Stability conditions for Seiberg-Witten quivers

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Summary

This thesis describes a connected component of the space of numerical stability conditions of certain CY3 triangulated categories using the period map of a meromorphic differential on a family of elliptic curves. The motivation for this result comes from studying meromorphic quadratic differentials on Riemann surfaces. On the one hand, a meromorphic quadratic differential on a Riemann surface defines a double cover, its spectral curve, together with a meromorphic abelian differential on it known as the Seiberg-Witten differential. On the other hand certain strata of meromorphic quadratic differentials determine a CY3 triangulated category such that the periods of the Seiberg-Witten differential define the central charge of a stability condition on the category.

The simplest examples of this construction involve two-dimensional strata of meromorphic quadratic differentials on the Riemann sphere in which case the spectral curves are elliptic curves. There are 10 such strata in bijective correspondence with the Painlevé equations whose families of spectral elliptic curves include the original examples of Seiberg-Witten curves and certain degenerations thereof. In these cases the periods of the Seiberg-Witten differential satisfy a hypergeometric differential equation, so that its period map is described by the Schwarz triangle theorem. In all but one of these examples this period map can be lifted to a map to a canonical connected component of the space of numerical stability conditions of the associated category.

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Chapter 1

Introduction

1.1 Context

This thesis explores a small corner of the landscape of mathematics which has arisen from string theory. Whilst the main body of the thesis can be read without any background in physics, we nevertheless give an overview of the physical motivations in order to put the results into their appropriate context. The reader who wishes to start with a mathematical description of the content of the thesis is advised to skip to Section 1.2.

String theory is an attempt to unify the theories of quantum mechanics and gravity. It postulates that classical particles which are point-like objects in the observable space-time are in fact higher-dimensional objects called branes in some extended version of space-time. One assumes that the extra dimensions of space-time are given by a compact manifold whose volume is so small as to not be directly observable. However the way in which the branes interact depends on the geometry of the extra dimensions.

String theory is far from realising the goal of describing a grand unified theory of the observable universe. Nevertheless theoretical physicists have studied more idealistic string theories which has given rise to a lot of interesting mathematics. The relevant context for the mathematics in this thesis is provided by the study of so-called type II supersymmetric string theories in 10 dimensions. The simplifying assumption of $N = 2$ supersymmetry means that the extra dimensions are a compact six-dimensional manifold X which admits a very special sort of geometry [Ler97]: that of a Calabi-Yau threefold.

There are two dual models of type II string theory, known as the A- and B-models, for which there exists a good mathematical description of the set of branes. They are the objects of a pair of triangulated categories $\mathcal{D}_A(X)$ and $\mathcal{D}_B(X)$ whose morphisms determine their interactions. The physical duality of these two models manifests itself in Kontsevich's homological mirror symmetry conjecture [Kon95] which postulates a duality $X \leftrightarrow X^\vee$ on the set of Calabi-Yau threefolds such that the triangulated categories $\mathcal{D}_A(X)$ and $\mathcal{D}_B(X^\vee)$ are equivalent.

The triangulated categories $\mathcal{D}_A(X)$ and $\mathcal{D}_B(X)$ depend only on the Kahler and complex geometry of the Calabi-Yau threefold X respectively. One may ask what is the physical significance of the remaining moduli of X, for example

the complex moduli of X in the A-model. It was found that a choice of complex structure on X determines a distinguished subset $\mathcal{P} \subset \mathcal{D}_A(X)$ of the objects of the triangulated category $\mathcal{D}_A(X)$ corresponding to the physically *stable* branes. This subset $\mathcal{P}$ can jump on codimension 1 walls of the moduli space $\mathcal{M}_{\mathbb{C}}(X)$ of complex structures on X along which stable branes can decay according to the triangulated structure of $\mathcal{D}_A(X)$.

For the A-model this can be understood using Π-stability of Douglas [Dou02] by considering the periods of the holomorphic volume form Ω_X specified by a complex structure on X. To a first approximation objects L of $\mathcal{D}_A(X)$ can be represented as Lagrangian submanifolds of X for which one can define a so-called *central charge*

$$Z(L) = \int_L \Omega_X$$

An object L is said to be semistable if it satisfies the BPS condition that its volume $|Z(L)|$ is minimal amongst all submanifolds representing the same homology class. The class $\mathcal{P}$ of semistable objects is the class of special Lagrangian submanifolds, along which the volume form Ω_X has a well-defined phase. The set $\mathcal{P}$ of semistable objects can jump along the loci in $\mathcal{M}_{\mathbb{C}}(X)$ where the phases of two special Lagrangian submanifolds coincide.

Bridgeland [Bri07] has abstracted Π-stability to the notion of a stability condition $(Z, \mathcal{P})$ on an arbitrary triangulated category $\mathcal{D}$. The set of stability conditions $\mathrm{Stab}(\mathcal{D})$ is a complex manifold modelled on the complex vector space of central charges $\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$. It admits actions of the autoequivalence group $\mathrm{Aut}(\mathcal{D})$ and the complex numbers $\mathbb{C}$ which act by rescaling the central charge. In the above context we expect to find an embedding

$$\mathcal{M}_{\mathbb{C}}(X) \to \mathbb{C} \backslash \mathrm{Stab}(\mathcal{D}_A(X)) / \mathrm{Aut}(\mathcal{D}_A(X))$$

A certain subset of stable branes persist in the so-called field theory limit where the effects of gravity become negligible to define BPS states of a gauge theory in four dimensions. It is remarkable that Seiberg and Witten [SW94] were able to describe the BPS states of the four-dimensional $N = 2$ supersymmetric $SU(2)$ Yang-Mills theories by purely field theoretic arguments. They did so by considering the periods of a meromorphic differential on an auxiliary family Σ of elliptic curves parameterised by the vacuum moduli space of the gauge theory.

With the advent of branes in string theory it was natural to try to find a string theoretic interpretation for the family of Seiberg-Witten curves and the Seiberg-Witten differential [Ler97]. For this one must first "geometrically engineer" the gauge theory by finding a Calabi-Yau threefold on which, in an appropriate limit in its moduli space, type II sting theory realises the gauge theory. For the Seiberg-Witten $SU(2)$ gauge theories the desired Calabi-Yau threefold in the A-model has a description as a K3 fibration over the projective line.

In the field theory limit, the generic fibre becomes a non-compact K3 surface isomorphic to the resolution of the Kleinian singularity associated with the gauge group $SU(2)$. The fibre degenerates to the singular surface over certain points of the projective line where the 2-sphere given by the exceptional divisor vanishes. The branes are Lagrangian 3-spheres which are fibrations over an open or closed curve in the projective line by the exceptional divisor of the K3 fibre.

One can obtain the Seiberg-Witten description of the BPS states by reducing the K3 fibre to a pair of points. The K3 fibration $X \to \mathbb{P}^1$ becomes the double cover $\Sigma \to \mathbb{P}^1$ expressing the Seiberg-Witten elliptic curve as a double cover over the projective line. The Lagrangian 3-spheres in X correspond to 1-cycles on the Seiberg-Witten curve Σ, and the holomorphic volume form Ω_X on X reduces to the meromorphic Seiberg-Witten differential λ on Σ.

The BPS states have the interpretation as geodesics of the metric induced by either the holomorphic volume form Ω on X or the Seiberg-Witten differential λ on Σ on the complement of their zeroes and poles on the projective line. A homology class γ in $H_3(X, \mathbb{Z})$ or $H_1(\Sigma, \mathbb{Z})$ supports a BPS state precisely when there exists a geodesic in $\mathbb{P}^1$ whose interior is disjoint from the zeroes of Ω or λ, which can be lifted along the projections $X \to \mathbb{P}^1$ and $\Sigma \to \mathbb{P}^1$ to a cycle representing γ.

Given the fundamental role of the work of Seiberg and Witten played in the understanding of the phenomenon of wall-crossing in the physics literature, it is somewhat surprising that it has not yet been interpreted in the mathematical formalism of stability conditions on an associated triangulated category. The main obstacle is finding a tractable description of the triangulated category $\mathcal{D}$ of BPS states; for a recent summary of approaches to this problem we refer to [yCDM$^+$13]. Following [BS13] we will describe $\mathcal{D}$ as a CY3 triangulated category $\mathcal{D}_{Q,W}$ associated to a mutation-equivalence class of quivers with potential (Q, W).

The idea of associating a mutation-equivalence class of quivers with potential to a certain four-dimensional $SU(2)$ $N = 2$ gauge theories is implicit in the work of Gaiotto, Moore and Neitzke [GMN09]. They study a class S of such theories which can be indexed by the set of marked surfaces with boundary, thereby making contact with cluster algebras associated to these surfaces. In the literature on cluster algebras it has recently been understood how to construct mutation-equivalence classes of quivers with potential from triangulations of marked bordered surfaces and how the mutation is "categorified" in $\mathcal{D}_{Q,W}$.

The vacuum moduli space of a theory of class S is given by the vector space of meromorphic quadratic differentials on the compactification C of the marked surface with boundary with fixed orders of poles determined by the marking. By considering the square root of such a quadratic differential u we obtain a spectral curve $\Sigma \to \mathbb{C}$ as a double cover of C and a meromorphic differential $\lambda = \sqrt{u}$ on Σ. For certain marked bordered surfaces whose compactification $C = \mathbb{P}^1$ is the projective line, this recovers the Seiberg-Witten curves and differential.

Bridgeland and Smith give a description of a distinguished component of the space of stability conditions $\mathrm{Stab}(\mathcal{D}_{Q,W})$ of the CY3 triangulated category $\mathcal{D}_{Q,W}$ by studying the geodesics of the metric on C induced by a quadratic differential. In particular they interpret a quadratic differential as a stability condition on $\mathcal{D}_{Q,W}$ and prove a bijection between a certain class of geodesics on the curve C and the set of semistable objects of the corresponding stability condition.

In this thesis we will give another description of the space of stability conditions of the CY3 categories $\mathcal{D}_{Q,W}$ associated to the Seiberg-Witten gauge theories and certain degenerations thereof. In doing so we recover the description of Bridgeland and Smith in these special cases, but via a different method which is logically independent of their results on geodesics of quadratic differentials. The method, which we outline in the next section, is in the spirit of the original

paper of Seiberg and Witten, studying the period map of the Seiberg-Witten differential on the family of Seiberg-Witten curves.

1.2   Content

The goal of the thesis is to describe a distinguished connected component of the space of numerical stability conditions of CY3 triangulated categories associated to seven mutation-equivalence classes of quivers. Each mutation-equivalence class contains an acyclic quiver whose underlying graph is either a finite or affine root system.

$$
1 \longrightarrow 0 \qquad 1 \longrightarrow 0 \longleftarrow 2 \qquad 1 \longrightarrow 0 \begin{smallmatrix} \swarrow 2 \\ \nwarrow 3 \end{smallmatrix}
$$

$$
1 \longrightarrow\!\!\!\!\rightarrow 0 \qquad \begin{smallmatrix} 2 \longrightarrow 0 \\ \searrow \quad \nearrow \\ 1 \end{smallmatrix} \qquad \begin{smallmatrix} 2 \longrightarrow 0 \\ \downarrow \quad \uparrow \\ 1 \longleftarrow 3 \end{smallmatrix} \qquad \begin{smallmatrix} 1 \quad\quad 4 \\ \searrow \quad \swarrow \\ 0 \\ \nearrow \quad \nwarrow \\ 2 \quad\quad 3 \end{smallmatrix}
$$
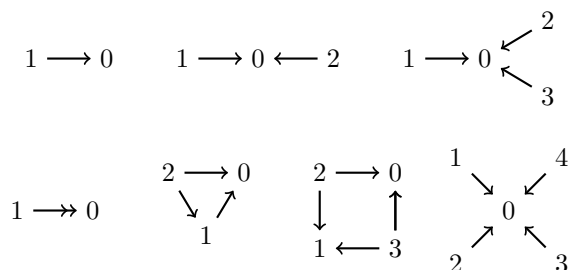
We consider the derived categories $\mathcal{D}_Q$ of finite-dimensional modules over the dg algebra which is the 3-Calabi-Yau completion in the sense of Keller [Kel11] of the path algebra of the quiver Q. This dg algebra is known as the Ginzburg algebra of the quiver Q following its construction in [Gin07] in the more general context of quivers with potential. The category $\mathcal{D}_Q$ has a canonical t-structure whose heart is the abelian category of finite-dimensional modules over the path algebra of Q.

The categories $\mathcal{D}_Q$ have been used to categorify the operation of mutation at a vertex of the quiver Q as defined in the literature on cluster algebras [FZ02]. Keller and Yang construct a pair of equivalences from $\mathcal{D}_Q$ to the derived category of finite dimensional modules of the dg Ginzburg algebra of the mutated quiver equipped with a suitable potential. Moreover the images of the heart of the natural t-structure on $\mathcal{D}_Q$ under this pair of equivalences are the left and right tilts of the heart with respect to the simple module associated to the vertex of the quiver at which we mutate.

The space of stability conditions $\mathrm{Stab}(\mathcal{D}_Q)$ of the triangulated category $\mathcal{D}_Q$ is a finite dimensional complex manifold modelled on the vector space of central charges $\mathrm{Hom}(K(\mathcal{D}_Q), \mathbb{C})$ where the Grothendieck group $K(\mathcal{D}_Q)$ is a free abelian group of rank the number of vertices of the quiver Q. The heart of the natural t-structure defines a domain in $\mathrm{Stab}(\mathcal{D}_Q)$ for which the central charges of the classes of its simple objects belong to the upper-half plane. The codimension one boundary components of this domain are labelled by its simple tilts whose domain shares the corresponding boundary component.

A well-studied family of mutation-equivalence classes of quivers is constructed by considering ideal triangulations of marked surfaces with boundary [FST08]. Each ideal triangulation of a given marked bordered surface defines a quiver whose vertices are the edges of the triangulation and arrows given by the adjacency matrix of its edges. The triangulation obtained by flipping one of its edges defines the mutation of the original quiver at the corresponding vertex.

Gaiotto, Moore and Neitzke [GMN09] have shown how to construct the ideal triangulations of marked surfaces with boundary by considering meromorphic quadratic differentials on the natural compactifiaction of the surface. They observe a wall-and-chamber structure on a certain stratum of the space of meromorphic quadratic differentials with simple zeroes and fixed orders of poles determined by the marking. The quadratic differentials in a given chamber define a so-called WKB ideal triangulation of the marked surface with boundary, and the WKB triangulations of two chambers sharing a wall differ by a flip.

Via the categorifcation of mutation by tilting in the category $\mathcal{D}_{Q,W}$ the wall-and-chamber structure on this stratum is reminiscent of the wall-and-chamber structure on its space of stability conditions. Bridgeland and Smith [BS13, Thm 1.3] show how a quadratic differential in this stratum defines a stability condition on $\mathcal{D}_{Q,W}$ up to the action of autoequivalences of $\mathcal{D}_{Q,W}$. Moreover they show [BS13, Thm 11.6] that the semistable objects of $\mathcal{D}_{Q,W}$ correspond to certain geodesics of a natural metric determined by the quadratic differential called saddle connections which connect two zeroes of the differential.

In this thesis we shall compute the distinguished connected component of the slice of numerical stability conditions of $\mathcal{D}_Q$ for the above seven quivers. A stability condition is numerical if its central charge $Z : K(\mathcal{D}) \to \mathbb{C}$ factors through the numerical Grothendieck group $N(\mathcal{D})$ obtained as the quotient of $K(\mathcal{D})$ by the radical of the Euler form of $\mathcal{D}$. When $\mathcal{D}$ is a CY3 category the Euler form is antisymmetric and so the rank of the numerical Grothendieck group is even.

The central charge of a numerical stability condition on $\mathcal{D}_Q$ determined by a quadratic differential u is given by considering the periods of a meromorphic 1-form $\lambda = \sqrt{u}$, called the Seiberg-Witten differential, on the spectral curve of the quadratic differential. The spectral curve is the double cover defined by $\sqrt{u}$ inside the twisted cotangent bundle, so that it is branched at the simple zeroes and poles of odd order. The Seiberg-Witten differential on the spectral curve is the restriction of the canonical differential on the cotangent bundle.

The dual graph of the WKB triangulation of a quadratic differential can be embedded in the surface as a graph of saddle connections whose edges connect a pair of simple zeroes of the quadratic differential. Each saddle connection lifts to a cycle on the spectral curve which is anti-invariant with respect to the involution on the spectral curve exchanging the two sheets of the double cover. This gives an identification of the numerical Grothendieck group $N(\mathcal{D}_Q)$ with the anti-invariant part of the first homology lattice of the spectral curve.

In particular a quadratic differential defines a *numerical* stability condition if and only if its Seiberg-Witten differential has zero residues so that its periods define a numerical central charge. For the above seven quivers the numerical Grothendieck group $N(D_Q)$ is of rank two and so the dimension of their space of numerical stability conditions is two. They correspond to all but one of the strata of residueless quadratic differentials on the projective line of dimension two whose orders of poles we list above.

There is a natural $\mathbb{C}^*$-action on the space of (residueless) quadratic differentials by multiplication which makes the map from quadratic differentials to stability conditions into an equivariant map. We find that the quotient of each two-dimensional stratum of residueless quadratic differentials by the $\mathbb{C}^*$-action is the complement of a finite collection of points Δ in a weighted projective line $\mathbb{P}(r, s)$ listed in the table below. Furthermore the sum of the number of orbifold

$$(4,4) \text{------} (3,4) \text{------} (3,3)$$

$$(4,1,1) \text{------} (3,1,1) \qquad (8) \text{------} (7)$$

$$(6,1) \text{------} (5,1)$$

$$\tilde{A}_3 \text{------} \tilde{A}_2 \text{------} \tilde{A}_1$$

$$\tilde{D}_4 \text{------} \tilde{D}_3 \qquad A_3 \text{------} A_2$$

$$D_4 \text{------} D_3$$
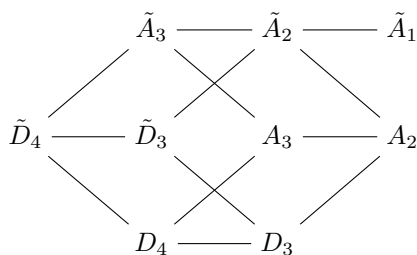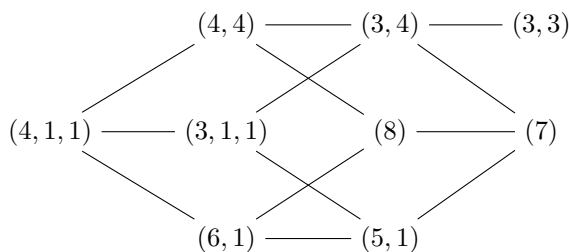
points and points in Δ is precisely three and so the fundamental group of the complement of Δ in $\mathbb{P}(r,s)$ is $\mathbb{Z}_r * \mathbb{Z}_s$, adopting the convention that $\mathbb{Z}_1 = \mathbb{Z}$

Q	A_2	A_3/D_3	D_4	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3/\tilde{D}_3$	$\tilde{D}_4$
(r,s)	$(2,3)$	$(1,2)$	$(1,3)$	$(2,1)$	$(3,1)$	$(2,1)$	$(1,1)$

The compactified spectral curves have first homology group of dimension two, and so we obtain a one-dimensional family of elliptic curves. The period map of the meromorphic differential λ has precisely three singularities, at the orbifold points and points of Δ, and so its image can be described using the classical Schwarz triangle theorem. Our main theorem asserts that the period map can be lifted to the universal cover $\widetilde{\mathrm{Stab}^0(\mathcal{D}_Q)}/\mathbb{C}$ a connected component of the space of numerical stability conditions.

Theorem 1.1. *There is a biholomorphic map*

$$\widetilde{\mathbb{P}(r,s)}\backslash\Delta \xrightarrow{\ f\ } \widetilde{\mathrm{Stab}^0(\mathcal{D}_Q)}/\mathbb{C}$$
$$p \searrow \qquad \downarrow \bar{Z}$$
$$\mathbb{P}\mathrm{Hom}(N(\mathcal{D}_Q),\mathbb{C})$$

lifting the period map p of the meromorphic differential λ of the family of spectral elliptic curves. It is equivariant with respect to the actions of $\mathbb{Z}_r * \mathbb{Z}_s$ on the left by deck transformations, and on the right by autoequivalences up to shift.

Furthermore in all but one case, that of $\tilde{A}_2$, we find that $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ is in fact simply connected and so we do not need to pass to a universal cover. In this case the group $\mathbb{Z}_r * \mathbb{Z}_s$ is isomorphic to the monodromy group of the family of elliptic curves, which is a congruence subgroup $\Gamma^0(N) < \mathrm{PSL}(2,\mathbb{Z})$ for some $N \le 4$.

1.3 Outlook

There is an interesting perspective on the above result through meromorphic $SL(2, \mathbb{C})$ Hitchin integrable systems $\mathcal{M} \to \mathcal{B}$. This integrable system has base $\mathcal{B}$, the vacuum moduli space of the gauge theory on $\mathbb{R}^4$, and becomes visible on compactification of one of these four directions to a circle. The fibres are Prym varieties of the spectral curves Σ_u and the pushforward of the Seiberg-Witten differential λ_u under the Abel-Jacobi map has the property that its exterior derivative agrees the holomorphic symplectic form on $\mathcal{M}$ [DM96a].

The base of an integrable system inherits an affine structure. One expects to find the more refined structure of an (almost) Frobenius manifold on spaces of stability conditions [Bri09a, Section 7]. In the A_2 case this almost Frobenius structure is considered further in joint work in progress with Bridgeland and Qiu [BQS], the periods of the Seiberg-Witten differential defining the so-called twisted periods of the Frobenius structure.

The $N = 2$ supersymmetry of the gauge theory means that the total space $\mathcal{M}$ of the Hitchin integrable system carries a hyperkahler metric. The main aim of the paper [GMN10] is to give a description of the hyperkahler metric via a family of Darboux coordinates for the holomorphic symplectic forms $\Omega(\zeta)$ on $\mathcal{M}$ parameterised by the twistor sphere $\mathbb{P}^1_\zeta$ for each $u \in \mathcal{B}$. These Darboux coordinates are required to have prescribed discontinuities along rays in $\mathbb{C}^*_\zeta$ for which there is a stable object of phase $\arg(\zeta)$ with respect to u, and asmyptotic behaviour as $\zeta \to 0$ governed by the central charge map.

Gaiotto, Moore and Neitzke conjecture that the Fock-Goncharov cluster $\mathcal{X}$-coordinates [FG09] on $\mathcal{M}$ have the required transformation law and asymptotics. The conjecture has not yet been verified in even the simplest examples; only a local model of the metric around the simplest singularity of the Hitchin system has been constructed. Recent work of Iwaki and Nakanishi [IN14] would appear using the WKB approximation to prove the desired asymptotics of the cluster coordinates for those local systems which can be represented as projective connections.

As a holomorphic symplectic manifold, $(\mathcal{M}, \Omega(\zeta))$ is isomorphic to a moduli space of $SL(2, \mathbb{C})$-local systems on $C \backslash P$ with appropriate Stokes data at each point of P. In the above examples the moduli space of local systems admits a description as an affine cubic surface [vdPS09]. It would appear, with the help of the recursion of [KS11] in the affine case, that the three coordinates on the cubic surface can be expressed in terms of cluster $\mathcal{X}$-coordinates. It would be interesting to see if this could be used to explicitly verify the GMN conjecture in these simplest examples.

Chapter 2

Background material

In this chapter we review some results about triangulated categories and stability conditions on them. In particular we study t-structures and how they relate to each other via tilting, which gives a certain combinatorial understanding of the space of stability conditions. We consider quivers with potential and their mutations, and state a result of Keller and Yang which categorifies this mutation procedure to a tilting equivalence of the derived categories of the Ginzburg dg algebras of the quivers with potential. In particular the derived category depends up to equivalence only on the mutation-equivalence class of the quiver with potential.

2.1 Triangulated categories, t-structures, torsion pairs and tilting

Triangulated categories were introduced by Verdier [Ver96] to axiomatise the properties of derived categories.

Definition 2.1. A triangulated category is an additive category $\mathcal{D}$ together with an invertible *shift functor* $[1] : \mathcal{D} \to \mathcal{D}$ and a collection of *distinguished triangles*

$$X \to Y \to Z \to X[1]$$

satisfying the following axioms

1. The triangle $X \xrightarrow{id} X \to 0 \to X[1]$ is distinguished

2. The rotation $Y \to Z \to X[1] \to Y[1]$ of a distinguished triangle is distinguished

3. Any morphism $X \xrightarrow{f} Y$ can be completed to a distinguished triangle whose third object we call a cone of f

4. The octahedral axiom [BBD82, Section 1.16]: given morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, this asserts the existence of cones Z', Y' and X' of f, $g \cdot f$ and f respectively forming a distinguished triangle $Z' \to Y' \to X' \to Z'[1]$ such that the four distinguished triangles fit into an octahedron-shaped commuting diagram.

The motivating example is the bounded derived category $\mathcal{D}^b(\mathcal{A})$ of an abelian category $\mathcal{A}$. The bounded derived category is the localisation of the homotopy category $\mathcal{K}(\mathcal{A})$ of chain complexes of objects in $\mathcal{A}$ with only finitely many non-zero cohomologies at the class of *quasi-isomorphisms*. Thus morphisms in $\mathcal{D}^b(\mathcal{A})$ are obtained by formally inverting all morphisms of $\mathcal{K}(\mathcal{A})$ which induce isomorphisms on the homology of the chain complex. The shift functor is the functor which moves everything in the chain complex one position to the left, and the distinguished triangles are those which are isomorphic to a mapping cone in the homotopy category.

One can draw analogies between the structure of a triangulated category and that of an abelian category. In fact we can view an abelian category $\mathcal{A}$ as embedded in its bounded derived category $\mathcal{D}^b(\mathcal{A})$ as the full subcategory of complexes concentrated in degree zero. Then a distinguished triangle whose three objects X, Y and Z all lie in $\mathcal{A}$ is precisely a short exact sequence in $\mathcal{A}$. In this case the octahedral axiom reduces to the third isomorphism theorem for $\mathcal{A}$.

By analogy to the Grothendieck group $K(\mathcal{A})$ of an abelian category, we can define a similar group for a triangulated category.

Definition 2.2. The Grothendieck group $K(\mathcal{D})$ of a triangulated category is the free abelian group on isomorphism classes $[X]$ of objects X of $\mathcal{D}$ modulo the relation

$$[Y] = [X] + [Z]$$

whenever $X \to Y \to Z \to X[1]$ is a distinguished triangle.

If the triangulated category $\mathcal{D}$ is Hom-finite, that is the graded vector space $\mathrm{Hom}_{\mathcal{D}}^*(X, Y) = \oplus_n \mathrm{Hom}_{\mathcal{D}}(X, Y[n])$ is finite dimensional for all X and Y, then $K(\mathcal{D})$ is a lattice with the *Euler pairing*

$$\chi(X, Y) = \sum_n \dim \mathrm{Hom}_{\mathcal{D}}(X, Y[n])$$

This is the case for the bounded derived category $\mathcal{D}^b(\mathcal{A})$ of an abelian category $\mathcal{A}$. The natural embedding $\mathcal{A} \hookrightarrow \mathcal{D}^b(\mathcal{A})$ then induces an isomorphism $K(\mathcal{A}) \to K(\mathcal{D})$ whose inverse is given by the alternating sum of the isomorphism classes of the cohomologies of the chain complex.

The embedding $\mathcal{A} \hookrightarrow \mathcal{D}^b(\mathcal{A})$ is the prototypical example of a heart of bounded t-structure on a triangulated category. The notion of a t-structure on a triangulated category $\mathcal{D}$ was introduced in [BBD82], and determines cohomology functors H^* on $\mathcal{D}$ whose images belong to an abelian category known as the heart of the t-structure.

Given a full subcategory $\mathcal{B}$ of a category $\mathcal{C}$, we denote by $\mathcal{B}^\perp$ the right orthogonal of $\mathcal{B}$ in $\mathcal{C}$, that is the full subcategory of $\mathcal{C}$ consisting of all objects $C \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(B, C) = 0$ for all $B \in \mathcal{B}$.

Definition 2.3. A t-structure on a triangulated category $\mathcal{D}$ is a full additive subcategory $\mathcal{T}$, stable under shift, such that for all $E \in \mathcal{D}$ there is a distinguished triangle

$$T \to E \to F \to T[1]$$

with $T \in \mathcal{T}$ and $F \in \mathcal{T}^\perp$.

We call the full subcategory $\mathcal{A} = \mathcal{T} \cap \mathcal{T}^{\perp}[1]$ the heart of the t-structure.

Proposition 2.4. *[BBD82, Théoreme 1.3.6] The heart $\mathcal{A}$ of a t-structure is an abelian category.*

We say that a t-structure is bounded if every object of $\mathcal{D}$ belongs to $\mathcal{T}[-n] \cap \mathcal{T}^{\perp}[n]$ for some n, that is the degree of the cohomology of any object with respect to the t-structure is bounded. We have the following characterisation of the additive subcategories of a triangulated category which occur as hearts of a bounded t-structure.

Proposition 2.5. *[Bay] A full additive subcategory $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{A}$ if and only if the following two conditions are satisfied*

- *$\mathrm{Hom}^n_{\mathcal{D}}(A_1, A_2) = 0$ for all objects $A_1, A_2 \in \mathcal{A}$ and all $n < 0$*

- *Every object $E \in \mathcal{D}$ has a filtration*

$$0 \longrightarrow E_1 \longrightarrow \ldots \longrightarrow E_{n-1} \longrightarrow E$$
$$\qquad\qquad F_1 \qquad\qquad\qquad\qquad\qquad F_n$$

   such that the $F_i \in \mathcal{A}[k_i]$ with $k_1 > \ldots > k_n$.

Indeed the F_i are precisely the non-zero cohomology objects of E with respect to the t-structure.

We can construct new t-structures from old ones by considering torsion pairs inside the heart of the t-structure.

Definition 2.6. A torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is a pair of full additive subcategories $(\mathcal{T}, \mathcal{F})$ such that $\mathcal{F} = \mathcal{T}^{\perp}$, and for all $A \in \mathcal{A}$ there is a short exact sequence

$$0 \to T \to A \to F \to 0$$

We note the similarity of this definition with that of a t-structure, in that both t-structures and torsion pairs give ways of decomposing an object in a triangulated and abelian category respectively into parts lying in a subcategory and its orthogonal complement.

Definition 2.7. The left and right tilts of the heart of a bounded t-structure $\mathcal{A} \subset \mathcal{D}$ at a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ are the full subcategories $\mathcal{A}^+$ and $\mathcal{A}^-$ of $\mathcal{D}$ consisting of objects E whose only non-zero cohomologies are

$$H^0(E) \in \mathcal{F}, H^1(E) \in \mathcal{T} \qquad \text{and} \qquad H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}$$

respectively.

This is a slight abuse of terminology in that the definition does not coincide with the left and right tilts of an abelian category in the sense of [HRO96]. The problem is that a heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$ is not necessarily faithful, that is the morphism spaces $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ and $\mathrm{Hom}_{\mathcal{D}}(A, B[n])$ are not necessarily isomorphic for all pairs of objects A, B of $\mathcal{A}$. Nevertheless we have the following analogue of their result.

Proposition 2.8. *[HRO96] The left and right tilts of a heart $\mathcal{A}$ are again hearts of bounded t-structures on $\mathcal{D}$.*

We will be particularly interested in the left and right *simple* tilts $\mathcal{A}_S^+$ and $\mathcal{A}_S^-$, that is when $\mathcal{T}$ or $\mathcal{F}$ respectively is the full subcategory of $\mathcal{A}$ consisting of direct sums of copies of the simple object S. We can compute the simple objects in the heart $\mathcal{A}_S^\pm$ when S is *rigid*, that is $\operatorname{Ext}^1_{\mathcal{A}}(S, S) = 0$.

Proposition 2.9. *Suppose S is a simple object of $\mathcal{A}$ satisfying $\operatorname{Ext}^1_{\mathcal{A}}(S, S) = 0$. For a simple object S_i of $\mathcal{A}$, denote by S_i^+ and S_i^- the universal extensions*

$$0 \to S_i \to S_i^+ \to \operatorname{Ext}^1_{\mathcal{A}}(S, S_i) \otimes S \to 0$$

and

$$0 \to \operatorname{Ext}^1_{\mathcal{A}}(S_i, S) \otimes S \to S_i^- \to S_i \to 0$$

respectively. Then the simple objects of the simple tilts $\mathcal{A}_S^\pm$ of $\mathcal{A}$ are $S[\mp 1]$ and $S_i^\pm$ for all simple objects S_i of $\mathcal{A}$ not equal to S.

Proof. We show only the statement for the simple objects of the left tilt $\mathcal{A}_S^+$, the proof for the right tilt being similar. Associated to any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}_S^+$, we have a long exact sequence in cohomology with respect to the original t-structure which is concentrated in degrees 0 and 1.

$$0 \to H^0(X) \to H^0(Y) \to H^0(Z) \to H^1(X) \to H^1(Y) \to H^1(Z) \to 0$$

The object Y is a simple object of $\mathcal{A}_S^+$ if and only if for every such short exact sequence, either $X = 0$ or $Z = 0$. Consider the object $Y = S[-1]$ whose cohomology is concentrated in degree 1. Then we have a four term exact sequence

$$0 \to H^0(Z) \to H^1(X) \to S \to H^1(Z) \to 0$$

Then as S is simple in $\mathcal{A}$, $H^1(Z)$ is either S or 0. In the first case we have $H^1(X) = H^0(Z) \in \mathcal{T} \cap \mathcal{F} = \{0\}$; thus $X = 0$ and $Z = S[-1]$. On the other hand if $H^1(Z) = 0$ then we have a short exact sequence

$$0 \to H^0(Z) \to H^1(X) \to S \to 0$$

whose last two terms belong to $\mathcal{T}$ and so $H^0(Z) \in \mathcal{T} \cap \mathcal{F} = \{0\}$. Thus $X = S[-1]$ and $Z = 0$ and we deduce that $S[-1]$ is simple in $\mathcal{A}_S^+$.

Now consider the universal extension $Y = S_i^+$, whose cohomology is concentrated in degree zero. We have the four-term exact sequence

$$0 \to H^0(X) \to S_i^+ \to H^0(Z) \to H^1(X) \to 0$$

The possibilities for $H^0(X)$ are the subobjects of S_i^+ in $\mathcal{A}$, namely S_i^+, S_i (supposing it is distinct from S_i^+) and 0. In the first case, we have $H^1(X) = H^0(Z) \in \mathcal{T} \cap \mathcal{F} = \{0\}$ and so $X = S_i^+$ and $Z = 0$. In the second case, we have an injection $\operatorname{Ext}^1_{\mathcal{A}}(S, S_i) \otimes S \hookrightarrow H^0(Z) \in \mathcal{F}$, which is a contradiction. Thirdly if $H^0(X) = 0$ then we have a short exact sequence

$$0 \to S_i^+ \to H^0(Z) \to H^1(X) \to 0$$

But then $H^1(X) = 0$ as $\operatorname{Ext}^1_{\mathcal{A}}(S_i^+, S) = 0$, so $X = 0$ and $Z = S_i^+$. Thus S_i^+ is simple. $\qquad\square$

2.2 Stability conditions

In this section we consider the space of stability conditions $\mathrm{Stab}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ introduced in [Bri07]. It is a complex manifold carrying an action of the autoequivalence group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$, covered by domains $U(\mathcal{A})$ of stability conditions supported on a given heart of a t-structure. In the case where a heart $\mathcal{A}$ has a finite length, the domain $U(\mathcal{A})$ has a simple description, and we study how the closure $\overline{U(\mathcal{A})}$ intersects $\overline{U(\mathcal{A}')}$ for hearts $\mathcal{A}'$ obtained from $\mathcal{A}$ by tilting. We find a wall-and-chamber structure whose chambers are the $U(\mathcal{A})$ and whose walls are given by simple tilts.

We first define the notion of a stability function on an abelian category $\mathcal{A}$. This is an axiomatisation [Rud97] of notions of stability used to construct nice moduli spaces of objects in abelian categories, such as slope-stability of coherent sheaves on a curve [HN75].

Definition 2.10. A stability function on an abelian category $\mathcal{A}$ is an additive function $Z : K(\mathcal{A}) \to \mathbb{C}$ such that

- For all non-zero objects $A \in \mathcal{A}$, the *phase*

$$\phi(A) = \frac{1}{\pi} \arg(Z([A]))$$

  lies in the interval $(0, 1]$, i.e $Z([A])$ belongs to the half-closed upper half plane $\bar{\mathfrak{h}} = \mathfrak{h} \cup \mathbb{R}_{<0}$.

- Z satisfies the Harder-Narasimhan property: say an object A in $\mathcal{A}$ is semistable if it has no non-zero subobjects S satisfying $\phi(S) > \phi(A)$. Then every A in $\mathcal{A}$ has a filtration

$$0 = A_0 \subset A_1 \ldots \subset A_n = A$$

  such that the quotients A_i/A_{i-1} have decreasing phase and are semistable.

We note that the semistable objects are precisely those with a trivial Harder-Narasimhan filtration. Thus a stability function gives a way of decomposing any object of $\mathcal{A}$ into its semistable parts ordered by their phases. We observe that the set of semistable objects can jump along codimension 1 loci in the space of stability functions where the phases of semistable objects align. Such loci are called *walls of marginal stability* or *walls of the first kind* inside the space of stability functions.

In the same way that t-structures and torsion pairs complement each other, so do the filtration of Proposition 2.5 and the Harder-Narisimhan filtration. This leads to a notion of stability condition on a triangulated category following [Bri07].

Definition 2.11. A stability condition on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z : K(\mathcal{A}) \to \mathbb{C}$ is a stability function on $\mathcal{A}$.

Remark 2.12. This is equivalent to the original definition of a stability condition [Bri07, Definition 1.1] in terms of slicings of $\mathcal{D}$ by [Bri07, Proposition 5.3]. Indeed if $\mathcal{A}$ is the heart of a bounded t-structure, then the natural map $K(\mathcal{A}) \to$

$K(\mathcal{D})$ is an isomorphism, so a stability function on $\mathcal{A}$ defines a *central charge* $Z : K(\mathcal{D}) \to \mathbb{C}$. The Harder-Narasimhan filtration of an object $E \in \mathcal{D}$ is then a splicing of the filtration of E by its cohomology objects in Proposition 2.5 and the Harder-Narasimhan filtration of the cohomology objects given by the stability function on $\mathcal{A}$.

The set $\mathrm{Stab}(\mathcal{D})$ of all stability conditions satisfying a technical condition called local finiteness [Bri07, Definition 5.7] has a metric topology described in [Bri07, Section 8]. Supposing further that $K(\mathcal{D})$ has finite rank, $\mathrm{Stab}(\mathcal{D})$ is in fact a complex manifold via the forgetful map to the complex vector space $\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ of central charges.

