FREE IDEMPOTENT GENERATED SEMIGROUPS

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Abstract

The study of the free idempotent generated semigroup $\mathrm{IG}(E)$ over a biordered set E began with the seminal work of Nambooripad in the 1970s and has seen a recent revival with a number of new approaches, both geometric and combinatorial. Given the universal nature of free idempotent generated semigroups, it is natural to investigate their structure. A popular theme is to investigate the maximal subgroups. It was thought from the 1970s that all such groups would be free, but this conjecture was false. The first example of a non-free group arising in this context appeared in 2009 in an article by Brittenham, Margolis and Meakin. After that, Gray and Ruškuc in 2012 showed that *any* group occurs as a maximal subgroup of some $\mathrm{IG}(E)$.

Following this discovery, another interesting question comes out very naturally: for a particular biordered E, which groups can be the maximal subgroups of $\mathrm{IG}(E)$? Several significant results for the biordered sets of idempotents of the full transformation monoid $\mathcal{T}_n$ on n generators and the matrix monoid $M_n(D)$ of all $n \times n$ matrices over a division ring D, have been obtained in recent years, which suggest that it may well be worth investigating maximal subgroups of $\mathrm{IG}(E)$ over the biordered set E of idempotents of the endomorphism monoid of an independence algebra of finite rank n.

To this end, we investigate another important example of an independence algebra, namely, the free (left) G-act $F_n(G)$ of rank n, where $n \in \mathbb{N}$, $n \geq 3$ and G is a group. It is known that the endomorphism monoid $\operatorname{End} F_n(G)$ of $F_n(G)$ is isomorphic to a wreath product $G \wr \mathcal{T}_n$. We say that the *rank* of an element of $\operatorname{End} F_n(G)$ is the minimal number of (free) generators in its image.

Let E be the biordered set of idempotents of $\operatorname{End} F_n(G)$, let $\varepsilon \in E$ be a rank r idempotent, where $1 \leq r \leq n$. For rather straightforward reasons it is known that if $r = n-1$ (respectively, n), then the maximal subgroup of $\mathrm{IG}(E)$ containing ε

is free (respectively, trivial). We show, in a transparent way, that, if $r = 1$ then the maximal subgroup of $\mathrm{IG}(E)$ containing ε is isomorphic to that of $\operatorname{End} F_n(G)$ and hence to G. As a corollary we obtain the 2012 result of Gray and Ruškuc that *any* group can occur as a maximal subgroup of *some* $\mathrm{IG}(E)$. Unlike their proof, ours involves a natural biordered set and very little machinery. However, for higher ranks, a more sophisticated approach is needed, which involves the presentations of maximal subgroups of $\mathrm{IG}(E)$ obtained by Gray and Ruškuc, and a presentation of $G \wr \mathcal{S}_r$, where $\mathcal{S}_r$ is the symmetric group on r elements. We show that for $1 \leq r \leq n-2$, the maximal subgroup of $\mathrm{IG}(E)$ containing ε is isomorphic to that of $\operatorname{End} F_n(G)$, and hence to $G \wr \mathcal{S}_r$. By taking G to be trivial, we obtain an alternative proof of the 2012 result of Gray and Ruškuc for the biordered set of idempotents of $\mathcal{T}_n$.

After that, we focus on the maximal subgroups of $\mathrm{IG}(E)$ containing a rank 1 idempotent $\varepsilon \in E$, where E is the biordered set of idempotents of the endomorphism monoid $\operatorname{End} \mathbf{A}$ of an independence algebra $\mathbf{A}$ of rank n with no constants, where $n \in \mathbb{N}$ and $n \geq 3$. It is proved that the maximal subgroup of $\mathrm{IG}(E)$ containing ε is isomorphic to that of $\operatorname{End} \mathbf{A}$, the latter being the group of all unary term operations of $\mathbf{A}$.

Whereas much of the former work in the literature of $\mathrm{IG}(E)$ has focused on maximal subgroups, in this thesis we also study the general structure of the free idempotent generated semigroup $\mathrm{IG}(B)$ over an arbitrary band B. We show that $\mathrm{IG}(B)$ is *always* a weakly abundant semigroup with the congruence condition, but not necessarily abundant. We then prove that if B is a quasi-zero band or a normal band for which $\mathrm{IG}(B)$ satisfying Condition (P), then $\mathrm{IG}(B)$ is an abundant semigroup. In consequence, if Y is a semilattice, then $\mathrm{IG}(Y)$ is adequate, that is, it belongs to the quasivariety of unary semigroups introduced by Fountain over 30 years ago. Further, the word problem of $\mathrm{IG}(B)$ is solvable if B is quasi-zero. We also construct a 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant.

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Preface

Let S be a semigroup with a set of idempotents $E = E(S)$. The structure of the set E can naturally be described as a *biordered set,* a notion arising as a generalisation of the semilattice of idempotents in inverse semigroups by Nambooripad [37]. Conversely, Easdown [10] shows that every biordered set E occurs as $E(S)$ for some semigroup S. Hence we lose nothing by assuming that a biordered set E is of the form $E(S)$ for a semigroup S. Further, a biordered set E has the property of being *regular* if and only if $E = E(S)$ is the set of idempotents of a regular semigroup S [37].

Let S be a semigroup and denote by $\langle E \rangle$ the subsemigroup of S generated by the set of idempotents $E = E(S)$ of S. If $S = \langle E \rangle$, then we say that S is *idempotent generated*. The significance of such semigroups was evident at an early stage: in 1966 Howie [28] showed that every semigroup may be embedded into one that is idempotent generated. To do so, he investigated the idempotent generated subsemigroups of transformation monoids, showing in particular that for the full transformation monoid $\mathcal{T}_n$ on n generators (where n is finite), the subsemigroup of singular transformations is idempotent generated. Erdos [12] proved a corresponding 'linearised' result, showing that the multiplicative semigroup of singular square matrices over a field is idempotent generated. An alternative proof of [12] was given by Dawlings [9], and the result was generalized to finite-dimensional vector spaces over division rings by Laffey [33]. Fountain and Lewin [17] subsumed these results into the wider context of endomorphism monoids of independence algbras. We note here that sets and vector spaces over division rings are examples of independence algebras, as are free (left) G-acts over a group G.

Given a biordered set E, i.e. a set E of idempotents of some semigroup S, there is a free object in the category of semigroups that are generated by E, called the *free idempotent generated semigroup* over E, given by the following presentation:

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\, e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle,$$

where $\overline{E} = \{\bar{e} : e \in E\}$. It is important to understand $\mathrm{IG}(E)$ if one is interested in understanding an arbitrary idempotent generated semigroup with a biordered set E. A current and interesting direction in this area is to investigate the maximal subgroups of $\mathrm{IG}(E)$.

In the first phase of the development of the subject several sets of conditions were found which imply freeness of maximal subgroups [41, 40, 35]. Therefore, it was conjectured that the maximal subgroups of $\mathrm{IG}(E)$ are *always* free [35]. However, this conjecture had been disproved by a counter-example provided by Brittenham, Margolis and Meakin [1]: they showed that $\mathbb{Z} \oplus \mathbb{Z}$, the free abelian group of rank 2, is a maximal subgroup of $\mathrm{IG}(E)$ for some particular biordered set E. Also, the paper [1] exhibited a strong relationship between maximal subgroups of $\mathrm{IG}(E)$ and algebraic topology: namely, it was shown that these groups are precisely fundamental groups of a complex naturally arising from S (called the *Graham-Houghton* complex of S). An unpublished non-free example of maximal subgroups of $\mathrm{IG}(E)$ of McElwee from the earlier part of 1970s was announced by Easdown in 2011 [11].

Motivated by the significant discovery in [1], Gray and Ruškuc [20] showed that *any* group occurs as the maximal subgroup of some $\mathrm{IG}(E)$. Their approach is to use existing machinery which affords presentations of maximal subgroups of semigroups, itself developed Ruškuc, and refine this to give presentations of $\mathrm{IG}(E)$, and then, given a group G, to carefully choose a biordered set E. Their techniques are significant and powerful, and have other consequences.

We would therefore be interested to know, given a special biordered set E, which kind of groups arise as the maximal subgroups of $\mathrm{IG}(E)$? Gray and Ruškuc [21] investigated the maximal subgroups of $\mathrm{IG}(E)$, where E is the biordered set of idempotents of the full transformation monoid $\mathcal{T}_n$ on n elements. It is proved that the maximal subgroup of $\mathrm{IG}(E)$ containing a rank r idempotent $\varepsilon \in E$, with $1 \leq r \leq n-2$, is isomorphic to that of $\mathcal{T}_n$, the latter being the symmetric group $\mathcal{S}_r$ on r elements. Dolinka [5] generalized their result to the biordered set of idempotents of the full partial transformation monoid $\mathcal{PT}_n$ on n elements. On the other hand, Brittenham, Margolis and Meakin [2] considered the biordered set E of the matrix monoid $M_n(D)$ of all $n \times n$ matrices over a division ring D. They have shown, by using a similar topological method to that of [1], that the maximal subgroup of $\mathrm{IG}(E)$ containing a rank 1 idempotent of E is isomorphic to the multiplicative group D^* of D. Then, by applying the presentation of $\mathrm{IG}(E)$ developed in [21], Dolinka and Gray [7] worked out the higher rank case for $M_n(D)$ with $r < n/3$. They proved that a maximal subgroup of $\mathrm{IG}(E)$ containing a rank r idempotent $\varepsilon \in E$ is isomorphic to that of $M_n(D)$, and hence to the general linear group $GL_r(D)$, where $r < n/3$. For the case $r \geq n/3$, the structure of maximal subgroups of $\mathrm{IG}(E)$ is still an open problem.

We have already mentioned in the beginning of the Preface that the full transformation monoid $\mathcal{T}_n$ and the matrix monoid $M_n(D)$ of all $n \times n$ matrices over a division ring D share several common pleasing properties, and this is the motivation of paper [22] by Gould, in which she introduced the investigation of the endomorphism monoid of universal algebra, called *an independence algebra* (also known as *a v^*-algebra*), which include sets, vector spaces and free G-acts, where G is a group. We will enlarge our discussion of independence algebras in Chapter 3.

Given the results obtained so far for the biordered sets of idempotents of $\mathcal{T}_n$, $\mathcal{PT}_n$ and $M_n(D)$, respectively, it is valuable to consider the structure of maximal subgroups of $\mathrm{IG}(E)$, where E is the biordered set of idempotents of the endomorphism monoid $\mathrm{End}\,\mathbf{A}$ of an independence algebra $\mathbf{A}$ of rank n. We start by looking at the case in which $\mathbf{A}$ has no constants. Some progress has been made for the rank 1 case, however for the higher rank cases, the whole picture of maximal subgroups of $\mathrm{IG}(E)$ seems not clear at all. Therefore we decided to transfer our attention to another important source of independence algebras, namely, free G-acts $F_n(G)$ of rank n, where G is a group, $n \in \mathbb{N}$ and $n \geq 3$. We succeed in giving a complete characterization of the maximal subgroups of $\mathrm{IG}(E)$, where E is the biordered set of idempotents of the endomorphism monoid $\mathrm{End}\,F_n(G)$ of $F_n(G)$.

The rest of this thesis is devoted to working on the general structure of the free idempotent generated semigroup $\mathrm{IG}(B)$ over a band B. It has been proved that $\mathrm{IG}(B)$ is always endowed with a significant property, namely, it is weakly abundant.

Now let me explain the main content of each chapter of this thesis:

Chapter 1: We will recall some basic definitions and results of semigroup theory which will be frequently used in the whole thesis.

Chapter 2: We briefly recall the definitions of $\mathcal{L}^*$ and $\mathcal{R}^*$, $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$, and the corresponding concepts of abundant semigroups and weakly abundant semigroups, both of which are generalizations of regular semigroups.

Chapter 3: The notion of *independence algebras* (also known as v^**-algebras*) and their endomorphism monoids will be recalled in this chapter. We remark here that independence algebras include sets, vector spaces and free G-acts, where G is a group; the latter are the main algebraic object we are concerned with in Chapters 5 and 6.

Chapter 4: We first give an overview of free idempotent generated semigroups $\mathrm{IG}(E)$ and several pleasant properties, particularly with respect to Green's relations. Then we recall the results that have been obtained so far in the current research direction of this area, namely, the maximal subgroups of $\mathrm{IG}(E)$.

Chapter 5: The aim of this chapter is to give an alternative proof, in a rather transparent way, for the result of the 2012 paper of Gray and Ruškuc [20], showing that *any* group occurs as a maximal subgroup of some $\mathrm{IG}(E)$. Their approach is to use existing machinery which affords presentations of maximal subgroups of semigroups. Our approach is to consider the biordered set E of non-identity idempotents of a wreath product $G \wr \mathcal{T}_n$, where G is a group and $\mathcal{T}_n$ is the full transformation monoid on n elements. It is known that $G \wr \mathcal{T}_n$ is isomorphic to the endomorphism monoid $\operatorname{End} F_n(G)$ of a free (left) G-act $F_n(G)$ on n generators (see, for example, [30, Theorem 6.8]), and this provides us with a convenient approach. Let $\varepsilon \in E$ be a rank 1 idempotent. We have shown, in a transparent way, that, the maximal subgroup of $\mathrm{IG}(E)$ containing ε is isomorphic to that of $\operatorname{End} F_n(G)$, and hence to G. We remark here that, although none of the technicalities involving presentations appear here explicitly, we nevertheless have made use of the essence of some of the arguments of [7, 20], and more particularly earlier observations from [3, 42] concerning sets of generators for subgroups.

Chapter 6: Here we continue the study of $\mathrm{IG}(E)$ over the biordered set E of idempotents of $\operatorname{End} F_n(G)$ with $n \in \mathbb{N}$ and $n \geq 3$. A complete description of maximal subgroups of $\mathrm{IG}(E)$ has been obtained. Unlike the proof of rank 1 case in Chapter 5, a more sophisticated approach is needed, in which we employ the generic presentation for maximal subgroups given in [20]. The main result we obtained in this chapter is that: for any rank r idempotent $\varepsilon \in E$, with $1 \leq r \leq n-2$, the maximal subgroup of $\mathrm{IG}(E)$ containing ε is isomorphic to that of $G \wr \mathcal{T}_n$. It is known that the latter is $G \wr \mathcal{S}_r$, where $\mathcal{S}_r$ is the symmetric group on n generators. Note that for $r = n-1$, the maximal subgroup is free, for the reason that there are no non-trivial singular squares in the $\mathcal{D}$-class D_r of $\mathrm{End}\, F_n(G)$; and for $r = 1$, the maximal subgroup is trivial. It is also worth remarking that if G is trivial, then $F_n(G)$ is essentially a set, so that $\mathrm{End}\, F_n(G) \cong \mathcal{T}_n$. Our work succeeds in extending the results of both [23] (which forms Chapter 5 of this thesis) and [21] (via a rather different strategy).

Chapter 7: In this chapter, our main concern is the biordered set E of idempotents of the endomorphism monoid $\mathrm{End}\, \mathbf{A}$ of an independence algebra $\mathbf{A}$ of rank n with no constants. Let $\varepsilon \in E$ be a rank r idempotent with $1 \leq r \leq n$. We know that the maximal subgroup of $\mathrm{End}\, \mathbf{A}$ containing ε is the automorphism monoid $\mathrm{Aut}\, \mathbf{A}$ of all automorphisms of $\mathbf{A}$. It turns out that an E-square in E is singular if and only if the idempotents involved form a rectangular band, and hence, if $r = n-1$ (respectively, n), then the maximal subgroup of $\mathrm{IG}(E)$ containing ε is free (respectively, trivial). It has been proved that if $r = 1$ then the maximal subgroup of $\mathrm{IG}(E)$ containing ε is isomorphic to that of $\mathrm{End}\, \mathbf{A}$, the latter is the group G of all unary term operations of $\mathbf{A}$.

Chapter 8: We completely change our view of point in this chapter by looking at the general structure of the free idempotent generated semigroup $\mathrm{IG}(B)$ over an arbitrary band B. We show that $\mathrm{IG}(B)$ is *always* a weakly abundant semigroup with the congruence condition, but not necessarily abundant. A 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant is given by the end of this chapter. Next, we focus on finding some special classes of band B for which $\mathrm{IG}(B)$ is abundant. We then prove that if B is a quasi-zero band or a normal band for which $\mathrm{IG}(B)$ satisfying Condition (P), then $\mathrm{IG}(B)$ is an abundant semigroup. In consequence, if Y is a semilattice, then $\mathrm{IG}(Y)$ is adequate, that is, it belongs to the quasivariety of unary semigroups introduced by Fountain over 30 years ago. Further, the word problem of $\mathrm{IG}(B)$ is solvable if B is quasi-zero.

Chapter 9: We will give a brief proposal for our further work.

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This thesis also marks the end of a chapter of my life. There is no doubt that these two and a half years has been one of the most valuable parts of my life. I am now ready to face every opportunity and every challenge in my future life!

Author's Declaration

I declare that this thesis is a record my two and a half years Ph.D. research work. My contribution to all the joint papers was substantial and significant.

Chapters 1 and 2 mainly present the basics of semigroup theory which are frequently used in the whole thesis. Chapter 3 is devoted to recalling the definition of independence algebras and their properties. Chapter 4 is a summary of the recent results obtained in the area of free idempotent generated semigroups. Chapters 5, 7 and 8 form my joint papers [23], [25] and [24] with Victoria Gould. Chapter 6 is a record of my joint paper [6] with Victoria Gould and Igor Dolinka. Chapter 9 is about some questions we would like to work with in the near future.

Chapter 1

Preliminaries I: Semigroup fundamentals

This chapter is devoted to reviewing the frequently used basic theory of semigroups in this thesis. All of the definitions and the results presented here are standard and can be found in [26], [37] and [10].

Throughout this thesis, mappings are written on the right of their arguments, so that the composition of mappings are from the left to the right.

1.1 Semigroups and binary relations

1.1.1 Semigroups

In this section, we will recall some basic definitions of semigroups, subsemigroups, ideals, morphisms, etc.

A *semigroup* is defined to be a non-empty set S together with a binary operation μ, i.e. a function $\mu : S \times S \longrightarrow S$, that satisfies the associative law: for all $a, b, c \in S$, the equation

$$((a, b)\mu, c)\mu = (a, (b, c)\mu)\mu$$

holds.

We usually write $(a, b)\mu$ as ab, so that the associative law can be expressed briefly as $(ab)c = a(bc)$ for all $a, b, c \in S$, and traditionally, we call the binary operation μ a *multiplication* on S.

If a semigroup S contains an element 1 with the property that, for all x in S, $x1 = 1x = x$, then we say that 1 is an *identity* of S, and that S is a *monoid*.

Note that every semigroup has at most one identity element, because, if both 1 and $1'$ are identities of S, then $1 = 1'1 = 1'$. Therefore if S is a monoid we may refer to *the* identity of S. If a semigroup S has no identity, then it is easy to form a monoid S^1 from S in which $S^1 = S \cup \{1\}$ with

$$1s = s1 = s \text{ for all } s \text{ in } S, \text{ and } 11 = 1.$$

In other words, every semigroup S may be embedded into a monoid S^1 (if S is a monoid then we take $S = S^1$).

A semigroup S with at least two elements is called a *semigroup with zero* if there exists an element 0 in S such that for all x in S, we have $0x = x0 = 0$, and we say that 0 is a *zero element* of S. Usually, S^0 means S with a zero adjoined whether or not S has one already.

Similarly, to the case of identities we can show that every semigroup S has at most one zero element, and moreover every semigroup S can be embedded into a semigroup S^0 with zero by putting $S^0 = S \cup \{0\}$ and define $0x = x0 = 00 = 0$, for all $x \in S$, if necessary.

An *idempotent* in a semigroup S is an element e such that $e^2 = ee = e$. We denote by $E(S)$ the set of all idempotents of S.

Let T be a non-empty subset of a semigroup S. We say that T is a *subsemigroup* of S if it is closed under the multiplication in S, i.e. for any $a, b \in T$, we have $ab \in T$. Further, T is called a *left ideal* (*right ideal*) of S, if for all $s \in S$ and $t \in T$, we have $st \in T$ ($ts \in T$). We therefore call the subset T a (*two-sided*) *ideal* of S if it is both a left ideal and a right ideal of S. We say that a subset T of S is a *generating set* of S if every element of S can be written as a product of elements in T. Clearly, every semigroup S has at least two ideals, i.e. the empty set $\emptyset$ and S itself. Note that if S is a monoid with identity 1, then an ideal A of S is equal to S if and only if $1 \in A$.

Let A be a non-empty subset of S and $\{A_i : \ i \in I\}$ the set of all ideals of S such that $A \subseteq A_i$, for each $i \in I$. Then it is easy to check that the intersection of A_i, denoted by $\bigcap\limits_{i \in I} A_i$, is again an ideal of S; moreover, it is the smallest ideal of S containing A.

Let A and B be subsets of a semigroup S. Then the *product* of A and B is defined as the set, denoted by AB, which consists of all products ab in S, where $a \in A$ and $b \in B$. Particularly, if $B = \{b\}$ or $A = \{a\}$ are singleton, then we may write AB as Ab or aB.

Suppose that T is a non-empty subset of S. Then we have that T is a subsemigroup of S if and only if $TT \subseteq T$, T is a left (right) ideal of S if and only if $ST \subseteq T$ ($TS \subseteq T$), and T is an ideal of S if and only if $TS \subseteq T$ and $ST \subseteq T$.

Let S and T be semigroups. Then a map $\phi : S \longrightarrow T$ is called a *morphism* if for all $a, b \in S$, $(ab)\phi = (a\phi)(b\phi)$. If ϕ is one-one, then we call it a *monomorphism*; if ϕ is onto, then we say that ϕ is an *epimorphism*, and ϕ is called an *isomorphism* if it is a bijection.

1.1.2 Binary relations

A *binary relation* ρ between two sets X and Y is a subset of $X \times Y$, i.e. a set of ordered pairs $(x, y) \in X \times Y$, and here we say that x and y are *ρ-related*.

For a binary relation $\rho \subseteq X \times Y$ we often write $x \; \rho \; y$ instead of $(x, y) \in \rho$. If X and Y are the same set, so that the relation ρ is a subset of $X \times X$, we say that ρ is a *binary relation* on X.

Note that the empty subset $\emptyset$ of $X \times X$ is included in every binary relation on X; and the whole set $X \times X$ includes every binary relation on X. The relation $X \times X$ is called the *universal* relation on X, in which every $x \in X$ is related to every $y \in X$. The *equality* or *diagonal* relation, denoted by 1_X on X, is defined as the set

$$1_X = \{(x, x) : \; x \in X\}.$$

Clearly, here two elements of X are related if and only if they are equal.

For each $\rho \subseteq X \times X$, we define ρ^{-1}, the *converse* of ρ, by

$$\rho^{-1} = \{(x, y) \in X \times X : \; (y, x) \in \rho\}.$$

Let $\mathcal{B}_X$ be the set of all binary relations on X and define a multiplication $\circ$ on $\mathcal{B}_X$ by the rule that, for all $\rho, \sigma \in \mathcal{B}_X$,

$$\rho \circ \sigma = \{(x, y) \in X \times X : \; (\exists z \in X)(x, z) \in \rho \text{ and } (z, y) \in \sigma\}.$$

Then we have the following lemma.

Lemma 1.1.1. [26] *Under the multiplication $\circ$ defined as above, the set $\mathcal{B}_X$ forms a semigroup.*

Before we introduce several important kinds of binary relations we are concerned with in this thesis, it is worth giving names to some special properties of binary relations on a set X. Let ρ be a binary relation on a set X. Then we say that:
(a) ρ is *reflexive*, if for all $x \in X$, $(x, x) \in \rho$;
(b) ρ is *symmetric*, if for all $x, y \in X$, $(x, y) \in \rho$ implies $(y, x) \in \rho$;
(c) ρ is *anti-symmetric*, if for all $x, y \in X$, $(x, y) \in \rho, (y, x) \in \rho$ imply $x = y$;
(d) ρ is *transitive*, if $(x, y) \in \rho, (y, z) \in \rho$ imply $(x, z) \in \rho$.

A relation ρ on a set X is called a *partial order* (*relation*) if it is reflexive, antisymmetric and transitive, and traditionally, we denote ρ by $\leq$, so that we can write $(x, y) \in \rho$ as $x \leq y$. We call a pair $(X, \leq)$ with $\leq$ a partial order on X a *partially ordered set.*

A *pre-order* on a set X is defined to be a reflexive and transitive relation ρ on X, and usually we denote ρ by $\preceq$, so that $(x, y) \in \rho$ can be written as $x \preceq y$.

An *equivalence* (*relation*) on a set X is defined to be a binary relation ρ which is reflexive, symmetric and transitive. We call a set $a\rho$ defined by

$$a\rho = \{b \in X : \ a \ \rho \ b\}$$

the *equivalence class* of a in X. It is known that an equivalence relation ρ on a set X partitions X. Conversely, corresponding to any partition of X, there exists an equivalence relation ρ on X.

Let $\{\rho_i : \ i \in I\}$ be a family of equivalence relations on a set X. Then it is easy to check that $\bigcap\limits_{i \in I} \rho_i$, the intersection of all ρ_i, $i \in I$, is also an equivalence relation on X. Further more, for any given relation ρ on X, clearly the family of all equivalence relations containing ρ is a non-empty set, as we certainly have $\rho \subseteq X \times X$, so that the intersection of these equivalence relations is again an equivalence relation, which it is the smallest equivalence relation containing ρ, and we call it the equivalence relation *generated by ρ,* and denoted by ρ^e.

However, this foregoing general description is not particularly useful, so we need a more practical method to find the equivalence relation ρ^e generated by a given binary relation ρ on a set X.

Let ρ be an arbitrary reflexive relation on a set X. For any $m \in \mathbb{N}$, we define

$$\rho^m = \underbrace{\rho \circ \rho \circ \cdots \circ \rho.}_{m \text{ times}}$$

Then we say that

$$\rho^\infty = \bigcup\{\rho^n : \ n \geq 1\}$$

is the *transitive closure* of the relation ρ. According to Howie [26], ρ^∞ is the smallest transitive relation on X containing ρ. Furthermore, we have:

Lemma 1.1.2. [26] *Let ρ be any fixed binary relation on X. Then the smallest equivalence relation on X containing ρ is given by*

$$\rho^e = (\rho \cup \rho^{-1} \cup 1_X)^\infty.$$

Suppose now that we have two binary relations ρ and σ on X, and we denote $(\rho \cup \sigma)^e$ by $\rho \vee \sigma$. Then we have the following useful lemma, the proof of which is straightforward.

Lemma 1.1.3. *Let ρ and σ be two equivalence relations on a set X such that $\rho \circ \sigma = \sigma \circ \rho$. Then $\rho \vee \sigma = \rho \circ \sigma = \sigma \circ \rho$.*

In semigroup theory, we are more interested in defining binary relations on semigroups, rather than just sets. We would therefore like to be able to say something about the interaction between the relation and the multiplication of a semigroup.

Let S be a semigroup with ρ a binary relation on S. Then we say that ρ is *left compatible* if

$$(\forall s, t, a \in S) \ (s, t) \in \rho \Longrightarrow (as, at) \in \rho$$

and ρ is *right compatible* if

$$(\forall s, t, a \in S) \ (s, t) \in \rho \Longrightarrow (sa, ta) \in \rho$$

and ρ is said to be *compatible* if

$$(\forall s, t, s', t' \in S) \ (s, t), (s', t') \in \rho \Longrightarrow (ss', tt') \in \rho.$$

We say that a left (right) compatible equivalence relation is a *left* (*right*) *congruence* on S, and a compatible equivalence relation is called a *congruence* on S.

Lemma 1.1.4. [26] *A relation ρ on a semigroup S is a congruence if and only if it is both a left and a right congruence.*

Now let ρ be an arbitrary binary relation on a semigroup S. It is clear that the family of all congruences containing ρ is non-empty, as we certainly have $\rho \subseteq X \times X$, and hence the intersection of these congruences is again a congruence, which is the smallest congruence on X containing ρ, denoted by $\rho^\sharp$. Then we have the following lemma as an analogous result for congruences of Lemma 1.1.2.

Lemma 1.1.5. [26] *For any fixed binary relation ρ on a semigroup S, the smallest congruence $\rho^\sharp$ containing ρ is defined by $\rho^\sharp = (\rho^c)^e$, i.e. the smallest equivalence relation containing ρ^c, where*

$$\rho^c = \{(xay, xby) : \ x, y \in S^1, (a, b) \in \rho\}.$$

We end this section by recalling the fundamental theorem of morphisms for semigroups.

Lemma 1.1.6. [26] *Let ρ be a congruence on a semigroup S. Then the set*

$$S/\rho = \{a\rho : \ a \in S\}$$

together with the multiplication defined by the rule that $(a\rho)(b\rho) = (ab)\rho$ forms a semigroup, and the mapping $\rho^\natural$ defined by

$$\rho^\natural : S \longrightarrow S/\rho, a \mapsto a\rho$$

is a morphism.

Now let ψ be a morphism from S to T. Then the relation

$$\ker \psi = \{(a, b) \in S \times S : \ a\psi = b\psi\}$$

is a congruence on S, $\operatorname{im} \psi$ is a subsemigroup of T, and $S/\ker \psi$ is isomorphic to $\operatorname{im} \psi$.

1.2 Green's relations and regular semigroups

1.2.1 Green's relations

We introduce an important tool for analyzing the ideals of a semigroup S and related notions of structure, called *Green's relations*, which are five equivalence relations that characterize the elements of S in terms of the principal ideals they generate.

The fundamental importance of Green's relations to the study of semigroups has led Howie to comment [27]:

"...on encountering a new semigroup, almost the first question one asks is 'What are the Green relations like'?"

Let a be an element of a semigroup S which may not contain an identity, so that Sa does not necessarily contain a. However, the following sets

$$S^1a = Sa \cup \{a\}, aS^1 = aS \cup \{a\}, S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$$

are all subsets of S containing a. Precisely, they are the smallest left, right and two-sided ideals of S containing a, respectively. We call S^1a the *principal left ideal generated by a*. Dually, aS^1 is the *principal right ideal generated by a* and S^1aS^1 is the *principal ideal generated by a*.

We now define relations $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ as follows: for any $a, b \in S$

$$a \leq_{\mathcal{L}} b \Longleftrightarrow S^1a \subseteq S^1b \text{ and } a \leq_{\mathcal{R}} b \Longleftrightarrow aS^1 \subseteq bS^1.$$

It is easy to check that $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ are two pre-orders on S. The pre-order $\leq_{\mathcal{J}}$ regarding the principal ideals of S can be defined by a similar way.

Note that if $e, f \in E(S)$, then we have that

$$e \leq_{\mathcal{L}} f \Longleftrightarrow ef = e \text{ and } e \leq_{\mathcal{R}} f \Longleftrightarrow fe = e.$$

We are now in a position to define Green's relations which were introduced by J.A. Green in 1951. The two most basic of Green's relations are $\mathcal{L}$ and $\mathcal{R}$, which are defined by the rule that for any $a, b \in S$

$$a \ \mathcal{L} \ b \Longleftrightarrow S^1a = S^1b \text{ and } a \ \mathcal{R} \ b \Longleftrightarrow aS^1 = bS^1.$$

Thus, a and b are $\mathcal{L}$-related if they generate the same principal left ideal, a and b are $\mathcal{R}$-related if they generate the same principal right ideal. It is easy to see that $\mathcal{L}$ is a right congruence on S and $\mathcal{R}$ is a left congruence on S.

The following lemma gives another characterization of $\mathcal{L}$ and $\mathcal{R}$ on S in terms of elements of S.

Lemma 1.2.1. [26] *Let a, b be elements of a semigroup S. Then $a \; \mathcal{L} \; b$ if and only if there exist $x, y \in S^1$ such that $xa = b, yb = a$. Dually, $a \; \mathcal{R} \; b$ if and only if there exist $u, v \in S^1$ such that $au = b, bv = a$.*

As the two-sided analogue of $\mathcal{L}$ and $\mathcal{R}$, we define an equivalence relation $\mathcal{J}$ on S by the rule that $a \; \mathcal{J} \; b$ if and only if $S^1aS^1 = S^1bS^1$, which is equivalent to the existence of $x, y, u, v \in S^1$ such that $xay = b$ and $ubv = a$.

If we denote the intersection of $\mathcal{L}$ and $\mathcal{R}$ by $\mathcal{H}$, then clearly $\mathcal{H}$ is an equivalence relation on S. The binary relation $\mathcal{L} \vee \mathcal{R}$ is denoted by $\mathcal{D}$. We use L_a, R_a, D_a, H_a and J_a to denote the $\mathcal{L}$-class, the $\mathcal{R}$-class, the $\mathcal{D}$-class, the $\mathcal{H}$-class and the $\mathcal{J}$-class of an element $a \in S$, respectively. Then we have:

Lemma 1.2.2. [26] *The relations $\mathcal{L}$ and $\mathcal{R}$ commute, i.e. $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, so that by Lemma 1.1.3, $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

It is easy to observe that $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. As $\mathcal{D}$ is the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$, we have $\mathcal{D} \subseteq \mathcal{J}$. Consequently, we have the following *Hasse diagram.*

$$\begin{tikzcd} & \mathcal{J} \arrow[d, dash] & \\ & \mathcal{D} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathcal{L} \arrow[dr, dash] & & \mathcal{R} \arrow[dl, dash] \\ & \mathcal{H} & \end{tikzcd}$$

Figure 1.1: Hasse diagram of Green's relations

Note that in a group G we have

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = G \times G.$$

It is clear that each $\mathcal{D}$-class in a semigroup S is a union of $\mathcal{L}$-classes and also a union of $\mathcal{R}$-classes. On the other hand, the intersection of an $\mathcal{L}$-class and an $\mathcal{R}$-class is either empty or is an $\mathcal{H}$-class. However, it follows from the definition of $\mathcal{D}$ and the fact $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ that

$$a \,\mathcal{D}\, b \Longleftrightarrow R_a \cap L_b \neq \emptyset \Longleftrightarrow L_a \cap R_b \neq \emptyset.$$

It is often useful to visualize a $\mathcal{D}$-class of a semigroup S using a so called *egg-box diagram.* An *egg-box diagram* of a $\mathcal{D}$-class D is a grid depicted by the figure below, whose rows represent $\mathcal{R}$-classes of D, columns represent $\mathcal{L}$-classes of D, and the cells of the grid represent $\mathcal{H}$-classes of D.

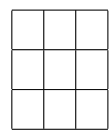

Figure 1.2: the egg-box of a typical $\mathcal{D}$-class

Lemma 1.2.3. [26] *If D is a $\mathcal{D}$-class of S and $a, b \in D$ are $\mathcal{R}$-related in S, say with $as = b$ and $bs' = a$ for some $s, s' \in S^1$, then the right translation $\rho_s : S \longrightarrow S$ defined by $x \mapsto xs$ maps L_a to L_b; $\rho'_s : S \longrightarrow S$ defined by $x \mapsto xs'$ maps L_b back to L_a; and the composition $\rho_s\rho'_s : S \longrightarrow S$ is the identity mapping on L_a. Furthermore, ρ_s is $\mathcal{R}$-class preserving in the sense that it maps each $\mathcal{H}$-class of L_a in a 1-1 manner onto the corresponding ($\mathcal{R}$-equivalent) $\mathcal{H}$-class of L_b.*

We remark here that a dual result holds for $\mathcal{L}$-classes.

Lemma 1.2.4. [26] *For each $\mathcal{H}$-class H of a $\mathcal{D}$-class D in S, we either have $H^2 \cap H = \emptyset$ or H is a subgroup; moreover, H is a subgroup if and only if H contains an idempotent of S, so that no $\mathcal{H}$-class can contain more than one idempotent.*

Given an idempotent $e \in E(S)$, let G be a subgroup of S containing e. For any $a \in G$, we know $a \,\mathcal{H}\, e$ in G, so that $a \,\mathcal{H}\, e$ in S, and hence $G \subseteq H_e$, where H_e is the $\mathcal{H}$-class of e in S. Therefore we have that H_e is a maximal subgroup of S containing e.

Lemma 1.2.5. [26] *Any two $\mathcal{H}$-classes of a $\mathcal{D}$-class D of S have the same cardinality; moreover, any two group $\mathcal{H}$-classes within the same $\mathcal{D}$-class are isomorphic.*

Lemma 1.2.6. [26] *If a and b are two elements in a $\mathcal{D}$-class D of S, then the product $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains an idempotent.*

1.2.2 Regular semigroups

We say that an element $a \in S$ is *regular* if there exists $x \in S$ such that $axa = a$. Note that $axa = a$ implies both ax and xa are idempotents, and $ax \; \mathcal{R} \; a \; \mathcal{L} \; xa$. A semigroup S is called a *regular semigroup* if all its elements are regular.

We remark here that for regular semigroups, Green's relations can be expressed in terms of S rather than S^1, since for each $a \in S$, $a \in aS, Sa$ and SaS.

Lemma 1.2.7. [26] *If a is a regular element of a semigroup S, then every element of D_a is regular.*

Hence, for every $\mathcal{D}$-class D in S, either all elements of D are regular or none of them are regular. We call a $\mathcal{D}$-class *$\mathcal{D}$-regular* if all its elements are regular. Note that for any idempotent $e \in S$, we have $eee = e$, so that every $\mathcal{D}$-class containing an idempotent is regular.

Lemma 1.2.8. [26] *For any idempotent $e \in S$, e is the left identity of its $\mathcal{R}$-class R_e and a right identity of its $\mathcal{L}$-class L_e.*

Lemma 1.2.9. [26] *A semigroup S is regular if and only if each $\mathcal{L}$-class and each $\mathcal{R}$-class contains an idempotent.*

An element $a' \in S$ is called an *inverse* of $a \in S$ if $aa'a = a$ and $a'aa' = a'$. Thus we have

$$aa' \; \mathcal{R} \; a \; \mathcal{L} \; a'a \text{ and } a'a \; \mathcal{R} \; a' \; \mathcal{L} \; aa'.$$

Clearly, an element with an inverse must be regular. Less obviously, every regular element has an inverse. For, suppose that a is regular. Then there exists $x \in S$ such that $axa = a$. By putting $a' = xax$, we have

$$aa'a = axaxa = axa = a, \; a'aa' = xaxaxax = xaxax = xax = a'.$$

For a given semigroup S, an element $s \in S$ need not necessarily have an inverse, or, if it does, it could have more than one. A semigroup S in which every element $s \in S$ has precisely one inverse, is called an *inverse semigroup*. A semigroup S is inverse if and only S is regular with commuting idempotents.

Lemma 1.2.10. [26] *Let a be an element of a regular $\mathcal{D}$-class D of a semigroup S. Then the following statements hold:*

(i) for any inverse a' of a, we have $a \mathcal{D} a'$ and $aa' \in R_a \cap L_{a'}$ and $a'a \in L_a \cap R_{a'}$;

(ii) if $e \in R_a$ and $f \in L_a$, then there exists an inverse a' of a in $R_f \cap L_e$ such that $aa' = e$ and $a'a = f$;

(iii) no $\mathcal{H}$-class contains more than one inverse of a.

1.2.3 Completely (0)- simple semigroups

A semigroup S without zero is called *simple* if it has no proper ideals. A semigroup S with zero is called *0-simple* if $S^2 \neq \{0\}$ and it has exactly two ideals, namely $\{0\}$ and S.

Note that S is simple if and only if $\mathcal{J} = S \times S$, and S with 0 is 0-simple if and only if $S^2 \neq \{0\}$ and it has two $\mathcal{J}$-classes, $\{0\}$ and $S \backslash \{0\}$.

Let $E(S)$ be the set of all idempotents of S. Define a binary relation $\leq$ on $E(S)$ by the rule that

$$e \leq f \text{ if and only if } ef = fe = e.$$

It is easy to check that $\leq$ is a partial order on $E(S)$.

A *completely simple semigroup* is defined to be a simple semigroup S with an idempotent e, which is *primitive within* $E(S)$, in the sense that for any idempotent $f \in E(S)$, $f \leq e$ implies $f = e$.

A semigroup S is called *completely 0-simple* if it is 0-simple and has an idempotent e, that is *primitive within the set of non-zero idempotents of* S, by which we mean, for any idempotent $f \neq 0$, $f \leq e$ implies $f = e$.

Lemma 1.2.11. [26] *Every completely 0-simple semigroup S is a regular semigroup with exactly two $\mathcal{D}$-classes, namely $\{0\}$ and $D = S \backslash \{0\}$. For any $a, b \in D$, then either $ab = 0$ or $ab \in R_a \cap L_b$, and the latter occurs if and only if $L_a \cap R_b$ contains an idempotent.*

1.2.4 The Rees theorem

Let G be a group with identity e and let I and Λ be non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in the 0-group $G^0 (= G \cup \{0\})$, and suppose that P

is *regular*, in the sense that no row or column of P entirely consists of zeros, i.e.

$$(\forall i \in I)(\exists \lambda \in \Lambda)\ p_{\lambda i} \neq 0 \text{ and } (\forall \lambda \in \Lambda)(\exists i \in I)\ p_{\lambda i} \neq 0.$$

Put $S = (I \times G \times \Lambda) \cup \{0\}$, and define a multiplication on the set S by

$$(i,a,\lambda)(j,b,\mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0 \end{cases}$$

and

$$(i,a,\lambda)0 = 0(i,a,\lambda) = 00 = 0.$$

We denote S under this multiplication by $\mathcal{M}^0[G; I, \Lambda; P]$, called the $I \times \Lambda$ *Rees matrix semigroup over G with regular sandwich matrix P.*

We have the following well known Rees Theorem.

Theorem 1.2.12. [26] *The semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ constructed in the above manner is a completely 0-simple semigroup; conversely, every completely 0-simple semigroup is isomorphic to one constructed in this way.*

We briefly recall how to shape a given completely 0-simple semigroup S into the form of a Rees matrix semigroup.

Let D be the unique non-zero $\mathcal{D}$-class of a completely 0-simple semigroup S. We use I to denote the $\mathcal{R}$-classes of D and Λ to denote the $\mathcal{L}$-classes of D, so that an $\mathcal{H}$-class which is the intersection of the $\mathcal{R}$-class R_i of D and the $\mathcal{L}$-class L_λ of D is denoted by $H_{i\lambda}$. We use $e_{i\lambda}$ to denote the unique identity in a group $\mathcal{H}$-class $H_{i\lambda}$. As D is a regular D-class, there must exist some group $\mathcal{H}$-class in D. Without loss of generality, we assume that $1 \in I \cap \Lambda$ and H_{11} is a group $\mathcal{H}$-class with identity e_{11}.

Now in a quite arbitrary manner, we choose an element $r_i \in H_{i1}$ for each $i \in I$, and an element $q_\lambda \in H_{1\lambda}$ for each $\lambda \in \Lambda$. For each $i \in I$ and each $\lambda \in \Lambda$, we define $p_{\lambda i}$ as $q_\lambda r_i$ if $p_{\lambda i} \in R_1 \cap L_1 = H_{11}$, otherwise $p_{\lambda i} = 0$, so that $P = (p_{\lambda i})$ is a matrix with entries in H_{11}^0. We need to argue here that $P = (p_{\lambda i})$ is regular. As D is a regular D-class, each $\mathcal{R}$-class and each $\mathcal{L}$-class contains at least one idempotent, respectively. Hence for each fixed $i \in I$, there exists some $\lambda \in \Lambda$ such that $H_{i\lambda}$ is a group $\mathcal{H}$-class, so that $0 \neq p_{\lambda i} \in H_{11}$. Also, for each fixed $\lambda \in \Lambda$, there exists some $i \in I$ such that $H_{i\lambda}$ is a group $\mathcal{H}$-class, so that $0 \neq p_{\lambda i} \in H_{11}$. Hence the sandwich matrix $P = (p_{\lambda i})$ is regular.

Now we have all the ingredients to construct a Rees matrix semigroup

$$\mathcal{M}^0(H_{11}; I, \Lambda; P) = (I \times H_{11} \times \Lambda) \cup \{0\}.$$

It is proved that the map

$$\psi : (I \times H_{11} \times \ \Lambda) \cup \{0\} \longrightarrow S$$

defined by

$$(i, a, \lambda)\psi = r_i a q_\lambda, 0\psi = 0$$

is an isomorphism.

Corresponding to completely simple semigroups, we have the following simplified version of the Rees Theorem.

Theorem 1.2.13. [26] *Let G be a group, let I and Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G. Let $S = I \times G \times \Lambda$, and define a multiplication on S by*

$$(i, a, \lambda)(j, b, \mu) = (i, a p_{\lambda j} b, \mu).$$

Then S is a completely simple semigroup.

Conversely, every completely simple semigroup is isomorphic to a semigroup constructed in this way.

We denote the semigroup $S = I \times G \times \Lambda$ with the multiplication given in Theorem 1.2.13 by

$$\mathcal{M}(G; I, \Lambda, P).$$

For further details, we refer readers to [26].

1.3 Free semigroups and presentations

1.3.1 Free semigroups

Let A be an alphabet. Let A^+ be the set of all finite, non-empty words $a_1 a_2 \cdots a_m$ in A. We say that two words $a_1 \cdots a_n, b_1 \cdots b_m \in A^+$ are *equal* if and only if $n = m$

and $a_i = b_i$ for all $1 \leq i \leq n$. Define a binary operation on A^+ by juxtaposition

$$(a_1 a_2 \cdots a_m)(b_1 b_2 \cdots b_n) = a_1 a_2 \cdots a_m b_1 b_2 \cdots b_n.$$

Then A^+ is a semigroup, called the *free semigroup* on A. Clearly, here A is the generating set of A^+. By adding an *empty word* (containing no letters at all) denoted by 1, into A^+, we obtain the *free monoid* $A^* = A^+ \cup \{1\}$.

An abstract way to define a free semigroup on A can be given as follows:

A semigroup F is called a *free semigroup* on a set A if we have the following:

(F1) there is a map $\alpha : A \longrightarrow F$;

(F2) for every semigroup S and every map $\phi : A \longrightarrow S$ there exists a unique morphism $\psi : F \longrightarrow S$ such that the following diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & F \arrow[dl, "\psi"] \\ S & \end{tikzcd}$$

Figure 1.3: the commutative diagram of free semigroups

commutes.

We claim that A^+ is a free semigroup on A in the sense of the above abstract definition of free semigroups.

We take the mapping $\alpha : A \longrightarrow A^+$ as the *standard embedding* of A into A^+, by which we mean that $a\alpha = a$, for each $a \in A$, i.e. associating each a in A with the corresponding one-letter word in A^+. Then for any given semigroup S and an arbitrary map $\phi : A \longrightarrow S$, we define $\psi : A^+ \longrightarrow S$ by

$$(a_1 a_2 \cdots a_m)\psi = (a_1\phi)(a_2\phi) \cdots (a_m\phi)$$

for all $a_1 a_2 \cdots a_m \in A^+$. It is easy to check that ψ is a unique morphism from A^+ to S such that $\alpha\psi = \phi$. Thus we have the following commuting diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & A^+ \arrow[dl, "\psi"] \\ S & \end{tikzcd}$$

Figure 1.4: the commutative diagram of A^+

We remark here that by Theorem 1.1.6 $S \cong A^+/\ker\psi$, so that we have the following lemma.

Lemma 1.3.1. *Every semigroup may be expressed up to an isomorphism as a quotient of a free semigroup by a congruence.*

1.3.2 Semigroup presentations

A *semigroup presentation* is an ordered pair $\langle A \mid R\rangle$, where R is a binary relation on A^+, i.e. $R \subseteq A^+ \times A^+$. An element a in A is called a *generator*, while a pair $(u, v) \in R$ is called a *defining relation*. Sometimes we write $u = v$ instead of $(u, v) \in R$. The *semigroup* defined by a *presentation* $\langle A \mid R\rangle$ is A^+/ρ, where ρ is the smallest congruence generated by R. A semigroup S is said to be defined by the presentation $\langle A \mid R\rangle$ if

$$S \cong A^+/\rho$$

or, equivalently, there is an epimorphism

$$\psi : A^+ \longrightarrow S \text{ with } \ker\psi = \rho.$$

We remark here that there is a similar theory of monoid presentations.

1.4 Biordered sets

1.4.1 Basic definitions

The concept of a biordered set was introduced by Nambooripad in an influential work [37] in the early 1970s, occurring in the description of the structure of the set of idempotents of a regular semigroup. The aim of this section is to recall the axioms defining biordered sets, and an extra axiom satisfied by what are called regular biordered sets.

Let E be a *partial algebra*, a set with a partial binary operation. Let D_E be the domain of the partial binary operation. Then we can regard D_E as a binary relation on E defined by $(e, f) \in D_E$ if and only if the product ef is defined in E.

Now we define two binary relations ω^r and ω^l on E in the following manner: for any $e, f \in E$

$$e \ \omega^r \ f \iff (e, f) \in D_E \text{ and } fe = e$$

and

$$e \ \omega^l \ f \iff (e, f) \in D_E \text{ and } ef = e.$$

We also define

$$R = \omega^r \cap (\omega^r)^{-1};\ L = \omega^l \cap (\omega^l)^{-1};\ \omega = \omega^r \cap \omega^l.$$

We put

$$\omega^r(e) = \{f \in E :\ f \ \omega^r \ e\};\ \omega^l(e) = \{f \in E :\ e \ \omega^l \ f\}.$$

A partial algebra E equipped with the above five binary relations is called a *biordered set* if it satisfies axioms (B1), (B21), (B22), (B31), (B32) and (B4) and their duals, for any $e, f, g, h \in E$.

(B1) ω^r and ω^l are pre-orders on E such that

$$D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}.$$

(B21) $f \in \omega^r(e)$ implies $f \ R \ fe \ \omega \ e$.

(B22) $f, g \in \omega^r(e)$ and $g \ \omega^l \ f$ imply $ge \ \omega^l \ fe$.

(B31) $g \ \omega^r \ f \ \omega^r \ e$ implies $gf = (ge)f$.

(B32) $f, g \in \omega^r(e)$ and $g \ \omega^l \ f$ imply $(fg)e = (fe)(ge)$.

Put $M(e, f) = \omega^l(e) \cap \omega^r(f)$, for any $e, f \in E$. Define a relation $\prec$ on $M(e, f)$ by the rule that

$$g \prec h \iff eg \ \omega^r \ eh,\ gf \ \omega^l \ hf.$$

It is easy to check that $\prec$ is a pre-order on $M(e, f)$.

The set

$$S(e,f) = \{h \in M(e,f) : \ g \prec h \text{ for all } g \in M(e,f)\}$$

is called the *sandwich set* of e and f in that order.

(B4) $f, g \in \omega^r(e)$ implies $S(f,g)e = S(fe, ge)$.

A biordered set E is said to be *regular* if $S(e,f) \neq \emptyset$ for all $e, f \in E$.

