Option Pricing and Hedging under Liquidity Costs

Zhidi Du

Leeds University Business School

University of Leeds

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The candidate confirms that the work submitted is her own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

The purpose of this thesis is to study the option pricing and hedging in an illiquid market. In order to decide the optimal strategy, we choose the maximisation of expected utility of terminal wealth as the identification tool. We design an efficient algorithm via the dynamic programming principle to compute the value function for European options, and calculate the optimal strategy numerically in the binomial market. Based on the numerical solution, we prove that the hedging strategy and the option prices would be distinctly identified by market parameters in the illiquid market. The study of option pricing as the function of initial number of shares allows us to observe a new phenomenon: curves of option price in the illiquid markets are intersected by the horizontal replicating price without transaction cost. And those intersections are very close to each other. That phenomenon implies that the price for selling call option can be lower than the replicating price under some conditions. We further observe the smile effect in the implied volatility and explain that the deeply smile of implied volatility in short-expiration options can be caused by the illiquidity effect in the market. Finally, we compare the implied volatilities which are given by the convex liquidity cost and the proportional transaction cost and prove that the convexity of liquidity cost can amplify the effect of transaction cost. We compare implied volatilities from the model to the real market quotes (S&P 500 index) and analyse how the market parameters affect on the implied volatility in the illiquid market. This comparison reveals an estimation of the level of liquidity in the real market.

Key Words: liquidity cost; hedging; option pricing; optimal strategy; dynamic programming; implied volatility;

Abbreviations

CARA	Constant Absolute Risk Aversion
CRRA	Constant Relative Risk Aversion
LOB	Limit Order Book
HJB	Hamilton-Jacobi-Bellman
RMS	Root Mean Squared
PTP	Price Transition Point
BM	Binomial Model
MC	Monte Carlo simulation

	Frequently Used Variables
T	Time Horizon
r	Riskfree Interest rate
m	Number of Time Periods
t_i	Time Variable at Period i
α	Liquidity Cost Effect
$\phi(\cdot)$	Liquidity Effect Function
γ	Risk Aversion Parameter
μ	Growth Rate of the Stock
σ	Volatility of the Stock
K	Strike Price
n	Number of Options
$\mathcal{C}$	European Contingent Claim
N^{path}	Number of Random Stock Paths
δt	Step Length
δr	Discretization Length
U	Utility Function
V	Value Function with a General Utility Function
$\tilde{V}$	Value Function with the CARA (Negative EXponential) Utility Function
W_n	Wealth Process at Time t_n in Perfect Liquid Market
$\tilde{W}_n$	Wealth Process at Time t_n in General Market
p^b, p^s	Reservation Buy Price and Reservation Sell Price
u, d	Parameters of the Binomial Model of the Stock Price
p	Probability of an Up Movement in the Binomial Model of Stock Price
x	Initial Amount of Shares in the Stock Account
c	Initial Amount of Cash in the Money Market Account
ε	Root Mean Squared Relative Error

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Chapter 1

Introduction

This chapter presents the motivation, the relevant research background of this thesis, the description of research problems, and the outline of the thesis.

1.1 Motivation and Research Background

How the liquidity in the market takes effect on the movement of price processes has become a popular discussion recently. There is no consensus for the definition of liquidity yet. We assume that the meaning of liquidity invokes the ease with which underlying assets can be purchased, or describes the ability to trade without causing changes of underlying prices. The literature of liquidity issue is inadequate to the combination of optimal portfolio choice problem and the option pricing in the binomial market with liquidity costs. The current popular direction on this liquidity issue is to model the illiquidity effect into the limit order book (LOB). When large orders come into the market, in order to trade smoothly and immediately, the bid-side price in LOB would go deeply and investors placing large orders have to purchase a price higher than the market price. Cetin & Rogers (2007) only solve the maximisation of expected utility of terminal wealth in discrete-time binomial model. Malo & Pennanen (2012) deal with the liquidation about the modelling assumptions. We are interested in classifying this problem as option pricing. Because of the gap between the optimal portfolio problem and the option pricing in the illiquid market, our thesis aims to incorporate the option pricing with the hedging of options to measure the illiquidity levels.

The study of liquidity concentrates on measuring the level of liquidity in the market. Until now, most papers in this field do not discuss the liquidity effect in combination of option pricing and hedging. This study has two main directions due to different assumption on whether trading amounts change the price of underlying asset. The first direction is the *temporary impact model*. The paper written by Cetin & Rogers (2007) describes a *temporary impact model* to solve the optimal portfolio choice problem. Besides the temporary impact model, another direction on the study of illiquidity models is the *feedback effect model*. The price in the feedback effect model immediately responds to the big trading amount and the effect of changed price will be constant until other large orders put into the market. The price in the temporary impact model is not affected by trading orders but the illiquidity in the market. Our thesis only considers the illiquidity effect in the temporary impact model. This kind of model was first introduced by Cetin *et al.* (2004). Cetin *et al.* (2004) prove that the martingale measure is unique in that model and any European claim approximately tends to its classical value. The liquidity costs in the temporary impact model of Cetin *et al.* (2004) can be avoided in trading strategies since every trade can be divided into many small trades. Cetin *et al.* (2006) provide some empirical evidence to show that the liquidity cost affects the option price in the illiquid market. Cetin *et al.* (2010) further figure out that the liquidity cost is the difference between the superreplicating value and the Black-Scholes value of the claim in the continuous-time superreplication model. Cetin & Rogers (2007) deal with the utility maximization problem of terminal wealth in a discrete-time model and prove the existence of optimal strategy. Based on the result of Cetin *et al.* (2004), Rogers & Singh (2010) show that the liquidity cost cannot be avoided in the pricing of European option hedging in an incomplete market. The recent research direction on the illiquidity is intended to model the bid-ask spread by the temporary impact model. Malo & Pennanen (2012) propose a new liquidity cost function into the above model to specialise the bid-ask spread in double auction market and prove that the liquidity cost retains the monotonicity as well as positivity of marginal prices of market orders.

1.2 Research Problems

In this section, we discuss four research problems for achieving the motivation of this thesis. These four problems are extended by the existing assumptions of Cetin & Rogers (2007). Based on these research problems, we address the gap between pricing of options and modelling of liquidity impact in the market.

Numerical implementation Papers concerning the liquidity impact in the temporary price impact model describe the assumptions of the model and mathematical analytical solution. However, it is very difficult to obtain the analytical solution in the liquidity research issue. We have to provide an efficient algorithm for computation and verification of numerical solution. In this thesis, we need to implement the model which is introduced by Cetin & Rogers (2007) efficiently. The challenging aspect of implementation is how to devise the computation of value function in dynamic programming approach. The normal implementation of dynamic programming approach is to construct a fixed-size matrix for storing the binomial model, which has a high requirement of memory. The purpose of this thesis is to provide an efficient algorithm which constructs an array for storing the binomial tree. How to identify the period number and the node position of the binomial tree in one array is the key challenge of the implementation.

Pricing of options in an illiquid market Studying pricing of options is important for examining the liquidity cost in the temporary impact model. As previous studies on this class of liquidity model shown, the illiquidity effect in the market can not be directly illustrated in the price of the underlying. However, the liquidity issue in the market definitely takes influence on option pricing. We can use the price of options to discuss the impact of liquidity. As so far, it is inadequate for studying the option price in the market with liquidity costs. The main research direction of liquidity issue is still the liquidation about assumptions both in discrete-time and continuous-time models. That stimulates our thesis to investigate the hedge strategy and option prices in the market with liquidity costs.

Hedging of options in an illiquid market Besides the research of option prices in the illiquid market, another way to measure the impact of liquidity in options is the hedging of options. A hedge in the financial market is used to reduce or eliminate some losses or gains. We know that there are different ways to define a hedge. Normally, a hedge is decided by taking offsetting position in the relevant security. A lot of attention in this thesis has been given to various hedging strategies according to different liquidity costs and other market parameters. We ensure that the liquidity cost would make the hedging strategy different. Hence, we can analyse the value of the hedge to quantity the liquidity cost. For the hedging of options, the popular research problems are what the reservation price (*utility-indifference pricing*) is and when we should use this approach to identify the liquidity cost in the market.

Application of implied volatility in an illiquid market In the research of option price, the implied volatility is a way to display the relative price of option. Investors, especially the informed traders, often quote option in terms of volatility. The related implied volatilities can be derived from actual transaction data. This causes a diversified research on the implied volatility. One popular direction of the implied volatility research concentrates on how jumps in the underlying asset price affect the *smile effect* of the implied volatility. Gatheral (2006) shows that jumps in the price of the underlying have greater effect on the curve of implied volatility for short-expiration options than for longer-dated options. That makes us wonder whether the deeply smiling can also be caused by the liquidity cost in the trading. Our thesis will extend the research on the implied volatility in the market with liquidity costs on Section 5.6.

1.3 Outline of the Thesis

This thesis contains six chapters. Chapter 1 is introduction. Chapter 2 reviews the relevant literature on liquidity issues in financial markets from economical and mathematical perspective. It presents the optimal portfolio decision problem and the pricing of options under different market frictions and relevant numerical methods. Chapter 3 introduces a paper written by Cetin & Rogers (2007) that pioneered a model with convex transaction cost (regarded as the liquidity cost) and provides some preliminary results as comparison between Cetin & Rogers (2007) and this thesis. Chapter 4 discusses a numerical approach taken in this thesis to solve the portfolio optimisation problem with liquidity costs. Chapter 5 makes an analysis of our numerical results. When market parameters change, option pricing and hedging in the model with liquidity cost are investigated. We find that in the market with liquidity cost, sellers of call option would be willing to offer option cheaper than in the liquid market. The last section examines the implied volatility smile resulting from reservation pricing of options under liquidity costs and investigates the comparison of implied volatility between in the real market quotes and in the model quotes. Chapter 6 summaries main findings of this thesis and introduces some further directions of research.

Chapter 2

Literature Review

This chapter provides an introduction of the relevant literature for the problem of optimal portfolio selection and pricing and hedging of options when the market is not perfectly liquid.

In Section 2.1, we survey some models of illiquidity that have an effect on the price of the underlying asset. The price impact is mainly divided into two groups: temporary impact on demand and supply, permanent impact on placing large orders. In short, if the price is only affected by the current trade, that is the temporary impact; if past trading decisions can take influence on the price, then it is the permanent impact. Besides discussing the illiquidity models, we introduce some popular applications caused by illiquidity models, i.e. the bid-ask spread, the price manipulation strategy and the optimal execution problem.

Section 2.2 reviews papers on the portfolio management without market frictions and with the transaction cost or liquidity constraints. The objective of portfolio optimisation studied in this thesis is to maximize the expected utility of terminal wealth. We separately survey papers on trading in the incomplete model with linear transaction cost and convex liquidity cost.

We introduce option pricing in the theoretical model and utility-based pricing in the illiquid market. Section 2.3 presents the process of option pricing in the Black-Scholes model and its approximation binomial model; also discusses a popular approach and the utility indifference pricing, to show pricing and hedging in the optimization model with liquidity cost.

The last Section 2.4 introduces representative numerical approaches for the optimization model with frictions since we do not know the analytical solution for them. Those numerical methods provide advice on generating algorithms to deal with the optimization problem in practice. The binomial model is for approximating Cetin & Rogers (2007) optimization model of stock price; the dynamic programming principle is the main tool for producing value function and corresponding optimal strategy of our numerical analysis; Monte Carlo simulation is to verify our implementation and to investigate the comparative analysis with different market parameters.

2.1 Models of Illiquidity

How liquidity influences price processes is one of major discussion in financial theory; and yet the study of liquidity is far less advanced. First of all, we need to understand what the liquidity is. It is not consensus on the definition of liquidity and how it is to be measured. Roughly speaking, the liquidity represents everything related to willingness of trading. In Rogers & Singh (2010), liquidity is thought of as a non-linear transaction cost incurred as rate of change of asset holding. Gokay *et al.* (2012) defined the liquidity as added costs per transaction associated to trading large quantities of the asset. The liquidity is usually measured by the liquidity risk which originates from asymmetric information and market structure. We survey two main models of illiquidity that exist a price impact on the underlying asset. The price impact in these models of illiquidity may be caused by a temporary imbalance between the demand and supply, or due to large orders which substantially affect the price of underlying asset. We call the former group as *temporary impact models* and the latter one as *permanent impact models*. We also introduce some applications based on the above illiqudity models, i.e. bid-ask spread and price manipulation strategy.

2.1.1 Liquidity

The liquidity is one of main arguments in market frictions. Although many definitions of liquidity exist but no one as the most precise standard, we denote the definition of liquidity as everything relating to willingness of trading. The most prevailing classification is to divide liquidity into market liquidity and funding liquidity. Market liquidity is the ease of trading an asset and funding liquidity is the ease of funds availability. These liquidities are mutually dependent, while traders provide market liquidity and their availability of funding is a determinant. On the other hand, traders funding negotiation ability is also linked to market liquidity. The relationship between these two liquidities is discussed in the paper by Brunnermeier & Pedersen (2009). The survey of liquidity models is based on the market liquidity in my thesis.

Concerning the funding liquidity, Drehmann & Nikolaou (2009) define that it mainly expresses the ability to operate funding obligations with immediacy.

Funding liquidity is mainly associated with the role of a bank. A bank is unable to afford the obligation of exchanging money if it lacks the funding liquidity. Another relating term funding liquidity risk is usually mentioned with the funding liquidity. Drehmann & Nikolaou (2009) also gave the definition of the funding liquidity risk, which represents the possibility that the bank cannot afford its obligations during a specific time period in the future. There exist two important distinctions to differ the funding liquidity and its risk. On one hand, funding liquidity expresses only two possibilities to the ability of affording obligations: can or cannot; however, funding liquidity risk indicates infinite possibilities as it refers to the future ability of funding. On the other hand, funding liquidity only relates to the funding at one moment; nonetheless, based on the definition of its risk, it is easy to understand the funding liquidity risk refers to one period of future time. In general, higher funding liquidity risk exists, less liquidity in the market which would cause wider range of the bid-ask spread as the expression of lack of liquidity.

Concerning the market liquidity, the lower market liquidity denotes the higher transaction costs and the higher expectation of return. Briefly speaking, the market liquidity has a negative correlation with costs of trading. The effect of market liquidity is regarded as a difficulty on immediacy cost when investors trade big volumes of the capital asset in small time periods. Then liquidity effects relate to the execution price of trading actions in financial markets. We assume that the liquidity cost is the difference between the actual execution price and the price under the perfectly liquid condition. Amihud & Mendelson (1991) divide the market liquidity costs into three categories: the market impact, the bid-ask spread and the cost for delay and search.

After the brief descriptions of funding and market liquidity, we discuss two categories of price impact models in the market liquidity (abbreviated as liquidity in the following).

2.1.2 Temporary Impact Models

First of all, we know that participants in the market are mainly divided into large traders and small traders. Large traders refer to price makers placing substantially large orders and their past trading decisions have a lasting impact on the price. We assume this price impact is permanent and the related models are called as *feedback effect models.* Smaller traders refer to price takers whose trading decisions and trading amount do not change the price. Their trades in current moment would affect the price as a shadow cost. We assume this price is temporary and call those models as *temporary impact models.* This subsection discusses the former category of models.

In order to understand the temporary impact model, we first interpret the role of small traders. Small traders in this thesis mean traders who invest a large volume and request more liquidities than the market liquidity to trade quickly. The existing market liquidity cannot cover the effect of their amount. However, trading amount from those small traders cannot change the price. Cetin *et al.* (2004) start the study of temporary impact model (they call *supply curve model*). They define the supply curve as the price per share in a function of transaction size. Cetin *et al.* (2004) focus on incorporating the liquidity effect into the option pricing theory and applied the liquidity risk into the arbitrary-free pricing theory, while the reaction function depends on the size of order from small traders but no lasting effect on the price process of the asset. Cetin *et al.* (2006) provide some empirical evidences to show that the liquidity cost affects the option price and it leads to changes of implied volatility. Recently, Cetin *et al.* (2010) further consider the super-replication model with the temporary price impact but under some constraints on the portfolio. They find a liquidity premium that is the difference between the super-replicating value and the Black-Scholes value of the claim. The paper by Cetin & Rogers (2007) and the paper by Rogers & Singh (2010) respectively proposed the temporary price impact model in discrete-time and continuous- time framework, respectively. Cetin & Rogers (2007) mainly survey the utility maximization problem of terminal wealth in a discrete-time market with liquidity cost. We will describe construction of Cetin-Rogers model (C-R model) in details in Chapter 3. Rogers & Singh (2010) investigated the temporary price impact model with large trades and explored consequences for European option hedging. This paper eliminates the feedback effect in the price and assumes no price impact on large orders. It reveals that incomplete market leads to no perfect replication, no unique replication price and related super-replication. Rogers & Singh (2010) chooses a more challenging portfolio setting that requires investors to achieve a random terminal wealth than the zero terminal wealth in Cetin & Rogers (2007).

Attention should be paid on the convexity of liquidity cost function in these temporary impact models. Cetin & Rogers (2007) consider the illiquidity in the market is a cost but it does not take effect on the price of underlying asset. It mentioned that the illiquidity effect is like a transaction cost but no one which is proportional to the amount traded. Malo & Pennanen (2012) also consider the total cost of a market order in the bid-ask spread as a convex increasing function of the order size. The convexity is an essential feature in the optimization of trading strategies. It leads to many important implications for risk management as well as for pricing and hedging, i.e. Edirisinghe *et al.* (1993), Huberman & Stanzl (2005), Alfonsi *et al.* (2010) and Alfonsi *et al.* (2012).

2.1.3 Feedback Effect Models

Permanent effect models refer to large traders who trade large enough amount that can affect the prices. This effect will persist a while until another big order comes into the market. Large traders are also called as price makers. Before surveying the permanent impact models, we discuss the characteristics of large traders first. The most important feature of a large trader is that his/her trading volume is large enough to change the price. Moreover, the large trader is different from the informed trader who has the advantage of asymmetric information. That is to say that large traders do no have more advantage on information than small traders. Recent papers on the liquidity effect from large traders modelled the permanent impact as the feedback effect.

The principle of the feedback effects model is that the current price of the asset immediately corresponds to the trading amount by the large size trader and the effect of changed price will be constant until another large order putting into the market. Due to the existence of different viewpoints of liquidity effects, there are different modelling approaches to explain the relationship between the asset price and the large size traders trading strategy in the asset. Jarrow (1992) introduced the discrete-time framework where large size traders manipulate the asset price and earn profit at no risk and he further studied the standard option pricing theory with the feedback effect in the derivative security market by Jarrow (1994). The paper by Bank & Baum (2004) accessing Jarrow (1992)s model and proposed the continuous-time framework of the feedback effects model. Frey & Stremme (1997) derives a reaction process for the permanent price impact process and investigates the price impact of dynamic hedging in discrete-time economy. This paper demonstrates that the feedback effect of those hedging strategies cannot replicate the payoff of claims. Schonbucher & Wilmott (2000) analysed, in a continuous-time model, the price dynamics under feedback effect in an illiquid financial market. Vath *et al.* (2007) solve a maximisation of expected utility in a continuous-time feedback effect model. Roch (2011) investigates the limit-order book models with permanent price impact. The limit-order book (LOB) is an important application of illiquidity models.

There exists several phenomenons in the market when the price is manipulated by some speculators. That is called *price manipulation* which is illegal. Price manipulation describes collusions among competitors to manipulate prices of the capital assets, for example these competitors negotiate agreements to sell one good at the same price or do not lower the price without notifying other collaborator. It is to be noted that price manipulation phenomenons are only incurred by large traders. The normal example for price manipulation is that the large trader can corner the market. Cornering the market means that large traders purchase enough amount of specified financial assets in order to manipulate the price. Due to the price manipulation, the most direct operation is to purchase a high percentage of one good in the spot market and store it secretly; at the same time, large traders take a significant long position in the future market. When the expiry date arrives, those traders who signed the short future contract may not find sufficient amount of the good, then it will result in the inflated price of that good and large traders can profit by the above described operations. More examples about the price manipulation and more details about different manipulation strategies can be found in Jarrow (1992). Jarrow (1992) provided different trading strategies to prove the existence of price manipulation. He explicitly indicated that given no arbitrage condition the price manipulation strategy exists in the execution of price makers; conversely, the manipulation strategy cannot exist by price takers due to the absence of market power. Following Jarrow (1992) and Jarrow (1994), Huberman & Stanzl (2004) generates the feedback effect model for price manipulation.

2.1.4 Applications of Illiquidity Models

In practice there exists some applications for illiqudity models. One of famous studies on the applications of illiquidity models is on the optimal execution problem. The optimal execution problem allocates large orders of risky assets with the aim of minimizing the expected liquidity costs. Many papers investigate the limit-order book (LOB), in particular the bid-ask spread, to present different optimal strategies between the temporary price impact and the feedback effect models.

The bid-ask spread occurs at different securities, such as stock, option or currency markets. Demsetz (1968) was the first researcher who formalized the bid-ask spread as transaction costs for measuring liquidity. From the prevailing thought perspective, the bid-ask spread is defined as the difference of prices quoted between an immediate buying and an immediate selling. Based on the definition by Harris (2003), the bid-ask spread is the price investors paying for liquidity. In particular, bid-ask spreads measure the liquidity costs from temporary price impact. Discussions on how the bid-ask spread affects the liquidity dominate the academic papers which analyse liquidity effects by small traders. The spread affects several aspects of trading. From individual investors perspective, when the spread is wide, providing liquidity is expensive, the market order strategy is costly and investors tend to submit limit orders; when the spread is narrow, providing liquidity is relative cheap and the market order attracts investors. From agents perspective, wide spreads make agents (large traders) profitable and attract more people to enter the market; narrow spreads may make agents no profit or not cover their expenses then they quit market.

Modelling the bid-ask spread by the temporary impact model is a popular research direction recently. Malo & Pennanen (2012) specialise the bid-ask spread in double auction market and propose a new liquidity model which additionally considers the market price and the ask or bid-side liquidity. Comparing to Cetin & Rogers (2007), the model in Malo & Pennanen (2012) retains the monotonicity as well as positivity of marginal prices of market orders. It also interprets the *crowding out effect* in illiquidity models.

Besides applications of illiquidity models in the stock and option markets, there exists some applications to present the liquidity cost in other underlying assets. In the bond market, some empirical evidence show the existence of liquidity cost. For example, comparing the treasury bills to the treasury bonds, Damodaran (2005) confirmed that the yield on the less liquid treasury bond is higher than the yield on the more liquid treasury bill. That difference between them is regarded as the measurement of illiquidity. Another application in bonds market to measure the liquidity is the corporate bond. Dick-Nielsen *et al.* (2012) compared corporate bonds spreads during $2005 - 2009$ and got a conclusion that less liquid bonds had much higher yield spreads than other liquid bonds. Papers on studying liquidity in bonds market found that liquidity takes influence on all bonds but it matters more with risky bonds than with less risky bonds. In the equity market, some papers test the turnover ratio to analyse the liquidity cost. For example, Haugen & Baker (1996) concludes that liquidity takes a significant impact on the returns, while less liquid equities have higher annual returns than more liquid equities. Pennanen & Penner (2010) studied option hedging of contingent claims of temporary impact models in currency market.

2.2 Portfolio Management under Market Frictions

When people trade in the financial market, the first note is to disperse their risks of expected return. Different assets obtain different levels of expected return risk. Possible combination of multi-asset investment may get a lower expected return risk than any individual assets. That fact promotes the development of portfolio management theory. A portfolio refers to a collection of investment assets. The primary motivation of portfolio selection theory (also called portfolio management) is trying to maximize the portfolio expected return given a specific amount of risk or minimize the portfolio risk given a specific expected return. The literature of portfolio management examines the optimal portfolio choice problem in two categories: the mean-variance analysis of Markowitz and the expected utility study of Merton. The objective of this thesis is to aim of maximizing expected utility.

2.2.1 Portfolio Selection without Frictions

The pioneering work of portfolio management was done by Markowitz (1952), who built a famous mean-variance model in one-period, and use the standard deviation of expected return as the proxy of risk. That paper considered both the expected return and the risk to get an efficient frontier without a risk-free asset. Any portfolio lying on that efficient frontier represents the best possible expected return in a given risk level. Based on the market equilibrium function, Sharpe (1964) extended the portfolio theory and established the capital asset pricing model (CAPM). Merton (1969) first studied the portfolio selection problem in continuous-time and figured out the explicit solution of optimal consumption strategy in the constant coefficient model. Many research works in 1980s have applied the martingale technique instead of the stochastic control theory to solve the optimal portfolio strategies problem, for example, Harrison & Pliska (1981), Cox & Huang (1989), Pliska (1986). The paper by Harrison & Pliska (1981) gave insight into the martingale measure technology, who assume that the price process exists an unique, equivalent martingale measure and can calculate the option pricing in complete market under this assumption. Cox & Huang (1989) adopted the martingale technique, which does not require the differentiable condition of utility function, and calculated one linear partial differential equation to get the optimal portfolio policy. The reason for developing the martingale approach is its easier computation than the dynamic programming in stochastic control theory.

However, when the market is not perfect or frictionless, the dynamic programming approach is more suitable than the martingale approach. The implementation of the model from Cetin & Rogers (2007) is based on the dynamic programming approach. We discuss this approach in Section 2.4.2 and Section 4.2.

The portfolio management concentrates on the *Consumption Problem* and the *Wealth Problem*. The consumption problem is aim to maximize the expected utility of consumption and the wealth problem is aim to maximize the expected utility of terminal wealth. Merton (1969) and Merton (1971) separately survey these two kinds of optimization problems. A large number of papers expanse this issue to many research directions. The current academic research focus on how to identify key aspects of real-world portfolio choice problems and to understand both qualitatively and quantitatively their roles in the optimal portfolio decisions of small and large traders (followed by Brandt (2009)).

Merton (1969) discusses the classical Merton utility maximization model. That model allows investors to observe dynamics of all state variables related to those portfolio decisions. Merton (1969) surveys an analytical solution for the optimal portfolio choice problem which combines both the consumption and wealth problems. Merton extended his portfolio problem in Merton (1971) and the portfolio problem can be applied to more general utility functions such as HARA utility function and the price process of assets is not necessary to follow the Brownian Motion movement. He worked out the explicit solution in the utility function which is one of HARA family and got a remarkable conclusion that the portion selection model. Merton (1969) and Merton (1971) provide a reliable benchmark in different utility functions for further study (e.g. Muthuraman & Kumar (2006), Liu & Yong (2005)). In Chapter 5, we compares the numerical solution of optimization model in the perfectly liquid market to solutions provided by Merton (1971) as the verification of correctness. Karatzas *et al.* (1986) further extended Merton's result which got the closed-form solution of general utility functions.

Nevertheless, the portfolio selection without frictions cannot be applied in the real market. For instance, on the assumption of portfolio selection in absence of transaction costs, the trading strategy with the fixed proportions of wealth could trade continuously. Followed the portfolio strategy by Merton (1971), the trade in the real market will rapidly bankrupt due to vast transaction costs. That limitation in the portfolio selection has inspired researchers to study the portfolio selection with transaction costs.

2.2.2 Portfolio Selection with Transaction Costs

Although Merton (1971)'s portfolio strategy is remarkable, this portfolio strategy will be useless when applied to the real market. In recent years, people have been interested in the study of portfolio management in imperfect market. An imperfect market includes any one or all imperfections as follows: restrictions on short selling of stock and borrowing of cash, existence of transaction costs, incomplete market, different lending rate and saving rate. The portfolio selection in absence of transaction costs predicts that investors can continually trade. However, investors in real markets usually reduce their trading frequencies and seek more available information to make their future trading decisions since investors' portfolio with transaction costs produces substantial utility loss.

The terminology *transaction cost* was created by Coase (1937). Based on this famous paper, the concept of transaction costs is regarded as the cost using the price mechanism. Magill & Constantinides (1976) start the investigation of the portfolio selection with transaction cost constraint, which specially stated that the no-transaction region is a wedge shape. In order to model the transaction costs more precisely, many papers studied the optimal portfolio decision with different kinds of transaction costs. For example, transaction costs are usually divided into either proportional component to changes in the total assets or fixed fraction of portfolio value. Davis & Norman (1990) used mathematical models to implement the structure among sell region, no-transaction region and buy region and figured out an explicit solution on how proportional transaction costs affect the portfolio selection. Morton & Pliska (1995) worked out one single return in the space which the optimal strategy with fixed transaction costs is to no trade.

The optimal portfolio decision problem divides into two categories, the consumption problem and the wealth problem. This thesis focuses on the wealth problem of maximizing expected utility of terminal wealth, so this subsection only discusses literature on the wealth problem. Sass (2005) examines the optimisation wealth problem with different utility functions based on a binomial model; Irle & Sass (2006) further investigate this problem both in fixed and proportional transaction costs. Atkinson & Quek (2012) discuss the optimal portfolio problem with large and small proportional transaction costs in discrete time.

The study of the optimal portfolio decision problem with transaction costs attracts the economical interesting but challenges the mathematical solution. The dynamic programming approach is a general numerical approach to deal with it. The difficulty in the implementation of dynamic programming is to reduce the dimensionality of the above problem. The CARA utility function can reduce one dimension of the optimization problem since the optimal portfolio in CARA utility function does not depend on the wealth invested in the risk-less asset (e.g. Cetin & Rogers (2007), Rogers & Singh (2010)).

2.2.3 Portfolio Selection under Liquidity Constraints

The previous section describes the portfolio selection with transaction cost, while investors act as price takers and the transaction cost constraint represents the explicit cost for transactions. Meanwhile, in real market the large size traders invest large volume to affect the price process; the medium size traders cannot change the price process but their immediate execution requirement requests the liquidity effect as the implicit cost of transactions. Hence, there has some papers considering the liquidity effects as market frictions into the portfolio selection theory. In general, the key idea of liquidity effect applied into the portfolio selection assumes the effect as the implicit cost in transactions. In the objective of portfolio selection, model the liquidity constraint as a proportional component of transaction costs.

The studies on portfolio selection with liquidity constraints have been extended as many research directions due to the different understanding of liquidity constraints. Longstaff (2001) has established a pioneering work on the valuation of illiquid assets in the optimal portfolio problem. He adopted the definition of liquidity effect as the thin-trading interpretation. The thin-trading describes a condition in which the trading activities are not enough active because of the lack of orders to raise the volume and usually happens at holidays. The main idea of Longstaff (2001) is the comparison the derived utility of wealth functions between optimal portfolio strategy without liquidity constraint and with liquidity constraint, while the liquidity constraint is specified as the borrowing and short-selling restrictions. In order to understand the optimal portfolio strategy with liquidity constraint better, we build a simple two-asset securities market following Longstaff (2001):

First of all, we denote one risky asset $S(t)$ in stock market and one unchanged risk-free asset $B(t)$ in bond market which assumes the interest rate equals to zero in bond and $B(t) = 1$. The wealth function $W(t)$ is formulated as $W(t) = X(t) \cdot S(t) + Y(t) \cdot B(t) = X(t) \cdot S(t) + Y(t)$, where $X(t)$ and $Y(t)$ represent the number of shares and bonds , respectively. We denote the logarithm form of wealth function to replace the wealth. Because of $\ln(0) = -\infty$, the wealth function must be restricted $W(t) > 0$ for all $t \in [0, T]$. Moreover, due to the unchanged price in risk-free asset, the wealth on the risky asset makes influence on the derived utility function; hence, we denote $w(t) = \frac{X(t) \cdot S(t)}{W(t)}$ as the portfolio weight in the risky asset. Then the objective is to maximize the derived utility of wealth function given by $J(W, t) = \max_{w(t)} E[\ln W(T)]$.

The following step is to apply the liquidity constraint into the above wealth function. The liquidity constraint only affect on the risky securities, therefore we model this restriction into the dynamics of number of shares as $dX(t) = \gamma(t)dt$. $\gamma(t) \in [-\alpha, \alpha]$ represents the upper and lower bounds on the number of shares. The number limitation $\alpha > 0$ indicates the immediacy execution in large volume. The new derived utility of wealth function with liquidity constraint depends on not only $W(t)$ and t but also $X(t)$ and $S(t)$, which is characterized as:

$$J(W, X, S, t) = \max_{w(t), \gamma(t)} E[\ln W(T)].$$

Longstaff (2001) solved this equation and obtained a key result that the optimal portfolio strategy with liquidity constraint should consider more on the portfolio weight $w(t)$ on the risky asset rather than the expected return of portfolio.

Kalin & Zagst (2004) also discussed a brief argument on optimal portfolio with liquidity restrictions. Their objective of model is to maximize the expected return of portfolio for a given risk. Considering the transaction cost as explicit cost and the liquidity constraint as implicit cost, they set up the objective function which is the optimal portfolio with maximized explicit and implicit costs for a given risk level and solved the unperturbed price which is the quote price prior to the execution. Vath *et al.* (2007) have built up the portfolio selection with liquidity effect model and obtained the explicit solution, while the liquidity effect in their paper is formalized as both long-term price impact and short-term price impact. Their objective function is the maximization of expected utility from the terminal wealth over finite time horizons. They firstly tried to implement the model with quasi-variational Hamilton-Jacobi-Bellman (HJB) inequality but adopted the viscosity solution to deal with because of none of smooth solution in quasi-variational HJB equation. HJB equation is one expression of standard finite difference method via dynamic programming methodology. Ang *et al.* (2011) investigate the illiquidity effect on the portfolio choice problem and find that the illiquidity takes a substantial influence on the optimal strategy and welfare.

2.3 Option Pricing

In finance theory, option is a derivative financial instrument. According to the definition of option by Hull (2003), an option establishes a contract between the buying and selling sides of an asset at a verified price during a specified time period. Until over the expiration date, the buyer of the option has the right not the obligation to purchase the financial securities or commodities at the verified price and the seller has the obligation to fulfil the transactions requested by the buyer. Because the seller of an option stands at the more severe position than the buyer, the seller usually adopts some hedging strategies to reduce the risk of option execution. The replication is the pricing and hedging derivative securities. The aim of hedging strategies makes the sellers wealth during the execution period of option pricing equal to the option value, which recoups the risk of selling options after the price of options increases. Besides the option execution and the hedging strategy, how to determine the fair price of the option is an important aspect of option theory, which is also called the option pricing.

2.3.1 Option Pricing without Frictions

Black & Scholes (1973) and Merton (1973) published a famous model, Black-Scholes model, to price options. The Black-Scholes Model figured out the replicating strategy of European call option, and demonstrated the initial wealth in the replicating strategy as the fair price of European call option. They indicated that the fair price of an option is independent of investors risk tolerance level. Another prevailing influence of the Black-Scholes model is an easy pricing of European call option represented.

Recently, main developments extend the Black-Scholes Model in two directions: Firstly, consider the option pricing mixing jump diffusions in the underlying asset processes and get the jump-diffusion model. Merton (1976) did the pioneering work which incorporated the jump diffusion processes into the option valuation and derived one option pricing formula with non- systematic jump risk. In the valuation theory, the non-systematic risk can be eliminated but the systematic cannot. Ahn (1992) extended Mertons jump-diffusion option pricing model with the systematic jump risk. Amin (1993) built an American option valuation model in discrete time to the case that underlying assets follow a systematic jump-diffusion process. Secondly, consider the option pricing with random volatility and get the stochastic volatility model. Hull & White (1987) allowed a stochastic volatility into the option pricing model and demonstrated the European call option prices are undervalued by Black-Scholes option price at the money options. Melino & Turnbull (1990) examined the option pricing model with stochastic volatility successful applying into the spot foreign currency options. Heston (1993) continued to study the option pricing model in bond and spot currency securities and worked out a new approach to derive the option pricing with random volatilities. That approach, square-root stochastic volatility model, informed almost option pricing biases to the dynamic of option prices and expected returns. Eraker *et al.* (2003) combined studies between the stochastic volatility and the jump-diffusion process applied in the Black-Scholes pricing model, and indicated that models without jump diffusions in the market volatility are unspecified.

In Chapter 5, we compare our hedging strategy (or call replicating strategy in other papers) in the perfectly liquid market to theoretical replicating strategy of C-R model and delta hedge strategy in Black-Scholes model. We characteristic the Black-Scholes option pricing model in here. The fundamental idea of Black-Scholes model is that based on the arbitrage-free assumption, investors adopt the delta hedge strategy to go into the risk-neutral world where all risky assets have the same expected return rate and the same risk-free interest rate.

The whole implementation and numerical analysis in this paper are based on the binomial option pricing model in discrete time. We use the binomial model to approximate the Black-Scholes model in continuous time. The arbitrage-free principle is the essential assumption of the binomial option pricing model. During the whole trading period $t \in [0, T]$, a portfolio strategy is called a self-financing strategy if there is no new money added or withdrawn from the portfolio. In order to understand the arbitrage-free principle, we define what arbitrage is first.

Definition 1 *An* ***arbitrage opportunity*** *exists in a self-financing portfolio strategy if it satisfies the following conditions: the initial wealth at time* t_0 *in the portfolio equals to* 0 *and the terminal wealth at time* t_T *in the portfolio is larger than* 0*, and the probability of the positive terminal wealth* $W_{t_T}(\Phi)$ *is greater than* 0*. The expressions of these conditions are given by:*

$$\begin{aligned} W_{t_0}(\Phi) &= 0; \\ W_{t_T}(\Phi) &\geq 0; \\ \mathbb{P}\{W_{t_T}(\Phi)\} &> 0. \end{aligned}$$

Then a market is called ***arbitrage-free*** when any self-financing consumption-portfolio strategy Φ does not have arbitrage opportunity at any time interval $[t_1, t_2] \subseteq [t_0, t_T]$.

We explain one simplest example in the binomial model, which is one-period and two-state model. It assumes that the market has two kind of assets: one risky asset S in stock and one risk-free asset B in bond market. The one-period indicates transactions happen at the initial time t_0 and the terminal time t_T and the two-state denotes the initial price of the risky asset S_0 has two possible prices at time t_T : upward price S_T^u or downward price S_T^d . Due to the uncertainty of price movements in the risky asset, if a trader takes the short position in stock, he/she needs to keep a long position in call option to hedge the strategy. The usual hedging operation is to invest appropriate shares and options in different direction, and how to figure out the appropriate portion of shares is called the delta hedge strategy. We describe the definition and formula of delta hedge strategy with details in Section 5.2.5, thus do not need to describe the delta hedging in here.

Another prevailing hedging strategy is to construct a portfolio which can completely replicate the option. Given the above market which consists in risky assets and risk-free assets, a formal replicating strategy is characterized as:

Lemma 1 *In a market with risky asset S and risk-free asset B, there exists a portfolio $\Phi = \Delta \cdot S + \beta \cdot B$ where Δ is the number of shares and β is the interest component while the value of this portfolio Φ at time t_T equals to the value of option C, which means*

$$\Phi = \Delta \cdot S_{t_T} + \beta \cdot B_{t_T} = C_{t_T}.$$

*Then the portfolio Φ is the **replication** of the option C.*

Applying the replicating strategy into the portfolio makes the expected return in the risky asset equalling to the return in the risk-free asset. We denote the market achieved the replication as a *risk-neutral world.*

Now we divide the trading period $[0, T]$ of the option into $N+1$ smaller time intervals, while $0 = t_0 < t_1 < ... < t_N = T$. The above Binomial option pricing model in one-period is extended into a complex binomial model in multi-period. We implement the binomial model as the approximation of the Black-Scholes model to generate the optimal strategy of C-R model.

A quite important improvement of option pricing in last century is the development of Black- Scholes option pricing model. The basic assumption of Black-Scholes model is to suppose the underlying assets moving as geometric Brownian motion. That implies the dynamic of risky asset, i.e. stock, is a random movement. In order to compute the implied volatility, we implement the Black-Scholes model in discrete-time and analyse the difference of implied volatility between the liquidity cost case and the linear transaction cost case in Section 5.6.

2.3.2 Utility Indifference Pricing

Real markets are imperfect and frictional. The market frictions (such as liquidity cost, portfolio constraints and differential borrowing and lending rate) make replication of the option impossible by Black-Scholes model. In other words, the replication price of an option cannot be computed by the Black-Scholes model. In practice, the lowest price which the seller may accept is always slightly higher than the highest price which the buyer would provide. The fair price of the option exists at the range between the ask price and the bid price and the range of bid-ask price is arbitrage-free. Besides the replication, another implication of option pricing is the utility maximization. Considering the option pricing in real markets, the utility-based pricing is one popular approach which is taken in an incomplete market. The principle of the utility-based pricing is a dynamic extension of the static certainty equivalence concept from economic consideration. Hodges & Neuberger (1989) started the pioneering work on the option pricing and hedging based on the utility-based pricing theory. In particular, the utility-based pricing is the most successful approach to model the arbitrage-free option pricing with the transaction costs. Considering the portfolio constraints (or called leverage constraints), the super-replication approach is adopted to work out the option pricing and hedging. An option is super-replicated when minimum hedging value of portfolio can guarantee over the payoff of the option. Broadie *et al.* (1998) investigated a hedging portfolio with convex constraints in continuous-time framework and demonstrated that the optimal cost of super-replication in such model equals to the option price without portfolio constraints.

Utility-based pricing model has been applied into many literature discussions in incomplete markets, which has been well implemented in the context of transaction costs and non-traded assets. For example, Henderson (2002) investigated the option pricing in the non-traded underlying assets, Hobson (2003) studied the utility-based pricing applied into the real options, Moller (2003) extended the utility-based pricing into a new field, the insurance contract. The relationship between the super-replication pricing and the utility-based pricing was worked out by Cheridito & Summer (2006). Kloppel & Schweizer (2007) represented the dynamic utility-based pricing for conditional convex risk measures, while the convex risk is originated from non-linear and monotonic properties of utility-based pricing. More recently, Ankirchner *et al.* (2010) are concentrating on the mathematical modelling of utility-based pricing, which obtained a conclusion that the partial replicated optimal hedging strategy of options can be represented as the product the price gradient multiplied the correlation coefficient. Their optimal replication strategy is derived from the forward-backward stochastic differential equation, the stochastic calculus of variations and Malliavin calculus.

Henderson & Hobson (2009) clearly interpreted the fundamental of utility-based pricing about the emerging field of utility-based pricing for option valuation in incomplete markets. We follow the key concept of Chapter 2 in Henderson & Hobson (2009) to explain the utility-based pricing model.

