Coalgebraic Cell Complexes

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Abstract

The main aim of this thesis is the definition of **CellCx**, the category of relative cell complexes generated from a given small category of generating maps. We establish sufficient conditions for this definition to work and give us a category that we can prove is equivalent to the left map category for the algebraic weak factorisation system (AWFS for short) generated by Garner's small object argument applied to the same generating maps. These sufficient conditions take the form of a special kind of nerve functor on the underlying category, and some properties the generating maps are required to satisfy with respect to that nerve functor. In particular, they isolate a special class of inclusion maps which we call *typical inclusions*; every cell complex will have an underlying map which is a typical inclusion.

We also give a survey of the current understanding of the semantic structure (left and right maps) that an AWFS determines. This includes a theorem that left and right map structures are always determined entirely by their lifting structures; this establishes that any AWFS is a fixed point for an adjunction which can be viewed as a higher order Galois connection.

Additionally, we prove that computads for globular operads are a special case of cell complexes. This is a result that has been present in the folklore for a while, but could not be fully formalised without the definition of cell complexes we establish here. Finally we consider the question of when cell complexes can themselves be expressed as presheaves; we prove a number of results for different examples and find some positive and some negative cases of this property. At the end we will connect this to the work of Batanin (see [Bat02]) in which he establishes conditions for this property to hold for computads.

Contents

Introduction

In this introduction we will first spend some time describing the general landscape of the subject and where this thesis sits in that picture. We will then turn to an overview of what is done in each chapter and how the different sections fit together. Hopefully this brief explanation will provide sufficient detail for a busy reader to get a quick understanding of the main features of the work, or to more efficiently navigate the rest of the thesis.

Background Context.

Despite the fact that this is certainly a thesis contained within the subject of category theory, most of the ideas we explore have their genesis somewhere in the field of homotopy theory. This is probably not that unusual; as we will see, homotopy theory and category theory have an illustrious history of mutual assistance and inspiration. The central and eponymous definition of *coalgebraic cell complexes* is essentially a more categorical version of the *relative cell complexes* that appear in Quillen's small object argument. The *algebraic weak factorisation systems* that we study here were first defined by Grandis and Tholen (see [GT06]) as a stricter version of the *weak factorisation systems* that are ubiquitous in model category theory.

We could continue listing the ideas that are derived from homotopy theory for some time; for this reason this context section will firstly concern itself with a survey of model category theory as developed by Quillen in [Qui67, Qui69] and presented in a very accessible form by Hovey in [Hov99]. We will then describe how the subject of algebraic factorisation systems grew out of model category theory, originally with Grandis and Tholen and then later in the work of Garner and Riehl. Finally we will quickly sketch the related field of higher category theory as one of the final aims of this thesis is to show how cell complexes and factorisation systems can be applied in this area.

Model Categories. The theory of model categories developed from the observation that the subject of homotopy theory can be studied mostly through the prism of three special classes of maps in the category of spaces: the *fibrations*, the *cofibrations*, and the *weak equivalences*. We will not formally define these, since we can actually make different choices depending on exactly what type of homotopy theory we wish to do. It suffices to say that fibrations are surjective maps that behave especially nicely, cofibrations are injective maps that behave especially nicely, and weak equivalences are maps that preserve all the homotopical information of a space (formally, they induce isomorphisms on homotopy groups).

The ‘especially nice’ behaviour that we mentioned above is defined in terms of *lifting properties*. A morphism f has the *left lifting property* with respect to a morphism g if, given any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet, \end{tikzcd}$$

there exists some dotted arrow as shown that makes both triangles commute. A sensible homotopy theory will demand that every cofibration has the left lifting property with respect to every trivial fibration, and dually every trivial cofibration should have the left lifting property with respect to every fibration (a trivial (co)fibration is one that it is also a weak equivalence).

Originally, homotopy theory was studied solely in the category of topological spaces. As the emphasis on fibrations, cofibrations and weak equivalences became more and more important, the subject gradually shifted towards the more general context of a *model category*. This concept was first explicitly defined by Quillen in the first chapter of [Qui67] as a way of expressing the connections between homotopy theory and homological algebra. In a model category we have two weak factorisation systems which determine between them the three classes of maps we care about; thus any morphism can be factorised into a cofibration followed by a trivial fibration, or a trivial cofibration followed by a fibration.

The basic aim of the definition of model categories is to provide sufficient structure to perform *localisation* with respect to the weak equivalences. The idea is that weakly equivalent spaces can be seen as ‘the same up to homotopy’, and one wishes to work with spaces in a way which only cares about homotopical differences. To localise with respect to the weak equivalences means to replace them freely with isomorphisms and obtain what we call the *homotopy category*. In general, the localisation process is a construction fraught with foundational difficulties; the model category axioms are designed to provide a way of avoiding these problems.

While the definition of a model category comes with a number of axioms which the factorisation systems must satisfy, in many examples the most difficult thing to establish is simply the existence of the factorisations themselves. The most common approach to the problem of constructing such factorisations is a somewhat daunting transfinite induction known as the *small object argument*; it appears originally in the second chapter of [Qui67].

The small object argument works for model categories that are defined in a specific way; we begin with two sets of morphisms J and J', which we call the *generating cofibrations* and the *generating trivial cofibrations*. We then use these sets to generate the classes of fibrations and trivial fibrations using the lifting properties. We also consider a special class of morphisms called the relative J-cell complexes: these are defined as all the morphisms that can be obtained from J using only the operations of pushout, composition (including transfinite composition) and coproducts. The relative J-cell complexes are important because they turn up very naturally in the small object argument. It constructs a factorisation by building a cell complex (which will be the left hand side of the factorisation) in a transfinite sequence of pushouts; at each stage we are explicitly adding new lifts to the right hand map, until after some ordinal number of steps (this number depends on the smallness conditions which give the argument its name) the right hand map has become a fibration. Since the class of (trivial) cofibrations is closed under pushout, composition and coproducts we can see that any J-cell complex is also a trivial cofibration—so the factorisation we have built is of the sort we require.

The small object argument allows us to construct standard model category structures on topological spaces, simplicial sets, chain complexes, and many other categories which seem to contain some notion of homotopy. One of the important early achievements of the subject was the proof that topological spaces and simplicial sets are *Quillen equivalent*. This is a formal way of expressing the fact that the homotopical information in each model category is essentially the same; in particular the localisations obtain equivalent homotopy categories.

The author's understanding of model categories is based primarily on the exposition in [Hov99]. Another good source for the basic material is [DS95].

Algebraic Weak Factorisation Systems. Historically speaking, the first type of factorisation system studied was the *orthogonal factorisation system*. This is a very strict structure, involving two classes of maps which have lifts (as defined above) against one another; however (in contrast to the weak factorisation systems used in model categories) for an orthogonal factorisation these lifts must be unique. See [FK72] for an early reference on orthogonal factorisation systems; the idea may originally have appeared in an unpublished article of Barr or in the related concept of *regular D-pairs* introduced by Ringel in [Rin70]. The next type of factorisation system was the *weak factorisation system* which became an integral part of model category theory. The paper [Bou77] was one of the first publications to discuss these factorisation systems, though they are also present (slightly less explicitly) in [Qui67].

Weak factorisation systems are great for doing homotopy theory, but the non-uniqueness of lifts has some undesirable consequences from a categorical perspective. In an orthogonal factorisation system, the left and right map classes have the nice property of being closed under all colimits and limits, respectively. In the weak case one loses this; *some* colimits and limits exist, but important cases (like coequalisers and equalisers) are not guaranteed. This was the original consideration which led Grandis and Tholen to propose (see [GT06]) a new type of factorisation system which would combine the best features of both orthogonal and weak factorisation systems. They called it the *natural weak factorisation system*; the name was gradually changed to algebraic weak factorisation system as the theory developed.

In an algebraic weak factorisation system, the lifts are not unique—instead, they are *specified* by an explicit choice of lifts for every left map and right map pair. A necessary complication arises because of this: the left and right maps are no longer simply classes of morphisms but categories of morphisms. We will view them as categories equipped with 'forgetful' functors to the arrow category, such as $U\colon \mathbf{L\text{-}Map} \to \mathcal{C}^{\mathbf{2}}$. A single morphism in $\mathcal{C}$ may have many *left or right map structures* and different structures will give different lifts between pairs. This extra complexity comes in a very satisfying package, however, because the algebraic weak factorisation system can be neatly expressed as a pair of endofunctors (L, R) on $\mathcal{C}^{\mathbf{2}}$ where L is a comonad and R is a monad. The comonad is the left hand side of the factorisation, and the left maps are simply its coalgebras; the definition is entirely symmetric so the monad is the right hand side and the right map category is its category of algebras.

Given a commutative square between a left map and a right map, the coalgebra and algebra structures can be used to construct a lift in an extremely natural way. In the case of a weak factorisation system the lifts had to be part of the definition, so it is a very pleasing feature of algebraic weak factorisation systems that everything drops out automatically from the algebraic structure. Another thing worth pointing out is the fact that orthogonal factorisation systems can now be viewed as a special case; if the comonad and monad are both required to be idempotent then the left and right map structures become unique and their lifts must also be so.

The discovery that really attracted interest to this new type of factorisation system was the adaptation of the small object argument to the new algebraic definition. Due to Garner (see [Gar07, Gar12b]) this *algebraic small object argument* made a number of improvements (at least from the categorical point of view) on the original argument. It showed how the construction was related to many other general transfinite constructions (studied in [Kel80], for example) and it forced the transfinite sequence to converge, thus eliminating the necessity of making an arbitrary choice of ordinal number at which to stop. Furthermore, it extended the possible input data from a set of generating cofibrations to a small category of generating cofibrations. Finally, with the new argument the resulting algebraic weak factorisation system possesses a reasonable universal property with respect to the original category of generators, making the whole construction much more natural from a category theoretic perspective.

Some applications of these factorisation systems in homotopy theory were developed by Riehl, who defined the notion of an *algebraic model structure* in which the weak factorisation systems are replaced with two algebraic ones. She developed this theory in [Rie11a], and went on to explore monoidal algebraic model structures in [Rie13b] (see also her Ph.D. Thesis [Rie11b]). She also worked together with Barthel in using the algebraic understanding of factorisation systems to help construct weak factorisation systems and hence model structures that had proved hard to construct using the regular small object argument (see [BR13]). Another good source for Riehl's approach to homotopy theory, including her use of algebraic factorisation systems, is her book [Rie13a].

Algebraic factorisation systems have also been used by Garner to help understand weak morphisms between higher categories (see [Gar08]) and, in computer science, to express the syntax of operation sequencing in an abstract mathematical framework (see [Gar12a]). The author's own contribution to the subject began with [Ath12], which proved a conjecture of Garner regarding the left maps for the standard algebraic factorisation system on the category **Top**. That paper provides the starting point for the work contained in this thesis; the content of Chapter 3 will be a generalisation of the conjecture and proof contained in the paper.

Higher Category Theory. The subject of higher category theory studies structures that are similar to categories in the sense that they have various types of arrows that can be composed together. However, these structures are more complicated because they are in some sense higher dimensional; a set is a 0-dimensional object because it consists only of points, and similarly a category can be viewed as 1-dimensional because there are morphisms connecting those points together. Higher categories take this further by introducing 2-morphisms between the morphisms, 3-morphisms between those, and so on as far as you wish to go.

We should first of all mention the deep and long-standing connection between higher category theory and homotopy theory. When one moves from the world of sets to the world of categories, one adds morphisms, which can be seen as paths between points. Following this philosophy, a 2-morphism in a 2-category is a 'path between paths', or, in other words, a homotopy. This continues to the higher dimensional morphisms, and can be made formal in the definition of the *fundamental n-groupoid* of a space. This is an n-category which has as objects the points of the space, as morphisms the paths between points, as 2-morphisms the homotopies between those, as 3-morphisms the homotopies between homotopies, and so on. Identities are defined as constant homotopies and composition is defined by concatenation. At the top dimension, we can quotient out the n-dimensional homotopies by the equivalence relation given by $(n+1)$-dimensional homotopies.

This example illustrates the central challenge of higher category theory: most of the morphisms in this n-category do not satisfy the associativity and unit axioms. They do, however, satisfy these axioms up to homotopy—or rather, up to morphisms in the next dimension up, called *coherence morphisms*. The existence of the coherence morphisms acts as a 'weakened' version of the normal axioms of category theory and hence we call such a higher category a *weak n-category*. Basically every actual example of a higher category that appears in practise is weak, so clearly to understand higher category theory we must understand weak n-categories.

Unfortunately, as we go up in dimension it becomes increasingly intractible to define the coherence morphisms and the axioms they must themselves satisfy; explicit hands-on definitions exist for dimension two [Lei98, Ben67], dimension three [Gur07, Gur13, GPS95] and even dimension four in a less complete, unpublished, form [Tri95]. Trimble's *tetracategories* are, however, more of an exercise in making a point than a serious attempt at a usable definition—by the time we are at dimension four the coherence data is so complex that it is a nightmare to simply write down, let alone work with! Thus more recently the main thrust of higher category theory has been to try finding clever ways to make general definitions that work for any dimension n.

We will not describe the different definitions in very much detail here; two good sources for a good overview and comparison are [Lei02] and [CL04]. As Leinster points out, the definitions can broadly be classified into two types, which we call *algebraic* and *non-algebraic*. In the non-algebraic definitions, an n-category is some kind of geometric data (such as a simplicial set, or multisimplicial set) that is required to satisfy the property that certain cells—composites, coherence cells, and so on—must exist. In the algebraic definitions, on the other hand, the composites and coherence cells are explicitly chosen by the algebraic structure of the n-category. Hence a pair of composable morphisms in a non-algebraic n-category may have several valid composites; but in an algebraic version we are given a particular choice which we could call *the* composite. This is philosophically very similar to the relation between (non-algebraic) weak factorisation systems and algebraic weak factorisation systems.

The definition that we care about most for the purposes of this thesis is the one due to Batanin [Bat98b], and refined by Leinster [Lei04a, Lei04b]. It involves the use of *globular operads*, which are a kind of operad defined using the presheaf category of *globular sets* (which is a generalisation to higher dimensions of the directed graphs that underly the usual notion of category). A globular operad essentially specifies all the operations possible in an n-category; this rather elegantly includes compositions, identities and coherence cells in a single notion of 'operation'. A fully weak n-category is defined as an algebra for the globular operad that is initial in the category of globular operads equipped with an extra piece of structure called a *contraction*—this is exactly the structure necessary to ensure that algebras behave sufficiently like categories.

The decision to use the initial object of the category of globular operads with contraction means that we end up with the 'weakest possible' notion of n-category based on globular operads. This is good in that it gives us the most general case, but also has a downside; generally speaking, the weaker a definition is the harder it will be to use in practise. In the two-dimensional world, bicategories are much harder to work with than strict 2-categories; similarly for three dimensions, strict 3-categories are not too bad, but tricategories are very complicated indeed. The difficulties in these low dimensional examples can be mitigated by the use of what are called *coherence theorems* which show that any weak bicategory or tricategory is equivalent (in some appropriate sense) to a stricter type of 2 or 3-category. In the case of bicategories, this takes us all the way to strict 2-categories, while for tricategories there are some weak behaviours the strict world cannot model; instead we use the intermediate notion of *Gray-category* which is only slightly weaker than the fully strict definition. The interested reader can consult [Gur13] for a complete picture of coherence theorems at dimensions 2 and 3.

Based on the situation for low-dimensional examples, one current problem in higher category theory is to prove some kind of general coherence theorem that works for any dimension n. The first question this suggests is to ask what sort of n-category would play the role that Gray-categories play at $n = 3$. Batanin coins the term *semi-strict* n-category, and makes a conjecture about the existence and definition of such a notion, in [Bat02]. He also conjectures that different levels of strictness in globular operads are related to the behaviour of *computads* for those globular operads. A computad is a generalised piece of generating data for an n-category; they were introduced by Street in [Str76] but extended by Batanin in [Bat98a] (see also [Gar08] for a cleaner definition). Computads are of particular interest here because they can be seen as a special sort of cell complex (this is the main point of Chapter 4) and at the end of Chapter 5 we will discuss the results of [Bat02].

Another ongoing subject of research is the problem of comparing the different definitions of higher category. Some comparisons at low dimensions can be found in [Lei02] and [Gur09]. There exist a few results for general n that deal with non-algebraic definitions; the situation for algebraic definitions is less good and there are very few known results that compare an algebraic definition with a non-algebraic definition. The Ph.D. Thesis of Cottrell [Cot13] demonstrates a comparison nerve functor between the definition of Penon (which is algebraic) and that of Tamsamani and Simpson (which is non-algebraic); it also includes in its introduction a good summary of the currently known comparison theorems.

The present thesis may contribute to this project indirectly; in it we suggest that cell complexes provide a logical generalisation of computads. Computads have helped us understand some of the algebraic definitions, so it could be valuable to find ways of defining analogous objects for the non-algebraic definitions.

We should end this discussion of higher category theory by mentioning a very active area of research—the subject of *homotopy type theory*. This is an area that builds upon Martin-Löf type theory to create a formal theory of homotopy types. The most natural models for this formal theory are so-called ∞-groupoids, which are ω-categories (defined in a non-algebraic way) such that all cells have inverses. This area demonstrates again the extremely close ties between higher category theory and homotopy theory; the axioms of homotopy type theory include some which imply the existence of a factorisation system similar to the ones found in model category theory. A good introduction for the interested reader is the recent book [UFP13].

Overview of Thesis.

In the rest of the introduction we will summarise the main points of each chapter in this thesis, and explain how they fit together. The first two chapters consist mainly of known background material. There is a little bit of original work here which we will point out; but for the most part these two chapters are an exposition of existing work. Some of it may not have been presented in quite this form before, and there are some parts that are so much part of the 'folklore' that it is very hard to find references!

The third chapter contains the essential core of the thesis—the definition of *cell complexes* and the proof of the main result, Theorem 3.5.1, which states that cell complexes are (given certain conditions) the left maps for a known algebraic weak factorisation system. This chapter can be viewed as analogous to the author's paper [Ath12], with everything being done in much greater generality. The fourth chapter is all about connecting the definition of cell complexes to the visibly similar definition of computads (as given in [Gar08]). In it we prove some general results about comparing alternative definitions of the category of cell complexes and we explain how computads fit into this framework.

The final chapter discusses the property of *corporeality* which expresses when we can prove that the category of cell complexes is equivalent to a presheaf category. In it we consider a number of different examples, including any presheaf category, various categories of spaces and, in the final section, n-categories according to the globular operad definition. In this final case we discuss the close links to [Bat02].

Chapter 1: Background Theory. This chapter is a tour of a number of different pieces of theory that will be used throughout the thesis. The first is the *nerve-realisation adjunction*, which is a standard (and very much part of the folklore) way of constructing an adjunction between a presheaf category and any cocomplete category. Section 1.1 provides the details of this construction. Section 1.2 is, as far as the author is aware, original work. We study some conditions that are sufficient to imply that the nerve-realisation adjunction is comonadic. One of the results in this section (Proposition 1.2.5) will be useful in Chapter 3. The rest is included for completeness and as a kind of warm-up or foreshadowing of the result in Chapter 3—the nerve-realisation adjunction can in fact be seen as a very special case of the cell complexes construction, and the result of Chapter 3 is also about checking an adjunction is comonadic.

Next we discuss the topic of *Galois connections*. These are also often seen as part of categorical folklore; a Galois connection is really just an adjunction between two pre-ordered sets, though they are often studied as an interesting type of structure in their own right. In particular we will consider how a Galois connection can be generated from a relation between two sets. This construction turns out to provide an illuminating perspective on factorisation systems of all three types.

In Section 1.4 we turn to the subject of factorisation systems, first considering orthogonal and then algebraic weak factorisation systems. This section is really just a brief introduction; algebraic weak factorisation systems are such a central concept that they are covered in much more detail in Chapter 2. The final section provides an introduction to the parts of higher category theory that will be necessary for the work in Chapters 4 and 5.

Chapter 2: Syntax and Semantics. In this chapter we home in on what is perhaps the most important concept for this thesis—the concept of an algebraic weak factorisation system. There are essentially two sides to such a factorisation system: the *syntax* is the monad and comonad that make up the actual factorisation data, while the *semantics* is the two categories of left and right maps that appear as the coalgebras and algebras.

We begin this chapter by focusing on the semantics side. In Section 2.1 we discuss the main properties and structures that the left and right map categories must have. Most importantly we describe the three 'colimit-like' constructions possible with left maps: colimits, pushouts along arbitrary maps and composition. In Section 2.2 we continue studying the semantics by proving that all of the (co)algebraic structure of a left or right map is captured by the solutions to lifting problems. This is an interesting result because it shows how a construction similar to generating a Galois connection from a relation can be used to construct an adjunction (which we call the *liftings adjunction*) that expresses the correspondence between left and right map categories. This allows us to view algebraic weak factorisation systems through the lens of lifting structures in the same way that we often view orthogonal or weak factorisation systems through the lens of lifting properties.

In the third section we consider another aspect of the semantics for an algebraic weak factorisation system. This is the construction of *weak morphisms* that the factorisation system allows; it is a sufficiently important part of the structure given by the factorisation system that the author feels this chapter would not be complete without this discussion. The work in this section is due to John Bourke (as far as the author is aware it is unpublished, though it is very closely related to the different sorts of morphisms studied in [Bou14]) and based on work by Garner (see [Gar08], and also [Gar12a]).

The final two sections are really the business end of this chapter since they discuss results that are vital for the work in Chapter 3. Section 2.4 is an exposition of the algebraic small object argument due to Garner, and as such all the material in it can be found in [Gar12b] (or in [Gar07], in a less developed form). The final section is spent proving Theorem 2.5.3, an adaptation of Beck's

Monadicity Theorem (see Chapter VI of [ML98]) to the special case of categories of left maps. Thus it gives a useful characterisation of when a category over $\mathcal{C}^{\mathbf{2}}$ is the left map category for some algebraic weak factorisation system. This final section is original work, though the theorem was also stated and proved independently by John Bourke; the two approaches are mostly the same, except for one small point of departure which we discuss at the end of the section.

Chapter 3: Cell Complexes. In the first two sections of this chapter we make the definition of **CellCx**, the category of cell complexes. We do this in two stages, of increasing sophistication; Section 3.1 restricts its attention to the case where the generating maps form a set, whereas Section 3.2 extends the definition to cases where the generating maps are considered as a small category over $\mathcal{C}^{\mathbf{2}}$. In each case, **CellCx** is defined using a transfinitely recursive iteration of the comma category construction on presheaves over the category of cell types $\mathcal{S}_{\mathcal{J}}$; given any cell complex A we define TA, called the *terminal layer* on A, to be the presheaf of all possible new cells that could be added to A. Then the data of an extra layer of cells on A is given by a presheaf of cells $\sigma \in \hat{\mathcal{S}}_{\mathcal{J}}$ together with a natural transformation $g\colon \sigma \to TA$ that tells us how those cells are glued onto A. This is an object of the comma category $(\hat{\mathcal{S}}_{\mathcal{J}} \downarrow T)$.

Section 3.3 introduces an axiomatisation of a structure on a category called a *typical nerve*. This is a nerve functor (in the sense of Section 1.1) that satisfies certain properties, and in particular induces a special class of inclusion maps called the *typical inclusions*. This is the structure on a category we require for the construction of **CellCx** to work properly with the small object argument; we finally describe some conditions a category of generating maps can satisfy with respect to the typical nerve, called the *typical conditions*, that are sufficient to prove the main result of this chapter. At the end of this section we prove two useful preliminary lemmas implied by the typical conditions: firstly Proposition 3.3.6, which says that any cell complex or subcomplex inclusion map is guaranteed to be a typical inclusion, and secondly Lemma 3.3.11 (the 'pullback lemma') which says that any cell complex morphism gives rise to a pullback square in $\mathcal{C}$.

Section 3.4 establishes a few properties of the category **CellCx**. The main aim here is to show it is complete and cocomplete; we do this by first observing that the functor ∂ which takes a cell complex to its base object is a bifibration. Finally, in Section 3.5 we check each of the conditions necessary to apply Theorem 2.5.3—most importantly, that $U\colon \mathbf{CellCx} \to \mathcal{C}^{\mathbf{2}}$ is conservative and preserves all equalisers. Once these checks have been done, we prove Theorem 3.5.1, the main result of the chapter (and indeed the thesis), which says that **CellCx** is exactly the category of left maps for the algebraic weak factorisation system given by the small object argument of Section 2.4.

Chapter 4: Computads. This is a shorter chapter in which we give the definition of *computads* for any globular operad, and show how they fit into the theory of Chapter 3. The first section simply states the definition, in much the same form as [Gar08] (a slightly neater presentation than [Bat98a], upon which it is based). The only novelty here is a note on the concept of *relative computads*; if a computad is viewed as a cell complex it will have a base space, which classically is always the empty n-category. In order to make the comparison with the definition of **CellCx** in Chapter 3 we will need to extend the usual notion of computad by allowing other base objects.

The only difficulty in comparing computads with cell complexes is that in a computad cells are adjoined strictly dimension by dimension, whereas the definition of **CellCx** simply adds as much as possible at each layer. This is not a genuine problem—it is really a question of adding the same cells but in a different order. To clarify this idea, in Section 4.2 we introduce the notion of a *stratification* on a category of cell complexes, which is a kind of 'normal form'; we prove a comparison theorem (Theorem 4.2.9) which works for any stratification and then establish that computads are a special case.

Chapter 5: Corporeality. In this final chapter we turn to a specific question one can ask about the category **CellCx**: when are all the fibres of ∂ (the subcategories of complexes which all have the same base object) equivalent to presheaf categories? This question is inspired by the interest there has been in the analogous question for the category of computads (which is, as we showed in Chapter 4, equivalent to the fibre of ∂ over the empty n-category). The question of when computads are presheaves has been studied, for example, in [Che12, Bat02].

Section 5.1 introduces the notion of *classifying complexes* and proves how the existence of sufficiently many classifying complexes implies that the fibres of **CellCx** are equivalent to presheaf categories. We use the term *corporeal* specifically to refer to a category, typical nerve and generating maps such that sufficient classifying complexes exist. This property says something rather strong about the geometric nature of colimits in that category; we spend a lot of this chapter seeing examples that are corporeal or that fail to be corporeal—in general, corporeality seems to imply a kind of rigidity or physicality of colimits that makes this property rather intriguing.

In Section 5.2 we prove that presheaf toposes are always corporeal. This is not surprising, but is a useful test case. The next section deals with topological spaces first, which fail to be corporeal for an interesting reason to do with infinite intersections of open sets. We then extend our attention to a general notion of *space-like* category, and come up with a simple condition that makes a space-like category corporeal; unfortunately the author is not currently aware of any actual examples of this, but we discuss the condition and conjecture that some nice category of corporeal spaces should exist.

The final section deals with categories of algebras for globular operads, and hence returns to the question of when computads are equivalent to presheaves. This problem has already been largely solved by Batanin (see [Bat02]) so there are no new theorems in this section. Instead, we discuss Batanin's results from the cell complexes perspective and hopefully shed some light on his approach. In particular, Batanin uses the notion of *strong regularity* for theories on the category of sets, and we prove in Lemma 5.4.2 that the condition of strong regularity is precisely equivalent to corporeality for a simple choice of generating map on the category of algebras.

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Chapter 1

Background Theory

In this chapter we will describe various pieces of standard category theory which make up the background for the results presented later in this thesis. There is a small amount of original work in Section 1.2. Besides that section, the chapter is entirely expository.

The first section introduces the nerve-realisation adjunction, which is fundamental to the definition of cell complexes in Chapter 3. The second section contains a few results about this construction, specifically regarding the question of when it produces a comonadic adjunction. This essentially foreshadows the results proved in the bulk of Chapter 3, which is also about checking that an adjunction is comonadic. That adjunction is constructed in a similar (though more complicated) way to the simple one presented here.

The next two sections are basically intended to introduce the notion of an *algebraic weak factorisation system*, which we will generally abbreviate to AWFS. The concept of a Galois connection, which is very interesting in its own right, will ultimately provide a rather nice conceptual approach to AWFS—though this will not become entirely clear until Chapter 2. Then, in the final section, we give some background in higher category theory (specifically the operadic definitions of higher categories) that will be useful in Chapter 5.

1.1 The Nerve-Realisation Adjunction

We begin by describing a very fundamental construction. It has been around for a long time and it underlies many other ideas in category theory; it will appear several times, in different contexts, in this thesis. It is one of most basic examples of a Kan extension; however, here we will discuss it from first principles without invoking too much abstract theory. We refer the reader to Chapter X of [ML98], which is devoted to Kan extensions in their full generality.

The nerve-realisation adjunction is now so entrenched in category theorists' general knowledge that it tends to be considered 'part of the folklore'. However, we should note that it was first introduced by Kan in [Kan58], while another important paper in the development of the idea is [DK84]. It was also studied by Isbell through the related concept of *adequacy* of subcategories—see [Isb60].

The nerve construction is about taking an arbitrary category and comparing it with some presheaf category; we obtain a functor which we think of as finding presheaf 'approximations' for objects in the original category. The realisation functor is the left adjoint to this approximation functor, which exists whenever the category in question has sufficient colimits.

The construction. We begin with an arbitrary category $\mathcal{C}$, the category we are interested in, and we consider any small category $\mathcal{S}$ together with a functor $I\colon \mathcal{S} \to \mathcal{C}$. We will sometimes call $\mathcal{S}$ the category of *shapes*. From this, we can determine a canonical functor $N\colon \mathcal{C} \to \hat{\mathcal{S}}$. (We will use the notation $\hat{\mathcal{S}}$ throughout to refer to the category of presheaves on $\mathcal{S}$—in other words, $\hat{\mathcal{S}} = [\mathcal{S}^{\mathrm{op}}, \mathbf{Set}]$.) To define N, we must first give a presheaf NA for each $A \in \mathcal{C}$. The formula

$$NA(j) = \mathcal{C}(Ij, A)$$

defines the sets that make up these presheaves. We must also define the restriction maps in each NA and see how N is itself a functor; this is easily done using precomposition and postcomposition functions. A morphism $a\colon j \to j'$ gives a function $NA(j') \to NA(j)$ by precomposition with Ia, and a morphism $f\colon A \to B$ gives a function $NA(j) \to NB(j)$ for any j by postcomposition with f. We call NA the *nerve* of the object A; the name comes from the example of nerves of categories, which we will see in a moment.

The left adjoint to N, when it exists, is called the *realisation* functor and will be written $\coprod$. There is a good reason for this coproduct notation; the realisation functor takes a presheaf σ to the colimit of the diagram

$$I_\sigma\colon \mathcal{E}_\sigma \to \mathcal{C}$$

where $\mathcal{E}_\sigma$ is the category of elements of σ and I_σ is defined by composing the projection functor onto $\mathcal{S}$ with the functor I. One can think of σ as containing instructions for building an object of $\mathcal{C}$; the realisation functor takes these instructions and uses a colimit to actually build the object. Of course, $\coprod$ only exists when $\mathcal{C}$ has sufficient colimits.

A 'geometric' intuition. In category theory, we take mathematical concepts such as groups and topological spaces that are traditionally studied as sets with structure, and replace them with abstract, structureless objects in a category. In this new way of thinking we cannot reason about the points in a space, or the elements in a group; instead we must reason about the continuous functions between spaces and the homomorphisms between groups.

For this reason, category theory is often described as mathematics done 'pointlessly'. However, this is not quite true; consider spaces viewed as objects of **Top** and you will see that the points are very much still there—they appear as the morphisms to a space from the terminal space. The difference is that now these points are joined by many other *figures* of different shapes, because we can reason in just the same way about morphisms to a space from the interval, or the circle, or any other space you choose!

It is a common technique to study an object in a category by considering the figures in it of certain shapes. Moreover, because these figures may intersect non-trivially, we should keep track of that information too. The construction above is exactly this idea made formal. We could call NA the presheaf of figures in A. It gives us a set of figures for each shape we are interested in, and furthermore it gives functions between these sets that tell us how the figures intersect. Let us see how this works with some explicit examples.

Representables. The simplest non-trivial example we could choose would be to make $\mathcal{S}$ the one-object category with a single morphism, and thus have I simply choose an object A of $\mathcal{C}$. Then the category $\hat{\mathcal{S}}$ is just **Set** itself, and the functor N, which takes an object of $\mathcal{C}$ to its set of A-figures, is more usually written as $\mathcal{C}(A, -)$. Thus we see that representable functors are a special case of nerve functors—or perhaps it is more true to say that a nerve functor is a kind of generalised representable functor.

Simplicial approximation. It is fascinating to think about all the possible choices of figure shapes in **Top** and the presheaves of figures they produce. Sometimes they capture very nearly all the information in a space, sometimes they can lose a lot—particularly in the case of the many pathological spaces encountered in point-set topology. The following example is the most commonly used approximation to topological spaces.

The *simplicial category*, Δ, is the category of finite, totally ordered sets (except for the empty set) and order preserving functions between them. We write $\mathbf{n}$ for the object of Δ with $n+1$ elements; this may seem odd, but the reason will become clear soon.

$$\begin{tikzcd}
\mathbf{0} \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathbf{1} \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \mathbf{2} \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift left=6] \arrow[r, shift right=6] & \mathbf{3} \ldots \arrow[l] \arrow[l, shift left=4] \arrow[l, shift right=4]
\end{tikzcd}$$

The arrows shown in this picture are the generating arrows of Δ—the order-preserving maps that either miss exactly one element, or double-up exactly one element. We call the former *coface* maps and the latter *codegeneracy* maps.

Now we define the functor $I \colon \Delta \to \mathbf{Top}$. It takes the object $\mathbf{n}$ to the n-dimensional *simplex*—that is the n dimensional disc, thought of as a polytope made out of the lower dimensional simplices; so $I(\mathbf{0})$ is the point, $I(\mathbf{1})$ the interval, $I(\mathbf{2})$ a triangle, $I(\mathbf{3})$ a tetrahedron and so on. In general we can write these explicitly as subsets of Euclidean space:

$$I(\mathbf{n}) = \{(x_i) \in \mathbb{R}^{n+1} \mid \sum x_i = 1, x_i \geq 0\}.$$

The $n+2$ coface maps $\mathbf{n} \to \mathbf{n+1}$ get mapped by I to the ways of including $I(\mathbf{n})$ in $I(\mathbf{n+1})$ as a face; explicitly, the jth one of these *face maps* acts by inserting a 0 in the jth coordinate. The *degeneracy maps* (the images of the $n+1$ codegeneracy maps under I) are slightly harder to describe. The jth degeneracy map squashes the jth face of $I(\mathbf{n+1})$ down into a single lower dimensional face of $I(\mathbf{n})$, while acting linearly on the rest of the space; explicitly we can write this map as

$$(x_1, x_2, \ldots x_{n+1}) \mapsto (x_1, x_2, \ldots x_{j-1}, x_j + x_{j+1}, x_{j+2}, \ldots x_{n+1}),$$

where we have reduced the dimension by one by adding together the jth and $(j+1)$th coordinates.

The category $\hat{\Delta}$ is called the category of *simplicial sets* and often written **SSet**, and the functor $N\colon \mathbf{Top} \to \mathbf{SSet}$ is called the *singular simplicial set* functor. In this case it has a left adjoint (**Top** is cocomplete) and this functor $\coprod\colon \mathbf{SSet} \to \mathbf{Top}$ is called the *geometric realisation* functor. This is the example that gave us the name 'realisation functor'.

What the singular simplicial set functor does is to look at all the simplex-shaped figures in a space, and consider how they join together as faces and degeneracies of one another. For most spaces, the number of such figures is uncountable, so the resulting simplicial set is very complicated.

The geometric realisation functor is somewhat easier to find simple finite examples for. If we consider an arbitrary simplicial set P, what $\coprod$ does is to use P as the 'specification' to build a colimit in **Top**. Each element of $P(\mathbf{0})$ specifies a point, each element of $P(\mathbf{1})$ specifies a line, and each element of $P(\mathbf{n})$ specifies an n-simplex. If we were simply to take the coproduct of this collection of points, lines, and so on, we would get a fractured realisation of P as a disjoint union of all these building blocks. But P also specifies, by way of the face and degeneracy maps, the instructions for gluing the points, lines and so on together; when we include all these functions in the diagram, the colimit will contain all the figure shapes specified, but glued together to make a geometric model of the simplicial set P. This adjunction is the underlying motivation for the idea that simplicial sets provide a combinatorial model for topological spaces; it is this that makes them one of the most important tools in modern algebraic topology.

The density comonad. It is worth considering briefly that when the left adjoint $\coprod$ exists, we obtain an endofunctor $N\coprod\colon \mathcal{C} \to \mathcal{C}$. Because of the adjunction, it has the structure of a comonad. It is known as the *density comonad* for the functor I.

We are particularly interested in the case when $\mathcal{S} \to \mathcal{C}$ is a full subcategory. One can then view the density comonad as a way of measuring how good the approximation is, and thus how well the objects in the subcategory can 'see' the whole category. In some cases, $N\coprod$ is actually the identity functor on $\mathcal{C}$; then we can see that the approximation was perfect, in the sense that it lost no information. In this case we say that $\mathcal{S}$ is a *dense subcategory* or an *adequate subcategory*. This is the notion studied in [Isb60].

Some examples involving Cat. One can define a simplicial approximation functor for **Cat** in a very similar way to that for **Top**.

Define a functor $I\colon \Delta \to \mathbf{Cat}$ that takes $\mathbf{n}$ to the category freely generated by n morphisms in a chain. Since the morphisms of Δ are order-preserving maps, we can have them act on the objects of the categories $I(\mathbf{n})$. The resulting maps on objects can be seen to be functors because of the existence of composites and identities.

This functor results in an adjunction $\coprod \dashv N$ between **Cat** and **SSet**; but this case differs from the topological simplicial approximation in a number of ways. Firstly, I is in fact the inclusion of Δ in **Cat** as a full subcategory. Furthermore, the density comonad $N\coprod$ is the identity, so Δ is an adequate subcategory of **Cat**. Hence the simplicial set $N\mathcal{A}$ for a small category $\mathcal{A}$ is a perfect representation of the category, which has historically been called the *nerve* of $\mathcal{A}$. However, it is interesting to note that using the entire category Δ

is actually unnecessary. Let us consider some much smaller subcategories.

Firstly, if we use only the object $\mathbf{0}$ of Δ, we get a functor $\mathbf{Cat} \to \mathbf{Set}$. Since $I(\mathbf{0})$ is the one-object category, the only figures our approximation sees are the object-shaped figures. Hence the functor we get takes a category to its underlying set of objects. This is an approximation of sorts, but it forgets all about the morphisms—so it loses rather a lot of information!

Next, consider using just the first two objects $\mathbf{0}$ and $\mathbf{1}$ of Δ and the three morphisms between them; presheaves on this two-object category are the same as digraphs (directed graphs) equipped with identities, known as *reflexive* digraphs. Our approximation will see object-shaped figures and arrow-shaped figures, and it will see how to get the identity arrow of an object. Thus the functor we get is the underlying reflexive digraph functor—it forgets the composition rule of a category, but remembers everything else.

Finally, what if we use the first three objects? Then we can see three different shapes of figures—object-shaped figures, arrow-shaped figures and composable pair-shaped figures. Most importantly, we can see how to compose a composable pair into an arrow. So in this case the approximation is perfect and the three-object subcategory in question is adequate.

1.2 Comonadicity of Nerve-Realisation

If from now on we assume that $\mathcal{C}$ is cocomplete, we can consider the nerve-realisation construction described above as an operation done on categories over $\mathcal{C}$. It extends $\mathcal{S}$ by freely adding all colimits—this gives the presheaf category $\hat{\mathcal{S}}$—and it replaces I with the realisation functor $\coprod$, which is the universal colimit preserving functor generated from I. One could view this as an attempt to freely generate a category of coalgebras over $\mathcal{C}$ from $\mathcal{S}$ and I. If we were to take any actual category of coalgebras for a comonad on $\mathcal{C}$ and choose a coalgebra structure for each object of $\mathcal{S}$, these choices would extend canonically to coalgebra structures for all presheaves in $\hat{\mathcal{S}}$ using colimits in the coalgebra category.

