\widetilde{U}_{1,0}\right)\\ &\qquad\qquad=g\left(\widehat{\widetilde{w}}'\left(\widehat{s}-\widetilde{s}\right)+\overline{w}\left(\widetilde{\overline{s}}'-\widehat{\overline{s}}'\right)\right),\end{aligned} \tag{4.99b}$$

$$p\left(u-\widetilde{u}\right)-q\left(u-\widehat{u}\right)+\widetilde{U}_{0,1}-\widehat{U_{0,1}}+gs\left(\widetilde{s}'-\widehat{s}'\right)+u\left(\widehat{u}-\widetilde{u}\right)=0, \tag{4.99c}$$

$$\begin{aligned}&g\left(\widehat{s}'w\left(p-\widehat{\widetilde{u}}\right)+\widehat{w}'w+\widehat{\widetilde{s}}'\,\widehat{w}\left(q-\underline{\widehat{u}}\right)-\widehat{\widetilde{s}}'\,\widehat{s}\left(3e-U_{0,1}-\underline{\widehat{U}}_{1,0}\right)\right.\\ &\left.-\widetilde{s}'w\left(q-\widetilde{\widehat{u}}\right)-\widetilde{w}'w-\widetilde{\widehat{s}}'\,\widehat{w}\left(p-\underline{\widetilde{u}}\right)+\widetilde{\widehat{s}}'\,\widehat{s}\left(3e-U_{0,1}-\underline{\widetilde{U}}_{1,0}\right)\right)=0,\end{aligned} \tag{4.99d}$$

$$\begin{aligned}&g\left(\widehat{s}'s\left(p-\widehat{\widetilde{u}}\right)+\widehat{w}'s-\widehat{\widetilde{s}}'\,\widehat{w}+\widehat{\widetilde{s}}'\,\widehat{s}\left(q+u\right)\right.\\ &\qquad\qquad\left.-\widetilde{s}'s\left(q-\widetilde{\widehat{u}}\right)-\widetilde{w}'s+\widetilde{\widehat{s}}'\,\widetilde{w}-\widetilde{\widehat{s}}'\,\widetilde{s}\left(p+u\right)\right)=0,\end{aligned} \tag{4.99e}$$

$$\begin{aligned}&g\biggl(-\widehat{s}'w\left(3e-\widehat{\overline{U}}_{0,1}-\widehat{\widehat{U}}_{0,1}\right)-\widehat{w}'w\Bigl(\bigl(p+\widehat{\overline{u}}\Bigr)+\widehat{s}\widehat{\widetilde{w}}'\left(3e-U_{0,1}-\underline{\widehat{U}}_{0,1}\right)\\ &\qquad-\widetilde{\widehat{w}}'\,\widehat{w}\bigl(\bigl(q-\underline{\widehat{u}}\bigr)+\widetilde{s}'w\left(3e-\widetilde{\overline{U}}_{0,1}-\widetilde{\widehat{U}}_{0,1}\right)+\widetilde{w}'w\Bigl(\bigl(q+\widetilde{\overline{u}}\Bigr)\\ &\qquad\qquad-\widetilde{s}\widetilde{\widehat{w}}'\left(3e-U_{0,1}-\underline{\widetilde{U}}_{0,1}\right)+\widetilde{\widehat{w}}'\,\widetilde{w}\bigl(\bigl(p-\underline{\widetilde{u}}\bigr)\biggr)=0,\end{aligned} \tag{4.99f}$$

$$\begin{aligned}&g\left(\widehat{s}'s\left(3e-\widehat{\overline{U}}_{0,1}-\widehat{\widehat{U}}_{0,1}\right)+\widehat{w}'s\left(p+\widehat{\overline{u}}\right)-\widehat{\widetilde{w}}'\,\widehat{w}+\widehat{\widetilde{w}}'\,\widehat{s}\left(q+u\right)\right.\\ &\left.-\widetilde{s}'s\left(3e-\widetilde{\overline{U}}_{0,1}-\widetilde{\widehat{U}}_{0,1}\right)-\widetilde{w}'s\left(q+\widetilde{\overline{u}}\right)+\widetilde{\widehat{w}}'\,\widetilde{w}-\widetilde{\widehat{w}}'\,\widetilde{s}\left(p+u\right)\right)=0,\end{aligned} \tag{4.99g}$$

$$s'\overline{w}=w'\overline{s}. \tag{4.99h}$$

Similar compatibility conditions also exist between the $\widetilde{\cdot}$ and $\dot{\cdot}$ shifts, and the $\widehat{\cdot}$ and $\dot{\cdot}$ shifts. It may be verified using MAPLE that the system (4.1) then follows from these compatibility conditions.

4.6 Dimensional Reduction and Degeneration

In this section, we explain how the lattice Potential KP equation and the Elliptic KdV system may be recovered under particular limits of the system (4.1).

4.6.1 Degeneration of the Elliptic Curve

Taking the limit $g \to 0$ causes the elliptic curve to degenerate, and from (4.1a) we recover the lattice potential KP equation[51],

$$\begin{aligned}(p-\widetilde{u})\left(q-r+\widetilde{\dot{u}}-\widetilde{\widehat{u}}\right)+(q-\widehat{u})\left(r-p+\widehat{\widetilde{u}}-\widehat{\dot{u}}\right)+\\ (r-\dot{u})\left(p-q+\dot{\widehat{u}}-\dot{\widetilde{u}}\right)=0,\end{aligned} \tag{4.100}$$

which first appeared in [108]. In contrast to the bilinear lattice KP of [56], this has a continuum limit directly to the potential KP equation.

4.6.2 Dimensional Reduction

As for the discrete elliptic KdV system, given in [112], this class of systems requires that the infinite matrix $\boldsymbol{C}$, and hence also the infinite matrix $\boldsymbol{U}$, are symmetric under transposition,

$${}^{t}\boldsymbol{C}=\boldsymbol{C} \Rightarrow \quad {}^{t}\boldsymbol{U}=\boldsymbol{U}, \tag{4.101}$$

i.e. $U_{i,j}=U_{j,i}$. This restriction ensures that the primed, and bar shifted, variables become equal to their unprimed, and unbarred, equivalents, respectively,

$$\begin{aligned} s'=s, \quad v'=v \quad w'=w,\\ \overline{u}\to u, \quad \overline{s}\to s \quad \overline{w}\to w,\end{aligned} \tag{4.102}$$

and results in a dimensional reduction,

$$r \to 0, \qquad\qquad \dot{u}=u, \tag{4.103}$$

from which the elliptic lattice KdV system is recovered from (4.1). Applying this dimensional reduction to (4.1a) gives

$$\left(p+q+u-\widehat{\widetilde{u}}\right)(p-q+\widetilde{u}-\widehat{u})=p^2-q^2+g\,(\widetilde{s}-\widehat{s})\left(\widehat{\widetilde{s}}-s\right). \tag{4.104}$$

For the next two equations in the elliptic KdV system we consider the alternative presentation (4.64, 4.66) for simplicity, as opposed to the presentation (4.1b, 4.1c). From equation (4.64) under the dimensional reduction it follows

$$((p+u)\,\widetilde{s}-(q+u)\,\widehat{s})\,\widehat{\widetilde{s}}-\left(\left(p-\widehat{\widetilde{u}}\right)\widehat{s}-\left(q-\widehat{\widetilde{u}}\right)\widetilde{s}\right)s=\left(\widehat{\widetilde{s}}-s\right)(\widetilde{w}-\widehat{w}). \tag{4.105}$$

Similarly, from equation (4.66), it follows

$$\left((p-\widetilde{u})+(q+\widetilde{u})\,\widehat{\widetilde{s}}\right)\widehat{s}-\left((p+\widehat{u})\,\widehat{\widetilde{s}}+(q-\widehat{u})\,s\right)\widetilde{s}=(\widehat{s}-\widetilde{s})\left(\widehat{\widetilde{w}}-w\right). \tag{4.106}$$

From equations (4.1d, 4.1e) it follows that

$$\left(p+u-\frac{\widetilde{w}}{\widetilde{s}}\right)\left(p-\widetilde{u}+\frac{w}{s}\right)=p^2-\left(\frac{1}{s\widetilde{s}}+3e+gs\widetilde{s}\right), \tag{4.107a}$$

$$\left(q+u-\frac{\widehat{w}}{\widehat{s}}\right)\left(q-\widehat{u}+\frac{w}{s}\right)=q^2-\left(\frac{1}{s\widehat{s}}+3e+gs\widehat{s}\right), \tag{4.107b}$$

respectively. Equations (4.104, 4.105, 4.106, 4.107a, 4.107b) entirely comprise the elliptic KdV system.

4.7 Soliton Type Solutions

As a concrete application of the infinite matrix scheme used to derive the lattice system, it is relatively straightforward to construct soliton type solutions. Introducing the $\mathcal{N}$ by $\mathcal{N}'$ matrix M, defined by

$$\boldsymbol{M}=\Omega(k_i,k'_j)\boldsymbol{r}\ {}^t\boldsymbol{s}, \tag{4.108}$$

with entries

$$M_{ij}=\frac{1-g/K_iK'_j}{k_i+k'_j}\rho_i\sigma_j, \qquad (i=1,\ldots,\mathcal{N};\,j=1,\ldots,\mathcal{N}'), \tag{4.109}$$

where the parameters of the solutions (k_i, K_i) and (k'_j, K'_j) are points on the elliptic curve Γ, as in (4.2), and the vectors $\boldsymbol{r}$, ${}^t\boldsymbol{s}$ are given by

$$\boldsymbol{r} = \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_{\mathcal{N}} \end{pmatrix}, \tag{4.110a}$$

$${}^t\boldsymbol{s} = \begin{pmatrix} \sigma_1, & \cdots & \sigma_{\mathcal{N}'} \end{pmatrix}. \tag{4.110b}$$

In order to obtain the soliton type solutions we take the infinite matrix $\boldsymbol{C}$ to be a finite rank, $\mathcal{N}'$ by $\mathcal{N}$ matrix of the form

$$\boldsymbol{C} = \sum_{i=1}^{\mathcal{N}'} \sum_{j=1}^{\mathcal{N}} \rho_i \boldsymbol{c}_{\kappa_i} \boldsymbol{c}_{\kappa} \ {}^t\boldsymbol{c}_{\kappa_j} \sigma_j. \tag{4.111}$$

We also define diagonal matrices

$$\boldsymbol{k} = \text{diag}(k_1, k_2, \ldots, k_{\mathcal{N}}), \qquad \boldsymbol{K} = \text{diag}(K_1, K_2, \ldots, K_{\mathcal{N}}), \tag{4.112a}$$

$$\boldsymbol{k'} = \text{diag}(k'_1, k'_2, \ldots, k'_{\mathcal{N}'}), \qquad \boldsymbol{K'} = \text{diag}(K'_1, K'_2, \ldots, K'_{\mathcal{N}'}). \tag{4.112b}$$

This leads to the following explicit solutions:

$$u = \ {}^t\boldsymbol{s}(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{Cr}, \tag{4.113a}$$

$$s = \ {}^t\boldsymbol{s}\boldsymbol{K}'^{-1}(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{Cr}, \tag{4.113b}$$

$$s' = \ {}^t\boldsymbol{s}(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{CK}^{-1}\boldsymbol{r}, \tag{4.113c}$$

$$h = \ {}^t\boldsymbol{s}\boldsymbol{K}'^{-1}(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{CK}^{-1}\boldsymbol{r}, \tag{4.113d}$$

$$v = 1 - \ {}^t\boldsymbol{s}\boldsymbol{K}'^{-1}\boldsymbol{k}'(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{Cr}, \tag{4.113e}$$

$$v' = 1 - \ {}^t\boldsymbol{s}(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{CkK}^{-1}\boldsymbol{r}, \tag{4.113f}$$

$$w = 1 + \ {}^t\boldsymbol{s}\boldsymbol{K}'^{-1}(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{Ckr}, \tag{4.113g}$$

$$w' = 1 + \ {}^t\boldsymbol{s}\,\boldsymbol{k}'(\boldsymbol{1} + \boldsymbol{CM})^{-1}\boldsymbol{CK}^{-1}\boldsymbol{r}. \tag{4.113h}$$

These solutions can be regarded as soliton type solutions in that upon the degeneration of the elliptic curve, they reduce to the Hirota-type solitons for the bilinear lattice KP of [56]. Note that, although the dynamics themselves, i.e. the evolution of the system through its independent variables, residing in the plane wave factors, ρ_i and σ_j, do not explicitly involve the elliptic curve, the soliton solutions are essentially dependent on the variables on the curve.

4.8 Conclusion

We have derived from a direct linearisation scheme a $3 + 1$ dimensional lattice system, naturally associated with an elliptic curve, as an extension of the lattice potential KP equation. We have also shown this system to be integrable through the existence of a Lax representation and soliton solutions. To our knowledge, this is the first lattice system associated with an elliptic curve that has been proposed in higher dimensions. In the next chapter, we will derive the system's continuous analogue.

Chapter 5

The Continuous Elliptic Kadomtsev-Petviashvili Equation

5.1 Introduction

Many discrete integrable lattice systems possess analogous compatible continuous systems, with these systems forming continuous symmetries for the lattice systems, whilst in turn these lattice systems constitute discrete symmetries for the corresponding continuous flows. In particular, the potential Kadomtsev-Petviashvili Equation,

$$\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u_x^2\right)_x = u_{yy}, \tag{5.1}$$

forms the continuous analogue to the lattice potential Kadomtsev-Petviashvili equation (4.1a). This elliptic lattice KP system is no different, in that it also possesses an analogous compatible continuous system. In this chapter, we provide some of the simplest of these continuous flows associated with the lattice system, and discuss some of the continuous system's properties. The system is an elliptic extension of the continuous (potential) KP equation. The only example of such a KP-type system in the continuous setting was given by Date, Jimbo and Miwa [41]. We discuss the difference between our system and the system given in [41] in section 5.5. There are various elliptic integrable systems of continuous type (including the Krichever-Novikov equation [80], the Landau-Lifshitz equation [85], Krichever's elliptic Toda [79] and various others) in $1+1$-dimensions.