Theorem 2.13. *[Bri07, Theorem 1.2] Suppose that $K(\mathcal{D})$ has finite rank. Restricted to any connected component of $\mathrm{Stab}(\mathcal{D})$, the forgetful map $Z : \mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ is a local homeomorphism onto its image, a linear subspace of $\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$*

We say a stability condition on $\mathcal{D}$ is *supported* on a heart $\mathcal{A}$ if it is of the form $(\mathcal{A}, Z)$ and write $U(\mathcal{A})$ for the set of stability conditions supported on $\mathcal{A}$. We have a simple description of the space of stability conditions supported on a heart $\mathcal{A}$ of finite length, for the Harder-Narasimhan property is then automatically satisfied by any function $K(\mathcal{A}) \to \mathbb{C}$.

Proposition 2.14. *[Bri09a, Lemma 5.2] Suppose a heart $\mathcal{A}$ has finite length $n = \mathrm{rk}(K(\mathcal{A}))$, i.e. it has precisely n simple objects $S_1, \ldots S_n$. Then $U(\mathcal{A}) \cong \bar{\mathfrak{h}}^n$, where $\bar{\mathfrak{h}} = \mathfrak{h} \cup \mathbb{R}_{<0}$ is the half closed upper-half plane as in Definiton 2.10.*

In particular we observe that the connected component of the space of stability conditions containing such a heart is locally homeomorphic to $\mathbb{C}^n$ for the image of the forgetful map Z must be the entire vector space $\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$. In this case we can also describe what happens on the codimension-one boundary components of $U(\mathcal{A})$ where the phase of a single simple object of $\mathcal{A}$ becomes 0 or 1.

Proposition 2.15. *[Bri09a, Lemma 5.5] Suppose that $\mathcal{A}$ has rigid simple objects $S_1, \ldots, S_n$. Then the codimension 1 boundary components of $U(\mathcal{A})$ are labelled by the pair of a simple object S_k of $\mathcal{A}$ and a sign $+$ or $-$, and consist of stability conditions for which $\phi(S_k) = 1$ or $\phi(S_k) = 0$ respectively. If $\sigma^{\pm}$ is contained in a unique such codimension one boundary component $(S_k, \pm)$ of $U(\mathcal{A})$, then a neighbourhood of $\sigma^{\pm}$ is contained in $\overline{U(\mathcal{A})} \cup \overline{U(\mathcal{A}_{S_k}^{\pm})}$ where $\mathcal{A}_{S_k}^{\pm}$ is the left or right simple tilt of $\mathcal{A}$ at S_k respectively.*

If the tilted heart is again of finite type, this process can be iterated, giving a wall-and-chamber structure on a subset of the space of stability conditions. We call these walls through which the supporting heart of a stability condition changes *walls of the second kind*. If the heart $\mathcal{A}$ has only finitely many indecomposables then Woolf [Woo10, Thm 2.17] has shown that the subset covered in this way is the entire connected component of $\mathrm{Stab}(\mathcal{D})$ containing $U(\mathcal{A})$. However this is not true in general as walls of the second kind can accumulate.

We will mainly be interested in the subset of the space of stability conditions which are *numerical*. Define the numerical Grothendieck group to be the quotient of the Grothendieck group by the left radical of the Euler form on $\mathcal{D}$.

Definition 2.16. The numerical Grothendieck group $N(\mathcal{D}) = K(\mathcal{D})/K(\mathcal{D})^\perp$ where $K(\mathcal{D})^\perp = \{[E] \in K(\mathcal{D}) \mid \chi([E],[F]) = 0 \,\forall F \in K(\mathcal{D})\}$

We note that the Euler form restricts to a non-degenerate bilinear form on the numerical Grothendieck group $N(\mathcal{D})$.

Definition 2.17. A stability condition is *numerical* if the central charge $Z : K(\mathcal{D}) \to \mathbb{C}$ factors through the numerical Grothendieck group $N(\mathcal{D})$.

We note that the numerical central charges $\text{Hom}(N(\mathcal{D}), \mathbb{C})$ form a linear subspace of $\text{Hom}(K(\mathcal{D}), \mathbb{C})$. Thus Theorem 2.13 holds with the appropriate modification, and provided that the heart $\mathcal{A}$ supports a numerical stability condition, we have that $U(\mathcal{A}) \cong \bar{\mathfrak{h}}^{\text{rk}(N(\mathcal{D}))}$ by Proposition 2.14.

Finally we recall two natural group actions on the space of stability conditions (cf [Bri07]) which preserve the space of numerical stability conditions. Firstly the group of autoequivalences $\text{Aut}(\mathcal{D})$ acts naturally on $\text{Stab}(\mathcal{D})$ via the map

$$\Phi(\mathcal{A}, Z) = (\Phi(\mathcal{A}), Z \cdot \phi^{-1})$$

where $\phi : K(\mathcal{D}) \to K(\mathcal{D})$ denotes the automorphism of the lattice $K(\mathcal{D})$ induced by the automorphism Φ.

As we are only interested in the action of the autoequivalence group on the space of stability conditions in this thesis, we will consider autoequivalences modulo the equivalence relation identifying two autoequivalences if they act trivially on the space of stability conditions. From now on we understand all references to the group of autoequivalences to refer to this quotient group.

There is also an action of the universal cover $\widetilde{\text{GL}^+(2,\mathbb{R})}$ of the group of orientation preserving maps of the plane, which acts on the central charge by postcomposition with the corresponding orientation preserving map of the plane $\mathbb{R}^2 \cong \mathbb{C}$. This action is not always free for the image of the central charge map can be contained in a line. However there is a free action of the subgroup $\mathbb{C} < \widetilde{\text{GL}^+(2,\mathbb{R})}$ which acts on the central charge via $\mathbb{C} = \widetilde{\mathbb{C}^*} \xrightarrow{\exp(i\pi-)} \mathbb{C}^*$ by rescaling.

Definition 2.18. A projective stability condition $\bar{\sigma}$ is an orbit of a stability condition σ under the free $\mathbb{C}$-action.

A projective stability condition $\bar{\sigma} \in \text{Stab}(\mathcal{D})/\mathbb{C}$ is a pair consisting of an equivalence class of hearts $\mathcal{A}$ under the $\mathbb{C}$-action, and a projectivised central charge $\overline{Z} \in \mathbb{P}\,\text{Hom}(K(\mathcal{D}), \mathbb{C})$. By Theorem 2.13, it is a complex manifold modeled on the projective space $\mathbb{P}\,\text{Hom}(K(\mathcal{D}), \mathbb{C})$. We note that an element n of the subgroup $\mathbb{Z} \subset \mathbb{C}$ acts as the autoequivalence $[n]$. Thus the space of projective stability conditions $\text{Stab}(\mathcal{D})/\mathbb{C}$ inherits an action of $\text{Aut}(\mathcal{D})/\mathbb{Z}[1]$.

2.3 Quivers with potential

In this section we introduce quivers with potential and their mutations.

A quiver Q is a directed graph. We will write Q_0 for the set of vertices of the underlying graph and Q_1 for the set of arrows. We then have a pair of natural maps $s, t : Q_1 \rightrightarrows Q_0$ called the source and target maps sending an arrow to the vertex at its tail and head respectively. A path of length l in Q is a word

$a_1 \ldots a_l$ in Q_1 such that $s(a_i) = t(a_{i-1})$. It is convenient to include the lazy paths e_i of length 0 at each vertex. The source and target maps then extend naturally to the set of all paths in Q.

Definition 2.19. The path algebra $\mathbb{C}Q$ of a quiver Q has underlying vector space the set of all paths of length ≥ 0. The composition of two paths is their concatenation if the target of the first path is the source of the second, and zero otherwise. A module over the path algebra is called a representation of Q.

The lazy paths e_i define a complete set of orthogonal idempotents for $A = \mathbb{C}Q$, so $A = \oplus Ae_i$ decomposes as a direct sum of projective modules $P_i = Ae_i$. Every module M over A then has a projective resolution

$$0 \to \oplus_{a \in Q_1} Ae_{t(a)} \otimes e_{s(a)} M \xrightarrow{\cdot a \cdot} \oplus_{i \in Q_0} Ae_i \otimes e_i M \to M \to 0$$

The abelian category $\mathcal{A} = \mathrm{mod}(\mathbb{C}Q)$ of finite-dimensional representations of Q is thus a hereditary category, i.e. $\mathrm{Ext}^i_{\mathcal{A}}(M, N) = 0$ unless $i = 0, 1$. The Euler form $\chi(-, -)$ on $K(\mathcal{A})$ can be computed to be the Ringel form

$$\langle M, N \rangle = \dim \mathrm{Hom}(M, N) - \dim \mathrm{Ext}^1(M, N)$$
$$= \sum_{i \in Q_0} \dim(e_i M) \dim(e_i N) - \sum_{a \in Q_1} \dim(e_{s(a)} M) \dim(e_{t(a)} N)$$

where the map is given by multiplication by the arrow a.

In particular if the quiver Q is acyclic, i.e. contains no oriented cycles, then the simple modules in $\mathcal{A}$ are the precisely one-dimensional representations $S_i = \mathbb{C}e_i$ at each vertex. The projective and simple modules then define dual bases of $K(\mathcal{A})$ with respect to the Euler form. The quiver Q can be recovered as the Ext-quiver of its category of representations.

Definition 2.20. Suppose $\mathcal{A}$ is a $\mathbb{C}$-linear abelian category of finite length. Then the Ext-quiver of $\mathcal{A}$ has vertices indexed by the set $\{S_i\}$ of simple objects in $\mathcal{A}$ and $\dim_{\mathbb{C}} \mathrm{Ext}^1_{\mathcal{A}}(S_i, S_j)$ arrows from S_j to S_i.

There is a mutation rule for quivers developed by Fomin and Zelevinsky in their work on cluster algebras [FZ02].

Definition 2.21. Let Q be a quiver without loops or 2-cycles. The mutation $\mu_k(Q)$ of a quiver Q at the vertex k is defined as follows

1. For every path $i \to k \to j$ of length two in Q, add an arrow $j \to i$

2. Reverse all arrows incident to k

3. Remove a maximal disjoint collection of 2-cycles

Consider an ordering of the set of vertices of a quiver Q with n vertices. There is a canonical bijection between the vertices of two quivers with potential related by a mutation at a given vertex. Thus the set of ordered quivers Q_n with n vertices has the structure of a homogeneous space for the action of the *mutation group M_n*.

Definition 2.22. The n-mutation group M_n is the semi-direct product of the free group F_n with the symmetric group S_n, acting by permutation on the generators of S_n.

We note that if the vertex k is a source or a sink of Q, then mutation at k underlies an equivalence of the derived categories of the path algebras of the quiver [Hap88]. Indeed there are a pair of derived equivalences given by tilting modules of [BGP73] such that the images of the standard heart $\mathcal{A} = \mathrm{mod}(\mathbb{C}Q)$ are the left and right tilted hearts $\mathcal{A}^+_{S_k} = \mathrm{mod}(\mathbb{C}\mu_k(Q))$ and $\mathcal{A}^-_{S_k} = \mathrm{mod}(\mathbb{C}\mu_k(Q))$ respectively.

Theorem 2.23. *[Hap88] Suppose k is a source or a sink of Q and let $\mathcal{A} = \mathrm{mod}(\mathbb{C}Q)$. Then there are a pair of derived equivalences $\Phi^{\pm}_{S_k} : \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A}^{\pm}_{S_k})$.*

More general mutations of quivers can be categorified by introducing a potential. This concept originated in the string theory literature (e.g. [DM96b]) where it is important in describing the relationship between "dual" theories described by quivers related by a mutation. It was used in [Gin07] in describing Calabi-Yau algebras and their mutations, and a general theory of mutations for quivers with potential was described in [DWZ08].

Definition 2.24. A potential W on a quiver Q is a $\mathbb{C}$-linear combination of cycles of Q up to cyclic equivalence, that is up to the relation $a_1 \ldots a_l \sim a_i \ldots a_l a_1 \ldots a_{i-1}$.

A choice of potential defines a quotient of the path algebra $\mathbb{C}Q$ of Q by the cyclic derivatives $\partial W/\partial a$ of W with respect to arrows $a \in Q_1$. The cyclic derivative is defined to be the sum of words in $\mathbb{C}Q$ obtained by deleting a from all words beginning with a in the cyclic equivalence class of words in W.

Definition 2.25. The Jacobi algebra of a quiver with potential (Q, W) is

$$ J(Q, W) = \frac{\hat{\mathbb{C}}Q}{\langle \frac{\partial W}{\partial a} | a \in Q_1 \rangle} $$

where $\hat{\mathbb{C}}Q$ denotes the completion of the path algebra of Q with respect to path-length. We say a quiver is Jacobi-finite if $J(Q, W)$ is finite-dimensional.

Note that if the quiver Q is acyclic then the only potential on Q is the zero potential. The Jacobi algebra is then isomorphic to the uncompleted path algebra $\mathbb{C}Q$ of the quiver Q.

We consider quivers with potential up to the notion of right-equivalence.

Definition 2.26. Two quivers with potential (Q, W) and (Q', W') are right-equivalent if they have the same vertex set and there is an algebra isomorphism $\mathbb{C}Q \to \mathbb{C}Q'$ which is the identity on vertices and maps W to W'.

The Jacobi algebras of two right-equivalent quivers with potential are isomorphic [DWZ08]. Up to right-equivalence, a quiver with potential can always be decomposed according to the following splitting theorem by "integrating out" quadratic terms in the potential.

We say a quiver with potential is *reduced* if the potential contains no 2-cycles. A quiver with potential is *trivial* if its Jacobi algebra is trivial. This is the case if and only if the arrows of the quiver come in pairs forming 2-cycles and the potential is the sum of quadratic terms given by these 2-cycles [DWZ08].

Theorem 2.27. *[DWZ08, Theorem 4.6] Any quiver with potential (Q, W) is right-equivalent to the direct sum of a reduced and trivial quiver with potential. Moreover the two quivers with potential in the decomposition are unique up to right-equivalence, and the Jacobi algebras of (Q, W) and its reduction are isomorphic.*

We note that the trivial part of a quiver with potential is not necessarily a maximal collection of 2-cycles, so the reduced part of the quiver with potential may contain 2-cycles.

Definition 2.28. The mutation $\mu_k(Q, W)$ of a quiver with potential (Q, W) without loops or 2-cycles at the vertex k is defined to be the reduced part of the quiver with potential obtained from (Q, W) as follows:

1. For every path $i \xrightarrow{a} k \xrightarrow{b} j$ of length two in Q, add an arrow $j \xrightarrow{[ab]} i$ and replace all occurrences of ab in W by $[ab]$

2. Replace every arrow a incident to k with its opposite a^*, and add a term $b^*a^*[ab]$ to the potential

We note that this is not a strict generalisation of the notion of quiver mutation of Fomin-Zelevinsky, for whether the mutation $\mu_k(Q, W)$ contains any 2-cycles depends on the potential.

Definition 2.29. A potential W on Q is non-degenerate if no iterated mutation of (Q, W) contains a 2-cycle.

The non-degeneracy condition holds outside a collection of hyperplanes in the space of potentials [DWZ08]. If a quiver with potential (Q, W) is non-degenerate then the underlying quiver of all of its iterated mutations agrees with the iterated mutations of the quiver Q in the sense of Definition 2.21. In the next section we will categorify mutation of quivers with potential using tilting of hearts in a triangulated category $\mathcal{D}_{Q,W}$. In particular we will see how to lift the M_n-action to the set of ordered hearts of the category $\mathcal{D}_{Q,W}$.

2.4   Ginzburg algebras

In this section we define a Calabi-Yau-3 triangulated category $\mathcal{D}_{Q,W}$ associated with a quiver with potential (Q, W) following Ginzburg [Gin07] whose space of stability conditions we study in this thesis. We will show how mutation of quivers with potential is categorified by tilting in $\mathcal{D}_{Q,W}$.

Given its string theoretic origin [DM96b], one might expect that the category $\mathrm{mod}(J(Q, W))$ is a Calabi-Yau-3 (CY3) category, that is there are functorial isomorphisms

$$\mathrm{Ext}^r(M, N) \cong \mathrm{Ext}^{3-r}(N, M)^*$$

Indeed there is a natural candidate for a projective resolution of the simple object S_i which is a complex whose terms are symmetric with respect to exchanging the degree r component with the $3 - r$ component.

$$P_i \xrightarrow{\cdot b} \oplus_{b:k \to i} P_k \xrightarrow{\cdot \partial W / \partial(ab)} \oplus_{a:i \to j} P_j \xrightarrow{\cdot a} P_i \to S_i$$

This is indeed a complex, as the two compositions are equal to the cyclic derivatives $\partial W/\partial b$ and $\partial W/\partial a$. However in general this complex is not exact; for example if the potential is zero. In this case the category is the category of finite dimensional modules over the path algebra, which is hereditary and so certainly not CY3.

One can however view the Jacobian algebra as an approximation of a dg algebra $\Gamma(Q, W)$ introduced by Ginzburg in [Gin07] in the sense that $H^0(\Gamma(Q, W)) = J(Q, W)$. This dg algebra is the CY3 completion of $J(Q, W)$ in the sense of Keller [Kel11] and Koszul dual to the cyclic completion of Ed Segal [Seg08] of the A_∞ completion of the algebra

$$\operatorname{Ext}^r(\oplus S_i, \oplus S_i) \oplus \operatorname{Ext}^{3-r}(\oplus S_i, \oplus S_i)$$

Indeed the definition is engineered such that for every arrow $i \to j$ of Q, which gives rise to an extension between simple objects S_i and S_j, there is a reverse arrow giving rise to an element of $\operatorname{Ext}^2(S_j, S_i)$. Similarly for every vertex i of Q there is the identity morphism $S_i \to S_i$ which is dual to a class in $\operatorname{Ext}^3(S_i, S_i)$ induced by a loop at the vertex.

Definition 2.30. The complete Ginzburg dg algebra $\Gamma(Q, W)$ of a quiver with potential (Q, W) has underlying graded algebra the completion of the graded path algebra of $\bar{Q}$ with respect to path length where $\bar{Q}$ is constructed from Q as follows:

- In degree 0, the arrows of Q

- In degree -1, the arrows $a^* : j \to i$ for each arrow $a : i \to j$ of Q

- In degree -2, the arrows $t_i : i \to i$ for each vertex i of Q

The differential is uniquely determined by the fact that it is linear, satisfies the Leibniz rule, and takes the following values on the arrows of $\bar{Q}$:

$$d(a) = 0 \qquad d(a^*) = \partial W/\partial a \qquad d(t_i) = \sum_{a \in Q_1} e_i[a, a^*]e_i$$

We now consider the derived category $\mathcal{D}_{fd}(\Gamma(Q, W))$ of the dg category of finite dimensional dg modules over the dg algebra $\Gamma(Q, W)$ in the sense of Keller [Kel94].

Proposition 2.31. *[Gin07] The triangulated category $\mathcal{D}_{Q,W} = \mathcal{D}_{fd}(\Gamma(Q, W))$ is CY3, that is there are functorial isomorphisms*

$$\operatorname{Hom}(E, F) \cong \operatorname{Hom}(F, E[3])^*$$

As the Ginzburg algebra is concentrated in degrees less than or equal to zero, it has a standard bounded t-structure [Kel12, Sect 4.2] whose heart $\mathcal{A}^0$ is isomorphic to the category $\operatorname{mod} J(Q, W)$ of nilpotent modules over the Jacobi algebra of the quiver with potential. If (Q, W) is Jacobi-finite, the simple modules of the heart $\mathcal{A}^0$ are the simple modules S_i of the Ginzburg algebra corresponding to each vertex. The S_i are so-called 3-spherical objects of the category $\mathcal{D}_{Q,W}$.

Definition 2.32. An object $E \in \mathcal{D}$ is n-spherical if $\operatorname{Hom}_{\mathcal{D}}^*(E, E) \cong H^*(S^n, \mathbb{C})$.

In a Calabi-Yau-n category, there is always an embedding $H^*(S^n, \mathbb{C}) \to$ $\operatorname{Hom}_{\mathcal{D}}^*(E, E)$ for any object E, so the spherical objects have the simplest possible Hom-complex. They provide autoequivalences of the category $\mathcal{D}$.

Theorem 2.33. *[ST01, Thm 1.2] For $E \in \mathcal{D}$ spherical, there exists an autoequivalence Tw_E of $\mathcal{D}$ called the* spherical twist at E, *such that*

$$\operatorname{Tw}_E(X) = \operatorname{Cone}(\operatorname{Hom}(E, X) \otimes E \xrightarrow{ev} X)$$

We will write $\operatorname{Sph}(\mathcal{D}_{Q,W})$ for the group of spherical twists of $\mathcal{D}_{Q,W}$ generated by the twists Tw_{S_i} at the simple objects of the standard heart $\mathcal{A}^0$. When the underlying graph of Q is a Dynkin diagram, Seidel and Thomas show that the spherical twists satisfy the corresponding braid relations.

Definition 2.34. Let Δ be a Dynkin diagram. Define the *braid group* $\operatorname{Br}(\Delta)$ to be the group generated by σ_i where i runs over the vertices of Δ subject to the relations

$$\sigma_i \sigma_j \ldots = \sigma_j \sigma_i \ldots$$

where the words on both sides have length m_{ij} given by

$$m_{ij} = \begin{cases} 2 & \text{if } i \text{ and } j \text{ are not connected} \\ 3 & \text{if } i \text{ and } j \text{ are connected by a single edge} \\ 4 & \text{if } i \text{ and } j \text{ are connected by a double edge} \\ 6 & \text{if } i \text{ and } j \text{ are connected by a triple edge} \\ \infty & \text{if } i \text{ and } j \text{ are connected by a pair of edges} \end{cases}$$

Suppose now that Δ is simply laced, that is has only single edges. A Δ-collection is a set of spherical objects $\{S_i\}$ for each vertex of Δ such that for distinct i and j, $\operatorname{Hom}_{\mathcal{D}}^*(S_i, S_j)$ is non-zero only if i and j are connected by a edge in which case it is one-dimensional and concentrated in either degree one or two. In particular the simple objects of the standard heart in the CY3 category $\mathcal{D}_Q = \mathcal{D}_{Q,0}$ for some orientation of Δ define a Δ-collection of spherical objects in $\mathcal{D}_Q$.

Theorem 2.35. *[ST01, Thm 1.2] A Δ-collection of spherical objects in $\mathcal{D}$ defines a weak $\operatorname{Br}(\Delta)$ action on $\mathcal{D}$.*

It is conjectured that this action is faithful, and this was proved in [ST01, Thm 1.3] for Dynkin diagrams of type A.

We can now give the result of Keller and Yang which categorifies mutations of quivers with potential via tilting in $\mathcal{D}_{Q,W}$ for a non-degenerate potential W. We first compute the simple objects of the simple tilts $\mathcal{A}_k^{\pm}$ of the standard heart $\mathcal{A}^0$.

Lemma 2.36. *[Kel11, Sect 4.3] The simple objects of $\mathcal{A}_k^+$ are $S_k[-1]$, the simple objects S_i of $\mathcal{A}$ with zero $\operatorname{Ext}^1(S_k, S_i)$ and the spherical twists $\operatorname{Tw}_{S_k}(S_i)$ of the remaining simple objects. The simple objects of $\mathcal{A}_k^-$ are obtained by applying Tw_{S_k} to the simple objects of $\mathcal{A}_k^+$.*

Proof. As the potential is non-degenerate, only one of the groups $\operatorname{Ext}^1(S_k, S_i) \cong \operatorname{Ext}^2(S_i, S_k)^*$ and $\operatorname{Ext}^1(S_i, S_k) \cong \operatorname{Ext}^2(S_k, S_i)^*$ is non-zero. Thus the universal extensions of the simple objects S_i in the statement of Proposition 2.9 are either $\operatorname{Tw}_{S_k}(S_i)$ or S_i according to whether the first or second group is non-zero. $\qquad \square$

The simple objects $\{S_i^+\}$ of the tilted heart $\mathcal{A}_k^+$ give a new basis of $K(\mathcal{D})$ which by Proposition 2.9 is given by

$$[S_k^+] = -[S_k] \qquad [S_i^+] = [S_i] + [\chi([S_i], [S_k])]_+ [S_k]$$

where $[x]_+$ denotes the positive part of x, i.e the minimum of 0 and x. The Ext-quiver of $\mathcal{A}_k^+$ is the mutation of the Ext-quiver of Q at k.

Theorem 2.37. *[KY11, Thm 3.2] There are a pair of equivalences*

$$\Phi_k^\pm : \mathcal{D}_{Q,W} \to \mathcal{D}_{\mu_k(Q,W)}$$

such that $\mathrm{Tw}_{S_k} \Phi_{S_k}^-$ *and* $\Phi_{S_k}^+$ *are isomorphic. Moreover* $\Phi_{S_k}^+(\mathcal{A}^0) = \mathcal{A}_{S_k}^+$ *and* $\Phi_{S_k}^-(\mathcal{A}^0) = \mathcal{A}_{S_k}^-$ *are the right and left simple tilts of* $\mathcal{A}^0$ *at* S_k.

We note that the Keller-Yang equivalences give a canonical bijection between the simple objects of the standard heart $\mathcal{A}^0$ and the simples of the tilted heart $\mathcal{A}_{S_k}^\pm$. Equivalently they give a canonical bijection of the vertices of the quivers $Q = Q(\mathcal{A}^0)$ and $\mu_k^\pm(Q) = Q(\mathcal{A}_{S_k}^\pm)$. We can lift the M_n-action on ordered quivers to an M_n-action on the set of ordered hearts of $\mathcal{D}_{Q,W}$ obtained from an ordering of the standard heart $\mathcal{A}^0$ by a finite sequence of tilts.

Definition 2.38. A heart $\mathcal{A}$ of $\mathcal{D}_{Q,W}$ is reachable if it can be obtained from the standard heart $\mathcal{A}^0$ by a finite sequence of simple tilts.

In other words reachable hearts are precisely the hearts which are images of the standard heart $\mathcal{A}^0$ under a composite of Keller-Yang equivalences of Theorem 2.37. An ordering of a reachable heart is an ordering of its simple objects or, equivalently, an ordering of the vertices of its quiver.

Definition 2.39. The set of ordered hearts H of a non-degenerate quiver with potential (Q, W) is the M_n-set of ordered reachable hearts of $\mathcal{D}_{Q,W}$ with M_n-structure given by simple tilts and permutations.

We note that autoequivalences Φ of $\mathcal{D}_{Q,W}$ which preserve the set of reachable hearts necessarily preserve the M_n-structure on H. We will study such autoequivalences via the induced automorphism $\Phi \in \mathrm{Aut}_{M_n}(H)$ of H.

Definition 2.40. An autoequivalence $\Phi \in \mathrm{Aut}(\mathcal{D}_{Q,W})$ is called reachable if the image $\Phi(\mathcal{A}^0)$ of the standard heart is obtained from $\mathcal{A}^0$ by a finite set of simple tilts.

We note that any reachable autoequivalence of $\mathcal{D}_{Q,W}$ necessarily preserves the quiver of a heart and so acts on H in a manner preserving the fibres of the map $H \to Q_n$. Examples of reachable autoequivalences are given by the spherical twists of the category $\mathcal{D}_{Q,W}$, for Lemma 2.36 implies that $\mathrm{Tw}_{S_k}(\mathcal{A}^0) = (\Phi_{S_k}^+)^2(\mathcal{A}^0)$. By Proposition 2.35, this yields an action of the braid group $\mathrm{Br}(Q)$ on the M_n-set of ordered reachable hearts H of $\mathcal{D}_{Q,W}$.

Remark 2.41. The above approach to categorification of mutations arose originally in the literature on cluster algebras. In particular the quotient of the set of ordered hearts H by the braid group action $\mathrm{Br}(Q)$ is isomorphic [Kel12, Theorem 5.6] to the M_n-set C of clusters with the cluster mutation rule of [FZ02]. The automorphism group $\mathrm{Aut}_{M_n}(C)$ has recently been studied by King and Pressland in [KP13].

In the sequel we will study the distinguished connected component of the space of stability conditions of $\mathcal{D}_{Q,W}$ which contains all stability conditions supported on the standard heart $\mathcal{A}_0$, and hence all stability conditions supported on reachable hearts.

Definition 2.42. The distinguished connected component $\mathrm{Stab}^*(\mathcal{D}_{Q,W})$ of the space of stability conditions of $\mathcal{D}_{Q,W}$ is the connected component of $\mathrm{Stab}(\mathcal{D}_{Q,W})$ containing the subset consisting of stability conditions supported on the hearts in H.

As we will study only the connected component $\mathrm{Stab}^*(\mathcal{D}_{Q,W})$ we will only be interested in the autoequivalences which preserve this component. Moreover we will factor out any autoequivalences which act trivially on $\mathrm{Stab}^*(\mathcal{D}_{Q,W})$.

Definition 2.43. The group $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$ is the subquotient of the autoequivalence group $\mathrm{Aut}(\mathcal{D}_{Q,W})$ consisting of those autoequivalences preserving the distinguished connected component $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$ modulo those acting trivially on it.

We note that reachable autoequivalences define elements of $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$ as they preserve H and so the distinguished component $\mathrm{Stab}^*(\mathcal{D}_{Q,W})$. This element is non-trivial if and only if it induces the trivial automorphism of H, for an autoequivalence fixing the simple objects of the standard heart $\mathcal{A}^0$ acts trivially on $\mathrm{Stab}^*(\mathcal{D}_{Q,W})$. In the next chapter we will outline a description of $\mathrm{Stab}^*(\mathcal{D}_{Q,W})$ and $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$ for almost all mutation-finite quivers with potential.

Chapter 3

Quadratic differentials

3.1 Mutation-finite quivers with potential

In this section we consider quivers with potential whose mutation-equivalence class is finite. We show, up to a small list of exceptions, that the set of finite mutation-equivalence classes is in bijection with the set of marked surfaces with boundary and that each quiver with potential corresponds to a unique ideal triangulation of the marked surface.

The almost bijection between mutation-finite quivers with potential and triangulations of marked surfaces with boundary has its origin in the theory of cluster algebras [FST08]. We begin by recalling the definition of a marked surface with boundary.

Definition 3.1. A marked surface with boundary (S, M) is a pair consisting of a smooth oriented surface S with possibly non-empty boundary ∂S, together with a non-empty set M of *marked* points of S such that each component of the boundary ∂S contains at least one point in M.

We will construct quivers with potential from triangulations of marked surfaces with boundary (S, M) whose vertices belong to the set of marked points M. The associated quiver with potential will only depend on the combinatorial type of the triangulation determined by the adjacency of its edges at vertices, and so is insensitive to an orientation-preserving diffeomorphism of the surface S which preserves the set of marked points M. This motivates the following slightly non-standard definition which is an equivalence class of ideal triangulations in the sense of [FST08] modulo the mapping class group $\mathrm{MCG}(S, M)$.

Definition 3.2. An ideal triangulation τ of (S, M) is an equivalence class modulo orientation-preserving diffeomorphisms of S preserving M of a maximal set of *edges*: real closed curves $\gamma : [0, 1] \to S$ with endpoints in M satisfying

- The interiors of the collection of edges do not intersect and are disjoint from M.

- Each edge is not null-homotopic relative to its endpoints in $S \backslash M$

- Each edge is not homotopic relative to its endpoints to a subset of the boundary ∂S whose interior is disjoint from M.

We note that the faces of an ideal triangulation are certainly three-sided for otherwise the collection of edges is not maximal. However it is possible that two of the edges bounding a given face coincide. The faces of a general ideal triangulation thus consist, in addition to usual triangles, of so-called self-folded triangles pictured below.
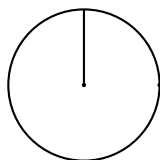
Figure 3.1: A self-folded triangle containing a double edge.

Definition 3.3. An ideal triangulation is non-degenerate if it contains no self-folded triangles.

Remark 3.4. In this chapter we will often work in the setting of non-degenerate ideal triangulations even though we require the extension of these ideas to encompass degenerate ideal triangulations. As in the literature on cluster algebras [FST08] this is achieved by considering so-called *tagged triangulations*. Where the exposition is simplified in this manner we make remarks regarding the correct generalisation and give references to the literature.

To an ideal triangulation we can associate an adjacency matrix recording how the edges of the ideal triangulation meet at vertices.

Definition 3.5. The adjacency matrix $B(\tau)$ of a non-degenerate ideal triangulation τ is given by the sum

$$B(\tau) = \sum_{\triangle} B^{\triangle}$$

over faces $\triangle$ of the ideal triangulation τ of the antisymmetric matrices $B^{\triangle}$ with entries

$$(b^{\triangle})_{ij} = \begin{cases} 1 & \text{if the edges } i \text{ and } j \text{ are edges of } \triangle \text{ in the clockwise order} \\ -1 & \text{if the edges } i \text{ and } j \text{ are edges of } \triangle \text{ in the anti-clockwise order} \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.6. For degenerate ideal triangulations we can apply the above definition having replaced each double edge of a self-folded triangle with the edge which encloses it.

An antisymmetric matrix B determines a quiver $Q(B)$ without loops or 2-cycles whose vertex set labelled by the rows (or columns) of B, and with b_{ij} arrows from i to j whenever b_{ij} is positive.

Definition 3.7. The quiver $Q(\tau)$ of an ideal triangulation τ is the quiver $Q(B(\tau))$ associated to its adjacency matrix $B(\tau)$.

The *flip* at an edge e of ideal triangulation τ is the unique ideal triangulation distinct from τ which contains all the edges of τ except e.
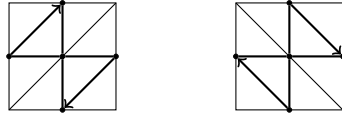
Figure 3.2: A flip of an ideal triangulation τ and mutation of the corresponding quiver $Q(\tau)$.

Lemma 3.8. *Suppose τ is a non-degenerate ideal triangulation of a marked surface with boundary with associated quiver $Q(\tau)$. Then the quiver associated to the flip of τ in one of its edges is the mutation of $Q(\tau)$ at the corresponding vertex.*

Remark 3.9. Flipping the double edge of a self-folded triangle in a degenerate ideal triangulation τ results in the same triangulation. On the other hand from Remark 3.6 the quiver $Q(\tau)$ associated to τ has a involution exchanging the vertices corresponding to the self-folded edge and the edge which encloses it. Thus the mutation of the quiver at the self-folded edge is equal to the quiver associated with the flip of the triangulation at the edge which encloses it. To extend the above Lemma to degenerate triangulations it is necessary to consider tagged triangulations, which treat the double edge and its enclosing edge on an equal footing.

As any two ideal triangulations are related by a sequence of flips we observe that the triangulations of any marked surface with boundary produce all quivers in an associated mutation-equivalence class. In fact almost all finite mutation-equivalence classes of quivers can be realised in this way.

Proposition 3.10. *[FST08] Suppose that a mutation-finite quiver has more than 2 vertices and is not mutation-equivalent to one of 11 exceptional quivers listed in [FST08]. Then its mutation-equivalence class is the set of quivers associated to a marked surface with boundary.*

In order to make use of the categorification of this mutation via tilting in CY3 triangulated categories outlined in Chapter 2 it is necessary to endow the quivers $Q(\tau)$ with a non-degenerate potential. With the exception of a finite number of marked surfaces with boundary this has been achieved by work of Labardini-Fragoso.

Theorem 3.11. *Suppose that (S, M) is not a closed genus zero surface with fewer than six punctures. There exists a non-degenerate Jacobi-finite quiver with potential associated to an ideal triangulation τ of (S, M) whose reduced part has the quiver $Q(\tau)$.*

If the ideal triangulation is non-degenerate the above quiver with potential can be constructed as follows:

- The quiver Q has arrows given by inscribing a clockwise-oriented triangle inside each face $\triangle$ of the triangulation and removing those incident to the boundary.

- The potential W is given by the sum

$$W = \sum_{\triangle} W_{\triangle} - \sum_{p} W_p$$

where $W_\triangle$ and W_p are the clockwise and anticlockwise oriented cycles of the quiver Q contained in the face $\triangle$ and surrounding the puncture p respectively.

We observe that the quiver Q is the same as the quiver $Q(\tau)$ provided that it contains no 2-cycles, which can arise if there are only two edges of the triangulation which are incident to a given puncture p. In this case there is a quadratic term W_p in the potential W so that the reduced quiver is equal to $Q(\tau)$. The non-degeneracy of the potential follows from showing that the quiver with potential associated to the flip of a triangulation is equal to the mutation of the quiver with potential at the corresponding vertex.

As mutation of quivers with potential is categorified via tilting of hearts in the CY3 categories $\mathcal{D}_{Q,W}$, we deduce that $\mathcal{D}_{Q,W}$ depends only on the mutation-equivalence class of (Q, W) for (almost all) finite mutation classes. Moreover we can use the relation to triangulations of marked surfaces with boundary to study the autoequivalence group $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$ for (almost all) mutation-finite quivers with potential. Indeed two hearts are related by an autoequivalence of $\mathcal{D}_{Q,W}$ if and only if they have the same quiver [GLFS13, Theorem 1.4], which occurs (with the exception of finitely many marked surfaces with boundary) if and only if they define the same triangulation [BS13, Proposition 8.5].

Bridgeland and Smith use the relation to the theory of cluster algebras to compute the group $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$. Indeed the quotient of the set of reachable hearts H by the action of the braid group $\mathrm{Br}(Q)$ by spherical twists is the set of clusters C. King and Pressland show that the automorphisms of the M_n-set C which preserve the quiver Q is isomorphic to the group of direct cluster automorphisms. In the case of a quiver associated to a triangulation of a marked surface (S, M) with boundary, this is known to be an extension of the mapping class group $\mathrm{MCG}(S, M)$.

Theorem 3.12. *[BS13, Thm 9.9] Suppose that (S, M) is an amenable marked surface with boundary in the sense of [BS13, Definition 9.3], i.e. it does not belong to a finite list of exceptional cases. Let (Q, W) be a quiver with potential associated to an ideal triangulation of the marked surface with boundary (S, M). There is a short exact sequence*

$$1 \to \mathrm{Sph}^*(\mathcal{D}_{Q,W}) \to \mathrm{Aut}^*(\mathcal{D}_{Q,W}) \to \mathrm{MCG}^\pm(S, M) \to 1$$

where $\mathrm{MCG}^\pm(S, M)$ is the signed mapping class group: the semi-direct product of the usual mapping class group $\mathrm{MCG}(S, M)$ acting on the group $\oplus_p \mathbb{Z}_2$ via its action on the set of punctures.

Remark 3.13. The extension $\mathrm{MCG}^\pm(S, M)$ of the mapping class by a permutation group in two letters at each puncture is natural in the context of tagged triangulations. Indeed a tagged triangulation may be viewed as an equivalence class of *signed triangulations* consisting of the data of a ideal triangulation together with a choice of sign at each puncture. Two signed triangulations are equivalent if they differ by changing the sign at punctures contained in self-folded triangles [BS13, Lemma 8.2], which may be thought of as exchanging the role of the double edge and the edge which encloses it.

