Model Theory, Algebra and Differential Equations

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The candidate confirms that the work submitted is his own, except where work which has formed part of jointly-authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Work from the following jointly-authored publications is included in this thesis:

- **Paper 1:** J. Nagloo and A. Pillay, On algebraic relations between solutions of a generic Painlevé equation, accepted in J. Reine Angew. Math.
- **Paper 2:** J. Nagloo and A. Pillay, On the algebraic independence of generic Painlevé transcendents, Compositio Mathematica, doi:10.1112/S0010437X13007525.

Chapters 3 and 4 are based on Paper 1, and Chapter 6 is based on Paper 2. The contribution of authors to the papers listed above consists of:

- **Paper 1:** A. Pillay provided a method that could be used to prove geometric triviality of the generic Painlevé equations and sketched such a proof for the second equation. J. Nagloo did the detailed investigation of the other Painlevé equations and use that method to prove geometric triviality of all the generic Painlevé equations. Other concepts were developed in discussions.
- **Paper 2:** J. Nagloo came up with a method that could be used to prove algebraic independence of the generic second Painlevé equations. A. Pillay noticed that this method could be adapted for the other equations. J. Nagloo investigated further and was able to prove algebraic independence (ω-categoricity in some cases) of the other generic Painlevé equations.

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Abstract

In this thesis, we applied ideas and techniques from model theory, to study the structure of the sets of solutions $X_I - X_{VI}$, in a differentially closed field, of the Painlevé equations. First we show that the generic $X_{II} - X_{VI}$, that is those with parameters in general positions, are strongly minimal and geometrically trivial. Then, we prove that the generic X_{II}, X_{IV} and X_V are strictly disintegrated and that the generic X_{III} and X_{VI} are ω-categorical. These results, already known for X_I, are the culmination of the work started by P. Painlevé (over 100 years ago), the Japanese school and many others on transcendence and the Painlevé equations. We also look at the non generic second Painlevé equations and show that all the strongly minimal ones are geometrically trivial.

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Chapter 1

Introduction

> *Les mathématiqués constituent un continent solidment agencé, dont tous les pays sont bien reliés les uns aux autres; l'oeuvre de Paul Painlevé est une ile originale et splendide dans l'océan voisin.*
>
> – Henri Poincaré

The Painlevé equations are second order ordinary differential equations and come in six families $P_I - P_{VI}$, where P_I consists of the single equation $y'' = 6y^2 + t$, and $P_{II} - P_{VI}$ come with some complex parameters:

$$
\begin{aligned}
P_{II}(\alpha): &\quad y'' = 2y^3 + ty + \alpha \\
P_{III}(\alpha,\beta,\gamma,\delta): &\quad y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_{IV}(\alpha,\beta): &\quad y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y} \\
P_V(\alpha,\beta,\gamma,\delta): &\quad y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} \\
&\quad + \delta\frac{y(y+1)}{y-1} \\
P_{VI}(\alpha,\beta,\gamma,\delta): &\quad y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
&\quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)
\end{aligned}
$$

They were isolated in the early part of the 20^{th} century, by Painlevé, with refinements by Gambier and Fuchs, as those ODE's of the form $y'' = f(y, y', t)$ (where f is rational over $\mathbb{C}$) which have the Painlevé property: any local analytic solution extends to a meromorphic solution on the universal cover of $P^1(\mathbb{C})\backslash S$, where S is the finite set of singularities of the equation (including the point at infinity if necessary).

Painlevé also believed that the solutions of the equations, at least for general values of the parameters, defined "new" special functions. He gave a definition of "known" functions and claimed to have proved that no solution of the first Painlevé equation is "known", that is the equation is "irreducible" (cf. [33]). Unfortunately, his definition lacked a bit of rigour and it took the work of Nishioka ([26]) and Umemura ([46]), after about 80 years, to clarify his notion of irreducibility. Of course, this special notion should not be confused with the usual notions of irreducibility of an algebraic (or differential algebraic) variety. In any case, in a series of papers by Okamoto, Nishioka, Noumi, Umemura, Watanabe, and others, the irreducibility of $P_I - P_{VI}$ outside special values of the complex parameters was established.

It turns out that what was actually proved by the Japanese school, is that, except for some special values of the parameters, the set of solutions of any of the Painlevé equations, considered as a differential algebraic variety or definable set in an ambient differentially closed field is strongly minimal in the sense of model theory. This is where the work in this thesis begins. Strong minimality is a fundamental notion in model theory. A definable set is said to be minimal if every definable subset is finite or cofinite and strongly minimal if it is minimal in every elementary extension. In differentially closed fields strongly minimal sets determine, in a precise manner, the structure of "finite rank" definable sets. It is no surprise then that a considerable amount of work had been devoted to further our understanding of them.

The deepest result in that direction, due to Hrushovski and Sokolovic [13], concerns the classification of strongly minimal sets. This classification, called the trichotomy theorem, asserts that there are only three types of strongly minimal sets: Type (i): those nonorthogonal to the constants; a version of algebraic integrability after base change. Type (ii): those closely related to the solution set $A^\sharp$ of a very special kind of ODE on a simple abelian variety A (of which $P_{VI}(0,0,0,1/2)$ is an example). Type (iii): those geometrically trivial: for equations of the form $y'' = f(y, y', t)$ (where f is rational over $\mathbb{C}$), geometric triviality means that for any distinct solutions $y_1, \ldots, y_n$, if $y_1, y_1', \ldots, y_n, y_n'$ are algebraically dependent over $\mathbb{C}(t)$, then already for some $1 \leq i < j \leq n$, y_i, y_i', y_j, y_j' are algebraically dependent over $\mathbb{C}(t)$.

So the first natural question to ask is: "Where do the strongly minimal Painlevé equations fit in the Hrushovski and Sokolovic classification?". This brings us to our first result (Propositions 4.3.6, 4.3.9, 4.3.12, 4.3.15 and 4.3.20):

Result A. *The generic Painlevé equations (that is those in the families $P_{II} - P_{VI}$ with algebraically independent complex parameters $\alpha, \beta, ..$), are geometrically trivial*

These equations give the first examples of geometrically trivial sets of order > 1. For the proof, we use the above-mentioned results of the Japanese school, together with additional techniques that allows us to rule out Type (i) and Type (ii) in the classification. One of the crucial ingredients is that for each of the Painlevé families $P_{II} - P_{VI}$, the set of complex tuples $(\alpha, \beta, \ldots)$ for which the corresponding equation is not strongly minimal has "infinitely many components".

Although the technique used in the proof of Result A is quite general and uniform, in the sense that it works for all the generic equations, it fails quite miserably for non generic parameters. Moreover, by adapting some techniques of Nishioka [27] in his study of P_I, we have been able to extend Result A for P_{II} to the non generic case (Propsition 5.2.2).

Result B. *The strongly minimal second Painlevé equations (i.e. whenever $\alpha \notin 1/2 + \mathbb{Z}$) are geometrically trivial*

Quite remarkably, using Result A and building on the ideas of its proof, we have also been able to prove what has been an old belief in the Painlevé theory (Propositions 6.2,2, 6.2.4, 6.2.7, 6.3.2 and 6.3.3):

Result C. *(i) Suppose $y_1, \ldots, y_n$ are distinct solutions of any generic P_{II}, P_{IV} or P_V. Then tr.deg $(\mathbb{C}(t)(y_1, y_1', \ldots, y_n, y_n')/\mathbb{C}(t)) = 2n$.*
(ii) Suppose $y_1, \ldots, y_k$ are distinct solutions of any generic P_{III} (repectively P_{VI}) such that tr.deg$(\mathbb{C}(t)(y_1, y_1', \ldots, y_k, y_k')/\mathbb{C}(t)) = 2k$. Then for all other solutions y, except for at most k (respectively 11k), tr.deg$(\mathbb{C}(t)(y_1, y_1', \ldots, y_k, y_k', y, y')/\mathbb{C}(t)) = 2(k+1)$.

Result C (i) means that the solution set (in a differentially closed field) of a generic P_{II}, P_{IV} or P_V is strictly disintegrated over $\mathbb{C}(t)$, while Result C (ii) means that the solution set of any generic P_{III} or P_{VI} is ω-categorical. It is worth mentioning that Result C is the culmination of the work started by P. Painlevé (over 100 years ago), the Japanese school and many others on transcendence and the Painlevé equations.

Let us finish by mentioning the structure of the thesis: In Chapter 2 we recall the necessary background in (ω-)stability theory and the model theory of differentially closed fields. We also give a detailed account of the proof of the trichotomy theorem mentioned above. In Chapter 3 we look at Umemura's irreducibility notion and explain how it translates to the model theoretic notion "analysability" and in particular point out its relation to strong minimality. Chapter 4, 5 and 6 are where the main results (A, B and C respectively) are proved. Finally in Chapter 7 we show how one can use the same techniques as in the proof of Result A (or C) to answer negatively, in most cases, the following question of P. Boalch: "Given any two generic Painlevé equations from any of the families $P_I - P_{VI}$, are there differential transformations between them?"

Chapter 2

Preliminaries

> *It is at first surprising that such a preliminary model-theoretic investigation of the basic geography of algebraic differential equations should discover Abelian varieties in a special role.*
>
> – Ehud Hrushovski

In this chapter we provide a summary of the model theoretic and differential algebraic notions that play an important role in the thesis. Our aim is to give an account of one of the most important results in the area, namely, the trichotomy for strongly minimal sets. This powerful result is at the heart of our work. We will assume that the reader has a working knowledge of the fundamentals of model theory and some understanding of the basics of algebraic geometry.

2.1 Stability theory

This section gives an overview of the basic machinery of ω-stability theory. Good references in this case are the book of Marker [19] and the online lecture notes of Pillay [41]. One can also find a more general treatment of stability theory in Pillay's book [35]. Examples to illustrate the various notions introduced here will only appear in the next section when we look at DCF_0.

Throughout L will be a countable language and T a complete L-theory. We also assume that T has elimination of imaginaries. Recall that this means that for any $\emptyset$-definable set Y, if E is a $\emptyset$-definable equivalence relation on Y, then there is a $\emptyset$-definable map $f_E : Y \to \mathcal{U}^m$, for some m, such that xEy if and only if $f_E(x) = f_E(y)$. In other words we can view Y/E as the definable set $f_E(Y)$.

2.1.1 Forking and Independence

Let $\mathcal{U}$, for now, be a κ-saturated, κ-strongly homogeneous model of T for a sufficiently big $\kappa > |T|$.

Definition 2.1.1. Let $Y \subseteq \mathcal{U}^n$ be a nonempty definable set

(i) The *Morley rank* of Y is defined inductively as follows:

- $RM(Y) \geq 0$;
- $RM(Y) \geq \alpha + 1$ if and only if there are definable subsets Y_i of Y for $i < \omega$ which are pairwise disjoint and such that $RM(Y_i) \geq \alpha$;
- For β a limit ordinal, $RM(Y) \geq \beta$ if and only if $RM(Y) \geq \alpha$ for all $\alpha < \beta$.

Then $RM(Y) = \alpha$ if $RM(Y) \geq \alpha$ and $RM(Y) \ngeq \alpha + 1$. We also write $RM(Y) = \infty$ if $RM(Y) \geq \alpha$ for all α.

(ii) If $RM(Y) = \alpha < \infty$ then the *Morley degree* of Y, $dM(Y)$, is the largest number d so that there are pairwise disjoint definable subsets Y_i of Y, with $RM(Y_i) = \alpha$ for $i = 1, \ldots, d$.

For a complete type p, the Morley rank and Morley degree are respectively defined as $RM(p) := inf\{RM(\phi(\mathcal{U})) : \phi \in p\}$ and $dM(p) = inf\{dM(\phi(\mathcal{U})) : \phi \in p, RM(\phi(\mathcal{U})) = RM(p)\}$ and if $dM(p) = 1$, p is said to be *stationary.*

Remark 2.1.2. Given a tuple $\overline{a}$ from $\mathcal{U}$, $RM(\overline{a}/B)$ (resp. $dM(\overline{a}/B)$) denotes the Morley rank (resp. Morley degree) of the type of $\overline{a}$ over B.

The Morley rank will allow us to define a good notion of independence and dimension. However, given an arbitrary T, it is not necessarily true that every definable set has ordinal valued Morley rank. This property is reserved for a special class of theories:

Definition 2.1.3. T is said to be *totally transcendental* if for every definable set Y, $RM(Y) < \infty$.

When working in a countable language, one can characterise totally transcendental theories in terms of counting types. We first need the following definition.

Definition 2.1.4. Let κ be an infinite cardinal.

(i) T is said to be κ*-stable* if for every A of cardinality at most κ, $|S_n(A)| \leq \kappa$. One usually says ω*-stable* instead of $\aleph_0$-stable.

(ii) T is said to be *stable* if it is κ-stable for some κ.

Theorem 2.1.5 ([19], Theorem 4.2.18 and 6.2.14). *The following are equivalent:*

(i) T is ω-stable.

(ii) T is totally transcendental.

(iii) T is κ-stable for all infinite κ.

As we shall see in the next section, it is easier to check whether or not a given theory is totally transcendental using the above characterisation. It turns out that the ω-stable theories are among the nicest in the class of stable theories. For example one has the following

Fact 2.1.6 ([19], Theorem 4.2.20 and 6.5.4). Assume that T is ω-stable. Then

(i) T has saturated models of size κ for each cardinal $\kappa > \aleph_0$.

(ii) T has prime models over any parameter set.

(iii) Any complete type $p(\overline{x}) \in S(A)$ is definable: for any formula $\varphi(\overline{x}, \overline{y})$, there is a formula $d_\varphi(\overline{y}) \in L_A$ such that for any $\overline{a} \in A$, $\varphi(\overline{x}, \overline{a}) \in p$ if and only if $\models d_\varphi(\overline{a})$.

The first assertion means that we can choose the size of $\mathcal{U}$ to be κ. The second assertion means that for any $A \subset \mathcal{U}$, there is $M_0 \models T$, with $A \subset M_0$, such that whenever $M \models T$ and $f : A \to M$ is partial elementary, there is an elementary $\hat{f} : M_0 \to M$ extending f. In ω-stable theories such an M_0 is unique up to isomorphisms fixing A.

Remark 2.1.7. The theory T is stable if and only if any complete type is definable.

From now on we assume that T is ω-stable and we fix a sufficiently saturated model $\mathcal{U}$ of T. The promised notion of independence is the following:

Definition 2.1.8. Suppose $A \subseteq B \subset \mathcal{U}$, $p \in S_n(A)$, $q \in S_n(B)$, and $p \subseteq q$. We say that q is a nonforking extension of p if $RM(q) = RM(p)$. If $\overline{a}$ is a tuple from $\mathcal{U}$ then we say that $\overline{a}$ is independent from B over A, written $\overline{a} \underset{A}{\downarrow} B$, if $tp(\overline{a}/B)$ is a nonforking extension of $tp(\overline{a}/A)$; that is, if $RM(\overline{a}/B) = RM(\overline{a}/A)$.

Theorem 2.1.9 ([19], Section 6.3). *Let $\overline{a}$ be a tuple and let $A \subseteq B$*

(i) There is $\overline{b}$ such that $tp(\overline{a}/A) = tp(\overline{b}/A)$ and $\overline{b} \underset{A}{\downarrow} B$.

(ii) If $C \supseteq B$, then we have $\overline{a} \underset{A}{\downarrow} C$ if and only if $\overline{a} \underset{A}{\downarrow} B$ and $\overline{a} \underset{B}{\downarrow} C$.

(iii) For another tuple $\overline{b}$, $\overline{a} \underset{A}{\downarrow} \overline{b}$ if and only if $\overline{b} \underset{A}{\downarrow} \overline{a}$.

(iv) $\overline{a} \underset{A}{\downarrow} B$ if and only if for all finite $B' \subseteq B$, $\overline{a} \underset{A}{\downarrow} B'$.

For obvious reasons (i), (ii), (iii) and (iv) are usually referred to as existence, transitivity, symmetry and finite character respectively. Also, for a type $p \in S(A)$, being stationary is equivalent to p having a unique nonforking extension to any $B \supseteq A$.

Definition 2.1.10. Let $Y \subseteq \mathcal{U}^n$ be definable. We say that $\overline{e}$ is a *canonical parameter* of Y if for all $\sigma \in Aut(\mathcal{U})$, σ fixes Y setwise if and only if σ fixes $\overline{e}$ pointwise.

As we assume that T has elimination of imaginaries, we have that any definable set has a canonical parameter in $\mathcal{U}$. Indeed, if Y is defined by $\varphi(x, \overline{a})$ we can let E be the equivalence relation $\overline{a}E\overline{b} \Leftrightarrow \forall x \left(\varphi(x, \overline{a}) \leftrightarrow \varphi(x, \overline{b})\right)$. Then $\overline{e} = f_E(\overline{a}/E)$ is a canonical parameter for Y. It is not hard to see that a canonical parameter is determined up to interdefinability. Now consider a stationary type $p \in S(A)$. A *canonical base* for p is a tuple fixed pointwise by the automorphisms of $\mathcal{U}$ that fix the global nonforking extension of p. As before, by elimination of imaginaries, a canonical base of p exists in $\mathcal{U}$ and is unique up to interdefinability. One usually writes $Cb(p)$ for the definable closure of any canonical base of p.

Lemma 2.1.11 ([41], **Lemma 2.38).** *Let $p \in S(A)$ be a stationary type. Then*

(i) $Cb(p) \subseteq dcl(A)$.

(ii) For any $B \subseteq A$, p does not fork over B if and only if $Cb(p) \subseteq acl(B)$.

(iii) $Cb(p)$ is interdefinable with a finite tuple.

Many of the properties discussed above hold for stable theories (cf. [35]) but we will not talk about this here.

2.1.2 Strongly minimal sets

We continue in similar settings as the previous section. So T is a countable complete ω-stable theory which eliminates imaginaries and $\mathcal{U}$ a sufficiently saturated model of T.

Definition 2.1.12. An infinite definable set Y in $\mathcal{U}$ is said to be *strongly minimal* if for every definable subset $Z \subseteq Y$, either Z or $Y \setminus Z$ is finite. Equivalently, Y is strongly minimal if and only if $RM(Y) = dM(Y) = 1$.

For a strongly minimal set $Y \subseteq \mathcal{U}$, if we let $acl_Y(A) = acl(A) \cap Y$, we have that (Y, acl_Y) forms a *pregeometry*, namely for any $A \subseteq Y$, the following holds (c.f [35]):

(i) $A \subseteq acl_Y(A)$ and $acl_Y(acl_Y(A)) = acl_Y(A)$.

(ii) If $a \in acl_Y(A \cup \{b\}) \setminus acl_Y(A)$, then $b \in acl_Y(A \cup \{a\})$.

(iii) If $a \in acl_Y(A)$, then there is some finite $A_0 \subseteq A$ such that $a \in acl_Y(A_0)$.

Property (iii) is known as the exchange property. Now, for any $A \subseteq Y$ and $B \subset \mathcal{U}$, we say that A is independent over B if for all $a \in A$, $a \notin acl_Y((A \setminus \{a\}) \cup B)$. Furthermore, $A_0 \subseteq A$ is a basis for A over B if A_0 is independent over B and $A \subseteq acl_Y(A_0 \cup B)$. It turns out that any two basis have the same cardinality and one writes $dim(A/B) = |A_0|$ for any basis A_0 of A over B. The upshot is that, if Y is a strongly minimal set defined over some B, then for any finite tuple $\overline{a}$ from Y and any $C \supseteq B$, $RM(\overline{a}/C) = dim(\overline{a}/C)$.

Definition 2.1.13. A pregeometry (Y, cl) is said to be

(i) *Modular*: if for all $b, c \in Y$ and $A \subseteq Y$, if $c \in cl(A, b)$, then there is an $a \in acl(A)$ such that $c \in cl(a, b)$;

(ii) *Locally modular*: if there is some $a \in Y$ such that $(Y, cl_{(a)})$ is modular.

(iii) *Geometrically trivial*: if $cl(A) = \cup_{a \in A} cl(\{a\})$ for all $A \subseteq Y$;

Here $(Y, cl_{(a)})$ is the pregeometry obtained after localising at a, that is for $B \subset \mathcal{U}$, $cl_{(a)}(B) = cl(\{a\} \cup B)$. Also it is not hard to see that geometric triviality implies modularity.

One can use the canonical base to give a different characterisation of modularity. We first need another definition

Definition 2.1.14. Let Y be an A-definable set. Then Y is said to be *one-based* if for every tuple $\overline{a}$ from Y and $B \supseteq A$, $Cb(tp(\overline{a}/acl(B))) \subseteq acl(A\overline{a})$

Fact 2.1.15 ([41], Lemma 3.32 and Theorem 3.35). Let Y be a strongly minimal set. Then

(i) If Y is modular, then Y is one-based.

(ii) Y is locally modular if and only if Y is one-based

The three typical examples of nonmodular, nontrivial modular and geometrically trivial strongly minimal sets are respectively algebraically closed fields, vector spaces and infinite sets with no structure. Zilber conjectured that these are "essentially" all there is.

Definition 2.1.16. Let Y and Z be strongly minimal sets defined over A and B respectively and denote by $\pi_1 : Y \times Z \to Y$ and $\pi_2 : Y \times Z \to Z$ the projections to Y and Z respectively. We say that Y and Z are nonorthogonal if there is some infinite definable relation $R \subset Y \times Z$ such that $\pi_{1\restriction R}$ and $\pi_{2\restriction R}$ are finite-to-one functions. We usually write $Y \not\perp Z$.

It is not hard to see that nonorthogonality is an equivalence relation for strongly minimal sets.

Zilber's Principle. *If Y is a non locally modular strongly minimal, then there is a strongly minimal algebraically closed field F, definable in $\mathcal{U}$, such that Y is nonorthogonal to F.*

Although the principle holds in many "important" examples, Hrushovski proved that it is false in general. It will nevertheless be true in the theory of differential closed fields of characteristic zero and this is crucial for the results in this thesis. On the other hand, note that for one-based groups and locally modular strongly minimal sets one has the following very general results.

Fact 2.1.17 ([35], Corollary 4.4.8 and Theorem 5.1.1).

(i) if G is a one-based group definable in $\mathcal{U}$ and $Y \subseteq G^n$ is definable, then Y is a finite Boolean combination of cosets of definable subgroups $H \leq G^n$.

(ii) Suppose $X \subseteq \mathcal{U}^n$ is a non geometrically trivial locally modular strongly minimal set. Then X is nonorthogonal to a definable modular strongly minimal group.

We finish this section by mentioning ω-categoricity, a property that is related to the "finer" structure of pregeometries. Recall that an infinite structure M in a countable language L is said to be *ω-categorical* if for each

n there are only finitely many $\emptyset$-definable subsets of M^n. The reason for the nomenclature is that M is ω-categorical if and only if $Th(M)$ has exactly one countable model, up to isomorphism. We want an analogous notion for definable sets.

Definition 2.1.18. Suppose $Y \subset \mathcal{U}^n$ is definable over some parameter set A. We say that Y is *ω-categorical in $\mathcal{U}$ over A* if there are finitely many subsets of Y^n which are definable over A, for each n.

As we are in the ω-stable context, this notion does not depend on the choice the parameter set A:

Lemma 2.1.19. *Suppose $Y \subset \mathcal{U}^n$ is definable, and let $\bar{b}, \bar{c}$ be finite tuples from $\mathcal{U}$ over which Y is definable. Then Y is ω-categorical in $\mathcal{U}$ over $\bar{b}$ iff Y is ω-categorical in $\mathcal{U}$ over $\bar{c}$. Therefore we just say that Y is ω-categorical in $\mathcal{U}$.*

Proof. It is enough (by adding parameters) to prove that if Y is $\emptyset$-definable, and ω-categorical over $\emptyset$ and $\bar{b}$ is any finite tuple from $\mathcal{U}$ then Y is ω-categorical over $\bar{b}$. Now by Fact 2.1.6, $tp(\bar{b}/Y)$ is definable over a tuple $\bar{e}$ of elements of Y. As ω-categoricity is preserved after naming a finite tuple from Y, we see that Y is ω-categorical over $\bar{e}$, so also over $\bar{b}$ (as every $\bar{b}$-definable subset of Y^m is $\bar{e}$-definable). □

For strongly minimal sets one can do better.

