Portfolio Choice Under Uncertainty

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Abstract

This thesis is about portfolio choice under ambiguity and risk. At its core is an experiment and a simulation, both concerning portfolio choice. The experiment is under ambiguity, in which the the probabilities of the states are not known to the subjects. We tested two multiple prior preference theories (MEU and α-MEU), both of which are fit significantly better than Expected Utility (EU) for around one third of the subjects, and better than Mean-Variance (MV) for the majority of the subjects. We also find that subjects have heterogenous beliefs about ambiguity, but on average they do a good job in guessing the true probabilities. The simulation is in the context of risk. Our interest here is in the specification of the stochastic process underlying our observations. The simulation led to a surprising result - the maximum likelihood estimation may suggest the wrong specification.

Contents

List of Figures

List of Tables

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More than three years ago I started my PhD with little knowledge about Economics, in the sense of understanding the intuition behind the mathematical equations. It is lucky to have a John Hey as my supervisor, who is stimulating and encouraging, who has taught me to think. Instead of thanking you John for teaching me how to do research, I more appreciate the spirit he passes to his students. That is the passion for research, the passion for life. I am glad I finally seem begin to think, begin to feel enthusiastic about research.

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Declaration of Authorship

I, Xueqi Dong, declare that this thesis titled, 'Portfoio Choice Under Uncertainty' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made reference to that.

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Chapter 1

Portfolio Choice under Ambiguity - Theory

1.1 Introduction

This chapter begins by presenting the classic individual portfolio choice problem in the context of *risk*. The environment is that there is one risk free asset, for which the end of period price is fixed, and several risky assets, for which the end of period prices are contingent on the possible states. The probabilities of each state are known to the individuals. The starting prices of all the assets are predetermined and individuals can buy or sell the assets using a cash endowment. Section 1.2 studies the scenario in which individuals can buy and sell any quantity of assets. Section 1.3 studies the scenario with a No-short-selling constraint, which means that individuals can only use their cash endowment to buy the assets and they can not borrow cash.

Clearly the optimal decision depends upon the preferences of the decision-maker. In this chapter, Expected Utility (EU) and Mean-Variance (MV) preferences are applied in this scenario. EU theory is widely used by economists to explain decision making under risk. It claims that individuals' decision rules can be described as maximising the sum of probability weighted von

Neumann-Morgenstern utility of every possible outcome (the EU theory assumes that the individuals obey certain axioms.) [1] While EU theory is widely favoured by economists, it is not the most preferred preference functional used in the financial professions. There people work directly with the expected return and variance of a gamble, and assume that decision-makers trade off the mean against the variance. In general, but not always, EU preferences are inconsistent with MV preferences and *vice versa*. However assuming MV preferences has advantages in that interesting analytical results can be obtained, which is often not possible when assuming EU preferences. MV preferences are also easier to handle when doing empirical work. For example, in the empirical study of the Capital Asset Pricing Model, the return on the asset is usually assumed to be normally distributed, and the corresponding return and variance can be derived from historical data. Though MV preferences may imply violation of statewise dominance (hence one might think it is not a *rational* preference), it is the building block for Modern Portfolio Theory (MPT). MPT was introduced by Markowitz (1952), who won a Nobel Prize for this research, and it has been widely used in practice.

The chapter then studies, in Section 1.4, the portfolio choice problem in the context of *ambiguity*. Ambiguity is distinguished from risk in the sense that the probabilities of each state happening are not known by the individuals. In both EU and MV, either a unique prior or a unique distribution is assumed. There are various characterisations of ambiguity and behaviour under ambiguity, and we will study one such characterisation: that with multiple priors - that is various *sets* of possible probabilities (in the case of risk, there is a unique member of this set). To be specific we will apply two multiple-prior models: the MaxMin Expected Utility (MEU) (Gilboa and Schmeidler 1989) model and the α-MaxMin Expected Utility (α-MEU) (Ghirardato et al. 2004)