The beginning of utility-based pricing is to understand what the utility function is. The utility function is the mathematical expression applied into the economic justification, while different utility functions represent different consumptions by orders of preference. In the utility-based pricing, utility functions are usually defined as twice continuously differentiable, strictly increasing and strictly concave. The strictly increasing property of utility function interprets investors higher preference on wealth than lower. The concavity property assumes investors are risk averse. The utility indifference prices consist with the utility indifference buy and sell price (or call reservation buy and sell prices). We define them separately in Section 3.3. In short, the utility indifference buy price is the highest price a buyer will purchase an asset and the corresponding sell price is the lowest price a seller will sell his asset. Considering the key term indifference into the definition of utility indifference prices, the utility indifference price is the price at which the expected utility under optimal portfolio strategy is indifference between the condition an investor does not buy the option and the condition the investor pay the price of option and exercise the option at the expiry date.

The utility-based pricing has been extended many implications in financial theory. As mentioned before, market frictions such as liquidity costs, non-traded assets have been applied into this approach of option pricing in incomplete market. Portfolio constraints are divided into many detailed frictions to apply into the utility-based pricing. We concentrate on option pricing with liquidity constraints.

In summary, the utility-based pricing can be regarded as the most successful option pricing approach applied into the incomplete market. From the microeconomics perspective, the utility-based pricing incorporates investors risk aversion level with the quantities of supply and demand. From the mathematical perspective, this approach can solve the non-linear pricing in the incomplete market and show explicit optimal hedging strategy based on the utility indifference prices. It is complex to get explicit solution so recent studies usually select the exponential utility functions to compute the optimal strategy in practice.

2.4 Numerical Methods

When Black & Scholes (1973) and Merton (1973) built up the Black-Scholes-Merton option pricing model, the most significant fruition is the calculation of payoff of European options. They used the analytic approach to obtain the explicit solution of European call option. However, due to non-existence of expiry date restriction in American options, the Black-Scholes model cannot provide analytic expressions for pricing of some complex options, e.g. American option, compound option or chooser option. The calculation complexity in option pricing with frictions is higher than in option pricing in the complete market. In incomplete market, the numerical approaches for option pricing have significant advantages on the calculation ability and operation speed over the analytic approaches over the analytic approaches.

The numerical approaches are mainly divided into discrete-time methods and continuous-time methods. Compared to continuous-time methods, discrete-time methods have easier programming operation and lower computation complexity that applies broadly in practice. Nonetheless, the accuracy of computation is the key limitation of discrete-time methods. The study of the continuous-time option pricing methods has difficulties on the convergence, boundary limit problem and the computation stability in theory. Empirically, some financial agencies are apt to use the prevailing and easy discrete-time option pricing methods. I generate the optimization model using the discrete-time methods in Chapter 4.

2.4.1 Binomial Model

There has several numerical approaches in discrete-time such as binomial-tree modelling, finite difference approach, Monte Carlo simulation, finite element method, domain decomposition method and so on. We choose the binomial-tree modelling as the generation of the optimization model in illiquid market and Monte Carlo simulation as the verification of dynamic programming in value function and boundaries of number of shares.

Generally speaking, the principle of binomial model approach is summarized as using large amounts of discrete binary motions of small amplitude to simulate continuous movements of underlying assets. The primary process of this approach has two main steps: the first step is to obtain the expected return of cash amount after exercise the option under the risk-neutral assumption and the second step is to calculate the discounted price of options. The binomial model in Chapter 4 is a CRR Binomial model which was developed by Ross *et al.* (1979). The primary classification of CRR binomial models are single-period pricing model and multi-periods pricing model. The single-period model is proposed on the arbitrage-free assumption to get the risk-neutral pricing of option. The multi-periods model partitions the whole trading period $t \in [0, T]$ into many smaller time intervals δt. Both of single-period and multi-period binomial tree models, the fundamental is the price process of underlying assets.

We define a risky asset in stock market to illustrate the structure of binomial tree. The non-recombining tree can be applied into either path-independent options or path-dependent options. Conversely, the recombining tree can only be used to price path-independent options and American options. However, when we price a path-independent option, it is no doubt to choose the recombining tree model because the recombining one is significantly better on the computational complexity than the non-recombining one.

In summary, the binomial tree model assumes that the price movements of risky assets consist of binary motions in small amplitude and adopt discrete random model to simulate the stochastic price processes of underlying assets. When the time interval $\delta t \to 0$, the binomial tree model convergences to the geometric Brownian motion, which is one approximation of Black-Scholes model.

2.4.2 Dynamic Programming

In the financial mathematics, we deal with the optimal portfolio choice problem by two approaches: the martingale approach and the dynamic programming approach. The reason for developing the martingale approach is its easier computation than the dynamic programming in stochastic control theory. However, when the market is not perfect or frictionless, the dynamic programming approach is more suitable than the martingale approach. This thesis investigates the optimal portfolio problem in the temporary impact model with liquidity cost. That means the market is not frictionless, hence our main tool is the dynamic programming method in the binomial tree setting.

The terminology "dynamic programming" was originally introduced in the 1940s by Richard Bellman. Bellman (1957) and Bertsekas (1976) started the mathematical theory of dynamic programming as a method for solving the dynamic optimisation problems. In short, dynamic programming breaks a complex problem down into simpler sub-problems. It solves every sub-problem exactly once, and therefore is more efficient in those cases where the sub-problems are not independent. The general schema of a dynamic programming solution includes four steps: structure, recursion, backward computation and construction of optimal solution. The first step, structure, is to characterise the structure of an optimal solution that the solution can be divided into optimal sub-problems. The second step, recursive computation, is to define the maximal or minimal value of that optimal solution in terms of optimal sub-problems. The third step is to compute the value of the optimisation in a backward table structure. The last step is to construct that optimal solution based on previous computed information. The step one to three describe the key features of a dynamic programming solution to an optimisation problem.

2.4.3 Monte Carlo

Monte Carlo simulation is a widely used computational algorithm that relies on repeated random sampling to obtain numerical results (summarized from Chapter 5, Hull (2003)). It is often dealt with problems of financial economics, e.g. optimization, numerical integration and generation of samples from a probability distribution. We describe an introduction to Monte Carlo simulation and the corresponding applications in financial area in this section.

In short, Monte Carlo methods are used to value and analyze instruments, portfolios and investments by simulating the various sources of uncertainty affecting their value, and then determining their average value over the range of resultant outcomes. Monte Carlo simulation generates a sequence, X_i, $i = 1, 2, ...,$ of independent identically distributed random variables with expected value μ and variance σ^2. We take the sample mean Y_i over i random draws, shown as: $Y_i = \frac{X_1+X_2+...+X_i}{i}$.

Monte Carlo simulation is a flexible method of estimating quantities that is applied to financial applications. There are three main applications for Monte Carlo method to finance: the first application is to determine the optimal strategy, which contains the computation of expectations (e.g. Detemple *et al.* (2003)); the second one is path-dependent option pricing, especially pricing American-style derivatives (e.g. Hull (2003)); the last popular application is that Monte Carlo simulation as a numerical tool to analyse optimal strategies and we employ this application in this research.

Chapter 3

Cetin & Rogers (2007) Model

This chapter describes a model of optimal portfolio choice. It mainly discusses the model developed by Cetin & Rogers (2007). In the Cetin & Rogers model (C-R model), an investor allocates the capital to a risk-less asset and a risky asset. The price of the risky asset follows a binomial model. When the investor changes his portfolio, he has to pay the stock price plus liquidity cost. The objective of C-R optimization model is to achieve the maximal expected utility of terminal wealth. Solving the optimization model requires determining the value function and the associated optimal portfolio strategy. We extend the optimization model without option hedging to the model with option hedging in the illiquid market. We choose the reservation price to measure the hedging of options.

3.1 Model Specification

Cetin & Rogers (2007) model is a two-asset model with multi periods. The multi-period model is obtained by the concatenation of many single period models. The trading period of single period model is the time between the beginning of trading t_i and the end of trading t_{i+1}, where the subscript of trading period represents the related date of the period. At the beginning t_i of the trading period, prices of the underlying assets are recorded and the investor chooses a portfolio and determines his investment; at the end of the period t_{i+1}, those prices of financial assets are recorded again and the investor gains a payoff corresponding to the value of the portfolio.

In the multi-period model trade takes place at dates t_i where $i \in \{0, 1, 2, ..., n\}$. During the whole period of trading dates, agents can observe prices of financial assets and gather information to adjust investments. These public information is represented mathematically by the concepts of σ-algebra and filtration.

Agents begin their trades at the initial date t_0 and end them at the terminal date t_N. We denote a probability space by $(\Omega, \mathcal{F}, \mathbb{P})$.

- $N+1$ trading dates: $t_0, t_1, ..., t_N$, while the whole period $[0, T]$ is discretized by $\delta t = t_{i+1} - t_i = \frac{T}{N}$.
- Ω is a finite state space with $k < \infty$ elements: $\Omega = \{w_1, w_2, ..., w_k\}$, where a simple w_i displays a possible state in the market.
- $\mathcal{F}_{t_i}$ is a σ-algebra of subsets of Ω. Set $F = \{\mathcal{F}_{t_i}\}_{i=0,...,N}$ a filtration in which $\mathcal{F}_{t_i} \in \mathcal{F}$ for each t_i and $t_1 \leq t_2 \Rightarrow \mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$.
- $\mathbb{P}$ is a probability measure on $\mathcal{F}$ and $\mathbb{P}(w_i) > 0$ for all $w_i \in \Omega$.

3.1.1 Representation of the Model

In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a two-asset model: agents allocate capitals to one risky stock and one risk free money market account. The money market account process is $B_{t_j} = B_{t_{j-1}} \cdot e^{r \cdot \delta t}$, $j = 1, 2, ..., N$ and $r \geq 0$ is a positive constant interest rate. Hence,

$$B_{t_n} = B_{t_0} \cdot e^{r \cdot \delta t \cdot n}, 1 \leq n \leq N.$$

The price of stock process $(S_{t_i})_{0 \leq i \leq N}$ follows a binomial model. We only denote two possibilities of the price movement corresponding to the previous price S_{t_i}: $S_{t_{i,u}} = S_{t_i} \cdot u$ is the price going up with the probability p and $S_{t_{i,d}} = S_{t_i} \cdot d$ is the price going down with the probability $(1 - p)$, where $d < 1 + r < u$. The price S_{t_j} at time t_j is determined by its previous price $S_{t_{j-1}}$ and the probability of the price movement u or d.

The stock price process $(S_{t_i})_{0 \leq i \leq N}$ is a Markov process.

Definition 2 *A stochastic process $(S_{t_n})_{t_n \geq t_0}$ with a filtration $F = \{\mathcal{F}_{t_i}\}_{i=0,...,N}$ in a finite space Ω is a **Markov process**, if for each $N \geq n \geq m \geq 0$ and any $x \in \mathbb{R}$*

$$\mathbb{P}(S_{t_n} = x \mid \mathcal{F}_{t_m}) = \mathbb{P}(S_{t_n} = x \mid S_{t_m}).$$

A Markov process is a stochastic process for which conditional on the present state, its future is independent of its past history. In other words, in order to predict what happens at time t_{n+1}, we only need to consider the state at time t_n, and states at time before t_n do not affect the state at t_{n+1}.

In the C-R model, the investor chooses a portfolio from a stock and a money market account. We describe a portfolio via the number of shares held and the amount of cash invested in the money market account. The number of shares before transaction is different from the number of shares after transaction at the same trading date. X_n^- and X_n^+ represent the numbers of shares before and after trading at the time t_n, respectively. We introduce a new variable, the change of number of shares at time t_n, $\Delta X_n = X_n^+ - X_n^-$ which is the transaction made at time t_n. In the money market account, the change of cash amount at time t_n includes two components: interest on the remaining cash from time t_{n-1} to time t_n and cash flows due to shares transaction at time t_n. We list Y_n^- and Y_n^+ to represent the cash holding before and after trading at time t_n. Then, $Y_n^+ = Y_n^- - S_{t_n} \cdot \Delta X_n = Y_{n-1}^+ \cdot e^{r\delta t} - S_{t_n} \cdot \Delta X_n$. Similarly, the cash holding Y_{n-1}^+ at time t_{n-1} can be showed as $Y_{n-1}^+ = Y_{n-2}^+ \cdot e^{r\delta t} - S_{t_{n-1}} \cdot \Delta X_{n-1}$. It can be concluded that if the change of number of shares ΔX_n at time t_n is known, it is possible to figure out the total amount of shares and the cash holding at time t_n. We call $(\Delta X_n)_{0 \leq n \leq N}$ a **trading strategy** for an agent in the multi-period model if ΔX_n is $\mathcal{F}_{t_n}$-adapted for each n.

We extend the above description to self-financing strategies to a market with transaction costs. A self-financing strategy is financially self-contained, i.e. any change in the value process is due to price movements and transaction costs paid. This implies $\Delta Y_n = Y_n^+ - Y_n^-$ is negative for buying stock and positive for selling stock. Therefore, the cash at time t_n after trading actions, Y_n^+, is equal to the cash at time t_n before trading actions Y_n^- plus the total cost in the share trading.

We propose that the total cost function $\mathcal{S}(\Delta, S_{t_n})$ is of the form

$$\mathcal{S}(\Delta, S_{t_n}) = \phi(\Delta X_n) \cdot S_{t_n}, \tag{3.1}$$

where ΔX_n is the change of number of shares at time t_n, S_{t_n} is the stock price at time t_n and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function such that $\phi(0) = 0$. The convexity of the total cost function $\mathcal{S}(\Delta, S_{t_n})$ is interpreted as modelling the increased difficulty of trading large number of shares.

Definition 3 *An illiquidity effect function $\phi : \mathbb{R} \to \mathbb{R}$ is a convex, increasing function that vanishes at the origin.*

Cetin & Rogers (2007) proposed the following illiquidity effect function:

$$\phi(\Delta X_n) = \frac{e^{\alpha \Delta X_n} - 1}{\alpha}, \tag{3.2}$$

where the parameter $\alpha > 0$ represents the degree of illquidity in the market. The larger the value of α, the less liquid the market. A perfectly liquid market can be achieved in the limit when α converges to 0. Recently, the similar liquidity cost functions in limit order market were proposed by Cont *et al.* (2010), Malo & Pennanen (2012).

Section 2.1 describes two principal illiquidity models and our viewpoint of illiquidity belongs to the temporary impact model. We assume that the illiquidity generates costs, but transaction amounts do not affect the price of the underlying asset. It affects the price at which an agent will trade the asset that reflects the depth of the limit order book. The faster an agent wants to buy/sell the asset, the deeper into the limit order book, and higher/lower will be the price for the later units of the asset bought/sold. Once a rapid transaction completes, we suppose that the limit order book will quickly fill up again and the transaction has no lasting effect on the price of underlying asset.

We now focus on the formula $Y_n^+ = Y_n^- + \Delta Y_n$ and incorporate the liquidity cost:

$$Y_n^+ \;=\; Y_n^- - \phi(\Delta X_n) \cdot S_{t_n}. \tag{3.3}$$

We use some examples to specify the liquidity cost in different illiquidity degrees. The simplest example is to regard the liquidity cost as zero which means in a perfectly liquid market. In this case, the liquidity cost $\phi(\Delta X_n)$ is equal to ΔX_n as $\lim_{\alpha \to 0} \frac{e^{\alpha \Delta X} - 1}{\alpha} = \Delta X$. A more realistic example is to model the transaction cost as a linear one, that is to say the transaction cost is proportional to the amount traded:

$$\phi_l(\Delta X_n) = \begin{cases} (1+\alpha) \cdot \Delta X_n & \text{for } \Delta X_n \geq 0 \text{ buy shares;} \\ (1-\alpha) \cdot \Delta X_n & \text{for } \Delta X_n \leq 0 \text{ sell shares.} \end{cases}$$

where α indicates the cost rate. Hence, with the proportional transaction cost, Equation 3.3 can be formulated as: $Y_n^+ = Y_n^- - \phi_l(\Delta X_n) \cdot S_{t_n}$. This formula indicates that transaction costs are proportional to the value of stock holdings. The type of transaction costs would affect the utility maximisation problem. Early studies of transaction costs concentrate on the optimisation problems with transaction costs providing understanding of the trading strategy in a market with transaction costs. Kamin (1975) examines the maximisation of expected utility of terminal wealth of a trader who trades in a two-asset model. Constantinides

(1976) extends this model to more general utility functions. We discuss the utility maximisation problem in details in the next subsection.

In the C-R model, the liquidity cost function is defined as a convex function (see Definition 3), in particular, this function covers the example of no transaction cost above. However, the linear transaction cost case does not satisfy the condition of the liquidity cost function.

We introduce a transaction cost function $\tilde{\phi}(|\Delta X_n|)$ here. Many papers define the transaction cost function $\tilde{\phi}(|\Delta X_n|)$ as a strictly concave and differentiable function. The liquidity cost function is proposed as a convex function, that we cannot regard the liquidity cost as a normal transaction cost. The transaction cost is a concave function of the traded volume of the risky asset. In order to relate common notation to the notation of the liquidity cost function, the liquidity cost function $\phi(\Delta X_n)$ also can be regarded as $\Delta X_n - \tilde{\phi}(|\Delta X_n|)$. Equation (3.3) is extended as

$$Y_n^+ = Y_n^- - \Delta X_n \cdot S_{t_n} + \tilde{\phi}(|\Delta X_n|) \cdot S_{t_n}. \tag{3.4}$$

In Equation (3.4) the term $\Delta X_n \cdot S_{t_n}$ expresses the value from the stock account into the bond account and the term $\tilde{\phi}(|\Delta X_n|) \cdot S_{t_n}$ represents the transaction costs paid for trading. In real world situation, the transaction costs $\tilde{\phi}(|\Delta X_n|) \cdot S_{t_n}$ is concave. As a result of this phenomenon, when the trading amount increases, the average cost per unit $(\tilde{\phi}(|\Delta X_n|) \cdot S_{t_n})/\Delta X_n$ will decrease.

The dynamics in our model is defined by three processes: X_n, Y_n and S_{t_n}. Self-financing condition allows us to reduce it to the initial position X_0^-, Y_0^-, trades in shares $(\Delta X_n)_{0 \le n \le N}$ and (S_{t_n}). The change of cash variable is decided by $\phi(\Delta X_n)$. According to the above conditions, the portfolio just relates to the change of numbers of shares, and we can say a self-financing strategy is only given by ΔX_n. In other words, choosing different changes of number of shares from the present time to the terminal t_N can decide different portfolio processes. Considering $X_{n+1}^- = X_n^+$ and $Y_{n+1}^- = Y_n^+ \cdot e^{r\delta t}$ into Formula (3.3), we get a general shares and cash holding formula respectively at time t_k and use a lemma to represent:

Lemma 2 *In a market with one risky stock and one money market account if a* ***trading strategy*** $(\Delta X_n)_{0\le n\le N}$ *is* ***self-financing*** *then for* $k \ge n$

$$X_k^+ = X_n^- + \sum_{j=n}^{k} \Delta X_j \tag{3.5}$$

$$Y_k^+ = Y_n^- \cdot e^{r\cdot\delta t\cdot(k-n)} - \sum_{j=n}^{k} \phi(\Delta X_j) \cdot S_{t_j} \cdot e^{r\cdot\delta t\cdot(k-n)} \tag{3.6}$$

Proof 1

$$
\begin{aligned}
X_k^+ &= X_k^- + \Delta X_k \\
&= X_{k-1}^+ + \Delta X_k \\
&= X_{k-1}^- + \Delta X_{k-1} + \Delta X_k \\
&= X_{k-2}^+ + \Delta X_{k-1} + \Delta X_k \\
&= X_{k-2}^- + \Delta X_{k-2} + \Delta X_{k-1} + \Delta X_k \\
&= ... \\
&= X_n^- + \Delta X_n + \Delta X_{n+1} + ... + \Delta X_k \\
&= X_n^- + \sum_{j=n}^{k} \Delta X_j,
\end{aligned}
$$

$$
\begin{aligned}
Y_k^+ &= Y_k^- - \phi(\Delta X_k) \cdot S_{t_k} \\
&= Y_{k-1}^+ \cdot e^{r\cdot\delta t} - \phi(\Delta X_k) \cdot S_{t_k} \\
&= (Y_{k-1}^- - \phi(\Delta X_{k-1}) \cdot S_{t_{k-1}}) e^{r\cdot\delta t} - \phi(\Delta X_k) \cdot S_{t_k} \\
&= Y_{k-1}^- \cdot e^{r\cdot\delta t} - \phi(\Delta X_{k-1}) \cdot S_{t_{k-1}} \cdot e^{r\cdot\delta t} - \phi(\Delta X_k) \cdot S_{t_k} \\
&= Y_{k-2}^+ \cdot e^{2r\cdot\delta t} - \phi(\Delta X_{k-1}) \cdot S_{t_{k-1}} \cdot e^{r\cdot\delta t} - \phi(\Delta X_k) \cdot S_{t_k} \\
&= (Y_{k-2}^- - \phi(\Delta X_{k-2}) \cdot S_{t_{k-2}}) \cdot e^{2r\cdot\delta t} \\
&\quad -\phi(\Delta X_{k-1}) \cdot S_{t_{k-1}} \cdot e^{r\cdot\delta t} - \phi(\Delta X_k) \cdot S_{t_k} \\
&= Y_{k-2}^- \cdot e^{2r\cdot\delta t} - \phi(\Delta X_{k-2}) \cdot S_{t_{k-2}} \cdot e^{2r\cdot\delta t} \\
&\quad -\phi(\Delta X_{k-1}) \cdot S_{t_{k-1}} \cdot e^{r\cdot\delta t} - \phi(\Delta X_k) \cdot S_{t_k} \\
&= \ldots \\
&= Y_n^- \cdot e^{r\cdot\delta t\cdot(k-n)} - \phi(\Delta X_n) \cdot S_{t_n} \cdot e^{r\cdot\delta t\cdot(k-n)} \\
&\quad -\ldots - \phi(\Delta X_{k-1}) \cdot S_{t_{k-1}} \cdot e^{r\cdot\delta t} - \phi(\Delta X_k) \cdot S_{t_k} \\
&= Y_n^- \cdot e^{r\cdot\delta t\cdot(k-n)} - \sum_{j=n}^{k} \phi(\Delta X_j) \cdot S_{t_j} \cdot e^{r\cdot\delta t\cdot(k-j)}.
\end{aligned}
$$

We consider the multi-period model in a perfectly liquid market as the benchmark case. The wealth dynamics of the trading strategy $(\Delta X_n)_{0\le n\le N}$ at any time t_n would not pay any transaction cost and the portfolio value only transfers between the stock and the money market account. The wealth process describes the total value of the portfolio at any time t_n. Denote by W_n^- and W_n^+ the wealth (book value) before and after trading at the same trading date t_n. In the case of no liquidity cost, the wealth process W_n^+ equals to W_n^- based on the choice of the trading strategy $(\Delta X_n)_{0\le n\le N}$. In what follows, we will call W_n^+ the **wealth process** and denote it by $W_n(\Delta X_n)$:

Definition 4 *The* ***wealth process*** *of strategy* (ΔX_n) *is given by*

$$W_n(\Delta X_n) = X_n^+ \cdot S_{t_n} + Y_n^+. \tag{3.7}$$

Lemma 3 *In a perfectly liquid market, the wealth process corresponding to the trading strategy* $\Delta X = (\Delta X_n)_{0 \leq n \leq N}$ *is given by*

$$\begin{aligned} W_n(\Delta X_n) &= (c - \Delta X_0 \cdot S_{t_0}) \cdot e^{r \cdot \delta t \cdot n} - \sum_{i=1}^{n} \Delta X_i \cdot S_{t_i} \cdot e^{r \cdot \delta t \cdot (n-i)} \\ &\quad + (x + \sum_{i=0}^{n} \Delta X_i) \cdot S_{t_n}, n = 0, ..., N. \end{aligned} \tag{3.8}$$

In the perfectly liquid market, an agent invests money into the strategy at time t_0, in the following moments he only rebalances his portfolio, neither withdrawing any money out of the market, nor investing new money into the market. The value of the portfolio immediately before transaction at time t_n equals to the value of the portfolio immediately after transaction at time t_n.

Applying the liquidity cost into the above market model, the dynamics of stock holdings and money market account would be different from the original ones in the benchmark case. The wealth difference between the benchmark case and the illiquid case depends on not only the liquidity cost $\phi(\cdot)$ but also on the trading strategy (ΔX_n).

We start from the money market account to explain how liquidity cost impacts on the wealth process. When the agent comes into the market before he starts transaction, the number of shares $\tilde{X_0^-} = x$ and $\tilde{Y_0^-} = c$; then the agent trades ΔX_0 shares there $\tilde{X_0^+} = x + \Delta X_0$ and $\tilde{Y_0^+} = c - \phi(\Delta X_0) \cdot S_{t_0}$. It is easy to see the difference between $Y_0^+ = c - \Delta X_0 \cdot S_{t_0}$ in the benchmark case and $\tilde{Y_0^+} = c - \phi(\Delta X_0) \cdot S_{t_0}$ in the illiquid case. The agent obtains $(x + \Delta X_0) \cdot S_{t_0}$ book value in his stock account after the transaction at time t_0. The sum of the money market account and the stock equals to $c - \phi(\Delta X_0) \cdot S_{t_0} + (x + \Delta X_0) \cdot S_{t_0}$ at the time t_0 after the transaction. The agent would release all share holdings at the terminal date t_N.

Lemma 4 *In a market with liquidity cost $\phi(\Delta X)$, the wealth process corresponding to the trading strategy $\Delta X = (\Delta X_n)_{0 \leq n \leq N}$ is given by*

$$\begin{aligned} \tilde{W_n}(\Delta X_n) &= (c - \phi(\Delta X_0) \cdot S_{t_0}) \cdot e^{r \cdot \delta t \cdot n} - \sum_{i=1}^{n} \phi(\Delta X_i) \cdot S_{t_i} \cdot e^{r \cdot \delta t \cdot (n-i)} \\ &\quad + (x + \sum_{i=0}^{n} \Delta X_i) \cdot S_{t_n}, n = 0, ..., N. \end{aligned} \tag{3.9}$$

For the numerical analysis, we assume that the liquidity cost is applied into all time periods except the terminal time period t_N. This is a common thing to not consider liquidity cost into the terminal time in the numerical research. We fully discuss the influence of liquidity cost applied or not into the terminal time period in Section 5.2.1.

3.1.2 Utility Maximization Problem

The investor manages a portfolio which consists of the risk-less asset (money market account) and the risky asset (stock) to achieve the maximal expected utility of terminal wealth. To solve the optimization model we need to determine the value function and the optimal portfolio strategy. Hence, the optimal portfolio problem is transferred to choose the best trading strategy in the value function from the initial time t_0 to the terminal time t_N. This problem needs a measure of performance to compute the best trading strategy. The maximization of the expected utility of terminal wealth is to represent the performance of the trading strategy (e.g. Rogers (2001)).

In order to find what kind of the trading strategy can be regarded as the optimal strategy, we measure the performance of a trading strategy in the following three factors: the quantitative amount of payoffs, the average performance of payoffs in different states and the consideration of the risk tolerance level in different traders.

The quantity standard is a direct way to perform the trading strategy. Every trader prefers a higher payoff than a lower one in real market. When a trading strategy has different payoffs in different states of the world, the method of purely comparing quantity of payoffs is not correct. In view of comparing the trading strategy with different payoffs in different states, the average performance is calculated through the expectation computation. The expectation corresponds to the expected value for the optimal portfolio problem. Nevertheless, this characteristic ignores traders' financial background and risk tolerance level. However, only computing the expectation of the terminal wealth may let the investor choose more risky strategy and result in a big loss. The flaw of using the expectation is that it does not consider the risk. In real market, the risk aversion of trades tends to increase corresponding to the increasing amount of money. In particular, it is significantly observed when the amount of money can affect traders' life if lose them. It is suitable to use a concave utility function to model the risk tolerance level. The concept of risk aversion is extensive used in economics and finance, which represents the tolerance level of wealth uncertainty.

Definition 5 *A function $U : \mathbb{R} \to \mathbb{R}$ is called a* ***utility function*** *if it is strictly increasing and strictly concave.*

The concavity of utility function means that

$$U(\lambda x + (1-\lambda)y) > \lambda U(x) + (1-\lambda)U(y),$$

where $\lambda \in [0,1]$. Below we list a few common utility functions:

- Logarithmic utility:$U(x) = \log(x)$
- Exponential utility:$U(x) = 1 - e^{-\lambda x}, \lambda > 0$
- Power utility:$U(x) = \frac{1}{\gamma}x^{\gamma}, \gamma \in [0,1]$

In the numerical analysis, we choose the exponential utility to analyse the optimal portfolio selection problem. It is because the form of the exponential utility allows us to eliminate one variable in the value function. Other utility functions such as the logarithmic or power utility functions do not benefit from dimensionality reduction, which increases the computational complexity considerably (from linear to quadratic). In Chapter 6, we discuss briefly how to deal with the CRRA utility functions.

Combining the three factors on performance of trading strategy, we translate the optimal portfolio problem into maximizing the expected utility of the wealth at t_N.

We know the key objective of the optimal portfolio problem is how to choose the best trading strategy $(\Delta X_j)_{0\leq j\leq N}$. Suppose the share position has to be liquidated at the terminal moment t_N. This means $X_N^+ = 0$ and Y_N^+ represents all wealth at time t_N. Based on general formula (3.5) and (3.6), amounts of share and cash holdings at terminal time t_N are as follows:

$$\begin{aligned} X_N^+ &= 0, \\ Y_N^+ &= Y_n^- \cdot e^{r\cdot\delta t\cdot(N-n)} - \sum_{j=n}^{N-1}\phi(\Delta X_j)\cdot S_{t_j}\cdot e^{r\cdot\delta t\cdot(N-j)} \\ &\quad -\phi(-X_n^- - \sum_{j=n}^{N-1}\Delta X_j)\cdot S_{t_N}. \end{aligned}$$

Hence, the terminal wealth $\tilde{W}_N$ is given by

$$\begin{aligned} \tilde{W}_N &= Y_n^- \cdot e^{r\cdot\delta t\cdot(N-n)} - \sum_{j=n}^{N-1}\phi(\Delta X_j)\cdot S_{t_j}\cdot e^{r\cdot\delta t\cdot(N-j)} \\ &\quad -\phi(-X_n^- - \sum_{j=n}^{N-1}\Delta X_j)\cdot S_{t_N} \end{aligned} \tag{3.10}$$

The utility maximization problem translates into finding a trading strategy ΔX_n s.t. $E(U(\tilde{W}_N))$ achieves a maximal value. Put formula (3.10) into the utility function, and define $\Phi_{t_n}(\cdot)$ as below:

$$\begin{aligned} \Phi_{t_n}(x,y,s,(\Delta X_j)) = \mathbb{E}\Big[U\Big(& y\cdot e^{r\cdot\delta t\cdot(N-n)} - \sum_{j=n}^{N-1}\phi(\Delta X_j)\cdot S_{t_j}\cdot e^{r\cdot\delta t\cdot(N-j)} \\ & -\phi(-x - \sum_{j=n}^{N-1}\Delta X_j)\cdot S_{t_N}\Big) \mid X_n^- = x, Y_n^- = y, S_{t_n} = s\Big]. \end{aligned} \tag{3.11}$$

Definition 6 *A trading strategy* (ΔX_j^*) *is called a solution for the optimal port-*

folio problem with utility function U, if

$$\Phi_{t_n}(x, y, s, (\Delta X_j^*)) \geq \Phi_{t_n}(x, y, s, (\Delta X_j)_{n \leq j \leq N-1}). \tag{3.12}$$

for any trading strategy $(\Delta X_j)_{n \leq j \leq N-1}$.

Following Definition 6 the solution for utility maximization problem at time t_n is based on choosing $(\Delta X_j)_{j=n}^{N-1}$. At every time moment t_n, we define the value function v_{t_n} by

$$v_{t_n}(x, y, s) = \max_{(\Delta X_n)} \Phi_{t_n}\Bigg(x, y, s, (\Delta X_j)\Bigg), \tag{3.13}$$

where variables x, y, s represent the number of shares, the wealth of cash holding and the stock price at time t_n.

3.2 Solution to Optimal Portfolio Problem

The utility maximization problem is usually solved by two analytical approaches: the stochastic dynamic programming approach and the martingale method. We choose the dynamic programming approach to deal with the optimization problem in the C-R model.

3.2.1 Dynamic Programming Equation

Combining the formula (3.12) and the value function (3.13) shows a fact, that the optimal strategy at time t_n is based on all its future optimal trades from time t_{n+1} to t_{N-1}. For the computation of v_{t_n} we use the dynamic programming method to solve the optimal portfolio problem. Now the dynamic programming framework is described as follows:

Denote $v_{t_n}(x, y, s)$ as the optimal value of the expected utility of terminal wealth, in which y is the cash wealth at time t_n, x is the number of shares at time t_n and s is the stock price:

$$v_{t_n}(x, y, s) = \max_{\Delta X} \mathbb{E}\Big[U(Y_N^- - \phi(-X_N)S_{t_N}) \mid X_n^- = x, Y_n^- = y, S_{t_n} = s\Big]. \quad (3.14)$$

With the self-financing condition, shown in formulae (3.5) and (3.6), the cash Y_{n+1}^+ at time t_{n+1} induced by an illiquidity term $\phi(\Delta X_n)$ corresponding to the cash Y_n^+ at time t_n is given by

$$Y_{n+1}^+ = Y_n^+ \cdot e^{r\cdot\delta t} - \phi(\Delta X_{n+1}) \cdot S_{t_{n+1}}. \quad (3.15)$$

Recall formula (3.11) and (3.15), the value function (3.14) is extended to the following form:

$$\begin{aligned} v_{t_n}(x, y, s) = \max_{\Delta X} \ \mathbb{E}\Big[U\Big(& Y_{n+1}^+ e^{r\cdot\delta t\cdot(N-n-1)} - \sum_{j=n+2}^{N-1} \phi(\Delta X_j) S_{t_j} e^{r\cdot\delta t\cdot(N-j)} \\ & -\phi(-X_{n+1}^+ - \sum_{j=n+2}^{N-1} \Delta X_j) S_{t_N}\Big) \mid X_n^- = x, Y_n^- = y, S_{t_n} = s\Big]. \quad (3.16) \end{aligned}$$

The expression of utility function in formula (3.16) shows how to calculate the terminal wealth when we know the stock holding, the cash holding and the corresponding stock price at any time t_n. In order to solve the value function at time t_n, we choose the dynamic programming principle to deal with the value function at time t_{n+1}. The dynamic programming method is a backward algorithm that the current value function is based on all its future value functions. Considering the information on time t_{n+1}, $X_{n+1}^{-} = X_n^{-} + \Delta x = x + \Delta x$ and $Y_{n+1}^{-} = Y_n^{+} \cdot e^{r\cdot\delta t} = (Y_n^{-} - \phi(\Delta x)\cdot S_{t_n})\cdot e^{r\cdot\delta t} = y\cdot e^{r\cdot\delta t} - \phi(\Delta x)\cdot S_{t_n}\cdot e^{r\cdot\delta t}$, we obtain:

$$\begin{aligned} v_{t_n}(x,y,s) &= \max_{\Delta x} \mathbb{E}\Big[v_{t_{n+1}}(x+\Delta x, y\cdot e^{r\cdot\delta t} - \phi(\Delta x)\cdot S_{t_n}\cdot e^{r\cdot\delta t}, S_{t_{n+1}}) \\ &\qquad \mid X_n^{-} = x, Y_n^{-} = y, S_{t_n} = s\Big]. \end{aligned} \tag{3.17}$$

Equation (3.17) shows that each transaction of the optimal strategy at time t_n can be derived from their next time period transaction at time t_{n+1}.

3.2.2 Value Function with Negative Exponential Utility

We implement an exponential utility function into the C-R model to investigate the optimal investment under the illiquidity constraints. In the money market account the interest rate r is constant. In the risky asset the stock price is denoted by S_{t_n} at time t_n. Due to the Markov property of the stock price processes, the price $S_{t_{n+1}}$ at time t_{n+1} is decided by the price S_{t_n} at time t_n. With the probability $p \in (0,1)$ the price is equal to $S_{t_{n+1}} = uS_{t_n}$, or with the probability $1-p \in (0,1)$ the price turns out $S_{t_{n+1}} = dS_{t_n}$, where $d < u$. Different utility functions lead to different value payoffs. We choose a negative exponential utility function $U(\tilde{W}_N) = -\exp(-\gamma\tilde{W}_N)$, where γ denotes the risk aversion parameter. The risk aversion $-U''(\cdot)/U'(\cdot)$ equals γ, so it is independent of the wealth. The bigger value of risk aversion parameter is, the more prudent trading strategies the trader selects.

In this specific setting, the dynamic programming equation (3.17) reads as follows:

$$\begin{aligned} v_{t_n}(x,y,s) &= \sup_{\Delta x}\Bigg\{p \cdot v_{t_{n+1}}\Bigg(x+\Delta x, y\cdot e^{r\delta t}-\phi(\Delta x)s\cdot e^{r\delta t}, us\Bigg) \\ &\quad +(1-p)\cdot v_{t_{n+1}}\Bigg(x+\Delta x, y\cdot e^{r\delta t}-\phi(\Delta x)s\cdot e^{r\delta t}, ds\Bigg)\Bigg\} \end{aligned} \tag{3.18}$$

For the negative exponential utility $U(\tilde{W}_N) = -\exp(-\gamma \tilde{W}_N)$, it can be simplified, i.e. we can eliminate one argument of $v_{t_n}(x,y,s)$.
Start from the terminal time t_N:

$$\begin{aligned} v_{t_N}(x,y,s) &= U(\tilde{W}_N) \\ &= -e^{-\gamma(y-\phi(-x)\cdot s)} \\ &= -e^{-\gamma y}\cdot e^{\gamma\phi(-x)s} \\ &= e^{-\gamma y}\cdot \hat{v}_{t_N}(x,s), \end{aligned}$$

where $\hat{v}_{t_N}(x,s) = -e^{\gamma\phi(-x)s}$.

Then, based on the formula (3.18) we can get:

$$
\begin{aligned}
v_{t_{N-1}}(x,y,s) &= \sup_{\Delta x}\Bigg\{p\cdot v_{t_N}\bigg(x+\Delta x, y\cdot e^{r\delta t}-\phi(\Delta x)\cdot s\cdot e^{r\delta t}, us\bigg) \\
&\quad +(1-p)\cdot v_{t_N}\bigg(x+\Delta x, y\cdot e^{r\delta t}-\phi(\Delta x)\cdot s\cdot e^{r\delta t}, ds\bigg)\Bigg\} \\
&= \sup_{\Delta x}\Bigg\{p\bigg(e^{-\gamma(ye^{r\delta t}-\phi(\Delta x)s)e^{r\delta t}}\bigg)\cdot\hat{v}_{t_N}(x+\Delta x, us) \\
&\quad +(1-p)\bigg(e^{-\gamma(ye^{r\delta t}-\phi(\Delta x)s)e^{r\delta t}}\bigg)\cdot\hat{v}_{t_N}(x+\Delta x, ds)\Bigg\} \\
&= \sup_{\Delta x}\Bigg\{p\cdot e^{-\gamma ye^{2r\delta t}}\cdot e^{\gamma e^{r\delta t}\phi(\Delta x)s}\cdot\hat{v}_{t_N}(x+\Delta x, us) \\
&\quad +(1-p)\cdot e^{-\gamma ye^{2r\delta t}}\cdot e^{\gamma e^{r\delta t}\phi(\Delta x)s}\cdot\hat{v}_{t_N}(x+\Delta x, ds)\Bigg\} \\
&= e^{-\gamma ye^{2r\delta t}}\sup_{\Delta x}\Bigg\{e^{\gamma e^{r\delta t}\phi(\Delta x)s}\bigg[p\hat{v}_{t_N}(x+\Delta x, us)+(1-p)\hat{v}_{t_N}(x+\Delta x, ds)\bigg]\Bigg\} \\
&= e^{-\gamma ye^{2r\delta t}}\cdot\hat{v}_{t_{N-1}}(x,s),
\end{aligned}
$$

where

$$
\hat{v}_{t_{N-1}}(x,s) = \sup_{\Delta x}\Bigg\{e^{\gamma e^{r\delta t}\phi(\Delta x)s}\bigg[p\hat{v}_{t_N}(x+\Delta x, us)+(1-p)\hat{v}_{t_N}(x+\Delta x, ds)\bigg]\Bigg\}
$$

Therefore, we define

$$
\begin{aligned}
\hat{v}_{t_k}(x,s) &= \sup_{\Delta x}\Bigg\{e^{\gamma\phi(\Delta x)se^{r\cdot\delta t\cdot(N-k)}}\bigg[p\hat{v}_{t_{k+1}}(x+\Delta x, us) \\
&\quad +(1-p)\hat{v}_{t_{k+1}}(x+\Delta x, ds)\bigg]\Bigg\}.
\end{aligned} \tag{3.19}
$$

Iteratively, we get

$$\begin{aligned} v_{t_k}(x,y,s) &= e^{-\gamma y e^{r\delta t(N-k)}} \sup_{\Delta x} \Bigg\{ e^{\gamma \phi(\Delta x) s e^{r\delta t(N-k)}} \\ & \Big[p\hat{v}_{t_{k+1}}(x+\Delta x, us) + (1-p)\hat{v}_{t_{k+1}}(x+\Delta x, ds) \Big] \Bigg\} \end{aligned} \tag{3.20}$$

Attention should be paid to the general equation (3.20), it is clear that all $\hat{v}_{t_k}(x,y,s)$ are based on choosing the best Δx which is the optimal trading strategy and $\hat{v}_{t_k}(x,y,s)$ only relates to two arguments X_k and S_{t_k}. Eliminating one argument Y_k can significantly simplify the computation complexity.