Lemma 2.1.20. *Let $Y \subset \mathcal{U}^n$ be a strongly minimal definable set. Then Y is ω-categorical if and only if for any finite tuple b from $\mathcal{U}$ over which Y is defined $acl(b) \cap Y$ is finite.*

Proof. If M is any structure then it is clear that M is ω-categorical just if for any finite tuple $\bar{a}$ from M, there are only finitely many $\bar{a}$-definable subsets of M. If M is also strongly minimal and $\bar{a}$ is a finite tuple from M, then as any $\bar{a}$-definable subset of M is finite or cofinite, there are only finitely many $\bar{a}$-definable subsets of M iff $acl(\bar{a})$ is finite. So the Lemma holds for a structure M in place of Y. The full statement follows as in the proof of the previous Lemma. □

Remark 2.1.21. If Y and Z are nonorthogonal strongly minimal sets in $\mathcal{U}$, then Y is ω-categorical iff Z is ω-categorical.

Finally one has the following general result of Zilber:

Theorem 2.1.22 ([35], Theorem 2.4.17). *If $X \subset \mathcal{U}^n$ is a definable, strongly minimal ω-categorical set, then X is modular.*

2.2 Differentially closed fields

The goal of this section is twofold. First, we will try to explain how all the abstract notions introduced in the previous section give rise to meaningful tools when applied to the concrete context of differential algebra. Secondly, we aim to give an account of the proof that Zilber's principle and the trichotomy theorem (Theorem 2.3.11) are true in DCF_0. As mentioned several times already, this will play an important role in our work.

2.2.1 Basic definitions and properties

Definition 2.2.1. A *differential field* (K, δ) is a field K equipped with a derivation $\delta : K \to K$, i.e. an additive group homomorphism satisfying the Leibniz rule

$$\delta(xy) = x\delta(y) + y\delta(x).$$

The field of *constants* C_K of K is defined set theoretically as $\{x \in K : \delta(x) = 0\}$. We usually write x' for $\delta(x)$ and $x^{(n)}$ for $\underbrace{\delta \dots \delta\delta}_{n}(x)$.

Example 2.2.2. $(\mathbb{C}(t), d/dt)$ the field of rational functions over $\mathbb{C}$ in a single indeterminate, where in this case, the field of constants is $\mathbb{C}$.

For each $m \in \mathbb{N}_{>0}$, associated with a differential field (K, δ), is the *differential polynomial ring* in m differential variables, $K\{\overline{X}\} = K[\overline{X}, \overline{X}', \dots, \overline{X}^{(n)}, \dots]$, where $\overline{X} = (X_1, \dots, X_m)$ and $\overline{X}^{(n)} = (X_1^{(n)}, \dots, X_m^{(n)})$. If $f \in K\{\overline{\overline{X}}\}$ is a differential polynomial, then the order of f, denoted $ord(f)$, is the largest n such that for some i, $X_i^{(n)}$ occurs in f.

Example 2.2.3. $f(X) = (X')^2 - 4X^3 - tX$ is an example of a differential polynomial in $\mathbb{C}(t)\{X\}$ and $ord(f) = 1$.

The analogue of algebraically closed fields in the differential context is defined as follows

Definition 2.2.4. A differential field (K, δ) is said to be *differentially closed* if for every $f, g \in K\{X\}$ such that $ord(f) > ord(g)$, there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.

We will later see an equivalent definition of a more geometric nature due to Pierce and Pillay. The one given above is due to Blum [1]. As a consequence of the definition, a differentially closed field is also algebraically closed. Differentially closed fields are the natural places for studying differential equations from an algebraic/geometric perspective and we shall say a little bit more about this now.

A differential ideal I in $K\{\overline{X}\}$ is an ideal which is closed under the derivation, that is $\delta(f) \in I$ for all $f \in I$. As with classical algebraic geometry, if $S \subseteq K\{\overline{X}\}$, by choosing $V_\delta(S) = \{x \in K^n : f(x) = 0 \ \forall \ f \in S\}$ as basic closed sets, we obtain a topology on K^n called the *Kolchin topology*. The Kolchin topology is Noetherian (cf. [18] Theorem 1.16 and 1.19):

Theorem 2.2.5 (Ritt-Raudenbush Basis Theorem). *Suppose (K, δ) is a differential field. Then*

(i) There are no infinite ascending chains of radical differential ideals in $K\{\overline{X}\}$. Equivalently, every radical differential ideal is finitely generated.

(ii) If $I \subset K\{\overline{X}\}$ is a radical differential ideal, there are distinct prime differential ideals $P_1, ..., P_r$ (unique up to permutation) such that $I = \bigcap_{i=1}^{r} P_i$.

One also has the analogue of Hilbert's Nullstellensatz (cf. [18] Corollary 2.6):

Theorem 2.2.6 (Seidenberg's Differential Nullstellensatz). *Let K be a differentially closed field. The map $I \to V_\delta(I)$ is a one to one correspondence between radical differential ideals and Kolchin closed sets.*

In many ways the point of view of Kolchin's differential algebraic geometry, which aims to study the solution sets of systems of differential algebraic equations, coincides with that of the model theory of differentially closed fields. The latter is of course the point of view we take in this thesis. So let us bring in model theory.

Our language is $L_\delta = (+, -, \cdot, \delta, 0, 1)$, the language of differential rings and we denote by DF_0 the theory of differential fields of characteristic zero. The axioms of DF_0 consist of the axioms for fields and the axioms for the derivation (expressed using δ). This theory can be quite wild: $(\mathbb{Q}, +, -, \cdot, 0, 1, \delta = 0)$ is an example of a differential field and so one gets non-computable definable sets.

Now, for each n, d_1 and $d_2 \in \mathbb{N}$, one can write down an L_δ-sentence that asserts that if f is a differential polynomial of order n and degree at most d_1 and g is a nonzero differential polynomial of order less than n and degree at most d_2, then there is a solution to $f(X) = 0$ and $g(X) \neq 0$. The theory obtain by adding to DF_0 all these L_δ-sentences is called the theory of differentially closed fields of characteristic 0, DCF_0. This theory is the model companion of DF_0, that is to say that any differential field embeds in a model of DCF_0 and DCF_0 is model complete. Moreover, we have the following:

Fact 2.2.7. DCF_0 is complete, eliminates quantifiers and imaginaries and is ω-stable.

Proof. The proof of completeness and quantifier elimination follows from a back-and-forth argument in two saturated models of DCF_0 (see [37] Theorem 1.8(b)). The proof of elimination of imaginaries can be found in [18] (Theorem 3.7). Finally to see that DCF_0 is ω-stable, one just need to use the fact that there is a bijection between $S_n(K)$ and $spec(K\{X_1, \ldots, X_n\})$, the space of prime differential ideals of $K\{\overline{X}\}$, and (using the Ritt-Raudenbush Basis Theorem) that $|spec(K\{\overline{X}\})| = |K\{\overline{X}\}| = |K|$. □

Quantifier elimination means that any definable set $Y \subseteq \mathcal{U}^n$, definable over a differential subfield K of $\mathcal{U} \models DCF_0$, is a finite boolean combination of Kolchin closed sets (over K). On the other hand as we have seen in the first section, ω-stability means that

1. Prime models exists: Let K be differential field. Then there exist a differentially closed field extension K^{diff} of K, called the differential closure of K, which embeds over K into any differentially closed extension of K and which is unique up to isomorphisms over K.

2. Saturated Models exists: We can work in a κ-saturated model of DCF_0 of cardinality κ (for some large cardinal κ) which will act as a universal domain for differential algebraic geometry in the sense of Kolchin. In particular, if $\mathcal{U}$ is such a saturated model and if K is a differential subfield of $\mathcal{U}$ of cardinality $< \kappa$ and L is a differential field extension of K of cardinality $< \kappa$, then there is an embedding of L into $\mathcal{U}$ over K.

3. Morley rank is well defined: To any definable set, one can associate a well-defined ordinal-valued dimension. Furthermore, if we let $\mathcal{U}$ be a saturated model of DCF_0 as described above, in DCF_0 the independence relation translates to: $\overline{a}$ is independent from B over A if $\langle A, \overline{a} \rangle$ is algebraically disjoint from $\langle A, B \rangle$ over $\langle A \rangle$, where $\langle A \rangle$ denotes the differential field generated by A, that is $\langle A \rangle = \mathbb{Q}(\{a, a', a'', \ldots : a \in A\})$.

Remark 2.2.8. For $\overline{a} \in \mathcal{U}^n$ and $K < \mathcal{U}$, we define $ord(\overline{a}/K)$ to be the transcendence degree of $K\langle \overline{a} \rangle = K(\overline{a}, \overline{a}', \overline{a}^{(2)}, \ldots)$ over K. And if $Y \subseteq \mathcal{U}^n$ is definable over K, we define the $ord(Y) = sup\{ord(\overline{a}/K) : \overline{a} \in Y\}$. One can show that for Y as above, $RM(Y) \leq ord(Y)$ and furthermore, $RM(Y) < \omega$ if and only if $ord(Y) < \omega$.

We say a few words about the field of constants of a differentially closed field as it will play an important role in later sections and chapters.

Fact 2.2.9. Let K be a differential field. Then C_K is relatively algebraically closed in K. Consequently, if K is algebraically closed as a field, so is C_K.

Proof. Suppose $a \in K$ is algebraic over C_K. Let $f(x) = \sum_{i=0}^{n} k_i x^i$ (in $C_K[x]$) be the minimum polynomial of a over C_K. So $f(a) = 0$. Since K has characteristic zero we have $\delta(f(a)) = (\sum_{i=0}^{n-1}(i+1)k_{i+1}a^i) \cdot \delta(a) = 0$. As $f(x)$ is the minimal polynomial of a we must have that $\delta(a) = 0$. □

So in particular if K is a differentially closed field, then C_K is algebraically closed as a field. More is true:

Theorem 2.2.10 ([37], Lemma 1.11). *Let K be a differentially closed field. Then C_K has no additional structure other than being an algebraically closed field. That is, a subset of C_K^n is definable over K if and only if it is definable (in the language of rings) over C_K.*

We finish this section by giving the algebraic characterisations of the model theoretic definable and algebraic closures.

Fact 2.2.11 ([37], Lemma 1.10). Let $K \models DCF_0$ and let A be a subset of K. Then

(i) $dcl(A)$ is the differential subfield of K generated by A, that is $dcl(A) = \langle A \rangle$.

(ii) $acl(A)$ is $dcl(A)^{alg}$, the field-theoretic algebraic closure of $dcl(A)$.

2.2.2 Finite Dimensional definable sets

We will now specialise to finite dimensional definable sets. We aim to introduce the category of algebraic δ-varieties and they turn out to be birationally equivalent to the category of finite dimensional Kolchin closed sets (see [10]). This in particular means that, every finite dimensional definable set can be expressed in terms of algebraic δ-varieties and this give a characterisation closer to geometry. We fix $(\mathcal{U}, \delta)$, a sufficiently saturated differentially closed field which we think of as a universal domain for differential fields as explained above.

Definition 2.2.12. A definable set $Y \subseteq \mathcal{U}^n$ is said to be *finite dimensional* if it has finite order, i.e. $ord(Y) < \omega$. Equivalently, Y is finite dimensional if $RM(Y) < \omega$.

Now suppose that K is a differential field and let $V \subseteq \mathcal{U}^n$ be an affine algebraic variety over K. The *shifted tangent bundle* is defined to be

$$T_\delta(V) = \{(a,u) \in \mathcal{U}^{2n} \ : \ a \in V, \sum_{i=1}^{n} \frac{\partial P}{\partial x_i}(a)u_i + P^\delta(a) = 0 \ \ for \ P \in I(V)\}$$

where $I(V) \subset K\{x_1, \ldots, x_n\}$ is the ideal of V and P^δ is the polynomial obtain by differentiating the coefficients of P. By definition, $T_\delta(V)$ is a Zariski closed subset of $\mathcal{U}^{2n}$. Of course, one can also define $T_\delta(V)$ for an abstract variety V. One just takes a covering of V by affine opens U_i and piece together the shifted tangent bundles $T_\delta(U_i)$ using the transition maps. One should also note that by construction, for $a \in V(\mathcal{U})$, $(a, \delta(a)) \in T_\delta(V)$. Before we proceed with the definition of an algebraic D-variety, let us give the promised geometric axioms for differentially closed fields (cf. [34]).

Theorem 2.2.13. *A differential field (K, δ) is differentially closed if and only if K is algebraically closed and for every irreducible affine algebraic variety $V \subseteq K^n$, if W is an irreducible affine subvariety of $T_\delta(V)$ such that the projection of W onto V is Zariski dense in V and U is a Zariski open subset of V, there exist $a \in U$ such that then $(a, \delta(a)) \in W$.*

In this form, it is not straightforward to see that this characterisation of DCF_0 is first order expressible. For example one needs to know that irreducibility of an affine variety is "definable". In any case this is explained in [34].

Let now $V \subseteq \mathcal{U}^n$ be an affine algebraic variety over a differential subfield K of $\mathcal{U}$. A *shifted vector field* on V over K is just a morphism $s : V \to T_\delta(V)$ which is also a section of the canonical projection $\pi : T_\delta(V) \to V$.

Definition 2.2.14. A pair (V, s), as given above, is called an *affine δ-variety* over K. Given such an affine δ-variety (V, s), we define $(V, s)^\delta$ to be the set $\{a \in V(\mathcal{U}) \ : (a, \delta(a)) = s(a)\}$. If the variety V is K-irreducible, we refer to (V, s) as a *K-irreducible* affine D-variety.

Remark 2.2.15. Given an affine δ-variety (V, s) over K, one can define a derivation δ_s on the coordinate ring $K[V]$ (also on $\mathcal{U}[V]$) as follows: Suppose

$s(\overline{x}) = (\overline{x}, s_1(\overline{x}), \ldots, s_n(\overline{x}))$ where $\overline{x} = (x_1, \ldots, x_n)$. Then for $f \in K[V]$ define $\delta_s(f)$ as

$$\delta_s(f) = f^{\delta}(\overline{x}) + \sum_{i=1}^{n} s_i(\overline{x}) \frac{\partial f}{\partial x_i}(\overline{x}).$$

It is not hard to check that δ_s is indeed a derivation on $K[V]$. Sometimes we will write (V, δ_s) instead of (V, s).

Clearly, $(V, s)^{\delta}$ is definable in $\mathcal{U}^n$. Furthermore one has the following:

Proposition 2.2.16. *Let K be a differential subfield of $\mathcal{U}$. Then*

(i) If (V, s) is an irreducible affine δ-variety, then $(V, s)^{\delta}$ is Zariski-dense in $V(\mathcal{U})$.

(ii) Any K-definable subset Y of $\mathcal{U}^n$ of finite Morley rank and Morley degree 1, is generically and up to definable bijection, of the form $(V, s)^{\delta}$ for some K-irreducible affine δ-variety (V, s).

(iii) Let $(V, \delta_s) = (V, s)$ be an algebraic δ-variety, and $a \in (V, s)^{\delta}$. Let $\mathcal{M}_{V,a}$ be the maximal ideal of $\mathcal{U}[V]$ at a. Then for each $r \geq 0$, $(\mathcal{M}_{V,a})^r$ is a differential ideal of the differential ring $(\mathcal{U}[V], \delta_s)$, namely $(\mathcal{M}_{V,a})^r$ is closed under δ_s.

Proof. The proof is folklore and we include it for completeness.
(i) We have that $\{s(b) : b \in V\}$ is closed and irreducible subvariety of $T_{\delta}V$ and projects onto V. Hence by Theorem 2.2.13, for any Zariski open subset U of V there is $a \in U$ such that $s(a) = (a, \delta(a))$ which is exactly what we had to prove.
(ii) Suppose $RM(Y) = l$ and let $p(\bar{x})$ be the unique generic type of Y over K (this is the type which contains Y and has Morley rank l). Let $\bar{a} \in \mathcal{U}^n$ be a realization of p so that in particular $tr.deg_K K(\bar{a}, \bar{a}', \ldots)$ is finite. We can hence suppose that $K\langle\bar{a}\rangle = K(\bar{a}, \bar{a}', \ldots, \bar{a}^{(r)})$ for some integer r and after renaming, we rewrite $K(\bar{a}, \bar{a}', \ldots, \bar{a}^{(r)})$ as $K(a_1, \ldots, a_m)$ where $m = (r+1)n$.

Now, by construction we have that for each $i = 1, \dots m$, $\delta(a_i) \in K(\bar{a})$, so that $\delta(a_i) = \frac{h_i(\bar{a})}{g_i(\bar{a})}$, with $g_i(\bar{a}) \neq 0$. We can then write $g(\bar{x}) = \prod_{i=1}^{m} g_i(\bar{x})$ and $f_i(\bar{x}) = h_i \prod_{j \neq i} g_j(\bar{x})$ to see that for each $i = 1, \dots m$, $\delta(a_i) = \frac{f_i(\bar{a})}{g(\bar{a})}$, $g(\bar{a}) \neq 0$.

Let now $\bar{b} = (b_1, \dots, b_{m+1})$ be a renaming of $(a_1, \dots, a_m, \frac{1}{g(\bar{a})})$ so that $\bar{a}$ and $\bar{b}$ are interdefinable over K (i.e. $K(\bar{a}) = K(\bar{b})$). Then for each $i = 1, \dots m$, $\delta(b_i) = s_i(\bar{b})$ for some polynomial $s_i \in K[x_1, \dots, x_{m+1}]$. For example for $i = 1, \dots, m$, we take $s_i(x_1, \dots, x_{m+1}) = x_{m+1} \cdot f_i(x_1, \dots, x_m)$.

Writing $s(\bar{x}) = (\bar{x}, s_1(\bar{x}), \dots, s_{m+1}(\bar{x}))$, we let V be the K-irreducible affine variety whose generic point (over K) is $\bar{b}$. So $s(\bar{b}) \in T_\delta(V)$ and we see that s is a section of $T_\delta(V) \to V$. Hence (V, s) is a K-irreducible affine D-variety and after removing from Y and $(V, s)^\delta$ definable sets of Morley rank $< l$ (for example where $s(x) \neq \delta(x)$ on V), $(V, s)^\delta$ is in definable bijection with Y. And we are done.

(iii) Keeping Remark 2.2.15 in mind, we see that for $a \in (V, s)^\delta$, $(\delta_s f)(a) = \delta(f(a))$. So if $f \in \mathcal{M}_{V,a}$, then as $f(a) = 0$ we have that $(\delta_s f)(a) = 0$. So $(\delta_s f) \in \mathcal{M}_{V,a}$ and $\mathcal{M}_{V,a}$ is closed under δ_s. A similar proof works for each $(\mathcal{M}_{V,a})^r$. □

Remark 2.2.17.

(i) From the proof of (ii) we see that $ord(Y) = dim(V)$.

(ii) From (iii) we get that δ_s equips each $\mathcal{U}$-vector space $V_r = \mathcal{M}_{V,a}/(\mathcal{M}_{V,a})^r$ with a δ-module structure (over $(\mathcal{U}, \delta)$): that is on V_r there is an additive homomorphism $\delta_s : V_r \to V_r$ satisfying $\delta_s(\lambda v) = \delta(\lambda)v + \lambda\delta_s(v)$ for all $\lambda \in \mathcal{U}$ and $v \in V$.

Finally, let us say a few words about the δ-subvarieties of a K-irreducible affine δ-variety (V, s).

Definition 2.2.18. By an *algebraic δ-subvariety* W of (V, s) we mean a subvariety W of V (defined over some $L > K$) such that $s_{\restriction W}$ is a section of $T_\delta(W) \to W$.

The following holds (cf. [10], Proposition 1.1)

Fact 2.2.19. Let (V, s) be an affine δ-variety. Then

(i) The map taking a Kolchin closed subset X of $(V, s)^\delta$ to its Zariski closure, establishes a bijection between the Kolchin closed subsets of $(V, s)^\delta$ and the algebraic δ-subvarieties of (V, s).

(ii) X is irreducible as a Kolchin closed set iff its Zariski closure is irreducible as a Zariski closed set.

Remark 2.2.20.

(i) The inverse of the above bijection takes an algebraic δ-subvariety Y of (V, s) to $Y \cap (V, s)^\delta$.

(ii) Fact 2.2.19(i) in particular means that a Kolchin closed subset Y of $(V, s)^\delta$ is of the form $W \cap (V, s)^\delta = (W, s_{\restriction W})^\delta$, where $W = Y^{zar} \subseteq V$.

2.3 Strongly minimal sets in DCF_0

In this section, we look at strongly minimal sets in DCF_0. We explain how to show that Zilber's Principle and the trichotomy theorem hold in the theory. Throughout we assume that $\mathcal{U}$ is a sufficiently saturated differentially closed field.

2.3.1 The dichotomy theorem

Let us start by characterising strongly minimal sets in the special case of δ-varieties.

Fact 2.3.1. Let (V, s) be an affine δ-variety. Then $(V, s)^\delta$ is strongly minimal if and only if V is positive-dimensional and (V, s) has no proper (irreducible) positive-dimensional algebraic δ-subvarieties.

Proof. This follows from Fact 2.2.19 and quantifier elimination for DCF_0. For example assuming the right hand side, Fact 2.2.19 implies that $(V, s)^{\delta}$ is infinite and has no proper infinite Kolchin closed sets. As any definable subset of $(V, s)^{\delta}$ is a finite Boolean combination of Kolchin closed sets we deduce strong minimality. Clearly it only suffices to consider irreducible Kolchin closed sets. □

Example 2.3.2. The field of constants $C_{\mathcal{U}}$ is strongly minimal.

Example 2.3.3. If f is an absolutely irreducible polynomial over $\mathcal{U}$ in 2 variables then the subset Y of $\mathcal{U}$ defined by $f(y, y') = 0$ is strongly minimal, of order 1.

Example 2.3.4. The subset of $\mathcal{U}$ defined by $\{yy'' = y', y' \neq 0\}$ is strongly minimal, of order 2. (See 5.17 of [18].)

Remark 2.3.5. Suppose Y_1 and Y_2 are nonorthogonal strongly minimal sets and that the relation $R \subset Y_1 \times Y_2$ is defined over some field K. Then by definition, for any generic $y \in Y_1$ there exist $z \in Y_2$ generic such that $(y, z) \in R$ and in that case $K\langle y \rangle^{alg} = K\langle z \rangle^{alg}$. So if Y_1 and Y_2 are nonorthogonal strongly minimal sets then $ord(Y_1) = ord(Y_2)$.

The field of constants, $C_{\mathcal{U}}$, is nonmodular. Indeed, if one consider algebraically independent points $a, b, c \in C_{\mathcal{U}}$ and let $d = ac + b$, then one has that $d \in acl(a, b, c)$ but there is no $b_1 \in acl(a, b)$ such that $d \in acl(b_1, c)$. It turns out that any nonmodular strongly minimal set is closely related to $C_{\mathcal{U}}$.

Theorem 2.3.6 (Zilber's Principle). *Suppose Y is strongly minimal. Then either Y is locally modular or Y is nonorthogonal to $C_{\mathcal{U}}$ (and not both).*

The first proof was found by Hrushovski and Sokolovic in [13] where they used the deep and difficult theorem on Zariski geometries from [14]. More recently, using the theory of (differential) jet spaces, Pillay and Ziegler in [42] found an alternative route that avoids Zariski geometries. We say a few words about the Pillay-Ziegler proof. To begin with, they show that DCF_0 has the *canonical base property* (CBP).

So what do canonical bases look like in DCF_0? As we have elimination of imaginaries, if Y is Kolchin closed, then Y has a canonical parameter in $\mathcal{U}$. If K is an algebraically closed differential subfield of $\mathcal{U}$ and $\overline{a} \in \mathcal{U}^n$ we want to understand $Cb(tp(\overline{a}/K))$, the canonical base of the type of $\overline{a}$ over K. We let $V_\delta(\overline{a}) \in \mathcal{U}^n$ be the differential locus of $\overline{a}$ over K. That is

$$V_\delta(\overline{a}) = \left\{\overline{b} \in \mathcal{U}^n : P(\overline{b}) = 0 \text{ for all } P \in K\{\overline{X}\} \text{ such that } P(\overline{a}) = 0\right\}.$$

Then $Cb(tp(\overline{a}/K))$ is a canonical parameter for $V_\delta(\overline{a})$. More generally, for $B \subset \mathcal{U}$, if $p \in S_n(B)$ is a stationary type, we define $Cb(p)$ as follows: We let $\overline{a} \in \mathcal{U}$ be a realisation of p and let $V_\delta(\overline{a}) \in \mathcal{U}^n$ be the differential locus of $\overline{a}$ over $\langle B\rangle$. Then again, $Cb(p)$ is a canonical parameter for $V_\delta(\overline{a})$.