An analogous continuous system to the elliptic lattice KP system (4.1), associated with the same elliptic curve (4.2), is given by

$$\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}\left(u_x\right)^2 + \frac{3}{2}gs_xs'_x\right)_x = \frac{3}{4}u_{yy} + \frac{3}{2}g\left(s'_xs_y - s'_ys_x\right), \tag{5.2a}$$

$$\begin{aligned}(ss')_t = \frac{1}{4}\left(s_{xxx}s' + s'_{xxx}s\right) + \frac{3}{2}u_x\left(ss'\right)_x - 3us_xs'_x + \frac{3}{4}\left(s's_y - ss'_y - s_xs'_x\right)_x \\ + \frac{3}{2}\left(us's_y - uss'_y + w'_ys - w_ys'\right) + \frac{3}{2}\left(w_xs'_x + w'_xs_x\right),\end{aligned} \tag{5.2b}$$

$$(ss')_y = 2sw'_x - 2s'w_x + 2u\left(s_xs' - s'_xs\right) + s_{xx}s' - s'_{xx}s, \tag{5.2c}$$

$$\left(u + \frac{w}{s}\right)_x + \left(u - \frac{w}{s}\right)^2 + \frac{w}{s}\left(u - \underline{u}\right) = \left(\frac{1}{\underline{s}'s} + 3e + gss'\right) + U_{1,0} - \underline{U}_{1,0}, \tag{5.2d}$$

$$\left(u + \frac{w'}{s'}\right)_x + \left(u - \frac{w'}{s'}\right)^2 + \frac{w'}{s'}\left(u - \overline{u}\right) = \left(\frac{1}{\overline{s}'s} + 3e + gss'\right) + U_{0,1} - \overline{U}_{0,1}, \tag{5.2e}$$

$$s'\overline{w} = w'\overline{s}. \tag{5.2f}$$

In fact, this can be obtained from a continuum limit of the lattice system. Note that this system is dependent on three continuous variables, x, y and t, and one discrete variable, $\boldsymbol{U} \mapsto \overline{\boldsymbol{U}}$. As in the previous discrete case in Chapter 4, the elimination of the variables $U_{1,0}$ and $U_{0,1}$ disguises the dependence on the elliptic modulus e. This may be achieved by differentiating both equations (5.2d) and (5.2e) with respect to x, and substituting in equations (5.41a) and (5.41b) below, respectively. Again the degeneration of the elliptic curve results in the potential KP equation (5.1), whilst the elliptic potential KdV equation [112] is recovered through a dimensional reduction.

5.2 Continuous Elliptic System

For the continuous case, we identify corresponding continuous flows to the the discrete flows discussed in chapter 4, and so the dynamics of the system are again encoded by the same parameter family of elements, $\boldsymbol{C} \in \mathcal{A}$, given by the formal integral

$$\boldsymbol{C} = \iint_D \mathrm{d}\mu\left(\lambda, \lambda'\right)\, \rho_\lambda \mathbf{c}_\lambda \quad {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'}, \tag{5.3}$$

but with the plane wave factors, ρ_k and $\sigma_{k'}$, now continuous exponential functions, given by

$$\rho_k = \exp\left(\sum_{j\in\mathbb{Z}} k^j x_j\right), \tag{5.4a}$$

$$\sigma_{k'} = \exp\left(-\sum_{j\in\mathbb{Z}} \left(-k'\right)^j x_j\right). \tag{5.4b}$$

The integral is again over an arbitrary domain in $\mathbb{C}^2$, D, on the space of variables λ, λ', with factors $\mathbf{c}_k$ and ${}^t\mathbf{c}_{k'}$, the infinite component vectors defined earlier in (4.12). Again, the integration measure, $\mathrm{d}\mu\left(\lambda, \lambda'\right)$, is in principle arbitrary, but we assume that basic operations, such as differentiation, and shifts, with respect to the parameters, commute with the integrations.

The dynamics of $\boldsymbol{C}$ themselves follow from the derivatives of these plane wave factors with respect to the variables x_i. The derivative of ρ_k with respect to x_i is given by

$$\begin{aligned} \frac{\partial}{\partial x_i}\left(\rho_k\right) &= \frac{\partial}{\partial x_i}\left(\exp\left(\sum_{j\in\mathbb{Z}} k^j x_j\right)\right), \\ &= \left(\frac{\partial}{\partial x_i}\left(\sum_{j\in\mathbb{Z}} k^j x_j\right)\right)\left(\exp\left(\sum_{j\in\mathbb{Z}} k^j x_j\right)\right), \\ &= k^i \rho_k. \end{aligned} \tag{5.5}$$

Similarly, for the derivative of $\sigma(k')$, we have

$$\frac{\partial}{\partial x_i}\left(\sigma_{k'}\right) = -\left(-k'\right)^i \sigma_{k'}. \tag{5.6}$$

From these derivatives of ρ_k and $\sigma_{k'}$, the derivative of $\boldsymbol{C}$ with respect to x_i then follows. This is given by

$$\begin{aligned}
\frac{\partial \boldsymbol{C}}{\partial x_i} &= \frac{\partial}{\partial x_i} \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \rho_\lambda \sigma_{\lambda'} \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'}, \\
&= \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \left(\frac{\partial \rho_\lambda}{\partial x_i} \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'} + \rho_\lambda \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \frac{\partial \sigma_{\lambda'}}{\partial x_i} \right), \\
&= \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \left(\lambda^i \rho_\lambda \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'} - \left(-\lambda'\right)^i \rho_\lambda \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'} \right), \\
&= \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \rho_\lambda \left(\lambda^i \mathbf{c}_\lambda\right) \ {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'} \\
&\qquad - \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \rho_\lambda \mathbf{c}_\lambda \left(\ {}^t\mathbf{c}_{\lambda'} \left(-\lambda\right)^i \right) \sigma_{\lambda'}, \\
&= \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \rho_\lambda \boldsymbol{\Lambda}^i \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'} \\
&\qquad - \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \rho_\lambda \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \left(- \ {}^t\boldsymbol{\Lambda}\right)^i \sigma_{\lambda'}, \\
&= \boldsymbol{\Lambda}^i \iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \rho_\lambda \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'} \\
&\qquad - \left(\iint_D \mathrm{d}\mu\left(\lambda,\lambda'\right) \rho_\lambda \mathbf{c}_\lambda \ {}^t\mathbf{c}_{\lambda'} \sigma_{\lambda'} \right) \left(- \ {}^t\boldsymbol{\Lambda}\right)^i, \\
&= \boldsymbol{\Lambda}^i \boldsymbol{C} - \boldsymbol{C} \left(- \ {}^t\boldsymbol{\Lambda}\right)^i.
\end{aligned} \tag{5.7}$$

Introducing, as before, the infinite matrix $\boldsymbol{U}$ which is related to $\boldsymbol{C}$ via the relationship

$$\boldsymbol{U} = \boldsymbol{C} - \boldsymbol{U}\boldsymbol{\Omega}\boldsymbol{C}, \tag{5.8}$$

it can be shown that

$$\frac{\partial \boldsymbol{U}}{\partial x_i} = \frac{\partial \boldsymbol{C}}{\partial x_i} - \frac{\partial \boldsymbol{U}}{\partial x_i} \boldsymbol{\Omega}\boldsymbol{C} - \boldsymbol{U}\boldsymbol{\Omega} \frac{\partial \boldsymbol{C}}{\partial x_i}, \tag{5.9}$$

and hence,

$$\frac{\partial \boldsymbol{U}}{\partial x_i} = \left(\mathbf{1} - \boldsymbol{U}\boldsymbol{\Omega}\right) \frac{\partial \boldsymbol{C}}{\partial x_i} \left(\mathbf{1} + \boldsymbol{\Omega}\boldsymbol{C}\right)^{-1}. \tag{5.10}$$

Indeed, from the relationship between $\boldsymbol{U}$ and $\boldsymbol{C}$, (5.8), we have

$$\begin{aligned}\boldsymbol{C} &= \boldsymbol{U} + \boldsymbol{U}\Omega\boldsymbol{C},\\ &= \boldsymbol{U}\left(\mathbf{1} + \Omega\boldsymbol{C}\right)\end{aligned} \tag{5.11}$$

or alternatively,

$$\begin{aligned}\boldsymbol{U} &= \boldsymbol{C} - \boldsymbol{U}\Omega\boldsymbol{C},\\ &= \left(\mathbf{1} - \boldsymbol{U}\Omega\right)\boldsymbol{C},\\ \Rightarrow \boldsymbol{C} &= \left(\mathbf{1} - \boldsymbol{U}\Omega\right)^{-1}\boldsymbol{U},\end{aligned} \tag{5.12}$$

and hence,

$$\begin{aligned}\left(\mathbf{1} + \boldsymbol{\Omega}\boldsymbol{C}\right)^{-1} &= \boldsymbol{U}^{-1}\left(\mathbf{1} - \boldsymbol{U}\Omega\right)\boldsymbol{U},\\ &= \mathbf{1} - \boldsymbol{\Omega}\boldsymbol{U}.\end{aligned} \tag{5.13}$$

It therefore follows that the derivative of $\boldsymbol{U}$ with respect to the variables x_i is given by

$$\frac{\partial \boldsymbol{U}}{\partial x_i} = \left(\mathbf{1} - \boldsymbol{U}\Omega\right)\frac{\partial \boldsymbol{C}}{\partial x_i}\left(\mathbf{1} - \boldsymbol{\Omega}\boldsymbol{U}\right). \tag{5.14}$$

In order to eliminate $\boldsymbol{C}$ entirely from the derivative $\partial\boldsymbol{U}/\partial x_i$, two further identities involving the Cauchy kernel, $\boldsymbol{\Omega}$, are required. The first is a generalisation of (4.24).

Proposition 5.2.1 *The following relationship between the raising operator, $\boldsymbol{\Lambda}$, and the Cauchy Kernel, $\boldsymbol{\Omega}$, holds for all $j \in \mathbb{N}$:*

$$\boldsymbol{\Omega}\boldsymbol{\Lambda}^j - \left(-\ {}^t\boldsymbol{\Lambda}\boldsymbol{\Omega}\right)^j = \sum_{i=0}^{j-1}\left(-\ {}^t\Lambda\right)^i\widehat{\boldsymbol{O}}\boldsymbol{\Lambda}^{j-1-i} =: \widehat{\boldsymbol{O}}_j. \tag{5.15}$$

Proof

Consider the operator acting on $\mathbf{c}(k)$ to the right, and its transpose to the left,

$${}^t\mathbf{c}(k')\left(\boldsymbol{\Omega}\boldsymbol{\Lambda}^j - \left(-\ {}^t\boldsymbol{\Lambda}\boldsymbol{\Omega}\right)^j\right)\mathbf{c}(k) = \ {}^t\mathbf{c}(k')\boldsymbol{\Omega}\boldsymbol{O}\left(k^j - (-k')^j\right)\mathbf{c}(k), \tag{5.16}$$

using (4.13). Using the general formula

$$a^n - b^n = (a-b)\sum_{i=0}^{n-1}a^{n-1-i}b^i, \tag{5.17}$$

for all $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, we have

$$ {}^{t}\mathbf{c}(k')\left(\mathbf{\Omega}\mathbf{\Lambda}^{j} - \left(-\ {}^{t}\mathbf{\Lambda}\mathbf{\Omega}\right)^{j}\right)\mathbf{c}(k) = \sum_{i=0}^{j-1}\ {}^{t}\mathbf{c}(k')\left(k+k'\right)\mathbf{\Omega}\boldsymbol{O}\left(-k'\right)^{i}k^{j-1-i}\mathbf{c}(k). \tag{5.18} $$

Now, by applying the definition of the Cauchy Kernel, (4.24), we have

$$ {}^{t}\mathbf{c}(k')\left(\mathbf{\Omega}\mathbf{\Lambda}^{j} - \left(-\ {}^{t}\mathbf{\Lambda}\mathbf{\Omega}\right)^{j}\right)\mathbf{c}(k) = \sum_{i=0}^{j-1}\ {}^{t}\mathbf{c}(k')\left(1 - \frac{g}{KK'}\right)\boldsymbol{O}\left(-k'\right)^{i}k^{j-1-i}\mathbf{c}(k), \tag{5.19} $$

from which the result follows, using the action of the raising operators on the monomial basis, (4.13). □

Proposition 5.2.2 *For all $i, j \in \mathbb{N}_0$, we have*

$$ \widehat{\boldsymbol{O}}_{j}\mathbf{\Lambda}^{i} + \left(-\ {}^{t}\mathbf{\Lambda}\right)^{j}\widehat{\boldsymbol{O}}_{i} = \widehat{\boldsymbol{O}}_{i+j}. \tag{5.20} $$

Proof

Substituting in the definition of $\widehat{\boldsymbol{O}}_{j}$, (5.15), we have

$$ \begin{aligned} \widehat{\boldsymbol{O}}_{j}\mathbf{\Lambda}^{i} + \left(-\ {}^{t}\mathbf{\Lambda}\right)^{j}\widehat{\boldsymbol{O}}_{i} &= \left(\sum_{k=0}^{j-1}\left(-\ {}^{t}\Lambda\right)^{k}\widehat{\boldsymbol{O}}\mathbf{\Lambda}^{j-1-k}\right)\mathbf{\Lambda}^{i} + \left(-\ {}^{t}\mathbf{\Lambda}\right)^{j}\left(\sum_{k=0}^{i-1}\left(-\ {}^{t}\Lambda\right)^{k}\widehat{\boldsymbol{O}}\mathbf{\Lambda}^{i-1-k}\right), \\ &= \sum_{k=0}^{j-1}\left(-\ {}^{t}\Lambda\right)^{k}\widehat{\boldsymbol{O}}\mathbf{\Lambda}^{i+j-1-k} + \sum_{k=0}^{i-1}\left(-\ {}^{t}\Lambda\right)^{j+k}\widehat{\boldsymbol{O}}\mathbf{\Lambda}^{i-1-k}. \end{aligned} \tag{5.21} $$

Relabelling the summation index for the second sum, by letting $k' = k + j$, we have

$$ \begin{aligned} \widehat{\boldsymbol{O}}_{j}\mathbf{\Lambda}^{i} + \left(-\ {}^{t}\mathbf{\Lambda}\right)^{j}\widehat{\boldsymbol{O}}_{i} &= \sum_{k=0}^{j-1}\left(-\ {}^{t}\Lambda\right)^{k}\widehat{\boldsymbol{O}}\mathbf{\Lambda}^{i+j-1-k} + \sum_{k'=0}^{i+j-1}\left(-\ {}^{t}\Lambda\right)^{k'}\widehat{\boldsymbol{O}}\mathbf{\Lambda}^{i+j-1-k'}, \\ &= \sum_{k=0}^{i+j-1}\left(-\ {}^{t}\Lambda\right)^{k}\widehat{\boldsymbol{O}}\mathbf{\Lambda}^{i+j-1-k}, \\ &= \widehat{\boldsymbol{O}}_{i+j}, \end{aligned} \tag{5.22} $$

as required. □

It now follows, using our earlier calculations for $\partial \boldsymbol{C}/\partial x_i$ and $\partial \boldsymbol{U}/\partial x_i$, (5.14,5.7), that

$$\begin{aligned} \frac{\partial \boldsymbol{U}}{\partial x_i} &= (\mathbf{1} - \boldsymbol{U\Omega}) \frac{\partial \boldsymbol{C}}{\partial x_i} (\mathbf{1} - \boldsymbol{\Omega U}), \\ &= (\mathbf{1} - \boldsymbol{U\Omega}) \left(\boldsymbol{\Lambda}^i \boldsymbol{C} - \boldsymbol{C} \left(-{}^t\boldsymbol{\Lambda}\right)^i \right) (\mathbf{1} - \boldsymbol{\Omega U}). \end{aligned} \tag{5.23}$$