Following this philosophy, it is then very natural to ask when the attempt succeeds: when is the nerve-realisation adjunction actually comonadic? This is the question we will answer in this section; it turns out that there are a few nice properties of I that will tell us when this happens. We will also consider the case when the adjunction is not itself comonadic, but can be made comonadic by restriction to a reflective subcategory of $\hat{\mathcal{S}}$. The results in this section will become relevant later on because they provide a nice microcosm for the theorem in Section 3.5.

We will now introduce some more notation: we will write $[s]$ for the representable presheaf given by applying the Yoneda embedding to $s \in \mathcal{S}$. By the Yoneda lemma, an element a of shape s in a presheaf σ can be thought of as a natural transformation $a\colon [s] \to \sigma$. Applying $\coprod$ to this gives the inclusion map of a into the colimit, and so we will generally write each such inclusion as

$$s \xrightarrow{\coprod a} \coprod \sigma.$$

There are three conditions that the choice of shapes in $\mathcal{C}$ will be required to satisfy. In order for these to make sense, the category $\mathcal{C}$ needs to have pullbacks, so from now on this will be assumed; generally in the examples $\mathcal{C}$ is complete so this will not be a problem. The first condition is a basic requirement of consistency between colimits built out of the chosen shapes and pullbacks. We will need it to prove that the realisation functor preserves equalisers.

Definition 1.2.1. Suppose σ is a presheaf on $\mathcal{S}$ and we have a morphism into the colimit

$$P \xrightarrow{\quad p \quad} \coprod \sigma.$$

We form a new diagram in $\mathcal{C}$ by pulling back every inclusion map into $\coprod \sigma$ along p; this is a diagram $\mathcal{E}_\sigma \to \mathcal{C}$ defined by taking an element a to the object $(p|a)$ given by

$$\begin{tikzcd}
(p|a) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & s \arrow[d, "\coprod a"] \\
P \arrow[r, "p"'] & \coprod \sigma.
\end{tikzcd}$$

We say that (for this choice of $\mathcal{S}$) *colimits commute with pullbacks* if P, together with the arrows on the left of each pullback square, is a colimit cone for the new diagram.

The second condition is one we will need to show that $\coprod$ is conservative. It will ensure that the shapes are chosen in such a way that every colimit is 'non-degenerate'; every element must contribute something distinct to the colimit.

Definition 1.2.2. We say a choice of shapes in $\mathcal{C}$ satisfies the *repeated element condition* if there does not exist any presheaf σ on $\mathcal{S}$ with two elements a and a' of the same shape in σ such that $\coprod a = \coprod a'$ as objects of the slice category $\mathcal{C}/\coprod \sigma$.

The final condition will be used in the proof that $\coprod$ preserves equalisers; what it says is that $\coprod$ preserves a very small (and easy to check) class of equalisers:

Definition 1.2.3. A *simple equaliser* in any presheaf category $\hat{\mathcal{S}}$ is one with a diagram of the form

$$\begin{tikzcd}
{[s]} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \sigma.
\end{tikzcd}$$

where σ is any presheaf and a and b are any two elements of the same shape in σ.

Definition 1.2.4. We will say that the functor $\coprod$ *preserves simple equalisers* if every equaliser of a diagram of the above form is mapped by $\coprod$ into an equaliser in $\mathcal{C}$.

Now that we know the appropriate conditions we will go ahead and prove the two main results of this section.

Proposition 1.2.5. *Suppose that $\mathcal{C}$ is cocomplete and has all pullbacks, and that the choice of shapes $I\colon \mathcal{S} \to \mathcal{C}$ satisfies the repeated element condition. Then the realisation functor $\coprod$ is conservative.*

Proof. Suppose that $f\colon \sigma \to \tau$ is any natural transformation between two presheaves on $\mathcal{S}$, and suppose that the morphism $\coprod f$ in $\mathcal{C}$ is an isomorphism, with inverse g. We wish to show that this means that f itself must be an isomorphism. For a morphism in a presheaf category, this means showing that f is bijective on elements.

First suppose f is not injective. So there are distinct elements a and a' in σ such that $f(a) = f(a')$. This implies that $\coprod f \circ \coprod a = \coprod f \circ \coprod a'$. Composing each side with the inverse g shows that $\coprod a = \coprod a'$, which contradicts the repeated element condition.

Now suppose that f is not surjective. So there exists some b in τ such that there is no a in σ with $f(a) = b$. We will construct a new presheaf τ' which is essentially the same as τ but with an extra copy of b freely adjoined. We define τ' using

$$\tau'(s) = \{(t, c) \mid t \in \tau(s), c\colon \mathcal{E}_\tau(t, b) \to \{0, 1\}\}$$

where $\mathcal{E}_\tau(t, b)$ is the set of morphisms in the category of elements, so the function c specifies a set of choices between the two copies of b. A morphism $\alpha\colon s' \to s$ in $\mathcal{S}$ acts in τ' as

$$\alpha(t, c) = (\alpha(t), \alpha^* c)$$

where $\alpha^* c$ is constructed by applying c to the obvious composite $t \to \alpha(t) \to b$ in $\mathcal{E}_\tau$.

Now observe that we have a few morphisms in the category of presheaves

$$\begin{tikzcd} \sigma \arrow[r, "f"] & \tau \arrow[r, bend left, "i_0"] \arrow[r, bend right, "i_1"'] & \tau' \arrow[l, "g"'] \end{tikzcd}$$

where i_0 and i_1 are the natural inclusions $b \mapsto b_0$ and $b \mapsto b_1$ respectively, and g is the natural map that projects both copies back onto the original b. Since b is not in the image of f, we have $i_0 f = i_1 f$, which implies that $\coprod i_0 = \coprod i_1$ because $\coprod f$ is an isomorphism. But then

$$\coprod b_0 = \coprod i_0 \circ \coprod b = \coprod i_1 \circ \coprod b = \coprod b_1$$

and this contradicts the repeated element condition. □

Proposition 1.2.6. *Suppose that $\mathcal{C}$ is cocomplete and has all pullbacks, and that the choice of shapes $I\colon \mathcal{S} \to \mathcal{C}$ gives colimits that commute with pullbacks. Suppose also that the realisation functor $\coprod$ preserves simple equalisers. Then it preserves all equalisers.*

Proof. Consider some pair of parallel morphisms in the presheaf category, f and g, and suppose that they have equaliser (ϵ, e) as shown below:

$$\begin{tikzcd} \epsilon \arrow[r, "e"] & \sigma \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \tau. \end{tikzcd}$$

We wish to show that $(\coprod \epsilon, \coprod e)$ is the equaliser of $\coprod f$ and $\coprod g$ in the category $\mathcal{C}$. We know exactly what ϵ is in set-theoretic terms; its elements are exactly the elements of σ that satisfy $f(a) = g(a)$, and e is just the inclusion map. Now suppose we have some other morphism in $\mathcal{C}$ that equalises $\coprod f$ and $\coprod g$, such as p in the diagram

$$\begin{tikzcd} \coprod \epsilon \arrow[r, "\coprod e"] & \coprod \sigma \arrow[r, shift left, "\coprod f"] \arrow[r, shift right, "\coprod g"'] & \coprod \tau. \\ P \arrow[u, dotted] \arrow[ur, "p"'] & & \end{tikzcd}$$

We need to construct the dotted morphism, and show it is unique.

We are going to approach this (using the assumption that colimits commute with pullbacks) by 'deconstructing' the object P as the colimit of the $(p|a)$ for all elements a in σ. When a has the property that $f(a) = g(a)$, we can consider it as an element of ϵ so we have the morphism

$$(p|a) \xrightarrow{\pi_s} s \xrightarrow{\coprod a} \coprod \epsilon.$$

What about an element a of σ which is not in the equaliser? Then we have two distinct maps $\coprod fa\colon s \to \coprod \tau$ and $\coprod ga\colon s \to \coprod \tau$. The equaliser of the two maps

$$\begin{tikzcd} {[s]} \arrow[r, shift left, "fa"] \arrow[r, shift right, "ga"'] & \tau \end{tikzcd}$$

in $\hat{\mathcal{S}}$ is a simple equaliser, which we will write $[fa, ga]$. The composite

$$[fa, ga] \longrightarrow [s] \xrightarrow{a} \sigma$$

equalises f and g, and hence induces a map $(fa, ga)\colon [fa, ga] \to \epsilon$.

By assumption, $\coprod$ preserves simple equalisers, so $\coprod[fa, ga]$ is the equaliser of $\coprod fa$ and $\coprod ga$. The projection map $\pi_s\colon (p|a) \to s$ equalises $\coprod fa$ and $\coprod ga$ because p equalises $\coprod f$ and $\coprod g$, so we obtain a map into $\coprod[fa, ga]$ and then compose

$$(p|a) \longrightarrow \coprod[fa, ga] \xrightarrow{\coprod(fa,ga)} \coprod \epsilon$$

to get a map into $\coprod \epsilon$. We note that in the case that $f(a) = g(a)$ we have $[fa, ga] \cong [s]$, so this map—which is defined for all a in σ—reduces to the map we defined earlier in this special case.

For any map $\alpha\colon t \to s$ in $\mathcal{S}$ there is an induced map $(p|\alpha(a)) \to (p|a)$, and we wish to check that these commute with the maps into $\coprod \epsilon$. To do this we observe there is also an induced map of presheaves $[f\alpha a, g\alpha a] \to [fa, ga]$, which commutes (via the inclusions) with $[\alpha]\colon [t] \to [s]$.

By assumption, the diagram consisting of the $(p|a)$ has P as its colimit, and we have just constructed a cone from this diagram to $\coprod \epsilon$. Thus the colimit condition induces a morphism

$$P \longrightarrow \coprod \epsilon$$

which commutes with $\coprod e$ since each of the maps from $(p|a)$ does. This is the dotted arrow we need in the original equaliser diagram; we must check it is unique. If we had some other map with the same property we could 'deconstruct' it into its action on the $(p|a)$; but each of these is uniquely determined by the equaliser condition on $\coprod[fa, ga]$. $\square$

Corollary 1.2.7. *Suppose that $\mathcal{C}$ is cocomplete and has all pullbacks, and that the choice of shapes $I\colon \mathcal{S} \to \mathcal{C}$ gives colimits that commute with pullbacks. Suppose also that the realisation functor $\coprod$ preserves simple equalisers, and that the repeated element condtion is satisfied. Then the nerve-realisation adjunction is comonadic.*

Proof. The two propositions above give the necessary conditions to apply (the dual of) Beck's monadicity theorem. □

Failure of the repeated element condition. It turns out that when the repeated element condition fails, but the other two conditions remain true, there is an interesting characterisation of the coalgebra category. In fact, it is the subcategory of sheaves for a particular Grothendieck topology on $\mathcal{S}$.

Definition 1.2.8. A *sieve* on $s \in \mathcal{S}$ is a subpresheaf of the representable presheaf $[s]$, say $c\colon C \rightarrowtail [s]$. We say a sieve is an *$I$-cover* if the image

$$\coprod C \xrightarrow{\coprod c} Is$$

is an isomorphism in $\mathcal{C}$.

Proposition 1.2.9. *The collection of sieves that are I-covers, as defined above, forms a Grothendieck topology on $\mathcal{S}$. This means the following three axioms are satisfied:*

1. *for each $s \in \mathcal{S}$ the maximal sieve on s is a cover,*
2. *given $\alpha\colon t \to s$ in $\mathcal{S}$ and a cover C on s, the pullback sieve defined as*
$$\alpha^* C = \{\beta\colon u \to t \mid \alpha \circ \beta \in C\}$$
is a cover,
3. *if C is a cover on s and Q is any sieve on s such that for all $\alpha \in C$, $\alpha^* Q$ is a cover, then Q is a cover too.*

Proof. We check each axiom in turn. The first one holds trivially, because the maximal sieve is just the identity on $[s]$.

We will use simple equalisers for the second and third axioms. First we note that for any sieve $C \rightarrowtail [s]$ we can form the pushout of this inclusion with itself

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d, "c"'] & {[s]} \arrow[d, "b"] \\ {[s]} \arrow[r, "a"'] \arrow[ur, phantom, "\ulcorner"] & J_C \end{tikzcd}$$

which will construct a presheaf J_C that has two specified elements, a and b, of shape s. One can easily see that (C, c) is the equaliser of a and b, as it is precisely the subpresheaf of elements $\alpha \in [s]$ such that $a(\alpha) = b(\alpha)$. We call a and b the *cokernal pair* of C. Note that because of the assumption that $\coprod$ preserves simple equalisers, and that fact that the cokernal pair of a sieve is always a simple equaliser diagram, C is a cover if and only if $\coprod a = \coprod b$.

Now to check the second axiom, first form the cokernal pair for C, and also observe that the pullback sieve is defined by an actual pullback in the presheaf category

$$\begin{tikzcd} \alpha^* C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[t]} \arrow[d, "{[\alpha]}"] & \\ C \arrow[r, "c"'] & {[s]} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & J_C \end{tikzcd}$$

so $\alpha^* C$ can also be thought of as the equaliser of $a \circ [\alpha]$ and $b \circ [\alpha]$. But $\coprod$ identifies these, so by the preservation of simple equalisers $\alpha^* C$ is a cover.

For the third axiom, let a and b be the cokernal pair of the sieve Q. We have that the pullback P in the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & {[t]} \arrow[d, "x"] & \\ & C \arrow[d, "c"] & \\ Q \arrow[r, "q"'] & {[s]} \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & J_Q \end{tikzcd}$$

is a cover on t for any t-shaped element x of C. Thus P is the equaliser of acx and bcx. By preservation of simple equalisers this implies that for all elements $x \in C$, $\coprod a \circ \coprod c \circ \coprod x = \coprod b \circ \coprod c \circ \coprod x$.

Since $\coprod C$ is defined as a colimit, this being true for every inclusion map $\coprod x$ implies that $\coprod a \circ \coprod c = \coprod b \circ \coprod c$. We are given that C is a cover on s, so $\coprod c$ is an isomorphism. This tells us that $\coprod a = \coprod b$ and therefore Q is a cover as required. □

Remark. The link between sieves and pairs of elements established by the cokernal pair construction and the preservation of simple equalisers shows that non-trivial covers correspond precisely to examples of the repeated element condition failing. Therefore in the case that the repeated element condition holds, we obtain the trivial Grothendieck topology on $\mathcal{S}$ where every presheaf is a sheaf and a coalgebra.

Definition 1.2.10. A presheaf σ on $\mathcal{S}$ is called a *sheaf* if for every cover C on s, every presheaf morphism $f\colon C \to \sigma$ has a unique extension

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "c"'] & \sigma \\ {[s]} \arrow[ur, dotted] & \end{tikzcd}$$

such that the diagram commutes. The sheaves form a full subcategory of $\hat{\mathcal{S}}$ which we write $\mathbf{Sh}_{\mathcal{C}}(\mathcal{S})$.

This is the usual definition of a sheaf over a category with a Grothendieck topology. There is a large literature of standard results about this situation; all the results we will actually use can be found, for example, in the second half of the third chapter of [MLM92]. The first thing we will note is the fact that $\mathbf{Sh}_{\mathcal{C}}(\mathcal{S})$ is a reflective subcategory of $\hat{\mathcal{S}}$—or, in other words, the inclusion functor has a left adjoint which we call the *associated sheaf* functor.

The next thing we should observe is that the right adjoint to $\coprod$, the functor $N\colon \mathcal{C} \to \hat{\mathcal{S}}$, factors through the subcategory $\mathbf{Sh}_{\mathcal{C}}(\mathcal{S})$. Indeed, given $A \in \mathcal{C}$ and some cover $c : C \rightarrowtail [s]$, clearly $\mathcal{C}(\coprod C, A)$ and $\mathcal{C}(Is, A)$ are in bijection because of the isomorphism $\coprod c$. But then applying the adjunction $\coprod \dashv N$, presheaf morphisms $C \to NA$ are in bijection with those $[s] \to NA$, which says precisely that NA is a sheaf. With this in mind, we will now write N as a functor $\mathcal{C} \to \mathbf{Sh}_{\mathcal{C}}(\mathcal{S})$.

Composing $\coprod$ with the inclusion functor will give the functor U below

$$\mathbf{Sh}_{\mathcal{C}}(\mathcal{S}) \underset{N}{\overset{U}{\rightleftarrows}} \mathcal{C}$$

and this gives an adjunction $U \dashv N$. We will now prove that this adjunction is comonadic. Since the comonad UN on $\mathcal{C}$ is the same as the original comonad, this means that $\mathbf{Sh}_{\mathcal{C}}(\mathcal{S})$ is the category of coalgebras for the nerve-realisation (or density) comonad generated by $I\colon \mathcal{S} \to \mathcal{C}$.

Proposition 1.2.11. *Suppose that $\mathcal{C}$ is cocomplete and has all pullbacks, and that the choice of shapes $I\colon \mathcal{S} \to \mathcal{C}$ gives colimits that commute with pullbacks. Suppose also that the realisation functor $\coprod$ preserves simple equalisers. Then the above construction gives a comonadic adjunction between the Grothendieck topos $\mathbf{Sh}_{\mathcal{C}}(\mathcal{S})$ and $\mathcal{C}$.*

Proof. It is known that any sheaf category is closed under limits in its presheaf category. This means that U preserves equalisers as a trivial consequence of $\coprod$ preserving equalisers. So all that remains to check, in order to apply Beck's comonadicity theorem, is the fact that U is conservative.

In order to prove this, suppose we have a sheaf σ, and are given two elements a and b of shape s in σ. If $\coprod a = \coprod b$ then the equaliser of a and b is a cover on s, say $c\colon C \rightarrowtail [s]$, such that $ac = ab$. By the uniqueness of the extensions specified by the sheaf property, this implies that $a = b$. What this tells us is that while not every presheaf satisfies the repeated element condition, those that are sheaves do satisfy it. We can now apply exactly the reasoning of Proposition 1.2.5 to show that U is conservative. $\square$

Notes on examples. Many of the common examples of nerve-realisation adjunctions are quite easily seen to be comonadic. When we consider a representable nerve functor, the conditions become a lot simpler to check and in particular the preservation of simple equalisers is trivial whenever $\mathcal{S}$ is discrete. So it is straightforward to see that **Set** is comonadic over both **Top** and **Cat** with the discrete object functors acting as forgetful functors. It is not much harder to check the conditions for $\Delta \to \mathbf{Top}$ and see that simplicial sets are comonadic over topological spaces.

Things do not work as well for categories. The moment we include the free-living morphism in our shape category, we can find colimits that don't commute with pullback. The colimit of two morphisms joined end to end contains the free composite of the two; we can define $P \to \coprod \sigma$ where P contains just one morphism that lives over that composite, as shown in the diagram

$$(a \xrightarrow{f} b) \xrightarrow{f \mapsto g \circ h} (x \xrightarrow{h} y \xrightarrow{g} z).$$

Then decomposing P and sticking it back together again will destroy this morphism and show that this is a counter-example. Hence directed graphs (and by the same logic, simplicial sets) are not comonadic over **Cat** using this particular realisation functor.

1.3 Galois Connections

In this section we will discuss a very nice piece of category theory that will provide an illuminating way to think about the factorisation systems in the following section. Recall that a partially ordered set can be viewed as a category with the property that every hom-set is either empty or a singleton. It is always interesting to consider concepts in category theory when we restrict to this world of posets. The idea of a Galois connection is exactly an adjunction between posets. These were studied extensively before category theory even appeared on the scene, the motivating example being the correspondence between field extensions and Galois groups obtained in Galois Theory.

Whenever we have an adjunction we want to know about its associated monad and comonad. In the world of posets, the idea of a monad is much simpler, but equally ubiquitous; a monad on a poset is called a *closure operation*. A closure operation is an order preserving map $c\colon P \to P$ that is *increasing* and *idempotent*, meaning that for any $x \in P$, $x \leq c(x)$ and $c(c(x)) = c(x)$.

A closure operation on a poset gives a subposet P^c consisting of the elements of P that are fixed by c; these are called the *closed* elements. (All the names are inspired by the basic example of closed sets in a topological space.) The poset of closed elements has the nice property that it is closed under arbitrary meets (or intersections). In the categorical sense, P^c is the category of algebras for the monad c.

The material in this section is often considered part of the folklore and can be found in many textbooks on category theory (see, for example, [ML98]) and on lattice theory. For an introduction which covers everything in this section from an entirely non-categorical perspective, the reader can consult [EKMS93].

The Galois connection generated by a relation. A rather nice way of making Galois connections is the following: begin with two sets A and B and a relation ρ between them. We will construct a galois connection

$$\begin{tikzcd} \mathcal{P}A \arrow[r, shift left, "\rho_*"] & \mathcal{P}B^{\mathrm{op}} \arrow[l, shift left, "\rho^*"] \end{tikzcd}$$

where $\mathcal{P}A$ and $\mathcal{P}B$ are the power sets of A and B, as follows:

$$\rho_*(X) = \{b \in B \mid \rho(a,b), \forall a \in X\}$$

$$\rho^*(Y) = \{a \in A \mid \rho(a,b), \forall b \in Y\}$$

These functions are both order-reversing, and we have $X \subseteq \rho^*\rho_*(X)$ and $Y \subseteq \rho_*\rho^*(Y)$ for all $X \subseteq A$ and $Y \subseteq B$, which means they constitute an adjunction between $\mathcal{P}A$ and $\mathcal{P}B^{\mathrm{op}}$. This gives us closure operations on both $\mathcal{P}A$ and $\mathcal{P}B$, and the poset of closed subsets of A is isomorphic to the poset of closed subsets of B—a Galois connection restricts to an isomorphism between the posets of closed elements.

There are many great examples, from many different fields of mathematics. Here are just a few.

Points and lines. A fun example: let A be the set of points in the plane and let B be the set of directed lines. Consider the relation "the point p is on the left of the line l". Then the closure operation gives the convex hull of any set of points.

Syntax and semantics. Here is a more serious example, from model theory. Consider a set of sentences in some language, and a set of structures which the language can describe. Let us say we have a relation v telling us whether or not a given sentence is true in a given structure. Then $v_*(A)$ is the set of structures in which the sentences of A hold. On the other hand, $v^*(B)$ is a maximal set of axioms which hold for all the structures in B. The closure operation on sets of sentences generates all theorems implied by a set of axioms, while the closed sets of structures are exactly those which are axiomatisable.

Algebra and geometry. Another serious example, which is fundamental to algebraic geometry, is this: consider n-dimensional affine space on one hand, and the set of n-variable polynomials on the other. There is a relation between a point p and a polynomial f given by $f(p) = 0$. The generated Galois connection has affine algebraic sets as its closed sets of points, paired to their ideals of vanishing functions.

Möbius transformations. Here is another fun example. Consider the Riemann sphere and the set of Möbius transformations. There is a relation that tells us when a given transformation fixes a given point on the sphere. The closed subsets of the sphere in the resulting Galois connection are single points, pairs of points, circles and of course the whole sphere.

Weak factorisation systems. In the theory of model categories, one is interested in certain classes of morphisms that have *lifting properties* with respect to one another. These lifting properties can be expressed as a relation between the class of all morphisms in a category and itself. Given two morphisms, f and g, the relation is satisfied between them if for any square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

such a dotted arrow exists (such that the two triangles both commute).

This relation can of course be used to construct a Galois connection, and the resulting fixed points are pairs of classes of morphisms $(\mathcal{L}, \mathcal{R})$ with the property that $\mathcal{L}$ is precisely the class of morphisms that lift against everything in $\mathcal{R}$, and $\mathcal{R}$ is precisely the class of morphisms that everything in $\mathcal{L}$ lifts against. Such a pair is exactly the underlying data required for a *weak factorisation system*—the only extra requirement is the property that any morphism can be factorised somehow as a map in $\mathcal{L}$ followed by a map in $\mathcal{R}$.

'Categorifying' the construction. A relation between two sets can always be viewed as a subset $R \subseteq A \times B$. Consider that, instead of sets, we have categories, and an arbitrary functor $R\colon \mathcal{R} \to \mathcal{A} \times \mathcal{B}$. The entire construction above can be made to work with categories over $\mathcal{A}$ and $\mathcal{B}$.

Given any functor $F\colon \mathcal{C} \to \mathcal{A}$, define a category $\mathcal{R}_*\mathcal{C}$ over $\mathcal{B}$ as follows: an object will be an object of $\mathcal{B}$, together with a choice of $r_c \in \mathcal{R}$ for every $c \in \mathcal{C}$, so that $R(r_c) = (F(c), b)$, and suitable choices for the morphisms in $\mathcal{C}$. This can be expressed as a functor $\mathcal{C} \to \mathcal{R}$ satisfying certain conditions. The morphisms of $\mathcal{R}_*\mathcal{C}$ will be morphisms of $\mathcal{B}$ that make the choices consistent.

We can define $\mathcal{R}^*$ similarly so as to obtain an adjunction

$$\begin{tikzcd} \mathbf{Cat}/\mathcal{A} \arrow[r, shift left, "\mathcal{R}_*"] \arrow[r, phantom, "\perp"] & \mathbf{Cat}^{\mathrm{op}}/\mathcal{B}. \arrow[l, shift left, "\mathcal{R}^*"] \end{tikzcd}$$

In the next section, we will see how we can construct such a 'categorified' version of the weak factorisation systems example above.

1.4 Factorisation Systems

In the last section we briefly mentioned weak factorisation systems, the main type of factorisation system studied in homotopy theory. Weak factorisation systems are historically very important and an absolutely vital building block of model category theory; despite this, they will not appear much in this thesis. Instead, it is the much stricter and more categorical notion of an *algebraic weak factorisation system* which takes centre stage. From now on, for the sake of brevity, we will use the abbreviation AWFS, which can be either singular or plural.

For a good background on weak factorisation systems and model categories, [Hov99] is probably the best source. The definition and motivation of AWFS is covered very well in the work of Emily Riehl; see [Rie13a] for general material and [Rie11a, Rie11b, Rie13b] for her work on using AWFS to build an algebraic version of model category theory. A lot of the material in this section can also be found in [Gar12b], or in the author's own [Ath12].

We begin by discussing yet another sort of factorisation system, the *orthogonal factorisation system*, which will be a very useful concept in its own right. It will also lead us naturally into the definition of an AWFS. Orthogonal factorisations systems are a much older idea than AWFS; the material here can be found in most textbooks on category theory. The paper [Kel80] includes the definition at the start and is also important due to its extensive use of the concept.

Definition 1.4.1. Given two morphisms f and g, we say that f is *orthogonal* to g if for any square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

the dotted arrow not only exists, but is unique.

Definition 1.4.2. An *orthogonal factorisation system* is a fixed point $(\mathcal{E}, \mathcal{M})$ for the Galois connection generated by the relation of orthogonality, with the additional property that every morphism can be factorised as a morphism in $\mathcal{E}$ followed by one in $\mathcal{M}$.

When working with orthogonal factorisation systems it is always useful to have the prototypical example in mind: the example on the category **Set** where $\mathcal{E}$ is epimorphisms and $\mathcal{M}$ is monomorphisms. This example leads to similar examples on many other categories; in particular in any topos or pretopos we can define an epi-mono factorisation system in precisely the same way, and in any regular category we can do the same but replace $\mathcal{E}$ with the class of regular epimorphisms. Practically every example of an orthogonal factorisation system at least 'feels like' some kind of inclusion coupled with some kind of projection. We even use the term *proper* to refer to orthogonal factorisation systems where $\mathcal{E}$ contains only epimorphisms and $\mathcal{M}$ contains only monomorphisms.

Let us consider some consequences of the definition above. First we'll consider the factorisations that are possible; by assumption every morphism has at least one factorisation $m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. However, if one assumes a second such factorisation $m' \circ e'$ it is immediate from the liftings in the two orthogonality squares produced that the two factorisations are isomorphic. So the factorisation rules are in fact determined up to isomorphism. A similar consideration of unique lifts allows us to show that this rule for factorising any morphism is actually functorial.

If $\mathbf{2}$ is the category with a single morphism and $\mathbf{3}$ is the category freely generated from two composable morphisms, there is a functor $D\colon \mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$ that defines composition in the category $\mathcal{C}$. A nice way to say that a factorisation system is functorial is to express it as a section for this composition functor—that is, a functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ which gives the identity on $\mathcal{C}^{\mathbf{2}}$ when composed with D. Given such a *functorial factorisation system* we can define the two halves of the factorisation as functors L and $R\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$, so that for any morphism f we have $f = Rf \circ Lf$. There is also a functor $M\colon \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ that gives the new object that appears in the 'middle' of the factorisation; the whole picture is given by the diagram, which shows how applying the functorial factorisation system to a morphism $(a, b)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ gives a morphism $(a, M(a, b), b)$ in $\mathcal{C}^{\mathbf{3}}$:

$$\begin{tikzcd} A \arrow[r, "Lf"'] \arrow[rr, bend left, "f"] \arrow[d, "a"'] & Mf \arrow[r, "Rf"'] \arrow[d, "M(a,b)"] & B. \arrow[d, "b"] \\ C \arrow[r, "Lg"] \arrow[rr, bend right, "g"'] & Mg \arrow[r, "Rg"] & D. \end{tikzcd}$$

Notice that R is automatically a pointed endofunctor, where the natural transformation $1 \Rightarrow R$ is given on f by $(Lf, 1_B)\colon f \to Rf$, and similarly L is a copointed endofunctor. Another consequence of the fact that factorisations must be unique up to isomorphism is that when we start with a morphism that is already in one of the two classes and factorise it, what we get must be (up to isomorphism) just the original morphism and an identity. This implies that the functors L and R are essentially idempotent, and furthermore that the two classes $\mathcal{E}$ and $\mathcal{M}$ are exactly their fixed points. This means that R is an idempotent monad whose category of algebras is $\mathcal{M}$, and dually L is an idempotent comonad whose category of coalgebras is $\mathcal{E}$.

AWFS. The last observation, that R and L are respectively an idempotent monad and comonad on $\mathcal{C}^{\mathbf{2}}$, leads us to consider a possible generalisation: we could remove the requirement of idempotence and see what happens. This is one way of motivating the definition of an algebraic weak factorisation system; we will see some other motivations later in this section.

Definition 1.4.3. An *algebraic weak factorisation system* on a category $\mathcal{C}$ is a pair $(\mathbb{L}, \mathbb{R})$ where $\mathbb{L} = (L, \vec{\epsilon}, \vec{\delta})$ is a comonad on $\mathcal{C}^{\mathbf{2}}$, $\mathbb{R} = (R, \vec{\eta}, \vec{\mu})$ is a monad on $\mathcal{C}^{\mathbf{2}}$, the copointed endofunctor $(L, \vec{\epsilon})$ together with the pointed endofunctor $(R, \vec{\eta})$ make up the data of a functorial factorisation system, and the pair satisfies the *distributivity axiom*, explained below.

The final condition will ensure that the monad and comonad behave properly with respect to one another. It follows from the monad laws that $\vec{\delta}$ must have trivial domain component and $\vec{\mu}$ must have trivial codomain component, so their components take the forms $(1, \delta_f)$ and $(\mu_f, 1)$:

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "Lf"'] \arrow[dr, phantom, "\vec{\delta}_f"] & \bullet \arrow[d, "LLf"] \\ Mf \arrow[r, "\delta_f"'] & MLf \end{tikzcd} \qquad \begin{tikzcd} MRf \arrow[r, "\mu_f"] \arrow[d, "RRf"'] \arrow[dr, phantom, "\vec{\mu}_f"] & Mf \arrow[d, "Rf"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

for some δ_f and μ_f. Then we can define a natural transformation $\Delta\colon LR \to RL$ with components given by (δ_f, μ_f) as shown in the square

$$\begin{tikzcd} Mf \arrow[r, "\delta_f"] \arrow[d, "LRf"'] & MLf \arrow[d, "RLf"] \\ MRf \arrow[r, "\mu_f"'] & Mf. \end{tikzcd}$$

The distributivity axiom says that this is a *distributive law* of the comonad over the monad, meaning that it commutes with the unit, counit, multiplication and comultiplication transformations.

The two types of factorisation system we introduced earlier were both defined as fixed points for certain Galois connections, together with the property that some factorisations exist. In the definition above, things have switched around, since the factorisation functors themselves have become the primary data of the definition. This makes sense because an AWFS is, after all, an algebraic object;

we would expect the monad and comonad to assume the leading roles in the definition. However, the other approach is still very much possible; in the next chapter we will consider these two philosophies in some detail.

Definition 1.4.4. We write **L-Map** for the category of coalgebras for the comonad $\mathbb{L}$ and we write **R-Map** for the category of algebras for the monad $\mathbb{R}$; we call the algebras *right maps* and the coalgebras *left maps.*

As always, a coalgebra is an object of $\mathcal{C}^{\mathbf{2}}$, $f\colon A \to B$, equipped with a structure map $f \to Lf$, which appears in this case as a map such as α in the following:

$$\begin{tikzcd} A \arrow[r, "Lf"'] & Mf \arrow[r, "Rf"'] & B, \arrow[l, bend right, "\alpha"'] \end{tikzcd}$$

which is a kind of 'partial inverse' to f. The map α must satisfy the three equations $\alpha \circ f = Lf$, $Rf \circ \alpha = 1_B$ and $M(1, \alpha) \circ \alpha = \delta_f \circ \alpha$; notice that the coalgebra axioms also force the domain part of the structure map to be the identity on A, and this is why we can represent the left map structure with just one morphism of $\mathcal{C}$. Since the notion of an AWFS is entirely symmetrical, the right maps can be described in exactly the same way, simply dualised.

From now on we will use a notational shorthand where instead of explicitly writing a map and its factorisation, we draw arrows going to and from the middle of an arrow to mean morphisms to and from the central object of that arrow's factorisation. Thus a left map will be drawn as

$$\begin{tikzcd} A \arrow[rr, "f"'] & {} & B \arrow[l, bend right, "\alpha"'] \end{tikzcd}$$

and the image of a morphism (a, b) in $\mathcal{C}^{\mathbf{2}}$ under the factorisation will be drawn as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "a"'] & {} \arrow[d, "M(a,b)"] & B \arrow[d, "b"] \\ C \arrow[rr, "g"'] & {} & D. \end{tikzcd}$$

Arrows to and from the one quarter point or three quarters point of an arrow mean the obvious thing, where the left or right part of a factorisation has been factorised again.

In order to justify the language we are using when we refer to algebras as 'right maps' and coalgebras as 'left maps', we would like to see what kind of lifting properties the left maps have with respect to the right maps. Interestingly, now that we have moved into the 'algebraic' world of AWFS, it is no longer appropriate to talk about lifting properties; what we have now are lifting *structures* instead. To see what this means, notice that given a left map (f, α), a right map (g, β) and any lifting problem between them

$$
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "b"'] & D,
\end{tikzcd}
$$

the three dotted arrows compose to give a lifting. This is not just an assertion that such a lifting exists; it is an explicit choice. If we were to choose a different left or right map structure on f or g, we would very possibly end up with a different solution to the lifting problem.

Morphisms of AWFS. There is a nice notion of morphism between two AWFS on the same category $\mathcal{C}$. We will define a morphism $\alpha\colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{L}', \mathbb{R}')$ to be a natural transformation between the central functors $\alpha\colon M \to M'$ which induces a comonad map $(1, \alpha)\colon \mathbb{L} \to \mathbb{L}'$ and a monad map $(\alpha, 1)\colon \mathbb{R} \to \mathbb{R}'$. Basically, for each $f\colon X \to Y$, we get

$$
\begin{tikzcd}
 & Mf \arrow[dr, "Rf"] \arrow[dd, "\alpha_f"] & \\
X \arrow[ur, "Lf"] \arrow[dr, "L'f"'] & & Y \\
 & M'f \arrow[ur, "R'f"'] &
\end{tikzcd}
$$

and everything that you would want to commute commutes. It is particularly useful to note that α lifts to $\alpha_l\colon \mathbf{L}\text{-}\mathbf{Map} \to \mathbf{L}'\text{-}\mathbf{Map}$, a morphism in $\mathbf{Cat}/\mathcal{C}^2$, and that dually it also lifts to $\alpha_r\colon \mathbf{R}'\text{-}\mathbf{Map} \to \mathbf{R}\text{-}\mathbf{Map}$.

Replacement functors. If the underlying category $\mathcal{C}$ has an initial object and a terminal object, then we can construct the *left* and *right replacement functors*, Q and S, by factorising the unique maps to and from the initial and terminal object—in exactly the same way as we construct the cofibrant and fibrant replacement functors for model categories. So $0 \to X$ factorises as $0 \to QX \to X$ and $X \to 1$ factorises as $X \to SX \to 1$. Because the functor L is a comonad, $Q\colon \mathcal{C} \to \mathcal{C}$ is a comonad too; similarly, S will be a monad.

Examples. We will finish this section by giving a quick tour of a few easily described examples of AWFS. Unfortunately, all the most interesting examples tend to require a bit of machinery to describe; the machinery in question is introduced in Section 2.4 so the more meaty examples cannot be considered until then.

For the moment, it is worthwhile considering a few extremely simple examples. For a given base category $\mathcal{C}$ we shall write $\mathbf{AWFS}(\mathcal{C})$ for the category of all AWFS on $\mathcal{C}$. We will now see how this category always has an initial object and a terminal object. While they are both trivial examples, it will shed some light on AWFS in general to consider them and their categories of left maps.

The initial AWFS on $\mathcal{C}$ is shown in the diagram

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, equal] \arrow[rr, "f"'] & & Y, \end{tikzcd}$$

and clearly satisfies the comonad and monad requirements. It is also easy to see that it is initial—the unique morphism to any other AWFS is given by the left hand maps of that other AWFS!

It is, however, a bit more interesting to consider the left map structures; a left map structure on f is given by a map $g\colon Y \to X$ making the necessary diagrams commute. If you work it out, this means that $fg = 1_Y$ and $gf = 1_X$: f has exactly one left map structure if and only if it is an isomorphism. This tells us another useful fact about left maps for any AWFS: there are always *trivial left map* structures on isomorphisms, defined as the images of the left maps of the initial AWFS under the induced functor between the left map categories.

The terminal AWFS is the symmetrically opposite one, given by

$$\begin{tikzcd} & Y \arrow[dr, equal] & \\ X \arrow[ur, "f"] \arrow[rr, "f"'] & & Y, \end{tikzcd}$$

which is again clearly an AWFS. The unique morphism from any other AWFS is given by the right maps. In this case there is a unique left map structure for every morphism of $\mathcal{C}$; this left map category is exactly the terminal object of $\mathbf{Cat}/\mathcal{C}^2$.

One can check that

$$\begin{tikzcd} & A \times B \arrow[dr, "\pi_B"] & \\ A \arrow[ur, "(1_A, f)"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

is an AWFS for any category with all finite products; the multiplication is a projection map and the comultiplication involves the diagonal $\Delta_A\colon A \to A \times A$. Here a left map structure on f is a choice of retract $\alpha\colon B \to A$. Right map structures for this AWFS are a little more subtle; one needs to consider the interaction with the multiplication. If we are dealing with the category **Set**, then it turns out that a right map is a surjection together with a consistent system of isomorphisms between fibres.

A similar game can be played with coproducts:

$$\begin{tikzcd} & A + B \arrow[dr, "(f, 1_B)"] & \\ A \arrow[ur, "i_A"] \arrow[rr, "f"'] & & B \end{tikzcd}$$

is an AWFS in a dual manner—now the multiplication will involve the codiagonal $\nabla_B\colon B + B \to B$ while the comultiplication is a simple coproduct inclusion. A right map structure on f is a choice of section. In the category of sets, there is precisely one left map structure on every injection. However, it is worth pointing out that a commutative square between injections is a left map morphism only when it is a pullback square.