Let E and F be two biordered sets. Then a mapping $\phi : E \longrightarrow F$ is called a *biordered set morphism* if for all $(e,f) \in D_E$ we have

$$(e\phi, f\phi) \in D_F \text{ and } (e\phi)(f\phi) = (ef)\phi$$

and further, if ϕ is bijective, then we call it a *biordered set isomorphism.*

1.4.2 Biordered sets and semigroups

Let S be a semigroup with a set $E = E(S)$ of idempotents. We call a pair $(e,f) \in E \times E$ a *basic pair* if one of the following equalities hold

$$ef = e, fe = e, ef = f \text{ or } fe = f.$$

We remark here that $(e,f) \in E \times E$ is a basic pair implies that both ef and fe are idempotents of E; for instance, if $ef = e$, then

$$(fe)(fe) = f(ef)e = fee = fe$$

so that fe is also an idempotent of E.

It was pointed out by Nambooripad [37] that the set E of idempotents of S forms a partial algebra with domain

$$D_E = \{(e,f) : \ (e,f) \text{ is a basic pair}\},$$

where for any $(e,f) \in D_E$, ef is defined to be the product ef in S, which is clearly an idempotent. Furthermore, there are two pre-orders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ defined on S.

Now we have the following significant results obtained by Nambooripad [37] and Easdown [10], which tells us that the concept of biordered set is a characterization of the partial algebras of idempotents of semigroups obtained as above.

Theorem 1.4.1. [37, 10] *For any semigroup S, the set $E = E(S)$ of idempotents of S forms a biordered set with respect to the pre-orders $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$ defined on E. Conversely, every biordered set E occurs as a set $E = E(S)$ of some semigroup S.*

Theorem 1.4.2. [37] *A biordered set E is regular if and only if $E = E(S)$ is the set of idempotents of a regular semigroup S.*

1.5 Bands

1.5.1 Basic definitions

Recall that an idempotent e of a semigroup is just an element with $e^2 = ee = e$. A *band* is defined as a semigroup S consisting entirely of idempotents. Traditionally, we use B to denote a band.

Now we introduce several special kinds of bands which are frequently mentioned in this thesis, including rectangular bands, semilattices, left normal bands, right normal bands, and normal bands.

Let B be a band. Then:

(i) B is a *rectangular band* if for any $e, f \in B$, $efe = e$;

(ii) B is a *semilattice* if for any $e, f \in B$, $ef = fe$;

(iii) B is a *left normal band* if for any $e, f, g \in B$, $efg = egf$;

(iv) B is a *right normal band* if for any $e, f, g \in B$, $efg = feg$;

(v) B is a *normal band* if for any $e, f, g \in B$, $efge = egfe$.

Recall that in Section 1.2.3, we have already defined a partial order $\leq$ on the set of idempotents of a semigroup S by the rule that, for any idempotents $e, f \in S$

$$e \ \leq \ f \Longleftrightarrow ef = fe = e$$

Hence $\leq$ is of course a partial order on B.

1.5.2 The decomposition theorem for bands

Let S be a semigroup and Y a semilattice. We say that S is a *semilattice Y of subsemigroups S_α, $\alpha \in Y$*, denoted by $S = \bigcup\limits_{\alpha \in Y} S_\alpha$ if the following hold:

(i) S is a disjoint union of subsemigroups S_α, where $\alpha \in Y$;

(ii) for any $\alpha, \beta \in Y$, $S_\alpha S_\beta \subseteq S_{\alpha\beta}$.

It follows immediately from the definition that if $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semilattice Y of subsemigroups S_α, $\alpha \in Y$, then for any $x \in S_\alpha, y \in S_\beta$, we have $xy \in S_{\alpha\beta}$. However, we do not know the location of xy within $S_{\alpha\beta}$; in other words, we know the 'gross' structure of S but not its 'fine' structure. For this purpose, we define the notion of strong semilattice of semigroups.

Suppose that we have a semilattice Y and a family of disjoint semigroups S_α indexed by Y, and suppose that, for all $\alpha \geq \beta$ in Y there exists a morphism

$$\phi_{\alpha,\beta} : \ S_\alpha \longrightarrow S_\beta$$

such that:

(S1) $(\forall \alpha \in Y)\ \phi_{\alpha,\alpha} = 1_{S_\alpha}$;

(S2) for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$.

Now we define a multiplication on the set $S = \bigcup_{\alpha \in Y} S_\alpha$ by the rule that for each $x \in S_\alpha$ and each $y \in S_\beta$,

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}),$$

where the multiplication on the right hand side is in $S_{\alpha\beta}$. Clearly, the operation extends the multiplication in each S_α, $\alpha \in Y$.

Lemma 1.5.1. [26] *Under the multiplication defined above, the set* $S = \bigcup_{\alpha \in Y} S_\alpha$ *forms a semigroup, called a strong semilattice* Y *of semigroups* $S_\alpha, \alpha \in Y$, *denoted by* $S = \mathcal{S}(Y; S_\alpha, \phi_{\alpha,\beta})$.

At times we will use this notations $S = \bigcup_{\alpha \in Y} S_\alpha$ to denote a semilattice Y of subsemigroups $S_\alpha, \alpha \in Y$, and $S = \mathcal{S}(Y; S_\alpha, \phi_{\alpha,\beta})$ to denote a strong semilattice Y of semigroups $S_\alpha, \alpha \in Y$, without specific comments.

It is worth pointing out here that if $S = \mathcal{S}(Y; S_\alpha, \phi_{\alpha,\beta})$ is a strong semilattice of Y of semigroups $S_\alpha, \alpha \in Y$, then it must be a semilattice Y of semigroups $S_\alpha, \alpha \in Y$. However, the converse, in general, is not true.

Now we are in the position to give the well known decomposition theorem of bands in terms of semilattices of semigroups.

Lemma 1.5.2. [26] *Let B be band. Then $B = \bigcup_{\alpha \in Y} B_\alpha$ is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, and the B_α's are the $\mathcal{D} = \mathcal{J}$-classes of B.*

Lemma 1.5.3. [26] *Let B be a normal band. Then $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ is a strong semilattice Y of rectangular bands B_α, $\alpha \in Y$, and the B_α's are the $\mathcal{D} = \mathcal{J}$-classes of B.*

Chapter 2

Preliminaries II: (Weakly) abundant semigroups

The aim of this chapter is to introduce two sets of binary relations, as analogues of the well known Green's relations. Corresponding to these binary relations, the notion of a (weakly) abundant semigroup is introduced in a very natural way, as a generalization of the notion of a regular semigroup. More details related to the content of chapter can be found in [14], [15], [16] and [32].

2.1 Abundant semigroups

2.1.1 The relations $\mathcal{L}^*$ and $\mathcal{R}^*$

Let S be a semigroup and $E = E(S)$ the set of all idempotents of S. A binary relation $\mathcal{L}^*$ on S is defined by the rule that $(a, b) \in \mathcal{L}^*$ if and only if the elements a and b are related by Green's relation $\mathcal{L}$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually.

From the definitions of $\mathcal{L}^*$ and $\mathcal{R}^*$, we easily deduce that

$$\mathcal{L} \subseteq \mathcal{L}^* \text{ and } \mathcal{R} \subseteq \mathcal{R}^*.$$

Obviously, $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations on S; furthermore, $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence. If S is regular, then in fact $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$. We denote the join of $\mathcal{L}^*$ and R^* by $\mathcal{D}^*$, while their intersection is denoted by $\mathcal{H}^*$. Note that unlike the case of Green's relations, generally $\mathcal{L}^* \circ \mathcal{R}^* \neq$

$\mathcal{R}^* \circ \mathcal{L}^*$. A binary relation $\mathcal{J}^*$ may also be defined on S, which is not required here. We refer readers to [14] and [15] for further details.

The following lemma gives another characterization of $\mathcal{L}^*$, clearly the dual holds for $\mathcal{R}^*$.

Lemma 2.1.1. [14] *Let S be a semigroup with $a, b \in S$. Then the following conditions are equivalent:*

(i) $a\ \mathcal{L}^*\ b$;

(ii) for all $x, y \in S^1$, $ax = ay$ if and only if $bx = by$.

As an easy but useful consequence of the above lemma, we have the following results (the duals hold for $\mathcal{R}^*$).

Lemma 2.1.2. [15] *Let S be a semigroup with $a \in S$ and $e^2 = e \in E(S)$. Then the following statements are equivalent:*

(i) $a\ \mathcal{L}^*\ e$;

(ii) $ae = a$ and for any $x, y \in S^1$, $ax = ay$ implies $ex = ey$.

Lemma 2.1.3. *Let S be a semigroup with $e, f \in E(S)$. Then $e\ \mathcal{L}\ f$ if and only if $e\ \mathcal{L}^*\ f$ and $e\ \mathcal{R}\ f$ if and only if $e\ \mathcal{R}^*\ f$.*

2.1.2 Abundant semigroups

Recall that a semigroup S is regular if and only if each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains an idempotent of S. Based on the relations $\mathcal{L}^*$ and $\mathcal{R}^*$, in a rather natural way, we introduce the concept of an *abundant semigroup*

Definition 2.1.4. [15] *A semigroup S is said to be abundant if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of S contains an idempotent of S.*

We call an abundant semigroup S *adequate* if its idempotents form a semilattice.

In view of the comment above, regular semigroups are abundant while inverse semigroups are adequate. But not all abundant semigroups are regular, for instance, a cancellative monoid is clearly abundant, but does not have to be regular. In the theory of abundant semigroups the relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{H}^*$, $\mathcal{D}^*$ play a role which is analogous to that of Green's relations in the theory of regular semigroups (see, for example [14] and [15]).

2.2 Weakly abundant semigroups

2.2.1 The relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$

A third set of relations, extending the stared versions of Green's relations, and useful for semigroups that are not abundant, were introduced in [32].

Let S be a semigroup with $E = E(S)$. The relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ on S are defined by the rule

$$a \,\widetilde{\mathcal{L}}\, b \iff (\forall e \in E(S))\ (ae = a \Leftrightarrow be = b)$$

and

$$a \,\widetilde{\mathcal{R}}\, b \iff (\forall e \in E(S))\ (ea = a \Leftrightarrow eb = b)$$

for any $a, b \in S$.

It follows directly from the definitions that

$$\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}} \text{ and } \mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}.$$

Moreover, if $a, b \in S$ are regular, then

$$a \,\mathcal{L}\, b \Longleftrightarrow a \,\mathcal{L}^*\, b \Longleftrightarrow a \,\widetilde{\mathcal{L}}\, b$$

and a dual holds for $\mathcal{R}, \mathcal{R}^*$ and $\widetilde{\mathcal{R}}$. Therefore if S is regular, then $\mathcal{L} = \mathcal{L}^* = \widetilde{\mathcal{L}}$ and $\mathcal{R} = \mathcal{R}^* = \widetilde{\mathcal{R}}$. As one might expect, the relations $\widetilde{\mathcal{H}}$, $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{J}}$ are also defined on S, and for details we recommend [32]. Whereas $\mathcal{L}^*$ and $\mathcal{R}^*$ are always right and left congruences on S, respectively, it is not necessarily true for $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$. In the following section, we will give an easy example to illustrate this.

The following result follows immediately from the definition of $\widetilde{\mathcal{L}}$. Of course, a dual result holds for $\widetilde{\mathcal{R}}$.

Lemma 2.2.1. [32] *Let S be a semigroup with $a \in S$ and $e^2 = e \in E(S)$. Then the following statements are equivalent:*

(i) $a \,\widetilde{\mathcal{L}}\, e$;

(ii) $ae = a$ and for any $f \in E(S)$, $af = a$ implies $ef = e$.

Lemma 2.2.2. *Let S be a semigroup with $e, f \in E(S)$. Then $e \,\mathcal{L}\, f$ if and only if $e \,\widetilde{\mathcal{L}}\, f$ and $e \,\mathcal{R}\, f$ if and only if $e \,\widetilde{\mathcal{R}}\, f$.*

2.2.2 Weakly abundant semigroups

As a generalisation of an abundant semigroup, we have the notion of a weakly abundant semigroup in terms of the relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$.

Definition 2.2.3. *A semigroup S is weakly abundant if each $\widetilde{\mathcal{L}}$-class and each $\widetilde{\mathcal{R}}$-class contains an idempotent.*

We say that a weakly abundant semigroup S satisfies the *congruence condition* if $\widetilde{\mathcal{L}}$ is a right congruence on S and $\widetilde{\mathcal{R}}$ is a left congruence on S.

We consider the following non-abundant but weakly abundant semigroup S for which we do not have the congruence condition.

Example 2.2.4. Let $S = \{e, 0, h, a, b\}$ be a semigroup with a multiplication given by the following table:

	e	h	a	b	0
e	e	h	a	b	0
h	h	h	0	0	0
a	a	0	0	0	0
b	b	0	0	a	0
0	0	0	0	0	0

Figure 2.1: a counterexample for the congruence condition

First, it is easy to check that the multiplication defined as above is associative, so that S forms a semigroup. Notice that

$$\mathcal{L}^* = \mathcal{R}^* = I_S.$$

Since $E(S) = \{e, 0, h\}$ and so not every $\mathcal{L}^*$-class of S contains an idempotent, we deduce that S is not abundant. On the other hand, we have

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{R}} = \{\{e, a, b\}, \{0\}, \{h\}\},$$

so that S is weakly abundant. However, S does not satisfy the congruence condition, since $e \,\widetilde{\mathcal{R}}\, a$ but he is not $\widetilde{\mathcal{R}}$-related to ha.

Lemma 2.2.5. *Let S be a semigroup, and let $a \in S$, $f^2 = f \in E(S)$ be such that $a \,\widetilde{\mathcal{R}}\, f$. Then a is not $\mathcal{R}^*$-related to f implies that a is not $\mathcal{R}^*$-related to any idempotent of S.*

Proof. Suppose that $a \mathcal{R}^* e$ for some idempotent $e \in E(S)$. Then $a \widetilde{\mathcal{R}} e$, as $\mathcal{R}^* \subseteq \widetilde{\mathcal{R}}$, so that $e \widetilde{\mathcal{R}} f$ by assumption, and so $e \mathcal{R} f$ by Lemma 2.2.2. Hence $a \mathcal{R}^* f$ as $\mathcal{R} \subseteq \mathcal{R}^*$, a contradiction. $\square$

Lemma 2.2.6. *Let S be a weakly abundant semigroup with $a \in S$ and $e^2 = e \in E(S)$ such that $a \widetilde{\mathcal{R}} e$. Then $a \mathcal{R}^* e$ if and only if for any $x, y \in S$, $xa = ya$ implies that $xe = ye$.*

Proof. Suppose that for all $x, y \in S$, if $xa = ya$ then $xe = ye$. By Lemma 2.1.2, we need only show that if $x \in S$ and $xa = a$, then $xe = e$. Suppose therefore that $x \in S$ and $xa = a$. As $a \widetilde{\mathcal{R}} e$, we have $xa = a = ea$, so that by assumption, $xe = ee = e$. $\square$

Chapter 3

Preliminaries III: Independence algebras and their endomorphism monoids

In this chapter, we introduce a kind of universal algebra, called an independence algebra. Our main focus is to study the endomorphism monoid of an independence algebra.

To do this, we start with two familiar kinds of independence algebras, namely, sets and vector spaces; and recall some common properties of the endomorphism monoids of them, i.e. full transformation monoids and full linear monoids. After this, we will formally give the definition of independence algebras, and the properties of the endomorphism monoids of independence algebras. Finally, we proceed with another class of independence algebras, namely, free G-acts over a group G, which are the main algebraic object we are working with in Chapters 5 and 6.

The following notational convention will be useful: for any $u, v \in \mathbb{N}$ with $u \leq v$ we will denote $\{u, u+1, \cdots, v-1, v\}$ and $\{u+1, \cdots, v-1\}$ by $[u, v]$ and (u, v), respectively.

We recommend [18], [30], [17], [22], [12] and [19] as references for Chapter 3.

3.1 Full transformation monoids

Let X be a non-empty set. The *full transformation monoid* on X, denoted by $\mathcal{T}(X)$, is defined to consist of all mappings from X into itself, with multiplication being composition of mappings, i.e. for any $\alpha, \beta \in \mathcal{T}(X)$, $x(\alpha\beta) = (x\alpha)\beta$, for all $x \in X$.

Note that the identity of $\mathcal{T}(X)$ is the identity mapping on X. We use $\mathcal{S}(X)$ to denote the *symmetric group* on X, which is the group of of all bijections from X into itself. We define the *rank* of an element α of $\mathcal{T}(X)$ to be the cardinality of $\operatorname{im}\alpha$. Then we have the following lemma.

Lemma 3.1.1. [26] *The full transformation monoid $\mathcal{T}(X)$ on X is a regular semigroup such that for all $\alpha, \beta \in \mathcal{T}(X)$, we have:*
(i) $\alpha\ \mathcal{L}\ \beta$ if and only if $\operatorname{im}\alpha = \operatorname{im}\beta$;
(ii) $\alpha\ \mathcal{R}\ \beta$ if and only if $\ker\alpha = \ker\beta$;
(iii) $\alpha\ \mathcal{D}\ \beta$ if and only of $\operatorname{rank}\alpha = \operatorname{rank}\beta$;
(iv) $\mathcal{D} = \mathcal{J}$.

If X is a set with n elements, then it will be convenient to write

$$X = \{1, 2, \cdots, n\}, \mathcal{T}(X) = \mathcal{T}_n, \mathcal{S}(X) = \mathcal{S}_r$$

and I_n to be the identity mapping of $\mathcal{T}_n$. We remark here that the maximal subgroup of $\mathcal{T}_n$ containing a rank r idempotent e, $1 \leq r \leq n$, is isomorphic to the symmetric group $\mathcal{S}_r$.

Let S be a semigroup with set $E = E(S)$ of idempotents, and let $\langle E\rangle$ denote the subsemigroup of S generated by E. We say that S is an *idempotent generated* semigroup if $S = \langle E\rangle$. This kind of semigroup plays an important role in semigroup theory and it occurs in many natural areas of mathematics. We remark here that the *singular subsemigroup* of $\mathcal{T}_n$ is such a one, by which we mean the subsemigroup of $\mathcal{T}_n$ consisting of all non-bijective mappings from the set $\{1, \cdots, n\}$ into itself.

Let $E = E(\mathcal{T}_n)$ be the set of idempotents of $\mathcal{T}_n$. In all what follows, it is convenient to exclude from consideration the identity mapping I_n of $\mathcal{T}_n$. Now we consider which elements of $\mathcal{T}_n$ form the idempotent generated subsemigroup $\langle E\rangle$ of $\mathcal{T}_n$.

Clearly, there are no elements in $\mathcal{S}_n$ (other than I_n) that can be expressed as a product of idempotents, so that $\langle E\rangle \subseteq \mathcal{T}_n \backslash \mathcal{S}_n$. In fact, the converse is also true.

Theorem 3.1.2. [28] *Let $\mathcal{T}_n$ be the full transformation monoid on n elements. Then the subsemigroup of $\mathcal{T}_n$ generated by its non-identity idempotents is $\mathcal{T}_n \backslash \mathcal{S}_n$. In fact, every elements of $\mathcal{T}_n \backslash \mathcal{S}_n$ is a product of idempotents with rank $n-1$.*

Theorem 3.1.3. [28] *Any (finite) semigroup can be embedded into a (finite) regular idempotent generated semigroup.*

3.2 Full linear monoids

Let $\mathbf{V}$ be a vector space over a field, and let $\operatorname{End}\mathbf{V}$ be the monoid of all linear maps $\alpha : \mathbf{V} \longrightarrow \mathbf{V}$ with multiplication being composition of mappings. Then we say that $\operatorname{End}\mathbf{V}$ is the *full linear monoid* of $\mathbf{V}$.

Let $\ker\alpha = \{v \in V : \ v\alpha = 0\}$ and $\operatorname{rank}\alpha$ be the dimension of the subspace $\operatorname{im}\alpha$ of $\mathbf{V}$. Then we have the following lemma.

Lemma 3.2.1. [26] *The full linear monoid* $\operatorname{End}\mathbf{V}$ *of* $\mathbf{V}$ *is a regular semigroup such that for all* $\alpha, \beta \in \operatorname{End}\mathbf{V}$, *we have:*

(i) $\alpha\ \mathcal{L}\ \beta$ *if and only of* $\operatorname{im}\alpha = \operatorname{im}\beta$;

(ii) $\alpha\ \mathcal{R}\ \beta$ *if and only* $\ker\alpha = \ker\beta$;

(iii) $\alpha\ \mathcal{D}\ \beta$ *if and only if* $\operatorname{rank}\alpha = \operatorname{rank}\beta$;

(iv) $\mathcal{D} = \mathcal{J}$.

Let $\mathbf{V}$ be an n dimensional vector space over a field F, where n is finite. Then it is well known that the full linear monoid $\operatorname{End}\mathbf{V}$ is isomorphic to the matrix monoid $M_n(F)$ of all $n \times n$ matrices over F. Moreover, we have the following result.

Lemma 3.2.2. [12] *Let $M_n(F)$ be the matrix monoid of all $n \times n$ matrices over a field F. Then the subsemigroup of $M_n(F)$ generated by its non-identity idempotents is $M_n(F)\backslash\mathcal{I}(M_n(F))$, where $\mathcal{I}(M_n(F))$ is the set of all $n\times n$ non-singular matrices of $M_n(F)$. In fact, every element of $M_n(F)\backslash\mathcal{I}(M_n(F))$ is a product of idempotents with rank $n-1$.*

We remark here that for any idempotent $e \in M_n(F)$ with $\operatorname{rank} e = r$, $1 \leq r \leq n$, the maximal subgroup with identity e, i.e. the $\mathcal{H}$-class of e in $M_n(F)$, is isomorphic to the r dimensional general linear group $GL_r(F)$, consisting of all $r\times r$ non-singular matrices over F. Note that if e is a rank 1 identity, then the maximal subgroup of $M_n(F)$ with identity e is therefore isomorphic to the multiplicative subgroup of F, i.e. the group of all units of F.

An alternative proof of [12] was given by Dawlings [9], and the result was generalized to finite dimensional vector spaces over division rings by Laffey [33].

3.3 Independence algebras

3.3.1 Basic definitions

'What then do vector spaces and sets have in common which forces End $\mathbf{V}$ and $\mathcal{T}(X)$ to support a similar pleasing structure'?

The above question was asked by Gould [22]. To answer it, she investigated the endomorphism monoid of a class of universal algebra, called an independence algebra (also known as a v^*-algebra), including sets, vector spaces, etc. For basic ideas of universal algebras we refer the reader to [36], [4] and [19].

For any $a_1, \cdots, a_n \in A$, a term operation built from these elements may be written as $t(a_1, \cdots, a_n)$. For any subset $X \subseteq A$, we use $\langle X \rangle$ to denote the universe of the subalgebra generated by X, consisting of all $t(x_1, \cdots, x_m)$, where $m \in \mathbb{N}^0$, $x_1, \cdots, x_m \in X$, and t is an m-ary term operation. A *constant* in an algebra $\mathbf{A}$ is the image of a basic nullary operation; an *algebraic constant* is the image of a nullary term operation. If $\mathbf{A}$ has any constants, then $\langle \emptyset \rangle$ denotes the subalgebra generated by them. Of course, $\langle \emptyset \rangle = \emptyset$ if and only if $\mathbf{A}$ has no algebraic constants, if and only if $\mathbf{A}$ has no constants.

We say that an algebra $\mathbf{A}$ satisfies the *exchange property* (EP), if for every subset X of A and all elements $x, y \in A$:

$$y \in \langle X \cup \{x\} \rangle \text{ and } y \notin \langle X \rangle \text{ implies } x \in \langle X \cup \{y\} \rangle.$$

A subset X of A is called *independent* if for each $x \in X$ we have $x \notin \langle X \backslash \{x\} \rangle$. We say that a subset X of A is a *basis* of $\mathbf{A}$ if X generates $\mathbf{A}$ and is independent.

Note that any algebra satisfying the exchange property has a basis, and in such an algebra a subset X is a basis if and only if X is a minimal generating set if and only if X is the maximal independent set. All bases of $\mathbf{A}$ have the same cardinality, called the *rank* of $\mathbf{A}$. Further, any independent subset X can be extended to be a basis of $\mathbf{A}$.

We say that a mapping θ from $\mathbf{A}$ into itself is an *endomorphism* if for any n-ary term operation $t(x_1, \cdots, x_n)$ we have

$$t(x_1, \cdots, x_n)\theta = t(x_1\theta, \cdots, x_n\theta);$$

if θ is bijective, then we call it an *automorphism.*

An algebra $\mathbf{A}$ satisfying the exchange property is called an *independence algebra* if it satisfies the *free basis property*, by which we mean that any map from a basis of $\mathbf{A}$ to $\mathbf{A}$ can be extended to an endomorphism of $\mathbf{A}$.

3.3.2 Endomorphism monoids of independence algebras

Let $\mathbf{A}$ be an independence algebra. Let $\operatorname{End}\mathbf{A}$ the *endomorphism monoid* of $\mathbf{A}$ and $\operatorname{Aut}\mathbf{A}$ the *automorphism group* of $\mathbf{A}$. We define the *rank* of an element $\alpha \in \operatorname{End}\mathbf{A}$ to be the rank of the subalgebra $\operatorname{im}\alpha$ of $\mathbf{A}$.

As a generalisation of Lemmas 3.1.1 and 3.2.1, we have the following result regarding $\operatorname{End}\mathbf{A}$.

Lemma 3.3.1. [22] *Let* $\mathbf{A}$ *be an independence algebra. Then* $\operatorname{End}\mathbf{A}$ *is a regular semigroup, and for any* $\alpha, \beta \in \operatorname{End}\mathbf{A}$, *the following statements are true:*

(i) $\alpha \,\mathcal{L}\, \beta$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$;

(ii) $\alpha \,\mathcal{R}\, \beta$ *if and only if* $\ker\alpha = \ker\beta$;

(iii) $\alpha \,\mathcal{D}\, \beta$ *if and only* $\operatorname{rank}\alpha = \operatorname{rank}\beta$;

(iv) $\mathcal{D} = \mathcal{J}$.

Let D_r be the $\mathcal{D}$-class of an arbitrary rank r element in $\operatorname{End}\mathbf{A}$. Then by Lemma 3.3.1, we have

$$D_r = \{\alpha \in \operatorname{End}\mathbf{A} : \operatorname{rank}\alpha = r\}.$$

Put $D_r^0 = D_r \cup \{0\}$ and define a multiplication on D_r^0 by:

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } \alpha, \beta \in D_r \text{ and } \operatorname{rank}\alpha\beta = r \\ 0 & \text{else} \end{cases}$$

Then according to [22], we have the following result.

Lemma 3.3.2. [22] *Under the above multiplication* $\cdot$ *given as above,* D_r^0 *is a completely 0-simple semigroup.*

It follows immediately from Rees Theorem that D_r^0 is isomorphic to some Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$. We remark here that if $\mathbf{A}$ has no constants, then

$$D_1 = \{\alpha \in \operatorname{End}\mathbf{A} : \operatorname{rank}\alpha = 1\}$$

forms a completely simple semigroup under the multiplication defined in End $\mathbf{A}$, so that D_1 is isomorphic to some Rees matrix semigroup $\mathcal{M}(G; I, \Lambda; P)$.

Given the results we obtained in Lemmas 3.1.2 and 3.2.2, we have the following generalisation for End $\mathbf{A}$.

Lemma 3.3.3. [17] *Let* $\mathbf{A}$ *be an independence algebra of rank* $n \in \mathbb{N}$. *Let* E *denote the non-identity idempotents of* End $\mathbf{A}$. *Then*

$$\langle E \rangle = \langle E_1 \rangle = \operatorname{End}\mathbf{A} \setminus \operatorname{Aut}\mathbf{A}$$

where E_1 *is the set of idempotents of rank* $n-1$ *in* End $\mathbf{A}$.

3.3.3 A classification of independence algebras with no constants

In this section, we recall part of the classification of independence algebras given by Urbanik in [43]. Note that in [43], an *algebraic constant* of an algebra is defined as the image of a constant term operation of $\mathbf{A}$, which is different with the one we introduced in Section 3.3.1. However, the following lemma illustrates that for independence algebras, these two notions coincide.

Proposition 3.3.4. *For any independence algebra* $\mathbf{A}$ *with* $|A| > 1$, *we have* $\langle \emptyset \rangle = C$, *where* C *is the collection of all elements* $u \in A$ *such that there is a constant term operation* $t(x_1, \cdots, x_n)$ *of* A *whose image is* u.

Proof. First, clearly we have $\langle \emptyset \rangle \subseteq C$. Suppose now that $|A| \neq 1$ and $A \neq \langle \emptyset \rangle$. Let $a \in C$. Then by the definition there exists a constant term operation $t(x_1, \cdots, x_n)$ with

$$\operatorname{im} t(x_1, \cdots, x_n) = \{a\}.$$

Here we put $s(x) = t(x, \cdots, x)$ and pick a fixed $x \in A \backslash \langle \emptyset \rangle$. Then

$$s(x) = t(x, \cdots, x) = a.$$

Suppose that $a \notin \langle \emptyset \rangle$. Clearly, $a \in \langle x \rangle$, so by the exchange property (EP) of $\mathbf{A}$, we have $x \in \langle a \rangle$, so that $x = u(a)$ for some unary term operation u of $\mathbf{A}$. Note that $a = s(a)$, so $x = u(s(a))$. As $\{x\}$ is independent, it can be extended to be a basis X of $\mathbf{A}$. Now we choose an arbitrary $b \in A$, and define an arbitrary mapping

$\theta : X \longrightarrow \mathbf{A}$ such that $x\theta = b$. Then by the free basis property of $\mathbf{A}$, θ can be extended to be an endomorphism $\overline{\theta}$ of $\mathbf{A}$ such that $x\overline{\theta} = b$. Then

$$b = x\overline{\theta} = u(s(a))\overline{\theta} = u(s(a\overline{\theta})) = u(a) = x.$$

As b is an arbitrary fixed element in A , we have $|A| = 1$, contradicting our assumption, so that $a \in \langle\emptyset\rangle$, and so $\langle\emptyset\rangle = C$ as required. □

The above result is also true for a larger class of universal algebra, called *a basis algebra*, which includes independence algebras. We refer the reader to [18] for details.

We are concerned with independence algebras with no constants in Chapter 7, so here we give the classification of independence algebras only in the case we have no constants. For the complete result we refer the reader to [44].

Theorem 3.3.5. [44] *Let* $\mathbf{A}$ *be an independence algebra of* rank n *with no constants, where* $n \geq 3$. *Then one of the following holds:*

(i) There exists a permutation group $\mathcal{G}$ *of the set* A *such that the class of all term operations of* A *is the class of all functions given by the following formula*

$$t(x_1, \cdots, x_m) = g(x_j), \ \ (m \in \mathbb{N}, 1 \leq j \leq m).$$

where $g \in \mathcal{G}$.

(ii) $\mathbf{A}$ *is an affine algebra, namely, there is a field* F *such that* $\mathbf{A}$ *is a vector space over* F *and further, there exists a linear subspace* $\mathbf{A}_0$ *of* $\mathbf{A}$ *such that the class of all term operations of* $\mathbf{A}$ *is the class of all functions defined as*

$$t(x_1, \cdots, x_n) = k_1x_1 + \cdots + k_nx_n + a$$

where $k_1, \cdots, k_n \in F$ *with* $k_1 + \cdots + k_n = 1 \in F$, $a \in \mathbf{A}_0$ *and* $n \geq 1$.

3.4 Free (left) G-acts and their endomorphism monoids

Let G be a group, $n \in \mathbb{N}, n \geq 3$, and let $F_n(G) = \bigcup_{i=1}^{n} Gx_i$ be a rank n free left G-act. We recall that, as a set, $F_n(G)$ consists of the set of formal symbols

$\{gx_i : g \in G, i \in [1,n]\}$, and we identify x_i with $1x_i$, where 1 is the identity of G. For any $g, h \in G$ and $1 \leq i, j \leq n$ we have that $gx_i = hx_j$ if and only if $g = h$ and $i = j$; the action of G is given by $g(hx_i) = (gh)x_i$. Let End $F_n(G)$ denote the endomorphism monoid of $F_n(G)$, i.e. the monoid of all endomorphisms of $F_n(G)$ under the binary operation given as composition of maps. The image of $\alpha \in \operatorname{End} F_n(G)$ being a (free) G-subact, we define the *rank* of α to be the rank of $\operatorname{im}\alpha$, namely, the minimal number of (free) generators in $\operatorname{im}\alpha$.

It was pointed out in [22] that free G-acts provide us with new examples of independence algebras. Hence, a direct application of Lemma 3.3.1 gives the characterisation of Green's relations on End $F_n(G)$.

Lemma 3.4.1. [22] *The endomorphism monoid* $\operatorname{End} F_n(G)$ *is a regular semigroup such that for all* $\alpha, \beta \in \operatorname{End} F_n(G)$*, we have:*

(i) $\alpha \ \mathcal{L} \ \beta$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$;

(ii) $\alpha \ \mathcal{R} \ \beta$ *if and only* $\ker\alpha = \ker\beta$;

(iii) $\alpha \ \mathcal{D} \ \beta$ *if and only if* $\operatorname{rank}\alpha = \operatorname{rank}\beta$;

(iv) $\mathcal{D} = \mathcal{J}$.

Next we aim to show that $\operatorname{End} F_n(G)$ is isomorphic to the wreath product $G \wr \mathcal{T}_n$ of G and $\mathcal{T}_n$. Recall from [30] that the *wreath product* $G \wr \mathcal{T}_n$ is defined to be a set

$$G^n \times \mathcal{T}_n = \{((g_1, \cdots, g_n), \overline{\alpha}) : (g_1, \cdots, g_n) \in G^n, \overline{\alpha} \in \mathcal{T}_n\}$$

with a multiplication given by

$$((g_1, \cdots, g_n), \overline{\alpha})((h_1, \cdots, h_n), \overline{\beta}) = ((g_1 h_{1\overline{\alpha}}, \cdots, g_n h_{n\overline{\alpha}}), \overline{\alpha}\ \overline{\beta}).$$

We know that each $\alpha \in \operatorname{End} F_n(G)$ depends only on its action on the free generators $\{x_i : i \in [1,n]\}$ and it is therefore convenient to write

$$x_j\alpha = w_j^\alpha x_{j\overline{\alpha}}$$

for $j \in [1,n]$. This determines a function $\overline{\alpha} : [1,n] \longrightarrow [1,n]$ and an element $\alpha_G = (w_1^\alpha, \ldots, w_n^\alpha) \in G^n$. It will frequently be convenient to express α as above as

$$\alpha = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ w_1^\alpha x_{1\overline{\alpha}} & w_2^\alpha x_{2\overline{\alpha}} & \ldots & w_n^\alpha x_{n\overline{\alpha}} \end{pmatrix}.$$

Theorem 3.4.2. [30] *The endomorphism monoid* $\operatorname{End} F_n(G)$ *of a free left* G*-act* $F_n(G)$ *of rank* n *is isomorphic to the wreath product* $G \wr \mathcal{T}_n$*, where the isomorphism* ψ *is defined by*

$$\psi : \operatorname{End} F_n(G) \longrightarrow G \wr \mathcal{T}_n, \ \alpha \mapsto (\alpha_G, \overline{\alpha}).$$

Lemma 3.4.3. [30] *The maximal subgroup of a rank* r *idempotent of* $\operatorname{End} F_n(G)$*, where* $1 \leq r \leq n$*, is isomorphic to the wreath product* $G \wr \mathcal{S}_r$*.*

We refer the reader to [30] for further details of the wreath product of monoids.

Chapter 4

Preliminaries IV: Free idempotent generated semigroups

The study of the free idempotent generated semigroup $\mathrm{IG}(E)$ over a biordered set E began with the seminal work [37] of Nambooripad in the 1970s and has seen a recent revival with a number of new approaches, both geometric and combinatorial. In this chapter, we will recall the basic definition of $\mathrm{IG}(E)$ and some pleasant properties, particularly with respect to Green's relations, and the structure of maximal subgroups of $\mathrm{IG}(E)$.

4.1 Basic definitions and properties

Let S be a semigroup and denote by $\langle E \rangle$ the subsemigroup of S generated by the set of idempotents $E = E(S)$ of S. Recall that S is *idempotent generated* if $S = \langle E \rangle$.

It follows immediately from the definition that for any pair $(e, f) \in E \times E$

$$\begin{array}{rcl} (e,f) \text{ is basic} & \Longleftrightarrow & \{e,f\} \cap \{ef, fe\} \neq \varnothing \\ & \Longleftrightarrow & e \leq_{\mathcal{R}} f \text{ or } f \leq_{\mathcal{R}} e \text{ or } e \leq_{\mathcal{L}} f \text{ or } f \leq_{\mathcal{L}} e \end{array}.$$

Further, $(e, f) \in E \times E$ is a basic pair implies that both ef and fe are idempotents of E.

Among the category of all idempotent generated semigroups whose biordered sets of idempotents are isomorphic to a given biordered set E, there is a free object called the *free idempotent generated semigroup* over E, given by the following presentation:

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\, e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \varnothing \rangle,$$

where $\overline{E} = \{\bar{e} : e \in E\}$.[1]

It is important to understand $\mathrm{IG}(E)$ if one is interested in understanding an arbitrary idempotent generated semigroup S with a biordered set $E = E(S)$ of idempotents. We begin with emphasising some pleasant properties of $\mathrm{IG}(E)$, particularly with respect to Green's relations, listed as (IG1)-(IG4) in the following:

(IG1) The natural map $\phi : \mathrm{IG}(E) \to S$, given by $\bar{e}\phi = e$, is a morphism onto $S' = \langle E \rangle$.

(IG2) The restriction of ϕ to the set of idempotents of $\mathrm{IG}(E)$ is a bijection onto E (and an isomorphism of biordered sets).

(IG3) The morphism ϕ induces a bijection between the set of all $\mathcal{R}$-classes (respectively $\mathcal{L}$-classes) in the $\mathcal{D}$-class of $\bar{e}$ in $\mathrm{IG}(E)$ and the corresponding sets in $\langle E \rangle$.

We pause here to remark that $\mathrm{IG}(E)$ can have non-regular $\mathcal{D}$-classes, even if E is a semilattice.

(IG4) The restriction of ϕ to $H_{\bar{e}}$ is a morphism onto H_e.

We clarify particularly that the property (IG1) follows directly from the definition of $\mathrm{IG}(E)$; (IG2) is proved in [37] and [10]; (IG3) is a corollary of [13]; (IG4) follows from (IG2).

Now let S be a regular semigroup. Then $E = E(S)$ is a regular biordered set, i.e. for any $(e, f) \in E \times E$, the sandwich set $S(e, f) \neq \varnothing$. The *free regular idempotent generated semigroup* over E, denoted by $\mathrm{RIG}(E)$, is defined to be the homomorphic image of $\mathrm{IG}(E)$ obtained by adding the relations

$$\overline{e}\ \overline{h}\ \overline{f} = \overline{e}\ \overline{f} \text{ whenever } h \in S(e, f).$$

There is a natural morphism from $\mathrm{IG}(E) \longrightarrow \mathrm{RIG}(E)$. Furthermore, $\mathrm{RIG}(E)$ has the following additional properties:

(RIG1) $\mathrm{RIG}(E)$ is a regular semigroup.

(RIG2) The natural morphism from $\mathrm{IG}(E)$ to $\mathrm{RIG}(E)$ induces an isomorphism

[1] It is more usual to identify elements of E with those of $\overline{E}$, but it helps the clarity of our later arguments to make this distinction.

between maximal subgroups of $\mathrm{IG}(E)$ and $\mathrm{RIG}(E)$ containing $\overline{e}$, where $\overline{e} \in \overline{E} = E$.

We remark here that the property (RIG1) is proved in [37], and (RIG2) follows from [1]. Obviously, $\mathrm{RIG}(E)$ satisfies properties (IG1)-(IG4).

4.2 Maximal subgroups of $\mathrm{IG}(E)$

4.2.1 A longstanding conjecture but a negative outcome

Given the universal nature of $\mathrm{IG}(E)$, it is natural to investigate its structure. A popular theme is to investigate the maximal subgroups of $\mathrm{IG}(E)$. A longstanding conjecture (which nevertheless seems not to have appeared formally until [35]), purported that all maximal subgroups of $\mathrm{IG}(E)$ were free. Several papers [35], [38], [41] and [40] established various sufficient conditions guaranteeing that all maximal subgroups are free.

For instance, in [40], Pastijn looked at the biordered set of idempotents of an arbitrary completely 0-simple semigroup. By employing Rees matrix semigroups, he has proved that:

Result 1 Let E be the biordered set of idempotents of a completely 0-simple semigroup S. Then the maximal subgroups of the non-zero $\mathcal{D}$-class of $\mathrm{IG}(E)$ are free groups and in the case E is finite, the maximal subgroup here has rank $(l_H - 1)(r_H - 1)$, where l_H and r_H are the numbers of $\mathcal{L}$-classes and $\mathcal{R}$-classes respectively of the $\mathcal{D}$-class D where H lies.

Also, in [35], McElwee generalized Pastijn's observation to a *locally trivial* biordered set E, by which we mean that all its principal ideals are singletons. It was pointed out in [35] that in a locally trivial biordered set E, $ef = e$ if and only if $fe = f$; and $fe = e$ if and only if $ef = f$. By showing that all maximal subgroups of $\mathrm{IG}(E)$ are fundamental groups of graphs, McElwee obtained the structure of $\mathrm{IG}(E)$ over a locally trivial biordered set E.

Result 2 Let E be a locally trivial biordered set. Then all the maximal subgroups of $\mathrm{IG}(E)$ are free. In the case where E is finite, each maximal subgroup H has rank $(l_H - 1)(r_H - 1) - d_H$, where l_H and r_H are the numbers of $\mathcal{L}$-classes and $\mathcal{R}$-classes of the $\mathcal{D}$-class D where H lies in, respectively; d_H is the number of non-group $\mathcal{H}$-classes of D.

Although this conjecture had been believed for more than 30 years, it was disproved by Brittenham, Margolis and Meakin [1] in 2009. In their paper they construct a special 72-element semigroup S, then by applying topological methods, they showed that $\mathrm{IG}(E)$ built over the biordered set E of idempotents of S has a free abelian group of rank 2, i.e. $\mathbb{Z} \oplus \mathbb{Z}$, as a maximal subgroup. Furthermore, [1] exhibited a strong relationship between maximal subgroups of $\mathrm{IG}(E)$ and algebraic topology: namely, it was shown that these groups are precisely the fundamental groups of a complex naturally arising from S (called the *Graham-Houghton* complex of S). An unpublished example of McElwee from the earlier part of 1970s was announced by Easdown in 2011 [11].

4.2.2 Singular squares of $\mathcal{D}$-classes

Before we proceed to Section 4.2.3, which explains a presentation of maximal subgroups of $\mathrm{IG}(E)$, we need the notion of singular squares.

Let E be a biordered set; from [10] we can assume that $E = E(S)$ for some semigroup S. An *E-square* is a sequence (e, f, g, h, e) of elements of E with

$$e \,\mathcal{R}\, f \,\mathcal{L}\, g \,\mathcal{R}\, h \,\mathcal{L}\, e.$$

We draw such an E-square as $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$. We call an E-square with one of the following forms a *trivial* E-square:

$$\begin{bmatrix} e & e \\ e & e \end{bmatrix} \text{ or } \begin{bmatrix} e & f \\ e & f \end{bmatrix} \text{ or } \begin{bmatrix} e & e \\ f & f \end{bmatrix}.$$

Lemma 4.2.1. *The elements of an E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ form a rectangular band (within S) if and only if one (equivalently, all) of the following four equalities holds:*

$$eg = f, ge = h, fh = e \text{ or } hf = g.$$

Proof. The necessity is clear. To prove the sufficiency, without loss of generality, suppose that the equality $eg = f$ holds. We need to prove

$$ge = h, fh = e, \text{ and } hf = g.$$

Notice that

$$gege = gfe = ge$$

so that ge is idempotent. But, as $f \in L_g \cap R_e$, we have $ge \in R_g \cap L_e$ by Lemma 1.2.6, which implies $ge = h$. Furthermore,

$$fh = fge = fe = e \text{ and } hf = heg = hg = g$$

and so $\{e, f, g, h\}$ is a rectangular band. □

We will be interested in rectangular bands in completely simple semigroups. The following lemma makes explicit ideas used implicitly elsewhere.

Lemma 4.2.2. *Let $\mathcal{M}^0 = \mathcal{M}^0(G; I, \Lambda; P)$ be a Rees matrix semigroup over a group G with sandwich matrix $P = (p_{\lambda i})$. For any $i \in I, \lambda \in \Lambda$ write $e_{i\lambda}$ for the idempotent $(i, p_{\lambda i}^{-1}, \lambda)$. Then an E-square $\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{j\lambda} & e_{j\mu} \end{bmatrix}$ is a rectangular band if and only if $p_{\lambda i}^{-1} p_{\lambda j} = p_{\mu i}^{-1} p_{\mu j}$.*

Proof. We have

$$\begin{aligned} e_{i\lambda} e_{j\mu} = e_{i\mu} &\Leftrightarrow (i, p_{\lambda i}^{-1}, \lambda)(j, p_{\mu j}^{-1}, \mu) = (i, p_{\mu i}^{-1}, \mu) \\ &\Leftrightarrow p_{\lambda i}^{-1} p_{\lambda j} p_{\mu j}^{-1} = p_{\mu i}^{-1} \\ &\Leftrightarrow p_{\lambda i}^{-1} p_{\lambda j} = p_{\mu i}^{-1} p_{\mu j}. \end{aligned}$$

The result now follows from Lemma 4.2.1. □

An E-square (e, f, g, h, e) is *singular* if, in addition, there exists $k \in E$ such that either:

$$\begin{cases} ek = e,\ fk = f,\ ke = h,\ kf = g \text{ or} \\ ke = e,\ kh = h,\ ek = f,\ hk = g. \end{cases}$$

We call a singular square for which the first condition holds an *up-down singular square*, and that satisfying the second condition a *left-right singular square*.

Note that all trivial E-squares are singular; for instance, if we have an E-square with the form $\begin{bmatrix} e & f \\ e & f \end{bmatrix}$, then we can take $k = f$, so that it is a left-right singular square; if we have an E-square with the form $\begin{bmatrix} e & e \\ f & f \end{bmatrix}$, then it is an up-down singular square by taking $k = e$.

Lemma 4.2.3. *If an E-square* $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ *is singular, then* $\{e, f, g, h\}$ *is a rectangular band.*

Proof. Suppose that $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$ is singular. If $k = k^2 \in E$ is such that

$$ek = e,\ fk = f,\ ke = h \text{ and } kf = g$$

then $eg = ekf = ef = f$. By Lemma 4.2.1, $\{e, f, g, h\}$ is a rectangular band. Dually for a left-right singular square. □

We remark here that the converse of Lemma 4.2.3 is not necessarily true; for this, we can consider a 4-element rectangular band E, say $\{e, f, g, h\}$, where $e \mathcal{R} f \mathcal{L} g \mathcal{R} h \mathcal{L} e$, obviously, there exists no idempotents of E to singularize our E-square $\begin{bmatrix} e & f \\ h & g \end{bmatrix}$.

4.2.3 A presentation of maximal subgroups of IG(E)

Motivated by the significant discovery in [1], Gray and Ruškuc [20] showed that *any* group occurs as a maximal subgroup of some IG(E). Their approach is to use existing machinery which affords presentations of maximal subgroups of semigroups, itself developed Ruškuc, and refine this to give presentations of IG(E), and then, given a group G, to carefully choose a biordered set E. Their techniques are significant and powerful, and have other consequences. We also remark here that the presentation obtained in [20] generalizes the corresponding result for regular semigroups proved by Nambooripad [37].

Let S be a semigroup with $E = E(S)$, let IG(E) be the corresponding free idempotent generated semigroup. For $e \in E$ we let $\overline{H}$ be the maximal subgroup of $\overline{e}$ in IG(E), (that is, $\overline{H} = H_{\overline{e}}$). We now recall the recipe for obtaining a presentation for $\overline{H}$ obtained by Gray and Ruškuc [20]; for further details, we refer the reader to that article.

We use J and Γ to denote the set of $\mathcal{R}$-classes and the set of $\mathcal{L}$-classes, respectively, in the $\mathcal{D}$-class $\overline{D} = D_{\overline{e}}$ of $\overline{e}$ in IG(E). In view of properties (IG1)-(IG4), J and Γ also label the set of $\mathcal{R}$-classes and the set of $\mathcal{L}$-classes, respectively, in the $\mathcal{D}$-class $D = D_e$ of e in S. For every $i \in J$ and $\lambda \in \Gamma$, let $\overline{H}_{i\lambda}$ and $H_{i\lambda}$ denote, respectively, the $\mathcal{H}$-class corresponding to the intersection of the $\mathcal{R}$-class indexed by i and the $\mathcal{L}$-class indexed by λ in $\mathrm{IG}(E)$, respectively S, so that $\overline{H}_{i\lambda}$ and $H_{i\lambda}$ are $\mathcal{H}$-classes of $\overline{D}$ and D, respectively. Where $\overline{H}_{i\lambda}$ (equivalently, $H_{i\lambda}$) contains an idempotent, we denote it by $\overline{e}_{i\lambda}$ (respectively, $e_{i\lambda}$). Without loss of generality we assume $1 \in J \cap \Gamma$ and $\overline{e} = \overline{e}_{11} \in \overline{H}_{11} = \overline{H}$, so that $e = e_{11} \in H_{11} = H$. For each $\lambda \in \Gamma$, we abbreviate $\overline{H}_{1\lambda}$ by $\overline{H}_\lambda$, and $H_{1\lambda}$ by H_λ and so, $\overline{H}_1 = \overline{H}$ and $H_1 = H$.

Let $\overline{h}_\lambda$ be an element in $\overline{E}^*$ such that $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$, for each $\lambda \in \Gamma$. The reader should be aware that this is a point where we are most certainly abusing notation: whereas $\overline{h}_\lambda$ lies in the free monoid on $\overline{E}$, by writing $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$ we mean that the image of $\overline{h}_\lambda$ under the natural map that takes $\overline{E}^*$ to (right translations in) the full transformation monoid on $\mathrm{IG}(E)$ yields $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$. In fact, it follows from (IG1)-(IG4) that the action of any generator $\overline{f} \in \overline{E}$ on an $\mathcal{H}$-class contained in the $\mathcal{R}$-class of $\overline{e}$ in $\mathrm{IG}(E)$ is equivalent to the action of f on the corresponding $\mathcal{H}$-class in the original semigroup S. Thus $\overline{H}_1\overline{h}_\lambda = \overline{H}_\lambda$ in $\mathrm{IG}(E)$ is equivalent to the corresponding statement $H_1h_\lambda = H_\lambda$ for S, where h_λ is the image of $\overline{h}_\lambda$ under the natural map to $\langle E\rangle^1$.