3.3 Reservation Pricing of Options

In the academic literature, reservation pricing of options is an important approach for the option pricing. The reservation pricing of options, also called the utility indifference pricing. The utility based valuation was pioneered by Hodges & Neuberger (1989) and further developed by Davis *et al.* (1993) in the optimization model with the finite time horizons. Damgaard (2003) and Damgaard (2006) discussed the problem of finding the reservation prices of European and American contingent claims in the model with transaction costs.

Generally speaking, the reservation price is the minimal (maximal) amount added to an option trader's initial wealth which allows him to attain the same utility that he would have attained without selling (buying) the option. Reservation prices consist of the reservation buy price and the reservation sell price. The reservation buy price is the price at which the agent is indifferent between buying and not buying the option. More precisely, the reservation buy price is defined as the price that the agent is indifferent between if he buys the option and holds to the expiry date of option in a portfolio maximization problem and if he only holds portfolios from the above portfolio maximization problem. We obtain the similar explanation for the reservation sell price, which is the price at which the agent is indifferent between selling and not selling options.

Let a function $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ determine the payoff of a path-independent European option, i.e. the option pays off $\mathcal{C}(S_{t_N})$ at time t_N. Generally speaking, the *reservation price* (also called *utility-indifference price*) is the price that the agent ensures the price per option results in no loss of utility compared with the alternative strategy of not writing or purchasing any option.

In order to define the reservation prices, we define the value function with shorting option at time t_0 by $v_{t_0}^{\mathcal{C}}$ and the value function with buying option at time t_0 by $v_{t_0}^{-\mathcal{C}}$.

Definition 7 *Consider an agent who only trades in the money market account and the stock with the initial wealth condition (x, c). The* ***reservation buy price*** *of an European option is the value $p^b \in \mathbb{R}$ that satisfies*

$$v_{t_0}^{-\mathcal{C}}(x, c - p^b, s) \quad = \quad v_{t_0}(x, c, s), \tag{3.21}$$

Definition 8 *Consider an agent who only trades in the money market account and the stock with the initial wealth condition (x, c). The* ***reservation sell price*** *of an European option is the value $p^s \in \mathbb{R}$ that satisfies*

$$v_{t_0}^{\mathcal{C}}(x, c + p^s, s) \quad = \quad v_{t_0}(x, c, s), \tag{3.22}$$

Therefore, depending on the formulas (3.21) and (3.22), we can compute a reservation price with respect to solve two portfolio maximization problems: one of problems is about a portfolio without option and the other one is about portfolio with options. The reservation pricing of options concerns the evaluation and hedging of contingent claims, when the liquidity cost or any transaction cost happens in the trading of underlying assets. The derivation of reservation price formula in the negative exponential utility is discussed in the following:

$$\begin{aligned} v_{t_0}^{-\mathcal{C}}(x, y-p^b, s) &= \sup_{\Delta X} \mathbb{E}\Big(U(\tilde{W}_N + \mathcal{C}(S_{t_N}))\Big) \\ &= e^{-\gamma y e^{r\cdot\delta t\cdot N}} \cdot \hat{v_{t_0}^{-\mathcal{C}}}(x, s), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} v_{t_0}(x, y, s) &= \sup_{\Delta X} \mathbb{E}(U(\tilde{W}_N)) \\ &= e^{-\gamma y e^{r\cdot\delta t\cdot N}} \cdot \hat{v_{t_0}}(x, s), \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} \hat{v}_{t_N}(x, s) &= -e^{\gamma\cdot\phi(-x)\cdot s}, \\ \hat{v_{t_N}^{-\mathcal{C}}}(x, s) &= e^{\gamma\cdot -\mathcal{C}(s)}\hat{v}_{t_N}(x, s), \\ \hat{v}_{t_n}(x, s) &= \sup_{\Delta X}\Big\{e^{\gamma\phi(\Delta x)se^{r\cdot\delta t\cdot N}}[p\hat{v}_{t_{n+1}}(x+\Delta X, us) + (1-p)\hat{v}_{t_{n+1}}(x+\Delta X, ds)]\Big\}. \end{aligned}$$

Notice that the derivative process of formula $\hat{v_{t_n}^{-\mathcal{C}}}(x, s)$ is identical as for $\hat{v}_{t_n}(x, s)$

in the previous subsection.

Substitute the formula (3.23) and (3.24) into the formula (3.21), we can obtain the equation of the reservation buy price:

$$p^b = -\frac{1}{\gamma e^{r\cdot\delta t\cdot N}}\log\frac{v_{t_0}^{-\mathcal{C}}(x,y,s)}{v_{t_0}(x,y,s)} = -\frac{1}{\gamma e^{r\cdot\delta t\cdot N}}\log\frac{\hat{v_{t_0}^{-\mathcal{C}}}(x,s)}{\hat{v_{t_0}}(x,s)} \tag{3.25}$$

Similarly, the reservation sell price of the contingent claim $\mathcal{C}$ satisfies the Equation (3.22):

$$v_{t_0}^{\mathcal{C}}(x, y + p^s, s) \quad = \quad v_{t_0}(x, y, s).$$

Based on the dynamic programming equation (3.20) in the exponential utility function, the detailed derivation process of reservation sell price p^s can be shown as:

$$\begin{aligned} v_{t_0}^{\mathcal{C}}(x, y+p^s, s) &= \sup_{\Delta X}\mathbb{E}\Big(U(\tilde{W}_N - \mathcal{C}(S_{t_N}))\Big) \\ &= e^{-\gamma y e^{r\cdot\delta t\cdot N}}\cdot e^{-\gamma p^s e^{r\cdot\delta t\cdot N}}\cdot \hat{v_{t_0}^{\mathcal{C}}}(x,s), \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} v_{t_0}(x,y,s) &= \sup_{\Delta X}\mathbb{E}(U(\tilde{W}_N)) \\ &= e^{-\gamma y e^{r\cdot\delta t\cdot N}}\cdot \hat{v_{t_0}}(x,s), \end{aligned} \tag{3.27}$$

where

$$
\begin{aligned}
\hat{v}_{t_N}(x,s) &= -e^{\gamma\cdot\phi(-x)\cdot s},\\
\hat{v^{\mathcal{C}}_{t_N}}(x,s) &= e^{\gamma\cdot\mathcal{C}(s)}\hat{v}_{t_N}(x,s),\\
\hat{v}_{t_n}(x,s) &= \sup_{\Delta X}\left\{e^{\gamma\phi(\Delta x)se^{r\cdot\delta t\cdot N}}[p\hat{v}_{t_{n+1}}(x+\Delta X,us)+(1-p)\hat{v}_{t_{n+1}}(x+\Delta X,ds)]\right\}
\end{aligned}
$$

Notice that the derivative process of formula $\hat{v^{\mathcal{C}}_{t_n}}(x,s)$ is identical as for $\hat{v}_{t_n}(x,s)$ in the previous subsection.

Substitute the formula (3.26) and (3.27) into the formula (3.22), we can obtain the expression of the reservation sell price:

$$
p^s = \frac{1}{\gamma e^{r\cdot\delta t\cdot N}}\log\frac{v^{\mathcal{C}}_{t_0}(x,y,s)}{v_{t_0}(x,y,s)} = \frac{1}{\gamma e^{r\cdot\delta t\cdot N}}\log\frac{\hat{v^{\mathcal{C}}_{t_0}}(x,s)}{\hat{v_{t_0}}(x,s)} \tag{3.28}
$$

3.4 Comparison with Cetin-Rogers Numerical Study

Our optimization model originates from Cetin & Rogers (2007)'s model. Before discussing the dynamics of the optimal strategy choice model, we firstly review Cetin and Rogers model (2007). Roughly speaking, the C-R model is a kind of the two-asset model with liquidity cost. We denote a probability space by $(\Omega,\mathcal{F},\mathbb{P})$ and there have $N+1$ trading dates $t_0,t_1,...,t_N$; the period $[0,T]$ is discretized by $\delta t = t_{i+1}-t_i = \frac{T}{N}$. In this two-asset model, an investor allocates his principal to a risk-less asset (a money cash account with a constant interest rate) and a risky asset (a stock). We approximate this two-asset model in an illiquid market. Zakamouline (2005) pointed out that the simplest Markov chain approximation consists of a discrete time equation for the amount of wealth in the money cash account and a binomial tree model of the stock price in the optimal portfolio problem. We formulate the money cash account process as:

$$B_{t_i} = e^{r\cdot(t_i - t_0)} \cdot B_{t_0}, 0 \leq i \leq N$$

where r is the constant interest rate. Using an appropriate binomial tree model (CRR binomial tree), we approximate the dynamics of the stock price process as:

$$S_{t_{i+1}} = S_{t_i} \cdot \omega(i), 0 \leq i \leq N$$

where S_{t_0} is a given constant and $(\omega(i))_{i=0,1,\ldots,N}$ is a sequence of i.i.d random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, taking two real values u and d with probability p and $1-p$, respectively.

$$\omega(i) = \begin{cases} u & \text{with probability} \quad p, \\ d & \text{with probability} \quad 1-p. \end{cases}$$

The three parameters u, d and p can determine the binomial tree model. We specify the choice of u, d and p under CRR binomial model setting (page 211 in Hull (2003))

$$u = e^{\sigma\sqrt{\delta t}}, d = e^{-\sigma\sqrt{\delta t}}, p = \frac{e^{\mu\delta t} - d}{u - d},$$

where μ, σ are the growth rate of the stock price and the volatility, respectively. Cetin & Rogers (2007) directly set u, d and p as constants $e^{0.1}$, $e^{-0.1}$ and 0.7 in their numerical study, respectively. We use the same constant parameters in this subsection for the comparison to Cetin and Rogers' numerical results.

To solve the utility maximization problem numerically, we assume that the investors have the negative exponential utility function $U(\Delta x) = -e^{-\gamma \Delta x}$. Formula (3.19) in Section 3.2 is the dynamic programming equation with negative exponential utility function.

Cetin & Rogers (2007) lists optimal strategies for utility maximisation with one written vanilla put option with the strike price K. Table 3.1 represents the default values of parameters in computation of optimal portfolio strategy. Table 3.2 lists various values in the strike price K and the risk aversion parameter γ. We compare the optimal strategy and the price of option between the C-R model and our algorithm, where Table 3.1 and 3.2 show the setting of the optimization model under the illiquidity constraint.

Table 3.1: Parameters with values in the optimal strategy comparison between C-R Model and our algorithm.

Parameters	Meaning	Value
T	Time Horizon	1 year
r	Interest Rate	0.05
S_0	Initial Stock Price	1.0
x	Initial Stock Holding	0
c	Initial Cash Holding	0
n	Number of Options	1

Table 3.2: Cases for optimal strategy comparison.

Parameters	Values		
	Case 1	Case 2	Case 3
K	1	1	1.1
γ	1	5	1

3.4.1 Expected Utility

We adopt the dynamic programming equation (3.19) to compute the optimal portfolio given by same parameters values in Cetin & Rogers (2007). Table 3.3 and 3.4 show the optimal strategy comparison between Cetin & Rogers (2007) and our implementation without or with liquidity cost, respectively. In short, the optimal strategy from our implementation has less fluctuation from any time t_n to its following time t_{n+1}. The values in the grey background of Table 3.3 and 3.4 give distinct evidence for proof of big fluctuation in the optimal strategy presented in Cetin & Rogers (2007).

In order to find the better 'optimal' strategy from the three conditions in Table 3.2, we investigate the optimal strategy in a three-period model for a trader who writes one European put option and holds zero initial positions both in the stock and the money market at time period t_0. The numerical algorithm provides the value of expected utility of terminal wealth which are compared to numerical results provided by Cetin & Rogers (2007). Table 3.3 and Table 3.4 give the optimal trading strategy $(\Delta X_n)^*_{0\leq n\leq N}$ with respect to different parameter conditions and show values of expected utility of terminal wealth in the last column. Notice that Table 3.3 corresponds to the optimal strategy in a perfectly liquid market ($\alpha = 0$), in this market the liquidity cost function $\phi(\Delta X) = \Delta X$. Values of strategy in Table 3.4 represent the existence of liquidity costs forces the trader to trade cautiously and less, in absolute quantities, compared to the perfectly liquid market. This behaviour even does not change when the risk aversion parameter γ significantly increases. Paying attention to values covered by the grey background in Table 3.3, unexpectedly large absolute quantities of trading amounts in the last period make us doubt the correctness of optimal strategy in the perfectly liquid market, which is provided in Cetin & Rogers (2007).

Table 3.3: Expected utility of terminal wealth and values of the optimal strategy with short one European put option under $\alpha = 0$ (the perfectly liquid market) in three-period model, while in each condition, the top row values of optimal strategy is collected from Cetin & Rogers (2007) and the bottom row values of optimal strategy is collected from our numerical data.

	S_{t_0}	$S_{t_{1,u}}$	$S_{t_{1,d}}$	$S_{t_{2,uu}}$	$S_{t_{2,ud}}$	$S_{t_{2,du}}$	$S_{t_{2,dd}}$	$\mathbb{E}U$
Case 1	-0.75	-0.60	-1.24	-8.86	-9.86	-9.90	-11.10	-1.31458
	-0.77	-0.63	-1.23	-0.49	-1.07	-1.07	-1.73	-1.01685
Case 2	-0.23	-0.11	-0.63	-2.96	-4.12	-4.12	-5.40	-2.13910
	-0.34	-0.22	-0.73	0. 10	-0.59	-0.59	-1.15	-1.01641
Case 3	-0.89	-0.64	-1.72	-8.9	-9.94	-9.94	-11.09	-1.33089
	-0.95	-0.75	-1.61	-0.49	-1.57	-1.57	-1.73	-1.03590

Table 3.3 clearly shows that our numerical results for number of shares are different to results provided by Cetin & Rogers (2007) in those three cases. In order to identify the difference between those two data sources, we pay attention to Case 1 firstly. It is reasonable that the investor would hold more shares if the current stock price has increased from the price at the previous time period; and vice versa. The numerical data from our numerical model is in accord with this phenomenon. However, the numerical data from that paper describes another story, especially at time t_2: the investor would sell a large quantity of shares no matter if the price moves to higher or lower. The huge fluctuations in stock holdings at time t_2 let us doubt the correctness of the data from that paper. In order to know which 'OPTIMAL' strategy is better, we compute the expected utility of terminal wealth. It is because that the optimal portfolio choice problem

in the model provided by Cetin & Rogers (2007) is based on the maximisation of expected utility of terminal wealth. The last columns in Table 3.3 and Table 3.4 both show that the strategy from our numerical data generates larger values in expected utility of terminal wealth. It reveals that the strategies provided by Cetin & Rogers (2007) are not true. We provide optimal strategies with higher values in expected utility of terminal wealth.

Table 3.4: Expected utility of terminal wealth and values of the optimal strategy with short one European put option under $\alpha = 0.05$ in three-period model, while in each condition, the top row values of optimal strategy is collected from Cetin & Rogers (2007) and the bottom row values of optimal strategy is collected from our numerical data.

	S_{t_0}	$S_{t_{1,u}}$	$S_{t_{1,d}}$	$S_{t_{2,uu}}$	$S_{t_{2,ud}}$	$S_{t_{2,du}}$	$S_{t_{2,dd}}$	$\mathbb{E}U$
Case 1	-0.32	-0.1	-0.62	-0.30	-0.34	-0.82	-0.86	-1.04279
	-0.16	-0.19	-0.26	-0.13	-0.17	-0.2	-0.23	-1.02466
Case 2	-0.24	-0.33	-0.57	-0.50	-0.64	-0.73	-0.87	-3.29088
	-0.20	-0.19	-0.41	-0.10	-0.24	-0.32	-0.45	-1.13891
Case 3	-0.36	-0.59	-0.70	-0.76	-0.84	-0.89	-0.89	-1.06730
	-0.20	-0.23	-0.33	-0.15	-0.22	-0.27	-0.26	-1.04878

3.4.2 Option Price

In Section 3.4.1, we discuss the comparison of strategies from Cetin & Rogers (2007) and the numerical data. The next analysis is to investigate the price of options in Cetin and Rogers model. We propose that a trader short one European put option and with zero initial number of shares and zero cash at time t_0. We define the reservation prices (utility-indifference prices) and the corresponding equations in Section 3.3. The price of options is calculated by the reservation prices with different initial number of shares.

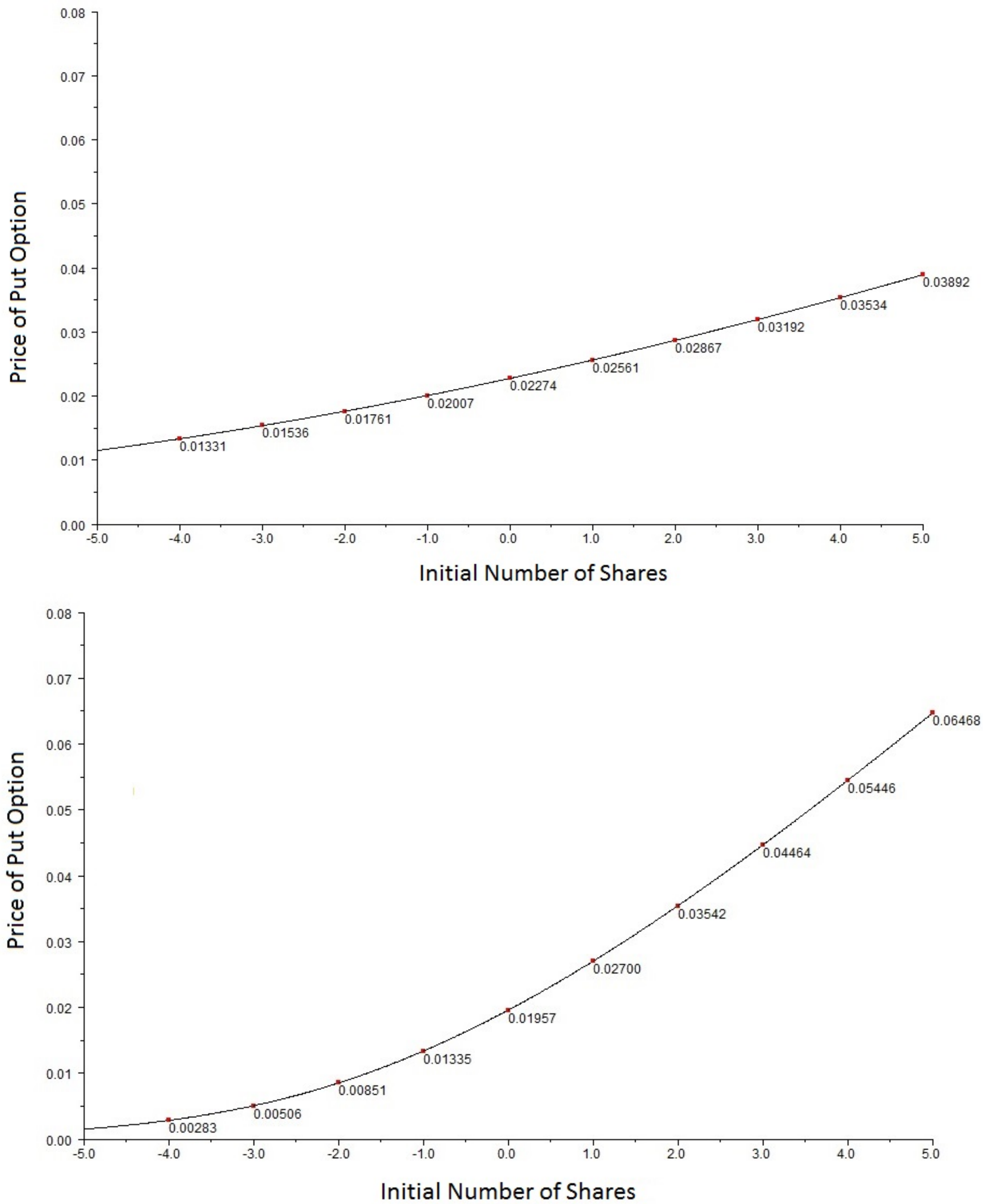


Figure 3.1: Price of put option in the function of the initial number of shares on the illiquid market with $\alpha = 0.05$. The strike price $K = 1$. The top panel corresponds to $\gamma = 1$ while the bottom panel corresponds to $\gamma = 5$.

Figure 3.1 and 3.2 present the reservation sell price as the function of initial number of shares (assuming no initial cash holding) with written one European

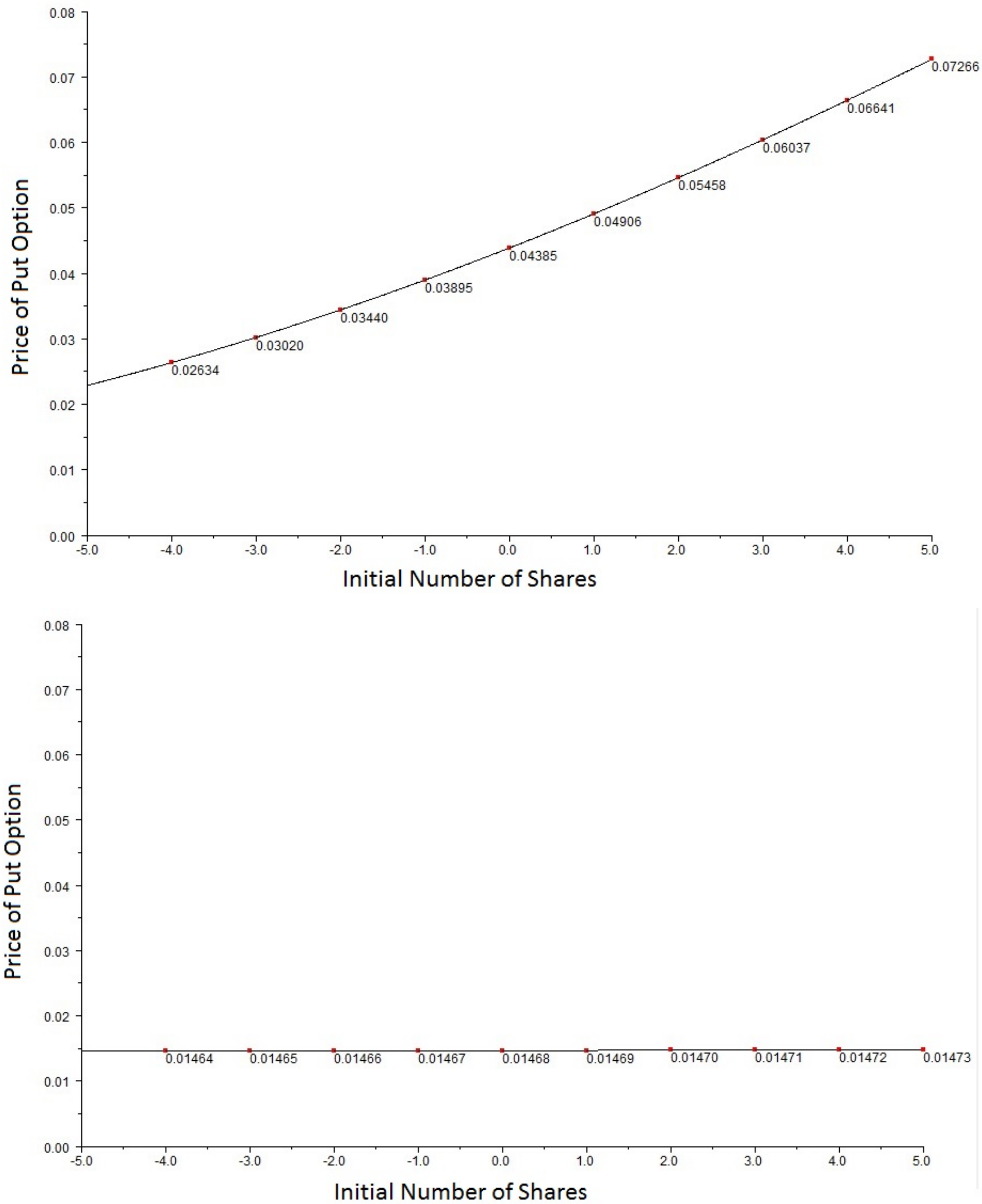


Figure 3.2: Price of put option in the function of the initial number of shares on the illiquid market with $\gamma = 1$. The top panel corresponds to the strike price $K = 1.1$ and $\alpha = 0.05$ while the bottom panel corresponds to $K = 1$ and $\alpha = 0.00005$.

put option. These two figures change strike price K, liquidity cost parameter α and risk aversion parameter γ to display the corresponding prices of put option.

As the liquidity parameter gets smaller, the change of price of option during the whole range of number of shares converges to zero, that the right plot in Figure 3.2 verifies this behaviour when α is close to zero (the perfectly liquid market).

Given the exponential utility function, Cetin and Rogers found the equation for reservation sell price as:

$$p' = \frac{1}{\gamma} \log \frac{v_{t_0}^{\mathfrak{C}}(x, y, s)}{v_{t_0}(x, y, s)} \tag{3.29}$$

3.5 Summary

In this chapter, we described the optimization problem of maximizing the expected utility of terminal wealth in the C-R model. First, we studied the model specification in the utility maximization problem. Second, we determine the dynamic programming approach as the solution of the utility maximization problem. Moreover, in order to solve the optimization model with liquidity cost, we determined the value function and the corresponding optimal portfolio strategy. Third, we investigated the reservation buy/sell price in the C-R model. We also compared the numerical result of the three-period model between Cetin & Rogers (2007) and our related implementation.

Chapter 4

Implementation

We have illustrated the optimal portfolio selection problem under illiquidity constraints introduced by Cetin & Rogers (2007) in Chapter 3. Due to no analytical solution of the value function in Cetin & Rogers (2007), we design an efficient algorithm to compute the value function numerically in an illiquid binomial market. A popular method to solve the portfolio optimisation problem numerically is dynamic programming method in discrete time. We apply dynamic programming method into the implementation of value function and optimal strategy. Before computing value function, we need to approximate the C-R model in the illiquid binomial market firstly.

This chapter is mainly divided into three parts: first, we design an efficient algorithm to compute the value function in binomial model via the dynamic programming principle; second, based on the optimisation problem in dynamic programming method, we compute the optimal strategy and verify whether the current range of stock holdings is sufficient; third, we discuss other numerical procedures which are Monte Carlo simulation as the verification of binomial model and the computation of implied volatility.

4.1 Binomial Model and the Choice of Parameters

This section focuses on selection of binomial model. We review some typical models and focus on how to construct the binomial model and the relevant choice of formula.

In an illiquid market, we have to solve the problem of optimal portfolio selection numerically by discrete time dynamic programming approach. It is a good way to employ a binomial model to approximate the dynamics of the stock price. The binomial appproximation has advantage on sampling a discrete time state space.

In order to deal with the optimal portfolio selection problem numerically, we need to discretize dynamics of stock state. It has two main steps in the discretization progress: the first step is to construct a binomial approximation of the stock price in the market; the second step is to set up the discrete vector of number of share holdings in each node of binomial model. We will introduce a binomial tree model to approximate the dynamics of the stock state.

First of all, we discretize the whole trading period $[0, T]$ by the *step size* $\delta t = \frac{T}{N}$, where N is the number of time steps. In the implementation of optimization model, we consider the number of periods m in the binomial tree model as the number of time steps. Thus, the smaller δt the more accurate the model since it has more time periods squeezed into the time period $[0, T]$. Denote by t_i the discretized time variable, which makes the time sequence in $t_0, t_1, ..., t_N$. In the money market account, the equation for the bond price process is

$$B_{t_i} = B_{t_0} exp(r \cdot i \cdot \delta t), 0 \leq i \leq N$$

where r is the constant interest rate.

The focus of discretization method is only on how to discretize the stock price and the stock holdings in the case of negative exponential utility function since the optimal strategy does not depend on the cash holdings in the money market account (risk-less asset). We explained why the optimal strategy is independent of the cash holdings and list the derivation process in Section 3.2.2.

The approximation of stock price process in risky asset is followed as:

$$S_{t_{i+1}} = S_{t_i} \cdot \omega(i), 0 \leq i \leq N$$

where S_{t_0} is a constant and $(\omega(i))_{i=0,1,...,N}$ is a sequence of i.i.d random variables taking two real values u and d with probability p and $1-p$, respectively. Ross *et al.* (1979) proposed the formulas for evaluating the probability of an up movement p^*, the parameters u and d as:

$$u = e^{\sigma\sqrt{\delta t}}, d = \frac{1}{u}, p^* = \frac{e^{r\delta t} - d}{u - d}.$$

Notice that p^* and $(1 - p^*)$ are called *risk neutral probabilities* for u and d. That implies that expected rate of return on the stock under these probabilities is the risk-free rate r. Moreover, investors are risk-neutral and do not require any risk premium for holding risky assets in such a risk-neutral world.

Besides the CRR model, another kind of binomial tree model is the He model developed by He (1990). The corresponding formulae for the He model are:

$$u^H = e^{(\mu-\frac{1}{2}\sigma^2)\delta t+\sigma\sqrt{\delta t}}, d^H = e^{(\mu-\frac{1}{2}\sigma^2)\delta t-\sigma\sqrt{\delta t}}, p^H = \frac{1}{2}.$$

We choose an appropriate CRR model to measure the dynamics of the underlying price.

$$u = e^{\sigma\sqrt{\delta t}}, d = \frac{1}{u}, p = \frac{e^{\mu\delta t} - d}{u - d}, \tag{4.1}$$

where μ is a known and constant drift of the stock price in the implementation. Comparing to the *risk neutral probability* p^*, the probability p in Equation (4.1) means that the expected rate of return on the stock is the drift of the stock price. Notice that parameters u depends on the *step size* δt. Hence, u, d and p would change by different number of period models. This treatment implies that we characterize the uncertainty about the dynamics of stock price by a binomial model with a dependable probability of an up movement. This binomial tree model is recombinant that makes the practical computation feasible, i.e. if the stock price moves up and then down, the price will be the same as if it had moved down and then up. Such two paths merge or recombine. This property reduced the number of nodes in the tree and the generation of the stock price in the recombining model leads to n nodes at the nth period. In reality, the order of magnitude of the widely used step length is from $\mathcal{O}(10^{-1})$ to $\mathcal{O}(10^{-3})$. We apply the higher order of magnitude $\mathcal{O}(10^{-2})$ in this implementation to get higher accuracy and balance the computational complexity. The computational complexity describes how the amount of resources required by the algorithm grows with the size of the problem. Two main types of resources are the memory and the running time of the program. In this subsection, the computational complexity refers to the number of periods in the binomial model. The property of recombination of nodes allows us to accelerate the running time of the implementation. The recombining property also makes that the value of the stock at each node can be calculated directly via formula of value function and does not require that the tree be built first. It significantly reduces the memory for dealing with huge data.

The second step of the discretization scheme is to allocate a discrete vector for storing the discretization of stock holdings that is measured in terms of physical units of stocks. We denote the value function vector by $\mathcal{V}$. There are $\ltimes + 1$ elements in the $\mathcal{V}$. The symbol $\ltimes$ equals to the range of stock holdings divided by the discretization *step size*. Denote by k the index of element of the $\mathcal{V}$-vector, where $k = 0, 1, ..., \ltimes$. The distance between the two adjacent elements is the discretization *step size*. We assume that the *step size* is a fixed value. Notice that in the traditional discretization method the $\mathcal{V}$-vector in each binomial tree node is identical. However, in our implementation the $\mathcal{V}$-vector in the first node arranges a different range of stock holdings besides others in the tree. We propose no initial stock holdings in the benchmark case of numerical analysis, that results the range of stock holdings including zero in the first node. We adopt a fixed narrowed range of stock holdings for other nodes, while the sufficient range of stock holdings are verified by Monte Carlo simulation. Why adopt a different range only in the first node will be shown and explained in an example in Section 5.2.2.

We implement the discretization of stock holdings in a matrix, J_{ij} for $i =$

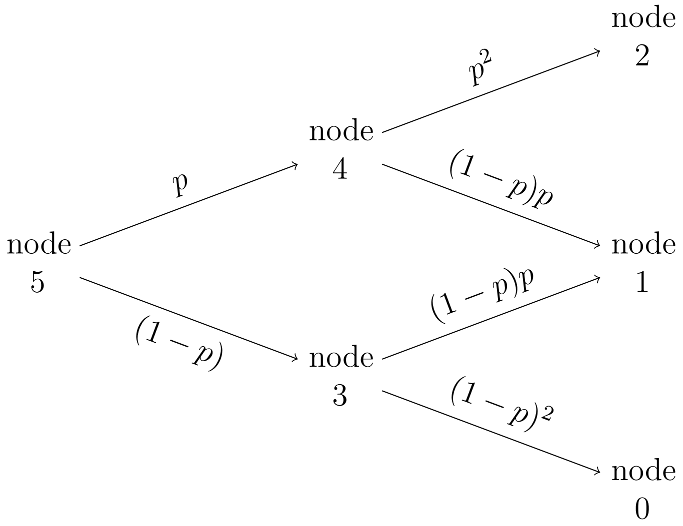


Figure 4.1: Two-periods recombining binomial model that shows the sequence order of array to store all nodes

$0, 1, ..., n-1$ and $j = 0, 1, ..., \ltimes$. The index i is the number of node in the binomial tree and j is the number of discretization of stock holdings. n elements in the representation of row index i are decided by the period number m in the binomial tree, that has $n = \frac{(m+1)(m+2)}{2}$. The discretization number $\ltimes$ is formulated as the range of stock holdings divided by the *step size* δt.

Notice that the left/right limit in the range of stock holdings is proposed as the lower/upper boundary, respectively. We sequence all nodes in the binomial tree from the last period to the former periods. The order of nodes in each period is from the smallest stock price node to the biggest stock price node. For instance, the first column of the matrix J_{ij}, where $i = 0$ and $j = 0, 1, ..., \ltimes$, stores all discretizations for node $S_{d,...,d}$. $S_{d,...,d}$ is the node having the smallest stock price in the whole binomial tree. Figure 4.1 illustrates how $J_{i\cdot}$ order the tree. Section 4.2 will explain the computation of value function in the binomial tree of the stock price.

4.2 Computation of the Value Function

Considering the computation of value functions and optimal portfolios, there are three main approaches (see Pliska (1997)): the conventional approach, the dynamic programming approach and the risk neutral computational approach. The first approach, a traditional way, refers to the differentiability of the utility function. This approach may be impractical when the number of variates in partial derivatives and equations in the simultaneous system are significantly large. The third approach involves the risk neutral probability measure. The principle of this approach is as follows: the key observation is that the trading strategy decides the objective, which is maximizing the expected utility function; compose the observation to obtain the set of the attainable wealth and then compute the optimal trading strategy with respect to the attainable wealth. The risk neutral computational approach can solve most of examples but the complicate solution as the disadvantage. We choose the dynamic programming approach to deal with the computation of value function.

The principle of dynamic programming can be concluded that knowing the value function at time t_n can determine the value function at time t_{n-1}. What the dynamic programming indeed does is substitute solving a sequence of single-period optimal decision problems for solving a multi-period decision problem. Davis *et al.* (1993) pioneered the numerical solution for the utility maximization problem with transaction costs.

Equation (3.17) expressed the general dynamic programming equation of computation of value function for the optimization model in the illiquid market (e.g. Cetin & Rogers (2007), Rogers & Singh (2010)). We derived the dynamic programming equation for the model with the negative exponential utility function in Section 3.2.2 and shown in equation (3.19). Equation (3.19) is implemented by a backward recursion algorithm that assumes knowing the value function for all states at the next time period. Using the backward recursion algorithm, we use an array $\mathcal{G}[i]$ to store the binomial tree of stock prices and $\mathcal{V}$ to store value function, where $i = 0, 1, ..., \frac{(m+1)(m+2)}{2} - 1$.

The construction of stock prices computation will be described in the following 3 steps:

1. Detect whether the current node is belong to the terminal period;
2. Compute the current stock price $S_{t_j} = S_0 u^j d^{m-j}$ when the number of node $j \in [0, m]$ and store it into the array $\mathcal{G}[j]$;
3. Use two for loops to detect the current period number represented by the index L and the position of node represented by the index k for other nodes except the terminal period, compute the current stock price $S_{t_k} = S_0 u^k d^{L-k}$ and then store into the array $\mathcal{G}[k]$;

The array $\mathcal{G}[i]$ store the information of each node about the current period number L and the current node number k, that corresponds to the computation of value functions $\mathcal{V}$-vector. Equation (3.19) determines the value function $\hat{v}_{t_k}(s)$ by computing the maximum of value functions in the binomial tree, given by choosing the trading strategy. We translate equation (3.19) into a numerical solution by

$$\mathcal{V}(i,j) = \sup_{\Delta x}\{\mathcal{Z}[p\mathcal{V}(i',j) + (1-p)\mathcal{V}(i^*,j)]\} \tag{4.2}$$

where $\mathcal{Z}$ is one factor in the dependence of cash amount at the current node and the indices i' and i^* represent the successive nodes of the current one in the binomial tree. We use the matrix J_{ij} to replace $\mathcal{V}$ and explain the construction of value functions as below:

1. Gather information about the current period number L and the current node number k;

2. Apply no liquidity cost into the computation of illiquid effect function when the current period is the terminal period $L = m$ and then compute value functions $\mathcal{V}(k, j)$ where $k \in [0, m]$;

3. Apply liquidity cost into the computation of illiquid effect function for nodes of other periods, detect the successive down node (i^*, j) by the relation $i^* = \frac{(L+1)L}{2} + k + L + 1$ and the successive up node (i', j) by the relation $i' = i^* + 1$ and then compute the corresponding value functions;

Notice that we assume applying liquidity cost into all time periods except the terminal time period t_T. From the mathematical perspective, we has noticed that there is a substantial effect on the strategy if we apply liquidity costs at the terminal time and illustrate this effect in Section 5.2.1. This effect is caused by the fact that liquidation of the whole portfolio is costly and spread over time. A lot of financial literatures in transaction costs assume that there is no cost at the terminal time (e.g. recent literatures Cetin *et al.* (2010) and Gokay *et al.* (2012) specify and explain the substantial effect). From the financial perspective, in real markets, financial institutions usually do not liquidate their whole portfolios due to they need to provide option payoffs or determine the total value of their portfolios. Section 5.2.1 illustrates how liquidity costs affect the strategy over time if applying the liquidity issues into the terminal time period.

4.3 Determination of the Optimal Strategy

Most of papers on determining optimal strategy measure stock holdings in terms of physical unit as our algorithm does. Zakamouline (2005) and Malo & Pennanen (2012) measured stock holdings in terms of monetary unit. In C-R model, the computation of optimal strategy is based on the maximal expected utility of terminal wealth. Equation (4.2) provides the numerical method of determining the optimal strategy. Notice that the method of solving the optimization model does not follow the recombining binomial tree since the existence of liquidity cost makes different value of optimal strategy at the same stock price node. The stock price that goes up at first and then goes down equals to the price that goes down at first and then goes up. However, the existence of liquidity costs in every trade, except transactions in the terminal time, makes the optimal strategy for the stock price going up and going down differ to the optimal strategy for the stock price going down and going up. We design an iteration algorithm to trace optimal strategies for all nodes in such a model that leads to 2^n nodes at the n^{th} step. The non-recombining construction of binomial model limits the iteration algorithm to deal with cases with large number of periods. For example, the model with number of periods $m = 50$ need to build $\sum_{i=0}^{50} 2^i$ nodes that is impossible to be implement by the computer. The following issue needed to consider in the method of determining the optimal strategy is how to produce optimal strategy in cases with large period number of periods binomial models. Generally speaking, we produce a random stock price path and search the corresponding transaction path which get a maximal expected utility of terminal wealth.

The method for searching optimal strategy is divided into the iteration algorithm for cases with small number of periods and the random path of optimal strategy for cases with large number of periods. We have stored all values of value function in matrix J_{ij} which contains information of optimal strategy along a selected stock price path.

- For cases with small number of periods:
  1. Build up an array $\mathcal{P}[k]$ that contains $\sum_{k=0}^{n} 2^k$ elements and will store values of optimal strategy into $\mathcal{P}[k]$;
  2. Search the entry point of the iteration algorithm in the position of array $\mathcal{G}[i]$, where i is decided by the difference between the initial stock holdings and the minimal value in the range of stock holding; notice that the *step size* δt decides the computation precision of the position i;
  3. Collect the corresponding value of value functions matrix J_{ij} where the index j is decided by the position of array $\mathcal{G}[i]$ from step 2;
  4. Calculate the positions of next successive nodes and recursively repeat step 2 until the last period;
- For cases with large number of periods:
  1. Generate a random number $\theta \in (0, 1)$ in every period to decide the corresponding path of stock prices; if θ is smaller than the probability

of stock price going-up, stock price goes up and other values of θ makes the stock price going down;

2. Select the corresponding transaction value from J_{ij} based on the current stock price S_i;

We propose the numerical solutions of the optimization model without liquidity costs as benchmarks. The first important reason is that we can test the current program by comparing our numerical solutions under full parameters known in the perfect liquid market with the analytical solution derived in Merton (1971). Section 5.2.4 shows the previous comparison in details. The second reason why list models without liquidity costs as benchmarks is that we can quantify the liquidity cost on the reservation price by comparing the option price with liquidity costs with that without liquidity cost and by comparing the corresponding hedging strategy (the definition will be introduced in Chapter 5) with different liquidity costs.

The cases with very small number of time steps can not simulate the optimization model with real market conditions. However, we list the comparative analysis of parameters both in models with small number and large number of time steps and compare how parameters affect the optimal strategy and the reservation price with different time horizons in Section 5.4.

The method of generating the optimal strategy by the corresponding stock price path only shows one trajectory in one time running of programme. That is impractical for applications. We design a Monte Carlo simulation to deal with large number of samples (e.g. $10,000$ samples) to present a large sample of optimal strategies.

4.4 Selection of the Range of Stock Holdings

When running the existing algorithm of binomial tree modelling, we do not provide a function to check whether the range of stock holdings is large enough. The sufficient range of stock holdings can decide the correctness of the optimal strategy. Because the binomial model produces all values of value function for all nodes, and some particular nodes have extreme values of stock prices which cause overflow or underflow of the range of stock holdings $\mathcal{V}$. In the existing algorithm, the factor of the value functions $\mathcal{V}$ always times by the related stock price S to correspond the maximization of expected utility of terminal wealth and the optimal strategy, while the dynamic programming equation (3.19) shows it. Due to extreme values of the stock prices, the computation of value functions or optimal strategy $\mathcal{V} \cdot S$ causes overflow or underflow the range of stock holdings. For instance, the large range of stock holdings contains either large positive values in the stock price path with many go-up movements or small negative values in the stock price path with many go-down movements. This drawback is that we select non-sufficient (or narrow) range of stock holdings to compute value function. We design a Monte Carlo simulation to determine the range of stock holdings. Besides the determination of the range of stock holdings, the Monte Carlo simulation is produced as the verification of the existing binomial model.