Remark 2.3.7. If Y is Kolchin closed, then the differential field $K = \langle \overline{e} \rangle$ generated by a canonical parameter $\overline{e}$ of Y is often called the *smallest* differential field of definition of Y. Namely, K is the smallest differential subfield of $\mathcal{U}$, such that Y can be defined by differential polynomials with coefficients from K.

Theorem 2.3.8 (Canonical base property, [42]). *Let Y be a finite-dimensional definable set, defined over an algebraically closed differential subfield K of $\mathcal{U}$. Let $a \in Y$ and let L be a algebraically closed differential subfield containing K. Let $\overline{b} = Cb(tp(a/L))$. Then there are finite tuples $\overline{c}$ in $C_{\mathcal{U}}$ and $\overline{d}$ in $\mathcal{U}$ such that $\overline{b}$ is independent from $\overline{d}$ over $K\langle a\rangle$ and $\overline{b} \in dcl(K, a, \overline{c}, \overline{d})$.*

Sketch of the proof of Theorem 2.3.8 We may assume that a is the generic point over K of $(V, s)^\delta$ for some affine δ-variety (V, s) over K (by applying the proof of Propostion 2.2.16(ii)). Let W be the algebraic-geometric locus of a over L. Then $(W, s_{\restriction W})$ is an algebraic δ-variety and a is a (differential) generic point over L of $(W, s_{\restriction W})^\delta$. It follows that $\overline{b} = Cb(tp(a/L))$ is interdefinable over K with the canonical parameter of W. More is true, if we let f_r be the map $\mathcal{M}_{V,a}/(\mathcal{M}_{V,a})^{r+1} \to \mathcal{M}_{W,a}/(\mathcal{M}_{W,a})^{r+1}$, then $\overline{b}$ is interdefinable over $K\langle a\rangle$ with the canonical parameter of $ker(f_r)$ for large enough r. But $ker(f_r)$ is a δ-submodule (see Remark 2.2.17(ii)) of $\mathcal{M}_{V,a}/(\mathcal{M}_{V,a})^{r+1}$ and using the theory of δ-modules one gets that after naming a "suitable" basis of $\mathcal{M}_{V,a}/(\mathcal{M}_{V,a})^{r+1}$ (say $\overline{e}_r$), the canonical parameter of $ker(f_r)$ is in $dcl(K, a, \overline{e}_r, \overline{c})$ for some tuple of constants $\overline{c}$. □

Proof of Theorem 2.3.6 We assume that Y is defined without parameters. If Y is nonmodular there are tuples $\overline{a}$ from Y and $B \subset \mathcal{U}$ such that $\overline{b} = Cb(tp(\overline{a}/acl(B)))$ is not contained in $acl(\overline{a})$. By the CBP, there are finite tuples $\overline{c}$ in $C_{\mathcal{U}}$ and $\overline{d}$ in $\mathcal{U}$ such that $\overline{b}$ is independent from $\overline{d}$ over $\overline{a}$ and $\overline{b} \in dcl(\overline{a}, \overline{c}, \overline{d})$. As $\overline{b}$ is essentially a tuple from Y ($\overline{b} \in acl(\overline{a}_1, \ldots, \overline{a}_r)$ for some independent realisations of $tp(\overline{a}/acl(B))$), this yields some nontrivial relation between Y and $C_{\mathcal{U}}$ giving nonorthogonality. □

2.3.2 The Classification of strongly minimal sets.

In the previous section we saw a neat characterisation of the non modular strongly minimal sets. What about the locally modular ones? As we will now see, the non trivial ones can also be classified up to nonorthogonality. From Fact 2.1.17(ii) we know that to understand/classify non trivial locally modular strongly minimal sets, one needs to identify and understand definable modular strongly minimal groups. Essentially everything is contained in the following (see [3] and [13]):

Fact 2.3.9. Let A be an abelian variety over $\mathcal{U}$. We identify A with its set $A(\mathcal{U})$ of $\mathcal{U}$-points. Then

(i) A has a (unique) smallest Zariski-dense definable subgroup, which we denote by $A^\sharp$ and called the *Manin kernel* of A.

(ii) $A^\sharp$ is finite-dimensional, $dim(A) \leq order(A^\sharp) \leq 2dim(A)$ and moreover $dim(A) = order(A^\sharp)$ if and only if A descends to $C_{\mathcal{U}}$ (in which case $A^\sharp = A(C_{\mathcal{U}})$).

(iii) If A is a simple abelian variety with $C_{\mathcal{U}}$-trace 0, then $A^\sharp$ is strongly minimal and modular (non trivial).

(iv) If A is an elliptic curve then $A^\sharp$ is strongly minimal, whether or not A is of $C_{\mathcal{U}}$-trace 0.

Remark 2.3.10. By an abelian variety A with $C_{\mathcal{U}}$-trace 0, we mean that A admits no non-zero algebraic homomorphisms to abelian varieties defined over $C_{\mathcal{U}}$.

Proof of Fact 2.3.9 The proof of (i) and (ii) can be found in the very good note [21]. We will only later give a geometric account of the construction of $A^\sharp$ as this is not required here.
We say a few words about (iii). So let A be a simple abelian variety over $\mathcal{U}$ which does not descend to $C_\mathcal{U}$. We want to see that $A^\sharp$ is strongly minimal and modular. Simplicity of A and (i) imply that $A^\sharp$ has no proper definable infinite subgroup, that is $A^\sharp$ is minimal . Let X be a strongly minimal definable subset of $A^\sharp$. We aim to understand $A^\sharp$ by studying the possible nature of X. If X were nonmodular then the dichotomy theorem together with some additional arguments (see [36]) yield that A descends to $C_\mathcal{U}$, contradiction. So X is modular. The minimality of $A^\sharp$ implies that $A^\sharp$ is contained in $acl(X)$ (together with finitely many additional parameters) and the modularity of X yields that $A^\sharp$ is a 1-based group and so (by Fact 2.1.17(i)) in particular up to finite Boolean combination any definable subset of $A^\sharp$ is a translate of a subgroup. Finally, again using the fact that $A^\sharp$ is minimal we have that $A^\sharp$ has no infinite co-infinite definable subset, so is strongly minimal, and of course modular. □

We can now state the most important result of this section.

Theorem 2.3.11 (**The trichotomy theorem**, [13]). *If $Y \in \mathcal{U}^n$ is strongly minimal, then exactly one of the following hold*

(i) Y is geometrically trivial, or

(ii) Y is non-orthogonal to the Manin kernel $A^\sharp$ of some simple abelian variety A of $C_\mathcal{U}$-trace zero, or

(iii) Y is non-orthogonal to the field of constants $C_\mathcal{U}$.

Proof. The arguments are given in [36], in the paragraphs leading up to Proposition 4.10 there, and we repeat/summarize them here. Firstly for a locally modular strongly minimal set Y in *any* structure, either Y is geometrically trivial or by Fact 2.1.17 Y is nonorthogonal to a definable modular strongly minimal group. So we may assume Y to be a strongly minimal modular group G (which has to be commutative, either by strong minimality or modularity). As discussed in [36] G definably embeds in a connected commutative algebraic group A without proper connected positive dimensional algebraic subgroups. So A is either the additive group, the multiplicative group, or a simple abelian variety. In the first two cases G is nonorthogonal to $\mathcal{C}_{\mathcal{U}}$, contradiction, so A must be a simple abelian variety. From Fact 2.3.9(i), strong minimality of G forces G to be $A^\sharp$ and by Fact 2.3.9(ii) A does not descend to $\mathcal{C}_{\mathcal{U}}$ (for then $A^\sharp = A(\mathbb{C})$ and G is nonorthogonal to $\mathcal{C}_{\mathcal{U}}$, contradiction again).

By definition modularity of $A^\sharp$ implies that (ii) and (iii) are mutually exclusive. On the other hand if G is a strongly minimal group defined over K, and a, b are mutually generic elements of G, then putting $c = a \cdot b$, the triple $\{a, b, c\}$ is a counterexample to the geometric triviality of G. So we see that (i) and (ii) are also mutually exclusive. □

Proposition 2.3.12 ([36], Proposition 4.10). *If A and B are two simple abelian varieties of $C_{\mathcal{U}}$-trace zero, then $A^\sharp$ and $B^\sharp$ are non-orthogonal if and only if A and B are isogenous.*

Of course one should note here that we do not have any general classification for geometrically trivial strongly minimal sets. Indeed, this issue can be seen as one of the motivating factor for the work on the Painlevé equations. For a while it was conjectured that ω-categoricity could be a characteristic feature of trivial sets. First note that one has the following

Lemma 2.3.13. *If $X \subset \mathcal{U}^n$ is a definable, strongly minimal ω-categorical set, then X is geometrically trivial.*

Proof. By Theorem 2.1.22 we know that X is modular. By Fact 2.1.17(ii), if X is not geometrically trivial, X is nonorthogonal to a strongly minimal group G which is also ω-categorical by Remark 2.1.21 as well as commutative. But G definably embeds in an algebraic group H hence (as characteristic is 0), G has only finitely many elements of any given finite order. This contradicts ω-categoricity. □

A beautiful result of Hrushovski [11] is that the converse holds for order 1 strongly minimal sets:

Fact 2.3.14 ([37], Corollary 1.82). Let $Y \subset \mathcal{U}^n$ be an order 1 strongly minimal set and assume it is orthogonal to the constants (and so geometically trivial). Then Y is ω-categorical.

This result of Hrushovski had given rise to a conjecture about geometrically trivial strongly minimal sets of arbitrary order (cf. [20]):

Conjecture 2.3.15. *In any differentially closed field, every geometrically trivial strongly minimal set is ω-categorical.*

It would seem however that Freitag and Scanlon [6] have recently found a counterexample by studying the differential equation satisfied by the j-function. In any case we shall later ask a related question that might still hold of trivial sets (see the end of Chapter 5).

Chapter 3

Irreducibility and Analysability

..., j'ai été conduit à une définition précise de l'irréductibilité, définition plus restreinte que celle qu'il faudrait adopter dans d'autres recherches, mais qui s'imposait ici.

– Paul Painlevé

3.1 Irreducibility

In this section, we look at Painlevé/Umemura's notion of classical functions and irreducibility. This also gives us the opportunity to introduce the Painlevé equations. Indeed Painlevé had them in mind when giving his definition.

3.1.1 The Painlevé transcendents

Consider the following algebraic differential equation

$$F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \dots, \frac{d^nx}{dt^n}\right) = 0 \tag{3.1.1}$$

where F is in the differential polynomial ring, $\mathbb{C}(t)\{x\}$, of $(\mathbb{C}(t), d/dt)$. In what follows we also let S denote the finite set of singularities of 3.1.1, that is the set of elements $t \in \mathbb{C}$ where the equation is not defined.

For $\bar{c} = (c_0, \dots, c_n) \in \mathbb{C}^{n+1}$ and $t_0 \in D$ (a domain in $\mathbb{C}$) such that $F(t, c_0, \dots, c_n) = 0$, one can find a local analytic solution $\varphi(t) = \varphi(t, t_0, \bar{c})$ to the initial value problem

$$\frac{d^i\varphi}{dt^i}(t_0) = c_i \quad (i = 0, \ldots, n).$$

The Painlevé property is a condition that guarantees the tame behaviour, under analytic (or meromorphic) continuation, of local solutions and is often used as criterion of "integrability" of differential equations:

Definition 3.1.1. The equation 3.1.1 is said to have the *Painlevé property* if any local analytic solution in a neighbourhood of any point $t_0 \in \mathbb{C} \setminus S$ can be analytically continued as a meromorphic function, along any path $\gamma \subset \mathbb{C} \setminus S$ starting at t_0.

One of the ongoing problems is to find all the algebraic differential equations which have the Painlevé property. In the case where $n = 1$ in 3.1.1, the work of Fuchs, Poincaré and Painlevé (cf. [33] and [43]) gives a full answer:

Fact 3.1.2. For $n = 1$, equation 3.1.1 has the Painlevé property if and only if it can be transformed, by a holomorphic change of the variable t and by a homographic change of the variable x with coefficients in $\mathcal{O}(D)$ (see Remark 3.1.3 below), into one of the following equations:

(i) The Riccati equation $x' = a(t)x^2 + b(t)x + c(t)$, where $a(t), b(t), c(t) \in \mathcal{O}(D)$; or

(ii) The equation of the Weierstrass $\wp$ function $(x')^2 = 4x^3 - g_2x - g_3$, where $g_2, g_3 \in \mathbb{C}$

where $\mathcal{O}(D)$ denotes the ring of holomorphic functions on D.

Remark 3.1.3. The change of variables $(t, x) \mapsto (T, X)$ mentioned above is simply given by

$$X(t) = \frac{\alpha(t)x(t) + \beta(t)}{\gamma(t)x(t) + \delta(t)} \text{ and } T = \theta(t)$$

where $\alpha\delta - \beta\gamma \neq 0$ and $\alpha(t), \beta(t), \gamma(t), \delta(t), \theta(t) \in \mathcal{O}(D)$

The case $n = 2$ was initiated by P. Painlevé with some refinements from his former student R. Gambier. For equations of the form $y'' = f(t, y, y')$, $f \in \mathbb{C}(t)\{y\}$, they came up with six equivalence classes under the transformations (as in Remark 3.1.3) of ODE which do not reduce to a first order or linear equation. The well-known representatives of these six classes are given in the following lists (where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$):

$$\begin{aligned}
P_I : \quad & y'' = 6y^2 + t \\
P_{II}(\alpha) : \quad & y'' = 2y^3 + ty + \alpha \\
P_{III}(\alpha, \beta, \gamma, \delta) : \quad & y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_{IV}(\alpha, \beta) : \quad & y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
P_V(\alpha, \beta, \gamma, \delta) : \quad & y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} \\
& \quad + \delta\frac{y(y+1)}{y-1} \\
P_{VI}(\alpha, \beta, \gamma, \delta) : \quad & y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
& \quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)
\end{aligned}$$

and are called the Painlevé equations.

One should note that R. Fuchs also independently discovered the sixth Painlevé equation, $P_{VI}(\alpha, \beta, \gamma, \delta)$, based on the notion of isomonodromic deformation.

The cases of second order higher degree and more general higher order equations are still open although there are partial results due to Chazy, Bureau and Cosgrove to name a few. We recommend the appendix of [4] for a very good survey.

The Painlevé equations are nowadays among the most studied algebraic differential equations. They have arisen in a variety of important physical applications including for example statistical mechanics, general relativity and fibre optics. However, from a differential algebraic perspective, it was Painlevé himself who made a claim that paved the way for the work in this thesis. Indeed, he gave a definition of "known" (or classical) functions and claimed that he proved the irreducibility of the first Painlevé equation with respect to known functions.

3.1.2 Classical functions and Irreducibility

Painlevé was aiming to show that the solutions to the Painlevé equations were really "new" special functions. It turned out though that Painlevé's definition lacked some rigour and this was taken up again the 1980s by Umemura and others who tried to give clear definitions of a "classical function" and an "irreducible ODE".

In this section, we aim to describe this theory of irreducibility as in [46]. This special notion of irreducibility should not be confused with the usual notions of irreducibility of a differential algebraic variety.

Let D be a domain in $\mathbb{C}$ and $\mathcal{F}(D)$ denote the differential field of meromorphic functions on D, equipped with the derivation d/dt. Note that $\mathbb{C}(t)$ is a differential subfield. We will consider only functions in $\mathcal{F}(D)$ where D may vary. We also may identify a function $f \in \mathcal{F}(D)$ with its restriction to a smaller domain D'. We will need the notion of the logarithmic derivative ∂ln_G corresponding to a connected complex algebraic group G:

Let TG be the tangent bundle of G. Then TG is also a connected complex algebraic group and is indeed a semidirect product of G with LG the Lie algebra of G. Note that the underlying vector space of LG is $\mathbb{C}^n$ (for some $n \in \mathbb{N}$). Let $F : D \to G$ be holomorphic. Then for $t \in G$, $F'(t) = dF/dt$ can be identified with a point in TG in the fibre above $F(t)$ and then $F'(t)F(t)^{-1}$ (multiplication in the sense of TG) lies in LG, and we define $\delta ln_G(F)$ to be the holomorphic function from D to LG whose value at t is precisely $F'(t)F(t)^{-1}$. Likewise if $F : D \to G$ is meromorphic $\delta ln_G(F) : D \to LG$ is meromorphic. As we shall later see, the logarithmic derivative δln_G can also be defined in any differential field.

Let us denote by K our base differential field $\mathbb{C}(t)$.

Definition 3.1.4. (a) Following Umemura [46] we give an inductive definition of a *classical function*.

(i) any $f \in K$ is classical.

(ii) Suppose that $f_1, .., f_n \in \mathcal{F}(D)$ are classical, and $f \in \mathcal{F}(D')$ (for some appropriate $D' \subseteq D$) is in the algebraic closure of the differential field generated by $\mathbb{C}(t)(f_1, .., f_n)$, then f is classical.

(iii) Let G be a connected complex algebraic group with $LG = \mathbb{C}^n$. Suppose $f_1, .., f_n \in \mathcal{F}(D)$ are classical, and for some $D' \subseteq D$, $f : D' \to G$ is a meromorphic function such that $\partial ln_G(f) = (f_1, .., f_n)$. Then the coordinate functions of f are classical.

(b) An equation $f(y, y', ...y^{(n)}) = 0$ (with coefficients from K) is then said to be *irreducible with respect to classical functions* if no solution is classical and so in particular there are no algebraic solutions.

In this form, it is not clear why any $f \in \mathcal{F}(D)$ satisfying the Definition 3.1.4(a) is a "known" function. So let us give some examples (we still write $(\mathbb{C}(t), d/dt)$ as (K, δ)):

Example 3.1.5. e^t is classical as it is a solution to $\frac{\delta y}{y} = 1$ and $y \mapsto \frac{\delta y}{y}$ is the logarithmic derivative on $(K, +)$

Example 3.1.6. The Weierstrass $\wp$ function is classical. It is the solution to $\frac{\delta x}{y} = 1$, where $y^2 = 4x^3 - g_2x - g_3$ and $g_2, g_3 \in \mathbb{C}$. As is well known the map $(x, y) \mapsto \frac{\delta x}{y}$ is the logarithmic derivative on the elliptic curve with affine part given by $y^2 = 4x^3 - g_2x - g_3$.

Example 3.1.7. Solutions of linear differential equations with classical coefficients are classical. To see this, recall that in matrix form, a linear differential equation (over K say) is given by $\delta Y = AY$ (or $\delta(Y)Y^{-1} = A$) on GL_n, where A is an $n \times n$ matrix over K and Y is a $n \times n$ matrix of unknowns ranging over GL_n. The map $Y \to \delta(Y)Y^{-1}$ is the logarithmic derivative on GL_n.

Consider now $P_{II}(-\frac{1}{2})$, the second Painlevé equation with parameter $\alpha = -\frac{1}{2}$, that is the equation $y'' = 2y^3 + ty - \frac{1}{2}$. It is not hard to see that any solution of the first order equation $y' = -y^2 - \frac{t}{2}$ is also a solution of $P_{II}(-\frac{1}{2})$. It should not matter whether or not $P_{II}(-\frac{1}{2})$ is irreducible with respect to classical functions as defined above: it satisfies a property that we would not like an irreducible equation to enjoy.

So, when considering nonlinear second (or higher) order equations, one needs to modify or extend the inductive Definition 3.1.4. This is exactly what Umemura did in [46]. We again consider functions in $\mathcal{F}(D)$ for varying D.

Definition 3.1.8. (a) We give an inductive definition of a 1-classical function:

(i) any $f \in K$ is 1-classical.

(ii) Suppose that $f_1, .., f_n \in \mathcal{F}(D)$ are classical, and $f \in \mathcal{F}(D')$ (for some appropriate $D' \subseteq D$) is in the algebraic closure of the differential field generated by $\mathbb{C}(t)(f_1, .., f_n)$, then f is 1-classical.

(iii) Let G be a connected complex algebraic group with $LG = \mathbb{C}^n$. Suppose $f_1, .., f_n \in \mathcal{F}(D)$ are classical, and for some $D' \subseteq D$, $f : D' \to G$ is a meromorphic function such that $\partial ln_G(f) = (f_1, .., f_n)$. Then the coordinate functions of f are 1-classical.

(iv) Suppose $f_1, .., f_n \in \mathcal{F}(D)$ are 1-classical, and $f \in \mathcal{F}(D)$ is a solution of an ODE $g(y, y') = 0$ where g has coefficients from $K(f_1, .., f_n)$. Then f is 1-classical.

(b) An ODE over K is said to be *irreducible with respect to* 1-*classical functions* if it has no 1-classical solutions.

Remark 3.1.9. The '1' in 1-classical corresponds to the fact that we include solutions of ODEs of order 1 in the definition of a 1-classical function (compare with Definition 3.1.4).

Of course one can define in a similar fashion irreducibility with respect to n-classical functions ($n \in \mathbb{N}$), but we will not need this here.

3.2 Analysability

In this section, we explain how to translate the above irreducibility notion into a more model theoretic language. We will of course use DCF_0 but also some differential Galois theory. Perhaps a natural question at this point is

"How do we move from thinking in term of functions to thinking in terms of points in a differential field?". This is how we will do so:

When required, we will fix a saturated model $(\mathcal{U}, \delta)$ of DCF_0 and assume that its cardinality is the continuum. Then the field of constants $C_{\mathcal{U}}$ of $\mathcal{U}$ can be identified with the field of complex numbers $\mathbb{C}$. If we let t denote an element of $\mathcal{U}$ such that $\delta t = 1$ then the differential field $(\mathbb{C}(t), d/dt)$ is a differential subfield of $\mathcal{U}$. Recall that the collection of all meromorphic functions on some open connected set $D \subseteq \mathbb{C}$, $\mathcal{F}(D)$ (equipped with d/dt), is a differential field containing $\mathbb{C}(t)$. It is true that $\mathcal{F}(D)$ has cardinality greater than the continuum so cannot be embedded in $\mathcal{U}$, but if K is any differential subfield of $\mathcal{F}(D)$ containing and countably generated over $\mathbb{C}(t)$ it will be embeddable in $\mathcal{U}$ over $\mathbb{C}(t)$. Hence any $f \in \mathcal{F}(D)$ can be assumed to be an element of $\mathcal{U}$.

3.2.1 Differential Galois theory

For the moment, as in the previous chapter, $\mathcal{U}$ will denote an arbitrary sufficiently saturated model of DCF_0. We begin by giving a more abstract definition of the logarithmic derivative. This can be found in many places but we recommend [37] which we follow. We fix (K, δ) a differential subfield of $\mathcal{U}$.

Let G be an algebraic group defined over C_K. The tangent bundle TG is also an algebraic group over C_K and we have the canonical projection $\pi : TG \to G$. As usual, we identify T_eG with the Lie algebra LG of G. We then also have the map $\pi_1 : TG \to LG$ taking $(g, v) \mapsto d(\rho^{g^{-1}})_g(v)$, where for $g \in G$, ρ^g denotes the right multiplication by g.

$$\begin{array}{ccc} T_gG & \xrightarrow{d(\rho^{g^{-1}})_g} & T_eG \\ \uparrow & & \uparrow \\ g & \xrightarrow{\rho^{g^{-1}}} & e \end{array}$$

(π, π_1) defines an isomorphism between TG and $LG \rtimes G$. On the other hand, using the homomorphism $\delta : G(\mathcal{U}) \to TG(\mathcal{U})$, $a \mapsto (a, \delta(a))$ we have

Definition 3.2.1. The (Kolchin) *logarithmic derivative* is the map

$$\begin{array}{rcl} \delta ln_G : G(\mathcal{U}) & \to & LG(\mathcal{U}) \\ a & \mapsto & \pi_1(a, \delta(a)) \end{array}$$

Remark 3.2.2. The fact that G is defined over the constants plays an important role in the above definition. If G was defined over an arbitrary differential field, δ is a map from $G(\mathcal{U})$ to the shifted tangent bundle $T_\delta G(\mathcal{U})$ rather than $TG(\mathcal{U})$. One then needs the notion of algebraic δ-group to be able to define the logarithmic derivative [38].