We say that $\{\overline{h}_\lambda \mid \lambda \in \Gamma\}$ forms a *Schreier system of representatives*, if every prefix of $\overline{h}_\lambda$ (including the empty word) is equal to some $\overline{h}_\mu$, where $\mu \in \Gamma$. Notice that the condition that $\overline{h}_\lambda\overline{e}_{i\mu} = \overline{h}_\mu$ is equivalent to saying that $\overline{h}_\lambda\overline{e}_{i\mu}$ lies in the Schreier system.

Define $K = \{(i,\lambda) \in J \times \Gamma : H_{i\lambda} \text{ is a group } \mathcal{H}\text{-class}\}$. Since D_e is regular, for each $i \in J$ we can find and fix an element $\omega(i) \in \Gamma$ such that $(i, \omega(i)) \in K$, so that $\omega : J \to \Gamma$ is a function. Again, for convenience, we take $\omega(1) = 1$.

Theorem 4.2.4. [20] *The maximal subgroup $\overline{H}$ of $\overline{e}$ in $\mathrm{IG}(E)$ is defined by the presentation*

$$\mathcal{P} = \langle F : \Sigma\rangle$$

with generators:

$$F = \{f_{i,\lambda} : \quad (i,\lambda) \in K\}$$

and defining relations Σ:

$(R1)$ $f_{i,\lambda} = f_{i,\mu} \quad (\overline{h}_\lambda\overline{e}_{i\mu} = \overline{h}_\mu)$;

$(R2)$ $f_{i,\omega(i)} = 1 \quad (i \in J)$;

$(R3)$ $f_{i,\lambda}^{-1}f_{i,\mu} = f_{k,\lambda}^{-1}f_{k,\mu} \quad \left(\begin{bmatrix} e_{i\lambda} & e_{i\mu} \\ e_{k\lambda} & e_{k\mu}\end{bmatrix} \text{ is a singular square}\right)$.

Note that if there are no non-trivial singular squares, then the maximal subgroup is free.

4.2.4 Every group arises as a maximal subgroup of some IG(E)

We have already mentioned in Section 4.2.3 that Gray and Ruškuc [20] gave a presentation of maximal subgroups of IG(E), and in the same paper, they constructed a special biordered set E, then by applying the presentation, they obtain the following significant results.

Theorem 4.2.5. [20]*Every group is a maximal subgroup of some free idempotent generated semigroup.*

Theorem 4.2.6. [20] *Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.*

Theorem 4.2.7. [20] *Every group is a maximal subgroup of some free regular idempotent generated semigroup.*

Theorem 4.2.8. [20] *Every finite group is a maximal subgroup of some free regular idempotent generated semigroup arising from a finite regular semigroup.*

The article [20] left open the question of whether every finitely presented group is a maximal subgroup of some free regular idempotent generated semigroup arising from a finite semigroup.

In 2013, Dolinka and Ruškuc [8] gave a positive answer to the above question, and an alternative proof of the result that every group can occur as a maximal subgroup of IG(E), where E may be taken to be a band with respect to an arbitrary fixed group.

The question remained of whether a group G occurs as a maximal subgroup of some IG(E) for a 'naturally occurring' E. The following chapter will answer this question in a positive way, and unlike the proofs in [20], those in Gould and Yang [23] involve a natural biordered set and very little machinery. The approach of [23] is to consider the biordered set E of non-identity idempotents of a wreath product $G \wr \mathcal{T}_n$. We refer the reader to Chapter 5 for details of this.

4.2.5 The results for $\mathcal{T}_n$, $\mathcal{PT}_n$ and $M_n(D)$

Following the discovery that every group can occur as a maximal subgroup of $\mathrm{IG}(E)$, another question comes out very naturally: For some particular biordered E, which groups can be the maximal subgroups of $\mathrm{IG}(E)$?

Gray and Ruškuc [21] investigated the maximal subgroups of $\mathrm{IG}(E)$, where E is the biordered set of idempotents of a full transformation monoid $\mathcal{T}_n$ on n elements. Recall that for any rank r idempotent $e \in \mathcal{T}_n$ with $1 \leq r \leq n$, the $\mathcal{D}$-class D of e is given by

$$D = \{\alpha \in \mathcal{T}_n : \operatorname{rank} \alpha = r\}.$$

We have already known from (IG4) that the maximal subgroup of $\mathrm{IG}(E)$ with identity e is the homomorphic preimage of $\mathcal{S}_r$. By using the presentation obtained in [20] and the standard Coxeter presentation of the symmetric group $\mathcal{S}_r$, Gray and Ruškuc [21] give a complete characterization of maximal subgroups of $\mathrm{IG}(E)$, where E is the biordered set of idempotents of $\mathcal{T}_n$.

Theorem 4.2.9. [21] *Let $\mathcal{T}_n$ be the full transformation semigroup on n elements, let E be its biordered set of idempotents, and let $e \in E$ with* $\operatorname{rank} r$*, where $1 \leq r \leq n-2$. Then the maximal subgroup of* $\mathrm{IG}(E)$ *containing $\overline{e}$ is isomorphic to the maximal subgroup of $\mathcal{T}_n$ containing e, and hence to the symmetric group $\mathcal{S}_r$.*

We remark here that if e is a rank $n-1$ idempotent, then the maximal subgroup of $\mathrm{IG}(E)$ containing $\overline{e}$ is a free group, as there are no non-trivial singular squares in the $\mathcal{D}$-class of e. If $e \in E$ has rank n, then the maximal subgroup of $\mathrm{IG}(E)$ containing $\overline{e}$ is the trivial group.

Subsequently, Dolinka [5] proved that the same result holds when $\mathcal{T}_n$ is replaced by $\mathcal{PT}_n$, where $\mathcal{PT}_n$ is the full monoid of partial transformations on n elements.

Another interesting strand of this popular theme is to consider the biordered set E of idempotents of the matrix monoid $M_n(D)$ of all $n \times n$ matrices over a division ring D. Let $e \in E$ with $\operatorname{rank} r$, where $1 \leq r \leq n$. It is known that the maximal subgroup of $M_n(D)$ containing e is the r dimensional general linear group $\mathrm{GL}_r(D)$. What are the relationships between maximal subgroups of $\mathrm{IG}(E)$ and $M_n(D)$ containing an idempotent $e \in E$? By using similar topological methods to those of [1], rather than the presentation obtained in [20], Brittenham, Margolis and Meakin [2] proved the following theorem:

Theorem 4.2.10. [2] *Let E be the biordered set of idempotents of $M_n(D)$, where D is a division ring, and let $e \in E$ be a rank 1 idempotent. Then the maximal subgroup of* $\mathrm{IG}(E)$ *containing $\overline{e}$ is isomorphic to that of $M_n(D)$, that is, the multiplicative group D^* of D.*

Dolinka and Gray [7] went onto generalise the result of [2] to higher rank r, where $r < n/3$.

Theorem 4.2.11. [7] *Let E be the biordered set of idempotents of $M_n(D)$, where D is a division ring, and let $e \in E$ be a rank r idempotent with $r < n/3$. Then the maximal subgroup of* $\mathrm{IG}(E)$ *containing $\overline{e}$ is isomorphic to the maximal subgroup of $M_n(D)$ containing e, and hence to the r dimensional general linear group $GL_r(D)$.*

So far, the structure of maximal subgroups of $\mathrm{IG}(E)$ containing $e \in E$, where $\operatorname{rank} e = r$ and $r \geq n/3$ remains unknown. It was conjectured in [2] that:

Conjecture [2] For any idempotent $e \in E$ with $\operatorname{rank} e = r$ and $r < n/2$, the maximal subgroup of $\mathrm{IG}(E)$ containing $\overline{e}$ is isomorphic to the maximal subgroup of $M_n(D)$ containing e.

Chapter 5

Every group occurs as a maximal subgroup of a natural $\mathrm{IG}(E)$

In this chapter, we will consider a free (left) G-act $F_n(G)$ of rank n, where G is a group and $3 \leq n \in \mathbb{N}$. We know from Theorem 3.4.2 that the endomorphism monoid $\operatorname{End} F_n(G)$ of $F_n(G)$ is isomorphic to the wreath product $G \wr \mathcal{T}_n$. Our main aim here is to show, in a transparent way, that for any idempotent $\varepsilon \in E$ lying in the minimal ideal (i.e. the rank 1 $\mathcal{D}$-class) of $G \wr \mathcal{T}_n$, the maximal subgroup of $\mathrm{IG}(E)$ containing ε is isomorphic to G, where E is the biordered set of idempotents of $G \wr \mathcal{T}_n$.

Our work in this chapter is analogous to that of Brittenham, Margolis and Meakin for rank 1 idempotents in full linear monoids [2]. As a corollary we obtain the result of Gray and Ruškuc [20] that *any* group can occur as a maximal subgroup of *some* free idempotent generated semigroup. Unlike their proof, ours involves a natural biordered set and very little machinery.

For the convenience of the reader we give the argument as presented in my joint paper with Gould [23], but for later use (particularly in Chapter 7) we make a series of observations on the more general context in which some of the lemmas hold.

In this chapter, we use greek letters to denote elements of $\operatorname{End} F_n(G)$, except the special $\mathbf{q}_\lambda, \mathbf{r}_i$ and $\mathbf{p}_{\lambda i}$ related to the Rees matrix semigroup representation of the completely simple semigroup D_1, which are denoted in bold. We want to keep the notations $\mathbf{q}_\lambda, \mathbf{r}_i$ and $\mathbf{p}_{\lambda i}$ here, as they are completely standard.

5.1 The structure of the rank-1 $\mathcal{D}$-class

We mentioned in Section 3.4 that elements $\alpha, \beta \in \operatorname{End} F_n(G)$ depend only on their action on the free generators $\{x_i : i \in [1,n]\}$ and therefore following our notation in Section 3.4 we have

$$\alpha = \begin{pmatrix} x_1 & \dots & x_n \\ w_1^\alpha x_{1\overline{\alpha}} & \dots & w_n^\alpha x_{n\overline{\alpha}} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} x_1 & \dots & x_n \\ w_1^\beta x_{1\overline{\beta}} & \dots & w_n^\beta x_{n\overline{\beta}} \end{pmatrix}$$

where $w_i^\alpha, w_i^\beta \in G$, for all $i \in \{1, \cdots, n\}$; and $\overline{\alpha}, \overline{\beta} \in \mathcal{T}_n$. Note that if α happens to be an idempotent then overline notation is *also* being used to denote $\overline{\alpha} \in \mathrm{IG}(E)$. However, it will always be clear from context what we mean by $\overline{\alpha}$.

Let ε be a rank 1 idempotent of $\operatorname{End} F_n(G)$. Then by Lemma 3.4.1 the $\mathcal{D}$-class of ε in $\operatorname{End} F_n(G)$, denoted by $D = D_1$ is given by

$$D = D_1 = \{\alpha \in \operatorname{End} F_n(G) \mid \operatorname{rank} \alpha = 1\}.$$

Clearly $\alpha, \beta \in D$ if and only if $\overline{\alpha}, \overline{\beta}$ are constant, and from Lemma 3.4.1 we have

$$\alpha \,\mathcal{L}\, \beta \Longleftrightarrow \operatorname{im} \overline{\alpha} = \operatorname{im} \overline{\beta}.$$

Lemma 5.1.1. *Let $\alpha, \beta \in D$ be as above. Then* $\ker \alpha = \ker \beta$ *if and only if* $(w_1^\alpha, \dots, w_n^\alpha)g = (w_1^\beta, \dots, w_n^\beta)$ *for some $g \in G$.*

Proof. Suppose $\ker \alpha = \ker \beta$. For any $i, j \in [1,n]$ we have

$$((w_i^\alpha)^{-1} x_i)\alpha = x_{i\overline{\alpha}} = ((w_j^\alpha)^{-1} x_j)\alpha$$

so that by assumption, $((w_i^\alpha)^{-1} x_i)\beta = ((w_j^\alpha)^{-1} x_j)\beta$. Consequently,

$$(w_i^\alpha)^{-1} w_i^\beta = (w_j^\alpha)^{-1} w_j^\beta = g \in G$$

and it follows that $(w_1^\alpha, \dots, w_n^\alpha)g = (w_1^\beta, \dots, w_n^\beta)$.

Conversely, if $g \in G$ exists as given then for any $u, v \in G$ and $i, j \in [1,n]$ we have

$$\begin{aligned}(ux_i)\alpha = (vx_j)\alpha &\Leftrightarrow uw_i^\alpha = vw_j^\alpha \Leftrightarrow uw_i^\alpha g = vw_j^\alpha g \\ &\Leftrightarrow uw_i^\beta = vw_j^\beta \Leftrightarrow (ux_i)\beta = (vx_j)\beta.\end{aligned}$$

The proof is completed. $\square$

We index the $\mathcal{L}$-classes in D by $J = [1, n] = \{1, \cdots, n\}$, where the image of $\alpha \in L_j$ is Gx_j, and we index the $\mathcal{R}$-classes of D by I, so that by Lemma 5.1.1, the set I is in bijective correspondence with G^{n-1}. From Lemma 3.3.2 we have that D is a completely simple semigroup. We denote by H_{ij} the intersection of the $\mathcal{R}$-class indexed by i and the $\mathcal{L}$-class indexed by j, and by ε_{ij} the identity of H_{ij}. For convenience we also suppose that $1 \in I \cap \Lambda$ and let

$$\varepsilon_{11} = \begin{pmatrix} x_1 & \cdots & x_n \\ x_1 & \cdots & x_1 \end{pmatrix}.$$

Clearly, for any given $i \in I, j \in J$ we have

$$\varepsilon_{1j} = \begin{pmatrix} x_1 & \cdots & x_n \\ x_j & \cdots & x_j \end{pmatrix} \text{ and } \varepsilon_{i1} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & a_{2i}x_1 & \cdots & a_{ni}x_1 \end{pmatrix},$$

where $a_{2i}, \cdots, a_{ni} \in G$.

Lemma 5.1.2. *Every $\mathcal{H}$-class of D is isomorphic to G.*

Proof. By standard semigroup theory, we know that any two group $\mathcal{H}$-classes in the same $\mathcal{D}$-class are isomorphic, so we need only show that H_{11} is isomorphic to G. By Lemma 5.1.1 an element $\alpha \in \operatorname{End} F_n(G)$ lies in H_{11} if and only if

$$\alpha = \alpha_g = \begin{pmatrix} x_1 & \cdots & x_n \\ gx_1 & \cdots & gx_1 \end{pmatrix}$$

for some $g \in G$. Clearly $\psi : H_{11} \to G$ defined by $\alpha_g \psi = g$ is an isomorphism. $\square$

By the results explained in Section 1.2.4, it follows that D is a completely simple semigroup, and hence it is isomorphic to some Rees matrix semigroup $\mathcal{M} = \mathcal{M}(H_{11}; I, J; P)$, where $P = (\mathbf{p}_{ji}) = (\mathbf{q}_j \mathbf{r}_i)$, and we can take

$$\mathbf{q}_j = \varepsilon_{1j} \in H_{1j} \text{ and } \mathbf{r}_i = \varepsilon_{i1} \in H_{i1}.$$

Since the $\mathbf{q}_j, \mathbf{r}_i$ are chosen to be idempotents, it is clear that $\mathbf{p}_{1i} = \mathbf{p}_{j1} = \varepsilon_{11}$ for all $i \in I, j \in J$.

Lemma 5.1.3. *For any $\alpha_{g_2}, \ldots, \alpha_{g_n} \in H_{11}$, we can choose $k \in I$ such that the kth column of P is $(\varepsilon_{11}, \alpha_{g_2}, \ldots, \alpha_{g_n})^T$.*

Proof. Choose $k \in I$ such that

$$\varepsilon_{k1} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_1 & g_2x_1 & \ldots & g_nx_1 \end{pmatrix}$$

(note that if $g_2 = \ldots = g_n = 1$ then $k = 1$). □

The above lemma essentially completely describes what the Rees structure matrix P looks like in this special case. The matrix P is given by taking the transpose of the matrix which has $|G|^{n-1}$ rows and n columns, the first row and the first column contains all 1s, and the remaining rows are given by listing all possible $|G|^{n-1}$ tuples of elements of G, each tuple occurring exactly once.

5.2 Singular squares of the rank-1 $\mathcal{D}$-class

We know that singular squares play a significant role in the structure of the free idempotent generated semigroups $\mathrm{IG}(E)$. By Lemma 4.2.3, every singular square forms a rectangular band, but as we remarked that the converse is not always true. In this section, we will show that in the rank-1 $\mathcal{D}$-class D of $\mathrm{End}\, F_n(G)$, singular squares are equivalent to rectangular bands. Note that this result also follows from Lemma 6.2.1, but here we would like to give a simple proof for our very special rank 1 case.

Lemma 5.2.1. *An E-square $\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}$ in D is singular if and only $\{\alpha, \beta, \gamma, \delta\}$ is a rectangular band.*

Proof. One direction follows immediately from Lemma 4.2.3.

Suppose that $\{\alpha, \beta, \gamma, \delta\}$ is a rectangular band. If $\alpha \,\mathcal{L}\, \beta$, then our E-square becomes $\begin{bmatrix} \alpha & \alpha \\ \gamma & \gamma \end{bmatrix}$ and taking $k = \gamma$ we see this is an up-down singular square.

Without loss of generality we therefore suppose that

$$\{1\} = \mathrm{im}\, \overline{\alpha} = \mathrm{im}\, \overline{\delta} \neq \mathrm{im}\, \overline{\beta} = \mathrm{im}\, \overline{\gamma} = \{2\}.$$

Following standard notation we write

$$x_i\alpha = w_i^\alpha x_1, x_i\delta = w_i^\delta x_1, x_i\beta = w_i^\beta x_2 \text{ and } x_i\gamma = w_i^\gamma x_2$$

for every $i \in [1, n]$. As α, β, δ and γ are idempotents, it is clear that

$$w_1^\alpha = w_1^\delta = 1 \text{ and } w_2^\beta = w_2^\gamma = 1.$$

Since $\{\alpha, \beta, \gamma, \delta\}$ is a rectangular band, we have $\alpha\gamma = \beta$ and so $x_1\alpha\gamma = x_1\beta$, that is, $w_1^\gamma = w_1^\beta$. Similarly, from $\gamma\alpha = \delta$, we have $w_2^\alpha = w_2^\delta$. Now we define $\theta \in \operatorname{End} F_n(G)$ by

$$x_i\theta = \begin{cases} x_1 & \text{if } i = 1; \\ x_2 & \text{if } i = 2; \\ w_i^\gamma x_2 & \text{else.} \end{cases}$$

Clearly θ is idempotent and since $\operatorname{im}\alpha$ and $\operatorname{im}\beta$ are contained in $\operatorname{im}\theta$ we have $\alpha\theta = \alpha$ and $\beta\theta = \beta$. Next we prove that $\theta\alpha = \delta$. Obviously, $x_1\theta\alpha = x_1\delta$ and $x_2\theta\alpha = x_2\delta$ from $w_2^\alpha = w_2^\delta$ obtained above. For other $i \in [1, n]$, we use the fact that from Lemma 5.1.1, there is an $s \in G$ with $w_i^\delta = w_i^\gamma s$, for all $i \in [1, n]$. Since

$$w_i^\delta (w_2^\alpha)^{-1} = w_i^\gamma s (w_2^\delta)^{-1} = (w_i^\gamma s)(w_2^\gamma s)^{-1} = w_i^\gamma (w_2^\gamma)^{-1} = w_i^\gamma$$

we have

$$x_i\theta\alpha = (w_i^\gamma x_2)\alpha = w_i^\gamma w_2^\alpha x_1 = w_i^\delta x_1 = x_i\delta$$

so that $\theta\alpha = \delta$. Since $\{\alpha, \beta, \gamma, \delta\}$ is a rectangular band it follows that

$$\theta\beta = \theta\alpha\beta = \delta\beta = \gamma.$$

Thus, by definition, $\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}$ is singular.

□

Lemma 5.2.2. *For any idempotents $\alpha, \beta, \gamma \in D$, $\alpha\beta = \gamma$ implies $\overline{\alpha}\overline{\beta} = \overline{\gamma}$ in* $\mathrm{IG}(E)$.

Proof. Since D is completely simple, we have $\alpha \,\mathcal{R}\, \gamma \,\mathcal{L}\, \beta$ and since every $\mathcal{H}$-class in D contains an idempotent, there exists some $\delta^2 = \delta \in D$ such that $\delta \in L_\alpha \cap R_\beta$.

We therefore obtain an E-square $\begin{bmatrix} \alpha & \gamma \\ \delta & \beta \end{bmatrix}$, which by Lemma 4.2.1 is a rectangular band. From Lemma 5.2.1 it is a singular square, so that from the property (IG2) of IG(E), we have that $\begin{bmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\delta} & \overline{\beta} \end{bmatrix}$ is also a singular square. By Lemma 4.2.3, $\overline{\alpha}\overline{\beta} = \overline{\gamma}$. □

Observation 5.2.3. *If S is a completely simple semigroup then S will not have any non-trivial singular squares, since there are no idempotents above to singularise. Thus, Lemma* 5.2.2 *is true for every completely simple semigroup in which rectangular bands are equivalent to singular squares.*

Note that the above comment really refers to completely simple $\mathcal{D}$-classes (principal factors) within a semigroup, which also applies to Observations 5.3.2, 5.3.5, 5.3.7 and 5.3.9.

5.3 A set of generators and relations of $\overline{H}$

The rest of this chapter is dedicated to proving that the maximal subgroup $H_{\overline{\varepsilon}_{11}}$ of IG(E) containing $\overline{\varepsilon}_{11}$ is isomorphic to the maximal subgroup $H_{\varepsilon_{11}}$ of End $F_n(G)$ containing ε_{11}, and hence by Lemma 5.1.2 to G. For ease of notation we denote $H_{\overline{\varepsilon}_{11}}$ by $\overline{H}$ and $H_{\varepsilon_{11}}$ by H.

As remarked earlier, although we do not directly use the presentations for maximal subgroups of semigroups developed in [3] and [42] and adjusted and implemented for free idempotent generated semigroups in [20], we are nevertheless making use of ideas from those papers. In fact, our work may be considered as a simplification of previous approaches, in particular [7], in the happy situation where a $\mathcal{D}$-class is completely simple, our sandwich matrix has the property of Lemma 5.1.3, and the singular squares are equivalent to rectangular bands.

We now locate a set of generators for $\overline{H}$. Observe first that for any $i \in I$ and $j \in J$

$$(\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11})(\overline{\varepsilon}_{1j}\overline{\varepsilon}_{i1}) = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{1j}\overline{\varepsilon}_{i1} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{i1} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{i1} = \overline{\varepsilon}_{11}$$

so that $\overline{\varepsilon}_{1j}\overline{\varepsilon}_{i1}$ is the inverse of $\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11}$ in $\overline{H}$; certainly then $\varepsilon_{1j}\varepsilon_{i1}$ is the inverse of $\varepsilon_{11}\varepsilon_{ij}\varepsilon_{11}$ in H.

In view of Lemma 1 of [29], which itself uses the techniques of [13], the next result follows from [3, Theorem 2.1]. Note that the assumption that the set of generators in [3] is finite is not critical.

Lemma 5.3.1. *Every element in $\overline{H}$ is a product of elements of the form $\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11}$ and $(\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11})^{-1}$, where $j \in J$ and $i \in I$.*

Observation 5.3.2. *Lemma* 5.3.1 *and the preceding comment hold for every completely simple semigroup.*

The next result is immediate from Lemma 5.2.2 and the observation preceding Lemma 5.3.1.

Lemma 5.3.3. *If $\varepsilon_{1j}\varepsilon_{i1} = \varepsilon_{11}$, then $\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}$.*

Lemma 5.3.4. *Let $i, l \in I$ and $j, k \in J$.*

(i) If $\varepsilon_{1j}\varepsilon_{i1} = \varepsilon_{1j}\varepsilon_{l1}$, that is, $\mathbf{p}_{ji} = \mathbf{p}_{jl}$ in the sandwich matrix P, then

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{lj}\overline{\varepsilon}_{11}.$$

(ii) If $\varepsilon_{1j}\varepsilon_{i1} = \varepsilon_{1k}\varepsilon_{i1}$, that is, $\mathbf{p}_{ji} = \mathbf{p}_{ki}$ in the sandwich matrix P, then

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ik}\overline{\varepsilon}_{11}.$$

Proof. (i) Notice that $\mathbf{p}_{1i}^{-1}\mathbf{p}_{1l} = \varepsilon_{11} = \mathbf{p}_{ji}^{-1}\mathbf{p}_{jl}$, so that from Lemma 4.2.2 we have that the elements of $\begin{bmatrix} \varepsilon_{i1} & \varepsilon_{ij} \\ \varepsilon_{l1} & \varepsilon_{lj} \end{bmatrix}$ form a rectangular band. Thus $\varepsilon_{ij} = \varepsilon_{i1}\varepsilon_{lj}$ and so from Lemma 5.2.2 we have that $\overline{\varepsilon}_{ij} = \overline{\varepsilon}_{i1}\overline{\varepsilon}_{lj}$. So,

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{i1}\overline{\varepsilon}_{lj}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{lj}\overline{\varepsilon}_{11}.$$

(ii) Here we have that $\mathbf{p}_{ji}^{-1}\mathbf{p}_{j1} = \mathbf{p}_{ki}^{-1}\mathbf{p}_{k1}$, so that $\begin{bmatrix} \varepsilon_{ij} & \varepsilon_{ik} \\ \varepsilon_{1j} & \varepsilon_{1k} \end{bmatrix}$ is a rectangular band and $\overline{\varepsilon}_{ij} = \overline{\varepsilon}_{ik}\overline{\varepsilon}_{1j}$. So,

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ik}\overline{\varepsilon}_{1j}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ik}\overline{\varepsilon}_{11}.$$

□

Observation 5.3.5. *Lemmas* 5.3.3 *and* 5.3.4 *are true for every completely simple semigroup $\mathcal{M} = \mathcal{M}(H_{11}; I, J; P)$ such that:* (i) *rectangular bands are equivalent* *to singular squares;* (ii) *the column* (p_{j1}) *and the row* (p_{1i}) *of* P *entirely consist of* e_{11}.

Lemma 5.3.6. *For any* $i, i' \in I$, $j, j' \in J$, *if* $\varepsilon_{1j}\varepsilon_{i1} = \varepsilon_{1j'}\varepsilon_{i'1}$, *that is* $\mathbf{p}_{ji} = \mathbf{p}_{j'i'}$ *in the sandwich matrix* P, *then*

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{i'j'}\overline{\varepsilon}_{11}.$$

Proof. Let $\alpha = \varepsilon_{1j}\varepsilon_{i1} = \varepsilon_{1j'}\varepsilon_{i'1}$. By Lemma 5.1.3 we can choose a $k \in I$ such that the kth column of P is $(\varepsilon_{11}, \alpha, \ldots, \alpha)$. Then $\mathbf{p}_{ji} = \mathbf{p}_{jk}$ and $\mathbf{p}_{j'k} = \mathbf{p}_{j'i'}$ (this is true even if j or j' is 1) and our hypothesis now gives that $\mathbf{p}_{jk} = \mathbf{p}_{j'k}$. The result now follows from three applications of Lemma 5.3.4. □

Observation 5.3.7. *Lemma* 5.3.6 *is true for every completely simple semigroup* $\mathcal{M} = \mathcal{M}(H_{11}; I, J; P)$ *such that:* (i) *rectangular bands are equivalent to singular squares;* (ii) *the column* (p_{j1}) *and the row* (p_{1i}) *of* P *entirely consist of* e_{11}; (iii) *for any* $i, i' \in I, j, j' \in J$ *with* $p_{ji} = p_{j'i'}$, *there exists* $k \in I$ *such that* $p_{ji} = p_{jk} = p_{j'k} = p_{j'i'}$.

In view of Lemma 5.3.6, we can define $w_\alpha = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11}$ where

$$\alpha = \varepsilon_{11}\varepsilon_{ij}\varepsilon_{11} = (\varepsilon_{1j}\varepsilon_{i1})^{-1} = (\mathbf{p}_{ji})^{-1}.$$

Of course, $\alpha = \alpha_g$ for some $g \in G$, and from Lemma 5.1.3, w_α is defined for any $\alpha \in H$.

Lemma 5.3.8. *With the notation given above, for any* $\alpha, \beta \in H$, *we have*

$$w_\alpha w_\beta = w_{\alpha\beta} \textit{ and } w_\alpha^{-1} = w_{\alpha^{-1}}.$$

Proof. By Lemma 5.1.3, P must contain columns

$$(\varepsilon_{11}, \alpha^{-1}, \beta^{-1}\alpha^{-1}, \cdots)^T \text{ and } (\varepsilon_{11}, \varepsilon_{11}, \beta^{-1}, \cdots)^T.$$

For convenience, we suppose that they are the i-th and l-th columns, respectively. So,

$$\mathbf{p}_{2i} = \varepsilon_{12}\varepsilon_{i1} = \alpha^{-1} \text{ and } \mathbf{p}_{3i} = \varepsilon_{13}\varepsilon_{i1} = \beta^{-1}\alpha^{-1}$$

and

$$\mathbf{p}_{2l} = \varepsilon_{12}\varepsilon_{l1} = \varepsilon_{11} \text{ and } \mathbf{p}_{3l} = \varepsilon_{13}\varepsilon_{l1} = \beta^{-1}.$$

It is easy to see that $\mathbf{p}_{2i}^{-1}\mathbf{p}_{2l} = \mathbf{p}_{3i}^{-1}\mathbf{p}_{3l}$. Then $\begin{bmatrix} \varepsilon_{i2} & \varepsilon_{i3} \\ \varepsilon_{l2} & \varepsilon_{l3} \end{bmatrix}$ is a rectangular band by Lemma 4.2.2. In the notation given above, we have

$$w_\alpha = \overline{\varepsilon}_{11}\overline{\varepsilon}_{i2}\overline{\varepsilon}_{11}, w_\beta = \overline{\varepsilon}_{11}\overline{\varepsilon}_{l3}\overline{\varepsilon}_{11} \text{ and } w_{\alpha\beta} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{i3}\overline{\varepsilon}_{11}.$$

By Lemma 5.2.2, $\overline{\varepsilon}_{12}\overline{\varepsilon}_{l1} = \overline{\varepsilon}_{11}$. We then calculate

$$\begin{aligned} w_\alpha w_\beta &= \overline{\varepsilon}_{11}\overline{\varepsilon}_{i2}\overline{\varepsilon}_{11}\overline{\varepsilon}_{11}\overline{\varepsilon}_{l3}\overline{\varepsilon}_{11} \\ &= \overline{\varepsilon}_{11}\overline{\varepsilon}_{i2}\overline{\varepsilon}_{12}\overline{\varepsilon}_{l1}\overline{\varepsilon}_{l3}\overline{\varepsilon}_{11} \\ &= \overline{\varepsilon}_{11}\overline{\varepsilon}_{i2}\overline{\varepsilon}_{l3}\overline{\varepsilon}_{11} \\ &= \overline{\varepsilon}_{11}\overline{\varepsilon}_{i3}\overline{\varepsilon}_{11} \qquad \left(\text{since } \begin{bmatrix} \varepsilon_{i2} & \varepsilon_{i3} \\ \varepsilon_{l2} & \varepsilon_{l3} \end{bmatrix} \text{ is a rectangular band}\right) \\ &= w_{\alpha\beta}. \end{aligned}$$

Finally, we show $w_\alpha^{-1} = w_{\alpha^{-1}}$. This follows since

$$\overline{\varepsilon}_{11} = w_{\varepsilon_{11}} = w_{\alpha^{-1}\alpha} = w_{\alpha^{-1}}w_\alpha.$$

□

Observation 5.3.9. *Lemma* 5.3.8 *is true for every completely simple semigroup* $\mathcal{M} = \mathcal{M}(H_{11}; I, J; P)$ *such that:* (i) *rectangular bands are equivalent to singular squares;* (ii) *for any* $i, i' \in I$ *and* $j, j' \in J$ *with* $p_{ji} = p_{j'i'}$ *we have that* $\overline{e}_{11}\overline{e}_{ij}\overline{e}_{11} = \overline{e}_{11}\overline{e}_{i'j'}\overline{e}_{11}$; (iii) *for any* $a, b \in H_{11}$, *there exist two columns of* P *with the following forms:*

$$(e_{11}, a^{-1}, b^{-1}a^{-1}, \cdots)^T \text{ and } (e_{11}, e_{11}, b^{-1}, \cdots)^T.$$

Note that (iii) *indicates that every element in* H_{11} *appears in* P.

It follows from Lemma 5.3.1 and Lemma 5.3.8 that any element of $\overline{H}$ can be expressed as $\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11}$ for some $i \in I$ and $j \in J$.

5.4 The main theorem

Now we are at the position of stating our main theorem in this chapter.

Theorem 5.4.1. *Let G be a group and let $F_n(G) = \bigcup_{i=1}^{n} Gx_i$ be a finite rank n free (left) G-act with $n \geq 3$, and let* $\operatorname{End} F_n(G)$ *the endomorphism monoid of $F_n(G)$. Let ε be an arbitrary rank 1 idempotent. Then the maximal subgroup of* $\mathrm{IG}(E)$ *containing $\overline{\varepsilon}$ is isomorphic to G.*

Proof. Without loss of generality we can take $\varepsilon = \varepsilon_{11}$. We have observed in Lemma 5.1.2 that $H = H_{\varepsilon_{11}}$ is isomorphic to G.

We consider the restriction of the natural morphism

$$\phi : \mathrm{IG}(E) \longrightarrow \operatorname{End} F_n(G)$$

as defined in the property (IG1) of $\mathrm{IG}(E)$; from the property (IG4) of $\mathrm{IG}(E)$,

$$\overline{\phi} = \phi|_{\overline{H}} : \overline{H} \longrightarrow H$$

is an onto morphism, where $\overline{H} = H_{\overline{\varepsilon}_{11}}$. We have observed that every element of $\overline{H}$ can be written as $\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11}$ for some $i \in I$ and $j \in J$. If

$$\varepsilon_{11} = \varepsilon_{11}\varepsilon_{ij}\varepsilon_{11} = (\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11})\overline{\phi}$$

then $\overline{\varepsilon}_{11}\overline{\varepsilon}_{ij}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}$ by Lemma 5.3.3. Thus $\overline{\phi}$ is an isomorphism and $\overline{H}$ is isomorphic to H. □

Corollary 5.4.2. *Any (finite) group G is a maximal subgroup of some free idempotent generated semigroup over a (finite) biordered set.*

We remark that in [20] it is proven that if G is *finitely presented*, then G is a maximal subgroup of $\mathrm{IG}(E)$ for some *finite* E: our construction makes no headway in this direction.

Chapter 6

Free idempotent generated semigroups: $\operatorname{End} F_n(G)$

Let $F_n(G) = \bigcup_{i=1}^{n} Gx_i$ be a rank n free left G-act with $n \in \mathbb{N}, n \geq 3$. Let $\operatorname{End} F_n(G)$ denote the endomorphism monoid of $F_n(G)$ (with composition left-to-right), which is isomorphic to a wreath product $G \wr \mathcal{T}_n$ by Theorem 3.4.2. Recall that the *rank* of an element $\alpha \in \operatorname{End} F_n(G)$ is defined to be the rank of $\operatorname{im} \alpha$.

The aim of this chapter is to extend the results of [23] (which forms Chapter 5 of this thesis) and [21], to show that for a rank r idempotent $\varepsilon \in \operatorname{End} F_n(G)$, with $1 \leq r \leq n-2$, we have that $H_{\overline{\varepsilon}}$ is isomorphic to H_ε and hence to $G \wr \mathcal{S}_r$. We have already remarked in the Introduction that unlike the proof of the rank 1 case in Chapter 5, a more sophisticated approach is needed.

We proceed as follows. In Section 7.1 we recall specific details concerning the structure of $\operatorname{End} F_n(G)$ and its $\mathcal{D}$-classes. In Section 6.2 we show how to use the generic presentation for maximal subgroups given in [20] and described in Chapter 4 to obtain a presentation of $H_{\overline{\varepsilon}}$; once these technicalities are in place we sketch the strategy employed in the rest of this chapter, and work our way through this in subsequent sections. By the end of Section 6.5 we are able to show that for $1 \leq r \leq n/3$, $H_{\overline{\varepsilon}} \cong H_\varepsilon$ (Theorem 6.5.3), a result corresponding to that in [7] for full linear monoids. To proceed further, we need more sophisticated analysis of the generators of $H_{\overline{\varepsilon}}$. Finally, in Section 6.8, we make use of the presentation of $G \wr \mathcal{S}_r$ given in [34] to show that we have the required result, namely that $H_{\overline{\varepsilon}} \cong H_\varepsilon$, for any rank r with $1 \leq r \leq n-2$ (Theorem 6.8.13). It is worth remarking that if G is trivial, then $F_n(G)$ is essentially a set, so that $\operatorname{End} F_n(G) \cong \mathcal{T}_n$. We are therefore able to recover, via a rather different strategy, the main result of [21].

6.1 The structure of a rank r $\mathcal{D}$-class

In this section, we are concerned with the structure of the $\mathcal{D}$-classes of $\operatorname{End} F_n(G)$.

Let $1 \leq r \leq n$ and set

$$D_r = \{\alpha \in \operatorname{End} F_n(G) \mid \text{rank } \alpha = r\},$$

that is, D_r is the $\mathcal{D}$-class in $\operatorname{End} F_n(G)$ of any rank r element. We let I and Λ denote the set of $\mathcal{R}$-classes and the set of $\mathcal{L}$-classes of D_r, respectively. Thus, I is in bijective correspondence with the set of kernels, and Λ with the set of images, of rank r endomorphisms, respectively. It is convenient to assume I *is* the set of kernels of rank r endomorphisms, and that

$$\Lambda = \{(u_1, u_2, \ldots, u_r) : 1 \leq u_1 < u_2 < \ldots < u_r \leq n\} \subseteq [1, n]^r.$$

Thus, it follows from Lemma 3.4.1 that $\alpha \in R_i$ if and only if $\ker \alpha = i$ and $\alpha \in L_{(u_1,\ldots,u_r)}$ if and only if

$$\operatorname{im} \alpha = Gx_{u_1} \cup Gx_{u_2} \cup \ldots \cup Gx_{u_r}.$$

For every $i \in I$ and $\lambda \in \Lambda$, we put $H_{i\lambda} = R_i \cap L_\lambda$ so that $H_{i\lambda}$ is an $\mathcal{H}$-class of D_r. Where $H_{i\lambda}$ is a subgroup, we denote its identity by $\varepsilon_{i\lambda}$. It is notationally standard to use the same symbol 1 to denote a selected element from both I and Λ. Here we let

$$1 = \langle (x_1, x_i) : r+1 \leq i \leq n \rangle \in I \text{ and } 1 = (1, 2, \ldots, r) \in \Lambda,$$

that is, the congruence generated by $\{(x_1, x_i) : r+1 \leq i \leq n\}$. Then $H = H_{11}$ is a group $\mathcal{H}$-class in D_r, with identity ε_{11}.

Continuing our notation in the previous chapters, a typical element of H looks like

$$\alpha = \begin{pmatrix} x_1 & x_2 & \ldots & x_r & x_{r+1} & \ldots & x_n \\ w_1^\alpha x_{1\overline{\alpha}} & w_2^\alpha x_{2\overline{\alpha}} & \ldots & w_r^\alpha x_{r\overline{\alpha}} & w_1^\alpha x_{1\overline{\alpha}} & \ldots & w_1^\alpha x_{1\overline{\alpha}} \end{pmatrix}$$

which in view of the following lemma we may abbreviate without further remark to:

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ w_1^\alpha x_{1\overline{\alpha}} & w_2^\alpha x_{2\overline{\alpha}} & \dots & w_r^\alpha x_{r\overline{\alpha}} \end{pmatrix}.$$

where here we are regarding $\overline{\alpha}$ as an element of $\mathcal{S}_r$.

Lemma 6.1.1. *The groups H and $\operatorname{Aut} F_r(G)$ are isomorphic under the map*

$$\begin{pmatrix} x_1 & \dots & x_r & x_{r+1} & \dots & x_n \\ w_1^\alpha x_{1\overline{\alpha}} & \dots & w_r^\alpha x_{r\overline{\alpha}} & w_1^\alpha x_{1\overline{\alpha}} & \dots & w_1^\alpha x_{1\overline{\alpha}} \end{pmatrix} \mapsto \begin{pmatrix} x_1 & \dots & x_r \\ w_1^\alpha x_{1\overline{\alpha}} & \dots & w_r^\alpha x_{r\overline{\alpha}} \end{pmatrix}.$$

Moreover, $\operatorname{Aut} F_r(G) \cong G \wr \mathcal{S}_r$.

Under this convention, the identity $\varepsilon = \varepsilon_{11}$ of H becomes

$$\varepsilon = \begin{pmatrix} x_1 & \dots & x_r \\ x_1 & \dots & x_r \end{pmatrix}.$$

With the aim of specialising the presentation given in Theorem 4.2.4, we locate and distinguish elements in $H_{1\lambda}$ and H_{i1} for each $\lambda \in \Lambda$ and $i \in I$. For any equivalence relation τ on $[1,n]$ with r classes, we write $\tau = \{B_1^\tau, \cdots, B_r^\tau\}$ (that is, we identify τ with the partition on $[1,n]$ that it induces). Let $l_1^\tau, \cdots, l_r^\tau$ be the minimum elements of $B_1^\tau, \cdots, B_r^\tau$, respectively. Without loss of generality we suppose that $l_1^\tau < \cdots < l_r^\tau$. Then $l_1^\tau = 1$ and $l_j^\tau \geq j$, for any $j \in [2,r]$. Suppose now that $\alpha \in \operatorname{End} F_n(G)$ and rank $\alpha = r$, that is, $\alpha \in D_r$. Then $\ker \overline{\alpha}$ has r equivalence classes. Where $\tau = \ker \overline{\alpha}$ we simplify our notation by writing $B_j^{\ker \overline{\alpha}} = B_j^\alpha$ and $l_j^{\ker \overline{\alpha}} = l_j^\alpha$. If there is no ambiguity over the choice of α we may simplify further to B_j and l_j.

Lemma 6.1.2. *Let $\alpha, \beta \in D_r$. Then $\ker \alpha = \ker \beta$ if and only if $\ker \overline{\alpha} = \ker \overline{\beta}$ and for any $j \in [1,r]$ there exists $g_j \in G$ such that for any $k \in B_j^\alpha = B_j = B_j^\beta$, we have $w_k^\alpha = w_k^\beta g_j$. Moreover, we can take $g_j = (w_{l_j}^\beta)^{-1} w_{l_j}^\alpha$ for $j \in [1,r]$.*

Proof. If $\ker \alpha = \ker \beta$, then clearly $\ker \overline{\alpha} = \ker \overline{\beta}$. Now for any $j \in [1,r]$ and $k \in B_j^\alpha = B_j^\beta$, we have that

$$((w_{l_j}^\alpha)^{-1} x_{l_j})\alpha = ((w_k^\alpha)^{-1} x_k)\alpha$$

and so

$$((w_{l_j}^\alpha)^{-1} x_{l_j})\beta = ((w_k^\alpha)^{-1} x_k)\beta,$$

giving that $w_k^\alpha = w_k^\beta((w_{l_j}^\beta)^{-1}w_{l_j}^\alpha)$. We may thus take $g_j = (w_{l_j}^\beta)^{-1}w_{l_j}^\alpha$.

Conversely, suppose that $\ker\overline{\alpha} = \ker\overline{\beta}$ (and has blocks $\{B_1, \cdots, B_r\}$) and for any $j \in [1, r]$ there exists $g_j \in G$ sastisfying the given condition. Let ux_h, vx_k be elements in $\operatorname{End} F_n(G)$. Then

$$\begin{aligned}(ux_h)\alpha = (vx_k)\alpha \;&\Leftrightarrow\; h, k \in B_j \text{ for some } j \in [1, r] \text{ and } uw_h^\alpha = vw_k^\alpha\\ &\Leftrightarrow\; h, k \in B_j \text{ for some } j \in [1, r] \text{ and } uw_h^\beta g_j = vw_k^\beta g_j\\ &\Leftrightarrow\; h, k \in B_j \text{ for some } j \in [1, r] \text{ and } uw_h^\beta = vw_k^\beta\\ &\Leftrightarrow\; (ux_h)\beta = (vx_k)\beta\end{aligned},$$

so that $\ker\alpha = \ker\beta$ as required. $\square$

For the following, we denote by $P(n, r)$ the set of equivalence relations on $[1, n]$ having r classes. Of course, $|P(n, r)| = S(n, r)$, where $S(n, r)$ is a Stirling number of the second kind, but we shall not need that fact here.

Corollary 6.1.3. *The map $\boldsymbol{\tau} : I \to G^{n-r} \times P(n, r)$ given by*

$$i\boldsymbol{\tau} = ((w_2^\alpha, \ldots, w_{l_2-1}^\alpha, w_{l_2+1}^\alpha, \ldots, w_{l_r-1}^\alpha, w_{l_r+1}^\alpha, \ldots, w_n^\alpha), \ker\overline{\alpha})$$

where $\alpha \in R_i$ and $w_{l_j}^\alpha = 1_G$, for all $j \in [1, r]$, is a bijection.

Proof. For $i \in I$ choose $\beta \in R_i$ and then define $\alpha \in \operatorname{End} F_n(G)$ by

$$x_k\alpha = w_k^\beta(w_{l_j}^\beta)^{-1}x_j,$$

where $k \in B_j^\beta$. It is clear from Lemma 6.1.2 that $\ker\alpha = \ker\beta$ and so $\alpha \in R_i$. Now

$$x_{l_j}\alpha = w_{l_j}^\beta(w_{l_j}^\beta)^{-1}x_j = x_j,$$

so that $i\boldsymbol{\tau}$ is defined. An easy argument, again from Lemma 6.1.2, gives that $\boldsymbol{\tau}$ is well defined and one-one.

For $\mu \in P(n, r)$ let $\nu_\mu : [1, n] \to [1, r]$ be given by $k\nu_\mu = j$ where $k \in B_j^\mu$. Now for $((h_1, \ldots, h_{n-r}), \mu) \in G^{n-r} \times P(n, r)$, define

$$\alpha = ((1_G, h_1, \ldots, h_{l_2^\mu-2}, I_G, h_{l_2^\mu-1}, \ldots, h_{l_r^\mu-r}, 1_G, h_{l_r^\mu-r+1}, \ldots, h_{n-r}), \nu_\mu)\boldsymbol{\psi}^{-1},$$

where $\boldsymbol{\psi}$ is defined in Theorem 3.4.2 on Page 34. It is clear that if $\alpha \in R_i$, then

$$i\boldsymbol{\tau} = ((h_1, \ldots, h_{n-r}), \mu).$$

Thus $\boldsymbol{\tau}$ is a bijection as required. $\square$

Corollary 6.1.4. *Let* Θ *be the set defined by*

$$\Theta = \{\alpha \in D_r : x_{l_j^\alpha}\alpha = x_j, j \in [1, r]\}.$$

Then Θ *is a transversal of the* $\mathcal{H}$*-classes of* L_1*.*

Proof. Clearly, $\operatorname{im}\alpha = Gx_1 \cup \cdots \cup Gx_r$, for any $\alpha \in \Theta$, and so that Θ is a subset of L_1.

Next, we show that for each $i \in I$, $|H_{i1} \cap \Theta| = 1$. Suppose that $\alpha, \beta \in \Theta$ and $\ker\alpha = \ker\beta$. Clearly $\ker\overline{\alpha} = \ker\overline{\beta}$ and so $B_j^\alpha = B_j = B_j^\beta$ for any $j \in [1, r]$, and by definition of Θ, $w_{l_j}^\alpha = w_{l_j}^\beta = 1_G$. It is then clear from Lemma 6.1.2 that for any $k \in B_j$ we have

$$x_k\alpha = w_k^\alpha x_j = w_k^\beta x_j = x_k\beta,$$

so that $\alpha = \beta$.

It only remains to show that for any $i \in I$ we have $|H_{i1} \cap \Theta| \neq \emptyset$. By Corollary 6.1.3, for $i \in I$ we can find $\alpha \in R_i$ such that $w_{l_j}^\alpha = 1_G$ for all $j \in [1, r]$. Composing α with $\beta \in \operatorname{End} F_n(G)$ where $x_{l_j\overline{\alpha}}\beta = x_j$ for all $j \in [1, r]$ and $x_k\beta = x_1$ else, we clearly have that $\alpha\beta \in H_{i1} \cap \Theta$. $\square$

For each $i \in I$, we denote the unique element in $H_{i1} \cap \Theta$ by $\mathbf{r}_i$. Notice that $\mathbf{r}_1 = \varepsilon$.

On the other hand, for $\lambda = (u_1, u_2, \ldots, u_r) \in \Lambda$, we define

$$\mathbf{q}_\lambda = \mathbf{q}_{(u_1,\cdots,u_r)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r & x_{r+1} & \cdots & x_n \\ x_{u_1} & x_{u_2} & \cdots & x_{u_r} & x_{u_1} & \cdots & x_{u_1} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ x_{u_1} & x_{u_2} & \cdots & x_{u_r} \end{pmatrix}.$$

It is easy to see that $\mathbf{q}_\lambda \in H_{1\lambda}$, as

$$\ker\mathbf{q}_\lambda = \langle (x_1, x_i) : r + 1 \leq i \leq n \rangle.$$

In particular, we have

$$\mathbf{q}_1 = \mathbf{q}_{(1,2,\cdots,r)} = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ x_1 & x_2 & \cdots & x_r \end{pmatrix} = \varepsilon.$$

It follows from Lemma 3.3.2 that $D_r^0 = D_r \cup \{0\}$ is a completely 0-simple semigroup with the multiplication defined by

$$\alpha \cdot \beta = \begin{cases} \alpha\beta & \text{if } \alpha, \beta \in D_r \text{ and } \operatorname{rank} \alpha\beta = r \\ 0 & \text{else} \end{cases}$$

We do not need to give the specifics of what the latter property entails, since, by the Rees Theorem (see [26, Chapter III]) and described in Chapter 1 as Theorem 1.2.12, D_r^0 is isomorphic to

$$\mathcal{M}^0 = \mathcal{M}^0(H; I, \Lambda; P),$$

where $P = (\mathbf{p}_{\lambda i})$ and $\mathbf{p}_{\lambda i} = (\mathbf{q}_\lambda \mathbf{r}_i)$ if $\operatorname{rank} \mathbf{q}_\lambda \mathbf{r}_i = r$, and is 0 else. Our choice of P will allow us at crucial points to modify the presentation given in Theorem 4.2.4.