Comparing to the determination of range of stock holdings in binomial model, we depict a method to exam the corresponding minimal and maximal values of stock holdings at each trajectory in Monte Carlo simulation. After running out all trajectories, we will get the number of trajectories which are hitting current range of stock holdings and the related minimal and maximal number of shares for those trajectories which did not hit current range. We check whether the range of stock holdings is sufficient at each node of trajectories. Hence, the construction of determining ranges starts after step 2 of Monte Carlo procedure in the previous subsection. In short, we construct a *for loop* to run all trajectories and check the range of stock holdings at each node of one trajectory. The procedure of determining range of stock holdings in one trajectory is as follows:

1. For the initial node, set the number of shares as the temporary minimal and maximal number of shares;

2. For other nodes except the terminal node, replace current number of shares to temporary minimal (maximal) number of shares if it is smaller (bigger) than the value in temporary variable; while the hitting boundary time would plus one time;

3. For the terminal node, only plus hitting times into the value which records the paths hitting time if current number of shares is smaller (bigger) than the temporary minimal (maximal) number of shares.

4.5 Other Numerical Procedures

After the description of designing the optimal strategy in binomial model, we develop other numerical procedures. The first procedure, the Monte Carlo, is a verification of binomial model for measuring the expected utility of terminal wealth. Another procedure is to compute the implied volatility in the illiquid binomial model.

4.5.1 Verification by Monte Carlo

The technique of Monte Carlo was first used by Boyle (1977) but it was computational inefficient. Boyle *et al.* (1997) further enhance the computation efficiency of the Monte Carlo method, meanwhile discuss its main advantages rather than other numerical methods and describe its applications in finance. Glasserman (2004) provides the implementation of Monte Carlo simulation in details. Cvitanic *et al.* (2003) develops an approximation computation of the optimal portfolio in the complete maket, that is computed by the covariation between optimal wealth and the uncertainty of shares. Detemple *et al.* (2003) also detects the optimal portfolio strategy but adopts different way that involves computing expectations.

The generation of optimal strategy in C-R model is based on the maximal expected utility of terminal wealth. The algorithm of binomial model shows the computation of maximization of expected utility and the method for choosing optimal strategy. We select the Monte Carlo simulation as the verification of the binomial model.

We characterize the procedure of Monte Carlo that simulates the computation of expected utility of terminal wealth to verify the Binomial model. In short, the Monte Carlo simulation generates the corresponding expectation value of terminal wealth based on each random trajectory and combine all samples with probabilities to obtain the estimation of expected utility of terminal wealth. We also construct 95% confidence interval to measure the accuracy of the estimation. The procedure of Monte Carlo simulation is as follows:

1. Construct a matrix M_{il} to store values of optimal strategy for all trajecto-

ries, where i refers to the number of period and l refers to the number of trajectories;

2. Generate an independent uniform $(0, 1)$ random variate U as a random number generator in each trajectory $(\cdot, l)$;

3. Based on random number from the generator at each node, the stock price in the successive node is decided;

4. Call value function matrix J_{ij} which was implemented in binomial model and select the corresponding transaction value from J_{ij};

5. Compute the estimation of expectation of terminal wealth EU_l;

6. Repeat steps 2-5 with corresponding modifications to obtain another trajectory $(\cdot, l + 1)$;

7. Compute the average expectation of terminal wealth for all trajectories, that equals to $EU_l^* = \frac{1}{l} \sum_{k=1}^{l} EU_l$;

8. Construct a 95% confidence interval to assess the qualify of the computation, that formula of confidence interval is given by

$$\left[EU_l^* - \frac{1.96 EU^d}{\sqrt{l}}, EU_l^* + \frac{1.96 EU^d}{\sqrt{l}} \right],$$

where EU^d represents the standard deviation of the difference between each estimation EU_l and the average estimation EU_l^*.

Notice that we can use confidence intervals with other confidence levels. Then we have different number instead of 1.96 in the example 95%.

4.5.2 Computation of Implied Volatility

In practice, the Black-Scholes model is one of the most widely used mathematical formulas to help investors find a fair price of an option. However, the Black-Scholes model is not a perfect tool to price options, especially for long-term maturities. In the Black-Scholes pricing model, the factor of transaction cost is disregarded to consider. Besides that, the Black-Scholes model cannot compute trading in continuous-time precisely. These problems make the mispricing of Black-Scholes model. Although this model has some drawbacks, we still use the Black-Scholes model to compute the price of an option. The implied volatility is used to represent the price of an option contract in the Black-Scholes model. The implied volatility of an option is that the volatility value of the underlying instrument which, when input in an option pricing model will return a theoretical value equal to the current market price of the option. The reason of using implied volatility to represent option price is that the range of option prices with different strike price is quite large. That makes hardly display all prices of options with different strike prices and maturities. Hence, the real market provides investors with both implied volatility percentages and the Black-Scholes price estimations of an option contract to show option prices.

In order to analyse the option price, we compute the implied volatility as one measurement to display the price with different maturities or strike prices. The implied volatility curve usually is represented as the volatility smile. It exhibits the volatility skew for call options. The skew property of implied volatility reveals that call options when deep-in-the-money have higher volatility than deep-out-of-the-money. In Section 5.6, we provide an analysis of how implied volatility is affected by market parameters in our model of the illiquid market. This subsection displays how to compute implied volatility for a given option price in the Black-Scholes model. In our further studies, we will use this algorithm to analyse reservation prices obtained from our model.

Consider a call option with strike price K and time to maturity T. Given r and S_0 the current stock price, the implied volatility σ_i is the value of the volatility parameter that the Black-Scholes price matches a given price $\mathcal{C}_m$ of the option:

$$\mathcal{C}_{BS}(r, S_0, T, K, \sigma_i) = \mathcal{C}_m, \tag{4.3}$$

where $\mathcal{C}_{BS}(r, S_0, T, K, \sigma_i)$ is the Black-Scholes pricing formula. The Black-Scholes formulae for the prices at t_0 of a European call option and a European put option are:

$$\mathcal{C}^c_{BS}(r, S_0, T, K, \sigma_i) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

and

$$\mathcal{C}^p_{BS}(r, S_0, T, K, \sigma_i) = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)Ke^{-rT}\Phi(d_2),$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}. \end{aligned}$$

The function $\Phi(\cdot)$ represents the cumulative probability distribution function for a standard normal distribution; other parameters in the above formulae are defined before.

Since the Black-Scholes price is be calculated by a known function of constant volatility, the equation above can be inverted to find the implied volatility σ_i. There does not exist the closed-form solution for Equation (4.3), thus we solve it numerically by using Newton-Raphson method. The Newton-Raphson method is easily used for European options since it requires to know vega. Vega is the derivative of the option value with respect to the volatility of the underlying asset, that measures sensitivity to volatility. For American options, since the option may be exercised in prior to the maturity, it does not require to know vega. Hence, the bisection method is used as the solution of finding volatility in American options.

We choose an iterative algorithm to implement the solution of implied volatility. The procedure of generation of implied volatility is as follows:

1. Initial a guess for the volatility: firstly, set up the guess σ_0 as $\frac{\mathcal{C}_m/S_{t_0}}{0.398\cdot\sqrt{T}}$ (followed by Odegaard (2007)); checking if the guess value is smaller than 0.3, if so a replacement of 0.3 value;

2. Iteratively, find a σ_{i+1} that satisfies

$$\sigma_{i+1} = \sigma_i + \frac{\mathcal{C}_m - \mathcal{C}_{BS}(r, S_0, T, K, \sigma_i)}{\nu(r, S_0, T, K, \sigma_i)},$$

where $\nu(r, S_0, T, K, \sigma_i) = \frac{1}{\sqrt{2\pi}} \cdot S_0 \exp(-rT) \cdot \exp(-0.5d_1^2) \cdot \sqrt{T}$ is the vega, until the required accuracy is reached:

$$|\mathcal{C}_m - \mathcal{C}_{BS}(r, S_0, T, K, \sigma_{i+1})| \leq \varepsilon$$

where ε is the acceptable error, set as 10^{-7} in the implementation.

In particular, the accuracy is quite important for computation precision. The accuracy in the current implementation is small enough for computation. Besides the accuracy, the iteration times also affect the precision of computation. We set up 100 times for this iteration algorithm; however in practice only $10-20$ times iteration can obtain the sufficient value of implied volatility since the volatility converges quickly.

4.6 Summary of Implementation of the Model

In this chapter, we design an efficient algorithm for computation of binomial model. That binomial model implementation shows the value function via the dynamic programming principle and the production of optimal strategy for all nodes in cases of small number of periods models. Another compulsory numerical procedure is the algorithm for Monte Carlo simulation. The Monte Carlo simulation can not only verify the correctness of the optimal strategy but also measure whether the range of stock holdings is sufficient for computing. A procedure of implied volatility in the illiquid binomial model is produced for resulting from reservation pricing of options under liquidity costs. We provide the relevant pseudo codes for different numerical procedure separately.

Chapter 5

Numerical Analysis

This chapter investigates the optimal hedging strategies of option traders under liquidity costs. It also examines the impact of reservation price by varying market parameters. This chapter consists of five parts. Firstly, we construct a benchmark case without option in the perfectly liquid market. Then we examine the dynamics of hedging strategies based on two particular multi-period models. Next we carry out an analysis of hedging strategies and reservation prices by five varying important market parameters in the market with liquidity cost. Besides we observe *Price Transition Points*. They are the intersections of the Black-Scholes price and the line of option price in the illiquid market. If investors' initial share holdings are greater than the *Price Transition Point*, they are willing to sell call options in lower price than the replicating price without transaction cost and take benefit from the written call option. Finally, we explore the implied volatility of call option under liquidity cost and compare the convex liquidity cost to the linear transaction cost when market prices between the liquidity cost and the transaction cost are matching at the price of at-the-money call option.

We compare the data of implied volatility from model quotes to the real market quotes. That comparison reveals the level of liquidity in the real market.

5.1 Parameters and Default Values

We simulate the model of optimal strategy with liquidity costs according to market parameters and discretization parameters. Table 5.1 lists the interpretation of these parameters and default values used in our numerical analysis. We carry out comparative analysis of the optimal strategy model with liquidity costs by simulating the model by varying parameter values. The corresponding values of these market parameters are expressed in Table 5.2.

Table 5.1: Interpretation of parameters and corresponding default values

Parameters	Meaning	Value
α	Illiquidity Effect	0.05
γ	Risk Aversion Parameter	1
μ	Growth Rate in Stock Price	0.15
σ	Stock Volatility	0.3
T	Time Horizon	1 year
r	Interest Rate	0.05
S_0	Initial Stock Price	1.0
x	Initial Stock Holding	0
c	Initial Cash Holding	0
n	Number of Options	1
K	Strike Price	1
N^{path}	Number of Random Stock Paths	10,000
m	Number of Time Periods	100
δt	Step Length	0.01
δr	Discretization Length	0.002

We assume that the zero initial number of shares and zero cash wealth in the C-R model with the negative exponential utility function. As discussed in

Table 5.2: Variation in parameters values in comparative analysis

Parameters	Values		
x	0	2	4
γ	1	2	4
K	1.2	1.3	1.4
T	0.5	1	5
n	1	2	5

Chapter 3, a particular property of the exponential utility function is that the optimal strategy for the risky asset is independent of the cash wealth. Therefore, we only consider the risky asset in this chapter.

5.2 Benchmark Cases in Perfectly Liquid Markets

Cetin & Rogers (2007) conducts a numerical analysis of a three-period model with writing European put options. In Section 3.4, using the same setting of market parameters, we compared corresponding numerical results with the results in Cetin & Rogers (2007). Based on that comparison, we compared the optimal strategies and the maximal expected utility of terminal wealth in a three-period model. In this section, we increase the number of periods m from 3 to 100. This shows more sensible strategy. However, it is too complicated to show all trajectories of the 100−period model by the binomial model. As proved in Section 4.5, Monte Carlo simulation is regarded as the correctness of the binomial model. For different situations, we display numerical data with the binomial model or Monte Carlo simulation in this section.

5.2.1 Illustration of Optimal Strategies

We assume that it is no option hedging and no liquidity cost in a frictionless market. In order to determine the influences of option hedging and liquidity, we analyse three kinds of benchmark cases: the case of the frictionless market, the case with option in the perfectly liquid market ($n = 1$ and $\alpha = 0$), and the case without option in the illiquid market ($n = 0$ and $\alpha > 0$). We randomly choose 200 samples to illustrate optimal strategies. Each sample indicates the optimal strategy, in terms of number of shares, according to the corresponding stock price trajectory. Figure 5.1 - 5.3 illustrate these three numerical cases.

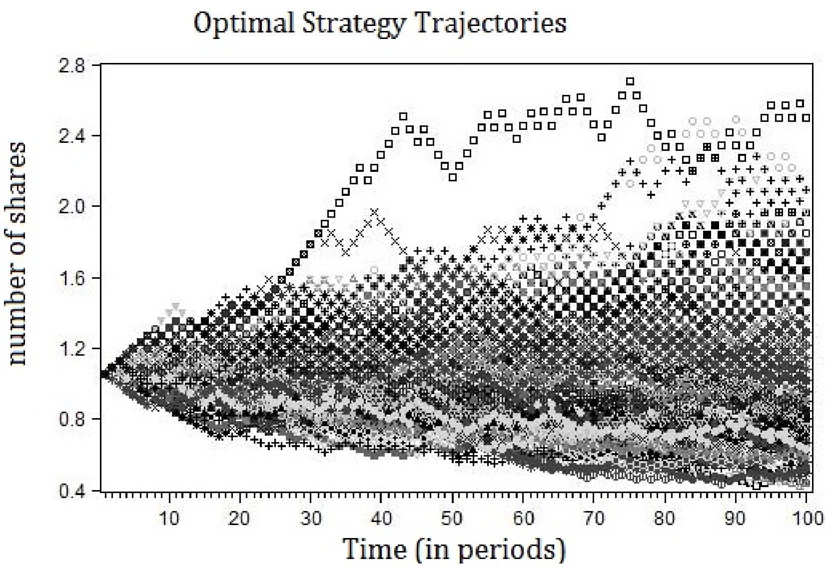


Figure 5.1: 200 trajectories of optimal strategy presented that the trader does not exercise any option in the perfectly liquid market (the liquidity parameter $\alpha = 0$). Other parameters are given in Table 5.1.

Figure 5.1 shows the optimal strategy changing over time in the perfectly liquid market without option. This figure expresses two properties of the range of stock holdings. First, for each period, there exists a small but sufficient range of stock holdings. We define that a sufficient range of stock holdings is the range which is small but sufficient for computing the correct the optimal strategy. It is apparent from this graph that almost all samples are located at the range $[0.4, 2.4]$ of stock holdings and only few samples are out of this range that we name them 'outliers'. We can enhance the computational efficiency and save the execution time by selecting a small but sufficient range of stock holdings. Whether the range of stock holdings is sufficient decides the correctness of optimal strategy. Our current numerical solution has implemented a way to measure the sufficient range. Second, the sufficient range fluctuates over time. The majority of those samples approximately displays a band with increasing width over time. That shape illustrates the minimal (or maximal) boundary of stock holdings at the current time (node) is close to the boundaries at its successive times (nodes). We will represent the histogram of boundaries of the stock holdings range in Section 5.2.2. Table 5.4 shows the minor difference of the expected utility of terminal wealth according to a sufficient range and a narrow range.

Figure 5.2 shows 200 samples of optimal strategy in the benchmark case with short one European call option and without liquidity cost. Comparing to Fig 5.1, this plot has two clear differences in the shape of samples on the optimal strategy. The shape on Figure 5.2 is approximately symmetric with respect to the horizontal line at the level of 1.68 shares. Another difference is a narrower range of stock holdings in the benchmark case with writing call option (Figure 5.2) than the benchmark case without option (Figure 5.1). The smaller band of the stock holdings is caused by the option impact. The reason is that investors who write call option are willing to trade fewer shares. We also test the optimal strategies in the benchmark case with short one European put option. The shape

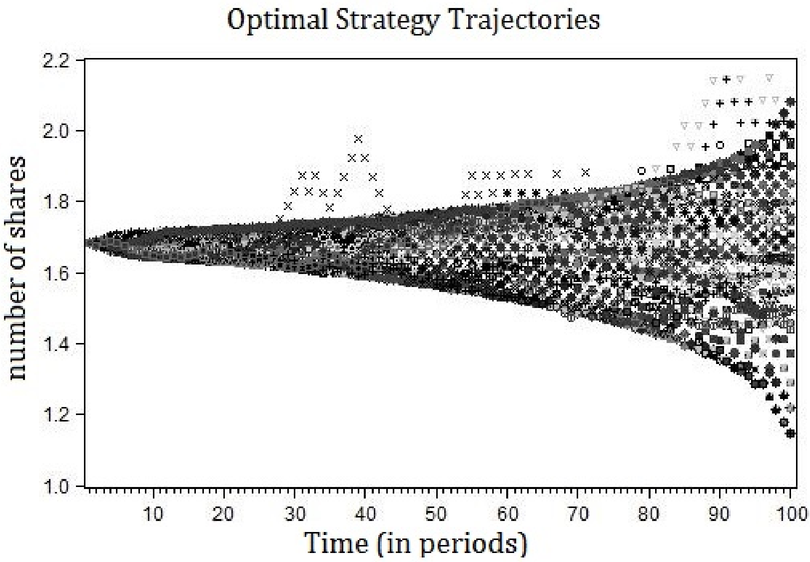
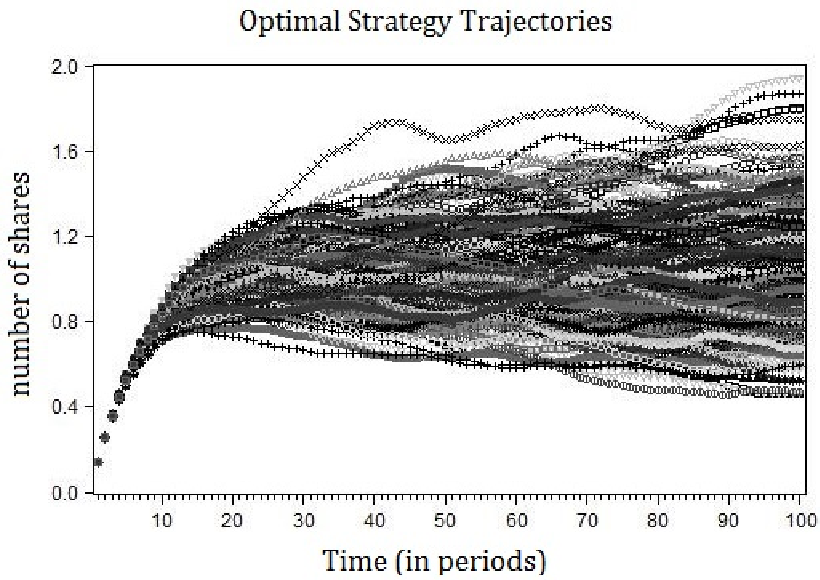


Figure 5.2: 200 trajectories of optimal strategy presented that the trader writes 1 call option in the perfectly liquid market (the liquidity parameter $\alpha = 0$). Other parameters are given in Table 5.1.

of put option of the range of stock holdings is similar to the shape of call option in Figure 5.2. The only difference between the model with call option (shown in Fig 5.2) and with put option is the range of stock holdings: for call option case, the sufficient range is $[1.0, 2.2]$; while for put option case, the sufficient range is from 0 to 1.2. The range difference between call and put option depends on the value of the strike price K. We only concentrate on the numerical analysis of written call option.

Figure 5.3 displays the optimal strategy samples in the benchmark case without option in the illiquid market: the top panel shows that the liquidity cost is applied into all periods except the terminal period of the model; the bottom panel shows that the liquidity cost is applied into all periods of the binomial model. In the numerical algorithm, we assume that the liquidity cost is applied into all time periods except the terminal time period t_T. We can explain this setting from two



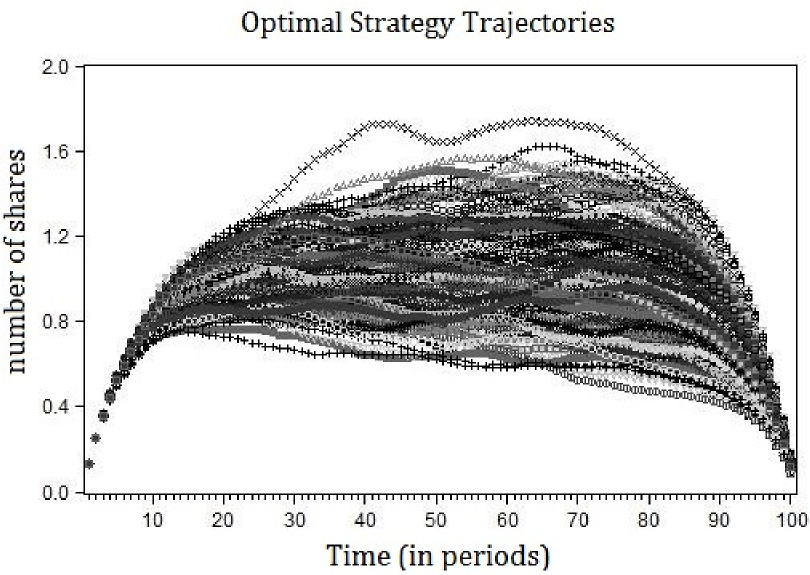


Figure 5.3: 200 trajectories of optimal strategy presented that the trader writes 1 put option in an illiquid market (the liquidity parameter $\alpha = 0.05$): no liquidity cost applied into the terminal time t_{100} in the top panel and liquidity cost applied into all time periods in the bottom panel. Other parameters are given in Table 5.1.

perspectives. From the mathematical perspective, there is a substantial effect on the strategy if we apply the liquidity cost into the terminal time (shown in the bottom panel of Fig 5.3). This effect is caused by the fact that the liquidation of the whole portfolio is costly and spread over time. Several financial papers on transaction costs assume that there is no cost at the terminal time (e.g. recent papers Cetin *et al.* (2010) and Gokay *et al.* (2012) specify and explain the substantial liquidity effect for the optimal strategy). From the financial perspective, in real markets, financial institutions usually do not liquidate their whole portfolios because they need to provide option payoff or determine the total value of their portfolios. A method called Mark-to-Market (MTM) is used to value positions and determine profits and losses based on the market price without taking into account the liquidity cost issues.

5.2.2 Choice of the Range of Stock Holdings

In the previous subsection, we chose 200 random trajectories to illustrate properties of optimal strategy in the multi-period model. However, those 200 selected trajectories cannot represent all properties of trajectories in the $100-$period model. The increasing simulation times in Monte Carlo computation can improve the estimation precision of the quantities. With some comparisons, we show that $10,000$ simulation times used in Monte Carlo algorithm is large enough to provide acceptable expected utility of terminal wealth. We also display the boundaries of stock holdings in histograms. Moreover, we choose a narrow range of stock holdings and compare the value of expected utility of terminal wealth and corresponding optimal strategies between the sufficient range and narrow range.

Expected Utility

Under the default parameters setting in Table 5.1, we use the binomial model to compute the expected utility of terminal wealth which equals to -0.945837. This value is adopted as the benchmark of computation precision. We use the Monte Carlo simulation to confirm the correctness of the binomial model. After running the Monte Carlo simulation with $N^{path} = 5,000, 10,000$ and $20,000$ separately, the estimate of expected utility (EU) and corresponding 95% confidence interval are shown in Table 5.3. From this table, for all of these three simulation times, the benchmark value from binomial model is contained in the 95% confidence intervals. This reveals the expected utility of terminal wealth in binomial model is correct. With consideration for both the computation precision and the computation complexity, $N^{path} = 10,000$ is a better choice: when $N^{path} = 5,000$, the 95% CI is too wide; while when $N^{path} = 20,000$, the execution time would be doubled of $N^{path} = 10,000$. Therefore, we select $N^{path} = 10,000$ as the default times of Monte Carlo simulation.

Table 5.3: Testing on Monte Carlo simulation times N^{path}

	Estimate of EU	95% Confidence Interval
N^{path} = 5,000	-0.947754	[-0.956917, -0.938591]
N^{path} = 10,000	-0.946602	[-0.952994, -0.94021]
N^{path} = 20,000	-0.946478	[-0.952982, -0.9449741]

The true EU in the binomial model is -0.945837.

As discussed in Section 5.2.1, besides a small number of outliers, most trajectories are located in a range narrower than the sufficient range. It might be possible to narrow the sufficient range besides little change in the expected utility of terminal wealth. We discuss the choice of the range of stock holdings in the following part.

Figure 5.4 and 5.5 in histograms show all lower/upper boundaries of 100-periods C-R model with no option and no liquidity impact. The left plot in Fig.5.4 indicates frequency of minimal number of shares occurred in 10,000 trajectories. We can see the range of lower boundary in all Monte Carlo runs is from 0.3 to 1.1. The right plot in Fig.5.4 represents cumulative frequency of lower boundary and shows the result of minimal number of shares (the value of horizontal axis) versus cumulative frequency (vertical ordinate). The cumulative frequency reveals the percentage of trajectories with minimal shares smaller or equal to the value of minimal number of shares on horizontal axis. The peak range of lower boundary is around [0.6, 1.1] which covers more than 95% of all trajectories.

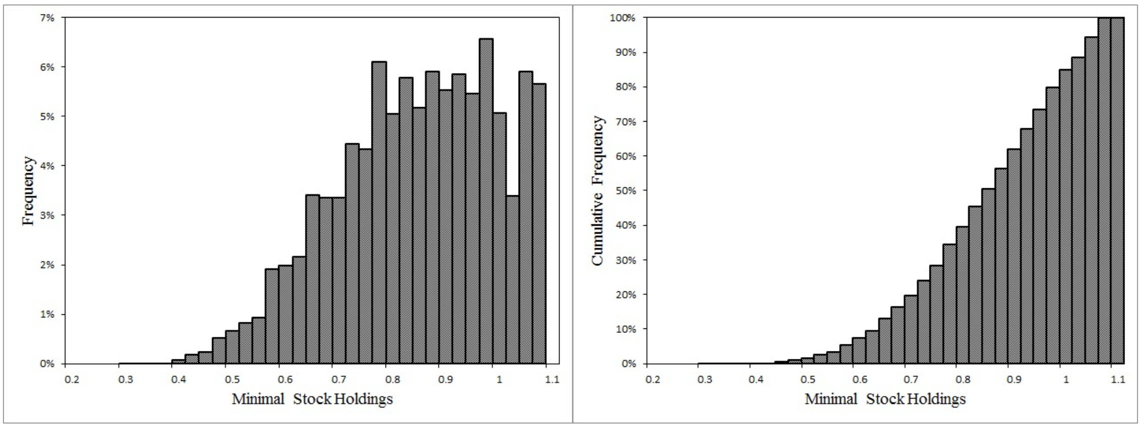


Figure 5.4: The frequency and cumulative frequency of lower boundary on the minimal number of shares in C-R model without option and no liquidity cost.

The left plot of Figure 5.5 shows the frequency of maximal number of shares occurred in 10,000 trajectories. From this plot, we can see the frequency reduces with the increasing maximal number of shares. Almost maximal number of shares in 10,000 trajectories happened at the range [1.04, 2.76]. There only exists 2 trajectories over this range. The right graph in Figure 5.5 represents the result of maximal number of shares (on horizontal axis) versus cumulative frequency (on vertical ordinate). The cumulative frequency shows the percentage of trajectories with maximal shares greater or equal to the value of maximal number of shares on horizontal ordinate. The frequency of trajectories with maximal shares no smaller than 1.88 is smaller than 5%. Based on Figures 5.4 and 5.5, approximate 90% trajectories are covered in the range of $[0.6, 1.8]$.

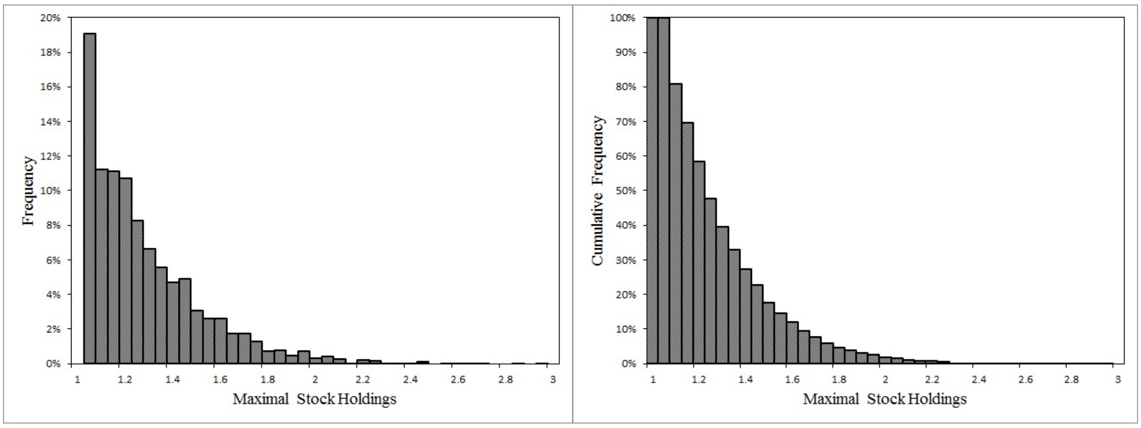


Figure 5.5: The frequency and cumulative frequency of upper boundary on the maximal number of shares in C-R model without option and no liquidity cost.

Table 5.4: Expected utility approximation versus the range of stock holdings

Range of Stock Holdings	EU_{BM}	Percentage Change
$[0.2, 3.1]^{\mathrm{S}}$	-0.945837	-
$[0.6, 3.1]^{\mathrm{N}}$	-0.945842	0.00053%
$[0.2, 1.8]^{\mathrm{N}}$	-0.945849	0.00127%
$[0.6, 1.8]^{\mathrm{N}}$	-0.946025	0.01988%

$^{\mathrm{S}}$ Sufficient Range of Stock Holdings.
$^{\mathrm{N}}$ Narrow Range of Stock Holdings.

Table 5.4 represents the expected utility of terminal wealth produced by binomial model under the benchmark case without option and no liquidity cost. This table includes the EU_{BM} of a sufficient range and three narrow ranges; the percentage change between the narrow ranges and the sufficient range are also displayed in last row. Based on the percentage changes, we find that shorten the range of stock holdings takes insignificant impact on the expected utility of terminal wealth. For example, the narrow range $[0.6, 1.8]^N$ covers 90% trajectories while the corresponding percentage change is less than 0.02%. We state that this is a good approximation compared to the sufficient range.

With the analysis above, we found that in the benchmark case without option, the narrow range takes little loss on the expected utility of terminal wealth. In order to figure out the change of reservation price on a narrow range, we further study the benchmark case with option. Table 5.5 lists values of EU_{BM} and reservation prices with written one call/put option separately. The narrow ranges in this table cover 90% trajectories in the sufficient range. The Black-Scholes prices are the verification of reservation prices. For the call option case, both of EU_{BM} and reservation prices are not changed when the range changes from $[-1, 4]$ to $[1.1, 2, 2]$. The percentage change between the Black-Scholes call price and the reservation call price in the sufficient range approximately equals to 0.4055%. While for the put option case, when the range is narrowed from $[-1, 4]$ to $[0.1, 1.2]$, the EU_{BM} is not changed but the reservation put price decreases 0.423%. The percentage change between the Black-Scholes put price and the reservation put price increases from 0.5852% to 1.0056% when the range of stock holdings is narrowed. Based on the above results, for both cases, with consideration of numerical errors, these reservation prices are acceptable. However, we suggest that for the put option case, it is better to use a sufficient range to obtain the reservation put price.

Table 5.5: Values of expected utility of terminal wealth and reservation prices in various ranges of stock holdings.

Option	Range of Shares	EU_{BM}	Reservation Price	Percentage Change
Call	$[-1, 4]^{\mathrm{S}}$	-1.0977	0.141723	0.4055%
	$[1.1, 2.2]^{\mathrm{N}}$	-1.0977	0.141723	0.4055%
Put	$[-1, 4]^{\mathrm{S}}$	-1.04284	0.0929528	0.5852%
	$[0.1, 1.2]^{\mathrm{N}}$	-1.04284	0.0925598	1.0056%

[S] Sufficient Range of Stock Holdings.
[N] Narrow Range of Stock Holdings.
The Black-Scholes call price is 0.1423 and the Black-Scholes put price is 0.0935.

Optimal Strategy

The following part is to discuss the optimal strategies for several particular stock price paths. We investigate the optimal strategies according to two typical stock price trajectories. We stimulate these stock price trajectories by a realization of the drift $\mu = 0.15$. Stock price path A is a trajectory of finishing out-of-money and stock price path B is a trajectory of finishing in-the-money. According to Table 5.5, when an investor trades with written one call option, the narrow range $[1.1, 2.2]^N$ obtains the identical expected utility of terminal wealth and identical reservation call prices. Therefore, we choose this shorten range of stock holdings to test the optimal strategies given by stock price path A and B. However, these optimal strategies by the shorten range of stock holdings are identical in this two examples. Therefore, it is meaningless to test the particular stock path trajectory since the sufficient range in particular trajectories is smaller than the narrow range for all trajectories in a very high probability. We only measure the choice of range of stock holdings in the expected utility of terminal wealth.

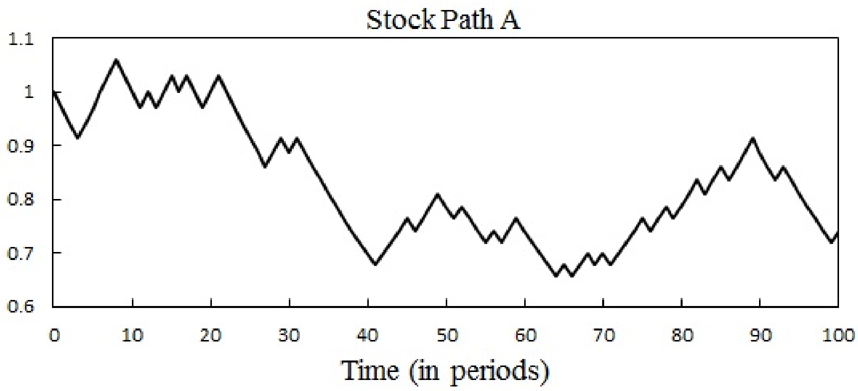


Figure 5.6: Stock Path A under the default parameters setting of Table 5.1.

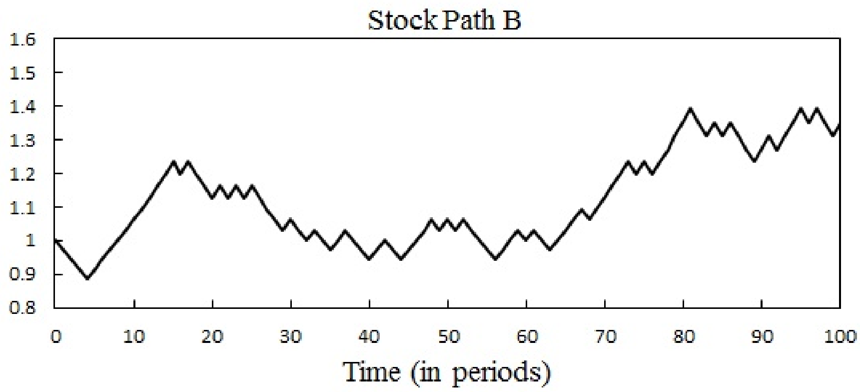


Figure 5.7: Stock Path B under the default parameters setting of Table 5.1.

5.2.3 Effect of the Number of Periods

The number of periods m will affect the C-R model in two factors: the size of the binomial model and the value of expected utility of terminal wealth. For the size of binomial model, the C-R model would be unrealistic if m is too small; while it will be quite time-consuming when m is too large. For the value of expected utility, with a larger number of periods, the C-R model would obtain a better performance on the expected utility of terminal wealth. With consideration both of these two factors, we need to find a suitable number of periods in this analysis.

We have simulated three models with different number of periods ($m = 50$, $m = 100$ and $m = 150$). This subsection shows the comparison of terminal wealth of these three models.

Histograms are traditionally used for density estimation. Using histograms allows us to provide the high quality of probability density estimation, specially in the case with long sequences. Figure 5.8 displays the histogram of terminal wealth in $10,000$ trajectories. The $100-$ period model and $150-$period model generate the approximate histogram rather than the $50-$period model. We provide the kernel estimator of the density function of the terminal wealth in Figure 5.9. The kernel estimators for $50-$period and $80-$period are clearly different. They have a tendency to move towards to estimators for $100-$period and $150-$period. Those densities both in the $100-$period and the $150-$period models are fitted very well. We conclude that choosing the $100-$period model as the default model in the numerical analysis rebalances the computation precision and the time consumption.

5.2.4 Comparison to Merton's Solution

After we fixed the default values of the model, the next step is to verify the correctness of the model. If we remove the liquidity cost from the C-R model, the wealth amount invested in shares would approximately equal to the solution from Merton (1969). This subsection explains the model of Merton (1969) without transaction cost and lists the comparison between wealth amount invested in shares from numerical solution and Merton's solution.

In the paper of Merton (1969), the dynamics of stock price follows a geometric Brownian motion. The drift μ and the volatility σ in Mertons model

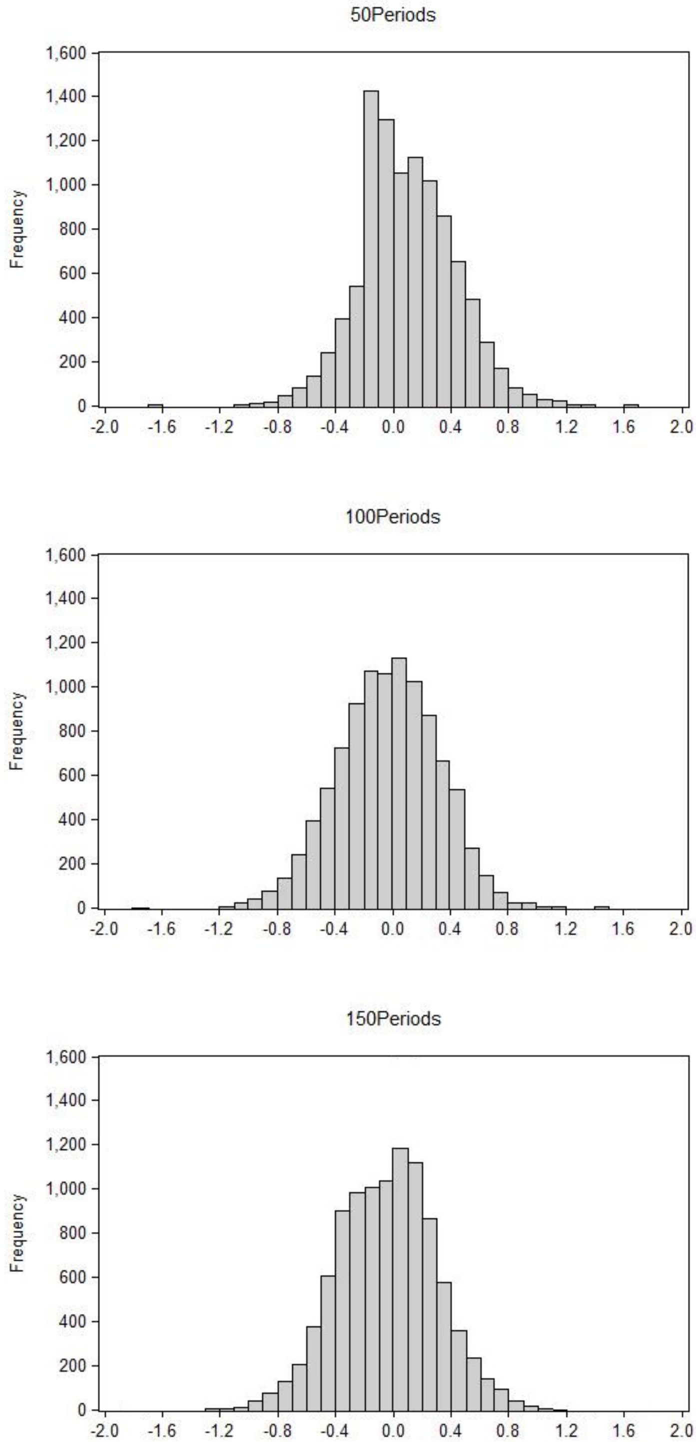


Figure 5.8: The frequency histograms for the terminal wealth in the 50−period, 100−period and 150−period models.

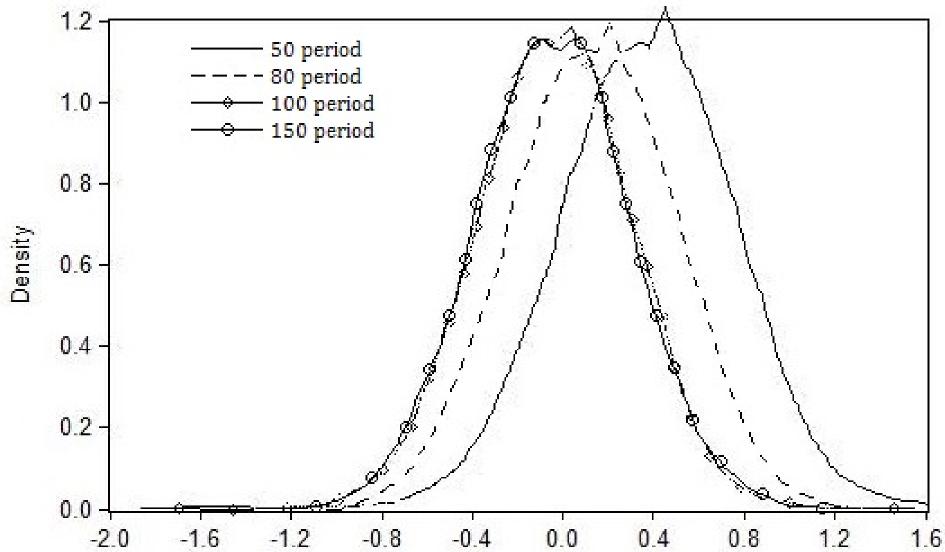


Figure 5.9: The density function for the terminal wealth (measured by the kernel estimator) in the 50−period, 80−period, 100−period and 150−period models.

which has full information are two constant parameters that characterize the dynamics of stock price. Merton's portfolio problem deals with an investor who must choose how to allocate his wealth between stocks and a risk-free asset in order to maximize the expected utility of the terminal wealth. When utility is given by $U(x) = -\frac{1}{\gamma}exp(-\gamma x)$, Henderson (2005) provides the explicit solution to the optimal portfolio (expressed in units of cash) for Merton (1969) problem is $\theta_t^M = \frac{\mu - r}{\gamma\sigma^2} \cdot exp(-r(T-t))$. We choose $\gamma = 1$ as the benchmark value in this chapter, hence the above solution can deal for our example utility $U(x) = -exp(-\gamma x)$.