Fact 3.2.3. Suppose G is as above. Then

(i) δln_G is surjective

(ii) $Ker(\delta ln_G)$ is precisely $G(\mathbb{C})$

Proof. (ii) $g \in Ker(\delta ln_G)$ iff $d(\rho^{g^{-1}})_g(\delta(g)) = 0$ iff $\delta g = 0$ iff $g \in G(\mathbb{C})$. □

The logarithmic derivative plays a crucial role in Kolchin's and more general treatments of differential Galois theory. Indeed in some way logarithmic differential equations are equations with good Galois theory. We will recall a few things that we need. (K, δ) denotes a fixed differential field.

Definition 3.2.4. A differential field extension L of K is said to be *strongly normal* if

(i) $C_L = C_K$ is algebraically closed;

(ii) L is finitely generated over K;

(iii) If $\sigma : \mathcal{U} \to \mathcal{U}$ is an automorphism fixing K, then $\langle L, C_\mathcal{U} \rangle = \langle \sigma L, C_\mathcal{U} \rangle$.

The following holds (see [16]).

Proposition 3.2.5.

(i) Suppose C_K is algebraically closed, G is a connected algebraic group over C_K and $b \in LG(K)$. Then there is a solution $\alpha \in G(\mathcal{U})$ of $\delta ln_G(-) = b$ such that $L = K(\alpha)$ is a strongly normal extension of K.

(ii) Conversely, if L is a strongly normal extension of K and K is algebraically closed, then there is a connected algebraic group G over C_K and $b \in LG(K)$ such that L is generated by a solution of $\delta ln_G(-) = b$.

It turns out that strongly normal extensions are examples of Galois extensions in the differential setting. Although not required for our work, we give here for completeness a summary of the fundamental theorem of differential Galois theory.

Definition 3.2.6. Let L be a strongly normal extension of K. We define the (full) differential Galois group of L over K, $Gal(L/K)$, to be the group of differential automorphisms of $\langle L, C_{\mathcal{U}} \rangle$ which fix $\langle K, C_{\mathcal{U}} \rangle$ pointwise, that is the group $Aut_{\delta}(\langle L, C_{\mathcal{U}} \rangle \,/\, \langle K, C_{\mathcal{U}} \rangle)$.

The proof of the following theorem can be found in [18].

Theorem 3.2.7. *Suppose L is a strongly normal extension of K and suppose K is algebraically closed. Then $Gal(L/K)$ is isomorphic to the group of $C_{\mathcal{U}}$-rational points of an algebraic group defined over C_K.*

The fundamental theorem of Kolchin is then the following (see [16] or the more general proof found in [39]):

Theorem 3.2.8. *Suppose L is a strongly normal extension of K and let H be the algebraic group over C_K given to us in Theorem 3.2.7. Let F be an intermediate differential field ($K \subseteq F \subseteq L$) and $H_F = \{g \in H(C_{\mathcal{U}}) : g(c) = c \text{ for all } c \in F\}$. Then*

(i) L is a strongly normal extension of F and H_F is the group of $C_{\mathcal{U}}$-rational points of an algebraic subgroup of H over C_K and is isomorphic to $Gal(L/F)$;

(ii) The correspondence F to H_F gives a 1 to 1 correspondence between the intermediate fields and the algebraic subgroups of H over C_K;

(iii) F is a strongly normal extension of K if and only if H_F is a normal subgroup of H ($(H/H_F)(C_{\mathcal{U}})$ is then isomorphic to $Gal(F/K)$).

Much of the theory described above has been generalised by A. Pillay where he replaces groups over the constants with algebraic δ-groups over differential fields. However we do not need this here.

At this point one can certainly give a characterisation of classical functions in terms of strongly normal extensions. This is already done in [46] and follows from the definitions. So we now assume $\mathcal{U}$ is a saturated model of DCF_0 of cardinality continuum.

Proposition 3.2.9. *$g \in \mathcal{F}(D)$ is classical if and only if g is contained in a differential field L which can be decomposed into a tower of differential field extensions*

$$K_0 = \mathbb{C}(t) \subseteq K_1 \subseteq \cdots \subseteq K_n = L$$

where for each i either (i) K_i is finitely generated and algebraic over K_{i-1} or (ii) K_i is a strongly normal extension of K_{i-1}.

Remark 3.2.10.

(i) The K_i's are taken to be differential subfields of the field of meromorphic functions over a domain $D' \subseteq D$.

(ii) Nishioka [28] calls a field L, with such a decomposition, a Painlevé-Umemura extension of $\mathbb{C}(t)$.

3.2.2 Internality and Analysability

We will now look at the very well known connection between differential Galois theory and the model theoretic notion internality. We start with some generalities and throughout K will denote a differential field with algebraically closed field of constants C_K. We also for the moment impose no condition on the cardinality of the saturated model $\mathcal{U}$

Definition 3.2.11. A partial type $\pi(x)$ over K is said to be internal to $C_{\mathcal{U}}$ if for some set B of parameters $\pi(\mathcal{U})$ is contained in $dcl(K, B, C_{\mathcal{U}})$, that is for every realisation a of $\pi(x)$ there is a tuple $\bar{c}$ from $C_{\mathcal{U}}$ such that $a \in dcl(K, B, \bar{c})$.

The proof of the following can be found in [45].

Proposition 3.2.12.

(i) *A type $p \in S(K)$ is internal to $C_{\mathcal{U}}$ if and only if there is some set of parameters B and some realisation e of p such that $e \in dcl(C_{\mathcal{U}}, B)$ and e is independent over K from $\langle K, B \rangle$.*

(ii) *A consistent formula $\varphi(x)$ is internal to $C_{\mathcal{U}}$ if and only if there is a definable surjection from $C_{\mathcal{U}}^n$ (for some $n \in \mathbb{N}$) onto $\varphi(\mathcal{U})$.*

We usually denote the definable surjection by $f(-, -, \overline{b})$ to show that $\overline{b} \in \mathcal{U}$ are the parameters needed to witness $C_{\mathcal{U}}$-internality of $\varphi(x)$. The tuple $\overline{b}$ is called a *fundamental system of solutions.*

Proposition 3.2.13. *Suppose $\varphi(x)$ is internal to $C_{\mathcal{U}}$. Then, the fundamental system of solutions can always be chosen as a tuple from $\varphi(K^{diff})$.*

Proof. Let $\Gamma \subseteq C_{\mathcal{U}}^n \times \varphi(\mathcal{U})$ denote the graph of the definable surjection $C_{\mathcal{U}}^n \to \varphi(\mathcal{U})$. So Γ is defined by $f(\overline{x}, y, \overline{b})$ with $\overline{b} \in \mathcal{U}$ a fundamental system of solutions.

Let N be the differential field generated by K, $C_{\mathcal{U}}$ and $\varphi(\mathcal{U})$. By ω-stability, $tp(\overline{b}/N)$ is definable with parameters from N. So $(\overline{c}, a) \in \Gamma$ if and only if $f(\overline{c}, a, -) \in tp(\overline{b}/N)$ if and only if $\models \phi_f(\overline{c}, a, \overline{d}, \overline{e})$ where $\overline{d}$ and $\overline{e}$ are tuples from $C_{\mathcal{U}}$ and $\varphi(\mathcal{U})$ respectively and ϕ_f is an L_K formula.

Finally as

$$\mathcal{U} \models \exists \overline{d} \exists \overline{e} \left(\bigwedge_i d_i' = 0 \wedge \bigwedge_j \varphi(e_j) \wedge \left(\forall a \exists \overline{c} (\phi_f(\overline{c}, a, \overline{d}, \overline{e}) \right) \right),$$

by model completeness

$$K^{diff} \models \exists \overline{d} \exists \overline{e} \left(\bigwedge_i d_i' = 0 \wedge \bigwedge_j \varphi(e_j) \wedge \left(\forall a \exists \overline{c} (\phi_f(\overline{c}, a, \overline{d}, \overline{e}) \right) \right).$$

Let $\overline{d}_0$ and $\overline{e}_0$ be tuples from $C_{K^{diff}} = C_K$ and $\varphi(K^{diff})$ witnessing the above. Then $\phi_f(\overline{x}, y, \overline{d}_0, z)$ is an L_K-formula such that $\phi_f(\overline{x}, y, \overline{d}_0, \overline{e}_0)$ defines the graph of a surjection $C_{\mathcal{U}}^n \to \varphi(\mathcal{U})$ and so witnesses $C_{\mathcal{U}}$-internality of $\varphi(x)$ with fundamental system of solutions $\overline{e}_0$. □

The connection between internality and differential Galois theory is striking but well known (see Section 18.3 of [44].):

Proposition 3.2.14. *Let $\overline{a}$ be a tuple from $\mathcal{U}$. Then $tp(\overline{a}/K)$ is isolated and internal to $C_{\mathcal{U}}$ if and only if $\overline{a}$ is contained in some strongly normal extension L of K.*

To find the link between model theory and classical functions one requires however the notion analysability. From now on, we assume that $\mathcal{U}$ is of cardinality continuum and $K = \mathbb{C}(t)$.

Definition 3.2.15. Let f be an element of $\mathcal{U}$ and let K be a differential subfield. $tp(f/K)$ is said to be analysable in the constants if (possibly passing to a larger universal differentially closed field $\mathcal{U}_1$), there are $a_0, a_1, .., a_n$ such that $f \in K\langle a_0, a_1, .., a_n\rangle^{alg}$ and for each i, either $a_i \in K\langle a_0, .., a_{i-1}\rangle^{alg}$ or $tp(a_i/K\langle a_0, .., a_{i-1}\rangle)$ is stationary and internal to the constants.

Remark 3.2.16.

(i) Suppose $a_0, .., a_n$ witness that $tp(f/K)$ is analysable in the constants. Then we may choose them (maybe enlarging the sequence) to be in $dcl(K, f)$

(ii) Analysability of $tp(a/K)$ in the constants is equivalent to every extension of this type being nonorthogonal to the constants.

Of course the connection between analysability and classical functions is now evident

Proposition 3.2.17. *A function $f \in \mathcal{F}(D)$ is classical if and only if in the structure $\mathcal{U}$, $tp(f/\mathbb{C}(t))$ is isolated and analysable in the constants $\mathcal{C}_{\mathcal{U}} = \mathbb{C}$.*

Proof. This follows by just putting together Proposition 3.2.9 and Proposition 3.2.14. □

As far as irreducibility is concerned:

Proposition 3.2.18. *Let* $f(y, y', .., y^{(n)}) = 0$ *(with* $n \geq 2$*) be an ODE over* $K = \mathbb{C}(t)$*, which has no algebraic (over* K*) solutions and whose solution set in* $\mathcal{U}$ *is strongly minimal. Then* $f(y, y', .., y^{(n)}) = 0$ *is irreducible with respect to classical functions.*

Proof. Let Y be the set of solution, in $\mathcal{U}$, of the equation $f(y, y', .., y^{(n)}) = 0$. As the $ord(Y) \geq 2$, using Remark 2.3.5 we have that Y is orthogonal to the constants. Hence since there are no K-algebraic solutions we obtain irreducibility. □

Of course instead of classical functions, one can also translate Umemura's 1-classical notion into model theoretic terminology and this is exactly what we need for the Painlevé equations.

Proposition 3.2.19. *A function* $f \in \mathcal{F}(D)$ *is* 1*-classical if and only if* $tp(f/K)$ *is isolated and analysable in the constants together with the family of all order* 1 *equations* $g(y, y') = 0$ *(with coefficients from* $\mathcal{U}$*).*

Proposition 3.2.20. *An ODE* $y'' = f(y, y')$ *over* K *is irreducible with respect to* 1*-classical functions if and only if it has no* K*-algebraic solutions and its solution set is strongly minimal.*

Proof. Let Y be the set of solution of the equation $y'' = f(y, y')$. As with the proof of Proposition 3.2.18, if we assume that Y is strongly minimal (and since $ord(Y) = 2$), Remark 2.3.5 gives that Y is orthogonal to the constants and to all order 1 equations. So if furthermore there are no algebraic solutions, one gets irreducibility with respect to 1-classical functions. Left-to-right follows as a consequence of the definitions. □

Chapter 4

Geometric triviality: The Generic cases

In the previous chapter we have seen Umemura's definition of a classical function. We have also shown how equipped with the notion analysability, one can give a model theoretic description of irreducibility. In this chapter we look at the work of the "Japanese school" on irreducibility and explain how they show that the Painlevé equations, except for some special sets of parameters, are strongly minimal. We then prove that the pregeometry attached to any generic member of the Painlevé family is geometrically trivial.

Throughout, $(\mathcal{U}, \delta)$ will be a saturated model of DCF_0 of cardinality the continuum (so $C_{\mathcal{U}} = \mathbb{C}$). We let t denote an element of $\mathcal{U}$ such that $\delta t = 1$ then the differential field $(\mathbb{C}(t), d/dt)$ is a differential subfield of $\mathcal{U}$.

4.1 Condition (J) and strong minimality

We ended the previous chapter by showing that one of the requirements for the Painlevé equations to be irreducible is that their solutions sets in $\mathcal{U}$ are strongly minimal. We give here a different characterisation of strong minimality in terms of the so called condition (J) of Umemura.

Let us begin by pointing out the Hamiltonian nature of the algebraic ∂-variety attached to P_I. Rewrite P_I as the system $y' = x$, $x' = 6y^2 + t$ in indeterminates y, x. Choosing $H(y, x, t)$ to be $\frac{1}{2}x^2 - 2y^3 + ty$, we see that the system can be written in Hamiltonian form as

$$\begin{aligned} y' &= \frac{\partial H}{\partial x} \\ x' &= -\frac{\partial H}{\partial y} \end{aligned} \tag{4.1.1}$$

So we obtain the algebraic δ-variety $(\mathbb{A}^2, s)$ where $s(y,x) = (x, 6y^2+t)$. Writing the vector field s as a derivation δ_s on $K[y,x]$ (for some/any differential field $(K,\delta) \geq (\mathbb{C}(t), d/dt)$) extending δ, we have that

$$\delta_s = \delta + x\frac{\partial}{\partial y} + (6y^2+t)\frac{\partial}{\partial x}.$$

where for $P \in K[y,x]$, $\delta(P)$ is the result of applying δ to the coefficients of P. δ_s is sometimes called a "Hamiltonian vector field". The solution set of P_I in $\mathcal{U}$ can be identified with $(\mathbb{A}^2, \delta_s)^\delta$.

Such a Hamiltonian δ-variety structure exists for all the Painlevé equations. As such, it will be a good idea to characterise strong minimality in the situation where (V, δ_s) is an algebraic δ-variety over K, where V is $\mathbb{A}^2$, affine 2-space and (K,δ) is any differential field extension of $(\mathbb{C}(t), d/dt)$). As explained before, δ_s is simply a derivation of the polynomial ring $K[x,y]$ extending δ, and for any $L > K$, $\delta_s \otimes_\delta L$ is the unique extension of δ_s to a derivation of $L[x,y]$ which extends δ. We may often notationally identify $\delta_s \otimes_\delta L$ with δ_s.

Corollary 4.1.1.

(i) In the above situation ($V = \mathbb{A}^2$) $(V,\delta_s)^\delta$ is strongly minimal if and only if there is no $L \geq K$ and nonconstant polynomial $P(x,y)$ over L such that $\delta_s(P) = GP$ for some polynomial $G(x,y)$ over L.

(ii) Moreover these conditions are also equivalent to each of

- (V,δ_s) *has no* 1*-dimensional algebraic* δ*-subvariety,*
- $(V,\delta_s)^\delta$ *has no order* 1 *definable subset.*

Proof. (i) Suppose first there is nonconstant $P(x,y)$ over $L > K$ such that $\delta_s(P) = GP$ for some G. Let $I \subseteq L[x,y]$ be the ideal generated by P. It follows that I is δ_s-invariant. So the radical $\sqrt{I}$ of I is also δ_s-invariant (see

Lemma 1.15 of [18]). So $\sqrt{I}$ is the ideal of an algebraic δ-subvariety W of V which has to be a curve. By Fact 2.3.1, $(V, D_V)^\partial$ is not strongly minimal.

Conversely if $(V, \delta_s)^\delta$ is not strongly minimal, then by Fact 2.3.1 (V, δ_s) has a proper positive-dimensional δ-subvariety, which can be assumed to be irreducible (see above) hence is an irreducible plane curve W defined over $L > K$ say. Then the ideal $I = I_L(W)$ is principal, generated by an irreducible polynomial $P(x, y)$ say. As I is δ_s-invariant it follows that $\delta_s(P) = GP$ for some G.

(ii) is clear. □

Definition 4.1.2. The δ-variety (V, δ_s) is said to satisfy (Umemura's) *condition (J)* if the right hand side of Corollary 4.1.1(i) holds.

The reader is directed to [46] for more about this property. In any case in a series of papers, the Japanese school showed that except for special sets of parameters, the (Hamiltonian) δ-varieties attached to the Painlevé equations satisfy condition (J) (and so are strongly minimal).

4.2 Definability of the $A \to A^\sharp$ functor

For the proof of our main result, we will need the following statement:

Lemma 4.2.1. *Let $\phi(x, y)$ be a formula in the language of rings $(+, -, \cdot, 0, 1)$, such that for each b, $\phi(x, b)$, if consistent, defines an abelian variety A_b. Then there is a formula $\psi(x, y)$ in our language L of differential rings, such that for each b, $\psi(x, b)$ defines $A_b^\sharp$.*

So here x, y are tuples of variables, and strictly speaking $\phi(x, y)$ is a pair of formulas, one for the underlying set (subset of a suitable projective space for example), and one for the graph of the group operation. The above result is implicit in Section 3.2 of [10], and may even be needed for key results in that paper, but as there is neither an explicit statement nor proof of Lemma 2.25 in [10] we take the opportunity to give a proof here.

The first ingredient is a "geometric" account of the construction of $A^\sharp$, appearing in many places including [10] and [21] (but possibly originating with Buium). We summarise the situation. Let A be an abelian variety over $\mathcal{U}$. Then A has a "universal vectorial extension" $\pi : \tilde{A} \to A$, that is $\tilde{A}$ is a commutative algebraic group which is an extension of A by a vector group (power of the additive group) and for any other such extension $f : B \to A$, there is a unique homomorphism $g : \tilde{A} \to B$ such that everything commutes. (Here we work in the category of algebraic groups.) By functoriality of $T_\delta(-)$, for any algebraic group G over $\mathcal{U}$, $T_\delta(G) \to G$ is a surjective homomorphism of algebraic groups with kernel a vector group, and moreover if $h : H \to G$ is a homomorphism of algebraic groups we obtain $T_\delta(h) : T_\delta(H) \to T_\delta(G)$. In particular, taking B to be $T_\delta(\tilde{A})$, and $f : B \to A$ the composition of $T_\delta(A) \to A$ with $T_\delta(\pi) : T_\delta(\tilde{A}) \to T_\delta(A)$, we obtain a regular homomorphism $g : \tilde{A} \to T_\delta(\tilde{A})$ which has to be a section of the canonical $T_\delta(\tilde{A}) \to \tilde{A}$. We note that:

Fact 4.2.2.

(i) $g : \tilde{A} \to T_\delta(\tilde{A})$ is the *unique* regular homomorphic section of $T_\delta(\tilde{A}) \to \tilde{A}$, and, just for the record

(ii) $(\tilde{A}, g)$ is an algebraic δ-group, in the obvious sense.

From Fact 4.2.2(ii), we obtain $(\tilde{A}, g)^\delta$ which is now a finite-dimensional *differential algebraic group*, and the main point is:

Fact 4.2.3. $A^\sharp = \pi((\tilde{A}, g)^\delta)$.

The second ingredient is Lemma 3.8 of [10], which we interpret as:

Fact 4.2.4. The map which takes $A \to \tilde{A}$ is definable in ACF_0. Namely let $\phi(x, y)$, $\theta(y)$ be formulas in the language of rings such that for all b satisfying $\theta(y)$ (in some ambient algebraically closed field of characteristic 0), $\phi(x, b)$ defines an abelian variety A_b. Then there is a formula $\chi(z, y)$ in the language of rings such that for all b satisfying θ, $\chi(z, b)$ defines $\widetilde{A_b}$ (and its canonical surjection π_b to A_b).

Proof of Lemma 4.2.1
We prove the equivalent statement:
(*) For any formula $\phi(x, y)$ in the language of rings and formula $\theta(y)$ in the

language of differential rings such that for each $b \in \mathcal{U}$ satisfying $\theta(y)$, $\phi(x,b)$ defines an abelian variety A_b, then there is a formula $\psi(x,y)$ in the language of differential rings such that for each b satisfying $\theta(y)$, $\psi(x,b)$ defines $(A_b)^\sharp$.

First let $\chi(z,y)$ be as in Fact 4.2.1. We prove (*) by induction on the Morley rank of $\theta(y)$. Suppose $RM(\theta(y)) = \alpha$. We may assume θ has Morley degree 1. Let b be a "generic point" of $\theta(y)$ over $\emptyset$, namely $\models \neg\nu(b)$ for any formula $\nu(y)$ without parameters, of Morley rank $< \alpha$. Then $\chi(z,b)$ defines $\widetilde{A_b}$ and $\pi_b : \widetilde{A_b} \to A_b$. Note that $T_\delta(\widetilde{A_y})$ and the canonical surjection $\lambda_y : T_\delta(\widetilde{A_y}) \to \widetilde{A_y}$ are also uniformly definable in y in the differentially closed field: by formula $\eta(w,y)$ say (in the language of differential rings). Let $s : \widetilde{A_b} \to T_\delta(\widetilde{A_b})$ be the unique regular homomorphic section of λ_b given by Fact 4.2.2. By uniqueness $s = s_b$ is definable (in the language of rings) over b, by a formula $\gamma(z,w,b)$ say.
Now consider the formula $\theta'(y)$ which expresses: $\theta(y)$ + "$\gamma(z,w,y)$ defines a homomorphic section of the map $\lambda_y : T_\delta(\widetilde{A_y}) \to \widetilde{A_y}$".
Note that $\models \theta'(b)$. Moreover whenever $\models \theta'(c)$, then $\gamma(z,w,c)$ defines the *unique* regular homomorphic section, say $s_c : \widetilde{A_c} \to T_\delta(\widetilde{A_c})$. Hence by Fact 4.2.3, $(A_c)^\sharp = \pi_c((\widetilde{A_c}, s_c)^\delta)$, and so is defined by a formula $\psi(x,c)$, where $\psi(x,y)$ does not depend on c. Hence (*) holds for $\theta'(y)$ in place of $\theta(y)$, but as $\theta'(y)$ is true of b, and b is "generic" for the Morley degree 1 formula $\theta(y)$, the formula $\theta(y) \wedge \neg\theta'(y)$ has Morley rank $< \alpha$, and we can use induction. This completes the proof of (*) and Lemma 4.2.1 . □

4.3 Strong minimality and geometric triviality

For each of the six families of Painlevé equations we will prove strong minimality and geometric triviality for an equation with generic parameters. We will say that an equation in one of these families is "generic" if the corresponding tuple of complex numbers is a tuple of algebraically independent transcendental complex numbers. For each of the families we will have to describe briefly relevant results by the "Japanese school".

4.3.1 The equation P_I.

As we have seen several times now, the first Painlevé equations P_I is given by $y'' = 6y^2 + t$. Prior to the work in this thesis, pretty much everything was already known for P_I:

Proposition 4.3.1. *The solution set of P_I is strongly minimal and ω-categorical.*

Strong minimality is given (using Corollary 4.1.1) by the following:

Fact 4.3.2. Let $K < L$ be differential fields containing $\mathbb{C}(t)$. Let $y \in L$ be a solution of $y'' = 6y^2 + t$. Then either $y \in K^{alg}$ or $tr.deg.(K\langle y\rangle/K) = 2$.