Finally, we can consider a way of generalising the product-based example. Given any monad T on a category with all finite products we can define an AWFS which acts as

$$\begin{tikzcd} & TA \times B \arrow[dr, "\pi_B"] & \\ A \arrow[ur, "(\eta_A, f)"] \arrow[rr, "f"'] & & B. \end{tikzcd}$$

This is interesting as it allows us to produce an AWFS whose right replacement monad is T; in fact, it is terminal among the AWFS which have this property.

1.5 Higher Categories

In this final section we turn away from factorisation systems for a moment and consider the subject of higher category theory, which provides us with the main examples we will be working with later in Chapters 4 and 5. The material in this section can all be found in [Lei04a] which is one of the best introductions to the subject. Other worthwhile references include [CL04] and [Lei02] which give a more general survey of the different definitions of higher categories. Two good sources specifically dealing with low dimensional examples (especially dimension three and questions of coherence) are [Gur07] and [Gur13].

The simplest notion of a higher dimensional category is that of a *strict n-category*. Strict 2-categories are categories enriched in **Cat**, strict 3-categories are categories enriched in **2Cat**; continuing this is a simple way of defining the category **nCat** of strict n-categories as a result of iterated enrichment. Since the theory of enriched categories is well understood (see [Kel80] for a comprehensive overview), and there is nothing particularly pathological about any of these examples, it follows that we actually understand strict n-categories extremely well.

However, these very strict objects are not all that useful. It turns out that in almost all 'real world' examples, some of the axioms will not hold in the strict sense. Instead, they hold in the weak sense that, rather than a simple equality, we obtain an isomorphism between the two cells we wish to compare. This weakness in examples forces us to consider subtler definitions of higher category which allow such behavior, and these turn out to be vastly more complicated and therefore less well understood than the strict version. In this section we consider one particular approach—the one we will be working with in Chapters 4 and 5—though there are many other definitions and it is not necessarily clear how they relate to one another.

Definition 1.5.1. A *globular set* is defined to be a presheaf on the category **Glob**, which has the natural numbers as its objects and morphisms as shown in

$$\begin{tikzcd} \mathbf{0} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbf{1} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbf{2} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbf{3} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \cdots, \end{tikzcd}$$

where the identities $ss = ts$ and $st = tt$ hold wherever they are well-defined. We write **GlobSet** for the category of globular sets. Note that the category **Glob** can be restricted to just the natural numbers up to n; we call a presheaf on this restricted category an *n-globular set* and we write **nGlobSet** for the category of such. If X is a globular set, we call the elements of $X(n)$ the *n-cells* of the globular set, and given an n-cell α we call $s(\alpha)$ and $t(\alpha)$ the *source* and *target* of α.

The reason we care about this particular collection of presheaf categories is that they are an obvious first answer to an interesting question: most objects in mathematics are naturally thought of as sets with structure, and this is formalised by the notion of monads and algebras; what should one replace the category of sets with in order to work with n-categories in the same way? Indeed, there is a natural forgetful functor from **nCat** to **nGlobSet**—as suggested by the terminology of 'n-cells', 'sources' and 'targets'. Furthermore, this forgetful functor has a left adjoint and the adjunction is monadic; we will write $T_n\colon \mathbf{nGlobSet} \to \mathbf{nGlobSet}$ for this monad, which we call the *free strict n-category* monad.

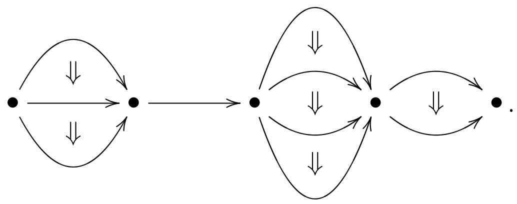
We note now that for any $m < n$ there is a natural functor $\mathbf{nGlobSet} \to \mathbf{mGlobSet}$ called a *truncation functor*, which simply removes all cells of dimension higher than m. Note also that **GlobSet** itself is exactly the categorical limit of the diagram formed by all the categories **nGlobSet** and all the truncation functors. Furthermore, the truncation functors commute with the monads T_n—if you form a free strict n-category and then remove cells of dimension higher than m you get the same result as you would if you first removed the cells and then generated the free strict m-category. This all means that we can extend the functors T_n to a monad T on the category **GlobSet**, which we call the *free strict ω-category* monad. In the proceeding discussion, we will use the monad T because it is obviously the most general case; for any n, the finite dimensional T_n can be obtained just by truncating the action of T.

Collections and operads. Recall that an *operad* (when the term is used without qualification) has as its underlying data a collection of sets $P(n)$ indexed by the natural numbers, where the elements of $P(n)$ are considered to represent operations of arity n. The *globular operads* that we will see in a moment are an example of generalised operads; one can use any monad with certain nice properties to provide a different collection of arities, and then define a type of operad with operations of those arities. To see this done in full detail the reader should consult [Lei04a]; here we will focus on this particular example where we start with the free strict ω-category monad T.

For general notions of higher category, the arities we care about are diagrams of cells which are all composable. So for example, in dimension one we have arities such as

$$\bullet \longrightarrow \bullet \longrightarrow \bullet, \qquad \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet,$$

and any other sequence of a finite number of morphisms, including the empty sequence (this allows the possibility of the 'nullary' operation that gives rise to identity morphisms). In dimension two, the arities must include both vertical and horizontal sequences of 2-cells, as well as more general combinations of horizontal and vertical such as



These arities can be seen to form a globular set themselves, and this object (while a bit fiddly to describe) is very straightforward to define: it is precisely the globular set $T1$, in other words the free strict ω-category generated by the terminal globular set. We call the cells of this globular set *pasting diagrams*.

Definition 1.5.2. A *globular collection* is a globular set over $T1$, in other words simply a globular set P and a morphism $p\colon P \to T1$.

The first thing we should note is that the category of globular collections (which is really just the slice category $\mathbf{GlobSet}/T1$) is a monoidal category; the unit is the collection specified by the unit of the monad, $\eta_1\colon 1 \to T1$, and given two collections (P, p) and (Q, q) we can define their tensor product as the left hand composite in the diagram

$$\begin{tikzcd}
 & & & P \otimes Q \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\
 & & TP \arrow[dl, "Tp"'] \arrow[dr, "T!"'] & & Q \arrow[dl, "q"] \\
 & T^2 1 \arrow[dl, "\mu_1"'] & & T1 & \\
T1. & & & &
\end{tikzcd}$$

There are a few diagram chases involved in showing that unit and associativity axioms hold up to isomorphism; a vital piece of information involved in this is the fact that T is a *cartesian monad*, meaning that it preserves small limits and the natural transformations η and μ have pullbacks for all their naturality squares.

A globular collection is the primary data we need to define a globular operad, since it contains a set of operations living over each pasting diagram, together with choices of sources and targets that are consistent with the arities. However, there is more to an operad than the sets of operations; one also needs rules for composing the operations. This is the point of the monoidal structure.

Definition 1.5.3. A *globular operad* is a monoid in the monoidal category of globular collections.

Taking this definition apart, we have some globular collection $p\colon P \to T1$ together with morphisms of collections $i\colon 1 \to P$ and $m\colon P \otimes P \to P$. Since the terminal globular set 1 has exactly one cell at each dimension, the morphism i tells us that P contains a specified trivial 'unary' operation at each dimension.

A cell of $P \otimes P$ is an element of the pullback in the diagram above; it involves an element of TP (which is a pasting diagram labelled with P-operations) together with another P-operation. The map m tells us how to compose this data into one big operation.

We need also to describe how the category of algebras for a globular operad is obtained—since the operad is meant to be some syntactic description of a theory of higher categories, we need to be able to extract the semantics and define the weak n-categories that arise. The simplest way to do this is to show how any globular operad gives rise to a monad on **GlobSet**.

Given any globular set A, we form $T_P A$ as the pullback

$$\begin{tikzcd} T_P A \arrow[r, "p_A"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TA \arrow[d, "T!"] \\ P \arrow[r, "p"'] & T1, \end{tikzcd}$$

which also defines a natural transformation $p\colon T_P \Rightarrow T$. One can see that our original morphism $p\colon P \to T1$ is actually the component of this natural transformation at the object 1, since $T_P 1$ is just P. One can check that this endofunctor T_P is a cartesian monad on **GlobSet**, and the transformation p is a cartesian monad morphism. This can be used as an alternative definition of a globular operad; however, the one we gave above has a certain conceptual advantage in that it explicitly contains a globular set of operations.

Definition 1.5.4. Given a globular operad P the category of *algebras* for P, which we will write **P-Alg**, is exactly the category of algebras for the associated cartesian monad T_P.

One particular thing we should point out about algebras for a globular operad is that there is always a functor **T-Alg** $\to$ **P-Alg**, obtained via the monad map $p\colon T_P \Rightarrow T$. A T-algebra structure $\alpha\colon TA \to A$ is taken to the composite $\alpha \circ p_A$, which one can check is indeed a valid P-algebra structure. When P is an operad for some notion of weak n-category, this functor expresses the fact that every strict category can be considered as a weak category that just happens to satisfy some extra axioms.

Contractible operads. The last sentence of the previous paragraph may prompt the question: what does it mean for an operad to describe some notion of weak n-category? It is certainly clear that we can build pathological globular operads that hardly behave like categories at all—adding extra operations at dimension zero is a good way to start! Therefore we do have to be careful about which operads we consider. We will now describe a way of characterising when the algebras behave like weak higher categories.

To begin with, it seems reasonable that in a weak category there should be at least one way of composing any pasting diagram of cells. In other words, if you can compose a collection of cells in a *strict* category you should be able to compose them in a weak category—though there may no longer be a unique choice of composition operation. To fix this possibility of non-uniqueness, we have a second reasonable request: given two different composites of the same pasting diagram at dimension k, there should be at least one $(k+1)$-cell connecting them. The notion of a *contraction* on a globular operad beautifully combines both requests into a single natural piece of data.

The next definition will use some important notation which we will end up using a great deal later on. We write G_k for the special globular set that is the representable object on the kth object of **Glob**—it contains a single k-cell and two cells of every dimension lower than k. We call it the ***k*-glob**. Another special globular set is the *boundary* of the k-glob, which we write ∂_k. It is just G_k with the single k-cell removed. The natural inclusion of the boundary into the glob is written $i_k \colon \partial_k \to G_k$; the diagram shows the inclusion of ∂_2 into the 2-glob:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \arrow[r, "i_2"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} .$$

The 0-glob G_0 is just a single 0-cell and its boundary ∂_0 is simply taken to be the empty globular set.

Definition 1.5.5. A *contraction* on a globular operad $p\colon P \to T1$ is a choice of lifting for every square

$$\begin{tikzcd} \partial_k \arrow[r, "a"] \arrow[d, "i_k"'] & P \arrow[d, "p"] \\ G_k \arrow[r, "b"'] \arrow[ur, dotted] & T1. \end{tikzcd}$$

Let us consider how the existence of a contraction on an operad P forces both of our requests to be satisfied. An induction on dimension shows how the first request is obtained: given any choice of pasting diagram in $T1$, this can be written as a map $a\colon G_k \to T1$. The source and target of a can be lifted to P by the induction assumption, and then a square can be formed so that a itself lifts. Thus we see that a contraction forces P to contain at least one operation for every pasting diagram, which was the first of our requests.

The second requests pertains to the existence of what are known as *coherence cells*—in other words, cells (usually isomorphisms or some weaker sort of equivalence) that compare composites which would have been forced to be equal in a strict category. Given a parallel pair of cells a_0 and a_1 in P such that $p(a_0) = p(a_1)$, the contraction lifts the identity on $p(a_0)$ to a cell between a_0 and a_1. This cell is another P-operation which has the effect of adding a coherence cell between every a_0-composite and its corresponding a_1-composite.

When we are dealing with a finite dimensional example there is a slight problem at the top dimension; you cannot compare two n-cells in an n-category without simply asking them to be equal. A nice little adaptation of the definition fixes this without really needing any extra ideas at all. For an n-dimensional globular operad $P \to T_n 1$, a contraction is defined exactly as above, except that we include $i_{n+1}\colon \partial_{n+1} \to G_{n+1}$ in the set of maps we lift against: ∂_{n+1} is the two parallel n-cells you would expect, and G_{n+1} is defined to be a single n-cell. The $(n+1)$-cell, which obviously cannot exist in an n-dimensional world, has been turned into an equality:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\Downarrow"] & \bullet \arrow[r, "i_{n+1}"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} .$$

The following proposition can be found in [Lei04a].

Proposition 1.5.6. *Define* **OC** *to be the category of globular operads equipped with contractions and globular operad morphisms that preserve the contractions. Then* **OC** *has an initial object that we will write as* (L, λ)*. Similarly, if* $\mathbf{OC}_n$ *is* n*-dimensional globular operads with contractions, an initial object exists and we will call it* (L_n, λ_n)*.*

The initial operad with contraction serves as our default definition of weak n-category; we will sometimes refer to L_n-algebras as *fully weak* n*-categories* to distinguish them from the various partially weakened notions one can also consider. These objects, due to the fact that the operad is initial, are as weak as one could possibly want; in fact, they are generally speaking weaker than is desirable (most of the time, the weaker a definition is, the harder it will be to work with). A lot of work has been done with the aim of finding other globular operads that are neither fully weak nor fully strict, and asking how much of the behaviour of fully weak categories can be captured by such semi-strict objects. We will return to this question in Chapter 5.

Low dimensional examples. We end this section with a quick tour of some low dimensional examples; since it is often hard to say anything explicit about weak n-categories in general, often considering $n = 2$ or $n = 3$ is the best way of seeing how something works in practice.

We may as well quickly consider $n = 1$ first of all. One could create an operad with contraction by adding extra (unary) operations at dimension zero which are then required to be isomorphic to the trivial one. However, the contraction then forces all morphisms to compose with strict associativity and unit axioms, so there is no way of obtaining anything really different to strict categories at this dimension.

At $n = 2$ things are still basically quite straightforward. We have strict 2-categories, and we have fully weak 2-categories given as the algebras for L_2. There is also the notion of *bicategories*—a definition of weak 2-category that was put together 'by hand' long before anyone tried to do anything clever for general n. Thankfully, L_2-algebras and bicategories turn out to be relatively similar; the only difference lies in the question of *bias*. Bicategories are called *biased* because they treat a single binary composition operation as primary. The algebras of L_2 are *unbiased* because the operad automatically includes a primary operation of composition for each natural number.

At dimension two, this is all made moot by the fact that in a very real sense, all notions of 2-category are basically the same: the *coherence theorem* for bicategories tells us that for $\mathcal{B}$ any bicategory (or fully weak 2-category) we can form a strict 2-category $\mathcal{B}'$ which is biequivalent to $\mathcal{B}$. Hence one can study the weakest version of 2-categories using only fully strict 2-categories, which are much easier to work with.

The world begins to be rather more complicated at $n = 3$. Again, of course, we have the extremes of fully strict and fully weak 3-categories; but now there are rather a lot of possibilities in between. As we had bicategories at dimension two, there is the notion of *tricategory* in dimension three. Tricategories are biased, but essentially just as weak as L_3-algebras. However, there are other notions of 3-category that are neither as weak as tricategories nor as strict as strict 3-categories: these include *Gray-categories* and a number of different notions of *weak-unit 3-categories*.

At dimension three the obvious coherence theorem no longer works; one can find tricategories that exhibit behaviour that is impossible in a strict 3-category. Another coherence theorem applies instead. While we cannot go all the way from tricategories to strict 3-categories, we can go most of the way—replace 'strict 3-category' with 'Gray-category' and the coherence theorem is true. This is one reason why Gray-categories are considered interesting. All in all, they are pretty strict, and therefore not too bad to work with; but they manage to be weak enough to capture the weakest possible behaviour at dimension three.

Chapter 2

Syntax and Semantics

In the first chapter we mentioned two ways of thinking about factorisation systems. One is in terms of the two types of morphism involved. In the case of a weak factorisation system or an orthogonal factorisation system these are simply two classes of morphisms—a fixed point for a certain Galois connection—and this is the approach usually taken. The other approach is to think primarily about the factorisation operation itself; this approach becomes more important as we move to the world of AWFS, since the factorisation functors contain the algebraic information. However, the old approach is still possible, and in many respects is even more important; the left and right maps now appear as the categories of coalgebras and algebras for our comonad and monad.

In general category theoretic parlance (championed by Lawvere in [Law63]) these two approaches correspond to what we call *semantics* and *syntax*. Whenever we consider an algebraic theory we view its category of algebras as the semantic realisation of the theory, and we call whatever algebraic data made up the theory in the first place the syntax; this is usually a monad, or some slightly more specific notion such as an operad or a Lawvere theory. For an AWFS, the relationship between syntax and semantics is especially interesting due to the richness of the algebraic structure we are working with.

In this chapter, we explore several aspects of this relationship. In the first section we will establish some important properties and structures that exist for the left and right maps of any AWFS. We will see that the semantics always fits certain patterns and in fact very different AWFS will seem to behave quite similarly from a sufficiently general point of view. In the second section we will follow these ideas further by describing how the semantic side of an AWFS is a fixed point for a certain adjunction. This links back to the world of weak factorisation systems, where we had a fixed point for a Galois connection.

The remaining sections are all about how to obtain the syntactic data of an AWFS if you start with some semantics in mind. In Section 2.4 we describe the small object argument, which generates an AWFS freely from a generating category of left maps. Section 2.5 is built around the theorem that describes how to apply the usual monadicity (or comonadicity) theorems in the context of an AWFS. Obviously one has to check a few extra properties to make sure that you are dealing with a genuine category of left or right maps, not just the algebras or coalgebras of some arbitrary monad or comonad on an arrow category.

A large part of this chapter is further background theory, though much more specialised than that in Chapter 1. The material in Section 2.2 is new in the sense that to the best of the author's knowledge it is previously unpublished (at least in this level of detail); it is based on conversations between the author and Richard Garner. The material in Section 2.5 is also new; it seems to have been discovered independently by both the author and John Bourke. There are a few points where the approaches are slightly different; we have hopefully made it clear what is the author's and what is Bourke's.

2.1 Left and Right Map Categories

For the whole of this section we will work with some arbitrary AWFS $(\mathbb{L}, \mathbb{R})$ on a category $\mathcal{C}$. We are going to study the categories **L-Map** and **R-Map**, both of which are categories over $\mathcal{C}^{\mathbf{2}}$ in the sense that there are forgetful functors $U_L\colon \mathbf{L\text{-}Map} \to \mathcal{C}^{\mathbf{2}}$ and $U_R\colon \mathbf{R\text{-}Map} \to \mathcal{C}^{\mathbf{2}}$. Mostly we will focus on **L-Map** and we will just call the forgetful functor U—everything that holds for **L-Map** will have a dual that is automatically true for **R-Map** due to the fact that the definition of an AWFS is entirely symmetric. Our choice to consider the left maps first may surprise some readers, as in the general literature the right maps (i.e. fibrations) are usually given more attention; we have a good reason though, as the category of cell complexes will be a left map category.

Let us first of all recall the definition of the left map category. **L-Map** is the Eilenberg-Moore category for the comonad $\mathbb{L}$, and hence we have the usual free-forgetful adjunction,

$$\mathbf{L\text{-}Map} \underset{K}{\overset{U}{\rightleftarrows}} \mathcal{C}^{\mathbf{2}},$$

where the composite UK is the endofunctor part of $\mathbb{L}$. This characterisation of **L-Map** immediately forces a few interesting properties.

Proposition 2.1.1. *A diagram D in* **L-Map** *has a colimit whenever the image of the diagram UD in $\mathcal{C}^{\mathbf{2}}$ has a colimit, and the functor U preserves all colimits that exist. Furthermore, U is faithful and conservative.*

The first part of this proposition is usually described by saying that U *creates* all colimits. It essentially means that **L-Map** has as many colimits as is possible, given a colimit preserving functor to $\mathcal{C}^{\mathbf{2}}$. For all the examples we care about $\mathcal{C}$ will be cocomplete, so this is actually an extremely academic point; given $\mathcal{C}$ cocomplete, which we will assume from now on, we can just say that **L-Map** is also cocomplete and U colimit-preserving. The second part of the proposition expresses the fact that a left map is a morphism with structure, so a morphism of left maps is given by a morphism in $\mathcal{C}^{\mathbf{2}}$ which happens to preserve the left map structures.

So far we have established that **L-Map** has all colimits, in the regular sense of building a colimit out of the objects and morphisms of **L-Map**. As a category of maps in $\mathcal{C}$, however, there are a few other colimit-like constructions available: these are pushforwards along arbitrary maps in $\mathcal{C}$ and composites, which we explore now. These constructions can all be found in [Ath12], and earlier in the work of Riehl [Rie11a].

Proposition 2.1.2. *Given a morphism $f\colon A \to B$ with a left map structure α, and any other morphism $p\colon A \to A'$, the pushout of f along p, which we write as p_*f, has a canonical left map structure called the* pushforward *of α along p and written as $p_*\alpha$. It is given by considering*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "p"'] & {} \arrow[d, "{M(p,f_*p)}"] & B \arrow[d, "f_*p"] \arrow[l, bend right, "\alpha"'] \\
A' \arrow[rr, "p_*f"'] & {} & B' \arrow[l, bend left, "p_*\alpha"]
\end{tikzcd}$$

and specifying the structure map $p_*\alpha$ as $[Lp_*f, M(p, f_*p) \circ \alpha]$. It has the universal property that given any left map morphism (p, q) from (f, α) to some (g, β) with $g\colon A' \to C$, the induced square $(1'_A, [g, q])$ is also a left map morphism.

Remark. It is worthwhile to note (it will reappear later) that the universal property of the pushforward left map makes the left map morphism $(f, \alpha) \to (p_*f, p_*\alpha)$ into a *weakly cocartesian* map with respect to the *boundary* functor $\partial\colon \mathbf{L\text{-}Map} \to \mathcal{C}$ that gives the domain of a left map. The fact that pushforwards along all maps in $\mathcal{C}$ exist implies that the functor ∂ is actually a Grothendieck opfibration. Making a choice of pushforwards corresponds to choosing an opcleavage for this opfibration.

It is worth reminding ourselves that the dual results will apply to **R-Map**. In other words, the codomain functor on right maps is a Grothendieck fibration as long as all pullbacks exist in $\mathcal{C}$.

Proposition 2.1.3. *Suppose we are given a pair of left maps (f, α) and (g, β) where f and g are composable. Then gf has the* composite left map structure *shown by the dotted arrows in the following diagram*

$$\begin{tikzcd}
A \arrow[rrrr, "gf"] \arrow[d, equal] & {} & {} & {} & C \arrow[d, equal] \arrow[l, dotted, bend right, "\mu_{gf}"'] \\
A \arrow[rr, "f"'] & {} \arrow[ur, bend left, "{M(1,g)}"] & B \arrow[rr, "g"'] \arrow[l, bend left, "\alpha"] & {} \arrow[u, dotted, "{M(M(1,g)\circ\alpha,1)}"] & C. \arrow[l, dotted, bend left, "\beta"]
\end{tikzcd}$$

We will write this left map structure as $(gf, \beta\bullet\alpha)$. This composition operation is associative, and unital with respect to the trivial left map structures on identities.

Remark. Recall that the *trivial* left map structure on any isomorphism was defined in Section 1.4 using the initial AWFS. We are now able to give an alternative definition: a trivial left map structure is any pushforward of an initial object in **L-Map**. The proof of the proposition is a rather extravagant diagram chase which is done fully in [Ath12], though the proposition was proved earlier by both Richard Garner and Emily Riehl—see [Gar12b] and [Rie11a].

One can view the category of left maps as a double category whose objects are the objects of $\mathcal{C}$, whose vertical maps are the ordinary morphisms of $\mathcal{C}$, whose horizontal maps are left maps and whose 2-cells are left map morphisms such as $(a, b)\colon (f, \alpha) \to (g, \beta)$ shown here:

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "g"'] & \bullet. \end{tikzcd}$$

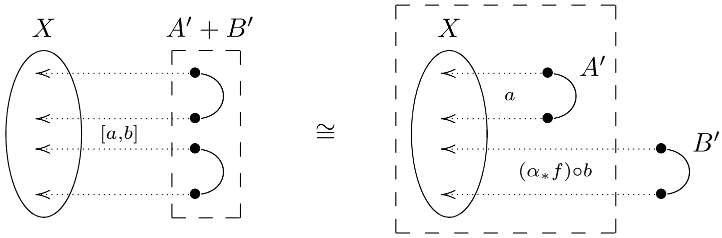
Note that it is an extra proposition to prove that this gives a full double category structure, since 2-cells have to be able to compose both vertically and horizontally. Vertical composition is just the usual composition of morphisms of left maps, but horizontal composition is an extra structure that one can check with a diagram chase. With this double category point of view the forgetful functor becomes a double functor from the double category of left maps to $\mathbf{Sq}(\mathcal{C})$, the *double category of squares* in $\mathcal{C}$, which has normal morphisms for both its horizontal and vertical maps, and simply commutative squares for its 2-cells.

Now that we have established the three main colimit-like operations that exist in **L-Map**, there is a final very important axiom that expresses how the three operations behave with respect to one another. The property they satisfy is called the *stacking property* and it is the basic principle underlying the definition of coalgebraic cell complexes. The proof can be found in [Ath12].

Proposition 2.1.4. *The composition rule in* **L-Map** *is well behaved with respect to coproducts and pushforwards in the following way: given $f\colon A \to A'$ and $g\colon B \to B'$ equipped with left map structures α and β, and maps $a\colon A \to X$ and $b\colon B \to X$, there is an isomorphism of left maps*

$$\big([a,b]_*(f+g), [a,b]_*(\alpha+\beta)\big) \cong \big((a_*f \circ b)_*g \circ a_*f, (a_*f \circ b)_*\beta \bullet a_*\alpha\big).$$

Remark. The proposition basically says that f and g can be 'glued on' to X in any order, or simultaneously by taking a coproduct first, and it makes no difference to the resulting left map. In the form of a picture:



This property immediately suggests the idea of expressing a left map in a normal form where it is decomposed into atoms (left maps that cannot be decomposed further) which are then expressed in layers, where each atom appears in the lowest layer possible. This normal form is precisely expressed in the definition of cell complex given in Chapter 3.

Of course, if we start with an arbitrary AWFS it is possible that there are no atoms. In this case the normal form will not work—this situation, in which objects have no smallest parts, is known in mereology by the lovely name *gunk*.

We have now built up a reasonably good picture of the category of left maps (and dually the category of right maps). Left maps can always be glued together in various ways so they always behave like cell complexes to some extent; this intuition will be very useful going forward. What we are currently missing, however, is arguably the most definitive behaviour of left and right maps: for a given AWFS, the left and right map categories determine one another completely because of the strict choice of liftings of every left map against every right map. This is the subject of the next section.

2.2 The Liftings Adjunction

Recall from Section 1.4 that for any AWFS if (f, α) is a left map and (g, β) is a right map, these structures give us a canonical choice of solution to every lifting problem between f and g. In other words, given any commutative square (a, b) between f and g we can construct a diagonal morphism as shown by the dotted arrows

$$\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\
{} \arrow[rr, dotted, "M(a,b)"] & & {} \arrow[u, dotted, bend right, "\beta"'] \\
B \arrow[rr, "b"'] \arrow[u, dotted, bend left, "\alpha"] & & D,
\end{tikzcd}$$

and this is a lifting in the sense that it causes both ‘triangles’ to commute (by which we mean $a = \delta \circ f$ and $g \circ \delta = b$, where δ denotes the ‘diagonal’ composite $\beta \circ M(a, b) \circ \alpha$). What we will see in this section is that a kind of converse is true. Basically, given any morphism f together with such choices of liftings against every right map, those choices themselves determine a left map structure on f. There are some subtleties involved, as the liftings have to be consistent with one another in various ways, but ultimately we will be able to define an adjunction—the *liftings adjunction* of the title—for which any AWFS is a fixed point.

First attempt. We make a first attempt to construct the liftings adjunction which is based on the construction described at the end of Section 1.3 as a ‘categorified’ Galois connection generated by a relation. For this construction we require the analogue of a relation between two sets; if we have two categories $\mathcal{A}$ and $\mathcal{B}$ we ask for a category over the product $\mathcal{A} \times \mathcal{B}^{\mathrm{op}}$. In this case, the two categories in question are two copies of the arrow category $\mathcal{C}^{\mathbf{2}}$, and we have

$$\mathcal{L} \longrightarrow \mathcal{C}^{\mathbf{2}} \times (\mathcal{C}^{\mathbf{2}})^{\mathrm{op}}.$$

The category $\mathcal{L}$ has as objects triples (f, g, ϕ) where f and g are maps in $\mathcal{C}$ and ϕ is a function which associates a lift $\phi(a, b)$ to every square between f and g. A morphism in $\mathcal{L}$ is given by $(a, b, c, d)\colon (f, g, \phi) \to (f', g', \phi')$ where $(a, b)\colon f \to f'$ and $(c, d)\colon g' \to g$ are commutative squares (morphisms of $\mathcal{C}^{\mathbf{2}}$), and for any

$(p, q)\colon f \to g$ as in

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[r, "p"] \arrow[d, "f'"] & \bullet \arrow[r, "c"] \arrow[d, "g'"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r, "b"'] & \bullet \arrow[r, "q"'] \arrow[ur, dotted, "\phi'(p,q)"] & \bullet \arrow[r, "d"'] & \bullet \end{tikzcd}$$

we have $\phi(cpa, dqb) = c \circ \phi'(p, q) \circ b$. This is essentially saying a morphism of $\mathcal{L}$ is a morphism in $\mathcal{C}^{\mathbf{2}} \times (\mathcal{C}^{\mathbf{2}})^{\mathrm{op}}$ that commutes with the lifting choices given by ϕ and ϕ'.

The construction uses this $\mathcal{L}$, which acts a bit like a relation between the arrow category and itself, to generate an adjunction

$$\mathbf{Cat}/\mathcal{C}^{\mathbf{2}} \underset{\mathcal{L}^*}{\overset{\mathcal{L}_*}{\rightleftarrows}} (\mathbf{Cat}/\mathcal{C}^{\mathbf{2}})^{\mathrm{op}}$$

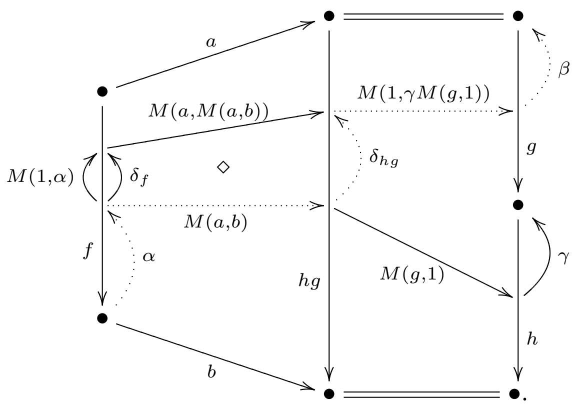
in a way analogous to the usual method of generating a Galois connection from a relation between sets. Thus if we start with some $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ on the left hand side of the adjunction, we apply the adjoint to get $\mathcal{L}_*(\mathcal{J}) \to \mathcal{C}^{\mathbf{2}}$, a category of maps with chosen liftings against everything in $\mathcal{J}$—liftings that are also required to be preserved by morphisms of $\mathcal{J}$. So far this appears to work very well as the algebraic version of the Galois connection used for weak factorisation systems.

However, a problem arises when we think about applying this adjunction to the left or right maps of an AWFS. Ideally, we would like $\mathcal{L}^*(\mathbf{R}\text{-}\mathbf{Map})$ to be isomorphic to $\mathbf{L}\text{-}\mathbf{Map}$. This does not always happen; it is possible to have a map with a choice of lifting against all the right maps that is not a left map. One can construct such a counterexample by noticing that the retract of a left map inherits liftings from that left map but is not necessarily a left map itself. This comes down to the difference between a coalgebra for the comonad and a coalgebra for the underlying copointed endofunctor: the latter is a more general class of objects—but they still have liftings against right maps, defined in the same way as above.

Fixing the problem. To sort this out we have to demand a little bit more consistency in our chosen liftings. What it is that sets apart the copointed endofunctor coalgebras that are also left maps? This is related to the question: what exactly does the comultiplication of the comonad do? The answer is hidden in the previous section, in the definition of composition of right and left maps. The comultiplication tells right maps how to compose. This suggests that the special feature of a 'genuine' left map is that it chooses liftings that are compatible with right map composition.

Lemma 2.2.1. *Given a left map (f, α) and two composable right maps (g, β) and (h, γ), there are two ways of defining the lift of f against hg: one can take the lift of (f, α) against the composite $(hg, \gamma \bullet \beta)$, or one can lift in two steps, first lifting against (h, γ) and then against (g, β). These two alternative lifts are equal.*

Proof. The single essential ingredient used by the proof is the comultiplication axiom for the coalgebra structure α. Let $(a, b)\colon f \to hg$ be the commutative square for which we want to solve the lifting problem. The 'one-step' approach defines the lift as the composite of the dotted arrows in



Using the naturality square for δ (marked with a diamond in the diagram) and the coalgebra axiom that $\delta_f \circ \alpha = M(1,\alpha) \circ \alpha$, we can turn this into the composite

$$\beta \circ M(1, \gamma M(g,1)) \circ M(a, M(a,b)) \circ M(1,\alpha) \circ \alpha.$$

Then we use functoriality of M a few times to see that this is equal to

$$\beta \circ M(a, \gamma M(ga,b)\alpha) \circ \alpha$$

which is the result we would have obtained using the alternative 'two-step' approach. $\square$

With this in mind, we will construct a new adjunction that includes a requirement that the lifting functions ϕ obey the compatibility with composition established in the lemma. First we need to establish the input data for either side of the adjunction, which is a category over $\mathcal{C}^{\mathbf{2}}$ that may come with some compositions defined.

Definition 2.2.2. A *category of partially composable maps* (which we will abbreviate CPCM) over $\mathcal{C}$ is a category over $\mathcal{C}^{\mathbf{2}}$, say

$$\mathcal{D} \xrightarrow{\quad U \quad} \mathcal{C}^{\mathbf{2}},$$

with a choice of identities $\mathbf{1}_A$ in $\mathcal{D}$ for some $A \in \mathcal{C}$ such that $U\mathbf{1}_A = 1_A$, and a choice of composites $d \bullet d'$ for some composable pairs in $\mathcal{D}$ such that $U(d \bullet d') = Ud \circ Ud'$. These compositions and identities satisfy associativity and unit axioms wherever those axioms can be defined.

We will write $\mathbf{CPCM}(\mathcal{C})$ for the category of CPCMs over $\mathcal{C}$. A morphism of $\mathbf{CPCM}(\mathcal{C})$ is a functor over $\mathcal{C}^{\mathbf{2}}$ which preserves all identities and composites which exist.

Definition 2.2.3. If a CPCM has all composites and identities we refer to it simply as a *category of composable maps*. These form a full subcategory of $\mathbf{CPCM}(\mathcal{C})$ which we will denote using $\mathbf{CCM}(\mathcal{C})$.

We note that $\mathbf{Cat}/\mathcal{C}^{\mathbf{2}}$ can also be viewed as a full subcategory of $\mathbf{CPCM}(\mathcal{C})$—the one containing just those CPCMs which happen to have no composites or identities at all.

We now obtain the *liftings adjunction*

$$\mathbf{CPCM}(\mathcal{C}) \underset{\mathcal{L}^*}{\overset{\mathcal{L}_*}{\rightleftarrows}} \mathbf{CPCM}(\mathcal{C})^{\mathrm{op}}$$

where $\mathcal{L}_*(\mathcal{D})$ is the category of maps with chosen liftings against $\mathcal{D}$ that are consistent with whatever compositions and identities $\mathcal{D}$ has. The easiest way to define this formally is as a full subcategory of the result of the earlier adjunction: an object of $\mathcal{L}_*(\mathcal{D})$ is a pair (g, Λ) with g any morphism of $\mathcal{C}$ and $\Lambda \colon \mathcal{D} \to \mathcal{L}$ a functor that chooses lifts against all the objects of $\mathcal{D}$ as shown:

$$\begin{tikzcd}
\bullet \arrow[r, "p"] \arrow[d, "Ud"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "q"'] \arrow[ur, dotted, "\phi_d(p,q)"] & \bullet,
\end{tikzcd}$$

so we must have that $\Lambda d = (Ud, g, \phi_d)$ for each $d \in \mathcal{D}$. The fact that Λ is a functor makes these lifts compatible with morphisms of $\mathcal{D}$. To make them compatible with the composition structure on $\mathcal{D}$ we add the condition that for any composite $d \bullet d'$ in $\mathcal{D}$ we have

$$\phi_d(\phi_{d'}(p, q \circ Ud), q) = \phi_{(d \bullet d')}(p, q)$$

which is shown in the diagram below; either side of the equation could label the lower dotted arrow

$$\begin{tikzcd}
\bullet \arrow[dr, "p"] \arrow[dd, "Ud'"'] & \\
& \bullet \arrow[dd, "g"] \\
\bullet \arrow[ur, dotted, "\phi_{d'}(p, q \circ Ud)"] \arrow[dd, "Ud"'] & \\
& \bullet. \\
\bullet \arrow[ur, "q"'] \arrow[uuur, dotted] &
\end{tikzcd}$$

We will observe a number of special properties that both $\mathcal{L}_*(\mathcal{D})$ and $\mathcal{L}^*(\mathcal{D})$ will have for any input $\mathcal{D}$. First of all, the forgetful functor $\mathcal{L}_*(\mathcal{D}) \to \mathcal{C}^{\mathbf{2}}$ is always faithful, since a morphism in $\mathcal{L}_*(\mathcal{D})$ is defined to be a morphism of $\mathcal{C}^{\mathbf{2}}$ with the property of preserving liftings. Secondly, if we are given two composable maps with lifting structures, we can define a lifting structure on the composite just by lifting one after the other; hence $\mathcal{L}_*(\mathcal{D})$ is always fully composable—it is a CCM over $\mathcal{C}$.

In fact we can go a little further. Recall that in Section 2.1 we pointed out that any category of left or right maps can be given the structure of a double category over $\mathbf{Sq}(\mathcal{C})$. It is easy to see that this is true also for $\mathcal{L}_*(\mathcal{D})$. It will be useful later to have a name for this property of a faithful CCM:

Definition 2.2.4. If $U\colon \mathcal{Q} \to \mathcal{C}^{\mathbf{2}}$ is a CCM over $\mathcal{C}$ such that U is faithful, we say that it has the *double category property* if, for any two composable pairs q, q', p and p' in $\mathcal{Q}$, if $(a,b)\colon Uq \to Up$ and $(b,c)\colon Uq' \to Up'$ as shown

$$\begin{tikzcd} \bullet \arrow[r, "Uq"] \arrow[d, "a"'] & \bullet \arrow[r, "Uq'"] \arrow[d, "b"] & \bullet \arrow[d, "c"] \\ \bullet \arrow[r, "Up"'] & \bullet \arrow[r, "Up'"'] & \bullet \end{tikzcd}$$

are given by $U\phi$ and $U\phi'$ for $\phi\colon q \to p$ and $\phi'\colon q' \to p'$ in $\mathcal{Q}$, then the entire square $(a,c)\colon Uq' \circ Uq \to Up' \circ Up$ is given by $U(\phi' \bullet \phi)$ for a map $\phi' \bullet \phi\colon q' \bullet q \to p' \bullet p$ in $\mathcal{Q}$.

We will now prove the main result of this section, cementing the connection between AWFS and the liftings adjunction.