Figure 5.10 shows the comparison of stock wealth between the numerical solution and Mertons solution, given by a particular stock path A (shown in Figure 5.6). We choose the default discretization step $\delta r = 0.002$ in the calculation of value functions that obtain a precise comparison. We apply the stock path A as the particular stock prices trajectory and generate the corresponding optimal

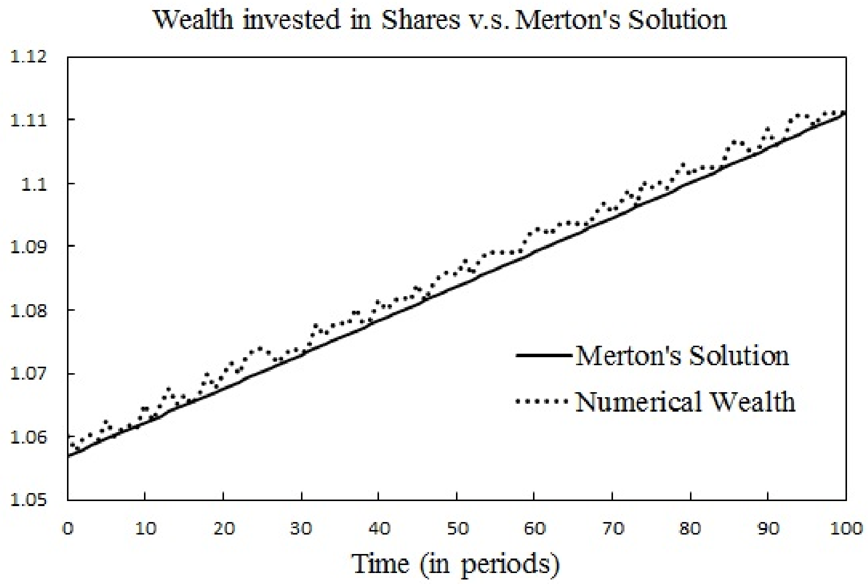


Figure 5.10: Comparison between wealth amount invested in shares over time (shown in dash line) and Merton's solution (shown in solid line) by stock price path A.

strategy by the Binomial Model algorithm. The wealth amount invested in shares over times (the dash line in Figure 5.10) is the product of the stock price A and the corresponding strategy values. From this plot, the difference between our numerical wealth and Merton's solution is quite minor. We provide the *root mean squared* (RMS) relative error ε as the error measure of the comparison of wealth amount. This RMS is calculated by Equation 5.1. The RMS relative error ε characterizes the computational error between the numerical solution (displayed in dash line in Figure 5.10) and Mertons solution to the optimal stock holdings (displayed in solid line in Figure 5.10).

The formula for RMS relative error follows as:

$$\varepsilon = \sqrt{\frac{1}{m}\sum_{i=0}^{m-1}(\frac{\tilde{W}_i - \theta_i^M}{\theta_i^M})^2}, \tag{5.1}$$

where the symbol $\tilde{W}_i$ is the wealth invested in shares over time t_i from the numerical results and θ_i^M represents the analytical result from Merton's solution in 1969. With this formula the RMS relative error for the average wealth amount invested in shares number is 6.8819×10^{-4}. This RMS value indicates that our numerical result is good enough as the approximation of Merton's solution in the perfectly liquid market.

In short, in this section we demonstrate the following points: first, there exists a sufficient range of stock holdings; second, shorten the sufficient range of stock holdings can decrease the expected utility of terminal wealth but enhance the computation efficiency; third, we also confirm the reasonable parameters in the default model; finally, we prove the correctness of the C-R model with comparing to Merton (1969).

5.2.5 Hedging Strategy

This section is concerned with computing the hedging strategy in the perfectly liquid market and compare it to the discrete delta hedge strategy. Hedging is a strategy designed to minimize exposure to the risk of the underlying asset. We define that in this paper a hedging strategy is the difference between the optimal strategy with and without an option. In order to compute the optimal strategy or the hedging strategy at a node of the binomial tree, we need to know the initial position in the risky asset before coming to that node. Intuitively, if investors know the initial number of shares at a node, they would know which path is followed up to that node. The delta hedge strategy in discrete version is the comparison of the hedging strategy. The key idea behind the Black-Scholes delta hedge strategy is to hedge perfectly the option by buying and selling the underlying asset and consequently eliminate risk in the continuous-time Black-Scholes model. In our case, the Black-Scholes strategy is considered as a strategy with European option on a non-dividend-paying stock. The formula of Black-Scholes strategy with call option is: $\Delta_c = \Phi(d_1(t))$, where symbol Φ represents the cumulative distribution function and $d_1(t)$ is defined as the following equation:

$$d_1(t) = \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

The formula of Black-Scholes strategy with put option is set as: $\Delta_p = \Phi(d_1(t)) - 1$. The range of Black-Scholes strategy with call is $(0,1)$, while the range of Black-Scholes strategy with put is $(-1,0)$. The Black-Scholes strategy is path-independent strategy: this strategy depends on the node of the binomial grid but not the path followed up to that node.

We select two particular stock price paths to produce the hedging strategy and the Black-Scholes delta hedge strategy. Stock price path A finishes out-of-money (shown in Fig.5.6) and stock price path B finishes in-the-money (shown in Fig.5.7). Figure 5.11 depicts the comparison between the hedging strategies without liquidity cost and the delta hedge strategy in discrete-time in these two paths. Both large panels display that the hedging strategy in the perfectly liquid market is very close to the Black-Scholes delta strategy. Regardless of the numerical error, we propose the hedging strategy in the perfectly liquid market is approximately identical to the Black-Scholes delta strategy. The corresponding small panels show the difference between the hedging strategy and the Black-Scholes strategy. Gokay *et al.* (2012) depict a different liquidity effect model (feedback

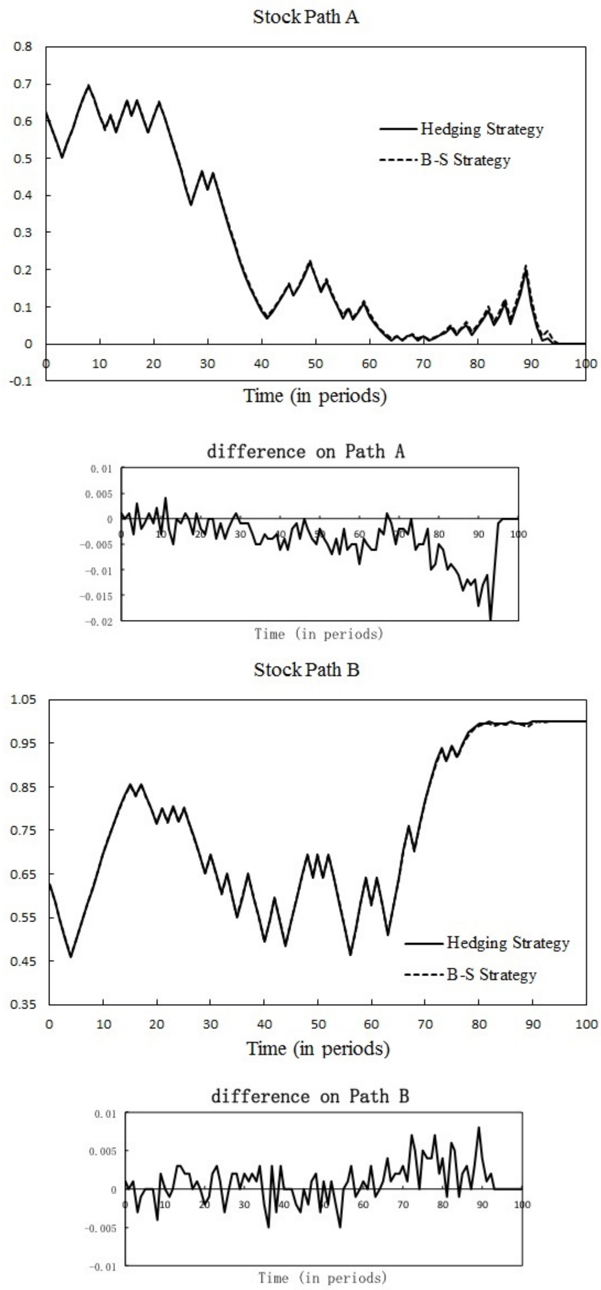


Figure 5.11: The top/bottom large panels are the comparison between hedging strategy without liquidity cost based on stock price path A or B (solid line) and Black-Scholes delta strategy (dash line). The small panels display the difference between the hedging strategy and the Black-Scholes strategy.

effect model) in a binomial structure and compare their optimal strategy to the Black-Scholes strategy as well. In this thesis, their optimal strategy also matches the Black-Scholes strategy well.

After demonstrate minor difference between the hedging strategy in the perfectly liquid market and the Black-Scholes delta strategy, we will discuss how the liquidity cost affects the hedging strategy by varying number of options in the next section.

5.3 Effect of Illiquidity on Hedging Strategies

In this section, we concentrate on comparing the hedging strategies with different liquidity costs in illiquid markets. The numerical analyses are illustrated by both of a three-period model and a $100-$period model in illiquid markets. The small model allows us to show all trajectories and how the illiquidity effect impacts them.

5.3.1 Three-Period Model

We discuss a three-period model followed by a non-recombining binomial tree in this subsection. This small number of period model is not enough to simulate changes of optimal strategy according to stock prices. However, it allows us to show all trajectories of optimal strategies and clearly illustrates how the liquidity cost affects the hedging strategy.

Figure 5.12 shows the non-recombining binomial tree of this three-period model. Subscripts in the stock price of each node mean the current period number and the price path: u indicates the price going-up and d indicates the price

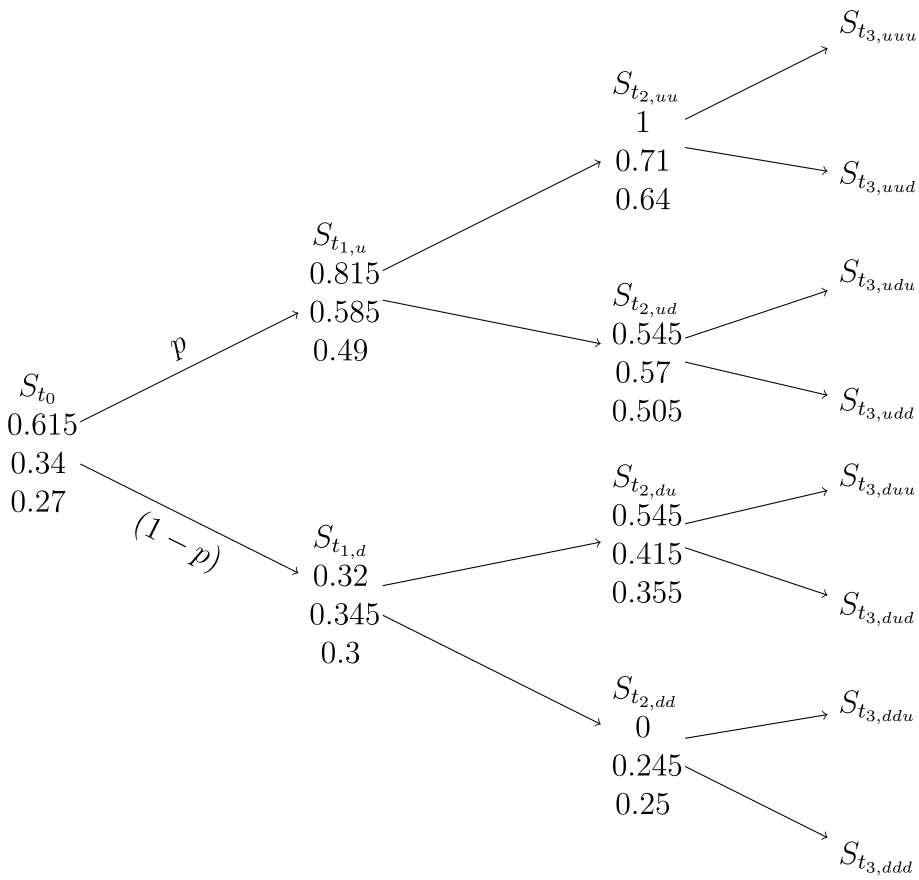


Figure 5.12: Three-period non-recombining binomial model shows all trajectories of hedging strategy: below the stock price of each node, the first value presents value of hedging strategy for $\alpha = 0$, the second one is the corresponding value for $\alpha = 0.05$ and the last value is the one for $\alpha = 0.1$.

going-down. Numbers below the stock price are the values of hedging strategy under the condition $\alpha = 0$, $\alpha = 0.05$ and $\alpha = 0.1$, respectively. We assume that investors liquidate their stock holdings at the terminal time. Hence, the path of optimal strategy would not be affected the stock price in the terminal period. We only concentrate on values of hedging strategy in the former 7 nodes (from period 0 to period 2 in Fig. 5.12). Based on Figure 5.12, we observe following phenomenons: first, values of hedging strategy at every node are not the same since the liquidity cost makes different strategies; second, no matter how much the liquidity cost in the market is, if the current stock price is bigger (smaller) than its successive nodes, the value of hedging strategy at the current node is bigger (smaller) than the value of its successive nodes; last, for nodes $S_{t_2,ud}$ and $S_{t_2,du}$ which have the identical stock price, they have same transactions in the perfectly liquid market but different transactions in the illiquid markets.

Table 5.6: Comparison of hedging strategies with short one European call option under $\alpha = 0$ (perfectly liquid market), $\alpha = 0.05$ (market with insignificant illiquidity effect) and $\alpha = 0.1$ (market with distinct illiquidity Effect) in three-period model.

	S_{t_0}	$S_{t_{1,u}}$	$S_{t_{1,d}}$	$S_{t_{2,uu}}$	$S_{t_{2,ud}}$	$S_{t_{2,du}}$	$S_{t_{2,dd}}$
$\alpha = 0$							
Strategy with Option	1.73	1.77	1.67	1.815	1.7	1.7	1.635
Strategy	1.115	0.955	1.35	0.815	1.155	1.155	1.635
Hedging Strategy	0.615	0.815	0.32	1	0.545	0.545	0
$\alpha = 0.05$							
Strategy with Option	0.915	1.355	1.265	1.5	1.485	1.425	1.375
Strategy	0.575	0.77	0.92	0.79	0.915	1.01	1.13
Hedging Strategy	0.34	0.585	0.345	0.71	0.57	0.415	0.245
$\alpha = 0.1$							
Strategy with Option	0.7	1.115	1.01	1.32	1.25	1.165	1.12
Strategy	0.43	0.625	0.71	0.68	0.745	0.81	0.87
Hedging Strategy	0.27	0.49	0.3	0.64	0.505	0.355	0.25

Table 5.6 lists the optimal strategy without option (named as 'Strategy' in table), the optimal strategy with option and the hedging strategy for all trajectories. We focus on two nodes $S_{t_2,ud}$ and $S_{t_2,du}$ which are identical on the stock price in Table 5.6. Focusing on the case in the perfectly liquid market $\alpha = 0$, those three types of strategies have the identical value between these two nodes. But values all change once applying the illiquidity effect into the model between these two particular nodes: the larger illiquidity effect α the more distinct change in values of strategies. The reason of the difference is that the values of strategies are decided by the stock price and the liquidity cost. In the perfectly liquid market, there is no liquidity cost, thus the strategy is controlled by the stock price only; in the illiquid market, both stock price and liquidity cost impact the strategy, hence the value of strategies in the identical stock price would be different from each other. For each node, the value of hedging strategy is decreased with the increasing liquidity cost. This is because the bigger liquidity cost raises the cost of each share. The data in the case of $\alpha = 0.1$ are all less than the corresponding values at the same node but with smaller liquidity costs. Those data in $\alpha = 0.1$ reveal that the illiquidity effect in the market reduces the share holdings at every trading moment and the change is not proportional. In a fixed wealth amount, the increased share cost means reduced transactions. That refers to lower values of hedging strategy.

The part above investigates the three-period model with shorting a single option. When the number of periods m and the number of option n change, the corresponding hedging strategies have to change. In the following subsection we will increase m and n to give a more practical model.

5.3.2 100 Period Model

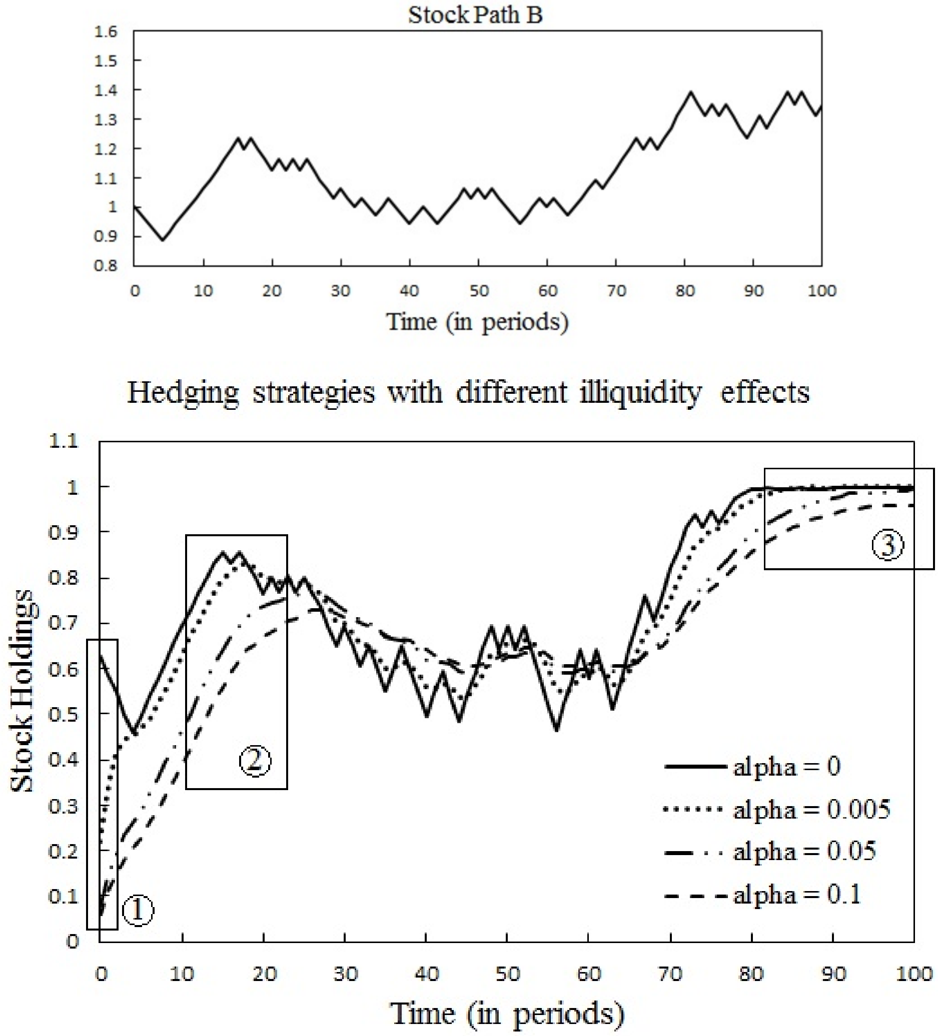


Figure 5.13: The stock holdings v.s. time (displayed in periods): the bottom panel shows hedging strategies with various liquidity costs based on the top panel stock price B

In this part we discuss changes of hedging strategies with different liquidity costs by stock price path B in a 100−period model. We produce corresponding hedging strategies based on different illiquidity effect parameter α. Figure 5.13 displays the stock price path B in the top panel and hedging strategies with various illiquidity effects in the bottom panel. The solid line in the bottom panel shows the difference of stock holdings between with and without option in perfectly liquid market, while fluctuations in the line response the ups and downs of the stock price B in the majority of the whole time, except the last periods. After the time period t_{80}, values of hedging strategy approximately stay at the upper limit 1 since the value of a option does not change more quickly than the value of the stock. The other cases (dash lines) of Figure 5.13 exhibit three levels of illiquidity effect into the original case. It illustrates that when we apply the liquidity cost into the model, the corresponding hedging (or optimal) strategy shows less fluctuations given by the same stock price changes and the lines are more smooth with larger value in α.

We check three parts which are drew in the rectangles in Figure 5.13 to characterise influences of liquidity costs into the hedging strategy. Part ① emphasises on the different initial stock holdings with various illiquidity effects. Applying the liquidity effect into the default model makes the initial stock holdings sharply decrease. The difference of the initial stock holdings between the case $\alpha = 0$ and $\alpha = 0.005$ illustrates the small liquidity effect is not trivial for the initial stock holdings rather than other periods of time. With respect to other bigger liquidity effect cases, the initial stock holdings keep reducing with diminished deceasing amount. This phenomenon shows the importance of the initial stock holdings in the market with illiquidity effect. Part ② depicts that the value of hedging strategy with larger liquidity cost is less than the value of hedging strategy with smaller liquidity cost. There are two reasons for this phenomenon: first, the liquidity cost makes the purchase less than the amount in perfectly liquid market since investors have to pay extra cost as the illiquidity effect; second, comparing to other markets, a market with larger α value shows a more significant lag of response of stock price movements. Referring to a fixed stock holdings, this market will need longer response time. Shown in Figure 5.13, the curve with larger α value stays at the lower position. Part ③ implies that the value of hedging strategy tends towards stability in the tail of the whole periods no matter how much the liquidity cost is. When increasing in the liquidity cost, the initial stock holdings are going close to zero and transactions in other periods are going to smooth over time. Figure 5.13 illustrates less stock holdings and transaction amount when the investor goes into a less liquid market.

We discuss the influence of the option on the hedging strategy in the multi-period model. We analyse how the number of options affects the hedging strategy given by the same liquidity cost consideration. Figure 5.14 displays the difference of hedging strategy between more options case and 1 option case under the liquidity effect $\alpha = 0.005$ and $\alpha = 0.05$, respectively. The number of call option in Figure 5.14, for instance 2 calls, denotes taking a short position of 2 calls in the hedging strategy. From the theoretical perspective, different number of options applying into the hedge would lead to different optimal strategies. It inspires us to check the hedging strategies with various number of options. In particular, hedging strategies with various number of options means that there are $n > 1$ options written in the hedging strategy and the responding hedging strategy is the difference of hedging strategy value, between value with n options and value with 1 option, divided by the number of options n. We use a terminology "n options solved" to represent the hedging strategies with various number of options. However, the hedging strategy per one option depending on 2 option solved and 5 options solved are quite close to each other that we cannot identify clearly. In

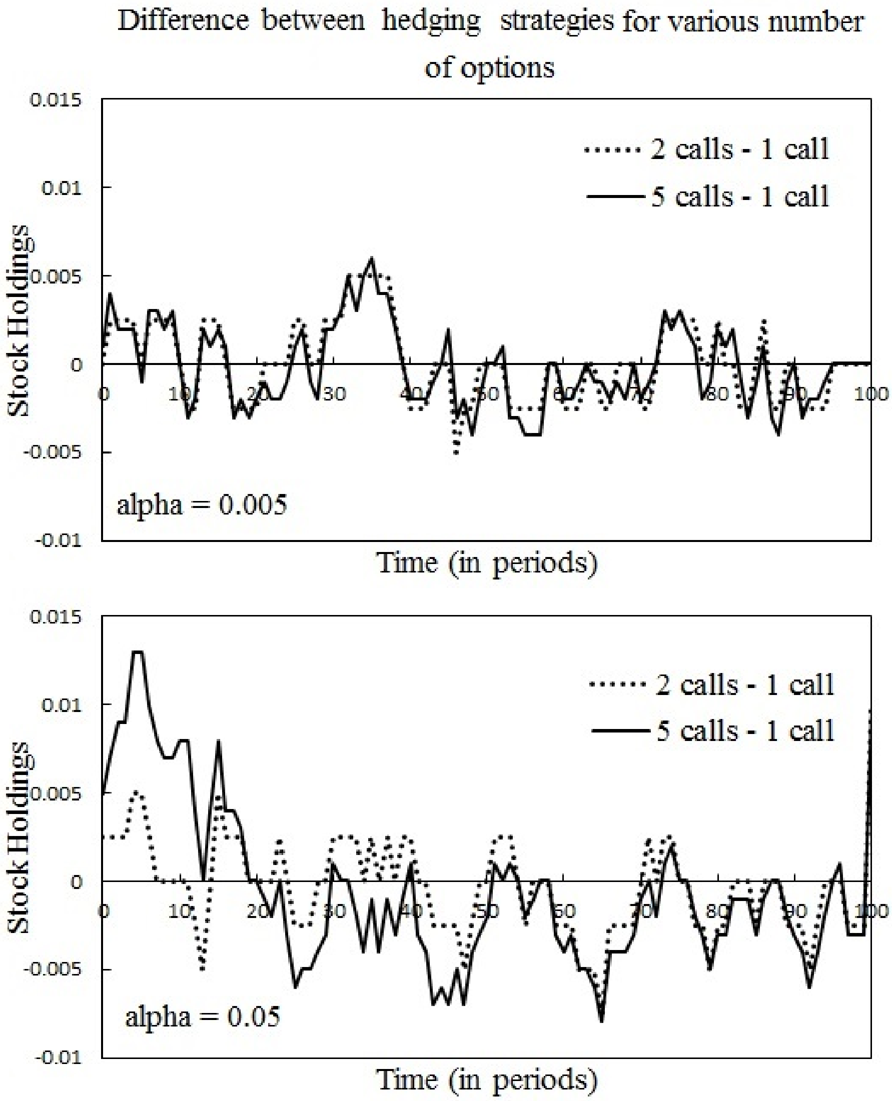


Figure 5.14: Differences of hedging strategies for various number of options: the annotation in the right top corner (e.g. 2 calls - 1 call) means the difference between the hedging strategy per one option depending on 2 options solved and the one depending on 1 option solved, other annotations follow the previous rule. The top panel shows examples in the case of $\alpha = 0.005$ and the bottom top shows the case of $\alpha = 0.05$.

order to display this difference, in Figure 5.14 we show the difference of hedging strategy per one option depending on positive options solved and the one depending on 1 option solved. The amplitudes (representations of the difference above) keep changing over time in both panels of Figure 5.14. The maximal value of amplitude in the top panel is around 0.005. This implies changing the number of options takes little influence on the hedging strategy in the market with small liquidity effect $\alpha = 0.005$. The bottom panel shows a comparison with same parameters setting but the illiquidity effect α expanding 10 times. Due to the increase of illiquidity effect, the bottom panel clearly displays that the amplitude of hedging strategies is much larger than the comparison in the top panel. We continue to check the difference of hedging strategy with larger illiquidity effect parameter, and we found that the amplitude would be larger for larger α. It illustrates that changing the number of options obvious affects the hedging strategy in an illiquid market, while in a liquid market such an impact is insignificant.

In summary, the liquidity cost impacts the hedging strategy in three factors: the value of hedging strategy, different transactions on the identical stock price and the lag response of the stock price movement.

5.4 Analysis of Hedging Strategies and Reservation Prices

We conduct comparative analysis to study the characteristics of the optimal strategy with liquidity costs. By using some stock paths, we illustrate the effects of five market parameters and the illiquidity effect parameter α on different optimal portfolio strategies. We examine the performance of the reservation price by changing market parameters in the illiquid market. In order to isolate and distinguish the individual effect of these parameters on the stability of the model, each of these parameters will consider separately.

5.4.1 Initial Portfolio

Some paper on portfolio selection with liquidity constraint have demonstrated that the illiquidity in the market can be regarded as a shadow cost in the whole trading (for example, Longstaff (2001), Cetin & Rogers (2007) and Rogers & Singh (2010)). The most direct way to identify how illiquidity affects portfolio decisions is to compare the investor's initial portfolio when there are restrictions to the initial portfolio chosen in the absence of restrictions. Because the investor in real market is not constrained in the choice of the initial portfolio; the illiquidity only takes influence on rebalancing the portfolio subsequently. By choosing the initial number of shares, investors can control the dynamics of wealth which to maximize expected utility. Due to adoption of negative exponential utility function, we analyse the optimal strategy and how reservation prices depend on initial portfolio in the illiquid market.

We first consider the hedging strategy in the perfectly liquid market. Intuitively, investors with different initial portfolios would do identical strategies in the market without frictions. It is not necessary to analyse the hedging strategy with different initial portfolios in the perfectly liquid market. But we check the difference between the hedging strategy at the zero initial number of shares and the theoretical replicating strategy. Figure 5.15 displays the difference between hedging strategy and replicating strategy. The gap between them goes narrower over time.

In order to check whether the initial portfolio indeed changes the optimal

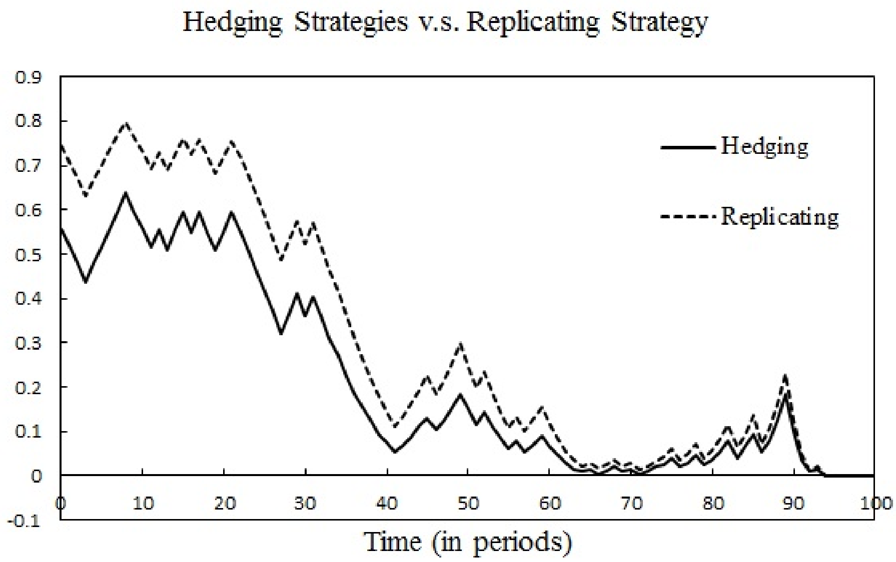


Figure 5.15: Comparison between the hedging strategy for 1 call option with strike $K = 1$ and theoretical replicating strategy. Graph for the Stock Path A (other parameters given in Table 5.1).

strategy, we apply two levels of illiquidity effects $\alpha = 0.05$ and $\alpha = 0.1$ into the default model and display the corresponding hedging strategies with different initial portfolios in Figure 5.16. As Figure 5.16 shown, investors with different initial portfolio condition would be willing to execute almost identical hedging strategy at the same time period. However, almost identical means some differences existing and those differences are slightly extended by increasing in the illiquidity effect. Various examples of stock paths reveal that the hedging strategy changes due to different liquidity effect parameter and other market parameters rather than the initial portfolio. Nevertheless, investors must have different optimal strategies if they obtain different initial portfolios. We continue to investigate the optimal strategy without option and the optimal strategy with option.

Figure 5.17 lists two panels which are optimal strategy without and with option in the market with $\alpha = 0.05$. First, we observe that optimal strategies

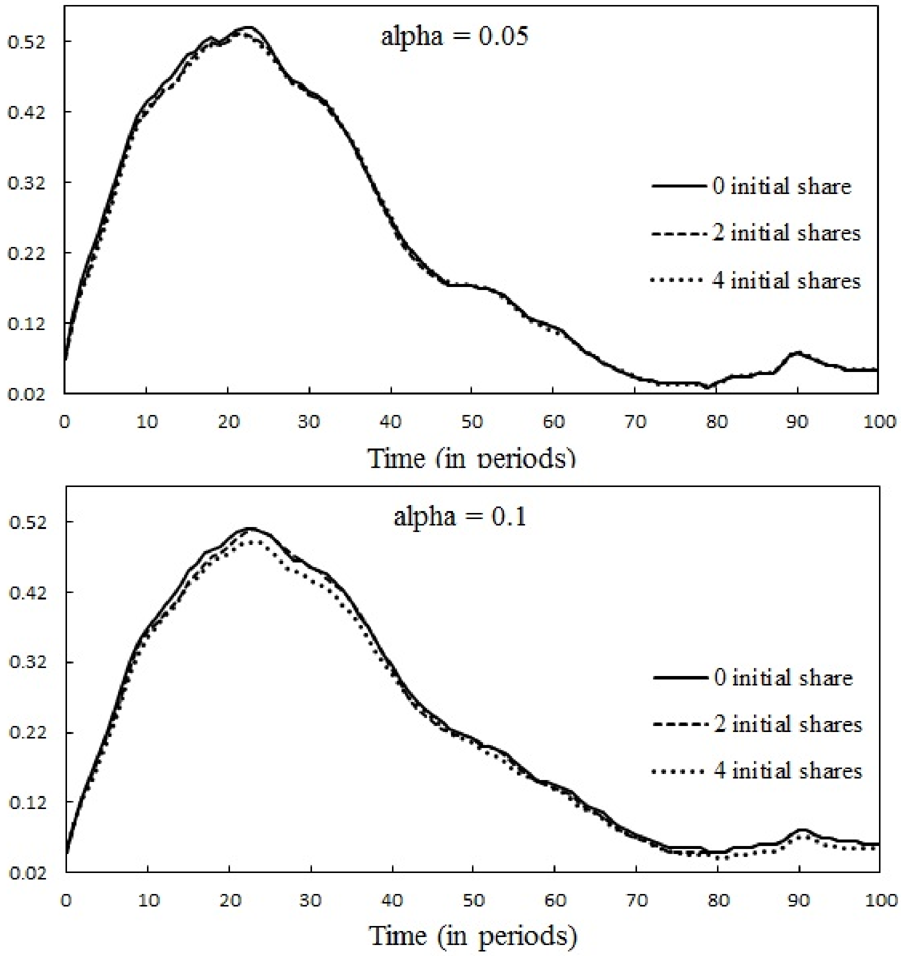


Figure 5.16: The hedging strategy for one call option with strike $K = 1$ versus time (calculated in periods) for different initial stock holdings. Graph for the Stock Path A (other parameters given in Table 5.1).

without option in different initial number of shares hold very different amounts of shares in the beginning of the whole trading period (e.g. the first 10 periods). These gaps among them are sharply reduced and tracing to be identical afterwards. The same observation happens in the optimal strategy with option. This observation reveals that the initial portfolio (e.g. the initial number of shares in the negative exponential utility function) indeed affects investors' trading strat-

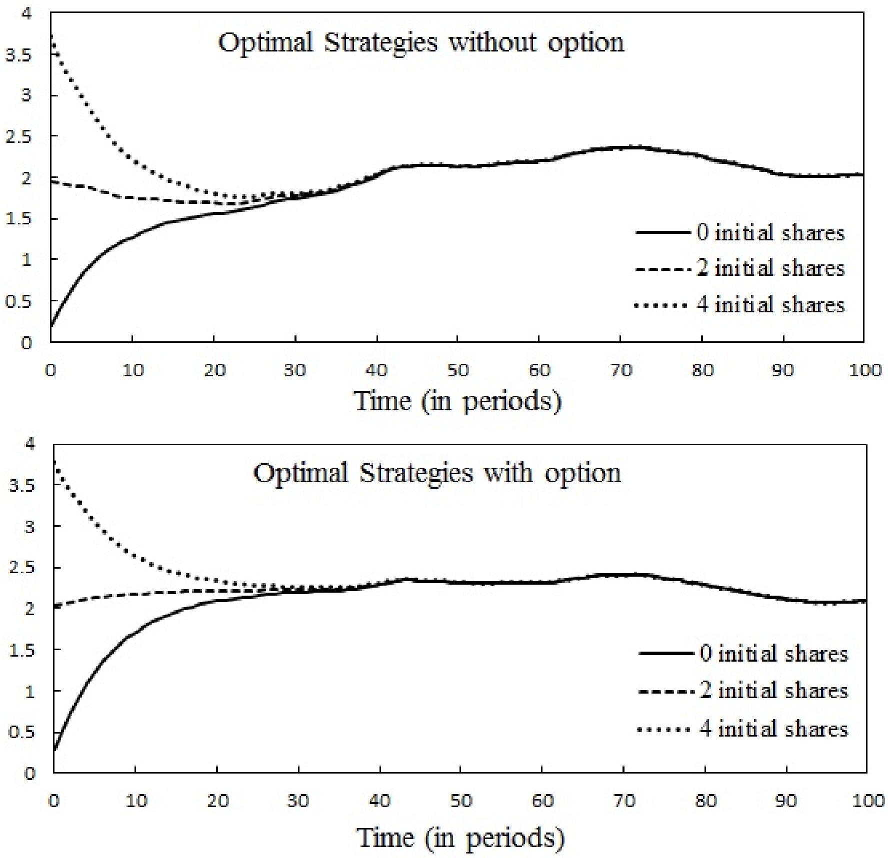


Figure 5.17: The stock holdings versus time (calculated in periods) changing on initial stock holdings by Stock Path A in the market with $\alpha = 0.05$ (other parameters given in Table 5.1).

egy, especially in the beginning of trading.

The next investigation is how the reservation prices depend on the initial portfolio. We proceed by defining two different portfolio optimization problems: the first problem deals with an investor who trades in the market for the stock account and the money market account, and who holds a long position of 1 call option; the second one is for the investor who trades in the stock and the bond, and who holds a short position of 1 call option. Those options in a long/short position have the same exercise price K and time to expiry T.

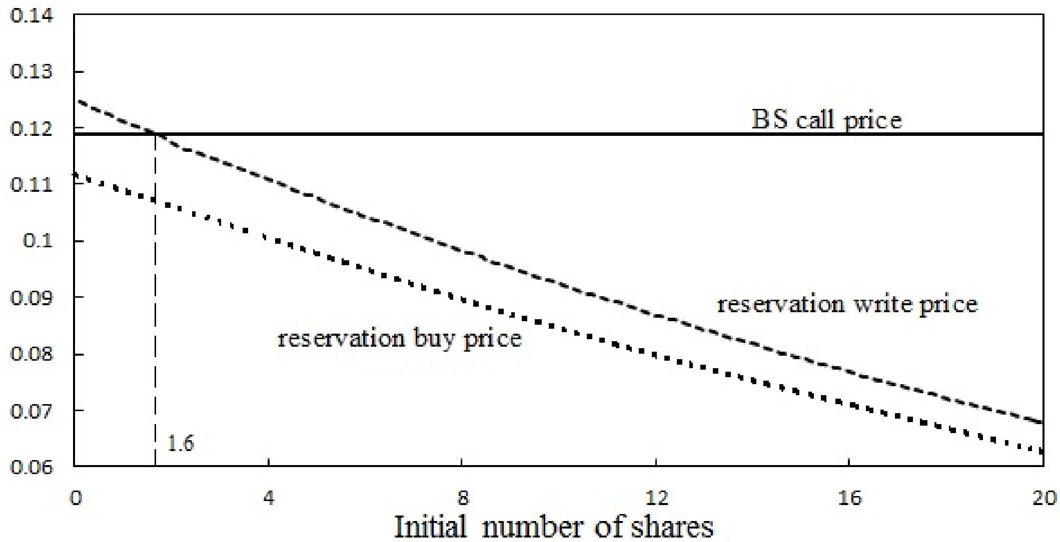


Figure 5.18: Reservation buy/sell prices of a European call option as functions of initial portfolio in the market with $\alpha = 0.05$ (shown in dashed lines) and the Black-Scholes call price in the perfectly liquid market (shown in solid line).

The reservation prices in the perfectly liquid market would be identical and independent of the initial number of shares. We analyse the reservation buy/sell price in the market with illiquidity effect. Figure 5.18 illustrates how the convergence pattern of reservation prices depend on the increase of the initial number of shares with $\alpha = 0.05$ in the negative exponential utility function. The Black-Scholes call price is the theoretical price of call option in the perfectly liquid market. We observe the reservation buy and write price are decreasing functions of the initial portfolio. Notice that, for very low levels of initial portfolio where the reservation write price (considered in the option writer's problem) is higher than the Black-Scholes call price without illiquidity costs, whereas for high levels initial portfolio approaches to horizontal axis. The line of reservation buy price shows that the investor hedges against the risk (e.g. here the risk aversion coefficient $\gamma = 1$) arising from holding the call option. However, the illiquidity cost makes hedging costly and consequently the investor deducts the hedging costs from the price he is willing to purchase the option. The reservation write price has the similar intuition for its decreasing function against the initial portfolio. In Figure 5.18, the gap between reservation write and buy price is the bid-ask spread. Damgaard (2006) examines the reservation prices with proportional transaction costs not the illiquidity costs and get the similar convergence pattern of the reservation prices against the initial portfolio. We focus on the option writer's problem in the hedging strategy and reservation prices in the following contents referred to other market parameters since writing a call option serves as a substitute for selling the stock.