6.2 A presentation of maximal subgroups of IG(E)

Our aim in this section is to specialise to $\operatorname{End} F_n(G)$ the presentation for the maximal subgroups of IG(E) obtained by Gray and Ruškuc [20].

For the remainder of this chapter, E will denote $E(\operatorname{End} F_n(G))$. In addition, for the sake of notational convenience, we now observe the accepted convention of dropping the overline notation for elements of $\overline{E}^*$. In particular, idempotents of IG(E) carry the same notation as those of $\operatorname{End} F_n(G)$; the context should hopefully prevent confusion.

We focus on the idempotent $\varepsilon = \varepsilon_{11}$ of Section 6.1. It follows immediately from Theorem 4.2.4 that the maximal subgroup $\overline{H}$ of IG(E) containing $\varepsilon \in E$ is defined by the presentation

$$\mathcal{P} = \langle F : \Sigma \rangle$$

with generators:

$$F = \{f_{i,\lambda} : \quad (i, \lambda) \in K\}$$

and defining relations Σ:

$(R1)$ $f_{i,\lambda} = f_{i,\mu} \quad (h_\lambda \varepsilon_{i\mu} = h_\mu)$;

$(R2)$ $f_{i,\omega(i)} = 1 \quad (i \in I)$;

$(R3)$ $f_{i,\lambda}^{-1} f_{i,\mu} = f_{k,\lambda}^{-1} f_{k,\mu} \quad \left(\begin{bmatrix} \varepsilon_{i\lambda} & \varepsilon_{i\mu} \\ \varepsilon_{k\lambda} & \varepsilon_{k\mu} \end{bmatrix} \text{ is a singular square} \right)$,

where the h_λ form a Scherier system of representatives, and $\omega : I \longrightarrow \Lambda$ is a function satisfying the conditions given on Page 41.

In order to specialise the above to E, our first step is to identify the singular squares.

Lemma 6.2.1. *An E-square $\begin{bmatrix} \gamma & \delta \\ \xi & \nu \end{bmatrix}$ is singular if and only if $\{\gamma, \delta, \nu, \xi\}$ is a rectangular band.*

Proof. The proof of necessity is trivial. We only need to show the sufficiency. Let $\{\gamma, \delta, \nu, \xi\}$ be a rectangular band so that

$$\gamma\nu = \delta, \nu\gamma = \xi, \delta\xi = \gamma \text{ and } \xi\delta = \nu.$$

Suppose

$$\operatorname{im}\gamma = \operatorname{im}\xi = \langle x_m \rangle_{m \in M} \text{ and } \operatorname{im}\delta = \operatorname{im}\nu = \langle x_n \rangle_{n \in N},$$

where $|M| = |N| = r$. Put $L = M \cup N$. Define a mapping $\theta \in \operatorname{End} F_n(G)$ by

$$x_i\theta = \begin{cases} x_i & \text{if } i \in L; \\ x_i\nu & \text{else.} \end{cases}$$

Since $\operatorname{im}\theta = \langle x_l \rangle_{l \in L}$ and for each $l \in L$, $x_l\theta = x_l$, we see that θ is an idempotent. It is also clear that $\gamma\theta = \gamma$ and $\delta\theta = \delta$, as $\operatorname{im}\gamma \cup \operatorname{im}\delta \subseteq \operatorname{im}\theta$.

Next, we will show $\theta\gamma = \xi$. If $i \in M$, then $x_i\theta\gamma = x_i\gamma = x_i = x_i\xi$. If $i \in N$, but $i \notin M$, then

$$x_i\theta\gamma = x_i\gamma = x_i\nu\gamma = x_i\xi.$$

If $i \notin L$, then

$$x_i\theta\gamma = x_i\nu\gamma = x_i\xi.$$

So, $\theta\gamma = \xi$. For the remaining equality $\theta\delta = \nu$ required in the definition of a singular square, since $\{\gamma, \delta, \nu, \xi\}$ is a rectangular band, we have

$$\theta\delta = \theta\gamma\delta = \xi\delta = \nu.$$

Hence we have proved that

$$\gamma\theta = \gamma, \delta\theta = \delta, \theta\gamma = \xi \text{ and } \theta\delta = \nu,$$

so that $\begin{bmatrix} \gamma & \delta \\ \xi & \nu \end{bmatrix}$ is an up-down singular square. □

The proof of Lemma 6.2.1 shows the following:

Corollary 6.2.2. *An E-square is singular if and only if it is an up-down singular square.*

The next corollary is immediate from Lemmas 4.2.2 and 6.2.1.

Corollary 6.2.3. *Let $P = (\mathbf{p}_{\lambda i})$ be the sandwich matrix of any completely 0-simple semigroup isomorphic to D_r^0. Then (R3) in Theorem 4.2.4 can be restated as:*

$(R3)$ $f_{i,\lambda}^{-1} f_{i,\mu} = f_{k,\lambda}^{-1} f_{k,\mu} \quad (\mathbf{p}_{\lambda i}^{-1}\mathbf{p}_{\lambda k} = \mathbf{p}_{\mu i}^{-1}\mathbf{p}_{\mu k})$.

For the presentation $\mathcal{P} = \langle F : \Sigma \rangle$ (we refer the reader to Chapter 4 for details) for our particular $\overline{H}$, we must define a Schreier system of words $\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$. In this instance, we can do so inductively, using the restriction of the lexicographic order on $[1, n]^r$ to Λ. Recall that we are using the same notation for $\mathbf{h}_\lambda \in E^*$ and its image under the natural morphism to the set of right translations of $\mathrm{IG}(E)$ and of $\mathrm{End}\, F_n(G)$.

First, we define

$$\mathbf{h}_{(1,2,\cdots,r)} = 1,$$

the empty word in E^*. Now let

$$(u_1, u_2, \ldots, u_r) \in \Lambda \text{ with } (1, 2, \cdots, r) < (u_1, u_2, \ldots, u_r)$$

and assume for induction that $\mathbf{h}_{(v_1,v_2,\ldots,v_r)}$ has been defined for all

$$(v_1, v_2, \ldots, v_r) < (u_1, u_2, \ldots, u_r).$$

Taking $u_0 = 0$ there must exist some $j \in [1, r]$ such that $u_j - u_{j-1} > 1$. Letting i be largest such that $u_i - u_{i-1} > 1$ observe that

$$(u_1, \dots, u_{i-1}, u_i - 1, u_{i+1}, \dots, u_r) < (u_1, u_2, \dots, u_r).$$

We now define

$$\mathbf{h}_{(u_1,\cdots,u_r)} = \mathbf{h}_{(u_1,\cdots,u_{i-1},u_i-1,u_{i+1},\cdots,u_r)}\alpha_{(u_1,\cdots,u_r)},$$

where $\alpha_{(u_1,\cdots,u_r)}$ is defined by

$$\begin{pmatrix} x_1 & \cdots & x_{u_1} & x_{u_1+1} & \cdots & x_{u_2} & \cdots & x_{u_{r-1}+1} & \cdots & x_{u_r} & x_{u_r+1} & \cdots & x_n \\ x_{u_1} & \cdots & x_{u_1} & x_{u_2} & \cdots & x_{u_2} & \cdots & x_{u_r} & \cdots & x_{u_r} & x_{u_r} & \cdots & x_{u_r} \end{pmatrix};$$

notice that $\alpha_{(u_1,\cdots,u_r)} = \varepsilon_{l(u_1,\dots,u_r)}$ where $\varepsilon_{l(u_1,\dots,u_r)}$ is an idempotent contained in the $\mathcal{H}$-class with kernel indexed by some $l \in I$ and image indexed by $(u_1, \dots, u_r)$.

Lemma 6.2.4. *For all* $(u_1, \dots, u_r) \in \Lambda$ *we have* $\varepsilon\mathbf{h}_{(u_1,\dots,u_r)} = \mathbf{q}_{(u_1,\dots,u_r)}$. *Hence right translation by* $\mathbf{h}_{(u_1,\cdots,u_r)}$ *induces a bijection from* $L_{(1,\cdots,r)}$ *onto* $L_{(u_1,\cdots,u_r)}$ *in both* $\operatorname{End} F_n(G)$ *and* $\operatorname{IG}(E)$.

Proof. We prove by induction on $(u_1, \dots, u_r)$ that $\varepsilon\mathbf{h}_{(u_1,\dots,u_r)} = \mathbf{q}_{(u_1,\dots,u_r)}$. Clearly the statement is true for $(u_1, \dots, u_r) = (1, \dots, r)$. Suppose now result is true for all $(v_1, \dots, v_r) < (u_1, \dots, u_r)$, so that

$$\varepsilon\mathbf{h}_{(u_1,\cdots,u_{i-1},u_i-1,u_{i+1},\cdots,u_r)} = \mathbf{q}_{(u_1,\cdots,u_{i-1},u_i-1,u_{i+1},\cdots,u_r)}.$$

Since $x_{u_j}\alpha = x_{u_j}$ for all $j \in \{1, \dots, r\}$ and $x_{u_i-1}\alpha = x_{u_i}$, it follows that

$$\begin{aligned} \varepsilon\mathbf{h}_{(u_1,\dots,u_r)} &= \varepsilon\mathbf{h}_{(u_1,\cdots,u_{i-1},u_i-1,u_{i+1},\cdots,u_r)}\alpha_{(u_1,\dots,u_r)} \\ &= \mathbf{q}_{(u_1,\cdots,u_{i-1},u_i-1,u_{i+1},\cdots,u_r)}\alpha_{(u_1,\dots,u_r)} \\ &= \mathbf{q}_{(u_1,\dots,u_r)} \end{aligned}$$

as required.

Since by definition, $\mathbf{q}_{(u_1,\dots,u_r)} \in L_{(u_1,\cdots,u_r)}$, the result for $\operatorname{End} F_n(G)$ follows from Green's Lemma (see, for example, [26, Chapter II]), and that for $\operatorname{IG}(E)$ by the comments in Section 4.2.3 that the action of any generators $\overline{f} \in \overline{E}$ on an $\mathcal{H}$-class contained in the $\mathcal{R}$-class $\overline{e}$ in $\operatorname{IG}(E)$ is equivalent to the action of f on the corresponding $\mathcal{H}$-class in End $F_n(G)$. $\square$

It is a consequence of Lemma 6.2.4 that $\{h_\lambda : \lambda \in \Lambda\}$ forms the required Schreier system for a presentation $\mathcal{P}$ for $\overline{H}$. It remains to define the function ω: we do so by setting

$$\omega(i) = (l_1^{\mathbf{r}_i}, l_2^{\mathbf{r}_i}, \ldots, l_r^{\mathbf{r}_i}) = (1, l_2^{\mathbf{r}_i}, \ldots, l_r^{\mathbf{r}_i})$$

for each $i \in I$. Note that for any $i \in I$ we have $\mathbf{q}_{\omega(i)}\mathbf{r}_i = \varepsilon$, i.e. $\mathbf{p}_{\omega(i),i} = \varepsilon$.

Definition 6.2.5. *Let $\mathcal{P} = \langle F : \Sigma \rangle$ be the presentation of $\overline{H}$ as in Theorem 4.2.4, where ω and $\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$ are given as above.*

Without loss of generality, we assume that $\overline{H}$ *is* the group with presentation $\mathcal{P}$.

In later parts of this work we will be considering for a non-zero entry $\phi \in P$, which $i \in I, \lambda \in \Lambda$ yield $\phi = \mathbf{p}_{\lambda i}$. For this and other purposes it is convenient to define the notion of district. For $i \in I$ we say that $\mathbf{r}_i$ lies in *district* $(l_1^{\mathbf{r}_i}, l_2^{\mathbf{r}_i}, \cdots, l_r^{\mathbf{r}_i})$ (of course, $1 = l_1^{\mathbf{r}_i}$). Note that the district of $\mathbf{r}_i$ is indeed determined by the kernel of the transformation $\overline{\mathbf{r}_i}$, and lying in the same district induces a partition of Θ.

Let us run an example, with $n = 9$ and $r = 3$ we can consider the following two partitions:

$$P_1 = \{\{1, 2, 8\}, \{3, 4, 7\}, \{5, 6, 9\}\}, P_2 = \{\{1, 4, 6\}, \{3, 2\}, \{5, 8, 9, 7\}\}.$$

Then for each of these partitions if we take the minimal entries in each class in both cases we get $1 < 3 < 5$, so these two partitions determine the same district.

The next lemma follows immediately from the definition of $\mathbf{r}_i, i \in I$.

Lemma 6.2.6. *For any $i \in I$, if $\mathbf{r}_i$ lies in district $(1, l_2, \cdots, l_r)$, then $l_s \geq s$ for all $s \in [1, r]$. Moreover, for $k \in [1, n]$, if $x_k\mathbf{r}_i = ax_j$, then $k \geq l_j$, with $k > l_j$ if $a \neq 1_G$.*

Proof. It is clear that $l_s \geq s$ for all $s \in [1, r]$. If $x_k\mathbf{r}_i = ax_j$, then $k \geq l_j$ because x_k and x_{l_j} belong to the same kernel class. If $a \neq 1_G$, then $x_k\mathbf{r}_i \neq x_j$ so $x_k \neq x_{l_j}$ and hence $k > l_j$. $\square$

We are interested in the non-zero entries $\phi \in H$ in the matrix P at this point. These are given by the $\mathbf{p}_{\lambda i} = \mathbf{q}_\lambda\mathbf{r}_i$ such that rank $\mathbf{q}_\lambda\mathbf{r}_i = r$. As indicated before

Lemma 6.1.1, we can write $\phi \in H$ as

$$\phi = \begin{pmatrix} x_1 & x_2 & \ldots & x_r \\ a_1 x_{1\overline{\phi}} & a_2 x_{2\overline{\phi}} & \ldots & a_r x_{r\overline{\phi}} \end{pmatrix}$$

where $\overline{\phi} \in \mathcal{S}_r$ and $(a_1, \ldots, a_r) \in G^r$.

In order for us to have rank $\mathbf{q}_\lambda \mathbf{r}_i = r$ it must be the case that $x_{u_1}, x_{u_2}, \cdots, x_{u_r}$ in the image of $\mathbf{q}_\lambda$ form a transversal of the kernel classes $B_1, B_2, \cdots, B_r$ of the element $\mathbf{r}_i$. This bijective correspondence between $\{x_{u_1}, x_{u_2}, \cdots, x_{u_r}\}$ and $\{B_1, B_2, \cdots, B_r\}$ induces a bijection from $[1, r]$ to $[1, r]$. In fact it defines precisely the element $\overline{\phi}$ of the symmetric group $\mathcal{S}_r$ where $\phi = \mathbf{p}_{\lambda i} = \mathbf{q}_\lambda \mathbf{r}_i$.

Suppose now that $\phi = \mathbf{p}_{\lambda i} \in P$, where $\lambda = (u_1, \ldots, u_r)$ and $\mathbf{r}_i$ lies in district $(l_1, \ldots, l_r)$. Then the u_j's and l_k's are constrained by

$$1 = l_1 < l_2 < \ldots < l_r,\ u_1 < u_2 < \ldots < u_r,$$

$$l_{j\overline{\phi}} \leq u_j \text{ for all } j \in [1, r] \text{ with } l_{j\overline{\phi}} < u_j \text{ if } a_j \neq 1_G,$$

and

$$l_k = u_j \text{ implies } k = j\overline{\phi} \text{ and } a_j = 1_G \text{ for all } k, j \in [1, r].$$

Conversely, if these constraints are satisfied by $l_1, \ldots, l_r, u_1, \ldots, u_r \in [1, n]$ with respect to some $\overline{\phi} \in \mathcal{S}_r$ and $(a_1, \ldots, a_r) \in G^r$, then it is easy to see that if $\xi \in \operatorname{End} F_n(G)$ is defined by

$$x_{l_k}\xi = x_k,\ x_{u_k}\xi = a_k x_{k\overline{\phi}},\ k \in [1, r]$$

and

$$x_j \xi = x_1 \text{ for } j \notin \{l_1, \ldots, l_r, u_1, \ldots, u_r\},$$

then $\xi = r_i$ for some $i \in I$, where r_i lies in district $(l_1, l_2, \cdots, l_r)$. Clearly, $\mathbf{p}_{\lambda i} = \phi$.

Lemma 6.2.7. *If $|G| > 1$ then every element of H occurs as an entry in P if and only if $2r \leq n$. If $|G| = 1$ then every element of H occurs as an entry in P if and only if $2r \leq n + 1$.*

Proof. Suppose first that $|G| > 1$. If $2r \leq n$, then given any

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ a_1 x_{1\overline{\alpha}} & a_2 x_{2\overline{\alpha}} & \dots & a_r x_{r\overline{\alpha}} \end{pmatrix}$$

in H, we can take

$$(l_1, \dots, l_r) = (1, 2, \dots, r) \text{ and } (u_1, \dots, u_r) = (r+1, \dots, 2r).$$

Conversely, if $2r > n$, then

$$\begin{pmatrix} x_1 & x_2 & \dots & x_r \\ a x_r & x_{r-1} & \dots & x_1 \end{pmatrix},$$

where $a \neq 1_G$. By the pigeon hole principal $r > n/2$ implies $u_1 < l_r$, and hence such element cannot lie in P.

Consider now the case where $|G| = 1$. If $2r \leq n+1$, then given any

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_r \\ x_{1\overline{\alpha}} & x_{2\overline{\alpha}} & \dots & x_{r\overline{\alpha}} \end{pmatrix}$$

in H, let $1\overline{\alpha} = t$ and choose

$$(l_1, \dots, l_r) = (1, \dots, r) \text{ and } (u_1, \dots, u_r) = (t, r+1, \dots, 2r-1).$$

It follows from the discussion preceding the lemma that $\alpha \in P$. Conversely, if $2r > n+1$, then

$$\begin{pmatrix} x_1 & x_2 & \dots & x_r \\ x_r & x_{r-1} & \dots & x_1 \end{pmatrix}$$

cannot lie in P, since now we would require $l_{1\overline{\alpha}} = l_r \leq u_1$.

□

We are now in a position to outline the proof of our main theorem, Theorem 6.8.13, which states that $\overline{H}$ is isomorphic to H, and hence to $G \wr \mathcal{S}_r$.

We first claim that for any $i, j \in I$ and $\lambda, \mu \in \Lambda$, if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$, then $f_{i,\lambda} = f_{j,\mu}$. We verify our claim via a series of steps. We first deal with the case where $\mathbf{p}_{\lambda i} = \varepsilon$ and here show that $f_{i,\lambda}$ (and $f_{j,\mu}$) is the identity of $\overline{H}$ (Lemma 6.3.1). Next, we verify the claim in the case where $\mu = \lambda$ (Lemma 6.4.1) or $i = j$ (Lemma 6.4.3). We then show that for $r \leq \frac{n}{2} - 1$, this is sufficient (via finite induction) to prove the claim holds in general (Lemma 6.5.1). However, a counterexample shows that for larger r this strategy will fail.

To overcome the above problem, we begin by showing that if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ is what we call a *simple form*, that is,

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \ldots & x_r \\ x_1 & x_2 & \cdots & x_{k-1} & x_{k+1} & x_{k+2} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \ldots & x_r \end{pmatrix},$$

for some $k \geq 1, m \geq 0, a \in G$, then $f_{i,\lambda} = f_{j,\mu}$. We then introduce the notion of *rising point* and verify by induction on the rising point, with the notion of simple form forming the basis of our induction, that our claim holds. As a consequence of our claim we denote a generator $f_{i,\lambda}$ with $\mathbf{p}_{\lambda i} = \phi$ by f_ϕ.

For $r \leq \frac{n}{2}$ it is easy to see that every element of H occurs as *some* $\mathbf{p}_{\lambda i}$ and for $r \leq \frac{n}{3}$ we have enough room for manoeuvre (the reader studying Sections 6.4 and 6.5 will come to an understanding of what this means) to show that $f_\phi f_\varphi = f_{\varphi\phi}$ and it is then easy to see that $\overline{H} \cong H$ (Theorem 6.5.3).

To deal with the general case of $r \leq n-2$ we face two problems. One is that for $r > \frac{n}{2}$, not every element of H occurs as some element of P and secondly, we need more sophisticated techniques to show that the multiplication in $\overline{H}$ behaves as we would like. To this end we show that $\overline{H}$ is generated by a restricted set of elements $f_{i,\lambda}$, such that the corresponding $\mathbf{p}_{\lambda i}$ form a standard set of generators of H (regarded as a wreath product). We then check that the corresponding identities to determine $G \wr S_r$ are satisfied by these generators, and it is then a short step to obtain our goal, namely, that $\overline{H} \cong H$ (Theorem 6.8.13). We note, however, that even at this stage more care is required than, for example, in the corresponding situation for $\mathcal{T}_n$ [21] or $\mathcal{PT}_n$ [5], since we cannot assume that G is finite. Indeed our particular choice of Schreier system will be seen to be a useful tool.

6.3 Identity generators

As stated at the end of Section 6.2, our first step is to show that if $(i, \lambda) \in K$ and $\mathbf{p}_{\lambda i} = \varepsilon$, then $f_{i,\lambda} = 1_{\overline{H}}$. Note that whenever we write $f_{i,\lambda} = 1_{\overline{H}}$ we mean that this relation can be deduced from the relations in the presentation $\langle F : \Sigma\rangle$. To this end we make use of our particular choice of Schreier system and function ω. The proof is by induction on $\lambda \in \Lambda$, where we recall that Λ is ordered lexicographically.

Lemma 6.3.1. *For any* $(i,\lambda) \in K$ *with* $\mathbf{p}_{\lambda i} = \varepsilon$, *we have* $f_{i,\lambda} = 1_{\overline{H}}$.

Proof. On page 63 we noted that $\mathbf{p}_{\omega(i),i} = \mathbf{q}_{\omega(i)}\mathbf{r}_i = \varepsilon$ for all $i \in I$. If $\mathbf{p}_{(1,2,\ldots,r)i} = \varepsilon$, that is, $\mathbf{q}_{(1,2,\cdots,r)}\mathbf{r}_i = \varepsilon$, then by definition of $\mathbf{q}_{(1,2,\cdots,r)}$ we have $x_1\mathbf{r}_i = x_1, \cdots, x_r\mathbf{r}_i = x_r$. Hence $\mathbf{r}_i$ lies in district $(1,2,\cdots,r)$, so that

$$\omega(i) = (1,2,\cdots,r).$$

Condition (R2) of the presentation $\mathcal{P}$ now gives that $f_{i,(1,2,\cdots,r)} = f_{i,\omega(i)} = 1_{\overline{H}}$.

Suppose now that $\mathbf{p}_{(u_1,u_2,\ldots,u_r)i} = \varepsilon$ where $(1,2,\ldots,r) < (u_1,u_2,\cdots,u_r)$. We make the inductive assumption that for any $(v_1,v_2,\cdots,v_r) < (u_1,u_2,\cdots,u_r)$, if $\mathbf{p}_{(v_1,v_2,\cdots,v_r)l} = \varepsilon$, for any $l \in I$, then $f_{l,(v_1,v_2,\ldots,v_r)} = 1_{\overline{H}}$.

With $u_0 = 0$, pick the largest number, say j, such that $u_j - u_{j-1} > 1$. By our choice of Schreier words, we have

$$\mathbf{h}_{(u_1,u_2,\cdots,u_r)} = \mathbf{h}_{(u_1,u_2,\cdots,u_{j-1},u_j-1,u_{j+1},\cdots,u_r)}\alpha_{(u_1,u_2,\cdots,u_r)},$$

where $\alpha_{(u_1,\cdots,u_r)}$ is defined by

$$\begin{pmatrix} x_1 & \cdots & x_{u_1} & x_{u_1+1} & \cdots & x_{u_2} & \cdots & x_{u_{r-1}+1} & \cdots & x_{u_r} & x_{u_r+1} & \cdots & x_n \\ x_{u_1} & \cdots & x_{u_1} & x_{u_2} & \cdots & x_{u_2} & \cdots & x_{u_r} & \cdots & x_{u_r} & x_{u_r} & \cdots & x_{u_r} \end{pmatrix}.$$

Suppose that the kernel of $\alpha_{(u_1,u_2,\cdots,u_r)}$ is l, so that $\alpha_{(u_1,u_2,\cdots,u_r)} = \varepsilon_{l(u_1,u_2,\ldots,u_r)}$.

By definition,

$$\mathbf{r}_l = \begin{pmatrix} x_1 & \ldots & x_{u_1} & x_{u_1+1} & \cdots & x_{u_2} & \cdots & x_{u_{r-1}+1} & \cdots & x_{u_r} & x_{u_r+1} & \cdots & x_n \\ x_1 & \ldots & x_1 & x_2 & \cdots & x_2 & \cdots & x_r & \cdots & x_r & x_r & \cdots & x_r \end{pmatrix}.$$

By choice of j we have $u_{j-1} < u_j - 1 < u_j$ so that $x_{u_j-1}\mathbf{r}_l = x_j$, giving

$$\mathbf{p}_{(u_1,u_2,\cdots,u_{j-1},u_j-1,u_{j+1},\cdots,u_r)l} = \varepsilon.$$

Since

$$(u_1, u_2, \cdots, u_{j-1}, u_j - 1, u_{j+1}, \cdots, u_r) < (u_1, u_2, \cdots, u_{j-1}, u_j, u_{j+1}, \cdots, u_r),$$

we call upon our inductive hypothesis to obtain

$$f_{l,(u_1,u_2,\cdots,u_{j-1},u_j-1,u_{j+1},\cdots,u_r)} = 1_{\overline{H}}.$$

On the other hand, we have

$$f_{l,(u_1,u_2,\cdots,u_r)} = f_{l,(u_1,u_2,\cdots,u_{j-1},u_j-1,u_{j+1},\cdots,u_r)}$$

by (R1), and so we conclude that $f_{l,(u_1,u_2,\cdots,u_r)} = 1_{\overline{H}}$.

Suppose that $\mathbf{r}_i$ lies in district $(l_1, l_2, \cdots, l_r)$. Since $\mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_i = \varepsilon$, we have $x_{u_k}\mathbf{r}_i = x_k$, so that $l_k \leq u_k$ by the definition of districts, for all $k \in [1, r]$. If $l_k = u_k$ for all $k \in [1, r]$, then

$$f_{i,(u_1,\ldots,u_r)} = f_{i,\omega(i)} = 1_{\overline{H}}$$

by $\mathcal{P}$. Otherwise, we let m be smallest such that $l_m < u_m$ and so (putting $u_0 = l_0 = 0$) we have

$$u_{m-1} = l_{m-1} < l_m < u_m.$$

Clearly

$$(u_1, u_2, \cdots, u_{m-1}, l_m, u_{m+1}, \cdots, u_r) \in \Lambda$$

and as $u_{m-1} < l_m < u_m$, we have $x_{l_m}\mathbf{r}_l = x_m$ by the definition of $\mathbf{r}_l$. We thus have the matrix equality

$$\begin{pmatrix} \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_l & \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_i \\ \mathbf{q}_{(u_1,u_2,\cdots,u_{m-1},l_m,u_{m+1},\cdots,u_r)}\mathbf{r}_l & \mathbf{q}_{(u_1,u_2,\cdots,u_{m-1},l_m,u_{m+1},\cdots,u_r)}\mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}.$$

Remember that we have already proven $f_{l,(u_1,u_2,\cdots,u_r)} = 1_{\overline{H}}$. Furthermore, as $l_m < u_m$ by assumption,

$$(u_1, u_2, \cdots, u_{m-1}, l_m, u_{m+1}, \cdots, u_r) < (u_1, u_2, \cdots, u_{m-1}, u_m, u_{m+1}, \cdots, u_r),$$

so that induction gives that

$$f_{i,(u_1,u_2,\cdots,u_{m-1},l_m,u_{m+1},\cdots,u_r)} = f_{l,(u_1,u_2,\cdots,u_{m-1},l_m,u_{m+1},\cdots,u_r)} = 1_{\overline{H}}.$$

From (R3) we deduce that $f_{i,(u_1,u_2,\cdots,u_r)} = 1_{\overline{H}}$ and the proof is completed. $\square$

6.4 Generators corresponding to the same rows or columns, and connectivity

The first aim of this section is to show that if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j} \neq 0$ where $\lambda = \mu$ or $i = j$, then $f_{i,\lambda} = f_{j,\mu}$. We begin with the more straightforward case, where $i = j$.

Lemma 6.4.1. *If $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu i}$, then $f_{i,\lambda} = f_{i,\mu}$.*

Proof. Let $\lambda = (u_1, \cdots, u_r)$ and $\mu = (v_1, \ldots, v_r)$. By hypothesis we have that

$$\mathbf{q}_{(u_1,\cdots,u_r)}\mathbf{r}_i = \mathbf{q}_{(v_1,\cdots,v_r)}\mathbf{r}_i = \psi \in H.$$

By definition of the $\mathbf{q}_\lambda$s we have $x_{u_j}\mathbf{r}_i = x_{v_j}\mathbf{r}_i$ for $1 \leq j \leq r$, and as rank $\mathbf{r}_i = r$ it follows that $u_j, v_j \in B^{\mathbf{r}_i}_{j'}$ where $j \mapsto j'$ is a bijection of $[1,r]$. We now define $\alpha \in \operatorname{End} F_n(G)$ by setting

$$x_{u_j}\alpha = x_j = x_{v_j}\alpha \text{ for all } j \in [1,r]$$

and

$$x_p\alpha = x_1 \text{ for all } p \in [1,n] \setminus \{u_1, \cdots, u_r, v_1, \cdots v_r\}.$$

Clearly $\alpha \in D_r$, indeed $\alpha \in L_1$. Since $w_m^\alpha = 1_G$ for all $m \in [1,n]$ and $\min\{u_j, v_j\} < \min\{u_k, v_k\}$ for $1 \leq j < k \leq r$, we certainly have that $\alpha = \mathbf{r}_l$ for some $l \in I$. By our choice of $\mathbf{r}_l$ we have the matrix equality

$$\begin{pmatrix} \mathbf{q}_{(u_1,\cdots,u_r)}\mathbf{r}_i & \mathbf{q}_{(u_1,\cdots,u_r)}\mathbf{r}_l \\ \mathbf{q}_{(v_1,\cdots,v_r)}\mathbf{r}_i & \mathbf{q}_{(v_1,\cdots,v_r)}\mathbf{r}_l \end{pmatrix} = \begin{pmatrix} \psi & \varepsilon \\ \psi & \varepsilon \end{pmatrix}.$$

Using Lemma 6.3.1 and (R3) of the presentation $\mathcal{P}$, we obtain

$$f_{i,(u_1,\cdots,u_r)} = f_{i,(v_1,\cdots,v_r)}$$

as required. $\square$

We need more effort for the case $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j}$. For this purpose we introduce the following notions of 'bad' and 'good' elements.

For any $i, j \in I$, suppose that $\mathbf{r}_i$ and $\mathbf{r}_j$ lie in districts $(1, k_2, \cdots, k_r)$ and $(1, l_2, \cdots, l_r)$, respectively. We call $u \in [1, n]$ a mutually *bad* element of $\mathbf{r}_i$ with respect to $\mathbf{r}_j$, if there exist $m, s \in [1, r]$ such that $u = k_m = l_s$, but $m \neq s$; all other elements are said to be mutually *good* with respect to $\mathbf{r}_i$ and $\mathbf{r}_j$. We call u a bad element of $\mathbf{r}_i$ with respect to $\mathbf{r}_j$ because, from the definition of districts, $\mathbf{r}_i$ maps x_{k_m} to x_m, and similarly, $\mathbf{r}_j$ maps x_{l_s} to x_s. Hence, if $u = k_m = l_s$ is bad, then it is impossible for us to find some $\mathbf{r}_t$ to make both $\mathbf{r}_i$ and $\mathbf{r}_j$ 'happy' in the point x_u, that is, for $\mathbf{r}_t$ (or, indeed, any other element of $\operatorname{End} F_n(G)$) to agree with both $\mathbf{r}_i$ and $\mathbf{r}_j$ on x_u.

Notice that if m is the minimum subscript such that $u = k_m$ is a bad element of $\mathbf{r}_i$ with respect to $\mathbf{r}_j$ and $k_m = l_s$, then s is also the minimum subscript such that l_s is a bad element of $\mathbf{r}_j$ with respect to $\mathbf{r}_i$. For, if $l_{s'} < l_s$ is a bad element of $\mathbf{r}_j$ with respect to $\mathbf{r}_i$, then by definition we have some $k_{m'}$ such that $l_{s'} = k_{m'}$ where $s' \neq m'$. By the minimality of m, we have $m' > m$ and so $l_{s'} = k_{m'} > k_m = l_s$, a contradiction. We also remark that since $l_1 = k_1 = 1$, the maximum possible number of bad elements is $r - 1$.

Let us run a simple example. Let $n = 7$ and $r = 4$, and suppose $\mathbf{r}_i$ lies in district $(1, 3, 4, 6)$ and $\mathbf{r}_j$ lies in district $(1, 4, 6, 7)$. By definition, $x_4\mathbf{r}_i = x_3$ and $x_6\mathbf{r}_i = x_4$, while $x_4\mathbf{r}_j = x_2$ and $x_6\mathbf{r}_j = x_3$. Therefore, $\mathbf{r}_i$ and $\mathbf{r}_j$ differ on x_4 and x_6, so that we say 4 and 6 are bad elements of $\mathbf{r}_i$ with respect to $\mathbf{r}_j$.

Lemma 6.4.2. *For any $i, j \in I$, suppose that $\mathbf{r}_i$ and $\mathbf{r}_j$ lie in districts $(1, k_2, \cdots, k_r)$ and $(1, l_2, \cdots, l_r)$, respectively. Let $\mathbf{q}_{(u_1, \cdots, u_r)}\mathbf{r}_i = \mathbf{q}_{(u_1, \cdots, u_r)}\mathbf{r}_j = \psi \in H$. Suppose $\{1, l_2, \cdots, l_s\}$ is a set of good elements of $\mathbf{r}_i$ with respect to $\mathbf{r}_j$ such that*

$$1 < l_2 < \cdots < l_s < k_{s+1} < \cdots < k_r.$$

Then there exists $p \in I$ such that $\mathbf{r}_p$ lies in district

$$(1, l_2, \cdots, l_s, k_{s+1}, \cdots, k_r)$$

and

$$\mathbf{q}_{(u_1, \cdots, u_r)}\mathbf{r}_p = \psi \text{ and } f_{p,(u_1, \cdots, u_r)} = f_{i,(u_1, \cdots, u_r)}.$$

Further, if $s = r$ then we can take $p = j$.

Proof. We begin by defining $\alpha \in D_r$, starting by setting $x_{k_m}\alpha = x_m$, $m \in [1, r]$. Now for $m \in [1, s]$ we put $x_{l_m}\alpha = x_m$. Notice that for $1 \le m \le s$, if $k_{m'} = l_m$ for $m' \in [1, r]$, then by the goodness of $\{1, l_2, \cdots, l_s\}$ we have that $m' = m$. We now set $x_{u_m}\alpha = x_{u_m}\mathbf{r}_i$ for $m \in [1, r]$. Again, we need to check we are not violating well-definedness. Clearly we need only check the case where $u_m = l_{m'}$ for some $m' \in [1, s]$, since here we have already defined $x_{l_{m'}}\alpha = x_{m'}$. We now use the fact that by our hypothesis, $x_{u_m}\mathbf{r}_i = x_{u_m}\mathbf{r}_j$ for all $m \in [1, r]$, so that $x_{u_m}\mathbf{r}_i = x_{u_m}\mathbf{r}_j = x_{l_{m'}}\mathbf{r}_j = x_{m'}$. Finally, we set $x_m\alpha = x_1$, for all $m \in [1, n] \setminus \{1, l_1, \cdots, l_s, k_2, \cdots, k_r, u_1, \cdots, u_r\}$.

We claim that $\alpha = \mathbf{r}_t$ for some $t \in I$. First, it is clear from the definition that $\alpha \in D_r$, indeed, $\alpha \in L_1$. We also have that for $1 \le m \le s$, $x_{l_m}\alpha = x_{k_m}\alpha = x_m$ and also for $s + 1 \le m \le r$, $x_{k_m}\alpha = x_m$. We claim that for $m \in [1, s]$ we have $l^\alpha_m = v_m$ where $v_m = \min\{k_m, l_m\}$ and for $m \in [s + 1, n]$ we have $l^\alpha_m = k_m$. It is clear that $1 = l^\alpha_1$. Suppose that for $m \in [2, r]$ we have $x_{u_q}\alpha = ax_m$. By definition, $x_{u_q}\mathbf{r}_i = ax_m = x_{u_q}\mathbf{r}_j$, so that $k_m, l_m \le u_q$ and our claim holds. It is now clear that $\alpha = \mathbf{r}_t$ for some $t \in I$ and lies in district $(v_1, \cdots, v_s, k_{s+1}, \cdots, k_r)$.

Having constructed $\mathbf{r}_t$, it is immediate that

$$\begin{pmatrix} \mathbf{q}_{(1,k_2,\cdots,k_r)}\mathbf{r}_i & \mathbf{q}_{(1,k_2,\cdots,k_r)}\mathbf{r}_t \\ \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_i & \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_t \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \psi & \psi \end{pmatrix},$$

so that in view of Corollary 6.2.3 and (R3) we deduce that

$$f_{i,(u_1,u_2\cdots,u_r)} = f_{t,(u_1,u_2,\cdots,u_r)}.$$

Notice now that if $s = r$ then

$$\begin{pmatrix} \mathbf{q}_{(1,l_2,\cdots,l_r)}\mathbf{r}_t & \mathbf{q}_{(1,l_2,\cdots,l_r)}\mathbf{r}_j \\ \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_t & \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_j \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \psi & \psi \end{pmatrix},$$

which leads to $f_{t,(u_1,u_2,\cdots,u_r)} = f_{j,(u_1,u_2,\cdots,u_r)}$, and so that $f_{j,(u_1,\cdots,u_r)} = f_{i,(u_1,\cdots,u_r)}$ as required.

Without the assumption that $s = r$, we now define $\mathbf{r}_p$ in a similar, but slightly more straightforward way, to $\mathbf{r}_t$. Namely, we first define $\beta \in \operatorname{End} F_n(G)$ by putting $x_{l_m}\beta = x_m$ for $m \in [1, s]$, $x_{k_m}\beta = x_m$ for $m \in [s+1, r]$, $x_{u_m}\beta = x_{u_m}\mathbf{r}_i$ for $m \in [1, r]$

and $x_m\beta = x_1$ for $m \in [1, n] \setminus \{1, l_1, \cdots, l_s, k_{s+1}, \cdots, k_r, u_1, \cdots, u_r\}$. It is easy to check that $\beta = \mathbf{r}_p$ where $\mathbf{r}_p$ lies in district $(1, l_2, \cdots, l_s, k_{s+1}, \cdots, k_r)$. Moreover, we have

$$\begin{pmatrix} \mathbf{q}_{(1,l_2,\cdots,l_s,k_{s+1},\cdots,k_r)}\mathbf{r}_t & \mathbf{q}_{(1,l_2,\cdots,l_s,k_{s+1},\cdots,k_r)}\mathbf{r}_p \\ \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_t & \mathbf{q}_{(u_1,u_2,\cdots,u_r)}\mathbf{r}_p \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon \\ \psi & \psi \end{pmatrix},$$

which leads to $f_{t,(u_1,u_2,\cdots,u_r)} = f_{p,(u_1,u_2,\cdots,u_r)}$, and so to $f_{p,(u_1,\cdots,u_r)} = f_{i,(u_1,\cdots,u_r)}$ as required. □

Lemma 6.4.3. *If $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j}$, then $f_{i,\lambda} = f_{j,\lambda}$.*

Proof. Suppose that $\mathbf{r}_i$ and $\mathbf{r}_j$ lie in districts $(1, k_2, \cdots, k_r)$ and $(1, l_2, \cdots, l_r)$, respectively. Let $\lambda = (u_1, \ldots, u_r)$ so that $\mathbf{q}_{(u_1,\cdots,u_r)}\mathbf{r}_i = \mathbf{q}_{(u_1,\cdots,u_r)}\mathbf{r}_j = \psi \in H$ say. We proceed by induction on the number of mutually bad elements. If this is 0, then the result holds by Lemma 6.4.2. We make the inductive assumption that if $\mathbf{p}_{\lambda l} = \mathbf{p}_{\lambda t}$ and $\mathbf{r}_l, \mathbf{r}_t$ have $k-1$ bad elements, where $0 < k \leq r-1$, then $f_{l,\lambda} = f_{t,\lambda}$.

Suppose now that $\mathbf{r}_j$ has k bad elements with respect to $\mathbf{r}_i$. Let s be the smallest subscript such that l_s is bad element of $\mathbf{r}_j$ with respect to $\mathbf{r}_i$. Then there exists some m such that $l_s = k_m$. Note, m is also the smallest subscript such that k_m is bad, as we explained before. Certainly $s, m > 1$; without loss of generality, assume $s > m$. Then $1 = l_1, l_2, \cdots, l_{s-1}$ are all good elements and

$$1 < l_2 < \cdots < l_{s-1} < k_s < \cdots < k_r.$$

By Lemma 6.4.2, there exists $p \in I$ such that $\mathbf{r}_p$ lies in district

$$(1, l_2, \cdots, l_{s-1}, k_s, \cdots, k_r),$$

$\mathbf{q}_{(u_1,\cdots,u_r)}\mathbf{r}_p = \psi$ and $f_{p,(u_1,\cdots,u_r)} = f_{i,(u_1,\cdots,u_r)}$.

We consider the sets B and C of mutually bad elements of $\mathbf{r}_j$ and $\mathbf{r}_p$, and of $\mathbf{r}_j$ and $\mathbf{r}_i$, respectively. Clearly $B \subseteq \{l_s, l_{s+1}, \cdots, l_r\}$. We have $l_s = k_m < k_s$, so that $l_s \notin B$. On the other hand if $l_q \in B$ where $s+1 \leq q \leq r$, then we must have $l_q = k_{q'}$ for some $q' \geq s$ with $q' \neq q$, so that $l_q \in C$. Thus $|B| < |C| = k$. Our inductive hypothesis now gives that $f_{p,(u_1,\cdots,u_r)} = f_{j,(u_1,\cdots,u_r)}$ and we deduce that $f_{i,(u_1,\cdots,u_r)} = f_{j,(u_1,\cdots,u_r)}$ as required. □

Definition 6.4.4. *Let $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$. We say that $(i, \lambda), (j, \mu)$ are* connected *if there exist*

$$i = i_0, i_1, \ldots, i_m = j \in I \text{ and } \lambda = \lambda_0, \lambda_1, \ldots, \lambda_m = \mu \in \Lambda$$

such that for $0 \leq k < m$ we have $\mathbf{p}_{\lambda_k i_k} = \mathbf{p}_{\lambda_k, i_{k+1}} = \mathbf{p}_{\lambda_{k+1} i_{k+1}}$.

The following picture illustrates that $(i, \lambda) = (i_0, \lambda_0)$ is connected to $(j, \mu) = (i_m, \lambda_m)$.

$$\begin{tikzcd}
\mathbf{p}_{\lambda_0 i_0} \arrow[r, dash] & \mathbf{p}_{\lambda i_1} \arrow[d, dash] & & & \\
& \mathbf{p}_{\lambda_1 i_1} \arrow[r, dash] & \mathbf{p}_{\lambda_1 i_2} \arrow[dr, dashed, dash] & & \\
& & & \mathbf{p}_{\lambda_{m-1} i_{m-1}} \arrow[r, dash] & \mathbf{p}_{\lambda_{m-1} i_m} \arrow[d, dash] \\
& & & & \mathbf{p}_{\lambda_m i_m}
\end{tikzcd}$$

Figure 6.1: the connectivity of (i, λ) and (j, μ)

Lemmas 6.4.1 and 6.4.3 now yield:

Corollary 6.4.5. *Let $i, j \in I$ and $\lambda, \mu \in \Lambda$ be such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$ where $(i, \lambda), (j, \mu)$ are connected. Then $f_{i,\lambda} = f_{j,\mu}$.*

6.5 The result for restricted r

We are now in a position to finish the proof of our first main result, Theorem 6.5.3, in a relatively straightforward way. Of course, in view of Theorem 6.8.13, it is not strictly necessary to provide such a proof here. However, the techniques used will be useful in the remainder of this chapter.

Let $\alpha = \mathbf{p}_{\lambda i} \in P$ and suppose that $\lambda = (u_1, \cdots, u_r)$ and $\mathbf{r}_i$ lies in district $(l_1, \cdots, l_r)$. Define

$$U(\lambda, i) = \{l_1, \cdots, l_r, u_1, \cdots, u_r\} \text{ and } S(\lambda, i) = [1, n] \setminus U,$$

where $U = U(\lambda, i)$.

Step D: moving ls down: Suppose that $l_j < t < l_{j+1}$ and $t \in S(\lambda, i)$. Define $\mathbf{r}_k$ by

$$x_t\mathbf{r}_k = x_{j+1} \text{ and } x_s\mathbf{r}_j = x_s\mathbf{r}_i \text{ for } s \neq t.$$

It is easy to see that $\mathbf{r}_k \in \Theta$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda k}$ and $\mathbf{r}_k$ lies in district

$$(l_1, \ldots, l_j, t, l_{j+2}, \ldots, l_r).$$

Clearly, (i, λ) is connected to (k, λ).

Step U: moving us up: Suppose that $u_j < t < u_{j+1}$ or $u_r < t$, where $t \in S(\lambda, i)$. Define $\mathbf{r}_m$ by

$$x_t\mathbf{r}_m = x_{u_j}\mathbf{r}_i \text{ and } x_s\mathbf{r}_m = x_s\mathbf{r}_i \text{ for } s \neq t.$$

It is easy to see that $\mathbf{r}_m \in \Theta$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda m}$ and $\mathbf{r}_m$ lies in district $(l_1, l_2, \ldots, l_r)$. Let

$$\mu = (u_1, \ldots, u_{j-1}, t, u_{j+1}, \ldots, u_r).$$

Clearly, $\mathbf{p}_{\lambda m} = \mathbf{p}_{\mu m}$ so that (i, λ) is connected to (m, μ).

Step U′: moving us down: Suppose that $t < u_{j+1}$ and $[t, u_{j+1}) \subseteq S(\lambda, i)$. Define $\mathbf{r}_l$ by

$$x_t\mathbf{r}_l = x_{u_{j+1}}\mathbf{r}_i \text{ and } x_s\mathbf{r}_m = x_s\mathbf{r}_i \text{ for } s \neq t.$$

It is easy to see that $\mathbf{r}_l \in \Theta$, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda l}$. Further, $\mathbf{r}_l$ lies in district $(l_1, l_2, \ldots, l_r)$ unless $u_{j+1} = l_{(j+1)\overline{\alpha}}$, in which case $l_{(j+1)\overline{\alpha}}$ is replaced by t. Let

$$\mu = (u_1, \ldots, u_j, t, u_{j+2}, \ldots, u_r);$$

clearly, $\mathbf{p}_{\lambda l} = \mathbf{p}_{\mu l}$, so that (i, λ) is connected to (l, μ).

Lemma 6.5.1. *Suppose that $n \geq 2r + 1$. Let $\lambda = (u_1, \cdots, u_r) \in \Lambda$, and $i \in I$ with $\mathbf{p}_{\lambda i} \in H$. Then we have that (i, λ) is connected to (j, μ) for some $j \in I$ and $\mu = (n - r + 1, \cdots, n)$. Consequently, if $\mathbf{p}_{\lambda i} = \mathbf{p}_{\nu k}$ for any $i, k \in I$ and $\lambda, \nu \in \Lambda$, then $f_{i,\lambda} = f_{k,\nu}$.*

Proof. Suppose that $\mathbf{r}_i$ lies in district $(l_1, \cdots, l_r)$. For the purposes of this proof, let

$$W(\lambda, i) = \sum_{k=1}^{r}(u_k - l_k);$$

clearly $W(\lambda, i)$ takes greatest value T where

$$(l_1, \ldots, l_r) = (1, \ldots, r) \text{ and } (u_1, \ldots, u_r) = (n - r + 1, \ldots, n).$$

Of course, here $W(\lambda, i)$ can be a negative integer, however, it has a minimal value that it can attain, i.e. is bounded below. We verify our claim by finite induction, with starting point T, under the reverse of the usual ordering on $\mathbb{Z}$. We have remarked that our result holds if $W(\lambda, i) = T$.

Suppose now that $W(\lambda, i) < T$ and the result is true for all pairs (ν, l) where $W(\lambda, i) < W(\nu, l) \leq T$.

If $u_r < n$, then as certainly $l_r \leq u_{r\overline{\mathbf{p}_{\lambda i}}^{-1}} \leq u_r$, we can apply Step U to show that (i, λ) is connected to (l, ν) where

$$\nu = (u_1, \ldots, u_{r-1}, u_r + 1)$$

and $\mathbf{r}_l$ lies in district $(l_1, \ldots, l_r)$. Clearly $W(\lambda, i) < W(\nu, l)$.

Suppose that $u_r = n$. We know that $l_1 = 1$, and by our hypothesis that $2r + 1 \leq n$, certainly $S(\lambda, i) \neq \emptyset$. If there exists $t \in S(\lambda, i)$ with $t < l_w$ for some $w \in [1, r]$, then choosing k with $l_k < t < l_{k+1}$, we have by Step D that (i, λ) is connected to (l, λ), where $\mathbf{r}_l$ lies in district $(l_1, \ldots, l_k, t, l_{k+2}, \ldots, l_r)$; clearly then $W(\lambda, i) < W(\lambda, l)$. On the other hand, if there exists $t \in S(\lambda, i)$ with $u_w < t$ for some $w \in [1, r]$, then now choosing $k \in [1, r]$ with $u_k < t < u_{k+1}$, we use Step U to show that (i, λ) is connected to (m, ν) where

$$\nu = (u_1, \ldots, u_{k-1}, t, u_{k+1}, \ldots, u_r),$$

and $\mathbf{r}_m$ lies in district $(l_1, \ldots, l_r)$. Again, $W(\lambda, i) < W(\nu, m)$.

The only other possibility is that $S(\lambda, i) \subseteq (l_r, u_1)$, in which case, $W(\lambda, i) = T$, a contradiction. $\square$

In view of Lemma 6.5.1 and Lemma 6.2.7 we may define, for $r \leq \frac{n-1}{2}$ and $\phi \in H$, an element $f_\phi \in \overline{H}$, where $f_\phi = f_{\lambda,i}$ for some (any) $(i, \lambda) \in K$ with $\mathbf{p}_{\lambda i} = \phi$.