Again, using the definition of $\boldsymbol{U}$, (5.8), this becomes

$$\frac{\partial \boldsymbol{U}}{\partial x_i} = \boldsymbol{\Lambda}^i \boldsymbol{U} - \boldsymbol{U} \left(- \ {}^t\boldsymbol{\Lambda}\right)^i + \boldsymbol{U} \left(\boldsymbol{\Omega\Lambda}^i - \left(- \ {}^t\boldsymbol{\Lambda}\right)^i \boldsymbol{\Omega} \right) \boldsymbol{U}. \tag{5.24}$$

From this it follows that

$$\begin{aligned} \frac{\partial}{\partial x_{i+j}} \boldsymbol{U} &= \boldsymbol{\Lambda}^{i+j} \boldsymbol{U} - \boldsymbol{U} \left(- \ {}^t\boldsymbol{\Lambda}\right)^{i+j} - \boldsymbol{U} \widehat{\boldsymbol{O}}_{i+j} \boldsymbol{U} \\ &= \left(\boldsymbol{\Lambda}^j - \boldsymbol{U} \widehat{\boldsymbol{O}}_j \right) \boldsymbol{\Lambda}^i \boldsymbol{U} - \boldsymbol{U} \left(-{}^t\boldsymbol{\Lambda}\right)^j \left(\left(-{}^t\boldsymbol{\Lambda}\right)^i + \widehat{\boldsymbol{O}}_i \boldsymbol{U} \right), \end{aligned} \tag{5.25}$$

which upon substituting $\boldsymbol{\Lambda}^i \boldsymbol{U}$ and $\boldsymbol{U} \left(-{}^t\boldsymbol{\Lambda}\right)^j$ using (5.24) implies

$$\frac{\partial}{\partial x_{i+j}} \boldsymbol{U} = \left(\boldsymbol{\Lambda}^j - \boldsymbol{U} \widehat{\boldsymbol{O}}_j \right) \left(\frac{\partial}{\partial x_i} \boldsymbol{U} \right) + \left(\frac{\partial}{\partial x_j} \boldsymbol{U} \right) \left(\left(- \ {}^t\boldsymbol{\Lambda}\right)^i + \widehat{\boldsymbol{O}}_i \boldsymbol{U} \right). \tag{5.26}$$

Denoting $x = x_1$, $y = x_2$ and $t = x_3$, it follows that

$$\boldsymbol{U}_x = \boldsymbol{\Lambda U} + \boldsymbol{U} \ \ {}^t\boldsymbol{\Lambda} - \boldsymbol{U} \widehat{\boldsymbol{O}}_1 \boldsymbol{U}. \tag{5.27}$$

Taking this as a starting point, we can simply read off

$$u_x = U_{1,0} + U_{0,1} - u^2 + gss'. \tag{5.28}$$

Further x-derivatives for the other variables given earlier follow by applying various powers of the operators $\boldsymbol{\Lambda}$ and $\boldsymbol{L}$, and their transposes. Applying $\boldsymbol{L}^{-1}$ to the left hand side gives

$$\begin{aligned} &\boldsymbol{L}^{-1} \boldsymbol{U} = \boldsymbol{L}^{-1} \boldsymbol{\Lambda U} + \boldsymbol{L}^{-1} \boldsymbol{U} \ \ {}^t\boldsymbol{\Lambda} - \boldsymbol{L}^{-1} \boldsymbol{U} \widehat{\boldsymbol{O}} \boldsymbol{U}, \\ &\Rightarrow s_x = w - v - s\,(u - gh), \end{aligned} \tag{5.29}$$

or ${}^t\boldsymbol{L}^{-1}$ to the right gives the s'_x derivative,

$$\begin{aligned} &\boldsymbol{U}\ {}^t\boldsymbol{L}^{-1} = \boldsymbol{\Lambda}\boldsymbol{U}\ {}^t\boldsymbol{L}^{-1} + \boldsymbol{U}\ {}^t\boldsymbol{\Lambda}\ {}^t\boldsymbol{L}^{-1} - \boldsymbol{U}\widehat{\boldsymbol{O}}\boldsymbol{U}\ {}^t\boldsymbol{L}^{-1}, \\ &\Rightarrow s'_x = w' - v' - s'\left(u - gh\right). \end{aligned} \tag{5.30}$$

Applying both $\boldsymbol{L}^{-1}$ to the left, and ${}^t\boldsymbol{L}^{-1}$ to the right, gives

$$\begin{aligned} &\boldsymbol{L}^{-1}\boldsymbol{U}\ {}^t\boldsymbol{L}^{-1} = \boldsymbol{L}^{-1}\boldsymbol{\Lambda}\boldsymbol{U}\ {}^t\boldsymbol{L}^{-1} + \boldsymbol{U}\ {}^t\boldsymbol{\Lambda}\ {}^t\boldsymbol{L}^{-1} - \boldsymbol{L}^{-1}\boldsymbol{U}\widehat{\boldsymbol{O}}\boldsymbol{U}\ {}^t\boldsymbol{L}^{-1}, \\ &\Rightarrow h_x = U_{-1,-2} + U_{-2,-1} - ss' + gh^2. \end{aligned} \tag{5.31}$$

Applying $\boldsymbol{L}^{-1}\boldsymbol{\Lambda}$ to the left, gives

$$\boldsymbol{L}^{-1}\boldsymbol{\Lambda}\boldsymbol{U} = \boldsymbol{L}^{-1}\boldsymbol{\Lambda}^2\boldsymbol{U} + \boldsymbol{L}^{-1}\boldsymbol{\Lambda}\boldsymbol{U}\ {}^t\boldsymbol{\Lambda} - \boldsymbol{L}^{-1}\boldsymbol{\Lambda}\boldsymbol{U}\widehat{\boldsymbol{O}}\boldsymbol{U}, \tag{5.32}$$

which when using equation (4.7) for the curve, becomes

$$-v_x = 3es + gU_{-4,0} + U{-1,1} + uv + gU_{-1,-2}s, \tag{5.33}$$

whereby $U_{-4,0}$ may be eliminated, using (4.52), giving

$$-v_x = v\left(u + g\underline{h}\right) + U_{-1,1} + gs\left(U_{-1,-2} + \underline{U}_{-2,-1}\right) + 3es. \tag{5.34}$$

Similarly, by applying ${}^t\boldsymbol{\Lambda}\ {}^t\boldsymbol{L}^{-1}$ to (5.27) we can derive a formula for v'_x,

$$-v'_x = v'\left(u + g\overline{h}\right) + U_{1,-1} + gs'\left(\overline{U}_{-1,-2} + U_{-2,-1}\right) + 3es'. \tag{5.35}$$

The derivative w_x follows by applying $\boldsymbol{L}^{-1}$ to the left and ${}^t\boldsymbol{\Lambda}^{-1}$ to the right of ${}^t\boldsymbol{\Lambda}$, giving

$$w_x = w\left(\underline{u} + gh\right) + U_{-1,1} - s\left(U_{0,1} + \underline{U}_{1,0}\right) + 3es, \tag{5.36}$$

whilst, in a similar manner, it can be shown that

$$w'_x = w'\left(\overline{u} + gh\right) + U_{1,-1} - s'\left(\overline{U}_{0,1} + U_{1,0}\right) + 3es'. \tag{5.37}$$

We can continue to use this formula to derive the other x-derivatives, but this results in the introduction of a lot of new variables, which we would then need to eliminate. We instead follow a different procedure. Taking as a starting point (5.26), we discover upon substitution from (5.24) that

$$\begin{aligned}\left(\frac{\partial}{\partial x_{i+j}}+\frac{\partial^2}{\partial x_j \partial x_i}\right)\boldsymbol{U} &= \left(\boldsymbol{\Lambda}^j-\boldsymbol{U}\widehat{\boldsymbol{O}}_j\right)\frac{\partial \boldsymbol{U}}{\partial x_i}+\frac{\partial \boldsymbol{U}}{\partial x_j}\left(\left(-\ {}^t\boldsymbol{\Lambda}\right)^i\right.\\ &\qquad \left.+\widehat{\boldsymbol{O}}_i\boldsymbol{U}\right)+\frac{\partial}{\partial x_{+j}}\left(\frac{\partial \boldsymbol{U}}{\partial x_i}\right),\\ &= \left(\boldsymbol{\Lambda}^j-\boldsymbol{U}\widehat{\boldsymbol{O}}_j\right)\frac{\partial \boldsymbol{U}}{\partial x_i}+\frac{\partial \boldsymbol{U}}{\partial x_j}\left(\left(-\ {}^t\boldsymbol{\Lambda}\right)^i+\widehat{\boldsymbol{O}}_i\boldsymbol{U}\right)\\ &\qquad +\frac{\partial}{\partial x_{+j}}\left(\boldsymbol{\Lambda}^i\boldsymbol{U}-\boldsymbol{U}\left(-\ {}^t\boldsymbol{\Lambda}\right)^i-\boldsymbol{U}\widehat{\boldsymbol{O}}_i\boldsymbol{U}\right),\\ &= \left(\boldsymbol{\Lambda}^j-\boldsymbol{U}\widehat{\boldsymbol{O}}_j\right)\frac{\partial \boldsymbol{U}}{\partial x_i}+\left(\boldsymbol{\Lambda}^i-\boldsymbol{U}\widehat{\boldsymbol{O}}_i\right)\frac{\partial \boldsymbol{U}}{\partial x_j}.\end{aligned} \tag{5.38}$$

Similarly, we also have

$$\left(\frac{\partial}{\partial x_{i+j}}-\frac{\partial^2}{\partial x_j \partial x_i}\right)\boldsymbol{U} = \frac{\partial \boldsymbol{U}}{\partial x_i}\left(\left(-\ {}^t\boldsymbol{\Lambda}\right)^j+\widehat{\boldsymbol{O}}_j\boldsymbol{U}\right)+\frac{\partial \boldsymbol{U}}{\partial x_j}\left(\left(-\ {}^t\boldsymbol{\Lambda}\right)^i+\widehat{\boldsymbol{O}}_i\boldsymbol{U}\right). \tag{5.39}$$

Hence, when setting $i=j=1$ we have

$$\left(\frac{\partial}{\partial x_2}+\frac{\partial^2}{\partial x_1^2}\right)\boldsymbol{U} = 2\left(\boldsymbol{\Lambda}-\boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\frac{\partial \boldsymbol{U}}{\partial x_1}, \tag{5.40a}$$

$$\left(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2}\right)\boldsymbol{U} = 2\frac{\partial \boldsymbol{U}}{\partial x_1}\left(-\ {}^t\boldsymbol{\Lambda}+\widehat{\boldsymbol{O}}_1\boldsymbol{U}\right), \tag{5.40b}$$

from which we can read off directly formulae for $\left(U_{1,0}\right)_x$ and $\left(U_{0,1}\right)_x$ derivatives,

$$u_y+u_{xx} = 2\left(U_{1,0}\right)_x-2uu_x+2gs's_x, \tag{5.41a}$$

$$u_y-u_{xx} = -2\left(U_{0,1}\right)_x+2uu_x-2gss'_x, \tag{5.41b}$$

respectively. As for the y-derivatives, summing (5.40a) and (5.40b) gives

$$\boldsymbol{U}_y = \left(\boldsymbol{\Lambda}\boldsymbol{U}-\boldsymbol{U}\ {}^t\boldsymbol{\Lambda}\right)_x+\boldsymbol{U}_x\widehat{\boldsymbol{O}}_1\boldsymbol{U}-\boldsymbol{U}\widehat{\boldsymbol{O}}_1\boldsymbol{U}_x, \tag{5.42}$$

from which we can read off similar formulae in the same manner that we did for the

x-derivatives:

$$u_y = \left(U_{1,0} - U_{0,1}\right)_x + g\left(s' s_x - s s'_x\right), \tag{5.43a}$$

$$h_y = \left(U_{-1,-2} - U_{-2,-1}\right)_x + g\left(s s'_x - s' s_x\right), \tag{5.43b}$$

$$s_y = -\left(v + w\right)_x + s_x\left(u + gh\right) - s\left(u + gh\right)_x, \tag{5.43c}$$

$$s'_y = \left(v' + w'\right)_x + s'\left(u + gh\right)_x - s'_x\left(u + gh\right), \tag{5.43d}$$

$$v_y = \left(U_{-1,1}\right)_x - \left(v u_x + u v_x\right) + g\left(s\left(U_{-1,-2}\right)_x - U_{-1,-2} s_x\right), \tag{5.43e}$$

$$v'_y = -\left(U_{1,-1}\right)_x + \left(u v'_x + v' u_x\right) + g\left(\left(U_{-2,-1}\right) s'_x - s'\left(U_{-2,-1}\right)_x\right), \tag{5.43f}$$

$$w_y = \left(U_{-1,1} - U_{-2,2}\right)_x + \left(U_{0,1} s_x - s\left(U_{0,1}\right)_x\right) + g\left(h w_x - h_x\left(w - 1\right)\right), \tag{5.43g}$$

$$w'_y = \left(U_{2,-2} - U_{1,-1}\right)_x + \left(U_{0,1} s'_x - s'\left(U_{0,1}\right)_x\right) + g\left(\left(w' - 1\right) h_x - h w'_x\right). \tag{5.43h}$$