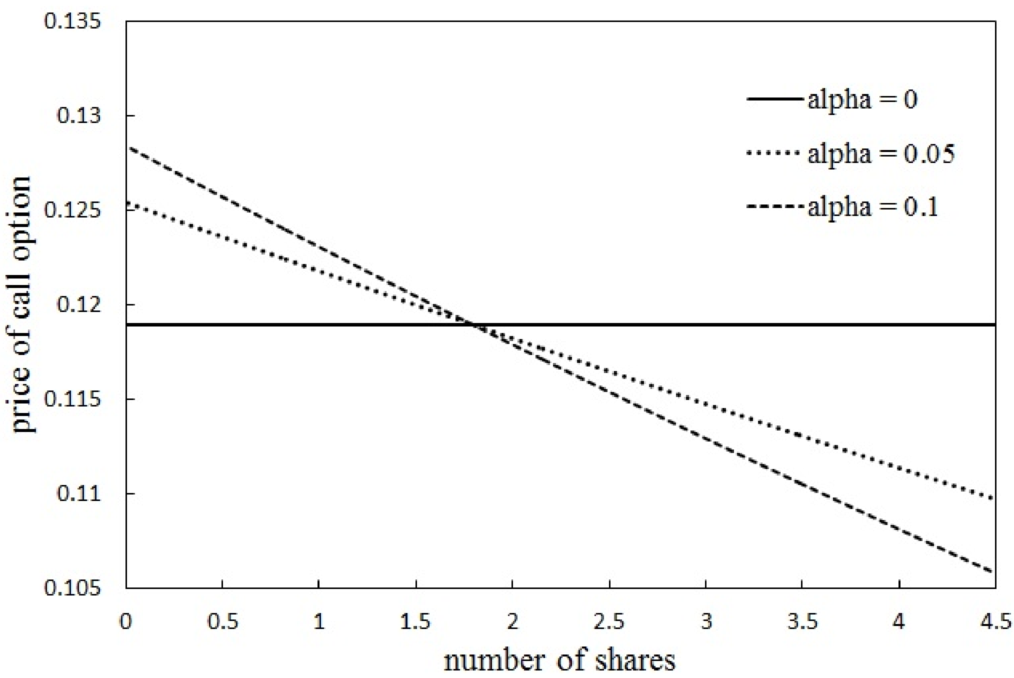


Figure 5.19: Reservation sell prices of a European call option as functions of initial portfolio with different illiquidity effects (shown in dashed lines) and the reservation price in the perfectly liquid market (shown in solid line).

The C-R model considers small investors since this model belongs to the temporary impact model which implies the trading amount would not take influence on the market price. Those large investors exactly affect the price when they trade. Another type of model named the feedback effect model is used for the large investors. Therefore, we only focus on a narrow range of initial portfolio to discuss the reservation price. Figure 5.19 displays the reservation write price on a small range of initial number of shares $[0, 4.5]$. We find that the reservation price at the zero initial number of share would be increasing with α. Reservation prices in the market with liquidity effects are all greater than the reservation prices in the perfectly liquid market. Once the initial number of shares increased, these prices of call option keep reducing and definitely cross the straight line of option price with $\alpha = 0$. Although these lines of option prices decrease linearly based on Figure 5.19, Figure 5.18 reveals the convex shape of option prices in illiquid markets. A very interesting observation in Figure 5.19 is that lines of option price with different liquidity effects seem to intersect the default line of option price in the perfectly liquid market at the same position. This explores that whether their intersections are identical and why there exists an extraordinary position where option price is identical no matter how much the illiquidity effect in the market. We will particularity study the reason that causes an intersection between the illiquid market and the perfectly liquid market in Section 5.5.

5.4.2 Risk Aversion

The C-R model characterises an investor's risk preference by a utility function with a risk aversion parameter. In this study we consider a negative exponential utility (CARA). Investors with CARA (constant absolute risk aversion) utility make the same decision about optimal stock position irrespective of wealth levels. The absolute risk aversion coefficient γ affects the investor's expected utility and optimal strategy. Investors with a higher risk aversion parameter γ allocate less wealth to stocks. In this subsection, we analyse the impact of risk aversion coefficient on the optimal strategy or the hedging strategy. We also examine the option price in the function of initial stock holdings under different liquidity costs and risk aversion coefficients.

We investigate the effect of increasing risk aversion on reservation price in a $100-$period economy for a trader with one short European call option. We pick the stock path B which is a stock trajectory finishing in-the-money and produce several hedging strategies in perfectly liquid market. We select three different values of risk aversion γ that equals to 1, 2 and 4, respectively. As shown in Figure 5.20, these corresponding hedging strategies are identical.

We set the solid line ($\gamma = 1$) in Fig. 5.20 as the default one and compare this line to the dash lines with higher risk aversion coefficients. This graph illustrates that hedging strategies with different risk aversion coefficients are almost identical over time. The difference among these trajectories of hedging strategy is caused by the numerical errors. Thus we could consider the hedging strategies with different risk aversion values are identical in the perfectly liquid market. In financial market, there exists two kinds of strategies: optimal strategy for the case without option, we name it *strategy* ⓐ; and replicating strategy for the option which can exactly replicate *strategy* ⓐ, we call it *strategy* ⓑ. In the perfectly liquid market, the utility corresponding to *strategy* ⓐ without option must be equal to the utility corresponding to the sum of *strategy* ⓐ and *strategy* ⓑ with

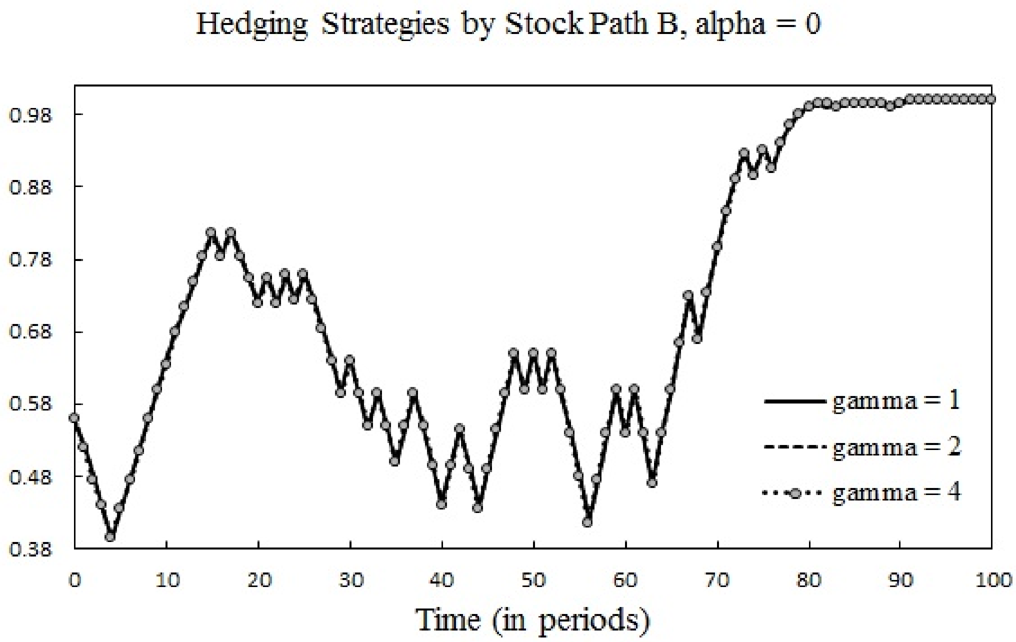


Figure 5.20: The stock holdings versus time (calculated in periods) changing on risk aversion coefficient, without liquidity cost, followed by Stock Path B (other parameters given in Table 5.1).

option. Intuitively, the sum of *strategy* ⓐ and *strategy* ⓑ totally remove the option effect on the optimal strategy. Hence, the hedging strategies with different risk aversion should be the same. It reveals that investors with the exponential utility make identical decisions irrespective of their risk aversion in the perfectly liquid market.

The following discussion is the influence of risk aversion parameter on the hedging strategy in the market with illiquidity effect. Once applying the illiquidity effect into the previous hedging strategies in Figure 5.20, those strategies would not be identical any more. As Figure 5.21 shown, all curves have more smooth fluctuations than those in the perfectly liquid market since investors with the same initial wealth have to pay the liquidity cost in the transactions. However,

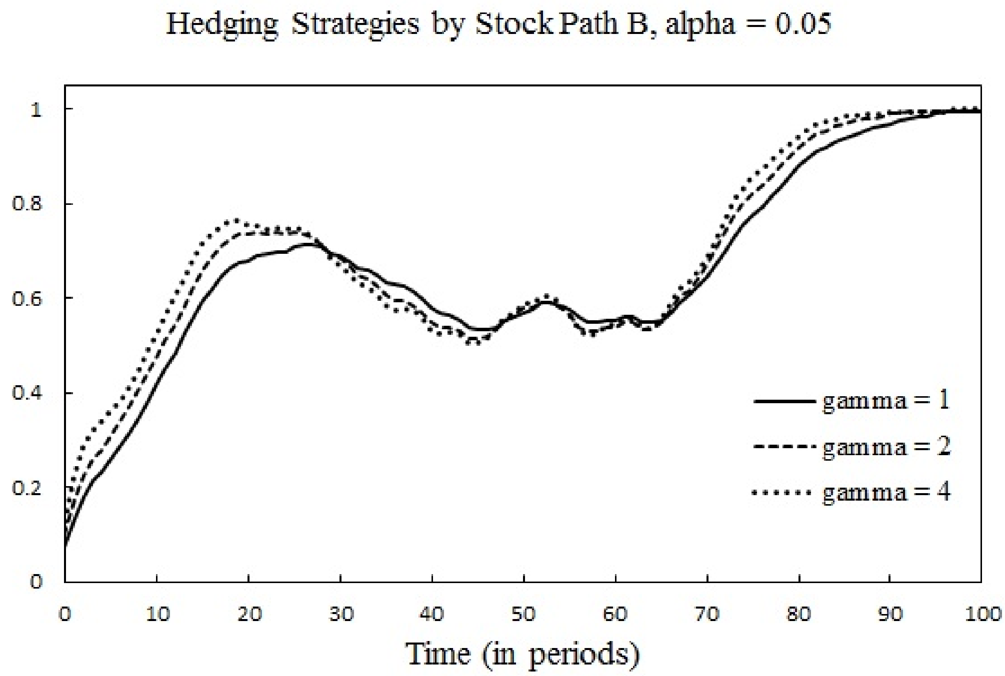


Figure 5.21: The stock holdings versus time (calculated in periods) changing on risk aversion coefficient, with liquidity effect $\alpha = 0.05$, followed by Stock Path B (other parameters given in Table 5.1).

investors with different risk tolerance levels trade different amount at the same time period. Investor with higher risk tolerance level (less value in γ) would hold less shares in the portfolio with short call option.

Many papers concentrate on the study of reservation price, especially in the CARA utility. When the risk aversion parameter goes into infinity, the reservation price tends to the superreplication price (the superreplication price means the price of hedging the option without any risk, quoted from Carassus & Rasonyi (2011)). Rouge & El-Karoui (2000) prove the convergence of reservation price with infinity risk aversion to the superreplication price; Carassus & Rasonyi (2011) built up the numerical applications in continuous-time model. Nevertheless, all these papers only consider the reservation price without liquidity costs.

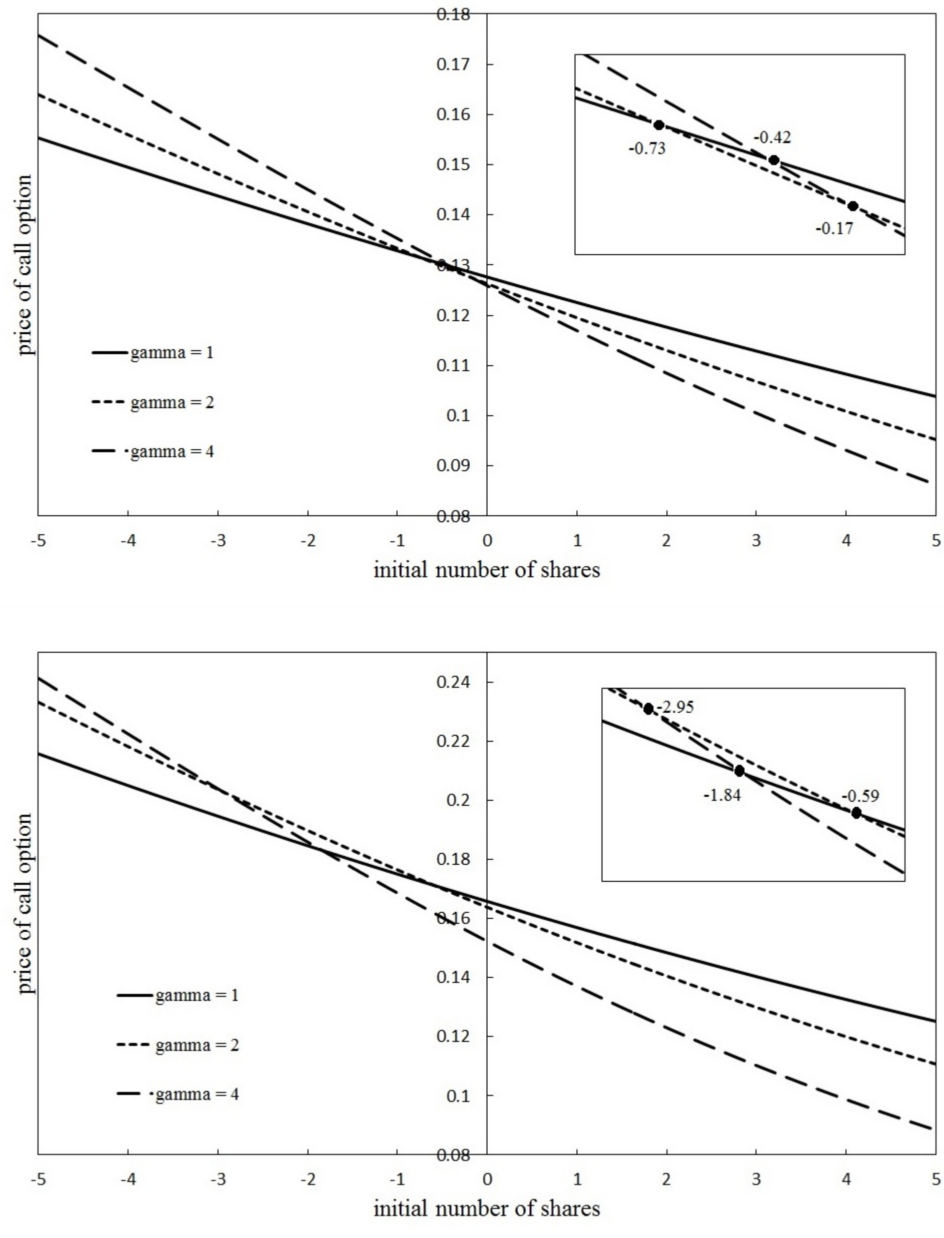


Figure 5.22: The price of call option versus the initial number of shares: The upper panel displays lines of option price with the liquidity effect $\alpha = 0.05$ and the lower panel shows lines of option price with the liquidity effect $\alpha = 0.1$ in 100−Periods Model (other parameters given in Table 5.1).

In this study, we analyse the reservation prices by changing the risk aversion levels in the illiquid market. Figure 5.22 displays lines of call option price under three risk tolerance levels. These two panels present changes in option price under $\alpha = 0.05$ and $\alpha = 0.1$, respectively. Concentrating on the case with the default risk aversion value $\gamma = 1$, the reservation price at zero initial stock holdings with the illiquidity effect $\alpha = 0.05$ is larger than the corresponding one with $\alpha = 0.1$. The lines of option price with different risk aversion parameters in the top panel intersect each other in a tiny area and the zoom-in box shows the exact intersections at the initial shares levels. Once applying a bigger illiquidity effect into the corresponding model, i.e. the illiquidity effect α increased from 0.05 to 0.1, we clearly see lines of option price would cross at very different initial shares levels (displayed in the bottom panel of Figure 5.22). The larger illiquidity effect the more distinct differences of intersections at the initial stock holdings levels. We will discuss more details about how lines of option price against the initial shares intersect each other and the corresponding financial contribution in Section 5.5.

5.4.3 Strike Price

In finance, the strike price of an option is the fixed price at which the option owner can buy (in a call option) or sell (in a put option) the underlying security. This subsection discusses the optimal strategy and reservation price for a trader writing one European call option. We select three different values of strike price K and display the corresponding hedging strategies by stock path B in Figure 5.23. The stock path B (shown in the top panel of Fig. 5.13) is a stock price trajectory finishing in-the-money. The terminal stock price in stock path B is

$S_T = 1.3$ so we select three other different strike prices which present in-the-money $K = 1.2$, at-the-money $K = 1.3$ and out-of-money $K = 1.4$ besides the default strike price $K = 1$, respectively, to check the change of hedging strategy.

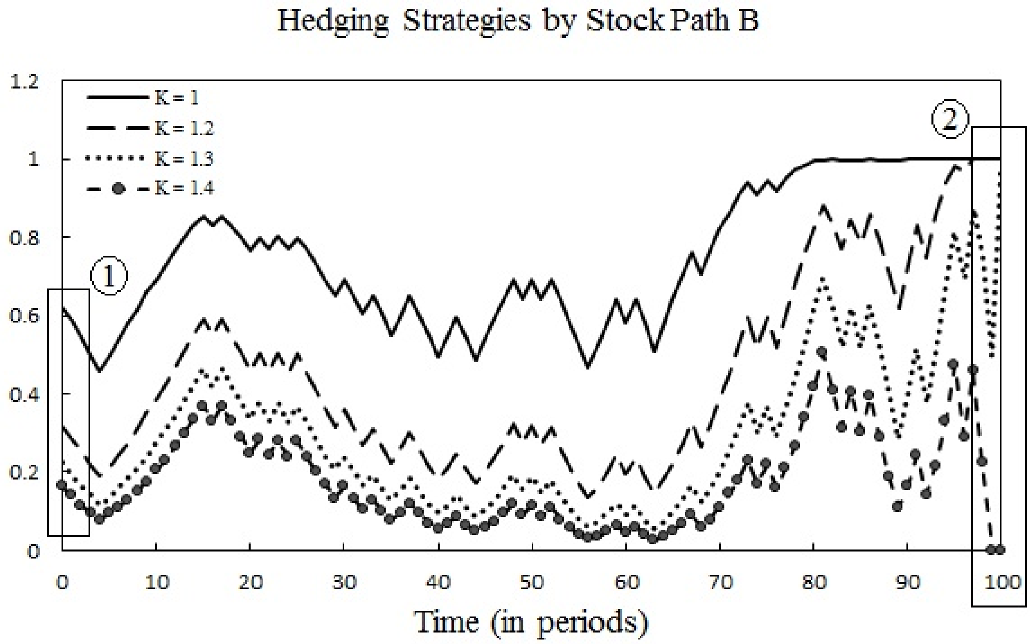


Figure 5.23: The stock holdings versus time (calculated in periods) changing on strike price, without liquidity cost, followed by Stock Path B (other parameters given in Table 5.1).

We explore two special parts which are drew in the rectangles in Figure 5.23 to characterise strike price on the hedging strategy. Part ① concentrates on the difference of the initial stock holdings in curves with different strike prices. It shows that a lower strike price in a portfolio with shorten one call option makes the investor hold more shares over time. The different values of the initial stock holdings disclose that the strike price is an important factor on the hedging strategy. Part ② highlights the stock holdings at the terminal time t_{100}. The significant differences of stock holdings happen at the final periods. Cases with the option in-the-money (e.g. $K = 1$ and $K = 1.2$) and at-the-money ($K = 1.3$)

would tend to the limit 1 since these trajectories can be exercised at the maturity date. However, the hedging strategy with the option out-of-money has more fluctuations at the final periods and tend to zero at the terminal time.

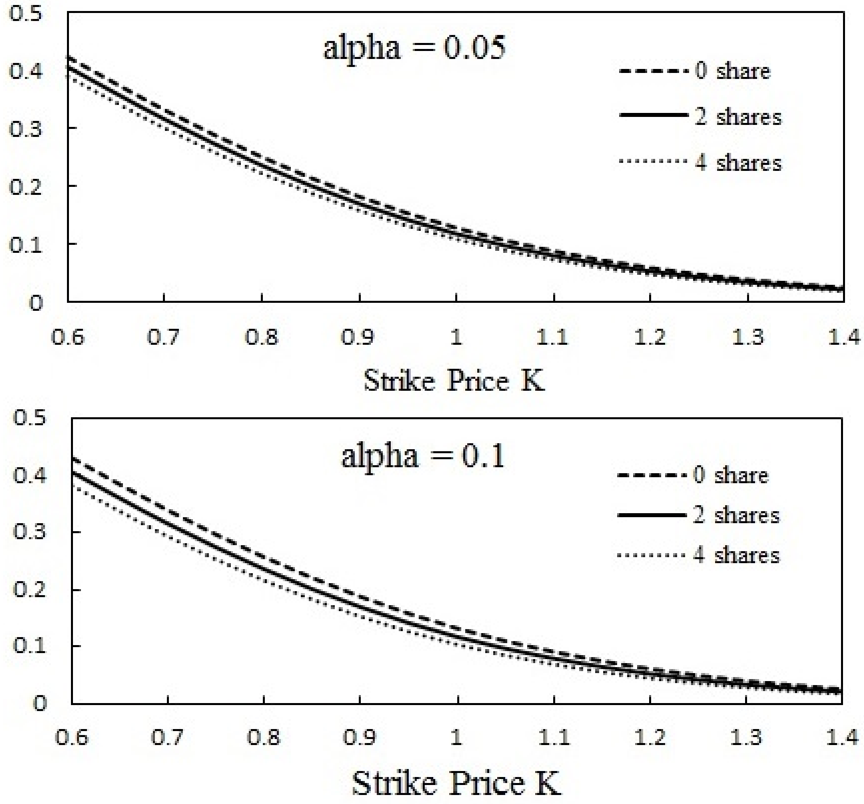


Figure 5.24: The price of call option versus the strike price K: panels display comparison of option price in different initial number of shares, with liquidity effect $\alpha = 0.05$ and $\alpha = 0.1$, respectively (other parameters given in Table 5.1).

The following investigation is the influence of strike price on the reservation price in the market with illiquidity effects. We test the illiquidity effect on the option price against various K. Figure 5.24 depicts lines of option price against the strike price under different initial shares. The top panel with less illiquidity effect $\alpha = 0.05$ shows narrower gaps among these lines of option price than the bottom panel in Figure 5.24. During the increase in the strike price, lines in both panels would be folded together and it is hard to identify the difference.

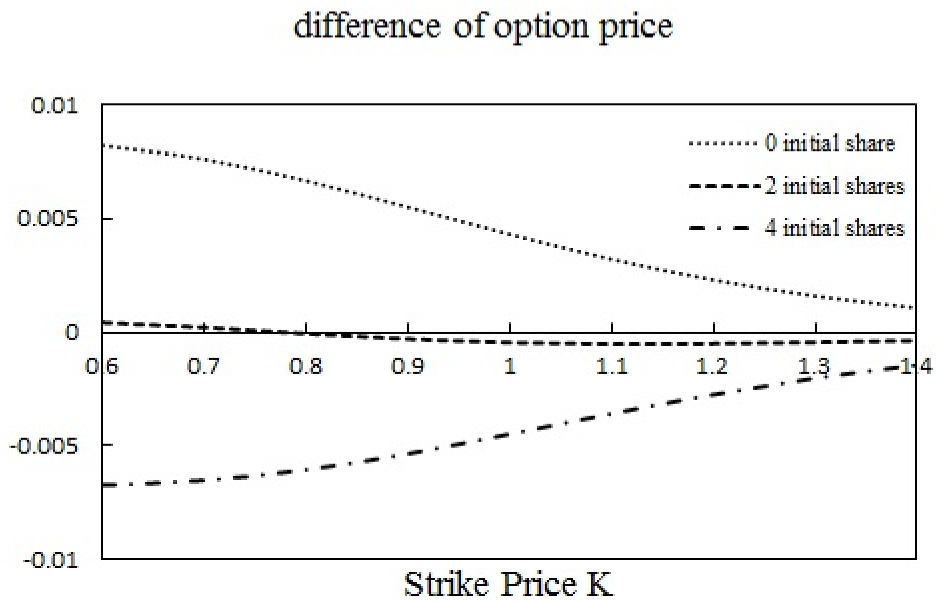


Figure 5.25: The difference of option price versus the strike price K with different initial number of shares. Other parameters are given in Table 5.1.

Then we plot the difference of option price under each initial portfolio condition between $\alpha = 0.05$ and $\alpha = 0.1$ in Figure 5.25. This figure illustrates trivial changes between the hedging strategies with illiquidity effect $\alpha = 0.05$ and $\alpha = 0.1$, no matter how much initial wealth the investor holds. Once the option is deep-out-of-money, the change of option price in different illiquid markets will tend to zero.

To sum up, we conduct the influence of strike price on the optimal strategy and the option price in this subsection. Because the strike price decides whether or not the option is exercised in the portfolio, it affects the optimal strategy indeed. In the study of option price, the price of call option is reduced by the increasing strike price no matter how much initial shares the investor holds.

5.4.4 Time Horizon

In finance, the price of an option has two key components: intrinsic value and time value. Intrinsic value of an option is the difference between the market price of the underlying security and the strike price of the option. It is usually called as the payoff of the option. The strike price K has been discussed in previous subsection. Numerically, time value is decided by the time to expiration and the option's volatility of the underlying security. We discuss the effect of time horizon T (time to expiration) in this subsection. The option's volatility is investigated in Section 5.6.

Generally speaking, long-maturity calls typically have more value than short-maturity calls since there is more time to have transactions that can occur to make them go in the money. For investors holdings the same initial wealth, different time horizons indicate strategies with different time lengths. It is meaningless to compare these hedging strategies. Therefore we examine the expected utility of terminal wealth with different liquidity effects. We compute the values of EU under different maturities, which perform changes of EU affected by time horizon T. Many papers quantify the performance of optimal strategy via maximal utility of terminal wealth or minimal loss in utility. Rogers (2001) discusses the wealth problem of maximizing the expected utility of terminal wealth in power utility form (CRRA utility). He defines the quantity of *efficiency* by comparing the expected utilities and illustrates that investors require longer time horizons of price data to obtain better performance of optimal strategy. The concept of *efficiency* in Rogers (2001) is defined as the difference of expected utility between complete and incomplete markets. Brendle (2006) also expresses the loss of utility both in negative exponential utility and power utility functions. We apply the illiquidity effect into the model and concentrate on the maximal expected utility of terminal wealth.

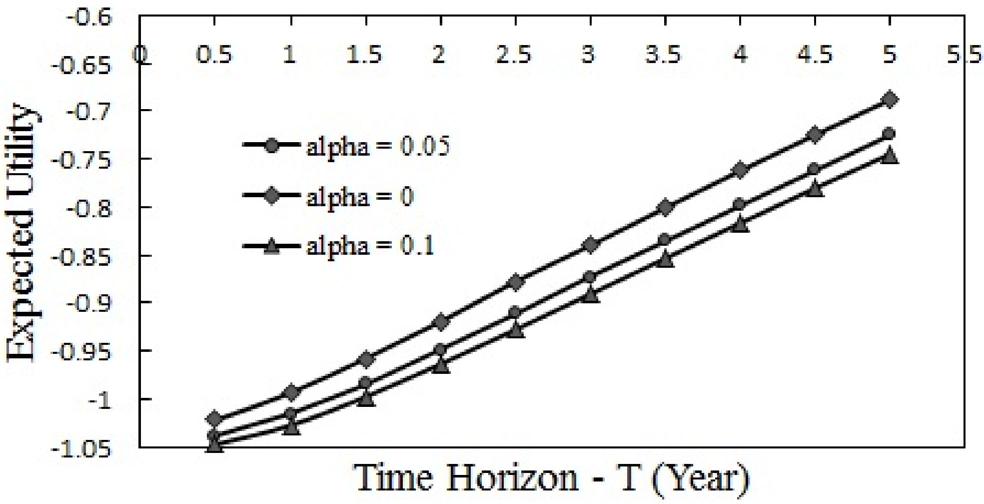


Figure 5.26: The expected utility of terminal wealth versus time to maturity T with the liquidity effect $\alpha = 0$, $\alpha = 0.05$ and $\alpha = 0.1$ (other parameters given in Table 5.1).

Figure 5.26 depicts the expected utility against the time length T. These go-upward values in expected utility reveal that investors obtain greater values of expected utility by increasing time horizons. Moreover, the bigger illiquidity effect α the smaller value in expected utility of terminal wealth since investors must pay extra costs as the illiquidity effect during the whole time length. To sum up, a long-term time horizon is beneficial to investors on wealth problem.

The next study in time horizon T concentrates on the price of call option with different initial number of shares in illiquid markets. As Figure 5.27 shown, prices

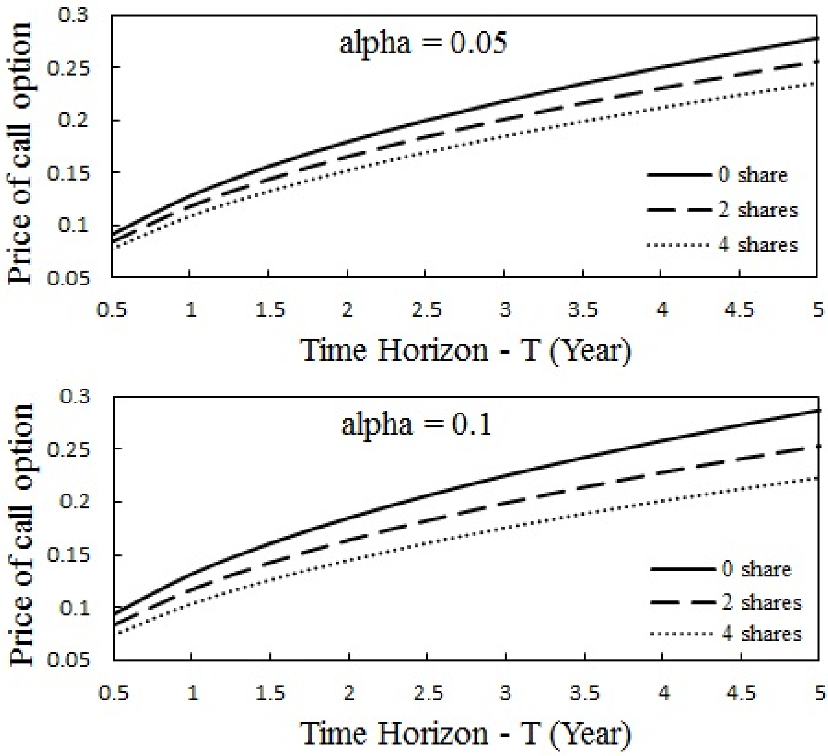


Figure 5.27: The price of call option versus time to maturity T: panels display comparison of option price in different initial number of shares, with liquidity effect $\alpha = 0.05$ and $\alpha = 0.1$, respectively (other parameters given in Table 5.1).

of call option tend to increase with expanding time horizons. This figure displays option price with time horizon from 0.5 to 5 years. If we apply a bigger number of time horizon, for example $T = 10$, the concave increasing tendency of the option price should be easier illustrated. The two panels with different illiquidity effect in Figure 5.27 both depict that investors with more initial stock holdings would afford the call option in lower price. The opposite situation happens at the case with investor written put options. Gaps among option price lines would go wider with longer time horizons, especially in less liquid market. It discloses that the larger the illiquidity is, the bigger the benefit of shorting call option might be. This phenomenon can be explained by the stock positions of the financial institution and the illiquidity on the market.

In short, an option portfolio with longer time horizon can produce higher expected utility of terminal wealth. The option price keeps increasing with longer time length and the illiquidity effect would amplify these increases. We conclude that long-term time horizon is beneficial to the portfolio performance.

5.4.5 Number of Options

Besides those basic characteristics of option discussed in previous subsections, we concentrate on the number of options. We analysed the hedging strategies per one option depending on different options solved with different illiquidity effects in Section 5.3.2. As Figure 5.14 shown, varying number of options takes little influence on hedging strategy of the particular stock price trajectory. However, the illiquidity effect would amplify the influence of number of options in the hedging strategy. This subsection we only examine how the number of options affects the option price.

Intuitively, comparing to people having small amount of initial shares, investors holding heavy initial shares would pay the call option per unit in lower price. Curves of option price in Figure 5.28 verify the above phenomenon. These two panels display that different number of options hedged in the portfolio change the option price. At the same initial wealth level, the option price by the investor with more options is higher than the one with less options. Comparing curves in Figure 5.28 and the straight line of option price in the perfectly liquid market, the option price rapidly decrease with the increasing initial stock holdings. The

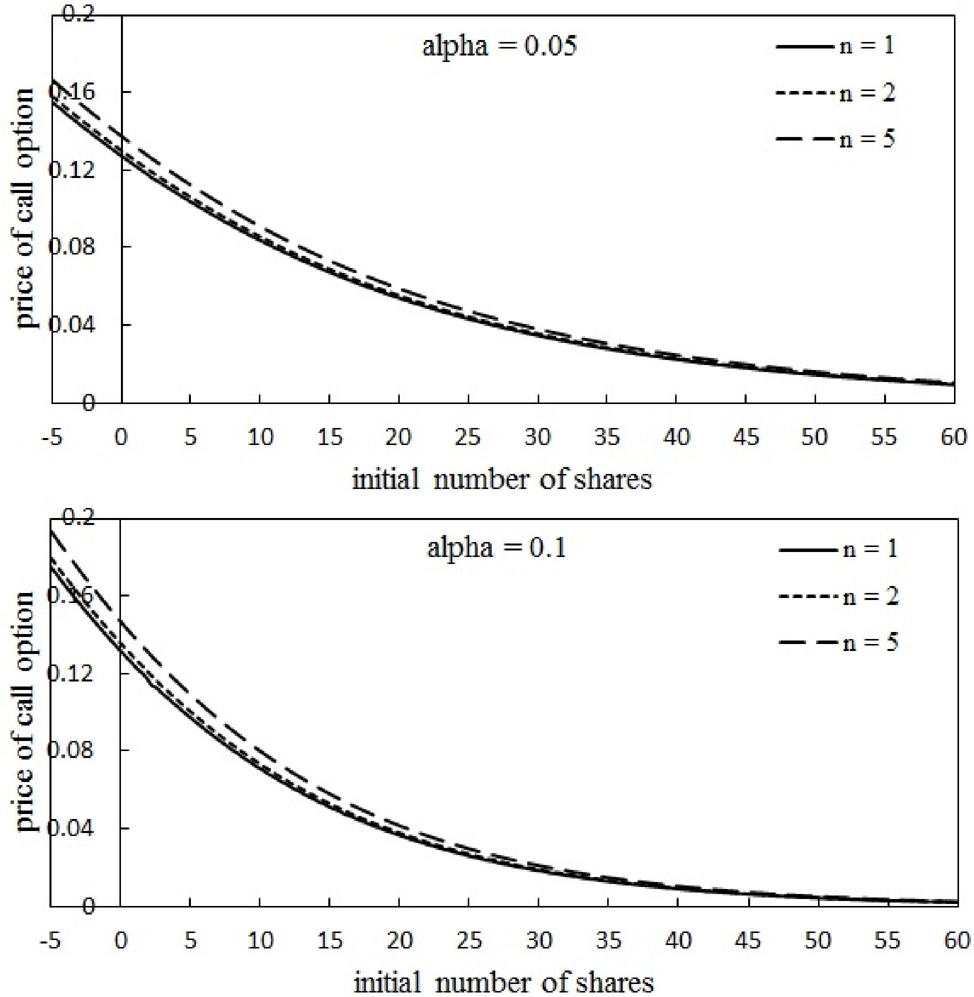


Figure 5.28: Effect of the number of call options with different illiquidity effects of reservation price.

only difference between curves in Figure 5.28 and the straight line is the value of illiquidity efect α. Thus, we regard that the illiquidity effect causes this change. The bigger illiquidity effect in the market the more distinct convex decrease in the option price against the initial stock holdings. Notice that we select a large range of initial number of shares to generate the option price. With this selection, we found that the gap between different curves decreases with increasing initial number of shares. The curves would be trended very close but not identical with each other. For investors holding a large amount of shares at the initial time, the number of option takes little influence on the option price.

However, based on same number of options, option prices with different illiquidity effects would intersect with each other. Intersections of option price caused by the illiquidity effect inspire us to get benefits of writing options in the C-R model. We will discuss this interesting circumstance in Section 5.5.

5.4.6 Put-Call Parity

Put-call parity is a relationship between the prices of European put and call option with the same strike price and the expiration date. Stoll (1969) first identified this relationship. This theorem states that a portfolio of long a call option and short a put option is equivalent to a single forward contract at the same strike price and expiry. In practice, due to the existence of transaction and liquidity costs, the put-call parity will not hold.

Applying the default values in Table 5.1 into the calculation of Black-Scholes model, we obtain that the Black-Scholes call price is 0.1423 and put price is 0.0935. We regard the difference between these two prices (0.0488) as the benchmark of put-call parity and compare it to reservation prices with different liquidity costs.

As Section 5.4.1 mentioned, the initial number of shares affects the option price in illiquid markets. Therefore, we focus on the difference of call price and put price with different initial portfolios. Table 5.7 lists comparison of difference of call and put price with different initial number of shares. The first row of Table 5.7 shows that the difference between the reservation call and put price increases with increasing α. Considering the zero initial number of shares, the call price in the market with larger α value is more expensive than the one with a smaller illiquidity effect value. This phenomenon is verified by many graphs in Section 5.4 (e.g. Figure 5.19). Reservation call prices in the illiquid market are higher than the replicating price without liquidity cost since that call price in the perfectly liquid market is what the investor's ultimate goal but hardly chased due to the illiquidity effect. Figure 5.29 shows the opposite situation happens at the price of put option. The reservation put price at the position of zero initial number of shares decreases with increasing illiquidity effect. Hence, the difference between call price and put price increases with α.

Table 5.7: Difference between reservation prices per call option and per put option when changes the initial number of shares (other parameters given in Table 5.1). The difference between the Black-Scholes call and put price equals to 0.0488.

Initial Shares	$\alpha = 0$	$\alpha = 0.005$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
0	0.04877	0.05147	0.05294	0.05945	0.06444
1	0.04877	0.04919	0.04944	0.05043	0.05141
2	0.04877	0.04693	0.04591	0.04154	0.03803
3	0.04877	0.04464	0.04241	0.0329	0.02577

Table 5.7 shows when the initial number of shares changes, the difference between call price and put price in different illiquid markets changes. The second row displays that when the initial number of shares equals to 1, the difference of reservation call and put price still increases with α but the increasing ratio is less than the first row. However, when the initial number of shares increases to 2 or more shares (the last two rows in this table), the difference between the reservation call and put price decreases with increasing α. Figures 5.19 and 5.29 show the price of call option and put option under the same parameters setting,

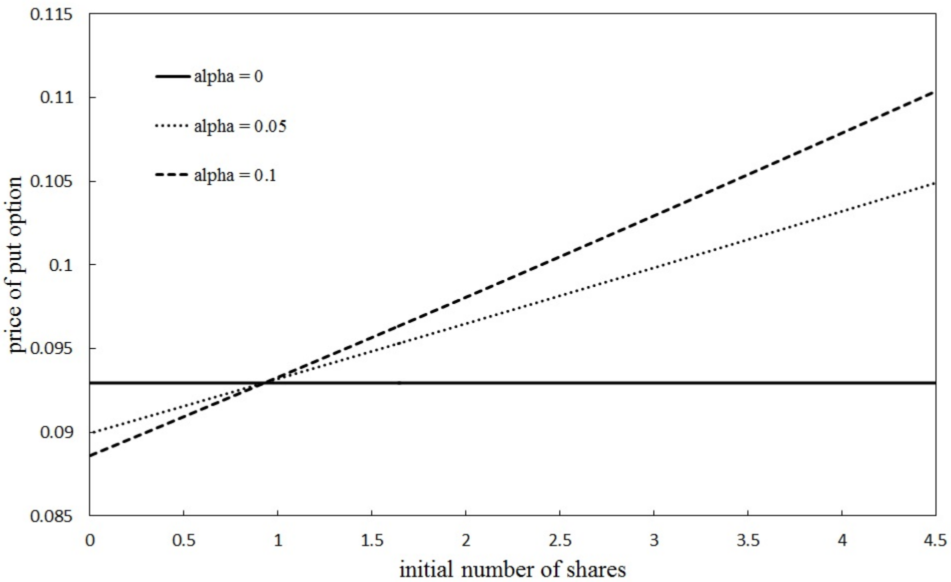


Figure 5.29: Reservation sell prices of a European put option as functions of initial portfolio with different illiquidity effects (shown in dashed lines) and the reservation price in the perfectly liquid market (shown in solid line).

respectively. The above difference in Table 5.7 can be explained by comparing these two figures. At the condition of 1 initial share, Figure 5.29 illustrates the corresponding prices are close to each other; the call option price in Figure 5.19 still increases by the increasing α. Hence, the difference between call and put price is affected by the call price at the condition of 1 initial share. For other initial number of shares cases, Figure 5.19 shows that call option price decreases with increasing in α; while Figure 5.29 displays the put option price increases with increasing α. Therefore, the last two rows in Table 5.7 demonstrate that the difference between call and put price reduces with the increasing illiquidity effect. It highlights again the importance of that point where option price in the illiquid market intersects the price in the perfectly liquid market. We particularly analyse that point and the corresponding option prices in the following section.

Table 5.8: Difference between reservation call price and put price when changes the number of options (other parameters given in Table 5.1). The difference between the Black-Scholes call and put price equals to 0.0488.

Number of Options	$\alpha = 0$	$\alpha = 0.005$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.1$
1	0.04877	0.05147	0.05294	0.05945	0.06444
2	0.04875	0.05175	0.05339	0.06061	0.06628
3	0.04754	0.05101	0.05291	0.06133	0.06798

Another interesting component that impacts the relation between the call and put price is the number of options sold. Table 5.8 compares a difference between call and put prices under different number of options applying into the pricing of option. From Table 5.8, no matter how much the number of options is, the difference of reservation price increases with the increasing illiquidity effect α. However, considering in the same illiquidity effect parameter, the number of options indeed affects the reservation prices. In the perfectly liquid market ($\alpha = 0$), there exists an effect that increasing in the number of options slightly reduce the difference of reservation prices. This effect is weakened by increasing α. If the value of α is big enough, for example $\alpha > 0.05$, the difference of reservation prices homogeneous increases by α.