Remark 3.14. King and Pressland [KP13, Corollary 7.3] show that for a mutation-finite quiver Q, the automorphisms of C which preserve the quiver Q are a subgroup of index 2 in $\mathrm{Aut}_{M_n}(C)$. The full group of automorphisms $\mathrm{Aut}_{M_n}(C)$ is

isomorphic to the group of indirect cluster transformations which preserve the quiver up to reversing the orientation of all its arrows. It is isomorphic to the extended signed mapping class group of (S, M) whose elements are the isotopy classes of (not necessarily orientation-preserving) diffeomorphisms of (S, M) together with a choice of sign at each puncture.

Correspondingly one could consider $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$ as an index two subgroup of the group of (anti-)autoequivalences of the category $\mathcal{D}_{Q,W}$ which in addition contains equivalences $\mathcal{D}_{Q,W} \to \mathcal{D}_{Q^{\mathrm{op}}, W^{\mathrm{op}}}$ where $(Q^{\mathrm{op}}, W^{\mathrm{op}})$ is the opposite quiver with potential obtained by reversing all arrows. An anti-autoequivalence acts as a reflection on $N(\mathcal{D}_{Q,W})$ and by -1 on the Euler form of $\mathcal{D}_{Q,W}$. This point of view is sometimes helpful in considering the autoequivalences of the Painlevé quivers as in Chapter 7.

3.2   Trajectories of quadratic differentials

In this section we consider the stratum of the space of meromorphic quadratic differentials on the natural compactification C of the marked surface with boundary (S, M) having poles specified by the marking M and only simple zeroes. We show how the trajectories of a generic quadratic differential on C in this stratum define an ideal triangulation of the associated marked bordered surface.

A holomorphic quadratic differential on a Riemann surface is a section of the second tensor power of its canonical line bundle. Holomorphic quadratic differentials on Riemann surfaces are much studied objects for, by Serre duality, they define cotangent vectors to the moduli space of complex structures on the underlying smooth surface. In this thesis we will consider meromorphic quadratic differentials with fixed orders of poles at specified marked points of the Riemann surface.

More precisely we will consider smooth oriented surfaces C endowed with an integer-weighted marking P: a collection of points $p_i \in C$ together with positive integers $m_i \in \mathbb{N}$. The data of a meromorphic quadratic differential on (C, P) will include a choice of complex structure on the surface C with the marked points p_i. Having chosen such a complex structure it will be convenient to let P denote the effective divisor $\sum_i m_i p_i$ on the Riemann surface which we continue to denote by the letter C.

Definition 3.15. A meromorphic quadratic differential u on (C, P) is a pair of a complex structure on (C, P) together with a holomorphic section $u \in H^0(C, \mathcal{K}_C^{\otimes 2}(P))$.

Viewing the set of meromorphic quadratic differentials on (C, P) as a vector bundle over the moduli space of complex structures $\mathcal{M}_{C,P}$ on the pair (C, P), it has the structure of a complex orbifold of dimension

$$\dim \mathcal{M}_{C,P} + \dim H^0(C, \mathcal{K}_C^{\otimes 2}(P)) = (3g(C) - 3 + |P|) + (3g(C) - 3 + \deg(P))$$
$$= 6g(C) - 6 + |P| + \deg(P)$$

This space is stratified by the orders of the zeroes and poles of the meromorphic quadratic differential on (C, P). We will be interested in the unique top dimensional stratum consisting of meromorphic quadratic differentials on C with poles of order precisely m_i at the marked points p_i and simple zeroes. It is

known that the top-dimensional stratum is connected by [KZ03] and it inherits the orbifold structure defined above.

Definition 3.16. The complex orbifold $\mathrm{Quad}(C, P)$ is the top-dimensional stratum of the space of meromorphic quadratic differentials on (C, P), for which the section $u \in H^0(C, \mathcal{K}_C^{\otimes 2}(P))$ has only simple zeroes which are disjoint from the set of marked points P.

We now associate to a marked surface with boundary (S, M) a pair (C, P) for which quadratic differentials in the stratum $\mathrm{Quad}(C, P)$ will be used to construct ideal triangulations of (S, M). We note that all points in P have weight at least two, so that the quadratic differentials in $\mathrm{Quad}(C, P)$ never have any simple poles.

Definition 3.17. The smooth oriented surface C with integer-weighted marking P associated with a marked surface with boundary (S, M) is the compactification of S obtained by contracting each boundary component to a point, together with the weighted marking consisting of:

- The punctures of (S, M) with weight two.

- The points of C corresponding to the boundary components S with weight two plus the number of marked points on the corresponding boundary component.

Our goal is to construct a triangulation of the marked surface (S, M) with boundary from a generic quadratic differential in the stratum $\mathrm{Quad}(C, P)$. Any meromorphic quadratic differential u on C determines a Riemannian metric on C away from its zeroes and poles by considering the absolute value of the bilinear form $u : TC \otimes TC \to \mathbb{C}$. Geodesics for this metric necessarily have a constant phase with respect to the quadratic differential u.

Proposition 3.18. *A trajectory of a meromorphic quadratic differential u on C is a maximal horizontal geodesic $\gamma : (0, 1) \to C \backslash P$ with respect to the metric defined by u on the complement of its zeroes and poles on C.*

We call the foliation of the surface C by trajectories the *horizontal foliation* of the meromorphic quadratic differential u on C. It is regular away from the zeroes and poles of the quadratic differential on whose complement the induced metric is defined. Some motivation for considering the stratum of quadratic differentials $\mathrm{Quad}(C, P)$ on C associated to (S, M) is provided by considering the local structure of the trajectories at these critical points.

Proposition 3.19. *[BS13, Section 3.3] Suppose that u is a meromorphic quadratic differential on C with a zero or pole at p and let z be a local coordinate centred at p. The quadratic differential u has the form*

$$h(z)\frac{dz^{\otimes 2}}{z^n}$$

for some holomorphic function h and integer n given by the order of vanishing of u at p, so that u has a zero or pole of order $|n|$ according to whether n is negative or positive. Then the horizontal foliation of u at p has the following properties.

$n < 2$ *There are $2 - n$ trajectories of u incident to p whose asymptotic directions differ by multiples of $\frac{2\pi}{2-n}$.*

$n = 2$ *There are either no trajectories incident to p or infinitely many according to whether $h(0)$ is real or not.*

$n > 2$ *There are infinitely many trajectories incident to p along $n - 2$ asymptotic directions whose angles differ by $\frac{2\pi}{n-2}$.*

Recall that a boundary component of a marked surface with boundary (S, M) gives rise to a point p in the associated pair (C, P) with weight $m > 2$. Moreover the number of marked points on the boundary component is equal to the number of asymptotic directions of trajectories incident to p identified in the above Proposition. In particular we can draw the horizontal foliation of a quadratic differential $u \in \mathrm{Quad}(C, P)$ on the marked surface with boundary (S, M). We draw each trajectory with endpoint a pole of order greater than two being drawn as incident to the marked point on the corresponding boundary component associated to its asymptotic direction.

For meromorphic quadratic differentials $u \in \mathrm{Quad}(C, P)$ belonging to the stratum associated to a marked surface with boundary (S, M) the only critical points of the foliation belonging to the first case of the Proposition are the simple zeroes, for there are no simple poles. The Proposition shows that there are precisely three trajectories of the quadratic differential u incident to each simple zero. For a generic quadratic differential in $\mathrm{Quad}(C, P)$ we will see that the zeroes are faces of a triangulation for which the three vertices are the endpoints of the three trajectories leaving the zero.

From this point forward we assume that (C, P) arises from a marked surface (S, M) with boundary so that the critical points of the horizontal foliation $u \in \mathrm{Quad}(C, P)$ are all simple poles or poles of order greater than two. We will say that a trajectory γ of a meromorphic quadratic differential u has a left or right endpoint respectively if the limits of $\gamma(t)$ as $t \to 0$ and $t \to 1$ exist. By definition such endpoints must belong to the set of zeroes and poles of the meromorphic quadratic differential u.

Definition 3.20. A trajectory of a meromorphic quadratic differential $u \in \mathrm{Quad}(C, P)$ is called

- *finite* if its endpoints are both zeroes.

- *separating* if its endpoints are a zero and a pole.

- *infinite* if its endpoints are both poles.

We note that finite trajectories have finite length with respect to the metric induced by the quadratic differential u, whereas the separating and infinite trajectories have infinite length, for poles of order greater than two are at infinite distance in the metric. A priori there is no reason to expect that both endpoints of a given trajectory exist. Gaiotto, Moore and Neitzke [GMN09, Section 6.3] discovered the strong restriction on the global structure of the horizontal foliation that in the absence of finite trajectories all trajectories have both endpoints.
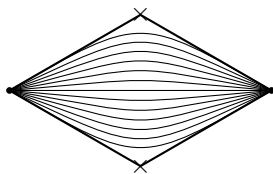
Figure 3.3: The foliation of a non-degenerate face of the dimer model where ●
is a marked point and × is a zero.

Proposition 3.21. *[BS13, Lemma 3.1] Suppose a quadratic differential* $u \in$
$\mathrm{Quad}(C, P)$ *has no finite trajectories. Then it has only infinite and separat-*
ing trajectories, and the separating trajectories divide the marked surface with
boundary (S, M) *into cells of two types:*

- Horizontal strips*: A quadrilateral of separating trajectories connecting a*
  pair of (possibly coincident) zeroes to a pair of distinct marked points as
  in Figure 3.2.

- Half-planes*: Bounded by two separating trajectories connecting a zero to*
  two consecutive marked points on a boundary component of S.

We will say that a quadratic differential is *generic* if it has no finite trajecto-
ries. This terminology is justified as the set of quadratic differentials with finite
trajectories is a real codimension 1 locus in $\mathrm{Quad}(C, P)$. Indeed a given finite
trajectory does not persist if the quadratic differential u is rotated to $e^{i\theta}u$ for
small values of the angle θ. We will study the components of this real codimen-
sion 1 locus and the jumps the graph of separating trajectories can undergo in
the next section.

The graph of separating trajectories of a quadratic differential $u \in \mathrm{Quad}(C, P)$
on (S, M) is a bipartite graph with vertex set the set of zeroes of the quadratic
differential u together with the set of marked points M of the surface S. We
note that the horizontal strips only intersect the boundary of S in the set of
marked points M, whereas the half-planes contain a component of $\partial S \backslash M$. It
is convenient to consider the subgraph which does not contain these boundary
faces.

Definition 3.22. The dimer model of a generic quadratic differential $u \in$
$\mathrm{Quad}(C, P)$ is the bipartite graph on (S, M) given by the subgraph of its graph
of separating trajectories consisting of those edges contained in at least one
horizontal strip.

The faces of the dimer model are all quadrilaterals although it is possible
that the edges of a face might not all be distinct, for it is possible that the two
zeroes in a horizontal strip coincide. In this case we say that the horizontal strip
is *self-folded*.

Definition 3.23. The dimer model of a quadratic differential is degenerate if
it contains a self-folded face.

Using Euler's formula it is easy to calculate the number of horizontal strips
of a generic quadratic differential $u \in \mathrm{Quad}(C, P)$. It is remarkable that the

number of faces of the dimer model is equal to the dimension of this stratum of the space of quadratic differentials.

Lemma 3.24. *[BS13, Lemma 3.2] The number of horizontal strips of a generic quadratic differential $u \in \mathrm{Quad}(C, P)$ is $6g(C) - 6 + |P| + \deg(P)$.*

We can define the so-called WKB triangulation of the surface S with vertices in M by considering the graph whose edges are given by choosing a single infinite trajectory inside each horizontal strip. The fact that the zeroes are trivalent vertices of the graph of separating trajectories ensures that the faces of this graph are indeed triangles. Moreover the fact that the number of edges of the WKB triangulation is equal to the number of horizontal strips can be used to show that it is in fact an ideal triangulation.

Proposition 3.25. *[BS13, Lemma 10.1] The WKB triangulation of a generic quadratic differential $u \in \mathrm{Quad}(C, P)$ is an ideal triangulation of the marked surface with boundary (S, M). Moreover it is degenerate if and only if the dimer model of u is.*

By Theorem 3.11 there is a quiver with potential (Q, W) associated to the WKB triangulation. If the WKB triangulation and dimer model are non-degenerate, the quiver Q coincides with the dual graph to the dimer model equipped with a natural orientation so that the cycles are oriented clockwise around every zero and anti-clockwise around every puncture. The dual of the whole graph of separating trajectories including its boundary faces is the quiver obtained by inscribing a clockwise-oriented 3-cycle inside each face of the triangulation.

One can also consider the dual graph to the WKB triangulation which we shall call the *ribbon graph*. Its vertices are the zeroes of the quadratic differential and two zeroes are connected by an edge if and only if there is a horizontal strip which contains them both. There is a natural embedding of the ribbon graph in the surface C where each edge is mapped to the unique (non-horizontal) geodesic of the metric induced by the quadratic differential contained in the given horizontal strip. If the triangulation is non-degenerate then the ribbon graph contains no loops.

In summary we have defined four graphs on the surface (S, M) associated to a generic quadratic differential $u \in \mathrm{Quad}(C, P)$: the dimer model, quiver, (WKB) triangulation and ribbon graph. Moreover if the dimer model is non-degenerate then they form two pairs of dual graphs in the rows of the following diagram. We have also indicated on the dashed arrows the bijections between vertices, edges and faces which exist between the corresponding graphs.

$$
\begin{array}{ccc}
\text{Dimer model} & \longleftrightarrow & \text{Quiver} \\
\text{f}\uparrow \quad {}^{\nwarrow}{}_{\text{-}}{}_{\text{-}}\text{v} \quad \text{f}{}_{\text{-}}{}_{\text{-}}{}^{\nearrow} \quad \uparrow\text{v} \\
\text{e}\downarrow \quad {}_{\swarrow}{}^{\text{-}}{}^{\text{-}}\text{e} \quad \text{e}{}^{\text{-}}{}^{\text{-}}{}_{\searrow} \quad \downarrow\text{e} \\
\text{Triangulation} & \longleftrightarrow & \text{Ribbon graph}
\end{array}
$$
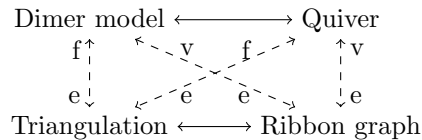
3.3 Quadratic differentials as stability conditions

In this section we show how to interpret quadratic differentials as stability conditions following Bridgeland and Smith [BS13]. We consider the wall-and-chamber

structure on the stratum $\mathrm{Quad}(C, P)$ whose walls are given by the codimension 1 loci along which the graph of separating trajectories jumps. Each chamber defines a quiver with potential (Q, W), and therefore a heart of the category $\mathcal{D}_{Q,W}$ up to an element of the autoequivalence group $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$. Given a $u \in \mathrm{Quad}(C, P)$ the lengths and phases of the geodesic embedding of the edges of the ribbon graph define a stability condition up to autoequivalence which is supported on the corresponding heart.

We have seen in the previous section that the number of horizontal strips of a generic quadratic differential $u \in \mathrm{Quad}(C, P)$ is equal to the dimension of this stratum of meromorphic quadratic differentials as a complex orbifold. In fact there is a system of local coordinates on $\mathrm{Quad}(C, P)$ associated to any ideal triangulation of (S, M) given by the spans of each horizontal strip of the corresponding dimer model.

Definition 3.26. The span of a horizontal strip is given by the integral

$$\int_e \sqrt{u} \in \mathfrak{h}$$

along the geodesically-embedded edge e of the ribbon graph contained in the horizontal strip of the square root of the quadratic differential u, where the sign is chosen such that the width belongs to $\mathfrak{h}$

We can choose a local coordinate z on C on which u restricted to the interior of the horizontal strip has the canonical form $dz^{\otimes 2}$. Moreover by transformations of the form $z \mapsto \pm z + \mathrm{cst}$ to which $dz^{\otimes 2}$ is insensitive we can arrange that the two zeros contained in the boundary of the horizontal strip occur at $z = \{0, w\}$ where $w \in \mathfrak{h}$ is the span of the horizontal strip. Bridgeland and Smith [BS13, Proposition 4.9] show how to construct a quadratic differential $u \in \mathrm{Quad}(C, P)$ with a given dimer model and arbitrary spans of the horizontal strips.

As the imaginary part of the span of a horizontal strip goes to zero, the horizontal strip degenerates onto the line $\Im(z) = 0$ whose span necessarily becomes a finite trajectory as the two zeroes cannot coincide. This finite trajectory connects the two zeroes which are the endpoints of the edge e of the ribbon graph contained in the degenerating horizontal strip. The non-generic quadratic differentials containing a finite trajectory connecting a pair of zeroes belong to the real codimension 1 locus in $\mathrm{Quad}(C, P)$ given by the condition

$$\int_e \sqrt{u} \in \mathbb{R}$$

Gaiotto, Moore and Neitzke [GMN09, Section 6.6] classify the real codimension 1 components of the locus of non-generic quadratic differentials in the stratum $\mathrm{Quad}(C, P)$ into three types of wall: *flips*, *pops* and *juggles*. If the triangulation of a generic $u \in \mathrm{Quad}(C, P)$ is non-degenerate then the wall corresponding to the above degeneration belongs to the class of flips. It is so-called because generic quadratic differentials on the other side of the wall have triangulations which differ by a flip in the edge contained in the degenerating horizontal strip.

In the degenerate case we must also consider pop walls where a self-folded horizontal strip degenerates to a finite trajectory which is a loop e from the unique zero of the horizontal strip to itself. The quadratic differentials on both

Dimer model	Quiver	Triangulation	Ribbon graph
Urban renewal	Mutation	Flip	Whitehead move

Table 3.1: Jumps in the four graphs of a generic quadratic differential associated to a flip wall in $\mathrm{Quad}(C, P)$

sides of a pop wall define the same degenerate WKB triangulation. Indeed we have already observed in Remark 3.9 that the flip of the double edge of a self-folded triangle does not change the ideal triangulation.

Any point on a juggle wall contains an open neighbourhood which contains infinitely many flip walls. Moreover Bridgeland and Smith show that removing the set of juggle walls from $\mathrm{Quad}(C, P)$ does not disconnect the space. In particular the flip walls define a wall-and-chamber structure on the connected dense open subset of $\mathrm{Quad}(C, P)$ given by the complement of the juggle walls. Within each chamber the generic quadratic differentials define the same WKB triangulation which undergoes a flip on passing through any of its walls. The jump in the other graphs defined by the quadratic differential is given in Table 3.1.

Bridgeland and Smith further show that a quadratic differential $u \in \mathrm{Quad}(C, P)$ defines a stability condition in the distinguished connected component $\mathrm{Stab}^*(\mathcal{D}_{Q,W})$ up to an element of the autoequivalence group $\mathrm{Aut}^*(\mathcal{D}_{Q,W})$. The chambers in $\mathrm{Quad}(C, P)$ are labelled by the quivers with potential in the mutation-equivalence class of (Q, W) and so define a heart of $\mathcal{D}_{Q,W}$ up to an autoequivalence. The spans of the horizontal strips of the quadratic differential define a central charge, associating to each vertex of the quiver the number

$$\int_e \sqrt{u} \in \mathfrak{h}$$

Theorem 3.27. *[BS13, Theorem 1.3] Suppose that (Q, W) is a quiver with potential associated to the marked surface with boundary (S, M). Then there is an isomorphism of orbifolds*

$$\mathrm{Quad}(S, M) \to \mathrm{Stab}^*(\mathcal{D}_{Q,W})/\mathrm{Aut}^*(\mathcal{D}_{Q,W})$$

where $\mathrm{Quad}(S, M)$ is the completion of the stratum $\mathrm{Quad}(C, P)$ by admitting quadratic differentials with either simple or double poles at the punctures of (S, M).

Bridgeland and Smith further describe the set of semistable objects of the category $\mathcal{D}_{Q,W}$ with respect to the stability condition associated to a generic quadratic differential $u \in \mathrm{Quad}(C, P) \subset \mathrm{Quad}(S, M)$. The stable objects in $\mathcal{D}_{Q,W}$ of a given phase θ correspond to the finite length geodesics of phase θ of the metric on $C \backslash P$ induced by the quadratic differential u whose endpoints are zeroes of u. This can be viewed as a mathematical formulation of the string-theoretic description of the BPS states of the gauge theories of class S in [GMN09].

As with the simplest gauge theories of this class considered by Seiberg and Witten, it is convenient to encode the lengths and phases of the geodesics in terms of the periods of a meromorphic differential on an auxiliary Riemann surface.

Definition 3.28. The spectral curve Σ_u of a quadratic differential $u \in \mathrm{Quad}(S, M)$ is the double cover $\Sigma_u \to C$

$$\{y^2 = u\} \subset \mathcal{K}_C(\lceil \tfrac{P}{2} \rceil)$$

where $\lceil \tfrac{P}{2} \rceil = \sum_i \lceil \tfrac{m_i}{2} \rceil p_i$.

The twisting of the cotangent bundle by $\lceil \tfrac{P}{2} \rceil$ ensures that the function y has only simple zeroes on Σ_u located at the branch points of the double cover given by the simple zeroes and odd-order poles of u. In particular the spectral curves Σ_u are smooth and topologically equivalent for $u \in \mathrm{Quad}(S, M)$.

There is a natural decomposition of the homology of the spectral curve Σ into the positive and negative eigenspaces of the induced action of the involution $\iota : \Sigma \to \Sigma$ which exchanges the two sheets of the spectral curve

$$H_1(\Sigma, \mathbb{Z}) = H_1(\Sigma, \mathbb{Z})^+ \oplus H_1(\Sigma, \mathbb{Z})^-$$

The edges e of the ribbon graph of a quadratic differential $u \in \mathrm{Quad}(C, P)$ lift to define 1-cycles γ_e on the spectral curve Σ which are anti-invariant with respect to the involution ι. The edges of the ribbon graph are in bijection with the vertices of the associated quiver with potential (Q, W) which label the simple objects S_e of the standard heart of the category $\mathcal{D}_{Q,W}$. We can therefore define a map $K(\mathcal{D}_{Q,W}) \to H_1(\Sigma, \mathbb{Z})^-$ which sends the basis of the Grothendieck group $K(\mathcal{D}_{Q,W})$, given by the classes of the simple objects $[S_e]$ of the standard heart, to the homology classes $[\gamma_e]$.

One can show that this map is independent of the bases determined by the choice of quadratic differential and respects the two antisymmetric bilinear forms given by the Euler form on $\mathcal{D}_{Q,W}$ and the non-degenerate intersection form on $H_1(\Sigma, \mathbb{Z})^-$.

Proposition 3.29. *The map $K(\mathcal{D}_{Q,W}) \to H_1(\Sigma, \mathbb{Z})^-$ factors through the numerical Grothendieck group $N(\mathcal{D}_{Q,W})$ inducing an isomorphism $N(\mathcal{D}_{Q,W}) \to H_1(\Sigma, \mathbb{Z})^-$.*

There is a canonical 1-form on the spectral curve Σ_u obtained by restriction of the tautological 1-form on the twisted cotangent bundle $\mathcal{K}_C(\lceil \tfrac{P}{2} \rceil)$ of C. This 1-form on Σ_u has the propery that its square agrees with the pullback of the quadratic differential u along the double cover $\Sigma_u \to C$.

Definition 3.30. The Seiberg-Witten differential λ_u on the spectral curve Σ_u is the meromorphic 1-form defined by $\sqrt{u}$.

We note that the integrals of the Seiberg-Witten differential $\lambda_u = \sqrt{u}$ around the cycles γ_e of Σ_u are equal to the spans of the corresponding horizontal strips. Thus a quadratic differential defines a numerical stability condition if and only if its Seiberg-Witten differential has zero residues, so that the period map

$$\int_\bullet \lambda_u : H_1(\Sigma, \mathbb{Z})^- \to \mathbb{C}$$

is well-defined.

Definition 3.31. A quadratic differential $u \in \mathrm{Quad}(S, M)$ is residueless if its Seiberg-Witten differential λ_u has zero residues.

We will denote by $\mathrm{Quad}^0(C, P) \subset \mathrm{Quad}(S, M)$ the subset of residueless quadratic differentials. As the residue of the Seiberg-Witten differential λ_u has opposite residues at the pre-images of the poles of even order of the quadratic differential u, the dimension of $\mathrm{Quad}^0(C, P)$ is

$$6g(C) - 6 + |P|_{odd} + \deg(P)$$

In Chapter 6 we will study the spaces of residueless quadratic differentials of dimension two.

Chapter 4

Picard-Fuchs equations of families of elliptic curves

4.1 Differential equations

In this section we consider linear homogeneous second-order differential equations on the complex plane with only regular singularities. At a non-singular point there exists a neighbourhood in which the solutions of the differential equation are analytic. A singular point is regular if the solutions grow at most polynomially in a neighbourhood of the singular point with rate of growth specified by the so-called characteristic exponents. We study the relation between the characteristic exponents, the monodromy of the solutions and their behaviour under pullback by a rational map.

Definition 4.1. A second-order linear homogeneous differential equation in the complex plane is an equation of the form

$$\frac{d^2 f}{dt^2} + r(t)\frac{df}{dt} + s(t)f = 0$$

for some meromorphic functions r and s of t.

A point t_0 in the complex plane at which the functions r and s do not have poles is called a *non-singular point* of the differential equation. We can find power series solutions of the differential equation with arbitrary $f(t_0)$ and $f'(t_0)$ which converge in a neighbourhood of t_0 to holomorphic functions. Such solutions can be analytically continued into at least the complement of the finite set S of *singular* points in $\mathbb{P}^1_t$ where at least one of r and s has a pole.

A general solution to the differential equation in $\mathbb{P}^1_t \backslash S$ can be written as a linear combination of a pair of solutions f_1 and f_2 if the vectors

$$\begin{pmatrix} f_1(t_0) \\ f_1'(t_0) \end{pmatrix} \qquad \begin{pmatrix} f_2(t_0) \\ f_2'(t_0) \end{pmatrix}$$

are linearly independent for some and hence all $t_0 \in \mathbb{P}^1_t \backslash S$. This is equivalent to the non-vanishing of the Wronskian

$$W(f_1, f_2) = \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}$$

on $\mathbb{P}^1_t \backslash S$. A pair $\{f_1, f_2\}$ satisfying this condition is called a *fundamental set of solutions* of the differential equation and provides a basis for the rank 2 local system of solutions to the differential equation over $\mathbb{P}^1 \backslash S$.

Definition 4.2. Given a fundamental set of solutions to a second-order differential equation, the monodromy matrix at a singular point $t_0 \in S$ is the matrix $M(t_0)$ representing the monodromy around t_0 in the local system of solutions with respect to this basis.

A *regular singularity* of the differential equation is characterised by the fact that the solutions are bounded by some fixed power of the local coordinate. In order to give a definition of regular singularity based solely on the differential equation itself, it is convenient to rewrite the differential equation using the differential operator $\vartheta_t = t\frac{d}{dt}$.

$$\vartheta^2 f + p\vartheta f + qf = 0$$

We note that this differential operator has the advantage that it is invariant under the transformation $t \mapsto 1/t$ and so we obtain a uniform characterisation which includes the point at infinity.

Definition 4.3. A point $t_0 \in S$ is regular if both coefficients p and q are analytic at t_0.

The asymptotic behaviour of a fundamental set of solutions at a singular point t_0 is governed by the *characteristic exponents*. The second-order differential equation can be rewritten in matrix form

$$\vartheta \left(\begin{array}{c} f \\ \vartheta f \end{array} \right) = \left(\begin{array}{cc} 0 & 1 \\ q & p \end{array} \right) \left(\begin{array}{c} f \\ \vartheta f \end{array} \right)$$

At a regular singular point t_0 we can evaluate the matrix A of coefficients at t_0 to obtain a matrix with complex entries called the residue matrix at t_0.

Definition 4.4. The characteristic exponents at a regular singular point $t_0 \in S$ are the eigenvalues of the residue matrix $A(t_0)$.

The characteristic exponents λ_1 and λ_2 determine the growth rates of a fundamental set of solutions at t_0. Indeed, as in e.g. [Yos87, Section 2.5], given a characteristic exponent λ one can postulate a solution of the form

$$(t - t_0)^\lambda \sum_{i=0}^{\infty} a_i (t - t_0)^i$$

As λ is an eigenvalue of the residue matrix $A(t_0)$, this satisfies the differential equation up to terms of higher order than λ in the local coordinate $t - t_0$ for arbitrary a_0. The coefficients a_i satisfy a linear recursion which can be solved to determine the remaining coefficients uniquely.

The solutions of the above form associated to the two characteristic exponents are linearly independent provided λ_1 and λ_2 do not differ by an integer. In the so-called *non-resonant case* where the monodromy matrix $M(t_0)$ and the matrix differential operator $\vartheta I - A$ can be diagonalised, a fundamental set of solutions is given by

$$(t - t_0)^{\lambda_1} h_1(t) \qquad (t - t_0)^{\lambda_2} h_2(t)$$

for some holomorphic functions $h_i(t)$ which do not vanish at t_0.

In the *resonant* case where $M(t_0)$ is not diagonalisable, the matrix differential equation cannot necessarily be diagonalised in a neighbourhood of t_0 as the two power series solutions are not necessarily linearly independent. In this case a fundamental set of solutions has the form

$$(t - t_0)^{\lambda_1} h_1(t) \log(t - t_0) \qquad (t - t_0)^{\lambda_2} h_2(t)$$

where $\lambda_1 \geq \lambda_2$.

We can also study the monodromy in the local system of solutions at a regular singular point $t_0 \in S$. Define the monodromy matrix $M(t_0)$ to be the matrix associated with the change in basis in the local system given by transporting the fundamental set of solutions around t_0.

Proposition 4.5. *[Yos87, Section 9.1] The eigenvalues of the monodromy operator $M(t_0)$ are the exponentials $e^{2\pi i \lambda_1}$ and $e^{2\pi i \lambda_2}$ of the characteristic exponents (λ_1, λ_2)*

We note that the eigenvalues of the monodromy matrix are equal precisely when the characteristic exponents differ by an integer. In this case there is resonance if and only if the monodromy matrix cannot be diagonalised.

In the following chapters we will often pull back a differential equation along a map $\mathbb{P}^1_u \to \mathbb{P}^1_t$. The characteristic exponents at a point u_0 in the preimage of a singular point t_0 can be computed from the degree of ramification at u_0. [1]

Lemma 4.6. *[Dor01, Lemma 4.2] Suppose $f : \mathbb{P}^1_u \to \mathbb{P}^1_t$ has degree of ramification d at a point u_0 in the preimage of a regular singular point t_0 of a differential equation on $\mathbb{P}^1_t$. Then u_0 is a regular singular point of the pulled-back equation and the characteristic exponents at u_0 are obtained by multiplying the characteristic exponents of the original equation by d.*

Via Proposition 4.5 this agrees with the monodromy of the pullback of the local system of solutions to the differential equation, for the monodromy at u_0 is the dth power of the monodromy at t_0. The pulled-back differential equation has additional singularities at points u in the preimage of the non-singular locus $\mathbb{P}^1_t \backslash S$ of the original differential equation with order of ramification greater than 1. Such singular points are necessarily *apparent* singular points around which the monodromy is the identity.

It is useful to think of a non-singular point $t_0 \in \mathbb{P}^1 \backslash S$ as having characteristic exponents 0 and 1 as one can check that the differential equation has matrix form

$$\vartheta \begin{pmatrix} f \\ \vartheta f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (t - t_0)^2 r & (t - t_0)s + 1 \end{pmatrix} \begin{pmatrix} f \\ \vartheta f \end{pmatrix}$$

for some r and s holomorphic at t_0. The eigenvalues of the coefficient matrix evaluated at t_0 are then 0 and 1. Adopting this convention, we can apply Lemma 4.6 equally to the apparent singularities in the preimage of a non-singular point, which thus have characteristic exponents 0 and d.

It is convenient to encode all of the data concerning the singularities of a differential equation with only regular singularities in the so-called *Riemann scheme*. This is a table listing the points $t^{(i)}$ in the singular set S together with their characteristic exponents $\lambda_1^{(i)}, \lambda_2^{(i)}$

---

[1] Our convention is that the degree of ramification of the map $x \to x^d$ at the origin is d as opposed to $d - 1$ which we call the degree of *excess* ramification.

$t^{(1)}$	$t^{(2)}$	$\ldots$	$t^{(n)}$
$\lambda_1^{(1)}$	$\lambda_1^{(2)}$	$\ldots$	$\lambda_1^{(n)}$
$\lambda_2^{(1)}$	$\lambda_2^{(2)}$	$\ldots$	$\lambda_2^{(n)}$

Table 4.1: The Riemann scheme of a second-order differential equation with regular singularities

The criterion for the existence of a differential equation with a given Riemann scheme is the Fuchs identity concerning the sum of the characteristic exponents. We notice that we can add any number of non-singular points with characteristic exponents 0 and 1 to the Riemann scheme without violating the identity.

Proposition 4.7. *[Yos87, Section 2.6] The table 4.1 is the Riemann scheme of a second-order linear homogeneous differential equation with only regular singularities if and only if the Fuchs identity*

$$\sum_{i=1}^{n} (\lambda_1^{(i)} + \lambda_2^{(i)}) = n - 2$$

is satisfied.

In general there are continuous families of differential equations with a given Riemann scheme. In the above situation the differential equation depends additionally on $n - 3$ so-called *accessory parameters*. We note the special case $n = 3$ where the differential equation is determined uniquely by its Riemann scheme. We shall study second-order differential equations with exactly three regular singularities in more detail in Section 4.4.

4.2   Elliptic surfaces

In this section we study families of elliptic curves over the projective line $\mathbb{P}^1$. Examples will be provided by spectral curves of one-dimensional families of quadratic differentials from the previous chapter. The j-invariant of a fibre of such a surface yields a j-map from the projective line to itself which determines the elliptic surface up to a so-called quadratic twist. We show how to determine the singular fibres of an elliptic surface using only the branching of the j-map and quadratic twists.

By a family of elliptic curves over the projective line, we mean a morphism $E \to \mathbb{P}^1$ whose generic fibre is a smooth elliptic curve over the field of rational functions in one variable. The total space of such a family is known in the literature as a Jacobian elliptic surface over the projective line, which we call simply an elliptic surface. In particular there is a section of the morphism $E \to \mathbb{P}^1$ given by the distinguished point o of the generic fibre.

The birational class of an elliptic surface determines and is determined by its generic fibre. Within each birational class there is, up to isomorphism respecting the fibration, a unique minimal model [Mir90, Cor II.1.3] with the following relative minimality property.

Definition 4.8. An elliptic surface $E \to \mathbb{P}^1$ is minimal if it contains no (-1)-curves in its fibres.

In particular there is a bijection between isomorphism classes of minimal elliptic surfaces and elliptic curves over the field of rational functions.

In order to understand when an elliptic surface is minimal we globalise the Weierstrass presentation of an elliptic curve.

Proposition 4.9. *[Mir90, Sect II.2] Given an elliptic curve (E, o) over a field of characteristic zero there exist sections*

$$z \in H^0(\mathcal{O}_E(2o)) \qquad y \in H^0(\mathcal{O}_E(3o)) \qquad a \in H^0(\mathcal{O}_E(4o)) \qquad b \in H^0(\mathcal{O}_E(6o))$$

satisfying the Weierstrass equation

$$y^2 = z^3 + az + b$$

Moreover such sections are unique up to multiplication by the second, third, fourth and sixth power of a unit of the base field.

An elliptic curve (E, o) in Weierstrass form is singular if and only if its discriminant
$$\Delta(E) = 4a^3 + 27b^2$$
vanishes. We note that this condition is independent of the choice of Weierstrass form as the discriminants of two such choices differ by multiplication by the twelfth power of a unit.

On an elliptic surface the Weierstrass representations of the individual fibres can be combined into a global Weierstrass representation as explained in [Mir90, Sect II.5]. One can construct sections of tensor powers of the so-called fundamental line bundle L, the dual of the normal bundle to the section o, which specialise to the sections y, z, a and b on each fibre. Such sections are unique up to multiplication by a non-zero regular function on the base $\mathbb{P}^1$, namely a non-zero complex number.

In particular the discriminant Δ defines a holomorphic section of $L^{\otimes 12}$ and so has degree twelve times the degree of the line bundle L. The vanishing locus of the discriminant Δ is precisely the set of points over which singular fibres of the elliptic surface lie. Contracting a (-1)-curve in a fibre E_p of the elliptic surface $E \to \mathbb{P}^1$ has the effect of twisting the fundamental line bundle L by $\mathcal{O}(-p)$ and decreasing the order of vanishing of the discriminant section Δ at p by twelve.

Conversely given an elliptic curve E over the function field $\mathbb{C}(t)$ of $\mathbb{P}^1_t$, there exists a unique relatively minimal compactification of E to a minimal elliptic surface $E \to \mathbb{P}^1_t$. This compactification has the property that the degree of the fundamental line bundle is minimal amongst all possible compactifications.

Given an elliptic surface $E \to \mathbb{P}^1_t$ the global Weierstrass construction yields a Weierstrass family $W \to \mathbb{P}^1_t$ given by the equation

$$y^2 = z^3 + a(t)z + b(t)$$

However the two surfaces E and W are not in general isomorphic. The Weierstrass fibration is by irreducible elliptic curves, whereas the original elliptic surface may have reducible fibres. On the other hand, whereas the original elliptic surface E is supposed to be smooth, the surface W may have singularities. In fact W is obtained from E by contracting all components of the reducible fibres which do not meet the section o.

The following proposition describes various equivalent characterisations of minimality of an elliptic surface $E \to \mathbb{P}^1$.

Proposition 4.10. *[Mir90, Sect III.3] The following are equivalent:*

1. *The elliptic surface $E \to \mathbb{P}^1$ is minimal.*

2. *The degree of the fundamental line bundle L is minimal amongst all birationally equivalent elliptic surfaces.*

3. *The associated Weierstrass family $W \to \mathbb{P}^1$ has only Kleinian singularities.*

The discriminant of an elliptic surface whose fundamental line bundle is of degree zero is a (non-vanishing) constant function and so is a product of an elliptic curve and the projective line. We will primarily be interested in the next simplest case, that of minimal elliptic surfaces with fundamental line bundle of degree 1. Such an elliptic surface is a rational elliptic surface and can be obtained by blowing up nine possibly infinitely near points in the projective plane [Dui10, Thm 9.1.3].

The possible singular fibres of a minimal elliptic surface have been classified by Kodaira. As suggested by the above theorem, this resembles an ADE classification, although there is some repetition for there are two non-isomorphic irreducible singular curves with a node and cusp respectively. The dual graph of the components of any singular fibre intersect is of affine type ADE, with the extending affine vertex corresponding to the unique component meeting the section.

The singular fibres of a minimal elliptic surface are determined by the monodromy in the local system given by the relative integer first homology of E. As a smooth elliptic curve is topologically a torus, the first homology $H_1(E, \mathbb{Z}) \cong \mathbb{Z}^2$ is a rank 2 unimodular lattice Γ with a skew-symmetric bilinear form given by Poincaré duality. The monodromy around a singular fibre gives a conjugacy class of the automorphism group $\mathrm{SL}(2, \mathbb{Z})$ of the lattice Γ, which can be represented by a matrix with determinant one.