Theorem 2.2.5. *For any AWFS $(\mathbb{L}, \mathbb{R})$ the pair $(\mathbf{L\text{-}Map}, \mathbf{R\text{-}Map})$ is a fixed point for the liftings adjunction.*

Proof. This simply amounts to showing that $\mathcal{L}_*(\mathbf{L\text{-}Map}) \cong \mathbf{R\text{-}Map}$, since the other side will then follow as it is precisely dual. We have already seen, in the previous section and then in Lemma 2.2.1, that any right map has consistent liftings against all left maps. To show the converse, let us assume that (f, ϕ) is some object of $\mathcal{L}_*(\mathbf{L\text{-}Map})$; in other words ϕ is some function that gives a lifting $\phi((g,\alpha), a, b)$ for every commutative square from a left map to f.

We will use the data contained in ϕ to construct a right map structure on f. The factorisation $Rf \circ Lf$ yields a commutative square

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "Lf"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "Rf"'] \arrow[ur, dotted, "\beta"'] & \bullet \end{tikzcd}$$

and $\beta = \phi((Lf, \delta_f), 1, Rf)$ is our prospective algebra structure for f. The two triangles in the diagram are the first two right map axioms; so far so straightforward.

The remaining axiom is $\beta \circ \mu_f = \beta \circ M(\beta, 1)$. To check this is a slightly tricky diagram chase; in fact it will be much easier if we first establish that the other lifts given by ϕ can all be derived from β in the usual way. For any commutative square (a,b) from a left map (g,α) to f, we consider that $(1,\alpha)\colon (g,\alpha) \to (Lg, \delta_g)$ is a left map morphism (using the coalgebra axioms) and $(a, M(a,b))\colon (Lg,\delta_g) \to (Lf, \delta_f)$ is clearly also a left map morphism. So in the diagram below the first two squares are both left map morphisms

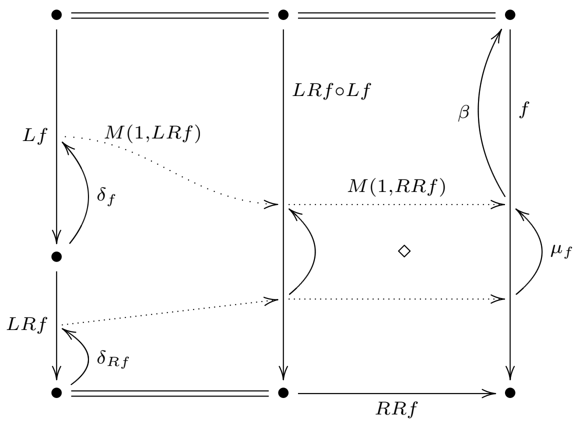
$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "g"'] & \bullet \arrow[r, "a"] \arrow[d, "Lg"'] & \bullet \arrow[r, equal] \arrow[d, "Lf"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "\alpha"'] & \bullet \arrow[r, "M(a,b)"'] & \bullet \arrow[r, "Rf"'] \arrow[ur, dotted, "\beta"'] & \bullet \end{tikzcd}$$

and the fact that ϕ is consistent with left map morphisms tells us that

$$\phi((g, \alpha), a, b) = \phi((Lf, \delta_f), 1, Rf) \circ M(a, b) \circ \alpha = \beta \circ M(a, b) \circ \alpha$$

which implies that β determines ϕ in the usual way.

Now we check the multiplication axiom. This works by considering the two ways of lifting the composite left map $(LRf \circ Lf, \delta_{Rf} \bullet \delta_f)$ against f. First we do this with a single application of ϕ



and the lift we obtain is

$$\beta \circ M(1, RRf) \circ \mu_{(LRf \circ Lf)} \circ M(M(1, LRf)\delta_f, 1) \circ \delta_{Rf}.$$

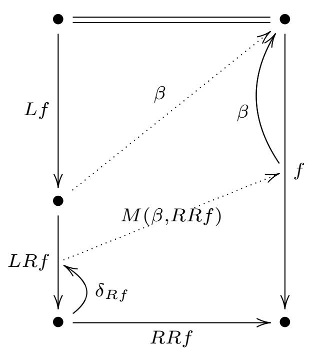
Using a naturality square for μ (marked with a diamond in the diagram), this simplifies to

$$\beta \circ \mu_f \circ M(M(1, Rf)\delta_f, RRf) \circ \delta_{Rf},$$

which, with the two immediate applications of one of the unit laws for the comonad, becomes $\beta \circ \mu_f$. On the other hand the original lift can alternatively be constructed using two applications of ϕ, lifting Lf first, and then LRf. This alternative approach gives

$$\beta \circ M(\beta, RRf) \circ \delta_{Rf}$$

as shown in the diagram



which, using the same unit law as before, simplifies to $\beta \circ M(\beta, 1)$.

This completes the process of checking that (f, β) is a bona fide object of **R-Map**. We also established that the lifting function ϕ was entirely determined by the algebra structure β it produces. This makes it easy to check that lift-preserving morphisms are the same as morphisms of right maps, so we have functors going either way. These form an isomorphism of categories; on both sides we have a strict equality of either algebra structure maps or lifting functions. $\square$

This theorem immediately tells us that the liftings adjunction has a lot of interesting fixed points; it is an intriguing question whether the adjunction is in fact idempotent. Certainly for a large class of inputs it reaches a fixed point (which also happens to be an AWFS) after a single step—we will see how to prove this in Section 2.4. However, we can also find examples of CPCMs which do not generate AWFS and therefore may not generate fixed points for the liftings adjunction; thus we cannot rule out the existence of some counterexample to idempotence.

2.3 Weak Morphisms

There is a final piece of 'semantic structure' that one can obtain from a general AWFS that we should briefly describe before continuing. There are two ways of getting at this structure, motivated by different philosophies, but the ultimate result is equivalent either way. Appealingly, one way uses the left hand side of the AWFS and the other way uses the right hand side, so there is a lovely balance to this section!

The idea is to construct a new notion of morphism on the category $\mathcal{C}$ based on the data of an AWFS on $\mathcal{C}$. There are a number of intuitions we should have in mind. Firstly, the construction is a little bit like the homotopy category of a model category; it is similar to the localisation obtained by asking that all right maps become isomorphisms. However, the right maps are only partially inverted—instead of becoming isomorphisms, they will become split epimorphisms. This gives the construction a rather different and subtler flavour to the construction of localisation.

Another perspective is that the new morphisms are some kind of 'weakened' version of normal morphisms; in the higher categorical examples in particular they behave very much like the various notions of weak functors between weak n-categories. A weak functor is still a morphism of the underlying globular sets, but it is not expected to preserve compositions precisely—only up to isomorphism.

Left approach. The approach that uses the left hand side of the AWFS is due to Garner, and is used in [Gar08] to define *homomorphisms* of higher categories. The idea is that a weak map from $A \to B$ is the same thing as a strict map from $QA \to B$, where QA is a kind of 'loosened' version of A. Of course, an AWFS gives us a formal notion for this 'loosening'—the left map replacement comonad.

Definition 2.3.1. Given any AWFS on a category $\mathcal{C}$ with an initial object, we define the *left weak maps category*, denoted by $\mathcal{C}_w$, to be the coKliesli category for the left map replacement comonad given by the AWFS.

It is clear that any strict map (ordinary morphism of $\mathcal{C}$) can be viewed as merely a special sort of weak map, by precomposing with the weak identity map (the counit of the comonad, $\epsilon\colon QA \to A$). Secondly, notice that the left map replacement comonad preserves the initial object of $\mathcal{C}$—this follows from the functoriality of the factorisation. Based on this, we see that $\mathcal{C}_w$ has the same initial object as $\mathcal{C}$. In many examples this is exactly what we should expect; the initial object is usually empty, so there is not really any room for weakness in the maps out of it.

Right approach. The other way of defining the weak maps category uses the intuition that we are trying to (at least partially) formally invert the right maps for the AWFS. If we simply formally adjoin inverses for all the objects of **R-Map** we immediately run into some problems with size; however, let us imagine the result anyway. A morphism in our new category is a zig-zag of morphisms where every second morphism is a right map pointing backwards. One could take some pullbacks and use the fact that right maps are composable to reduce this zig-zag to a single span in which the left hand map is a right map. This motivates our next definition.

This second approach is due to Bourke; as far as the author is aware, he has not published the material yet, but similar ideas can be found in [Gar12a] and in [Bou14] where they have some applications to 2-categorical algebra.

Definition 2.3.2. Given any AWFS on a category $\mathcal{C}$ with pullbacks, we define the bicategory of *right map spans*, denoted by $\mathbf{Span}_R$, to be the bicategory with the same objects as $\mathcal{C}$, with morphisms from A to B given by spans such as

$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[loop left, "\alpha"] & \\ A & & B \end{tikzcd}$$

and with 2-morphisms given by morphisms of spans that are also right map morphisms. Composition in $\mathbf{Span}_R$ can be defined by pullback, and this works due to the properties of the category of right maps.

Definition 2.3.3. The *right weak maps category*, denoted by $\mathcal{C}_{w'}$, is the result of quotienting $\mathbf{Span}_R$ out by all of its 2-morphisms.

Theorem 2.3.4. *There is an identity on objects isomorphism of categories between $\mathcal{C}_w$ and $\mathcal{C}_{w'}$.*

Proof. We need to construct an isomorphism

$$\mathcal{C}_w(A,B) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \mathcal{C}_{w'}(A,B)$$

for every pair of objects in $\mathcal{C}$, and then check that these isomorphisms preserve identities and composition. The map ϕ is very easy to construct: we are given a map $QA \to B$ and we always have $\epsilon_A\colon QA \to A$, which is a right map. This pair is a right map span from A to B with QA at its apex.

To construct ψ we will use the liftings structure as shown in the diagram

$$\begin{tikzcd}
 & 0 \arrow[dl, "!"'] \arrow[dr, "!"] & & \\
QA \arrow[rr, dotted] \arrow[dr, "\epsilon_A"'] & & \bullet \arrow[dl, "f"] \arrow[dr, "g"] & \\
 & A & & B
\end{tikzcd}$$

where the dotted arrow is the lift of $!\colon 0 \to QA$ against f, and the image of ψ is defined to be the composite of the dotted arrow with g.

It follows from the algebra and coalgebra axioms that the lift in the square $(Lf, Rf)\colon Lf \to Rf$ is always the identity map 1_{Mf}. This implies that $\psi\phi$ is trivial, since the dotted arrow in the above diagram will be 1_{QA}. To check that $\phi\psi$ is trivial we have to check that the two spans in the above diagram are in the same equivalence class. This follows from the fact that the dotted arrow gives a right map morphism from ϵ_A to f. This is a short diagram chase involving the comultiplication axiom for f.

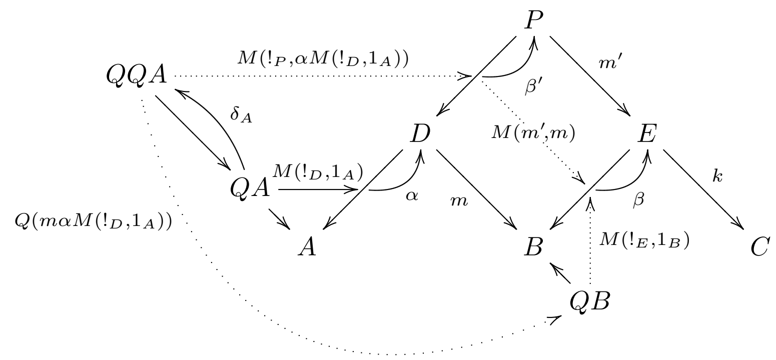
The fact that the isomorphisms preserve identities is immediate: notice that the spans $(A, 1_A, 1_A)$ and $(QA, \epsilon_A, \epsilon_A)$ are both valid representatives for the identity equivalence class in $\mathbf{Span}_R(A, A)$. It is a little more involved to check that they preserve composition, though we only have to show this for one of the isomorphisms; we will do ψ. We take two composable spans and first form the pullback shown in the diagram

$$\begin{tikzcd}
 & & P \arrow[dl, "\beta'"] \arrow[dr, "m'"] & & \\
 & D \arrow[dl, "\alpha"] \arrow[dr, "m"] & & E \arrow[dl, "\beta"] \arrow[dr, "k"] & \\
A & & B & & C
\end{tikzcd}$$

in which β' is the pullback right map structure obtained from β. We then apply ψ to each span separately, using the axiom $M(1, Rf) \circ \delta_f = 1_{Mf}$ in each case to simplify things:

$$\begin{tikzcd}
 & & & P \arrow[dl, "\beta'"] \arrow[dr, "m'"] & & \\
QA \arrow[rr, "M(!_D, 1_A)"] \arrow[dr] & & D \arrow[dl, "\alpha"] \arrow[dr, "m"] & & E \arrow[dl, "\beta"] \arrow[dr, "k"] & \\
 & A & & B & & C. \\
 & & & & QB \arrow[ul] \arrow[uu, "M(!_E, 1_B)"'] &
\end{tikzcd}$$

Now we want to show that composing these two maps in the coKleisli category gives the same result as applying ψ to the pullback span. Both of these are shown in the diagram



as the two ways of getting from QA to C; they both begin with δ_A and end with $k \circ \beta$. The coKleisli composition is given by the lower dotted arrows while the upper dotted arrows show the result of applying ψ to the whole span. So the problem comes down to proving that the square of dotted arrows is commutative, which follows by expressing $Q(m\alpha M(!_D, 1_A))$ in terms of M and using functoriality a few times. □

It is interesting to take particular note of one major asymmetry in the two definitions of weak maps: the left approach uses only the replacement comonad, whereas the right approach uses the entire category of right maps. There are examples where two different AWFS have the same replacement comonad, in which case the category of weak maps is the same for both. However, the categories of right maps will be different, so $\mathbf{Span}_R$ will also be different in either case. The quotienting out of $\mathbf{Span}_R$ potentially loses a vast amount of information.

We can also comment briefly on the dual construction, which demonstrates an isomorphism between the Kleisli category of a right map replacement monad and a category of cospans. It is tempting to speculate that it may be of interest to computer scientists that the Kleisli category can be given this alternative expression. Especially so if the left maps can be thought of as cell complexes—arguably a kind of abstract syntax for sequencing operations, some of which can be done in parallel and some of which need to happen in a specific order. Some related applications of AWFS in computer science have already been explored in [Gar12a].

2.4 The Algebraic Small Object Argument

The original small object argument of Quillen is the cornerstone of the theory of weak factorisation systems, as it is how practically all of the interesting examples are constructed. One begins with a small set of morphisms J in a cocomplete category $\mathcal{C}$ in which one wishes to construct a weak factorisation system. Applying the Galois connection described in Section 1.3, one obtains a fixed point consisting of two classes of maps $\mathcal{L}_J$ and $\mathcal{R}_J$. Now one wishes to show that this fixed point constitutes a weak factorisation system: that there exists a factorisation for every morphism of $\mathcal{C}$ into a member of $\mathcal{L}_J$ followed by a member of $\mathcal{R}_J$.

The set J also generates a class of maps $\mathbf{Cell}_J$ called the *relative J-cell complexes*, which are defined as all maps in $\mathcal{C}$ that can be obtained from J using only coproducts, composition and pushouts along any other morphism in $\mathcal{C}$. It follows from a straightforward argument that you can construct liftings of any relative cell complex against any map in $\mathcal{R}_J$, and thus $\mathbf{Cell}_J \subset \mathcal{L}_J$. The small object argument is a transfinite construction that takes any morphism f of $\mathcal{C}$ and factorises it into a relative cell complex followed by a member of $\mathcal{R}_J$. This demonstrates that $(\mathcal{L}_J, \mathcal{R}_J)$ is a weak factorisation system.

Because the ordinary small object argument only really cares about demonstrating the existence of some factorisation, it does not need to worry about constructing the factorisation in a particularly controlled way. Therefore it is rather wasteful; the relative cell complex constructed ends up being much larger than it needs to be, the transfinite recursion does not converge (so it has to be terminated at an arbitrary sufficiently large ordinal number), and there is no universal property. The *algebraic small object argument* which was described by Richard Garner in [Gar12b] was originally suggested as a way of fixing these unsatisfactory features; the fact that it happens to produce an AWFS was an interesting side-effect. For us, however, this is the key feature!

Now we will give a quick description of this revised small object argument. As before, we have a complete and cocomplete category $\mathcal{C}$. Now, instead of a set of maps in $\mathcal{C}$ we will consider a small category $\mathcal{J}$ together with a functor $I\colon \mathcal{J} \to \mathcal{C}^2$. The objects of $\mathcal{J}$ are the *generating left maps*. Already this is more general than the original small object argument; there may be morphisms between generating left maps and there may be more than one generating left map structure on a given morphism of $\mathcal{C}$.

We also call the objects of $\mathcal{J}$ *cells*—the name comes from the basic examples in topological spaces, where they will be the inclusions of spheres into discs. Following this intuition futher, we will sometimes refer to the domains of these maps as their *boundaries*.

Step 1: the density comonad. The first step in our construction of the factorisation is simply to form the density comonad on $\mathcal{C}^2$ that is created by $\mathcal{J}$—this was defined in section 1.1. We will describe it explicitly here though, since it is not obvious exactly what it does.

Given a morphism of $\mathcal{C}$, $f\colon X \to Y$, we consider the category $\mathcal{S}$ of squares from generating left maps to f—in other words, objects are commuting squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "Ij"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

where j is some object of $\mathcal{J}$. Morphisms between squares are morphisms in $\mathcal{J}$ that commute with the squares. The category $\mathcal{S}$ is small because $\mathcal{J}$ is small, and it comes with a functor $\mathcal{S} \to \mathcal{C}^2$. We take the colimit of this functor in $\mathcal{C}^2$; this is the image of the density comonad on f. Write it as Df. In most of the cases we consider, $\mathcal{J}$ will be discrete, so Df will be a huge coproduct built out of all the $\mathcal{J}$-shaped cells in f.

Step 2: normalise the domain. In step 1 we created Df, which is (usually) a very disjoint colimit of all the cells possible in f. The domain of Df is a disjoint collection of boundaries, which is not much good as we want Df to be the left hand side of a factorisation—we want its domain to be X.

Fortunately, this can be easily arranged! The counit of the comonad D gives a commutative square from Df to f, which we can abuse as follows:

$$\begin{tikzcd}
\mathrm{dom}(Df) \arrow[r] \arrow[d, "Df"'] & X \arrow[r, equal] \arrow[d, "L_0 f"] & X \arrow[d, "f"] \\
\mathrm{cod}(Df) \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & M_0 f \arrow[r, "R_0 f"'] & Y.
\end{tikzcd}$$

It is straightforward to show that L_0 is also a comonad. It has the advantage of being domain preserving, but aside from that it looks very much like D—all the cells are still there, but their boundaries have been squashed back together to make X. The factorisation of f into $L_0 f$ and $R_0 f$ is our first approximation to the final factorisation.

Step 3: iterate, carefully. We have constructed a functor $R_0 \colon \mathcal{C}^2 \to \mathcal{C}^2$, and we can see that it is a pointed endofunctor. There is a general technique in category theory to construct the *free monad* on a pointed endofunctor; this is done by iterating the functor. In the case of R_0, it is possible to show that the property that the other half of the factorisation is a comonad is preserved by this iteration process—hence we end up with a factorisation consisting of a comonad and a monad.

We will briefly explain this iteration more explicitly. For some ordinal number α assume R_α, L_α and M_α are all already defined. In the diagram

$$\begin{tikzcd}
X \arrow[r, "L_\alpha f"] \arrow[d, "f"'] & M_\alpha \arrow[r, shift left, "L_0 R_\alpha f"] \arrow[r, shift right, "M_\alpha(L_\alpha f, 1_Y)"'] \arrow[d, "R_\alpha f"'] & M_0 R_\alpha f \arrow[r] \arrow[d, "R_0 R_\alpha f"] & M_{\alpha+1} f \arrow[d, "R_{\alpha+1} f"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y,
\end{tikzcd}$$

$M_{\alpha+1} f$ is defined as the coequaliser of the two canonical maps $R_\alpha f \to R_0 R_\alpha f$. We can then define $L_{\alpha+1} f$ as the composite of the entire top row in the diagram.

The coequaliser we take here has the effect of removing the redundancies that were a problem in the normal small object argument. The object $M_0 R_\alpha f$ has two copies of a lot of cells that we only need one copy of; the two maps of the coequaliser match up these doubles perfectly, so when we take the coequaliser we get a new version of $M_0 R_\alpha f$ with the unnecessary cells removed. In the case of a limit ordinal λ, the factorisation (L_λ, R_λ) is defined in the obvious way using transfinite composition.

In the presence of certain smallness conditions, similar to those required for the usual small object argument to work, the sequence of factorisations that we have constructed converges. We call the resulting functors L, M and R. They form an AWFS, in which the objects of $\mathcal{J}$ have canonical left map structures.

Expression as a universal property. This construction, while obviously very close in spirit to the original small object argument, is now so natural that we would hope to be able to express the properties of the resulting AWFS in a single universal property. This is exactly what we will now do!

Definition 2.4.1. Given a small category over $\mathcal{C}^2$, say $I\colon \mathcal{J} \to \mathcal{C}^2$, and an AWFS $(\mathbb{L}, \mathbb{R})$ on $\mathcal{C}$, we say that $(\mathbb{L}, \mathbb{R})$ is *free* on $\mathcal{J}$ if there is a functor $\eta\colon \mathcal{J} \to$ **L-Map** over $\mathcal{C}^2$ with the property that, given any other AWFS $(\mathbb{L}', \mathbb{R}')$ and a functor $F\colon \mathcal{J} \to$ **L′-Map**, there is a unique morphism of AWFS $\phi\colon (\mathbb{L}, \mathbb{R}) \to (\mathbb{L}', \mathbb{R}')$ for which $F = \phi_l \circ \eta$.

It is a theorem (see [Gar12b] for the full details) that the AWFS generated from $\mathcal{J}$ by the revised small object argument is free on $\mathcal{J}$; in fact we end up with a partial adjunction between the category of AWFS and $\mathbf{Cat}/\mathcal{C}^2$.

There is another nice property an AWFS can have with respect to $\mathcal{J}$: we say that $(\mathbb{L}, \mathbb{R})$ is *algebraically free* if there is a functor $\eta\colon \mathcal{J} \to$ **L-Map** over $\mathcal{C}^2$ that induces an isomorphism of categories between **R-Map** and $\mathcal{L}_*(\mathcal{J})$, which we defined in Section 2.2. This is exactly saying that if we begin with $\mathcal{J}$, viewed as a CPCM with no compositions or identities, and apply the liftings adjunction, we obtain (**L-Map**, **R-Map**). Algebraically free implies free, but the converse may not be true. However, in the case of the small object argument above, the AWFS produced is both free and algebraically free.

2.5 Characterising Left Maps

In this section we conclude our exploration of the relationship between semantics and syntax by asking an obvious question: if you start with some possible semantics, how can you tell that there is some syntax for which it is the semantics? More specifically, given a category of composable maps over $\mathcal{C}$, what are sufficient and necessary conditions for it to be equivalent to the category of left maps for some AWFS? The answer to this question extends Beck's Monadicity Theorem to the world of AWFS.

For the rest of this section we will now assume that we have a category of composable maps over $\mathcal{C}$ which we will write

$$\mathcal{Q} \xrightarrow{\quad U \quad} \mathcal{C}^{\mathbf{2}}.$$

There are essentially two parts of the theorem. The first part is checking that $\mathcal{Q}$ is equivalent to a category of coalgebras for a comonad on $\mathcal{C}^{\mathbf{2}}$, and this part is basically a simple application of the dual of the monadicity theorem. The second part, in which we will find a few subtleties, involves making sure that the comonad in question forms part of an AWFS. The second part rests upon the following lemma due to Richard Garner, which can be found (in its dual form) in [BR13].

Lemma 2.5.1. *If $\mathbb{L}$ is a domain preserving comonad on $\mathcal{C}^{\mathbf{2}}$, whose category of coalgebras has a given CCM structure with the double category property, then there is an AWFS $(\mathbb{L}, \mathbb{R})$ where $\mathbb{R}$ is defined using the counit of $\mathbb{L}$, and the composition structure of left maps for $(\mathbb{L}, \mathbb{R})$ coincides with the one given.*

One can see from this lemma that there are two small obstacles to simply sticking the two existing results together and calling it a theorem. The first obstacle is the requirement that the comonad be domain preserving; the second obstacle is the necessary CCM structure on the coalgebras—we will have to check that the CCM structure on $\mathcal{Q}$ can be satisfactorily moved across the equivalence.

We will now give a complete list of all the conditions the theorem requires. The first three will be recognisable as exactly the requirements for the dual of the monadicity theorem. The remaining three are extra conditions required to overcome the obstacles we mentioned above. They are extremely easy to check; in most examples they are practically automatic.

L1 The functor U has a right adjoint.

L2 The functor U is conservative.

L3 The category $\mathcal{Q}$ has, and U preserves, all equalisers of U-split pairs.

L4 The identities for the CCM structure on $\mathcal{Q}$ have the following universal property: for any $q \in \mathcal{Q}$ and commutative square such as

$$\begin{tikzcd} A \arrow[r, equal, "U\mathbf{1}_A = 1_A"] \arrow[d, "a"'] & A \arrow[d] \\ B \arrow[r, "Uq"'] & C, \end{tikzcd}$$

there is a unique morphism $\langle a \rangle \colon \mathbf{1}_A \to q$ in $\mathcal{Q}$ living over the square.

L5 The domain functor $\partial \colon \mathcal{Q} \to \mathcal{C}$, which takes an object $q \in \mathcal{Q}$ to the domain of its image under U, is an isofibration. This means that given $q \in \mathcal{Q}$ and an isomorphism $a \colon \partial q \cong A$ in $\mathcal{C}$, there exists a unique isomorphism $\psi \colon q \cong q'$ in $\mathcal{Q}$ such that the domain part of $U\psi$ is a.

L6 The CCM structure on $\mathcal{Q}$ has the double category property given in Definition 2.2.4.

The following lemma will allow us to ensure that the comonad we are dealing with satisfies the condition of being domain preserving.

Lemma 2.5.2. *Given conditions* **L1**, **L4** *and* **L5** *we can choose a right adjoint K in such a way that the comonad UK is strictly domain preserving.*

Proof. From condition **L1**, we know that a right adjoint exists; let K' be some right adjoint for U. Given any morphism $f \colon A \to B$ in $\mathcal{C}$, we have $UK'f \colon A' \to B'$ and a commutative square $\epsilon_f \colon UK'f \to f$. For any $X \in \mathcal{C}$, we know from **L4** that morphisms $X \to A'$ are in bijection with morphisms $\mathbf{1}_X \to K'f$ in $\mathcal{Q}$. By the adjunction, morphisms $\mathbf{1}_X \to K'f$ in $\mathcal{Q}$ are in bijection with commutative squares $1_X \to f$, which are really just morphisms $X \to A$ in $\mathcal{C}$. Thus we see that the domain part of ϵ_f induces a bijection $\mathcal{C}(X, A') \cong \mathcal{C}(X, A)$ for every $X \in \mathcal{C}$, which means it is an isomorphism. We will denote it using $e_f \colon A' \cong A$ for clarity.

Now we can define a new functor $K \colon \mathcal{C}^{\mathbf{2}} \to \mathcal{Q}$ using the condition **L5**. Take the object $K'f$ and use the isofibration structure to transfer it along the isomorphism e_f; the result of this is what we will call Kf. We define the action of K on morphisms by composition with the isomorphisms; it is straightforward to see that this is well-defined, using the naturality of ϵ. Functoriality is similarly very quick to check. It is immediate that UK is a domain preserving endofunctor on $\mathcal{C}^{\mathbf{2}}$.

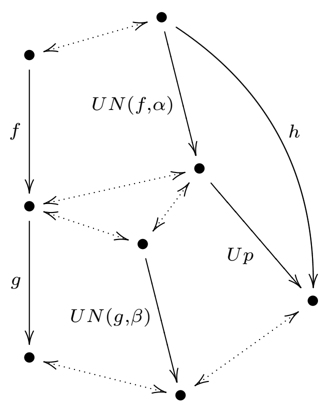
Note that the isomorphism $\psi_f \colon K'f \cong Kf$, given to us by the isofibration structure, is the unique isomorphism of that form which has e_f as its boundary part. Using ψ_f, one can define a unit and counit that make U and K adjoint; furthermore, both of these natural tranformations have trivial domain parts, meaning that UK is strictly domain preserving as a comonad. $\square$

Having carefully chosen a particular right adjoint in this way, we can proceed with the application of the monadicity theorem. Conditions **L1**, **L2** and **L3** are exactly the conditions required by the standard theorem to show that the functor M induced by the universal property of an Eilenberg-Moore category is one half of an equivalence between $\mathcal{Q}$ and **UK-Coalg**.

$$
\begin{tikzcd}
\mathcal{Q} \arrow[dd, shift right, "U"'] \arrow[dr, dotted, shift right, "M"'] & \\
& \mathbf{UK\text{-}Coalg} \arrow[ul, dotted, shift right, "N"'] \arrow[dl, shift right, "U'"'] \\
\mathcal{C}^{\mathbf{2}} \arrow[uu, shift right, "K"'] \arrow[ur, shift right, "K'"'] &
\end{tikzcd}
$$

It is important to note that while M is necessarily a functor over $\mathcal{C}^{\mathbf{2}}$, meaning that $U'M = U$, the other half of the equivalence, N, need not have that property. In fact it is generally to be expected that there may be morphisms of $\mathcal{C}$ which have no $\mathcal{Q}$-map structures, but which do have coalgebra structures; the equivalence does not necessarily restrict to an equivalence between the fibres over each object of $\mathcal{C}^{\mathbf{2}}$.

Now we must define a CCM structure on **UK-Coalg** using the one on $\mathcal{Q}$. Identities are very easy; we just apply M to each of the identities in $\mathcal{Q}$. For composition, suppose we are given a pair of composable coalgebras (f, α) and (g, β). First we need to apply N to both, giving two $\mathcal{Q}$-maps. However, these are only composable up to isomorphism, so we use **L5** to change the domain of $N(g, \beta)$, obtaining a new $\mathcal{Q}$-map p which we can compose with $N(f, \alpha)$. Applying M to the result gives some coalgebra (h, γ), such that h is isomorphic to $g \circ f$, so we can transfer the coalgebra structure γ across the isomorphism to obtain $(g \circ f, \beta \bullet \alpha)$. This is shown in the diagram



where the dotted arrows represent isomorphisms.

Finally we must check that this composition structure is associative and unital, and that it satisfies the double category condition. The general strategy for all three of these is the same; in each case we obtain two new maps with coalgebra structures (h, γ) and (h', γ'), and we want to check that a given commutative square between h and h' is a coalgebra morphism. This follows from **L6** each time. For the double category condition this coalgebra morphism $(h, \gamma) \to (h', \gamma')$ then induces a coalgebra morphism between the two composites themselves. In checking associativity and unitality the given commutative square is an isomorphism, so the fact that it is a coalgebra morphism shows that (h, γ) and (h', γ') are isomorphic as coalgebras. This implies that the coalgebra structures they induce on the composites are equal.

Theorem 2.5.3 (Characterisation of left map categories.)**.** *Let $U\colon \mathcal{Q} \to \mathcal{C}^{\mathbf{2}}$ be a CCM over $\mathcal{C}$ that satisfies the six conditions* **L1** *to* **L6***. Then there exists an AWFS* $(\mathbb{L}, \mathbb{R})$ *such that there is an equivalence*

$$\mathcal{Q} \underset{M}{\overset{N}{\rightleftarrows}} \mathbf{L\text{-}Map}$$

where the functor M is a morphism of CCMs over $\mathcal{C}$.

Note on an alternative approach. There is an alternative way of handling the question of making sure that UK is a domain preserving comonad. This approach is due to John Bourke, who proved the theorem independently of the author; his proof is largely very similar to the author's, but we will quickly remark on this one aspect where the two proofs diverge. In Bourke's approach we replace conditions **L4** and **L5** with a single alternative condition:

L4/5 The functor U is a *discrete pushout-opfibration*. This means that $\mathcal{Q}$ has pushforwards in the same sense that we showed for left map categories in Section 2.1.

Bourke's proof that UK is domain preserving uses the orthogonal factorisation system on $\mathcal{C}^{\mathbf{2}}$ given by $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is all pushout squares and $\mathcal{M}$ is all squares with isomorphisms for their domains. It is generally true that if the forgetful functor on a category of coalgebras for a comonad is a discrete $\mathcal{E}$-opfibration for some orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$, then the counit maps ϵ_A for that comonad are all elements of $\mathcal{M}$. Specialising this result to the case at hand, it implies that given **L4/5**, the comonad is essentially domain preserving.

Both **L4** and **L5** can be derived as special cases of **L4/5**. While the converse, that $\mathbf{L4} + \mathbf{L5} \Rightarrow \mathbf{L4/5}$, is clearly true in the presence of the other conditions, one can easily construct a counterexample that shows it is not necessarily true in general. For instance, let $\mathcal{Q}$ be a CCM on $\mathcal{C}$ such that there is only one non-trivial $\mathcal{Q}$-map up to isomorphism; this $\mathcal{Q}$-map can have pushouts along isomorphisms (to satisfy **L5**) without needing to have all pushouts (as **L4/5** would imply).

Categories of lifting structures. We end this chapter by considering $\mathcal{D} \to \mathcal{C}^{\mathbf{2}}$ to be an arbitrary CPCM and looking at the properties of $\mathcal{L}^*(\mathcal{D})$. How close do we come to being able to apply the theorem in this extremely general case?

The conditions **L4** and **L5** are easy to see, and we have already noted the fact that **L6** and **L2** hold—see Section 2.2. We will not give the proofs here, but one can also establish **L2**, and similarly that $\mathcal{L}^*(\mathcal{D})$ has all small colimits and the forgetful functor preserves them too; it also has pushforwards. Essentially, $\mathcal{L}^*(\mathcal{D})$ has all the properties and structures you would expect from a category of left maps, apart from one very important exception: the existence of a right adjoint.

This observation demonstrates the importance of the small object argument. The small object argument fills this gap by constructing the factorisation system, and hence the right adjoint (or, from the dual perspective, it constructs a left adjoint for the forgetful functor $\mathcal{L}_*(\mathcal{J}) \to \mathcal{C}^{\mathbf{2}}$).

Chapter 3

Cell Complexes

In this chapter we reach the main core of this thesis—the definition of the category of cell complexes generated by some category of morphisms. This generalises the work in the paper [Ath12], in which cell complexes were defined for a single set of generating maps in the category of topological spaces. Here we extend the definition so that it works in any complete and cocomplete category equipped with some simple structure. In the paper the definition was set up in a way that depended on the generating maps forming a set; here we will give a more complicated definition that works for any small category of generating maps that satisfy some conditions described in Section 3.3. We will prove that when those conditions hold, the resulting category of cell complexes satisfies the requirements of Theorem 2.5.3 and hence is equivalent to a category of left maps; furthermore we will see that these are the left maps for the AWFS generated from the same category of generating maps using Garner's algebraic small object argument.

The first two sections of this chapter approach the definition at two levels of sophistication: the first deals merely with a set of generating maps, the second extends this to a category of generating maps. The definition in Section 3.2 is the main one which we will use in the rest of this chapter and beyond, and Section 3.1 can easily be recovered as a special case of it. It is worth doing the simpler case first, however, as it seriously improves the readability of the chapter.

In the third section we introduce some structure and conditions on the underlying category and the choice of generating maps. We define the notion of a *typical nerve* which is a nerve functor on a category with special properties; it leads us to identify a class of maps which we call *typical inclusions*. This is the structure we need to define certain conditions on a category of generating maps called the *typical conditions*. When these conditions hold it will allow us to prove an important lemma called the *pullback lemma* (Lemma 3.3.11) which is necessary to make the definitions in the first two sections work; formally speaking the recursive definition of **CellCx** is not completed until we prove this lemma. In the fourth section we establish a few basic properties of the category of cell complexes, and in the fifth we will prove the main result, applying Theorem 2.5.3.

3.1 A Set of Generators

The special case where we have a category of generating maps which is discrete (and can therefore be viewed simply as a set) is one that includes most of the examples we are interested in. It also allows a rather simpler definition of cell complexes, and as such it provides a useful warm-up to the more general definition in the next section. The basic approach is the same in both definitions, so the intuitions introduced in this section are worth keeping in mind later in the chapter.

Recall the observations made in Section 2.1, where we saw how a category of left maps always has pushouts, composites and colimits—and that these three ways of joining left maps together behave consistently with one another according to the stacking property (see Proposition 2.1.4). Given any left map, it is natural to ask how it can be decomposed into simpler left maps. In general there may not be a canonical way to do this; but in the case of cell complexes the generating maps form a collection of simplest possible complexes and every complex can be decomposed into some combination of these. The stacking property tells us that this decomposition can be performed in layers and this observation is the central intuition behind the definition in this section.

Single-layer cell complexes. A cell complex with only one layer is one that can be formed by taking a base space and gluing a set of cells onto it "all in one go". This concept of simultaneous gluing can be formalised by expressing the complex as the pushout of a coproduct of generating maps.

We will work over some cocomplete category $\mathcal{C}$, and consider some discrete category of generating maps $I\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$. In a single-layer cell complex, the cells are all entirely independent of one another, so one can describe the complex simply using a set of cells together with instructions for how they are each glued onto the base space. If $Ij\colon \partial j \to \overline{j}$ is a generating map and a is a cell of shape j in a single-layer cell complex over the base object X, the necessary information to see how a is glued onto X is just a morphism $g_a\colon \partial j \to X$ which we call the *binding map* of a.

The set of cells we consider in the paragraph above is really a presheaf over $\mathcal{J}$ since each cell has a *shape* given by an object of $\mathcal{J}$. We want to capture the notion of such a presheaf together with a bit of extra information (the binding map) attached to each cell. This data is all very nicely encapsulated in the formalism of a comma category (or Artin gluing) of a certain functor $T\colon \mathcal{C} \to \hat{\mathcal{J}}$.

Definition 3.1.1. The *terminal layer functor* $T\colon \mathcal{C} \to \hat{\mathcal{J}}$ takes an object X to the presheaf of possible binding maps into X. This means that

$$TX(j) = \mathcal{C}(\partial j, X),$$

and this is made into a functor in the obvious way by postcomposition.

Definition 3.1.2. The category of *height one cell complexes* generated by $\mathcal{J}$, which we write as $\mathbf{CellCx}_1$, is defined to be the comma category $(\hat{\mathcal{J}} \downarrow T)$.

Let us unpack this definition and recall what the comma category construction does. An object of $\mathbf{CellCx}_1$ is given by a triple (X, σ, g) where X is an object of $\mathcal{C}$ (the *base object*), σ is a presheaf of sets over $\mathcal{J}$ (which specifies a set of cells of each shape), and g is a presheaf morphism $g\colon \sigma \to TX$. Hence g is a collection of functions that assign to every cell $a \in \sigma(j)$ an element of $TX(j)$—it specifies the binding maps of cells. Given a cell $a \in \sigma$, we will write s_a for the shape of a (the object of $\mathcal{J}$ which a lives over) and g_a for the binding map.

Definition 3.1.3. The *underlying map functor* $U_1\colon \mathbf{CellCx}_1 \to \mathcal{C}^{\mathbf{2}}$ is defined using pushouts. Given a height one complex (X, σ, g), we define $U_1(X, \sigma, g)$ to be the bottom morphism in the pushout square

$$\begin{tikzcd}
\coprod_{a\in\sigma} \partial s_a \arrow[r, "\coprod_{a\in\sigma} I s_a"] \arrow[d, "\langle g \rangle"'] & \coprod_{a\in\sigma} \overline{s_a} \arrow[d] \\
X \arrow[r] & \overline{(X, \sigma, g)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

in which the map $\langle g \rangle$ is the map out of the coproduct induced by the collection of maps $g_a\colon \partial s_a \to X$. A morphism of $\mathbf{CellCx}_1$ is a morphism of base objects together with a function between the sets of cells. This function induces a map between the upper parts of the pushout squares for each complex, and this induces a commutative square between the images under U_1 which makes U_1 a functor.