As for the t-derivatives, from (5.26), setting $i = 1$ and $j = 2$, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial x_3} + \frac{\partial^2}{\partial x_i \partial x_j}\right) &= \left(\boldsymbol{\Lambda}^2 - \boldsymbol{U}\widehat{\boldsymbol{O}}_2\right)\frac{\partial \boldsymbol{U}}{\partial x_1} + \left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\frac{\partial \boldsymbol{U}}{\partial x_2}, \\
&= \left(\boldsymbol{\Lambda}^2 - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\boldsymbol{\Lambda} + \boldsymbol{U}\ \ {}^t\boldsymbol{\Lambda}\widehat{\boldsymbol{O}}_1\right)\frac{\partial \boldsymbol{U}}{\partial x_1} \\
&\qquad + \left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\frac{\partial \boldsymbol{U}}{\partial x_2}, \\
&= \Bigg(\left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)^2 + \left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\boldsymbol{U}\widehat{\boldsymbol{O}}_1 \\
&\qquad + \boldsymbol{U}\ \ {}^t\boldsymbol{\Lambda}\widehat{\boldsymbol{O}}_1\Bigg)\frac{\partial \boldsymbol{U}}{\partial x_1} \\
&\quad + \left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\frac{\partial \boldsymbol{U}}{\partial x_2}, \\
&= \left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\left(\left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\frac{\partial \boldsymbol{U}}{\partial x_1} + \frac{\partial \boldsymbol{U}}{\partial x_2}\right) + \frac{\partial \boldsymbol{U}}{\partial x_1}\widehat{\boldsymbol{O}}_1\frac{\partial \boldsymbol{U}}{\partial x_1},
\end{aligned} \tag{5.44}$$

from which, using (5.40a), we get

$$\left(\frac{\partial}{\partial x_3} + \frac{\partial^2}{\partial x_1 \partial x_2}\right)\boldsymbol{U} = \frac{1}{2}\left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_1\right)\left(3\frac{\partial \boldsymbol{U}}{\partial x_2} + \frac{\partial^2 \boldsymbol{U}}{\partial x_1^2}\right)\boldsymbol{U} + \frac{\partial \boldsymbol{U}}{\partial x_1}\widehat{\boldsymbol{O}}_1\frac{\partial \boldsymbol{U}}{\partial x_1}. \tag{5.45}$$

Similarly, we can show that

$$\left(\frac{\partial}{\partial x_3}-\frac{\partial^2}{\partial x_1\partial x_2}\right)\boldsymbol{U} \\ =\frac{1}{2}\left(\left(3\frac{\partial \boldsymbol{U}}{\partial x_2}-\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}\right)\boldsymbol{U}\right)\left(-\ {}^t\boldsymbol{\Lambda}+\widehat{\boldsymbol{O}}_1\boldsymbol{U}\right)+\frac{\partial \boldsymbol{U}}{\partial x_1}\widehat{\boldsymbol{O}}_1\frac{\partial \boldsymbol{U}}{\partial x_1}. \tag{5.46}$$

These equations imply, in particular,

$$s_t=-\frac{3}{2}v_y-\frac{1}{2}v_{xx}-\frac{3}{2}su_y-\frac{1}{2}su_{xx} \\ +\frac{1}{2}gh\left(3s_y+s_{xx}\right)+s_xu_x-gh_xs_x-s_{xy}, \tag{5.47a}$$

$$s'_t=\frac{3}{2}w'_y+\frac{1}{2}w'_{xx}-\frac{3}{2}us'_y-\frac{1}{2}us'_{xx} \\ +\frac{1}{2}gs'\left(3h_y+h_{xx}\right)+s'_xu_x-gh_xs'_x-s'_{xy}, \tag{5.47b}$$

$$s_t=-\frac{3}{2}w_y+\frac{1}{2}w_{xx}+\frac{3}{2}us_y-\frac{1}{2}us_{xx} \\ +\frac{1}{2}gs\left(-3h_y+h_{xx}\right)+s_xu_x-gh_xs_x+s_{xy}, \tag{5.47c}$$

$$s'_t=\frac{3}{2}v'_y+\frac{1}{2}v'_{xx}+\frac{3}{2}s'u_y-\frac{1}{2}s'u_{xx} \\ +\frac{1}{2}gh\left(-3s'_y+s'_{xx}\right)+s'_xu_x-gh_xs'_x+s'_{xy}. \tag{5.47d}$$

For the other t-derivatives, it is neater to follow another method. Differentiating (5.45) with respect to x_1 implies

$$\frac{\partial^2\boldsymbol{U}}{\partial x_1\partial x_3}=-\frac{\partial^3\boldsymbol{U}}{\partial x_1^2\partial x_2}+\frac{1}{2}\left(3\frac{\partial \boldsymbol{U}}{\partial x_2}+\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}\right)\frac{\partial \boldsymbol{\Lambda U}}{\partial x_1} \\ -\frac{1}{2}\frac{\partial \boldsymbol{U}}{\partial x_1}\widehat{\boldsymbol{O}}_1\left(3\frac{\partial}{\partial x_2}+\frac{\partial^2}{\partial x_1^2}\right)\boldsymbol{U}-\frac{1}{2}\boldsymbol{U}\widehat{\boldsymbol{O}}_1\left(3\frac{\partial \boldsymbol{U}}{\partial x_2}+\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}\right)\frac{\partial \boldsymbol{U}}{\partial x_1} \\ +\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}\widehat{\boldsymbol{O}}_1\frac{\partial \boldsymbol{U}}{\partial x_1}+\frac{\partial \boldsymbol{U}}{\partial x_1}\widehat{\boldsymbol{O}}_1\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}. \tag{5.48}$$

Substituting in $\frac{\partial \boldsymbol{\Lambda U}}{\partial x_1}$, from (5.40a), gives

$$\frac{\partial^2\boldsymbol{U}}{\partial x_1\partial x_3}=-\frac{\partial^3\boldsymbol{U}}{\partial x_1^2\partial x_2}+\frac{1}{4}\left(3\frac{\partial}{\partial x_2}+\frac{\partial^2}{\partial x_1^2}\right)\left(\frac{\partial}{\partial x_2}+\frac{\partial^2}{\partial x_1^2}\right)\boldsymbol{U} \\ +\frac{1}{2}\left(3\frac{\partial}{\partial x_2}+\frac{\partial^2}{\partial x_1^2}\right)\left(\boldsymbol{U}\widehat{\boldsymbol{O}}_1\frac{\partial \boldsymbol{U}}{\partial x_1}\right)-\frac{1}{2}\frac{\partial \boldsymbol{U}}{\partial x_1}\widehat{\boldsymbol{O}}_1\left(3\frac{\partial}{\partial x_2}+\frac{\partial^2}{\partial x_1^2}\right)\boldsymbol{U} \\ -\frac{1}{2}\boldsymbol{U}\widehat{\boldsymbol{O}}_1\left(3\frac{\partial \boldsymbol{U}}{\partial x_2}+\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}\right)\frac{\partial \boldsymbol{U}}{\partial x_1}+\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}\widehat{\boldsymbol{O}}_1\frac{\partial \boldsymbol{U}}{\partial x_1}+\frac{\partial \boldsymbol{U}}{\partial x_1}\widehat{\boldsymbol{O}}_1\frac{\partial^2\boldsymbol{U}}{\partial x_1^2}, \tag{5.49}$$

which reduces to

$$\left(\boldsymbol{U}_t - \frac{1}{4}\boldsymbol{U}_{xxx} - \frac{3}{2}\boldsymbol{U}_x\widehat{\boldsymbol{O}}_1\boldsymbol{U}_x\right)_x = \frac{3}{4}\boldsymbol{U}_{yy} + \frac{3}{2}\left(\boldsymbol{U}_y\widehat{\boldsymbol{O}}_1\boldsymbol{U}_x - \boldsymbol{U}_x\widehat{\boldsymbol{O}}_1\boldsymbol{U}_y\right). \tag{5.50}$$

From this, we can simply read off the remaining t-derivatives:

$$\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}\left(u_x\right)^2 + \frac{3}{2}gs_xs'_x\right)_x = \frac{3}{4}u_{yy} + \frac{3}{2}g\left(s'_xs_y - s'_ys_x\right), \tag{5.51a}$$

$$\left(h_t - \frac{1}{4}h_{xxx} - \frac{3}{2}s_xs'_x + \frac{3}{2}g\left(h_x\right)^2\right)_x = \frac{3}{4}h_{yy} + \frac{3}{2}\left(s'_xs_y - s'_ys_x\right), \tag{5.51b}$$

$$\begin{aligned}&\left(s_t - \frac{1}{4}s_{xxx} - \frac{3}{2}s_xu_x + \frac{3}{2}gh_xs_x\right)_x \\ &\qquad = \frac{3}{4}s_{yy} + \frac{3}{2}\left(s_y\left(u+gh\right)_x - s_x\left(u+gh\right)_y\right),\end{aligned} \tag{5.51c}$$

$$\begin{aligned}&\left(s'_t - \frac{1}{4}s'_{xxx} - \frac{3}{2}s'_xu_x + \frac{3}{2}gh_xs'_x\right)_x \\ &\qquad = \frac{3}{4}s'_{yy} + \frac{3}{2}\left(s'_x\left(u+gh\right)_y - s'_y\left(u+gh\right)_x\right),\end{aligned} \tag{5.51d}$$

$$\begin{aligned}&\left(v_t - \frac{1}{4}v_{xxx} - \frac{3}{2}v_xu_x + \frac{3}{2}g\left(U_{-1,-2}\right)_xs_x\right)_x \\ &\qquad = \frac{3}{4}v_{yy} + \frac{3}{2}\left(v_yu_x - v_xu_y\right) + \frac{3}{2}g\left(\left(U_{-1,-2}\right)_xs_y - \left(U_{-1,-2}\right)_ys_x\right),\end{aligned} \tag{5.51e}$$

$$\begin{aligned}&\left(v'_t - \frac{1}{4}v'_{xxx} - \frac{3}{2}v'_xu_x - \frac{3}{2}g\left(U_{-2,-1}\right)_xs'_x\right)_x \\ &\qquad = \frac{3}{4}v'_{yy} + \frac{3}{2}\left(u_yv'_x - u_xv'_y\right) + \frac{3}{2}g\left(\left(U_{-2,-1}\right)_ys_x - \left(U_{-2,-1}\right)_xs_y\right),\end{aligned} \tag{5.51f}$$

$$\begin{aligned}&\left(w_t - \frac{1}{4}w_{xxx} - \frac{3}{2}s_x\left(U_{0,1}\right)_x + \frac{3}{2}gh_xw_x\right)_x \\ &\qquad = \frac{3}{4}w_{yy} + +\frac{3}{2}\left(s_y\left(U_{0,1}\right)_x - s_x\left(U_{0,1}\right)_y\right) + \frac{3}{2}g\left(h_xw_y - h_yw_x\right),\end{aligned} \tag{5.51g}$$

$$\begin{aligned}&\left(w'_t - \frac{1}{4}w'_{xxx} - \frac{3}{2}s'_x\left(U_{1,0}\right)_x + \frac{3}{2}gh_xw'_x\right)_x \\ &\qquad = \frac{3}{4}w'_{yy} + +\frac{3}{2}\left(s'_x\left(U_{1,0}\right)_y - s_y\left(U_{1,0}\right)_x\right) + \frac{3}{2}g\left(h_yw'_x - h_xw'_y\right).\end{aligned} \tag{5.51h}$$

The equations involving the t-derivatives of u, s, s' and h, (5.51a, 5.51c, 5.51d, 5.51b) provide a closed system of equations in terms of these variables. However, this is simply a covering system, in the sense that it is not dependent upon the elliptic curve. We instead provide a closed system in terms of the u, s, s' and w, which does have a dependence on the curve. This system is found by eliminating all other variables. Again this system also includes (5.51a), as it is already written in the required parameters. As for the rest of the system, we take as a starting point the four equations for s_x, s_y, s'_x and s'_y, (5.29, 5.43c, 5.30, 5.43d), from which we eliminate the variables v, v' and h. In order to do so, we differentiate the s_x, and s'_x, equations with respect to x, and solve for v_x, and v'_x, respectively, giving

$$v_x = w_x - s_x x - s u_x - u s_x + g s h_x + g h s_x, \tag{5.52a}$$

$$v'_x = w'_x - s'_x x - s' u_x - u s'_x + g s' h_x + g h s'_x. \tag{5.52b}$$

Substituting these terms into the s_y and s'_y formulae to completely eliminate all v and v' terms gives

$$s_y = -2w_x + s_{xx} + 2u s_x - 2g s h_x, \tag{5.53a}$$

$$s'_y = -2w'_x - s'_{xx} - 2u s'_x + 2g s' h_x, \tag{5.53b}$$

after which, eliminating h_x, implies

$$\left(ss'\right)_y = 2s w'_x - 2s' w_x + 2u\left(s_x s' - s s'_x\right) + s_{xx} s' - s s'_{xx}. \tag{5.54}$$

Similarly, using the formulae (5.47c, 5.47b) for s_t and s'_t, we can eliminate h using (5.53a, 5.53b). If we rearrange (5.47c, 5.47b) to give

$$\begin{aligned} 3gsh_y = 2s_{xy} - 2s_t - 3w_y + w_{xx} + 3us_y - us_{xx} \\ + gsh_{xx} + 2s_x u_x - 2gh_x s_x, \end{aligned} \tag{5.55a}$$

$$\begin{aligned} 3gs'h_y = 2s'_{xy} + 2s'_t - 3w'_y - w'_{xx} + 3us'_y + us'_{xx} \\ - gs'h_{xx} - 2s'_x u_x + 2gh_x s'_x, \end{aligned} \tag{5.55b}$$

and multiply (5.55a) by s', and (5.55a) by s, respectively, these may be equated to give

$$\begin{aligned} \left(ss'\right)_t = s_{xy}s' - ss'_{xy} + \frac{1}{2}\left(w_{xx}s' + sw'_{xx}\right) + \frac{3}{2}\left(sw'_y - w_y s'\right) + \frac{3}{2}u\left(s_y s'' - ss'_y\right) \\ - \frac{1}{2}u\left(s_{xx}s' + ss'_{xx}\right) + u_x\left(ss'\right)_x + \frac{1}{2}g\left(s'\left(sh_x\right)_x + s'\left(s'h_x\right)_x\right) - \frac{3}{2}gh_x\left(ss'\right)_x. \end{aligned} \tag{5.56}$$

Eliminating h and its derivatives, again using (5.53a, 5.53b), we get

$$\begin{aligned}(ss')_t = \frac{1}{4}\left(s_{xxx}s' + s'_{xxx}s\right) + \frac{3}{2}u_x\left(ss'\right)_x - 3us_xs'_x + \frac{3}{4}\left(s's_y - ss'_y - s_xs'_x\right)_x \\ + \frac{3}{2}\left(s's_y - ss'_y + w'_ys - w_ys'\right) + \frac{3}{2}\left(w_xs'_x + w'_xs_x\right). \quad (5.57)\end{aligned}$$