In Table 4.2 we give Kodaira's notation for each type of singular fibre [Kod63], along with the ADE type, the order of vanishing of the discriminant and a representative of the conjugacy class of the monodromy in Γ. We note that the singular fibres come in $*$-pairs, listed above in the same columns. The monodromy around the two singular fibres differs by $-I$, the negative of the identity matrix, and so in particular are equal in $\mathrm{PSL}(2, \mathbb{Z})$. It is convenient to denote a smooth fibre by I_0, so that the singular fibre I_0^* with monodromy $-I$ also has a $*$-partner.

The singular fibres can be grouped into two types according to whether the contraction of the components not meeting the section o is a nodal or cuspidal curve on which the group law degenerates to the action of an additive or multiplicative group respectively. The *additive* singular fibres are $\mathrm{I}_n^{(*)}$ and have monodromy operator of infinite order. The remaining *multiplicative* singular fibres of types $\mathrm{II}^{(*)}, \mathrm{III}^{(*)}, \mathrm{IV}^{(*)}$ have finite-order monodromy.

The isomorphism class of an elliptic curve is determined by its J-invariant

$$J(E) = 4a^3/\Delta$$

$(\mathrm{I}_n)_{\{n \geq 1\}}$	II	III	IV
A_n	A_0	A_1	A_2
n	2	3	4
$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$(\mathrm{I}_n^*)_{\{n \geq 0\}}$	IV^*	III^*	II^*
D_{n+4}	E_6	E_7	E_8
$n+6$	8	9	10
$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

Table 4.2: Kodaira's list of singular fibres

The definition is independent of the Weierstrass representation as both the numerator and denominator scale by the twelfth power of a unit. An elliptic curve in Weierstrass form is singular precisely when the discriminant is zero, so irreducible singular elliptic curves satisfy $J(E) = \infty$.

The affine line is a coarse moduli space of (smooth) elliptic curves but there is no universal family lying over it. Indeed any elliptic curve has an automorphism of order two which acts on the section y by multiplication by -1, exchanging the two sheets of the double cover $E \to \mathbb{P}^1_z$ given by the Weierstrass equation.

Furthermore there are two isomorphism classes of elliptic curves with additional symmetry represented by the following Weierstrass equations

$$y^2 = z^3 + z \qquad y^2 = z^3 + 1$$

The former has J-invariant 0 and automorphism group of order four; the latter has J-invariant 1 and automorphism group of order six.

Throughout the rest of the thesis we find it more convenient to work with a different coordinate j on the projective line $\mathbb{P}^1_J$ obtained by performing the fractional-linear transformation

$$j = \frac{J}{J-1} = \frac{4a^3}{27b^2}$$

which is a more immediate invariant of an elliptic curve written in Weierstrass form. The special elliptic curves with automorphism group of order four and six occur at $j = \infty$ and $j = 0$, and the singular irreducible elliptic curves have $j = 1$.

Definition 4.11. The j-map of an elliptic surface $\mathcal{E} \to \mathbb{P}^1_t$ is the map $\mathbb{P}^1_t \to \mathbb{P}^1_j$ sending a point t to the value of the j-invariant of the fibre E_t.

Although there is no universal family of elliptic curves over $\mathbb{P}^1_j \setminus \{1\}$, there are elliptic surfaces with identity j-map which have (reducible) singular fibres over the special points ∞ and 0.

Example 4.12. Consider the following "universal Weierstrass family" of elliptic curves over $\mathbb{C}_j$

$$y^2 = z^3 - 3jz + 2j$$

It clearly has trivial j-map, and discriminant

$$\Delta = 108j^2(j-1)$$

It can be compactified to a minimal Weierstrass family whose discriminant vanishes to order $12-2-1=9$ at $j=\infty$. Resolving the singularities at $j=\infty$ and $j=0$ we obtain a rational elliptic surface $U \to \mathbb{P}^1$ with precisely three singular fibres. Over $j=1$ the surface U has an additive singular fibre already present in the Weierstrass family which in Kodaira's list is denoted by I_1. The monodromy around this fibre is given by the unipotent matrix given by the Picard-Lefschetz formula for the monodromy in terms of the vanishing cycle

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The multiplicative singular fibres around $j=\infty$ and $j=0$ of types III^* and type II respectively have monodromy which generates the automorphism group of the special elliptic curves with these j-invariants

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Given a family of elliptic curves $E \to \mathbb{P}^1_t$ and a rational map $f : \mathbb{P}^1_u \to \mathbb{P}^1_t$ we can consider the pulled back family $f^*E \to \mathbb{P}^1_u$. The fundamental line bundle of f^*E is $L^{\otimes d}$ where L is the fundamental line bundle of E and d is the degree of the map f. Similarly the order of vanishing of the discriminant section at a point u_0 in $\mathbb{P}^1_u$ is obtained by multiplying the degree of ramification of f at u_0 by the order of vanishing of the discriminant $t_0 = f(u_0)$ at its image in $\mathbb{P}^1_t$.

The monodromy in the pullback $f^*\Gamma$ of the local system of first homology Γ is also determined by the branching of the map f. The conjugacy class in $\mathrm{SL}(2,\mathbb{Z})$ giving the monodromy around u_0 is the dth power of the conjugacy class corresponding to the monodromy at $t_0 = f(u_0)$. As the monodromy determines the type of singular fibre in a minimal elliptic surface, the singular fibres of the minimal model $(f^*E)^{(\min)} \to \mathbb{P}^1_u$ can be computed from the branching of f.

We remark that the pull-back of a minimal elliptic surface is not necessarily again minimal. The fibre at a point in the preimage of an I_n or I_n^* fibre with ramification degree ν is indeed of type $I_{\nu n}$ or $I_{\nu n}^*$ respectively. However the multiplicative singular fibres all have monodromy of finite-order and so the pulled-back surface can have trivial monodromy at a point in the preimage. Such a surface cannot be minimal as the discriminant section does not vanish at a smooth fibre.

Passing to the minimal model of f^*E by contracting any (-1)-curves in the fibres reduces the order of vanishing of the discriminant section by a multiple of twelve. Supposing the multiplicative singular fibre at a point p in E has type given in the table below, the type of the singular fibre at a point in its preimage with ramification degree ν is that corresponding to $\nu \operatorname{ord}_p(\Delta)$ modulo twelve.

$\operatorname{ord}(\Delta)$ (mod 12)	0	2	4	6	8	10
Fibre type	I_0	II	IV	I_0^*	IV^*	II^*

$\operatorname{ord}(\Delta)$ (mod 12)	0	3	6	9
Fibre type	I_0	III	I_0^*	III^*

One can consider pulling back the "universal" family $U \to \mathbb{P}^1_j$ along the j-map of an elliptic surface $E \to \mathbb{P}^1_t$. The pulled-back family has the same j-map as E but is not necessarily birational to E. Indeed the j-map of an elliptic surface $E \to \mathbb{P}^1_t$ is identical to the j-map of the corresponding Weierstrass family. The j-map is clearly left unchanged by rescaling a and b by the second and third power respectively of a rational function q in t. The elliptic surface

$$y^2 = z^3 + a(t)q(t)^2 z + b(t)q(t)^3$$

is birational to $E \to \mathbb{P}^1_t$ if and only if q is a square in $\mathbb{C}(t)$. Indeed Weierstrass representations of the generic fibre are unique up to multiplication of a and b by a fourth and sixth power of a rational function respectively.

Definition 4.13. Two elliptic surfaces differ by a quadratic twist by a rational function q if their generic fibres are isomorphic over at most a quadratic field extension of the field of rational functions $\mathbb{C}(t)$.

Such a quadratic field extension is necessarily generated by the square root of a rational function q. Note that if q is a square the field extension is trivial and so the two elliptic surfaces are birational. In studying the birational class of the resulting elliptic surface we may therefore assume that q has only simple zeroes and poles. Over $\mathbb{C}(t)(\sqrt{q})$ the elliptic curves with Weierstrass presentations

$$y^2 = z^3 + az + b \qquad y^2 = z^3 + q^2 az + q^3 b$$

are isomorphic by rescaling the sections by the appropriate powers of $\sqrt{q}$ and so the corresponding elliptic surfaces differ by a quadratic twist. Conversely if the generic fibres of two elliptic surfaces are isomorphic over quadratic field extension of $\mathbb{C}(t)$ then there exist Weierstrass presentations of the generic fibres of the above form for some generator q of the extension.

The above discussion gives the following characterisation of elliptic surfaces with the same j-map.

Proposition 4.14. *Two elliptic surfaces have the same j-map if and only if they differ by a quadratic twist.*

In particular we see that the "universal" family $U \to \mathbb{P}^1_j$ is universal up to quadratic twists. We note that we have already seen how to compute the singular fibres and monodromy of the pull-back $j^*U \to \mathbb{P}^1_j$ of the universal family along the j-map of the elliptic surface. Therefore understanding the effect of a quadratic twist on the singular fibres (or equivalently the monodromy) of an elliptic surface would give an effective way of computing the singular fibres (or monodromy) of an arbitrary elliptic surface.

Lemma 4.15. *Suppose two minimal elliptic surfaces differ by a quadratic twist by a rational function q. Then the local systems given by their relative first homology Γ differ by the transformation $(-I)^{\mathrm{ord}(q)}$. Equivalently, the fibres of the elliptic surfaces differ by replacing a fibre with its $*$-partner wherever q has a zero or pole.*

An important corollary of this is that the image of the monodromy of an elliptic surface under the natural map $\mathrm{SL}(2, \mathbb{Z}) \to \mathrm{PSL}(2, \mathbb{Z})$ is determined by

its j-map for quadratic twists only change the monodromy by powers of the matrix $-I$.

As an example of the above ideas we study a family of elliptic curves as a quadratic twist of a pullback of the "universal" family $U \to \mathbb{P}^1_j$ along its j-map.

Example 4.16. Consider the Weierstrass family over $\mathbb{C}_b$

$$y^2 = z^3 - 3z + 2b$$

It has discriminant $\Delta = b^2 - 1$ and has two nodal I_1 singular fibres at $b = \pm 1$. Its minimal completion is a rational elliptic surface $\Sigma_b \to \mathbb{P}_1$ with a type II^* singular fibre at ∞.

The j-map of the surface $\Sigma \to \mathbb{P}^1_b$ is given by $j = b^{-2}$, a degree two map ramified to order two at $b = 0$ and $b = \infty$. The surface $j^*U \to \mathbb{P}^1_b$ also has two nodal I_1 singular fibres at $b = \pm 1$, the preimages of $j = 1$. The unique point in the j-map preimage of $j = 0$ is $b = \infty$, where j^*U has a singular fibre of type IV as the discriminant vanishes to order $2 \times 2 = 4$. Similarly the minimal model of j^*U has a singular fibre of type I_0^* at $b = 0$ as the discriminant of j^*U vanishes to order $2 \times 9 = 18$ there.

Thus we have a pair of elliptic surfaces Σ and $(j^*U)^{\min}$ over $\mathbb{P}^1_b$ with identical j-map $j = b^2$ and singular fibres

$$
\begin{array}{c|ccc}
b & \infty & 1 & -1 \\
\hline
\Sigma_b & II^* & I_1 & I_1
\end{array}
\qquad
\begin{array}{c|cccc}
b & \infty & 0 & 1 & -1 \\
\hline
(j^*U)^{\min}_b & IV & I_0^* & I_1 & I_1
\end{array}
$$

The two families differ by a quadratic twist by $q = b$, which exchanges the smooth fibre Σ_0 with its $*$-partner I_0^* and the fibre of type II^* at ∞ with its $*$-partner IV.

The two families $U \to \mathbb{P}^1_j$ and $\Sigma \to \mathbb{P}^1_b$ will serve as running examples in the next section where we consider differential equations satisfied by the periods of differentials on families of elliptic curves.

4.3   Picard-Fuchs equations

In this section we consider the Picard-Fuchs differential equation satisfied by the periods of the holomorphic differential on families of elliptic curves over the projective line. We see that the periods satisfy a second-order differential equation with respect to a generator of the function field with regular singularities precisely at the points over which singular fibres lie. We show how to compute the characteristic exponents of the Picard-Fuchs equations, which characterise the asymptotic behaviour of solutions at the singular points, by understanding how they change under pullback and quadratic twist.

We recall how the periods of a meromorphic residueless differential η on a family of elliptic curves $E \to \mathbb{P}^1_t$ satisfy a second-order differential equation in t with regular singularities. We will see that meromorphic residueless differentials provide representatives of classes in the complex de Rham cohomology group $H^1_{dR}(E, \mathbb{C})$ of a smooth elliptic curve, which is isomorphic to $H^1(E, \mathbb{C}) = \mathrm{Hom}(H_1(E, \mathbb{Z}), \mathbb{C})$ by taking their periods. The natural connection in the holomorphic vector bundle with fibre $H^1_{dR}(E_t), \mathbb{C})$ over $\mathbb{P}^1_t \setminus S$ induced by

the local system $H_1(E, \mathbb{Z})$ extends with regular singularities over the singular set.

The de Rham theorem asserts a quasi-isomorphism between the constant sheaf $\mathbb{C}_X$ on a smooth manifold X and its de Rham complex $(\Omega_X^{\cdot}, d)$ of sheaves of complex-valued differential forms with the exterior derivative. In other words, defining the de Rham cohomology $H_{dR}^{\cdot}(X)$ to be the hypercohomology with respect to the de Rham complex, the map

$$H_{dR}^i(X) \to H^i(X, \mathbb{C})$$

$$\sigma \mapsto \int_{\cdot} \sigma$$

given by evaluating the differential i-form σ along codimension i cycles is an isomorphism.

On the other hand one can show that a residueless meromorphic 1-form on a Riemann surface X differs from a complex valued 1-form by the exterior derivative of a meromorphic function. This yields an isomorphism between $H_{dR}^1(X)$ and the vector space of residueless meromorphic differentials on X modulo those which are exact, which is at the origin of the algebraic de Rham theory. In the sequel we will we will be interested in representing cohomology classes by meromorphic 1-forms.

We now consider a family of elliptic curves $\pi : E \to \mathbb{P}^1_t$ with singular set S. The first higher direct image of the constant sheaf $\mathbb{C}_E$ defines a locally constant sheaf of vector spaces on $\mathbb{P}^1 \backslash S$ with fibre $H^1(E_t, \mathbb{C})$. The induced connection in the holomorphic vector bundle H over $\mathbb{P}^1_t$ with fibre $H_{dR}^1(X)$ is known as the Gauss-Manin connection. It will not extend as a holomorphic connection to the whole of $\mathbb{P}^1$ due to the monodromy in the relative cohomology bundle.

It is important that the Gauss-Manin connection extends with regular singularities over the singular set S, that is

$$\nabla_{GM} : H \to H \otimes \Omega_{\mathbb{P}^1}^{1;0}(logS)$$

Thus given a choice of extension of the holomorphic structure on H to the whole of the projective line, we have a regular endomorphism of H obtained by contracting ∇_{GM} with the vector field $(t - t_0)\partial_{t-t_0}$ for $t_0 \in S$. In an analogous way to a regular singularity of a differential equation, we can then define the characteristic exponents of ∇_{GM} whose exponentials agree with the eigenvalues of the monodromy of the connection.

We now consider the covariant derivatives of a section η_t of the relative cohomology bundle represented by a family of residueless meromorphic differentials on the fibres E_t. As $H^1(E_t, \mathbb{C})$ is a two-dimensional vector space, the vectors $\eta_t, \nabla_{GM}(t\partial_t)(\eta_t)$ and $\nabla_{GM}^2(t\partial t)(\eta_t)$ are linearly dependent in each fibre, which is a two-dimensional vector space. We have three linearly dependent sections of the holomorphic vector bundle H over $\mathbb{P}^1_t \backslash S$ which can be paired with a homology class $\gamma \in H_1(E, \mathbb{Z})$ to give linearly dependent functions

$$\int_\gamma \eta_t \qquad t\frac{d}{dt}\int_\gamma \eta_t \qquad t^2\frac{d^2}{dt^2}\int_\gamma \eta_t$$

over the function field $\mathbb{C}(t)$. Thus the periods of η_t are solutions to a second-order differential equation of the form

$$\vartheta^2 f + p\vartheta f + q = 0$$

for some rational functions p and q in t. The Riemann bilinear relations show that the periods of η with respect to a basis of the homology lattice Γ have non-vanishing Wronskian and so are a fundamental set of solutions for this differential equation.

Definition 4.17. The Picard-Fuchs equation of a residueless abelian differential η on a family of elliptic curves $E \to \mathbb{P}^1_t$ is the second-order linear homogeneous differential equation satisfied by the periods of η.

The regularity of the extension of the Gauss-Manin connection implies that Picard-Fuchs equations have only regular singularities.

Proposition 4.18. *The Picard-Fuchs equation of a residueless differential η on a family $E \to \mathbb{P}^1_t$ has only regular singularities.*

Proof. We can rewrite the Picard-Fuchs equation in the matrix form

$$\vartheta \begin{pmatrix} f \\ \vartheta f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix} \begin{pmatrix} f \\ \vartheta f \end{pmatrix}$$

The coefficient matrix agrees with the matrix of the endomorphism $\nabla_{GM}(t\partial_t)$ with respect to the basis $\{\eta, \nabla_{GM}(t\partial_t)\eta\}$ of H. As the extension of ∇_{GM} over the singular points S has regular singularities, so the differential equation has only regular singularities. $\qquad\square$

There is a distinguished line $H^0(E, \omega_E) \subset H^1(E, \mathbb{C})$ spanned by the unique non-zero holomorphic differential ω on the elliptic curve E up to scale. In a Weierstrass representation for E it is given by the expression

$$\omega = \frac{dz}{y}$$

For an elliptic surface $E \to \mathbb{P}^1$ this lifts to a section of the dual L^* of the fundamental line bundle which restricts to a non-vanishing holomorphic differential ω_t on each fibre. We call the Picard-Fuchs equation for the holomorphic differential ω on an elliptic surface $E \to \mathbb{P}^1_t$ simply the Picard-Fuchs equation.

We would like to compute the exponents of the Picard-Fuchs equation of the families $E \to \mathbb{P}^1_t$ of elliptic curves. Our strategy will be to pull back the Picard-Fuchs equation of the "universal" family $U \to \mathbb{P}^1_j$ along the j-map, using Lemma 4.6 to compute the exponents of the Picard-Fuchs equations of the pull-back family $j^*U \to \mathbb{P}^1_t$. This family is related to $E \to \mathbb{P}^1_t$ by a quadratic twist whose effect on the exponents of the Picard-Fuchs equations is easy to compute.

To perform this programme it is necessary to understand the ramification of the j-map. Over $j = 1$ the branching is determined by the singular fibre over a point in the preimage; the ramification degree is n precisely when there is a singular fibre of type I_n over the preimage. Over $j = \infty$ or $j = 0$ the type of singular fibre, or equivalently the monodromy, determines the degree of ramification only modulo 2 and 3 respectively, the orders in $\mathrm{PSL}(2, \mathbb{Z})$ of the monodromy operator of the universal family at these points. The j-map might also be branched to arbitrary degree over the complement of $\{\infty, 0, 1\}$ which cannot be detected by considering the monodromy.

Definition 4.19. The j-map of a minimal elliptic surface $E \to \mathbb{P}^1_t$ is minimally ramified if it is branched only over $\{\infty, 0, 1\}$ and has degree of ramification at singular points according to the singular fibre in the following table.

j	∞		0			1
	III/III*	I$_0$/I$^*_0$	II/IV*	IV/II*	I$_0$/I$^*_0$	I$_n$/I$^*_n$
	1	2	1	2	3	n

We observe that the Picard-Fuchs equation of an elliptic surface is an invariant of its birational class and so we are free to work with the minimal model. In this case Duistermaat has computed the exponents of the Picard-Fuchs equations at each type of singular fibre given the ramification degree of the j-map at this point. We recall the special case below where the j-map is minimally ramified.

Lemma 4.20. *[Dui10, Lemma 6.2.38] Suppose the minimal elliptic surface $E \to \mathbb{P}^1_t$ has minimally ramified j-map and $t_0 \in S$ does not belong to the support of the divisor of the fundamental line bundle L. Then the exponents of the Picard-Fuchs equation of $E \to \mathbb{P}^t_1$ are determined by the type of singular fibre at t_0 according to the following table (cf Table 6.2.39 in [Dui10])*

I$_n$	II	III	IV
0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$
0	$-\frac{1}{6}$	$-\frac{1}{4}$	$-\frac{1}{3}$

I$^*_n$	IV*	III*	II*
$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{6}$
$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{3}{4}$	$-\frac{5}{6}$

If a point $t_0 \in S$ belongs to the support of the divisor of the fundamental line bundle L then the section ω of L^* viewed as a function on $\mathbb{P}^1_t$ has a zero or pole at t_0. Thus if $L = \mathcal{O}_{\mathbb{P}^1}(nt_0)$ then the exponents of the Picard-Fuchs equations differ from those in the table by adding n to both exponents. If the j-map is not minimally ramified then the exponents differ from those in the table by adding the degree of excess ramification to the top exponent in the table.

Example 4.21. Consider the "universal" family of elliptic curves already considered in example 4.12 with singular fibres III*, II, I$_1$ at $\{\infty, 0, 1\}$. Its j-map is the identity and so is clearly minimally ramified and has fundamental line bundle $L = \mathcal{O}_{\mathbb{P}^1}(\infty)$.

The Riemann scheme of the Picard-Fuchs equation can thus be read off from the table of Lemma 4.20, adding one to the exponents at ∞ where the section ω of $L^* = \mathcal{O}_{\mathbb{P}_1}(-\infty)$ viewed as a function has a simple zero.

j	∞	0	1
	$\frac{3}{4}$	$\frac{1}{6}$	0
	$\frac{1}{4}$	$-\frac{1}{6}$	0

We note that this is a Riemann scheme of a second-order linear homogeneous differential equation with regular singularities as the exponents sum to one and so satisfy the Fuchs relation of Proposition 4.7. Furthermore as the differential equation has precisely three singularities, there are no accessory parameters and so the equation is determined by the Riemann scheme.

Using the Picard-Fuchs equation for the "universal" family $U \to \mathbb{P}_j^1$ we want to compute the Riemann scheme of an arbitrary elliptic surface by pulling back along its j-map and performing a quadratic twist. It remains to understand how performing a quadratic twist by the square root of a rational function q changes the characteristic exponents.

Lemma 4.22. *Suppose two elliptic surfaces differ by a quadratic twist by $\sqrt{q}$. Then the exponents at a point $t_0 \in \mathbb{P}^1$ differ by the addition of $-\frac{1}{2}\operatorname{ord}_{t_0}(q)$.*

Proof. The quadratic twist by q rescales the section ω of L^* by the inverse of $\sqrt{q}$. □

The number of singular points of the Picard-Fuchs equation can be changed by a quadratic twist. Indeed a quadratic twist can send a fibre of type I_0^* at which the exponents are both half-integer to its $*$-partner I_0, a smooth fibre at which the exponents are both integer. If these exponents are 0 and 1 then this is a non-singular point of the resulting Picard-Fuchs equation. An example in which this phenomenon occurs is in the calculation of the Picard-Fuchs equations of the elliptic surface considered in Example 4.16.

Example 4.23. We compute the exponents of the Picard-Fuchs equation of the family $\Sigma_b \to \mathbb{P}_b^1$ whose singular fibres were computed in Example 4.16. The j-map of this family is the degree 2 map $j = b^{-2}$, which is ramified to degree 2 at $b = \infty$ and $b = 0$. Thus the Picard-Fuchs equation of the pull back $j^*U \to \mathbb{P}_b^1$ of the universal family has Riemann scheme

$$
\begin{array}{c|cccc}
b & 0 & \infty & 1 & -1 \\
\hline
 & \frac{3}{2} & \frac{1}{3} & 0 & 0 \\
 & \frac{1}{2} & -\frac{1}{3} & 0 & 0
\end{array}
$$

The family $\Sigma_b \to \mathbb{P}_b^1$ is obtained from the pulled-back family by the quadratic twist by $q = b$. This quadratic twist has the effect of decreasing the exponents at $b = 0$ by $\frac{1}{2}$ and increasing the exponents at $b = \infty$ by the same amount. The exponents of the Picard-Fuchs twisted surface $\Sigma_b \to \mathbb{P}_b^1$ at $b = 0$ are 0 and 1 and so $b = 0$ is a non-singular point. Thus the Picard-Fuchs equation of the family $\Sigma_b \to \mathbb{P}_b^1$ has Riemann scheme

$$
\begin{array}{c|ccc}
b & \infty & 1 & -1 \\
\hline
 & \frac{5}{6} & 0 & 0 \\
 & \frac{1}{6} & 0 & 0
\end{array}
$$

It is easy to see that in this case the j-map of the minimal elliptic surface $\Sigma_b \to \mathbb{P}_b^1$ is minimally ramified. Thus having already computed the singular fibre types in Example 4.16 we can also read off the exponents from the table in Lemma 4.20. Indeed the section ω_b of L^* has only a simple zero at $b = \infty$ are those in the table corresponding to the singular fibres II^*, I_1 and I_1, shifted by one at $b = \infty$. This agrees with the Riemann scheme given above.

The Picard-Fuchs equations of the two families $U \to \mathbb{P}_j^1$ and $\Sigma \to \mathbb{P}_b^1$ both have precisely three regular singularities and so are determined by their Riemann scheme. The next section studies such differential equations in more detail and in particular the map to the projective line given by the ratio of a pair of independent solutions to the differential equation. This provides a geometric description of the so-called period map of the Picard-Fuchs equations of these families of elliptic curves.

4.4 Hypergeometric equations and Schwarz triangles

In this section we study the second-order differential equations on the projective line with precisely three regular singularities. Such differential equations are rigid in the sense that they are determined by their characteristic exponents and so their global monodromy is determined by the local monodromy at the three singular points.

We show that any such differential equation is projectively equivalent to a hypergeometric equation, which serves as a normal form for properties of the differential equation which are invariant under rescaling the dependent variable by a rational function. In particular we discuss the Schwarz triangle theorem, which describes the image of the ratio of a pair of linearly independent solutions of the hypergeometric equation via a curvilinear triangle, with angles given by the differences in characteristic exponents at the three singular points.

Definition 4.24. A quasi-hypergeometric differential equation is a second-order linear homogeneous differential equation on the projective line with precisely three regular singularities.

Local systems on the complement of three points in the projective line are rigid in the sense that they are determined by the local monodromy at the three singular points. Correspondingly a quasi-hypergeometric differential equation is determined, up to a Möbius transformation in the dependent variable, by the characteristic exponents at the three singular points. This lack of additional so-called accessory parameters was known to Riemann, who introduced the following projective normal form for quasi-hypergeometric differential equations.

Definition 4.25. The Riemann hypergeometric differential equation with parameters $a, b, c \in \mathbb{C}$ is the unique differential equation on $\mathbb{P}^1_t$ with Riemann scheme

$$
\begin{array}{c|ccc}
t & \infty & 0 & 1 \\
\hline
 & a & 0 & 0 \\
 & b & 1-c & c-a-b
\end{array}
$$

This equation is given by

$$t(\vartheta - a)(\vartheta - b)f = \vartheta(\vartheta - c + 1)f$$

The sum of the characteristic exponents in the above Riemann scheme is equal to 1 as required by the Fuchs identity of Proposition 4.7. It is clear that the set of parameters is in bijection with the set of possible values of the four characteristic exponents which are not fixed to be zero. The reason for parameterising Riemann hypergeometric equations in this way is that the classical Gaussian hypergeometric series $F(a, b, c, t)$ is a solution in a neighbourhood of $t = 0$ of the hypergeometric differential equation with the corresponding parameters.

We notice that the Riemann scheme of the Riemann hypergeometric equation is distinguished amongst quasi-hypergeometric differential equations in that it has singular points $\{\infty, 0, 1\}$ and a zero characteristic exponent at 0 and 1. In the sequel we will consider quasi-hypergeometric differential equations with a singular point at ∞ which have a zero characteristic exponent at each of the two

remaining singular points t_0 and t_1. We abuse terminology slightly by calling such differential equations hypergeometric; they can be transformed into the usual Gauss hypergeometric equation by a harmless linear transformation of the dependent variable t.

Definition 4.26. A quasi-hypergeometric equation is called hypergeometric with parameters $a, b, c \in \mathbb{C}$ if it has Riemann scheme of the form

$$
\begin{array}{c|ccc}
t & \infty & t_0 & t_1 \\
\hline
& a & 0 & 0 \\
& b & 1-c & c-a-b
\end{array}
$$

A quasi-hypergeometric differential equation can be transformed into a hypergeometric equation by multiplying f by an appropriate rational function in t whose divisor is supported on the three singular points $\{\infty, t_0, t_1\}$. Explicitly we have the equality of Riemann schemes

$$
(t-t_0)^{-\beta_1}(t-t_1)^{-\gamma_1}
\begin{array}{c|ccc}
t & \infty & t_0 & t_1 \\
\hline
& \alpha_1 & \beta_1 & \gamma_1 \\
& \alpha_2 & \beta_2 & \gamma_2
\end{array}
=
\begin{array}{c|ccc}
t & \infty & t_0 & t_1 \\
\hline
& \alpha_1 + \beta_1 + \gamma_1 & 0 & 0 \\
& \alpha_2 + \beta_1 + \gamma_1 & \beta_2 - \beta_1 & \gamma_2 - \gamma_1
\end{array}
$$

This transformation does not change the ratio of a fundamental set of solutions to the differential equation. Two differential equations are said to be *projectively equivalent* if they differ by such a transformation.

Proposition 4.27. *The projective equivalence class of a quasi-hypergeometric differential equation is labelled by a triple (α, β, γ) of differences in characteristic exponents at the three singular points.*

In particular the projective equivalence class of a quasi-hypergeometric differential equation contains 24 hypergeometric differential equations.

Theorem 4.28. *[Neh52, p.207] (Schwarz triangle theorem) Suppose we have a quasi-hypergeometric differential equation with real exponent differences (α, β, γ). Consider the map $s : \mathbb{P}^1 \backslash \widetilde{\{\infty, 0, 1\}} \to \mathbb{P}^1$ given by the ratio of two linearly independent solutions to the quasi-hypergeometric differential equation. Then s maps any copy of the upper-half plane $\mathfrak{h} \subset \mathbb{P}^1 \backslash \{\infty, 0, 1\}$ conformally onto the interior of a curvilinear triangle with angles $(\pi\alpha, \pi\beta, \pi\gamma)$. Furthermore s is a biholomorphism onto its image if and only if $\alpha, \beta, \gamma \in 1/\mathbb{N} \cup \{0\}$*

Example 4.29. The classical example of such a quasi-hypergeometric differential equation is the Picard-Fuchs equation of the "universal" family $U \to \mathbb{P}^1_j$ whose Riemann scheme is given in Example 4.21. The differences in the characteristic exponents at the singular points $j = \{\infty, 0, 1\}$ are $\frac{1}{2}$, $\frac{1}{3}$ and 0 respectively. The image of the period map of ω is the hyperbolic plane $\mathfrak{h}$ which is tiled by reflections of the Schwarz triangle with angles $(\pi/2, \pi/3, 0)$.

An important property of the hypergeometric equations is that they can be easily integrated. In fact the derivative of any solution to a hypergeometric differential equation with parameters (a, b, c) is itself a solution to the shifted hypergeometric differential equation with the shifted parameters $(a+1, b+1, c+1)$.

Proposition 4.30. *Suppose that the derivative Df satisfies the hypergeometric differential equation with parameters (a, b, c). Then the function f satisfied the hypergeometric differential equation $(a - 1, b - 1, c - 1)$.*

As an example of the above Proposition we consider the meromorphic differential $\lambda = y\, dz$ on an elliptic curve in Weierstrass form. The differential λ has the property that its covariant derivative $\nabla_{GM}(\partial_b)\lambda$ is the holomorphic differential ω. Thus the periods of the meromorphic differential λ satisfy the integral of the Picard-Fuchs equation with respect to b.

Example 4.31. We compute the differential equation satisfied by the periods of the meromorphic differential λ_j on the "universal" family $U \to \mathbb{P}^1_j$. The meromorphic differential λ_j pulls back to the meromorphic differential λ_b on $\Sigma \to \mathbb{P}^1_b$ whose periods satisfy the integral of the hypergeometric equation in 4.23 with respect to b. It has Riemann scheme given by

$$
\begin{array}{c|ccc}
b & \infty & 1 & -1 \\
\hline
 & -\frac{1}{6} & 1 & 1 \\
 & -\frac{5}{6} & 0 & 0
\end{array}
$$

The Riemann scheme of the differential equation satisfied by the periods of λ_j can now be obtained by performing a quadratic twist to get $j^*U \to \mathbb{P}^1_b$ and pushing down along the j-map $j : \mathbb{P}^1_b \to \mathbb{P}^1_j$ given by $j = b^{-2}$. As the Seiberg-Witten differential defines a section of the fifth power of the fundamental line bundle L, the effect of the quadratic twist by b^{-1} is to shift the characteristic exponents at 0 and ∞ by $-\frac{5}{2}$ and $\frac{5}{2}$ respectively. Thus the periods of the Seiberg-Witten differential λ on $j^*U \to \mathbb{P}^1_b$ satisfy the differential equation with Riemann scheme

$$
\begin{array}{c|cccc}
b & 0 & \infty & 1 & -1 \\
\hline
 & -\frac{3}{2} & \frac{7}{3} & 1 & 1 \\
 & -\frac{5}{2} & \frac{5}{3} & 0 & 0
\end{array}
$$

Finally pushing down along the j-map gives the Riemann scheme

$$
\begin{array}{c|ccc}
j & \infty & 0 & 1 \\
\hline
 & -\frac{3}{4} & \frac{7}{6} & 1 \\
 & -\frac{5}{4} & \frac{5}{6} & 0
\end{array}
$$

Chapter 5

Stability conditions for the A_2 quiver

In this chapter we compute a connected component $\mathrm{Stab}^*(\mathcal{D}_{A_2})$ of the space of stability conditions for the category $\mathcal{D}(A_2)$ associated to the pole divisor $P = (7\infty)$ on $\mathbb{P}^1$ as in [Sut11]. The quotient $\mathrm{Stab}^*(\mathcal{D}_{A_2})/\mathrm{Sph}^*(D_{A_2})$ is isomorphic to the space of cubic polynomials of the form $z^3 + az + b$ which have simple zeroes, that is the complement of the discriminant locus $\Delta = \{a^2 = b^3\}$ inside $\mathbb{C}^2$.

   This will serve to illustrate some of the methods of the following Chapters 6 and 7 where we compute a connected component of two-dimensional spaces of numerical stability conditions associated to the Painlevé pole divisors on $\mathbb{P}^1$. Our method is to work modulo the natural $\mathbb{C}^*$-actions on the spaces $\mathrm{Quad}(\mathbb{P}^1, (7\infty))$ and $\mathrm{Stab}^*(\mathcal{D}_{A_2})/\mathrm{Aut}^*(\mathcal{D}_{A_2})$ considered in the previous chapter. We show that we can lift the period map of the Seiberg-Witten differential on the family of spectral elliptic curves over $(\widetilde{\mathbb{C}^2\backslash\Delta})/\mathbb{C}^*$ to a biholomorphism onto $\mathrm{Stab}^*(\mathcal{D}_{A_2})/\mathbb{C}$.

5.1 Autoequivalences

We begin by computing the group of Keller-Yang equivalences $\mathrm{Aut}^*(\mathcal{D}_{A_2})$. The marked bordered surface associated to the pair $(\mathbb{P}^1, (7\infty))$ is the pentagon, that is a disc with five marked points on the boundary. We draw one of the five triangulations of the pentagon and the associated quiver with potential, the A_2 quiver, in Figure 5.1. The five triangulations all differ by an element of the mapping class group $\mathrm{MCG}(\varhexagon) \cong \mathbb{Z}_5$ which consists of the rotations of the pentagon.

   From Proposition 3.12 and the faithfulness of the action of the braid group on $\mathcal{D}_{A_2}$ by spherical twists, we have a short exact sequence

$$1 \to \mathrm{Sph}(\mathcal{D}_{A_2}) \to \mathrm{Aut}^*(\mathcal{D}_{A_2}) \to \mathrm{MCG}^{\bowtie}(\varhexagon) \to 1$$

As there are no punctures, the signed mapping class group is isomorphic to
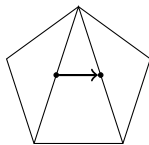
Figure 5.1: The A_2 quiver associated to a triangulation of the pentagon

$\mathbb{Z}/5\mathbb{Z}$. Further, by Theorem 2.35 we have an isomorphism

$$\mathrm{Br}_3 \to \mathrm{Sph}(\mathcal{D}_{A_2})$$
$$\sigma_1 \to \Phi_S$$
$$\sigma_2 \to \Phi_T$$

where S and T are the objects of $\mathcal{D}_{A_2}$ given by the simple representations of A_2 corresponding to the vertices at the head and tail of the arrow respectively.

The centre of the braid group is freely generated by the element $u = (\sigma_1\sigma_2)^3$ by [KT08]. We use this to compute the centre of $\mathrm{Sph}(\mathcal{D}_{A_2})$.

Lemma 5.1. *The centre of $\mathrm{Sph}(\mathcal{D}_{A_2})$ is generated by $[-5]$.*

Proof. It suffices to compute the images of the simple objects S and T for an autoequivalence acting trivially on the simple objects of a heart of finite length also acts trivially on the space of stability conditions. Denote by E and X the unique up to isomorphism non-trivial extensions of S by T and T by $S[1]$ respectively. Then

$$
\begin{array}{ccccccccccccccc}
S & \mapsto & X & \mapsto & T[-1] & \mapsto & T[-3] & \mapsto & E[-3] & \mapsto & S[-3] & \mapsto & S[-5] \\
T & \mapsto & T[-2] & \mapsto & E[-2] & \mapsto & S[-2] & \mapsto & S[-4] & \mapsto & X[-4] & \mapsto & T[-5]
\end{array}
$$

as required. $\square$

We claim that the group of Keller-Yang equivalences $\mathrm{Aut}^*(\mathcal{D}_{A_2})$ is also isomorphic to the braid group Br_3. Indeed we prove that the braid group fits into the following exact sequence. As the braid relation is homogeneous in the generators σ_1 and σ_2, there is a well-defined word-length map on the braid group Br_3.

Lemma 5.2. *There is a short exact sequence*

$$1 \to \mathrm{Br}_3 \to \mathrm{Br}_3 \xrightarrow{l} \mathbb{Z}_5 \to 1$$

where the map l is the word-length map in the generators σ_1 and σ_2 modulo 5.