Finally we can quickly note that U_1 has a right adjoint, which we call the *free cell complex* functor and write K_1. For any morphism $f\colon A \to B$ in $\mathcal{C}$, the free cell complex $K_1 f$ has base object A. For its presheaf part it has the presheaf of all commutative squares

$$\begin{tikzcd}
\partial j \arrow[r] \arrow[d, "Ij"'] & A \arrow[d, "f"] \\
\overline{j} \arrow[r] & B,
\end{tikzcd}$$

and we can specify that the binding map of each square is the top morphism $\partial j \to A$.

Now, any map from another cell complex (X, σ, g) into $K_1 f$ gives a function from σ to the presheaf of such commutative squares; this determines a map $\overline{s_a} \to B$ for each $a \in \sigma$, which gives a morphism in $\mathcal{C}^{\mathbf{2}}$ of the form $U(X, \sigma, g) \to f$. Furthermore, this is a bijective correspondence since any such map $U(X, \sigma, g) \to f$ determines precisely the morphisms $\overline{s_a} \to B$ for each of its cells, and thus specifies a unique cell complex morphism $(X, \sigma, g) \to K_1 f$. Hence we can see that U_1 and K_1 form an adjunction.

Multiple-layer cell complexes. To build an appropriate category of cell complexes (and create a category that has some chance of being a category of left maps) we must be able to compose our maps. Since composing two single-layer complexes may result in some cells that are joined onto other cells, we will end up with some cell complexes that cannot be described by a single layer. For full generality (depending on the choice of generating maps) we may need to consider complexes with any transfinite number of layers.

For this reason the definition now becomes recursive. We assume that we have defined *height* α complexes, where α can be any ordinal number, and we will construct another *terminal layer functor* $T_\alpha\colon \mathbf{CellCx}_\alpha \to \hat{\mathcal{J}}$. Then we can define height $\alpha+1$ complexes using the same kind of comma category construction as we used previously.

Considering this iterated comma category construction, we will end up thinking of a cell complex as a base object followed by a sequence of presheaves. The length of the sequence is the height of the complex; in particular we should point out that the initial case of the recursion is just a base object and an empty sequence—in other words, $\mathbf{CellCx}_0 \cong \mathcal{C}$. We will generally write a cell complex over X in the form $(X,\vec{\sigma})$ where $\vec{\sigma}$ represents the sequence of presheaves.

Another piece of notation we should introduce immediately is the following: if β is some ordinal less than the height of $(X,\vec{\sigma})$, we will write $(X,\vec{\sigma}|_\beta)$ for the *β-abbreviation* of $(X,\vec{\sigma})$, by which we mean the height β cell complex we get by removing layers from $(X,\vec{\sigma})$. The abbreviation $(X,\vec{\sigma}|_\beta)$ is naturally a subcomplex of $(X,\vec{\sigma})$ so there is an inclusion morphism which we will denote by

$$\overline{(X,\vec{\sigma}|_\beta)} \xrightarrow{i_{\beta\alpha}} \overline{(X,\vec{\sigma})}.$$

We will see how this is formally defined in the following paragraphs; it is worthwhile introducing the notation up front since the definition is highly recursive.

Definition 3.1.4. The *terminal layer functor* $T_\alpha\colon \mathbf{CellCx}_\alpha \to \hat{\mathcal{J}}$ takes a complex $(X,\vec{\sigma})$ to the presheaf of possible *new* binding maps into the object $\overline{(X,\vec{\sigma})}$. This means that if we start with the presheaf $\mathcal{C}(\partial(-),\overline{(X,\vec{\sigma})})$, then $T_\alpha(X,\vec{\sigma})$ is the subpresheaf of this consisting of only the binding maps which cannot be factored through any lower layer of the complex. So formally

$$T_\alpha(X,\vec{\sigma})(j) = \Big\{ g\colon \partial j \to \overline{(X,\vec{\sigma})} \,\Big|\, \forall\beta<\alpha,\, \forall k\colon \partial j \to \overline{(X,\vec{\sigma}|_\beta)},\, g \neq i_{\beta\alpha}\circ k \Big\}.$$

Again, this is made into a functor in the obvious way by postcomposition; we must check that postcomposition with any cell complex morphism $(f,\vec{\theta})\colon (X,\vec{\sigma}) \to (Y,\vec{\tau})$ preserves the set of binding maps that cannot be factored through any lower layer. To do this we use the pullback lemma (Lemma 3.3.11) which we will prove later. It implies that if the composite

$$\partial j \xrightarrow{g} \overline{(X,\vec{\sigma})} \xrightarrow{\overline{(f,\vec{\theta})}} \overline{(Y,\vec{\tau})}$$

factors through a lower layer of $(Y,\vec{\tau})$ then the binding map g itself factors through a lower layer of $(X,\vec{\sigma})$.

The fact that we have used a lemma from the end of Section 3.3 to make sure this definition makes sense may seem rather strange; in fact, it is just part of the recursion. There is an induction hypothesis that we have proved the pullback lemma for height α complexes before we move on to defining height $\alpha+1$, and this is not a problem because everything in Section 3.3 will work exactly the same for the category of complexes up to any ordinal height.

There is a good reason for the extra condition that the binding maps do not factor through any lower layer. If we relaxed this condition we could have cells in the third layer of a complex that could really be moved into the second layer using the stacking property; this condition enforces the 'normal form' we are working with where every cell has to be in the lowest layer it can be in.

Definition 3.1.5. The category of *height* $\alpha + 1$ *cell complexes*, which we will write $\mathbf{CellCx}_{\alpha+1}$, is defined to be the comma category $(\hat{\mathcal{J}} \downarrow T_\alpha)$.

Definition 3.1.6. The *underlying map functor* $U_{\alpha+1}\colon \mathbf{CellCx}_{\alpha+1} \to \mathcal{C}^{\mathbf{2}}$ is defined in a similar way to U_1 above. Let $(X, \vec{\sigma})$ be a height $\alpha + 1$ complex. Its image under $U_{\alpha+1}$ is constructed using the composite

$$X \xrightarrow{U_\alpha(X,\vec{\sigma}|_\alpha)} \overline{(X, \vec{\sigma}|_\alpha)} \xrightarrow{\quad\psi\quad} \overline{(X, \vec{\sigma})}$$

where the map ψ is the result of treating the top layer of $(X, \vec{\sigma})$ as if it were a single-layer complex over $\overline{(X, \vec{\sigma}|_\alpha)}$ and then applying U_1.

The inclusion map $i_{\alpha(\alpha+1)}$ is defined to be equal to ψ above. Each other inclusion map $i_{\beta\alpha}$ is defined as a composite (possibly transfinite) of these 'single-step' inclusion maps.

Again, we wish to check that $U_{\alpha+1}$ has a right adjoint which we will denote by $K_{\alpha+1}$. This is not hard to construct; given any map $f\colon A \to B$ in $\mathcal{C}$, the first step is to apply K_α to f to obtain a height α complex. We must then add one new layer to this in order to get $K_{\alpha+1}f$. Our immediate instinct is to apply the functor K_1 to the map $\epsilon\colon \overline{K_\alpha f} \to B$ given by the counit of the α-level adjunction. This is the right kind of idea, but one can see that the result contains cells that disobey our important rule—they have binding maps that can be factored through abbreviations of $K_\alpha f$.

The solution to this is not particularly subtle; we just take the subcomplex of $K_1\epsilon$ that contains only the legal cells. This is now a valid extra layer we can add to $K_\alpha f$, and doing so provides our definition of $K_{\alpha+1}f$. Since any map from a height $\alpha+1$ complex into $K_{\alpha+1}f$ will only be able to hit the cells in $K_1\epsilon$ that are legal anyway, it is not hard to see that we have a right adjoint to $U_{\alpha+1}$.

The limit ordinal case. The case of a cell complex of limit ordinal height λ is very straightforward. First of all, note that for each ordinal α the category $\mathbf{CellCx}_\alpha$ has an inclusion functor into $\mathbf{CellCx}_{\alpha+1}$, which we will denote by

$$\mathbf{CellCx}_\alpha \xrightarrow{I_{\alpha(\alpha+1)}} \mathbf{CellCx}_{\alpha+1}.$$

This is defined simply by taking each height α complex $(X, \vec{\sigma})$ to the initial (in other words, empty) presheaf in $T_\alpha(X, \vec{\sigma})$. Basically, every height α complex can be thought of as a height $\alpha + 1$ complex whose top layer is empty; we will also compose these inclusion functors together to get $I_{\beta\alpha}\colon \mathbf{CellCx}_\beta \to \mathbf{CellCx}_\alpha$ for any pair of ordinals $\beta < \alpha$. Then the category $\mathbf{CellCx}_\lambda$ is just the colimit of the sequence of inclusion functors $I_{\beta\alpha}$ for all $\beta < \alpha < \lambda$.

To define the underlying map of a height λ complex $(X, \vec{\sigma})$, we first obtain the underlying maps $U_\alpha(X, \vec{\sigma}|_\alpha)$ of the abbreviations for every $\alpha < \lambda$. For any pair of these abbreviations given by $\beta < \alpha < \lambda$ there is the inclusion map $i_{\beta\alpha}\colon \overline{(X, \vec{\sigma}|_\beta)} \to \overline{(X, \vec{\sigma}|_\alpha)}$ such that $U_\alpha(X, \vec{\sigma}|_\alpha)$ is equal to the composite

$$X \xrightarrow{U_\beta(X,\vec{\sigma}|_\beta)} \overline{(X, \vec{\sigma}|_\beta)} \xrightarrow{i_{\beta\alpha}} \overline{(X, \vec{\sigma}|_\alpha)}.$$

This shows that sequence of abbreviations gives a colimit diagram that defines a transfinite composite of the inclusion maps. We define the underlying map of the entire complex, $U_\lambda(X, \vec{\sigma})\colon X \to \overline{(X, \vec{\sigma})}$, to be this transfinite composite. The right adjoint of this is defined simply as the limit of the sequence of right adjoints for each ordinal less than λ: the free height λ complex on f is the object of $\mathbf{CellCx}_\lambda$ defined by the sequence $(K_\alpha f)_{\alpha<\lambda}$.

The total category of cell complexes. So far we have treated cell complexes of different heights as entirely different cases; for our definition to make sense we will need to have a single category that contains all cell complexes of all heights. This is itself rather easy, but we do run into complications when we try to extend the functors K_α into a free total cell complex functor K. The problem is that some choices of generating maps may lead to complexes which can be arbitrarily high—and since Kf must be universal it has to be as high as any of them, so it cannot exist! For this reason, we care a lot about the existence of a maximum height for cell complexes; this property of the generating maps is analogous to the smallness conditions required for the small object argument to work.

Definition 3.1.7. We define the *total category of cell complexes*, which we denote by $\mathbf{CellCx}$. It has as its objects all cell complexes of any height. The morphisms $(A, \vec{\sigma}) \to (B, \vec{\tau})$, where these are complexes of height α and β respectively, are the morphisms in $\mathbf{CellCx}_\alpha$ from $(A, \vec{\sigma})$ to $(B, \vec{\tau}|_\alpha)$ if $\alpha \le \beta$, or $(A, \vec{\sigma})$ to $I_{\beta\alpha}(B, \vec{\tau})$ if $\beta < \alpha$.

One can clearly use the sequence of functors U_α to obtain a single underlying map functor $U\colon \mathbf{CellCx} \to \mathcal{C}^\mathbf{2}$ in such a way that they commute with the inclusion functors $\mathbf{CellCx}_\alpha \to \mathbf{CellCx}$. As we observed above, this functor U does not necessarily have a right adjoint. The following condition is enough to ensure that it does.

Definition 3.1.8. We say that a (discrete) category of generating maps $\mathcal{J}$ has λ as a *height ceiling* if T_λ is the constant functor that takes every object in $\mathbf{CellCx}_\lambda$ to the empty presheaf.

Having λ as a height ceiling implies that any higher ordinal is also a height ceiling. It also forces layers above λ in any complex to be trivial, and hence tells us that every complex is isomorphic to one of height λ or lower. This means that the categories $\mathbf{CellCx}_\lambda$ and $\mathbf{CellCx}$ are equivalent, and in turn implies that we can construct the total right adjoint K simply by using K_λ.

Basically any example we care about in practice will have a height ceiling; the classic example on topological spaces has ω as a height ceiling—this follows from the usual smallness conditions that are part of the small object argument (see [Hov99] for detail). Similarly, all the n-categorical examples we will be interested in will have ω as a height ceiling, because any cell in a free n-category is the composite of a finite number of generating cells. One can also construct examples with 1 as a height ceiling—this happens, for example, if every generating map has empty domain. In this case the entire construction reduces to the nerve-realisation adjunction discussed in Section 1.1.

3.2 A Category of Generators

We will now repeat the work in the previous section without the requirement that $\mathcal{J}$ be a discrete category—so the initial data we begin with is any small category over $\mathcal{C}^{\mathbf{2}}$ which we will always write as $I\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$. The case where $\mathcal{J}$ is not discrete proves to be rather more complex and potentially confusing than the simpler case; hopefully the framework we established in the last section will serve as a reference point and keep an overall picture of the definition accessible.

Single-layer cell complexes. Again we begin by considering cell complexes that can be described using only one pushout of a single colimit of generating maps. For the case of complexes with only one layer of cells, the fact that $\mathcal{J}$ is a category does not really change the definition. Instead of just a set of cells with different shapes we have a set of cells together with some data telling us how they fit together using the morphisms of $\mathcal{J}$; but this data is still all captured perfectly by the notion of a presheaf.

Definition 3.2.1. The *terminal layer functor* $T\colon \mathcal{C} \to \hat{\mathcal{J}}$ takes an object X to the presheaf of possible binding maps into X. This means that

$$TX(j) = \mathcal{C}(\partial j, X),$$

and this is a presheaf over $\mathcal{J}$ by precomposition with the boundary part of each map $\alpha\colon j \to k$. As before, T is made into a functor by postcomposition.

Definition 3.2.2. The category of *height one cell complexes* generated by $\mathcal{J}$, which we write as $\mathbf{CellCx}_1$, is defined to be the comma category $(\hat{\mathcal{J}} \downarrow T)$.

At this point in the definition, everything works just as it did in the previous section. We continue using the same notation and terminology; so if (X, σ, g) is a complex we call X the *base object* and the natural transformation $g\colon \sigma \to TX$ gives each cell a a *binding map* g_a. Now we also have *restriction maps*—for any cell a and morphism $\alpha\colon j \to s_a$ in $\mathcal{J}$ there is a cell $\alpha(a)$ of shape j which we call the *restriction of a along α.*

The adjunction between the *underlying map functor* U_1 and the *free cell complex functor* K_1 is defined almost exactly as before. The pushout square to define U_1 uses colimits rather than coproducts, and the set of commutative squares

$$\begin{tikzcd} \partial j \arrow[r] \arrow[d, "Ij"'] & A \arrow[d, "f"] \\ \overline{j} \arrow[r] & B, \end{tikzcd}$$

is made into a presheaf over $\mathcal{J}$ by defining restriction maps using precomposition.

We should point out that for a given base object X, the slice category over TX (which is the subcategory of single-layer complexes on X) is itself a presheaf category with one object for every element of TX. From this perspective, U_1 and K_1 can be constructed as a nerve-realisation adjunction. This alternative approach to defining $\mathbf{CellCx}_1$ might seem easier as it makes the adjunction fall out trivially; however, the approach we are taking has the advantage that all complexes are defined as presheaves over the same category, which will make life simpler later on.

Downward cell functions. The main subtlety we will now have to contend with as we define cell complexes with multiple layers is the fact that restriction maps do not necessary respect the layers. One can easily visualise the situation where a cell has a binding map $g_a \colon \partial j \to \overline{(X, \vec{\sigma})}$ that only factors through layer one, but when restricted along α we get $g_{\alpha(a)}$ given by the composite

$$\partial k \xrightarrow{\quad \partial\alpha \quad} \partial j \xrightarrow{\quad g_a \quad} \overline{(X, \vec{\sigma})}$$

which can easily factor through X if ∂k is a subobject of ∂j (which will generally be the case). In this case, we cannot represent layer two as a presheaf in $\hat{\mathcal{J}}$ because the restriction $\alpha(a)$ is not in layer two—it must be in layer one by the rule saying cells must be in the lowest possible layer.

To solve this issue, we encode some of the restriction maps (the ones that change layer) with an extra piece of information called the *downward cell function* (which will be formally defined in Definition 3.2.4). Note that, as the name would suggest, these 'layer traversing' restriction maps can only go downwards in the complex. The downward cell function is encoded in the same way as the binding map of a cell, using the presheaf $T(X, \vec{\sigma})$ and the natural transformation g. We also need to change the category over which we take presheaves so that it is more flexible—since some restriction maps are missing from its point of view—and to do this we can use the notion of a *sieve*.

Definition 3.2.3. For any small category $\mathcal{J}$ we define the *category of sieves* on $\mathcal{J}$, which we will write $\mathcal{S}_{\mathcal{J}}$, as follows:

- an object of $\mathcal{S}_{\mathcal{J}}$ is a strict subobject of a representable in the presheaf category $\hat{\mathcal{J}}$,
$$A \to [s],$$
- a morphism between two such sieves is a pullback square
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[s]} \arrow[d, "{[\alpha]}"] \\ B \arrow[r] & {[t]} \end{tikzcd}$$
in the presheaf category, for any morphism $\alpha \colon s \to t$ in $\mathcal{J}$.

It is tempting to think of this construction as some kind of completion—one considers the canonical inclusion map $J \colon \mathcal{J} \to \mathcal{S}_{\mathcal{J}}$ which takes each object s in $\mathcal{J}$ to the initial sieve, $\varnothing \to [s]$. However, it is easily seen that the construction is not idempotent; the category $\mathcal{S}_{\mathcal{S}_{\mathcal{J}}}$ is not equivalent to $\mathcal{S}_{\mathcal{J}}$. For example, it contains a new copy of every sieve in the original sieve category. A better way to think about $\mathcal{S}_{\mathcal{J}}$ is as a category of '$\mathcal{J}$-objects with structure'; an object of

$\mathcal{S}_{\mathcal{J}}$ is an object of $\mathcal{J}$ together with the structure of a sieve on that object. This makes the definition of morphisms in $\mathcal{S}_{\mathcal{J}}$ less opaque—a morphism between two sieves is a 'structure preserving' morphism between their carrier $\mathcal{J}$-objects.

The category $\mathcal{S}_{\mathcal{J}}$ is exactly the right one to consider as a general category of cell shapes. When a cell is simple and has a trivial downward cell function, it is represented as an element over one of the trivial sieves in the image of $J\colon \mathcal{J} \to \mathcal{S}_{\mathcal{J}}$. However, when a cell has some non-trivial downward cell function, this defines a collection of cells inside it that are really in lower layers of the complex. This collection of cells is precisely a sieve, so we represent the cell as an element over that sieve. The sieve tells us exactly which restriction maps are missing.

Multiple-layer cell complexes. We are now equipped to make the recursive definition of cell complexes with any transfinite number of layers. As before, we assume everything is done for some ordinal number α, and proceed to make the necessary definitions for $\alpha+1$. We will define the terminal layer functor again; though this time it takes values in $\hat{\mathcal{S}}_{\mathcal{J}}$ rather than $\hat{\mathcal{J}}$.

The same intuition—and notation—as before can be used for this definition. A cell complex consists of a base object followed by a sequence of presheaves, where each element represents a single cell and comes with the extra information of a binding map and downward cell function given by the natural transformation $g\colon \sigma_\alpha \to T_\alpha(X, \vec{\sigma}|_\alpha)$. Again we can talk about the *β-abbreviation* of a height α complex for any $\beta < \alpha$, and again we will have inclusion maps

$$(X, \vec{\sigma}|_\beta) \xrightarrow{i_{\beta\alpha}} (X, \vec{\sigma}|_\alpha)$$

for any pair of ordinals $\beta < \alpha$ both less than or equal to the height of $(X, \vec{\sigma})$.

Definition 3.2.4. The *terminal layer functor* $T_\alpha\colon \mathbf{CellCx}_\alpha \to \hat{\mathcal{S}}_{\mathcal{J}}$ takes a complex $(X, \vec{\sigma})$ to the presheaf of possible cells in the next layer—it encodes the data of binding maps and downward cell functions. This means that an element of $T_\alpha(X, \vec{\sigma})(A \to [j])$ is given by a pair (g, κ). The *binding map* g is a morphism $\partial j \to \overline{(X, \vec{\sigma})}$ such that

$$\forall \beta < \alpha,\, \forall k\colon \partial j \to \overline{(X, \vec{\sigma}|_\beta)},\; g \neq i_{\beta\alpha} \circ k,$$

just as in the previous section. The *downward cell function* κ assigns to each morphism $\phi\colon k \to j$ in the sieve $A \to [j]$ a cell $\kappa(\phi)$ of shape k in some lower layer of the complex; we can write it as

$$\kappa\colon A \to \sum_{\beta \leq \alpha} \sigma_\beta$$

where the sum on the right represents the set of all cells in the complex so far.

Both sides of the arrow above can be seen as presheaves on $\mathcal{J}$ (A by definition, the set of all cells using Proposition 3.2.6 below) so we ask κ to be a morphism of presheaves on $\mathcal{J}$. This implies that if $\psi\colon h \to k$ is another morphism of $\mathcal{J}$, then

$$\kappa(\psi \circ \phi) = \psi(\kappa(\phi)),$$

where $\psi(\kappa(\phi))$ is the restriction of $\kappa(\phi)$ along ψ, defined using Proposition 3.2.6. We also require κ to respect the binding maps in the sense that

$$g_{\kappa(\phi)} = g \circ \partial\phi.$$

Now we can easily make $T_\alpha(X, \vec{\sigma})$ into a presheaf on $\hat{\mathcal{S}}_{\mathcal{J}}$; the restriction of (g, κ) along some morphism in $\mathcal{S}_{\mathcal{J}}$ given by the morphism ϕ in $\mathcal{J}$ has binding map $g \circ \partial\phi$ and downward cell function that takes ψ to $\kappa(\phi \circ \psi)$. Finally, we can see that T_α is a functor: if $(f, \vec{\theta})\colon (X, \vec{\sigma}) \to (Y, \vec{\tau})$ is a morphism in $\mathbf{CellCx}_\alpha$ then $T_\alpha(f, \vec{\theta})$ takes (g, κ) to an element with binding map $\overline{(f, \vec{\theta})} \circ g$ and downward cell function $\vec{\theta} \circ \kappa$. Again, we must use the pullback lemma (Lemma 3.3.11) to check that the new binding map is still legal; this works in exactly the same way as in Section 3.1.

Definition 3.2.5. The category of *height* $\alpha + 1$ *cell complexes*, which we denote by $\mathbf{CellCx}_{\alpha+1}$, is defined to be the comma category $(\hat{\mathcal{S}}_{\mathcal{J}} \downarrow T_\alpha)$.

For any morphism ϕ in $\mathcal{J}$ we can talk about the *restriction* $\phi(a)$ of a cell a along ϕ, and this will sometimes mean a cell in the same layer using the presheaf structure and sometimes mean a cell in a lower layer using the downward cell function; see the following Proposition.

Proposition 3.2.6. *Given a height α cell complex $(X, \vec{\sigma})$, we can partition all of its cells into sets*

$$S(j) = \sum_{\substack{\beta \leq \alpha \\ (A \to [j]) \in \hat{\mathcal{S}}_{\mathcal{J}}}} \sigma_\beta(A \to [j])$$

of cells with the same underlying shape in $\mathcal{J}$. Then S is a presheaf over $\mathcal{J}$, where restriction maps are defined either using the presheaf structures at each layer or using the downward cell functions.

Proof. First we check that these restriction maps are well defined; given a cell whose shape in $\mathcal{S}_{\mathcal{J}}$ is $A \to [j]$, the maps $\phi\colon k \to j$ into j are split into two disjoint sets. Some are in A, in which case they do not produce morphisms of sieves because when we pullback along an element of the sieve we get the identity morphism on $[k]$ which is not a strict subobject of $[k]$. Those that are not in A, however, do produce morphisms in $\mathcal{S}_{\mathcal{J}}$. Thus the restriction maps for our presheaf structure are well-defined, because we can always use either the downward cell function or the presheaf structure, but never both.

After this check all that is left to see is the functoriality part of being a presheaf. For those restriction maps that stay in one layer this follows immediately from the presheaf structure of the layer. For those that go downwards in the cell complex functoriality is one of the two conditions on downward cell functions given in the definition above. $\square$

The next step in our definition is as usual to construct the underlying map functor $U_{\alpha+1}\colon \mathbf{CellCx}_{\alpha+1} \to \mathcal{C}^{\mathbf{2}}$. This is made slightly more complicated by the downward cell functions, since when we add a cell there may be parts of it that are really in lower layers—so we need to make our pushout in such a way that we do not add on an extra copy of some earlier cells.

In order to do this, consider some object of $\mathcal{S}_{\mathcal{J}}$ that we will write $A \to [j]$. We will begin by thinking about exactly what a cell of that shape adds to the underlying object $\overline{(A, \vec{\sigma})}$ of a cell complex. The overall shape of this cell is j, so we begin with the map $Ij\colon \partial j \to \overline{j}$. However, for each $\phi\colon k \to j$ which is in A, we get a piece of $\overline{j}$ (given by $\overline{\phi}\colon \overline{k} \to \overline{j}$) that must already appear in the complex. We must somehow put these $\overline{k}$ shaped pieces in the boundary of our map.

Definition 3.2.7. If $A \to [j]$ is a sieve over $j \in \mathcal{J}$, define the *underlying map* of the sieve to be the dotted arrow induced by the pushout square

$$\begin{tikzcd} \coprod\limits_{(\phi\colon k\to j)\in A} \partial k \arrow[rr, "\coprod Ik"] \arrow[d, "\langle \partial\phi \rangle"'] & & \coprod\limits_{(\phi\colon k\to j)\in A} \overline{k} \arrow[d] \arrow[ddr, "\langle \overline{\phi} \rangle"] & \\ \partial j \arrow[rr] \arrow[drrr, "Ij"'] & & \partial A \arrow[dr, dotted] \arrow[ul, phantom, "\ulcorner", very near start] & \\ & & & \overline{j}, \end{tikzcd}$$

in which the coproduct symbols represent colimits taken over the diagram that contains an object for each $\phi\colon k \to j$ in A, and a morphism from $\phi\colon k \to j$ to $\phi'\colon k' \to j$ is simply any morphism $\psi\colon k \to k'$ in $\mathcal{J}$ such that $\phi = \phi' \circ \psi$. The triangular brackets represent maps out of these colimits induced by cocones. We will denote the underlying map of the sieve by $I(A \to [j])\colon \partial A \to \overline{j}$.

We now observe that any morphism of sieves gives us a morphism of pushout diagrams like the one above, and hence a commutative square

$$\begin{tikzcd} \partial A \arrow[r] \arrow[d] & \overline{j} \arrow[d, "\overline{\alpha}"] \\ \partial B \arrow[r] & \overline{k}. \end{tikzcd}$$

Furthermore, the functor T_α gives exactly the data needed to define a map $\partial A \to \overline{(X, \vec{\sigma})}$ for each element of $T_\alpha(X, \vec{\sigma})(A \to [j])$. The binding map is a morphism $g\colon \partial j \to \overline{(X, \vec{\sigma})}$, while the downward cell function gives us

$$\langle \iota_{\kappa(\phi)} \rangle\colon \coprod_{(\phi\colon k\to j)\in A} \overline{k} \to \overline{(X, \vec{\sigma})}$$

in which ι_a represents the inclusion map of $\overline{s_a}$ into the colimit $\overline{(X, \vec{\sigma})}$. These two morphisms induce a map $h_{(g,\kappa)}\colon \partial A \to \overline{(X, \vec{\sigma})}$ using the pushout square that defines ∂A.

Definition 3.2.8. We define the *underlying map functor* $U_{\alpha+1}\colon \mathbf{CellCx}_{\alpha+1} \to \mathcal{C}^{\mathbf{2}}$ using a pushout square to construct the new layer and composing this with the result of U_α applied to the α-abbreviation. Given a height $\alpha+1$ cell complex

$(X, \vec{\sigma})$ we use the following pushout

$$\begin{tikzcd}
\coprod_{a \in \sigma_\alpha} \partial S_a \arrow[r, "\coprod_{a \in \sigma_\alpha} I(S_a \to [s_a])"] \arrow[d, "\langle h_{(g_a, \kappa_a)} \rangle"'] & \coprod_{a \in \sigma_\alpha} \overline{s_a} \arrow[d] \\
\overline{(X, \vec{\sigma}|_\alpha)} \arrow[r, "\lambda"'] & \overline{(X, \vec{\sigma})}
\end{tikzcd}$$

to construct λ, the map underlying the top layer of $(X, \vec{\sigma})$. Note that we use the notation S_a to mean the sieve that gives the shape of a in $\mathcal{S}_\mathcal{J}$, while we have continued using s_a to mean the shape of a in $\mathcal{J}$. Also note that the coproduct signs are being used to denote colimits defined in the same way as for single-layer complexes; one considers the category of elements of σ_α as a diagram in $\mathcal{C}^\mathbf{2}$ using the underlying maps of sieves.

Having defined λ, the morphism $U_{\alpha+1}(X, \vec{\sigma})$ is defined as the composite

$$X \xrightarrow{U_\alpha(X, \vec{\sigma}|_\alpha)} \overline{(X, \vec{\sigma}|_\alpha)} \xrightarrow{\lambda} \overline{(X, \vec{\sigma})}.$$

We can make this into a functor in the same way as we did the underlying map functor in the previous section; a function between the presheaves of cells induces maps between the colimits and this gives a morphism of pushout squares.

We will now construct the right adjoint to this underlying map functor, which we write as $K_{\alpha+1}$ and call the *free cell complex functor*. If $f \colon X \to Y$ is any morphism in $\mathcal{C}$, we begin by applying K_α to obtain a height α cell complex $K_\alpha f$. Now we must define a new layer on this complex. Consider the presheaf σ on $\mathcal{S}_\mathcal{J}$ with $\sigma(A \to [j])$ given by the set of commutative squares

$$\begin{tikzcd}
\partial A \arrow[r] \arrow[d, "I(A \to [j])"'] & \overline{K_\alpha f} \arrow[d, "\epsilon"] \\
\overline{j} \arrow[r] & Y,
\end{tikzcd}$$

where as before $\epsilon \colon \overline{K_\alpha f} \to Y$ is the map given by the counit of the adjunction between U_α and K_α. For each element of σ we consider the top morphism $\partial A \to \overline{K_\alpha f}$. In some cases, this can be specified by a binding map and downward cell function; we take the subpresheaf consisting of such cases together with the obvious natural transformation to $T_\alpha K_\alpha f$. This defines $K_{\alpha+1} f$.

A similar argument to those used in previous sections will show that this $K_{\alpha+1}$ is first of all a functor $\mathcal{C}^\mathbf{2} \to \mathbf{CellCx}_{\alpha+1}$ and secondly the right adjoint to $U_{\alpha+1}$. Basically, any morphism of cell complexes into $K_{\alpha+1}$ can only hit the 'legal' elements that are contained in the subpresheaf.

Completing the definition. The remaining steps necessary to complete the definition of the *total category of cell complexes* $\mathbf{CellCx}$ are performed in exactly the same way as in the previous section. The inclusion functors

$$\mathbf{CellCx}_\beta \xrightarrow{I_{\beta\alpha}} \mathbf{CellCx}_\alpha$$

are once again defined by adding empty layers to the shorter cell complex. The limit ordinal case once again uses the colimit of this sequence of inclusion functors for $\beta < \alpha < \lambda$ and the transfinite composite of morphisms in $\mathcal{C}$. Similarly, the total category definition is precisely the same as Definition 3.1.7.

Again, the existence of a total right adjoint $K\colon \mathcal{C}^{\mathbf{2}} \to \mathbf{CellCx}$ depends on the condition of a *height ceiling*, an ordinal number at which the functors T_α become trivial. This is formalised in precisely the same manner as that given in Definition 3.1.8.

3.3 Typical Inclusions

In the previous two sections we were slightly vague in our assumptions; we worked with a category $\mathcal{C}$ which we asked to be cocomplete and we also assumed that we could prove the pullback lemma. It turns out that proving the pullback lemma requires quite a few conditions and some special structure on $\mathcal{C}$, which we will introduce in this section. These conditions will also allow us to check that **CellCx** has the basic properties we would expect, and ultimately they will let us prove that we have defined the category of left maps for the AWFS generated by $\mathcal{J}$.

The main source of issues that can prevent us from being able to prove the pullback lemma is the inclusion of generating maps that are not sufficiently injective; for example, simply considering the generating map $\{a, b\} \to \{*\}$ in **Set** will provide a rather pathological example. The basic point is that for cell complexes to behave as one would expect the process of adding a cell should mean strictly adding to the complex's underlying object; if we have non-injective generators then sometimes adding a cell actually means reducing the underlying object by identifying parts of it. This really messes up our geometric intuition about cell complexes and (usually) causes the pullback lemma to fail; for this reason, the main substance of this section will be to establish a characterisation of inclusion maps that are 'sufficiently injective'.

It is worth saying that throughout this section there are two main examples to have in mind. The first is topological spaces, where the 'nice' inclusions we care about are the subspace inclusions. The second is categories, or some globular definition of higher categories, where the nice inclusions are simply all the subcategory inclusions. These are the examples we actually want to use later on, and they serve as the archetypes for the definitions we make here.

Definition 3.3.1. We call a subcategory $\mathcal{N}$ of the arrow category $\mathcal{C}^{\mathbf{2}}$ a category of *typical inclusions* if

- any equaliser morphism in $\mathcal{C}$ is in $\mathcal{N}$ (or equivalently $\mathcal{N}$ contains the regular monomorphisms),
- $\mathcal{N}$ is a category of composable maps over $\mathcal{C}$, as in Section 2.1,
- $\mathcal{N}$ is closed under pushout and pullback,
- if we have an equaliser

$$E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y,$$

together with a morphism $m\colon E \to E'$ that is in $\mathcal{N}$, taking the pushout of the whole equaliser diagram along m, as shown below

$$\begin{tikzcd}
E \arrow[r, "e"] \arrow[d, "m"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d] & Y \arrow[d] \\
E' \arrow[r, "e'"] \arrow[ur, phantom, very near end, "\ulcorner"] & X' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[ur, phantom, very near end, "\ulcorner"] & Y',
\end{tikzcd}$$

where the rightmost map is defined as the pushout of m along fe (or ge) and the maps f' and g' are induced by the universal property of X', gives another equaliser diagram,

- any small diagram in $\mathcal{C}^{\mathbf{2}}$ which contains only objects and morphisms in $\mathcal{N}$ has a colimit which is also an object of $\mathcal{N}$, and finally,
- given any morphism $(a, b)\colon n \to m$ in $\mathcal{N}$, in the diagram

$$\begin{tikzcd}
A \arrow[r, "n"] \arrow[d, "a"'] & B \arrow[d] \arrow[ddr, "b"] & \\
C \arrow[r] \arrow[drr, "m"'] \arrow[ur, phantom, very near end, "\ulcorner"] & E \arrow[dr, dotted] & \\
 & & D
\end{tikzcd}$$

the dotted arrow is also an object of $\mathcal{N}$.

This definition gives the basic axioms of the class of inclusions we want to choose our generating maps from. They are mostly pretty sensible things for a class of inclusions to have; the fourth bullet point is a special case of limits commuting with colimits that will be necessary to prove that U preserves equalisers in Section 3.5. We might observe that the final bullet point is similar in some respects to the pushout-product axiom which appears as Definition 4.2.1 in [Hov99], though the author is not aware of any real connection.

We also need the typical inclusions we use to be characterised in a particular way. Recall that if $\mathcal{S}$ is a small category with a functor $\mathcal{S} \to \mathcal{C}$ we can form a nerve functor $V\colon \mathcal{C} \to \hat{\mathcal{S}}$. If $\mathcal{S}$ and its inclusion functor are chosen nicely, this nerve functor can behave in some respects like the underlying set functor on topological spaces, and it will induce a category of typical inclusions.

Definition 3.3.2. A nerve functor $V\colon \mathcal{C} \to \hat{\mathcal{S}}$ is called a *typical nerve* if it induces a class of typical inclusions, which we will write $\mathcal{N}_V$, as follows

- a monomorphism $m\colon A \to B$ in $\mathcal{C}$ is an object of $\mathcal{N}_V$ if it is terminal among the monomorphisms into B that are mapped to Vm by V;
- a square between two such monics in $\mathcal{C}$ is a morphism of $\mathcal{N}_V$ if it satisfies the final bullet point in Definition 3.3.1;

and furthermore, we have two extra conditions:

- any pushout square between two typical inclusions is also a pullback square, and

- the functor V preserves transfinite composites.

Remark. Firstly, we should note that the first of the two extra conditions implies that any morphism of $\mathcal{N}_V$ is a pullback square, because it can be written as a composite of a pushout between two typical inclusions and a triangle of typical inclusions (which we consider as a square whose top morphism is an identity). This is shown in the following diagram, where typical inclusions are denoted by $\mapsto$ arrows:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, |->] & \bullet \arrow[r, equal] \arrow[d, |->] & \bullet \arrow[d, |->] \\
\bullet \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & \bullet \arrow[r, |->] & \bullet .
\end{tikzcd}$$

The fact that the latter square is a pullback follows simply from the properties of monomorphisms.

Secondly, it is worth pointing out that in order to check the first of the two conditions we only need to check that the square is a pullback when V is applied to it. One can use a combination of the fact that V preserves limits (it is a right adjoint) and the universal property of any typical inclusion to see that V reflects pullback squares between typical inclusions.

The following property of typical inclusions will be very useful in Proposition 3.3.9.

Proposition 3.3.3. *Given a typical nerve V and any object X, the counit map $\epsilon_X \colon \coprod VX \to X$ has the left lifting property with respect to every typical inclusion: that is, for any square*

$$\begin{tikzcd}
\coprod VX \arrow[r] \arrow[d, "\epsilon_X"'] & A \arrow[d, |->, "m"] \\
X \arrow[r] \arrow[ur, dotted] & B
\end{tikzcd}$$

in which m is a typical inclusion, there exists a unique dotted arrow as shown such that the two triangles commute.

Proof. This is partly a consequence of the universal property of typical inclusions, but also requires the fact that typical inclusions are preserved by pullback. First we take the pullback inside the square given above to get

$$\begin{tikzcd}
\coprod VX \arrow[dr, dotted] \arrow[drr] \arrow[ddr, "\epsilon_X"'] & & \\
& P \arrow[r] \arrow[d, |->, "p"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, |->, "m"] \\
& X \arrow[r] & B .
\end{tikzcd}$$

Then we consider the nerve functor V applied to the left hand side of the diagram; the map $Vp \colon VP \to VX$ is an injective map of presheaves. But even more than this, each element $s \to X$ of VX appears inside $\coprod VX$ and can hence be factored through P—this implies that Vp is surjective as well, and hence it is an isomorphism.