As for relations involving the curve, we consider $\left(u + \frac{w}{s}\right)_x$ using (5.28, 5.29, 5.36), the equations we have for their individual x-derivatives,

$$\begin{aligned}\left(u + \frac{w}{s}\right)_x &= u_x + \frac{w_x}{s} - \frac{ws_x}{s^2}, \\ &= \left(\frac{1}{\underline{s}'s} + 3e + gss'\right) - \left(u - \frac{w}{s}\right)^2 + \frac{w}{s}\left(u - \underline{u}\right) + U_{1,0} - \underline{U_{1,0}},\end{aligned} \quad (5.58)$$

from which, differentiating, and substituting in $(U_{1,0})_x$ from (5.41a), gives

$$\begin{aligned}\left(\frac{1}{2}\left(\underline{u} + \frac{w}{s}\right)_x + \left(\underline{u} - \frac{w}{s}\right)^2\right)_x + \left(\frac{1}{2}\left(u + \frac{w}{s}\right)_x + \left(u - \frac{w}{s}\right)^2\right)_x + \frac{1}{2}\left(\underline{u} - u\right)_y \\ = \left(\frac{1}{s\underline{s}'} + 3e + gss'\right)_x - g\left(s's_x - \underline{s}'\underline{s}_x\right). \quad (5.59)\end{aligned}$$

This completes the derivation of the closed system of equations (5.2). As in the discrete case, there also exists an analogous continuous system of equations in terms of h, s, s', v and v' shadowing the system (5.2). This shadow system is derived in a similar method, except this time it is the variables u, w, w', $U_{0,1}$ and $U_{1,0}$ that are eliminated. The system is as follows:

$$\left(h_t - \frac{1}{4}u_{xxx} - \frac{3}{2}s_x s'_x + \frac{3}{2}gh_x^2\right)_x = \frac{3}{4}h_{yy} + \frac{3}{2}g\left(s'_x s_y - s'_y s_x\right), \tag{5.60a}$$

$$(ss')_t = \frac{1}{4}\left(s_{xxx}s' + s'_{xxx}s\right) - \frac{3}{2}gh_x (ss')_x + 3ghs_x s'_x + \frac{3}{4}\left(ss'_y - s's_y - s_x s'_x\right)_x$$
$$+ \frac{3}{2}gh\left(s's_y - ss'_y + v'_y s - v_y s'\right) - \frac{3}{2}\left(v_x s'_x + v'_x s_x\right), \tag{5.60b}$$

$$(ss')_y = 2sv'_x - 2s'v_x + 2gh\left(s_x s' - s'_x s\right) + s'_{xx}s - s_{xx}s', \tag{5.60c}$$

$$\left(gh + \frac{v}{s}\right)_x - \left(gh + \frac{v}{s}\right)\left(gh + \frac{v}{s}\right) + \frac{v}{s}\left(gh + g\underline{h}\right)$$
$$= -\left(\frac{1}{\underline{s}'s} + 3e + gss'\right) + g\left(U_{-2,-1} - \underline{U}_{-2,-1}\right), \tag{5.60d}$$

$$\left(gh + \frac{v'}{s'}\right)_x - \left(gh - \frac{v'}{s'}\right)\left(gh + \frac{v'}{s'}\right) + \frac{v'}{s'}\left(gh + g\overline{h}\right)$$
$$= -\left(\frac{1}{\overline{s}s'} + 3e + gss'\right) + g\left(U_{-1,-2} - \overline{U}_{-1,-2}\right), \tag{5.60e}$$

$$s'\overline{v} = v'\overline{s}. \tag{5.60f}$$

This system is again the result of the automorphism (4.76), with the resultant system (5.60) being analogous to its discrete counterpart (4.75).

We will now consider some of the properties of the system (5.2). The system (5.2) is integrable by construction, and in fact admits soliton solutions of the same from as in Chapter 4, but with $\rho$and σ replaced by their continuous analogues.

5.3 Lax Representation

The integrability of of the system (5.2) can be shown by the fact that it admits a Lax triplet. Again, in order to obtain a Lax representation for the system, the infinite component vectors introduced earlier in (4.84), that is

$$\boldsymbol{u}_k \equiv \left(\mathbf{1} - \boldsymbol{U}\boldsymbol{\Omega}\right)\rho_k \boldsymbol{c}_k. \tag{5.61}$$

are considered.

Proposition 5.3.1 *The derivatives of $\boldsymbol{u}_k$ with respect to the variables x_i are given by*

$$\frac{\partial}{\partial x_j}\mathbf{u}_k = \left(\boldsymbol{\Lambda} - \boldsymbol{U}\widehat{\boldsymbol{O}}_j\right)\mathbf{u}_k. \tag{5.62}$$

Proof

From definition (5.61), we have

$$\frac{\partial}{\partial x_j}\mathbf{u}_k = -\frac{\partial \boldsymbol{U}}{\partial x_j}\boldsymbol{\Omega}\rho_k\boldsymbol{c}_k + (\mathbf{1} - \boldsymbol{U}\boldsymbol{\Omega})\frac{\partial \rho_k}{\partial x_j}\boldsymbol{c}_k. \tag{5.63}$$

Recalling (5.5, 4.13, 5.24),

$$\frac{\partial \rho_k}{\partial x_j} = k^j\rho_k, \tag{5.64a}$$

$$\boldsymbol{\Lambda}^i\boldsymbol{c}_k = k^i\boldsymbol{c}_k, \tag{5.64b}$$

$$\frac{\partial \boldsymbol{U}}{\partial x_j} = \boldsymbol{\Lambda}^j\boldsymbol{U} - \boldsymbol{U}\left(-\ {}^t\boldsymbol{\Lambda}\right)^j + \boldsymbol{U}\left(\boldsymbol{\Omega}\boldsymbol{\Lambda}^j - \left(-\ {}^t\boldsymbol{\Lambda}\right)^j\boldsymbol{\Omega}\right)\boldsymbol{U}, \tag{5.64c}$$

we have

$$\frac{\partial}{\partial x_j}\mathbf{u}_k = \left(\boldsymbol{\Lambda}^j - \boldsymbol{U}\left(\boldsymbol{\Omega}\boldsymbol{\Lambda}^j - \left(-\ {}^t\boldsymbol{\Lambda}\right)^j\boldsymbol{\Omega}\right)\right)\mathbf{u}_k. \tag{5.65}$$

Now applying the definition (5.15),

$$\widehat{\boldsymbol{O}}_j = \boldsymbol{\Omega}\boldsymbol{\Lambda}^j - \left(-\ {}^t\boldsymbol{\Lambda}\right)^j\boldsymbol{\Omega}, \tag{5.66}$$

the result follows. □

Now as a result, we have

$$\frac{\partial}{\partial x_{i+j}}\boldsymbol{u}_k = \left(\boldsymbol{\Lambda}^{i+j} - \boldsymbol{U}\widehat{\boldsymbol{O}}_{i+j}\right)\boldsymbol{u}_k, \tag{5.67}$$

where, recalling (5.20), we also have

$$\widehat{\boldsymbol{O}}_{i+j} = \widehat{\boldsymbol{O}}_j\boldsymbol{\Lambda}^i + \left(-\ {}^t\Lambda\right)^j\widehat{\boldsymbol{O}}_i, \tag{5.68}$$

which may be rewritten as

$$\begin{aligned}\frac{\partial}{\partial x_{i+j}}\boldsymbol{u}_k = \Big(&\left(\boldsymbol{\Lambda}^j - \boldsymbol{U}\widehat{\boldsymbol{O}}_j\right)\left(\boldsymbol{\Lambda}^i - \boldsymbol{U}\widehat{\boldsymbol{O}}_i\right)\\ &+ \left(\boldsymbol{\Lambda}^j\boldsymbol{U} - \boldsymbol{U}\left(-\ {}^t\boldsymbol{\Lambda}\right)^j - \boldsymbol{U}\widehat{\boldsymbol{O}}_j\boldsymbol{U}\right)\widehat{\boldsymbol{O}}_i\Big)\boldsymbol{u}_k.\end{aligned} \tag{5.69}$$

Recalling (5.24), we obtain

$$\frac{\partial}{\partial x_j}\boldsymbol{U} = \boldsymbol{\Lambda}^j\boldsymbol{U} - \boldsymbol{U}\left(-\ {}^t\boldsymbol{\Lambda}\right)^j - \boldsymbol{U}\widehat{\boldsymbol{O}}_j\boldsymbol{U}, \tag{5.70}$$

and using (5.62, 5.69), this reduces to

$$\begin{aligned}\frac{\partial}{\partial x_{i+j}}\boldsymbol{u}_k &= \left(\left(\boldsymbol{\Lambda}^j - \boldsymbol{U}\widehat{\boldsymbol{O}}_j\right)\frac{\partial \boldsymbol{u}_k}{\partial x_i} + \frac{\partial \boldsymbol{U}}{\partial x_j}\widehat{\boldsymbol{O}}_i\right)\boldsymbol{u}_k, \\ &= \frac{\partial^2 \boldsymbol{u}_k}{\partial x_j \partial x_i} + \frac{\partial \boldsymbol{U}}{\partial x_i}\widehat{\boldsymbol{O}}_j\boldsymbol{u}_k + \frac{\partial \boldsymbol{U}}{\partial x_j}\widehat{\boldsymbol{O}}_i\boldsymbol{u}_k.\end{aligned} \tag{5.71}$$

Setting $\boldsymbol{\varphi} = (\varphi_0, \varphi_1)^t = (\boldsymbol{u}_0, \boldsymbol{u}_1)^t$, as in Chapter 4, this gives the Lax triplet to be

$$\boldsymbol{\varphi}_y = \boldsymbol{\varphi}_{xx} + A\boldsymbol{\varphi} + B\underline{\boldsymbol{\varphi}}, \tag{5.72a}$$

$$\boldsymbol{\varphi}_x = J\overline{\boldsymbol{\varphi}} + C\boldsymbol{\varphi} + D\underline{\boldsymbol{\varphi}}, \tag{5.72b}$$

$$\boldsymbol{\varphi}_t = \boldsymbol{\varphi}_{xy} + E\boldsymbol{\varphi} + F\boldsymbol{\varphi}_x + G\underline{\boldsymbol{\varphi}}, \tag{5.72c}$$

where

$$J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.73a}$$

$$A = \begin{pmatrix} 2u_x & 0 \\ 2\left(U_{1,0}\right)_x & 0 \end{pmatrix}, \tag{5.73b}$$

$$B = \begin{pmatrix} -2gs'_x w & 2gs'_x s \\ -2gw'_x w & 2gw'_x s \end{pmatrix}, \tag{5.73c}$$

$$C = \begin{pmatrix} -u & 1 \\ 3e - U_{1,0} - \overline{U}_{0,1} & \overline{u} \end{pmatrix}, \tag{5.73d}$$

$$D = \begin{pmatrix} gws' & -gss' \\ gww' & -gsw' \end{pmatrix}, \tag{5.73e}$$

$$E = \begin{pmatrix} u_y - \left(U_{0,1}\right)_x - \frac{u\left(s'_y + v'_x\right)}{s'} & u_x + \frac{s'_y + v'_x}{s'} \\ \left(U_{1,0}\right)_y - \left(U_{1,1}\right)_x - \frac{u\left(\left(U_{1,-1}\right)_x - w'_y\right)}{s'} & \left(U_{1,0}\right)_x + \frac{v'_y - \left(\left(U_{1,-1}\right)_x\right.}{s'} \end{pmatrix}, \tag{5.73f}$$

$$F = \begin{pmatrix} \frac{-s'_y - v'_x}{s'} & 0 \\ \frac{\left(U_{1,-1}\right)_x - w'_y}{s'} & 0 \end{pmatrix}, \tag{5.73g}$$

$$G = \begin{pmatrix} -gs'_x U_{1,-1} & -gs'_x v \\ -gw'_x U_{-1,1} & -gw'_x v \end{pmatrix}. \tag{5.73h}$$

Equations for the system follow from the compatibility conditions between each pair $(\boldsymbol{\varphi}_x, \boldsymbol{\varphi}_t)$, $(\boldsymbol{\varphi}_x, \boldsymbol{\varphi}_y)$ and $(\boldsymbol{\varphi}_y, \boldsymbol{\varphi}_t)$. For compatibility between (5.72b, 5.72a), we consider the difference

$$\begin{aligned} \boldsymbol{\varphi}_{yx} - \boldsymbol{\varphi}_{xy} =& \boldsymbol{\varphi}_{xxx} + A\boldsymbol{\varphi}_x + \left(A_x - C_y\right)\boldsymbol{\varphi} + B\underline{\boldsymbol{\varphi}}_x + \left(B_x - D_y\right)\underline{\boldsymbol{\varphi}} \\ &- J\overline{\boldsymbol{\varphi}}_y - C\boldsymbol{\varphi}_y - D\underline{\boldsymbol{\varphi}}_y. \end{aligned} \tag{5.74}$$

Using the formulae (5.72b, 5.72b) all derivatives of $\boldsymbol{\varphi}$ on the right hand side of (5.74) may be eliminated to leave a result in terms of only $\boldsymbol{\varphi}$ and its shifts, given by

$$\begin{aligned}\boldsymbol{\varphi}_{yx}-\boldsymbol{\varphi}_{xy}=&\left(C_{xx}+A_x-C_y-J\overline{B}-CA+2C_xC+AC+2D_xJ+BJ\right)\boldsymbol{\varphi}\\&+\left(D_{xx}+B_x-D_y-CB-D\underline{A}+2C_xD+AD+2D_x\underline{C}+B\underline{C}\right)\underline{\boldsymbol{\varphi}}\\&+\left(2D_x\underline{D}+B\underline{D}-D\underline{B}\right)\underline{\underline{\boldsymbol{\varphi}}}.\end{aligned} \tag{5.75}$$