[1] Usually the axioms are described as the following (where P denotes a risky gamble)

- Axiom 1 Completeness Either $P_1 \succ P_2$ or $P_1 \prec P_2$ or $P_1 = P_2$
- Axiom 2 Transitivity If $P_1 \succeq P_2$ and $P_2 \succeq P_3$ then $P_1 \succeq P_3$
- Axiom 3 Continuity If $P_1 \succeq P_2 \succeq P_3$ then there exists a value $\pi \in [0, 1]$ that $P_1\pi + P_3(1 - \pi) = P_2$
- Axiom 4 Independence If $P_1 \succeq P_2$ then for any $\pi \in [0, 1]$

model in a portfolio choice problem. α-MEU is a generalisation of MEU, and both of these two preferences are built 'on top of' EU theory. The study of ambiguity in portfolio choice may help us to understand puzzles which cannot be explained by classical finance theory, such as the equity premium puzzle. This latter refers to the phenomenon that the returns for stocks in excess of government bonds are too great to be rationalised.

1.2 Unconstrained Portfolio Choice in the context of risk

1.2.1 Introduction

The general environment is as follows. There are several assets with starting prices predetermined, and with end-of-period prices depending on which state occurs out of all the possible states. There is one risk free asset, for which the end of period price is the same for all possible states, as a fixed interest rate is assumed. The others are risky assets, for which end-of-period prices are contingent on the state which occurs. For any one state that occurs, all the end-of-period prices of the risky assets are determined.The risky assets may be correlated in terms of their end-of-period prices. The end-of-period price could also be considered as the end-of-period price plus a dividend as sometimes dividend are paid on some stocks. But these two interpretations do not make a real difference to the theoretical analysis. Individuals are given a certain amount of cash endowment at the start, and then they can buy or sell the assets. In this thesis, we exclusively study the case in which there are 2 risky assets and 1 risk free asset. This is for the following reasons. The study of portfolio choice are mainly focused on two aspects. One is how individuals allocate cash to the risk free asset and the risky assets, and the other is how they diversify in risky assets. Diversification means hedging risk by constructing a portfolio containing several risky assets, as risky assets may be correlated. For example, the famous CAPM model (which assumes that individuals have mean variance preferences) claims that every individual wants to choose a portfolio with the same weights in risky assets, that is market portfolio; the heterogeneity in risk aversion only influences the proportion in the risk free asset and in market portfolio;. So a risk free asset and also at least two risky assets should be involved in this study. As the analysis in the case of two risky assets already captures the essence of portfolio choice theory, no more should be added, since this would make the problem more complicated in a unnecessary way. Also, I want to test the theoretical results in the experiment. As to the number of possible states, we chose to study the case where there are 3 possible states. First, at least two states should be involved as otherwise the risky assets are not risky anymore when only one state can occur. But two states are not informative enough in terms of revealing individuals' preferences (as we shall show later). Second, the problem should be as simple as possible for the subjects. Then it seems natural to choose 3 states. Although it is more explicit if we model the payoffs for the assets in different states by prices, relative returns are used instead in this chapter. This leads to a much conciser theoretical analysis and does not alter the basic problem. For example[2], if the opening prices for risky asset 1 and risky asset 2 and risk free asset are 4, 2 and 1 units of cash respectively, individuals should have exactly the same preferences when told that the end of period prices to be 8, 3 and 1 units of cash respectively or the relative returns are 1, 0.5 and 0 respectively. We also introduce another simplification by assuming that the opening prices for all the assets are 1 unit in cash since the opening price does not really matter as long as the relative return is given. Furthermore the relative return for the risk free asset is nomalised to 0. Note in this case the risk free asset and cash are equivalent. So in future the risk free asset is just called cash.