To sum up, the put-call parity would not be held in the illiquid market. Considering different initial number of shares, we found that the relation between the difference of reservation prices and the illiquidity effect α would be affected by intersections of option prices. It inspires us to concentrate on the study of intersections of option price with different liquidity costs.

5.5 Analysis of Price Transition Point for Option Price

In this section, by examining effects of changing the liquidity cost and the risk aversion parameter, we are able to evaluate the approximate intersection point of reservation prices and the price of option changes. By decreasing the discretization step to $\delta r = 0.0005$ (the default discretization step is $\delta r = 0.002$), we have a better precision of computation. We decrease this parameter in order to analyse the intersection points in the graphs of option prices with various illiquidity effect parameters. The intersection points are observed since lines of option price with illiquidity effect parameters always cross the horizontal line of option price without liquidity cost. We call those intersection points 'Price Transition Point' (PTP). In this section we discuss some phenomenons about the PTP in three main parts. First, we discuss the position of PTP and study hedging strategy in a single period model. Second, we investigate different positions of PTPs in a two-period model and show some special properties which are succeeded from the single period model; third, we compare intersections of reservation price in different period models and discuss general interpretations of reservation price in multi-period model.

5.5.1 Single-Period Model

Section 5.4 gives a comparative analysis of the reservation price under five changing model parameters. We observe that curves of option price with different liquidity cost would be intersected. These intersections (we call them PTPs) are very close to each other in the graph of option prices. In order to carry out the special meaning of these PTPs in the option price versus the initial stock holdings, we analyse the reservation price from the simplest model - the single period model. Figure 5.30 presents the price of call option versus the initial number of shares in the single period model. Based on three different values of illiquidity effect α, lines of call option prices $\alpha = 0.05$ and $\alpha = 0.1$ intersect with $\alpha = 0$ line (which is the horizontal line at the level 0.148885) at one point approximately. With current discretization step size $\delta r = 0.0005$, those price transition points are quite close to each other but unfortunately they can not be regarded as identical. When the discretization step is not precise enough, e.g. $\delta r = 0.05$, it is difficult to observe that these PTPs are different from each other. That is why we choose $\delta r = 0.0005$ in this section rather than the default $\delta r = 0.002$. At the right top corner of Figure 5.30, we zoom in the small area of these PTPs to display differences clearly. In the enlarged region, $\alpha = 0.05$ line crosses $\alpha = 0$ line at the level 2.1038. Notice that this is a better precise value than the scale used in this graph ($\delta r = 0.0005$). Current implementation cannot provide such precise value automatically. We use the linear interpolation method to get this value.

Figure 5.30 displays call option prices in the function of the initial number of shares, varying the illiquidity effect parameter α. Based on the definition of reservation price in Chapter 3, we know that in perfectly liquid market, the reservation price is independent of the initial portfolio. Thus, the price of option in the initial stock holdings function is exhibited as a straight line. The lines of option price with liquidity effects show that, comparing with the default one without illiquidity effect, the prices to the left of the corresponding PTP are

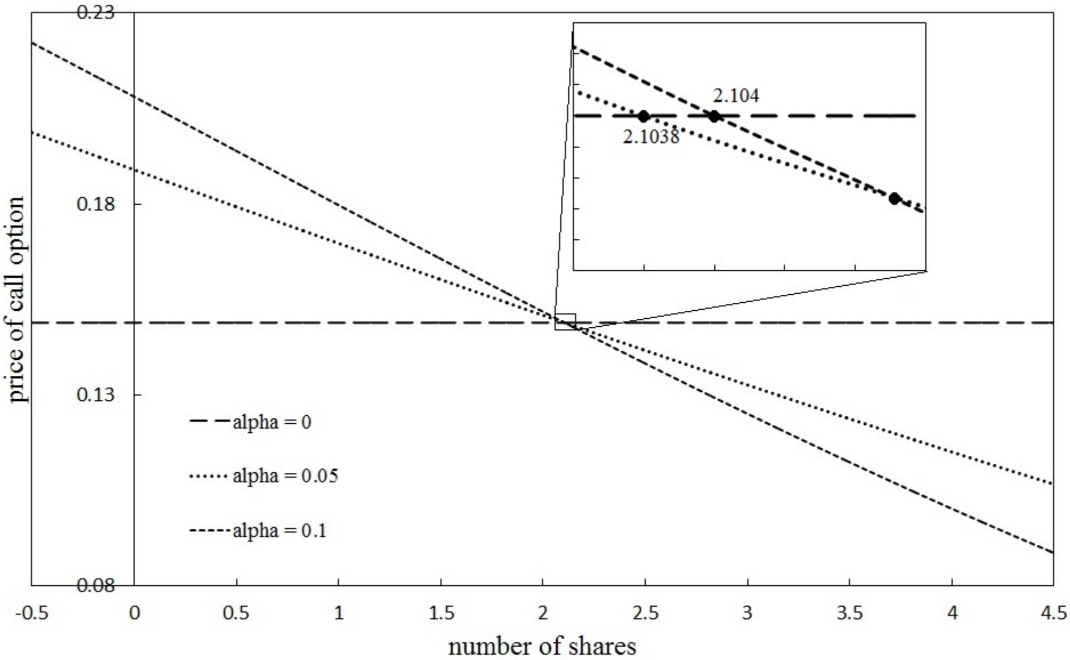


Figure 5.30: Price of option v.s. initial number of shares: The price transition point in price of call option is given by the default risk aversion $\gamma = 1$ and changing liquidity effects; the enlarged region shows the PTPs between in the illiquid market and in the perfect liquid market, respectively.

higher and the prices to the right of the PTP are lower. Moreover, the higher the liquidity effect, the steeper the price of option. Applying more liquidity effects into the calculation of reservation prices, lines of option price become steeper since the larger liquidity effect makes the hedge more costly. There exists similar PTP of price for put option case: prices of put option increase with increasing initial stock holdings, except the line with $\alpha = 0$. The plot of put option case is shown in Figure 5.29.

Although the PTPs do not locate at the same position in Figure 5.30, we observe an interesting phenomenon: once applying the illiquidity effect into the model, if there are investors with the initial number of shares bigger than the PTP (number of shares 2.104 in Figure 5.30), then these investors can purchase call options for lower prices than people with same initial shares in the perfectly liquid market. And vice versa, investors with initial position smaller than the PTP 2.104 are willing to purchase call option for higher prices than people in the market without illiquidity effect. This observation reflects that in real world, traders at greater initial positions in shares (amount exceeding the corresponding PTP) are happy to write call options in lower prices. Moreover, comparing to investors in markets with smaller α, the investors in larger α market whose initial shares are more than the PTP would short options in a lower price. That highlights the importance of the position of PTP. It reveals that the larger the illiquidity is, the bigger the benefit of shorting call options might be. We will explain why the illiquidity effect influences the price of option at the end of this subsection.

In the single period model, differences between any two PTPs are small. We propose the value 2.104 (the approximation of all PTPs) as the initial stock holdings in single period model. Then we pick up four initial stock positions symmetric around this PTP 2.104 plus this point to check corresponding hedging strategies. Table 5.9 lists these five selected initial stock holdings and their corresponding hedging strategy in the single period model. The data in Table 5.9 indicates that the hedging strategy for the PTP takes the maximal value in these selected points. Moreover, values in other cases of the initial stock holdings display symmetrically decreasing besides this PTP.

Section 5.4.2 shows that the risk aversion parameter affects the position of PTPs. The next investigation is to check the price of call option in the function of initial stock holdings with a larger γ. We test the price of call option under the high risk aversion condition $\gamma = 6$ in Figure 5.31. This graph reveals that

Table 5.9: Comparison on hedging strategy in selected initial stock holdings points. Other parameters are given in Table 5.1.

		Initial	Stock	Holdings	
α	0.104	1.104	2.104	3.104	4.104
0	0.5745	0.5745	0.5745	0.5745	0.5745
0.05	0.3645	0.369	0.37	0.369	0.364
0.1	0.259	0.2695	0.273	0.2695	0.2585

the prices of call option under the high risk aversion coefficient, varying illiquidity effects, still exist PTPs close to each other. Moreover, Figure 5.32 compares corresponding hedging strategies between in the low and high risk aversion coefficients.

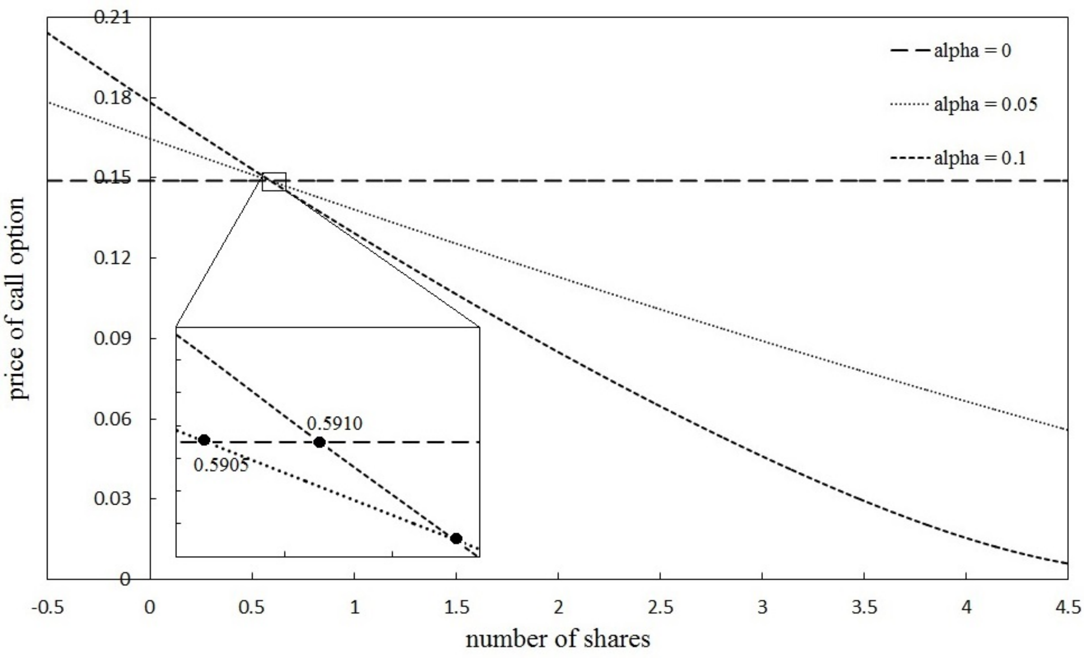


Figure 5.31: Price of option v.s. initial number of shares: The price transition point in price of call option is given by a high risk aversion $\gamma = 6$ and changing liquidity effects; the enlarged region shows the PTPs between in the illiquid market and in the perfect liquid market, respectively.

Comparing Figure 5.30 and 5.31, we observe three properties of reservation price. First, changes in risk aversion coefficient do not affect the reservation price if the market is perfectly liquid. Based on Equation (3.25) (the equation of reservation buy price in Chapter 3) we know that the change in risk aversion coefficient is recovered by changes both in value function with and without option. In Figure 5.20 of Section 5.4.2, we provide an evidence of identical hedging strategies by changing risk aversion values in the perfectly liquid market. This property reveals that, in the perfectly liquid market, investors pay the same option price no matter how much initial wealth and risk tolerance level are. Second, the approximation of the PTP is changed with different values of the risk aversion coefficient. Comparing Figure 5.30 and 5.31, we explore that the position of PTP moves from 2.104 to 0.591. With an increasing γ, the initial number of shares of PTP would decrease. Third, comparing to lines in Figure 5.30, the concavity of option price lines in Figure 5.31 is more distinct. Figure 5.30 and 5.31 both reveal that if the risk aversion parameter increases, the initial number of shares of the PTP would decrease and the concavity of the option price line would be easier to observe.

Figure 5.32 characterises the hedging strategy in the function of the initial number of shares: the top panel is for $\gamma = 1$ and the bottom panel is for $\gamma = 6$. Both panels in this figure demonstrate that the maximal value of hedging strategy is produced by the initial number of share of PTP. Especially, the concave and symmetric properties of the hedging strategy in the high risk aversion coefficient condition is more distinct rather than in the low risk aversion condition .

Besides the symmetric and concave properties in the hedging strategy, we are more interested in a phenomenon: lines of option price in the illiquid market cross

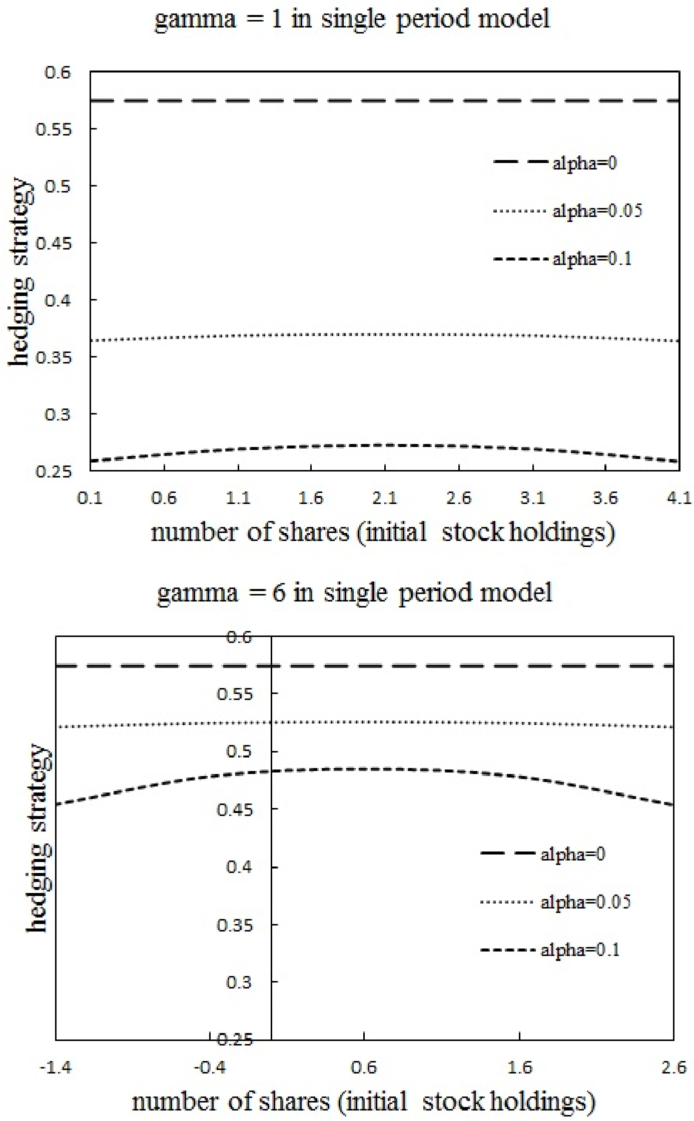


Figure 5.32: Maximum of hedging strategy versus initial number of shares: the existence of PTP in reservation prices varying by liquidity effects in $\gamma = 6$.

with the line in the perfectly liquid market (as Figure 5.30 and 5.31 shown). These two figures illustrate some benefits provided by the portfolio of writing an option in the illiquid market. Once we find the position of the PTP, it is beneficial for the writer of the option since the price is lower to the right of the PTP; whereas to the left of the PTP, it is bad for the writer of the option then the option price is higher. Generally speaking, the optimal strategy without option in perfectly liquid market is what the investor aims at, but he is prevented from doing it due to trading costs (liquidity costs). A written option allows the investor to remove some kind of costs since a short option has some characteristics of selling shares. That is the reason why we study writing a call option in the portfolio.

In real world, it is common that investors are willing to sell certain options for a price which seems lower than the price implied by the market. This is often caused by the need to close some certain open positions, e.g. to close a long call option or to pair a short put besides with a short call. However, Figure 5.30 and 5.31 both reveal that this phenomenon can also be caused by the initial portfolio of the financial institution and the illiquidity effect in the market.

In order to interpret why the benefits above are caused by the initial stock position and the illiquidity effect, we compare the optimal strategy with illiquidity effect to the one without it. The optimal strategy without option in the perfectly liquid market is what the investor aims at. We propose that the ultimate goal is the optimal strategy without illiquidity effect and the replicating strategy for the option (optimal strategy with option). They display straight lines in the graph of option price against the initial portfolio. Unfortunately the investor is prevented from doing the optimal strategy without any illiquidity effect because of the existence of trading costs in the market. Those downward skew lines of option price in the illiquid market present the hedge costly. Table 5.10 shows optimal strategies without and with option by varying illiquidity effect α and the differences between these illiquidity effects.

We start from the middle column (initial number of shares $x = 2.104$). In

Table 5.10: Comparison of optimal strategies in the function of initial stock holdings. Other parameters are given in Table 5.1.

		Strategy	without	Option	
Conditions	0.104	1.104	2.104	3.104	4.104
$\alpha = 0$	1.816	1.816	1.816	1.816	1.816
$\alpha = 0.05$	1.203	1.5625	1.9185	2.2755	2.64
$DIFF^1$	-0.613	-0.2535	0.1025	0.4595	0.824
$\alpha = 0.1$	0.9075	1.442	1.967	2.2755	3.041
$DIFF^2$	-0.9085	-0.374	0.151	0.68	1.225
		Strategy	with	Option	
$\alpha = 0$	2.3905	2.3905	2.3905	2.3905	2.3905
$\alpha = 0.05$	1.5675	1.9315	2.2885	2.6445	3.004
$DIFF^{1^c}$	-0.823	-0.459	-0.102	0.254	0.6135
$\alpha = 0.1$	1.1665	1.7115	2.24	2.7655	3.2995
$DIFF^{2^c}$	-1.224	-0.679	-0.1505	0.375	0.909

this column, the difference of optimal strategy without option between $\alpha = 0.05$ and $\alpha = 0$ is 0.1025 and the corresponding difference of optimal strategy with option is -0.102. This comparison verifies that the difference of optimal strategy with option would be smaller than the difference of optimal strategy without option since writing option removes some costs and the investor is willing to sell more shares. Besides this possible interpretation, we concentrate on the data of optimal strategy without option to explain the benefits of writing an option. From the first two columns in Table 5.10, we can see that those initial number of shares which are smaller than the PTP (initial number of shares 2.104) in illiquid markets can generate values of optimal strategy smaller than values in $\alpha = 0$. In other words, the smaller values of optimal strategy means less trading amount. As Figure 5.30 shown, the price to the left of the PTP is higher than the price in $\alpha = 0$, which is bad for the writer of the option. The low trading amount (value of optimal strategy) could be considered as a reflection of this bad influence on the price. The values of optimal strategy decrease with increasing illiquidity effect parameter. Looking at the last two columns in Table 5.10, when the initial number of shares are bigger than the value in PTP, the trading amounts are larger than the amount in $\alpha = 0$ and the corresponding prices of option are lower (to the right of the PTP in Figure 5.30). That is beneficial for the writer of the option.

In Table 5.10, $DIFF^1$ (or $DIFF^2$) indicates the difference of optimal strategy between $\alpha = 0.05$ (or $\alpha = 0.1$) and $\alpha = 0$. Based on the first two column in Table 5.10, we explore that values in $DIFF^1$ row are greater than the corresponding values in $DIFF^2$ row. While from the last two columns, we observe that the values in $DIFF^2$ row are greater than that in $DIFF^1$ row. The description above represents that when the illiquidity effect is large, both good and bad influences are amplified. The intuition behind the data in the optimal strategy with option is similar to the strategy without option. Considering the illiquidity effect into the market, investors hedged by written call options have to hold less number of shares than in the perfectly liquid market. The larger illiquidity effect makes the stock holding less since investors need to pay transaction costs at every trade. Therefore, we regard that the illiquidity effect reduces the option price when the investor holds sufficient shares (larger than the shares by PTP) in the initial time. We compute the expected utility of terminal wealth for the case with and without option followed by Table 5.10.

In summary, this subsection demonstrates that there does not exist an exact intersection but many PTPs are close to each other. Investors who holds more initial shares than the value of PTP can take benefit through writing the option. In following subsection, we will show how the PTPs move and explain the importance of observing the PTP in the two period model.

5.5.2 Two-Period Model

Once we have understood the reservation price and the hedging strategy given by the single period model, it is easy to extend the model to the two-period model so that options with maturity in two or multi periods can be examined. Comparing to the single period model, the two-period model with a chain of hedging strategy values depicts more details of characteristics of option pricing. Investors release all stock holdings at the terminal time, thus the following analysis of hedging strategy focus on all nodes except the nodes in the last period.

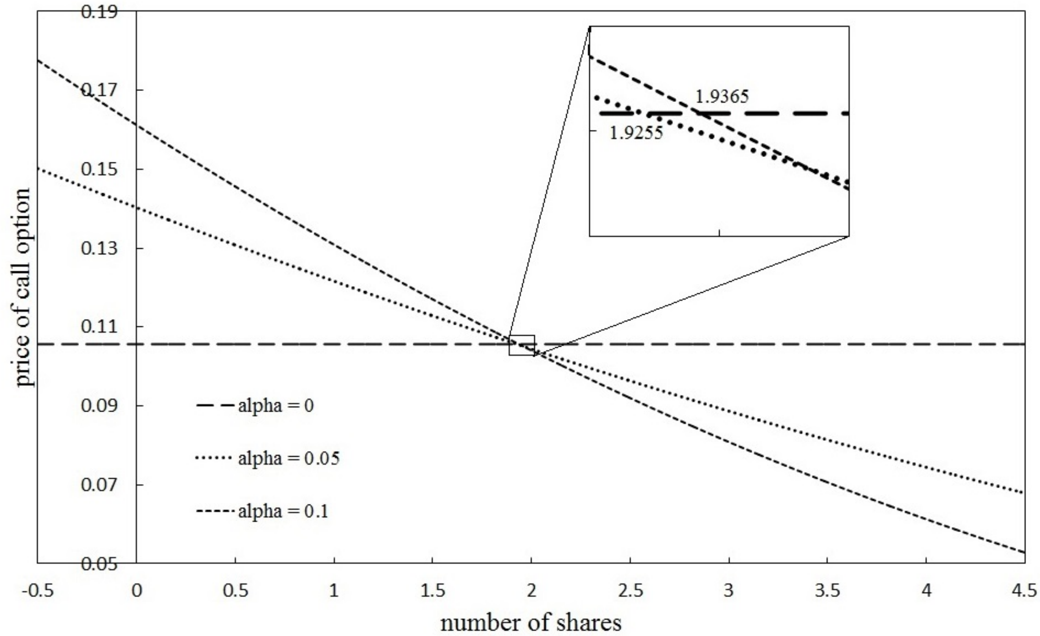


Figure 5.33: Price of option versus initial number of shares in a two-period model: the price transition point in price of call option is given by changing liquidity effects; the enlarged region shows the PTPs between in the illiquid market and in the perfectly liquid market.

Figure 5.33 displays lines of call option price versus the initial stock holdings in two-period model. The enlarged region in Figure 5.33 shows that the lines of option price with different levels of liquidity cost indeed intersect with the line without liquidity cost, separately. The PTP between $\alpha = 0$ line and $\alpha = 0.05$ line is at the level of 1.9255 shares and another PTP between $\alpha = 0$ line and $\alpha = 0.1$ line is at the level of 1.9365 shares. The difference between these two PTPs is 0.011. This difference distinctly shows that PTPs between lines of option price with illiquidity effects and the default line in the perfectly liquid market are not identical. By comparing this value to the difference between PTPs in Figure 5.30 (0.0002), we notice that the difference between PTPs in two-period model is larger than the one in single period model. Moreover, in multi-period model (show in Section 5.5.3), this corresponding difference increases with the expanding number of periods in the model.

In single period model, we checked the hedging strategy given by different initial stock holdings and found that the PTP at the level of 2.104 shares indicates the maximal value of hedging strategy (shown in Figure 5.32). We conduct a similar analysis in the two-period model to check whether this special phenomenon exists as well. Because of the trading restriction on the terminal time, releasing all stock holdings at the terminal period, we do not consider the transaction in the terminal time. Table 5.11 displays 5 selected initial stock holdings and corresponding values of hedging strategy at the period 0 and 1, with three levels of illiquidity effect α.

We select a middle value 1.93 between those two PTPs shown in Fig.5.33 as the initial number of shares. The next step is to select other values in the initial stock holdings which are symmetric distributed besides 1.93 shares. Table 5.11

Table 5.11: Comparison on hedging strategy in the function of initial stock holdings in the two-period model with $\gamma = 1$ and other parameters given in Table 5.1.

			Initial	Stock	Holdings	
Node	α	-0.07	0.93	1.93	2.93	3.93
	0	0.553	0.553	0.553	0.553	0.553
S_{t_0}	0.05	0.3715	0.35	0.3305	0.3095	0.2895
	0.1	0.2975	0.2755	0.252	0.225	0.199
	0	1	1	1	1	1
$S_{t_{1,u}}$	0.05	0.702	0.6925	0.6835	0.673	0.6628
	0.1	0.5351	0.534	0.5195	0.5005	0.4825
	0	0	0	0	0	0
$S_{t_{1,d}}$	0.05	0.218	0.2025	0.191	0.1785	0.167
	0.1	0.2195	0.202	0.185	0.165	0.145

expresses stable values in the hedging strategy without liquidity cost. It reveals that the line of hedging strategy in the function of the initial stock holdings keeps straight in the two-period model. This property is also involved in the single period model. The value 0.553 at the node S_{t_0} is slightly smaller than the value 0.5745 at the initial node in single period model (shown in Table 5.9) since more periods would reduce the initial value of hedging strategy. This phenomenon is more distinct in multi-period model (e.g. $m = 100$). The value one at the node $S_{t_{1,u}}$ and zero at the node $S_{t_{1,d}}$ in Table 5.11 are reflected by the payoff of call option at the maturity time according to the corresponding stock price. Another observation in Table 5.11 is that values of hedging strategy descend progressively with increasing values of initial stock holdings, at all nodes. We do not know why the beautiful phenomenon that maximal hedging strategy in the approximate intersection disappear unexpectedly from single period model to two-period model. It does not appear in the multi-period model as well. Nevertheless, the existing of PTPs in Figure 5.33 characteristics some benefits provided by the option writer's problem with illiquidity effect in two-period model as well. The analysis of two-period model gives us further research issue: there is a phenomenon that investors with more initial portfolios will sell the call option in a price which is lower than the option price in the perfectly liquid market. This phenomenon is caused by liquidity cost in the market. The intuition here is that the presence of liquidity costs makes hedging costly when the investor only holds a small amount of shares (less than the PTP shares); consequently, the investor will deduct the hedging costs from the option price if his initial portfolio is greater than the PTP's initial shares.

To sum up, this subsection demonstrates no precise intersection but an approximation existed in the graph of option price against the initial share holdings, and symmetric convex shape of hedging strategy (maximal value in the hedging strategy) disappears suddenly.

5.5.3 Multi-Period Model

In this part we analyse option price in multi-period model with different liquidity costs and explain how the liquidity costs affect the option price. This model is more realistic than the single-period model.

Figure 5.34 represents the price of call option in the function of the initial stock holdings in a $100-$period model. Numbers in the enlarged region in Fig.5.34 specify the length of two PTPs with different liquidity costs. The $100-$period model has one feature which is different from single and two-period model: the PTP between $\alpha = 0$ line and $\alpha = 0.1$ line is at the level of 1.795 shares; this value is less than the value of PTP between $\alpha = 0$ line and $\alpha = 0.05$ line (at the level

1.796 shares). This difference makes the PTP between $\alpha = 0.05$ line and $\alpha = 0.1$ line stand above the line of option price without liquidity cost. Lines of option price in the 100−period model are similar as graphs in single and two-period model. It reveals that in both single and multi-period models, lines of option price with different liquidity costs indeed intersect with the option price in the perfectly liquid market.

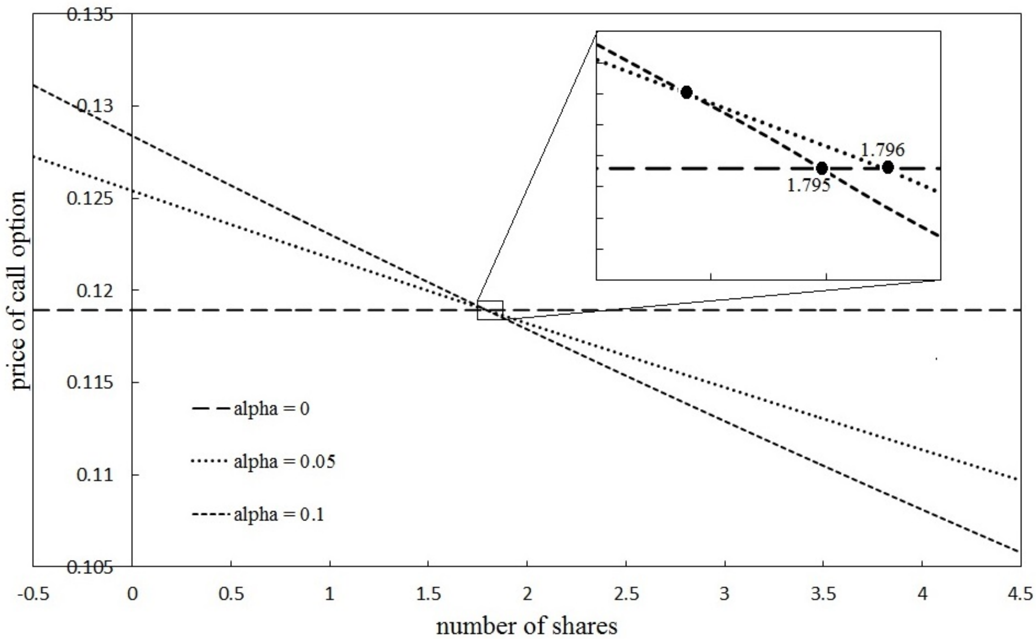


Figure 5.34: Price of option versus initial number of shares in a 100-period model: the price transition point in price of call option is given by changing liquidity effects; the enlarged region shows the PTPs between in the illiquid market and in the perfectly liquid market.

Table 5.12 lists values of price transition points in different number of period models. The first comprehension from Table 5.12 is that the value of PTP decreases with increasing number of periods. The last row in Table 5.12 displays the difference between $\alpha = 0.05$ and $\alpha = 0.1$ and there is no homogeneous tendency on these differences. We assume that the shifts of these differences are caused by the computation error of the current implementation.

Table 5.12: Intersection point between with liquidity cost (shown in the first row) and without liquidity cost ($\alpha = 0$) in models with different number of periods, with respect to discretization step size $\delta r = 0.0005$ and other parameters given in Table 5.1. The last row displays the difference of intersection points between $\alpha = 0.05$ and $\alpha = 0.1$.

	$m = 1$	$m = 2$	$m = 50$	$m = 100$
$\alpha = 0.05$	2.1038	1.9255	1.7992	1.796
$\alpha = 0.1$	2.104	1.9365	1.7995	1.795
DIFF	0.0002	0.011	0.0003	0.001

To summarise, this section examines the price of option in the function of the initial stock holdings with different liquidity costs both in single and multi-period models. First, graphs establish a phenomenon that lines of option price with illiquidity effects intersect with the default option price ($\alpha = 0$). Second, we compare the case of option price with liquidity cost to the case without it. Based on this comparison, investors holding more initial shares are able to hedge the written call (put) option in lower (bigger) price since the payoff of call (put) option is analogous to trade less (more) shares.

5.6 Illustration of Implied Volatility Curves

In recent years the Black-Scholes formula is rarely used to price options since a lot of options with a wide range of strikes and expiries are so liquid that the market price cannot be disputed. The volatility implied by the Black-Scholes formula is a common way to display the market prices of liquid options. We discuss the implied volatility in this section. Before understanding the meaning of implied volatility, we first discuss what historical volatility is. Historical volatility is the realized volatility of a underlying asset over a given time period, stated in terms of annualized standard deviation of the market prices as a percentage of the underlying asset. The historical volatility is calculated from the past returns of a security. Different from historical volatility, the implied volatility is a forward-looking and subjective measure. Due to being forward-looking, the implied volatility tends to lead backward-looking historical volatility reading. The implied volatility is widely regarded as the option market's forecast of future return volatility over the remaining life of the relevant option(quoted from Christensen & Prabhala (1998)). However, it is not possible to give a closed form formula directly for implied volatility in terms of option price. We choose Newton's method as the root finding technique in the implementation to solve the calculation of implied volatility.

5.6.1 Illiquidity Effect on Implied Volatility

Past papers concentrate on the time maturity behaviour of the implied volatility. For example, Rogers & Tehranchi (2010) deal with the long-term maturity behaviour of implied volatility and Forde *et al.* (2010) investigate the asymptotic formulae for implied volatility in the Heston model. Figure 5.35 shows curves of the implied volatility in the function of strike price K with different maturities. The top panel displays implied volatility curves with shorter maturities (0.1 to 0.5) and the bottom panel shows implied volatilities for larger maturities (0.6 to 1). Notice that the curve of implied volatility gets flatter when the maturity increases. With increasing in the strike price, the corresponding option would change from deep in-the-money to at-the-money and values of implied volatilities rapidly fall down to around the level of 30%. Within a shorter maturity, this drop is distinct. Figure 5.35 shows half U-shape curves showing high implied volatilities for in-the-money call options and low volatilities for at-the-money options. Many papers find empirical evidences of this phenomenon (e.g. Christensen & Prabhala (1998), Grover & Thomas (2012) and Chaudhury (2011)). Figure 5.35 proves that the skewed shape of implied volatility becomes flatter at longer maturities (Rogers & Tehranchi (2010) prove this phenomenon as well). Notice that the line of implied volatility in the short-term maturity (e.g. $T = 0.1$) in Figure 5.35 shows a slightly upward skew towards the right. This is caused by a numerical error since the binomial model of stock price in our implementation is a discrete approximation of the Black-Scholes model.

When observed the 'smile effect' of the implied volatility in Figure 5.35, we investigate whether the liquidity cost, the main issue in our research, influences that 'smile effect'. Past papers discuss the reason of 'smile effect' in stochastic volatility, human behaviours and transaction costs (e.g. Gatheral (2006), Boyle & Vorst (1992)). The 'smile effect' is a result of an empirical observation of the option's implied volatility in the function of strike price, with the same expiration date. Recent research of the implied volatility issue concentrate on how jumps in prices of underlying affect the 'smile effect'. In real market, the price of underlying asset does not diffuse smoothly that looks like jumps. Jumps in the stock prices can explain the volatility smile since they produce the steep short skewness that exists in the market. However, the liquidity cost, the convex transaction costs, also may result in the steep skew at the short-term of the implied volatility. Cetin *et al.* (2006) provides empirical evidence to show that the liquidity cost affects option price, and their results demonstrate changes in the implied volatility. The

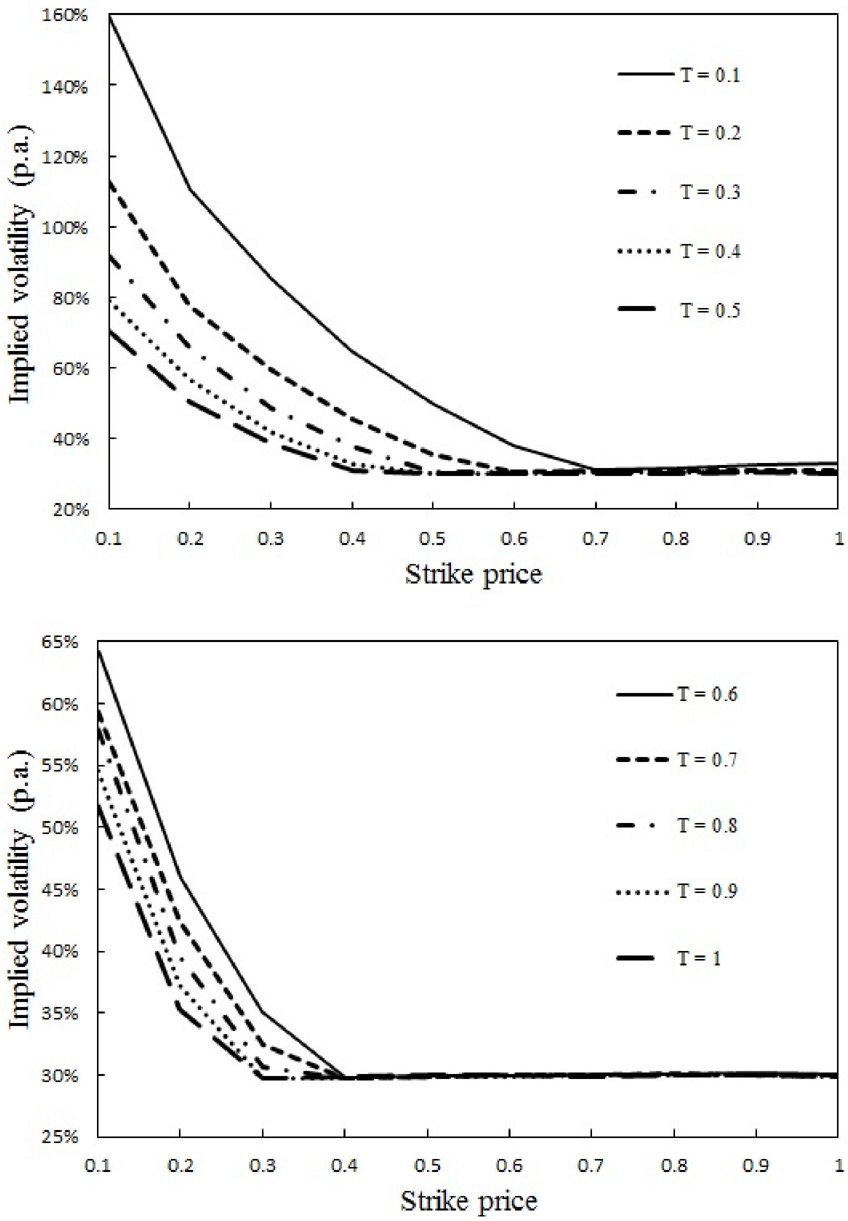


Figure 5.35: Implied volatilities against strike price with different maturities by writing one call option in the perfectly liquid market $\alpha = 0$: the top panel shows implied volatility with maturity from $T = 0.1$ to $T = 0.5$; the bottom panel shows i.v. with $T = 0.6$ to $T = 1.0$.

implied volatility's significant level for option price is illustrated by the implied volatilities given by corresponding liquidity effect parameter α by Cetin *et al.*

(2006).

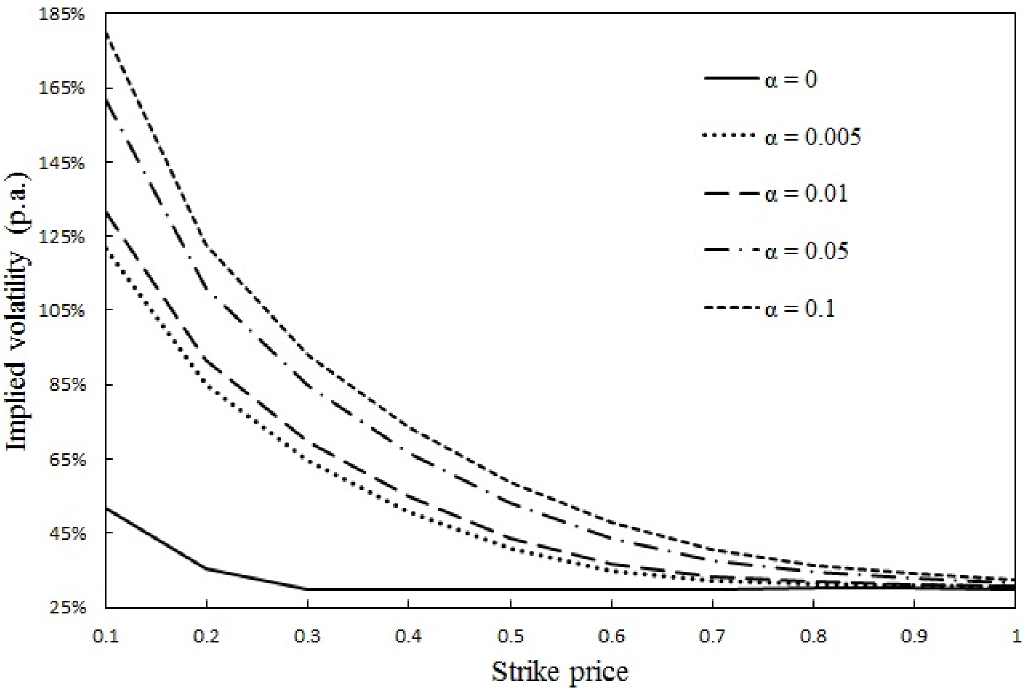


Figure 5.36: Implied volatilities against strike price with different α. Other parameters are given in Figure 5.1).

Figure 5.36 shows five implied volatility curves for various values of the illiquidity effect parameter α. It is clear that the implied volatility curves are steeper when the market has less liquidity (higher value in α). These results are consistent with the findings of Cetin *et al.* (2006) who show that with an increase in liquidity, there will be a decline in the level of the implied volatility curve. Given that the implied volatility is positive linked with the option price, the observation in Figure 5.36 implies that low liquidity leads to high option prices. This graph implies that the change of implied volatilities does not have to be caused by jumps in underlying asset price but can also be caused by the illiquidity effect in the market.

Considering the illiquidity issue on the implied volatility, we firstly try to match the market price of short one at-the-money call option ($K = 1$) with the linear transaction cost. One root-finding method, the bisection method, is used to find the relation between the convex liquidity cost and the linear transaction cost. The bisection method repeatedly bisects an interval and then selects a subinterval in which a root must lie for further processing. Applying a function of the bisection method described above into the implementation, it is simple and robust to match the market price of at-the-money option between the convex condition and the linear condition. This method is just a rough approximation to a solution which is used as a starting point for more rapidly converging methods (quoted from page 31 of Burden & Faires (1985)).

Table 5.13: Comparison of illiquidity effect α and linear transaction cost when they match the price of at-the-money call option $K = 1$ and the maturity $T = 1$.