Note that this has no $\overline{\overline{\boldsymbol{\varphi}}}$ terms as a result of J being nilpotent. In order to achieve compatibility, each of these coefficients must equate to zero giving a series of equations. The coefficient of $\underline{\underline{\boldsymbol{\varphi}}}$ yields

$$2D_x\underline{D}+B\underline{D}-D\underline{B}=2g^2\begin{pmatrix}s'\\w'\end{pmatrix}\frac{\partial}{\partial x}\left(\begin{pmatrix}w & -s\end{pmatrix}\begin{pmatrix}\underline{s}'\\\underline{w}'\end{pmatrix}\right)\begin{pmatrix}\underline{w} & -\underline{s}\end{pmatrix}, \tag{5.76}$$

and hence,

$$\frac{\partial}{\partial x}\left(w\underline{s}'-s\underline{w}'\right)=0, \tag{5.77}$$

the derivative of (5.2f) with respect to x. From the coefficient of $\boldsymbol{\varphi}$, we recover

$$u_{xx}+u_y+2uu_x-2\left(U_{1,0}\right)_x+2gs'_xs=0, \tag{5.78a}$$

$$\overline{u}_{xx}-\overline{u}_y+2\overline{u}\overline{u}_x-2\left(\overline{U}_{0,1}\right)_x-2g\overline{s}'_x\overline{s}=0, \tag{5.78b}$$

$$\begin{aligned}&\left(\left(U_{1,0}\right)_{xx}+\left(U_{1,0}\right)_y+2U_{1,0}u_x+2gw'_xs-6eu_x-2\left(U_{2,0}\right)_x\right)\\&-\left(\left(\overline{U}_{0,1}\right)_{xx}-\left(\overline{U}_{0,1}\right)_y+2\overline{U}_{0,1}\overline{u}_x-2g\overline{s}'_xw-6e\overline{u}_x-2\left(\overline{U}_{0,2}\right)_x\right)\\&\qquad+2\left(U_{2,0}-\overline{U}_{0,2}+u\overline{U}_{0,1}-\overline{u}U_{1,0}\right)=0,\end{aligned} \tag{5.78c}$$

whereby the first two equations are (5.41a) and (5.41b), respectively, and the third is a combination of the first with $\boldsymbol{\Lambda}$ applied to the left, and the second with ${}^t\boldsymbol{\Lambda}$ applied to the right, together with (4.58). Finally, the coefficient of $\underline{\varphi}$ yields

$$g\left(s\left(s'_{xx}+s'_y+2us'_x-2w'_x-2gs'h_x\right)\right.$$
$$\left.-s'\left(s_{xx}-s_y+2us_x-2w_x-2gsh_x\right)\right)=0, \tag{5.79a}$$
$$g\left(s'\left(w_{xx}-w_y+2s_xU_{0,1}-2gwh_x-6es_x-2\left(U_{-2,2}\right)_x\right)\right.$$
$$-w\left(s'_{xx}+s'_y+2us'_x-2w'_x-2gs'h_x\right)$$
$$\left.+2s'\left(U_{-2,2}-w\underline{u}+s\underline{U}_{1,0}\right)_x\right)=0, \tag{5.79b}$$
$$g\left(w'\left(w_{xx}-w_y+2s_xU_{0,1}-2gwh_x-6es_x-2\left(U_{-2,2}\right)_x\right)\right.$$
$$-w\left(w'_{xx}+w'_y+2s'_xU_{0,1}-2gw'h_x-6es'_x-2\left(U_{2,-2}\right)_x\right)$$
$$\left.+2w'\left(U_{-2,2}-\underline{u}w+s\underline{U}_{1,0}\right)_x-2w\left(U_{2,-2}-\overline{u}w'+s'\overline{U}_{0,1}\right)_x\right)=0, \tag{5.79c}$$
$$g\left(s\left(w'_{xx}+w'_y+2s'_xU_{0,1}-2gw'h_x-6es'_x-2\left(U_{2,-2}\right)_x\right)\right.$$
$$-w'\left(s_{xx}-s_y+2us_x-2w_x-2gsh_x\right)$$
$$\left.+2s\left(U_{2,-2}-\overline{u}w'+s'\overline{U}_{0,1}\right)_x\right)=0. \tag{5.79d}$$

Note that these four equations all follow from combinations of (5.40a, 5.40b), together with (4.54, 4.55).

Similarly, compatibility between (5.72a, 5.72c) results in

$$
\begin{aligned}
&(E_{xx} + A_t + AE - A_{xy} - E_y - FA_x - A_xA - EA - B_x \\
&\qquad\qquad + (2F_x - A_x)\left(J\overline{D} + C_x + C^2 + DJ\right) \\
&\qquad\qquad + (2E_x + F_{xx} + AF - A_y - F_y - FA)\,C \\
&\qquad\qquad\qquad\qquad + (2G_x + B\underline{F} - FB - B_y)\,J)\,\boldsymbol{\varphi} \\
&+ (G_{xx} + AG + B_t + B\underline{E} - B_{xy} - G_y \\
&\qquad\qquad - FB_x - EB - G\underline{A} - A_xB - B_x\underline{A} \\
&\qquad\qquad + (2F_x - A_x)\,(CD + D_x + D\underline{C}) \\
&\qquad\qquad + (2E_x + F_{xx} + AF - A_y - F_y - FA)\,D \\
&\qquad\qquad + (2G_x + B\underline{F} - FB - B_y)\,\underline{C} \\
&\qquad\qquad\qquad\qquad -B_x\left(JD + \underline{C}_x + \underline{C}^2 + \underline{D}J\right))\,\underline{\boldsymbol{\varphi}} \\
&+ \left(B\underline{G} - B_x\underline{B} - G\underline{B} - B_x\left(\underline{CD} + \underline{D}_x + \underline{D\underline{C}}\right)\right. \\
&\qquad\qquad\qquad\qquad \left. + (2G_x + B\underline{F} - FB - B_y)\,\underline{D}\right)\underline{\underline{\boldsymbol{\varphi}}} = 0.
\end{aligned}
\tag{5.80}
$$

Likewise, the compatibility between (5.72b, 5.72c) gives

$$
\begin{aligned}
&\Big(-J\overline{A}_x + J\overline{E} - J\overline{AC} - J\overline{FC} \\
&\qquad + \big(2A_x + F_x + E - CA - CF - J\overline{B} \\
&\qquad\qquad +3C_{xx} + AC + F\underline{C} + 3C_xC\big)\, J\Big)\, \overline{\boldsymbol{\varphi}} \\
&+ \Big(A_{xx} + E_x - C_t - J\overline{B}_x - J\overline{G} - CA_x - CE + C_{xxx} \\
&\qquad + AC_x + FC_x + 3C_x^2 - J\overline{AD} - J\overline{FD} \\
&\qquad + \big(2A_x + F_x + E - CA - CF - J\overline{B} + 3C_{xx} \\
&\qquad +AC + F\underline{C} + 3C_xC + BJ + 3D_xJ\big)\, C \\
&\qquad + \big(2B_x + G - CB - D\underline{A} - D\underline{F} + 3D_{xx} + AD \\
&\qquad\qquad +FD + 3C_xD + B\underline{C} + 3D_x\underline{C}\big)\, J\Big)\, \boldsymbol{\varphi} \\
&+ \Big(B_{xx} + G_x - D_t - CB_x - CG - D\underline{A}_x - D\underline{E} + D_{xx} \\
&\qquad + AD_x + FD_x + 3C_xD_x + B\underline{C}_x + 3D_x\underline{C}_x \\
&\qquad + \big(2A_x + F_x + E - CA - CF - J\overline{B} + 3C_{xx} \\
&\qquad +AC + F\underline{C} + 3C_xC + BJ + 3D_xJ\big)\, D \\
&\qquad + \big(2B_x + G - CB - D\underline{A} - D\underline{F} + 3D_{xx} + AD \\
&\qquad +FD + 3C_xD + B\underline{C} + 3D_x\underline{C}\big)\, \underline{C} \\
&\qquad\qquad + \big(B\underline{D} + 3D_x\underline{D} - D\underline{B}\big)\, \underline{\underline{C}}J\Big)\, \underline{\boldsymbol{\varphi}} \\
&+ \Big(B\underline{D}_x + 3\underline{D}_x^2 - D\underline{B}_x - D\underline{G} \\
&\qquad + \big(2B_x + G - CB - D\underline{A} - D\underline{F} + 3D_{xx} + AD \\
&\qquad +FD + 3C_xD + B\underline{C} + 3D_x\underline{C}\big)\, \underline{D} \\
&\qquad\qquad + \big(B\underline{D} + 3D_x\underline{D} - D\underline{B}\big)\, \underline{\underline{C}}\Big)\, \underline{\underline{\boldsymbol{\varphi}}} \\
&+ \Big(\big(B\underline{D} + 3D_x\underline{D} - D\underline{B}\big)\, \underline{\underline{D}}\Big)\, \underline{\underline{\underline{\boldsymbol{\varphi}}}}.
\end{aligned}
\tag{5.81}
$$

The remainder of the system may be extracted from the equations that follow from these compatibility conditions.

5.4 Dimensional Reduction and Degeneration

As in the lattice case, the Potential KP equation and the Elliptic KdV system may be recovered as particular limits of the system (5.2), which we now describe.

5.4.1 Degeneration of the Elliptic Curve

Taking the limit $g \to 0$ causes the elliptic curve to degenerate, and from (5.2a) we recover the lattice potential KP equation, (5.1),

$$\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}\left(u_x\right)^2\right)_x = \frac{3}{4}u_{yy}. \tag{5.82}$$

5.4.2 Dimensional Reduction

By taking particular symmetries, the continuous elliptic KdV system[112], may also be recovered. The KdV class of systems requires that the matrix $\boldsymbol{C}$, and hence $\boldsymbol{U}$ is symmetric under transpositions,

$$^{t}\boldsymbol{C} = \boldsymbol{C} \Rightarrow \quad {}^{t}\boldsymbol{U} = \boldsymbol{U}, \tag{5.83}$$

i.e. $U_{i,j} = U_{j,i}$. This restriction ensures that the primed variables are equal to their unprimed equivalents,

$$s' = s, \quad v' = v \quad w' = w, \tag{5.84}$$

as do the barred variables,

$$\overline{u} = u, \quad \overline{s} = s \quad \overline{w} = w. \tag{5.85}$$

This then results in a dimensional reduction to the elliptic KdV system.

Equation (5.2f) then becomes trivial, whilst (5.2c) then ensures that $s_y = 0$, and hence $u_y = w_y = 0$, removing all y-dependence from the system, resulting in the dimensional reduction required to give the Elliptic KdV system. This causes the system to reduce to

$$\left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}\left(u_x\right)^2 + \frac{3}{2}gs_x^2\right)_x = 0 \tag{5.86}$$

$$s_t = \frac{1}{4}s_{xxx} + \frac{3}{2}s_x\left(u_x - \frac{us_x}{s} - \frac{1}{2}\frac{s_{xx}}{s} + \frac{w_x}{s}\right) \tag{5.87}$$

$$\left(\left(u + \frac{w}{s}\right)_x + \left(u - \frac{w}{s}\right)^2\right)_x = \left(\frac{1}{s^2} + 3e + gs^2\right)_x \tag{5.88}$$

which, upon integration of (5.86) and (5.88), is the elliptic KdV system of [112] up to constants of integration.

5.5 Date, Jimbo and Miwa's elliptic KP system

The system (5.2) is reminiscent of, but different from, an elliptic generalisation of the KP equation given by Date, Jimbo and Miwa in [41]. Their system is given by

$$\overline{u} - u = v_x, \tag{5.89a}$$

$$\frac{3}{4}u_{yy} = \left(u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u_x^2 + 3c^2\mathrm{e}^{v-\underline{v}}\right)_x, \tag{5.89b}$$

$$v_t = \frac{3}{2}v_xv_y - \frac{1}{2}v_{xxx} - \frac{1}{2}v_x^3 - \frac{3}{2}\left(a^2+b^2\right)v_x + \frac{3}{2}\left(\overline{u}_y + u_y\right) + \frac{3}{2}v_x\left(\overline{u}_x + u_x\right), \tag{5.89c}$$

where u and v together with their shifts are variables, and a, b and c are parameters related by $c = (a^2 - b^2)/4$. They provide a Lax triplet for the system given by

$$\varphi_y = -\varphi_{xx} + \alpha\varphi + 2\beta\overline{\varphi}, \tag{5.90a}$$

$$\varphi_t = \varphi_{xxx} + \gamma\varphi_x + \delta\varphi + \epsilon\overline{\varphi}, \tag{5.90b}$$

$$\varphi_{xx} + \underline{v}_x\varphi_x + \mu\varphi = \beta\overline{\varphi} + c\underline{\varphi}, \tag{5.90c}$$

with coefficients

$$\alpha = -2u_x + \frac{1}{2}\left(a^2 + b^2\right), \tag{5.91a}$$

$$\beta = ce^{v-\underline{v}}, \tag{5.91b}$$

$$\gamma = 3u_x, \tag{5.91c}$$

$$\delta = \frac{3}{2}\left(u_x x - u_y\right), \tag{5.91d}$$

$$\epsilon = 3\underline{v}_x\beta, \tag{5.91e}$$

$$\mu = \frac{1}{2}\left(\underline{v}_x - \underline{v}_y + 3u_x - a^2 - b^2\right). \tag{5.91f}$$

However, this Lax triplet they provide does not correspond to the system that they give.

Proof

Considering the compatibility between φ_{yt} and φ_{ty}, and eliminating all derivatives of higher order than one using (5.90c), gives the condition

$$\begin{aligned}
&\left(\alpha_t - \delta_{xx} - \alpha_{xxx} - \gamma\alpha_x + \frac{2c\beta\overline{\epsilon}}{\overline{\beta}} - \delta_y - 2c\epsilon - 12c\beta_x\right)\varphi \\
&+\left(\frac{2\beta\overline{\epsilon}}{\overline{\beta}} - 2\epsilon - 6\beta_x\right)\overline{\varphi}_y \\
&+\Bigg(\alpha\epsilon - \epsilon_{xx} + 2\beta_t + 2\beta\overline{\delta} - \frac{2\beta\overline{\epsilon\mu}}{\overline{\beta}} - \epsilon_y - 2\beta_{xxx} - 2\gamma\beta_x - 4\beta\gamma_x - 2\beta\delta \\
&\qquad -6\beta\alpha_x - \frac{2\overline{\alpha}\beta\overline{\epsilon}}{\overline{\beta}} + \overline{\alpha}\epsilon + 6\overline{\alpha}\beta_x + 2\overline{\mu}\epsilon + 12\overline{\mu}\beta_x\Bigg)\overline{\varphi}.
\end{aligned} \tag{5.92}$$

Equations (5.89a) and (5.89b) follow from the coefficients of $\overline{\varphi}_y$ and φ, respectively. However, the coefficient of $\overline{\varphi}$ gives

$$\begin{aligned}
v_t - \underline{v}_t = &-\frac{1}{2}\left(v_{xxx} - \underline{v}_{xxx}\right) + \frac{3}{2}\left(v_x v_y - \underline{v}_x\underline{v}_y\right) + \frac{3}{2}\left(\left(\overline{u}_y - u_y\right) - \left(u_y - \underline{u}_y\right)\right) \\
&+\frac{3}{2}\left(v_x - \underline{v}_x\right)\left(a^2 + b^2\right) - \frac{3}{2}\left(\overline{u}_x v_x - \underline{u}_x\underline{v}_x\right) \\
&+\frac{1}{2}\underline{v}_x^3 - \frac{3}{2}v_x^2 + v_x^3 + \frac{3}{2}v_x\underline{v}_x - \frac{3}{2}v_x^2\underline{v}_x,
\end{aligned} \tag{5.93}$$

and not (5.89c) as required. □

Our understanding is that the difference between the system given in [41] and (5.2) is in the choice of Cauchy kernel. Whereas our system has a Cauchy kernel related to the elliptic KdV system, given by

$$\Omega(k,k') = \frac{k-k'}{K-K'} = \frac{1-g/\left(KK'\right)}{k+k'}, \tag{5.94}$$

the system given in [41] has a Cauchy kernel related to the Landau-Lifshitz equation, given by

$$\Omega(k,k') = \frac{K-K'}{k+k'} = \frac{k-k'}{1-g/\left(KK'\right)}. \tag{5.95}$$

These Cauchy kernels are related, but it is not yet obvious how this relation manifests itself into a relation at the level of the equations. This will be the subject of further investigation.