Proof. It is clear that the map l is surjective, so it remains to prove that its kernel K is isomorphic to the braid group Br_3. The kernel is generated by words of length five in the generators σ_1, σ_2. We claim that K is in fact generated by the two words of length five

$$\sigma_1^{-1}u = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 \qquad \sigma_2^{-1}u = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1$$

Indeed as u is central any word of length five can be expressed as a word in the elements $\sigma_1^{-1}u$, $\sigma_2^{-1}u$ and u^5 and their inverses. Furthermore

$$u^5 = u^6 u^{-1} = u^6(\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2)^{-1} = (\sigma_1^{-1}u)(\sigma_2^{-1}u)(\sigma_1^{-1}u)(\sigma_2^{-1}u)(\sigma_1^{-1}u)(\sigma_2^{-1}u)$$

so K is in fact generated by $\sigma_1^{-1}u$ and $\sigma_2^{-1}u$. As u is central it is clear that the generators $\sigma_1^{-1}u$ and $\sigma_2^{-1}u$ of K satisfy the braid relations. We claim that the map $\mathrm{Br}_3 \to K$ given by these generators is in fact an isomorphism. Suppose w is a word in the generators of K which is equal to the identity element. Then as u is central, we can express the word $w = u^m w'$ where m is the length of the word w, and w' is the corresponding word in the generators σ_1^{-1}, σ_2^{-1} of the original braid group.

If the length m of the word w is zero, then $w = w'$ is itself a word in the generators σ_1 and σ_2 of the braid group and so is trivial if and only if it can be made trivial by applying the braid relations. If the length m is non-zero, then $w' = u^{-m}$ and so the word w' belongs to the centre of the braid group, which is freely generated by $u^{-1} = (\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2)^{-1}$, which is a word of length six in the generators σ_1 and σ_2. This is a contradiction as $u^{-1} \neq u^{-6}$. $\qquad\square$

To compute the autoequivalence group we consider the images of the standard heart $\mathcal{A}^0$ under the simple tilts.

Lemma 5.3. *The Keller-Yang autoequivalences associated to simple tilts of the standard heart $\mathcal{A}^0$ satisfy*

$$(\Phi_S^+)^{-1}(\mathcal{A}^0) = \mathrm{Tw}_S\mathrm{Tw}_T[2](\mathcal{A}^0) = \Phi_T^-(\mathcal{A}^0)$$

and

$$\Phi_S^-(\mathcal{A}^0) = \mathrm{Tw}_T\mathrm{Tw}_S\mathrm{Tw}_T[3](\mathcal{A}^0) = (\Phi_T^+)^{-1}(\mathcal{A}^0)$$

Proof. The actions of the composites $\mathrm{Tw}_S\mathrm{Tw}_T$ and $\mathrm{Tw}_T\mathrm{Tw}_S\mathrm{Tw}_T$ of spherical twists on the simple objects S and T is given in the proof of Lemma 5.1. The action of the simple tilts on the simple objects S and T of $(\mathcal{A}^0)$ can be computed by Lemma 2.36 and one observes that the images of the simple objects agree. $\quad\square$

Thus we see that $\mathrm{Aut}^*(\mathcal{D}_{A_2})$ is generated by the elements $\Sigma = \Phi_S\Phi_T[2]$ and $\Upsilon = \Phi_T\Phi_S\Phi_T[3]$.

Lemma 5.4. *The generators Σ and Υ of $\mathrm{Aut}^*(\mathcal{D}_{A_2})$ satisfy the relation $\Sigma^3 = [1] = \Upsilon^2$.*

Proof. As the shift functor commutes with spherical twists, we have that

$$\Sigma^3 = (\Phi_S\Phi_T)^3[6] = \Upsilon^2$$

Furthermore by Lemma 5.1 we have that $(\Phi_S\Phi_T)^3[6] = [-5][6] = [1]$. $\qquad\square$

In fact

$$\langle \alpha, \beta \mid \alpha^2 = \beta^3 \rangle$$

is an alternative presentation of the braid group on three strings [KT08]. The original generators of the braid group are given in terms of the new generators as $\sigma_1 = \beta^{-1}\alpha$ and $\sigma_2 = \alpha\beta^{-1}$. Thus $\mathrm{Aut}^*(\mathcal{D}_{A_2})$ is alternatively generated by the generators $\Phi_S[1] = \Sigma^{-1}\Upsilon$ and $\Phi_T[1] = \Upsilon\Sigma^{-1}$ which can immediately be seen to satisfy the braid relations.

Lemma 5.5. *The map $f : \mathrm{Br}_3 \to \mathrm{Aut}^*(\mathcal{D}_{A_2})$ sending (σ_1, σ_2) to $(\Phi_S[1], \Phi_T[1])$ is an isomorphism.*

Proof. We show that the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Br}_3 & \longrightarrow & \mathrm{Br}_3 & \longrightarrow & \mathbb{Z}_5 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Sph}(\mathcal{D}_{A_2}) & \longrightarrow & \mathrm{Aut}^*(\mathcal{D}_{A_2}) & \longrightarrow & \mathrm{MCG}^{\bowtie}(\varhexagon) & \longrightarrow & 1
\end{array}
$$

is commutative, whence the result follows from the five lemma. The second square is commutative as the maps $\mathrm{Br}_3 \to \mathbb{Z}_5$ and $\mathrm{Aut}^*(\mathcal{D}_{A_2}) \to \mathrm{MCG}^{\bowtie}(\varhexagon) \cong \mathbb{Z}_5$ both send the respective generators (σ_1, σ_2) and $(\Phi_S[1], \Phi_T[1])$ to $1 \in \mathbb{Z}_5$. $\qquad\square$

As we will work primarily with the projective stability conditions, we will consider the quotient group $\mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}[1]$ by the free subgroup generated by the shift functor $[1] = (\Phi_S[1]\Phi_T[1])^3$. The isomorphism of the previous lemma identifies this subgroup with the centre of the braid group $\mathbb{Z} \subset \mathrm{Br}_3$.

Proposition 5.6. *The quotient on the braid group Br_3 by its centre $\mathbb{Z}$ is isomorphic to $\mathrm{PSL}(2, \mathbb{Z})$.*

Proof. In the alternative presentation of the braid group Br_3, the central element is generated by $\alpha^2 = \beta^3$. The result is then immediate from the standard presentation of $\mathrm{PSL}(2, \mathbb{Z})$

$$\mathrm{PSL}(2, \mathbb{Z}) = \langle A, B \mid A^2 = 1 = B^3 \rangle$$

where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$\qquad\square$

We now give an alternative way of computing the autoequivalence group $\mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}[1]$ modulo the shift functor which reflects the methods used in Chapter 7. Recall that we have an isomorphism $K(\mathcal{D}_{A_2}) \cong \mathbb{Z}[S] \oplus \mathbb{Z}[T]$, and so we have a map $\phi : \mathrm{Aut}^*(\mathcal{D}_{A_2}) \to \mathrm{PSL}(2, \mathbb{Z})$ sending an autoequivalence to its action on the lattice $\mathbb{Z}^2$. This map factors through the quotient of $\mathrm{Aut}^*(\mathcal{D}_{A_2})$ by the group generated by the shift functor as $[1]$ acts on $K(\mathcal{D}_{A_2})$ by $-I$, the identity in $\mathrm{PSL}(2, \mathbb{Z})$.

By Lemma 5.4, the group $\mathrm{Aut}^*(\mathcal{D}_{A_2})$ is generated by the autoequivalences Υ and Σ whose second and third powers respectively are equal to the shift functor. Thus we have a map $\mathbb{Z}_2 * \mathbb{Z}_3 \to \mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}[1]$ and a commutative triangle

$$
\begin{array}{ccc}
\mathbb{Z}_2 * \mathbb{Z}_3 & \longrightarrow & \mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}[1] \\
 & \searrow & \downarrow {\scriptstyle \phi} \\
 & & \mathrm{PSL}(2, \mathbb{Z})
\end{array}
$$

whose third arrow $\mathbb{Z}_2 * \mathbb{Z}_3 \to \mathrm{PSL}(2, \mathbb{Z})$ is given by the composition of the other two. It is an isomorphism giving the standard presentation of $\mathrm{PSL}(2, \mathbb{Z})$ by generators and relations in the proof of Proposition 5.6, for the generators Υ and Σ of $\mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}$ act on $K(\mathcal{D}_{A_2}) \cong \mathbb{Z}[S] \oplus \mathbb{Z}[T]$ as the matrices A and B. Thus we deduce that the other two maps are isomorphisms, and in particular there is an isomorphism $\mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}[1] \to \mathrm{PSL}(2, \mathbb{Z})$.

5.2 *V*-domains

In this section we construct a fundamental domain for the action of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$ on $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$. We have seen in the previous section that the set of reachable projective hearts is a torsor for the group $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$. We will define when a projective stability condition is supported on a projective heart so that the *V*-domain consisting of those projective stability conditions supported on a given projective heart is a fundamental domain for the action.

The following slightly non-standard definition makes precise the desired properties of a fundamental domain. In particular we prefer to work with closed domains.

Definition 5.7. Suppose a group G acts on a topological space X. We say a closed set $V \subset X$ is a fundamental domain for the action of G on X if

- The G-translates of V cover X

- The G-translates of the interior of V are disjoint

Recall that a projective heart $\overline{\mathcal{A}}$ is an equivalence class of hearts with respect to the action of the subgroup of the autoequivalence group generated by the shift functor. The notion of a stability condition being supported on a heart does not immediately descend to a projective stability condition being supported on a projective heart as there are non-projectively equivalent hearts which support a lift of a given projective stability condition. We will choose the projective heart on which the cone spanned by the stable objects of the projective heart has minimal angle.

Definition 5.8. The width w of a stability condition $\sigma = (\mathcal{A}, Z)$ is the difference between the phases of the semistable objects of $\mathcal{A}$ of maximal and minimal phase.

Given a projective stability condition $\overline{\sigma}$ we note that the width of a stability condition in the $\mathbb{C}$-orbit of $\overline{\sigma}$ depends only on the projective equivalence class $\overline{\mathcal{A}}$ of its heart. As the set of phases supporting a semistable object with respect to a projective stability condition is not dense in the circle, the infimum of the widths of all lifts of $\overline{\sigma}$ is achieved by the lifts supported on some projective equivalence class of hearts $\overline{\mathcal{A}}$.

Definition 5.9. A projective stability condition $\overline{\sigma}$ is supported on a projective heart $\overline{\mathcal{A}}$ if the width of a lift of $\overline{\sigma}$ to a stability condition supported on a lift of $\overline{\mathcal{A}}$ is minimal amongst all lifts of $\overline{\sigma}$.

As the minimal width can be potentially achieved on more than one projective heart, so a projective stability condition can be supported on more than one projective heart. We say that a projective stability condition is strictly supported on a projective heart if it is the unique heart which realises this minimum.

Definition 5.10. The V-domain $V(\overline{\mathcal{A}}) \subset \mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ of a projective heart $\overline{\mathcal{A}}$ consists of all projective stability conditions supported on $\overline{\mathcal{A}}$.

We will call a strict V-domain the subset of a V-domain consisting of those projective stability conditions strictly supported on the projective heart.

We wish to describe the V-domain of the standard projective heart $\overline{\mathcal{A}}^0$. We first consider walls of marginal stability in the space of stability conditions $\mathrm{Stab}^*(\mathcal{D})$ which are characterised by the fact that the image of the central charge map is contained in a line. This condition descends to give walls of marginal stability in $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ which all have image contained in the great circle $\mathcal{W}$ in $\mathbb{P}\mathrm{Hom}(\mathcal{K}(\mathcal{D}), \mathbb{C})$ along which $\overline{Z}$ is real. Within each chamber the set of hearts supporting a stability condition in the $\mathbb{C}$-orbit of a given projective stability condition is constant.

The V-domain $V(\overline{\mathcal{A}}^0)$ intersects a unique wall of marginal stability, along which $Z(S)$ and $Z(T)$ are contained in the same ray and the extension E destabilises. The locus described by this wall of marginal stability $W(\overline{\mathcal{A}}^0)$ (along which $\overline{Z} \in \mathbb{R}_{>0}$) divides the V-domain into two regions in which E is stable and unstable respectively. The set of projective hearts supporting a lift of a projective stability condition in the unstable and stable regions is $\{\overline{\mathcal{A}}^0, \overline{\Upsilon}(\overline{\mathcal{A}}^0)\}$ and $\{\overline{\mathcal{A}}^0, \overline{\Sigma}^{-1}(\overline{\mathcal{A}}^0), \overline{\Sigma}(\overline{\mathcal{A}}^0)\}$ respectively.

We note that in order to describe the unstable and stable regions of the V-domain of the standard heart it suffices to describe their respective images in $\mathbb{P}\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$. Indeed every projective stability condition contained in such a region has the same set of stable objects and so such a projective stability condition is determined by its projective central charge. The images of the unstable and stable regions are contained in the hemispheres $\mathfrak{h}^-$ and $\mathfrak{h}^+$ respectively of the decomposition

$$\mathbb{P}\mathrm{Hom}(K(\mathcal{D})) = \mathfrak{h}^- \cup \mathcal{W} \cup \mathfrak{h}^+$$

To the hemispheres $\mathfrak{h}^-$ and $\mathfrak{h}^+$ we can associate a root system R_- and R_+ consisting of the classes in $K(\mathcal{D})$ of stable objects. In the unstable region $\mathfrak{h}^-$ the root system is of type $A_1 \times A_1$ with roots $\{\pm S, \pm T\}$. In the stable region $\mathfrak{h}^+$ the root system is of type A_2 consisting of the classes of the simple objects S and T of $\overline{\mathcal{A}}^0$, their stable extension of class $[E] = [S] + [T]$ and their negatives. We note that any projective heart supporting a lift in the unstable or stable region determines a choice of simple roots in the corresponding root system.

The two simple objects S and T of the heart $\mathcal{A}^0$ each determine a pair of simple tilts of $\mathcal{A}^0$, with precisely one tilted heart supporting a lift of a stability condition in the stable region and one supporting a lift in the unstable region. We consider the codimension one locus consisting of projective stability conditions whose widths of the lifts to $\mathcal{A}^0$ and one of its simple tilts are equal.

Lemma 5.11. *Suppose the width of $\overline{\sigma} \in \mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ is minimised on the hearts $\mathcal{A}$ and $\mu_X^+(\mathcal{A})$ for some simple object X of $\mathcal{A}$. Denote by Y_- and $Y_+ = \mu_X^+(Y_-)$ denote the other simple object of $\mathcal{A}$ and $\mu_X^+(\mathcal{A})$ respectively. Then the projective central charge $\overline{Z}$ of $\overline{\sigma}$ lies on the orthogonal circle to W passing through the points where $\overline{Z}([X])$ and $\overline{Z}([Y_+] + [Y_-])$ vanish.*

Proof. Consider a lift of $\overline{\sigma}$ for which the phase of X is equal to 1. The condition that the widths are equal is that $\phi(X) - \phi(Y_-) = \phi(Y_+) - \phi(X[-1])$ so $\phi(Y_-) + \phi(Y_+) = 1$. As Y_+ is the universal extension of Y_- by X, this implies that the

lengths of $Z(Y_-)$ and $Z(Y_+)$ are equal. Thus the vectors $Z(X)$ and $Z(Y_-) + Z(Y_+)$ are orthogonal
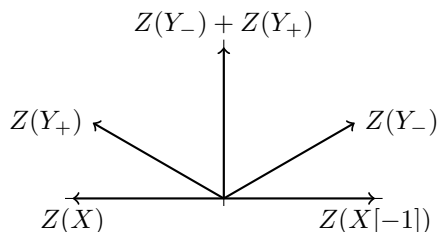
Thus within the unstable and stable half-planes $\mathfrak{h}_-$ and $\mathfrak{h}^+$ we have arcs of the circles W_S^-, W_T^- and W_S^+, W_T^+ associated to the simple tilts. We denote by V^- and V^+ the intersection of the closed half-planes bounded by these circles containing the wall of marginal stability W_m with the half planes $\mathfrak{h}^-$ and $\mathfrak{h}^+$ respectively. Thus V^- and V^+ consist of projective stability conditions whose width in $\overline{\mathcal{A}}^0$ is less than or equal to the width in any of its simple tilts.

As inversion in a circle preserves any orthogonal circle, inversion in the circles W_S^-, W_T^- and W_S^+, W_T^+ preserves the unstable and stable half-planes $\mathfrak{h}^-$ and $\mathfrak{h}^+$ respectively. Moreover as each pair of circles meets in twice the Coxeter angle of the corresponding root system R_- and R_+, the domains V^- and V^+ are fundamental domains for the corresponding action of the index 2 rotation subgroups $\mathbb{Z}_2 < W(R^-)$ and $\mathbb{Z}_3 < W(R^+)$ of the Weyl groups generated by the Coxeter elements w_- and w_+ respectively.

Lemma 5.12. *The action of the groups generated by w_- and w_+ on $\mathfrak{h}^-$ and $\mathfrak{h}^+$ agrees with the induced action of the subgroups of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$ generated by the elements $\overline{\Upsilon}$ and $\overline{\Sigma}$ respectively.*

Proof. This is immediate from the induced action of $\overline{\Upsilon}$ and $\overline{\Sigma}$ on the lattice $K(\mathcal{D})$. $\qquad\square$

This yields a description of the *V*-domain within the unstable and stable half-planes.

Lemma 5.13. *The intersections $V^\pm(\overline{\mathcal{A}}^0)$ of the V-domain $V(\overline{\mathcal{A}}^0)$ with the unstable and stable half-planes $\mathfrak{h}^-$ and $\mathfrak{h}^+$ are equal to V^- and V^+.*

Proof. By definition the intersections $V^\pm(\overline{\mathcal{A}}^0)$ of the *V*-domain with $\mathfrak{h}^-$ and $\mathfrak{h}^+$ are contained in V^- and V^+. Conversely $\mathfrak{h}^-$ and $\mathfrak{h}^+$ are covered by the closed domains $V^-(\overline{\Upsilon}^i\overline{\mathcal{A}}^0) = w_-^i V^-(\overline{\mathcal{A}}^0)$ and $V^+(\Sigma^i\overline{\mathcal{A}}^0) = w_+^i V^+(\overline{\mathcal{A}}^0)$ respectively. As $V^\pm$ are fundamental domains for the action on $\mathfrak{h}^\pm$ of the rotation groups generated by $w_\pm$ we have $V^\pm \subset V^\pm(\overline{\mathcal{A}}^0)$ and hence equality. $\qquad\square$

As a consequence we have an explicit description of the standard *V*-domain as the disjoint union

$$V(\overline{\mathcal{A}}^0) = V^+ \cup \mathcal{W}^0 \cup V^-$$

where $\mathcal{W}^0 \subset \mathcal{W}$ is the arc of the equator along which $\overline{Z}$ is real and negative. We note that the *V*-domain $V(\overline{\mathcal{A}}^0)$ projects isomorphically onto its image $\Diamond$ under $\overline{Z}$ which we have drawn in Figure 5.2 on the decomposition $\mathbb{P}^1 = \mathfrak{h}^+ \cup \mathcal{W} \cup \mathfrak{h}^-$.
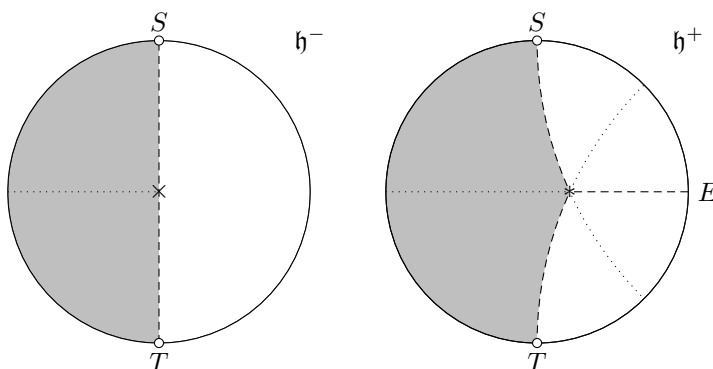
Figure 5.2: The projection $\lozenge$ under $\overline{Z}$ of the fundamental domain $V(\overline{\mathcal{A}}^0)$ for the action of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$ on $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$.

Proposition 5.14. *The V-domain $V(\overline{\mathcal{A}}^0)$ is a fundamental domain for the action of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$ on $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$.*

Proof. The V-domain $V(\overline{\mathcal{A}}^0)$ is closed in $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ as it is defined as the (infinite) intersection of closed sets. To see that the $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$-translates of $V(\overline{\mathcal{A}}^0)$ cover the connected component $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ we must show that any $\overline{\sigma} \in \mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ is supported on a projective heart $\overline{\mathcal{A}} = \overline{\Phi}(\overline{\mathcal{A}}^0)$ for some $\overline{\Phi} \in \mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$. As the reflection hyperplanes of $R^\pm$ are locally finite we can find a path from a point in $V(\overline{\mathcal{A}}^0)$ to $\overline{\sigma}$ passing through finitely many walls corresponding to simple tilts. The composite of the relevant Keller-Yang equivalences then gives the desired autoequivalence.

The interior of the V-domain is contained in the strict V-domain. The local (indeed global) finiteness of the reflection hyperplanes of $R^\pm$ exhibits the strict V-domain as the intersection of finitely many open half-planes. Thus the interior of the V-domain is equal to the strict V-domain and so its $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$-translates are clearly disjoint. Furthermore the boundary of the V-domain is contained in the reflection hyperplanes $W_S^\pm$, $W_T^\pm$ and so is of codimension one. $\qquad\square$

A fundamental domain for the action of a group with finitely many codimension one faces determines a finite presentation of the group according to the identifications of points in the boundary of the fundamental domain. The above fundamental domain yields the presentation $\mathrm{Aut}(\mathcal{D})/\mathbb{Z} \cong \mathbb{Z}_2 * \mathbb{Z}_3$ given by the generators $\overline{\Upsilon}$ and $\overline{\Sigma}$ which identify the two sides of $V(\overline{\mathcal{A}}^0)$ in $\mathfrak{h}^-$ and $\mathfrak{h}^+$ respectively. Moreover the intersections of the two side pairs are stabilised by the subgroup generated by this pair of generators.

Lemma 5.15. *Let $D = V(\overline{\mathcal{A}}^0)$ be the standard fundamental domain for the action of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$ on $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$. Suppose x and y belong to D and let $g \in \mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$. Then $x = gy$ if and only if one of the following occurs:*

- *$x = y$ and g is the identity*

- *$x = y = \overline{\sigma}_-$ and g belongs to the subgroup generated by $\overline{\Upsilon}$*

- $x = y = \overline{\sigma}_+$ *and g belongs to the subgroup generated by $\overline{\Sigma}$*

- $x, y \in \partial D \cap \mathfrak{h}^- \backslash \overline{\sigma}^-$ *and $x = \overline{\Upsilon} y$*

- $x, y \in \partial D \cap \mathfrak{h}^+ \backslash \overline{\sigma}^+$ *and $x = \overline{\Sigma} y$ or $x = \overline{\Sigma}^{-1} y$*

Proof. The first statement is clear. There is no $g \in \text{Aut}^*(\mathcal{D})/\mathbb{Z}$ for which $x \in \mathfrak{h}^\pm$ and $y \in \mathfrak{h}^\mp$. Moreover if $x, y \in \mathfrak{h}^-$ then $g \in \langle \overline{\Upsilon} \rangle$ and if $x, y \in \mathfrak{h}^+$ then $g \in \langle \overline{\Sigma} \rangle$. The second and third statements follow from the fact that $\overline{\sigma}^\pm$ is fixed by $w^\pm$.

If x and y belong to $\partial \Diamond \cap \mathfrak{h}^\pm \backslash \overline{\sigma}^\pm$ and g is not the identity then x and y belong to distinct components of $\partial \Diamond \cap \mathfrak{h}^\pm \backslash \overline{\sigma}^\pm$. These two components are mapped onto each other by $w^\pm$. As $w^\pm$ does not fix any point in $\partial \cap \mathfrak{h}^\pm \backslash \overline{\sigma}^\pm$ the last two claims follow. $\square$

The fact that the codimension 1 boundary components of $V(\overline{\mathcal{A}}^0)$ are paired by the generators Σ and Υ is a manifestation of the fact that the anti-autoequivalence $T : \mathcal{D}_Q \to \mathcal{D}_{Q^{\mathrm{op}}}$ acts on $V(\overline{\mathcal{A}}^0)$ identifying the paired sides. The quotient Δ has three sides corresponding to the anti-autoequivalences T, Φ_S^+ and Φ_T^+ which act on $N(\mathcal{D})$ by reflections. Thus $\Delta \subset \mathbb{P}^1$ is a Schwarz triangle with monodromy group given by $\text{Aut}^0(\mathcal{D}_{A_2})/\mathbb{Z}$.

5.3 Families of spectral curves

In this section we study the family of spectral elliptic curves parameterised by the two-dimensional family of quadratic differentials on the projective line with a single pole of order 7 at the point ∞. We find a double cover of the $\mathbb{C}^*$-action on the space of quadratic differentials by rescaling, which lifts to the total space of the family. The quotient by this $\mathbb{C}^*$-action is the universal family $\mathcal{W} \to \mathbb{P}(4,6) \backslash \delta$ of elliptic curves.

We compute the space of quadratic differentials $\text{Quad}(\mathbb{P}^1, 7(\infty))$ on the Riemann sphere with simple zeroes and a single pole of order 7. There is a four dimensional space of sections of the bundle $K_{\mathbb{P}^1}^{\otimes 2}(7\infty) = \mathcal{O}_{\mathbb{P}^1}(3\infty)$ given by

$$(A + Bz + Cz^2 + Dz^3)\, dz^{\otimes 2}$$

There is a two dimensional group of conformal automorphisms of the Riemann sphere fixing the marked point ∞, given by the Möbius maps $z \mapsto pz + q$ fixing the point ∞. We use this freedom to fix the coefficients of z^3 and z^2 to be 1 and 0 respectively. Thus we have a two-dimensional space of quadratic differentials

$$u_{a,b} = (z^3 + az + b)\, dz^{\otimes 2}$$

parameterised by the vector space $\mathbb{C}^2_{a,b}$. The discriminant locus $\Delta = \{4a^3 = 27b^2\}$ is given by the vanishing of the discriminant of the cubic polynomial. The link of the singularity of Δ at the origin is the trefoil knot. The fundamental group of $\mathbb{C}^2_{a,b} \backslash \Delta$ is thus the braid group on three strings Br_3, the knot group of the trefoil. From the description at the end of the previous chapter, we should identify this with the group of spherical twists $\text{Sph}(\mathcal{D}_{A_2}) \cong \text{Br}_3$.

There is a $\mathbb{C}^*$-action on the space of quadratic differentials parameterised by $\mathbb{C}^2_{a,b} \backslash \Delta$ by $(z, a, b) \mapsto (\zeta z, \zeta^2 a, \zeta^3 b)$ which rescales the quadratic differential by ζ^5. In particular there is a residual freedom in this parameterisation as the

$\mathbb{Z}/5\mathbb{Z}$-orbits define the same quadratic differential. Thus the space of quadratic differentials $\mathrm{Quad}(\mathbb{P}^1, 7\infty)$ is given by

$$\mathrm{Quad}(\mathbb{P}^1, 7\infty) = \frac{\mathbb{C}^2_{a,b} \backslash \Delta}{\mathbb{Z}/5\mathbb{Z}}$$

The group $\mathbb{Z}/5\mathbb{Z}$ is precisely the mapping class group of the pentagon. Thus the fundamental group of the space of quadratic differentials fits into the exact sequence

$$1 \to \mathrm{Br}_3 \to \pi_1(\mathrm{Quad}(\mathbb{P}^1, 7\infty)) \to \mathbb{Z}/5\mathbb{Z} \to 1$$

and can be identified with the autoequivalence group $\mathrm{Aut}^*(\mathcal{D}_{A_2})$.

The quotient of $\mathrm{Quad}(\mathbb{P}^1, 7\infty) = (\mathbb{C}^2_{a,b} \backslash \Delta)/(\mathbb{Z}/5\mathbb{Z})$ by this $\mathbb{C}^*$-action is the complement of a point δ, the image of Δ, in the weighted projective line $\mathbb{P}(2,3)$. We consider the coordinate $j = (-4a^3)/(27b^2)$ on the $\mathbb{P}(2,3)$ so the orbifold points $\times$ and $*$ of order 2 and 3 are at the points $j = \infty$ and $j = 0$ respectively, and the point δ is at $j = 1$.

We expect the fundamental group of the quotient $\mathbb{P}(2,3) \backslash \delta$ to be given by the quotient of $\pi_1(\mathrm{Quad}(\mathbb{P}^1, 7\infty)) \cong \mathrm{Br}_3$ by the free subgroup generated by central element given by a loop around the discriminant Δ. Of course this can only hold if we consider the orbifold fundamental group which takes into account the stabiliser groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ at the two orbifold points $\times$ and $*$. We can compute the (orbifold) fundamental group directly using an orbifold version of van Kampen's theorem.

Lemma 5.16. *The map $\mathbb{Z}_2 * \mathbb{Z}_3 \to \pi_1(\mathbb{P}(2,3) \backslash \delta)$ sending the generators to clockwise loops around the orbifold points $\times$ and $*$ is an isomorphism.*

We now consider the family of compactified spectral curves given by the family of quadratic differentials over $\mathbb{C}^2_{a,b}$. This is a family of genus 1 curves branched over the three simple zeroes and the odd-order pole at infinity given by

$$y^2 = z^3 + az + b$$

The point at infinity gives a section of this family over $\mathbb{C}^2_{a,b}$, giving a family of elliptic curves $\mathcal{W} \to \mathbb{C}^2_{a,b}$ which is nothing but the Weierstrass family. Moreover the discriminant of this family is equal to the discriminant Δ of the space of cubic polynomials $z^3 + az + b$.

There is a $\mathbb{C}^*$-action on the total space of the Weierstrass family given by

$$\zeta(y, z, a, b) \mapsto (\zeta^3 y, \zeta^2 z, \zeta^4 a, \zeta^6 b)$$

whose quotient is the universal family of elliptic curves $\mathcal{W} \to \mathbb{P}(4,6) \backslash \delta$. We note that this $\mathbb{C}^*$-action is a double cover of the $\mathbb{C}^*$-action on $\mathbb{C}^2_{a,b}$ given by the rescaling action on quadratic differentials considered above whose quotient is $\mathbb{P}(2,3)$. The automorphism group $\mathbb{Z}_2$ of this cover gives rise to the generic stabiliser of $\mathbb{P}(4,6)$ and acts on the spectral curves by exchanging the two sheets.

In particular we note the smooth family $\mathcal{W} \to \mathbb{C}^2_{a,b} \backslash \Delta$ does not descend along the $\mathbb{C}^*$-quotient with weights 2 and 3 to a smooth family of elliptic curves over the quotient $\mathbb{P}(2,3) \backslash \delta$. However the projectivisation of the local system given by the first homology of the fibres of $\mathcal{W}$ does indeed descend to $\mathbb{P}(2,3) \backslash \delta$ and whose monodromy defines a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$. We thus have a monodromy map m fitting into the commutative diagram

$$\mathbb{Z}_2 * \mathbb{Z}_3 \longrightarrow \pi_1(\mathbb{P}(2,3)\backslash\delta)$$

$$\downarrow m$$

$$\mathrm{PSL}(2,\mathbb{Z})$$

whose diagonal arrow gives the standard presentation of $\mathrm{PSL}(2,\mathbb{Z})$ and therefore has all arrows isomorphisms.

5.4 Period maps and stability conditions

In this section we complete the proof of the main theorem in the case of the A_2 quiver.

We wish to find a family of elliptic curves $E \to \mathbb{P}^1_j$ over the projective line with precisely three singular fibres at $j = \{\infty, 0, 1\}$ whose period map agrees with that of the universal family $\mathcal{W} \to \mathbb{P}(4,6)\backslash\delta$. In this way we can study the periods of the holomorphic and Seiberg-Witten differentials of the universal family using the methods of Chapter 4. As the isomorphism class of an elliptic curve is determined by its j-invariant, the family E must have identity j-map.

As $\mathcal{W} \to \mathbb{P}(4,6)\backslash\delta$ is the universal family of elliptic curves such a family is obtained by pulling-back this family along a section $\pi : \mathbb{P}^1_j \to \mathbb{P}(4,6)$ of the map $\mathbb{P}(4,6) \to \mathbb{P}^1_j$ from the base of the universal family to its coarse moduli space. There are natural choices of section of the form

$$[(-3j)^p : (2j)^q] : \mathbb{P}^1_j \to \mathbb{P}(4,6)$$

where x and y are integers satisfying the condition

$$3p - 2q = 1$$

There are infinitely many solutions to such an equation of the form

$$p = 1 + 2n \qquad q = 1 + 3n$$

for some integer n. The minimal solution for which p and q are both positive is $p = q = 1$ giving the family

$$y^2 = z^3 - 3jz + 2j$$

which we recognise as the "universal" family $U \to \mathbb{P}^1_j$ of Example 4.12.

In studying the period map of a differential on the universal family of elliptic curves $\mathcal{W} \to \mathbb{P}(4,6)\backslash\delta$ we can use the period map of the differential on the "universal" family $U \to \mathbb{P}^1_j$. Indeed the coarse moduli map $\mathbb{P}(4,6) \to \mathbb{P}^1_j$ and the section $[-3j : 2j] : \mathbb{P}^1_j \to \mathbb{P}(4,6)$ intertwine the two multivalued period maps. Moreover these maps and the respective period maps both factor through $\mathbb{P}(2,3)\backslash\delta$ by forgetting the generic stabiliser and the same section $[-3j : 2j] : \mathbb{P}^1_j \to \mathbb{P}(4,6)$ respectively.

Denote by $\Lambda \cong \mathbb{Z}^2$ the symplectic lattice equipped with a standard basis e_1, e_2 for which the pairing $\langle e_1, e_2 \rangle = 1$. There is a standard coordinate on $\mathbb{P}^1 = \mathbb{P}\operatorname{Hom}(\Lambda, \mathbb{C})$ given by the ratio $[f(e_1) : f(e_2)]$ of the evaluations of a homomorphism $f \in \operatorname{Hom}(\Lambda, \mathbb{C})$ on the two basis elements e_1 and e_2. This coordinate is independent of the change of basis achieved by multiplying by $-I$ and so depends only on the underlying projective local system.

Choosing a symplectic basis $\{\alpha, \beta\}$ of the local system $H_1(\mathcal{W}, \mathbb{Z})$ gives an identification of $\mathbb{P}\operatorname{Hom}(H_1(\mathcal{W}, \mathbb{Z})) \cong \mathbb{P}\operatorname{Hom}(\Lambda, \mathbb{C})$ at each point on the universal cover $\widetilde{\mathbb{P}(2,3)}\backslash\delta$. Thus the periods of the Seiberg-Witten differential λ define a map

$$p = \left[\int_\alpha \lambda : \int_\beta \lambda \right] : \widetilde{\mathbb{P}(2,3)}\backslash\delta \to \mathbb{P}\operatorname{Hom}(\Lambda, \mathbb{C})$$

Proposition 5.17. *Consider the fundamental domain D for the action of the fundamental group on the universal cover of the orbifold $\mathbb{P}(2,3)\backslash\delta$ obtained by making branch cuts along the line $j \in \mathbb{R}$ connecting the two orbifold points to δ. The period map of the Seiberg-Witten differential of the universal family $\mathcal{W} \to \mathbb{P}(4,6)\backslash\delta$ maps D biholomorphically onto the double $\Diamond$ of the Schwarz triangle with angles $(\pi/2, \pi/3, \pi)$ at the vertices (i, ρ, ∞) along the edge connecting the first two vertices.*

Proof. The period map p of the Seiberg-Witten differential of $\mathcal{W} \to \mathbb{P}(4,6)\backslash\delta$ agrees with the period map of the "universal" family $U \to \mathbb{P}^1_j$. By the Schwarz triangle theorem (Theorem 4.28) the image of p restricted to the upper half-plane $\mathfrak{h} \subset D$ is a Schwarz triangle with angles determined by the differences $(1/2, 1/3, 1)$ in the characteristic exponents in Example 4.31. The triangle has angles $\pi/2$ and $\pi/3$ at the images $i = p(\infty)$ and $\rho = p(0)$ of the two orbifold points and angle π at $p(\delta) = \infty$ where the period of λ with respect to the cycle α vanishes.

Moreover we can analytically continue the map p restricted to $\mathfrak{h} \subset D$ to the entire fundamental domain such that its image $\Diamond$ is the reflection of the Schwarz triangle in the edge connecting the vertices i and ρ. $\square$

We can also identify the local system $K(\mathcal{D})$ with Λ at every point of $\operatorname{Stab}^*(\mathcal{D})/\mathbb{C}$ by using the basis of simples in the heart on which the projective stability condition is supported. With this identification we notice that the domain $\Diamond \subset \mathbb{P}\operatorname{Hom}(\Lambda, \mathbb{C})$ is the bijective image of both:

- The fundamental domain D of the action of $\pi_1(\widetilde{\mathbb{P}(2,3)}\backslash\delta)$ on $\widetilde{\mathbb{P}(2,3)}\backslash\delta$ under the period map p

- The fundamental domain $V(\overline{\mathcal{A}}^0)$ for the action of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$ on $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ under the forgetful map $\overline{Z}$

Our strategy in the proof of the main theorem will be to extend this identification of fundamental domains to a map $\pi_1(\mathbb{P}(2,3)\backslash\delta) \to \mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ equivariantly with respect to the $\mathbb{Z}_r * \mathbb{Z}_s$-actions summarised in the following proposition.

Proposition 5.18. *There is a commutative diagram of groups*

$$\pi_1(\mathbb{P}(2,3)\backslash\delta) \xrightarrow{\;\varphi\;} \mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}$$
$$\searrow{\scriptstyle m} \qquad \downarrow{\scriptstyle \phi}$$
$$\mathrm{PSL}(2,\mathbb{Z})$$

whose vertices are all isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ and whose arrows are all isomorphisms, identifying the following pairs of generators

- *The loops $\gamma_\times, \gamma_*$ around the two orbifold points $\times$ and $*$ of $\mathbb{P}(2,3)\backslash\delta$*

- *The autoequivalences $\overline{\Upsilon}, \overline{\Sigma}$ of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$*

- *The matrices A and B of the standard presentation of $\mathrm{PSL}(2,\mathbb{Z})$.*

Proof. We define the map $\varphi : \pi_1(\mathbb{P}(2,3)\backslash\delta) \to \mathrm{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}$ by mapping the generators $\gamma_\times, \gamma_*$ to $\overline{\Upsilon}, \overline{\Sigma}$. The maps m and ϕ are taken to be those appearing in the commutative diagrams at the end of Sections 5.3 and 5.1 respectively under which the generators are mapped to the matrices A and B as desired. $\qquad\square$

We are now in a position to prove the main result.

Theorem 5.19. *There is a biholomorphic map*

$$\widetilde{\mathbb{P}(2,3)}\backslash\delta \xrightarrow{\;f\;} \mathrm{Stab}^*(\mathcal{D}_{A_2})/\mathbb{C}$$
$$\searrow{\scriptstyle p} \qquad \downarrow{\scriptstyle \overline{Z}}$$
$$\mathbb{P}\,\mathrm{Hom}(K(\mathcal{D}_{A_2}),\mathbb{C})$$

lifting the period map p of the Seiberg-Witten differential λ of the family of spectral elliptic curves $\mathcal{W} \to \mathbb{P}(4,6)\backslash\delta$. It is equivariant with respect to the actions of $\mathbb{Z}_2 * \mathbb{Z}_3$ on the left by deck transformations, and on the right by autoequivalences up to shift.