α	0.0005	0.005	0.01	0.05	0.1
linear	0.00074	0.00347	0.00518	0.01185	0.0169

Table 5.13 displays comparison between the illiquidity cost (the top row) and the linear transaction cost (the bottom row) in the case that investors write a call option at $K = 1$ and $T = 1$. In order to figure out the relation between the convex cost and the linear cost, we select one set ($\alpha = 0.05$ and $linear = 0.01185$) as an example to discuss. Figure 5.37 displays the implied volatilities from $K = 0.4$ to $K = 1$ under the condition of zero initial shares. The top and bottom panes show 1 and 5 options, respectively. As Figure 5.37 shown, in the zero initial shares condition, the gap between the implied volatility curves is narrow with shorten one option; the gap sharply increases with shorten 5 options. In particular, the implied volatilities with shorten 5 options are no longer close to each other at

the strike price $K = 1$. That implies when we amplifies the number of options, the corresponding market price calculated by the liquidity cost function does not match the market price by the linear transaction cost.

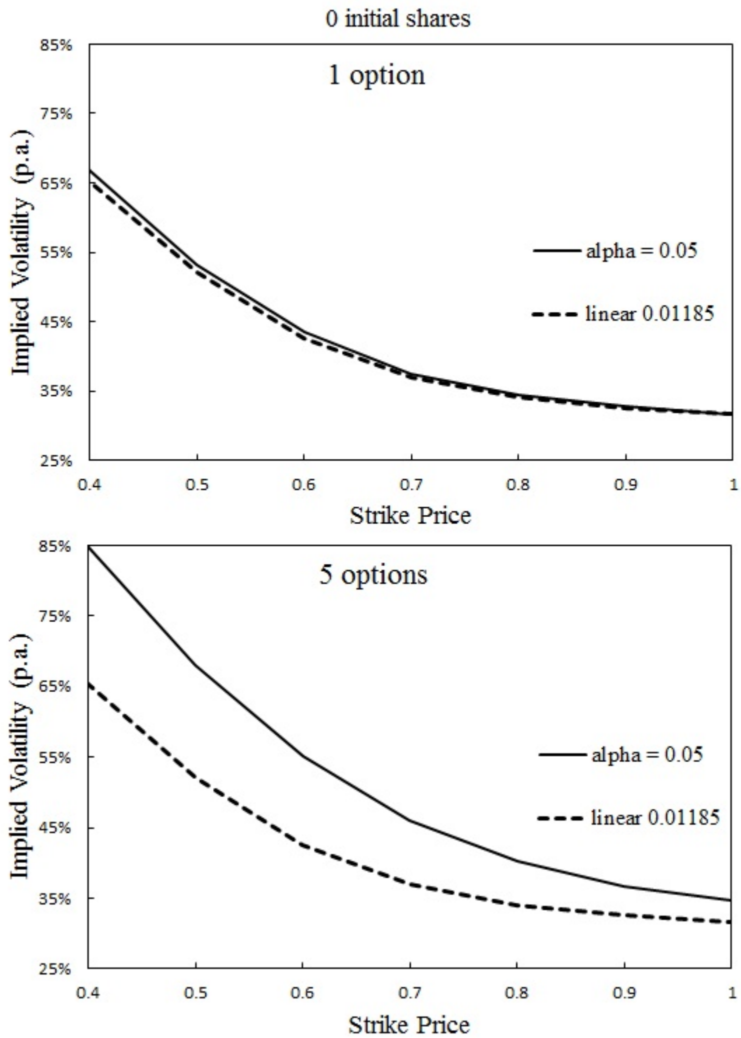


Figure 5.37: Comparison of implied volatility between illiquidity cost and linear transaction cost for one option and for five options under zero initial shares. Other parameters are given in Figure 5.1).

Figure 5.38 shows the implied volatilities when the initial shares increases to 3 shares. For the top panel, we observe a narrow gap of implied volatilities

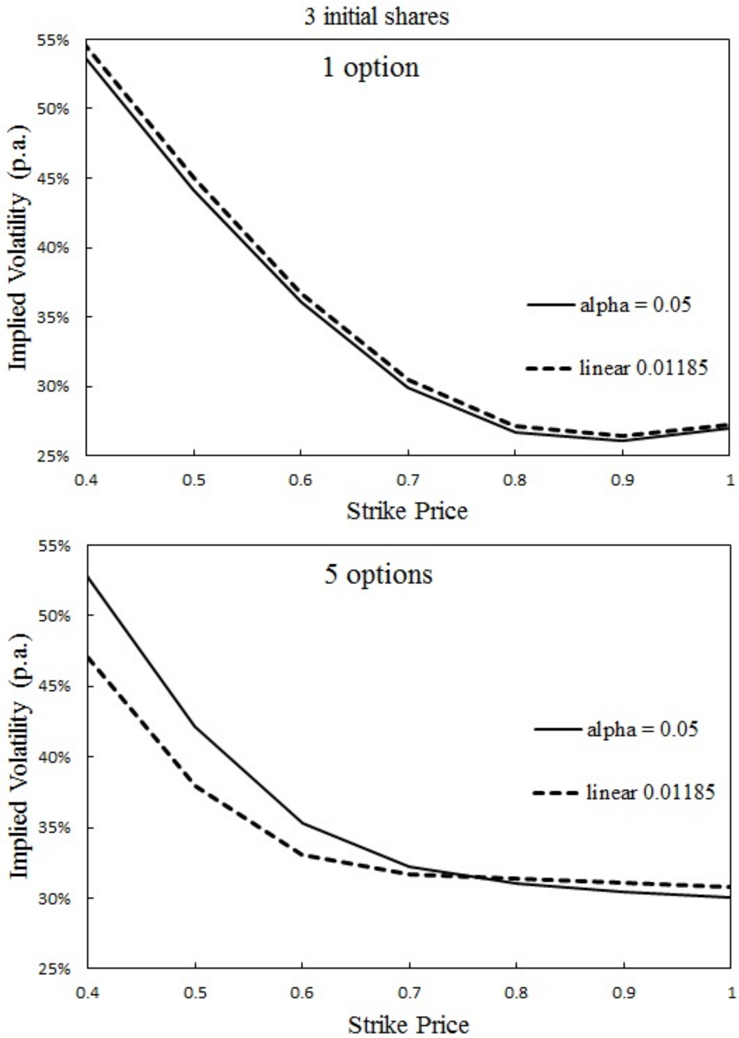


Figure 5.38: Comparison of implied volatility between illiquidity cost and linear transaction cost for one option and for five options under 3 initial shares. Other parameters are given in Figure 5.1).

between the convex and the linear one as well. But the curves are distinctly shown a upward skew when the option goes to the position of at-the-money. For the bottom panel, curves of implied volatility intersect between the strike price $K = 0.7$ and $K = 0.8$. Comparing Figure 5.37 to Figure 5.38, we conclude that the convexity of liquidity cost amplifies the effect of proportional transaction costs. It inspires us that some of the general observations from previous sections should also hold for linear transaction costs provided that the trading was in small quantities.

5.6.2 Comparison with Market Data

We discuss the change of implied volatility in our model of illiquid market in the previous subsection. It inspires us to check the difference of implied volatility between the real market quotes and the model quotes. In this subsection, we analyse how the implied volatility is affected by market parameters in our model of the illiquid market. We compare the model quotes to the market quotes (S&P 500 index with one month Call maturity) to get a reasonable estimation of level of liquidity in the real market.

S&P 500 index options are used in our comparison since the data is readily available and the options market is very active. We select the data of European call option with one month maturity in S&P 500 index. The data for the market example include the average price quotes and the corresponding implied volatility quotes for European call options during the period $10^{th}, March$ to $11^{th}, April, 2014$. The average price is the middle value of bid and ask prices. This example is a European call option with dividend yield 2.0065%. We select the one month LIBOR interest rate 0.155% in March, 2014 as the interest rate for computing call option price.

The analysis of implied volatility in the previous subsection is based on the binomial model without dividend. We modify the computation of CRR binomial model in the implementation: the probability of price movement of underlying

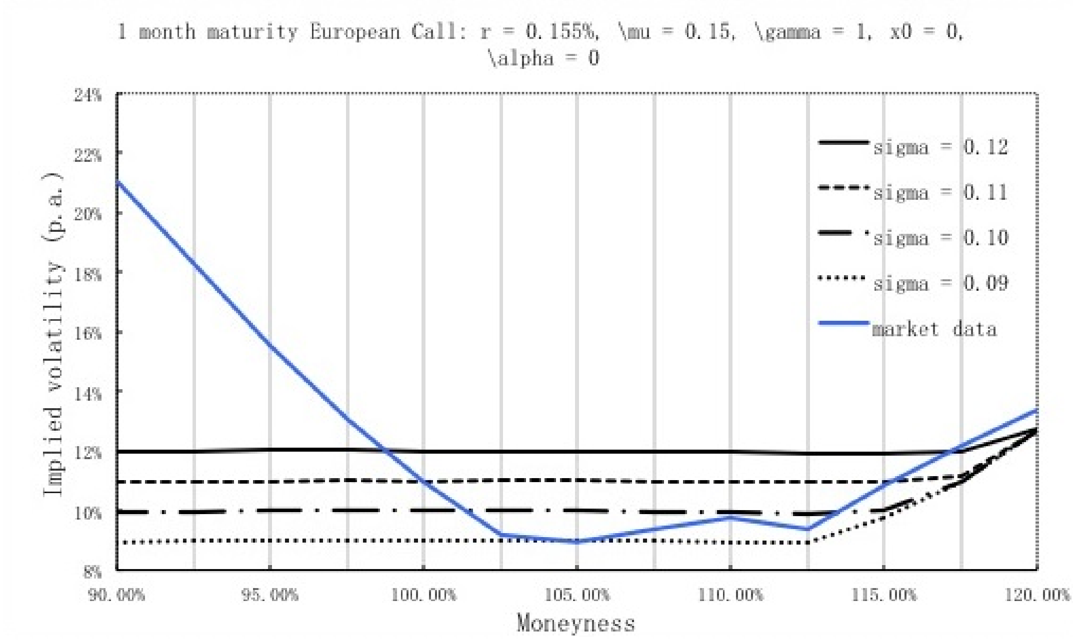


Figure 5.39: Comparison of implied volatilities both in real market quotes and in model quotes: the blue curve is the market quotes for implied volatilities from S&P 500 index European call option prices for the April 2014 option expiration; the other curves are Black-Scholes implied volatilities computed from our model with different volatility σ values.

asset $p_d = \frac{e^{(\mu-q)\Delta t}-d}{u-d}$ to replace the probability in the case without dividend $p = \frac{e^{\mu\Delta t}-d}{u-d}$, where q represents the dividend yield. We match the quoted prices as closely as possible with different moneyness from the example. Moneyness is a term describing the relationship between the strike price of an option and the spot price of its underlying security. We only concentrate on the range of $90\% - 120\%$ moneyness in the comparison. Because investors usually trade options in this range in the real market. Figure 5.39 displays the market quotes of implied volatilities in Black-Scholes pricing model and several implied volatilities under different market volatilities in our model.

Figure 5.39 illustrates big difference of implied volatilities between in market quotes and in our model data. The blue curve of market quotes is selected as the benchmark. We display changes of implied volatilities in the model by varying market parameters and compare them to the benchmark. All curves of implied volatilities with different σ values in our model of perfectly liquid market ($\alpha = 0$) are flat lines with a slight upward ending in moneyness 120%. Due to unknown of market volatility in the real market, we try different values of σ in the model computation. Cases $\sigma = 0.09$ and $\sigma = 0.10$ have similar values of implied volatilities to the benchmark in the moneyness range $102.5\% - 120\%$. Considering liquidity cost into the model would make implied volatility curves go upward in the range $90\% - 100\%$ (as Fig.5.36 indicated).

The analysis of implied volatility is the discussion of the liquidity cost on the option prices. Figure 5.40 shows the comparison of implied volatilities in the market with different level of liquidity. These two panels have the same setting of market parameters except the volatility σ: the one month LIBOR interest rate $r = 0.155\%$, the drift $\mu = 0.15$, the risk aversion parameter $\gamma = 1$ and zero initial number of shares $x_0 = 0$. The top panel is for $\sigma = 0.09$ and the bottom one is for $\sigma = 0.10$. Both in these two panels, the solid lines are the curves of implied volatilities from our model with level of liquidity $\alpha = 0.003$ and the dash lines are the ones with level of liquidity $\alpha = 0.004$. Comparing Fig.5.40 to Fig.5.39, we find that applying the liquidity cost into the model would increase the value of implied volatilities at the both sides of moneyness 100% (at-the-money). It makes the volatility smile in the illiquid market. The curves of implied volatilities from our model in the top panel simulate approximately the market quote (the blue line) better than curves in the bottom panel. This indicates that $\sigma = 0.09$ is a good estimation of market volatility in S&P 500 index. We set up $\sigma = 0.09$ in the following analysis of other market parameters. The difference of implied

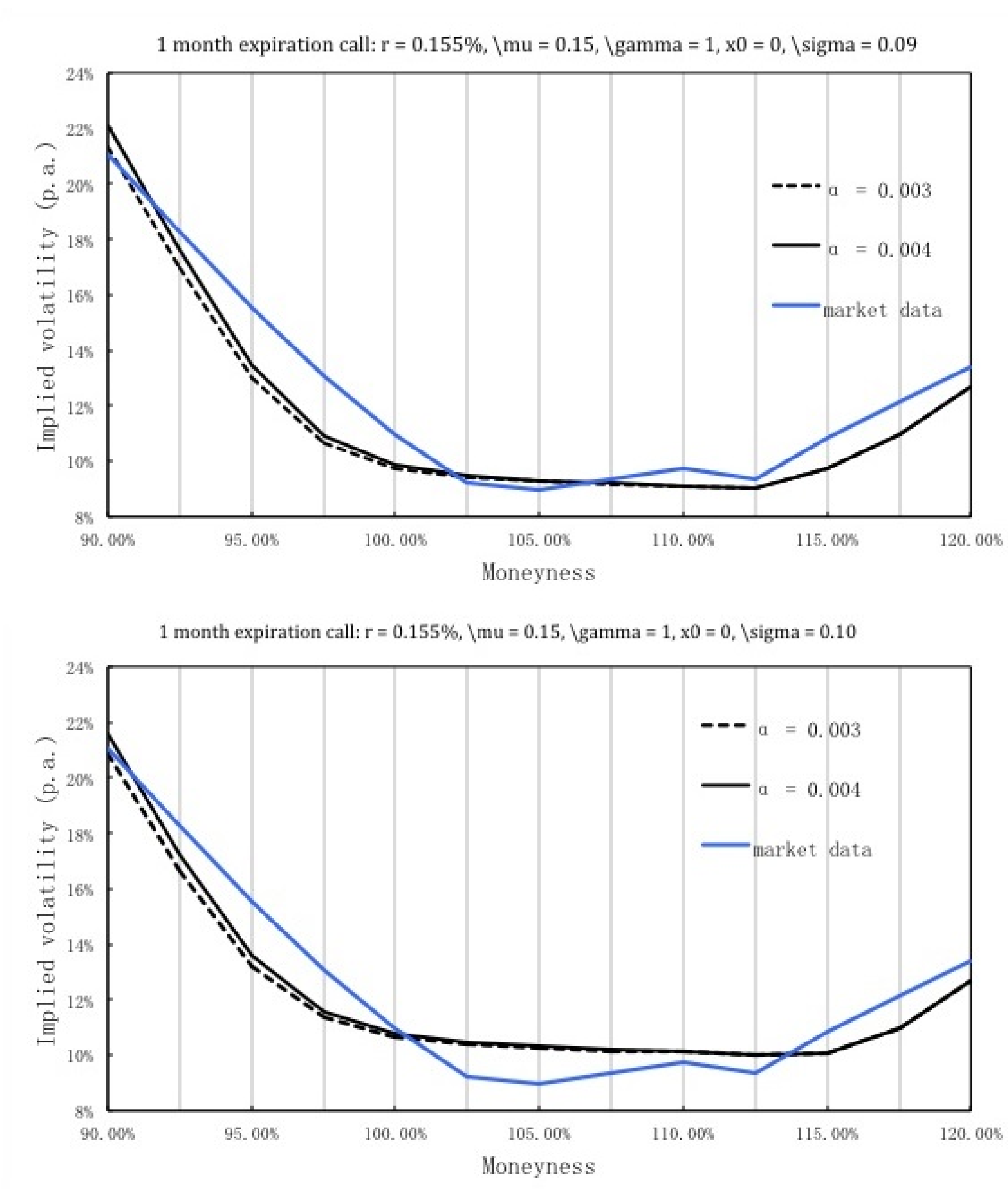


Figure 5.40: Comparison of implied volatilities both in real market replication and in model quotes: the blue curve displays the market quotes from S&P 500 index European call option prices for the April 2014 option expiration; the top panel is for Case $\sigma = 0.09$ and the bottom panel is for Case $\sigma = 0.10$.

volatilities between the solid line and the dash line confirms that less liquidity in the market (higher value in α) makes higher implied volatility. This phenomenon makes sense in the real market because less liquidity refers to higher liquidity cost and higher purchase price of call options. The top panel in Fig.5.40 reveals that the study of implied volatilities from our model would indicate the real market quotes in a certain extent. Moreover, the liquidity parameter $\alpha = 0.003$ to 0.004 is a good indication for the level of liquidity in the real market.

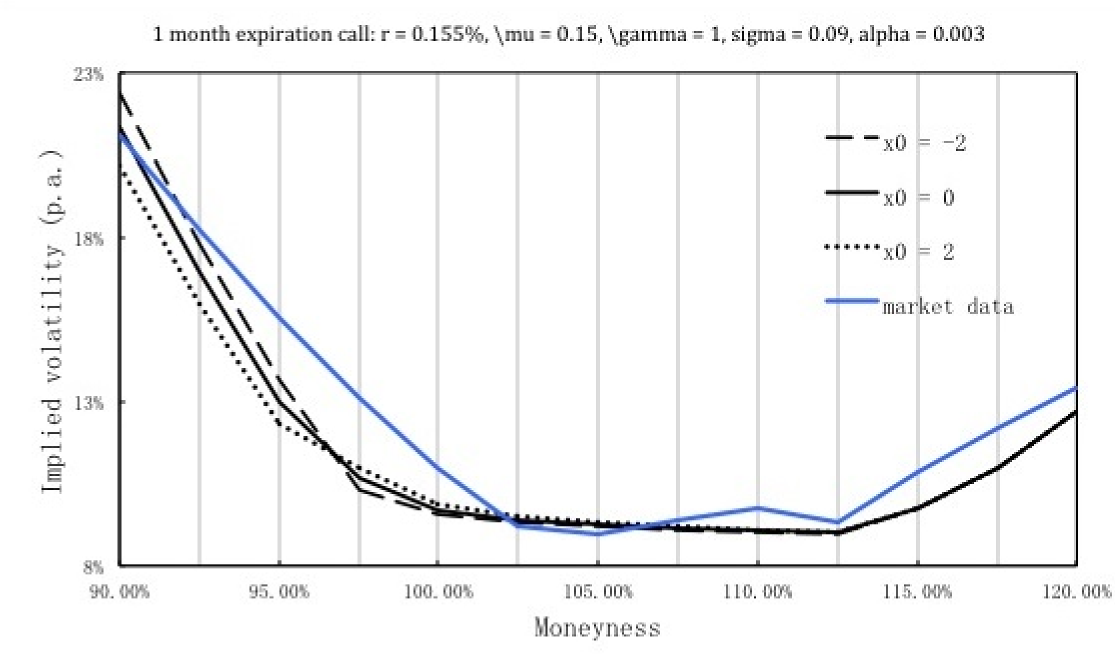


Figure 5.41: Comparison of implied volatilities both in real market replication and in model quotes: the blue curve displays market quotes from S&P 500 index European call option prices for the April 2014 option expiration; the black curves are implied volatility lines with different initial number of shares in our model.

Figure 5.41 represents changes of implied volatility with different initial number of shares in the market with $\alpha = 0.003$. We observe that the implied volatility with more initial number of shares is smaller than the one with less initial number of shares when the moneyness $< 97\%$. During the increasing in the moneyness, the implied volatility with more initial portfolio is bigger than others with less initial portfolio. The difference of implied volatility among those cases would

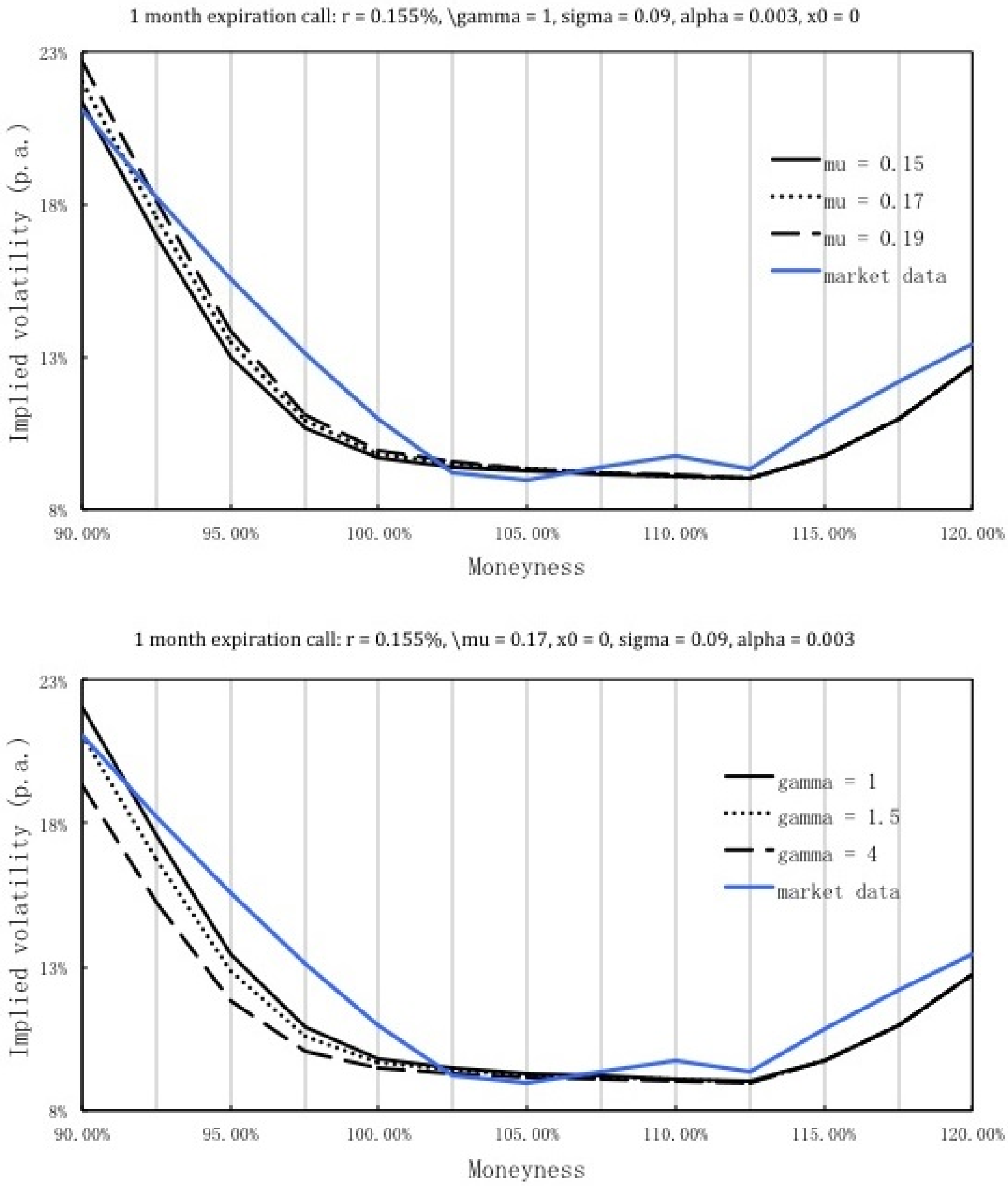


Figure 5.42: Comparison of implied volatilities both in real market replication and in model quotes: the blue curve displays market quotes from S&P 500 index European call option prices for the April 2014 option expiration; black curves in the top panel display the change of implied volatility by varying the stock price drift μ and black curves in the bottom panel display the change of implied volatility by varying the risk aversion parameter γ.

tend to zero with the increasing moneyness value. This observation reflects the property of PTP in Section 5.5. We test the value of PTP in the model as described in Fig.5.41, which is between 1 and 2 initial number of shares. Because the value of implied volatility is monotonic to the price of options. Figure 5.41 confirms the existence of PTP again.

The top panel and the bottom panel in Figure 5.42 separately display the change of implied volatility by varying the stock price drift μ and the risk aversion parameter γ. The top panel shows that the implied volatility is monotonically increasing with the stock price drift. Moreover, the drift $\mu = 0.15$ to 0.17 is a good estimation for S&P 500 index at March to April, 2014. The bottom panel represents that the risk aversion parameter γ affects the implied volatility and the price of options clearly.

Based on Figure 5.39 to Figure 5.42, we can conclude that all market parameters take sensitive affects on the implied volatility when options are in-the-money rather than when they are out-of-the-money. We provide an estimation of the level of liquidity $\alpha = 0.003$ for the real market.

5.7 Summary

We described the dynamics of the C-R model of optimal strategy with illiquidity effect in this chapter. First, we specify the market parameters and their corresponding default values. Second, we introduced how to decide the default values of the benchmark case. Third, we concentrated on the generation of hedging strategy in the small period and large period models. Fourth, we illustrated how the market parameters impact on the optimal strategies and corresponding reservation prices in the default model with illiquidity effect. Once investigated different dynamics of market impact on the reservation price, we observe an important phenomenon in the option price graph: we explain the reason that both of the illiquidity effect in the market and investors' stock positions take good influence on the option price (lower prices than the one in the perfectly liquid market) when the initial stock position is greater than the *Price Transition Point* of shares in the market without liquidity cost. In addition, we analysed the corresponding value of hedging strategy in the PTP position both in single period and multi-period models. Finally, we explore the implied volatility in the model without and with liquidity cost. We compare the convex liquidity cost to the linear transaction cost when they are matching at the price of at-the-money option and conclude that the convexity of liquidity cost amplifies the effect of proportional transaction costs. Moreover, we compare the data of implied volatilities from the model to the real market S&P 500 index. The comparison expresses that the model has a good potential to simulate the real market quotes. Most importantly, this comparison reveals a good estimation of the level of liquidity in the real market, which is around $\alpha = 0.003$.

Chapter 6

Conclusions

In this thesis we study an optimal portfolio choice problem in the model introduced by Cetin & Rogers (2007). The objective is to investigate the option prices in the market with liquidity costs. We examine the hedging of option to measure the liquidity in the market. The maximisation of the expected utility of terminal wealth is as a tool to identify the optimal portfolio choice problem.

In the C-R model, the capital is allocated to a risk-less asset and a risky asset. When the investor changes his portfolio, he has to pay the liquidity cost. The liquidity cost is a non-linear (convex) transaction cost depending on the volume of trades. The larger the volume, the deeper we have to reach an order book to find buyers/sellers. This increases the cost of transaction compared to trading at the quoted market price. In order to implement the model introduced by Cetin & Rogers (2007), we design an efficient numerical algorithm for the computation of the value function via the dynamic programming principle. Moreover, in order to understand the impact of liquidity, we present an analysis of numerical results for hedging and pricing of options with different liquidity costs. Our numerical analysis shows both changes in hedging strategy and option prices in the illiquid market with varying market parameters. We also identify how the initial portfolio and the number of options affect the put-call parity and pricing of options. In the analysis of option prices, we observe that the horizontal line of replication price without liquidity cost is indeed intersected by curves of option prices in the illiquid markets. Intersections responding to different liquidity cost levels are very close to each other. We name such intersections as *Price Transition Points* (PTPs). This phenomenon illustrate that if an investor's initial number of shares is larger than PTPs, he will be willing to sell options at a lower price than the replication price in the perfectly liquid market. To the best of our knowledge this is a new observation. We regard this phenomenon as the impact of liquidity in the market and interpret it from two aspects: the initial portfolio of the financial institution and the illiquidity effect in the market. In order to understand the pricing of options, we show a comparison of implied volatility between the convex liquidity cost case and the proportional transaction cost case. We prove that the change of implied volatilities does not have to be caused by jumps in underlying asset price but can also be caused by illiquidity effect in the market. That comparison shows that the liquidity cost can amplify the transaction cost. We compare the implied volatility from the model quotes to the one from the real market quotes. That comparison reveals that the current model has the potential to simulate the real market and also indicates the level of liquidity in the real market.

In the following, we explain more details about the main numerical procedures and relevant numerical analysis of this thesis. Section 6.1 shows some outlook for further research.

Numerical approach In order to provide the optimal strategy numerically, we design an efficient algorithm to compute the value function in the illiquid binomial model. We also implement numerical procedures for calculating the relevant implied volatility and Monte Carlo simulation. In the algorithm of value function in binomial model, we select the dynamic programming approach as the numerical tool. The normal dynamic programming backward algorithm is used to construct a matrix for storing all nodes in the binomial tree. However, in order to improve the computation efficiency, we build up an array for storing all nodes. First, we store nodes in the sequence that from the terminal period to the initial period. In the same period, nodes having smaller stock prices would be first ordered than those having larger stock prices. The challenging component of the existing binomial implementation is to identify the period number and the node position in every period. The next step for computation of value function is to produce an array for every variable in the previous array. That is for the discretization of the range of stock holdings. In other words, the selection of optimal strategy is structured by a matrix, which is much faster than the normal matrix $\times$ matrix. Monte Carlo simulation is produced by two reasons: the determination of range of stock holdings and the verification for binomial model. Whether the range of stock holdings is sufficient decides the correctness of the optimal strategy. We have to identify the range in every trading period. Any algorithm is needed to verify the correctness. Based on Monte Carlo simulation, we can not only prove the correctness of expected utility of terminal wealth from the binomial model but also know the sufficient range of stock holdings in any random stock price trajectory.

Hedging and pricing in the illiquid market We discussed the importance of research on hedging and pricing of options in the illiquid market. Based on the numerical data from the existing implementation, we conducted analysis by changing market parameters and revealed the characteristics of hedging and pricing of options in the market with liquidity costs. In order to identify how the liquidity cost affects the portfolio selection, we compared the prices of option with different initial portfolios in the illiquid market; we found that reservation prices in the illiquid market are reducing with increasing initial portfolios. That is because the illiquidity makes hedging of options costly and consequently the investor has to deduct the hedging cost from the price which he is willing to pay for the option. Considering the risk aversion coefficient, we observed that it has no impact on hedging strategies in the perfectly liquid market; but it has a big influence on hedging strategies in the illiquid market. In addition, in the analysis of option pricing with varying risk aversion coefficient, we observed that curves of option prices in the market with liquidity cost are indeed intersected by the horizontal replicating price without liquidity cost. We call these intersections as *Price Transition Points* and the PTPs are very close to each other. Another aspect of comparative analysis is about changes of option parameters: the strike price, the time horizon and the number of options. The strike price affects the optimal strategy significantly and the price of call option is reduced by increasing strike prices; a longer time length for trading is benefit on the option pricing besides the illiquidity in the market would amplify that benefit; we proved that the number of options affects the hedging and pricing of options, especially in the illiquid market. Moreover, we investigated the put-call parity and proved that it would not be held in the illiquid market and particularly affected by the initial portfolio and the number of options. The whole part of analysis inspires us to study the price transition point which exist in the curves of option price with different liquidity costs and the replicating price in the perfectly liquid market.

Price Transition Point By examining effects of liquidity costs, we evaluated the price transition point (PTP) which is determined by the reservation prices in the perfectly liquid market and in the illiquid market. We separately investigated the corresponding hedging strategies with different initial portfolios and found that the hedging strategy obtains a maximal value if its related initial portfolio equals to the number of PTP. However, this nice phenomenon only exists in the single period model. The existence of PTPs in different levels of liquidity shows that the illiquidity effect in the market makes option prices decrease with the increasing initial portfolios. It reveal that option sellers in the illiquid market are willing to sell call options at lower prices than the replicating price in the perfectly liquid market; the constraint is that sellers have to hold the initial number of shares larger than the PTP. This phenomenon is reflected in the real market, traders at larger initial positions in shares are happy to write call options in lower prices. It highlights that once the investor holds greater initial number of shares than the PTP, in the illiquid market there exists call options in lower prices than the market price in a complete liquid market. This phenomenon implies that sellers of call option holding higher initial shares than PTPs are willing to purchase in lower price than the replicating price to reduce the illiquidity in the market; whereas it is bad for option seller whose initial shares are less than the PTPS since the price is higher than the replicating price and they have to afford the liquidity costs as well. We explained this observation from the initial portfolio and the illiquidity effect perspectives.

Application of implied volatility Many papers show that jumps in the price of the underlying have greater effect on the curve of the implied volatility for short-expiration options than for long-expiration options. Our numerical solution of implied volatilities in the market with liquidity costs proves that the deeply smile in the implied volatility can also be caused by the liquidity cost. We explored the implied volatility with liquidity cost and compared the convex liquidity cost to the linear transaction cost when they are matching at the price of at-the-money call option. With higher liquidity costs, curves of implied volatility tend to be steeper since the liquidity cost indeed makes the market price of option higher. In addition, we compared the corresponding implied volatilities between the liquidity cost case and the proportional transaction cost case when we changed both the number of options and the initial portfolio. Those comparisons reveal that the convexity of liquidity cost amplifies the effect of proportional transaction cost. Moreover, we concentrate on comparing the model data of implied volatility to the market quotes. That comparison shows that the current model provides good simulation of implied volatility to S&P 500 index. We discuss the change of implied volatility by five varying market parameters and observe an estimation of the level of liquidity in the real market.

6.1 Further Research

Calibration to market data Some papers had measured the illiquidity effect in the limit order market. For example, Malo & Pennanen (2012) proposed a parametric approach for temporary impact modelling of bid-ask spread. Their resulting models are able to calibrate and to analyse for stochastic stock price processes. We can extend the study to find whether the illiquidity effects in Cetin & Rogers' model can fit to the empirical data in the London Stock Exchange. We can explore whether the illiquidity calibrated to stock market data is coherent with the one implied by observed option prices. In the analysis of option price in the market with liquidity cost, we observed some surprising behaviours of option sellers with regards to their pricing decisions. We are able to calibrate this model with real market price and conduct a thorough empirical study. The aim is to check whether those unusual phenomenons exist in the real market. In both above directions, we can measure the liquidity quantitatively in the empirical analysis. Many papers on the liquidity literature thought that it is hard to choose the value of liquidity parameter as the reflection of realistic levels of liquidity. The paper by Rogers & Singh (2010) introduced an example that a quant with experience of trading equities thought "*a dawn raid of* 10% *liquidity cost or so of the shares will probably propel the market* 15% *higher*". We can examine the difference between our numerical solution and the empirical data in order to measure the realistic level of liquidity cost.

Implementation for the CRRA utility functions Further research on a general utility function would grateful help the optimal portfolio selection problem. Many papers adopted the CARA utility function (negative exponential utility) as the example to calculate the value function in the market since this specific form of utility allows us to eliminate one parameter of the value function (the risk-less asset wealth). The relevant dynamic programming modelling is much more computationally efficient. Once the utility function is chosen as the CRRA utility functions (e.g., the power or logarithmic utility functions), we have to compute the value function in full dimensions. For the exponential utility function, in each node of the binomial tree we have to store an array with values of the value function for all numbers of shares from a prespecified grid. However, for the CRRA utility functions, one more dimension is needed, resulting in a matrix at each node of the binomial tree. This will increase the memory requirement as well as the computation time precisely as many times as the size of the grid of the additional variable. One possible way to deal with the CRRA utility is to use a new discretization method. That method had been introduced in Palczewski *et al.* (2013). They described a dimensionality reduction, non-equidistant discretization method which provides a variable range of stock holdings for different nodes.

Considering the logarithm or power utility function, we have to ensure that the terminal wealth is positive. For the portfolio without option, it is simple to solve by only taking long positions in shares and bonds. For the portfolio with option, we need to verify the value of terminal wealth. One possible way is that once the terminal wealth is negative, we choose that value to minus infinity. That would give the algorithm a signal to abandon those trading strategies that lead to negative wealth. It might also happen that for some initial positions we cannot get the corresponding reservation prices.

Appendices

Appendix A

Algorithms for Implementation

A.1 Construction of stock prices in binomial model

Algorithm 1 shows the pseudo code of stock price process in binomial model.

Algorithm 1 Construction of Stock Prices in Binomial Model

Input: u, d, S_0, T, m
Set $\delta t = T/m$
Set $n = (m+1)(m+2)/2$
for $i = 0$ to $n-1$ **do**
    $S_i = S_0 \cdot u^i \cdot d^{m-i}$
end for

Algorithm 2 Dynamic Programming of Value Function

Input: u, d, $prob$, $S[0]$, m
Set num_discretization = (range_max - range_min) / δr + 1
for $n = 0$ to m **do** /* last period*/
    pos = (m + 1)(m + 2) / 2 + n
    S[pos] = S[0] $\cdot$ u^n $\cdot$ d^{m-n}
    for $i = 0$ to $num\_discretization$ **do** /*discretization*/
        Set x = range_min + δr $\cdot$ i
        Set $\phi = -x$
        Set $V^*[pos][i] = -exp(\gamma \cdot \phi \cdot S[pos])$
    end for

    for $L = m - 1$ to 0 **do** /*current period number*/
        Set interest = $\exp(r \cdot T/m \cdot (m - L))$
        for $n = 0$ to L **do** /*node number in current period*/
            pos = (L + 1) L / 2 + n
            S[pos] = S[0] $\cdot$ u^n $\cdot$ d^{L-n}
            Set pos_d = pos + L + 1
            Set pos_u = pos_d + 1

            for $i = 0$ to $num\_discretization$ **do** /*discretization*/
                Set x = range_min + δr $\cdot$ i
                for $j = 0$ to $num\_discretization$ **do**
                    Set x = range_min + δr $\cdot$ j - x
                    Set $\phi = \phi(x))$
                    Set $c = exp(\gamma \cdot \phi \cdot S[pos] \cdot interest)$
                    Set $V[pos][j] = c\left\{prob \cdot V[pos\_u][j] + (1-prob) \cdot V[pos\_d][j]\right\}$
                    if $j == 0$ **then**
                    else/*store maximal value function*/
                        if $V[pos][j] > V^*[pos][j]$ **then**
                            Set $V^*[pos][j] = V[pos][j]$
                        end if
                    end if
                end for
            end for

        end for
    end for
end for
Output: $V^*\left[(m+1)(m+2)/2\right]\left[num\_discretization\right]$

A.2 Dynamic programming of value function

A.3 Recursive algorithm for checking optimal strategy

Algorithm 3 **calc_strategy(L, strategy_pos, m, min_x, δr)**: this recursive function is for Checking Optimal Startegy in All Nodes of non-recombining tree.

```
Find start point of this recursion:
current_pos = (initial shares - min_x) / δr
strategy[strategy_pos] = ΔX[current_pos]

if current level L < number of periods m then
    strategy_pos_d = 2 strategy_pos + 1
    strategy_pos_u = strategy_pos_d + 1

    calc_strategy(L + 1, strategy_pos_d, m, min_x, δr)
    calc_strategy(L + 1, strategy_pos_u, m, min_x, δr)
end if
```

A.4 Determination of ranges of stock holdings

A.5 Verification of expected utility by Monte Carlo

A.6 Computation of implied volatility

Algorithm 4 **Determination of Ranges of Stock Holdings**

```
Input: min_x, max_x
for k = 1 to l do
    for i = 1 to m do
        if i == 1 then
            range_min = strategy_i
            range_max = strategy_i
        else
            if i == m then
                if min_x ≤ strategy_i ≤ min_x + 0.5δr then
                    Set lower_hit = lower_hit + 1
                else
                    if max_x − 0.5δr ≤ strategy_i ≤ max_x then
                        Set upper_hit = upper_hit + 1
                    end if
                end if

            else
                if strategy_i < range_min then
                    range_min = strategy_i
                end if
                if strategy_i > range_max then
                    range_max = strategy_i
                end if
                if min_x ≤ strategy_i ≤ min_x + 0.5δr then
                    Set lower_hit = lower_hit + 1
                else
                    if max_x − 0.5δr ≤ strategy_i ≤ max_x then
                        Set upper_hit = upper_hit + 1
                    end if
                end if
            end if
        end if
    end for
end for
Output: rang_min, range_max, lower_hit, upper_hit
```

Algorithm 5 **Verification of Expected Utility by Monte Carlo**

Input: u, d, $prob$;
for $k = 1$ to l **do**
  for $i = 1$ to m **do**
    Compute an $U(0,1)$ sample ϵ
    if $\epsilon < prob$ **then**
      $S_{i+1} = S_i \cdot u$
    else
      $S_{i+1} = S_i \cdot u$
    end if

    Set $W_i = W_i - S_i \cdot \phi \cdot \exp(r \cdot \frac{T}{m} \cdot (m - i))$
  end for

  Set $EU_k = -\exp(-\gamma \cdot W_{m,k})$
end for

$EU^* = \frac{1}{l} \sum_{k=1}^{l} EU_k$
Output: EU^*

Algorithm 6 Generation of Implied Volatility

Input: $\mathcal{C}_m$, $accuracy$
Initialise $\sigma = \mathcal{C}_m / (0.398 \cdot S_0 \cdot \sqrt{T})$
if $\sigma < 0.3$ **then** $\sigma = 0.3$
end if
for $i = 1$ to 100 **do**
  $\mathcal{C}_{BS}$ = Function of Black-Scholes (σ)
  $diff = \mathcal{C}_m$ - $\mathcal{C}_{BS}$
  if abs($diff$) $< accuracy$ **then**
    return σ
  else
    Set $d_1 = (\ln(S_0/K) + r \cdot T)/(\sigma \cdot \sqrt{T} + 0.5 \cdot \sigma \cdot \sqrt{T})$
    Set $\kappa = 1/\sqrt{2\pi} S_0 \exp(-rT) \exp(-0.5 d_1^2) \sqrt{T}$
    Set $\sigma = \sigma + diff/\kappa$
  end if
end for
Output: σ

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