Chapter 6

Conclusion

6.1 Discussion

There are a number of results to this thesis. Following an introductory chapter, which outlines the background areas and results relevant to spherical trigonometry, multi-dimensional integrability in terms of the d-simplex equations, spherical volumes and the KP equation, and particle models as the reductions of the KP equation, the first new results are found in Chapter 2.

The nested structure of hyperspherical trigonometry is generalised to any dimension using higher-dimensional vector products. This results in relations between the various angles and sides governing hyperspherical simplices, as generalisations of the main formulae of spherical trigonometry discussed in Section 1.1. Whilst some of the formulae in the four-dimensional case are already known, next to nothing had been done in the higher-dimensional cases. The inter-relations between the formulae, however, become increasingly complicated. The chapter also contains a novel connection between the formulae governing four-dimensional hyperspherical geometry and elliptic functions. This connection is via the generalised Jacobi functions, defined through Abelian integrals associated with a double cover of an elliptic curve. These generalised Jacobi functions are expressible in terms of the usual Jacobi elliptic functions, but with two different moduli k_1 and k_2 [116]. This connection is analogous to the connection between the formulae of spherical trigonometry and the Jacobi elliptic functions discussed in Section 1.1. As such these functions may be of interest for the study of certain elliptic integrable models where solutions in terms of elliptic functions with different moduli appear, for example the $Q4$ lattice equation of the ABS classification [10], and the Dell model [96, 29, 95, 30, 15].

In fact, in Chapter 3, we show that these generalised Jacobi functions provide a more natural parameterisation for the Hamiltonian of the two-particle Dell model. A higher-dimensional Euler top is derived, and also solved in terms of these generalised Jacobi functions. This higher-dimensional Euler top is shown to be equivalent to the two-particle Dell model.

Chapter 4 provides the derivation, from a direct linearisation scheme, of a $(3 + 1)$-dimensional lattice system, naturally associated with an elliptic curve, as an extension of the lattice potential KP equation. We show this system to be integrable through the existence of a Lax representation and soliton solutions. To our knowledge, this is the first integrable lattice system proposed in higher dimensions. An analogous continuous system with three continuous variables and one discrete variable is derived in Chapter5. This continuous system is reminiscent of, but different from, an elliptic generalisation of the KP equation given in [41]. Our understanding is that the difference is in the choice of Cauchy kernel. Whereas our system has a Cauchy kernel related to the elliptic KdV system, we believe that the system in [41] has a Cauchy kernel related to the Landau-Lifshitz equation.

6.2 Future Work

Having derived the formulae between the variables determining hyperspherical simplices, our attention turns to the d-simplex equations. As discussed in Section 1.3.1, the tetrahedron equation has solutions in terms of spherical trigonometry. As the d-simplex equations are higher-dimensional generalisations of the tetrahedron equation, it is natural to think that their solutions may be in terms of higher-dimensional generalisations of spherical trigonometry, i.e. hyperspherical trigonometry. Further investigation is needed to see if this really is the case, and if it is, whether it is the volume of these spherical simplices that plays the key role. The recently established connection between volume forms for hyperspherical tetrahedra and dilogarithms, or Lobachevsky functions, discussed in Section 1.3.2, seems to suggest that there are connections with Lagrangians of certain integrable systems.

The link between addition formulae and hyperspherical geometry suggests that discrete integrable models, such as a discretisation of the two particle Dell model, may be derived exploiting this connection. This is a link that has been previously exploited in the spherical case by Petrera and Suris [122] in producing an integrable map, in the sense of multi-dimensional consistency, based upon the cosine and polar cosine rules. They have shown this map to be related to the Kahan-Hirota-Kimura discretisation of the Euler top [70, 55]. This discretisation of quadratic vector fields was first discovered, although unpublished, by Kahan [70], and later rediscovered independently by Hirota and Kimura [55]. The discretisation was later generalised to a large number of integrable quadratic vector fields by Hone and Petrera [119], and Petrera, Pfadler and Suris [120]. It is more possible that a discretisation of the two particle Dell model may more likely lead to an n-particle Dell model, than in the continuous case.

Alternatively, it may be possible to derive the discrete many-particle Dell model as a reduction of the elliptic KP system derived in Chapters 4 and 5. As discussed in

Section 1.4, the discrete Calogero-Moser model follows from pole solutions of the semi-discretised KP equation, and the discrete Ruijsenaars-Schneider model as a reduction of the fully discrete KP equation. As an elliptic generalisation of the KP equation, it is expected that an elliptic model, potentially the Dell model, will follow from a reduction of the elliptic KP system.

Lastly, how the Cauchy kernels used in elliptic KP system and in [41] relate remains to be understood. Whereas our system has a Cauchy kernel related to the elliptic KdV system, the system in [41] has a Cauchy kernel related to the Landau-Lifshitz equation. These Cauchy kernels are related, but it is not yet obvious how this relation manifests itself into a relation at the level of the equations comprising the two systems.

Bibliography

[1] N.H. Abel, *Recherches sur les fonctions elliptiques*, Journal für die reine und angewandte Mathematik **2(2)** (1827), 101–181.

[2] N.H. Abel, *Recherches sur les fonctions elliptiques*, Journal für die reine und angewandte Mathematik **3(2)** (1828), 160–190.

[3] N.H. Abel, *Precis dune théorie des fonctions elliptiques*, Journal für die reine und angewandte Mathematik **4(4)** (1829), 236–277, 309–348.

[4] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York **9th ed.** (1972).

[5] N. Abromisov and A. Mednykh, *Volumes of polytopes in spaces of constant curvature*, arXiv: 1302.4919 (2013).

[6] P.B. Acosta-Humánez, M. Aranda, and R. Núñez, *Some remarks on a generalized vector product*, Rev. Integr. Temas. Mat. **29(2)** (2011), 151–162.

[7] V.E. Adler, *Bäcklund transformation for the Krichever-Novikov equation*, International Mathematics Research Notices **1** (1998), 1–4.

[8] V.E. Adler, *Discretizations of the Landau-Lifshits equation*, Theoretical and Mathematical Physics **124** (2000), 897–908.

[9] V.E. Adler, A.I. Bobenko, and Y.B. Suris, *Classication of Octahedron-Type Equations*, International Mathematics Research Notices **8** (2012), 1822–1889.

[10] V.E Adler, A.I. Bobenko, and Yu. B. Suris, *Classification of Integrable Equations on Quad-Graphs, the Consistency Approach*, Commun. Math. Phys. **233** (2002), 513–543.

[11] V.E Adler, A.I. Bobenko, and Yu. B. Suris, *Classification of Integrable Cases*, Functional Analysis and its Applications **43(1)** (2009), 3–17.

[12] V.E. Adler and R.I. Yamilov, *Explicit auto-transformations of integrable chains*, Journal of Physics A **27** (1994), 477–492.

[13] H. Airault, H. McKean, and J. Moser, *Rational and elliptic solutions of the Korteweg de Vries equation and a related many-body problem*, Commun. Pure Appl. Math. **30** (1977), 95–148.

[14] N.I. Akheizer, *Elements of the theory of elliptic functions*, American Mathematical Society, U.S.A. (1990).

[15] G. Aminov, A. Mironov, A. Morozov, and A. Zotov, *Three-particle Integrable Systems with Elliptic Dependence on Momenta and θ Function Identities*, Phys. Lett. B. **726** (2013), 802–808.

[16] V. Arnol'd, *Mathematical Methods in Classical Mechanics*, Graduate Texts in Mathematics 60, Springer-Verlag, New York **2** (1989).

[17] M. Audin, *Spinning tops : a course on integrable systems*, Cambridge University Press, Cambridge (1996).

[18] H.F. Baker, *Abelian Functions: Abel's Theorem and the Allied Theory, Including the Theory of the Theta Functions*, Cambridge University Press, New York (1995).

[19] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London (1982).

[20] R.J. Baxter, *On Zamolodchikov's solution of the tetrahedron equations*, Commun. Math. Phys. **88** (1983), 185–202.

[21] R.J. Baxter, *The Yang-Baxter Equations and the Zamolodchikov Model*, Physica D **18** (1986), 321–347.

[22] V.V Bazhanov and R.J. Baxter, *New solvable lattice models in three-dimensions*, J. Statist. Phys. **69** (1992), 453–585.

[23] V.V. Bazhanov, V.V. Mangazeev, and S.M. Sergeev, *Free fermions on a three-dimensional lattice and tetrahedron equations*, Nuclear Phys. B **230** (1984), 435–454.

[24] V.V. Bazhanov and S.M. Sergeev, *Zamolodchikov's tetrahedron equation and hidden structure of quantum groups*, J. Phys. A **39** (2009), 3295–3310.

[25] V.V Bazhanov and Y.G. Stroganov, *Free Fermions on a 3-dimensional lattice and tetrahedron equations*, Nuclear Physics B **230** (1984), 435–454.

[26] A.I. Bobenko and Yu.B. Suris, *Integrable Systems on Quad-Graphs*, International Mathematics Research Notices **11** (2002), 573–611.

[27] A.V. Borisov and I.S. Mamaev, *Poisson structures and Lie-algebras in Hamiltonian mechanics*, University of Udmurtiya, Izhevsk (1999).

[28] H. Braden, A. Marshakov, A. Mironov, and A. Morozov, *On Double-Elliptic Integrable Systems. 1. A Duality Argument for the case of SU(2)*, Nucl. Phys. B **573** (2000), 553–572.

[29] H.W. Braden, A. Gorsky, A. Odesskii, and V. Rubtsov, *Double-elliptic dynamical systems from generalized Mukai-Sklyanin algebras*, N. Phys. B **633** (2002), 414–442.

[30] H.W. Braden, A. Marshakov, A. Mironov, and A. Morozov, *On Double-Elliptic Integrable Systems. 1. A Duality Argument for the case of SU(2)*, Nucl. Phys. B **573** (2000), 553–572.

[31] D. Bressoud, *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*, MAA Spectrum, Washington D.C. (1999).

[32] J. Browne, *Grassman algebra volume 1: Foundations*, Barnard Publishing (2012).

[33] M.A. Cagnoli, *Trigonométrie Rectiligne et Sphérique*, Courcier, Paris (1808).

[34] F. Calogero, *Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials*, J. Math. Phys. **12** (1971), 419–436.

[35] H.W. Capel, G.L. Wiersma, and F.W. Nijhoff, *Linearizing integral transform for the multicomponent lattice KP*, Physica A **138** (1986), 76–99.

[36] A. Cayley, *An Elementary Treatise on Elliptic Functions (2d. ed.)*, G.Bell and Sons, London (1895).

[37] Y. Cho and H. Kim, *On the volume formula for hyperbolic tetrahedra*, Discrete Comput. Geom. **22** (1999), 347–366.

[38] D.V. Chudnovsky and G.V. Chudnovsky, *Pole expansions of nonlinear partial differential equations*, Nuovo Cim. **40B** (1977), 339–353.

[39] H.S.M. Coxeter, *Regular Polytopes*, 3rd ed. Dover Publications, New York (1973), 125.

[40] H. Crabtree, *An elementary treatment of the theory of spinning tops and gyroscopic motion*, Longmans Green, London (1909).

[41] E. Date, M. Jimbo, and T. Miwa, *Method for Generating Discrete Soliton Equations V.*, Journal of the Physical Society of Japan **52** (1983), 766–771.

[42] J.B.J. Delambre, *Connaissance des temps ou des mouvements célestes*, Bureau des Longitudes, France (1809), 445.

[43] D.A. Derevnin and A.D. Mednykh, *A Formula for the Volume of a Hyperbolic Tetrahedron*, Russ. Math. Surv. **60** (2005), 346–348.

[44] D.A. Derevnin, A.D. Mednykh, and M.G. Pashkevich, *On the Volume of Symmetric Tetrahedron*, Siberian Math. J. **45** (2004), 840–848.

[45] P. Desnanot, *Complément de la théorie des èquations du Premier Degrè*, Private Publication, Paris (1819).

[46] F. Eriksson, *The Law of Sines for Tetrahedra and n-Simplices*, Geometriae Dedicata **7** (1978), 71–80.

[47] L. Euler, *Theoria Motus Corporum Solidorum seu Rigidorum*, Opera Omnia **2(3)** (1765).

[48] L. Euler, *Trigometeria spaerica universe, ex primis principiis breviter et dilucide derivata*, Acta Academiae Scientiarum Petropolitanae **3** (1782), 72–86.

[49] L.D. Faddeev, *Recent Advances in Field Theory and Statistical Mechanics*, Les Houches Conference Proceedings (1984), 561.

[50] D. Fairlie, *An Elegant Integrable System*, Phys. Lett. A **119(9)** (1987), 438–440.

[51] W. Feng and S. Zhao, *Generalized Cauchy matrix approach for lattice KP-type equations*, Commun. Nonlinear. Sci. Numer. Simulat. **18** (2013), 1652–1664.

[52] G. Frobenius, *Anwendungen der Determinantentheorie auf die Geometrie des Masses*, Journal für die reine und angewandte Mathematik **73** (1875), 184–247.

[53] H. Grassman, *Die Lineale Ausdehnungslehre ein neuer Zweig der Mathematik*, O. Wigand, Leipzig (1844).

[54] A.G. Greenhill, *The applications of elliptic functions*, MacMillan, London (1892).

[55] K. Hirota, R.and Kimura, *Discretization of the Euler top*, Jour. Phys. Soc. Jap. **69** (2000), 627–630.

[56] R. Hirota, *Discrete Analogue of a Generalised Toda Equation*, Journal of Physical Society of Japan **50** (1981), 3785–3791.

[57] J.G. Hocking and G.S. Young, *Topology*, Dover Publications, New York (1988), 17.

[58] D.D. Holm, *Geometric Mechanics Part I: Dynamics and Symmetry*, Imperial College Press, London (2008).

[59] P.I. Holod, *Hidden symmetry of the Landau-Lifshitz equation, hierarchy of higher equations, and the dual equation for an asymmetric chiral field*, Funktional Anal. i. Prilozhen **70(1)** (1987), 11–19.