Lemma 6.5.2. *Let $r \leq n/3$. Then for any $\phi, \theta \in H$, we have*

$$f_{\phi\theta} = f_\theta f_\phi \text{ and } f_{\phi^{-1}} = f_\phi^{-1}.$$

Proof. Since $n \geq 3$ and $r \leq n/3$ we deduce that $2r+1 \leq n$. Define $\mathbf{r}_i$ by

$$x_j\mathbf{r}_i = x_j, j \in [1,r];\ x_j\mathbf{r}_i = x_{j-r}\phi\theta, j \in [r+1,2r];\ x_j\mathbf{r}_i = x_{j-2r}\theta, j \in [2r+1,3r]$$

and

$$x_j\mathbf{r}_i = x_1, j \in [3r+1,n].$$

Clearly, $\mathbf{r}_i \in \Theta$ and $\mathbf{r}_i$ lies in district $(1,\cdots,r)$. Next we define $\mathbf{r}_l$ by

$$x_j\mathbf{r}_l = x_j, j \in [1,r];\ x_j\mathbf{r}_l = x_{j-r}\phi, j \in [r+1,2r];\ x_j\mathbf{r}_l = x_{j-2r}, j \in [2r+1,3r];$$

and

$$x_j\mathbf{r}_l = x_1, j \in [3r+1,n].$$

Again, $\mathbf{r}_l$ is well defined and lies in district $(1,\cdots,r)$. By considering the submatrix

$$\begin{pmatrix} \mathbf{q}_{(r+1,\cdots,2r)}\mathbf{r}_l & \mathbf{q}_{(r+1,\cdots,2r)}\mathbf{r}_i \\ \mathbf{q}_{(2r+1,\cdots,3r)}\mathbf{r}_l & \mathbf{q}_{(2r+1,\cdots,3r)}\mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \phi & \phi\theta \\ \varepsilon & \theta \end{pmatrix},$$

of P, Corollary 6.2.3 gives that $f_{i,(r+1,\cdots,2r)} = f_{i,(2r+1,\cdots,3r)}f_{l,(r+1,\cdots,2r)}$, which in our new notation says $f_{\phi\theta} = f_\theta f_\phi$, as required.

Finally, since

$$1_{\overline{H}} = f_\varepsilon = f_{\phi\phi^{-1}} = f_{\phi^{-1}}f_\phi,$$

we have $f_{\phi^{-1}} = f_\phi^{-1}$. □

Theorem 6.5.3. *Let $r \leq n/3$. Then $\overline{H}$ is isomorphic to H under $\boldsymbol{\psi}$, where $f_\phi\boldsymbol{\psi} = \phi^{-1}$.*

Proof. We have that $\overline{H} = \{f_\phi : \phi \in H\}$ by Lemma 6.5.2 and $\boldsymbol{\psi}$ is well defined, by Lemma 6.5.1. By Lemma 6.2.7, $\boldsymbol{\psi}$ is onto and it is a homomorphism by Lemma 6.5.2. Now $f_\phi\boldsymbol{\psi} = \varepsilon$ means that $\phi = \varepsilon$, so that $f_\phi = 1_{\overline{H}}$ by Lemma 6.3.1. Consequently, $\boldsymbol{\psi}$ is an isomorphism as required. □

6.6 Non-identity generators with simple form

First we explain the motivation for this section. It follows from Section 6.5 that for any r and n with $n \geq 2r+1$, all entries in the sandwich matrix P are connected. However, this connectivity will fail for higher ranks. Hence, the aim here is to identify the connected entries in P in the case of higher rank. It turns out that entries with *simple form* are always connected. For the reason given in the abstract, we know that for $r = n-1$ the maximal subgroup is free, and for $r = n$ it is trivial. Hence from now on we may assume that $1 \le r \le n-2$.

We run an easy example to explain the lack of connectivity for $r \ge n/2$.

Let $n = 4$, $r = 2$, and

$$\alpha = \begin{pmatrix} x_1 & x_2 \\ ax_1 & bx_2 \end{pmatrix},$$

with $a, b \neq 1_G \in G$. It is clear from Lemma 3.10 that there exists $i \in I$, $\lambda \in \Lambda$ such that $\alpha = \mathbf{p}_{\lambda i} \in P$, in fact we can take

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & ax_1 & bx_2 \end{pmatrix}$$

and $\lambda = (3, 4)$.

How many copies of α occur in the sandwich matrix P? Suppose that $\alpha = \mathbf{p}_{\mu j}$ where $\mathbf{r}_j$ lies in district (l_1, l_2) and $\mu = (u_1, u_2)$. Since $\overline{\alpha}$ is the identity of $\mathcal{S}_2$, and $a, b \neq 1_G$, we must have $1 = l_1 < l_2$, $u_1 < u_2$, $l_1 < u_1$, $l_2 < u_2$ and $\{l_1, l_2\} \cap \{u_1, u_2\} = \varnothing$. Thus the only possibilities are

$$(l_1, l_2) = (1, 2), (u_1, u_2) = (3, 4) = \lambda$$

and

$$(l_1, l_2) = (1, 3), (u_1, u_2) = (2, 4) = \mu.$$

In the first case, $\alpha = \mathbf{p}_{\lambda i}$ and in the second, $\alpha = \mathbf{p}_{\mu j}$ where

$$\mathbf{r}_j = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & ax_1 & x_2 & bx_2 \end{pmatrix}.$$

Clearly then, $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j} \in H$ but (i, λ) is *not* connected to (j, μ).

We know from Lemma 6.2.7, that in case $r \ge n/2$, not every element of H lies in P. However, we are guaranteed that certainly all elements with simple form

$$\phi = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \dots & x_r \\ x_1 & x_2 & \cdots & x_{k-1} & x_{k+1} & x_{k+2} & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \dots & x_r \end{pmatrix},$$

where $k \geq 1, m \geq 0, a \in G$, lie in P. In particular, we can choose

$$\mathbf{r}_{l_0} = \begin{pmatrix} x_1 & x_2 & \cdots & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & \cdots & x_{r+1} & x_{r+2} & \cdots & x_n \\ x_1 & x_2 & \cdots & \cdots & x_{k+m} & ax_k & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}$$

and $\mu_0 = (1, \cdots, k-1, k+1, \cdots, r+1)$ to give $\mathbf{p}_{\mu_0 l_0} = \mathbf{q}_{\mu_0}\mathbf{r}_{l_0} = \phi$. We now proceed to show that if $\mathbf{p}_{\lambda i} = \phi \neq \varepsilon$, then (i, λ) is connected to (j, μ_0) for some $j \in I$ and hence to (l_0, μ_0).

Lemma 6.6.1. *Let $\varepsilon \neq \phi$ be as above and suppose that $\phi = \mathbf{p}_{\lambda i} \in H$ where $\lambda = (u_1, \cdots, u_r)$ and $\mathbf{r}_i$ lies in district $(l_1, \cdots, l_r)$. Then (i, λ) is connected to (j, μ_0) for some $j \in I$.*

Proof. Notice that as $\phi = \mathbf{p}_{\lambda i}$, we have $x_{u_k} r_i = x_k \phi$, so that $x_{u_k} r_i = x_{k+1}$ if $m > 0$, and so $u_k \geq l_{k+1} > l_k$ by Lemma 6.2.6; or if $m = 0$ and $a \neq 1_G$, $x_{u_k} r_i = ax_k$ so that $u_k > l_k$ by Lemma 6.2.6 again. Further, from the constraints on $(l_1, \cdots, l_r)$ it follows that

$$l_1 < l_2 < \cdots < l_{k-1} < l_k < u_k.$$

We first ensure that (i, λ) is connected to some (j, κ) where

$$\kappa = (1, \ldots, k-1, u_k, \ldots, u_r),$$

by induction on $(l_1, \cdots, l_{k-1}) \in [1, n]^r$ under the lexicographic order.

If $(l_1, \cdots, l_{k-1}) = (1, \cdots, k-1)$, then clearly $(i, \lambda) = (i, \kappa)$. Suppose now that $(l_1, \cdots, l_{k-1}) > (1, \cdots, k-1)$ and the result is true for all

$$(l'_1, \cdots, l'_{k-1}) \in [1, n]^r \text{ where } (l'_1, \cdots, l'_{k-1}) < (l_1, \cdots, l_{k-1}),$$

namely, if $\mathbf{p}_{\eta l} = \phi$ with $\mathbf{r}_l$ in district $(l'_1, \cdots, l'_r)$, then (l, η) is connected to some (j, κ).

By putting $\nu = (l_1, \cdots, l_{k-1}, u_k, \cdots, u_r)$ we have $\mathbf{p}_{\nu i} = \mathbf{p}_{\lambda i}$. Since

$$(l_1, \cdots, l_{k-1}) > (1, \cdots, k-1),$$

there must be a $t \in (l_s, l_{s+1}) \cap S(\nu, i)$ for some $s \in [0, k-2]$, where $l_0 = 0$. We can use Step D to move l_{s+1} down to t, obtaining $\mathbf{r}_p$ in district $(l_1, \ldots, l_s, t, l_{s+2}, \ldots, l_r)$ such that $\mathbf{p}_{\nu i} = \mathbf{p}_{\nu p}$. Clearly $(l_1, \cdots, l_s, t, l_{s+2}, \cdots, l_{k-1}) < (l_1, \cdots, l_{k-1})$, so that by induction (p, ν) (and hence (i, λ)) is connected to some (j, κ).

We now proceed via induction on $(u_k, \ldots, u_r) \in [k+1, n]^r$ under the lexicographic order to show that (j, κ) is connected to some (l, μ) where

$$\mu = (1, \cdots, k-1, k+1, \cdots, r+1).$$

Clearly, this is true for $(u_k, \ldots, u_r) = (k+1, \cdots, r+1)$.

Suppose that $(u_k, \ldots, u_r) > (k+1, \ldots, r+1)$, and the result is true for all $(v_k, \cdots, v_r) \in [k+1, n]^r$ where $(v_k, \cdots, v_r) < (u_1, \cdots, u_r)$. Then we define $\mathbf{r}_w$ by:

$$x_l \mathbf{r}_w = x_l, l \in [1, k],\ x_{u_l} \mathbf{r}_w = x_{u_l} \mathbf{r}_j, l \in [k, r] \text{ and } x_v \mathbf{r}_w = x_1 \text{ for all other } x_v.$$

It is easy to see that $\mathbf{r}_w \in \Theta$, $\mathbf{r}_w$ lies in district

$$(1, 2, \cdots, k, u_k, \cdots, u_{k+m-1}, u_{k+m+1}, \cdots, u_r)$$

and $\mathbf{p}_{\kappa j} = \mathbf{p}_{\kappa w}$. There must be a $t < u_h$ for some $h \in [k, r]$ with $[t, u_h) \subseteq S(\kappa, w)$. By Step U$'$, we have that (w, κ) is connected to (v, ρ) where

$$\rho = (1, \ldots, k-1, u_k, \ldots, u_{h-1}, t, u_{h+1}, \ldots, u_r).$$

Clearly,

$$(u_k, \ldots, u_{h-1}, t, u_{h+1}, \ldots, u_r) < (u_k, \cdots, u_{h-1}, u_h, u_{h+1}, \cdots, u_r),$$

so that by induction (v, ρ) is connected to (l, μ). The proof is completed. $\square$

The following corollary is immediate from Lemma 6.3.1 , Corollary 6.4.5 and Lemma 6.6.1.

Corollary 6.6.2. *Let $\mathbf{p}_{\lambda i} = \mathbf{p}_{\nu k}$ have simple form. Then $f_{i,\lambda} = f_{k,\nu}$.*

6.7 Non-identity generators with arbitrary form

Our aim here is to show that for any $\alpha \in H$, if $i, j \in I$ and $\lambda, \mu \in \Lambda$ with

$$\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j} = \alpha \in H,$$

then $f_{i,\lambda} = f_{j,\mu}$. This property of α is called *consistency*. Notice that Corollary 6.6.2 tells us that all elements with simple form are consistent.

Before we explain the strategy in this section, we run the following example by the reader, which shows that if $|G| > 1$, we cannot immediately separate an element $\alpha \in \overline{H}$ into a product, $\beta\gamma$ or $\gamma\beta$, where β is essentially an element of $\mathcal{S}_r$, and $\overline{\gamma}$ is the identity in $\mathcal{S}_r$.

Let $1_G \neq a$, $n = 6$ and $r = 4$, so that $\alpha = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & ax_2 & x_4 & x_1 \end{pmatrix} \in H$. By putting

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 & ax_2 & x_4 & x_1 \end{pmatrix}$$

and $\lambda = (3, 4, 5, 6)$, clearly we have $\mathbf{p}_{\lambda i} = \alpha$.

Next we argue that $i \in I$ and $\lambda \in \Lambda$ are unique such that $\mathbf{p}_{\lambda i} = \alpha$. Let $\mu = (u_1, u_2, u_3, u_4)$ and $\mathbf{r}_j$ lie in district (l_1, l_2, l_3, l_4) with $\mathbf{p}_{\mu j} = \alpha$; we show that $\mathbf{r}_j = \mathbf{r}_i$ and $\mu = \lambda$. Since $x_{u_1}\mathbf{r}_j = x_1\alpha = x_3$ by assumption, we must have $l_1 < l_2 < l_3 \leq u_1$, so that $u_1 \geq 3$. As $3 \leq u_1 < u_2 < u_3 < u_4 \leq n = 6$, we have $\mu = (u_1, u_2, u_3, u_4) = (3, 4, 5, 6) = \lambda$, and $(l_1, l_2) = (1, 2)$. Clearly then $\mathbf{r}_j = \mathbf{r}_i$.

Certainly $\alpha = \gamma\beta = \beta\gamma$ with

$$\gamma = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_2 & x_4 & x_1 \end{pmatrix}, \beta = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & ax_2 & x_3 & x_4 \end{pmatrix}.$$

Our question is, can we find a sub-matrix of P with one of the following forms:

$$\begin{pmatrix} \gamma & \alpha \\ \varepsilon & \beta \end{pmatrix} \text{ or } \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}.$$

Clearly, here the answer is in the negative, as it is easy to see from the definition $\mathbf{r}_i$ that there does not exist $\nu \in \Lambda$ with $\mathbf{p}_{\nu i} = \beta$ or $\mathbf{p}_{\nu i} = \gamma$.

Now it is time for us to explain our trick of how to split an arbitrary element α in H into a product of elements with simple form (defined in the previous section), and moreover, how this splitting matches the products of generators $f_{i,\lambda}$ in $\overline{H}$.

Our main strategy is as follows. We introduce a notion of 'rising point' of $\alpha \in H$. Now, given $\mathbf{p}_{\lambda i} = \alpha$, we decompose α as a product $\alpha = \beta\gamma$ *depending only on* α such that γ is an element with simple form, $\beta = \mathbf{p}_{\lambda j}$ has a lower rising point than α, $\gamma = \mathbf{p}_{\mu i}$ for some $j \in I, \mu \in \Lambda$ such that our presentation gives

$f_{i,\lambda} = f_{i,\mu} f_{j,\lambda}$.

Definition 6.7.1. *Let $\alpha \in H$. We say that α has rising point $r+1$ if $x_m\alpha = ax_r$ for some $m \in [1,r]$ and $a \neq 1_G$; otherwise, the rising point is $k \leq r$ if there exists a sequence*

$$1 \leq i < j_1 < j_2 < \cdots < j_{r-k} \leq r$$

with

$$x_i\alpha = x_k, x_{j_1}\alpha = x_{k+1}, x_{j_2}\alpha = x_{k+2}, \cdots, x_{j_{r-k}}\alpha = x_r$$

and such that if $l \in [1,r]$ with $x_l\alpha = ax_{k-1}$, then if $l < i$ we must have $a \neq 1_G$.

Now let me explain how one would go about computing the rising point value of an element α i.e. what would be the algorithm for computing it, which would convince our readers that the rising point value is uniquely determined by α.

To compute the rising point value k of an element $\alpha \in G \wr \mathcal{S}_r$ one does the following:

(1) First look at the unique ax_r in the image of α. If $a \neq 1_G$ then set $k = r+1$.

(2) Otherwise, look to the left of x_r and see if ax_{r-1} appears to the left in the image. If it does, check the value of a in ax_{r-1}. If $a = 1$ then repeat the process of looking left.

(3)Carrying out this process eventually one of two things must happen, either

(I) we stop because we reach some ax_{k-1} with $a \neq 1_G$. Then we say the rising point value is k. Or

(II) we reach $ax_k = 1x_k$ and do not see ax_{k-1} to the left so the process stops and the rising point value is k.

Now let us consider the symmetric group $\mathcal{S}_5$ and take the following two permutations:

$$\alpha_1 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_2 & x_4 & x_5 \end{pmatrix}, \alpha_2 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_4 & x_5 & x_3 \end{pmatrix}.$$

To find the rising point value of α_1, first we find where is our x_5 in the image, then look to the left we find our x_4, then look for the left of x_4, we find x_3, however, x_2 is to the right of x_3, and hence the rising point value of α_1 is 3. For α_2, by observing that x_4 is to the left of x_5 in the image, but x_3 is to the right of x_4, we deduce the rising point of α_2 is 4.

It is easy to see that the only element with rising point 1 is the identity of H, and elements with rising point 2 have either of the following two forms:

(i) $\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ ax_1 & x_2 & \cdots & x_r \end{pmatrix}$, where $a \neq 1_G$;

(ii) $\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_r \\ x_2 & x_3 & \cdots & x_k & ax_1 & x_{k+1} & \cdots & x_r \end{pmatrix}$, where $k \geq 2$.

Note that both of the above two forms are the so called simple forms; however, elements with simple form can certainly have rising point greater than 2, indeed, it can be $r+1$. From Lemma 6.3.1 and Corollary 6.6.2 we immediately deduce:

Corollary 6.7.2. *Let $\alpha \in H$ have rising point 1 or 2. Then α is consistent.*

Next, we will see how to decompose an element with a rising point at least 3 into a product of an element with a lower rising point and an element with simple form.

Lemma 6.7.3. *Let $\alpha \in H$ have rising point $k \geq 3$. Then α can be expressed as a product of some $\beta \in H$ with rising point no more than $k-1$ and some $\gamma \in H$ with simple form.*

Proof. Case (0) By definition of rising point, if $k = r+1$, then we have $x_m\alpha = ax_r$ for some $a \neq 1_G$ and $m \in [1, r]$. We define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{r-1} & x_r \\ x_1 & x_2 & \cdots & x_{r-1} & ax_r \end{pmatrix}$$

and β by $x_m\beta = x_r$ and for other $j \in [1, r]$, $x_j\beta = x_j\alpha$. Clearly, $\alpha = \beta\gamma$, γ is a simple form. Further, as in the image of β we have x_r so that by the algorithm we compute the rising point value, we know that β has rising point no greater than r.

On the other hand, if $k \leq r$ there exists a sequence

$$1 \leq i < j_1 < j_2 \cdots < j_{r-k} \leq r$$

with

$$x_i\alpha = x_k, x_{j_1}\alpha = x_{k+1}, x_{j_2}\alpha = x_{k+2}, \cdots, x_{j_{r-k}}\alpha = x_r$$

such that if $l \in [1, r]$ with $x_l\alpha = ax_{k-1}$, then if $l < i$ we must have $a \neq 1_G$. We proceed by considering the following cases:

Case (i) If $l < i$, so that $a \neq 1_G$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & ax_{k-1} & x_k & \cdots & x_r \end{pmatrix}$$

and put $\beta = \alpha\gamma^{-1}$. It is easy to check that $x_l\beta = x_l\alpha\gamma^{-1} = x_{k-1}$ and $x_p\beta = x_p\alpha$, for other $p \in [1, r]$.

Case (ii) If $i < l < j_1$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & x_{k+1} & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & x_k & ax_{k-1} & x_{k+1} & \cdots & x_r \end{pmatrix}$$

and again, we put $\beta = \alpha\gamma^{-1}$. By easy calculation we have

$$x_i\beta = x_{k-1}, x_l\beta = x_k, x_{j_1}\beta = x_{k+1}, \cdots, x_{j_{r-k}}\beta = x_r$$

and for other $p \in [1, r]$, $x_p\beta = x_p\alpha$.

Case (iii) If $j_{r-k} < l$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & x_{k+1} & \cdots & x_{r-1} & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & x_k & x_{k+1} & x_{k+2} & \cdots & x_r & ax_{k-1} \end{pmatrix}$$

and again, we define $\beta = \alpha\gamma^{-1}$. It is easy to see that

$$x_i\beta = x_{k-1}, x_{j_1}\beta = x_k, x_{j_2}\beta = x_{k+1}, \cdots, x_{j_{r-k}}\beta = x_{r-1}, x_l\beta = x_r$$

and for other $p \in [1, r]$, $x_p\beta = x_p\alpha$.

Case (iv) If $j_u < l < j_{u+1}$ for some $u \in [1, r-k-1]$, then define

$$\gamma = \begin{pmatrix} x_1 & x_2 & \cdots & x_{k-2} & x_{k-1} & x_k & \cdots & x_{k+u-1} & x_{k+u} & x_{k+u+1} & \cdots & x_r \\ x_1 & x_2 & \cdots & x_{k-2} & x_k & x_{k+1} & \cdots & x_{k+u} & ax_{k-1} & x_{k+u+1} & \cdots & x_r \end{pmatrix}$$

and again, we put $\beta = \alpha\gamma^{-1}$. Then we have

$$x_i\beta = x_{k-1}, x_{j_1}\beta = x_k, \ldots, x_{j_u}\beta = x_{k+u-1},$$

and

$$x_l\beta = x_{k+u}, x_{j_{u+1}}\beta = x_{k+u+1}, \ldots, x_{j_{r-k}}\beta = x_r$$

and for other $p \in [1, r]$, $x_p\beta = x_p\alpha$.

In each of Cases $(i) - (iv)$ it is clear that γ has simple form, $\alpha = \beta\gamma$ and β has a rising point no more than $k - 1$. The proof is completed. $\square$

Note that in each of Cases $(ii) - (iv)$ of Lemma 6.7.3, that is, where $i < l$, we have $x_p\beta = x_p\alpha$ for all $p < i$.

Lemma 6.7.4. *Let $\alpha, \beta, \gamma \in H$ with $\alpha = \beta\gamma$ and β, γ consistent. Suppose that whenever $\alpha = \mathbf{p}_{\lambda j}$, we can find $(t, \lambda), (j, \mu) \in K$ with $\beta = \mathbf{p}_{\lambda t}, \gamma = \mathbf{p}_{\mu j}$ and $f_{j,\lambda} = f_{j,\mu} f_{t,\lambda}$. Then α is consistent.*

Proof. Let α, β, γ satisfy the hypotheses of the lemma. If $\alpha = \mathbf{p}_{\lambda j} = \mathbf{p}_{\lambda' j'}$, then by assumption we can find

$$(t, \lambda), (j, \mu), (t', \lambda'), (j', \mu') \in K$$

with

$$\beta = \mathbf{p}_{\lambda t} = \mathbf{p}_{\lambda' t'}, \gamma = \mathbf{p}_{\mu j} = \mathbf{p}_{\mu' j'}, f_{j,\lambda} = f_{j,\mu} f_{t,\lambda} \text{ and } f_{j',\lambda'} = f_{j',\mu'} f_{t',\lambda'}.$$

The result now follows from the consistency of β and γ. $\square$

Proposition 6.7.5. *Every $\alpha \in P$ is consistent. Further, if $\alpha = \mathbf{p}_{\lambda j}$ then $f_{j,\lambda}$ is equal in $\overline{H}$ to a product $f_{i_1,\lambda_1} \cdots f_{i_k,\lambda_k}$, where $\mathbf{p}_{\lambda_t, i_t}$ is an element with simple form, $t \in [1, k]$.*

Proof. We proceed by induction on the rising point of α. If α has rising point 1 or 2, and $\mathbf{p}_{\lambda i} = \alpha$, then the result is true by Corollary 6.7.2 and the comments preceding it. Suppose for induction that the rising point of α is $k \geq 3$, and the result is true for all $\beta \in H$ with rising point strictly less than k and all $f_{i,\mu} \in F$ where $\mathbf{p}_{\mu i} = \beta$.

We proceed on a case by case basis, using γ and β as defined in Lemma 6.7.3. Since γ has simple form, it is consistent by Corollary 6.6.2 and as β has rising point strictly less than k, β is consistent by our inductive hypothesis.

Suppose that $\alpha = \mathbf{p}_{\lambda j}$ where $\lambda = (u_1, \ldots, u_r)$ and $\mathbf{r}_j$ lies in district $(l_1, \ldots, l_r)$.

Case (0) If $k = r + 1$, then we have $x_m\alpha = ax_r$ for some $a \neq 1_G$. We now define $\mathbf{r}_t$ by $x_{u_m}\mathbf{r}_t = x_r$ and $x_s\mathbf{r}_t = x_s\mathbf{r}_j$, for other $s \in [1, n]$. As $x_{u_m}\mathbf{r}_j = ax_r$, it is easy to see that $\mathbf{r}_t \in \Theta$. Notice that $l_{r-1} < l_r = l_{m\overline{\alpha}} < u_m$. Then by setting $\mu = (1, l_2, \cdots, l_{r-1}, u_m)$ we have

$$\begin{pmatrix} \mathbf{p}_{\lambda t} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu t} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

and our presentation gives $f_{j,\lambda} = f_{j,\mu} f_{t,\lambda}$.

We now suppose that $k \leq r$. By definition of rising point there exists a sequence

$$1 \leq i < j_1 < j_2 \cdots < j_{r-k} \leq r$$

such that

$$x_i\alpha = x_k, x_{j_1}\alpha = x_{k+1}, x_{j_2}\alpha = x_{k+2}, \cdots, x_{j_{r-k}}\alpha = x_r$$

such that if $l \in [1, r]$ with $x_l\alpha = ax_{k-1}$, then if $l < i$ we must have $a \neq 1_G$.

We consider the following cases:

Case (i) If $l < i$ we define $\mathbf{r}_t$ by $x_{u_l}\mathbf{r}_t = x_{k-1}$ and for other $p \in [1, n]$, $x_p\mathbf{r}_t = x_p\mathbf{r}_j$. As by assumption $x_{u_l}\mathbf{r}_j = x_l\alpha = ax_{k-1}$, clearly $\mathbf{r}_t \in \Theta$. Then by putting

$$\mu = (1, l_2, \cdots, l_{k-2}, u_l, u_i, u_{j_1}, \cdots, u_{j_{r-k}})$$

we have

$$\begin{pmatrix} \mathbf{p}_{\lambda t} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu t} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

which implies $f_{j,\lambda} = f_{j,\mu} f_{t,\lambda}$.

Case (ii) If $i < l < j_1$ we define $\mathbf{r}_s$ by

$$x_p\mathbf{r}_s = x_p\mathbf{r}_j \text{ for } p < u_i,\ x_{u_w}\mathbf{r}_s = x_w\beta \text{ for } i \leq w \leq r$$

and

$$x_v\mathbf{r}_s = x_1 \text{ for all other } v \in [1, n].$$

We must argue that $\mathbf{r}_s \in \Theta$. Note that from the comment following Lemma 6.7.3, for any $v < i$ we have that

$$x_{u_v}\mathbf{r}_s = x_{u_v}\mathbf{r}_j = x_v\alpha = x_v\beta,$$

so that in particular, rank $\mathbf{r}_s = r$. Further,

$$x_{u_i}\mathbf{r}_s = x_i\beta = x_{k-1}, x_{u_l}\mathbf{r}_s = x_l\beta = x_k$$

and

$$x_{u_{j_1}}\mathbf{r}_s = x_{j_1}\beta = x_{k+1}, \ldots, x_{u_{j_{r-k}}}\mathbf{r}_s = x_{j_{r-k}}\beta = x_r$$

so that

$$\langle x_{u_i}, x_{u_l}, x_{u_{j_1}}, \cdots, x_{u_{j_{r-k}}}\rangle\mathbf{r}_s = \langle x_{k-1}, x_k, \cdots, x_r\rangle.$$

Thus for any $v \neq \{i, l, j_1, \cdots, j_{r-k}\}$, $x_{u_v}\mathbf{r}_s = x_v\beta \in \langle x_1, \cdots, x_{k-2}\rangle$.

As $x_{u_i}\mathbf{r}_j = x_k$, we have $1 = l_1 < l_2 < \cdots < l_{k-1} < l_k \leq u_i$. Let h be the largest number with

$$1 = l_1 < l_2 < \cdots < l_{k-1} < l_k < l_{k+1} < \cdots < l_{(k-1)+h} < u_i.$$

Clearly here we have $h \in [0, r-k+1]$. Now we claim that $\mathbf{r}_s \in \Theta$ and lies in district

$$(l_1, l_2, \cdots, l_{(k-1)+h}, u_{j_h}, u_{j_{h+1}}, \cdots, u_{j_{r-k}}).$$

To simplify our notation we put

$$(l_1, l_2, \cdots, l_{(k-1)+h}, u_{j_h}, u_{j_{h+1}}, \cdots, u_{j_{r-k}}) = (z_1, z_2, \cdots, z_{(k-1)+h}, z_{k+h}, \cdots, z_r),$$

where $j_0 = l$. Clearly, by the definition of $\mathbf{r}_s$, we have $x_{z_v}\mathbf{r}_s = x_v$ for all $v \in [1, r]$. Hence, to show $\mathbf{r}_s \in \Theta$, by the definition we only need to argue that for any $m \in [1, n]$ and $b \in G$, $x_m\mathbf{r}_s = bx_t$ implies $m \geq z_t$.

Suppose that $t \in [1, (k-1)+h]$, so that $z_t = l_t < u_i$. If $m < z_t$, then from the definition of $\mathbf{r}_s$ we have $x_m\mathbf{r}_s = x_m\mathbf{r}_j$, so that $x_m\mathbf{r}_j = bx_t$. As $\mathbf{r}_j \in \Theta$ and $x_{l_t}\mathbf{r}_j = x_t$, we have $z_t = l_t \leq m$, a contradiction, and we deduce that $m \geq z_t$.

Suppose now that $t \in [k+h, r]$. Note that $m \geq u_i$; because, if $m < u_i$, then $x_m\mathbf{r}_j = x_m\mathbf{r}_s = bx_t$. As $\mathbf{r}_j \in \Theta$, $l_t \leq m < u_i$ and so $t \leq (k-1)+h$, a contradiction. Thus $m \geq u_i$. Now, by the definition of $\mathbf{r}_s$, we know there is exactly one possibility that $x_m\mathbf{r}_s = bx_t$ with $t \in [k+h, r]$, that is, $x_{z_t}\mathbf{r}_s = x_t$, so that $m = z_t$ and $b = 1$. Thus $\mathbf{r}_s \in \Theta$.

Now set

$$\eta = (1, l_2, \cdots, l_{k-2}, u_i, u_l, u_{j_1}, \cdots, u_{j_{r-k}})$$

then we have
$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\eta s} & \mathbf{p}_{\eta j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix},$$
which implies $f_{j,\lambda} = f_{j,\eta} f_{s,\lambda}$.

Case (iii) If $j_{r-k} < l$, then, defining $\mathbf{r}_s$ as in *Case (ii)*, a similar argument gives that $\mathbf{r}_s \in \Theta$ and $x_{u_v}\mathbf{r}_s = x_v\beta$ for all $v \in [1, r]$ (of course here β is defined differently to that given in *Case (ii)* and the district of $\mathbf{r}_s$ will have a different appearance.). Moreover, by setting

$$\delta = (1, l_2, \cdots, l_{k-2}, u_i, u_{j_1}, \cdots, u_{j_{r-k}}, u_l)$$

we have
$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\delta s} & \mathbf{p}_{\delta j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}.$$
implying $f_{j,\lambda} = f_{j,\delta} f_{s,\lambda}$.

Case (iv) If $j_u < l < j_{u+1}$ for some $u \in [1, r-k-1]$, then again by defining $\mathbf{r}_s$ as in *Case (ii)*, we have $\mathbf{r}_s \in \Theta$ and $x_{u_v}\mathbf{r}_s = x_v\beta$ for all $v \in [1, r]$. Take

$$\sigma = (1, l_2, \cdots, l_{k-2}, u_i, u_{j_1}, u_{j_u}, u_l, u_{j_{u+1}}, \cdots, u_{j_{r-k}}).$$

Then we have
$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\sigma s} & \mathbf{p}_{\sigma j} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$
so that $f_{j,\lambda} = f_{j,\sigma} f_{s,\lambda}$.

In each of the cases above, the consistency of α follows from Lemma 6.7.4. The result now follows by induction. $\square$

In view of Lemma 6.7.4, we can now denote all generators $f_{i,\lambda}$ with $\mathbf{p}_{\lambda i} = \alpha$ by f_α, where $(i, \lambda) \in K$.

6.8 The main theorem

Our eventual aim is to show that $\overline{H}$ is isomorphic to H and hence to the wreath product $G \wr \mathcal{S}_r$. With this in mind, given the knowledge we have gathered concerning the generators $f_{i,\lambda}$, we first specialise the general presentation given in

Theorem 4.2.4 to our specific situation.

We will say that for $\phi, \varphi, \psi, \sigma \in P$ the quadruple $(\phi, \varphi, \psi, \sigma)$ is *singular* if

$$\phi^{-1}\psi = \varphi^{-1}\sigma$$

and we can find $i, j \in I, \lambda, \mu \in \Lambda$ with

$$\phi = \mathbf{p}_{\lambda i}, \varphi = \mathbf{p}_{\mu i}, \psi = \mathbf{p}_{\lambda j} \text{ and } \sigma = \mathbf{p}_{\mu j}.$$

In the sequel, we denote the free group on a set X by $\tilde{X}$. For convenience, we use, for example, the same symbol $f_{i,\lambda}$ for an element of $\widetilde{F}$ and $\overline{H}$. We hope that the context will prevent ambiguities from arising.

Lemma 6.8.1. *Let $\overline{\overline{H}}$ be the group given by the presentation $\mathcal{Q} = \langle S : \Gamma\rangle$ with generators:*

$$S = \{f_\phi : \ \phi \in P\}$$

and with the defining relations Γ :

$(P1)$ $f_\phi^{-1} f_\varphi = f_\psi^{-1} f_\sigma$ where $(\phi, \varphi, \psi, \sigma)$ is singular;

$(P2)$ $f_\varepsilon = 1$.

Then $\overline{\overline{H}}$ is isomorphic to $\overline{H}$.

Proof. From Theorem 4.2.4, we know that $\overline{H}$ is given by the presentation

$$\mathcal{P} = \langle F : \Sigma\rangle,$$

where $F = \{f_{i,\lambda} : \ (i, \lambda) \in K\}$ and Σ is the set of relations as defined in (R1), (R2) and (R3), and where the function ω and the Schreier system $\{\mathbf{h}_\lambda : \lambda \in \Lambda\}$ are fixed as in Section 6.2. Note that (R3) is reformulated in Corollary 6.2.3.

By freeness of the generators we may define a morphism

$$\boldsymbol{\theta} : \widetilde{F} \to \overline{\overline{H}}, f_{i,\lambda}\boldsymbol{\theta} = f_\phi,$$

where $\phi = \mathbf{p}_{\lambda i}$. We show that $\Sigma \subseteq \ker \boldsymbol{\theta}$. It is clear from (P1) that relations of the form (R3) lie in $\ker \boldsymbol{\theta}$.

Suppose first that $\mathbf{h}_\lambda \varepsilon_{i\mu} = \mathbf{h}_\mu$ in $\overline{E}^*$. Then $\varepsilon\mathbf{h}_\lambda \varepsilon_{i\mu} = \varepsilon\mathbf{h}_\mu$ in $\operatorname{End} F_n(G)$, so that from Lemma 6.2.4, $\mathbf{q}_\lambda \varepsilon_{i\mu} = \mathbf{q}_\mu$. Hence $\mathbf{q}_\mu \mathbf{r}_i = \mathbf{q}_\lambda \varepsilon_{i\mu}\mathbf{r}_i = \mathbf{q}_\lambda \mathbf{r}_i$, so that $\mathbf{p}_{\mu i} = \mathbf{p}_{\lambda i}$

and $f_{i,\lambda}\boldsymbol{\theta} = f_{i,\mu}\boldsymbol{\theta}$. Now suppose that $i \in I$; we have remarked that $\mathbf{p}_{\omega(i)i} = \varepsilon$, so that $f_{i,\omega(i)}\boldsymbol{\theta} = f_{\varepsilon} = 1\boldsymbol{\theta}$.

We have shown that $\Sigma \subseteq \ker \boldsymbol{\theta}$ and so there exists a morphism

$$\overline{\boldsymbol{\theta}} : \overline{H} \to \overline{\overline{H}}, f_{i,\lambda}\overline{\boldsymbol{\theta}} = f_{\phi}$$

where $\phi = \mathbf{p}_{\lambda i}$.

Conversely, we define a map

$$\boldsymbol{\psi} : \widetilde{S} \to \overline{H}, f_{\phi}\boldsymbol{\psi} = f_{i,\lambda},$$

where $\phi = \mathbf{p}_{\lambda i}$. By Lemma 6.7.4, $\boldsymbol{\psi}$ is well defined. Since $f_{\varepsilon}\boldsymbol{\psi} = f_{i,\lambda}$ where $\mathbf{p}_{\lambda i} = \varepsilon$, we have $f_{\varepsilon}\boldsymbol{\psi} = 1_{\overline{H}}$ by Lemma 6.3.1. Clearly relations (P1) lie in $\ker \boldsymbol{\psi}$, so that $\Gamma \subseteq \ker \boldsymbol{\psi}$. Consequently, there is a morphism

$$\overline{\boldsymbol{\psi}} : \overline{\overline{H}} \to \overline{H}, f_{\phi}\overline{\boldsymbol{\psi}} = f_{i,\lambda},$$

where $\phi = \mathbf{p}_{\lambda i}$.

It is clear that $\overline{\boldsymbol{\theta}}\,\overline{\boldsymbol{\psi}}$ and $\overline{\boldsymbol{\psi}}\,\overline{\boldsymbol{\theta}}$ are, respectively, the identity maps on the generators of $\overline{H}$ and $\overline{\overline{H}}$, respectively. It follows immediately that they are mutually inverse isomorphisms. □

We now recall the presentation of $G \wr \mathcal{S}_r$ obtained by Lavers [34]. In fact, we translate his presentation to one for our group H.

We begin by defining the following elements of H: for $a \in G$ and for $1 \leq i \leq r$ we put

$$\iota_{a,i} = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_r \\ x_1 & \cdots & x_{i-1} & ax_i & x_{i+1} & \cdots & x_r \end{pmatrix};$$

for $1 \leq k \leq r-1$ we put

$$(k\ k{+}1 \cdots k{+}m) = \begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & \cdots & x_{k-1} & x_{k+1} & \cdots & x_{k+m} & x_k & x_{k+m+1} & \cdots & x_r \end{pmatrix}$$

and we denote $(k\ k+1)$ by τ_k.

It is clear that G^r has presentation $\mathcal{V} = \langle Z :\ \Pi \rangle$, with generators

$$Z = \{\iota_{a,i} : i \in [1, r], a \in G\}$$

and defining relations Π consisting of (W4) and (W5) below. Using the standard Coxeter presentation for $\mathcal{S}_r$ with generators the transpositions τ and relations $(W1)$, $(W2)$ and $(W3)$, we employ the recipe of [34] to obtain:

Lemma 6.8.2. *The group H has a presentation $\mathcal{U} = \langle Y : \Upsilon \rangle$, with generators*

$$Y = \{\tau_i, \iota_{a,j} : \ 1 \le i \le r-1, \ 1 \le j \le r, a \in G\}$$

and defining relations Υ:

$(W1)$ $\tau_i \tau_i = 1$, $1 \le i \le r-1$;
$(W2)$ $\tau_i \tau_j = \tau_j \tau_i$, $j \pm 1 \ne i \ne j$;
$(W3)$ $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$, $1 \le i \le r-2$;
$(W4)$ $\iota_{a,i} \iota_{b,j} = \iota_{b,j} \iota_{a,i}$, $a, b \in G$ *and* $1 \le i \ne j \le r$;
$(W5)$ $\iota_{a,i} \iota_{b,i} = \iota_{ab,i}$, $1 \le i \le r$ *and* $a, b \in G$;
$(W6)$ $\iota_{a,i} \tau_j = \tau_j \iota_{a,i}$, $1 \le i \ne j, j+1 \le r$;
$(W7)$ $\iota_{a,i} \tau_i = \tau_i \iota_{a,i+1}$, $1 \le i \le r-1$ *and* $a \in G$.

Now we turn to our maximal subgroup $\overline{H}$. From Lemma 6.8.1, we know that $\overline{H}$ is isomorphic to $\overline{\overline{H}}$, and it follows from the definition of the isomorphism and Proposition 6.7.5 that

$$\overline{\overline{H}} = \langle f_\alpha : \ \alpha \text{ has simple form} \rangle.$$

We now simplify our generators further. For ease in the remainder of this chapter, it is convenient to use the following convention: for $u, v \in [1, r+2]$ with $u < v$, we denote by $\neg(u, v)$ the r-tuple

$$(1, \cdots, u-1, u+1, \ldots, v-1, v+1, \cdots, r+2).$$

Lemma 6.8.3. *Consider the element*

$$\alpha = \begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & \cdots & x_r \\ x_1 & \cdots & x_{k-1} & x_{k+1} & \cdots & x_{k+m} & a x_k & x_{k+m+1} & \cdots & x_r \end{pmatrix}$$

in simple form, where $m \ge 1$. Then $f_\alpha = f_\gamma f_\beta$ in $\overline{\overline{H}}$, where

$$\beta = \iota_{a,k+m} \text{ and } \gamma = (k \ \ k+1 \ \ \cdots \ \ k+m).$$

Proof. Define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_k & ax_k & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(k, k+m+1)$ and $\mu = \neg(k, k+m+2)$. Then $\mathbf{p}_{\lambda t} = \alpha$ and $\mathbf{p}_{\mu t} = \gamma$.

Next we define $\mathbf{r}_s$ by

$$\begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k-1} & x_{k-1} & x_k & \cdots & x_{k+m-1} & x_{k+m} & ax_{k+m} & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \beta$ and $\mathbf{p}_{\mu s} = \varepsilon$. Notice that $\alpha = \beta\gamma$ and

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

which implies $f_\alpha = f_\gamma f_\beta$. $\square$

Lemma 6.8.4. *Let $\alpha = (k\ k+1\ \cdots\ k+m)$ where $m \geq 1$. Then, in $\overline{\overline{\overline{H}}}$, we have $f_\alpha = f_{\tau_k} f_{\tau_{k+1}} \cdots f_{\tau_{k+m-1}}$.*

Proof. We proceed by induction on m: clearly the result is true for $m = 1$. Assume now that $m \geq 2$, $\alpha = (k\ k+1\ \cdots\ k+m)$ and that

$$f_{(k\ k+1\ \cdots\ k+s)} = f_{\tau_k} f_{\tau_{k+1}} \cdots f_{\tau_{k+s-1}}$$

for any $s < m$. It is easy to check that $\alpha = \tau_{k+m-1}\gamma$, where

$$\gamma = (k\ k+1\ \cdots\ k+m-1).$$

Now we define $\mathbf{r}_j$ by

$$\begin{pmatrix} x_1 & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k+m-1} & x_k & x_{k+m} & x_k & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(k, k+m)$ and $\mu = \neg(k, k+m+2)$. Then $\mathbf{p}_{\lambda j} = \alpha$ and $\mathbf{p}_{\mu j} = \gamma$.

Next we define $\mathbf{r}_l$ by

$$\begin{pmatrix} x_1 & \cdots & x_{k-1} & x_k & x_{k+1} & \cdots & x_{k+m} & x_{k+m+1} & x_{k+m+2} & x_{k+m+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{k-1} & x_k & x_k & \cdots & x_{k+m-1} & x_{k+m} & x_{k+m-1} & x_{k+m+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_{k+m-1}$ and $\mathbf{p}_{\mu l} = \varepsilon$. Thus we have

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu l} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \tau_{k+m-1} & \alpha \\ \varepsilon & \gamma \end{pmatrix}$$

implying $f_\alpha = f_\gamma f_{\tau_{k+m-1}}$ and so $f_\alpha = f_{\tau_k} \cdots f_{\tau_{k+m-1}}$, using our inductive hypothesis applied to γ. $\square$

It follows from Lemmas 6.8.3 and 6.8.4 that

$$\overline{\overline{H}} = \langle f_{\tau_i}, f_{\iota_{a,j}} \,:\, 1 \leq i \leq r-1, 1 \leq j \leq r, a \in G \rangle.$$

Now it is time for us to find a series of relations satisfied by these generators. These correspond to those in Lemma 6.8.2, with the exception of a twist in (W5).

Lemma 6.8.5. *For all $i \in [1, r-1]$, $f_{\tau_i} f_{\tau_i} = 1$, and so $f_{\tau_i}^{-1} = f_{\tau_i}$.*

Proof. Notice that $\tau_i \tau_i = \varepsilon$. First we define $\mathbf{r}_s$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, i+1)$ and $\mu = \neg(i, i+3)$. Then $\mathbf{p}_{\lambda s} = \tau_i$ and $\mathbf{p}_{\mu s} = \varepsilon$.

Next, we define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda t} = \varepsilon$ and $\mathbf{p}_{\mu t} = \tau_i$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \tau_i & \varepsilon \\ \varepsilon & \tau_i \end{pmatrix}$$

which implies $f_{\tau_i} f_{\tau_i} = 1$. $\square$

Lemma 6.8.6. *For any $j \pm 1 \neq i \neq j$ we have $f_{\tau_i} f_{\tau_j} = f_{\tau_j} f_{\tau_i}$.*

Proof. Without loss of generality, suppose that $i > j$ and $i \neq j+1$. First, define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{j+1} & x_{j+2} & x_{j+3} & \cdots x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j+1} & x_j & x_{j+2} & \cdots x_{i-1} & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Note that if $i = j+2$ then the section from $j+3$ to i is empty. Let $\lambda = \neg(j, i+1)$ and $\mu = \neg(j, i+3)$, so that $\mathbf{p}_{\lambda t} = \tau_i \tau_j$ and $\mathbf{p}_{\mu t} = \tau_j$. Next define $\mathbf{r}_s$ by

$$\begin{pmatrix} x_1 & \cdots & x_j & x_{j+1} & x_{j+2} & \cdots x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_j & x_j & x_{j+1} & \cdots x_{i-1} & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \tau_i$ and $\mathbf{p}_{\mu s} = \varepsilon$. Thus we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \tau_i & \tau_i\tau_j \\ \varepsilon & \tau_j \end{pmatrix}$$

implying $f_{\tau_i\tau_j} = f_{\tau_j} f_{\tau_i}$.

To complete the proof, we define $\mathbf{r}_l$ by

$$\begin{pmatrix} x_1 & \cdots & x_{j+1} & x_{j+2} & x_{j+3} & \cdots x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j+1} & x_j & x_{j+2} & \cdots x_{i-1} & x_i & x_i & x_{i+1} & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_j$. Put $\eta = \neg(j+2, i+1)$. Then $\mathbf{p}_{\eta l} = \varepsilon$ and $\mathbf{p}_{\eta t} = \tau_i$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \tau_j & \tau_j\tau_i \\ \varepsilon & \tau_i \end{pmatrix}$$

which implies $f_{\tau_j\tau_i} = f_{\tau_i} f_{\tau_j}$, and hence $f_{\tau_j} f_{\tau_i} = f_{\tau_i} f_{\tau_j}$. □

Lemma 6.8.7. *For any $i \in [1, r-2]$ we have $f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} = f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}}$.*

Proof. Let $\rho = \tau_{i+1}\tau_i = (i\ i+1\ i+2)$ so that $\rho^2 = (i\ i+2\ i+1)$.

First, we show that $f_{\rho^2} = f_\rho f_\rho$. For this purpose, we define $\mathbf{r}_j$ by

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+1} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, i+1)$ and $\mu = \neg(i, i+4)$, so that $\mathbf{p}_{\lambda j} = (i\ i+2\ i+1) = \rho^2$ and $\mathbf{p}_{\mu j} = (i\ i+1\ i+2) = \rho$. Next we define $\mathbf{r}_l$ by

$$\begin{pmatrix} x_1 & \cdots & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_i & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = (i\ i+1\ i+2) = \rho$ and $\mathbf{p}_{\mu l} = \varepsilon$, so here we have

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda j} \\ \mathbf{p}_{\mu l} & \mathbf{p}_{\mu j} \end{pmatrix} = \begin{pmatrix} \rho & \rho^2 \\ \varepsilon & \rho \end{pmatrix}$$

Hence we have $f_{\rho^2} = f_\rho f_\rho$.

Secondly, we show that $f_\rho = f_{\tau_i} f_{\tau_{i+1}}$. Note that $\tau_{i+1}\rho = \tau_i$. Now we define $\mathbf{r}_s$

by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+2} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\nu = \neg(i, i+2)$ and $\xi = (i, i+4)$. Then $\mathbf{p}_{\nu s} = \tau_i$ and $\mathbf{p}_{\xi s} = \rho = (i\ i+1\ i+2)$. Next, define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & x_{i+2} & x_{i+1} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\nu t} = \tau_{i+1}$ and $\mathbf{p}_{\xi t} = \varepsilon$, and so we have

$$\begin{pmatrix} \mathbf{p}_{\nu t} & \mathbf{p}_{\nu s} \\ \mathbf{p}_{\xi t} & \mathbf{p}_{\xi s} \end{pmatrix} = \begin{pmatrix} \tau_{i+1} & \tau_i \\ \varepsilon & \rho \end{pmatrix}$$

implying $f_{\tau_i} = f_\rho f_{\tau_{i+1}}$, so $f_\rho = f_{\tau_i} f_{\tau_{i+1}}$ by Lemma 6.8.5.

Finally, we show that $f_{\rho^2} = f_{\tau_{i+1}} f_{\tau_i}$. Note that $\rho^2 = (i\ i+2\ i+1) = \tau_i \tau_{i+1}$. Define $\mathbf{r}_u$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & x_{i+5} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_i & x_{i+1} & x_{i+3} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\tau = \neg(i, i+1)$ and $\delta = \neg(i+1, i+3)$. Then $\mathbf{p}_{\tau u} = \rho^2$ and $\mathbf{p}_{\delta u} = \tau_{i+1}$. Define $\mathbf{r}_v$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\tau v} = \tau_i$ and $\mathbf{p}_{\delta v} = \varepsilon$, so we have

$$\begin{pmatrix} \mathbf{p}_{\tau v} & \mathbf{p}_{\tau u} \\ \mathbf{p}_{\delta v} & \mathbf{p}_{\delta u} \end{pmatrix} = \begin{pmatrix} \tau_i & \rho^2 \\ \varepsilon & \tau_{i+1} \end{pmatrix}$$

Hence $f_{\rho^2} = f_{\tau_{i+1}} f_{\tau_i}$. We now calculate:

$$f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} = f_{\tau_i} f_{\rho^2} = f_{\tau_i} f_\rho f_\rho = f_{\tau_i} f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}} = f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}},$$

the final step using Lemma 6.8.5. $\square$

We warn the reader that the relation we find below is a twist on that in (W5).