Fact 4.3.2 is attributed to Kolchin in [18] (Theorem 5.18) and to Kolchin-Kovacic in [46] (Lemma 0). It was also rediscovered by Nishioka [26]. In any case [46] gives a complete proof.

Painlevé proved that P_I has no solution algebraic over $\mathbb{C}(t)$ and again a complete proof appears in [46] (Lemma 0.8). Nishioka [27] (Theorem 1) proves:

Fact 4.3.3. If $K < L$ are differential fields containing $\mathbb{C}(t)$ and $y_1, .., y_n$ are distinct solutions of P_I in L each of which is not in K^{alg}, then $y_1, y_1', ...y_n, y_n'$ are algebraically independent over K.

This gives ω-categoricity and geometric triviality of P_I.

4.3.2 The family P_{II}.

For $\alpha \in \mathbb{C}$, $P_{II}(\alpha)$ is the following equation

$$y'' = 2y^3 + ty + \alpha.$$

Defining x to be $y' + y^2 + t/2$, we obtain the following equivalent (Hamiltonian) system:

$$S_{II}(\alpha)\left\{\begin{array}{lcl} y' & = & x - y^2 - \frac{t}{2} \\ x' & = & 2xy + \alpha + \frac{1}{2}. \end{array}\right.$$

We write $\delta(\alpha)$ for the corresponding vector field (as a derivation)

$$\delta + (x - y^2 - \frac{t}{2})\frac{\partial}{\partial y} + (2xy + \alpha + \frac{1}{2})\frac{\partial}{\partial x}.$$

As mentioned before the solution set of $P_{II}(-1/2)$ is not strongly minimal: the ODE $y' = -y^2 - t/2$ defines a proper infinite differential algebraic subvariety. Equivalently the curve defined by $x = 0$ is an algebraic δ-subvariety of $(\mathbb{A}^2, \delta(-1/2))$.

In general there are "Backlund transformations" which take solutions of $S_{II}(\alpha)$ to solutions of $S_{II}(-1-\alpha)$, $S_{II}(\alpha - 1)$ and $S_{II}(\alpha + 1)$. From (I) and (II) on p.160 of [47] we see:

Fact 4.3.4. For $\alpha \in \mathbb{C}$, $\delta(\alpha)$ satisfies condition (J) if and only if $\alpha \notin \frac{1}{2} + \mathbb{Z}$.

So from Corollary 4.1.1, we see:

Corollary 4.3.5. *For $\alpha \in \mathbb{C}$, the solution set (in $\mathcal{U}$) of $P_{II}(\alpha)$ is strongly minimal if and only if $\alpha \notin \frac{1}{2} + \mathbb{Z}$. Moreover if $\alpha \in \frac{1}{2} + \mathbb{Z}$ then the solution set of $P_{II}(\alpha)$ contains a definable subset of order* 1.

We now give the main result in the case of P_{II}. Its proof is the model for all subsequent proofs of the main result for $P_{III} - P_{VI}$,

Proposition 4.3.6. *Let $\alpha \in \mathbb{C}$ be transcendental. Then (the solution set of) $P_{II}(\alpha)$ is strongly minimal and geometrically trivial.*

Proof. Let α be transcendental and let $Y(\alpha)$ denote the solution set of $P_{II}(\alpha)$. By Corollary 4.3.5, $Y(\alpha)$ is strongly minimal. As $ord(Y(\alpha)) = 2$, $Y(\alpha)$ is orthogonal to the differential algebraic variety $\mathbb{C}$ (see Remark 2.3.5). So if $Y(\alpha)$ is not geometrically trivial, then by the trichotomy theorem Theorem 2.3.11, $Y(\alpha)$ has to be nonorthogonal to $A^\sharp$ for some simple abelian variety A with $\mathbb{C}$-trace 0.

Claim I. A is an elliptic curve.
Proof. By Remark 2.3.5 $ord(A^\sharp) = 2$. So by Fact 2.3.9(ii), $dim(A) \leq 2$. If $dim(A) = 2$ then by Fact 2.3.9(ii) again A descends to $\mathbb{C}$, a contradiction, so $dim(A) = 1$ and A is an elliptic curve.

So A is the solution set of $y^2 = x(x-1)(x-a)$ for some $a \in \mathcal{U} \setminus \mathbb{C}$ (in fact we could choose $a \in \mathbb{C}(t)^{alg}$ but this does not simplify the argument). Let us rewrite A as E_a. Applying Lemma 4.2.1 to the family of elliptic curves in Legendre form: $\{E_b : y^2 = x(x-1)(x-b) : b \neq 0, 1\}$, we obtain a formula $\psi(x, y, z)$ (in the language of differential rings) such that $\psi(x, y, b)$ defines $E_b^\sharp$ (for $b \neq 0, 1$). Now the nonorthogonality of $Y(\alpha)$ and $E_a^\sharp$ is witnessed by some definable set $Z \subset Y(\alpha) \times E_a^\sharp$ which, without loss of generality projects onto each of $Y(\alpha)$, $E_a^\sharp$ and moreover such that each of these projections has fibres of cardinality $\leq k$ for some fixed k. Now Z is defined by some formula $\chi(-, -, c)$ where we witness the parameters in the formula by c.

Claim II. There is an L-formula $\rho(w, u, v)$ with additional parameter t (where u is possibly a tuple of variables) such that for any α_1, c_1, a_1 from $\mathcal{U}$, $\mathcal{U} \models \rho(\alpha_1, c_1, a_1)$ if and only α_1 is a constant, a_1 is not a constant, and $\chi(-, -, c_1)$ defines a subset of $Y(\alpha_1) \times E_{a_1}^\sharp$ which projects onto each of $Y(\alpha_1)$, $E_{a_1}^\sharp$, and with all fibres of cardinality at most k.

Proof. This follows from the existence of the formula $\psi(x, y, z)$, i.e. uniform definability of $E_b^\sharp$ as b varies. (The additional parameter t in ρ is there because $P_{II}(\alpha)$ has parameter t in addition to α.)

So by Claim II, we have that $\mathcal{U} \models \rho(\alpha, a, c)$, in particular taking $\eta(w)$ to be the formula $\exists u, v(\rho(w, u, v))$, we have that $\mathcal{U} \models \eta(\alpha)$.

Claim III. $\models \eta(\alpha_1)$ for all but finitely many $\alpha_1 \in \mathbb{C}$.

Proof. This is because $\eta(w)$ is over t, α is independent from t over $\emptyset$ and $\mathbb{C}$ is strongly minimal. A little more slowly: By strong minimality of the field $\mathcal{C}_\mathcal{U} = \mathbb{C}$ of constants, $\eta(w)$ defines either a finite or cofinite subset of $\mathbb{C}$. Suppose for the sake of contradiction that it defines a finite subset. Then as $\models \eta(\alpha)$, we would conclude that $\alpha \in acl(t)$ in $\mathcal{U}$. But $acl(t)$ is simply the field-theoretic algebraic closure of the differential field $\mathbb{Q}(t)$ generated by t, which clearly does not contain the transcendental element $\alpha \in \mathbb{C}$. So $\eta(w)$ defines a cofinite subset of $\mathbb{C}$ as required.

By Claim III we conclude that $\models \eta(\alpha_1)$ for some $\alpha_1 \in 1/2 + \mathbb{Z}$. Hence by the definition of $\eta(w)$ there is $a_1 \notin \mathbb{C}$ and some finite-to-finite definable relation

R between $Y(\alpha_1)$ and $E^\sharp_{a_1}$. By Corollary 4.3.5 $Y(\alpha_1)$ contains an order 1 definable subset Z say. Let K be a countable differential field over which all the data $Y(\alpha_1)$, $E^\sharp_{a_1}, Z, R$ are defined. Let z be a generic point of Z over K , and let (x, y) a point of $E^\sharp_{a_1}$ such that $R(z,(x,y))$. Then $tr.deg(K\langle z\rangle/K) = 1$ (as Z has order 1 and $z \notin K^{alg}$), and $tr.deg(K\langle x, y\rangle/K) = 2$ (as by Fact 2.3.9(iii) $E^\sharp_{a_1}$ is strongly minimal of order 2 and $(x, y) \notin K^{alg}$). We now have a contradiction, because R witnesses that (x, y) is in the algebraic closure of K, z, which we know to be the field-theoretic algebraic closure of $K\langle z\rangle$. □

4.3.3 The family P_{III}.

On the face of it, the family P_{III} is a 4-parameter family: where $P_{III}(\alpha, \beta, \gamma, \delta)$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ is given by the following

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}.$$

Okamoto [32] rewrites the equation as a 2-parameter Hamiltonian system which is then further studied by Umemura and Watanabe [48]. We give a quick summary. Okamoto replaces t^2 by t and ty by t to obtain the "equivalent" family:

$$P_{III'}(\alpha, \beta, \gamma, \delta): \qquad y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{y^2}{4t^2}(\gamma q + \alpha) + \frac{\beta}{4t} + \frac{\delta}{4y}$$

and we are reduced to showing that for algebraically independent $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the solution set of $P_{III'}(\alpha, \beta, \gamma, \delta)$ is strongly minimal and geometrically trivial. In particular we can assume that neither γ nor δ are 0.

Okamoto then points out that for $\lambda, \mu \in \mathbb{C}$, the transformation taking y to λy and t to μt takes $P_{III'}(\alpha, \beta, \gamma, \delta)$ to $P_{III'}(\lambda\alpha, \mu\lambda^{-1}\beta, \lambda^2\gamma, \mu^2\lambda^{-2}\delta)$.

Hence taking $\lambda^2 = 4/\gamma$ and $\mu^2 = 1/\gamma\delta$ (assuming as above that $\gamma\delta \neq 0$), this transformation takes $P_{III'}(\alpha, \beta, \gamma, \delta)$ to $P_{III'}(\lambda\alpha, \mu\lambda^{-1}\beta, 4, -4)$, and moreover if $\alpha, \beta, \gamma, \delta$ are algebraically independent, so are $\lambda\alpha, \mu\lambda^{-1}\beta$. Hence the family $P_{III'}(\alpha, \beta, \gamma, \delta)$ can be replaced by the family $P_{III'}(\alpha, \beta, 4, -4)$ and we are reduced to showing that for α, β algebraically independent the solution set of $P_{III'}(\alpha, \beta, 4, -4)$ is strongly minimal and geometrically trivial. Finally $P_{III'}(\alpha, \beta, 4, -4)$ can be written as an "Hamiltonian δ-variety"

$$S_{III'}(v_1, v_2) \begin{cases} y' &= \frac{1}{t}(2y^2x - y^2 + v_1 y + t) \\ x' &= \frac{1}{t}(-2yx^2 + 2yx - v_1 x + \frac{1}{2}(v_1 + v_2)) \end{cases}$$

where $\alpha = 4v_2$ and $\beta = -4(v_1 - 1)$.

(Note that α and β are algebraically independent if and only if v_1 and v_2 are.)

The Hamiltonian is here:

$$H(v_1, v_2) = \frac{1}{t}\left[y^2x^2 - \left(y^2 - v_1y - t\right)x - \frac{1}{2}(v_1 + v_2)y\right]$$

and the Hamiltonian vector field on $\mathbb{A}^2$ is:

$$\delta(v_1, v_2) = \delta + \frac{1}{t}(2y^2x - y^2 + v_1y + t)\frac{\partial}{\partial y} + \frac{1}{t}(-2yx^2 + 2yx - v_1x + \frac{1}{2}(v_1 + v_2))\frac{\partial}{\partial x}$$

From [48] one now extracts:

Fact 4.3.7. $\delta(v_1, v_2)$ satisfies condition (J) if and only if $v_1 + v_2 \notin 2\mathbb{Z}$ and $v_1 - v_2 \notin 2\mathbb{Z}$.

Commentary. This is contained in the statements of Theorem 1.2(i) and (ii), Proposition 2.1 and Corollary 2.5 of [48]: 1.2(i) and (ii) say that if either $v_1 + v_2$ or $v_1 - v_2$ are in $2\mathbb{Z}$ then the solution set of the system $S_{III'}(v_1, v_2)$ has (many) proper differential algebraic subvarieties (hence by Corollary 4.1.1) $\delta(v_1, v_2)$ does not satisfy condition (J). On the other hand Proposition 2.1 says that if $\delta(v_1, v_2)$ satisfies condition (J), then for some integers $i, j, h \geq 0$, such that not both $i, j = 0$ we have $i(v_1 + v_2) + j(v_1 - v_2) + 2h(1 - v_1) = 0$. Strictly speaking this Proposition 2.1 studies the "Hamiltonian vector field" corresponding to a different δ-variety structure on $\mathbb{A}^2$, namely

$$\begin{cases} \delta_1(y) &= 2y^2x - y^2 + v_1y + t \\ \delta_1(x) &= -2yx^2 + 2yx - v_1x + \frac{1}{2}(v_1 + v_2) \end{cases}$$

where δ_1 is the derivation $t\delta$ (or "$t\frac{d}{dt}$) on $\mathcal{U}$ with respect to which $\mathcal{U}$ is also saturated, differentially closed, but it is clear that δ-subvarieties with respect to the original δ-structure correspond to δ_1-subvarieties with the new δ_1-structure.

Corollary 4.3.8. *The solution set of $S_{III'}(v_1, v_2)$ is strongly minimal if and only if $v_1 + v_2 \notin 2\mathbb{Z}$ and $v_1 - v_2 \notin 2\mathbb{Z}$.*

We conclude our main result for P_{III}:

Proposition 4.3.9. *Let v_1, v_2 be algebraically independent. Then (the solution set of) $S_{III'}(v_1, v_2)$ is strongly minimal and geometrically trivial.*

Proof. Let $Y(v_1, v_2)$ denote the solution set of $S_{III'}(v_1, v_2)$. Strong minimality is by Corollary 4.3.8. Suppose that geometric triviality fails. We copy the proof of Proposition 4.3.6. So $Y(v_1, v_2)$ is nonorthogonal to $E_a^\sharp$ for an elliptic curve $E_a : y^2 = x(x-1)(x-a)$ with $a \notin \mathbb{C}$. Using definability of the family $E_b^\sharp$, we express nonorthogonality of $Y(v_1, v_2)$ to some $E_b^\sharp$ by a formula $\theta(v_1, v_2, t)$. As v_2 is independent from v_1, t, there are cofinitely many $v \in \mathbb{C}$ such that $\theta(v_1, v, t)$ holds. In particular we can find such $v \in v_1 + 2\mathbb{Z}$, and we have a contradiction as in Proposition 4.3.6 (using now Fact 4.3.7) □

4.3.4 The family P_{IV}.

$P_{IV}(\alpha, \beta)$, $\alpha, \beta \in \mathbb{C}$ is given by the following equation

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

Following [31] and [47] the equations can be rewritten in the form

$$S_{IV}(v_1, v_2, v_3) \begin{cases} y' &= 2xy - y^2 - 2ty + 2(v_1 - v_2) \\ x' &= 2xy - x^2 + 2tx + 2(v_1 - v_3) \end{cases}$$

where v_1, v_2, v_3 are complex numbers satisfying $v_1 + v_2 + v_3 = 0$.

(*) Here $\alpha = 3v_3 + 1$ and $\beta = -2(v_2 - v_1)^2$.

Let $\mathcal{V}$ be the plane defined by $v_1 + v_2 + v_3$=0 and throughout let $\mathbf{v}$ denote a tuple (v_1, v_2, v_3) from $\mathcal{V}$. So the algebraic independence of α and β means that the corresponding $\mathbf{v}$ is a generic point on $\mathcal{V}$. Let $\delta(\mathbf{v})$ be the derivation (Hamiltonian vector field) corresponding to $S_{IV}(\mathbf{v})$. Let $\mathcal{W} = \{\mathbf{v} \in \mathcal{V} : v_1 - v_2 \in \mathbb{Z}\} \cup \{\mathbf{v} \in \mathcal{V} : v_2 - v_3 \in \mathbb{Z}\} \cup \{\mathbf{v} \in \mathcal{V} : v_3 - v_1 \in \mathbb{Z}\}$. Then the analysis in [47], specifically Theorem 3.2, Proposition 3.5, and Corollary 3.9 yields

Fact 4.3.10. For $\mathbf{v} \in \mathcal{V}$, $\delta(\mathbf{v})$ satisfies condition (J) if and only if $\mathbf{v} \notin \mathcal{W}$.

Corollary 4.3.11. *For $\mathbf{v} \in \mathcal{V}$, the solution set of $P_{IV}(\mathbf{v})$ is strongly minimal if and only if $\mathbf{v} \notin \mathcal{W}$.*

Proposition 4.3.12. *Let $Y(\alpha,\beta)$ denote the solution set of $P_{IV}(\alpha,\beta)$. Suppose $\alpha,\beta\in\mathbb{C}$ are algebraically independent. Then $Y(\alpha,\beta)$ is strongly minimal and geometrically trivial.*

Proof. Let $(v_1,v_2,v_3)\in\mathcal{V}$ correspond to (α,β) as in (*) above. So v_1 and v_2 are algebraically independent and $v_3=-v_1-v_2$. It suffices to work with the solution set $Y(\mathbf{v})$ of $S_{IV}(\mathbf{v})$. Strong minimality is by Corollary 4.3.11. Again if geometric triviality fails this is witnessed by the truth of a formula $\theta(v_1,v_2,t)$, where $\theta(u,w,t)$ expresses the existence of a differential algebraic correspondence between $Y(u,w,-u-w)$ and some $E_b^\sharp$ with $b\notin\mathbb{C}$. The independence of v_2 over v_1,t implies that $\models\theta(v_1,w,t)$ for all but finitely many $w\in\mathbb{C}$. So we can find such $w\in v_1+\mathbb{Z}$ giving a contradiction to Corollary 4.3.11. □

4.3.5 The family P_V

We recall that $P_V(\alpha,\beta,\gamma,\delta)$, $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, is given by the following equation

$$y''=\left(\frac{1}{2y}+\frac{1}{y-1}\right)(y')^2-\frac{1}{t}y'+\frac{(y-1)^2}{t^2}\left(\alpha y+\frac{\beta}{y}\right)+\gamma\frac{y}{t}+\delta\frac{y(y+1)}{y-1}$$

The situation is similar to the case of P_{III} above, in that, in the light of certain transformations preserving the equation, P_V can be written as a 3-parameter family. The analysis is carried out by Okamoto[30] and then Watanabe [49] and again we give a summary:

First $P_V(\alpha,\beta,\gamma,\delta)$ can be rewritten as the Hamiltonian system

$$y'=\frac{\partial H}{\partial x},x'=-\frac{\partial H}{\partial y}$$

where the Hamiltonian polynomial is

$$H_V(\kappa_0,\kappa_1,\theta,\eta)=\frac{1}{t}\left[y(y-1)^2x^2-\big(\kappa_0(y-1)^2+\theta y(y-1)+\eta ty\big)x+\kappa(y-1)\right]$$

and

$$\alpha=\frac{1}{2}\kappa_1^2,\quad\beta=-\frac{1}{2}\kappa_0^2,\quad\gamma=-\eta(\theta+1),\quad\delta=-\frac{1}{2}\eta^2,\quad\kappa=\frac{1}{4}(\kappa_0+\theta)^2-\frac{1}{4}\kappa_1^2.$$

Okamoto points out that the transformation $(y,x,H,t)\to(y,x,\lambda H,\lambda^{-1}t)$ and $\eta\to\lambda^{-1}\eta$ "preserves the system" for any nonzero λ. As we may assume

$\delta \neq 0$ (so also $\eta \neq 0$) we may choose $\lambda = \eta$, and hence the transformation takes δ to $-1/2$ (and η to 1). Hence we need only consider the system $P_V(\alpha, \beta, \gamma, -\frac{1}{2})$, and prove that for α, β, γ algebraically independent, the solution set of $P_V(\alpha, \beta, \gamma, -\frac{1}{2})$ is strongly minimal and geometrically trivial.

Let us now define:

$$v_1 = -\frac{1}{4}(2\kappa_0 + \theta), \quad v_2 = \frac{1}{4}(2\kappa_0 - \theta), \quad v_3 = \frac{1}{4}(2\kappa_1 + \theta), \quad v_4 = -\frac{1}{4}(2\kappa_1 - \theta),$$

then the vector $\mathbf{v} = (v_1, v_2, v_3, v_4)$ is on the complex hyperplane $\mathcal{V}$ in $\mathbb{C}^4$ defined by $v_1 + v_2 + v_3 + v_4 = 0$. So our family is now parameterised by $\mathcal{V}$.

Secondly make the following substitution: replace $y(y-1)^{-1}$ by y, and $-y(y-1)^2 x + (v_3 - v_1)(y-1)$ by x.

Then our system can be written as:

$$S_V(\mathbf{v}) \begin{cases} y' &= \frac{1}{t}(2y^2x - 2yx + ty^2 - ty + (v_1 - v_2 - v_3 + v_4)y + v_2 - v_1) \\ x' &= \frac{1}{t}(-2yx^2 + x^2 - 2txy + tx - (v_1 - v_2 - v_3 + v_4)x + (v_3 - v_1)t) \end{cases}$$

and note that $\alpha = \frac{1}{2}(v_3 - v_4)^2$, $\beta = -\frac{1}{2}(v_2 - v_1)^2$ and $\gamma = 2v_1 + 2v_2 - 1$.
Let $\delta(\mathbf{v})$ be the corresponding (Hamiltonian) vector field $\delta(\mathbf{v}) = \delta + (\frac{1}{t}(2x^2x - 2yx + ty^2 - ty + (v_1 - v_2 - v_3 + v_4)y + v_2 - v_1))\frac{\partial}{\partial y} + (\frac{1}{t}(-2yx^2 + x^2 - 2txy + tx - (v_1 - v_2 - v_3 + v_4)x + (v_3 - v_1)t)\frac{\partial}{\partial x}$.

Hence we want to prove that for generic $\mathbf{v} \in \mathcal{V}$, $S_V(\mathbf{v})$ is strongly minimal and geometrically trivial.

Consider the union of lines in $\mathcal{V}$ given by

$$\begin{aligned} \mathcal{W} &= \{v \in \mathcal{V} :\ v_1 - v_2 \in \mathbb{Z}\} \cup \{v \in \mathcal{V} :\ v_1 - v_3 \in \mathbb{Z}\} \cup \{v \in \mathcal{V} :\ v_1 - v_4 \in \mathbb{Z}\} \\ &\cup \{v \in \mathcal{V} :\ v_1 - v_3 \in \mathbb{Z}\} \cup \{v \in \mathcal{V} :\ v_2 - v_4 \in \mathbb{Z}\} \cup \{v \in \mathcal{V} :\ v_3 - v_4 \in \mathbb{Z}\}, \end{aligned}$$

The following is proved in [49] (Theorem 1.2, Proposition 2.1, and Corollary 2.6):

Fact 4.3.13. For $\mathbf{v} \in \mathcal{V}$, $\delta(\mathbf{v})$ satisfies condition (J) if and only if $\mathbf{v} \notin \mathcal{W}$.

Commentary. Strictly speaking Watanabe considers rather the vector field corresponding to the δ_1-variety structure on $\mathbb{A}^2$:

$$\begin{cases} \delta_1(y) &= 2x^2x - 2yx + ty^2 - ty + (v_1 - v_2 - v_3 + v_4)y + v_2 - v_1) \\ \delta_1(x) &= -2yx^2 + x^2 - 2txy + tx - (v_1 - v_2 - v_3 + v_4)x + (v_3 - v_1)t) \end{cases}$$

where $\delta_1 = t\delta$.

But as in the case of P_{III} above, this suffices.

Corollary 4.3.14. *For $\mathbf{v} \in \mathcal{V}$, the solution set of $S_V(\mathbf{v})$ is strongly minimal if and only if $\mathbf{v} \notin \mathcal{W}$.*

We conclude, as previously:

Proposition 4.3.15. *For $v_1, v_2, v_3 \in \mathbb{C}$ algebraically independent, the solution set of $S_V(\mathbf{v})$ is strongly minimal and geometrically trivial. Hence by the reductions above, for algebraically independent $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ the solution set of $P_V(\alpha, \beta, \gamma, \delta)$ is strongly minimal and geometrically trivial.*

4.3.6 The family P_{VI}

Recall that $P_{VI}(\alpha, \beta, \gamma, \delta)$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, is given by the following equation

$$\begin{aligned} y'' =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right) \end{aligned}$$

The equation is studied by Okamoto [29] followed by Watanabe [50]. Again we summarise what we need.