[60] M. Horibe and K. Shigemoto, *On solutions of tetrahedron equations based on Korepanov mechanism*, Progr. Theoret. Phys. **93** (1995), 871–878.

[61] M. Horibe and K. Shigemoto, *The structure of the Bazhanov-Baxter model and a new solution of the tetrahedron equation*, Progr. Theoret. Phys. **102** (1999), 221–236.

[62] J.O. Irwin, *Note on the Addition Formula for the Jacobian Elliptic Functions and their Connexion with the Theory of Correlation, in Statistics*, The Mathematical Gazette **49** (1965), 425–427.

[63] R. Ivanov, *Hamiltonian formulation and integrability of a complex symmetric nonlinear system*, Phys. Lett. A **350** (2006), 232–235.

[64] C.G. Jacobi, *De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas tranformandis, quae solis quadratis variabilium constant*, Journal für die reine und angewandte Mathematik **12** (1833), 1–89.

[65] C.G.J. Jacobi, *De functionibus duarum variabilium quadrupliciter periodicis, quibus theori transcendetium Abelianarum innititur*, J. F. Math. **13** (1834), 55–78.

[66] M.T. Jaekel and J.M. Maillard, *Symmetry relations in exactly soluble models*, J. Phys. A **15** (1982), 1309–1325.

[67] K. Jänich, *Vector Analysis*, Springer-Verlag, New York (2001).

[68] P. Jennings and F. Nijhoff, *On an elliptic extension of the Kadomtsev-Petviashvili equation*, J. Phys. A **47(5)** (2014).

[69] B.B. Kadomtsev and V.I Petviashvili, *On the stability of solitary waves in weakly dispersive media*, Sov. Phys. Dokl. **15** (1970), 539–541.

[70] W. Kahan, *Unconventional Numerical Methods for Trajectory Calculations*, Unpublished lecture notes (1993).

[71] M. Kapranov and V.A. Voevodsky, *2-categories and Zamolodchikov tetrahedra equations, Algebraic groups and their generalizations: quantum and infnite-dimensional methods*, Proc. Sympos. Pure Math **56** (1994), 177–259.

[72] R.M. Kashaev, I.G. Korepanov, and S.M. Sergeev, *The functional tetrahedron equation*, Teoret. Mat. Fiz **117** (1998), 370–384.

[73] R.M. Kashaev, V.V. Mangazeev, and Yu.G. Stroganov, *Spatial symmetry, local integrability and tetrahedron equations in the Baxter-Bazhanov model*, Int. J. Mod. Phys. A **8** (1993), 587.

[74] D. Kazhdan and Y. Soibelman, *Representations of the quantized function algebras, 2-categoriesand Zamolodchikov tetrahedra equation*, The Gel'fand Mathematical Seminars, Birkhäuser, Boston (1990), 163–171.

[75] I.G. Korepanov, *Tetrahedral Zamolodchikov algebra and the two-layer flat model in statistical mechanics*, Modern Phys. Lett. B **3** (1989), 201–206.

[76] I.G. Korepanov, *Tetrahedral Zamolodchikov algebras corresponding to Baxter's L-operators*, J. Stat. Phys. **71** (1989), 85–97.

[77] I.G. Korepanov, *Tetrahedron equation and the algebraic geometry*, Teor. Polya i Statist. Fiz. **12** (1994), 137–149, 262.

[78] I.M. Krichever, *Rational solutions of the Kadomtsev-Petviashvili equation and integrable systems of N particles on a line*, Funct. Anal. Appl. **12** (1978), 59–61.

[79] I.M. Krichever, *Elliptic analog of the Toda lattice*, Int. Math. Res. Not. **2000** (2000), 383–412.

[80] I.M. Krichever and S.P. Novikov, *Holomorphic Bundles over Algebraic Curves, and Nonlinear Equations*, Russ. Math. Surv. **35** (1980), 53–80.

[81] I.M. Krichever and S.P. Novikov, *Algebras of Virasoro type, Riemann surfaces and structures of the theory of solitons*, Funktional Anal. i. Prilozhen **21(2)** (1987), 46–63.

[82] I.M. Krichever and S.P. Novikov, *Virasoro type algebras, Riemann surfaces and strings in Minkowski space*, Funktional Anal. i. Prilozhen **21(4)** (1987), 47–61.

[83] I.M. Krichever and S.P. Novikov, *Algebras of Virasoro type, energy-momentum tensors and decompositions of operators on Riemann surfaces*, Funktional Anal. i. Prilozhen **23(1)** (1989), 19–33.

[84] J.L. Lagrange, *Theorie de fonctions analytiques*, Imprimerie de la Republique, Paris (1825).

[85] L.D. Landau and E. Lifschitz, *On the theory of the dispersion of magnetic permeability in ferromagnetic bodies*, Phys. Zeitsch. der Sow. **8** (1935), 135–169.

[86] A.M. Legendre, *Traité des Fonctions elliptiques*, Imprimerie de Huzard-Courcier, Paris (1826).

[87] S.A.J. l'Huillier, *Analogies entre les triangles, rectangles, rectlignes et sphérique*, Annales de mathématiques pures et appliquées **1** (1811), 197–201.

[88] S. Lobb and F. Nijhoff, *Lagrangian multiforms and multidimensional consistency*, J. Phys. A: Math. Theor. **42** (2009), 454013.

[89] J.M. Maillet and F. Nijhoff, *Multidimensional lattice integrability and the simplex equations*, CERN-TH-5202-88, C88-07-04.1 (1988).

[90] J.M. Maillet and F. Nijhoff, *Integrability for multidimensional lattice models*, Phys. Lett. B **224(4)** (1989), 389–396.

[91] J.M Maillet and F.W. Nijhoff, *The tetrahedron equation and the four-simplex equation*, Fermi-Lab Publication **88/71-T** (1988).

[92] V.V. Mangazeev and Yu.G. Stroganov, *Elliptic solution for modifed tetrahedron equations*, Modern Phys. Lett. A **8** (1993), 3475–3482.

[93] J. McMahon, *Hyperspherical Goniometry; and its application to Correlation Theory for N Variables*, Biometrika **XV** (1923), 173–208.

[94] D. Minic and C.H. Tze, *Nambu Quantum Mechanics: A Nonlinear Generalization of Geometric Quantum Mechanics*, Phys. Lett. B **536** (2002), 305–314.

[95] A. Mironov and A. Morozov, *Commuting Hamiltonians from Seiberg-Witten Θ-Functions*, Phys. Lett. B **475** (2000), 71–76.

[96] A. Mironov and A. Morozov, *Double Elliptic Systems: Problems and Perspectives*, arXiv:hep-th/0001168v1 (2000).

[97] T. Miwa, *On Hirota's Difference Equations*, Proc. Japan. Acad. A **58** (1982), 9–12.

[98] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. **16** (1975), 197–220.

[99] J. Murakami and A. Ushijima, *A Volume Formula for Hyperbolic Tetrahedra in terms of Edge Lengths*, J. Geom. **83** (2005), 153–163.

[100] J. Murakami and M. Yano, *On the Volume of a Hyperbolic and Spherical Tetrahedron*, Comm. Anal. Geom. **13** (2005), 379–400.

[101] Y. Nambu, *Generalized Hamiltonian Dynamics*, Physical Review D **7** (1973), 2405–2412.

[102] J. Napier, *Mirifici Logarithmorum Canonis Descriptio*, A. Hart, Edinburgh **2** (1614).

[103] F. Nijhoff, *Lax pair for the Adler (lattice Krichever-Novikov) system*, Phys. Lett. A **297** (2002), 49–58.

[104] F. Nijhoff and A.J. Walker, *The discrete and continuous Painlevé VI hierarchy and the Garnier systems*, Glasgow Mathematical Journal **43A** (2001), 109–123.

[105] F.W. Nijhoff, *Lax pair for the Adler (lattice Krichever-Novikov) system*, Phys. Lett. A **297** (2002), 49–58.

[106] F.W. Nijhoff and H.W. Capel, *The direct linearisation approach to hierarchies of integrable PDEs in* $2+1$ *dimensions: I. Lattice equations and the differential-difference hierarchies*, Inv. Problems **6** (1990), 567–590.

[107] F.W. Nijhoff, H.W. Capel, and G.L. Wiersma, *Integrable lattice systems in two and three dimensions*, Geometric Aspects of the Einstein Equations and Integrable Systems, Lecture Notes in Physics, Springer Verlag, Berlin/New York (1985), 263–302.

[108] F.W. Nijhoff, H.W. Capel, G.L. Wiersma, and G.R.W. Quispel, *Bäcklund Transformations and Three-Dimensional Lattice Equations*, Phys. Lett. A **105** (1984), 267–272.

[109] F.W. Nijhoff and G.D. Pang, *A time-discretized version of the Calogero-Moser model*, Phys. Lett. A **191** (1994), 101–107.

[110] F.W. Nijhoff and G.D. Pang, *Discrete-time Calogero-Moser Model and Lattice KP Equations*, Symmetries and Integrability of Difference Equations, Montral, CRM Lecture Notes and Proceedings Series **9** (1996), 253–264.

[111] F.W. Nijhoff and V. Papageorgiou, *Lattice Equations associated with the Landau-Lifschitz Equations*, Phys. Lett. A **141** (1989), 269–274.

[112] F.W. Nijhoff and S. Puttock, *On a two-parameter extension of the lattice KdV system associated with an elliptic curve*, Journal of Nonlinear Mathematical Physics **10** (2003), 107–123.

[113] F.W. Nijhoff, G.R.W. Quispel, and H.W. Capel, *Direct Linearisation of Difference-Difference equations*, Phys. Lett. A **97** (1983), 125–128.

[114] F.W. Nijhoff, O. Ragnisco, and V. Kuznetsov, *Integrable Time-Discretisation of the Ruijsenaars-Schneider Model*, Commun. Math. Phys. **176** (1996), 681–700.

[115] M. Noumi, *Painlevé Equations through Symmetry*, AMS Translations of Mathematical Monographs, AMS **223** (2000).

[116] M. Pawellek, *On a Generalisation of Jacobi's Elliptic functions and the Double Sine-Gordon Kink Chain*, J. Math. Phys. **52** (2011).

[117] J.H.H. Perk and H. Au-Yang, *Yang-Baxter Equations*, Encyclopedia of Mathematical Physics, Elsevier Science, Oxford **5** (2006), 465–473.

[118] E. Peschl, *Winkelrelationen am Simplex und die Eulersche Charakteristik*, Münch. Akad. Sb. **25** (1955), 319–345.

[119] M. Petrera and A. Hone, *Three-dimensional discrete systems of Hirota-Kimura type and deformed Lie-Poisson algebras*, Journal of Geometric Mechanics **1** (2009), 55–85.

[120] M. Petrera, A. Pfadler, and Y.B. Suris, *On integrability of Hirota-Kimura type discretizations*, Regular & Chaotic Dynamics **16** (2011), 245–289.

[121] M. Petrera and Yu. Suris, *S. V. Kovalevskaya system, its generalisation and discretisation*, Front. Math. China **8(5)** (2013), 1047–1065.

[122] M. Petrera and Yu.B. Suris, *Spherical Geometry and Integrable Systems*, Geometriae Dedicata **10.1007/s10711-013-9843-4** (2012).

[123] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. **170** (1986), 370–405.

[124] K. Sato, *Spherical Simplices and their Polars*, Quart. J. Math. **58** (2007), 107–126.

[125] L. Schläfli, *Theorie der Vielfachen Kontinuität*, Gesammelte Mathematische Abhandlungen, Birkhauser, Basel (1950).

[126] S. Sergeev, *Geometry of Quadrilateral Nets: Second Hamiltonian Form*, J. Geom. & Phys. **59** (2009), 1150–1154.

[127] S. Sergeev, *Mathematics of Quantum Integrable Systems in Multidimensional Discrete Space-Time*, Preprint **Version 4** (2009).

[128] E.K. Skyanin, *On complete integrability of the Landau-Lifschitz equation*, Preprint, LOMI E-3-79, Leningrad (1979).

[129] Y. Soibelman, *Algebra of functions on a compact quantum group and its representations*, Algebra i Analiz **2** (1990), 190–212.

[130] B. Sturmfels, *Algorithms in Invariant Theory*, Texts and Monographs in Symbolic Computation, Springer-Verlag, Vienna (1993).

[131] K. Takasaki, *Geometry of Universal Grassmann Manifold from Algebraic Point of View*, Reviews in Mathematical Physics **1(1)** (1989), 1–146.

[132] L. Takhtajan, *On Foundation of the Generalized Nambu Mechanics*, Comm. Math. Phys. **160** (1994), 295–315.

[133] I. Todhunter and J.G. Leatham, *Spherical Trigonometry*, McMillan and Co., London (1914).

[134] E.B. Vinberg, *Volume of non-Euclidean Polyhedra*, Russ. Math. Surv. **48** (1993), 15–45.

[135] K. Weierstrass, *Werke*, **I** (1894, 1895), 1–49, 245–255, 257–309.

[136] E. T. Whittaker and G. N. Watson, *A Course in Modern Analysis*, Cambridge University Press, Cambridge **4th ed.** (1990).

[137] S. Winitzki, *Linear Algebra via Exterior Products*, Lulu Publishing, Raleigh (2010).

[138] C.N. Yang, *Some Exact Results for the Many-Body Problem in one Dimension with Repulsive Delta-Function Interaction*, Physical Review Letters **19** (1967), 1312–1315.

[139] S. Yoo-Kong, S. Lobb, and F. Nijhoff, *Discrete-time Calogero-Moser system and Lagrangian 1-form structure*, J. Phys. A: Math. Theor. **44** (2011), 365203.

[140] S. Yoo-kong and F. Nijhoff, *Discrete-time ruijsenaars-schneider system and lagrangian 1-form structure*, arXiv:1112.4576v2 (2013).

[141] A.B. Zamolodchikov, *Factorised S-matrices and exactly solvable lattice models*, Soviet Scientific Reviews A **2** (1980), 100–135.

[142] A.B. Zamolodchikov, *Tetrahedra equations and integrable systems in three-dimensional space*, Sov. Phys. JETP **52(2)** (1980), 325–336.

[143] A.B. Zamolodchikov, *Tetrahedron Equations and the Relativistic S-Matrix of Straight-Strings in 2+1 Dimensions*, Comm. Math. Phys. **79** (1981), 489–505.

[144] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Factorized S-Matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models*, Annals of Physics **120** (1979), 253–291.

[145] T. Zehrt, *Polytopal Complexes in Space of Constant Curvature*, PhD. Thesis, Basel (2003).