Proof. Denote by D the fundamental domain for the action of $\pi_1(\widetilde{\mathbb{P}(2,3)}\backslash\delta)$ on the universal cover obtained by making branch cuts along the line $j \in \mathbb{R}$ between δ and the two orbifold points. Let $D' = V(\overline{\mathcal{A}}^0)$ be the fundamental domain for the action of $\mathrm{Aut}^*(\mathcal{D})/\mathbb{Z}$ on $\mathrm{Stab}^*(\mathcal{D})/\mathbb{C}$ considered in Proposition 5.14.

By Proposition 5.17 the period map p maps D bijectively onto the double of the Schwarz triangle with angles $(\pi/2, \pi/3, \pi)$ at (i, ρ, ∞) along the edge connecting the first two vertices. This agrees with the image of D' under $\overline{Z}$ which restricts to a bijection on D'. Thus we can define $f : D \to D'$ by $\overline{Z}_{|D'}^{-1} \cdot p_{|D}$ which by definition makes the above diagram commute.

We can extend the above f to the desired map $f : \widetilde{\mathbb{P}(2,3)}\backslash\delta \to \text{Stab}^*(\mathcal{D})/\mathbb{C}$ by equivariance with respect to the $\mathbb{Z}_2 * \mathbb{Z}_3$-actions by deck transformations and autoequivalences up to shift respectively. These two actions are intertwined by the map φ of Lemma 5.18 which identifies the natural pairs of generators on both sides: the clockwise loops around the two orbifold points $\times$ and $*$ and the autoequivalences $\overline{\Upsilon}$ and $\overline{\Sigma}$.

We claim that the map f is well-defined, for which it suffices to show that the map $f : D \to D'$ respects the identification of the points on the boundaries with respect to the two group actions identified via the map φ. The branch cuts giving rise to the domain D are so chosen such that the identif

As the map $\varphi : \pi_1(\widetilde{\mathbb{P}(2,3)}\backslash\delta) \to \text{Aut}^*(\mathcal{D})/\mathbb{Z}$ is an isomorphism the map $f : \widetilde{\mathbb{P}(2,3)}\backslash\delta \to \text{Stab}^*(\mathcal{D})/\mathbb{C}$ is bijective. As φ commutes with the maps m and ϕ it follows that f commutes with the local isomorphisms p and $\overline{Z}$. We conclude that f is a biholomorphism and it has the desired equivariance property by construction.

$\square$

Remark 5.20. This description of the space of stability conditions is a consequence of the main theorem of [BS13] and this particular example is discussed in Section 12.1 of loc.cit. A proof that $\text{Stab}^*(\mathcal{D})/\mathbb{C} \cong \mathfrak{h}$ also appears in an early version of [Qiu11] which relies on deforming the description given in [Bri09b, Tho06] of the distinguished connected component of the space of stability conditions of the CY2 category associated to the A_2 singularity.

Remark 5.21. It is possible to reformulate the above theorem to include the missing order 2 symmetry of the heart groupoid which exchanges the order of the two simple objects S and T. In this case one must extend the group of deck transformations by the order 2 reflection along the real axis in $\mathbb{P}(2,3)$. The modular group $\text{PSL}(2,\mathbb{Z})$ should be replaced by the extended modular group, which contains the order 2 element

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which exchanges the order of the basis elements of the lattice.

The relevant fundamental domains with respect to the extended groups project to the Schwarz triangle $\Delta_{(\frac{1}{2},\frac{1}{3},1)} \subset \mathbb{P}^1$. The extended modular group can be interpreted as the group generated by reflections in the three edges of the Schwarz triangle.

Remark 5.22. One could attempt to reformulate this theorem by considering the universal cover of the base $\mathbb{P}(4,6)\backslash\delta$ of the Weierstrass family of elliptic curves. In this case one should construct a commutative triangle of groups

$$\pi_1(\widetilde{\mathbb{P}(4,6)}\backslash\delta) \xrightarrow{\varphi} \text{Aut}^*(\mathcal{D}_{A_2})/\mathbb{Z}[2]$$
$$\searrow_{m} \qquad \downarrow_{\phi}$$
$$\text{SL}(2,\mathbb{Z})$$

lifting the commutative triangle of Lemma 5.1. By this we mean that we recover the diagram in Lemma 5.1 on taking the quotients of the three groups by the

respective subgroups of order 2 generated by the generic stabiliser of $\mathbb{P}(4,6)$, the shift functor $[1]$ and the negative $-I$ of the identity matrix.

The natural candidate for the map φ is given by sending the two generators of $\pi_1(\widetilde{\mathbb{P}(4,6)}\backslash\delta) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ to the generators Υ and Σ lifting the map in Lemma 5.1. Taking m to be the monodromy of the universal family $\mathcal{W} \to \mathbb{P}(4,6)\backslash\delta$ and ϕ to be given by the induced action of an autoequivalence on the lattice $\Lambda = K(\mathcal{D})$ we find the surprising conclusion that the triangle does not commute. In particular the actions of monodromy around $\times$ and the autoequivalence Υ on the lattice Λ agree only up to sign.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

One remedy to this problem is to take m to be the monodromy of the elliptic surface $U \to \mathbb{P}^1_j$. As U has the starred fibre III^* over $j = \infty$, the monodromy around $j = \infty$ differs from that of the universal family by multiplication by $-I$. However this is unnatural in the context of [BS13] for the universal family arises directly as a quotient of the family of spectral elliptic curves over $\mathrm{Quad}(\mathbb{P}^1_z, 7\infty)$ by a $\mathbb{C}^*$-action, whereas the family $U \to \mathbb{P}^1_j$ admits no immediate interpretation in terms of quadratic differentials.

One might speculate that this is a manifestation of the phenomenon of orientation data in the theory of invariants of CY3 categories. In particular one might expect to find a quadratic refinement $q : \Lambda \to \mathbb{Z}_2$ which intertwines the monodromy of the two families of elliptic curves U and $\mathcal{W}$. We will return to this in Chapter 7.

Chapter 6

Painlevé divisors

In this chapter we focus on a class of examples of construction in the previous section given by certain pole divisors P on $\mathbb{P}^1$ which we call the Painlevé divisors. These are the pole divisors P on $\mathbb{P}^1$ for which the space $\mathrm{Quad}^0(C, P)$ of residueless quadratic differentials has dimension two. This condition means that the numerical Grothendieck group $N(\mathcal{D})$ of the associated category has rank two, isomorphic as a lattice to the homology of the spectral elliptic curve.

There is a natural action of $\mathbb{C}^*$ on these two-dimensional spaces of residueless quadratic differentials by rescaling, and the quotient is (a gerbe over) a weighted projective line. We show that the periods of the Seiberg-Witten differential λ satisfy a quasi-hypergeometric differential equation over its coarse moduli space. The classical Schwarz triangle theorem then gives us a complete understanding of the image and monodromy of the period map of λ.

6.1 The Painlevé pole divisors

As we computed in Chapter 3, the dimension of the space of residueless quadratic differentials $\mathrm{Quad}^0(\mathbb{P}^1, P)$ on $\mathbb{P}^1$ with poles in P is $-6 + \deg(P) + |P|_{odd}$. From this formula we can list the ten pole divisors whose space of residueless quadratic differentials has dimension two, which we call the Painlevé divisors.

We arrange the ten divisors in the following diagram, where the columns from left to right contain the divisors P for which the space of quadratic differentials has $4, 3, 2, 1$ and 0 residue parameters (cf Proposition 2.13) at its poles respectively. We draw a line between two pole divisors if one can be viewed as the degeneration of the other in the following sense: either two poles of the quadratic differentials coincide, or a zero coincides with an even-order pole leaving a pole of order 1 fewer. Each of these operations reduces the number of residue parameters by one.

Using the techniques of the Chapter 3, we can construct a mutation equivalence class of quivers with potential associated to each pole divisor. There is a unique root system which is the underlying graph of a quiver in each mutation equivalence class, which is either finite, affine or elliptic. We give the corresponding *Painlevé diagram* of mutation classes in Figure 6.2, labelled by the root system in each class. Note that the columns now have the interpretation of the dimension of the kernel of the Euler form of the root system. The de-

$$(4,4) \text{———} (3,4) \text{———} (3,3)$$

$$(2,2,2,2) \text{———} (4,2,2) \text{———} (3,2,2) \quad (8) \text{———} (7)$$

$$(6,2) \text{———} (5,2)$$
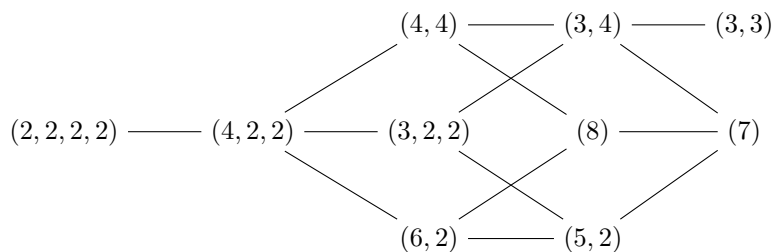
Figure 6.1: Degenerations of four regular singularities on the Riemann sphere

$$\tilde{A}_3 \text{———} \tilde{A}_2 \text{———} \tilde{A}_1$$

$$\tilde{\tilde{D}}_4 \text{———} \tilde{D}_4 \text{———} \tilde{D}_3 \quad A_3 \text{———} A_2$$

$$D_4 \text{———} D_3$$
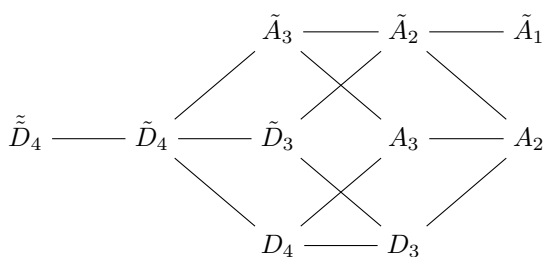
Figure 6.2: Degenerations of the Painlevé diagrams

generation arises from deleting a vertex of some quiver in the original mutation class.

Two pairs of pole divisors give identical diagrams, namely the pair (8) and $(5,2)$ and the pair $(4,4)$ and $(3,2,2)$. We nevertheless give the corresponding diagrams the distinct labels A_3 and D_3 and $\tilde{A}_3$ and $\tilde{D}_3$. Indeed the diagrams A_n and D_n can be realised by the pole divisors $(n+5)$ and $(n+2,2)$ respectively. The quivers given by the alternating orientation of edges of the cyclic graph $\tilde{A}_n$ are realised by the pole divisors $(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$, whereas those with underlying graph $\tilde{D}_n$ are realised by $(n,2,2)$.

We fix a labelling of the vertices and orientation of the arrows of each of the Painlevé diagrams to give a labelled *Painlevé quiver* in each mutation equivalence class. The finite quivers A_2, A_3/D_3 and D_4 are drawn in Figure 6.3, the affine quivers $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3/\tilde{D}_3$ and $\tilde{D}_4$ in Figure 6.4 and the sole elliptic quiver $\tilde{\tilde{D}}_4$ in Figure 6.5.

Remark 6.1. We call these diagrams the Painlevé diagrams as they are related to the Painlevé equations which describe isomonodromic deformations of mero-

$$1 \longrightarrow 0 \qquad 1 \longrightarrow 0 \longleftarrow 2 \qquad 1 \longrightarrow 0 \begin{array}{c} \nearrow 2 \\ \searrow 3 \end{array}$$
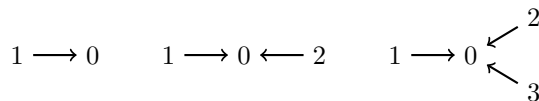
Figure 6.3: The finite Painlevé quivers

$$2 \longrightarrow 0 \qquad 2 \longrightarrow 0 \qquad \begin{matrix} 1 & & 4 \end{matrix}$$

$$1 \longrightarrow 0 \qquad \searrow \quad \nearrow \qquad \downarrow \qquad \uparrow \qquad \searrow \quad \swarrow$$

$$1 \qquad 1 \longleftarrow 3 \qquad 0$$

$$\begin{matrix} & \nearrow & \searrow \\ 2 & & 3 \end{matrix}$$
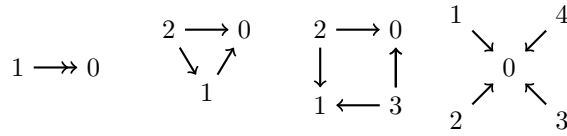
Figure 6.4: The affine Painlevé quivers

$$5 \lllgtr \begin{matrix} 4 \\ 3 \\ 2 \\ 1 \end{matrix} \ggtr 0$$
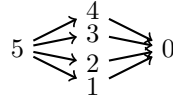
Figure 6.5: The elliptic Painlevé quiver

morphic flat $SL(2, \mathbb{C})$-connections on the Riemann sphere. Indeed the divisors P are precisely those for which the connections whose determinant 1-form is a quadratic differential with poles in P have a 1-dimensional family of isomonodromic deformations. The ten corresponding Painlevé equations were considered in [OO06, vdPS09], and in the notation of [vdPS09] we have the corresponding diagram of Painlevé equations in Figure 6.1. The columns correspond to the number of parameters of the Painlevé equation, and the degeneration procedure is known as confluence [MR12].

To each Painlevé divisor, we associate in this Chapter the following data:

- A two-dimensional family of quadratic differentials $u = u_{a,b}(z)dz^{\otimes 2}$ with poles in P, zero residues and simple zeroes parameterised by the complement of a discriminant locus Δ in $\mathbb{C}^2_{a,b}$ of a polynomial $p_{a,b}(z)$ (Table 6.1)

- A family of spectral elliptic curves $\mathcal{E} \to \mathbb{C}^2 \backslash \Delta$ given by $y^2 = p_{a,b}(z)$ in Table 6.1 with a $\mathbb{C}^*$-action with weights $(2r, 2s)$ on (a, b) lifting the $\mathbb{C}^*$-action above on the base $\mathbb{C}^2 \backslash \Delta$. The quotient is a family of elliptic curves $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$.

- A map $\pi_1(\mathbb{P}(r, s) \backslash \Delta) \cong \mathbb{Z}_r * \mathbb{Z}_s \to \Gamma(\mathcal{E})$, the monodromy group of the

$$P_{III}^{(D_6)} \text{——} P_{III}^{(D_7)} \text{——} P_{III}^{(D_8)}$$

$$P_{VI} \text{——} P_V \text{——} P_V^{deg} \qquad P_{II} \text{——} P_I$$

$$P_{IV} \text{——} P_{II}^{FN}$$

Figure 6.6: Degenerations of the Painlevé equations

family of elliptic curves $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \Delta$ which is a finite index subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ given in Table 6.1.

- A Schwarz triangle, the image of the upper-half plane $\mathfrak{h} \subset \mathbb{P}(r, s) \backslash \Delta$ under the period map p of the Seiberg-Witten differential on the family $\mathcal{E}$, the angles of which are given in Proposition 6.8.

6.2   Quadratic differentials

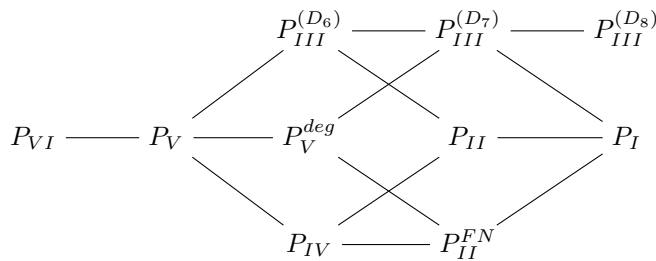In this section we study the family of spectral elliptic curves $\mathcal{E} \to \mathbb{C}^2_{a,b} \backslash \Delta$ given by a family of residueless quadratic differentials over $\mathbb{P}^1$ with a Painlevé pole divisor P parameterised by $\mathbb{C}^2_{a,b}$ with discriminant locus Δ. We define a $\mathbb{C}^*$-action with even integer weights $2r$ and $2s$ on the family whose quotient is a family $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$ over the complement of finitely many points in the weighted projective line, generalising the construction of the Weierstrass family of elliptic curves in Chapter 5. We study the monodromy $\Gamma(\mathcal{E})$ of this family which is a finite index congruence subgroup of the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

Recall that a quadratic differential $u \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{\otimes 2}(P))$ with poles in P and simple zeroes can be viewed as a holomorphic section of the square of the line bundle $\mathcal{K}_{\mathbb{P}^1}(\lceil \frac{P}{2} \rceil)$ with only simple zeroes. We note that for the Painlevé pole divisors P the degree of the divisor $\lceil \frac{P}{2} \rceil$ is always four. Thus we can view a quadratic differential $u \in \mathrm{Quad}^s(\mathbb{P}^1, P)$ as defining a section of $\mathcal{O}(4)$ with only simple zeroes, which we represent as a polynomial p of degree at most four with only simple zeroes. These zeroes occur at the odd-order poles and simple zeroes of u.

In Table 6.1 we give a family of polynomials $p_{a,b}(z)$ parameterised by $\mathbb{C}^2_{a,b}$ for each Painlevé quiver Q. In the complement of the discriminant locus $\Delta \subset \mathbb{C}^2_{a,b}$ these polynomials define residueless quadratic differentials $u \in \mathrm{Quad}^0(\mathbb{P}^1, P)$ with simple zeroes. We denote by r and s the coprime integers whose ratio is the ratio of weights of a and b under the $\mathbb{C}^*$-action on $\mathbb{C}^2_{a,b} \backslash \Delta$ associated to the rescaling action on quadratic differentials.

The compactified spectral curve associated to a residueless quadratic differential on $\mathbb{P}^1$ with simple zeroes and poles in P is given by the double cover

$$y^2 = p(z)$$

inside the total space of the line bundle $\mathcal{K}_{\mathbb{P}}^1(\lceil \frac{P}{2} \rceil)$. It is branched over the odd-order poles of P and the simple zeroes of the quadratic differential specified by p. For the Painlevé pole divisors P, the polynomials $p_{a,b}(z)$ for $(a, b) \in \mathbb{C}^2 \backslash \Delta$ define a family of smooth genus 1 curves $\mathcal{E}$ as double covers of $\mathbb{P}^1$ branched over the points where the quadratic differential has a zero or an odd-order pole.

There is a $\mathbb{C}^*$-action on the total space of the family of spectral elliptic curves $\mathcal{E} \to \mathbb{C}^2_{a,b} \backslash \Delta$ with weights $2r$ and $2s$ on a and b. The quotient is a family of elliptic curves $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$ over the complement of finitely many points δ in the weighted projective line $\mathbb{P}(2r, 2s)$.

We have already seen an example of this construction in the A_2 case in Chapter 5. The family of cubic polynomials $p_{a,b}(z) = z^3 + az + b$ over $\mathbb{C}^2 \backslash \{4a^3 = 27b^2\}$ defines a family of quadratic differentials $u_{a,b}(z) = (z^3 + az + b) \, dz^{\otimes 2}$ with three simple zeroes and a single pole of order 7 at $z = \infty$. The rescaling action

Q	p	r	s	Δ	$\Gamma(\mathcal{E})$
A_2	$z^3 - 3az + 2b$	2	3	$a^3 - b^2$	$\Gamma_0(1)$
A_3	$z^4 + 2az^2 + b$	1	2	$b(a^2 - b)^2$	$\Gamma_0(2)$
D_3	$z((z+a)^2 - b)$	1	2	$b(a^2 - b)^2$	$\Gamma_0(2)$
D_4	$z((z+a)^2(z+4a) - 4b)$	1	3	$b(a^3 - b)^3$	$\Gamma_0(3)$
$\tilde{A}_1$	$z^3 + bz^2 + az$	2	1	$a^2(b^2 - a)$	$\Gamma_0(2)$
$\tilde{A}_2$	$2z^3 - 3bz^2 + a$	3	1	$a(b^3 - a)$	$\Gamma_0(1)$
$\tilde{A}_3$	$z^4 + bz^2 + a$	2	1	$a(b^2 - a)^2$	$\Gamma_0(2)$
$\tilde{D}_3$	$(z+b)(z^2 - a)$	2	1	$a(b^2 - a)^2$	$\Gamma_0(2)$
$\tilde{D}_4$	$(z^2 - a)(z^2 - b)$	1	1	$ab(b - a)^4$	$\Gamma_0(4)$
$\tilde{\tilde{D}}_4$	$bz(z - 1)(z - a)$	0	1	$a^2(a - 1)^2 b^4$	$\Gamma(2)$

Table 6.1: Families of quadratic differentials for the Painlevé quivers

has weights 2 and 3 on a and b and the quotient family of spectral curves is just the universal family of elliptic curves $\mathcal{W} \to \mathbb{P}(4,6) \backslash \delta$.

We may choose the coefficients of the polynomials $p_{a,b}(z)$ carefully such that the discriminant Δ has a nice form: it is a constant multiple of the function given in Table 6.1. This function descends to a function δ on $\mathbb{P}(r,s)$ whose zero set belongs to the set $t = \{0, 1, \infty\}$ with respect to the natural choices of coordinate $t = [a^s : b^r]$ or its reciprocal $t = [b^r : a^s]$. Moreover these three points are precisely the union of the orbifold points of $\mathbb{P}(r,s)$ and the zero set of δ, i.e. δ vanishes at $t = 0, \infty$ precisely when the corresponding r or s is equal to one.

The set δ is precisely the preimage of the point $j = 1$ under the j-map of the family $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$. The points of δ are known as *cusps* of the base of the family $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$. Their number is equal to the number of distinct irreducible factors of the discriminant Δ. The multiplicity of a factor is known as the *width* of the corresponding cusp.

Finally in Table 6.1 we have listed the subgroup $\Gamma(\mathcal{E})$ of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ whose congruence class corresponds to the monodromy of the family of spectral elliptic curves $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$. The monodromy groups are all of index at most six in the modular group $\mathrm{PSL}(2, \mathbb{Z})$ and are all congruence subgroups of level N at most four, meaning that they contain the kernel $\Gamma(N)$ of the homomorphism $\mathrm{PSL}(2, \mathbb{Z}) \to \mathrm{PSL}(2, \mathbb{Z}/N\mathbb{Z})$.

Apart from $\tilde{D}_4$ for which the monodromy is the principal congruence subgroup $\Gamma(2)$, the monodromy groups are all given by the preimage $\Gamma_0(N)$ of the unipotent matrices in $\mathrm{PSL}(2, \mathbb{Z}/N\mathbb{Z})$ under the map $\mathrm{PSL}(2, \mathbb{Z}) \to \mathrm{PSL}(2, \mathbb{Z}/N\mathbb{Z})$, i.e.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$

The congruence class in $\mathrm{PSL}(2, \mathbb{Z})$ of the monodromy of $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$ can be determined from the discriminant Δ in Table 6.1 with the help of a table of subgroups of the modular group $\mathrm{PSL}(2, \mathbb{Z})$ as, for example, in [CP03, Tables 2 and 4]. We will compute the monodromy by considering the j-map of the family $\mathcal{E} \to \mathbb{P}(2r, 2s) \backslash \delta$, which factors through the map

$$j : \mathbb{P}(r,s) \backslash \delta \to \mathbb{P}(2,3) \backslash \{1\}$$

As we have seen in Chapter 5 the map $\pi_1(\mathbb{P}(2,3)\backslash\{1\}) \to \mathrm{PSL}(2,\mathbb{Z})$ given by the monodromy of the universal family $\mathcal{W}$ of elliptic curves is an isomorphism. Thus the monodromy map of the family $\mathcal{E} \to \mathrm{PSL}(2,\mathbb{Z})$ is given by the pushforward of the fundamental group along the j-map

$$j_* : \pi_1(\mathbb{P}(r,s)\backslash\delta) \to \pi_1(\mathbb{P}(2,3)\backslash\{1\}) \cong \mathrm{PSL}(2,\mathbb{Z})$$

We can now study the monodromy through the branching of the j-map. We show that with the exception of $\tilde{A}_2$, the families of elliptic curves $\mathcal{E} \to \mathbb{P}(2r,2s)\backslash\delta$ have unramified j-map.

Lemma 6.2. *With the exception of $\tilde{A}_2$, the j-map $j : \mathbb{P}(r,s)\backslash\delta \to \mathbb{P}(2,3)\backslash\{1\}$ of the families of elliptic curves $\mathcal{E} \to \mathbb{P}(2r,2s)\backslash\delta$ specified by Table 6.1 is a regular covering map.*

Proof. We can compute the degree of the j-map by considering its extension $j : \mathbb{P}(r,s) \to \mathbb{P}(2,3)$ over the preimage of the point $j = 1$. The preimage consists of the set of cusps δ with ramification degree the width of the cusp, which are determined by the factorisation of the discriminant given in Table 6.1. The degree d of the j-map is obtained as the sum of all the cusp widths. It is now remains to check that the equality $\chi(\mathbb{P}(r,s)\backslash\delta) = d\chi(\mathbb{P}(2,3)\backslash\{1\})$ holds; we omit the necessary computation. $\square$

The Galois correspondence asserts that the map

$$j_* : \pi_1(\mathbb{P}(r,s)\backslash\delta) \to \pi_1(\mathbb{P}(2,3)\backslash\{1\})$$

is the embedding of a normal subgroup of index the degree of the map j. We conclude that, with the exception of the $\tilde{A}_2$ family, the monodromy group of the families of elliptic curves $\mathcal{E} \to \mathbb{P}(2r,2s)\backslash\delta$ is a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ of index the degree of the j-map. Together with the cusp widths, i.e. the branching over $j = 1$, this is enough to determine the subgroup in the list [CP03, Table 2].

Example 6.3. We study the monodromy of the $\tilde{A}_2$ family $\mathcal{E} \to \mathbb{P}(6,2)\backslash\delta$. There are two cusps, both of width one, and so the degree of the j-map is two. However we can compute

$$\chi(\mathbb{P}(3,1)\backslash\delta) = \chi(\mathbb{P}^1) - |\delta| - \left(1 - \frac{1}{3}\right) = 2 - 2 - \frac{2}{3} = -\frac{2}{3}$$

which is not equal to $2\chi(\mathbb{P}(2,3)\backslash\{1\}) = 2(-\frac{1}{6}) = -\frac{1}{3}$. The j-map is ramified to degree 2 at the orbifold point of order 3 and so the degree of excess ramification of the j-map is $\frac{1}{3}$ accounting for the above discrepancy.

The monodromy map is given by sending the generators γ_∞, γ_0 of $\pi_1(\mathbb{P}(3,1)\backslash\delta) \cong \mathbb{Z}_3 * \mathbb{Z}$ to the elements B^2 and $(BA)^{-1}$ of $\mathrm{PSL}(2,\mathbb{Z})$ where A and B denote the generators of order 2 and 3 of $\mathrm{PSL}(2,\mathbb{Z})$ as given in the proof of Proposition 5.6. This is clearly a surjective map and has a non-trivial kernel, containing the element $(\gamma_\infty \gamma_0^{-1})^2$.

We can summarise the above computation of the monodromy groups in the following Proposition, which for the A_2 family yields the commutative triangle at the end of Section 5.3.

Proposition 6.4. *For each Painlevé quiver there is a commutative triangle of groups*

$$\begin{array}{ccc} \mathbb{Z}_r * \mathbb{Z}_s & \longrightarrow & \pi_1(\mathbb{P}(r,s)\backslash\delta) \\ & \searrow & \downarrow {\scriptstyle m} \\ & & \Gamma(\mathcal{E}) \end{array}$$

whose diagonal map is given as the composition of the presentation of $\pi_1(\mathbb{P}(r,s)\backslash\delta)$ and the monodromy map m. Moreover, with the exception of $\tilde{A}_2$, all the arrows are isomorphisms.

6.3 Seiberg-Witten curves

In this section we study the period map of the Seiberg-Witten differential λ on the families of elliptic curves $\mathcal{E} \to \mathbb{P}(2r, 2s)\backslash\delta$ associated to each of the Painlevé quivers in the previous section. We construct a rational elliptic surface $E \to \mathbb{P}^1$ with the same period map over the coarse moduli space of $\mathbb{P}(2r, 2s)$ with precisely three singular fibres at the orbifold points and the points of δ. The period map is then completely determined by computing the characteristic exponents of the Picard-Fuchs equation of λ on $E \to \mathbb{P}^1$.

We recall from Chapter 3 that the Seiberg-Witten differential λ on the spectral elliptic curve is given by the restriction of the tautological 1-form on the total space of the twisted cotangent bundle $T^*_{\mathbb{P}^1}(\lceil \frac{P}{2} \rceil)$. We can also characterise λ as the differential form on the family of spectral elliptic curves $\mathcal{E} \to \mathbb{C}^2_{a,b}\backslash\Delta$ whose covariant derivative with respect to the Gauss-Manin connection along the vector field ∂_b is the holomorphic differential ω.

Proposition 6.5. *The Seiberg-Witten differential λ satisfies $\nabla^{GM}(\partial_b)(\lambda) = \omega$.*

Proof. The Seiberg-Witten differential on the family of spectral elliptic curves $\mathcal{E} \to \mathbb{C}^2_{a,b}\backslash\Delta$ has the form

$$\lambda = \frac{y}{f(z)} dz$$

where $f(z) \in H^0(\mathcal{O}_{\mathbb{P}^1}(\lceil \frac{P}{2} \rceil))$ is the unique, up to scale, holomorphic section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(\lceil \frac{P}{2} \rceil)$. It is easy to verify that $f(z)$ is the coefficient of b in the expression $p_{a,b}(z)$ in Table 6.1. Thus we have

$$\nabla^{GM}(\partial_b)(\lambda) = \frac{\partial_b(p_{a,b}(z))}{f(z)} \frac{dz}{y} = \frac{dz}{y} = \omega$$

$\square$

In order to study the period maps of the holomorphic and Seiberg-Witten differentials on the families $\mathcal{E} \to \mathbb{P}(2r, 2s)\backslash\delta$ of spectral elliptic curves we construct an auxiliary family of elliptic curves $E \to \mathbb{P}^1$ over the coarse moduli space. By construction the j-maps of these rational elliptic surfaces $E \to \mathbb{P}^1$ can be identified with the j-maps of the families of spectral curves. Thus the period maps of a given differential form on these two families E and $\mathcal{E}$ can also be identified, so we can study the periods of ω and λ on E instead.

We will construct the rational elliptic surface $E \to \mathbb{P}^1$ by pulling back the elliptic surface $\mathcal{E} \to \mathbb{P}(2r, 2s)\backslash\delta$ along a section τ of the natural coarse moduli map $\mathbb{P}(2r, 2s) \to \mathbb{P}^1$ forgetting the stacky structure. We note that there are a pair of reciprocal natural coordinates $[a^s : b^r]$ and $[b^r : a^s]$ on the coarse moduli space $\mathbb{P}^1$ for which the set of orbifold and singular points of the family $\mathcal{E} \to \mathbb{P}(2r, 2s)\backslash\infty$ is $\{\infty, 0, 1\}$. Our choice of section will also determine a choice t of one of these two coordinates.

The forgetful map $t : \mathbb{P}(2r, 2s) \to \mathbb{P}^1_t$ and its section τ intertwine the j-maps of the families E and $\mathcal{E}$ of elliptic curves which, as we have previously observed, factor through $\mathbb{P}(r, s)$. We have an identical diagram expressing the relationship between the period maps of differential forms on E and $\mathcal{E}$ whose restriction to the generic fibre is identical. In the sequel we will denote such differential forms on E and $\mathcal{E}$ by the same symbol so that in particular ω and λ denote the holomorphic and Seiberg-Witten differentials.
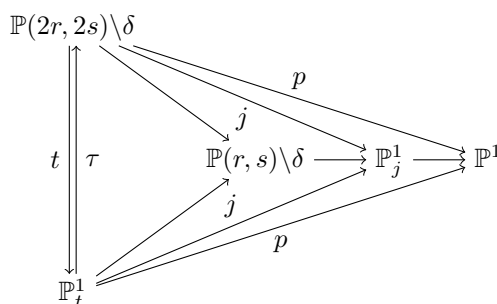
As in Section 5.4 we consider sections $\tau : \mathbb{P}^1 \to \mathbb{P}(2r, 2s)\backslash\delta$ of the form

$$a = t^p \qquad b = t^q$$

for some non-negative integers p, q and one of the two natural coordinates t on $\mathbb{P}^1$. There is a section of this form precisely when the equation

$$ps - qr = \pm 1$$

is satisfied with the sign determined by the choice of coordinate t.

We choose the section τ for which the sum $p + q$ is minimal. As r and s are coprime there is a unique such minimal solution to the equation which thereby fixes a choice of coordinate t on $\mathbb{P}^1$. We observe that the choice of t is in fact determined by whether the associated Painlevé quiver is isomorphic to its opposite or not. If $Q \cong Q^{\mathrm{op}}$, we have $t = [a^s : b^r]$, whereas if $Q \not\cong Q^{\mathrm{op}}$ we have $t = [b^r : a^s]$.

Pulling back the family of spectral elliptic curves $\mathcal{E} \to \mathbb{P}(2r, 2s)\backslash\delta$ along $\tau : \mathbb{P}^1_t \to \mathbb{P}(2r, 2s)\backslash\delta$ gives an elliptic curve E over the ring of integers $\mathbb{C}[t]$ of the function field $\mathbb{C}(t)$ of the form

$$y^2 = p_t(z)$$

where $p_t(z)$ is given by substituting $a = t^s$ and $b = t^r$ in the expression $p_{a,b}(z)$ in Table 6.1. We list the coordinate t, the polynomial $p_t(z)$ and its discriminant Δ in Table 6.2.

Q	p	t	Δ	E_∞	E_0	E_1
A_2	$z^3 - 3tz + 2t$	$[a^3 : b^2]$	$t^2(t-1)$	III*	II	I$_1$
A_3	$z^4 + 2z^2 + t$	$[b : a^2]$	$t(t-1)^2$	III*	I$_1$	I$_2$
D_3	$z((z+1)^2 - t)$	$[b : a^2]$	$t(t-1)^2$	III*	I$_1$	I$_2$
D_4	$z((z+1)^2(z+4) - 4t)$	$[b : a^3]$	$t(t-1)^3$	IV*	I$_1$	I$_3$
$\tilde{A}_1$	$z^3 + 2z^2 + tz$	$[a : b^2]$	$t^2(t-1)$	III*	I$_2$	I$_1$
$\tilde{A}_2$	$2z^3 - 3z^2 + t$	$[a : b^3]$	$t(t-1)$	II*	I$_1$	I$_1$
$\tilde{A}_3$	$z^4 + 2z^2 + t$	$[a : b^2]$	$t(t-1)^2$	III*	I$_1$	I$_2$
$\tilde{D}_3$	$(z^2 - t)(z + 1)$	$[a : b^2]$	$t(t-1)^2$	III*	I$_1$	I$_2$
$\tilde{D}_4$	$(z^2 - 1)(z^2 - t)$	$[a : b]$	$t(t-1)^4$	I$_1^*$	I$_1$	I$_4$
$\tilde{\tilde{D}}_4$	$z(z - 1)(z - t)$	$[a : 1]$	$t^2(t-1)^2$	I$_2^*$	I$_2$	I$_2$

Table 6.2: The families $E \to \mathbb{P}^1_t$ and their singular fibres

The elliptic surface $E \to \mathbb{P}^1_t$ is defined to be the minimal compactification of the curve E over $\mathbb{C}(t)$. As E is in fact defined over the ring of integers $\mathbb{C}[t]$, the discriminant Δ is a polynomial in t which by inspection has degree less than twelve. Thus the minimal compactification is a rational elliptic surface with fundamental line bundle $L = \mathcal{O}_{\mathbb{P}^1_t}(\infty)$ whose singular fibres we also list in Table 6.2.

As the j-map of $E \to \mathbb{P}^1_t$ can be identified with that of $\mathcal{E} \to \mathbb{P}(2r, 2s)\backslash\delta$, the monodromy group of the elliptic surface $E \to \mathbb{P}^1_t$ is given by the subgroup $\Gamma(\mathcal{E}) < \mathrm{PSL}(2, \mathbb{Z})$ in Table 6.1. Furthermore it follows from the study of the ramification of the j-map of the family of spectral curves $\mathcal{E}$ that the j-map of E is minimally ramified in the sense of Definition 4.19.

Proposition 6.6. *The rational elliptic surfaces $E \to \mathbb{P}^1_t$ are minimally ramified.*

Proof. We compute the ramification of the j-map $\mathbb{P}^1_t \to \mathbb{P}^1_j$ from its factorisation through the j-map $\mathbb{P}(r, s) \to \mathbb{P}(2, 3)$ via the section $\tau : \mathbb{P}^1_t \to \mathbb{P}(r, s)$ and the forgetful map $f : \mathbb{P}(2, 3) \to \mathbb{P}^1_j$.

$$\mathbb{P}^1_t \xrightarrow{\tau} \mathbb{P}(r, s) \xrightarrow{j} \mathbb{P}(2, 3) \xrightarrow{f} \mathbb{P}^1_j$$

By Lemma 6.2 the restriction of the j-map $j : \mathbb{P}(r, s)\backslash\delta \to \mathbb{P}(2, 3)\backslash\{1\}$ to the complement of the discriminant locus δ is an orbifold covering map, with the sole exception of the $\tilde{A}_2$ family. In the latter case we have seen in Example 6.3 that the sole ramification point of the map is the unique preimage of the $\mathbb{Z}_3$-orbifold point where the ramification degree is equal to the degree of the map which is two.

It is clear that the map $\mathbb{P}^1_t \to \mathbb{P}^1_j$ is not ramified outside $j = \{\infty, 0, 1\}$ for the branching of the maps τ and f occurs only at the orbifold points which are in the preimage of $j = \{\infty, 0\}$ and the map $\mathbb{P}(r, s) \xrightarrow{j} \mathbb{P}(2, 3)$ is unramified over the point $j = 1$. Furthermore the ramification of $\mathbb{P}^1_t \to \mathbb{P}^1_j$ over points in the preimage $j = \{\infty, 0\}$ is as follows:

$j = \infty$ The ramification degree is 1 at any singular point and 2 at any regular point.

$j = 0$ The ramification degree is 1 at any singular point with the exception of the $\tilde{A}_2$ case where it is 2, and 3 at any regular point.

Referring to the singular fibres of the surface $E \to \mathbb{P}^1_t$ given in Table 6.2 we see immediately that it is minimally ramified. $\square$

It follows that from Lemma 4.20 the Picard-Fuchs equation for the holomorphic differential ω on $E \to \mathbb{P}^1_t$ is quasi-hypergeometric with singularities only at $t = \{\infty, 0, 1\}$. The period map of ω is thus specified by the differences in the characteristic exponents at the three singular points, which we list in the following Proposition.