Lemma 6.8.8. *For all $i \in [1, r]$, $a, b \in G$, $f_{\iota_{b,i}} f_{\iota_{a,i}} = f_{\iota_{ab,i}}$, and so $f_{\iota_{a,i}}^{-1} = f_{\iota_{a^{-1},i}}$.*

Proof. Define $\mathbf{r}_j$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & bx_i & abx_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, i+2)$ and $\mu = \neg(i, i+1)$, then $\mathbf{p}_{\lambda j} = \iota_{b,i}$ and $\mathbf{p}_{\mu j} = \iota_{ab,i}$. Next, we define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda t} = \varepsilon$ and $\mathbf{p}_{\mu t} = \iota_{a,i}$, so we have

$$\begin{pmatrix} \mathbf{p}_{\mu t} & \mathbf{p}_{\mu j} \\ \mathbf{p}_{\lambda t} & \mathbf{p}_{\lambda j} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{ab,i} \\ \varepsilon & \iota_{b,i} \end{pmatrix}$$

implying $f_{\iota_{ab,i}} = f_{\iota_{b,i}} f_{\iota_{a,i}}$. □

Lemma 6.8.9. *For all $i \neq j$ and $a, b \in G$ we have $f_{\iota_{a,i}} f_{\iota_{b,j}} = f_{\iota_{b,j}} f_{\iota_{a,i}}$.*

Proof. Without loss of generality, suppose that $i > j$. Recall that $\iota_{a,i}\iota_{b,j} = \iota_{b,j}\iota_{a,i}$. First define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j-1} & x_j & bx_j & x_{j+1} & \cdots & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(j, i+1)$ and $\mu = \neg(j, i+2)$. Then $\mathbf{p}_{\lambda t} = \iota_{a,i}\iota_{b,j}$ and $\mathbf{p}_{\mu t} = \iota_{b,j}$.

Next, we define $\mathbf{r}_s$ by

$$\begin{pmatrix} x_1 & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j-1} & x_j & x_j & x_{j+1} & \cdots & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \iota_{a,i}$ and $\mathbf{p}_{\mu s} = \varepsilon$. Thus we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{a,i}\iota_{b,j} \\ \varepsilon & \iota_{b,j} \end{pmatrix}$$

implying $f_{\iota_{b,j}} f_{\iota_{a,i}} = f_{\iota_{a,i}\iota_{b,j}}$.

Define $\mathbf{r}_l$ by

$$\begin{pmatrix} x_1 & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{j-1} & x_j & bx_j & x_{j+1} & \cdots & x_i & x_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \iota_{b,j}$. On the other hand, by putting $\eta = \neg(j+1, i+1)$ we have $\mathbf{p}_{\eta l} = \varepsilon$ and $\mathbf{p}_{\eta t} = \iota_{a,i}$, and so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \iota_{b,j} & \iota_{b,j}\iota_{a,i} \\ \varepsilon & \iota_{a,i} \end{pmatrix}$$

which implies $f_{\iota_{b,j}\iota_{a,i}} = f_{\iota_{a,i}} f_{\iota_{b,j}}$, and hence $f_{\iota_{a,i}} f_{\iota_{b,j}} = f_{\iota_{b,j}} f_{\iota_{a,i}}$. □

Lemma 6.8.10. *For any i, j with $i \neq j, j+1$ and $a \in G$ we have $f_{\iota_{a,i}} f_{\tau_j} = f_{\tau_j} f_{\iota_{a,i}}$.*

Proof. Suppose that $i < j$; the proof for $j < i$ is entirely similar. Then

$$\iota_{a,i}\tau_j = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_{j+1} & x_{j+2} & \cdots & x_r \\ x_1 & \cdots & x_{i-1} & ax_i & x_{i+1} & \cdots & x_{j-1} & x_{j+1} & x_j & x_{j+2} & \cdots & x_r \end{pmatrix}.$$

Define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_j & x_{j+1} & x_{j+2} & x_{j+3} & x_{j+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & ax_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_{j+1} & x_j & x_{j+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Let $\lambda = \neg(i, j+1)$ and $\mu = \neg(i+1, j+1)$. Then $\mathbf{p}_{\lambda t} = \iota_{a,i}\tau_j$ and $\mathbf{p}_{\mu t} = \tau_j$.

Define $\mathbf{r}_s$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_j & x_{j+1} & x_{j+2} & x_{j+3} & x_{j+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & ax_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_j & x_{j+1} & x_{j+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \iota_{a,i}$ and $\mathbf{p}_{\mu s} = \varepsilon$. Hence we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{a,i}\tau_j \\ \varepsilon & \tau_j \end{pmatrix}$$

implying $f_{\iota_{a,i}\tau_j} = f_{\tau_j} f_{\iota_{a,i}}$.

Next we define $\eta = \neg(i, j+3)$, so that $\mathbf{p}_{\eta t} = \iota_{a,i}$. Now let $\mathbf{r}_l$ be

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_j & x_{j+1} & x_{j+2} & x_{j+3} & x_{j+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & \cdots & x_{j-1} & x_j & x_{j+1} & x_j & x_{j+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_j$ and $\mathbf{p}_{\eta l} = \varepsilon$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \tau_j & \tau_j\iota_{a,i} \\ \varepsilon & \iota_{a,i} \end{pmatrix}$$

implying $f_{\tau_j \iota_{a,i}} = f_{\iota_{a,i}} f_{\tau_j}$, so $f_{\tau_j} f_{\iota_{a,i}} = f_{\iota_{a,i}} f_{\tau_j}$. $\square$

Lemma 6.8.11. *For any $i \in [1, r-1]$ and $a \in G$ we have $f_{\iota_{a,i}} f_{\tau_i} = f_{\tau_i} f_{\iota_{a,i+1}}$.*

Proof. We have

$$\iota_{a,i}\tau_i = \begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & \cdots & x_r \\ x_1 & \cdots & x_{i-1} & ax_{i+1} & x_i & x_{i+2} & \cdots & x_r \end{pmatrix} = \tau_i \iota_{a,i+1}.$$

Define $\mathbf{r}_t$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & ax_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Define $\lambda = \neg(i, i+1)$ and $\mu = \neg(i, i+2)$. Then $\mathbf{p}_{\lambda t} = \iota_{a,i}\tau_i$ and $\mathbf{p}_{\mu t} = \tau_i$. Define $\mathbf{r}_s$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & ax_i & x_{i+1} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda s} = \iota_{a,i}$ and $\mathbf{p}_{\mu s} = \varepsilon$, so we have

$$\begin{pmatrix} \mathbf{p}_{\lambda s} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\mu s} & \mathbf{p}_{\mu t} \end{pmatrix} = \begin{pmatrix} \iota_{a,i} & \iota_{a,i}\tau_i \\ \varepsilon & \tau_i \end{pmatrix}$$

so $f_{\iota_{a,i}\tau_i} = f_{\tau_i} f_{\iota_{a,i}}$.

Now put $\eta = \neg(i+1, i+3)$, so that $\mathbf{p}_{\eta t} = \iota_{a,i+1}$. Define $\mathbf{r}_l$ by

$$\begin{pmatrix} x_1 & \cdots & x_{i-1} & x_i & x_{i+1} & x_{i+2} & x_{i+3} & x_{i+4} & \cdots & x_{r+2} & x_{r+3} & \cdots & x_n \\ x_1 & \cdots & x_{i-1} & x_i & x_i & x_{i+1} & x_i & x_{i+2} & \cdots & x_r & x_1 & \cdots & x_1 \end{pmatrix}.$$

Then $\mathbf{p}_{\lambda l} = \tau_i$ and $\mathbf{p}_{\eta l} = \varepsilon$, so

$$\begin{pmatrix} \mathbf{p}_{\lambda l} & \mathbf{p}_{\lambda t} \\ \mathbf{p}_{\eta l} & \mathbf{p}_{\eta t} \end{pmatrix} = \begin{pmatrix} \tau_i & \tau_i\iota_{a,i+1} \\ \varepsilon & \iota_{a,i+1} \end{pmatrix}$$

so that $f_{\tau_i \iota_{a,i+1}} = f_{\iota_{a,i+1}} f_{\tau_i}$. Thus $f_{\tau_i} f_{\iota_{a,i}} = f_{\iota_{a,i+1}} f_{\tau_i}$ and so $f_{\iota_{a,i}} f_{\tau_i} = f_{\tau_i} f_{\iota_{a,i+1}}$, bearing in mind Lemmas 6.8.5 and 6.8.8. $\square$

We denote by Ω all the following relations we have obtained so far on the set of generators

$$T = \{f_{\tau_i}, f_{\iota_{a,j}} :\ 1 \le i \le r-1, 1 \le j \le r, a \in G\}$$

of $\overline{\overline{H}}$:

$(T1)$ $f_{\tau_i} f_{\tau_i} = 1$, $1 \le i \le r-1$.

$(T2)$ $f_{\tau_i} f_{\tau_j} = f_{\tau_j} f_{\tau_i}$, $j \pm 1 \ne i \ne j$.

$(T3)$ $f_{\tau_i} f_{\tau_{i+1}} f_{\tau_i} = f_{\tau_{i+1}} f_{\tau_i} f_{\tau_{i+1}}$, $1 \le i \le r-2$.

$(T4)$ $f_{\iota_{a,i}} f_{\iota_{b,j}} = f_{\iota_{b,j}} f_{\iota_{a,i}}$, $a, b \in G$ and $1 \le i \ne j \le r$.

$(T5)$ $f_{\iota_{b,i}} f_{\iota_{a,i}} = f_{\iota_{ab,i}}$, $1 \le i \le r$ and $a, b \in G$.

$(T6)$ $f_{\iota_{a,i}} f_{\tau_j} = f_{\tau_j} f_{\iota_{a,i}}$, $1 \le i \ne j, j+1 \le r$.

$(T7)$ $f_{\iota_{a,i}} f_{\tau_i} = f_{\tau_i} f_{\iota_{a,i+1}}$, $1 \le i \le r-1$ and $a \in G$.

Note that the relations $(T1) - (T7)$ match exactly the relations $(W1) - (W7)$ on page 89.

We now have all the ingredients in place to prove the following.

Proposition 6.8.12. *The group $\overline{\overline{H}}$ with a presentation $\mathcal{Q} = \langle S : \Gamma \rangle$ of Lemma 6.8.1 is isomorphic to the presentation $\mathcal{U} = \langle Y :\ \Upsilon \rangle$ of H given in Lemma 6.8.2, so that $\overline{H} \cong H$.*

Proof. We define a map $\boldsymbol{\theta} : \widetilde{Y} \longrightarrow \overline{\overline{H}}$ by

$$\tau_i \boldsymbol{\theta} = f_{\tau_i}^{-1} (= f_{\tau_i}),\ \iota_{a,j} \boldsymbol{\theta} = f_{\iota_{a,j}}^{-1} (= f_{\iota_{a^{-1},j}})$$

where $1 \le i \le r-1,\ 1 \le j \le r, a \in G$. Now we claim that $\Upsilon \subseteq \ker \boldsymbol{\theta}$. Clearly, the relations corresponding to $(W1) - (W4)$ and $(W6)$ and $(W7)$ lie in $\ker \boldsymbol{\theta}$. Moreover, considering $(W5)$

$$(\iota_{a,i} \iota_{b,i}) \boldsymbol{\theta} = \iota_{a,i} \boldsymbol{\theta} \iota_{b,i} \boldsymbol{\theta} = f_{\iota_{a,i}}^{-1} f_{\iota_{b,i}}^{-1} = f_{\iota_{a^{-1},i}} f_{\iota_{b^{-1},i}} = f_{\iota_{b^{-1}a^{-1},i}} = f_{\iota_{(ab)^{-1},i}} = \iota_{ab,i} \boldsymbol{\theta}$$

so that $\Upsilon \subseteq \ker \boldsymbol{\theta}$, and hence there exists a well defined morphism $\overline{\boldsymbol{\theta}} : H \longrightarrow \overline{\overline{H}}$ given by $\tau_i \overline{\boldsymbol{\theta}} = f_{\tau_i}^{-1}$ and $\iota_{a,j} \overline{\boldsymbol{\theta}} = f_{\iota_{a,j}}^{-1}$, where $1 \le i \le r-1,\ 1 \le j \le r, a \in G$.

Conversely, we define $\boldsymbol{\psi} : \widetilde{S} \longrightarrow H$ by $f_\phi \boldsymbol{\psi} = \phi^{-1}$. We show that $\Gamma \subseteq \ker \boldsymbol{\psi}$. Clearly, $f_\varepsilon \boldsymbol{\psi} = \varepsilon^{-1} = \varepsilon = 1\boldsymbol{\psi}$. Suppose that $(\phi, \varphi, \psi, \sigma)$ is singular, which gives

$\phi\varphi^{-1} = \psi\sigma^{-1}$. Then

$$(f_\phi^{-1} f_\varphi)\boldsymbol{\psi} = (f_\phi\boldsymbol{\psi})^{-1} f_\varphi\boldsymbol{\psi} = \phi\varphi^{-1} = \psi\sigma^{-1} = (f_\psi\boldsymbol{\psi})^{-1} f_\sigma\boldsymbol{\psi} = (f_\psi^{-1} f_\sigma)\boldsymbol{\psi}$$

so $\Gamma \subseteq \ker\boldsymbol{\psi}$. Thus there exists a well defined morphism $\overline{\boldsymbol{\psi}} : \overline{\overline{H}} \longrightarrow H$ given by $f_\phi\overline{\boldsymbol{\psi}} = \phi^{-1}$. Then

$$\tau_i\overline{\boldsymbol{\theta}}\;\overline{\boldsymbol{\psi}} = f_{\tau_i}^{-1}\overline{\boldsymbol{\psi}} = (f_{\tau_i}\overline{\boldsymbol{\psi}})^{-1} = \tau_i$$

and

$$\iota_{a,i}\overline{\boldsymbol{\theta}}\;\overline{\boldsymbol{\psi}} = f_{\iota_{a,i}}^{-1}\overline{\boldsymbol{\psi}} = (f_{\iota_{a,i}}\overline{\boldsymbol{\psi}})^{-1} = \iota_{a,i}$$

hence $\overline{\boldsymbol{\theta}}\;\overline{\boldsymbol{\psi}}$ is the identity mapping, and so $\overline{\boldsymbol{\theta}}$ is one-one. Since T is a set of generators for $\overline{\overline{H}}$, it is clear that $\overline{\boldsymbol{\theta}}$ is onto, and so

$$\overline{H} \cong \overline{\overline{H}} \cong H \cong G \wr S_r.$$

$\square$

We can now state the main theorem of this chapter:

Theorem 6.8.13. *Let* $\operatorname{End} F_n(G)$ *be the endomorphism monoid of a free* G*-act* $F_n(G)$ *on* n *generators, where* $n \in \mathbb{N}$ *and* $n \geq 3$*, let* E *be the biordered set of idempotents of* $\operatorname{End} F_n(G)$*, and let* $\mathrm{IG}(E)$ *be the free idempotent generated semigroup over* E*.*

For any idempotent $\varepsilon \in E$ *with rank* r*, where* $1 \leq r \leq n-2$*, the maximal subgroup* $\overline{H}$ *of* $\mathrm{IG}(E)$ *containing* $\overline{\varepsilon}$ *is isomorphic to the maximal subgroup* H *of* $\operatorname{End} F_n(G)$ *containing* ε *and hence to* $G \wr \mathcal{S}_r$*.*

Note that if ε is an idempotent with rank n, that is, the identity map, then $\overline{H}$ is the trivial group, since it is generated (in $\mathrm{IG}(E)$) by idempotents of the same rank. On the other hand, if the rank of ε is $n-1$, then $\overline{H}$ is the free group as there are no non-trivial singular squares in the $\mathcal{D}$-class of ε in $\operatorname{End} F_n(G)$.

Finally, if G is trivial, then $\operatorname{End} F_n(G)$ is essentially $\mathcal{T}_n$, so we deduce the following result from [21].

Corollary 6.8.14. [21] *Let* $n \in \mathbb{N}$ *with* $n \geq 3$ *and let* $\mathrm{IG}(E)$ *be the free idempotent generated semigroup over the biordered set* E *of idempotents of* $\mathcal{T}_n$*.*

For any idempotent $\varepsilon \in E$ with rank r, where $1 \leq r \leq n-2$, the maximal subgroup $\overline{H}$ of $\mathrm{IG}(E)$ *containing $\overline{\varepsilon}$ is isomorphic to the maximal subgroup H of $\mathcal{T}_n$ containing ε, and hence to $\mathcal{S}_r$.*

Chapter 7

Free idempotent generated semigroups: $\operatorname{End} \mathbf{A}$

We have already remarked in Chapter 3 that independence algebras include sets, vector spaces and free G-acts over a group G. The significant results for the biordered sets of idempotents of the full transformation monoid $\mathcal{T}_n$ on n elements, the full linear monoid $M_n(D)$ of all $n \times n$ matrices over a division ring D and the endomorphism monoid $\operatorname{End} F_n(G)$ of a free (left) G-act $F_n(G)$, suggest that it may well be worth investigating maximal subgroups of $\mathrm{IG}(E)$, where E is the biordered set of idempotents of the endomorphism monoid $\operatorname{End} \mathbf{A}$ of an independence algebra $\mathbf{A}$ of rank n, where $n \in \mathbb{N}$ and $n \geq 3$.

Given the diverse methods needed in the biordered sets of idempotents of $\mathcal{T}_n$, $M_n(D)$ and $\operatorname{End} F_n(G)$, it would be very hard to find a unified approach to the biordered set of idempotents of $\operatorname{End} \mathbf{A}$. However, we show that for the case where $\mathbf{A}$ has no constants, the maximal subgroup of $\mathrm{IG}(E)$ containing a rank 1 idempotent $\varepsilon \in E$ is isomorphic to that of $\operatorname{End} \mathbf{A}$, and the latter is the group G of all unary term operations of $\mathbf{A}$. Hence, our work here clearly is a generalization of the result obtained in Chapter 5.

7.1 Unary term operations and rank-1 $\mathcal{D}$-classes

Throughout this chapter, we use $\mathbf{A}$ to denote an independence algebra of rank $n \geq 3$ with no constants, i.e. $\langle \emptyset \rangle = \emptyset$. Let $X = \{x_1, \cdots, x_n\}$ be a basis of $\mathbf{A}$, so that $\mathbf{A} = \langle X \rangle$ and X is independent.

We first recall the following fact observed by Gould [22], the proof of which follows from the free basis property of independence algebras.

Lemma 7.1.1. [22] *Let $Y = \{y_1, \cdots, y_m\}$ be an independent subset of an independence algebra $\mathbf{A}$, where $m \in \mathbb{N}$. Then for any m-ary term operations s and t, we have that $s(y_1, \cdots, y_m) = t(y_1, \cdots, y_m)$ implies*

$$s(a_1, \cdots, a_m) = t(a_1, \cdots, a_m)$$

for all $a_1, \cdots, a_m \in A$, so that $s = t$.

Now we put G to be the set of all unary term operations of $\mathbf{A}$. Then we have the following lemma.

Lemma 7.1.2. *For any independence algebra $\mathbf{A}$ of rank $n \geq 3$ with no constants, the set G of all unary term operations of $\mathbf{A}$ forms a group under composition of functions.*

Proof. Clearly, the identity unary term operation, denoted by $1_{\mathbf{A}}$, is contained in G. Let t be an arbitrary unary term operation of $\mathbf{A}$. Then for any $x \in A$, we have $t(x) \in \langle x \rangle$ and $t(x) \notin \langle \emptyset \rangle = \emptyset$. By the exchange property (EP) of independence algebras, we have that

$$x \in \langle t(x) \rangle, \text{ i.e. } x = st(x)$$

for some unary term operation s. As $\{x\}$ is independent, we have $st \equiv 1_{\mathbf{A}}$ by Lemma 7.1.1. Hence we have $t(x) = tst(x)$, and since $\{t(x)\}$ is independent, it again follows from Lemma 7.1.1 that $ts \equiv 1_{\mathbf{A}}$, so that G is a group. □

Let $\operatorname{End} \mathbf{A}$ be the endomorphism monoid of $\mathbf{A}$ and let ε be a rank 1 idempotent of $\operatorname{End} \mathbf{A}$. Then it follows immediately from Lemma 3.3.1 that the $\mathcal{D}$-class of ε is given by

$$D = D_\varepsilon = \{\alpha \in \operatorname{End} \mathbf{A} : \operatorname{rank} \alpha = 1\}$$

which is a completely simple semigroup by Lemma 3.3.2, so that each $\mathcal{H}$-class of D is a group.

The following lemma gives a characterisation of the $\mathcal{R}$-classes of D in terms of unary term operations of $\mathbf{A}$.

Lemma 7.1.3. *For any $\alpha, \beta \in D$ with $\operatorname{im} \alpha = \langle y_1 \rangle$ and $\operatorname{im} \beta = \langle y_2 \rangle$, suppose that $x_i\alpha = s_i(y_1)$ and $x_i\beta = t_i(y_2)$, where $i \in [1, n]$ and $s_i, t_i \in G$. Then $\ker \alpha = \ker \beta$*

if and only if there exists some unary term operation $q \in G$ such that $s_i = t_i q$, for all $i \in [1, n]$.

Proof. Necessity: Suppose that $y_1\alpha = s(y_1)$ and $y_1\beta = t(y_2)$, where $s, t \in G$. Then for any $i \in [1, n]$

$$s_i^{-1}(x_i)\alpha = s_i^{-1}s_i(y_1) = y_1 = s^{-1}s(y_1) = s^{-1}(y_1)\alpha$$

so that $(s_i^{-1}(x_i), s^{-1}(y_1)) \in \ker\alpha$. By assumption we have $\ker\alpha = \ker\beta$ so that

$$s_i^{-1}(x_i)\beta = s^{-1}(y_1)\beta, \text{ i.e. } s_i^{-1}t_i(y_2) = s^{-1}t(y_2).$$

As $\{y_2\}$ is independent, we have that $s_i^{-1}t_i = s^{-1}t$ by Lemma 7.1.1, and so $s_i = t_i(s^{-1}t)^{-1}$. Then by taking $q = (s^{-1}t)^{-1}$ we have $s_i = t_i q$, for all $i \in [1, n]$.

Sufficiency: Let q be an unary term operation on $\mathbf{A}$ such that $s_i = t_i q$, for all $i \in [1, n]$. Suppose now that

$$u(x_1, \cdots, x_n)\alpha = v(x_1, \cdots, x_n)\alpha.$$

Then we have

$$u(s_1(y_1), \cdots, s_n(y_1)) = v(s_1(y_1), \cdots, s_n(y_1)).$$

By assumption

$$u(t_1q(y_1), \cdots, t_nq(y_1)) = v(t_1q(y_1), \cdots, t_nq(y_1)).$$

As $\{q(y_1)\}$ is independent, it follows from Lemma 7.1.1 that

$$u(t_1(y_2), \cdots, t_n(y_2)) = v(t_1(y_2), \cdots, t_n(y_2)),$$

so $\ker\alpha \subseteq \ker\beta$. Dually, since G is a group, we can show that $\ker\beta \subseteq \ker\alpha$, so that $\ker\alpha = \ker\beta$ as required. □

It follows from Lemma 7.1.3 that the index set I of $\mathcal{R}$-classes of D is in bijective correspondence with G^{n-1}.

Let I index the $\mathcal{R}$-classes in D, Λ index the $\mathcal{L}$-classes in D, so that $H_{i\lambda}$ denote the $\mathcal{H}$-class of D which is the intersection of R_i and L_λ. Note that $H_{i\lambda}$ is a group, and we use $\varepsilon_{i\lambda}$ to denote the identity of $H_{i\lambda}$, for all $i \in I$ and all $\lambda \in \Lambda$. It is notationally standard to use the same symbol 1 to denote a selected element from both I and Λ, and here we let

$$1 = \langle x_1 \rangle \in \Lambda \text{ and } \langle (x_1, x_i) : 1 \leq i \leq n \rangle \in I,$$

the latter of which is the congruence generated by $\{(x_1, x_i) : 1 \leq i \leq n\}$. Then the identity of the group $\mathcal{H}$-class H_{11} is

$$\varepsilon_{11} = \begin{pmatrix} x_1 & \cdots & x_n \\ x_1 & \cdots & x_1 \end{pmatrix}.$$

As we pointed out before, the group $\mathcal{H}$-classes of D are the maximal subgroups of $\operatorname{End} \mathbf{A}$ with a rank 1 idempotent, and moreover, by standard semigroup theory, all group $\mathcal{H}$-classes in D are isomorphic, we only need to show that H_{11} is isomorphic G. For notation convenience, put $H = H_{11}$ and $\varepsilon = \varepsilon_{11}$. In what follows, we denote an element $\begin{pmatrix} x_1 & \cdots & x_n \\ s(x_1) & \cdots & s(x_1) \end{pmatrix} \in \operatorname{End} \mathbf{A}$ by α_s, where $s \in G$.

Lemma 7.1.4. *The maximal subgroup H with a rank 1 idempotent ε in* $\operatorname{End} \mathbf{A}$ *is isomorphic to G.*

Proof. It follows from Lemma 3.3.1 that

$$\alpha \in H \Longleftrightarrow \alpha = \alpha_s = \begin{pmatrix} x_1 & \cdots & x_n \\ s(x_1) & \cdots & s(x_1) \end{pmatrix}$$

for some unary term operation $s \in G$. Define a mapping

$$\phi : H \longrightarrow G, \begin{pmatrix} x_1 & \cdots & x_n \\ s(x_1) & \cdots & s(x_1) \end{pmatrix} \mapsto s.$$

Clearly, ϕ is an isomorphism (note that composition in G is right to left), so that $H \cong G$ as required. □

Since the $\mathcal{D}$-class D of $\operatorname{End} \mathbf{A}$ is a completely simple semigroup, we have that D is isomorphic to some Rees matrix semigroup $\mathcal{M}(H; I, \Lambda; P)$. Next we will choose and fix $P = (\mathbf{p}_{\lambda i})$ with $\mathbf{p}_{\lambda i} = \mathbf{q}_\lambda \mathbf{r}_i$, where $\mathbf{r}_i \in H_{i1}$ and $\mathbf{q}_\lambda \in H_{1\lambda}$.

Let $X = \{x_1, \cdots, x_n\}$ be a basis of $\mathbf{A}$. We have already assumed that $1 \in I \cap \Lambda$ are such that

$$1 = \langle x_1 \rangle \in \Lambda \text{ and } 1 = \langle (x_1, x_2), \cdots, (x_1, x_n) \rangle \in I.$$

Here we take $\mathbf{r}_i = \varepsilon_{i1}$ and $\mathbf{q}_\lambda = \varepsilon_{1\lambda}$, for each $i \in I$ and each $\lambda \in \Lambda$. Note that an element $\alpha \in \operatorname{End} \mathbf{A}$ with $\operatorname{im} \alpha = \langle x_1 \rangle$ is an idempotent if and only if $x_1 \alpha = x_1$, so that for each $i \in I$, we must have

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_{i_2}(x_1) & \cdots & s_{i_n}(x_1) \end{pmatrix}$$

where $s_{i_2}, \cdots, s_{i_n} \in G$.

On the other hand, for each $\lambda \in \Lambda$, choose a generator y of λ, so $\lambda = \langle y \rangle$ and then choose t with $y = t(x_1, \cdots, x_n)$. Then we put $t'(x) = t(x, \cdots, x)$ and $s_t = (t')^{-1}$. Then we define

$$\mathbf{q}_\lambda = \begin{pmatrix} x_1 & \cdots & x_n \\ s_t(y) & \cdots & s_t(y) \end{pmatrix}.$$

Obviously, we have $\ker \mathbf{q}_\lambda = \langle (x_1, x_2), \cdots, (x_1, x_n) \rangle$ and $\operatorname{im} \mathbf{q}_\lambda = \lambda$, so $\mathbf{q}_\lambda \in H_{1\lambda}$. It follows from

$$y\mathbf{q}_\lambda = t(x_1, \cdots, x_n)\mathbf{q}_\lambda = t(s_t(y), \cdots, s_t(y)) = t'(s_t(y)) = y$$

that $\mathbf{q}_\lambda$ is an idempotent of $H_{1\lambda}$. Since each group $\mathcal{H}$-class contains exactly one idempotent, we deduce that $\mathbf{q}_\lambda = \varepsilon_{1\lambda}$. This also implies that $\mathbf{q}_\lambda$ does not depend on our choice of the generator y.

Note that we must have special elements $\lambda_1, \cdots, \lambda_n$ of Λ such that $\lambda_k = \langle x_k \rangle$, for all $k = 1, \cdots, n$. To simplify our notation, at times we put $k = \lambda_k$, for all $k = 1, \cdots, n$. Clearly, we have

$$\mathbf{q}_k = \varepsilon_{1k} = \begin{pmatrix} x_1 & \cdots & x_n \\ x_k & \cdots & x_k \end{pmatrix}$$

for all $k = 1, \cdots, n$.

We now aim to look into the structure of the sandwich $P = (\mathbf{p}_{\lambda i})$. Let $\mathbf{r}_i$ and

$\mathbf{q}_\lambda$ be defined as above. Then we have

$$\begin{aligned}\mathbf{p}_{\lambda i} &= \begin{pmatrix} x_1 & \cdots & x_n \\ s_t(y) & \cdots & s_t(y) \end{pmatrix}\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_{i_2}(x_1) & \cdots & s_{i_n}(x_1) \end{pmatrix} \\ &= \begin{pmatrix} x_1 & \cdots & x_n \\ s_t t(x_1, s_{i_2}(x_1), \cdots, s_{i_n}(x_1)) & \cdots & - \end{pmatrix}\end{aligned}$$

Particularly, if $\lambda = 1$ then

$$\begin{aligned}\mathbf{p}_{1i} &= \begin{pmatrix} x_1 & \cdots & x_n \\ x_1 & \cdots & x_1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_{i_2}(x_1) & \cdots & s_{i_n}(x_1) \end{pmatrix} \\ &= \begin{pmatrix} x_1 & \cdots & x_n \\ x_1 & \cdots & x_1 \end{pmatrix} = \alpha_{1_{\mathbf{A}}} = \varepsilon_{11}\end{aligned}$$

and if $\lambda = k$ with $k \in \{2, \cdots, n\}$, then

$$\begin{aligned}\mathbf{p}_{ki} &= \begin{pmatrix} x_1 & \cdots & x_n \\ x_k & \cdots & x_k \end{pmatrix}\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_{i_2}(x_1) & \cdots & s_{i_n}(x_1) \end{pmatrix} \\ &= \begin{pmatrix} x_1 & \cdots & x_n \\ s_{i_k}(x_1) & \cdots & s_{i_k}(x_1) \end{pmatrix} = \alpha_{s_{i_k}}\end{aligned}.$$

For convenience, we arrange the rows $(\mathbf{p}_{1i}), (\mathbf{p}_{2i}), \cdots, (\mathbf{p}_{ni})$ to be the first row, second row,$\cdots$, n-th row of the sandwich matrix $P = (\mathbf{p}_{\lambda i})$. Notice that

$$(\mathbf{p}_{1i}) = (\mathbf{q}_1\mathbf{r}_i) = (\varepsilon_{11}\varepsilon_{i1}) = (\varepsilon_{11}) = (\alpha_{1_{\mathbf{A}}})$$

and

$$(\mathbf{p}_{\lambda 1}) = (\mathbf{q}_\lambda\mathbf{r}_1) = (\varepsilon_{1\lambda}\varepsilon_{11}) = (\varepsilon_{11}) = (\alpha_{1_{\mathbf{A}}}).$$

Furthermore, P has the following nice property.

Lemma 7.1.5. *For any $\alpha_{s_2}, \cdots, \alpha_{s_n} \in H_{11}$ with $s_2, \cdots, s_n \in G$, there exists some $k \in I$ such that the k-th column of the sandwich matrix P is $(\alpha_{1_{\mathbf{A}}}, \alpha_{s_2}, \cdots, \alpha_{s_n}, \cdots)$.*

Proof. To show this, we only need to take $\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_2(x_1) & \cdots & s_n(x_1) \end{pmatrix}$. Then the i-th column is

$$(\mathbf{p}_{1i}, \mathbf{p}_{2i}, \cdots, \mathbf{p}_{ni}, \cdots)^T = (\alpha_{1_{\mathbf{A}}}, \alpha_{s_2}, \cdots, \alpha_{s_n}, \cdots)^T.$$

□

7.2 Singular squares of the rank-1 $\mathcal{D}$-class

Our main aim in this section is to locate singular squares of the rank r $\mathcal{D}$-class D of $\operatorname{End}\mathbf{A}$, where $r \geq 1$. However, as mentioned in the beginning of this section, our concern is the case $r = 1$.

Lemma 7.2.1. *An E-square $\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}$ in D is singular if and only if $\{\alpha, \beta, \gamma, \delta\}$ froms a rectangular band.*

Proof. The necessity follows directly from Lemma 4.2.3. Suppose that $\{\alpha, \beta, \gamma, \delta\}$ is a rectangular band in D. Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ generated by $\operatorname{im}\alpha \cup \operatorname{im}\beta$, i.e. $B = \langle \operatorname{im}\alpha \cup \operatorname{im}\beta \rangle$. Suppose that $\mathbf{B}$ has a basis U. As $\mathbf{A}$ is an independence algebra, any independent subset of A can be extended to be a basis of $\mathbf{A}$, so that we can extend U to be a basis $U \cup W$ of $\mathbf{A}$.

Now we define an element $\sigma \in \operatorname{End}\mathbf{A}$ by

$$x\sigma = \begin{cases} x & \text{if } x \in U; \\ x\gamma & \text{if } x \in W \end{cases}$$

Notice that, for any $x \in A$, $x\gamma \in \operatorname{im}\gamma = \operatorname{im}\beta \subseteq B$ and $\sigma|_B = I_B$, so that $\sigma^2 = \sigma$ is an idempotent of $\operatorname{End}\mathbf{A}$. Clearly, we have

$$\alpha\sigma = \alpha \text{ and } \beta\sigma = \beta.$$

On the other hand, since for any $x \in U$, $x\sigma\alpha = x = x\delta$ and for any $x \in W$, $x\sigma\alpha = x\gamma\alpha = x\delta$, we have that $\sigma\alpha = \delta$. Further, for any $x \in U$, $x\sigma\beta = x = x\gamma$ and for any $x \in W$, $x\sigma\beta = x\gamma\beta = x\gamma$, so $\sigma\beta = \gamma$. Hence, $\begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}$ is a singular square in D. □

The following result can be obtained from Lemma 7.2.1 and Observation 5.2.3.

Lemma 7.2.2. *For any idempotents $\alpha, \beta, \gamma \in D$, $\alpha\beta = \gamma$ implies $\overline{\alpha}\overline{\beta} = \overline{\gamma}$.*

7.3 A set of generators and relations of $\overline{H}$

The aim of this chapter is to show that the maximal subgroup $\overline{H} = H_{\overline{\varepsilon}_{11}}$ of $\mathrm{IG}(E)$ containing $\overline{\varepsilon}_{11}$ is isomorphic to the maximal subgroup $H = H_{\varepsilon_{11}}$ of End $\mathbf{A}$ containing ε_{11}. In this section, we will determine a set of generators of $\overline{H}$ and find out a series of relations satisfied by these generators.

It is clear that the $\mathcal{D}$-class D is completely simple, so that Observation 5.3.2 leads to the following comment and Lemma 7.3.1.

For each $i \in I$ and each $\lambda \in \Lambda$, we have

$$(\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11})^{-1} = \overline{\varepsilon}_{1\lambda}\overline{\varepsilon}_{i1}.$$

Lemma 7.3.1. *Every element in $\overline{H}$ is a product of elements of the form $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11}$ and $(\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11})^{-1}$, where $i \in I$ and $\lambda \in \Lambda$.*

We have already noticed that the first row $(\mathbf{p}_{1i})$ and the first column $(\mathbf{p}_{\lambda 1})$ of the sandwich matrix $P = (\mathbf{p}_{\lambda i})$ consist entirely of ε, so by Lemma 7.2.1 and Observation 5.3.5 we have the following three lemmas.

Lemma 7.3.2. *If $\varepsilon_{1\lambda}\varepsilon_{i1} = \varepsilon_{11}$, then $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}$.*

Lemma 7.3.3. *For any $\lambda \in \Lambda$ and $i, j \in I$, $\varepsilon_{1\lambda}\varepsilon_{i1} = \varepsilon_{1\lambda}\varepsilon_{j1}$, i.e. $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j}$, implies $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{j\lambda}\overline{\varepsilon}_{11}$.*

Lemma 7.3.4. *For any $\lambda, \mu \in \Lambda$ and $i \in I$ $\varepsilon_{1\lambda}\varepsilon_{i1} = \varepsilon_{1\mu}\varepsilon_{i1}$, i.e. $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu i}$, implies $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{i\mu}\overline{\varepsilon}_{11}$.*

Now we divide the sandwich matrix $P = (\mathbf{p}_{\lambda i})$ into two blocks, say a *good* block and a *bad* block. Here the so called *good* block consists of all rows $(\mathbf{p}_{ki})$, where $k \in [1, n]$, and of course, the rest of P forms the *bad* block.

For any $i, j \in I$ and $\lambda, \mu \in \{1, \cdots, n\}$ with $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$, it follows from Lemma 7.1.5 that there exists $l \in I$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda l} = \mathbf{p}_{\mu l} = \mathbf{p}_{\mu j}$. Hence, we have the following result by Lemma 7.2.1 and Observation 5.3.7 in terms of the good block of P.

Lemma 7.3.5. *For any $i, j \in I$ and $\lambda, \mu \in \{1, \cdots, n\}$, $\varepsilon_{1\lambda}\varepsilon_{i1} = \varepsilon_{1\mu}\varepsilon_{j1}$, i.e. $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$, implies $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{j\mu}\overline{\varepsilon}_{11}$.*

If the bad block does not exist in P, clearly we directly have Theorem 7.3.12 without any more effort. Suppose now that the bad block does exists, so our task now is to deal with its elements. The main strategy here is to find a 'bridge' to connect the bad block and the good block, in the sense that, for each $\lambda \in \Lambda, i \in I$, to try to find some $k \in \{1, \cdots, n\}$, $j \in I$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j} = \mathbf{p}_{kj}$. For this purpose, we consider the following cases:

Lemma 7.3.6. *Suppose that we have*

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_2(x_1) & \cdots & s_n(x_1) \end{pmatrix}$$

for some $i \in I$ and $\lambda = \langle y \rangle$ with $y = t(x_{l_1}, \cdots, x_{l_k})$ such that

$$1 = l_1 < \cdots < l_k \leq n \text{ and } k < n.$$

Then there exists some $j \in I$ and $m \in [1, n]$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j} = \mathbf{p}_{mj}$.

Proof. By assumption, we have

$$\mathbf{p}_{\lambda i} = \begin{pmatrix} x_1 & \cdots & x_n \\ s_t t(x_1, s_{l_2}(x_1), \cdots, s_{l_k}(x_1)) & \cdots & - \end{pmatrix}.$$

Define $\mathbf{r}_j$ by $x_1\mathbf{r}_j = x_1$, $x_{l_2}\mathbf{r}_j = s_{l_2}(x_1), \cdots, x_{l_k}\mathbf{r}_j = s_{l_k}(x_1)$, $x_m\mathbf{r}_j = x_1\mathbf{p}_{\lambda i}$, for any $m \in [1, n] \setminus \{l_1, l_2, \cdots, l_k\}$. Note that such m must exist as by assumption we have $1 = l_1 < \cdots < l_k \leq n$ and $k < n$. Then we clearly have $\mathbf{p}_{\lambda j} = \mathbf{p}_{\lambda i}$ and

$$\mathbf{p}_{mj} = \begin{pmatrix} x_1 & \cdots & x_n \\ x_1\mathbf{p}_{\lambda i} & \cdots & - \end{pmatrix},$$

and hence we have $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j} = \mathbf{p}_{mj}$. $\square$

Lemma 7.3.7. *Let*

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_2(x_1) & \cdots & s_n(x_1) \end{pmatrix}$$

for some $i \in I$, $\lambda = \langle y \rangle$ with $y = t(x_{l_1}, \cdots, x_{l_k})$ such that $1 \neq l_1 < \cdots < l_k \leq n$. Then there exists some $j \in I$ and $m \in [1, n]$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j} = \mathbf{p}_{mj}$.

Proof. It follows from our assumption that

$$\mathbf{p}_{\lambda i} = \begin{pmatrix} x_1 & \cdots & x_n \\ s_t t(s_{l_1}(x_1), \cdots, s_{l_k}(x_1)) & \cdots & - \end{pmatrix}.$$

Let $w(x) = s_t t(s_{l_1}(x), \cdots, s_{l_k}(x))$. Then as $s_t = (t')^{-1}$, we have

$$w(x) = s_t t'(w(x)) = s_t t(w(x), \cdots, w(x)).$$

Let $\mathbf{r}_j = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & w(x_1) & \cdots & w(x_1) \end{pmatrix}$. Then

$$\mathbf{p}_{\lambda j} = \begin{pmatrix} x_1 & \cdots & x_n \\ s_t t(w(x_1), \cdots, w(x_1)) & \cdots & - \end{pmatrix} = \begin{pmatrix} x_1 & \cdots & x_n \\ w(x_1) & \cdots & w(x_1) \end{pmatrix}$$

and

$$\mathbf{p}_{2j} = \begin{pmatrix} x_1 & \cdots & x_n \\ w(x_1) & \cdots & w(x_1) \end{pmatrix}$$

so that we have $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j} = \mathbf{p}_{2j}$ as required. $\square$

Now we are only left with the case such that $y = t(x_1, \cdots, x_n)$ is *truly* n-ary, in the sense that there exists no proper subset X' of the basis $X = \{x_1, \cdots, x_n\}$ such that $y \in \langle X' \rangle$, where we need more effort.

Let G be the group of all unary term operations on an independence algebra $\mathbf{A}$ of finite rank $n \geq 3$ with no constants, and let $s_2, \cdots, s_{n-1}$ be arbitrary chosen and fixed unary term operations. Define a mapping θ as follows:

$$\theta : G \longrightarrow G, u(x) \longmapsto t(x, s_2(x), \cdots, s_{n-1}(x), u(x)).$$

Lemma 7.3.8. *The mapping θ defined as above is one-one.*

Proof. Let $X = \{x_1, \cdots, x_n\}$ be a basis of $\mathbf{A}$. First, we claim that

$$\{x_1, \cdots, x_{n-1}, t(x_1, \cdots, x_n)\}$$

is an independent subset of A. Since $t(x_1, \cdots, x_n)$ is truly n-ary, we have that $t(x_1, \cdots, x_n) \notin \langle x_1, \cdots, x_{n-1} \rangle$. Suppose that $x_1 \in \langle x_2, \cdots, x_{n-1}, t(x_1, \cdots, x_n) \rangle$. Then as $x_1 \notin \langle x_2, \cdots, x_{n-1} \rangle$, by the exchange property (EP), we must have that

$t(x_1, \cdots, x_n) \in \langle x_1, \cdots, x_{n-1} \rangle$, a contradiction. As any n-element independent set forms a basis of $\mathbf{A}$, we have

$$\mathbf{A} = \langle x_1, \cdots, x_{n-1}, t(x_1, \cdots, x_n) \rangle$$

and so $x_n = w(x_1, \cdots, x_{n-1}, t(x_1, \cdots, x_n))$ for some n-ary term operation w. Let u and v be unary term operations such that $u(x)\theta = v(x)\theta$. Then by the definition of θ, we have

$$t(x, s_2(x), \cdots, s_{n-1}(x), u(x)) = t(x, s_2(x), \cdots, s_{n-1}(x), v(x)).$$

On the other hand, it follows from Lemma 7.1.1 that

$$u(x) = w(x, s_2(x), \cdots, s_{n-1}(x), t(x, s_2(x), \cdots, s_{n-1}(x), u(x))$$

and

$$v(x) = w(x, s_2(x), \cdots, s_{n-1}(x), t(x, s_2(x), \cdots, s_{n-1}(x), v(x)).$$

Therefore, we have $u(x) = v(x)$, so that θ is one-one. $\square$

Corollary 7.3.9. *If* $\mathbf{A}$ *is a finite independence algebra, then the mapping* θ *defined as above is onto.*

If $\mathbf{A}$ is infinite, so far we have not found a direct way to show that the mapping θ defined as above is onto, and in this case we need the classification described in Theorem 3.3.5. As we assumed that the bad block does exists in P, we have that $\mathbf{A}$ is an affine algebra. Then the following lemma holds.

Lemma 7.3.10. *If* $\mathbf{A}$ *is an affine algebra, then the mapping* θ *defined as above is onto.*

Proof. Let $\mathbf{A}^0$ be a subalgebra of $\mathbf{A}$ satisfying the condition stated in Theorem 3.3.5. Let $t(x_1, \cdots, x_n)$ be a truly n-ary term operation with $s_2, \cdots, s_{n-1} \in G$. Then we have

$$t(x_1, \cdots, x_n) = k_1 x_1 + \cdots + k_n x_n + a$$

and

$$s_2(x) = x + a_2, \cdots, s_{n-1}(x) = x + a_{n-1}$$

where for all $i \in [1,n]$, $k_i \neq 0$, $k_1 + \cdots + k_n = 1$ and $a, a_2, \cdots, a_{n-1} \in A^0$. For any unary term operation $v(x) = x + b \in G$ with $b \in A^0$, by putting $s_n(x) = x + a_n$, where

$$a_n = k_n^{-1}(b - k_2 a_2 - \cdots - k_{n-1}a_{n-1} - a) \in A^0$$

we have $t(x, s_2(x), \cdots, s_{n-1}(x), s_n(x)) = v(x)$, and hence θ is onto. $\square$

Lemma 7.3.11. *Let*

$$\mathbf{r}_i = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ x_1 & s_2(x_1) & \cdots & s_n(x_1) \end{pmatrix}$$

for some $i \in I$ and let $\lambda = \langle y \rangle$, where $y = t(x_1, \cdots, x_n)$ is a truly n-ary term operation on $\mathbf{A}$. Then there exists some $j \in I$ such that $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j} = \mathbf{p}_{2j}$.

Proof. By assumption, we have

$$\mathbf{p}_{\lambda i} = \begin{pmatrix} x_1 & \cdots & x_n \\ s_t t(x_1, s_2(x_1), \cdots, s_n(x_1)) & \cdots & - \end{pmatrix}.$$

Put $w(x) = s_t t(x_1, s_2(x_1), \cdots, s_n(x_1))$. It follows from Lemma 7.3.10 that the mapping

$$\theta : G \longrightarrow G, k(x) \longmapsto t(x, w(x), \cdots, w(x), k(x))$$

is onto, so that there exists some $h(x) \in G$ such that

$$t(x, w(x), \cdots, w(x), h(x)) = s_t^{-1}(w(x))$$

and so

$$w(x) = s_t t(x, w(x), \cdots, w(x), h(x)).$$

Let

$$\mathbf{r}_j = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ x_1 & w(x_1) & \cdots & w(x_1) & h(x_1) \end{pmatrix}.$$

Then

$$\mathbf{p}_{\lambda j} = \begin{pmatrix} x_1 & \cdots & x_n \\ s_t t(x_1, w(x_1), \cdots, w(x_1), h(x_1)) & \cdots & - \end{pmatrix} = \begin{pmatrix} x_1 & \cdots & x_n \\ w(x_1) & \cdots & - \end{pmatrix}$$

and clearly,

$$\mathbf{p}_{2j} = \begin{pmatrix} x_1 & \cdots & x_n \\ w(x_1) & \cdots & - \end{pmatrix}.$$

Therefore, we have $\mathbf{p}_{\lambda i} = \mathbf{p}_{\lambda j} = \mathbf{p}_{2j}$. □

Lemma 7.3.12. *For any $i, j \in I$ and $\lambda, \mu \in \Lambda$, $\varepsilon_{1\lambda}\varepsilon_{i1} = \varepsilon_{1\mu}\varepsilon_{j1}$, i.e. $\mathbf{p}_{\lambda i} = \mathbf{p}_{\mu j}$, implies $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{j\mu}\overline{\varepsilon}_{11}$.*

Proof. By the above discussion, we have

$$\mathbf{p}_{\lambda l} = \mathbf{p}_{ml} = \mathbf{p}_{\lambda i} \text{ and } \mathbf{p}_{\mu k} = \mathbf{p}_{sk} = \mathbf{p}_{\mu j}$$

for some $l, k \in I$ and $m, s \in [1, n]$. Then it follows from Lemmas 7.3.3 and 7.3.4 that

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{l\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\ \overline{\varepsilon}_{lm}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11}$$

and

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{k\mu}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ks}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{j\mu}\overline{\varepsilon}_{11}.$$

From $\mathbf{p}_{ml} = \mathbf{p}_{sk}$, we have $\overline{\varepsilon}_{11}\overline{\varepsilon}_{lm}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{ks}\overline{\varepsilon}_{11}$ by Lemma 7.3.5, so that

$$\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}\overline{\varepsilon}_{j\mu}\overline{\varepsilon}_{11}$$

as required. □

Following the fact we obtained in Lemma 7.3.12, we denote the generator $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11}$ with $\mathbf{p}_{\lambda i} = \alpha^{-1}$ by w_α, where $\alpha \in H$.

Now we show that an analogous result of Lemma 5.3.8 holds for independence algebras. It follows from our assumption $n \geq 3$ and Lemma 7.1.5 that for any $\alpha, \beta \in H$, the sandwich matrix P has two columns with the following forms:

$$(\varepsilon_{11}, \alpha^{-1}, \beta^{-1}\alpha^{-1}, \cdots)^T \text{ and } (\varepsilon_{11}, \varepsilon_{11}, \beta^{-1}, \cdots)^T.$$

Therefore, by Lemmas 7.2.1, 7.3.12 and Observation 5.3.9 we have:

Lemma 7.3.13. *For any $\alpha, \beta \in H$, $w_\alpha w_\beta = w_{\alpha\beta}$ and $w_{\alpha^{-1}} = w_\alpha^{-1}$.*

7.4 The main theorem

As we stated in the beginning of this chapter that our main aim is to characterize the maximal subgroup of $\mathrm{IG}(E)$ containing a rank 1 idempotent $\varepsilon \in E$, where E is the biordered set of idempotents of $\operatorname{End} \mathbf{A}$. First, it follows from Lemmas 7.3.1 and 7.3.13 that

$$\overline{H} = \{\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} : \ i \in I, \lambda \in \lambda\}.$$

Now we are in the position to state our main theorem.