We have seen that the typical inclusion $p\colon P \to X$ is mapped to an isomorphism by V. Together with the universal property of typical inclusions, this tells us that p itself is an isomorphism, allowing us to construct the dotted arrow $X \to A$. The uniqueness requirement is given by the fact that m is a typical inclusion and hence a monomorphism. □

From now on, we will work over a pair $(\mathcal{C}, V)$ consisting of a complete and cocomplete category $\mathcal{C}$ equipped with a typical nerve V. One can quite easily verify that in the category of topological spaces the forgetful functor to **Set** is a typical nerve, whose typical inclusions are exactly the subspace inclusions. Similarly, for categories the underlying graph functor is the typical nerve that we wish to use. For higher categories the generalisation of the underlying graph functor, the underlying globular set functor, will also in general be a typical nerve.

We must now consider what conditions a category of generating maps $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ should satisfy with respect to the typical nerve. First of all, we need every generating map to be a typical inclusion. We also need the morphisms between generating maps to be morphisms of typical inclusions. This is very easy to formalise simply by asking that the functor $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ factor through the inclusion $\mathcal{N}_V \to \mathcal{C}^{\mathbf{2}}$; however, this is not quite enough to prove Proposition 3.3.6.

Definition 3.3.4. A category of generating maps $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ is *distinguishable* with respect to the typical nerve V if the underlying map functor $I\colon \mathcal{S}_{\mathcal{J}} \to \mathcal{C}^{\mathbf{2}}$ on the category of sieves factors through the category $\mathcal{N}_V$, and no sieve $A \to [j]$ has an underlying map that is an isomorphism.

Note that as $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ can be factored through the category of sieves using the trivial sieves, distinguishability implies that each generating map is itself a non-trivial typical inclusion. Proposition 3.3.6 is now almost automatic; first we should define exactly what we mean by a *subcomplex*.

Definition 3.3.5. A *subcomplex* of a complex $(A, \vec{\sigma})$ is a subobject of $(A, \vec{\sigma})$ in the category $\mathbf{CellCx}(A)$. This is the subcategory of **CellCx** consisting just of complexes over A and morphisms with 1_A as their base object part. (Recall that a *subobject* of some object X in any given category is an isomorphism class of monomorphisms into X.)

One could simply consider all monomorphisms in **CellCx** as subcomplexes, but this would introduce annoying complications because of subcomplexes that only contain part of the base object; the definition above will make life a great deal simpler later on. Any subcomplex is determined by some subset of the set of all cells in a complex; this follows from the fact that a morphism of presheaves is a monomorphism if each of its components is an injection of sets. Of course, not every subset gives a subcomplex, since some subsets will not define valid cell complexes.

Proposition 3.3.6. *Given the assumption that $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ is distinguishable with respect to V, any strict subcomplex inclusion map is a non-trivial typical inclusion.*

Proof. Any subcomplex inclusion map can clearly be defined as a transfinite composite of pushouts of underlying maps for sieves. To add some extra cell a which is not in the subcomplex, the pushout we use is

$$\begin{tikzcd}
\partial A \arrow[r, "I(A\to[s_a])"] \arrow[d, "(g_a,\langle\iota_{\phi(a)}\rangle_{\phi\in A})"'] & \overline{s_a} \arrow[d] \\
\overline{(A,\vec{\tau})} \arrow[r] & \overline{(A,\vec{\tau}+a)} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where $A \to [s_a]$ is the sieve of parts of a that are in the subcomplex already. A simply induction argument will show that each layer can be filled in using a sequence of such pushouts, and then a transfinite composition will allow us to do each layer in turn. Since $\mathcal{N}_V$ is closed under pushout and closed under transfinite composition, the proposition follows immediately. □

Remark. It follows straight away from this proposition that the underlying morphism of any complex (or any individual layer of a complex) is a typical inclusion, as these are just special cases of subcomplex inclusions.

The following definition will provide a very useful way to study maps into cell complexes. It is also vital to defining the final condition we will ask our generating maps to satisfy.

Definition 3.3.7. We consider some cell complex $(A, \vec{\sigma})$, and any morphism $x: P \to \overline{(A, \vec{\sigma})}$ in $\mathcal{C}$. The subcomplexes of $(A, \vec{\sigma})$ form a poset, so we can ask if there is a initial object in the subposet of subcomplexes with the property that x factors through their inclusion maps. If this exists, we call it the *minimal subcomplex* determined by x. We will denote it by $\mu(x)$.

We do not need all minimal subcomplexes to exist and behave well; it is simpler to restrict our attention to certain maps into $\overline{(A, \vec{\sigma})}$. Recall that the nerve V is constructed from the category of shapes $\mathcal{S}$ and its functor into $\mathcal{C}$. We call the objects in the image of this functor *shape* objects, and it is only morphisms from these into $\overline{(A, \vec{\sigma})}$ that we actually worry about. Any morphism from a shape object into $\overline{(A, \vec{\sigma})}$ corresponds to an element in the typical nerve $V\overline{(A, \vec{\sigma})}$, so it makes sense to focus on these particular morphisms.

Definition 3.3.8. Given a category of generating maps, we say that it has *consistent subcomplexes* whenever

- any morphism $x: X \to \overline{(A, \vec{\sigma})}$ into a cell complex has a minimal subcomplex,
- given any morphism of complexes as in the diagram

$$\begin{tikzcd}
A \arrow[r, "U(A,\vec{\sigma})"] \arrow[d, "f"'] & \overline{(A,\vec{\sigma})} \arrow[d, "\overline{(f,\vec{\theta})}"] & X \arrow[l, "x"'] \\
B \arrow[r, "U(B,\vec{\tau})"'] & \overline{(B,\vec{\tau})} &
\end{tikzcd}$$

with a morphism from X as shown, we have

$$\mu(\overline{(f,\vec{\theta})} \circ x) = \vec{\theta}(\mu(x))$$

where we can demand strict equality by considering them both as subcomplexes of $(B, \vec{\tau})$.

The following proposition will make the condition of having consistent subcomplexes much easier to check in practice, by allowing us to restrict our attention to a small set of objects in $\mathcal{C}$.

Proposition 3.3.9. *A weaker (and somewhat easier to check) alternative to the above is the condition that a category of generating maps has* consistent subcomplexes with respect to shape objects*. This is the same as the condition in Definition 3.3.8 above, except that we only consider morphisms $x\colon s \to \overline{(A, \vec{\sigma})}$ out of objects s in the image of the typical nerve V. In fact, when the category of generating maps is distinguishable with respect to V, these two conditions are equivalent.*

Proof. Assume that some category of generating maps $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ is distinguishable with respect to the typical nerve V, and that it has consistent subcomplexes with respect to shape objects. We simply have to check the conditions of Definition 3.3.8 for a morphism out of any object

$$X \xrightarrow{\quad x \quad} \overline{(A, \vec{\sigma})}.$$

The first thing we can do is apply the nerve functor to X to obtain a presheaf VX over the category of shape objects $\mathcal{S}$. We will write $\mathcal{E}_{VX}$ for the category of elements of this presheaf.

Notice first of all that there is a natural functor $D\colon \mathcal{E}_{VX} \to \mathcal{C}/X$, since each element of VX specifies a shape object s together with a map $s \to X$. Composing each of these maps with x gives morphisms from the shape objects into $\overline{(A, \vec{\sigma})}$, and by assumption such morphisms have consistent minimal subcomplexes. This constructs a functor $\mathcal{E}_{VX} \to \mathbf{SubCx}(A, \vec{\sigma})$ into the poset of subcomplexes of $(A, \vec{\sigma})$. We can form the colimit of this diagram in $\mathbf{SubCx}(A, \vec{\sigma})$—it is given by taking the union of all the subsets of cells that determine each subcomplex.

Call this colimit subcomplex $\mu(x)$. We will show that it is indeed the minimal subcomplex of x. We can form a square

$$\begin{tikzcd}
\coprod VX \arrow[r] \arrow[d, "\epsilon_X"'] & \overline{\mu(x)} \arrow[d, "\overline{n}", tail] \\
X \arrow[r, "x"'] \arrow[ur, dotted] & \overline{(A, \vec{\sigma})}
\end{tikzcd}$$

in which n is the subcomplex inclusion map for $\mu(x)$ and the morphism at the top is induced by the maps $s \to \overline{\mu(x)}$ for each element in the presheaf VX. We note that because of Proposition 3.3.6 the map $\overline{n}$ is a typical inclusion; this means we must have a unique dotted arrow $X \to \overline{\mu(x)}$ as shown in the diagram, using Proposition 3.3.3.

To see that $\mu(x)$ is indeed the minimal subcomplex with such a map out of X, suppose we had another: $X \to \overline{M} \mapsto \overline{(A, \vec{\sigma})}$. Then we have a map $s \to \overline{M}$ for any element $a\colon s \to X$ in VX, and this induces a subcomplex inclusion $\mu(ax) \to M$. Since $\mu(x)$ is the colimit of these subcomplexes, this induces the required map $\mu(x) \to M$ and gives it the necessary uniqueness property.

We must also show that these minimal subcomplexes are consistent. For any morphism of cell complexes $(f, \vec{\theta})$ the identity

$$\mu(\overline{(f, \vec{\theta})} \circ x) = \vec{\theta}(\mu(x))$$

holds by assumption for each of the shape objects given by elements of VX. Since the minimal subcomplexes we care about are formed as unions of these, their consistency follows because any morphism of cell complexes preserves unions (since it is simply a function of sets of cells). $\square$

Definition 3.3.10. Assume we are working over the pair $(\mathcal{C}, V)$. We say a category of generating maps satisfies the *typical conditions* or is *typical* if it is distinguishable and has consistent subcomplexes with respect to V.

The typical conditions will prove to be sufficient for our needs. They ensure that **CellCx** behaves roughly according to our intuitions. Most importantly, they are enough to finally prove the vital pullback lemma (and thus actually complete the recursive definition that we started in Section 3.1).

Lemma 3.3.11 (Pullback Lemma)**.** *Given any morphism $(f, \vec{\theta})\colon (A, \vec{\sigma}) \to (B, \vec{\tau})$ in* **CellCx***, every square in $\mathcal{C}$ of the form*

$$\begin{tikzcd}
\overline{(A,\vec{\sigma}|_{\alpha})} \arrow[r] \arrow[d, "\overline{(f,\vec{\theta}|_{\alpha})}"'] & \overline{(A,\vec{\sigma})} \arrow[d, "\overline{(f,\vec{\theta})}"] \\
\overline{(B,\vec{\tau}|_{\alpha})} \arrow[r] & \overline{(B,\vec{\tau})}
\end{tikzcd}$$

for some ordinal α, where the top and bottom are the natural inclusion maps, is a pullback square in $\mathcal{C}$.

Proof. Consider a pair of morphisms from a shape object s as shown below

$$\begin{tikzcd}
s \arrow[dr, dotted] \arrow[drr, "x"] \arrow[ddr, "y"'] & & \\
& \overline{(A,\vec{\sigma}|_{\alpha})} \arrow[r] \arrow[d, "\overline{(f,\vec{\theta}|_{\alpha})}"] & \overline{(A,\vec{\sigma})} \arrow[d, "\overline{(f,\vec{\theta})}"] \\
& \overline{(B,\vec{\tau}|_{\alpha})} \arrow[r] & \overline{(B,\vec{\tau})}
\end{tikzcd}$$

such that the outer square commutes. The diagram shows that $\overline{(f, \vec{\theta})} \circ x$ factors through the subcomplex $(B, \vec{\tau}|_{\alpha})$, meaning that its minimal subcomplex has height α or less. By the consistency of minimal subcomplexes,

$$\mu(\overline{(f, \vec{\theta})} \circ x) = \vec{\theta}(\mu(x)),$$

so we can see that $\mu(x)$ is also height α or less. But x factors through $\mu(x)$ by definition, so x must factor through $\overline{(A, \vec{\sigma}|_\alpha)}$ as shown in the diagram by the dotted arrow. The fact that the bottom arrow is a typical inclusion for a typical nerve (and hence a monomorphism—see Definition 3.3.2) shows that this dotted arrow commutes with y, and the same argument for the top arrow (which is also a typical inclusion) shows that it is unique.

This shows that the square is mapped to a pullback square by the nerve functor; and as we showed in the remark after Definition 3.3.2 this implies it is a pullback square in $\mathcal{C}$, because of the universal property of typical inclusions. □

3.4 Basic Properties

In this section we will take the definition of **CellCx** and establish some of the basic properties this category possesses. From now on, we work with the assumption that $\mathcal{C}$ comes equipped with a typical nerve V, and that $\mathcal{J}$ satisfies the typical conditions. The first aim is to show that **CellCx** is complete and cocomplete. On the way to showing this, we will examine the functor $\mathbf{CellCx} \to \mathcal{C}$ that gives the base object of a complex; this functor turns out to be a bifibration and this structure involves useful adjunctions between different fibres (categories of cell complexes with a given base object). The second aim will be to describe the composition structure on **CellCx**.

Bifibration structure. As we mentioned in the paragraph above, we are interested in the functor that gives the base object of a complex, which we call the *base object* functor (or sometimes the *boundary* functor) on **CellCx**, and denote using the boundary symbol that we already use for single cells

$$\partial \colon \mathbf{CellCx} \to \mathcal{C}.$$

We will write $\mathbf{CellCx}(A)$ for the fibre of ∂ over A—the category of complexes with base object A and morphisms with boundary part 1_A. Recall the following definition:

Definition 3.4.1. A functor $P \colon \mathcal{A} \to \mathcal{B}$ is a *Grothendieck fibration* if for every map $f \colon A \to B$ in $\mathcal{B}$ and $E \in \mathcal{A}$ such that $PE = B$ there exists some *cartesian* morphism $f' \colon F \to E$ in $\mathcal{A}$ such that $Pf' = f$.

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "P"'] & G \arrow[rr, bend left, "k"] \arrow[r, dotted, "\exists! g'"'] & F \arrow[r, "f'"'] & E \\
\mathcal{B} & PG \arrow[rr, bend left, "Pk"] \arrow[r, "g"'] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

A morphism $f' \colon F \to E$ is *cartesian* if for every $k \colon G \to E$ and $g \colon PG \to A$ such that $f \circ g = Pk$ there exists an unique $g' \colon G \to F$ in $\mathcal{A}$, as shown by the dotted arrow in the digram, such that $Pg' = g$ and $f' \circ g' = k$.

We will show that ∂ is a bifibration, meaning it is both a Grothendieck fibration and opfibration, the dual concept. In order to do this, we are going to define an adjunction

$$\mathbf{CellCx}(A) \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathbf{CellCx}(B)$$

for every morphism $f\colon A \to B$ in $\mathcal{C}$, and then check that these adjunctions compose functorially up to isomorphism. It is a classical result that there is an equivalence of 2-categories between the 2-category of (cloven) Grothendieck fibrations on $\mathcal{C}$ and the 2-category of pseudofunctors from $\mathcal{C}^{\mathrm{op}}$ to the 2-category of categories. Section 2.1.4 (and in particular Proposition 2.1.25) of [Mic10] extend this classical result by considering (bicloven) bifibrations and pseudofunctors from $\mathcal{C}$ into the 2-category of categories with adjunctions as the morphisms. For our purposes all we really need to know is that the structure with adjunctions $f_* \dashv f^*$ for every morphism in $\mathcal{C}$ is an alternative way of demonstrating a bifibration structure on ∂.

Pushforwards. We will construct the functor $f_*\colon \mathbf{CellCx}(A) \to \mathbf{CellCx}(B)$ first and we will usually call it the *pushforward* functor along f. Recall that for any category of left maps, pushforwards must exist; the pushforwards we construct now will turn out to be the same ones that the general theory of left maps provides in the case that **CellCx** is a category of left maps. This means, in particular, that when composed with the underlying map functor U, f_* becomes the usual pushout functor in the category $\mathcal{C}^{\mathbf{2}}$. In fact, the domain functor $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is also a bifibration, and the functor U preserves all of the bifibration structure.

Given a cell complex over A, say $(A, \vec{\sigma})$, the pushforward complex $f_*(A, \vec{\sigma})$ is defined quite straightforwardly layer by layer. The presheaf part of each layer in $f_*(A, \vec{\sigma})$ is exactly the same as that in the original; we simply change the binding maps. At the first layer this is simply a case of composing each binding maps with f to get the new binding map into B. Thus if (A, σ, g) is an object of $\mathbf{CellCx}_1$, expressed fully as an object in the comma category $(\hat{\mathcal{S}}_{\mathcal{J}} \downarrow T_0)$, we can write

$$f_*(A, \sigma, g) = (B, \sigma, T_0 f \circ g),$$

since the functor T_0 applied to the morphism f has the effect of composing each binding map with f. Note that $(1_\sigma, T_0 f)$ is a canonical cell complex morphism $(A, \sigma, g) \to f_*(A, \sigma, g)$ and its action on the cells is trivial in the sense that it is represented by the identity on σ.

At subsequent layers we can inductively assume we already have a canonical cell complex morphism $\phi_\alpha\colon (A, \vec{\sigma}|_\alpha) \to f^*(A, \vec{\sigma}|_\alpha)$ with this trivial action on cells. We can thus obtain a new binding map for each cell $a \in \sigma_\alpha$ by composing g_a with $\overline{\phi_\alpha}$, and note that we do not need to change the downward cell functions. Formally, we write this in precisely the same way as above: if $((A, \vec{\sigma}|_\alpha), \sigma_\alpha, g_\alpha)$ is an object of the comma category $(\hat{\mathcal{S}}_{\mathcal{J}} \downarrow T_\alpha)$, we write

$$f_*((A, \vec{\sigma}|_\alpha), \sigma_\alpha, g_\alpha) = (f_*(A, \vec{\sigma}|_\alpha), \sigma_\alpha, T_\alpha \phi_\alpha \circ g_\alpha),$$

since the functor T_α applied to ϕ_α will compose each binding map with $\overline{\phi_\alpha}$ while not changing the downward cell functions (since ϕ_α acts trivially on the presheaves of cells). Again, $(1_{\sigma_\alpha}, T_\alpha\phi_\alpha)$ is the canonical cell complex morphism we want for our next $\phi_{\alpha+1}$.

One can see that a morphism of complexes $(A,\vec{\sigma}) \to (B,\vec{\tau})$ which has f as its boundary part consists of exactly the same data as a morphism $f_*(A,\vec{\sigma}) \to (B,\vec{\tau})$ which has 1_B as its boundary part. In both cases we are specifying a function from the set of cells of $(A,\vec{\sigma})$ to the set of cells of $(B,\vec{\tau})$ such that the binding maps commute with f.

Pullbacks. The pushforward functor creates a new complex with a set of cells that is isomorphic to the set of cells of the original complex. The pullback functor is rather different however; the new complex may contain a large number of cells, or no cells at all, that correspond to a single cell in the original complex. One can view the pullback as containing one copy of every possible cell that can be mapped into the original complex along f.

To formalise this, let $(B,\vec{\tau})$ be a complex over B. Again we will define $f^*(B,\vec{\tau})$ layer by layer. Working inductively, assume we have done so up to some α, so $(f,\vec{\theta}|_\alpha)$ is a cell complex morphism from $f^*(B,\vec{\tau}|_\alpha)$ to $(B,\vec{\tau}|_\alpha)$. A cell in the next layer of the pullback complex is given by a triple (a,g,κ) where a is a cell in the next layer of $(B,\vec{\tau})$, g is any legal binding map such that

$$\overline{(f,\vec{\theta}|_\alpha)} \circ g = g_a,$$

and κ is any legal downward cell function such that

$$\vec{\theta} \circ \kappa = \kappa_a.$$

This clearly gives a presheaf on $\mathcal{S}_{\mathcal{J}}$—we lift the restriction maps from $(B,\vec{\tau})$—and we obviously have a natural transformation to $T_\alpha f^*(B,\vec{\tau}|_\alpha)$. The next layer of θ is then given by the obvious map $(a,g,\kappa) \mapsto a$.

Again, consider a morphism of complexes $(A,\vec{\sigma}) \to (B,\vec{\tau})$ which has f as its boundary part. Now we claim this consists of precisely the same data as a morphism $(A,\vec{\sigma}) \to f^*(B,\vec{\tau})$ with 1_A as its boundary part; such a morphism takes a cell a in $(A,\vec{\sigma})$ and chooses a triple (b,g,κ) as in the definition above. But the g and the κ part of this triple are actually completely determined by the binding map and downward cell function of the cell a, so we end up with just the data of a function between the sets of cells such that binding maps commute with f.

We have now basically established the adjunction between f_* and f^*, since we have shown that

$$\mathbf{CellCx}(B)(f_*P,Q) \cong \mathbf{CellCx}(P,f,Q) \cong \mathbf{CellCx}(A)(P,f^*Q),$$

where the notation $\mathbf{CellCx}(P,f,Q)$ is used to mean the subset of morphisms in $\mathbf{CellCx}(P,Q)$ mapped to f by ∂. The remaining naturality conditions are straightforward to check.

Limits and colimits. When it comes to showing that **CellCx** is a category of left maps, we only really need colimits—there is no necessity for a category of left maps to possess limits. However, in the case of cell complexes, both limits and colimits exist and are rather nicely behaved; furthermore, the processes involved in constructing them are basically the same (though obviously dual to one another). For this reason, we may as well discuss both at the same time.

To help prove that **CellCx** is complete and cocomplete, we will use the following general result about bifibrations.

Lemma 3.4.2. *Given a bifibration $\mathcal{A} \to \mathcal{C}$, the category $\mathcal{A}$ is cocomplete if $\mathcal{C}$ is cocomplete and each fibre is cocomplete. Dually, the same result holds for completeness.*

Proof. Suppose we are given a diagram functor $\mathcal{D} \to \mathcal{A}$; write each object in the image as (A_d, σ_d) for $d \in \mathcal{D}$. We will show that the colimit of the diagram is given by

$$\coprod_{d\in\mathcal{D}} (A_d, \sigma_d) = (\coprod_{d\in\mathcal{D}} A_d, \coprod_{d\in\mathcal{D}} (i_d)_*\sigma_d),$$

where for each $d \in \mathcal{D}$, i_d is the inclusion $A_d \to \coprod_{d\in\mathcal{D}} A_d$. Note we are using the symbol $\coprod$ as notation for any colimit, not just for coproducts, as indicated by the fact that d ranges over a category rather than a set.

This colimit works by first taking the colimit of the underlying objects A_d in $\mathcal{C}$. Then each σ_d, which is an object in the fibre over A_d, is canonically transferred to the fibre over $\coprod A_d$ using the adjunction $i_* \dashv i^*$ that lives over the inclusion map. Now that all the σ_d have been 'moved' into the same category, we can take their colimit, and we are done.

We must also specify the inclusion maps into this colimit, which have the form

$$(A_c, \sigma_c) \xrightarrow{(i_c, l_c)} (\coprod_{d\in\mathcal{D}} A_d, \coprod_{d\in\mathcal{D}} (i_d)_*\sigma_d)$$

where $l_c \colon \sigma_c \to (i_c)^*\big(\coprod_{d\in\mathcal{D}} (i_d)_*\sigma_d\big)$ is a morphism in the fibre over the object A_c, which we define as the composite

$$\sigma_c \xrightarrow{\eta_{\sigma_c}} (i_c)^*(i_c)_*\sigma_c \xrightarrow{(i_C)^*(n_{\sigma_c})} (i_c)^*\big(\coprod_{d\in\mathcal{D}} (i_d)_*\sigma_d\big)$$

in which n_{σ_c} is the inclusion of $(i_c)_*\sigma_c$ into the colimit in the fibre category.

Suppose now that we have another cocone over the diagram, which is given by its vertex object (Q, τ) and a collection of morphisms $(f_d, \phi_d) \colon (A_d, \sigma_d) \to (Q, \tau)$. We immediately get a unique map $\langle f \rangle \colon \coprod_{d\in\mathcal{D}} A_d \to Q$ induced by the colimit in $\mathcal{C}$. To show our definition of the colimit is correct, we also exhibit the composite

$$\coprod_d (i_d)_*\sigma_d \xrightarrow{\coprod_d (i_d)_*\phi_d} \coprod_d (i_d)_* f_d^*\tau \xrightarrow{\cong} \coprod_d (i_d)_*(i_d)^*\langle f \rangle^*\tau \xrightarrow{\langle \epsilon \rangle} \langle f \rangle^*\tau$$

which we will call α. We claim α is the unique such map in the fibre over the colimit that produces a morphism in the bifibred category that commutes with the cocones. To check the commutativity is a straightforward diagram chase. To see that the morphism is unique, let us suppose that there were some other morphism $\beta \colon \coprod_d (i_d)_*\sigma_d \to \langle f \rangle^*\tau$ that also commuted with the cocones. When we write out explicitly what this commutativity means, we see that it implies that

$$(i_c)^*\big(\beta \circ n_{\sigma_c}\big) \circ \eta_{\sigma_c} = (i_c)^*(\alpha \circ n_{\sigma_c}) \circ \eta_{\sigma_c}$$

for all $c \in \mathcal{D}$. Via the adjunction, this is equivalent to $\beta \circ n_{\sigma_c} = \alpha \circ n_{\sigma_c}$, and since the n_c are the inclusions for a colimit in the fibre category, this implies that $\alpha = \beta$. $\square$

We know (by assumption) that $\mathcal{C}$ is complete and cocomplete, so in order to apply the lemma above we just have to check that each fibre is complete and cocomplete. In fact, we can use the lemma repeatedly to do an induction argument on the height of our cell complexes. Extending this argument to the total category **CellCx** is easy for colimits, but requires an extra condition in order for all limits to exist.

Proposition 3.4.3. *The category* $\mathbf{CellCx}_\alpha$ *is complete and cocomplete for any ordinal number* α. *The global category* **CellCx** *will also have all small colimits; it possesses all small limits if and only if it has a height ceiling.*

Proof. We will use a slight generalisation of the fact that ∂ is a bifibration. If ∂_α is the functor that takes any cell complex and returns the α-abbreviation of that complex, ∂_α is also a bifibration; this can be seen using the same construction that we made above for ∂. Note in particular that ∂ itself is just the special case ∂_0.

If we have a height α cell complex $(A, \vec{\sigma})$, consider the fibre of $\mathbf{CellCx}_{\alpha+1}$ over $(A, \vec{\sigma})$. This category is complete and cocomplete since it is equivalent to the slice category $\hat{\mathcal{S}}_{\mathcal{J}}/T_\alpha(A, \vec{\sigma})$ (the fibres of any comma category are slice categories). This allows us to use Lemma 3.4.2 to do an induction step; if $\mathbf{CellCx}_\alpha$ is complete and cocomplete then so is $\mathbf{CellCx}_{\alpha+1}$ because of the bifibration ∂_α. For the case of a limit ordinal we see that the limits and colimits can be defined pointwise, so we have proved the first part of the proposition.

To extend this to the total category, in **CellCx** we have all colimits because they can also be defined pointwise. The only possible issue is the height of the result; for colimits this is not a problem as clearly the colimit will have height no greater than the maximum height of the diagram complexes. However, for limits when we define the pointwise limit of a diagram of cell complexes there is nothing to limit the height of the result. Indeed, it is easy to find examples (consider, for instance, the terminal object of **CellCx**) where the result needs to have height greater than or equal to any other complex. This gives the only if part of the proposition. $\square$

Composition of cell complexes. We will now finish this section by showing how to compose cell complexes. Given two cell complexes $(A, \vec{\sigma})$ and $(B, \vec{\tau})$ where $B = \overline{(A, \vec{\sigma})}$, we will see how to create a composite complex $(A, \vec{\sigma} \bullet \vec{\tau})$. This essentially works by going through the cells of $(B, \vec{\tau})$ and looking at the binding map of each, deciding which is the lowest layer of the new complex we can put it in, and adding it to that layer of $(A, \vec{\sigma})$. To do this formally we will use induction on the height of the second complex.

To begin with, assume that $(B, \vec{\tau})$ has height one. Then it is given by a single presheaf $\tau \in \hat{\mathcal{J}}$ of cells equipped with binding maps. For each cell $b \in \tau$, let γ_b be the height of the smallest abbreviation of $(A, \vec{\sigma})$ that the binding map $g_b \colon \partial s_b \to \overline{(A, \vec{\sigma})}$ can be factored through. We add each b to the layer σ_{γ_b} of $(A, \vec{\sigma})$. For each layer this is achieved simply using coproducts of presheaves; the restriction maps between the new cells are induced by restriction maps in τ, and in some places this will involve downward cell functions. To see that the underlying map of the new composite complex is isomorphic to the composite of the underlying maps of the two original complexes is a straightforward equivalence of two colimits.

Now consider the case that $(B, \vec{\tau})$ has some ordinal height $\alpha + 1$, and inductively assume that we have defined composites for complexes up to height

α. The final layer, τ_α, is a presheaf on $\mathcal{S}_\mathcal{J}$ with binding maps into $\overline{(B, \vec{\tau}|_\alpha)}$, which is by an induction hypothesis isomorphic to $\overline{(A, \vec{\sigma} \bullet \vec{\tau}|_\alpha)}$. Therefore we can perform the same construction on the presheaf τ_α; for each $b \in \tau_\alpha$ let γ_b be the height of the shortest abbreviation of $(A, \vec{\sigma} \bullet \vec{\tau}|_\alpha)$ that g_b factors through. Again we can add each b to the layer given by γ_b using coproducts of presheaves, and we get restriction maps and downward cell functions induced from the ones in $(B, \vec{\tau})$.

Proposition 3.4.4. *The definition of composite cell complexes described in the paragraphs above gives $U\colon$ **CellCx** $\to \mathcal{C}^\mathbf{2}$ the structure of a category of composable maps over $\mathcal{C}$.*

Proof. Identities are given by trivial cell complexes in which all layers are empty. The associativity and unit axioms are straightforward to check. $\square$

3.5 Cell Complexes are Left Maps

We are now ready to prove the main result of this chapter and show that **CellCx** is equivalent to the category of left maps for the AWFS generated from $\mathcal{J}$ using the algebraic small object argument. This will be a simple application of Theorem 2.5.3. The first condition for that theorem to hold is the existence of a right adjoint for U, which we have already constructed (provided that **CellCx** has a height ceiling). We will now check each of the other five conditions in turn.

Preservation of equalisers. Consider an equaliser of complexes of the form

$$\begin{tikzcd} (E, \vec{\gamma}) \arrow[r, "(e,\vec{\epsilon})"] & (X, \vec{\sigma}) \arrow[r, shift left, "(f,\vec{\theta})"] \arrow[r, shift right, "(g,\vec{\lambda})"'] & (Y, \vec{\tau}), \end{tikzcd}$$

and suppose we have some $x\colon s \to \overline{(X, \vec{\sigma})}$, such that $\overline{(f, \vec{\theta})} \circ x = \overline{(g, \vec{\lambda})} \circ x$. Then using the consistency of subcomplexes, we have

$$\vec{\theta}(\mu(x)) = \vec{\lambda}(\mu(x)),$$

which means that every cell in $\mu(x)$ equalises $\vec{\theta}$ and $\vec{\lambda}$. This means that $\mu(x)$ is contained in the subcomplex $e_*(E, \vec{\gamma})$ of $(X, \vec{\sigma})$. Thus x can be factored through $\overline{e_*(E, \vec{\gamma})}$, and this factorisation is unique because the subcomplex inclusion is a typical inclusion.

Now notice that the diagram

$$\begin{tikzcd} \overline{(E, \vec{\gamma})} \arrow[r, "\overline{(e,1)}"] & \overline{e_*(E, \vec{\gamma})} \arrow[r, shift left, "\overline{(f,1)}"] \arrow[r, shift right, "\overline{(g,1)}"'] & \overline{f_* e_*(E, \vec{\gamma})} \cong \overline{g_* e_*(E, \vec{\gamma})}, \end{tikzcd}$$

is itself an equaliser—because it can be expressed as the pushout of an equaliser along the typical inclusion $E \to \overline{(E, \vec{\gamma})}$ we can use the fourth bullet point in Definition 3.3.1. The factorisation of x through $\overline{e_*(E, \vec{\gamma})}$ must equalise $\overline{(f, 1)}$ and $\overline{(g, 1)}$, so we can see that x factors uniquely through $\overline{(E, \vec{\gamma})}$.

We have demonstrated the equaliser property holds for maps into $\overline{(X, \vec{\sigma})}$ out of shape objects, which implies that V maps the original diagram to an equaliser in the presheaf category $\hat{\mathcal{S}}$. Since $\overline{(e, \vec{\epsilon})}$ is a typical inclusion (it is the composite of a regular monomorphism and a subcomplex inclusion), it is terminal amongst monomorphisms that V maps to $V\overline{(e, \vec{\epsilon})}$. But V preserves limits, so the actual equaliser gets mapped to $V\overline{(e, \vec{\epsilon})}$ too; this implies that $\overline{(E, \vec{\gamma})}$ is isomorphic to the actual equaliser, and hence an equaliser itself.

Equalisers in the arrow category $\mathcal{C}^{\mathbf{2}}$ are defined pointwise and we already know that E is the equaliser of f and g. So we have proved that U preserves all equalisers in **CellCx**.

Conservativity. Given some ordinal α and any height α cell complex $(A, \vec{\sigma})$ we will write $\mathbf{CellCx}_{\alpha+1}(A, \vec{\sigma})$ for the fibre of $\partial_\alpha \colon \mathbf{CellCx}_{\alpha+1} \to \mathbf{CellCx}_\alpha$ over $(A, \vec{\sigma})$. This is the category of extra layers we can add to $(A, \vec{\sigma})$. We will write

$$U_{(A,\vec{\sigma})} \colon \mathbf{CellCx}_{\alpha+1}(A, \vec{\sigma}) \to \overline{(A, \vec{\sigma})}/\mathcal{C}$$

for the functor that gives the underlying map of just the final layer. Now $\mathbf{CellCx}_{\alpha+1}(A, \vec{\sigma})$ is equivalent to $\hat{\mathcal{S}}_{\mathcal{J}}/T_{\alpha+1}(A, \vec{\sigma})$, which is a presheaf category itself; furthermore, it is apparent that $U_{(A,\vec{\sigma})}$ is actually a realisation functor.

This means we can apply Proposition 1.2.5; what we need to show is simply that $U_{(A,\vec{\sigma})}$ satisfies the repeated element condition. To this end, let a and b be two distinct cells with the same shape, binding map and downward cell function in some object X of $\mathbf{CellCx}_{\alpha+1}(A, \vec{\sigma})$. Considering a and b as morphisms from a representable into X, we can take their equaliser. Because they are distinct, this is a strict subobject of the representable, so Proposition 3.3.6 means it is mapped by $U_{(A,\vec{\sigma})}$ to a strict typical inclusion. But as we just showed, $U_{(A,\vec{\sigma})}$ must preserve equalisers, so the fact this is not an isomorphism implies that a and b have distinct images under $U_{(A,\vec{\sigma})}$, and this is precisely the repeated element condition. This proves conservativity for the single-layer functor $U_{(A,\vec{\sigma})}$.

It is now straightforward to use the pullback lemma to extend this result to $U \colon \mathbf{CellCx} \to \mathcal{C}^{\mathbf{2}}$. Suppose we have a morphism $(f, \vec{\theta})$ of cell complexes such that $U(f, \vec{\theta})$ is an isomorphism. This means that f and $\overline{(f, \vec{\theta})}$ are certainly isomorphisms; but using the lemma it also implies that every 'intermediate' map $\overline{(f, \vec{\theta}|_\alpha)}$ is also an isomorphism, since it is a pullback of one. Now we can see that each θ_α is realised as an isomorphism, so the conservativity for layers we just proved tells us that each θ_α is itself an isomorphism of presheaves. This is exactly what it means for $(f, \vec{\theta})$ to be an isomorphism of cell complexes.

Remaining conditions. There are three remaining conditions we must check in order to apply Theorem 2.5.3, but each of these will be very simple. The condition written as **L4** in Section 2.5 is trivial to see just by looking at the definition of cell complex morphism—a morphism out of a trivial complex is just a boundary part and nothing else. The condition **L5**, which says that ∂ is an isofibration, follows immediately from the bifibration structure.

The condition **L6** asks that **CellCx** satisfy the double category property. We have now proved U is conservative, so it is also faithful (using the fact that it has a right adjoint and that $\mathcal{C}$ is complete). Since the set of cells in a composite complex is the disjoint union of the two sets of cells, it is clear how we should define the horizontal composite of two morphisms. One has to check that the binding maps work properly, but this is straightforward.

Theorem 3.5.1. *Let* $I: \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ *be a small category of generating maps satisfying the typical conditions with respect to some typical nerve on* $\mathcal{C}$ *and such that a height ceiling exists for the category of* $\mathcal{J}$*-cell complexes. Let* $(\mathbb{L}, \mathbb{R})$ *be the AWFS generated from* $\mathcal{J}$ *using the algebraic small object argument. Then there is a morphism of CCMs over* $\mathcal{C}$

$$\mathbf{CellCx} \longrightarrow \mathbf{L}\text{-}\mathbf{Map}$$

which is also an equivalence of categories.

Proof. Based on the conditions that we have checked in this section, we can apply Theorem 2.5.3 to see that the claim holds with respect to some AWFS on $\mathcal{C}$. We have only to check that this AWFS is the one generated from $\mathcal{J}$ using the algebraic small object argument. This is basically proof by examination—we look at the comonad UK on $\mathcal{C}^{\mathbf{2}}$ and it is clearly equivalent to the comonad $\mathbb{L}$.

Another way to check this is to demonstrate that the AWFS corresponding to **CellCx** has the same universal property as $(\mathbb{L}, \mathbb{R})$. This universal property of being free with respect to $\mathcal{J}$ (see Definition 2.4.1) says that any functor $\mathcal{J} \to \mathbf{L}'\text{-}\mathbf{Map}$ over $\mathcal{C}^{\mathbf{2}}$ into another category of left maps extends uniquely to a morphism of AWFS that has as its left map part a functor $\mathbf{CellCx} \to \mathbf{L}'\text{-}\mathbf{Map}$. This universal property holds for **CellCx** because every cell complex is canonically expressed as a composite of pushouts of colimits of maps in $\mathcal{J}$. $\square$

Chapter 4

Computads

We now return to the subject of higher category theory and globular operads. In this chapter, we will examine the definition of what are called *computads*, a generalisation of globular sets designed to capture the best kind of data from which one can generate a n-category. It will be immediately clear that this definition is rather similar in many respects to the definition of cell complexes in the previous chapter; we will make this intuition formal by showing how computads are equivalent to cell complexes given by a specific choice of generating maps in a category of n-categories.

The notion of a computad was first introduced by Ross Street, who defined computads for strict 2-categories (see [Str76]). They were generalised to higher dimensions in the strict case, and then Michael Batanin extended the concept so that it would work for any globular operad whatsoever. Batanin's definition was first given in [Bat98a] and can be found in [Bat02]. However, the definition we give here is closer to the approach of Richard Garner's simpler reformulation, which can be found in [Gar08].

4.1 Definition

Recall that given a globular operad P, for any n-globular set A we can generate a P-algebra freely from A. The basic observation that leads us to the concept of a computad is that the 'free' categories obtained from the globular sets are not quite as general as we would like. Notice that the source and target of a generating cell in such a free category are always generating cells themselves—this comes from the globular set structure we started with. What if we want some generating cells whose source and target are composites of lower dimensional generating cells? There is no reason why we cannot have such generators; a computad is a piece of data in which we give generators of this more general type.

Before we proceed with the definition let us establish some useful pieces of notation. From now on, P is assumed to be some globular operad, which we will generally take to be contractible—note however that the whole of this chapter works for any globular operad at all. With P assumed, when we talk about n-categories (unless we qualify by saying strict or weak n-categories) we will mean algebras for the n-truncation of P, which is the composite $\mathrm{tr} \circ P \circ I$ as shown below

$$\begin{tikzcd} \mathbf{nGSet} \arrow[r, shift left, "I"] & \mathbf{GSet} \arrow[l, shift left, "\mathrm{tr}"] \arrow[loop right, "P"] \end{tikzcd}$$

where tr is the truncation functor that forgets cells above dimension n and I is the obvious inclusion functor that creates a globular set with no cells above dimension n. Of course, for a given P we usually only care about a specific dimension.