Proposition 6.7. *The differences in the characteristic exponents of the Picard-Fuchs equation for the holomorphic differential ω on $E \to \mathbb{P}^1_t$ at the points $t = \{\infty, 0, 1\}$ are as follows*

A_2	A_3/D_3	D_4	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3/\tilde{D}_3$	$\tilde{D}_4$	$\tilde{\tilde{D}}_4$
$(\frac{1}{2}, \frac{1}{3}, 0)$	$(\frac{1}{2}, 0, 0)$	$(\frac{1}{3}, 0, 0)$	$(\frac{1}{2}, 0, 0)$	$(\frac{2}{3}, 0, 0)$	$(\frac{1}{2}, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$

Proof. The fact that the j-map is minimally ramified shows that the Picard-Fuchs equation for ω has no apparent singularities. The characteristic exponents at the three singular points can be read off from Lemma 4.20, remembering to add 1 to the exponents at ∞ for $\omega \in H^0(\mathcal{O}_{\mathbb{P}^1_t}(-\infty))$ viewed as a function on $\mathbb{P}^1_t$ has a simple zero there. The Riemann schemes of the Picard-Fuchs equations are given in the first column of Table A.1 from which the differences in exponents are easily computed. $\square$

We have drawn the Schwarz triangles of the Picard-Fuchs equations for ω on the families $E \to \mathbb{P}^1_t$ associated to each Painlevé quiver in Table A in the Appendix.

We note that with the exception of the $\tilde{A}_2$-case the differences in exponents are all reciprocals of elements of the set $\mathbb{N} \cup \{\infty\}$. Thus by the Schwarz triangle theorem, the period map $\widetilde{\mathbb{P}(r, s)}\backslash\delta \to \mathbb{P}^1$ of the holomorphic differential ω is a bijection onto its image, the upper half-plane $\mathfrak{h} \subset \mathbb{P}^1$. This also follows from applying the modularity condition for an elliptic surface in [Dor01] which states that the period map of the holomorphic differential is orbifold-uniformising precisely when its j-map is minimally ramified and it contains no fibres of type II* or IV.

In the $\tilde{A}_2$ case the elliptic surface has a singular fibre of type II* at $t = \infty$ where the difference in exponents is $\frac{2}{3}$. Thus the analytic continuation of the period map to the universal cover $\widetilde{\mathbb{P}(3, 1)}\backslash\delta \to \mathbb{P}^1$ is not injective for the reflections of the Schwarz triangle overlap each other. This is another manifestation of the non-injectivity of the monodromy map $\pi_1(\mathbb{P}(3, 1)\backslash\delta) \to \mathrm{PSL}(2, \mathbb{Z})$ observed in Example 6.3.

We now compute the characteristic exponents of the Picard-Fuchs equation of the Seiberg-Witten differential λ on $E \to \mathbb{P}^1_t$. A priori we know that the monodromies of the Picard-Fuchs equations for λ and ω agree, and so the differences in the characteristic exponents of the equations can differ only by integers. In order to compute the exponents however, it is necessary to exploit the relationship between λ and ω established in Proposition 6.5, as we did in Chapter 4 for the A_2 family $U \to \mathbb{P}^1_j$.

Proposition 6.8. *The differences in the characteristic exponents of the Picard-Fuchs equation for the Seiberg-Witten differential λ on $E \to \mathbb{P}^1_t$ at the points $t = \{\infty, 0, 1\}$ are as follows*

A_2	A_3/D_3	D_4	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3/\tilde{D}_3$	$\tilde{D}_4$	$\tilde{\tilde{D}}_4$
$(\frac{1}{2}, \frac{1}{3}, 1)$	$(\frac{1}{2}, 1, 1)$	$(\frac{1}{3}, 1, 1)$	$(\frac{1}{2}, 0, 1)$	$(\frac{1}{3}, 0, 1)$	$(\frac{1}{2}, 0, 1)$	$(0, 1, 1)$	$(0, 0, 0)$

Proof. We consider the rational elliptic surfaces $\Sigma \to \mathbb{P}^1_b$, the minimal compactification of the slice of the family of spectral elliptic curves $\mathcal{E} \to \mathbb{C}^2_{a,b} \backslash \Delta$ given by setting $a = 1$. As the generic fibre is defined over $\mathbb{C}[b]$ it is easy to see that these are also rational elliptic surfaces with fundamental line bundle $L = \mathcal{O}_{\mathbb{P}^1_b}(\infty)$.

We have a t-map $\mathbb{P}^1_b \to \mathbb{P}^1_t$ given by $t = b^{\pm r}$ according to the expression of the t-coordinate given in Table 6.2. The family of elliptic curves $\Sigma \to \mathbb{P}^1_b$ thus differs by a quadratic twist from the pull-back of the family $E \to \mathbb{P}^1_t$ along the t-map, as shown in the diagram below. The quadratic twist q can be determined explicitly by, for example, considering the ratio of the determinants of the two surfaces $\Sigma \to \mathbb{P}^1_b$ and $t^*E \to \mathbb{P}^1_b$.

$$
\begin{array}{ccccc}
\Sigma & \xleftarrow{\ \ q\ \ } & t^*E & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^1_b & =\!=\!= & \mathbb{P}^1_b & \xrightarrow{\ \ t\ \ } & \mathbb{P}^1_t
\end{array}
$$

We now proceed as follows:

- We compute the characteristic exponents of the Picard-Fuchs equation of the holomorphic differential ω on $\Sigma \to \mathbb{P}^1_b$ by pulling back those on $E \to \mathbb{P}^1_t$ along the t-map $t = b^{\pm r}$ and performing a quadratic twist.

- We compute the characteristic exponents of the Picard-Fuchs equation of the Seiberg-Witten differential λ on $\Sigma \to \mathbb{P}^1_b$ from those for the holomorphic differential ω using the relationship $\nabla^{GM}(\partial_b)(\lambda) = \omega$.

- We compute the characteristic exponents of the Picard-Fuchs equation of the Seiberg-Witten differential λ on $E \to \mathbb{P}^1_t$ by performing a quadratic twist and pushing forward along the map given by $t = b^{\pm r}$.

The first step is an easy application of Lemma 4.6 to pull-back the exponents of the Picard-Fuchs equation of ω on $E \to \mathbb{P}^1_t$ along the t-map $t = b^{\pm r}$ whose ramification is easy to compute. The two elliptic surfaces $\Sigma \to \mathbb{P}^1_b$ and $t^*E \to \mathbb{P}^1_b$ differ by a quadratic twist by q, whose effect is to shift the characteristic exponents by $-\frac{deg(q)}{2}$, for ω is a section of the dual of the fundamental line bundle $L = \mathcal{O}(1)$. In other words the holomorphic differential ω has weight -1 with respect to the $\mathbb{C}^*$-action on the family of spectral elliptic curves.

The third step is nothing but the reverse of this process. It is easy to compute the weight w of the Seiberg-Witten differential λ with respect to the $\mathbb{C}^*$-action from the explicit form given in Lemma 6.5. Thus the shift in the characteristic exponents when reversing the quadratic twist is given by $\frac{-w \deg(q)}{2}$. Finally we push forward the Picard-Fuchs equation for the Seiberg-Witten differential

λ along the t-map, dividing the characteristic exponents by the degree of its ramification.

The second step involves integrating the Picard-Fuchs equation for ω with respect to the coordinate b to obtain the Picard-Fuchs equation for λ. If, as for the A_2 family, the Picard-Fuchs equation for ω on $\Sigma \to \mathbb{P}^1_b$ is hypergeometric, we can integrate it using Lemma 4.30. There are only two exceptional cases where the Picard-Fuchs equation of the holomorphic differential ω on $\Sigma \to \mathbb{P}^1_b$ is not hypergeometric:

$\tilde{A}_2$ The family $\Sigma \to \mathbb{P}^1_b$ has four singular fibres and the Picard-Fuchs equation for ω has an additional apparent singular point at $b = 0$. We will show how to integrate this Picard-Fuchs equation in Example 6.9.

$\tilde{\tilde{D}}_4$ The family $\Sigma \to \mathbb{P}^1_b$ is isotrivial with only two singular fibres of type I_0^* at $b = \infty$. Moreover the preimage of the Riemann sphere $\mathbb{P}^1_t$ under $b = t^0$ is just a point. In this case the family $E \to \mathbb{P}^1_t$ is independent of b and so the holomorphic and Seiberg-Witten differentials agree up to scale. Thus the differences in the characteristic exponents of the Picard-Fuchs equations for ω and λ are the same.

For convenience we collect the Riemann schemes for the four Picard-Fuchs equations associated to each quiver in Table A.1. The differences in the characteristic exponents of the Picard-Fuchs equation for λ on $E \to \mathbb{P}^1_t$ can be read off from the last column.

$\square$

We have drawn the Schwarz triangles for the Picard-Fuchs equations for λ in Table A in the Appendix. We note that with the exception of $\tilde{\tilde{D}}_4$ they are not identical to those for the holomorphic differential ω, and are no longer contained in the half-plane $\mathfrak{h} \subset \mathbb{P}^1$. A description of the relationship between the two Schwarz triangles is given in [Yos05].

Example 6.9. We show how to compute the characteristic exponents of the Picard-Fuchs equation of the Seiberg-Witten differential λ on the rational elliptic surface $E \to \mathbb{P}^1_t$ associated to $\tilde{A}_2$. As the j-map of the surface is minimally ramified, the Picard-Fuchs equation of the holomorphic differential has the Riemann scheme

t	∞	0	1
	$\frac{5}{6}$	0	0
	$\frac{1}{6}$	0	0

To compute the Riemann scheme of the Picard-Fuchs equation of ω on $\Sigma \to \mathbb{P}^1_b$ we must pull-back $E \to \mathbb{P}^1_t$ along the map $t = b^{-3}$. The resulting differential equation has five singular points: one at $b = \infty$, the unique preimage of $t = 0$, three at cube roots of unity, the preimages of $t = 1$, and an apparent singularity at the $b = 0$, the unique preimage of $t = \infty$.

b	∞	0	1	ρ	ρ^2
	0	$\frac{5}{2}$	0	0	0
	0	$\frac{1}{2}$	0	0	0

The surface $\Sigma_b \to \mathbb{P}^1_b$ differs from the surface $t^*E \to \mathbb{P}^1_b$ by a quadratic twist by $q = \sqrt{b}$. Thus the Picard-Fuchs equation for the holomorphic differential ω on $\Sigma \to \mathbb{P}^1_b$ has the following Riemann scheme.

b	∞	0	1	ρ	ρ^2
	$\frac{1}{2}$	2	0	0	0
	$\frac{1}{2}$	0	0	0	0

We claim that the above differential equation is in fact the derivative of an equation with only four regular singularities at infinity and the third roots of unity and no singularity, apparent or otherwise, at zero. Moreover apart from the singularity at infinity, one of the characteristic exponents at each of the other singularities is zero. This is the natural generalisation of the hypergeometric condition in Definition 4.25 to differential equations with four regular singularities; a normal form for such a differential equation is the so-called Heun equation whose Riemann scheme is

b	∞	0	1	a
	α	0	0	0
	β	$1-\gamma$	$1-\delta$	$1-\epsilon$

To specify such a Heun equation it does not suffice to give its characteristic exponents; in the classical language there is a single accessory parameter q. The derivative of the Heun differential equation for generic values of the accessory parameter has an apparent singularity whose location $b = \frac{q}{\alpha}\beta$ is determined by the accessory parameter. The Riemann scheme of the derivative of the Heun equation is

b	∞	$\frac{q}{\alpha}\beta$	0	1	a
	$\alpha+1$	2	0	0	0
	$\beta+1$	0	$-\gamma$	$-\delta$	$-\epsilon$

We observe that the Riemann scheme of the Picard-Fuchs equation for ω on $\Sigma \to \mathbb{P}^1_b$ is of this form after a harmless linear transformation in the dependent variable. In this case the locations of four non-apparent singular points of the Riemann scheme of the Picard-Fuchs equation for ω on $\Sigma_b \to \mathbb{P}^1$ form a so-called equianharmonic quadruple. Furthermore the apparent singular point lies in the distinguished position given by the mean of the three regular singular points other than ∞ at which the characteristic exponents are all the same.

The Heun equation whose derivative has such a specially symmetric Riemann scheme of this form is described in some detail by Maier [Mai05]. He shows that such a differential equation arises as a cubic transformation of a hypergeometric differential equation. For the Picard-Fuchs equation of the Seiberg-Witten differential λ on $\Sigma \to \mathbb{P}^1$ whose Riemann scheme is

b	∞	1	ρ	ρ^2
	$-\frac{1}{2}$	0	0	0
	$-\frac{1}{2}$	1	1	1

this transformation produces the Picard-Fuchs equation for λ on $E \to \mathbb{P}^1_t$.

The Seiberg-Witten differential

$$\lambda = \frac{y}{z^2} dz$$

has weight 1 with respect to the $\mathbb{C}^*$-action on the family of spectral curves. Thus the Riemann scheme for the Seiberg-Witten differential on $t^* E \to \mathbb{P}^1_b$ is given by shifting the characteristic exponents by $\frac{1}{2}$ at ∞ and $-\frac{1}{2}$ at 0.

b	∞	0	1	ρ	ρ^2
	0	$\frac{1}{2}$	0	0	0
	0	$-\frac{1}{2}$	1	1	1

Pushing forward along the t-map $t = b^{-3}$ gives the Riemann scheme for the Picard-Fuchs equation for λ on $E \to \mathbb{P}^1_t$.

t	∞	0	1
	$\frac{1}{6}$	0	0
	$-\frac{1}{6}$	0	1

Chapter 7

Stability conditions for the Painlevé quivers

In this chapter we complete the proof of the main theorem describing the canonical connected component $\text{Stab}^0(\mathcal{D}_Q)$ of the space of numerical stability conditions of the Painlevé quivers (except $Q = \tilde{D}_4$) via periods of the Seiberg-Witten differentials studied in the previous chapter.

We define a subquotient $\text{Aut}^0(\mathcal{D}_Q)$ of the autoequivalence group consisting of autoequivalences preserving the component $\text{Stab}^0(\mathcal{D})$ modulo those acting trivially on it. We find that a fundamental domain for the action of $\text{Aut}^0(\mathcal{D})/\mathbb{Z}$ on $\text{Stab}^0(\mathcal{D})/\mathbb{C}$ has image the double $\Diamond_Q$ of a Schwarz triangle in the space of projective numerical central charges $\mathbb{P}^1$. Together with a presentation $\mathbb{Z}_r * \mathbb{Z}_s \to \text{Aut}^0(\mathcal{D})/\mathbb{Z}$ we can define the desired lift of the period map to $\text{Stab}^0(\mathcal{D})/\mathbb{C}$ by equivariance with respect to the natural $\mathbb{Z}_r * \mathbb{Z}_s$-actions.

7.1 Numerical stability conditions

In this section we study the combinatorial structure of the connected component $\text{Stab}^0(\mathcal{D}_Q)$ of numerical stability conditions containing those supported on the standard heart $\mathcal{A}^0$. We note that this connected component is empty unless the standard heart $\mathcal{A}^0$ is numerical. In this case we see that the numerical U-domains of two hearts meet in codimension 1 if and only if they are related by a composite of simple tilts at all simple objects in a given numerical equivalence class.

Recall that each finite-length heart $\mathcal{A}$ defines a cone $C(\mathcal{A}) \subset \text{Hom}(K(\mathcal{D}), \mathbb{C})$ of maximal dimension in the vector space of central charges consisting of those for which the central charge of the simple objects of $\mathcal{A}$ belongs to the upper-half plane. Such a heart $\mathcal{A}$ defines a chamber $U(\mathcal{A}) \cong \mathfrak{h}^n$ in the distinguished connected component of the space of stability conditions $\text{Stab}^*(\mathcal{D})$. Two such chambers intersect in codimension 1 if and only if they are related by a simple tilt at one of the simple objects of the heart $\mathcal{A}$, which corresponds to extremal ray of the dual cone $C(\mathcal{A})^* \subset K(\mathcal{D}) \otimes \mathbb{C}$.

Similarly a finite-length heart defines a cone $C^0(\mathcal{A}) \subset \text{Hom}(N(\mathcal{D}), \mathbb{C})$ by intersecting the cone $C(\mathcal{A})$ with the space of numerical central charges. However

this cone is not necessarily of maximal dimension for restriction of the cone $C(\mathcal{A})$ to the vector subspace of numerical central charges may not necessarily result in a slice of maximal dimension.

Definition 7.1. A finite-length heart $\mathcal{A}$ is said to be numerical if its numerical cone of central charges $C^0(\mathcal{A}) = C(\mathcal{A}) \cap \mathrm{Hom}(N(\mathcal{D}), \mathbb{C})$ has dimension equal to the rank of the numerical Grothendieck group $N(\mathcal{D})$.

It is easy to check that a heart $\mathcal{A}$ is numerical if and only if any element of $K(\mathcal{D})$ in the kernel of the Euler form of $\mathcal{D}$ has both positive and negative coefficients when expressed in the basis of simple objects of the heart $\mathcal{A}$. The codimension 1 boundary components of the numerical cone $C^0(\mathcal{A})$ are in bijection with the extremal rays of the dual cone $C^0(\mathcal{A})^* \subset N(\mathcal{D}) \otimes \mathbb{C}$. For a numerical heart $\mathcal{A}$ each ray of the cone $C^0(\mathcal{A})^*$ can be labelled by the possibly multiple simple objects of heart $\mathcal{A}$ whose corresponding extremal ray in $C(\mathcal{A})^*$ projects to the given ray in $C^0(\mathcal{A})^*$.

Definition 7.2. A numerical equivalence class ν of simple objects of a heart $\mathcal{A}$ is a subset of the set of simple objects of $\mathcal{A}$ consisting of those which define the same class in $N(\mathcal{D})$.

We consider the tilts of a numerical heart $\mathcal{A}$ with respect to torsion pairs for which either the torsion or torsion-free part is generated by the simple objects in a numerical equivalence class.

Definition 7.3. The left and right numerically simple tilts $\mu_\nu^\pm(\mathcal{A})$ of a heart $\mathcal{A}$ at the numerical equivalence class ν are the tilted hearts associated to the torsion pairs

$$\mathcal{T} = \langle S \mid S \in \nu \rangle \qquad \mathcal{F} = \{X \mid \mathrm{Hom}(S, X) = 0 \,\forall\, S \in \nu\}$$

and

$$\mathcal{F} = \langle S \mid S \in \nu \rangle \qquad \mathcal{T} = \{X \mid \mathrm{Hom}(X, S) = 0 \,\forall\, S \in \nu\}$$

respectively.

We have the following criterion which determines when the numerically simple tilts of a numerical heart $\mathcal{A}$ are themselves numerical.

Lemma 7.4. *The hearts $\mu_{\nu_k}^\pm(\mathcal{A})$ are numerical if and only if the numerical equivalence class ν_k defines an extremal ray of the cone $C^0(\mathcal{A})^*$.*

Proof. An element of $K(\mathcal{D})$ in the kernel of the Euler form is necessarily of the form

$$\sum_i a_i [\nu_i]$$

where

$$[\nu_i] = \sum_{S_j \in \nu_i} [S_j]$$

are the sums over numerical equivalence classes of the classes of the simple objects $[S_i]$ of $\mathcal{A}$. The numerical equivalence classes ν_i' of the numerically tilted heart are in bijection with those of ν_i related by the analogue of Lemma 2.36. In particular we have that

$$[\nu_k'] = -[\nu_k] \qquad [\nu_i'] = [\nu_i] + c_i[\nu_k] \quad \text{for } i \neq k$$

for some non-negative coefficients c_i.

A heart is numerical if and only if there exists an element of $K(\mathcal{D})$ in the radical of the Euler form whose the coefficients a_i do not all have the same sign. Rewriting the element of $K(\mathcal{D})$ in terms of the classes $[\nu_i']$ of the tilted heart, we observe that the only coefficient which can possibly change sign is a_k. Thus the tilted heart can only be non-numerical if there exists an element of the kernel whose only negative coefficient is a_k, i.e. if the ray ν_k is not extremal.

Conversely if ν_k is not extremal there exists an element of the kernel whose only non-negative coefficient is a_k. It is clear from the above relations that the coefficient of $[\nu_k']$ in its reexpression in terms of the basis of simple objects $[S_i']$ has a positive coefficient. Thus the tilted heart is not extremal. □

In light of the previous Proposition we make the following Definition

Definition 7.5. A numerical equivalence class ν of a numerical heart $\mathcal{A}$ is called numerically tiltable if it defines an extremal ray of the cone $C^0(\mathcal{A})^*$.

The numerical U-domain $U^0(\mathcal{A})$ of a numerical heart $\mathcal{A}$ is defined to be the subset of the space of numerical stability conditions which are supported on $\mathcal{A}$. We have an analogue of Bridgeland's result in the non-numerical setting specifying when the closures of two numerical U-domains intersect in codimension 1.

Proposition 7.6. *The numerical U-domains of two numerical hearts intersect in codimension* 1 *in* $\mathrm{Stab}^0(\mathcal{D})$ *if and only if they are related by a numerically simple tilt.*

Proof. An extremal ray of the cone $C^0(\mathcal{A})^*$ labelled by the numerical equivalence class ν corresponds to a pair of codimension 1 boundary components of the cone of numerical central charges $C^0(\mathcal{A})$ supported on $\mathcal{A}$ which it shares with the cone $C^0(\mu_\nu^\pm \mathcal{A})$ of the respective numerical tilts. Moreover by the previous Lemma the extremal rays are labelled by the numerically tiltable equivalence classes. The result now follows by a similar argument to the proof of Lemma 2.15. □

7.2 Numerical autoequivalences

In this section we study the distinguished connected component of the space of numerical stability conditions $\mathrm{Stab}^0(\mathcal{D}_{Q,W})$ containing those supported on the standard heart. We consider the hearts supporting a numerical stability condition in this component and the numerical tilts which relate them. Composites of the associated Keller-Yang equivalences provide autoequivalences of the category $\mathcal{D}_{Q,W}$ which preserve the connected component $\mathrm{Stab}^0(\mathcal{D}_{Q,W})$.

Definition 7.7. The numerical autoequivalence group $\mathrm{Aut}^0(\mathcal{D}_{Q,W})$ is the subquotient of the group of autoequivalences $\mathrm{Aut}(\mathcal{D}_{Q,W})$ preserving the distinguished connected component $\mathrm{Stab}^0(\mathcal{D}_{Q,W})$ of numerical stability conditions of $\mathcal{D}_{Q,W}$.

As in the non-numerical setting we will have a wall-and-chamber structure on the space of numerical stability conditions whose chambers are labelled by numerical hearts which are related to the standard heart by successive numerical tilts.

Definition 7.8. A heart $\mathcal{A}$ of $\mathcal{D}_{Q,W}$ is called numerically reachable if it can be obtained from the standard heart $\mathcal{A}^0$ by a finite sequence of numerical tilts at numerically tiltable equivalence classes.

We note that by Proposition 7.4 any numerically reachable heart is necessarily numerical. However it is not true that any reachable numerical heart of $\mathcal{D}_{Q,W}$ is numerically reachable, for it might not be obtained by a sequence of *numerical* tilts at numerically tiltable equivalence classes. For example the image of a standard heart under a spherical twist at a simple object is again numerical for it leaves the quiver of the heart unchanged, but it is only numerically reachable if the simple object defines a numerically tiltable equivalence class.

We denote by N the set of all numerically reachable hearts of the category $\mathcal{D}_{Q,W}$ and consider the Keller-Yang equivalences of $\mathcal{D}_{Q,W}$ which preserve N. An important property of the categories $\mathcal{D}_{Q,W}$ is that any reachable heart has an Ext-quiver with no 2-cycles. As a consequence we deduce that any pair of distinct simple objects in a numerical equivalence class of a reachable heart are orthogonal in $\mathcal{D}_{Q,W}$.

Lemma 7.9. *Let $\nu = \{S_j\}$ be a numerical equivalence class of a reachable heart $\mathcal{A}$ in $\mathcal{D}_{Q,W}$. Then $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{Q,W}}(S_i, S_j) = 0$ whenever $i \neq j$.*

Proof. Suppose S_i and S_j are two simple objects in the numerical equivalence class ν. As $\mathcal{D}_{Q,W}$ is a CY3 category we know that the morphism complex $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{Q,W}}(S_i, S_j)$ is concentrated in degrees 1 and 2. As the class $[S_i] - [S_j]$ belongs to the kernel of the Euler form, we have that

$$0 = \chi([S_i] - [S_j], [S_j]) = \dim \mathrm{Hom}^1_{\mathcal{D}}(S_j, S_i) - \dim \mathrm{Hom}^2_{\mathcal{D}}(S_j, S_i)$$
$$= \dim \mathrm{Ext}_{\mathcal{A}}(S_j, S_i) - \dim \mathrm{Ext}_{\mathcal{A}}(S_i, S_j)$$

As the Ext-quiver of $\mathcal{A}$ has no 2-cycles, we conclude that the morphism complex $\mathrm{Hom}^{\cdot}_{\mathcal{D}_{Q,W}}(S_i, S_j) = 0$. $\square$

An immediate corollary is that the tilted heart $\mu_{S_i}(\mathcal{A})$ for $S_i \in \nu$ contains *as simple objects* the remaining simple objects of $\mathcal{A}$ in the numerical equivalence class ν. We now show that the tilts with respect to simple objects in the same numerical equivalence class commute.

Proposition 7.10. *The simple tilts $\mu^{\pm}_{S_k}$ for S_k belonging to a numerical equivalence class ν of $\mathcal{A}$ commute, and their product $\prod_{\nu} \mu^{\pm}_{S_k} = \mu^{\pm}_{\nu}$ is equal to the numerically simple tilt at ν.*

Proof. The number of arrows between a given vertex of the Ext-quiver of the heart $\mathcal{A}$ and any vertex in ν is equal. In particular the Ext-quiver of the heart $\mathcal{A}$ has no arrows between any vertices corresponding to the simple objects $S_k \in \nu$. Thus the arrows incident to the other vertices of ν of the quiver obtained by mutating at a vertex corresponding to a particular simple object $S_k \in \nu$ are unchanged.

Fix a given order of the simple objects in the numerical equivalence class ν and denote by Tw_{ν} the product $\mathrm{Tw}_{\nu} = \prod_{\nu} \mathrm{Tw}_{S_k}$. By Lemma 2.36 we have that the simple objects of the composite $\prod_{\nu} \mu^{\pm}_{S_k}(\mathcal{A})$ are

$$\{S_k[\pm] : S_k \in \nu\} \cup \{\mathrm{Tw}_{\nu}(S_i) : \pm\chi(S_i, S_k) \geq 0\} \cup \{S_i : \pm\chi(S_k, S_i) \geq 0\}$$

By [ST01, Lemma 2.11] the spherical twists at orthogonal simple objects commute. We conclude that the set of simple objects of the heart $\prod_\nu \mu_{S_k}^\pm(\mathcal{A})$ is independent of the order of the product, and so the tilts with respect to simple objects in the numerical equivalence class commute. $\qquad\square$

We will consider the composite of the Keller-Yang equivalences over all simple objects in a numerical equivalence class, which send the standard heart $\mathcal{A}^0$ to its numerical tilt.

Definition 7.11. The numerical Keller-Yang equivalences $\Phi_\nu^\pm : \mathcal{D}_{Q,W} \to \mathcal{D}_{\mu_\nu(Q,W)}$ associated to the numerical equivalence class ν are defined to be the composites $\prod_{S_i \in \nu} \Phi_{S_k}^\pm$.

Numerical Keller-Yang equivalences with respect to numerically tiltable equivalence classes preserve N. In particular composites of such numerical Keller-Yang equivalences which preserve the quiver Q define elements of the numerical autoequivalence group $\mathrm{Aut}^0(\mathcal{D})$.

Definition 7.12. The reachable numerical autoequivalence group $\mathrm{Aut}_N^0(\mathcal{D})$ is the subgroup of $\mathrm{Aut}^0(\mathcal{D})$ which preserves the set of numerically reachable hearts N.

In the next section we will compute the group $\mathrm{Aut}_N^0(\mathcal{D}_Q)$ of reachable numerical autoequivalences for the categories $\mathcal{D}_Q$ associated to the Painlevé quivers Q with the exception of $\tilde{\tilde{D}}_4$.

7.3 Painlevé quivers

In this section we consider the numerical Keller-Yang equivalences associated to numerical simple tilts of the Painlevé quivers with the exception of $Q = \tilde{\tilde{D}}_4$. We use the labelling the remaining seven finite and affine Painlevé quivers given in Figures 6.3 and 6.4.

Remark 7.13. One can easily check that none of the four quivers with potential in the mutation equivalence class of $\tilde{\tilde{D}}_4$ are numerical, and so the corresponding distinguished connected components $\mathrm{Stab}^0(\mathcal{D}_{Q,W})$ are empty. The hearts of $\mathcal{D}_{\tilde{\tilde{D}}_4}$ which do support numerical stability conditions are not of finite-length and so the analysis of the previous two sections is not applicable.

As the numerical Grothendieck group $N(\mathcal{D}_Q)$ has rank two for the Painlevé quivers Q, the cone $C^0(\mathcal{A}^0)^*$ spanned by the numerical equivalence classes ν of simple objects must have precisely two extremal rays. Moreover as the corresponding numerical equivalence classes generate the numerical Grothendieck group, there are arrows in the quiver Q between simple objects of the classes. We denote the two classes by ν_+ and ν_- such that the arrows in Q all point from vertices of the former to the latter.

We list the vertices belonging to the numerical equivalence classes $\nu_\pm$ for each of the Painlevé quivers in the Table below. We notice that every vertex of each Painlevé quiver belongs to one of the two numerical equivalence classes $\nu_\pm$ with the exception of $\tilde{A}_2$. In this case each vertex belongs to its own numerical equivalence class, but the ray in $N(\mathcal{D})$ corresponding to the simple object S_2 is not extremal for we have the relation $[S_2] = [S_1] + [S_3]$.

As in Chapter 5 we note that there is a unique wall of marginal stability in the numerical U-domain of any numerical heart $\mathcal{A}$ of $\mathcal{D}_{Q,W}$. On one side of the wall, the only stable objects are shifts of the simple objects in the numerical equivalence classes $\nu_{\pm}$ of $\mathcal{A}^0$, and on the other side the stable objects are all the indecomposable objects of the heart $\mathcal{A}$. We will call a numerical tilt at the numerically tiltable classes $\nu_{\pm}$ (un)stable respectively according to the side of the wall of marginal stability on which the corresponding codimension 1 wall of the numerical U-domain of $\mathcal{A}$ lies.

We denote by $\Phi_{\nu_{\pm}}^{\pm}$ the four numerical Keller-Yang equivalences which realise the four numerical tilts $\mu_{\nu_{\pm}}^{\pm}$ of a numerical heart $\overline{\mathcal{A}}$. We call a numerical Keller-Yang equivalence $\Phi_{\nu_{\pm}}^{\pm}$ (un)stable if the corresponding numerical tilt is.

Proposition 7.14. *The numerical Keller-Yang equivalence $\Phi_{\nu_{\pm}}^{\pm}$ is stable respectively unstable according to whether the product of the two signs is $+$ or $-$.*

Proof. This follows immediately from our convention that the numerical equivalence classes $\nu_{\pm}$ are labelled such that the arrows in the quiver of the numerical heart $\mathcal{A}$ point from ν_{-} to ν_{+}. $\qquad\square$

If a numerical heart $\mathcal{A}' = \Phi_{\nu_{\pm}}^{\pm}(\mathcal{A})$ is obtained from $\mathcal{A}$ by a numerical tilt, then the original heart $\mathcal{A} = \Phi_{\nu_{\mp}}^{\mp}(\mathcal{A}')$ is recovered by applying the Keller-Yang equivalence with the two opposite signs. The set of numerically reachable hearts N is therefore given the structure of a homogeneous set for the free group on two generators by the action of the numerical Keller-Yang equivalences $\Phi_{\nu_{\pm}}^{\pm}(\mathcal{A})$.

In Table 7.1 we have computed the action $\phi_{\nu_{\pm}}^{\pm}$ of the numerical Keller-Yang equivalences $\Phi_{\nu_{\pm}}^{\pm}$ of the standard heart $\mathcal{A}^0$ of each of the Painlevé quivers on $N(\mathcal{D})$. We list the change of basis matrix from the basis $\{\nu_+, \nu_-\}$ of numerical equivalence classes of the standard heart $\mathcal{A}^0$ to the basis $\{\nu'_+, \nu'_-\} = \{\nu_-, \nu_+\}$ of the tilted heart. By the previous paragraph the square of these matrices is the identity, so they define reflections of the lattice $N(\mathcal{D})$.

By definition mutation at either of the numerical equivalence classes $\nu_{\pm}$ reverses all arrows between vertices in the two numerical equivalences classes. In the case of the Painlevé quivers, we observe further that either mutation sends the quiver Q to its opposite Q^{op}. Indeed this is immediate in the case of the bipartite quivers and is easily checked for the remaining $\tilde{A}_2$ quiver.

In the sequel we will make the distinction between the two classes of the Painlevé quiver Q according to whether Q is isomorphic to its opposite Q^{op} or not.

$Q \cong Q^{\mathrm{op}} : A_2, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{D}_3$

$Q \not\cong Q^{\mathrm{op}} : A_3, D_3, D_4, \tilde{D}_4$

If Q is isomorphic to its opposite then there exists an autoequivalence $T : \mathcal{D}_{Q^{\mathrm{op}}} \to \mathcal{D}_Q$ which sends the simple objects in the numerical equivalence classes $\nu_{\pm}$ of the standard heart of $\mathcal{D}_{Q^{\mathrm{op}}}$ to the classes $\nu_{\mp}$ of the standard heart of $\mathcal{D}_Q$. Any autoequivalence with these properties defines the same element of the group of numerically reachable autoequivalences $\mathrm{Aut}_N^0(\mathcal{D})$.

Proposition 7.15. *For the finite and affine Painlevé quivers Q we have either*

Q	ν_+	ν_-	$\phi^+_{\nu_+}$	$\phi^+_{\nu_-}$	$\phi^-_{\nu_+}$	$\phi^-_{\nu_-}$
A_2	$\{0\}$	$\{1\}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$
A_3	$\{0\}$	$\{1,2\}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
D_3	$\{0\}$	$\{1,2\}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
D_4	$\{0\}$	$\{1,2,3\}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}$
$\tilde{A}_1$	$\{0\}$	$\{1\}$	$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
$\tilde{A}_2$	$\{0\}$	$\{2\}$	$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
$\tilde{A}_3$	$\{0,2\}$	$\{1,3\}$	$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
$\tilde{D}_3$	$\{0,2\}$	$\{1,3\}$	$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
$\tilde{D}_4$	$\{0\}$	$\{1,2,3,4\}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}$

Table 7.1: The action on $N(\mathcal{D})$ of the numerical tilt of the standard heart of $\mathcal{D}_Q$

$Q = Q^{\mathrm{op}}$ *The group* $\mathrm{Aut}^0_N(\mathcal{D}_Q)$ *is generated by the four elements* $T \cdot \Phi^{\pm}_{\nu_{\pm}}$.

$Q \neq Q^{\mathrm{op}}$ *The group* $\mathrm{Aut}^0_N(\mathcal{D}_Q)$ *is generated by the sixteen elements* $\Phi^{\pm}_{\nu_{\pm}} \cdot \Phi^{\pm}_{\nu_{\pm}}$.

Proof. A composite of numerical Keller-Yang equivalences is an equivalence $\mathcal{D}_Q \to \mathcal{D}_Q$ or $\mathcal{D}_Q \to \mathcal{D}_{Q^{\mathrm{op}}}$ which preserves N according to the parity of the length of the composition. It defines an autoequivalence of the category $\mathcal{D}_Q$, after composition with the standard equivalence T if required, if and only if the image of the standard heart $\mathcal{A}^0$ has the same quiver.

Conversely the image of the standard heart under any autoequivalence of $\mathcal{D}_{Q,W}$ which preserves N acts on the standard heart $\mathcal{A}^0$ by a composite of Keller-Yang equivalences. Thus the group $\mathrm{Aut}^0_N(\mathcal{D}_Q)$ consists of composites of Keller-Yang equivalences and the standard equivalence T which preserve the quiver Q. As T commutes with the Keller-Yang equivalences the group $\mathrm{Aut}^0_N(\mathcal{D}_Q)$ is generated by the composites given in the statement of the Proposition. $\square$

For the Painlevé quivers the shift functor is always numerically reachable for the shift $\mathcal{A}^0[1]$ of the standard heart is realised by a composite of unstable tilts. Indeed the shift functor is equal to the composite $\Phi^+_{\nu_-}\Phi^+_{\nu_-}$ when the quiver is bipartite, and $\mathcal{A}^0[1] = (\Phi^+_{\nu_-})^3(\mathcal{A}^0)$ for the sole non-bipartite Painlevé quiver $Q = \tilde{A}_2$. We can show that the quotient of the group of reachable numerical autoequivalences $\mathrm{Aut}^0_N(\mathcal{D})/\mathbb{Z}$ by the shift functor is generated by a pair $\overline{\Sigma}$ and $\overline{\Upsilon}$ of autoequivalences up to shift.

Lemma 7.16. *The quotient group* $\mathrm{Aut}^0_N(\mathcal{D})/\mathbb{Z}$ *is generated by the equivalence classes of the numerically reachable autoequivalences* Σ *and* Υ *as follows.*

- *If $Q \cong Q^{\mathrm{op}}$*

$$\Sigma = T \cdot \Phi^+_{\nu_+} \qquad \Upsilon = T \cdot \Phi^+_{\nu_-}$$

- *If $Q \not\cong Q^{\mathrm{op}}$*

$$\Sigma = \Phi^-_{\nu_-} \Phi^-_{\nu_-} \qquad \Upsilon = \Phi^-_{\nu_-} \Phi^+_{\nu_-}$$

Proof. We observe that the inverses of the above autoequivalences are obtained by reversing all of the signs of the Keller-Yang equivalences. If $Q \cong Q^{\mathrm{op}}$, the result now follows immediately from the previous Proposition which states that $\mathrm{Aut}^0_N(\mathcal{D})$ is generated by Σ and Υ and their inverses.

If $Q \not\cong Q^{\mathrm{op}}$ the previous Proposition implies that $\mathrm{Aut}^0_N(\mathcal{D}_Q)$ is generated by the sixteen composites $\Phi^\pm_{\nu_\pm} \Phi^\pm_{\nu_\pm}$ for all possible choices of sign. However using the relations $\Phi^\pm_{\nu_\pm} \Phi^\mp_{\nu_\mp} = \mathrm{id}$ and $\Phi^+_{\nu_-} \Phi^+_{\nu_-} = [1]$ one can check that $\mathrm{Aut}^0_N(\mathcal{D})/\mathbb{Z}$ is generated by the two elements $\Phi^-_{\nu_-} \Phi^-_{\nu_-}$ and $\Phi^-_{\nu_+} \Phi^-_{\nu_-}$. $\qquad\square$

For the A_2 quiver we have already seen how to express the generators Σ and Υ as a product of spherical twists and the shift functor. For the Painlevé quivers Q which are not isomorphic to their opposite this is also possible for we have

$$\Upsilon = \mathrm{Tw}_-^{-1} \qquad \Sigma = \mathrm{Tw}_-^{-1}\mathrm{Tw}_+^{-1}$$

The remaining Painlevé quivers $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3 = \tilde{D}_3$ have an underlying graph which is a cycle. In this case the group $\mathrm{Aut}^0_N(\mathcal{D})/\mathbb{Z}$ is not generated by spherical twists, although they generate the group together with the autoequivalence $\Delta = T \cdot \Phi^+_{\nu_-}$. It acts on the simple objects of the standard heart in their natural cyclic order by shifting the simple objects in the numerical equivalence class ν_- by [1] and rotating the result objects clockwise in the cyclic order by one position.