Given $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, let $\alpha_0, \alpha_1, \alpha_3, \alpha_4$ be complex numbers such that $\alpha = \frac{1}{2}\alpha_1^2$, $\beta = -\frac{1}{2}\alpha_4^2$, $\gamma = \frac{1}{2}\alpha_3^2$ and $\delta = \frac{1}{2}(1 - \alpha_0^2)$. In what follows we also let $\boldsymbol{\alpha}$ be the tuple $(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$.

Then the equation $P_{VI}(\alpha, \beta, \gamma, \delta)$ can be written as the (Hamiltonian) system:

$$S_{VI}(\boldsymbol{\alpha}) \begin{cases} y' &= \frac{1}{t(t-1)}(2xy(y-1)(y-t) - \{\alpha_4(y-1)(y-t) + \alpha_3 y(y-t) \\ & \quad +(\alpha_0 - 1)y(y-1)\}) \\ x' &= \frac{1}{t(t-1)}(-x^2(3y^2 - 2(1+t)y + t) + x\{2(\alpha_0 + \alpha_3 + \alpha_4 - 1)y \\ & \quad -\alpha_4(1+t) - \alpha_3 t - \alpha_0 + 1\} - \alpha_2(\alpha_1 + \alpha_2)) \end{cases}$$

where $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$ and where $\delta(\boldsymbol{\alpha}) = \partial + (\frac{1}{t(t-1)}(2xy(y-1)(y-t) - \{\alpha_4(y-1)(y-t) + \alpha_3 y(y-t) + (\alpha_0 - 1)y(y-1)\}))\frac{\partial}{\partial y} + (\frac{1}{t(t-1)}(-x^2(3y^2 - 2(1+t)y+t) + x\{2(\alpha_0+\alpha_3+\alpha_4-1)y - \alpha_4(1+t) - \alpha_3 t - \alpha_0 + 1\} - \alpha_2(\alpha_1+\alpha_2)))\frac{\partial}{\partial x}$ is the derivation giving the corresponding δ-variety structure on $\mathbb{A}^2$.

Let us note that any solution y of $P_{VI}(\alpha, \beta, \gamma, \delta)$ yields a unique solution (y, x) of $S_{VI}(\boldsymbol{\alpha})$. The only possible solutions (y, x) of $S_{VI}(\boldsymbol{\alpha})$ not of this form are when $y = 0, 1, t$ and such solutions will exhibit non strong minimality of $S_{VI}(\bar{\alpha})$ (even though $P_{VI}(\alpha, \beta, \gamma, \delta)$ may be strongly minimal).

As before, we take a close look at the work of Watanabe on P_{VI}. First, note that the relation between the parameters $\boldsymbol{\alpha}$ above and the parameters (a_1, a_2, a_3, a_4) in Watanabe's Hamiltonian vector field for P_{VI} (beginning of section 3 of [49]) is: $\alpha_4 = a_3 + a_4$, $\alpha_3 = a_3 - a_4$, $\alpha_0 = 1 - a_1 - a_2$, and $\alpha_1 = a_1 - a_2$. However in [49], instead of only working with $\delta(\boldsymbol{\alpha})$ and the δ-variety structure on $\mathbb{A}^2$, Watanabe works with the so called Okamoto space of initial conditions. This roughly corresponds to patching six copies $U_i = \{(y_i, x_i)\}$, $i = 0, 1, 2, 3, 4, \infty$, of $\mathbb{A}^2$ with some specified patching rule (for example on $U_0 \cap U_\infty$: $y_0 y_\infty = 1$ and $y_0 x_0 + y_\infty x_\infty = -\alpha_2$). On each U_i is also defined a derivation $\delta_i(\boldsymbol{\alpha})$. These derivations respect the patching rule and Watanabe looks for condition (J) for this entire "system". But we only work with $\delta_0 = \delta$ here and hence we shall only focus on what we need. So let

	α_0	α_1	α_2	α_3	α_4	y	x
s_0	$-\alpha_0$	α_1	$\alpha_2 + \alpha_0$	α_3	α_4	y	$x - \frac{\alpha_0}{y-t}$
s_1	α_0	$-\alpha_1$	$\alpha_2 + \alpha_1$	α_3	α_4	y	x
s_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$	$\alpha_3 + \alpha_2$	$\alpha_4 + \alpha_2$	$y + \frac{\alpha_2}{x}$	x
s_3	α_0	α_1	$\alpha_2 + \alpha_3$	$-\alpha_3$	α_4	y	$x - \frac{\alpha_3}{y-1}$
s_4	α_0	α_1	$\alpha_2 + \alpha_4$	α_3	$-\alpha_4$	y	$x - \frac{\alpha_4}{y}$

Table 4.1: Some Backlund Transformations for S_{VI}

us start by describing some "Backlund transformations" for $S_{VI}(\bar{\alpha})$. These map solutions of a given S_{VI} equation to solutions of the same equation with different values of parameters $\boldsymbol{\alpha}$, but clearly may be undefined at certain solutions. The list of the Backlund transformations we are interested in are given in Table 4.1. The five transformations s_0, s_1, s_2, s_3, s_4 generate a group $\mathcal{W}$ which is isomorphic to the affine Weyl group of type D_4 and which is sometimes referred to as Okamoto's affine D_4 symmetry group. By definition, the reflecting hyperplanes of Okamoto's affine D_4 action are given by the affine linear relations
$\alpha_i = n$ for $i = 0, 1, 3, 4$ and $n \in \mathbb{Z}$,
as well as
$\alpha_0 \pm \alpha_1 \pm \alpha_3 \pm \alpha_4 = 2n + 1$ for $n \in \mathbb{Z}$.

Let $\mathcal{M}$ be the union of all these hyperplanes. Then the following is proved in [50] (Theorem 2.2, Proposition 3.1, and Corollary 3.7).

Fact 4.3.16. For $(\alpha_0, \alpha_1, \alpha_3, \alpha_4) \in \mathbb{C}^4$, if $(\alpha_0, \alpha_1, \alpha_3, \alpha_4) \notin \mathcal{M}$ then $\delta(\boldsymbol{\alpha})$ satisfies condition (J).

Commentary. Again Watanabe works instead with $t(t-1)\delta(\boldsymbol{\alpha})$, but it suffices, as at the end of the Commentary in Subsection 4.3.3.

Corollary 4.3.17. *If $(\alpha_0, \alpha_1, \alpha_3, \alpha_4) \notin \mathcal{M}$, then the solution set of $S_{VI}(\boldsymbol{\alpha})$ is strongly minimal.*

Clearly this is not enough for us to prove our main result. Indeed we need to find a Zariski dense subset (of $\mathbb{C}^4$) of parameters where condition (J) fails.

Remark 4.3.18. Let us write $t_0 = s_0 s_2 (s_1 s_3 s_4 s_2)^2$, $t_1 = s_1 s_2 (s_0 s_3 s_4 s_2)^2$, $t_3 = s_3 s_2 (s_0 s_1 s_4 s_2)^2$, and $t_4 = s_4 s_2 (s_0 s_1 s_3 s_2)^2$. Then for $i = 0, 1, 3, 4$, and parameters $(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$, $t_i(\alpha_i) = \alpha_i - 2$, and $t_i(\alpha_j) = \alpha_j$. Hence the orbit of $(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ under $\mathcal{W}$ includes $\{(\alpha_0 - 2\mathbb{Z}, \alpha_1 - 2\mathbb{Z}, \alpha_3 - 2\mathbb{Z}, \alpha_4 - 2\mathbb{Z})\}$.

	α_0	α_1	α_2	α_3	α_4
s_2	α_0+α_2	α_1+α_2	-α_2	α_3+α_2	α_4+α_2
$s_4 s_2$	α_0+α_2	α_1+α_2	α_4	α_3+α_2	-α_4-α_2
$s_3 s_4 s_2$	α_0+α_2	α_1+α_2	α_4+α_3+α_2	-α_3-α_2	-α_4-α_2
$s_1 s_3 s_4 s_2$	α_0+α_2	-α_1-α_2	1-α_0	-α_3-α_2	-α_4-α_2
$(s_1 s_3 s_4 s_2)^2$	α_0+α_2+1-α_0	α_1+α_2-1+α_0	1-α_0-α_2	α_3+α_2-1+α_0	α_4+α_2-1+α_0
$s_2(s_1 s_3 s_4 s_2)^2$	2-α_0	α_1	α_0+α_2-1	α_3	α_4
$s_0 s_2(s_1 s_3 s_4 s_2)^2$	α_0-2	α_1	α_2+1	α_3	α_3

Table 4.2: The transformation t_0

Using the transformation t_0 we get the following:

Proposition 4.3.19. *If $\alpha_1, \alpha_3, \alpha_4 \in \mathbb{C}$ are transcendental and algebraically independent, then for $\alpha_0 \in 2\mathbb{Z}$, $S_{VI}(\boldsymbol{\alpha})$ is not strongly minimal.*

Proof. Let $\alpha_1, \alpha_3, \alpha_4$ be transcendental and algebraically independent. Note that then α_2 is also transcendental. One can easily check that $S_{VI}(0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is not strongly minimal; the curve defined by $y = t$ is an algebraic ∂-subvariety of $(\mathbb{A}^2, D(0, \alpha_1, \alpha_2, \alpha_3, \alpha_4))$.

As a consequence $s_0 : x \to x - \frac{\alpha_0}{y-t}$ is undefined on these solutions. However all the other s_i's, as well as t_0, are well defined. We want to worry now about successively applying t_0. But note that the occurrence of $y = 0$, $y = 1$, $y = t$ and $x = 0$ (where the s_i's are not well defined) implies that $\alpha_4 = 0$, $\alpha_3 = 0$, $\alpha_0 = 0$, and $\alpha_2 = 0$ respectively. Looking at Table 4.2, we see that this cannot happen on successively applying t_0, as $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are transcendental and we are done. □

Proposition 4.3.20. *If $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are algebraically independent (and transcendental), then the solution set of $P_{VI}(\alpha, \beta, \gamma, \delta)$ is strongly minimal and geometrically trivial.*

Proof. It is enough to work with the solution set of the system $S_{VI}(\boldsymbol{\alpha})$ and note that $\alpha_0, \alpha_1, \alpha_3, \alpha_4$ are algebraically independent, so by Corollary 4.3.17 its solution set $Y(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ is strongly minimal. If it is not geometrically trivial, then by independence of $\alpha_0, \alpha_1, \alpha_3, \alpha_4, t$ in DCF_0, for all but finitely many $c \in \mathbb{C}$, $Y(c, \alpha_1, \alpha_3, \alpha_4)$ is in finite-to-finite relation with some $E_b^\sharp$ with $b \notin \mathbb{C}$. Choosing such $c \in 2\mathbb{Z}$ gives a contradiction to Proposition 4.3.19 □

4.3.7 Further Remarks

First note that the methods above show that for the families P_{III}, P_V and P_{VI}, the solution set of an equation is (strongly minimal and) geometrically trivial as long as the transcendence degree of the tuple $\alpha, \beta, \gamma, \delta$ is at least 2.

We would also guess that in fact the solution set of any of the Painlevé equations $P_{II} - P_V$ is geometrically trivial whenever it is strongly minimal, but this is not given by our methods (we have only succeeded to prove it for the second Painlevé equation using a different idea).

Note that the model-theoretic content of our methods is:

Proposition 4.3.21. *Let $y'' = f(y, y', b_1, .., b_n)$ where $f(-,-,-..)$ is a rational function (over $\mathbb{Q}$) and $b_1, .., b_n \in \mathcal{U}$. Suppose that the solution set of the equation is strongly minimal and* not *geometrically trivial. Then there is a formula $\theta(y_1, .., y_n)$(without parameters) true of $b_1, .., b_n$ such that for any $c_1, .., c_n$ satisfying θ the solution set of $y'' = f(y, y', c_1, .., c_n)$ is strongly minimal (and nontrivial).*

Likewise if $X(\bar{b})$ is a strongly minimal set of order 2 which is defined over $\bar{b}$ and is not geometrically trivial, then there will be a formula $\theta(\bar{y})$ true of $\bar{b}$ such that for any $\bar{c}$ satisfying θ, $X(\bar{c})$ is strongly minimal.

The next remark concerns "downward semi-definability of Morley rank", as discussed in [12]. The point is that each of the Painlevé families witnesses non downward semi-definability of Morley rank. For example for $P_{II}(\alpha)$ the analysis of [47] that we cite also gives that for $\alpha \in 1/2 + \mathbb{Z}$, the solution set of $P_{II}(\alpha)$ contains infinitely many (in fact a definable family of) order 1 definable sets, hence has Morley rank 2. So:

Fact 4.3.22. For $\alpha \in \mathbb{C}$ transcendental, the solution set $Y(\alpha)$ of $P_{II}(\alpha)$ has Morley rank 1, but for every formula $\theta(x)$ true of α there is α_1 satisfying $\theta(x)$ such that Y_{α_1} has Morley rank > 1,

This fact, together with the proof of Lemma 1.1 in [12], shows that there is a $\emptyset$-definable set X of order 3 in DCF_0 with Morley rank different from Lascar rank. This answers (in a negative sense) Question 2.9 of [12].

Finally one might wonder whether our soft proofs of geometric triviality of certain order 2 equations might apply to higher order equations. But an obstacle is precisely non downward semi-definability of Morley rank for more general families $A^\sharp$, as given in [12]: There is a definable (in DCF_0) family $(A_b : b \in B)$ (with B of finite Morley rank) family of 2-dimensional abelian varieties such for generic $b \in B$, A_b is simple (hence $A_b^\sharp$ is strongly minimal), but for every $\emptyset$-definable subset $B' \subset B$ there is $b' \in B'$ such that $A_{b'}$ is not simple (hence $A_{b'}^\sharp$ has Morley rank 2).

Chapter 5

Geometric triviality: Non Generic P_{II}

In this Chapter, we extend the result in Chapter 4 for $P_{II}(\alpha)$ to the non generic strongly minimal case. That is we show that for $\alpha \notin 1/2 + \mathbb{Z}$, $P_{II}(\alpha)$ is geometrically trivial. Our proof is a uniform one, in the sense that it works for all $\alpha \notin 1/2 + \mathbb{Z}$ (so including the generic case). For the proof of the main result, we give a new criteria for deciding whether strongly minimal sets in DCF_0 are geometrically trivial.

5.1 Correspondences on $A^\sharp$

We will in this section make use of the notion of multiplicity of a type, so we say a few words in the strongly minimal setting. Suppose Y is a strongly minimal set in $\mathcal{U}$ and K is any differential field over which Y is defined. If two elements y and z of Y are such that $y \in K\langle z\rangle^{alg}$ (and so $z \in K\langle y\rangle^{alg}$), then $mult(y/K\langle z\rangle)$ will denote the finite number of elements of Y that realise $tp(y/K\langle z\rangle)$. In other words, $mult(y/K\langle z\rangle)$ is the degree of the minimal polynomial of y over $K\langle z\rangle$.

It is well known that finite-to-finite correspondences (indeed finite-to-one) exist for the Manin kernels: Let A be a simple abelian variety and let $Y = A^\sharp$ be its Manin kernel. For each $n \in \mathbb{N}$, we have the multiplication-by-n map, $n : A \to A$. This map is surjective and n^{2d}-to-1 (where d is the dimension of

A). So for any generic point $a \in Y$, $mult(n \cdot a/a) = 1$ while $mult(a/n \cdot a) > 1$ (as long as $n > 1$).

The same is true for any non-trivial locally modular strongly minimal set:

Proposition 5.1.1. *Suppose that Y is a non-trivial locally modular strongly minimal set in $\mathcal{U}$. Then there exists a differential field K such that for any generic point y of Y over K, there exist $z \in Y$ generic over K, such that $z \in K\langle y\rangle^{alg}$ and $z \notin K\langle y\rangle$.*

Proof. Assume for contradiction that the conclusion of the proposition does not hold. Now as Y is non-trivial locally modular, from Theorem 2.3.11, there exist a simple Abelian variety A not defined over $\mathbb{C}$ such that Y is nonorthogonal to $X = A^\sharp$. We work throughout over the differential field K over which nonorthogonality occurs. Also, in what follows Y_{gen} will be the set of generic points of Y (same for X_{gen}).

Let $y \in Y_{gen}$ and let $Z = acl(y) \cap (Y_{gen} \cup X_{gen})$. Note that as Y and X are nonorthogonal, $Z \cap X$ is non empty. Also if $e \in Z \cap X$, then for any $n \in \mathbb{N}_{>1}$ we have that $n \cdot e \in Z \cap X$ (since $n \cdot e \in acl(e) = acl(y)$). We show that if $d, e \in Z \cap X$, then $mult(e/d) = mult(d/e)$. This contradicts the case when $d = n \cdot e$ as discussed above and we are done. We prove this using several claims.

Let $a \in Z \cap Y$ and $d, e \in Z \cap X$.

Claim 1: $mult(e/a) = mult(d/a)$ and $mult(a/e) = mult(a/d)$. Indeed $mult(d/a)$ and $mult(a/d)$ does not depend on the choice of a or d.

Proof: First note that by construction (and nonorthogonality) a,d and e are all interalgebraic. Indeed any two elements of Z are.

We first prove that $mult(e/a) = mult(d/a)$. So suppose $D = \{d = d_1, \ldots, d_k\}$ is the set of realisations of $tp(d/a)$, i.e $mult(d/a) = k$. Now as $tp(e) = tp(d)$, it is not hard to see that there is $b \in Y_{gen}$ and $E = \{e = e_1, \ldots, e_k\} \subseteq X_{gen}$ such that E is the set of realisations of $tp(e/b)$. But then $K\langle a\rangle^{alg} = K\langle b\rangle^{alg}$ and hence by assumption $K\langle a\rangle = K\langle b\rangle$. So for any $\sigma \in Aut(\mathcal{U}/a)$, $\sigma(b) = b$

and hence $\sigma(E) = E$. In particular E is definable over a and we have that E is the set of realisations of $tp(e/a)$, i.e $mult(e/a) = k$.

We now prove that $mult(a/e) = mult(a/d)$. Suppose $mult(a/d) = l$ and that $\phi(x, d)$ isolates $tp(a/d)$, i.e $\models \exists^{=l} y\phi(y, d)$. As $tp(e) = tp(d)$, we have that $\models \exists^{=l} y\phi(y, e)$. Choose $c \in Y_{gen}$ such that $\models \phi(c, e)$. Then $K\langle a\rangle^{alg} = K\langle c\rangle^{alg}$ and hence by assumption $K\langle a\rangle = K\langle c\rangle$, that is there is a L_∂-formula $\theta(x, y)$ such that $\models \theta(a, c) \wedge \exists^{=1} x\theta(x, c) \wedge \exists^{=1} y\theta(a, y)$. Let $\psi(x, e)$ be the L_∂-formula $\exists y\phi(y, e) \wedge \theta(x, y)$. By construction $\models \exists^{=l} x\psi(x, e)$ and it follows that $\psi(x, e)$ isolates $tp(a/e)$, i.e $mult(a/e) = l$.

Claim 2: $mult(d/a, e) = mult(e/a, d)$.

Proof: Since

$$\begin{aligned} mult(de/a) &= mult(e/a) \cdot mult(d/a, e) \\ &= mult(d/a) \cdot mult(e/a, d), \end{aligned}$$

using Claim 1 we are done.

Claim 3: $mult(da/e) = mult(ea/d)$.

Proof: As before, since

$$\begin{aligned} mult(da/e) &= mult(a/e) \cdot mult(d/e, a) \\ mult(ea/d) &= mult(a/d) \cdot mult(e/a, d), \end{aligned}$$

using both Claim 1 and Claim 2 we are done.

Finally

Claim 4: $mult(d/e) = mult(e/d)$.

Proof: This time we use

$$\begin{aligned} mult(da/e) &= mult(d/e) \cdot mult(a/d, e) \\ mult(ea/d) &= mult(e/d) \cdot mult(a/d, e). \end{aligned}$$

So from Claim 3 the result follows □

So if Y is a non-trivial locally modular strongly minimal set, over some differential field K, there exist generic definable finite-to-finite correspondences $Y \to Y$ that are not generic definable permutations of Y. The aim of the next section will hence be to show that such is not the case for the strongly minimal second Painlevé equations.

5.2 Geometric triviality and the second Painlevé equations

In this section we look at the second Painlevé equation, $y'' = 2y^3 + ty + \alpha$, $\alpha \in \mathbb{C}$ and denote by $Y(\alpha) \subseteq \mathcal{U}$ its solution set. As we have seen in the previous chapter the following is true (Corollary 4.3.5 and Proposition 4.3.6):

Fact 5.2.1.

(i) $Y(\alpha)$ is strongly minimal if and only if $\alpha \notin \frac{1}{2} + \mathbb{Z}$.

(ii) For $\alpha \notin \mathbb{Q}^{alg}$, $Y(\alpha)$ is geometrically trivial.

We now aim to extend Fact 5.2.1(ii) to all $\alpha \notin \frac{1}{2} + \mathbb{Z}$. Before we proceed recall that for a field K, $K((X))$ denotes the field of formal Laurent series in variable X, while $K\langle\langle X\rangle\rangle$ denotes the field of formal Puiseux series, i.e. the field $\bigcup_{d\in\mathbb{N}} K\left(\left(X^{1/d}\right)\right)$. It is well known that if K is algebraically closed of characteristic 0 then so is $K\langle\langle X\rangle\rangle$ - it is the algebraic closure of $K((X))$ (cf. [5] Corollary 13.15).

Proposition 5.2.2. *For $\alpha \notin \frac{1}{2} + \mathbb{Z}$, $Y(\alpha)$ is geometrically trivial.*

Proof. We fix $\alpha \notin \frac{1}{2} + \mathbb{Z}$. First note that by Remark 2.3.5, $Y(\alpha)$ is orthogonal to $\mathbb{C}$. So from Proposition 5.1.1 we only have to prove that for any differential field K over which $Y(\alpha)$ is defined, if y and z are two generic elements (over K) of $Y(\alpha)$, then if $K(y, y')^{alg} = K(z, z')^{alg}$, then $K(y, y') = K(z, z')$.

So let K be any differential field containing $\mathbb{C}(t)$ and let $y, z \in Y(\alpha)$ (generic) be such that $z \in K(y, y')^{alg}$. Letting K_1 denote the algebraic closure of $K(y')$ in $\mathcal{U}$, we regard z as algebraic over $K_1(y)$ and we first aim to show that $z \in K_1(y)$. As $z \in K_1(y)^{alg}$ for any $\beta \in K_1$, z can be seen as an element of $K_1\langle\langle y - \beta\rangle\rangle$, so that there exists $e \in \mathbb{N}$ such that $z \in K_1\left(\left((y-\beta)^{1/e}\right)\right)$. A simpler way of saying the above (*à-la-Nishioka*) is that we look at expansions in a local parameter τ at $\beta \in K_1$ given by

$$y = \beta + \tau^e \qquad\qquad z = \sum_{i=r}^{\infty} a_i \tau^i \quad (a_r \neq 0)$$

with e the ramification exponent. Of course our intention is to show that for every choice of $\beta \in K_1$, we have that $e = 1$ and we are done.

Differentiating we have

$$\begin{aligned} e\tau^{e-1}\tau' &= y' - \beta' \\ &= y' - \beta^* - \beta_{y'}(2y^3 + ty + \alpha) \\ &= y' - \beta^* - \beta_{y'}(2\beta^3 + t\beta + \alpha) + \\ &\quad +(6\beta^2 + t)\beta_{y'}\tau^e + 6\beta\beta_{y'}\tau^{2e} + \beta_{y'}\tau^{3e} \end{aligned} \tag{5.2.1}$$

where "*" indicates the extension on K_1 of the derivation on $K[y']$ given by $(\sum c_i y'^i)^* = \sum c_i' y'^i$ and $\beta_{y'} = \frac{\partial \beta}{\partial y'}$.