In Chapter 1 we used $i_k\colon \partial_k \to G_k$ to represent certain morphisms of globular sets. From now on we will use this notation to represent morphisms of n-categories instead—precisely the n-categories we get by freely generating from the globular sets we had before. So previously we wrote G_k for the representable presheaf on the kth object of **Glob**; now it will denote the n-category freely generated from that globular set. Similarly ∂_k denoted the globular set obtained by taking the representable presheaf G_k and removing its single k-cell; from now on ∂_k denotes the n-category freely generated from that globular set. One cannot really give a formal description of the precise cells in these n-categories because they depend entirely on the choice of globular operad P—some choices of P will add many identity cells onto the globular sets we have started with, while others will leave them almost unchanged.

In particular, G_0 is the free n-category on a single object and ∂_0 will be the empty n-category. As an example, the morphism $i_2\colon \partial_2 \to G_2$ looks like

$$\begin{tikzcd} \bullet \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & \bullet & {} \arrow[r, mapsto, "i_2"] & {} & \bullet \arrow[r, bend left, "i_2(s)", ""{name=U, below}] \arrow[r, bend right, "i_2(t)"', ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

together with all the extra composites and identities that will be determined by the choice of P.

We are now ready to give the definition of a computad. It is recursive on dimension, so we begin with a 0-computad, the data necessary to generate a 0-category. This first step is obviously entirely trivial; but it is the absolutely vital foundation for the rest of the recursion.

Definition 4.1.1. A *0-computad* is a set. We write **0Comp** for the category of 0-computads, which is isomorphic to the category of sets. There is an adjunction

$$\begin{tikzcd} \mathbf{0Comp} \arrow[r, shift left, "F_0"] & \mathbf{nCat} \arrow[l, shift left, "U_0"] \end{tikzcd}$$

where F_0A is the n-category freely generated by the set of objects A, and $U_0\mathcal{B}$ is the underlying set of objects of the n-category $\mathcal{B}$.

We now need to describe the successor step of the recursion. For this we must imagine that we have already defined the category $(\mathbf{k-1})\mathbf{Comp}$ of $(k-1)$-computads, and that we have given an adjunction $F_{k-1} \dashv U_{k-1}$ like the one above. Based on this, a k-computad must be the most general possible data to build an n-category up to dimension k. The first part of such data should clearly be a choice of some $(k-1)$-computad; the whole point of computads is that one can build an n-category one dimension at a time, so to build up to dimension k we start by building up to dimension $(k-1)$.

We now merely have to give generating data at dimension k. This should take the form of some generating k-cells, and since we do not want to limit ourselves in any way, these generating cells should be attached onto the n-category we have built so far in any way possible. This means we need to use the free functor F_{k-1} to give us our n-category so far; then a cell is 'attached on' using a morphism from its boundary, which is why the maps $i_k\colon \partial_k \to G_k$ will prove to be important.

Definition 4.1.2. A *k-computad* is given by a triple (A, S_k, γ_k) where A is a $(k-1)$-computad, S_k is a set we call the set of *k-cells*, and γ_k is a function that gives for every element $s \in S_k$ a morphism

$$\partial_k \xrightarrow{\gamma_k(s)} F_{k-1}A$$

which we call the *binding map* for s. We write **kComp** for the category of k-computads, where a morphism is given by a morphism of $(k-1)$-computads and a function between the sets of k-cells such that the binding maps commute.

There is a very apparent similarity to the definition of cell complexes in the last chapter. We should note in particular that the category **kComp** defined above can easily be expressed as the comma category $\mathbf{Set}/T_k$ where

$$T_kA = \mathbf{nCat}(\partial_k, F_{k-1}A).$$

To complete the recursion we must also give the free functor to n-categories. While we are at it we may as well define the whole adjunction

$$\mathbf{kComp} \underset{U_k}{\overset{F_k}{\rightleftarrows}} \mathbf{nCat}.$$

Firstly, $U_k\mathcal{B}$ is given by $(U_{k-1}\mathcal{B}, \sigma_k\mathcal{B}, \lambda_k\mathcal{B})$ where $\sigma_k\mathcal{B}$ is simply the set of all k-cells in $\mathcal{B}$, and $\lambda_k\mathcal{B}$ takes a k-cell s to the composite map

$$\partial_k \xrightarrow{g_s} \mathcal{B} \xrightarrow{\eta_\mathcal{B}} F_{k-1}U_{k-1}\mathcal{B}$$

where g_s is the map specifying the actual source and target of s in $\mathcal{B}$. Secondly, to define $F_k(A, S_k, \gamma_k)$ we have to somehow freely adjoin the new cells onto $F_{k-1}A$. Clearly this should involve a pushout; to be specific, the following one:

$$\begin{tikzcd} S_k \cdot \partial_k \arrow[r, "\overline{\gamma_k}"] \arrow[d, "S_k \cdot i_k"'] & F_{k-1}A \arrow[d] \\ S_k \cdot G_k \arrow[r] & F_k(A, S_k, \gamma_k). \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The dots represent coproducts indexed by the set S_k and the map $\overline{\gamma_k}$ is obtained from γ_k using the bijection

$$\mathbf{Set}[S_k, \mathbf{nCat}(\partial_k, F_{k-1}A)] \cong \mathbf{nCat}(S_k \cdot \partial_k, F_{k-1}A).$$

We will not reproduce here a full proof that $F_k \dashv U_k$ is an adjunction. It follows fairly quickly from the universal property of the pushout and the induction hypothesis of the adjunction for $(k-1)$-computads.

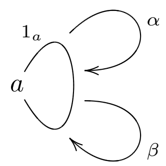
Some low dimensional examples. Let us unpack the first few steps of this definition for some specific choices of P. First of all, we shall note that for any reasonable globular operad the category **1Comp** is nothing new. Considering that ∂_1 is just a pair of objects, the function γ_1 will assign a single source and a single target object to each 1-cell, yielding exactly the structure of a directed graph or 1-globular set. This should not be a surprise, since it is pretty clear that we will not find any more general data than this from which to generate a category of the usual 1-dimensional sort.

When we move to dimension two we have to decide between strict and weak. For both of these, the category **2Comp** can be described as a presheaf category; we will primarily discuss the strict version as it is a little simpler. A 2-computad consists of a 1-computad, which we know to be a directed graph, together with some 2-cells. The function γ_2 specifies source and target of each 2-cell, and these are any two parallel morphisms in the free category generated by the directed graph. So we see that the set of 2-cells can be partitioned into a collection of sets indexed by pairs of natural numbers (n, m) where n is the number of generating morphisms composed in the source, and m the number composed in the target.

This gives us a presheaf category description of **2Comp**. Each pair of natural numbers is considered a different 'shape' for this presheaf category—these shapes are sometimes called *computopes*. The author is not aware who first introduced the notion of computopes, though it may have been Makkai (see [Mak05]). As well as the pairs of natural numbers that constitute the 2-dimensional computopes, there is a single 0-dimensional computope $\underline{0}$ and a single 1-dimensional computope $\underline{1}$. The category of computopes has two morphisms $\underline{1} \to \underline{0}$ and for natural numbers n and m, it has $n + m$ morphisms $(n, m) \to \underline{1}$.

We can attempt to extend this notion of computopes to dimension three, hoping to describe **3Comp** as a presheaf category, and for weak 3-categories this will be successful. However, we run into difficulties in the strict case. To see why, we will consider a particular 3-computad that ought to be a 3 dimensional computope, but observe that it exhibits behaviour that cannot be modelled in a presheaf category.

Let us describe this 3-computad, which we will call A. We specify that A_0, the set of objects or 0-cells, is the singleton set $\{a\}$. The set of 1-cells, A_1, is defined to be the empty set. For A_2 we take a two element set, $\{\alpha, \beta\}$, and note that both α and β have to have the identity morphism 1_a as their source and target. Finally, we prescribe a single 3-cell, χ, whose source is the horizontal composite $\alpha * \beta$, and whose target is the identity on 1_a, which we will write 1_{1_a}. Here is a diagram showing the computad A up to dimension 2.



Now the key observation is that since this is a strict 3-computad, the source and target of χ are defined by morphisms into the strict 3-category generated by A_0, A_1 and A_2. In this strict 3-category, an Eckmann-Hilton argument shows that the horizontal composites $\alpha * \beta$ and $\beta * \alpha$ are equal. This means that when we try to consider A as a computope, we find ourselves unable to define the 'boundary' morphisms from A to the 2-computope which describes both α and β. Of course, this does not constitute a proof that strict 3-computads are not a presheaf category; it merely highlights where the problem arises. For a full proof, one can consult [Che12] or [MZ08].

Relative computads. Before we begin the comparison of computads and cell complexes, we have to make another quick observation about the definition above. Recall that a cell complex is built by adjoining cells to some starting object in the category $\mathcal{C}$; this 'boundary' is free to be any object whatsoever. In model category theory, the class of maps with cell complex structure is often called the class of *relative cell complexes* due to this arbitrary choice of boundary. On the other hand the definition of computads, as it is usually given, always begins with the empty set.

It is very simple remove this limitation, and obtain a definition of a *relative computad*. However, we will not make a formal definition now, since the abstract theory presented in the next section will do it for us.

4.2 Comparison of Normal Forms

We will now attack the question: in what sense is the definition of computads a special case of the definition of cell complexes? There are very obvious similarities in the definitions; they are both recursive and each step in the recursion is defined as a comma category over the previous step. They both involve a nerve-realisation adjunction at each step. They both come with left adjoints that are constructed using a sequence of pushouts.

However, the definition of a computad is not just a special case of a cell complex. There is something very different about the way the layers of a computad work—in a cell complex, each layer contains cells of all the same shapes; in a computad, the shapes available change fundamentally at each layer. In a cell complex one cannot have a cell in layer five whose binding map factors through layer two; in a computad one can easily construct such an example by creating a 5-cell whose source and target are identity maps generated from 2-cells.

Despite these differences, with a bit of thought it seems intuitively obvious that k-computads are 'equivalent', in some sense, to cell complexes on **nCat** generated by the discrete category of maps containing $i_m\colon \partial_m \to G_m$ for $0 \leq m \leq k$. One can quite easily check that the underlying class of maps in both cases is the same. The central observation is that one can find a computad with exactly the same cells as a given cell complex—the only thing that changes in this construction is that they end up expressed in a different order. In this section we will show how this works in a very general context using a theory of normal forms for cell complexes.

We begin by showing how one can take any cell complex and define a partial order on the set of cells. This order will capture the intuitive notion that some pairs of cells have to be added to the complex in a specific order while other pairs are completely independent of one another. The definition will depend heavily on the idea of subcomplexes; recall that a subcomplex of $(A, \vec{\sigma})$ is another complex $(A, \vec{\sigma}')$ with a monomorphism $(1_A, \vec{\iota})\colon (A, \vec{\sigma}') \to (A, \vec{\sigma})$. Every subcomplex is determined by the subset of cells it contains; however, there are clearly some subsets that do not constitute subcomplexes.

Definition 4.2.1. For any two cells a and b we say that a *depends on* b, which we will write $b < a$, if b is an element of every subcomplex which g_a, the binding map of a, factors through.

Proposition 4.2.2. *The relation $a \leq b$, which holds when either b depends on a or they are equal, gives a partial ordering on the set of all cells in a cell complex. Furthermore we can use the Pullback Lemma from Section 3.5 to show that this partial ordering defines a functor $P\colon \mathbf{CellCx} \to \mathbf{Poset}$.*

Proof. First, note that this means a (strict) partial ordering rather than a pre-order structure. Consider that g_b always factors through the abbreviation of the cell complex just below b; this is the subcomplex $(A, \vec{\sigma}|_{l(b)})$ where we use $l(b)$ to mean the layer of b. Using this observation we see that $a < b$ implies that $l(a) < l(b)$ (the layer of a is strictly lower than that of b). The antisymmetry of the relation $a \leq b$ follows immediately.

To prove transitivity we will use the fact that $a < b$ implies that a is in every subcomplex that b is in. To see this, assume $a < b$ and suppose b is in the subcomplex $(A, \vec{\sigma}')$. Then $(A, \vec{\sigma}'|_{l(b)})$ is also a subcomplex, and g_b factors through it, implying that it contains a. But then the original subcomplex $(A, \vec{\sigma}')$ must also contain a.

We have now shown that $\leq$ is a partial order; we must check that this gives a functor $P\colon \mathbf{CellCx} \to \mathbf{Poset}$. We have the action of P on objects, and the action on morphisms is clearly given by the function between the sets of cells. Hence all we need to do is check that for any cell complex morphism the cell function θ preserves the dependence structure.

Let $(f, \vec{\theta})\colon (A, \vec{\sigma}) \to (B, \vec{\tau})$ be a cell complex morphism; given $a < b$ we must show that $\theta(a) < \theta(b)$. Suppose we have a subcomplex $(B, \vec{\tau}') \to (B, \vec{\tau})$ which $g_{\theta(b)}$ factors through. We take the preimage $\theta^{-1}(B, \vec{\tau}')$ of this subcomplex. By the Pullback Lemma, the square shown in the diagram

$$\begin{tikzcd}
& \partial s_b \arrow[dl, equal] \arrow[d, dotted] \arrow[dr, "g_b"] & \\
\partial s_{\theta(b)} \arrow[dr, "g_{\theta(b)}"'] & \overline{\theta^{-1}(B, \vec{\tau}')} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \overline{(A, \vec{\sigma})} \arrow[d, "\overline{(f, \vec{\theta})}"] \\
& \overline{(B, \vec{\tau}')} \arrow[r] & \overline{(B, \vec{\tau})}
\end{tikzcd}$$

is a pullback, and therefore the dotted map induced by the universal property shows that g_b factors through the preimage. Therefore, since b depends on a, we know that a is in the preimage; this implies that $\theta(a)$ is in the subcomplex $(B, \vec{\tau}')$, and we have shown that $\theta(b)$ depends on $\theta(a)$. So θ preserves the order relation and therefore P is a functor as promised. $\square$

We should not allow ourselves to become confused by the fact that there is another related partial order on the set of cells. We call it the *subcomplex partial order* and we will write it as $a \leq_s b$; it holds whenever every subcomplex containing b also contains a. We showed in the proof of the last proposition that $a \leq b$ implies $a \leq_s b$, but the converse is definitely not true. The set of cells of a cell complex always has the structure of a presheaf on $\mathcal{J}$, the category of generating maps. If $a = \phi(b)$ for some morphism ϕ of $\mathcal{J}$, then $a \leq_s b$. On the other hand, $a = \phi(b)$ does not tell us anything about whether b depends on a or not.

We now make the main definition of this section, which uses the order relation of dependence to formalise the idea of a normal form for cell complexes.

Definition 4.2.3. A *stratification* or *normal form* on a category of cell complexes **CellCx** is a collection of conservative order preserving maps

$$\lambda_{(A,\vec{\sigma})} \colon P(A, \vec{\sigma}) \to \mathbf{Ord}$$

into the ordered class of ordinal numbers that satisfies the naturality condition $\lambda_{(A,\vec{\sigma})} = \lambda_{(B,\vec{\tau})} \circ P(f, \vec{\theta})$ for any cell complex morphism $(f, \vec{\theta}) \colon (A, \vec{\sigma}) \to (B, \vec{\tau})$. In this context conservative means what it does if we consider the posets as categories; that is, if $a \leq b$ and $\lambda(a) = \lambda(b)$, then $a = b$.

We begin by remarking that there is a natural pointwise partial order on the class of stratifications. This partial order structure has an initial object, which is clearly the stratification given by the function l that specifies which layer a cell is in. We call this the *standard stratification*, since it is one we have used so far in all our treatment of cell complexes. The point of this section is basically to demonstrate that one could have started with any stratification at all and defined cell complexes using it instead; any such choice will lead to an equivalent category. When cell complexes are viewed as left maps for an AWFS the choice of stratification is entirely a matter of convention and notation.

Suppose we have some stratification λ on **CellCx**. We will give an alternative definition of what we will call λ-cell complexes. Fortunately we do not have to start all over again; we can use the definition of **CellCx** that we already have to give us a leg up.

Definition 4.2.4. The category of *height zero λ-cell complexes*, which we write $\mathbf{CellCx}_0^\lambda$, is isomorphic to $\mathcal{C}$. We define the functor

$$V_0^\lambda \colon \mathbf{CellCx}_0^\lambda \to \mathbf{CellCx}$$

by taking the height zero cell complex on the given object of $\mathcal{C}$.

Definition 4.2.5. Given any ordinary cell complex $(A, \vec{\tau})$, define the *presheaf of extra cells* on $(A, \vec{\tau})$, which we will write as $E(A, \vec{\tau})$, as a presheaf on $\mathcal{S}_{\mathcal{J}}$ defined as follows:

- an element of $E(A, \vec{\tau})(U \to [j])$ is an isomorphism class of subcomplex inclusions
$$(A, \vec{\tau}) \to (A, \vec{\tau}+)$$
such that if we consider the sets of cells as $\mathcal{J}$-presheaves (as described in Proposition 3.2.6), $\vec{\tau}+$ is isomorphic to a pushout
$$\begin{tikzcd} U \arrow[r] \arrow[d] & {[j]} \arrow[d] \\ \vec{\tau} \arrow[r] & \vec{\tau}+ \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$
for some morphism of presheaves $U \to \vec{\tau}$,

- given a morphism of sieves

  $$\phi\colon (U \to [j]) \to (V \to [k])$$

  and an element of $E(A, \vec{\tau})(V \to [k])$, one can see there is a unique element of $E(A, \vec{\tau})(U \to [j])$ such that the pushout squares of the two elements commute with the map ϕ. This defines the restriction along ϕ.

In the following definitions we are establishing the successor step in the recursive definition of λ-cell complexes. So we assume throughout that for some ordinal α we already have the category $\mathbf{CellCx}^\lambda_\alpha$ of height α λ-cell complexes, and also the functor

$$V^\lambda_\alpha\colon \mathbf{CellCx}^\lambda_\alpha \to \mathbf{CellCx}$$

which allows us to convert any λ-cell complex (so far) into a normal cell complex.

Definition 4.2.6. For any height α λ-cell complex $(A, \vec{\sigma})$, define the *terminal λ-layer* $T^\lambda_\alpha(A, \vec{\sigma})$ as the subpresheaf of the presheaf of extra cells $EV^\lambda_\alpha(A, \vec{\sigma})$ consisting of elements such that all the 'new' cells which are not in the subobject $V^\lambda_\alpha(A, \vec{\sigma})$ get mapped to α by the stratification map λ. Formally speaking $T^\lambda_\alpha(A, \vec{\sigma})(U \to [j])$ is the subset of $EV^\lambda_\alpha(A, \vec{\sigma})(U \to [j])$ that contains elements with the property that, given any representative subcomplex inclusion $(A, \vec{\sigma}) \to (A, \vec{\sigma}+)$, every cell x in $\vec{\sigma}+$ satisfies $(\lambda(x) = \alpha + 1) \vee (x \in \vec{\sigma})$.

The naturality of λ tells us that this is a valid subpresheaf, and that it defines a functor

$$T^\lambda_\alpha\colon \mathbf{CellCx}^\lambda_\alpha \to \hat{\mathcal{S}}_\mathcal{J}.$$

Definition 4.2.7. The category of *height* $(\alpha + 1)$ *λ-cell complexes*, written $\mathbf{CellCx}^\lambda_{\alpha+1}$, is defined as the comma category $(\hat{\mathcal{S}}_\mathcal{J} \downarrow T^\lambda_\alpha)$.

Definition 4.2.8. We will define the functor

$$V^\lambda_{\alpha+1}\colon \mathbf{CellCx}^\lambda_{\alpha+1} \to \mathbf{CellCx}.$$

Let $(A, \vec{\sigma})$ be any object of $\mathbf{CellCx}^\lambda_{\alpha+1}$. Recall that for any sieve $U \to [j]$ an element of $T^\lambda_\alpha(A, \vec{\sigma}|_\alpha)(U \to [j])$ is an isomorphism class of morphisms of ordinary cell complexes out of $V^\lambda_\alpha(A, \vec{\sigma}|_\alpha)$. Thus given the top layer of $(A, \vec{\sigma})$—a presheaf σ_α over $T^\lambda_\alpha(A, \vec{\sigma}|_\alpha)$—we can construct a diagram using a representative of the isomorphism class for each element in each $\sigma_\alpha(U \to [j])$. The diagram contains $V^\lambda_\alpha(A, \vec{\sigma}|_\alpha)$ together with a morphism out of it for each cell in σ_α; constructing the colimit of this diagram in $\mathbf{CellCx}$ is essentially taking a large many-legged pushout. We define $V^\lambda_{\alpha+1}(A, \vec{\sigma})$ to be this colimit.

We can handle limit ordinals in exactly the same way that we did for the original definition of $\mathbf{CellCx}$, and we define $\mathbf{CellCx}^\lambda$ similarly. The following theorem gives us the result we want—that all stratifications lead to equivalent categories of cell complexes.

Theorem 4.2.9. *For any stratification λ on $\mathbf{CellCx}$, the functor*

$$V^\lambda\colon \mathbf{CellCx}^\lambda \to \mathbf{CellCx}$$

is one half of an equivalence of categories.

Proof. The other half of the equivalence will be the functor $W^\lambda\colon \mathbf{CellCx} \to \mathbf{CellCx}^\lambda$, which is defined as the colimit of a recursively constructed ordinal sequence of functors W^λ_α. We will construct these functors at the same time as proving inductively that for each ordinal α, we have

$$V^\lambda(W^\lambda_\alpha(A,\vec{\sigma})) \cong (A, \vec{\sigma}|_{\lambda<\alpha})$$

where the notation on the right hand side means the subcomplex containing all cells x with $\lambda(x) < \alpha$, and similarly

$$W^\lambda_\alpha(V^\lambda(B,\vec{\tau})) \cong (B, \vec{\tau}|_\alpha).$$

Now assume these have been defined for some ordinal α. To define $W^\lambda_{\alpha+1}(A,\vec{\sigma})$ we take the set of cells a in $(A,\vec{\sigma})$ such that $\lambda(a) = \alpha$ and partition it into a collection of sets with the structure of a presheaf on $\mathcal{S}_{\mathcal{J}}$ together with a morphism to $T^\lambda_\alpha(W^\lambda_\alpha(A,\vec{\sigma}))$. This is straightforward to do, because $T^\lambda_\alpha(W^\lambda_\alpha(A,\vec{\sigma}))$ is defined using complexes containing $V^\lambda(W^\lambda_\alpha(A,\vec{\sigma}))$ as a subcomplex; by the induction hypothesis $V^\lambda(W^\lambda_\alpha(A,\vec{\sigma}))$ is isomorphic to $(A,\vec{\sigma}|_{\lambda<\alpha})$, so each cell with $\lambda(a) = \alpha$ gives us exactly what we need—a complex with $(A,\vec{\sigma}|_{\lambda<\alpha})$ as a subcomplex containing all cells except some new ones given by the pushout of a sieve.

We must then check that

$$V^\lambda(W^\lambda_{\alpha+1}(A,\vec{\sigma})) \cong (A, \vec{\sigma}|_{\lambda<\alpha+1}).$$

The left hand side is defined as a colimit in $\mathbf{CellCx}$. The colimit diagram includes a morphism out of $(A,\vec{\sigma}|_{\lambda<\alpha})$ for each cell a such that $\lambda(a) = \alpha$, and it contains a morphism for each pair of such cells with $a = \phi(a')$ for some ϕ in $\mathcal{J}$. Therefore it is clear that this colimit reconstructs a cell complex isomorphic to $(A,\vec{\sigma}|_{\lambda<\alpha+1})$.

The other identity,

$$W^\lambda_{\alpha+1}(V^\lambda(B,\vec{\tau})) \cong (B, \vec{\tau}|_{\alpha+1}),$$

is straightforward when we consider the definition of $W^\lambda_{\alpha+1}$. The left hand side is given by a presheaf over $T^\lambda_\alpha(W^\lambda_\alpha(V^\lambda(B,\vec{\tau})))$ which is isomorphic to $T^\lambda_\alpha(B,\vec{\tau}|_\alpha)$ by the induction hypothesis; the elements of this presheaf are given by the cells in $V^\lambda(B,\vec{\tau})$ that λ maps to α, and these are in bijection with the cells in τ_α.

For a limit ordinal we define W^λ_α as the colimit of the functors W^λ_β for $\beta < \alpha$. Then the two identities follow immediately from the induction hypotheses for all such β. Finally, W^λ is defined by taking colimits that go sufficiently far along the ordinal numbers in each case. When the two identities are extended to W^λ, they become a straightforward assertion that W^λ and V^λ form an equivalence between $\mathbf{CellCx}$ and $\mathbf{CellCx}^\lambda$. □

Now that we are equipped with the notion of stratifications and the alternative definitions of $\mathbf{CellCx}$ that they give, the case of computads becomes a simple example of the general theory. We will consider $\mathcal{J}$ to be the discrete category of maps $i_m\colon \partial_m \to G_m$ for $0 \le m \le k$ in the category of n-categories. This satisfies the conditions necessary for the Pullback Lemma, so it gives a well defined category of cell complexes with a functor $P\colon \mathbf{CellCx} \to \mathbf{Poset}$.

Definition 4.2.10. For $\mathcal{J}$ as given above, we write λ_d for the *dimension stratification*, which is given by defining $\lambda_d(a)$ to be the dimension of the cell a—in other words, if the shape $s_a \in \mathcal{J}$ is i_m, then $\lambda_d(a) = m$.

To see that λ_d is a valid stratification, consider two cells a and b in $(A, \vec{\sigma})$ such that $a < b$. No matter which globular operad we are using, it is a fact that adjoining a cell of dimension m will not add any cells of dimension lower than m to an n-category. Therefore it makes sense to consider the $\lambda_d(b)$-*skeleton* of $(A, \vec{\sigma})$, which is the subcomplex of cells with dimension lower that b. Furthermore, the fact tells us that the map g_b factors through this subcomplex, implying that it contains a. Hence $\lambda_d(a) < \lambda_b(b)$. The naturality condition is an immediate consequence of the fact that cell complex morphisms preserve the shapes of cells.

Proposition 4.2.11. *The category* $\mathbf{CellCx}^{\lambda_d}$ *is what we will take as our formal definition of the category of relative k-computads. This makes sense: the fibre of* $\mathbf{CellCx}^{\lambda_d}$ *over the empty n-category is isomorphic to* $\mathbf{kComp}$.

Proof. At each stage of the recursion, $T_n^{\lambda_d}(A, \vec{\sigma})$ contains elements of the shape i_n only, and it contains exactly one such element for every morphism $\partial_n \to (A, \vec{\sigma})$. Based on this observation, the definition of a k-computad is clearly a special case of Definition 4.2.7. □

Corollary 4.2.12. *The category of relative k-computads is equivalent to the category* $\mathbf{CellCx}$ *generated from the maps* $i_m \colon \partial_m \to G_m$ *for* $0 \le m \le k$.

Chapter 5

Corporeality

In the last chapter, we considered the category of computads for some globular operad, showing that it is equivalent to the category of cell complexes for the natural category of generating maps in the category of n-categories. This observation forms the starting point for the investigation of this chapter, where we will consider some more general theory about cell complexes and then apply it to the particular case of computads. A question of particular interest in the study of computads is the one considered by Batanin in [Bat02]: when is the category of computads a presheaf category, and what does this tell us about the globular operad in question? (We should point out that this question was also studied earlier for particular cases by Makkai [MZ08]; also see [Che12]). Batanin answers the first part of the question by giving a simple characterisation that uses the concept of *slices* of an operad; we will see how this works in detail later in this chapter.

For an answer to the second part of the question he makes a conjecture; essentially, he suggests that a globular operad has the property that its computads form a presheaf category precisely when it has algebras that are weak enough to model homotopy n-types. This conjecture seems to work for all the examples we understand well enough to be able to tell. For instance, strict 3-categories are too strict and strict 3-computads are not presheaves. Meanwhile, Gray-categories are weak enough and Gray-computads do form a presheaf category. The conjecture also has a strong intuitive appeal; in cases where both properties fail, the counterexamples that show this are constructed in a very similar way.

In this chapter we will consider the obvious generalisation of the first part of the question to the world of cell complexes. We will ask when the category of cell complexes is locally a presheaf category—that is, the cell complexes over a particular base object form a presheaf category. This is property we refer to as *corporeality*. In the first section we will consider a general method of checking when a category of cell complexes is corporeal. The second section will then apply this theory to a particularly simple example; we will prove that any presheaf category gives cell complexes that are presheaves themselves. In Section 5.3 we will find that cell complexes on topological spaces fail to be presheaves, though for a very different sort of reason than the reason 3-computads fail. We will see that this pathological behaviour is to do with a particular part of the definition of topological spaces, and we will go on to discuss a general notion of *space-like* category, isolating a condition that is sufficient to make such a space-like category corporeal.

In the last section we return to the world of relative computads. We will study the results of [Bat02] through the lens of cell complexes and hopefully shed some light on his main theorem using our new language. There are no new results in this final section, but it will discuss Batanin's work from a novel angle; in many respects the approach he takes and the condition his result requires appear extremely natural from the cell complexes perspective.

5.1 Cell Classifiers

We should begin by observing how the category of cell complexes over a given base space A, which we will denote by $\mathbf{CellCx}(A)$, is already very similar in many respects to a presheaf category. A cell complex consists of a set of cells which have different shapes and can be joined together in different ways. Within a single layer of the complex the relationships between cells are exactly those between the elements of a presheaf. The difficulty we encounter concerns the relationship between cells in different layers.

Recall the discussion of computopes in Section 4.1. A 2-dimensional computope was given by a pair of natural numbers; this approach works because a 2-dimensional cell is determined by a binding map $g\colon \partial_2 \to FA$ where A is a 1-computad, and the set of such maps can be partitioned into a collection of sets indexed by pairs of natural numbers. This partitioning is performed by composing with the unique map to the terminal computad,

$$\partial_2 \xrightarrow{\quad g \quad} FA \xrightarrow{\quad F! \quad} F1,$$

and noticing that the set of maps $\partial_2 \to F1$ is isomorphic to the set of pairs of natural numbers. We will call the resulting map into $F1$ the *pattern* of g.

The important property of 2-computads that leads to the realisation as a presheaf category is the fact that every such pattern has a *classifying computad*. This is a computad $\chi(g)$ such that morphisms of computads $\chi(g) \to B$ are in bijection with maps $\partial_2 \to FB$ which have the same pattern as g, where B can be any other computad. Because such maps correspond to all possible 2-cells in any computad, we can immediately construct a set of computads called *2-cell classifiers* such that any 2-cell in a computad B corresponds to a map from a 2-cell classifier into B. It is then straightforward to check that these 2-cell classifiers (together with the obvious 1-cell and 0-cell classifiers) are the representable objects in a presheaf category structure on $\mathbf{2Comp}$.

Generalising this approach. We will now see how we can define the concepts in the paragraph above for any category of cell complexes, and the existence of sufficiently many classifying complexes will be enough to prove that $\mathbf{CellCx}(A)$ is a presheaf category. While ultimately we will only care about morphisms into a cell complex that come from a boundary ∂j for some generating map j, we could theoretically have any object appear as such a boundary. Hence the first few definitions will operate with a morphism $x\colon X \to \overline{(A, \vec{\sigma})}$ for any object $X \in \mathcal{C}$ whatsoever.

Definition 5.1.1. Given some cell complex $(A, \vec{\sigma})$ and a morphism $x\colon X \to \overline{(A, \vec{\sigma})}$, the *pattern* of x, which we will write $\pi(x)$, is the composite morphism

$$X \xrightarrow{\quad x \quad} \overline{(A, \vec{\sigma})} \xrightarrow{\quad \bar{!} \quad} \overline{(A, 1)},$$

where we have used the unique morphism of cell complexes into the terminal complex over A.

The terminal complex over A is given by the sequence of terminal layers—its first layer is $T_0 A$, and each subsequent layer is defined by applying T_α to the complex so far (just as for any comma category $(\hat{\mathcal{S}} \downarrow T)$ the terminal object of the fibre over A is the identity map $1_{TA}\colon TA \to TA$ viewed as an object of the comma category). One can imagine the terminal complex over A in the same way one thinks about the terminal object of a presheaf category; it has precisely one cell in every possible position.

The pattern of a map x tells us how x behaves on each sort of cell—the only additional information we need to determine the map completely is data telling us which cells it chooses to do that behavior on. It follows from the definition that if there exists a morphism between two cell complexes that commutes with morphisms x and x' out of X, then they must have the same pattern—we have $\pi(x) = \pi(x')$. Hence the notion of pattern is a way of decomposing the category of maps from X into its connected components.

The following definition can be viewed as a natural follow-on from the definition of minimal subcomplexes in Section 3.5. The minimal subcomplex was the initial object of the poset of subcomplexes that a given map into a complex factors through; now we will define the classifying complex as the initial object of a category which is analogous to the poset of subcomplexes: the slice category of cell complex morphisms into the particular cell complex we are studying.

Definition 5.1.2. We will write $\mathcal{P}(x)$ for the category of maps from X possessing the same pattern as x; this is one connected component of the category mentioned in the paragraph above. An object is a map $k\colon X \to \overline{(A, \vec{\tau})}$ such that $\pi(k) = \pi(x)$. A morphism between $k\colon X \to \overline{(A, \tau)}$ and $k'\colon X \to \overline{(A, \tau')}$ is a morphism of cell complexes $\beta\colon (A, \vec{\tau}) \to (A, \vec{\tau}')$ such that in the diagram

$$\begin{tikzcd}
 & \overline{(A, \vec{\tau})} \arrow[dd, "\overline{\beta}"'] \arrow[dr, "\bar{!}"] & \\
X \arrow[ur, "k"] \arrow[dr, "k'"'] & & \overline{(A, 1)} \\
 & \overline{(A, \vec{\tau}')} \arrow[ur, "\bar{!}"'] &
\end{tikzcd}$$

both triangles commute. If the category $\mathcal{P}(x)$ has an initial object we write it as $\theta_x\colon X \to \overline{\chi(x)}$ and we call $\chi(x)$ the *classifying complex* of x.

If the classifying complex for some pattern p exists, then every map from X to a cell complex which has this pattern is determined by a cell complex morphism out of $\chi(p)$. The existence of classifying complexes is a very powerful property to have.

Definition 5.1.3. Suppose we have a complete and cocomplete category $\mathcal{C}$, a typical nerve $V\colon \mathcal{C} \to \hat{\mathcal{S}}$ and a category of generating maps $\mathcal{J}$ which satisfies the typical conditions. If X is any object of $\mathcal{C}$, we say that the triple $(\mathcal{C}, V, \mathcal{J})$ is *corporeal with respect to* X if the category **CellCx** generated by $\mathcal{J}$ has classifying complexes for all maps from X. Often we can say that a pair $(\mathcal{C}, V)$ is *universally corporeal with respect to* X, meaning that it is corporeal for any choice of $\mathcal{J}$ that satisfies the typical conditions.

The property of being corporeal with respect to X tells us that when viewed through the object X the category $\mathcal{C}$ behaves with a kind of rigidity; we will now sketch a simple example in a category that fails to be corporeal to try and illustrate this point and motivate the name 'corporeal'. Consider the category of abelian groups, and the object $\mathbb{Z}$. A simple category of cell complexes here is generated by the single homomorphism that adjoins one new element to a group freely. In this case the number 1 is a height ceiling for **CellCx** and any complex is given by a group G and a set A; its underlying morphism is then the inclusion map from G into the abelian group generated by freely adjoining A new elements to G. A very simple complex is given by starting with the trivial group and adjoining two elements a and b. Clearly we obtain the free abelian group on two letters; now consider the homomorphism $\mathbb{Z} \to F(\{a, b\})$ given by $1 \mapsto ab$. We claim that this map has no classifying complex.

The reason the category we get in this case has no initial object is that one can switch the cells a and b without changing the morphism $1 \mapsto ab$. If we think about this example for a few moments we realise that there is something extremely non-physical about this behaviour—from the perspective of the element ab, the two cells have somehow been mixed together to the extent that they are indistinguishable. When a category is corporeal with respect to an object X such confusing mixing of cells is impossible (or at least, not detectable by X). In a corporeal category every cell knows its place. We would remark—for any reader who wishes to see this example done more rigorously—that it is a special case of Lemma 5.4.2.

The following theorem could be proved as a straightforward corollary of general results that are in the literature, since it is basically founded on the fact that a comma category $(\hat{\mathcal{S}} \downarrow T)$ is a presheaf category when T is *familially representable*. The interested reader can consult [CJ95]. For our purposes now we will give a hands-on proof, since it will not take too long and is probably more illuminating.

Theorem 5.1.4. *Suppose we have a complete and cocomplete category $\mathcal{C}$ and a typical nerve $V \colon \mathcal{C} \to \hat{\mathcal{S}}$. Let $\mathcal{J}$ be a category of generating maps satisfying the typical conditions. If $(\mathcal{C}, V, \mathcal{J})$ is corporeal with respect to every object that appears as a boundary ∂j for some $j \in \mathcal{J}$, then* **CellCx**(A) *is equivalent to a presheaf category for any object A in $\mathcal{C}$.*

Proof. We fix some object A of $\mathcal{C}$. We will construct the category of cell classifiers that makes **CellCx**(A) into a presheaf category. First we consider the terminal object of **CellCx**(A), which is denoted by $(A, 1)$. Now define a presheaf P on $\mathcal{J}$, which we call the presheaf of patterns, by setting

$$P(j) = \mathcal{C}(\partial j, \overline{(A, 1)}),$$

and defining restriction maps by precomposition.

For any pattern $p \in P(j)$, the complex $\chi(p)$ has a canonical map $\theta_p \colon \partial j \to \overline{\chi(p)}$. We define the *cell classifier* $\chi(p)_+$ by adjoining a single extra cell of shape

j onto $\chi(p)$ along the map θ_p—formally this can be done using pushforward and then composition of cell complexes which was explored in Section 3.4. Furthermore, the map θ_p clearly cannot be factored through any abbreviation of $\chi(p)$, since if it could that abbreviation would be a counterexample to the universal property of a classifying complex. This means that the height of $\chi(p)_+$ is exactly one more than the height of $\chi(p)$, and that the top layer of $\chi(p)_+$ is given by a representable presheaf in $\mathcal{S}_{\mathcal{J}}$.

There is a cell in the top layer of $\chi(p)_+$ which acts as the apex of the entire complex; nothing in $\chi(p)_+$ exists that is not necessary for the existence of this apex cell. We call the apex c_p, and note that the previous sentence can be formalised using the order relation of dependence that we used in Section 4.2; we can say that for every cell $a \in \chi(p)_+$, either c_p depends on a, or a is the restriction of c_p along some map in $\mathcal{J}$.

We define the category of cell classifiers, which we will write as $\mathcal{K}(A)$, simply by taking the full subcategory of $\mathbf{CellCx}(A)$ containing $\chi(p)_+$ for every pattern p. Since these are the objects that will end up being the representables of our presheaf category of complexes it makes sense for us to define $\mathcal{K}(A)$ as a full subcategory of $\mathbf{CellCx}(A)$—the inclusion map $\mathcal{K}(A) \to \mathbf{CellCx}(A)$ will ultimately be a Yoneda embedding. We will now show directly that the category of presheaves on $\mathcal{K}(A)$ is isomorphic to the category $\mathbf{CellCx}(A)$. We already have a subcategory inclusion $\mathcal{K}(A) \to \mathbf{CellCx}(A)$ that is given by the definition of $\mathcal{K}(A)$. This induces a nerve-realisation adjunction

$$[\mathcal{K}(A)^{\mathrm{op}}, \mathbf{Set}] \underset{N_A}{\overset{\coprod_A}{\rightleftarrows}} \mathbf{CellCx}(A)$$

in the usual manner; we require to show that N_A and $\coprod_A$ form an equivalence of categories in this case.