Theorem 7.4.1. *Let* $\operatorname{End} \mathbf{A}$ *be the endomorphism monoid of an independence algebra* $\mathbf{A}$ *of rank* $n \geq 3$ *with no constants, let* E *be the biordered set of idempotents of* $\operatorname{End} \mathbf{A}$*, and let* $\mathrm{IG}(E)$ *be the free idempotent generated semigroup over* E*. Then for any rank 1 idempotent* $\varepsilon \in E$*, the maximal subgroup* $\overline{H}$ *of* $\mathrm{IG}(E)$ *containing* $\overline{\varepsilon}$ *is isomorphic to the maximal subgroup of* $\operatorname{End} \mathbf{A}$ *containing* ε*, and hence to the group* G *of all unary term operations of* $\mathbf{A}$*.*

Proof. As all group $\mathcal{H}$-classes in the same $\mathcal{D}$-class are isomorphic, we only need to show that $\overline{H} = H_{\overline{\varepsilon}_{11}}$ is isomorphic to G.

Let $\overline{\phi}$ be the restriction of the natural map $\phi : \mathrm{IG}(E) \longrightarrow \langle E \rangle$ defined in Property (IG1). Then by (IG4), we know that

$$\overline{\phi} : \overline{H} \longrightarrow H, \ \overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} \mapsto \varepsilon_{11}\varepsilon_{i\lambda}\varepsilon_{11}$$

is an onto morphism. Furthermore, $\overline{\phi}$ is one-one, because if we have

$$(\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11})\,\overline{\phi} = \varepsilon_{11}$$

then $\varepsilon_{11}\varepsilon_{i\lambda}\varepsilon_{11} = \varepsilon_{11}$ and by Lemma 7.3.2, $\overline{\varepsilon}_{11}\overline{\varepsilon}_{i\lambda}\overline{\varepsilon}_{11} = \overline{\varepsilon}_{11}$. We therefore have $\overline{H} \cong H \cong G$. $\square$

Chapter 8

Free idempotent generated semigroups over bands

Whereas much of the former work in the literature of $\mathrm{IG}(E)$ has focused on the maximal subgroups, the aim of this chapter is to investigate the general structure of $\mathrm{IG}(B)$ for a band B. Our main result is that for an arbitrary band B, $\mathrm{IG}(B)$ is a weakly abundant semigroup with the congruence condition.

We proceed as follows. In Section 8.1 we recall some basics of reduction systems. We briefly describe how $\mathrm{IG}(B)$ naturally can be induced by a noetherian reduction system $(\overline{B}^+, \longrightarrow)$. In Section 8.2, we begin our investigation of $\mathrm{IG}(B)$ by looking at a semilattice Y. We prove that every element of $\mathrm{IG}(Y)$ has a unique normal form. We then use this to show that $\mathrm{IG}(Y)$ is abundant, and hence adequate. We remark here that this result can be obtained as a corollary of Proposition 8.6.2, however, the straightforward proof makes clear the strategies we subsequently use in other contexts. In Section 8.3, we show that for any rectangular band B, $\mathrm{IG}(B)$ is regular. We then proceed to look at a general band B in Section 8.4. Unlike the case of semilattices and rectangular bands, here we may lose uniqueness of normal forms. To overcome this problem, the concept of *almost normal form* is introduced. It is proved that for any band B, $\mathrm{IG}(B)$ is a weakly abundant semigroup with the congruence condition, but need not be abundant.

We then consider some sufficient conditions for $\mathrm{IG}(B)$ to be abundant. In Section 8.5, we introduce a class of bands B, which are in general not normal, for which the word problem of $\mathrm{IG}(B)$ is solvable. Then in Section 8.6, we show that if B is a quasi-zero band or a normal band for which $\mathrm{IG}(B)$ satisfies a condition we label (P), then $\mathrm{IG}(B)$ is an abundant semigroup. We then find two classes of normal bands satisfying Condition (P). One would naturally ask here whether $\mathrm{IG}(B)$ is abundant for an arbitrary normal band B. In Section 8.7, we construct a 10-element normal band with 4 $\mathcal{D}$-classes for which $\mathrm{IG}(B)$ is not abundant.

8.1 Reduction systems

The aim of this section is to recall the definition of reduction systems and their properties. As far as possible we follow standard notation and terminology, as may be found in [39].

Let A be a set of objects and $\longrightarrow$ a binary relation on A. We call the structure $(A, \longrightarrow)$ a *reduction system* and the relation $\longrightarrow$ a *reduction relation.* The reflexive, transitive closure of $\longrightarrow$ is denoted by $\xrightarrow{*}$, while $\xleftrightarrow{*}$ denotes the smallest equivalence relation on A that contains $\longrightarrow$. We denote the equivalence class of an element $x \in A$ by $[x]$. An element $x \in A$ is said to be *irreducible* if there is no $y \in A$ such that $x \longrightarrow y$; otherwise, x is *reducible.* For any $x, y \in A$, if $x \xrightarrow{*} y$ and y is irreducible, then y is a *normal form* of x. A reduction system $(A, \longrightarrow)$ is *noetherian* if there is no infinite sequence $x_0, x_1, \cdots \in A$ such that for all $i \geq 0$, $x_i \longrightarrow x_{i+1}$.

We say that a reduction system $(A, \longrightarrow)$ is *confluent* if whenever $w, x, y \in A$ are such that $w \xrightarrow{*} x$ and $w \xrightarrow{*} y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by the figure below on the left, and $(A, \longrightarrow)$ is *locally confluent* if whenever $w, x, y \in A$, are such that $w \longrightarrow x$ and $w \longrightarrow y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by the figure below on the right.

$$\begin{tikzcd} & w \arrow[dl, "*"'] \arrow[dr, "*"] & \\ x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\ & z & \end{tikzcd} \qquad \begin{tikzcd} & w \arrow[dl] \arrow[dr] & \\ x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\ & z & \end{tikzcd}$$

Figure 8.1: confluence and local confluence

Lemma 8.1.1. [39] *Let $(A, \longrightarrow)$ be a reduction system. Then the following statements hold:*

(i) If $(A, \longrightarrow)$ is noetherian and confluent, then for each $x \in A$, $[x]$ contains a unique normal form.

(ii) If $(A, \longrightarrow)$ is noetherian, then it is confluent if and only if it is locally confluent.

Let E be a biordered set. Recall that we denote the free semigroup on $\overline{E} = \{\overline{e} : e \in E\}$ by $\overline{E}^{+}$.

Lemma 8.1.2. *Let E be a biordered set, and let R be the relation on $\overline{E}^{+}$ defined by*

$$R = \{(\bar{e}\bar{f}, \overline{ef}) : \ (e, f) \text{ is a basic pair}\}.$$

Then $(\overline{E}^{+}, \longrightarrow)$ forms a noetherian reduction system, where $\longrightarrow$ is defined by

$$u \longrightarrow v \Longleftrightarrow (\exists\ (l, r) \in R)\ (\exists\ x, y \in \overline{E}^{+})\ u = xly \text{ and } v = xry.$$

Proof. The proof follows directly from the definitions of the reduction system and the binary relation $\longrightarrow$. □

Note that the smallest equivalence relation $\overset{*}{\longleftrightarrow}$ on $\overline{E}^{+}$ is exactly the congruence generated by R. Obviously, the free idempotent generated semigroup $\mathrm{IG}(E)$ is given by a noetherian reduction system $(\overline{E}^{+}, \longrightarrow)$.

8.2 Free idempotent generated semigroups over semilattices

We start our investigation of free idempotent generated semigroups $\mathrm{IG}(B)$ over bands B, by looking at the special case of semilattices. Throughout this section we will use the letter Y to denote a semilattice. It is proved that $\mathrm{IG}(Y)$ is an adequate semigroup; however, it need not be regular.

Lemma 8.2.1. *Let Y be a semilattice. Then every element in $\mathrm{IG}(Y)$ has a unique normal form.*

Proof. By Lemma 8.1.1, to show the required result we only need to argue that $(\overline{Y}^{+}, \longrightarrow)$ is locally confluent. For this purpose, it is sufficient to consider an arbitrary word of length 3, say $\overline{e}\ \overline{f}\ \overline{g} \in \overline{Y}^+$, where e, f and f, g are comparable. There are four cases, namely, $e \leq f \leq g$, $e \geq f \geq g$, $e \leq f, f \geq g$ and $e \geq f, f \leq g$, for which we have the following 4 diagrams:

$$
\begin{tikzcd}[column sep=small]
 & \overline{e}\ \overline{f}\ \overline{g} \arrow[dl] \arrow[dr] & \\
\overline{e}\ \overline{g} \arrow[dr] & & \overline{e}\ \overline{f} \arrow[dl] \\
 & \overline{e} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \overline{e}\ \overline{f}\ \overline{g} \arrow[dl] \arrow[dr] & \\
\overline{f}\ \overline{g} \arrow[dr] & & \overline{e}\ \overline{g} \arrow[dl] \\
 & \overline{g} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \overline{e}\ \overline{f}\ \overline{g} \arrow[dl] \arrow[dr] & \\
\overline{e}\ \overline{g} \arrow[dr, "*"] & & \overline{e}\ \overline{g} \arrow[dl, "*"'] \\
 & \overline{e}\ \overline{g} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \overline{e}\ \overline{f}\ \overline{g} \arrow[dl] \arrow[dr] & \\
\overline{f}\ \overline{g} \arrow[dr] & & \overline{e}\ \overline{f} \arrow[dl] \\
 & \overline{f} &
\end{tikzcd}
$$

Figure 8.2: the confluence of $\mathrm{IG}(Y)$ over a semilattice Y

Thus $(\overline{Y}^+, \longrightarrow)$ is locally confluent, so that every element in $\mathrm{IG}(Y)$ has a unique normal form. $\square$

Note that an element $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(Y)$ is in normal form if and only if x_i and x_{i+1} are incomparable, for all $i \in [1, n-1]$. By uniqueness of normal forms in $\mathrm{IG}(Y)$, we can easily deduce that two words of $\mathrm{IG}(B)$ are equal if and only the corresponding normal forms of them are identical word in $\overline{E}^+$, and hence the word problem of $\mathrm{IG}(Y)$ is solvable.

Proposition 8.2.2. *The free idempotent generated semigroup* $\mathrm{IG}(Y)$ *over a semilattice* Y *is adequate.*

Proof. We begin with considering a product $(\overline{x_1} \cdots \overline{x_n})(\overline{y_1} \cdots \overline{y_m})$, where $\overline{x_1} \cdots \overline{x_n}$, $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(Y)$ are in normal form. Either x_n, y_1 are incomparable, $x_n \geq y_1$ or $x_n \leq y_1$. In the first case it is clear that $\overline{x_1} \cdots \overline{x_n}\ \overline{y_1} \cdots \overline{y_m}$ is a normal form. If $x_n \geq y_1$, then either $\overline{x_1} \cdots \overline{x_{n-1}}\ \overline{y_1} \cdots \overline{y_m}$ is in normal form, or y_1 and x_{n-1} are comparable. If y_1 and x_{n-1} are comparable, then $y_1 < x_{n-1}$, for we cannot have $x_{n-1} \leq y_1$ else $x_{n-1} \leq x_n$, a contradiction. Continuing in this manner we obtain $(\overline{x_1} \cdots \overline{x_n})(\overline{y_1} \cdots \overline{y_m})$ has normal form $\overline{x_1} \cdots \overline{x_{t-1}}\ \overline{y_1} \cdots \overline{y_m}$, where $1 \leq t \leq n$, $x_n, \cdots, x_t \geq y_1$, and either $t = 1$ (in which case $\overline{x_1}\ \cdots\ \overline{x_{t-1}}$ is the empty product) or x_{t-1}, y_1 are incomparable. Similarly, if $x_n \leq y_1$, then $(\overline{x_1} \cdots \overline{x_n})(\overline{y_1} \cdots \overline{y_m})$ has normal form $\overline{x_1} \cdots \overline{x_n}\ \overline{y_{t+1}} \cdots \overline{y_m}$, where $1 \leq t \leq m$, $x_n \leq y_1, \cdots y_t$, and $t = m$ or x_n, y_{t+1} are incomparable.

Suppose now that $\overline{x_1} \cdots \overline{x_n}, \overline{z_1} \cdots \overline{z_k}$ and $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(Y)$ are in normal form such that

$$\overline{x_1} \cdots \overline{x_n}\ \overline{y_1} \cdots \overline{y_m} = \overline{z_1} \cdots \overline{z_k}\ \overline{y_1} \cdots \overline{y_m}$$

in IG(Y). Here we assume $n, k \geq 0$ and $m \geq 1$. We proceed to prove that

$$\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$$

in IG(Y). If $n = k = 0$ there is nothing to show. Note that the result is clearly true if $m = 1$, so in what follows we assume $m \geq 2$.

First we assume that $n \geq 1$ and $k = 0$ (i.e. $\overline{z_1} \cdots \overline{z_k}$ is empty), so that

$$\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} \cdots \overline{y_m} = \overline{y_1} \cdots \overline{y_m}.$$

In view of Lemma 8.2.1, x_n and y_1 must be comparable. If $x_n \geq y_1$, then it follows from the above observation that $y_1 \leq x_1, \cdots, x_n$, so that $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{y_1}$. On the other hand, if $x_n \leq y_1$, then

$$\overline{x_1} \cdots \overline{x_n}\, \overline{y_{t+1}} \cdots \overline{y_m} = \overline{y_1} \cdots \overline{y_m}$$

for $1 \leq t \leq m$ such that $x_n \leq y_1, \cdots, y_t$ and $t = m$ or x_n, y_{t+1} are incomparable. Then $x_n = y_t$, so that to avoid the contradiction $y_t \leq y_{t-1}$ we must have $t = 1$. Clearly then $n = 1$ and $x_1 = x_n = y_1$ so that $\overline{x_1}\, \overline{y_1} = \overline{y_1}$. Hence certainly holds for $n + k + m \leq 3$.

Suppose that $n+k+m \geq 4$ and the result is true for all $n'+k'+m' < n+k+m$. Recall we are assuming that $m \geq 2$ and in view of the above we may take $n, k \geq 1$.

If x_n, y_1 and z_k, y_1 are incomparable pairs, then it follows from uniqueness of normal form that $k = n$ and $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$.

Suppose now that $y_1 \leq x_n$. Then

$$\overline{x_1} \cdots \overline{x_{n-1}}\, \overline{y_1} \cdots \overline{y_m} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1} \cdots \overline{y_m}$$

so that our induction gives us

$$\overline{x_1} \cdots \overline{x_{n-1}}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$$

and hence $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$. A similar result holds for the case $y_1 \leq z_k$.

Suppose now that $y_1 \nleq x_n$ and $y_1 \nleq z_k$ and at least one of x_n, y_1 or z_k, y_1 are comparable. Without loss of generality assume that $x_n < y_1$. As above

$x_n \leq y_1, \cdots, y_t$ for some $1 \leq t \leq m$ with $t = m$ or x_n, y_{t+1} incomparable. Further, there is an r with $0 \leq r \leq m$ such that $z_k \leq y_1, \cdots, y_r$ and $r = m$ or z_k, y_{r+1} incomparable. Thus both sides of

$$\overline{x_1} \cdots \overline{x_n}\, \overline{y_{t+1}} \cdots \overline{y_m} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_{r+1}} \cdots \overline{y_m}$$

are in normal form and so $n - t = k - r$. If $n > k$, then $r < t$, so $x_n = y_t$. To avoid the contradiction $y_t \leq y_{t-1}$, we must have $t = 1$, but then $x_n = y_1$ a contradiction. Similarly, we can not have $k > n$. Hence $n = k$, and hence $\overline{x_1} \cdots \overline{x_n} = \overline{z_1} \cdots \overline{z_k}$, so that certainly $\overline{x_1} \cdots \overline{x_n}\, \overline{y_1} = \overline{z_1} \cdots \overline{z_k}\, \overline{y_1}$ as required. Hence, we have that $\mathrm{IG}(Y)$ is abundant. It then follows from Property (IG1) that the biordered set of idempotents of $\mathrm{IG}(Y)$ is isomorphic to Y, which is a semilattice, so that $\mathrm{IG}(Y)$ is adequate. $\square$

We remark here that Proposition 8.2.2 can also be obtained as a corollary of Proposition 8.6.2, but for the sake of our readers, we have proved this special case to outline our strategy in a simple case.

Example 8.2.3. Consider a semilattice $Y = \{e, f, g\}$ with $e, f \geq g$ and e, f incomparable.

First, we observe that

$$\mathrm{IG}(Y) = \{\overline{e}, \overline{f}, \overline{g}, (\overline{e}\, \overline{f})^n, (\overline{f}\, \overline{e})^n, (\overline{e}\, \overline{f})^n\, \overline{e}, (\overline{f}\, \overline{e})^n\, \overline{f} : \ n \in \mathbb{N}\}.$$

It is easy to check that for any $n \in \mathbb{N}$, the element $(\overline{e}\, \overline{f})^n \in \mathrm{IG}(Y)$ is not regular, as for any $w \in \mathrm{IG}(Y)$, we have $(\overline{e}\, \overline{f})^n w (\overline{e}\, \overline{f})^n = \overline{g}$ if w contains $\overline{g}$ as a letter; otherwise $(\overline{e}\, \overline{f})^n w (\overline{e}\, \overline{f})^n = (\overline{e}\, \overline{f})^m$ for some $m \geq 2n \in \mathbb{N}$. Therefore, $\mathrm{IG}(Y)$ is not a regular semigroup.

On the other hand, by Proposition 8.2.2 we have that $\mathrm{IG}(Y)$ is an abundant semigroup. Furthermore,

$$\mathcal{R}^* = \{\{\overline{e}, (\overline{e}\, \overline{f})^n, (\overline{e}\, \overline{f})^n\, \overline{e} : \ n \in \mathbb{N}\}, \{\overline{f}, (\overline{f}\, \overline{e})^n, (\overline{f}\, \overline{e})^n\, \overline{f} : \ n \in \mathbb{N}\}, \{\overline{g}\}\}$$

and

$$\mathcal{L}^* = \{\{\overline{e}, (\overline{f}\, \overline{e})^n, (\overline{e}\, \overline{f})^n\, \overline{e} : \ n \in \mathbb{N}\}, \{\overline{f}, (\overline{e}\, \overline{f})^n, (\overline{f}\, \overline{e})^n\, \overline{f} : \ n \in \mathbb{N}\}, \{\overline{g}\}\}$$

Note that we have
$$\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{R}^* \circ \mathcal{L}^*$$
in IG(Y), and there are two $\mathcal{D}^*$-classes of IG(Y), namely,
$$\{\overline{g}\} \text{ and } \{\overline{e}, (\overline{e}\ \overline{f})^n, (\overline{e}\ \overline{f})^n\ \overline{e}, \overline{f}, (\overline{f}\ \overline{e})^n, (\overline{f}\ \overline{e})^n\ \overline{f} : n \in \mathbb{N}\},$$
the latter of which can be depicted by the following so called *egg-box* picture:

$\overline{e}, (\overline{e}\ \overline{f})^n\overline{e}$	$(\overline{e}\ \overline{f})^n$
$(\overline{f}\ \overline{e})^n$	$\overline{f}, (\overline{f}\ \overline{e})^n\overline{f}$

Figure 8.3: the egg-box of Example 8.2.3

8.3 Free idempotent generated semigroups over rectangular bands

In this section we are concerned with the free idempotent generated semigroup IG(B) over a rectangular band B. Recall from [26] that a band B is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, and the B_α's are the $\mathcal{D} = \mathcal{J}$-classes of B. Thus $B = \bigcup_{\alpha \in Y} B_\alpha$ where each B_α is a rectangular band and $B_\alpha B_\beta \subseteq B_{\alpha\beta}$, $\forall \alpha, \beta \in Y$. At times we will use this notation without specific comments. We show that IG(B) is a regular semigroup. It follows that if B is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, then any word in $\overline{B_\alpha}^+$ is regular in IG(B).

Lemma 8.3.1. *Let B be a rectangular band. Then every element in* IG(B) *has a unique normal form.*

Proof. We have already remarked that the reduction system $(\overline{B}^+, \longrightarrow)$ induced by IG(B) is noetherian, so that according to Lemma 8.1.1, to show the uniqueness of normal form of elements in IG(B), we only need to prove that $(\overline{B}^+, \longrightarrow)$ is locally confluent.

For this purpose, it is sufficient to consider an arbitrary word of length 3, say $\overline{e}\ \overline{f}\ \overline{g} \in \overline{Y}^+$, where e, f and f, g are comparable. Clearly, there are four cases, namely, $e\ \mathcal{L}\ f\ \mathcal{L}\ g$, $e\ \mathcal{R}\ f\ \mathcal{R}\ g$, $e\ \mathcal{L}\ f\ \mathcal{R}\ g$ and $e\ \mathcal{R}\ f\ \mathcal{L}\ g$. Then we have the following 4 diagrams:

$$
\begin{tikzcd}[column sep=small]
& \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & \\
\overline{e}\,\overline{g} \arrow[dr] & & \overline{e}\,\overline{f} \arrow[dl] \\
& \overline{e} &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & \\
\overline{f}\,\overline{g} \arrow[dr] & & \overline{e}\,\overline{g} \arrow[dl] \\
& \overline{g} &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & \\
\overline{e}\,\overline{g} \arrow[dr, "*"] & & \overline{e}\,\overline{g} \arrow[dl, "*"'] \\
& \overline{e}\,\overline{g} &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & \\
\overline{f}\,\overline{g} \arrow[dr] & & \overline{e}\,\overline{f} \arrow[dl] \\
& \overline{f} &
\end{tikzcd}
$$

Figure 8.4: the confluence of $\mathrm{IG}(B)$ over a rectangular band B

Hence (B^*, R) is locally confluent. □

Lemma 8.3.2. *Suppose that B is a rectangular band and $\overline{u_1}\ \cdots\ \overline{u_n} \in \mathrm{IG}(B)$. Then we have $\overline{u_n}\ \mathcal{L}\ \overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}$, and hence $\mathrm{IG}(B)$ is a regular semigroup.*

Proof. Let $w = \overline{u_1}\ \cdots\ \overline{u_n} \in \mathrm{IG}(B)$. First we claim that

$$\overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}\ \cdots\ \overline{u_{n-1}}.$$

Observe that $(u_n, u_n u_{n-1})$ and $(u_{n-1}, u_n u_{n-1})$ are both basic pairs. Hence we have

$$\begin{aligned}
\overline{u_1}\ \cdots\ \overline{u_{n-1}}\ \overline{u_n}\ \overline{u_n u_{n-1}} &= \overline{u_1}\ \cdots\ \overline{u_{n-1}}\ \overline{u_n u_n u_{n-1}} \\
&= \overline{u_1}\ \cdots\ \overline{u_{n-1}}\ \overline{u_n u_{n-1}} \\
&= \overline{u_1}\ \cdots\ \overline{u_{n-1} u_n u_{n-1}} \\
&= \overline{u_1}\ \cdots\ \overline{u_{n-1}},
\end{aligned}$$

so that $\overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}\ \cdots\ \overline{u_{n-1}}$. By finite induction we obtain that $\overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{R}\ \overline{u_1}$.

Similarly, we can show that $\overline{u_1}\ \cdots\ \overline{u_n}\ \mathcal{L}\ \overline{u_n}$. Certainly then $\mathrm{IG}(B)$ is regular. □

Corollary 8.3.3. *Let B be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Then for any $x_1, \cdots, x_n \in B_\alpha$, $\overline{x_1}\ \cdots\ \overline{x_n}$ is a regular element of* $\mathrm{IG}(B)$.

Proof. It is clear from the presentations of $\mathrm{IG}(B_\alpha)$ and $\mathrm{IG}(B)$ that there is a well defined morphism

$$\overline{\psi} : \mathrm{IG}(B_\alpha) \longrightarrow \mathrm{IG}(B),\ \text{ are such that } \overline{e}\,\overline{\psi} = \overline{e}$$

for each $e \in B_\alpha$. It follows from Lemma 8.3.2 that for any $x_1, \cdots, x_n \in B_\alpha$, $\overline{x_1} \ \cdots \ \overline{x_n}$ is regular in $\mathrm{IG}(B_\alpha)$. Since $\overline{\psi}$ preserves the regularity, we have that $(\overline{x_1} \ \cdots \ \overline{x_n})\, \overline{\psi} = \overline{x_1} \ \cdots \ \overline{x_n}$ is regular in $\mathrm{IG}(B)$. □

8.4 Free idempotent generated semigroups over bands

Our aim here is to investigate the general structure of $\mathrm{IG}(B)$ for an arbitrary band B. We prove that for any band B, $\mathrm{IG}(B)$ is a weakly abundant semigroup with the congruence condition. However, we demonstrate a band B for which $\mathrm{IG}(B)$ is not abundant.

Lemma 8.4.1. *Let S and T be semigroups with biordered sets of idempotents $U = E(S)$ and $V = E(T)$, respectively, and let $\theta : S \longrightarrow T$ be a morphism. Then the map from $\overline{U}$ to $\overline{V}$ defined by $\overline{e} \mapsto \overline{e\theta}$, for all $e \in U$, lifts to a well defined morphism $\overline{\theta} : \mathrm{IG}(U) \longrightarrow \mathrm{IG}(V)$.*

Proof. Since θ is a morphism by assumption, we have that (e, f) is basic in U implies $(e\theta, f\theta)$ is basic in V, so that there exists a morphism $\overline{\theta} : \mathrm{IG}(U) \longrightarrow \mathrm{IG}(V)$ defined by $\overline{e}\, \overline{\theta} = \overline{e\theta}$, for all $e \in U$. □

Let B be a band. Write B as a semilattice Y of rectangular bands B_α, $\alpha \in Y$. The mapping θ defined by

$$\theta : B \longrightarrow Y, x \mapsto \alpha$$

where $x \in B_\alpha$, is a morphism with kernel $\mathcal{D}$. Hence, by applying Lemma 8.4.1 to our band B and the associated semilattice Y, we have the following corollary.

Corollary 8.4.2. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a semilattice Y of rectangular bands B_α, $\alpha \in Y$. Then a map $\overline{\theta} : \mathrm{IG}(B) \longrightarrow \mathrm{IG}(Y)$ defined by*

$$(\overline{x_1} \ \cdots \ \overline{x_n})\, \overline{\theta} = \overline{\alpha_1} \ \cdots \ \overline{\alpha_n}$$

is a morphism, where $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$.

To proceed further we need the following definition of *left to right significant indices* of elements in $\mathrm{IG}(B)$, where B is a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$.

Let $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with $x_i \in B_{\alpha_i}$, for all $1 \leq i \leq n$. Then a set of numbers

$$\{i_1, \cdots, i_r\} \subseteq [1, n] \text{ with } i_1 < \cdots < i_r$$

is called the *left to right significant indices* of $\overline{x_1} \cdots \overline{x_n}$, if these numbers are picked out in the following manner:

i_1 : the largest number such that $\alpha_1, \cdots, \alpha_{i_1} \geq \alpha_{i_1}$;

k_1 : the largest number such that $\alpha_{i_1} \leq \alpha_{i_1}, \alpha_{i_1+1}, \cdots, \alpha_{k_1}$.

We pause here to remark that $\alpha_{i_1}, \alpha_{k_1+1}$ are incomparable. This is because, if $\alpha_{i_1} \leq \alpha_{k_1+1}$, we add 1 to k_1, contradicting the choice of k_1; and if $\alpha_{i_1} > \alpha_{k_1+1}$, then $\alpha_1, \cdots, \alpha_{i_1}, \cdots, \alpha_{k_1} \geq \alpha_{k_1+1}$, contradicting the choice of i_1. Now we continue our process:

i_2 : the largest number such that $\alpha_{k_1+1}, \cdots, \alpha_{i_2} \geq \alpha_{i_2}$;

k_2 : the largest number such that $\alpha_{i_2} \leq \alpha_{i_2}, \alpha_{i_2+1}, \cdots, \alpha_{k_2}$.

$\vdots$

i_r : the largest number such that $\alpha_{k_{r-1}+1}, \cdots, \alpha_{i_r} \geq \alpha_{i_r}$;

$k_r = n$: here we have $\alpha_{i_r} \leq \alpha_{i_r}, \alpha_{i_r+1}, \cdots, \alpha_n$. Of course, here we may have $i_r = k_r = n$.

Corresponding to the so called left to right significant indices $i_1, \cdots, i_r$, we have

$$\alpha_{i_1}, \cdots, \alpha_{i_r} \in Y.$$

We claim that for all $1 \leq s \leq r-1$, α_{i_s} and $\alpha_{i_{s+1}}$ are incomparable. If not, suppose that there exists some $1 \leq s \leq r-1$ such that $\alpha_{i_s} \leq \alpha_{i_{s+1}}$. Then $\alpha_{i_s} \leq \alpha_{k_s+1}$ as $\alpha_{i_{s+1}} \leq \alpha_{k_s+1}$, a contradiction; if $\alpha_{i_s} \geq \alpha_{i_{s+1}}$, then we have

$$\alpha_{i_{s+1}} \leq \alpha_{i_{s+1}}, \alpha_{i_{s+1}-1}, \cdots, \alpha_{k_{s-1}+1}, \text{ with } k_0 = 0$$

contradicting our choice of i_s. Therefore, we deduce that $\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}}$ is the unique normal form of $\overline{\alpha_1} \cdots \overline{\alpha_n}$ in $\mathrm{IG}(Y)$.

We can use the following *Hasse diagram* to depict the relationship among $\alpha_{i_1}, \cdots, \alpha_{i_r}$:

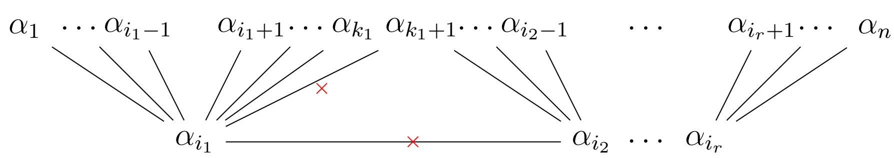

Figure 8.5: Hasse diagram illustrating significant indices

Dually, we can define the *right to left significant indices* $\{l_1, \cdots, l_s\} \subseteq [1, n]$ of $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$, where $l_1 < \cdots < l_s$. Note that as $\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}}$ must equal to $\overline{\alpha_{l_1}} \cdots \overline{\alpha_{l_s}}$ in $\overline{B}^+$, we have $r = s$.

Lemma 8.4.3. *Let $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with $x_i \in \alpha_i$, for all $i \in [1, n]$, and left to right significant indices $i_1, \cdots, i_r$. Suppose also that $\overline{y_1} \cdots \overline{y_m} \in \overline{B}^+$ with $y_i \in \beta_i$, for all $i \in [1, m]$, and left to right significant indices $l_1, \cdots, l_s$. Then*

$$\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$$

in $\mathrm{IG}(B)$ *implies $s = r$ and $\alpha_{i_1} = \beta_{l_1}, \cdots, \alpha_{i_r} = \beta_{l_r}$.*

Proof. It follows from Lemma 8.4.2 and the discussion above that

$$\overline{\alpha_{i_1}} \cdots \overline{\alpha_{i_r}} = \overline{\alpha_1} \cdots \overline{\alpha_n} = \overline{\beta_1} \cdots \overline{\beta_m} = \overline{\beta_{l_1}} \cdots \overline{\beta_{l_s}}$$

in $\mathrm{IG}(Y)$. By uniqueness of normal form in $\mathrm{IG}(Y)$, we have that $s = r$ and $\alpha_{i_1} = \beta_{l_1}, \cdots, \alpha_{i_r} = \beta_{l_r}$. $\square$

In view of the above observations, we introduce the following notions.

Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$, and let $w = \overline{x_1} \cdots \overline{x_n}$ be a word in $\overline{B}^+$ with $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$. Suppose that w has left to right significant indices $i_1, \cdots, i_r$. Then we call the natural number r the *Y-length*, and $\alpha_{i_1}, \cdots, \alpha_{i_r}$ the *ordered Y-components* of the equivalence class of w in $\mathrm{IG}(B)$.

In all what follows whenever we write $w \sim w'$ for $w, w' \in \overline{B}^+$, we mean that the word w' can be obtained from the w from a single splitting step or a single squashing step.

Lemma 8.4.4. *Let $\overline{x_1} \cdots \overline{x_n} \in \overline{B}^+$ with left to right significant indices $i_1, \cdots, i_r$, where $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$. Let $\overline{y_1} \cdots \overline{y_m} \in \overline{B}^+$ be an element obtained from*

$\overline{x_1}\ \cdots\ \overline{x_n}$ from a single step, and suppose that the left to right significant indices of $\overline{y_1}\cdots\ \overline{y_m}$ are $j_1, \cdots, j_r$. Then for all $l \in [1, r]$, we have

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \ \cdots\ \overline{x_{i_l}}\ \overline{u}$$

and $y_{j_l} = u' x_{i_l} u$, where $u' = \varepsilon$ or $u' \in B_\sigma$ with $\sigma \geq \alpha_{i_l}$, and either $u = \varepsilon$, or $u \in B_\delta$ for some $\delta > \alpha_{i_l}$, or $u \in B_{\alpha_{i_l}}$ and there exists $v \in B_\theta$ with $\theta > \alpha_{i_l}$, $vu = u$ and $uv = x_{i_l}$.

Proof. Suppose that we split $x_k = uv$ for some $k \in [1, n]$, where uv is a basic product with $u \in B_\mu$ and $v \in B_\tau$, so that $\alpha_k = \mu\tau$. Then

$$\overline{x_1}\ \cdots\ \overline{x_n} \sim \overline{x_1}\ \cdots\ \overline{x_{k-1}}\ \overline{u}\ \overline{v}\ \overline{x_{k+1}}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}.$$

If $k < i_l$, then clearly $y_{j_l} = x_{i_l}$ and

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{k-1}}\ \overline{u}\ \overline{v}\ \overline{x_{k+1}} \cdots\ \overline{x_{i_l}} = \overline{x_1}\ \ \cdots\ \overline{x_{i_l}},$$

so we may take $u = u' = \varepsilon$.

If $k = i_l$, then $\mu\tau = \alpha_{i_l}$. If $\mu \geq \tau$, then $y_{j_l} = v$ and again

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots \overline{x_{i_l-1}}\ \overline{u}\ \overline{v} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}.$$

As $x_{i_l} = uv\ \mathcal{L}\ v$, we have $y_{j_l} = v = vx_{i_l}$. Also, $x_{i_l} = uv = uy_{j_l}$.

On the other hand, if $\mu < \tau$, then $y_{j_l} = u$. As uv is a basic product, we have that $uv = u = x_{i_l}$ or $vu = u$. If $uv = u = x_{i_l}$, then

$$\overline{y_1} \cdots \overline{y_{j_1}} = \overline{x_1} \cdots \overline{x_{i_l-1}}\ \overline{u} = \overline{x_1} \cdots \overline{x_{i_l}},$$

and $y_{j_l} = u = uv = x_{i_l}$. If $vu = u$, then as $x_k = uv\ \mathcal{R}\ u$ and $u = uvu$,

$$\overline{y_1} \cdots \overline{y_{j_1}} = \overline{x_1} \cdots\ \overline{x_{i_l-1}}\ \overline{u} = \overline{x_1} \cdots\ \overline{x_{k-1}}\ \overline{uv}\ \overline{u} = \overline{x_1} \cdots\ \overline{x_{i_l}}\ \overline{u}$$

and $y_{j_l} = x_{i_l} u$ where $vu = u$. Also,

$$\overline{x_1} \cdots\ \overline{x_{i_l}} = \overline{x_1} \cdots\ \overline{x_{k-1}}\ \overline{uv} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}\ \overline{u}\ \overline{v} = \overline{y_1}\ \cdots\ \overline{y_{j_l}}\ \overline{v}$$

and $x_{i_l} = y_{j_l}v$.

Finally, suppose that $k > i_l$. Then it is obviously that $j_l = i_l$, $x_{i_l} = y_{j_l}$ and

$$\overline{y_1}\ \cdots\ \overline{y_{j_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}.$$

□

It follows immediately from Lemma 8.4.4 that

Corollary 8.4.5. *Suppose that* $\overline{y_1}\cdots\ \overline{y_m} = \overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ *with left to right significant indices* $j_1, \cdots, j_r$ *and* $i_1, \cdots, i_r$*, respectively, and suppose* $x_i \in B_{\alpha_i}$ *for all* $i \in [1, n]$*. Then for all* $l \in [1, r]$*, we have*

$$\overline{y_1}\ \cdots\ \overline{y_{i_l}} = \overline{x_1}\ \cdots\ \overline{x_{i_l}}\ \overline{u_1}\ \overline{u_2}\cdots\ \overline{u_s}$$

and $y_{j_l} = u'_s \cdots u'_1 x_{i_l} u_1 \cdots u_s$*, where for all* $t \in [1, s]$*,* $u'_t = \varepsilon$ *or* $u'_t \in B_{\sigma_t}$ *for some* $\sigma_t \geq \alpha_{i_l}$*, and either* $u_t = \varepsilon$ *or* $u_t \in B_{\delta_t}$ *for some* $\delta_t > \alpha_{i_l}$*, or* $u_t \in B_{\alpha_{i_l}}$ *and there exists* $v_t \in B_{\theta_t}$ *with* $\theta_t > \alpha_{i_l}$ *and* $v_t u_t = u_t$*. Consequently,* $\overline{y_1}\ \cdots\ \overline{y_{j_l}}\ \mathcal{R}\ \overline{x_1}\ \cdots\ \overline{x_{i_l}}$*, and hence* $y_1\ \cdots\ y_{j_l}\ \mathcal{R}\ x_1\ \cdots\ x_{i_l}$*.*

Proof. The proof follows from Lemma 8.4.4 by finite induction. □

Note that the duals of Lemma 8.4.4 and Corollary 8.4.5 hold for right to left significant indices.

From Lemmas 8.2.1 and 8.3.1, we know that every element in $\mathrm{IG}(B)$ has a unique normal form, if B is a semilattice or a rectangular band. However, it may not true for an arbitrary band B, even if B is normal. Recall that a normal band

$$B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$$

is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, such that for all $\alpha \geq \beta$ in Y there exists a morphism $\phi_{\alpha,\beta} : B_\alpha \longrightarrow B_\beta$ such that

(B1) for all $\alpha \in Y$, $\phi_{\alpha,\alpha} = 1_{B_\alpha}$;

(B2) for all $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$, $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$,

and for all $\alpha, \beta \in Y$ and $x \in B_\alpha, y \in B_\beta$,

$$xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}).$$

Example 8.4.6. Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice $Y = \{\alpha, \beta, \gamma, \delta\}$ of rectangular bands B_α, $\alpha \in Y$ (see the figure below), such that $\phi_{\alpha,\beta}$ is defined by $a\phi_{\alpha,\beta} = b$, the remaining morphisms being defined in the obvious unique manners.

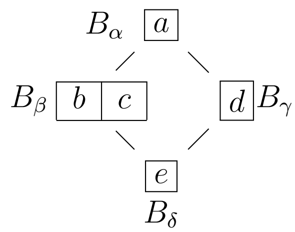

Figure 8.6: the semilattice decomposition structure of Example 8.4.6

By an easy calculation, we have

$$\overline{c}\ \overline{d} = \overline{c}\ \overline{ad} = \overline{c}\ \overline{a}\ \overline{d} = \overline{ca}\ \overline{d} = \overline{b}\ \overline{d}$$

in $\mathrm{IG}(B)$, so that not every element in $\mathrm{IG}(B)$ has a unique normal form.

We pause here to make a comment on the above example. It was shown by Pastign [41] that for any normal band B, if one work with $\mathrm{RIG}(B)$ rather than $\mathrm{IG}(B)$ then one does get a complete rewriting system and unique normal forms. This result contrasts nicely with our example showing that for normal bands B normal forms are not necessarily unique in $\mathrm{IG}(B)$.

Lemma 8.4.7. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$. Let $\overline{x_1}\ \cdots\ \overline{x_n} \in \mathrm{IG}(B)$ with $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$, and let $y \in B_\beta$ with $\beta \leq \alpha_i$, for all $i \in [1, n]$. Then in $\mathrm{IG}(B)$ we have*

$$\overline{x_1}\ \cdots\ \overline{x_n}\ \overline{y} = \overline{x_1 \cdots x_n y x_n \cdots x_1}\ \cdots\ \overline{x_{n-1} x_n y x_n x_{n-1}}\ \overline{x_n y x_n}\ \overline{y}$$

and

$$\overline{y}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{y}\ \overline{x_1 y x_1}\ \overline{x_2 x_1 y x_1 x_2}\ \cdots\ \overline{x_n \cdots x_1 y x_1 \cdots x_n}.$$

Proof. First, we notice that for any $x \in B_\alpha, y \in B_\beta$ such that $\alpha \geq \beta$, we have $yx\ \mathcal{R}\ y$, so that (y, yx) is a basic pair and $(yx)y = y$. On the other hand, as

$(yx)x = yx$, we have that (x, yx) is a basic pair, so that

$$\overline{x}\ \overline{y} = \overline{x}\ \overline{(yx)y} = \overline{x}\ \overline{yx}\ \overline{y} = \overline{xyx}\ \overline{y}.$$

Thus, the first required equality follows from the above observation by finite induction. Dually, we can show the second one. □

Corollary 8.4.8. *Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band and let $\overline{x_1}\ \cdots\ \overline{x_n}$ be an element of* IG(B) *such that $x_i \in B_{\alpha_i}$, for all $i \in [1, n]$. Let $y \in B_\beta$ with $\beta \leq \alpha_i$, for all $i \in [1, n]$. Then*

$$\overline{x_1}\ \cdots\ \overline{x_n}\ \overline{y} = \overline{x_1\phi_{\alpha_1,\beta}}\ \cdots\ \overline{x_n\phi_{\alpha_n,\beta}}\ \overline{y}$$

and

$$\overline{y}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{y}\ \overline{x_1\phi_{\alpha_1,\beta}}\ \cdots\ \overline{x_n\phi_{\alpha_n,\beta}}.$$

Corollary 8.4.9. *Let $B = \bigcup\limits_{\alpha \in Y} B_\alpha$ be a chain Y of rectangular bands B_α, $\alpha \in Y$. Then* IG(B) *is a regular semigroup.*

Proof. Let $\overline{u_1}\ \cdots\ \overline{u_n}$ be an element in IG(B). From Lemma 8.4.7 it follows that $\overline{u_1}\ \cdots\ \overline{u_n}$ can be written as an element of IG(B) in which all letters come from B_γ, where γ is the minimum of $\{\alpha_1, \cdots, \alpha_n\}$, so that $\overline{u_1}\ \cdots\ \overline{u_n}$ is regular by Lemma 8.3.3. □

Given the above observations, we now introduce the idea of *almost normal form* for elements in IG(B).

Definition 8.4.10. *An element $\overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ is said to be in almost normal form if there exists a sequence*

$$1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n$$

with

$$\{x_1, \cdots, x_{i_1}\} \subseteq B_{\alpha_1}, \{x_{i_1+1}, \cdots, x_{i_2}\} \in B_{\alpha_2}, \cdots, \{x_{i_{r-1}+1}, \cdots x_n\} \subseteq B_{\alpha_r}$$

where α_i, α_{i+1} are incomparable for all $i \in [1, r-1]$.

It is obvious that the element $\overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ defined as above has left to right significant indices $i_1, i_2, \cdots, i_{r-1}, i_r = n$ (right to left significant indices

$1, i_1+1, \cdots, i_{r-2}+1, i_{r-1}+1)$, Y-length r and ordered Y-components $\alpha_1, \cdots, \alpha_r$. Note that, in general, the almost normal forms of elements of $\mathrm{IG}(B)$ are not unique. Further, if $\overline{x_1} \ \cdots \ \overline{x_n} = \overline{y_1} \ \cdots \ \overline{y_m}$ are in almost normal form, then they have the same Y-length and ordered Y-components, but the left to right significant indices of them can be quite different.

The next result is immediate from the definition of significant indices and Lemma 8.4.7.

Lemma 8.4.11. *Every element of* $\mathrm{IG}(B)$ *can be written in almost normal form.*

We have the following lemma regarding the almost normal form of the product of two almost normal forms.

Lemma 8.4.12. *Let* $\overline{x_1} \ \cdots \ \overline{x_n} \in \mathrm{IG}(B)$ *be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$ and ordered Y-components $\alpha_1, \cdots, \alpha_r$, and let* $\overline{y_1} \ \cdots \ \overline{y_m} \in \mathrm{IG}(B)$ *be in almost normal form with Y-length s, left to right significant indices $l_1, \cdots, l_s = m$ and ordered Y-components $\beta_1, \cdots, \beta_s$. Then (with $i_0 = 0$)*

(i) α_r and β_1 incomparable implies that $\overline{x_1} \ \cdots \ \overline{x_{i_r}} \ \overline{y_1} \ \cdots \ \overline{y_{l_s}}$ *is in almost normal form;*

(ii) $\alpha_r \geq \beta_1$ implies

$$\overline{x_1} \cdots \overline{x_{i_t}} \ \overline{x_{i_t+1} \cdots x_{i_r} y_1 x_{i_r} \cdots x_{i_t+1}} \ \cdots \ \overline{x_{i_r} y_1 x_{i_r}} \ \overline{y_1} \ \cdots \ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1} \ \cdots \ \overline{x_{i_r}} \ \overline{y_1} \ \cdots \ \overline{y_{l_s}}$*, for some $t \in [0, r-1]$ such that $\alpha_r, \cdots, \alpha_{t+1} \geq \beta_1$ and $t = 0$ or α_t, β_1 are incomparable;*

(iii) $\alpha_r \leq \beta_1$ implies

$$\overline{x_1} \ \cdots \ \overline{x_{i_r}} \ \overline{y_1 x_{i_r} y_1} \ \cdots \ \overline{y_{l_v} \cdots y_1 x_{i_r} y_1 \cdots y_{l_v}} \ \overline{y_{l_v+1}} \ \cdots \ \overline{y_{l_s}}$$

is an almost normal form of the product $\overline{x_1} \ \cdots \ \overline{x_{i_r}} \ \overline{y_1} \ \cdots \ \overline{y_{l_s}}$ *for some $v \in [1, s]$ such that $\alpha_r \leq \beta_1, \cdots, \beta_v$ and $v = s$ or β_{v+1}, α_r are incomparable;*

Proof. Clearly, the statement (i) is true. We now aim to show (ii). Since $\alpha_r \geq \beta_1$, we have

$$\overline{x_{i_{r-1}+1}} \ \cdots \ \overline{x_{i_r}} \ \overline{y_1} = \overline{x_{i_{r-1}+1} \cdots x_{i_r} y_1 x_{i_r} \cdots x_{i_{r-1}+1}} \ \cdots \ \overline{x_{i_r} y_1 x_{i_r}} \ \overline{y_1}$$

by Corollary 8.4.7. Consider α_{r-1} and β_1, then we either have $\alpha_{r-1} \geq \beta_1$ or they are incomparable, as $\alpha_{r-1} < \beta_1$ would imply $\alpha_r > \alpha_{r-1}$, which contradicts the almost normal form of $\overline{x_1} \cdots \overline{x_{i_r}}$. By finite induction we have that

$$\overline{x_1} \cdots \overline{x_{i_t}}\ \overline{x_{i_t+1} \cdots x_{i_r} y_1 x_{i_r} \cdots x_{i_t+1}} \cdots \overline{x_{i_r} y_1 x_{i_r}}\ \overline{y_1} \cdots \overline{y_{l_s}}$$

is an almost normal form of $\overline{x_1} \cdots \overline{x_{i_r}}\ \overline{y_1} \cdots \overline{y_{l_s}}$ for some $t \in [0, r-1]$ such that $\alpha_r, \cdots, \alpha_{t+1} \geq \beta_1$ and $t = 0$ or α_t, β_1 are incomparable. Similarly, we can show (iii). $\square$

Theorem 8.4.13. *Let B be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$. Then* $\mathrm{IG}(B)$ *is a weakly abundant semigroup with the congruence condition.*

Proof. Let $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, and Y-components $\alpha_1, \cdots, \alpha_r$. Clearly $\overline{x_1}\ \overline{x_1} \cdots \overline{x_n} = \overline{x_1} \cdots \overline{x_n}$. Let $e \in B_\delta$ be such that $\overline{e}\ \overline{x_1} \cdots \overline{x_n} = \overline{x_1} \cdots \overline{x_n}$. Then by Corollary 8.4.2, that applying $\overline{\theta}$ we have $\overline{\delta}\ \overline{\alpha_1} \cdots \overline{\alpha_r} = \overline{\alpha_1} \cdots \overline{\alpha_r}$. It follows from Lemma 8.2.1 that $\delta \geq \alpha_1$, so that by Corollary 8.4.5 we have

$$ex_1 \cdots x_{i_1}\ \mathcal{R}\ x_1 \cdots x_{i_1}.$$

On the other hand, $x_1 \cdots x_{i_1}\ \mathcal{R}\ x_1$ so that $ex_1\ \mathcal{R}\ x_1$, thus we have $x_1 \leq_{\mathcal{R}} e$. Thus $\overline{e}\ \overline{x_1} = \overline{ex_1} = \overline{x_1}$. Therefore $\overline{x_1} \cdots \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{x_1}$. Dually, $\overline{x_1} \cdots \overline{x_n}\ \widetilde{\mathcal{L}}\ \overline{x_n}$, so that $\mathrm{IG}(B)$ is a weakly abundant semigroup as required.

Next we show that $\mathrm{IG}(B)$ satisfies the congruence condition.

Let $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(B)$ be defined as above and let $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length u, left to right significant indices $l_1, \cdots, l_u = m$ and ordered Y-components $\beta_1, \cdots, \beta_u$. From the above discussion and the property (IG2), we have $\overline{x_1} \cdots \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{y_1} \cdots \overline{y_m}$ if and only if $x_1\ \mathcal{R}\ y_1$. Suppose now that $x_1\ \mathcal{R}\ y_1$, so that $\alpha_1 = \beta_1$. Let $\overline{z_1} \cdots \overline{z_s} \in \mathrm{IG}(B)$, where, without loss of generality, we can assume it is in almost normal form with Y-length t, left to right significant indices $j_1, \cdots, j_t = s$, and Y-components $\gamma_1, \cdots, \gamma_t$. We aim to show that

$$\overline{z_1} \cdots \overline{z_s}\ \overline{x_1} \cdots \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{z_1} \cdots \overline{z_s}\ \overline{y_1} \cdots \overline{y_m}.$$

We consider the following three cases.