Letting $\gamma = y' - \beta^* - \beta_{y'}(2\beta^3 + t\beta + \alpha)$, we have

Claim 1: $\gamma \neq 0$

Proof: For contradiction, suppose that $\gamma = 0$, that is

$$y' - \beta^* = \beta_{y'}(2\beta^3 + t\beta + \alpha) \tag{5.2.2}$$

Let $F \in K[y, y']$ be an irreducible polynomial such that $F(\beta, y') = 0$. Then

$$(F(\beta, y'))^* = F^*(\beta, y') + \beta^* F_y(\beta, y') = 0 \tag{5.2.3}$$

$$\frac{\partial}{\partial y'}(F(\beta, y')) = F_{y'}(\beta, y') + \beta_{y'} F_y(\beta, y') = 0. \tag{5.2.4}$$

If we multiply 5.2.4 by $2\beta^3 + t\beta + \alpha$ and use 5.2.2, we have

$$(2\beta^3 + t\beta + \alpha)F_{y'}(\beta, y') + (y' - \beta^*)F_y(\beta, y') = 0.$$

So, together with 5.2.4 we get

$$F^*(\beta, y') + y' F_y(\beta, y') + (2\beta^3 + t\beta + \alpha)F_{y'}(\beta, y') = 0.$$

In other words $(\delta F)(\beta, y') = 0$, so that F divides its derivative δF (i.e $\delta F = GF$ for some $G \in K[y, y']$). This contradicts condition (J) and hence strong minimality of $Y(\alpha)$ as per Corollary 4.1.1 and the claim is proved.

Hence from equation 5.2.1, $\gamma \neq 0$ implies that $\tau' = e^{-1}\gamma\tau^{1-e} + A$ where

$$A = e^{-1}\tau^{1-e}\left((6\beta^2 + t)\beta_{y'}\tau^e + 6\beta\beta_{y'}\tau^{2e} + \beta_{y'}\tau^{3e}\right)$$

and from this we get that

$$\begin{aligned}(\tau^i)' &= i\tau^{i-1}\tau' \\ &= i\tau^{i-1}e^{-1}\gamma\tau^{1-e} + A' \\ &= \frac{i\gamma}{e}\tau^{i-e} + A'\end{aligned}$$

and similarly

$$(\tau^i)'' = \frac{i(i-e)}{e^2}\gamma^2\tau^{i-2e} + A''.$$

Now from

$$z'' = 2z^3 + tz + \alpha \qquad \text{and} \qquad z = \sum_{i=r}^{\infty} a_i\tau^i$$

we have

$$\sum_{i=r}^{\infty} a_i \frac{i(i-e)}{e^2}\gamma^2\tau^{i-2e} + \cdots = 2\sum_{i,j,k} a_i a_j a_k \tau^{i+j+k} + \cdots . \qquad (5.2.5)$$

Using this we prove a couple of claims. One should of course notice that the "ignored" part does no play a role in our calculations below.

Claim 2: If $e > 1$ then $r < 0$.

Proof: Let $e > 1$ and assume $r \geq 0$. If for all $l \in \{r, r+1, \ldots\}$ we have that $e \mid l$, then it is not hard to see that the ramification exponent must be one (i.e $e = 1$), a contradiction. So choose $l \in \{r, r+1, \ldots\}$ least such that $e \nmid l$ and $a_l \neq 0$. First, note that since $l - 2e < l$ and $e \nmid l - 2e$, we have that $a_{l-2e} = 0$ (same for $a_{l-e} = 0$) or else this contradicts that l is least such. So in what follows one does not need to worry about the other coefficients in 5.2.5.

So if we look at the coefficient of τ^{l-2e} on the LHS of 5.2.5, we see that

$$a_l \frac{l(l-e)}{e^2}\gamma^2 \neq 0.$$

This implies that the coefficient on the RHS of τ^{i+j+k} for some $i, j, k \geq r$ with $i+j+k = l-2e$ must be non-zero. However for any such, since $i+j+k < l$ and $e \nmid i+j+k$, we have that e does not divide at least one them, say $i < l$. But then $a_i = 0$ (as l was chosen to be the least with this property) and so $a_i a_j a_k = 0$, a contradiction.

Claim 3: The case $e > 1$ and $r < 0$ leads to a contradiction.

Proof: So this time suppose $e > 1$ and $r < 0$. From the least powers of τ in 5.2.5 (which is $3r$ on the R.H.S) we have $r - 2e = 3r$, that is $r = -e$, and from the coefficients of τ^{r-2e} we get

$$a_r \frac{r(r-e)}{e^2}\gamma^2 = 2a_r^3$$

so that $a_r = \pm\gamma$, since $a_r \neq 0$.

So again choose $l \in \{r, r+1, \ldots\}$ least such that $e \nmid l$ and $a_l \neq 0$. The coefficient of $\tau^{l-2e} = \tau^{l+2r}$ on the LHS of 5.2.5 is

$$a_l \frac{l(l-e)}{e^2}\gamma^2 \neq 0.$$

On the RHS we see that the coefficient of $\tau^{l-2e} = \tau^{l+2r}$ should be $6a_r^2 a_l$. (Indeed, $e \nmid i+j+k$ means that e does not divide at least one of them, say i. Then $e \nmid i$, $a_i \neq 0$ means either $i = l$ or $i > l$. But $i > l$ implies that $j + k < 2r$ a contradiction. So $i = l$ and hence $j = k = r$).

Hence

$$a_l \frac{l(l-e)}{e^2}\gamma^2 = 6a_r^2 a_l = 6\gamma^2 a_l$$

and we see that either $l = -2e$ or $l = 3e$, contradicting $e \nmid l$ and we are done

Hence $e = 1$, and since β was arbitrary, the ramification exponent at every $\beta \in K_1$ is 1. So $z \in K_1(y)$.

Finally, letting K_0 denote the algebraic closure of $K(y)$ in $\mathcal{U}$, one can show similarly that $z \in K_0(y')$. Indeed the calculation above seems to work for any strongly minimal equation $y'' = f(y, y', t)$, with f polynomial with coefficient in $\mathbb{C}$. Since $K_1(y) \cap K_0(y') = K(y, y')$, we have shown that $z \in K(y, y')$.

Changing the role of y and z, we also have $y \in K(z, z')$ and hence $K(y, y') = K(z, z')$. $\square$

5.3 Further comments

Recall that a strongly minimal set Y in $\mathcal{U}$ is said to be unimodular if for any n and any differential field K over which Y is defined, if $y_1, \ldots, y_n$,$x_1, \ldots, x_n \in Y$ are such that $\overline{y}$ and $\overline{x}$ are interalgebraic over K (i.e $K\langle\overline{y}\rangle^{alg} = K\langle\overline{x}\rangle^{alg}$), and $RM(\overline{y}/K) = RM(\overline{x}/K) = n$, then $mult(\overline{y}/K\langle\overline{x}\rangle) = mult(\overline{x}/K\langle\overline{y}\rangle)$. Here $RM(\overline{y}/K) = n$ means that $y_1, .., y_n$ together with all their derivatives $y_i^{(j)}$ are algebraically independent over K.

Remark 5.3.1.

(i) This notion, unimodularity, is not restricted to DCF_0 and indeed makes sense for any strongly minimal set in any complete theory T (see for example Definition 2.4.2 in [35]).

(ii) Any ω-categorical strongly minimal set in DCF_0 (or in any stable theory) is unimodular (cf. [35] Remark 2.4.2).

It is not hard to check that for $\alpha \notin 1/2+\mathbb{Z}$, since generic correspondences of $X_{II}(\alpha)$ are generic permutations, $X_{II}(\alpha)$ is unimodular. On the other hand, we have seen that the Manin kernels are not unimodular. Hrushovski ([8]) pointed out that for strongly minimal sets, unimodularity is preserved under nonorthogonality (and the proof of Proposition 5.1.1 is basically a special case). So we could have also deduced in that way that $X_{II}(\alpha)$, $\alpha \notin 1/2 + \mathbb{Z}$, is geometrically trivial. More importantly though, we have that in DCF_0 any unimodular strongly minimal set is geometrically trivial. We conjecture that the converse, a weakening of Conjecture 2.3.15, is also true:

Conjecture 5.3.2. *In any differentially closed field, every geometrically trivial strongly minimal set is unimodular.*

Chapter 6

Algebraic independence: The generic cases

In this chapter we show that that if $y'' = f(y, y', t, \alpha, \beta, \ldots)$ is a generic Painlevé equation from among the classes II, IV and V, and if $y_1, \ldots, y_n$ are distinct solutions, then $tr.deg\,(\mathbb{C}(t)(y_1, y_1', \ldots, y_n, y_n')/\mathbb{C}(t)) = 2n$. For generic Painlevé III and VI, we show that their solution sets are ω-categorical The results confirm old beliefs about the Painlevé transcendents and is a culmination of the work started by Painlevé over 100 years ago.

6.1 A few remarks

As in the previous chapters $(\mathcal{U}, \delta)$ will be a saturated differentially closed field of cardinality continuum with field of constants $\mathbb{C}$ and t will denote an element of $\mathcal{U}$ with $\delta(t) = 1$. Throughout F_0 will be a finitely generated subfield of $\mathbb{C}$, K_0 will denote the differential field $F_0(t)$ and K will usually denote the field $\mathbb{C}(t)$. Note that when working with one of the Painlevé equation $y'' = f(y, y', t, \alpha, \beta, \gamma, \delta)$, F_0 will be the subfield of $\mathbb{C}$ generated over $\mathbb{Q}$ by the $\alpha, \beta, \gamma, \delta$.

Remark 6.1.1. Suppose that Y is the solution set of one of the generic Painlevé equations. If $y_1, .., y_n \in Y$ are mutually generic over K_0, that is $tr.deg\,(K_0(y_1, y_1', \ldots, y_n, y_n')/K_0) = 2n$, then they are also mutually generic over K. In particular if $y \in Y$ is in K^{alg} then it is already in K_0^{alg}.

This is because Y, as we have seen in the Chapter 4, is strongly minimal and of order 2 and so, by Theorem 2.3.11 and Remark 2.3.5, is orthogonal to the constants $\mathbb{C}$.

Definition 6.1.2. Let Y be a K_0-definable set and let $L > K_0$. We say that Y is *strictly disintegrated* over L, if whenever $y_1, .., y_n \in Y$ are distinct, then they are mutually generic over L.

So in particular we want to show that the solution sets of generic Painlevé equations from among the classes II, IV and V are strictly disintegrated over K. Note that Fact 4.3.3 shows that P_I is strictly disintegrated over any differential field K containing $\mathbb{C}(t)$

Remark 6.1.3.

(i) Strict disintegratedness over L of Y implies strong minimality of Y. It also implies that no solution is in L^{alg}.

(ii) As in the previous remark, if Y is the solution set of one of the generic Painlevé equations, we have that Y is strictly distintegrated over K_0 if and only if it is strictly disintegrated over K.

(iii) Strict disintegratedness of Y over L implies that any permutation of Y extends to an automorphism of the differential field $\mathcal{U}$ which fixes L pointwise.

6.2 Generic Painlevé equations P_{II}, P_{IV} and P_V

In this section we prove that the solution set of each of the generic Painlevé equations P_{II}, P_{IV} and P_V is strictly disintegrated over $\mathbb{C}(t)$: if $y_1, ..., y_n$ are distinct solutions, then $y_1, y_1', ..., y_n, y_n'$ are algebraically independent over $\mathbb{C}(t)$.

For each of the families we will first describe the results on the classification of algebraic solutions and then make use of those results to prove strict disintegratedness. There is essentially just one common argument; using the information that in each of the families, there is a Zariski-dense subset of the parameter space for which the corresponding equation has a unique algebraic (over $\mathbb{C}(t)$) solution. We will go through the details in the case of P_{II}, giving sketches in the remaining cases.

6.2.1 The family P_{II}

For $\alpha \in \mathbb{C}$, $P_{II}(\alpha)$ is given by the following equation

$$y'' = 2y^3 + ty + \alpha.$$

or by the equivalent Hamiltonian system:

$$S_{II}(\alpha)\begin{cases} y' &= x - y^2 - \frac{t}{2} \\ x' &= 2xy + \alpha + \frac{1}{2}. \end{cases}$$

It is not difficult to see that $(y, x) = (0, t/2)$ is a rational solution of $S_{II}(0)$. The work of Murata in [22] shows that this is the only algebraic solution. However we also have "Backlund transformations" that send solutions of $S_{II}(\alpha)$ to those of $S_{II}(-1-\alpha)$, $S_{II}(\alpha-1)$ and $S_{II}(\alpha+1)$. We have from [22] and [47]:

Fact 6.2.1. $P_{II}(\alpha)$ has an algebraic over $\mathbb{C}(t)$ solution iff $\alpha \in \mathbb{Z}$. Furthermore, this solution is unique.

We can now prove our main result:

Proposition 6.2.2. *Let $\alpha \in \mathbb{C}$ be generic (i.e. transcendental). Then the solution set $X(\alpha)$ of $P_{II}(\alpha)$ is strictly disintegrated over $K = \mathbb{C}(t)$.*

Proof. By Proposition 4.3.6 and Remark 6.1.1 it suffices to prove that any two elements of $X(\alpha)$ are mutually generic over $K_0 = \mathbb{Q}(\alpha, t)$. Let $y \in X(\alpha)$ (so generic over K_0). We want to show that $acl_{X(\alpha)}(K_0, y) = \{y\}$. For a contradiction suppose there is $z \in acl_{X(\alpha)}(K_0, y)$, with $z \neq y$. Let the formula $\phi(\alpha, t, u, v)$ witness this, i.e. $\mathcal{U} \models \phi(\alpha, t, y, z)$ and for any α_1, y_1, z_1 such that $\mathcal{U} \models \phi(\alpha_1, t, y_1, z_1)$ we have that $z_1 \in acl(\mathbb{Q}(\alpha_1, t, y_1))$. Now as all elements of $X(\alpha)$ are generic over K_0, strongly minimality means that in

particular they all satisfy the same formulas as y over K_0.
Hence: $\mathcal{U} \models \sigma(\alpha, t)$ where $\sigma(\alpha, t)$ is

$$\forall u \, (u \in X(\alpha)) \to \exists v (u \neq v \wedge v \in X(\alpha) \wedge \phi(\alpha, t, u, v)))$$

As explained in Chapter 4, $\mathcal{U} \models \sigma(\alpha_1, t)$ for all but finitely many $\alpha_1 \in \mathbb{C}$. So for some $n \in \mathbb{Z}$, $\sigma(n, t)$ is true in $\mathcal{U}$; that is

$$\forall u \, (u \in X(n)) \to \exists v (u \neq v \wedge v \in X(n) \wedge \phi(n, t, u, v))) \,.$$

However, choosing u to be the unique algebraic (over $\mathbb{C}(t)$) element of $X(n)$ (from Fact 6.2.1), we obtain another distinct algebraic (over $\mathbb{C}(t)$) element of $X(n)$, a contradiction. □

6.2.2 The family P_{IV}

For $\alpha, \beta \in \mathbb{C}$, the fourth Painlevé equation is

$$y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}.$$

From the work of Murata [22] (see also [7]) we have the following:

Fact 6.2.3. P_{IV} has algebraic solutions if and only if α, β satisfy one of the following conditions:

(i) $\alpha = n_1$ and $\beta = -2(1 + 2n_2 - n_1)^2$, where $n_1, n_2 \in \mathbb{Z}$;

(ii) $\alpha = n_1$, $\beta = -\frac{2}{9}(6n_2 - 3n_1 + 1)^2$, where $n_1, n_2 \in \mathbb{Z}$.

Furthermore the algebraic solutions for these parameters are unique.

Proposition 6.2.4. *The solution set $X(\alpha, \beta)$ of $P_{IV}(\alpha, \beta)$, $\alpha, \beta \in \mathbb{C}$ algebraically independent, is strictly disintegrated over $\mathbb{C}(t)$.*

Proof. Again it suffices to work over $K_0 = \mathbb{Q}(t, \alpha, \beta)$. Let $y \in X(\alpha, \beta)$. We want to show that $acl_{X(\alpha,\beta)}(K_0, y) = \{y\}$. Suppose for a contradiction there is $z \in acl_{X(\alpha,\beta)}(K, y)$, with $z \neq y$. As before this is witnessed by a formula

$\phi(\alpha, \beta, t, u, v)$, and again as all solutions of $X(\alpha, \beta)$ are generic over K_0, the following sentence $\sigma(t, \alpha, \beta)$ is true in $\mathcal{U}$:

$$\forall u\, (u \in X(\alpha, \beta)) \to \exists v(u \neq v \wedge v \in X(\alpha, \beta) \wedge \phi(\alpha, \beta, t, u, v)))\,.$$

Since α and β are mutually generic (and by Fact 6.2.3(i)) we can first choose $n_1 \in \mathbb{Z}$, then $n_2 \in \mathbb{Z}$ such that $\sigma(t, n_1, -2(1 + 2n_2 - n_1)^2)$ is true in $\mathcal{U}$ and $X(n_1, -2(1 + 2n_2 - n_1)^2)$ has a unique algebraic (over $\mathbb{C}(t)$) point. As in the P_{II} case we get a contradiction. $\square$

6.2.3 The family P_V

The fifth Painlevé equation $P_V(\alpha, \beta, \gamma, \delta)$ is given by

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

For our purposes it is enough to restrict to the case when $\delta \neq 0$, in which case all algebraic (over $\mathbb{C}(t)$) solutions are rational (see [15] and [7]). We let $\lambda_0 = (-2\delta)^{-1/2}$, fixing $-\pi < arg(\lambda_0) < \pi$, and with the same references we have:

Fact 6.2.5. P_V with $\delta \neq 0$ has an algebraic solution if and only if for some branch of λ_0, one of the following holds with $m, n \in \mathbb{Z}$:

(i) $\alpha = \frac{1}{2}(m + \lambda_0\gamma)^2$ and $\beta = -\frac{1}{2}n^2$ where $n > 0$, $m + n$ is odd, and $\alpha \neq 0$ when $|m| < n$;

(ii) $\alpha = \frac{1}{2}n^2$ and $\beta = -\frac{1}{2}(m + \lambda_0\gamma)^2$ where $n > 0$, $m + n$ is odd, and $\beta \neq 0$ when $|m| < n$;

(iii) $\alpha = \frac{1}{2}a^2$, $\beta = -\frac{1}{2}(a + n)^2$ and $\lambda_0\gamma = m$, where $m + n$ is even and a arbitrary;

(iv) $\alpha = \frac{1}{8}(2m + 1)^2$, $\beta = -\frac{1}{8}(2n + 1)^2$ and $\lambda_0\gamma \notin \mathbb{Z}$.

Remark 6.2.6.

(i) In case (iv) the algebraic solution is unique. This is also true for most of the other cases (see [15]) except for:

(ii) In case (i) or (ii), if $\lambda_0\gamma \in \mathbb{Z}$ then there are at most two algebraic solutions. Specifically, if $\alpha\beta \neq 0$, there are exactly two; otherwise there is only one.

Proposition 6.2.7. *The solution set $X(\alpha, \beta, \gamma, \delta)$ of $P_V(\alpha, \beta, \gamma, \delta)$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ algebraically independent, is strictly disintegrated over $\mathbb{C}(t)$.*

Proof. Assuming not, as in the earlier cases we find a sentence $\sigma(t, \alpha, \beta, \gamma, \delta)$ expressing that for any solution u of $X(\alpha, \beta, \gamma, \delta)$ there is another solution $v \neq u$ which is algebraic over $\mathbb{Q}(t, \alpha, \beta, \gamma, \delta, u)$. By Fact 4.11 (applied twice), we first find $r = \frac{1}{8}(2m+1)^2$ for some $m \in \mathbb{Z}$, and then $s = -\frac{1}{8}(2n+1)^2$ for some $n \in \mathbb{Z}$ such that $\mathcal{U} \models \sigma(t, r, s, \gamma, \delta)$, and obtain (since γ and δ are algebraically independent) a contradiction to Fact 6.2.5 and Remark 6.2.6(i). $\square$

6.3 Generic P_{III} and P_{VI}.

We do not, currently, have any reason to believe that the results for generic P_{II}, P_{IV} and P_V do not hold for generic P_{III} and P_{VI}. But our methods, involving the description of algebraic solutions, as parameters vary, yield a weaker statement: the solution sets of generic P_{III} and P_{VI} are ω-categorical, as in Definition 2.1.18.

6.3.1 The family P_{III}

As discussed in Chapter 4, on the face of it, the family P_{III} is a 4-parameter family: where $P_{III}(\alpha, \beta, \gamma, \delta)$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ is given by the following

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}.$$

However Okamoto [32] (see also [23]) shows that for the case $\gamma\delta \neq 0$, it is enough to rewrite the equation as a 2-parameter family. Indeed it is not difficult to check that that for $\lambda, \mu \in \mathbb{C}$, the transformation taking y to λy and t to μt takes $P_{III}(\alpha, \beta, \gamma, \delta)$ to $P_{III}(\lambda\mu\alpha, \mu\lambda^{-1}\beta, \lambda^2\mu^2\gamma, \mu^2\lambda^{-2}\delta)$.

Hence taking $\mu^4 = -16/\gamma\delta$ and $\lambda^2 = 4/\gamma\mu^2$ (assuming as above that $\gamma\delta \neq 0$), this transformation takes $P_{III}(\alpha, \beta, \gamma, \delta)$ to $P_{III}(\lambda\mu\alpha, \mu\lambda^{-1}\beta, 4, -4)$, and moreover if $\alpha, \beta, \gamma, \delta$ are algebraically independent, so are $\lambda\mu\alpha, \mu\lambda^{-1}\beta$. Hence the family $P_{III}(\alpha, \beta, \gamma, \delta)$ can be replaced by the family $P_{III}(\alpha, \beta, 4, -4)$ and we are reduced to showing that for α, β algebraically independent the solution set of $P_{III}(\alpha, \beta, 4, -4)$ is ω-categorical. Finally, in the study of algebraic solutions one also replaces α, β by new parameters v_1, v_2, with $\alpha = -4v_2$ and $\beta = 4(v_1 + 1)$ and α, β are algebraically independent if and only if v_1, v_2 are.

Murata in [23] gives a classification of all the algebraic solutions of $P_{III}(v_1, v_2)$:

Fact 6.3.1.

(i) $P_{III}(v_1, v_2)$ has algebraic solutions if and only if there exists an integer n such that $v_2 - v_1 - 1 = 2n$ or $v_2 + v_1 + 1 = 2n$.

(ii) If $P_{III}(v_1, v_2)$ has algebraic solutions, then the number of algebraic solutions is two or four. $P_{III}(v_1, v_2)$ has four algebraic solutions if and only if there exist two integers n and m such that $v_2 - v_1 - 1 = 2n$ and $v_2 + v_1 + 1 = 2m$.

He also shows that all the algebraic solutions are rational and gives an explicit description of the solutions (see Proposition 3.11 of [23]). From this we easily get:

Proposition 6.3.2. *Let X be the solution set of $P_{III}(\alpha, \beta, \gamma, \delta)$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are algebraically independent (and transcendental). Then for any $y \in X$, $acl_X(K, y)$ is finite, where $K = \mathbb{C}(t)$. Consequently as X is geometrically trivial, X is ω-categorical.*

Proof. We only need to work with $X(v_1, v_2)$ the solution set of $P_{III}(v_1, v_2)$, v_1, v_2 in $\mathbb{C}$ algebraically independent. By Remark 6.1.1 again, it is enough to prove the result over $K_0 = \mathbb{Q}(t, v_1, v_2)$.

Let $y \in X(v_1, v_2)$. We claim that $acl_{X(v_1,v_2)}(K_0, y)$ has cardinality at most 2 (including y itself). If not, as before we have a formula $\sigma(v_1, v_2, t)$ expressing that for any solution $y \in X(v_1, v_2)$, there are at least 2 other solutions in the algebraic closure of $K_0(y, y')$. Then $\mathcal{U} \models \sigma(v_1, c, t)$ is true for all but finitely many $c \in \mathbb{C}$.