First, consider any cell complex $(A, \vec{\sigma})$ containing some cell a of shape j. The pattern of a is given by composing the binding map $\partial j \to \overline{(A, \vec{\sigma})}$ with the unique complex morphism to $(A, 1)$, and we will write it $p(a)$. There is a cell complex morphism $\chi(p(a)) \to (A, \vec{\sigma})$ induced by the binding map of a. This extends to a morphism $\hat{a}\colon \chi(p(a))_+ \to (A, \vec{\sigma})$ which takes $c_{p(a)}$ to a, and this morphism $\hat{a}$ is unique with this property. This justifies the name 'cell classifier'; cell complex morphisms out of $\chi(p)_+$ are in bijection with cells whose pattern is p.

This tells us that the nerve functor applied to a cell complex gives a presheaf $N_A(A, \vec{\sigma})$ whose elements are in bijection with the cells of $(A, \vec{\sigma})$ partitioned according to their patterns. Applying the realisation functor to this gives us

$$\textstyle\coprod_A N_A(A, \vec{\sigma}) \xrightarrow{\epsilon_A} (A, \vec{\sigma}),$$

which we must show to be an isomorphism of cell complexes. We write the element of $N_A(A, \vec{\sigma})$ that corresponds to a cell a as $\hat{a}$; the map ϵ_A is clearly surjective, because every cell a is in the image of $\hat{a}$.

Furthermore, assume that a is in the image of some other part of the colimit, say $a = \hat{b}(d)$, for some cell $d \in \chi(p(b))_+$; then d is itself classified by a map $\hat{d}\colon \chi(p(a))_+ \to \chi(p(b))_+$ (d has the same pattern as a because cell complex morphisms preserve patterns of cells). This $\hat{d}$ is a morphism of $\mathcal{K}(A)$, and we now get that $\hat{a} = \hat{d}(\hat{b})$ in the presheaf $N_A(A, \vec{\sigma})$. So a copy of $\hat{d}$ appears in the colimit diagram which puts d in the same equivalence class as $c_{p(a)}$; this reasoning shows that ϵ_A is also injective, and this is enough to make it an isomorphism.

Given any presheaf X on $\mathcal{K}(A)$, we want to show that

$$X \xrightarrow{\eta_A} N_A \coprod_A X$$

is an isomorphism of presheaves. If x and y are elements of $X(\chi(p)_+)$, the colimit $\coprod_A X$ has two distinct copies of $\chi(p)_+$ corresponding to x and y; furthermore, the uniqueness part of the universal property of cell classifiers shows that any pair of morphisms in the colimit diagram out of these two copies of $\chi(p)_+$ must take the apex of each to a different cell. This shows that $\eta_A(x)$ and $\eta_A(y)$ are distinct, so η_A is an injection.

On the other hand, any cell a in the colimit must arise from some cell b in some $\chi(p)_+$ in the colimit diagram; then applying the restriction map for $\hat{b}$ in X we find an element $x \in X$ such that $\eta_A(x) = \hat{a}$. This proves η_A is surjective, so it is an isomorphism, and we have shown that this nerve-realisation adjunction is really an equivalence. □

Remark. We note that the converse of this theorem is very straightforward. Given the assumption that **CellCx**(A) is a presheaf category, one immediately obtains cell classifiers as they are the representables. Then every morphism $x \colon \partial j \to \overline{(A, \sigma)}$ gives a new cell on (A, σ), which has a cell classifier, and this cell classifier can be beheaded to obtain the classifying complex of x.

This theorem gives us a very clear strategy for proving that a given category of cell complexes is locally a presheaf category. In the next two sections we will apply this technique to some categories we are interested in, with varying degrees of success.

5.2 Corporeality of Presheaf Toposes

We begin our investigation of corporeality with the simplest choice possible—that of letting $\mathcal{C}$ itself be a presheaf category and then using the obvious typical nerve given by the identity map. In this case it is still not obvious that universal corporeality holds, but it will not be very difficult to prove. Furthermore, this section will allow us to introduce a few general methods for checking corporeality; we begin with two very useful lemmas.

Lemma 5.2.1. *If a triple $(\mathcal{C}, V, \mathcal{J})$ is corporeal with respect to every object in a (small) diagram, it is also corporeal with respect to the colimit of the diagram.*

Proof. Let $D \colon \mathcal{D} \to \mathcal{C}$ be some small diagram in $\mathcal{C}$, and let $\coprod D$ be its colimit. We assume that for each $d \in \mathcal{D}$, the category $\mathcal{C}$ is corporeal with respect to the object $D(d)$. Now consider some map $x \colon \coprod D \to \overline{(A, \vec{\sigma})}$ into any cell complex generated from $\mathcal{J}$. We will show that x has a classifying complex and hence that $\mathcal{C}$ is corporeal with respect to $\coprod D$ too.

For each $d \in \mathcal{D}$, the inclusion map $i_d \colon D(d) \to \coprod D$ is part of the colimit data. By the assumption of corporeality with respect to $D(d)$, we can construct a classifying complex $\chi(x \circ i_d)$ for each part of the colimit. Furthermore, for any map $\phi \colon d \to d'$ in $\mathcal{D}$, the morphism $D(\phi) \colon D(d) \to D(d')$ induces a cell complex morphism $\chi(x \circ i_d) \to \chi(x \circ i_{d'})$. This gives us a $\mathcal{D}$-shaped diagram $D_x \colon \mathcal{D} \to \mathbf{CellCx}(A)$ in the category of cell complexes over A.

The obvious approach is now to take the colimit $\coprod D_x$ of this diagram; this is the classifying complex $\chi(x)$ which we want. To see this, one simply considers the colimit property. Any object of the category $\mathcal{P}(x)$ is the apex of a unique cocone out of the diagram D_x such that everything commutes; hence there is a unique map from $\coprod D_x$ to any other object of $\mathcal{P}(x)$. □

The next lemma establishes an inductive approach to proving that any category is universally corporeal. The point is that it is generally easier to check the existence of classifying complexes for height one cell complexes than it is for general cell complexes. Fortunately, we can use Theorem 5.1.4 to 'bootstrap' this restricted existence property up into full corporeality; the trick is to use the presheaf expression of $\mathbf{CellCx}(A)$ to express any cell complex as a height one cell complex for a different category of generating maps. This method usually only works when we want to show *universal* corporeality, since changing the category of generating maps may change the boundary objects. We will however see one example, Proposition 5.3.1, where this approach can be used with a subclass of objects rather than all objects.

Lemma 5.2.2. *Suppose any map into a height one complex from any object of $\mathcal{C}$ has a classifying complex. Then the category $\mathcal{C}$ is universally corporeal with respect to all objects.*

Proof. We fix a category of generating maps $\mathcal{J}$ and proceed by induction. Suppose that the corporeality property is satisfied for cell complexes up to some ordinal height α. Then, if we choose a base object A, we can use the proof of Theorem 5.1.4 to construct a category of cell classifiers up to height $\alpha + 1$. We write this category as $\mathcal{K}(A)_{\alpha+1}$ and note that it is naturally viewed as a category over $\mathcal{C}^{\mathbf{2}}$.

Furthermore, we note that the category $\mathcal{K}(A)_{\alpha+1} \to \mathcal{C}^{\mathbf{2}}$ inherits the typical conditions for a category of generating maps from $\mathcal{J}$. Firstly, any map in the image of $\mathcal{K}(A)_{\alpha+1}$ is a $\mathcal{J}$-cell complex so it is a typical inclusion; this also applies to all subcomplex inclusions so $\mathcal{K}(A)_{\alpha+1}$ is distinguishable with respect to the typical nerve (see Definition 3.3.4). Secondly, $\mathcal{K}(A)_{\alpha+1}$-cell complexes have minimal subcomplexes because $\mathcal{J}$-cell complexes have minimal subcomplexes, and these are consistent by the same reasoning (see Definition 3.3.8).

Finally, we remark that any $\mathcal{J}$-cell complex of height $\alpha + 1$ or less can be expressed as a presheaf on $\mathcal{K}(A)_{\alpha+1}$, and therefore as a height one $\mathcal{K}(A)_{\alpha+1}$-cell complex with the rather strange property that every binding map is the identity on A. Conversely, any height one $\mathcal{K}(A)_{\alpha+1}$-cell complex with that property determines a $\mathcal{J}$-cell complex of height $\alpha + 1$ or less. Since any map into a height one complex has a classifying complex this equivalence shows that the corporeality property is satisfied up to height $\alpha + 1$.

The limit case of the induction is a straightforward colimit argument. The classifying complex of a map $x \colon X \to \overline{(A, \vec{\sigma})}$ into a complex of height λ is the colimit of the sequence of classifying complexes of pullbacks of x along the abbreviation inclusions $\overline{(A, \vec{\sigma}|_\beta)} \to \overline{(A, \vec{\sigma})}$ for all $\beta < \lambda$. □

The two lemmas we have just proved lead very naturally to the main result of this section:

Theorem 5.2.3. *Given a presheaf category $\hat{\mathcal{S}}$ and the typical nerve given by the identity functor on it, this pair $(\hat{\mathcal{S}}, 1_{\hat{\mathcal{S}}})$ is universally corporeal with respect to any presheaf.*

Proof. Using Lemma 5.2.1 it will be sufficient to show universal corporeality with respect to the representable presheaves. Using Lemma 5.2.2 we only need to check the existence of classifying complexes for height one cell complexes. Given a presheaf A on $\mathcal{S}$ we will consider $\mathbf{CellCx}_1(A)$; these are themselves simply presheaves on a category we will write $\mathcal{J}_A$ which has an object for every morphism $\partial j \to A$ from the boundary of some generating map to A.

Consider such a cell complex (A, σ) together with a morphism $x\colon [s] \to \overline{(A, \sigma)}$ from a representable into it. As usual we can use the Yoneda lemma to consider this map x as a shape-s element of the presheaf $\overline{(A, \sigma)}$. We also have the typical conditions on the category of generating maps, so we can form the minimal subcomplex $\mu(x)$. Now either $\mu(x)$ is empty, in which case we trivially have a classifying complex which is also empty, or we can use our understanding of colimits in presheaf categories to see that x must factor through the inclusion map of some cell $a \in \mu(x)$ into the colimit $\overline{\mu(x)}$.

We will write s_a for the shape of this cell in the category $\mathcal{J}_A$, and then $\overline{s_a}$ is the result of a single pushout of one cell glued onto A. We will see that this choice of factorisation is unique and that it is determined by the subcomplex $\mu(x)$. Firstly, the map $a\colon [s_a] \to \mu(x)$ is surjective in the category $\hat{\mathcal{J}}_A$ since if it were not we could find a smaller minimal subcomplex for x by taking the subcomplex of $\mu(x)$ given by its image. Secondly, there can only be one surjective map into $\mu(x)$ from a representable object; if two existed we could consider them as elements of $\mu(x)$, and each being surjective would show we could factor either one through the other. This would result in an isomorphism; but $\mathcal{J}_A$ cannot contain any isomorphisms due to the typical conditions. So our choice of a is determined entirely by $\mu(x)$.

If we had two different ways to factorise x through the same cell $\overline{s_a} \to \overline{\mu(x)}$ that would give us two elements a and b of the presheaf $\overline{s_a}$ with the same image under ι_a. But this would imply, based on our understanding of colimits in presheaf categories, that a and b appear in different cells of $[s_a]$ which are mapped to the same cell in $\mu(x)$; this would allow us to construct a smaller minimal subcomplex. Hence the factorisation of x through the cell a is unique.

Now we consider composing with the unique map into the terminal cell complex $(A, 1)$. Consistency of minimal subcomplexes implies that $\mu(!x)$ is the image of $\mu(x)$ under the presheaf morphism $!$, which means the pattern of x factors through a cell in $(A, 1)$ of the same shape s_a. A similar argument can be used for any map into a cell complex with the same pattern to see that all such maps factor uniquely through the representable complex $[s_a]$. Hence, this is clearly the classifying complex of the map x. □

5.3 Failure of Corporeality for Spaces

In this section we attempt to apply the methods introduced in the last section to the category **Top** of topological spaces and continuous maps. Interestingly, we will not get very far. We will see why this is, and it will show us another fundamentally different way in which a category can fail to be corporeal. We will then spend a bit of time discussing *space-like* categories that behave similarly to

Top; we will see that in general we can come very close to proving that they are universally corporeal. We will also present a condition on a space-like category to make it corporeal—this condition will be easy to state, but unfortunately the author does not know of many examples; we will see one example, which is quite interesting but perhaps rather strange as a category of 'spaces'.

Topological spaces. The strongest proposition we can prove for topological spaces is one that applies only with respect to discrete spaces and generating maps with discrete spaces as their boundaries. We will discuss afterwards how the property fails for non-discrete spaces.

Proposition 5.3.1. *The triple* $(\mathbf{Top}, U, \mathcal{J})$ *is corporeal with respect to all discrete spaces, for* $\mathcal{J}$ *any category of generating maps that satisfies the typical conditions and the condition that all of its boundaries are discrete.*

Proof. By Lemma 5.2.1, it is sufficient to prove corporeality with respect to the one point space. We can also use a slightly altered form of Lemma 5.2.2 to see that it is sufficient to show that classifying complexes exist for maps into height one complexes—this works because any $\mathcal{J}$-cell complex can be expressed as the pushforward of a complex whose base space is discrete.

The vital observation that makes this proof work is the fact that the underlying set functor $U\colon \mathbf{Top} \to \mathbf{Set}$ is colimit preserving. Given any category of generating maps in **Top** we can apply U to get a category of generating maps in **Set**; one can check that this new category of generating maps also satisfies the typical conditions. The fact that U preserves colimits means that any cell complex over **Top** is mapped by U to the underlying map of precisely the same cell complex over **Set**. We use Theorem 5.2.3 to get classifying complexes in the world of sets; since we are considering only maps out of discrete spaces, this gives us classifying complexes in the world of topological spaces too. □

Let us now consider why topological spaces fail to be any more corporeal than this. We will create a simple counterexample that uses only the closed interval—not a space which is generally known for its pathology! First, let $\mathcal{J}$ be the one object category of generating maps containing only the map $\{*\} \to [0,1]$ which takes $*$ to one end point, say 0, of the closed interval. We will find a continuous map $[0,1] \to \overline{(\{*\},1)}$ which has no classifying complex.

First we note that the complex $(\{*\},1)$ has a single cell, so $\overline{(\{*\},1)}$ is isomorphic to $[0,1]$. The endomorphism on $[0,1]$ which we want is given by

$$f(x) = (1-x) \cdot \left| \sin\left(\pi \log_2\left(\frac{1}{1-x}\right)\right)\right|$$

which when plotted appears as a sequence of sinusoidal bumps which get faster and faster, but also smaller and smaller, as we approach 1. We have chosen to use the base 2 logarithm in order to ensure that each bump is half the size of the previous one—this makes the rest of the counterexample easier. We claim that the morphism f has no classifying complex. If it had one we would expect a countably infinite set of cells, since each of the bumps in f could, theoretically, appear in a different cell of $\chi(f)$. Thus simply considering limits in the category of sets indicates that the classifying complex must be $(\{*\},\omega)$. (There is only one possible binding map, so ω is really just a set of cells, one for every natural number.) Furthermore,

$$\theta_f\colon [0,1] \to \overline{(\{*\},\omega)}$$

must be the function which behaves exactly like f except that it jumps to the next cell every time it passes a point of the form $\frac{1}{2^n}$.

This would be a lovely classifying complex were it not for the small issue of θ_f failing to be continuous. We see this by constructing an open neighbourhood U around the point $*$ as follows: for any cell c_n given by a natural number $n \in \omega$, the intersection $U \cap c_n$ is the half open interval $[0, \frac{1}{2^{n+1}})$; this completely determines U. The preimage of U under θ_f can be seen to be missing a part of every bump; therefore there are points arbitrarily close to 1 which are not in $\theta_f^{-1}(U)$. But 1 *is* contained in $\theta_f^{-1}(U)$, which means it cannot be an open set. Therefore, θ_f is not continuous.

The key aspect of the category of topological spaces which makes this counterexample possible is the fact that topological spaces have no scale. The way we defined the open set U was basically to say "for each cell, include half as much in U as we did of the last cell". The issue arises because from the perspective of topological spaces there is no such notion of size—since the intersections $U \cap c_n$ are disjoint (except at the point 0) they cannot be compared. For the original function f we can say that the bumps get smaller as we approach 1, but for the function θ_f we cannot say such a thing because we no longer have any way of comparing the sizes of the bumps.

Generalised spaces. Based on the reasoning of the last paragraph the suggestion is that we should consider some other sort of 'spaces'—perhaps metric spaces or something similar. Instead of focusing on a specific example, we will consider a general notion of *space-like* category and see what conditions will ensure corporeality. What we mean by 'space-like' in this context is captured by a typical nerve to the category of sets which satisfies a straightforward property.

Definition 5.3.2. A *space-like* category is a complete and cocomplete category $\mathcal{C}$ equipped with a typical nerve $U\colon \mathcal{C} \to \mathbf{Set}$ that is faithful and preserves colimits.

This is immediately enough to prove the analogue of Proposition 5.3.1. We define the *discrete* spaces to be the objects in the image of the realisation functor from **Set**; then $\mathcal{C}$ is automatically corporeal with respect to any discrete space and category of generating maps with discrete boundaries.

Note that the requirement that U be faithful allows us to view spaces strictly as 'sets with structure', since a morphism between two spaces is determined entirely by its action on the points of the spaces (the elements of the underlying sets). We should also point out that one could use a stronger notion of space-like that replaces the colimit preserving property with the requirement that U has a right adjoint—this would essentially be asking for the existence of indiscrete spaces. For the purposes of this section, however, the weaker condition is strong enough so we will stick to it.

We will now proceed to try and prove that an arbitrary space-like category $\mathcal{C}$ is universally corporeal. Along the way we will discover the single problematic step and establish the extra condition to fix it. Assume we are given an arbitrary category of generating maps $\mathcal{J}$ in $\mathcal{C}$ satisfying the typical conditions and we must demonstrate that all classifying complexes exist for height one $\mathcal{J}$-cell complexes (we are using Lemma 5.2.2). Recall that a classifying complex is by definition the initial object of a certain category $\mathcal{P}(x)$. The strategy we take is straightforward; we will first define a natural coreflective subcategory of the category $\mathcal{P}(x)$. Then we will attempt to show that this subcategory is a poset, that it is complete and that it is small—if these three things were true it would have an initial object formed by taking the limit of the entire category (formally, the limit of the diagram given by the identity functor from the category to itself).

The subcategory. First of all, let $x\colon X \to \overline{(A, \vec{\sigma})}$ be any map into the body of some height one cell complex. Recall that $\mathcal{P}(x)$ is defined as the category of such maps with the same pattern as x. So an object of $\mathcal{P}(x)$ is any map $k\colon X \to \overline{(A, \vec{\tau})}$ such that $\pi(k) = \pi(x)$, and a morphism in $\mathcal{P}(x)$ is a morphism of cell complexes such that the maps out of X commute. Obviously, one can create new objects in this category by simply adding extra cells to the complex that have absolutely nothing to do with the morphism from X. These objects tell us nothing new; they are just unhelpful—so we will consider the subcategory of $\mathcal{P}(x)$ we get by avoiding such unnecessary cells.

To define this subcategory we can make use of the notion of minimal subcomplexes which come as part of the typical conditions (see Definitions 3.3.7 and 3.3.8). Given any object of $\mathcal{P}(x)$ one can take the minimal subcomplex of the map out of X and this gives a new object of $\mathcal{P}(x)$. We write this operation as M. The consistency of these minimal subcomplexes shows that M is an endofunctor, and clearly it is idempotent. Furthermore, the fact that it is a subcomplex means that there is an inclusion map $i_{(k,(A,\vec{\tau}))}\colon M(k, (A, \vec{\tau})) \to (k, (A, \vec{\tau}))$ for any object of $\mathcal{P}(x)$.

This is sufficient to see that M is an idempotent comonad. We write $\mathcal{P}'(x)$ for its category of fixed points; this is the subcategory we wish to consider. An object of $\mathcal{P}'(x)$ is a map $k\colon X \to \overline{(A, \vec{\tau})}$ such that $\mu(k)$ is the whole of $(A, \vec{\tau})$. Hence it is an object of $\mathcal{P}(x)$ with no superfluous cells, which is exactly the type of object we care about. If we prove that $\mathcal{P}'(x)$ has an initial object it follows immediately that it is also an initial object for $\mathcal{P}(x)$.

Showing $\mathcal{P}'(x)$ is a poset. Suppose that θ and θ' are any two morphisms of the form

$$(k, (A, \vec{\tau})) \to (h, (A, \vec{\gamma}))$$

in the category $\mathcal{P}'(x)$. For any cell a in the complex $(A, \vec{\tau})$ we can take the smallest subcomplex containing a which we will write $\nu(a)$; every cell in this subcomplex is either a or the result of some restriction map, $\phi(a)$. Now we claim that any single point (by which we mean an element of the set $U\overline{(A, \vec{\tau})}$, which can be written as a morphism $\{*\} \to \overline{(A, \vec{\tau})}$ where $\{*\}$ is the one shape object for the typical nerve U) has a subcomplex of the form $\nu(a)$ for its minimal subcomplex. This follows from the assumption that $\mathcal{C}$ is space-like, because each subcomplex inclusion is injective and U preserves colimits.

The minimal subcomplex $\mu(k)$ can be expressed as the union of the $\mu(k(p))$ for all $p \in UX$. Hence, for any cell $a \in (A, \vec{\tau})$ we can find a point $p \in UX$ such that a is in the subcomplex $\mu(k(p))$, using the fact that $(k, (A, \vec{\tau}))$ is in the subcategory $\mathcal{P}'(x)$ and therefore $\mu(k)$ is the whole complex. Now we use the result of the last paragraph to see that $\mu(k(p))$ is equal to $\nu(b)$ for some cell

$b \in (A, \vec{\tau})$, and therefore there is a map ϕ in $\mathcal{J}$ such that $a = \phi(b)$. We have $\overline{\theta} \circ k = h = \overline{\theta'} \circ k$, so by consistency of subcomplexes,

$$\theta(\mu(k(x))) = \mu(\overline{\theta}(k(x))) = \mu(h(x)) = \mu(\overline{\theta'}(k(x))) = \theta'(\mu(k(x))),$$

which implies that $\theta(b) = \theta'(b)$. Since $a = \phi(b)$, this further implies that $\theta(a) = \theta'(a)$, and since a was chosen arbitrarily, we have shown that $\theta = \theta'$. Hence there is at most one morphism between any two objects of $\mathcal{P}'(x)$, meaning it is a pre-ordered set.

Showing $\mathcal{P}'(x)$ is small. Let $(k, (A, \vec{\tau}))$ be an object of $\mathcal{P}'(x)$. Since it has no superfluous cells, $(A, \vec{\tau})$ is the union of the subcomplexes $\mu(k(p))$ for all $p \in UX$. Each of these subcomplexes can be written as $\nu(a)$ for a cell $a \in (A, \vec{\tau})$ and the cardinality of $\nu(a)$ is limited by the cardinality of the representable presheaf $[s_a]$ where s_a is the shape of the cell a. Since there is only a set of points in X—that is, UX is a set—this argument gives a maximum cardinal number of cells in any object of $\mathcal{P}'(x)$. Hence it is a small category.

Showing $\mathcal{P}'(x)$ is complete. Because we know that $\mathcal{P}'(x)$ is a poset, it is sufficient to check that all small products exist. Given any set of objects $\{(k_i, (A, \sigma_i)) \mid i \in I\}$ in $\mathcal{P}'(x)$, this gives us a set of objects (A, σ_i) in $\mathbf{CellCx}(A)$. We can form the product of this set; since all the cell complexes involved have height one we are really just taking the product of a set of presheaves. Hence a cell in this product is just a choice of cells, one in each σ_i, with the same shapes and binding maps. For any point $p \in UX$, we get a minimal complex of the form $\nu(a_i)$ for some $a_i \in \sigma_i$. These a_i all have the same shape and binding map since they all live over the same cell in $(A, 1)$; hence they correspond to a single cell a in the product. We do this for each point in X and it allows us to construct a function of underlying sets $k \colon UX \to U\big(\overline{(A, \prod \sigma_i)}\big)$ into the product.

This is where we encounter the problem that causes this argument to fail for topological spaces (and probably many other space-like categories): there is no way to show that this function k is actually the underlying function of some map of spaces. In the case of **Top** one can form a countably infinite product and construct k which fails to be continuous (in much the same way that θ_f in the counterexample we gave earlier failed to be continuous). In the last paragraph we showed that $\overline{(A, \prod \sigma_i)}$ has the same underlying set as the limit in $\mathcal{C}$ of the diagram with each of the $\overline{(A, \sigma_i)}$ and their maps to $\overline{(A, 1)}$. We need to show, however, that they are the same space, not just that they have the same underlying set.

Definition 5.3.3. We say that a space-like category $\mathcal{C}$ is *docile* if for any realisation functor $\coprod \colon \hat{\mathcal{T}} \to \mathcal{C}$ and any small diagram of presheaves $D \colon \mathcal{D} \to \hat{\mathcal{T}}$, if $\coprod(\lim(D))$ and $\lim(\coprod \circ D)$ have the same underlying set then they are also isomorphic as objects of $\mathcal{C}$.

The space $\overline{(A, \prod \sigma_i)}$ can be constructed as the realisation of a limit of presheaves in exactly the way specified by the definition above; hence if $\mathcal{C}$ is docile we can show that k does indeed define a morphism of spaces. This gives a new object of $\mathcal{P}(x)$. It may contain superfluous cells, however, so the next step is to apply the comonad M to obtain an object of $\mathcal{P}'(x)$. A standard argument for reflective subcategories confirms that this object has the universal property of a product with respect to morphisms from objects of the subcategory; thus we have shown that $\mathcal{P}'(x)$ is complete.

Theorem 5.3.4. *A space-like category that is docile is universally corporeal with respect to every object.*

Clearly this property of docility has just been invented in a very ad hoc manner. To suggest that it is not just some completely bizarre requirement we would briefly observe that while **Top** clearly fails to be docile in general, the equivalent property does hold for finite limit diagrams; the failure for infinite diagrams corresponds to the fact that an infinite intersection of open sets is not necessarily open.

This suggests that we could consider the subcategory of topological spaces which have the extra property that every intersection of open sets is open, rather than just the finite intersections. This subcategory has in fact been studied quite a lot; these spaces are called *Alexandrov spaces*. It is a very standard result that the category of Alexandrov spaces is equivalent to the category of pre-ordered sets (see for example [Joh86])—this works by observing that the upwardly closed sets in a poset form a topology. One can easily check that the category of Alexandrov spaces (or pre-ordered sets) is space-like and docile, so it is an example of the theorem above—although it is a bit strange as a category of spaces.

One could also consider the category of metric spaces and short maps, which is space-like and seems at first like another possible solution to the problem encountered with **Top**. However, upon some reflection it is not hard to find a counterexample that shows this is not docile—and once you know where to look it is then trivial to construct a category of generating maps which causes metric spaces to fail to be corporeal. This time the problem arises because of metric spaces with points that are in different connected components but which are not infinitely far apart. Based on this, one might conjecture that some docile category of geodesic metric spaces exists.

5.4 Corporeal Globular Operads

In this section we will consider the corporeality of categories of higher categories given by globular operads. As already discussed, different globular operads will give different results to the question of whether computads are presheaves; therefore some types of n-category will be corporeal and some will not. The question of which globular operads turn out to be corporeal has already been answered in the work of Batanin—see [Bat02]. He proves that computads for a globular operad are presheaves (and therefore the category is corporeal) exactly when the operad has a strongly regular theory for each of its slices.

This terminology will be explained, and we will see how this characterisation appears quite natural when viewed through the language of cell complexes. In particular the property of a theory being strongly regular is equivalent to a certain category being corporeal in a specific context. Thus the approach Batanin takes can be seen as restricting the problem to proving corporeality in a few very special cases.

There are no new theorems in this section. The aim here is rather to study the existing work of Batanin through the lens of cell complexes and hopefully see how his theorem can be more intuitively understood when seen in this general context.

Strongly regular theories. We begin by studying the essential link between corporeality and the property of strong regularity. This is extremely simple and really involves nothing more than unpacking the definition of strong regularity, which we will give first. This theory of strong regularity and its connection to another property called *familial representability* can be found in [CJ95]—Lemma 5.4.2 below is an immediate corollary of results found in that paper, but we will give a full proof here for the sake of clarity.

Definition 5.4.1. A theory on the category of sets is *strongly regular* if it can be presented using only relations in which the same variables appear on both sides, in the same order, without repetition.

For example, the theory of monoids is strongly regular, since $(ab)c = a(bc)$, $a1 = a$ and $1a = a$ clearly satisfy the property described. On the other hand, the theory of abelian groups requires relations such as $ab = ba$ and $aa^{-1} = 1$, both of which fail the test. It turns out that strongly regular theories are exactly those that can be given by (non-symmetric) operads on **Set**—see [Lei04a] for more detail on this subject.

Lemma 5.4.2. *A theory F on* **Set** *is strongly regular if and only if the triple* $(\mathbf{F\text{-}Alg}, U, \{0 \to F1\})$*, in which U is the underlying set functor, is corporeal with respect to the one point set.*

Proof. A cell complex generated from this single generating morphism is just an F-algebra with a set of new elements freely adjoined. Therefore each element of a cell complex can be expressed as an operation in F applied to a set of elements, some from the original F-algebra, and some which are new ones that have been adjoined by the cell complex. However, this expression may not be unique—there may be two or more such expressions with a chain of relations in the theory connecting them.

The condition of F being strongly regular implies that this sequence of relations cannot alter the set of elements that go into the operation; neither can they change the order of the set—in other words, no matter how many different expressions of the element there may be, they all use the same set of cells in the same order. This immediately gives a classifying complex for the element.

The converse is very straightforward to show, since if $(\mathbf{F\text{-}Alg}, U, \{0 \to F1\})$ is corporeal with respect to the singleton set we can construct a strongly regular presentation of F as follows: first, there is an operation for every pattern $\{*\} \to \overline{(0,1)}$, and the arity of each operation is given by the cardinality of its classifying complex. Then to compose n operations ϕ_1 to ϕ_n of arities m_1 to m_n with one operation ψ of arity n we form the morphism

$$\begin{tikzcd}
\{*\} \arrow[r, "\psi(a_1,a_2,\ldots,a_n)"] & F\{a_1, a_2, \ldots, a_n\} \arrow[d, "a_i \mapsto \phi_i(b_{i1}, b_{i2}, \ldots, b_{im_i})"] \\
& F\{b_{ij} \mid 1 \le i \le n, 1 \le j \le m_i\}
\end{tikzcd}$$

and use its pattern to define the composite operation $\psi(\phi_1, \phi_2 \ldots \phi_n)$. What we have is a non-symmetric operad, and hence a strongly regular theory which we can easily see gives the same monad as F. $\square$

This lemma suggests how Batanin's condition might be connected to the notion of corporeality. In order to see how this works, we first have to establish the terminology of *slices* of a globular operad.

Definition 5.4.3. Given a globular operad P, define the kth slice of P to be the monad on **Set** given by the composite

$$\mathbf{Set} \xrightarrow{\Sigma^k} \mathbf{nGSet} \xrightarrow{P_k} \mathbf{nGSet} \xrightarrow{\mathbf{nGSet}(G_k,-)} \mathbf{Set}.$$

The first step Σ^k means suspending k times (the suspension functor replaces an n-globular set X with an $(n+1)$-globular set that has a single 0-cell and higher cells given by moving everything in X up one dimension). The second step is just the kth truncation of P which we defined near the beginning of Section 4.1. The final step returns the set of k-cells in the resulting n-category (which is trivial up to dimension k). It is not particularly hard to show this is a monad; the interested reader will find more detail in [Bat02].

We can now see how Batanin's condition—that all the slices of the operad up to the $(n-1)$th slice are strongly regular—translates directly into a statement about corporeality in **nCat** for a very restricted class of cell complexes. If we consider cell complexes over the terminal n-category $\mathbf{CellCx}(1)$, and we restrict ourselves to the single generating map $\partial_k \to G_k$, we find that each cell complex is just given by a set of k-cells. Furthermore, the n-categories generated by these cell complexes are given by algebras for the kth slice of P. So Batanin's condition is equivalent to saying that $\mathbf{CellCx}_{\{\partial_k \to G_k\}}(1)$ has all classifying complexes. His theorem can be restated as follows:

Theorem 5.4.4 (Equivalent to Theorem 5.2 in [Bat02])**.** *Suppose that for a globular operad P the cell complex category $\mathbf{CellCx}_{\{\partial_k \to G_k\}}(1)$ has all classifying complexes for $0 \leq k \leq n-1$. Then $(\mathbf{nCat}, U, \{\partial_i \to G_i\}_{0 \leq i \leq n})$ is corporeal with respect to any n-category that contains no non-trivial cells at the top dimension.*

For practical purposes we only care about being corporeal with respect to the globs G_k for $k \leq n-1$, but Lemma 5.2.1 gives us the rest at no extra cost. It is not surprising that the corporeality fails when we introduce a non-trivial cell at the top dimension; the top dimension of an n-category is always strict by necessity, so the kind of behaviour we saw earlier for strict 3-categories can easily occur. However, this does not matter from the point of view of computads, since the boundaries of the generating maps do not involve any top-dimension cells.

Bibliography

[Ath12] T. Athorne, *The coalgebraic structure of cell complexes*, Theory and Applications of Categories **26** (2012), no. 11, 304–330.

[Bat98a] M. A. Batanin, *Computads for finitary monads on globular sets*, Contemporary Mathematics **230** (1998), 37–57.

[Bat98b] ———, *Monoidal globular categories as a natural environment for the theory of weak n-categories*, Adv. Math. **136** (1998), 39–103.

[Bat02] ———, *Computads and slices of operads*, ArXiv Mathematics e-prints: math/0209035 (2002).

[Ben67] J. Benabou, *Introduction to bicategories, part I*, Reports of the Midwest Category Seminar **47** (1967), 1–77.

[Bou77] A. K. Bousfield, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra **9** (1977), 207–220.

[Bou14] J. Bourke, *Two dimensional monadicity*, Advances in Mathematics **252** (2014), 708–747.

[BR13] T. Barthel and E. Riehl, *On the construction of functorial factorizations for model categories*, Algebraic Geometric Topology **13** (2013), 1089–1124.

[Che12] E. Cheng, *A direct proof that the category of 3-computads is not cartesian closed*, ArXiv e-prints: 1209.0414 (2012).

[CJ95] A. Carboni and P. Johnstone, *Connected limits, familial representability and artin glueing*, Mathematical Structures in Computer Science **5** (1995), 441–459.

[CL04] E. Cheng and A. Lauda, *Higher dimensional categories: an illustrated guidebook*, `cheng.staff.shef.ac.uk/guidebook` (prepared for IMA workshop), August 2004.

[Cot13] T. Cottrell, *Comparing algebraic and non-algebraic foundations of n-category theory*, Ph.D. thesis, University of Sheffield, September 2013.

[DK84] W. G. Dwyer and D. M. Kan, *Singular functors and realization functors*, Nederl. Akad. Wetensch. Indag. Math. **46** (1984), 147–153.

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[EKMS93] M. Erné, J. Koslowski, A. Melton, and G. E. Strecker, *A primer on galois connections*, Annals of the New York Academy of Sciences **704** (1993), 103–125.

[FK72] P. J. Freyd and G. M. Kelly, *Categories of continuous functors I*, J. Pure Appl. Algebra **2** (1972), 169–191.

[Gar07] R. Garner, *Cofibrantly generated natural weak factorisation systems*, ArXiv Mathematics e-prints: math/0702290 (2007).

[Gar08] ———, *Homomorphisms of higher categories*, U. U. D. M. Report **47** (2008).

[Gar12a] ———, *An abstract view on syntax with sharing*, J. Log. Comput. **22** (2012), no. 6, 1427–1452.

[Gar12b] ———, *Understanding the small object argument*, Applied Categorical Structures **20** (2012), 103–141.

[GPS95] R. Gordon, A. J. Power, and R. Street, *Coherence for tricategories*, vol. 117, Memoirs of the American Mathematical Society, 1995.

[GT06] M. Grandis and W. Tholen, *Natural weak factorization systems*, Archivum Mathematicum **42** (2006), 397–408.

[Gur07] N. Gurski, *An algebraic theory of tricategories*, Ph.D. thesis, University of Chicago, March 2007.

[Gur09] ———, *Nerves of bicategories as stratified simplicial sets*, Pure Appl. Algebra **213** (2009), 927–946.

[Gur13] ———, *Coherence in three-dimensional category theory*, Cambridge university press, 2013.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs Series, American Mathematical Society, 1999.

[Isb60] J. Isbell, *Adequate subcategories*, Illinois J. Math. **4** (1960), 541–552.

[Joh86] P. T. Johnstone, *Stone spaces*, Cambridge University Press, 1986.

[Kan58] Daniel M. Kan, *Functors involving c.s.s. complexes*, Transactions of the American Mathematical Society **87** (1958), no. 2, 330–346.

[Kel80] G. M. Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on.*, Bull. Austral. Math. Soc. **22** (1980), 1–83.

[Law63] F. W. Lawvere, *Functorial semantics of algebraic theories*, Proceedings of the National Academy of Science **50** (1963), no. 5, 869–872.

[Lei98] T. Leinster, *Basic bicategories*, ArXiv Mathematics e-prints: math/9810017 (1998).

[Lei02] ______, *A survey of definitions of n-category*, Theory and Applications of Categories **10** (2002), no. 1, 1–70.

[Lei04a] ______, *Higher operads, higher categories*, Cambridge University Press, 2004.

[Lei04b] ______, *Operads in higher-dimensional category theory*, Theory and Applications of Categories **12** (2004), no. 3, 73–194.

[Mak05] M. Makkai, *The word problem for computads*, `www.math.mcgill.ca/makkai/computad.zip`, 2005.

[Mic10] N. Michel, *Categorical foundations for k-theory*, Ph.D. thesis, École Polytechnique Fédérale de Lausanne, 2010.

[ML98] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, Springer, 1998.

[MLM92] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic*, Springer, 1992.

[MZ08] M. Makkai and M. Zawadowski, *The category of 3-computads is not cartesian closed*, J. Pure Appl. Algebra **212** (2008), no. 11, 2543–2546.

[Qui67] D. G. Quillen, *Homotopical algebra*, Springer, 1967.

[Qui69] ______, *Rational homotopy theory*, Ann. of Math. **90** (1969), 209–295.

[Rie11a] E. Riehl, *Algebraic model structures*, New York Journal of Mathematics **17** (2011), 173–231.

[Rie11b] ______, *Algebraic model structures*, Ph.D. thesis, University of Chicago, 2011.

[Rie13a] ______, *Categorical homotopy theory*, Cambridge University Press, 2013.

[Rie13b] ______, *Monoidal algebraic model structures*, J. Pure Appl. Algebra **217** (2013), 1069–1104.

[Rin70] C. M. Ringel, *Diagonalisierungspaare I*, Math Z. **117** (1970), 249–266.

[Str76] R. Street, *Limits indexed by category-valued 2-functors*, J. Pure Appl. Algebra **8** (1976), 149–181.

[Tri95] T. Trimble, *Notes on tetracategories*, `math.ucr.edu/home/baez/trimble/tetracategories.html`, 1995.

[UFP13] *Homotopy type theory: Univalent foundations of mathematics*, The Univalent Foundations Program, Institute for Advanced Study, `homotopytypetheory.org/book`, 2013.