(i) If $\alpha_1 = \beta_1, \gamma_t$ are incomparable, then it is clear that

$$\overline{z_1}\ \cdots\ \overline{z_s}\ \overline{x_1}\ \cdots\ \overline{x_n} \text{ and } \overline{z_1}\ \cdots\ \overline{z_s}\ \overline{y_1}\ \cdots\ \overline{y_m}$$

are in almost normal form, so clearly we have

$$\overline{z_1}\ \cdots\ \overline{z_s}\ \overline{x_1}\ \cdots\ \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{z_1}\ \mathcal{R}\ \overline{z_1}\ \cdots\ \overline{z_s}\ \overline{y_1}\ \cdots\ \overline{y_m}.$$

(ii) If $\beta_1 = \alpha_1 \leq \gamma_1$, then by Lemma 8.4.12

$$\overline{z_1}\ \cdots\ \overline{z_s}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{z_1}\ \cdots\ \overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_s x_1 z_s \cdots z_{j_v+1}}\ \cdots\ \overline{z_s x_1 z_s}\ \overline{x_1}\ \cdots\ \overline{x_n}$$

and

$$\overline{z_1}\ \cdots\ \overline{z_s}\ \overline{y_1}\ \cdots\ \overline{y_m} = \overline{z_1}\ \cdots\ \overline{z_{j_v}}\ \overline{z_{j_v+1}\cdots z_s y_1 z_s \cdots z_{j_v+1}}\ \cdots\ \overline{z_s y_1 z_s}\ \overline{y_1}\ \cdots\ \overline{y_m}$$

where $v \in [0, t-1]$, $\gamma_{v+1}, \cdots, \gamma_t \geq \alpha_1 = \beta_1$ and γ_v, β_1 are incomparable or $v = 0$. Clearly, the right hand sides are in almost normal form.

If $v \geq 1$, then clearly the required result is true, as the above two almost normal forms begin with the same idempotent. If $v = 0$, then we need to show that

$$z_1 \cdots z_s x_1 z_s \cdots z_1\ \mathcal{R}\ z_1 \cdots z_s y_1 z_s \cdots z_1$$

Since $x_1\ \mathcal{R}\ y_1$, it follows from the structure of B that

$$z_1 \cdots z_s x_1 z_s \cdots z_1\ \mathcal{R}\ z_1 \cdots z_s x_1\ \mathcal{R}\ z_1 \cdots z_s y_1\ \mathcal{R}\ z_1 \cdots z_s y_1 z_s \cdots z_1$$

as required.

(iii) If $\beta = \alpha_1 \geq \gamma_1$, then by Lemma 8.4.12

$$\overline{z_1}\ \cdots\ \overline{z_s}\ \overline{x_1}\ \cdots\ \overline{x_n} = \overline{z_1}\ \cdots\ \overline{z_s}\ \overline{x_1 z_s x_1}\ \cdots\ \overline{x_{i_k} \cdots x_1 z_s x_1 \cdots x_{i_k}}\ \overline{x_{i_k+1}}\ \cdots\ \overline{x_n}$$

and

$$\overline{z_1}\ \cdots\ \overline{z_s}\ \overline{y_1}\ \cdots\ \overline{y_m} = \overline{z_1}\ \cdots\ \overline{z_s}\ \overline{y_1 z_s y_1}\ \cdots\ \overline{y_{l_p} \cdots y_1 z_s y_1 \cdots y_{l_p}}\ \overline{y_{l_p+1}}\ \cdots\ \overline{y_m},$$

where $k \in [1, r]$, $\alpha_1, \cdots, \alpha_k \geq \gamma_1$, and α_{k+1}, γ_1 are incomparable or $k = r$, and

$p \in [1, u]$, $\beta_1, \cdots, \beta_p \geq \gamma_1$, and β_{p+1}, γ_1 are incomparable or $p = u$. Clearly, the right hand sides are in almost normal form, so that

$$\overline{z_1}\ \cdots\ \overline{z_s}\ \overline{x_1}\ \cdots\ \overline{x_n}\ \widetilde{\mathcal{R}}\ \overline{z_1}\ \widetilde{\mathcal{R}}\ \overline{z_1}\ \cdots\ \overline{z_s}\ \overline{y_1}\ \cdots\ \overline{y_m}.$$

Similarly, we can show that $\widetilde{\mathcal{L}}$ is a right congruence, so that $\mathrm{IG}(B)$ is a weakly abundant semigroup satisfying the congruence condition. This completes the proof. $\square$

We finish this section by constructing a band B for which $\mathrm{IG}(B)$ is not an abundant semigroup.

Example 8.4.14. Let $B = B_\alpha \cup B_\beta \cup B_\gamma$ be a band with the semilattice decomposition structure and the multiplication table defined by

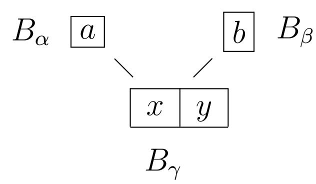

Figure 8.7: the semilattice decomposition structure of Example 8.4.14

	a	b	x	y
a	a	y	x	y
b	y	b	x	y
x	x	y	x	y
y	y	y	x	y

Figure 8.8: the multiplication table of Example 8.4.14

First, it is easy to check that B is indeed a semigroup. We now show that $\mathrm{IG}(B)$ is not abundant by arguing that the element $\overline{a}\,\overline{b} \in \mathrm{IG}(B)$ is not $\mathcal{R}^*$-related to any idempotent of $\overline{B}$. It follows from Theorem 8.4.13 that $\overline{a}\,\overline{b}\ \widetilde{\mathcal{R}}\ \overline{a}$. However, $\overline{a}\,\overline{b}$ is not $\mathcal{R}^*$-related to $\overline{a}$, because

$$\overline{x}\,\overline{a}\,\overline{b} = \overline{y} = \overline{y}\,\overline{a}\,\overline{b} \text{ but } \overline{x}\,\overline{a} = \overline{x} \neq \overline{y} = \overline{y}\,\overline{a},$$

so that from Lemma 2.2.5, $\overline{a}\,\overline{b}$ is not $\mathcal{R}^*$-related to any idempotent of $\overline{B}$, and hence $\mathrm{IG}(B)$ is not an abundant semigroup.

8.5 Free idempotent generated generated semigroups over quasi-zero bands

In this section we will introduce a class of bands B for which the word problem of $\mathrm{IG}(B)$ is solvable. Further, in Section 8.6, we will show that for any quasi-zero band B, the semigroup $\mathrm{IG}(B)$ is abundant.

Definition 8.5.1. *Let B be a semilattice Y of rectangular bands $B_\alpha, \alpha \in Y$. We say that B is a quasi zero band if for all $\alpha, \beta \in Y$ with $\beta > \alpha$, $u \in B_\alpha$ and $v \in B_\beta$, we have $uv = vu = u$.*

It is easy to deduce that if B is quasi-zero, then for any $\alpha, \beta \in Y$ with $\alpha < \beta$, $u \in B_\alpha$ and $v \in B_\beta$, the products uv and vu are basic.

Lemma 8.5.2. *Let B be a quasi-zero band, and let $\overline{x_1} \cdots \overline{x_n}$, $\overline{y_1} \cdots \overline{y_m}$ be elements of $\mathrm{IG}(B)$ with left to right significant indices $i_1, \cdots, i_r$; $j_1, \cdots, j_r$, respectively. If $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$, then for any $l \in [1, r]$, $\overline{x_1} \cdots \overline{x_{i_l}} = \overline{y_1} \cdots \overline{y_{j_l}}$.*

Proof. Suppose that $x_i \in B_{\alpha_i}$ for all $i \in [1, r]$. It is enough to consider a single step, say,

$$\overline{x_1} \cdots \overline{x_n} \sim \overline{w_1} \cdots \overline{w_s}.$$

Suppose that the significant indices of $\overline{w_1} \cdots \overline{w_s}$ are $k_1, \cdots, k_r$. By Lemma 8.4.4, for any $l \in [1, r]$, we have

$$\overline{w_1} \cdots \overline{w_{k_l}} = \overline{x_1} \cdots \overline{x_{i_l}}\, \overline{u}$$

and $w_{k_l} = u' x_{i_l} u$, where $u' = \varepsilon$ or $u' \in B_\sigma$ with $\sigma \geq \alpha_{i_l}$, and either $u = \varepsilon$, or $u \in B_\delta$ for some $\delta > \alpha_{i_l}$, or $u \in B_{\alpha_{i_l}}$ and there exists $v \in B_\theta$ with $\theta > \alpha_{i_l}$, $vu = u$ and $uv = x_{i_l}$. By the comment proceeding Lemma 8.5.2 we see that in each case, $\overline{x_{i_l}}\, \overline{u} = \overline{x_{i_l}}$, so that clearly, $\overline{w_1} \cdots \overline{w_{k_l}} = \overline{x_1} \cdots \overline{x_{i_l}}$. $\square$

Lemma 8.5.3. *Let B be a quasi-zero band, let $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant $i_1, \cdots, i_r = n$ and ordered Y-components $\alpha_1, \cdots, \alpha_r$, and let $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length s, left to right significant indices $j_1, \cdots, j_s = m$ and ordered Y-components $\beta_1, \cdots, \beta_s$. Then $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m}$ in $\mathrm{IG}(B)$ if and only if $r = s$, $\alpha_l = \beta_l$ and $\overline{x_{i_{l-1}+1}} \cdots \overline{x_{i_l}} = \overline{y_{j_{l-1}+1}} \cdots \overline{x_{j_l}}$ in $\mathrm{IG}(B)$, for each $l \in [1, r]$, where $i_0 = j_0 = 0$.*

Proof. The sufficiency is obvious. Suppose now that $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$ in $\mathrm{IG}(B)$. Then it follows from Lemma 8.4.3 that $r = s$ and $\alpha_i = \beta_i$ for all $i \in [1, r]$. From Lemma 8.5.2, we have that $\overline{x_1}\ \cdots\ \overline{x_{i_l}} = \overline{y_1}\ \cdots\ \overline{y_{j_l}}$ in $\mathrm{IG}(B)$, for all $l \in [1, r]$. Then by the dual of Lemma 8.5.2, $\overline{x_{i_{l-1}+1}}\ \cdots\ \overline{x_{i_l}} = \overline{y_{j_{l-1}+1}}\ \cdots\ \overline{x_{j_l}}$ in $\mathrm{IG}(B)$. □

Lemma 8.5.4. *Let B be a quasi-zero band and $w = \overline{x_1}\ \cdots\ \overline{x_n} \in \overline{B}^+$ with $x_i \in B_{\alpha_i}$ for each $i \in [1, n]$. Suppose that there exists an $\alpha \in Y$ such that for all $i \in [1, n]$, $\alpha_i \geq \alpha$ and there is at least one $j \in [1, n]$ such that $\alpha = \alpha_j$. Suppose also that p is a word in $\overline{B}^+$ obtained by a single step (a basic pair splitting or squashing) on w. Then we have that $w' = p'$ in $\mathrm{IG}(B_\alpha)$, where w' and p' are words obtained by deleting all letters in w and p which do not lie in B_α.*

Proof. Suppose that we split $x_k = uv$ for some $k \in [1, n]$, where $u \in B_\nu$ and $v \in B_\tau$. Then we have

$$w = \overline{x_1}\ \cdots\ \overline{x_{k-1}}\ \overline{x_k}\ \overline{x_{k+1}}\ \cdots\ \overline{x_n} \sim \overline{x_1}\ \cdots\ \overline{x_{k-1}}\ \overline{u}\ \overline{v}\ \overline{x_{k+1}}\ \cdots\ \overline{x_n} = p.$$

If $\alpha_k > \alpha$, then $\nu, \tau > \alpha$. Hence $w' = p'$ in $\overline{B_\alpha}^+$; of course, they are also equal in $\mathrm{IG}(B_\alpha)$.

If $\alpha_k = \alpha$ and $\mu = \tau = \alpha$, then $u\ \mathcal{L}\ v$ or $u\ \mathcal{R}\ v$, so that uv is basic in B_α. In this case, $\overline{x_k} = \overline{uv} = \overline{u}\ \overline{v}$ in $\mathrm{IG}(B_\alpha)$, so that certainly,

$$p' = (\overline{x_1}\ \cdots\ \overline{x_{k-1}})'\ \overline{u}\ \overline{v}\ (\overline{x_{k+1}}\ \cdots\ \overline{x_n})' = (\overline{x_1}\ \cdots\ \overline{x_{k-1}})'\ \overline{x_k}\ (\overline{x_{k+1}}\ \cdots\ \overline{x_n})' = w'$$

in $\mathrm{IG}(B_\alpha)$.

If $\alpha_k = \alpha$ and $\nu > \tau = \alpha$, then we have $x_k = uv = v$ as B is a quasi-zero band, so that

$$\begin{aligned} p' &= (\overline{x_1}\ \cdots\ \overline{x_{k-1}})'\ (\overline{u}\ \overline{v})'\ (\overline{x_{k+1}}\ \cdots\ \overline{x_n})' \\ &= (\overline{x_1}\ \cdots\ \overline{x_{k-1}})'\ \overline{v}\ (\overline{x_{k+1}}\ \cdots\ \overline{x_n})' \\ &= (\overline{x_1}\ \cdots\ \overline{x_{k-1}})'\ \overline{x_k}\ (\overline{x_{k+1}}\ \cdots\ \overline{x_n})' \\ &= w' \end{aligned}$$

in $\overline{B_\alpha}^+$, so that certainly $p' = w'$ in $\mathrm{IG}(B_\alpha)$.

A similar argument holds if $\alpha_k = \alpha$ and $\alpha = \nu < \tau$. □

Lemma 8.5.5. *Let B be a quasi-zero band and let $x_1, \cdots, x_n, y_1, \cdots, y_m \in B_\alpha$ for some $\alpha \in Y$. Then with $w = \overline{x_1}\ \cdots\ \overline{x_n}$ and $p = \overline{y_1}\ \cdots\ \overline{y_m}$ we have $w = p$ in $\mathrm{IG}(B_\alpha)$ if and only if $w = p$ in $\mathrm{IG}(B)$.*

Proof. The sufficiency is clear, as any basic pair in B_α is basic in B. Conversely, if $w = p$ in $\mathrm{IG}(B)$, there exists a finite sequence

$$w = w_0 \sim w_1 \sim w_2 \cdots \sim w_{s-1} \sim w_s = p.$$

Let w_0', $w_1', w_2', \cdots, w_{s-1}', w_s'$ be the words obtained by deleting letters x within the word such that $x \in B_\beta$ with $\beta \neq \alpha$. From Lemma 8.5.4, we have that

$$w_0' = w_1' = w_2' = \cdots = w_{s-1}' = w_s'$$

in $\mathrm{IG}(B_\alpha)$. Note that $w_0' = w_0 = w \in \overline{B_\alpha}^+$ and $w_s' = w_s = p \in \overline{B_\alpha}^+$, so that $w = p$ in $\mathrm{IG}(B_\alpha)$. $\square$

Lemma 8.5.6. *Let B be a quasi-zero band. Then the word problem of $\mathrm{IG}(B)$ is solvable.*

Proof. The result is immediate from Lemmas 8.3.1, 8.5.3 and 8.5.5. $\square$

8.6 Free idempotent generated semigroups with Condition (P)

From the above discussion, we know that for any band B, the semigroup $\mathrm{IG}(B)$ is always weakly abundant with the congruence condition, but not necessarily abundant. The aim of this section is devoted to finding some special kinds of bands B for which $\mathrm{IG}(B)$ is abundant.

Definition 8.6.1. *We say that the semigroup $\mathrm{IG}(B)$ satisfies Condition (P) if for any two almost normal forms $\overline{u_1} \cdots \overline{u_n} = \overline{v_1} \cdots \overline{v_m} \in \mathrm{IG}(B)$ with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$ and $l_1, \cdots, l_r = m$, respectively, and ordered Y-components $\alpha_1, \cdots, \alpha_r$, the following statements (with $i_0 = l_0 = 0$) hold:*

(i) $u_{i_s} \mathcal{L} v_{l_s}$ implies $\overline{u_1} \cdots \overline{u_{i_s}} = \overline{v_1} \cdots \overline{v_{l_s}}$, for all $s \in [1, r]$.

(ii) $u_{i_t+1} \mathcal{R} v_{l_t+1}$ implies $\overline{u_{i_t+1}} \cdots \overline{u_n} = \overline{v_{l_t+1}} \cdots \overline{v_m}$, for all $t \in [0, r-1]$.

Proposition 8.6.2. *Let B be a band for which $\mathrm{IG}(B)$ satisfies Condition (P). In addition, suppose that B is normal (so that $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$) or quasi-zero. Then $\mathrm{IG}(B)$ is an abundant semigroup.*

Proof. Let $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. By Theorem 8.4.13, $\overline{x_1} \cdots \overline{x_{i_r}} \ \widetilde{\mathcal{R}} \ \overline{x_1}$. We aim to show that $\overline{x_1} \cdots \overline{x_{i_r}} \ \mathcal{R}^* \ \overline{x_1}$. From Lemma 2.2.6, we only need to show that for any two almost normal forms $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ with Y-length m, left to right significant indices $l_1, \cdots, l_s = m$, and ordered Y-components $\beta_1, \cdots, \beta_s$, and $\overline{z_1} \cdots \overline{z_h} \in \mathrm{IG}(B)$ with Y-length t, left to right significant indices $j_1, \cdots, j_t = h$, and ordered Y-components $\gamma_1, \cdots, \gamma_t$, we have that

$$\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1} \cdots \overline{x_{i_r}} = \overline{y_1} \cdots \overline{y_{l_s}} \, \overline{x_1} \cdots \overline{x_{i_r}}$$

implies that $\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1} = \overline{y_1} \cdots \overline{y_{l_s}} \, \overline{x_1}$.

Suppose now that

$$\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1} \cdots \overline{x_{i_r}} = \overline{y_1} \cdots \overline{y_{l_s}} \, \overline{x_1} \cdots \overline{x_{i_r}}.$$

We consider the following cases:

(i) If γ_t, α_1 and β_s, α_1 are incomparable, then both sides of the above equality are in almost normal form, so that by Condition (P)

$$\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1} \cdots \overline{x_{i_1}} = \overline{y_1} \cdots \overline{y_{l_s}} \, \overline{x_1} \cdots \overline{x_{i_1}}.$$

Since $\overline{x_1} \cdots \overline{x_{i_1}} \ \mathcal{R} \ \overline{x_{i_1}}$ by Lemma 8.3.2, we have $\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1} = \overline{y_1} \cdots \overline{y_{l_s}} \, \overline{x_1}$.

(ii) If $\gamma_t \leq \alpha_1$ and β_s, α_1 are incomparable, then by Lemma 8.4.12, we know that $\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1} \cdots \overline{x_{i_r}}$ has an almost normal form

$$\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} \, \overline{x_{i_v+1}} \cdots \overline{x_{i_r}},$$

for some $v \in [1, r]$, where $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable. Hence we have

$$\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} \, \overline{x_{i_v+1}} \cdots \overline{x_{i_r}} = \overline{y_1} \cdots \overline{y_{l_s}} \, \overline{x_1} \cdots \overline{x_{i_r}}.$$

Note that both sides of the above equality are in almost normal form. It follows from Corollary 8.4.2 that

$$(\overline{z_1} \cdots \overline{z_{j_t}} \, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} \, \overline{x_{i_v+1}} \cdots \overline{x_{i_r}})\overline{\theta} = (\overline{y_1} \cdots \overline{y_{l_s}} \, \overline{x_1} \cdots \overline{x_{i_r}})\overline{\theta}$$

and so

$$\overline{\gamma_1} \cdots \overline{\gamma_t}\, \overline{\alpha_{v+1}} \cdots \overline{\alpha_r} = \overline{\beta_1} \cdots \overline{\beta_s}\, \overline{\alpha_1} \cdots \overline{\alpha_r}.$$

As $v \geq 1$, $\gamma_t = \alpha_v$. To avoid contradiction, $v = 1$, so $x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1} = x_{i_1}$, and hence by Condition (P)

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1}} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_1}}.$$

and so

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_1}} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_1}}$$

so that $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} = \overline{y_1} \cdots \overline{y_{l_1}} \cdots \overline{y_{l_s}}\, \overline{x_1}$.

(iii) If $\gamma_t \leq \alpha_1$ and $\beta_s \leq \alpha_1$, then by Lemma 8.4.12 we have the following two almost normal forms for $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_r}}$ and $\overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_r}}$, namely,

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\, \overline{x_{i_v+1}} \cdots \overline{x_{i_r}}$$

where $v \in [1, r]$ such that $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable, and

$$\overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1 y_{l_s} x_1} \cdots \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}\, \overline{x_{i_u+1}} \cdots \overline{x_{i_r}}$$

where $u \in [1, r]$ with $\beta_s \leq \alpha_1, \cdots, \alpha_u$ and $u = r$ or β_s, α_{u+1} are incomparable. Hence by Corollary 8.4.2,

$$\overline{\gamma_1} \cdots \overline{\gamma_t}\, \overline{\alpha_{v+1}} \cdots \overline{\alpha_r} = \overline{\beta_1} \cdots \overline{\beta_s}\, \overline{\alpha_{u+1}} \cdots \overline{\alpha_r}$$

If $v > u$, then $\gamma_t = \alpha_v$, to avoid contradiction $v = 1$, so $u = 0$, contradiction. Similarly, $v < u$ is impossible. If $v = u$, then $t = s$ and $\beta_s = \gamma_t$. If B is a normal band satisfying Condition (P),

$$x_1 z_{j_t} x_1 = x_1 \phi_{\alpha_1, \gamma_t} = x_1 \phi_{\alpha_1, \beta_s} = x_1 y_{l_s} x_1$$

$$\vdots$$

$$x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v} = x_{i_v} \phi_{\alpha_v, \gamma_t} = x_{i_u} \phi_{\alpha_u, \beta_s} = x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}$$

so that by Condition (P), we have

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}$$

$$= \overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_1 y_{l_s} x_1} \ \cdots \ \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}.$$

On the other hand, we have

$$\overline{x_1 z_{j_t} x_1} \ \cdots \ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} = \overline{x_1 y_{l_s} x_1} \ \cdots \ \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}}$$

which is $\mathcal{R}$-related to $x_1 z_{j_t} x_1$, and so

$$\overline{z_1} \ \cdots \ \overline{z_{j_t}} \ \overline{x_1 z_{j_t} x_1} = \overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_1 y_{l_s} x_1},$$

and hence

$$\overline{z_1} \ \cdots \ \overline{z_{j_t}} \ \overline{x_1} = \overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_1}.$$

Suppose now that B is a quasi-zero band. First suppose that $v = u = 1$. Then by Lemma 8.5.2 we have

$$\overline{z_1} \ \cdots \ \overline{z_{j_t}} \ \overline{x_1 z_{j_t} x_1} \ \cdots \ \overline{x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1}}$$

$$= \overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_1 y_{l_s} x_1} \ \cdots \ \overline{x_{i_1} \cdots x_1 y_{l_s} x_1 \cdots x_{i_1}}$$

and so

$$\overline{z_1} \ \cdots \ \overline{z_{j_t}} \ \overline{x_1} \ \cdots \ \overline{x_{i_1}} = \overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_1} \ \cdots \ \overline{x_{i_1}}$$

so that

$$\overline{z_1} \ \cdots \ \overline{z_{j_t}} \ \overline{x_1} = \overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_1}.$$

Suppose now that $v = u > 1$. By assumption $\beta_s = \gamma_t \leq \alpha_1, \cdots, \alpha_v$. We claim that there exists no $j \in [1, v]$ such that $\gamma_t = \alpha_j$; otherwise we will have α_j, α_{j+1} are comparable if $v > j$ or α_v, α_{v-1} are comparable if $v = j$. Hence

$$\gamma_t = \beta_s < \alpha_1, \cdots, \alpha_v.$$

Since B is a quasi-zero band, we have

$$\overline{z_1} \ \cdots \ \overline{z_{j_t}} \ \overline{x_1 z_{j_t} x_1} \ \cdots \ \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}} \ \overline{x_{i_v+1}} \ \cdots \ \overline{x_{i_r}} = \overline{z_1} \ \cdots \ \overline{z_{j_t}} \ \overline{x_{i_v+1}} \ \cdots \ \overline{x_{i_r}}$$

and

$$\overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_1 y_{l_s} x_1} \ \cdots \ \overline{x_{i_u} \cdots x_1 y_{l_s} x_1 \cdots x_{i_u}} \ \overline{x_{i_u+1}} \ \cdots \ \overline{x_{i_r}} = \overline{y_1} \ \cdots \ \overline{y_{l_s}} \ \overline{x_{i_v+1}} \ \cdots \ \overline{x_{i_r}}$$

so that it follows from Lemma 8.5.2 that

$$\overline{z_1} \cdots \overline{z_{j_t}} = \overline{y_1} \cdots \overline{y_{l_s}}$$

and so certainly

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1}.$$

(iv) If $\gamma_t \leq \alpha_1$ and $\beta_s \geq \alpha_1$, then by Lemma 8.4.12 we have the following two almost normal forms for $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_r}}$ and $\overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_r}}$, namely,

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_v} \cdots x_1 z_{j_t} x_1 \cdots x_{i_v}}\, \overline{x_{i_v+1}} \cdots \overline{x_{i_r}}$$

for some $v \in [1, r]$ with $\gamma_t \leq \alpha_1, \cdots, \alpha_v$ and $v = r$ or γ_t, α_{v+1} are incomparable, and

$$\overline{y_1} \cdots \overline{y_{l_u}}\, \overline{y_{l_u+1} \cdots y_{l_s} x_1 y_{l_s} \cdots y_{l_u+1}} \cdots \overline{y_{l_s} x_1 y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_r}}$$

for some $u \in [0, s-1]$ with $\beta_{u+1}, \cdots, \beta_s \geq \alpha_1$ and β_u, α_1 are incomparable or $u = 0$. It follows from Corollary 8.4.2 that

$$\overline{\gamma_1} \cdots \overline{\gamma_t}\, \overline{\alpha_{v+1}} \cdots \overline{\alpha_r} = \overline{\beta_1} \cdots \overline{\beta_u}\, \overline{\alpha_1} \cdots \overline{\alpha_r}.$$

Note that both sides of the above are normal forms of IG(Y). As $v \geq 1$, $\gamma_t = \alpha_v$, so that to avoid contradiction we have $v = 1$ and so $x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1} = x_{i_1}$, and hence by Condition (P)

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1 z_{j_t} x_1} \cdots \overline{x_{i_1} \cdots x_1 z_{j_t} x_1 \cdots x_{i_1}}$$

$$= \overline{y_1} \cdots \overline{y_{l_u}}\, \overline{y_{l_u+1} \cdots y_{l_s} x_1 y_{l_s} \cdots y_{l_u+1}} \cdots \overline{y_{l_s} x_1 y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_1}}$$

and so

$$\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_1}} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_1}},$$

which implies $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} = \overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1}$.

(v) If $\gamma_t \geq \alpha_1$ and $\beta_s \geq \alpha_1$, then by Lemma 8.4.12 we have the following two almost normal forms for $\overline{z_1} \cdots \overline{z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_r}}$ and $\overline{y_1} \cdots \overline{y_{l_s}}\, \overline{x_1} \cdots \overline{x_{i_r}}$, namely,

$$\overline{z_1} \cdots \overline{z_{j_v}}\, \overline{z_{j_v+1} \cdots z_{j_t} x_1 z_{j_t} \cdots z_{j_v+1}} \cdots \overline{z_{j_t} x_1 z_{j_t}}\, \overline{x_1} \cdots \overline{x_{i_1}} \cdots \overline{x_{i_r}}$$

for some $v \in [0, t-1]$ such that $\gamma_{v+1}, \cdots, \gamma_t \geq \alpha_1$ and γ_v, α_1 are incomparable or $v = 0$, and

$$\overline{y_1} \cdots \overline{y_{l_u}}\ \overline{y_{l_u+1} \cdots y_{l_s} x_1 y_{l_s} \cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s} x_1 y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $u \in [0, s-1]$ such that $\beta_{u+1}, \cdots, \beta_s \geq \alpha_1$ and β_u, α_1 are incomparable or $u = 0$. Hence by Condition (P),

$$\overline{z_1} \cdots \overline{z_{j_v}}\ \overline{z_{j_v+1} \cdots z_{j_t} x_1 z_{j_t} \cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t} x_1 z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

$$= \overline{y_1} \cdots \overline{y_{l_u}}\ \overline{y_{l_u+1} \cdots y_{l_s} x_1 y_{l_s} \cdots y_{l_u+1}}\ \cdots\ \overline{y_{l_s} x_1 y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}},$$

so that

$$\overline{z_1} \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1} \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

and hence $\overline{z_1} \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1} \cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

(vi) If $\gamma_t \geq \alpha_1$ and β_s, α_1 are incomparable, then by Lemma 8.4.12

$$\overline{z_1} \cdots \overline{z_{j_v}}\ \overline{z_{j_v+1} \cdots z_{j_t} x_1 z_{j_t} \cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t} x_1 z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

$$= \overline{y_1} \cdots \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}\ \cdots\ \overline{x_{i_r}}$$

for some $v \in [0, t-1]$ with $\gamma_{v+1}, \cdots, \gamma_t \geq \alpha_1$ and γ_v, α_1 are incomparable or $v = 0$. Note that both sides of the above equality are in almost normal form. Again by Condition (P)

$$\overline{z_1} \cdots \overline{z_{j_v}}\ \overline{z_{j_v+1} \cdots z_{j_t} x_1 z_{j_t} \cdots z_{j_v+1}}\ \cdots\ \overline{z_{j_t} x_1 z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1} \cdots \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

so that

$$\overline{z_1} \cdots\ \overline{z_{j_t}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}} = \overline{y_1} \cdots\ \overline{y_{l_s}}\ \overline{x_1}\ \cdots\ \overline{x_{i_1}}$$

and hence $\overline{z_1} \cdots\ \overline{z_{j_t}}\ \overline{x_1} = \overline{y_1} \cdots\ \overline{y_{l_s}}\ \overline{x_1}$.

From the above discussion, we can deduce that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{R}^*\ \overline{x_1}$, and similarly we can show that $\overline{x_1}\ \cdots\ \overline{x_{i_r}}\ \mathcal{L}^*\ \overline{x_{i_r}}$, so that $\mathrm{IG}(B)$ is an abundant semigroup. $\square$

We now aim to find examples of normal bands B for which $\mathrm{IG}(B)$ satisfies Condition (P), so that by Proposition 8.6.2, $\mathrm{IG}(B)$ is abundant.

A band $B = \bigcup_{\alpha \in Y} B_\alpha$ is called a *simple band* if it is a semilattice Y of rectangular bands B_α, $\alpha \in Y$, where B_α is either a left zero band or a right zero band.

Lemma 8.6.3. *Let $B = \bigcup_{\alpha \in Y} B_\alpha$ be a simple band and let $e \in B_\alpha$ and $f \in B_\beta$. Then (e, f) is a basic pair in B if and only (α, β) is a basic pair in Y, i.e. if and only if α and β are comparable in Y.*

Proof. Since the necessity is clear, we are left with showing the sufficiency. Without loss of generality, suppose that $\alpha \leq \beta$. Then $ef, fe \in B_\alpha$. As B is a simple band, we have B_α is either a left zero band or a right zero band. If B_α is a left zero band, then $e(ef) = e$, i.e. $ef = e$, so (e, f) is a basic pair. If B_α is a right zero band, then $(fe)e = e$, i.e. $fe = e$, which again implies that (e, f) is a basic pair. $\square$

It follows from Lemma 8.6.3 that for a simple band B, every element $\overline{x_1} \cdots \overline{x_n}$ of $\mathrm{IG}(B)$ has a special normal form (of course, which may not unique), say, $\overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ with y_i and y_{i+1} incomparable, for all $i \in [1, m-1]$.

Lemma 8.6.4. *Let B be a simple band. Then* $\mathrm{IG}(B)$ *satisfies Condition (P).*

Proof. Let $\overline{x_1} \cdots \overline{x_n} = \overline{y_1} \cdots \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, $j_1, \cdots, j_r = m$, respectively, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. It then follows from Corollary 8.4.5 that for all $s \in [1, r]$,

$$\overline{y_1} \cdots \overline{y_{j_s}} = \overline{x_1} \cdots \overline{x_{i_s}}\, \overline{e_1} \cdots \overline{e_m} \text{ (in which we remove the empty word)}$$

where for all $k \in [1, m]$, $e_k \in B_{\delta_k}$ with $\delta_k \geq \alpha_{i_s}$. By Lemma 8.6.3, we have

$$\overline{x_{i_s}}\, \overline{e_1} \cdots \overline{e_m} = \overline{x_{i_s} e_1 \cdots e_m},$$

so that if we assume $x_{i_s} \mathcal{L}\, y_{j_s}$, then

$$\begin{aligned} \overline{y_1} \cdots \overline{y_{j_s}} &= \overline{y_1} \cdots \overline{y_{j_s}}\, \overline{x_{i_s}} \\ &= \overline{x_1} \cdots \overline{x_{i_s} e_1 \cdots e_m}\, \overline{x_{i_s}} \\ &= \overline{x_1} \cdots \overline{x_{i_s} e_1 \cdots e_m x_{i_s}} \\ &= \overline{x_1} \cdots \overline{x_{i_s}}. \end{aligned}$$

Together with the dual, we have shown that $\mathrm{IG}(B)$ satisfies Condition (P). $\square$

Corollary 8.6.5. *Let B be a simple normal band. Then* $\mathrm{IG}(B)$ *is abundant.*

Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a normal band. We say that B is a *trivial normal band* if for every $\alpha \in Y$, there exists a $a_\alpha \in B_\alpha$ such that for all $\beta > \alpha$, $x\phi_{\beta,\alpha} = a_\alpha$.

Lemma 8.6.6. *Let* $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ *be a trivial normal band. Then* $\mathrm{IG}(B)$ *satisfies Condition* (P).

Proof. First note that since B is a trivial normal band, there exists $a_\alpha \in B_\alpha$ be such that for any $\beta > \alpha$ and $u \in B_\beta$, $u\phi_{\beta,\alpha} = a_\alpha$.

Let $\overline{x_1} \ \cdots \ \overline{x_n} = \overline{y_1} \ \cdots \ \overline{y_m} \in \mathrm{IG}(B)$ be in almost normal form with Y-length r, left to right significant indices $i_1, \cdots, i_r = n$, $j_1, \cdots, j_r = m$, respectively, and ordered Y-components $\alpha_1, \cdots, \alpha_r$. It follows from Corollary 8.4.5 that

$$\overline{y_1} \ \cdots \ \overline{y_{j_l}} = \overline{x_1} \ \cdots \ \overline{x_{i_l}} \ \overline{u_1} \ \cdots \ \overline{u_s} \text{ (in which we remove the empty word)}$$

such that for all $k \in [1, s]$ we have $u_k \in B_{\delta_k}$ with $\delta_k > \alpha_{i_l}$, so that $u_k\phi_{\delta_k,\alpha_{i_l}} = a_{\alpha_{i_l}}$; or $u_k \in B_{\alpha_{i_l}}$ with $v_k u_k = u_k$ for some $v_k \in B_{\eta_k}$ such that $\eta_k > \alpha_{i_l}$, and in this case we have $a_{\alpha_{i_l}} u_k = u_k$, so that $a_{\alpha_{i_l}} \ \mathcal{R} \ u_k$. Thus the idempotents $u_1\phi_{\delta_1,\alpha_{i_l}}, \cdots, u_s\phi_{\delta_s,\alpha_{i_l}}$ are all $\mathcal{R}$-related, and so

$$\overline{x_{i_l}} \ \overline{u_1} \ \cdots \ \overline{u_s} = \overline{x_{i_l}} \ \overline{u_1\phi_{\delta_1,\alpha_{i_l}}} \ \cdots \ \overline{u_s\phi_{\delta_s,\alpha_{i_l}}} = \overline{x_{i_l}} \ \overline{u_1\phi_{\delta_1,\alpha_{i_l}} \cdots u_s\phi_{\delta_s,\alpha_{i_l}}}.$$

On the other hand, we have $y_{j_l} = u'_s \cdots u'_1 x_{i_l} u_1 \cdots u_s$, where $u'_k \in B_{\sigma_k}$ with $\sigma_k \geq \alpha_{i_l}$. Hence if we assume that $x_{i_l} \ \mathcal{L} \ y_{j_l}$, then $x_{i_l} = x_{i_l} u_1 \cdots u_s$, and so $x_{i_l} = x_{i_l}(u_1\phi_{\delta_1,\alpha_{i_l}}) \cdots (u_s\phi_{\delta_s,\alpha_{i_l}})$, so that

$$\overline{x_{i_l}} \ \overline{u_1\phi_{\delta_1,\alpha_{i_l}} \cdots u_s\phi_{\delta_s,\alpha_{i_l}}} = \overline{x_{i_l}(u_1\phi_{\delta_1,\alpha_{i_l}}) \cdots (u_s\phi_{\delta_s,\alpha_{i_l}})} = \overline{x_{i_l}}.$$

Hence $\overline{y_1} \ \cdots \ \overline{y_{j_l}} = \overline{x_1} \ \cdots \ \overline{x_{i_l}}$ as required. $\square$

Corollary 8.6.7. *Let* $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ *be a trivial normal band. Then* $\mathrm{IG}(B)$ *is an abundant semigroup.*

8.7 A normal band B for which $\mathrm{IG}(B)$ is not abundant

From Section 8.6, we know that the free idempotent idempotent generated semigroup $\mathrm{IG}(B)$ over a normal band B satisfying Condition (P) is an abundant semigroup. Therefore, one would like to ask whether for any normal band B, $\mathrm{IG}(B)$ is abundant. In this section we will construct a 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant.

Throughout this section, B denotes a normal band $\mathcal{B}(Y; B_\alpha, y\phi_{\alpha,\beta})$.

Lemma 8.7.1. *Let B be a normal band, and let $x \in B_\beta, y \in B_\gamma$ with $\beta, \gamma \geq \alpha$. Then (x, y) is a basic pair implies $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair and*

$$(x\phi_{\beta,\alpha})(y\phi_{\gamma,\alpha}) = (xy)\phi_{\delta,\alpha},$$

where δ is minimum of β and γ.

Proof. Let (x, y) be a basic pair with $x \in B_\beta, y \in B_\gamma$. Then β, γ are comparable. If $\beta \geq \gamma$, then we either have $xy = y$ or $yx = y$. If $xy = y$, then $(x\phi_{\beta,\gamma})y = y$, so

$$y\phi_{\gamma,\alpha} = ((x\phi_{\beta,\gamma})y)\phi_{\gamma,\alpha} = (x\phi_{\beta,\alpha})(y\phi_{\gamma,\alpha}),$$

so $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair. If $yx = y$, then $y(x\phi_{\beta,\gamma})$, so

$$y\phi_{\gamma,\alpha} = (y(x\phi_{\beta,\gamma}))\phi_{\gamma,\alpha} = (y\phi_{\gamma,\alpha})(x\phi_{\beta,\alpha}),$$

so that $(x\phi_{\beta,\alpha}, y\phi_{\gamma,\alpha})$ is a basic pair.

A similar argument holds if $\gamma \geq \beta$. The final part of the lemma is clear. $\square$

Lemma 8.7.2. *Let $\overline{u_1} \cdots \overline{u_n} \in \mathrm{IG}(B)$ with $u_i \in B_{\alpha_i}$ and $\alpha_i \geq \alpha$ for all $i \in [1, n]$. Suppose that $\overline{v_1} \cdots \overline{v_m} \in \mathrm{IG}(B)$ with $v_i \in B_{\beta_i}$ for all $i \in [1, m]$ is an element obtained by single step on $\overline{u_1} \cdots \overline{u_n}$ (note that $\beta_i \geq \alpha$, for all $i \in [1, m]$). Then in $\mathrm{IG}(B_\alpha)$ we have*

$$\overline{u_1\phi_{\alpha_1,\alpha}} \cdots \overline{u_n\phi_{\alpha_n,\alpha}} = \overline{v_1\phi_{\beta_1,\alpha}} \cdots \overline{v_m\phi_{\beta_m,\alpha}}.$$

Proof. Suppose that $u_i = xy$ is a basic product with $x \in B_\delta, y \in B_\eta$, for some $i \in [1, n]$. Note that the minimum of δ and η is α_i. Then

$$\overline{u_1} \cdots \overline{u_n} \sim \overline{u_1} \cdots \overline{u_{i-1}}\ \overline{x}\ \overline{y}\ \overline{u_{i+1}} \cdots \overline{u_n}.$$

If follows from Lemma 8.7.1 that in $\mathrm{IG}(B_\alpha)$

$$\begin{aligned}
\overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}} &= \overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{u_i\phi_{\alpha_i,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}} \\
&= \overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{x\phi_{\delta,\alpha}y\phi_{\eta,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}} \\
&= \overline{u_1\phi_{\alpha_1,\alpha}}\ \cdots\ \overline{u_{i-1}\phi_{\alpha_{i-1},\alpha}}\ \overline{x\phi_{\delta,\alpha}}\ \overline{y\phi_{\eta,\alpha}}\ \overline{u_{i+1}\phi_{\alpha_{i+1},\alpha}}\ \cdots\ \overline{u_n\phi_{\alpha_n,\alpha}}
\end{aligned}$$

as required. □

Corollary 8.7.3. *Let $x_1, \cdots, x_n, y_1, \cdots, y_m \in B_\alpha$. Then $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$ in $\mathrm{IG}(B_\alpha)$ if and only if the equality holds in $\mathrm{IG}(B)$.*

Proof. The necessity is obvious, as any basic pair in B_α must also be basic in B. Suppose now that we have

$$\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$$

in $\mathrm{IG}(B)$. Then there exists a sequence of transitions

$$\overline{x_1}\ \cdots\ \overline{x_n} \sim \overline{u_1}\ \cdots\ \overline{u_s} \sim \overline{v_1}\ \cdots\ \overline{v_t} \sim\ \cdots\ \sim \overline{w_1}\ \cdots\ \overline{w_l} \sim \overline{y_1}\ \cdots\ \overline{y_m},$$

using basic pairs in B. Note that all idempotents involved in the above sequence lie in B_β for some $\beta \geq \alpha$, so that successive applications of Lemma 8.7.2 give $\overline{x_1}\ \cdots\ \overline{x_n} = \overline{y_1}\ \cdots\ \overline{y_m}$ in $\mathrm{IG}(B_\alpha)$. □

We remark here that for an arbitrary band B, Corollary 8.7.3 need not be true.

Example 8.7.4. Let $B = B_\alpha \cup B_\beta$ be a band with the semilattice decomposition structure and the multiplication table defined by

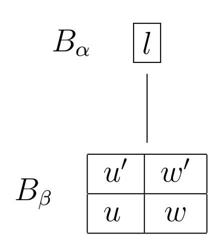

Figure 8.9: the semilattice decomposition structure of Example 8.7.4

	l	u	w	u'	w'
l	l	u'	w'	u'	w'
u	u	u	w	u	w
w	w	u	w	u	w
u'	u'	u'	w'	u'	w'
w'	w'	u'	w'	u'	w'

Figure 8.10: the multiplication table of Example 8.7.4

It is easy to check that B forms a band. By the uniqueness of normal forms in $\mathrm{IG}(B_\beta)$, we have $\overline{u'}\ \overline{w} \neq \overline{w'}$ in $\mathrm{IG}(B_\beta)$. However in $\mathrm{IG}(B)$ we have

$$\begin{aligned}
\overline{u'}\ \overline{w} &= \overline{u'l}\ \overline{w} \\
&= \overline{u'}\ \overline{l}\ \overline{w} \quad \text{(as } (u', l) \text{ is a basic pair)} \\
&= \overline{u'}\ \overline{lw} \quad \text{(as } (l, w) \text{ is a basic pair)} \\
&= \overline{u'}\ \overline{w'} \\
&= \overline{w'}
\end{aligned}$$

With the above preparations, we now construct a 10-element normal band B for which $\mathrm{IG}(B)$ is not abundant.

Example 8.7.5. Let $B = \mathcal{B}(Y; B_\alpha, \phi_{\alpha,\beta})$ be a strong semilattice $Y = \{\alpha, \beta, \gamma, \delta\}$ of rectangular bands (see the figure below), where $\phi_{\alpha,\beta} : B_\alpha \longrightarrow B_\beta$ is defined by

$$a\phi_{\alpha,\beta} = e, b\phi_{\alpha,\beta} = f, c\phi_{\alpha,\beta} = g, d\phi_{\alpha,\beta} = h$$

the remaining morphisms being defined in the obvious unique manner.

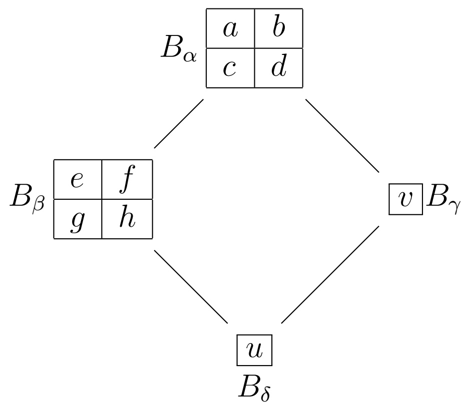

Figure 8.11: the semilattice decomposition structure of Example 8.7.5

Now we consider an element $\overline{e}\ \overline{v} \in \mathrm{IG}(B)$, then we have

$$\begin{aligned}
\overline{e}\ \overline{v} &= \overline{e}\ \overline{dv} \\
&= \overline{e}\ \overline{d}\ \overline{v} \quad \text{(as } (d,v) \text{ is a basic pair)} \\
&= \overline{e}\ \overline{h}\ \overline{v} \quad \text{(as } \overline{e}\ \overline{d} = \overline{e}\ \overline{d\phi_{\alpha,\beta}} = \overline{e}\ \overline{h} \text{ by Corollary 8.4.8)} \\
&= \overline{e}\ \overline{h}\ \overline{av} \\
&= \overline{e}\ \overline{h}\ \overline{a}\ \overline{v} \quad \text{(as } (a,v) \text{ is a basic pair)} \\
&= \overline{e}\ \overline{h}\ \overline{e}\ \overline{v} \quad \text{(as } \overline{h}\ \overline{a} = \overline{h}\ \overline{a\phi_{\alpha,\beta}} = \overline{h}\ \overline{e} \text{ by Corollary 8.4.8)}
\end{aligned}$$

However, $\overline{e}\ \overline{h}\ \overline{e} \neq \overline{e}$ in $\mathrm{IG}(B_\beta)$ by the uniqueness of normal forms, so by Corollary 8.7.3, we have $\overline{e}\ \overline{h}\ \overline{e} \neq e$ in $\mathrm{IG}(B)$, which implies $\overline{e}\ \overline{v}$ is not $\mathcal{R}^*$-related to $\overline{e}$. On the other hand, we have known from Theorem 8.4.13 that $\overline{e}\ \overline{v}\ \widetilde{\mathcal{R}}\ \overline{e}$, so that by Lemma 2.2.5 that $\overline{e}\ \overline{v}$ is not $\mathcal{R}^*$-related any idempotent of B, so that $\mathrm{IG}(B)$ is not an abundant semigroup.

Chapter 9

A plan for further work

Let me finish the writing of my PhD thesis by giving a brief proposal for further work.

First of all, as we have already seen, for the biordered sets E of idempotents the full transformation monoid $\mathcal{T}_n$ on n elements and the endomorphism monoid $\operatorname{End} F_n(G)$ of a rank n free (left) G-act $F_n(G)$, the maximal subgroups of $\mathrm{IG}(E)$ containing a rank r idempotent $\varepsilon \in E$, where $1 \leq r \leq n-2$, are isomorphic to that of the original semigroup, which are known to be $\mathcal{S}_r$ and $G \wr \mathcal{S}_r$, respectively. However, the result for the matrix monoid $M_n(D)$ of all $n \times n$ matrices over a division ring D has only been obtained for r restricted to $r < n/3$. Hence, my next focus is to investigate the higher rank cases for $M_n(D)$, and for some very special reason, I will start with the case in which $r = 2$ and $n = 6$. I have obtained the result for the case $r = 2$ and $n = 4$ by hand calculation, in which I can show that the maximal subgroup here is isomorphic to the 2 dimensional general linear group $\mathrm{GL}_2(D)$.

Secondly, we would like to continue the study of maximal subgroups of $\mathrm{IG}(E)$, where E is the biordered set of idempotents of the endomorphism monoid $\operatorname{End} \mathbf{A}$ of an independence algebra $\mathbf{A}$ of finite rank n. In Chapter 7 we deal with a very special case, namely, independence algebras with no constants and $r = 1$; we therefore are far from getting a whole picture for the maximal subgroups in terms of a general independence algebra. The diverse methods needed in the biordered sets of $\mathcal{T}_n$, $M_n(D)$ and $\operatorname{End} F_n(G)$ suggest that it would be very hard to find a unified approach to $\operatorname{End} \mathbf{A}$. However, given the main strategies we have applied in the work of free left G-acts, it is hopeful to work out a general proof for independence algebras with no constants, with rank $r \leq n/3$, by some similar discussions to that of $\operatorname{End} F_n(G)$, and in the mean time, we would get a set of generators for all elements of $\operatorname{Aut}\mathbf{B}$ in the sandwich matrix, where $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ with rank r. The problem for higher rank is, first finding a presentation for $\operatorname{Aut}\mathbf{B}$ and then showing that it gives us the maximal subgroup of $\mathrm{IG}(E)$ with a rank r idempotent.

After that, we intend to change our main focus to the general structure of a free idempotent generated semigroup $\mathrm{IG}(E)$ over a biordered E. In this thesis, we considered a very special kind of biordered set, namely, bands, and it is proven that for any band B, $\mathrm{IG}(B)$ is always a weakly abundant semigroup with the congruence condition. Does this result hold for an arbitrary biordered set? Or perhaps for an arbitrary regular biordered set?

Another question in this direction is the word problem of $\mathrm{IG}(B)$, where B is a band. Does $\mathrm{IG}(B)$ always have a solvable word problem, for a finite band B, and if not, is the word solvable in $\mathrm{IG}(B)$ equivalent to that of the maximal subgroups? A recent but not yet published work of Dolinka, Gray, and Ruškuc gives a negative answer to both of the above questions! However, the whole story has not been finished... What would happen for a normal band? Recently, we have worked out a special kind of normal bands on which $\mathrm{IG}(B)$ has a solvable word problem. So, it would be interesting to investigate the word problem of $\mathrm{IG}(B)$ over a normal band B.

An interesting and valuable question provided by Professor John Fountain is: Let E be the biordered set of a regular ring R. Then what can we say on $\mathrm{IG}(E)$ here? It would be a completely new direction in the study of biordered sets and free idempotent generated semigroups.

The final thing I would like to say is about the freeness of maximal subgroups of $\mathrm{IG}(E)$. We have already known that for the biordered set E of idempotents of a completely 0-simple semigroup S, the maximal subgroups of $\mathrm{IG}(E)$ here are free groups. Would the same be true for 0-simple semigroups?

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