In Table 7.2 we collect the explicit expressions for the generators Σ and Υ in terms of spherical twists, the shift functor, and the autoequivalence Δ. We have also computed their actions σ and υ on the numerical Grothendieck group. In the last column we have given the congruence class $\Gamma(Q) < \mathrm{PSL}(2, \mathbb{Z})$ of the subgroup which they generate, which we note is the same congruence subgroup of the modular group as the monodromy $\Gamma(\mathcal{E})$ of the associated family $\mathcal{E}$ of spectral elliptic curves.

We can now prove an analogue of the result at the end of Section 5.1 for the Painlevé quivers.

Proposition 7.17. *For each of the finite and affine Painlevé quivers there is a commutative triangle of groups*

$$
\begin{array}{ccc}
\mathbb{Z}_r * \mathbb{Z}_s & \longrightarrow & \mathrm{Aut}^0_N(\mathcal{D}_Q)/\mathbb{Z}[1] \\
& \searrow & \downarrow{\scriptstyle\phi} \\
& & \Gamma(Q)
\end{array}
$$

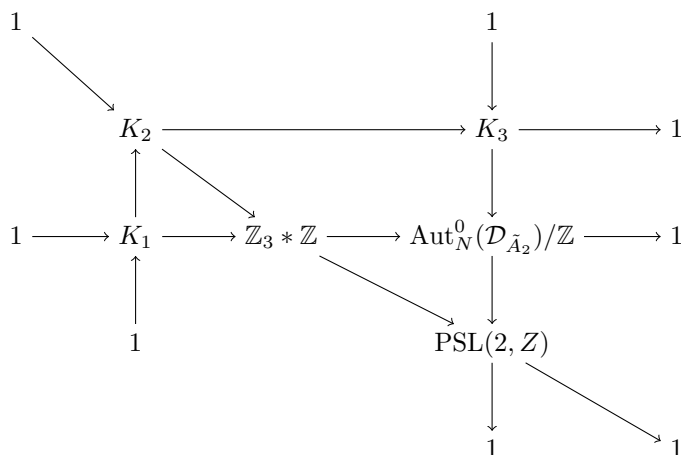*where ϕ is the map sending a numerically reachable autoequivalence $\mathrm{Aut}^0_N(\mathcal{D})/\mathbb{Z}$ to its action on $N(\mathcal{D})$ and the remaining maps are given by sending the generators of $\mathbb{Z}_r * \mathbb{Z}_s$ to (Υ, Σ) and (υ, σ) respectively. The maps are all surjective and are isomorphisms with the exception of the $\tilde{A}_2$ quiver.*

Q	Σ	Υ	σ	υ	Γ_Q
A_2	$\mathrm{Tw}_-\mathrm{Tw}_+[2]$	$\mathrm{Tw}_+\mathrm{Tw}_-\mathrm{Tw}_+[3]$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\Gamma(1)$
A_3	$\mathrm{Tw}_-^{-1}\mathrm{Tw}_+^{-1}$	Tw_-	$\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\Gamma_0(2)$
D_3	$\mathrm{Tw}_-^{-1}\mathrm{Tw}_+^{-1}$	Tw_-	$\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	$\Gamma_0(2)$
D_4	$\mathrm{Tw}_-^{-1}\mathrm{Tw}_+^{-1}$	Tw_-	$\begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}$	$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$	$\Gamma_0(3)$
$\tilde{A}_1$	$\mathrm{Tw}_-^{-1}\Delta^{-1}$	Δ	$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\Gamma_0(2)$
$\tilde{A}_2$	$\mathrm{Tw}_-^{-1}\Delta^{-1}$	Δ	$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$	$\Gamma(1)$
$\tilde{A}_3$	$\mathrm{Tw}_-^{-1}\Delta^{-1}$	Δ	$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\Gamma_0(2)$
$\tilde{D}_3$	$\mathrm{Tw}_-^{-1}\Delta^{-1}$	Δ	$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\Gamma_0(2)$
$\tilde{D}_4$	$\mathrm{Tw}_-^{-1}\mathrm{Tw}_+^{-1}$	Tw_-	$\begin{pmatrix} 1 & -4 \\ 1 & -3 \end{pmatrix}$	$\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$	$\Gamma_0(4)$

Table 7.2: The generators Σ and Υ of the group $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}[1]$ and their action on $N(\mathcal{D}_Q)$.

Proof. It is easy to check that the autoequivalences Υ and Σ have orders r and s where they are greater than one and infinite order when they are equal to one. Thus we have a commutative diagram of groups all of whose arrows are surjections. Furthermore with the exception of the $\tilde{A}_2$ case the diagonal map in Proposition 6.4 is an isomorphism, from which we deduce that all arrows are isomorphisms. $\square$

Example 7.18. We study the commutative triangle of groups in Proposition 7.17 in the exceptional case of the $\tilde{A}_2$ quiver where the diagonal arrow $\mathbb{Z}_3 * \mathbb{Z} \to \mathrm{PSL}(2,\mathbb{Z})$ is not an isomorphism. In this case we will see that none of the arrows is in fact an isomorphism. Consider the diagram

$$
\begin{array}{ccccc}
1 & & & 1 & \\
\downarrow & & & \downarrow & \\
K_2 & \longrightarrow & K_3 & \longrightarrow & 1 \\
\uparrow & & \downarrow & & \\
1 \longrightarrow K_1 \longrightarrow & \mathbb{Z}_3 * \mathbb{Z} & \longrightarrow \mathrm{Aut}_N^0(\mathcal{D}_{\tilde{A}_2})/\mathbb{Z} & \longrightarrow & 1 \\
\uparrow & & \downarrow & & \\
1 & & \mathrm{PSL}(2,Z) & & \\
& & \downarrow & \searrow & \\
& & 1 & & 1
\end{array}
$$

We have seen that there is a map $\mathrm{Br}(\tilde{A}_2) \to \mathrm{Aut}_N^0(\mathcal{D}_{\tilde{A}_2})/\mathbb{Z}$ from the affine braid group to the quotient of the numerically reachable autoequivalence group by $\mathbb{Z}$. We note that the kernel K_1 contains the free subgroup $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ generated by the three elements corresponding to the braid relations. Furthermore by [Bri05, Proposition 2.2] we have a short exact sequence

$$1 \to \mathbb{Z} * \mathbb{Z} * \mathbb{Z} \to \mathrm{Br}(\tilde{A}_2) \to \mathrm{PSL}(2,\mathbb{Z}) \to 1$$

whose second map factors through

$$\mathrm{Br}(\tilde{A}_2) \to \mathrm{Aut}_N^0(\mathcal{D}_{\tilde{A}_2}) \xrightarrow{\phi} \mathrm{PSL}(2,\mathbb{Z})$$

The free subgroup $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ is the image of the subgroup of the kernel of the map $\mathbb{Z}_3 * \mathbb{Z} \to \mathrm{PSL}(2,\mathbb{Z})$ generated by the elements $(\gamma_\infty \cdot \gamma_0)^2$ and their conjugates by $\gamma_\infty^{\pm 1}$ identified in Example 6.3.

7.4 V-domains and fundamental domains

For the finite and affine Painlevé quivers we show that the connected component $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ is the union of the numerically reachable V-domains and that they intersect in codimension 1 if their hearts are related by a numerical tilt. A fundamental domain for the action of $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$ is either a V-domain or a union of two neighbouring V-domains according to whether the quiver is isomorphic to its opposite or not. We identify this fundamental domain with the double $\diamondsuit_Q$ of the Schwarz triangle.

The description of the V-domains of the standard heart $V(\overline{\mathcal{A}}^0)$ will follow Section 5.2 in the A_2 case. We first observe that the V-domain is contained in the set of projective stability conditions for which the set of stable objects is a subset of the irreducible objects of $\mathcal{A}^0$. Furthermore it contains a single wall of marginal stability along which all extensions of the simple objects destabilise, separating the V-domain into the stable and unstable regions where the stable objects up to shift are the irreducible and simple objects of $\mathcal{A}^0$ respectively.

We can identify the V-domain $V(\overline{\mathcal{A}}^0)$ under the projective central charge map with a closed subset of

$$\mathbb{P}^1 = \mathfrak{h}^+ \cup \mathcal{W} \cup \mathfrak{h}^-$$

where $\mathcal{W} \subset \mathbb{P}^1$ is the equator containing the image of the wall of marginal stability. We describe the *half V-domains* $V^{\pm}(\overline{\mathcal{A}}^0) = \mathfrak{h}^{\pm} \cap V(\overline{\mathcal{A}}^0)$ given by the intersection of the V-domain with the stable and unstable half-planes $\mathfrak{h}^{\pm}$. The whole V-domain is obtained by gluing the half V-domains along the part of the equator $\mathcal{W}$ in which they meet.

Each half V-domain $V^{\pm}(\overline{\mathcal{A}}^0)$ is contained in the domains $V^{\pm}$ respectively consisting of projective stability conditions in $\mathfrak{h}^{\pm}$ whose width is no larger on $\overline{\mathcal{A}}^0$ than on the stable and unstable tilts respectively of $\overline{\mathcal{A}}^0$ with respect to the two numerical equivalence classes $\nu_{\pm}$. By Lemma 5.11, the two codimension 1 boundary components of each of the domains $V^{\pm}$ are contained in circles.

Definition 7.19. The bounding circles of the half V-domains $V^{\pm}(\overline{\mathcal{A}}^0)$ are the two circles containing the boundary components of $V^{\pm}$.

Using the explicit description of the bounding circles provided by Lemma 5.11, one can give a description of the boundary of the domains $V^{\pm}$.

Proposition 7.20. *Let $c_{\nu_{\pm}}^{\pm}$ be the non-negative integers given by the maximal off-diagonal entry in the matrices $\phi_{\nu_{\pm}}^{\pm}$ given in Table 7.1. Denote by $d^{\pm} = c_{\nu_+} c_{\nu_-}$ the product of the two coefficients $c_{\nu_{\pm}}^{\pm}$ whose product of signs is equal to $+$ or $-$ respectively. Then the boundary of each of the domains $V^{\pm}$ is described by one of the three cases below according to the value of $d^{\pm}$*

$d^{\pm} = 0$ *The two bounding circles coincide.*

$1 \leq d^{\pm} \leq 3$ *The two bounding circles meet at a unique point $\sigma^{\pm}$ in the interior of $\mathfrak{h}^{\pm}$.*

$d^{\pm} = 4$ *The two bounding circles meet at a unique point on the boundary of $\mathfrak{h}^{\pm}$.*

Proof. By Lemma 5.11 the bounding circles are given by the loci

$$Z(\nu_+) \perp 2Z(\nu_-) + c_{\nu_+} Z(\nu_+) \qquad Z(\nu_-) \perp 2Z(\nu_+) + c_{\nu_-} Z(\nu_-)$$

We notice that the boundary circles coincide if and only if both coefficients c_{ν_+} and c_{ν_-} are equal to zero in which case they define the circle $Z(\nu_+) \perp Z(\nu_-)$. The two bounding circles meet in the closure of $\mathfrak{h}^{\pm}$ if and only if the inequality

$$\frac{c_{\nu_+}}{2} \leq \frac{2}{c_{\nu_-}}$$

is satisfied, with equality if and only if their unique intersection point is on the boundary. $\qquad \square$

The above Proposition is reminiscent of the classification of the rank 2 root systems R of finite and affine type according to the off-diagonal elements (c_+, c_-) of its Cartan matrix.

(c_+, c_-)	$(0,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(2,2)$	$(1,4)$
R	$A_1 \times A_1$	A_2	B_2	G_2	$\tilde{A}_1$	$\tilde{BC}_1$

We note that for the bipartite Painlevé quivers Q, the root systems $R^{\pm}$ associated to the unstable and stable regions $\mathfrak{h}^{\pm}$ are $R^- = A_1 \times A_1$ and R^+ obtained by folding the quiver with respect to the automorphisms of the quiver permuting the vertices in the two numerical equivalence classes. For $Q = \tilde{A}_2$, we find that $R^- = A_2$ and $R^+ = \tilde{A}_1$

Proposition 7.21. *The set of classes in $N(\mathcal{D}_Q)$ of the stable objects of $\mathcal{D}_Q$ in the stable and unstable regions $\mathfrak{h}^\pm$ is in bijection with the set of roots of the corresponding root system $R^\pm$.*

Proof. The action of the stable and unstable Keller-Yang equivalences on $N(\mathcal{D})$ given in Table 7.1 corresponding to the two simple tilts is that of the generators of the Coxeter group of the root system $R^\pm$ in its standard representation by reflections. $\qquad\square$

It will be helpful to identify the roots of the root systems $R^\pm$ with the points on the boundary $\mathcal{W}$ of the hemispheres $\mathfrak{h}^\pm$ where the projective central charge of the corresponding stable object vanishes. We remark that there is no projective stability condition in the closure of the domains $\mathfrak{h}^\pm \subset \mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ with these projective central charges.

By definition the half V-domains $V^\pm(\overline{\mathcal{A}}^0)$ are contained in the domains $V^\pm$ bounded by the bounding circles. We now establish the converse inclusion.

Proposition 7.22. *The half V-domains $V^\pm(\overline{\mathcal{A}}^0)$ are equal to the domains $V^\pm$ bounded by the bounding circles.*

Proof. We claim that any projective stability condition $\overline{\sigma}$ in $\mathfrak{h}^\pm$ is supported on a projective heart $\overline{\mathcal{A}}^0$ which is the image of the standard heart under a finite number of stable, respectively unstable, numerical tilts. Indeed $\mathfrak{h}^\pm$ is by definition covered by the V-domains of the projective hearts supporting a stability condition in the $\mathbb{C}$-orbit of a stability condition supported on $\mathcal{A}^0$. There is at most one such projective heart $\overline{\mathcal{A}}$ which is not numerically reachable corresponding to the unique imaginary root of the root system $R^\pm$ if it exists, but the width of any projective stability condition $\overline{\sigma}$ on $\overline{\mathcal{A}}$ is 1 and so it is not supported on $\overline{\mathcal{A}}$.

As $V(\Phi\overline{\mathcal{A}}) = \phi V(\overline{\mathcal{A}})$ it suffices to show that the interiors of the translates of the closed domains $V^\pm \subset \mathfrak{h}^\pm$ under the reflection group in $\mathrm{PSL}(2, \mathbb{Z})$ generated by the two stable, respectively unstable, reflections given in Table 7.1 are disjoint. This follows immediately from the fact that there is no element of the root system $R^\pm$ defining a point on the boundary of $\mathfrak{h}^\pm$ which is contained in the domain $V^\pm$, for all stable objects of $\overline{\mathcal{A}}^0$ belong to the cone $C^0(\mathcal{A}^0)^* \subset N(\mathcal{D}) \otimes \mathbb{C}$ spanned by $[\nu_+]$ and $[\nu_-]$. $\qquad\square$

We thus have an explicit description of the V-domain $V(\overline{\mathcal{A}}^0)$ of the standard heart $\overline{\mathcal{A}}^0$ as the disjoint union

$$V(\overline{\mathcal{A}}^0) = V^+ \cup \mathcal{W}^0 \cup V^-$$

of its intersections $V^\pm = V(\overline{\mathcal{A}}^0) \cap \mathfrak{h}^\pm$ with the stable and unstable half-planes $\mathfrak{h}^\pm$ and with the equator $\mathcal{W}^0 = V(\overline{\mathcal{A}}^0) \cap \mathcal{W}$ consisting of points where $\overline{Z}$ is real and negative. Its boundary $\partial V(\overline{\mathcal{A}}^0) = \partial V^+ \cup \partial V^-$ consists of four codimension 1 components corresponding to the four simple numerical tilts of $\overline{\mathcal{A}}$.

Proposition 7.23. *The distinguished connected component $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ is covered by the V-domains of the numerically reachable hearts.*

Proof. We must show that the domain V given by the union of the V-domains of the numerically reachable hearts is both open and closed. The key observation is that the reflection hyperplanes of the two root systems $R^{\pm}$ are locally-finite in $\mathfrak{h}^{\pm}$. In particular the walls of the V-domains of the numerically reachable hearts are locally finite in $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$. This means that given any point in $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ we can find an open neighbourhood which intersects only finitely many of these walls.

The domain V is open because given a projective stability condition supported on a numerically reachable heart there exists an open neighbourhood for which the projective stability conditions are supported on hearts obtained by a finite sequence of numerical tilts. The domain V is also closed, for any limiting point of a sequence of projective stability conditions in $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ is supported on any of the projective hearts supporting a projective stability condition in a sufficiently small open neighbourhood. $\qquad\square$

An immediate corollary of this is that all numerical autoequivalences of the categories $\mathcal{D}_Q$ are in fact numerically reachable.

Corollary 7.24. *The subgroup $\mathrm{Aut}^0_N(\mathcal{D}_Q)$ of numerically reachable autoequivalences of $\mathrm{Aut}^0(\mathcal{D}_Q)$ is equal to the entire group.*

Proof. By the previous Proposition any autoequivalence of the category $\mathcal{D}_Q$ which preserves $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ also preserves the set N of numerically reachable hearts. $\qquad\square$

Furthermore we can now describe a fundamental domain $\lozenge_Q$ for the action of $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}$ on $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$. According to whether Q is isomorphic to its opposite, it consists of a single V-domain or the union of two V-domains of hearts whose quivers have the opposite orientation.

Proposition 7.25. *A fundamental domain D_Q for the action of $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}$ on $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ is given by*

$Q \cong Q^{\mathrm{op}}$ *The V-domain $V(\overline{\mathcal{A}}^0)$ of the standard heart $\overline{\mathcal{A}}^0$*

$Q \not\cong Q^{\mathrm{op}}$ *The union $V(\overline{\mathcal{A}}^0) \cup V(\Phi^+_{\nu_+}\overline{\mathcal{A}}^0)$ of the V-domains of the standard heart and one of its simple tilts.*

Proof. Every point in $\mathrm{Stab}^0(\mathcal{D})/\mathbb{C}$ belongs to the V-domain of a numerically reachable heart and so is a translate of a point in D_Q by an element of $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$. The interiors of the $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$ are disjoint as the interiors of the V-domains of the set of numerically reachable hearts are. $\qquad\square$

We note that the fundamental domain D_Q is adapted to the presentation of $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z} \cong \mathrm{Aut}^0_N(\mathcal{D}_Q)$ given in Proposition 7.17. In particular the generators Σ and Υ of $\mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$ identify the boundary components of D_Q as in Lemma 5.15.

Lemma 7.26. *Let D_Q be the standard fundamental domain for the action of $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}$ on $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ given in Proposition 7.25. Suppose x and y belong to D_Q and let $g \in \mathrm{Aut}^*(\mathcal{D}_Q)/\mathbb{Z}$. Then $x = gy$ if and only if one of the following occurs:*

- $x = y$ *and g is the identity*

- $x = y = \overline{\sigma}_-$ *if this point exists and g belongs to the subgroup generated by* $\overline{\Upsilon}$

- $x = y = \overline{\sigma}_+$ *if this point exists and g belongs to the subgroup generated by* $\overline{\Sigma}$

- $x, y \in \partial D_Q \cap \mathfrak{h}^- \backslash \overline{\sigma}_-$ *and $x = \overline{\Upsilon}y$ or $x = \overline{\Upsilon}y$.*

- $x, y \in \partial D_Q \cap \mathfrak{h}^+ \backslash \overline{\sigma}_+$ *and $x = \overline{\Sigma}y$ or $x = \overline{\Sigma}^{-1}y$*

Proof. If $Q \cong Q^{\mathrm{op}}$ this follows from the same argument as in Lemma 5.15. If $Q \not\cong Q^{\mathrm{op}}$ then the V-domain $V(\overline{\mathcal{A}}^0)$ is triangular for the bounding circles of V^- coincide as $\Phi^+_{\nu_-}\Phi^+_{\nu_-} = [1]$. The boundary of D_Q therefore consists of four codimension 1 components consisting of the boundary edges of the domains $V(\overline{\mathcal{A}}^0)$ and $V(\Phi^+_{\nu_+}\overline{\mathcal{A}}^0)$ with the exception of the one they have in common. As $\Phi^+_{\nu_+}$ is a stable equivalence, each hemisphere $\mathfrak{h}^\pm$ contains a pair of the boundary components of D_Q, which are identified by $\overline{\Upsilon}$ and $\overline{\Sigma}$ respectively. Furthermore $\overline{\sigma}_-$ does not exist and any autoequivalence fixing $\overline{\sigma}_+$ sends $\overline{\mathcal{A}}^0$ to a heart obtained by successive stable tilts and so belongs to the subgroup generated by $\overline{\Sigma}$. $\quad\square$

As we observed for $Q = A_2$, the side pairing is indicative of the fact that the domain $\Diamond_Q = \overline{Z}(D_Q) \subset \mathbb{P}^1$ onto which D_Q projects biholomorphically under $\overline{Z}$ is the double of a Schwarz triangle. We observe that $\Diamond_Q$ is the double of the Schwarz triangle given by the image of a branch of the period map of the Seiberg-Witten differential of the associated family of spectral elliptic curves.

Proposition 7.27. *The domain $\Diamond_Q = \overline{Z}(D_Q)$ is the double of the corresponding Schwarz triangle given in Proposition 6.8 along the edge connecting the vertices corresponding to the first two exponents.*

Proof. The domain $\Diamond_Q \subset \mathbb{P}^1$ is the double of a Schwarz triangle Δ whose triangle group is generated by the actions on $N(\mathcal{D})$ of the following equivalences $\mathcal{D}_Q \to \mathcal{D}_{Q^{\mathrm{op}}}$ along the edge corresponding to the last equivalence in the list

$Q \cong Q^{\mathrm{op}}$ $\Phi^+_{\nu^+}$, $\Phi^+_{\nu_-}$ and T

$Q \not\cong Q^{\mathrm{op}}$ $\Phi^-_{\nu_-}$, $\Phi^-_{\nu^+}$ and $\Phi^+_{\nu_+}$

Furthermore as the monodromy group $\Gamma(Q)$ is equal to $\Gamma(\mathcal{E})$, the Schwarz triangle has angles specified by the exponents (α, β, γ) of Proposition 6.8 up to shifting the differences in the exponents by an integer. One can verify that in fact the angles correspond to the exponents (α, β, γ) precisely:

$Q \cong Q^{\mathrm{op}}$ The two vertices of the edge of Δ along which the Schwarz triangle is doubled are located at the two intersection points of the bounding circles of V^- and V^+ respectively. The edges meet in the angles $\frac{\pi}{r}$ and $\frac{\pi}{s}$ where r and s are greater than one, and 0 if r respectively s is equal to one. The third vertex is at an intersection of one of the bounding circles of V^+ with one of the bounding circles of V^- at a point on the equator, making angle π.

$Q \not\cong Q^{\mathrm{op}}$ Two of the vertices of Δ occur at the intersection points of the coincident bounding circles of V^- with the two bounding circles of V^+ each making angle π. The third vertex, which belongs to the doubled edge, is located at the intersection of the bounding circles of V^+, with angle $\frac{\pi}{s}$ or 0 according to whether s is greater than, or equal to, one.

$\square$

This Proposition identifies the image under $\overline{Z}$ of the fundamental domain D_Q of the action of $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}[1]$ on $\mathrm{Stab}(\mathcal{D}_Q)/\mathbb{C}$ with the image $\Diamond_Q$ of a branch of the period map $p : \widetilde{\mathbb{P}(r,s)}\backslash\delta \to \mathbb{P}^1$. In the next section we will use this to lift p to a map $f : \widetilde{\mathbb{P}(r,s)}\backslash\delta \to \mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ to the distinguished component of the space of projective numerical stability conditions.

7.5 Proof of main theorem

In this section we complete the proof of the main theorem using the method employed in the proof of Theorem 5.19 in the A_2 case. The strategy is to extend the identification of fundamental domains by equivariance with respect to the $\mathbb{Z}_r * \mathbb{Z}_s$-actions of Proposition 6.4 and 7.17.

Proposition 7.28. *There is a commutative diagram of groups*

$$\pi_1(\widetilde{\mathbb{P}(r,s)}\backslash\delta) \xrightarrow{\varphi} \mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}$$
$$\underset{m}{\searrow} \qquad \downarrow{\phi}$$
$$\Gamma(Q)$$

consisting of surjective maps identifying the following pairs of generators

- *The loops γ_0, γ_∞ around the points $t = 0$ and $t = \infty$ of $\mathbb{P}(r,s)\backslash\delta$*
- *The autoequivalences $\overline{\Upsilon}, \overline{\Sigma}$ of $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}$*
- *The matrices υ and σ of $\Gamma(Q)$ given in Table 7.2.*

Moreover if $Q \neq \tilde{A}_2$ then all the arrows are isomorphisms.

Proof. We define the map $\varphi : \pi_1(\widetilde{\mathbb{P}(r,s)}\backslash\delta) \to \mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}$ on generators so that it sends γ_0 and γ_∞ to $\overline{\Upsilon}$ and $\overline{\Sigma}$. By the commutativity of the triangles in the Propositions 6.4 and 7.17, the map φ commutes with the maps m and ϕ. As m and ϕ are isomorphisms if $Q \neq \tilde{A}_2$ it follows that φ is an isomorphism with the exception of the $\tilde{A}_2$ case. $\square$

We can now prove the main theorem.

Theorem 7.29. *There is a holomorphic covering map*

$$\widetilde{\mathbb{P}(r,s)}\backslash\delta \xrightarrow{f} \mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$$
$$\underset{p}{\searrow} \qquad \downarrow{\bar{Z}}$$
$$\mathbb{P}\,\mathrm{Hom}(N(\mathcal{D}_Q), \mathbb{C})$$

lifting the period map p of the Seiberg-Witten differential λ of the family of elliptic curves $\mathcal{E} \to \mathbb{P}(2r, 2s)\backslash\delta$. It is equivariant with respect to the actions of $\mathbb{Z}_r * \mathbb{Z}_s$ on the left by deck transformations, and on the right by $\mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}[1]$.

Remark 7.30. In all but the $\tilde{A}_2$ case we have observed in Proposition 7.17 that the map $\mathbb{Z}_r * \mathbb{Z}_s \to \mathrm{Aut}^0(\mathcal{D})/\mathbb{Z}$ is in fact an isomorphism, and so the lift f of the period map in the above theorem is in fact a biholomorphism.

Proof. Consider the fundamental domain D for the action of $\mathbb{Z}_r * \mathbb{Z}_s$ on $\widetilde{\mathbb{P}(r, s)}\backslash\delta$ obtained by making branch cuts along the intervals $t \in (0, 1)$ and $t \in (1, \infty)$. By Theorem 4.28 the image of the branch of the period map p is the double $\Diamond \subset \mathbb{P}^1$ of the Schwarz triangle with exponents given in Proposition 6.8 along the edge connecting the vertices corresponding to the first two exponents. By Proposition 7.27 this agrees with the image under $\overline{Z}$ of the fundamental domain D_Q for the action of $\mathrm{Aut}^0(\mathcal{D}_Q)/C$ on $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$.

Define the map $f_{|D} = \overline{Z}_{|\Diamond}^{-1} \cdot p_{|D} : D \to D_Q$ which by definition commutes with the maps p and $\overline{Z}$. Consider its extension $f : \widetilde{\mathbb{P}(r, s)}\backslash\delta \to \mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ with respect to the $\mathbb{Z}_r * \mathbb{Z}_s$ actions on both sides. This map is well-defined for the identification of points on the boundary of D by γ_0 and γ_∞ agrees with the identification of the boundary of $D_Q = f(D)$ by $\overline{\Upsilon} = \varphi(\gamma_0)$ and $\overline{\Sigma} = \varphi(\gamma_\infty)$.

As the map $\varphi : \pi_1(\mathbb{P}(r, s)\backslash\delta) \to \mathrm{Aut}^0(\mathcal{D}_Q)/\mathbb{Z}$ commutes with the maps m and ϕ, the map f commutes with the local isomorphisms p and $\overline{Z}$ and is therefore holomorphic. As φ is surjective it is also a covering map with automorphism group the kernel of φ. □

Remark 7.31. There can be more than one connected component of the space of numerical stability conditions inside the connected component of the space of stability conditions containing the standard heart. Indeed in the finite cases A_3 and D_4 it is shown in [Qiu12] that there are $\mathrm{Br}(A_3)/\mathrm{Br}(B_2)$ and $\mathrm{Br}(D_4)/\mathrm{Br}(G_2)$ many connected components respectively.

Appendix A

Appendix

Below we draw the image of a fundamental domain for the action of $\mathrm{Aut}^0(\mathcal{D}_Q)$ on $\mathrm{Stab}^0(\mathcal{D}_Q)/\mathbb{C}$ under the projective central charge map $\overline{Z}$ to $\mathbb{P}^1$ for each of the Painlevé quivers Q. We picture the sphere as two hemispheres which are identified along the equator where the image of $\overline{Z}$ is contained in a line. At points $\circ$ on the equator which do not belong to the fundamental domain we have indicated the class in $N(\mathcal{D}_Q)$ which vanishes there.

The fundamental domain coincides with the image of a branch of the period map of the Seiberg-Witten differential λ on the family of elliptic curves $E \to \mathbb{P}^1$. The image is the double of a Schwarz triangle along the dashed line whose angles are given by the differences in the characterstic exponents in the last column of Table A.1.
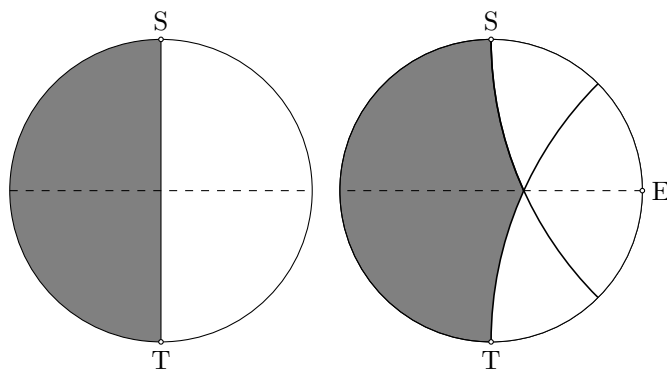
Figure A.1: Fundamental domain for the quiver of type A_2
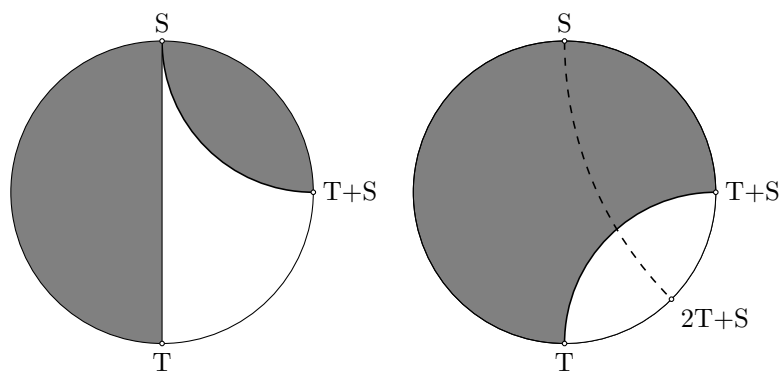
Figure A.2: Fundamental domain for the quivers of type A_3/D_3
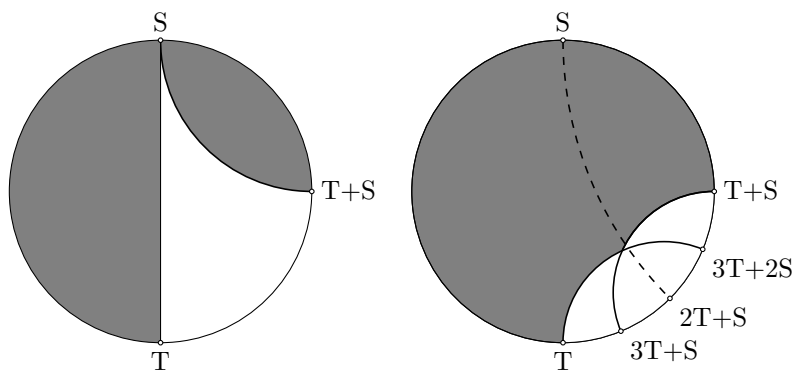
Figure A.3: Fundamental domain for the quiver of type D_4

Figure A.4: Fundamental domain for the quivers of types $\tilde{A}_1$ and $\tilde{A}_3/\tilde{D}_3$

Figure A.5: Fundamental domain for the quiver of type $\tilde{A}_2$

Figure A.6: Fundamental domain for the quiver of type $\tilde{D}_4$

Figure A.7: Fundamental domain for the quiver of type $\tilde{\tilde{D}}_4$

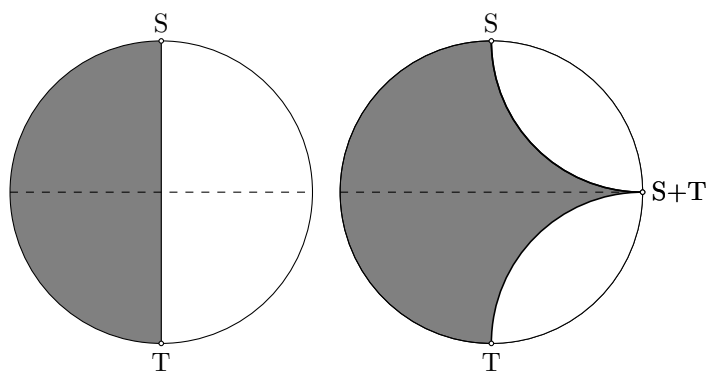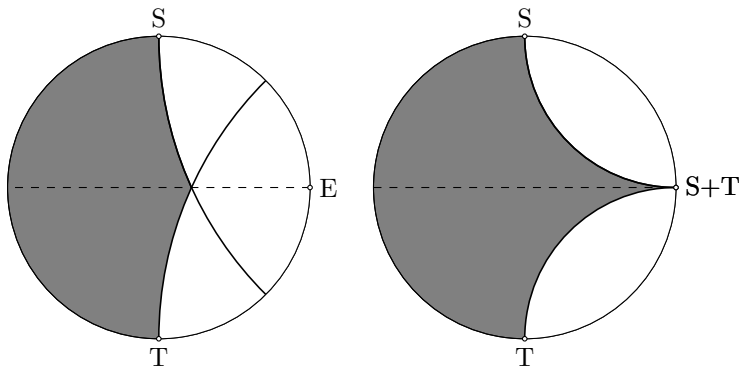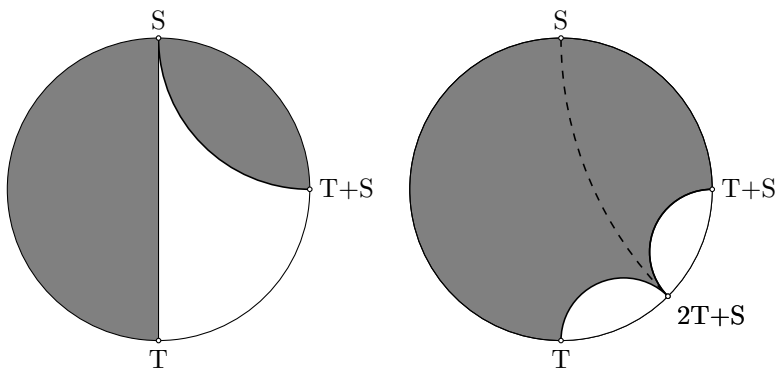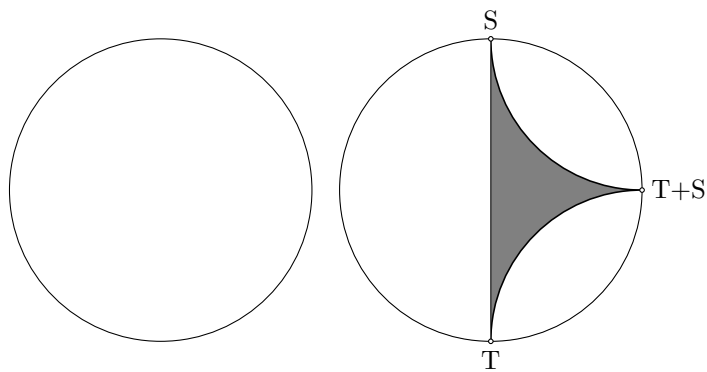|	$(E \to \mathbb{P}^1_t, \omega)$	$(\Sigma \to \mathbb{P}^1_b, \omega)$	$(\Sigma \to \mathbb{P}^1_b, \lambda)$	$(E \to \mathbb{P}^1_t, \lambda)$
A_2	$\begin{smallmatrix} \frac{3}{4} & \frac{1}{6} & 0 \\ \frac{1}{4} & -\frac{1}{6} & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{5}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{6} & 0 & 0 \\ -\frac{5}{6} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{3}{4} & \frac{7}{6} & 0 \\ -\frac{5}{4} & \frac{5}{6} & 1 \end{smallmatrix}$
A_3	$\begin{smallmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{3}{4} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{3}{4} & 1 & 1 \end{smallmatrix}$
D_3	$\begin{smallmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{3}{4} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{4} & 0 & 0 \\ -\frac{3}{4} & 1 & 1 \end{smallmatrix}$
D_4	$\begin{smallmatrix} \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 1 \end{smallmatrix}$
$\tilde{A}_1$	$\begin{smallmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 0 & 1 \end{smallmatrix}$
$\tilde{A}_2$	$\begin{smallmatrix} \frac{5}{6} & 0 & 0 \\ \frac{1}{6} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & 2 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{6} & 0 & 0 \\ -\frac{1}{6} & 0 & 1 \end{smallmatrix}$
$\tilde{A}_3$	$\begin{smallmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 0 & 1 \end{smallmatrix}$
$\tilde{D}_3$	$\begin{smallmatrix} \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{4} & 0 & 0 \\ -\frac{1}{4} & 0 & 1 \end{smallmatrix}$
$\tilde{D}_4$	$\begin{smallmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 1 \end{smallmatrix}$	$\begin{smallmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 1 \end{smallmatrix}$
$\tilde{\tilde{D}}_4$	$\begin{smallmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{smallmatrix}$	$\begin{smallmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{smallmatrix}$

Table A.1: Riemann schemes of the Picard-Fuchs equations for the holomorphic and Seiberg-Witten differentials ω and λ on the elliptic surfaces E and Σ

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