So we can find such c with $c + v_1 + 1 \in 2\mathbb{Z}$. By Fact 6.3.1 and the fact that v_1 is transcendental, $X(v_1, c)$ has only two algebraic (over $\mathbb{C}(t)$) solutions. As before, this gives a contradiction. □

6.3.2 The family P_{VI}

$P_{VI}(\alpha, \beta, \gamma, \delta)$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, is given by the following equation

$$\begin{aligned} y'' =& \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y+1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right) \end{aligned}$$

As for P_{III}, our result is the following:

Proposition 6.3.3. *Let $X = X(\alpha, \beta, \delta, \gamma)$ be the solution set of $P_{VI}(\alpha, \beta, \delta, \gamma)$, where $\alpha, \beta, \delta, \gamma$ are algebraically independent, transcendental complex numbers. Then for any $y \in X$, $acl_X(K, y)$ is finite, where $K = \mathbb{C}(t)$. Consequently as X is geometrically trivial, X is ω-categorical.*

We will prove the proposition by again making use of part of the classification of algebraic solutions of P_{VI} (see [17]). However to state the result we need, we first recall a few facts about P_{VI}.

Recall from Chapter 4 that in its hamiltonian form, P_{VI} is given by

$$S_{VI}(\bar{\alpha}) \begin{cases} \partial y &= dH/dx \\ \partial x &= -dH/dy \end{cases}$$

where $H(\bar{\alpha}) = \frac{1}{t(t-1)}(y(y-1)(y-t)x^2 - x\{\alpha_4(y-1)(y-t) + \alpha_3 y(y-t) + (\alpha_0 - 1)y(y-1) + \alpha_2(\alpha_2 + \alpha_1)(y-t))$ and $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$. The parameters $\alpha, \beta, \delta, \gamma$ of P_{VI} are related to the $\bar{\alpha}$ as follows: $\alpha = \frac{1}{2}\alpha_1^2$, $\beta = -\frac{1}{2}\alpha_4^2$, $\gamma = \frac{1}{2}\alpha_3^2$ and $\delta = \frac{1}{2}(1 - \alpha_0^2)$.

Let us note that any solution y of $P_{VI}(\alpha, \beta, \gamma, \delta)$ yields a unique solution (y, x) of $S_{VI}(\bar{\alpha})$. The only possible solutions (y, x) of $S_{VI}(\bar{\alpha})$ not of this form are when $y = 0, 1, t$ and such solutions will exhibit non strong minimality of $S_{VI}(\bar{\alpha})$ (even though $P_{VI}(\alpha, \beta, \gamma, \delta)$ may be strongly minimal).

Let $\mathcal{M}$ and $\mathcal{W}$ be as defined in Chapter 4 Section 3.6. As proven in [49] by Watanabe (see Theorem 2.1(v)), and discussed in Chapter 4, if $(\alpha_0, \alpha_1, \alpha_3, \alpha_4) \notin \mathcal{M}$ then the solution set of $S_{VI}(\bar{\alpha})$ is strongly minimal (equivalently Umemura's condition (J) holds). We have already described some "Backlund transformations" for $S_{VI}(\bar{\alpha})$ in Chapter 4 (see Remark 4.3.18 in particular). Here are a few more properties of these transformations:

Remark 6.3.4.

(i) If $(\alpha_0, \alpha_1, \alpha_3, \alpha_4) \notin \mathcal{M}$ then its orbit under $\mathcal{W}$ also avoids $\mathcal{M}$.

(ii) If $\bar{\alpha} = (\alpha_0, \alpha_1, \alpha_3, \alpha_4) \notin \mathcal{M}$, then each s_i, $i = 0, 1, 2, 3, 4$ establishes a *bijection* between the solutions of $S_{VI}(\bar{\alpha})$ and $S_{VI}(s_i(\bar{\alpha}))$.

Proof. (ii) This is because the solution sets of both $S_{VI}(\bar{\alpha})$ and $S_{VI}(s_i(\bar{\alpha}))$ are strongly minimal, hence neither has a solution of form (y, x) with $y = 0, 1$ or t, or $x = 0$. So s_i is defined on all solutions. Using the fact that s_i^2 is the identity for each i we obtain the desired conclusion. □

The key result is Boalch's "generic icosahedral solution": see Section 6 of [2].

Fact 6.3.5. The equation $S_{VI}(1/2, -1/5, 1/3, 2/5)$ has exactly 12 algebraic solutions (of course all in $\mathbb{Q}(t)^{alg}$). Moreover $(1/2, 4/5, 1/3, 2/5) \notin \mathcal{M}$.

By Remark 6.3.4 we conclude:

Corollary 6.3.6. *Let* $\bar{\alpha} \in \{(1/2 - 2\mathbb{Z}, -1/5 - 2\mathbb{Z}, 1/3 - 2\mathbb{Z}, 2/5 - 2\mathbb{Z})$. *Then* $S_{VI}(\bar{\alpha})$ *has precisely* 12 *algebraic solutions (again necessarily in* $\mathbb{Q}(t)^{alg}$*).*

This is enough for us to prove our result:

Proof of Proposition 6.3.3. Let $\alpha, \beta, \gamma, \delta$ be algebraically independent and transcendental constants. The solutions of $P_{VI}(\alpha, \beta, \gamma, \delta)$ are in bijection with those of the $S_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ (where the α_i are related to $\alpha, \beta, \gamma, \delta$ as stated above) via $y \to (y, x)$. (Because $\alpha_0, \alpha_1, \alpha_3, \alpha_4$ are also algebraically independent, so $\bar{\alpha} \notin \mathcal{M}$ and $S_{VI}(\bar{\alpha})$ is strongly minimal.) Without ambiguity we denote a solution of a system $S_{VI}(-)$ by y. Let now $X(\bar{\alpha})$ denote the solution set of $S_{VI}(\bar{\alpha})$ (likewise for other parameters). Let $y \in X(\bar{\alpha})$. As before (using Remark 6.1.1), it is enough to work over $K_0 = \mathbb{Q}(\bar{\alpha}, t)$. We know y (like all elements of $X(\bar{\alpha})$) is generic over K_0.

Claim. $acl_{X(\bar{\alpha})}(K_0, y)$ has cardinality at most 12 (including y itself).

Proof. The same argument as before: if not then we find a true sentence $\sigma(\alpha_0, \alpha_1, \alpha_3, \alpha_4, t)$ expressing that for any solution y of $S_{VI}(\bar{\alpha})$, there are at least 12 other solutions in the algebraic closure of $\mathbb{Q}(\bar{\alpha}, t, y, y')$ (i.e. in $(\mathbb{Q}(\bar{\alpha}, t, y, y')^{alg})$. We hence can choose (one by one), $(r_0, r_1, r_3, r_4) \in \{(1/2 - 2\mathbb{Z}, -1/5 - 2\mathbb{Z}, 1/3 - 2\mathbb{Z}, 2/5 - 2\mathbb{Z})\}$ such that $\mathcal{U} \models \sigma(r_0, r_1, r_3, r_4, t)$. But then choosing y to be one of the algebraic solutions of $S_{VI}(r_0, r_1, r_3, r_4)$ we obtain at least 12 other algebraic solutions, contradicting Corollary 6.3.6. This proves the claim and the Proposition. □

Let us finish by pointing out how Result C (ii) in Chapter 1 relates to Proposition 6.3.2 and Proposition 6.3.3. We give the argument for P_{III} and the same will work for P_{VI}: Let X be the solution set of $P_{III}(\alpha, \beta, \gamma, \delta)$, where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are algebraically independent (and transcendental) and let $y \in X$. The proof of Proposition 6.3.2 shows that the cardinality of $acl_X(\mathbb{C}(t), y)$ is at most 2 (including y itself). Hence for all other points $z \in X$, except for at most 1, y, y', z, z' are algebraically independent over $\mathbb{C}(t)$. The full statement of Result C (ii) for P_{III} then follows from geometric triviality of X.

Chapter 7

On Transformations in the Painlevé Family

In this chapter we look at other natural and related questions concerning algebraic relations between solutions of generic Painelevé equations from different families. The techniques we use are very similar to those employed in Chapter 4 and 6. Throughout we assume that $(\mathcal{U}, \delta)$ is a saturated model of DCF_0 of cardinality the continuum and t is an element of $\mathcal{U}$ satisfying $\delta t = 1$

7.1 A remark on weak orthogonality

As we have seen in the previous chapters, an important feature of the Painlevé family is the existence of Backlund transformations. For example for $Y_{II}(\alpha)$, the solution set of $P_{II}(\alpha)$, we have a bijection $T_+ : Y_{II}(\alpha) \to Y_{II}(\alpha+1)$, where

$$T_+(y) = -y + \frac{\alpha - 1/2}{y' + x^2 + t/2}$$

Clearly, the existence of T_+ means that for generic α, $Y_{II}(\alpha)$ is nonorthogonal to $Y_{II}(\alpha+1)$. However, they are nonorthogonal in a very special way, namely one does not require extra parameters (other that α and t) to witness that they are nonorthogonal. It turns out that this is no coincidence but first we need a definition.

Definition 7.1.1. Two strongly minimal sets Y_1 and Y_2 (both defined over $\emptyset$) are said to be *non weakly orthogonal* if they are nonorthogonal, that is there is an infinite finite-to-finite relation $R \subseteq Y_1 \times Y_2$, and the formula defining R has no parameters.

Fact 7.1.2 ([35], Corollary 2.5.5). Let Y_1 and Y_2 be modular strongly minimal sets (both defined over $\emptyset$). Assume that they are nonorthogonal. Then they are non weakly orthogonal, that is there are $a \in Y_1 \setminus acl(\emptyset)$ and $b \in Y_2 \setminus acl(\emptyset)$ such that $a \in acl(b)$ (and so $b \in acl(a)$)

So in particular if two generic members of the Painlevé family are nonorthogonal, then they are non weakly orthogonal (as they are modular). Our aim in this chapter is to show that distinct generic Painlevé equations (from any of the families) are orthogonal! As we have seen in Chapter 4 and 6, when working with the generic Painlevé equations, we can restrict our attention to a special form of the equations. So, unless otherwise stated, in this chapter $Y_{III'}(v_1, v_2)$, $Y_{IV}(v_1, v_2)$, $Y_V(v_1, v_2, v_3)$ and $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ will respectively denote the solution sets of the following form of the Painlevé equations (in Hamiltonian form):

$$S_{III'}(\overline{v}) \quad \begin{cases} y' &= \frac{1}{t}(2y^2x - y^2 + v_1y + t) \\ x' &= \frac{1}{t}(-2yx^2 + 2yx - v_1x + \frac{1}{2}(v_1 + v_2)) \end{cases}$$

$$S_{IV}(\overline{v}) \quad \begin{cases} y' &= 2xy - y^2 - 2ty + 2(v_1 - v_2) \\ x' &= 2xy - x^2 + 2tx + 2(v_1 - v_3) \\ v_3 &= -v_1 - v_2 \end{cases}$$

$$S_V(\overline{v}) \quad \begin{cases} y' &= \frac{1}{t}(2y^2x - 2yx + ty^2 - ty + (v_1 - v_2 - v_3 + v_4)y \\ & \quad +v_2 - v_1) \\ x' &= \frac{1}{t}(-2yx^2 + x^2 - 2txy + tx - (v_1 - v_2 - v_3 + v_4)x \\ & \quad +(v_3 - v_1)t) \\ v_4 &= -v_1 - v_2 - v_3 \end{cases}$$

$$S_{VI}(\overline{\alpha}) \quad \begin{cases} y' &= \frac{1}{t(t-1)}(2xy(y-1)(y-t) - \{\alpha_4(y-1)(y-t) + \alpha_3 y(y-t) \\ & \quad +(\alpha_0 - 1)y(y-1)\}) \\ x' &= \frac{1}{t(t-1)}(-x^2(3y^2 - 2(1+t)y + t) + x\{2(\alpha_0 + \alpha_3 + \alpha_4 - 1)y \\ & \quad -\alpha_4(1+t) - \alpha_3 t - \alpha_0 + 1\} - \alpha_2(\alpha_1 + \alpha_2)) \\ \alpha_2 &= 1/2 - (\alpha_0 + \alpha_1 + \alpha_3 + \alpha_4)/2 \end{cases}$$

Y_I and $Y_{II}(\alpha)$ are taken to be those of the usual form of P_I and $P_{II}(\alpha)$.

7.2 The special case of P_I

We show in section that Y_I is orthogonal to all the other generic Painlevé equations. We give a very detailed proof of the first proposition as this will be a model for many of the other cases.

Proposition 7.2.1. *Y_I is orthogonal to any generic $Y_{II}(\alpha)$.*

Proof. Let $\alpha \in \mathbb{C}$ be generic and for contradiction, suppose that Y_I is non weakly orthogonal to $Y_{II}(\alpha)$ (this suffices by Fact 7.1.2). By definition, this means that there exists a finite-to-finite definable relation $R \subseteq Y_I \times Y_{II}(\alpha)$ and that R is defined over $\mathbb{Q}(\alpha, t)^{alg}$. Since both Y_I and $Y_{II}(\alpha)$ are strictly disintegrated (by Fact 4.3.3 and Proposition 6.2.2) and they have no elements in $\mathbb{C}(t)^{alg}$, it is not hard to see however that R is the graph of a bijection between Y_I and $Y_{II}(\alpha)$. Let us suppose that R is defined by $\varphi(x, y, \alpha, t)$ and let $\sigma(u, v)$ be the L_δ formula $\forall x \exists^{=1} y \varphi(x, y, u, v) \wedge \forall y \exists^{=1} x \varphi(x, y, u, v)$. So $\mathcal{U} \models \sigma(\alpha, t)$ and by construction, $\mathcal{U} \models \sigma(\tilde{\alpha}, t)$ implies that $\tilde{\alpha} \in \mathbb{C}$ and Y_I is in bijection with $Y_{II}(\tilde{\alpha})$.

As $\mathbb{C}$ is strongly minimal and α generic, as argued before, $\sigma(\tilde{\alpha}, t)$ is true for all but finitely many $\tilde{\alpha} \in \mathbb{C}$. In particular for some $\alpha_0 \in 1/2 + \mathbb{Z}$, we have that $\mathcal{U} \models \sigma(\alpha_0, t)$ and hence Y_I is in bijection with $Y_{II}(\alpha_0)$ (and $Y_{II}(\alpha_0)$ is not strongly minimal by Corollary 4.3.5). By the same argument as in the final paragraph of the proof of Corollary 4.3.6 we get a contradiction. □

Similarly one has the following

Proposition 7.2.2. *Y_I is orthogonal to the generic $Y_{III'}(v_1, v_2)$, $Y_{IV}(v_1, v_2)$, $Y_V(v_1, v_2, v_3)$ and $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$.*

Proof. In the case of the generic $Y_{IV}(v_1, v_2)$ and $Y_V(v_1, v_2, v_3)$ the same proof as the one given above works. One only need to replace $1/2 + \mathbb{Z}$ by the appropriate exceptional sets. For $Y_{IV}(v_1, v_2)$ and $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ the only other slight modification is to replace the definable bijection R by one-to-finite maps (as the sets are only ω-categorical). But this does not pose any real problems. □

7.3 Orthogonality in the remaining cases

Although the idea of the proofs are the same, when trying to prove for example that generic $Y_{II}(\alpha)$ is orthogonal to generic $Y_{III}(v_1, v_2)$ one needs to be more careful as the parameters are not necessarily assumed to be mutually generic. We start again with an easy case.

Proposition 7.3.1. *The generic $Y_{II}(\alpha)$ is orthogonal to the generic $Y_V(v_1, v_2, v_3)$ and $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$.*

Proof. We look at the case of generic $Y_V(v_1, v_2, v_3)$ and the same will work for Y_{VI} (although below we will see a different method of proof).
Case (i): α, v_1 v_2 and v_3 are mutually generic. Then one uses the same proof as that of Proposition 7.2.1.
Case (ii): $\alpha \in \mathbb{Q}(v_1, v_2, v_3)^{alg}$. For contradiction, suppose that $Y_{II}(\alpha)$ is non weakly orthogonal to $Y_V(v_1, v_2, v_3)$. So we have as before a formula $\sigma(u, w_1, w_2, w_3, x)$ such that $\mathcal{U} \models \sigma(\alpha, v_1, v_2, v_3, t)$ and this expresses that $Y_{II}(\alpha)$ is in bijection with $Y_V(v_1, v_2, v_3)$. We then have to consider three sub-cases:
Sub-case (i): $\alpha \notin \mathbb{Q}(v_i, v_j)^{alg}$ for any $i \neq j$. All we have to do is quantify over v_3 say, that is we use the fact that $\mathcal{U} \models \exists v_3 \sigma(\alpha, v_1, v_2, v_3, t)$. As $\alpha \notin \mathbb{Q}(v_1, v_2)^{alg}$ and $\mathbb{C}$ is strongly minimal we can as before find $\tilde{v}_1 \in v_2 + \mathbb{Z}$ such that $\mathcal{U} \models \exists v_3 \sigma(\alpha, \tilde{v}_1, v_2, v_3, t)$. Choosing any $\tilde{v}_3 \in \mathbb{C}$ witnessing this, we get a bijection between $Y_{II}(\alpha)$ and $Y_V(\tilde{v}_1, v_2, \tilde{v}_3)$ (but again $Y_V(\tilde{v}_1, v_2, \tilde{v}_3)$ is not strongly minimal by Corollary 4.3.14). This gives our desired contradiction.
Sub-case (ii): $\alpha \in \mathbb{Q}(v_1, v_2)^{alg}$ say but not in $\mathbb{Q}(v_1)^{alg}$ and $\mathbb{Q}(v_2)^{alg}$. First note that as v_1, v_2 and v_3 are mutually generic we have that α is not in $\mathbb{Q}(v_3)^{alg}$, $\mathbb{Q}(v_1, v_3)^{alg}$ or $\mathbb{Q}(v_2, v_3)^{alg}$. So this time we quantify over v_2 say and look at $\mathcal{U} \models \exists v_2 \sigma(\alpha, v_1, v_2, v_3, t)$. This allows us again to find $\tilde{v}_1 \in v_3 + \mathbb{Z}$ such that $\mathcal{U} \models \exists v_2 \sigma(\alpha, \tilde{v}_1, v_2, v_3, t)$. Finally any $\tilde{v}_2 \in \mathbb{C}$ making this true leads us to our contradiction.
Sub-case (iii): $\alpha \in \mathbb{Q}(v_1)^{alg}$ say. This time all we have to do is to quantify over v_1 and the same argument works. □

For the other cases, we need to change a little bit our strategy. We will this time use the results on algebraic solutions of the Painlevé equations.

Proposition 7.3.2. *The generic $Y_{II}(\alpha)$ is orthogonal to the generic $Y_{III'}(v_1, v_2)$.*

Proof. We will work instead with $Y_{III}(v_1, v_2)$, the solution set of P_{III} given in the form

$$y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{4}{t}(v_1 + 1 - v_2 y^2) + 4y^3 - \frac{4}{y}.$$

For contradiction, suppose that $Y_{II}(\alpha)$ is non weakly orthogonal to $Y_{III}(v_1, v_2)$. As before, we get a formula $\sigma(u, w_1, w_2, x)$ such that $\mathcal{U} \models \sigma(\alpha, v_1, v_2, t)$ and this expresses that there is a "1 to ≤ 2" map between $Y_{II}(\alpha)$ and $Y_{III}(v_1, v_2)$. This follows from strict disintegratedness of $Y_{II}(\alpha)$ and ω-categoricity of $Y_{III}(v_1, v_2)$ (more precisely the proof of Proposition 6.3.2 gives that for any $y \in Y_{III}(v_1, v_2)$ the cardinality of $acl(\mathbb{C}(t), y) \cap Y_{III}(v_1, v_2)$ is at most 2).

We then quantify over α, that is use that $\mathcal{U} \models \exists\alpha\sigma(\alpha, v_1, v_2, t)$. As v_1 and v_2 are mutually generic, we can first choose $\tilde{v}_1 \in (v_2 - 1) + 2\mathbb{Z}$ (and have $\mathcal{U} \models \exists\alpha\sigma(\alpha, \tilde{v}_1, v_2, t)$) and then $\tilde{v}_2 \in \mathbb{Z}$ so that $\mathcal{U} \models \exists\alpha\sigma(\alpha, \hat{v}_1, \tilde{v}_2, t)$ where $\hat{v}_1 = \tilde{v}_2 - 1 + 2m$ for some $m \in \mathbb{Z}$. Choosing any $\tilde{\alpha} \in \mathbb{C}$ witnessing this, we have that there is a "1 to ≤ 2" map between $Y_{II}(\tilde{\alpha})$ and $Y_{III}(\hat{v}_1, \tilde{v}_2)$. By Fact 6.2.1, $Y_{II}(\tilde{\alpha})$ contain at most 1 algebraic solution whereas by Fact 6.3.1(ii), $Y_{III}(\hat{v}_1, \tilde{v}_2)$ has exactly 4 algebraic solutions. So we get a contradiction. □

Similarly one has the following

Proposition 7.3.3.

(i) The generic $Y_{III'}(v_1, v_2)$ is orthogonal to the generic $Y_{IV}(w_1, w_2)$.

(ii) The generic $Y_{IV}(w_1, w_2)$ is orthogonal to the generic $Y_V(v_1, v_2, v_3)$.

Proof. For (i) the exact same proof as Proposition 7.3.2, where one just needs to quantify over w_1 and w_2, works.

For (ii), as mentioned in Section 4.3.5 of Chapter 4, one can instead work with $X_V(\alpha,\beta,\gamma)$ the solution set of $P_V(\alpha,\beta,\gamma,-1/2)$. Arguing by contradiction, we have $\mathcal{U} \models \sigma(w_1, w_2, \alpha, \beta, \gamma, t)$ witnessing that there is a bijection between $Y_{IV}(w_1, w_2)$ and $X_V(\alpha,\beta,\gamma)$. Again by quantifying over w_1 and w_2 and moving α, β and γ into an appropriate set where X_V has 2 algebraic solutions (as given to us by 6.2.6(ii)), we are done as $Y_{IV}(w_1, w_2)$ can only have at most 1 algebraic solution. □

Finally we look at orthogonality to the sixth Painlevé equation. We will need results about algebraic solutions (over $\mathbb{C}(t)$) of Y_{VI}.

It is well known that for $\alpha_0 = \alpha_1 = \alpha_3 = \alpha_4 = 0$, Y_{VI} has infinitely many algebraic solutions over $\mathbb{C}(t)$. Indeed, $Y_{VI}(0,0,0,0)$ can be identified with the Manin kernel $E_t^\sharp$ of the elliptic curve $E_t:\ y^2 = x(x-1)(x-t)$. Algebraic solutions then corresponds to torsion points E_t^{tor}. Furthermore, by applying Backlund transformations (see [29]) one has the following:

Fact 7.3.4. For $\alpha_0, \alpha_1, \alpha_3, \alpha_4 \in 1/2 + \mathbb{Z}$, $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ has infinitely many algebraic solutions over $\mathbb{C}(t)$.

Remark 7.3.5. Similarly, if $\alpha_i \in \mathbb{Z}$ and $\alpha_0 + \alpha_1 + \alpha_3 + \alpha_4 \in 2\mathbb{Z}$ there are infinitely many algebraic solutions.

From this we easily get

Proposition 7.3.6. *The generic $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ is orthogonal to the generic $Y_{III'}(v_1, v_2)$, $Y_{IV}(v_1, v_2)$ and $Y_V(v_1, v_2, v_3)$.*

Proof. One just uses the same trick as the proof of Proposition 7.3.2: We want to prove that the generic $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ is orthogonal to $Y(\overline{v})$, where $Y(\overline{v})$ is any of the above generic sets.

Arguing by contradiction we get $\mathcal{U} \models \sigma(\overline{v}, \alpha_0, \alpha_1, \alpha_3, \alpha_4, t)$ witnessing that there is a finite-to-finite map between $Y_{VI}(\alpha_0, \alpha_1, \alpha_3, \alpha_4)$ and $Y(\overline{v})$. Quantifying over $\overline{v}$, we can move $\alpha_0, \alpha_1, \alpha_3$ and α_4 one by one into the set of half integers. On one side we then have infinitely many algebraic solutions whereas on the other side $Y(\overline{\nu})$ (for any $\overline{\nu}$) can only have finitely many. A contradiction. □

Note that it would seem that none of the above arguments can be use to prove that generic $Y_{II}(\alpha)$ are orthogonal to generic $Y_{IV}(v_1, v_2)$ and similarly that generic $Y_{III'}(v_1, v_2)$ are orthogonal to generic $Y_V(v_1, v_2, v_3)$.

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