1.2.2 The Basic Scenario

Assume that there are two risky assets $i \in (1, 2)$, both priced at 1 unit of cash. The relative return for risky assets are contingent on the state $j \in (1, 2, 3)$. Individuals are endowed with a certain amount of cash e to buy

[2] In this example only one state is assumed, as it is only for illustrating the idea.

the risky assets. They can spend all their cash or they can keep as much as they want. In this basic scenario short selling in any asset or cash is allowed. Denoting by $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$ the portfolio choice [3] for and the cash holding as c_0 as the remaining cash.

$$c_0 = e - c_1 - c_2 \tag{1.1}$$

The interest rate is 0, so the cash holding is the same at the end of period. Denoting by d_{ij} the return for risky asset i if state j occurs, the return table for the risky assets can be written as

	state1	state2	state3
asset1	d_{11}	d_{12}	d_{13}
asset2	d_{21}	d_{22}	d_{23}

and it can be written in a concise matrix form

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} \tag{1.2}$$

Then the portfolio payoff for each state j is

$$\begin{aligned} w_j &= c_1 d_{1j} 1 + c_2 d_{2j} 1 + e \\ &= \sum_{i \in (1,2)} c_i d_{ij} \end{aligned} \tag{1.3}$$

The probabilities of the possible states are denoted by the vector

$$P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}' \tag{1.4}$$

where $'$ means a transpose operator for a vector/matrix. This notation is used throughout the thesis.

[3] In this thesis, in order to simplify the notation, the expression 'portfolio choice' always refers to the allocations to the risky/ambiguous assets, as the allocation to cash is implicitly decided by the budget constraint (1.1)

1.2.3 Mean Variance (MV) Preferences

Mean Variance preferences assumes that individuals trade-off the mean against the variance of a portfolio. Given the return table (1.2) and probability vector (1.4), the mean return of asset i is

$$\mu_i = d_{i1}p_1 + d_{i2}p_2 + d_{i3}p_3 \tag{1.5}$$

the variance of the return for asset i is

$$\sigma_i^2 = (d_{i1} - \mu_i)^2 p_1 + (d_{i2} - \mu_i)^2 p_2 + (d_{i3} - \mu_i)^2 p_3 \tag{1.6}$$

and the covariance for returns of the two assets is

$$\sigma_{12} = (d_{11} - \mu_1)(d_{21} - \mu_2)p_1 + (d_{11} - \mu_1)(d_{21} - \mu_2)p_2 + (d_{11} - \mu_1)(d_{21} - \mu_2)p_3 \tag{1.7}$$

For conciser writing, denote the return vector by

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \tag{1.8}$$

and the covariance matrix by

$$\Omega = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \tag{1.9}$$

For any allocation $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$ the mean of portfolio (relative) payoff is $C\mu$ and the variance of portfolio (relative) payoff is $C\Omega C'$. Mean-Variance preferences assumes that individuals maximise the following function

$$\begin{aligned} U &= e + C\mu - \frac{1}{2} r C \Omega C' & (1.10) \\ &= e + (c_1\mu_1 + c_2\mu_2) - \frac{1}{2} r (c_1^2\sigma_1^2 + 2c_1c_2\sigma_{12} + c_2^2\sigma 2^2) & (1.11) \end{aligned}$$

which is a linear combination of the mean and the variance of the portfolio payoff [4]. Notice that $r \geq 0$ is the risk aversion parameter and represents the degree an individual would penalize a portfolio for its risk. The first order conditions for the maximisation of the function (1.10) are

$$\begin{aligned}\frac{dU}{dc_1} &= \mu_1 - r(c_1\sigma_1^2 + c_2\sigma_{12}) \\ \frac{dU}{dc_2} &= \mu_2 - r(c_2\sigma_2^2 + c_1\sigma_{12})\end{aligned}$$

hence the optimal allocation is given by

$$C^* = \frac{1}{r}(\Omega^{-1}\mu)'. \tag{1.12}$$

Conditions for the existence of an optimal allocation The optimal solution (1.12) is an expression involving the inverse of the covariance matrix Ω, so its determinant should not be equal to 0; that is

$$\sigma_{12}^2 - \sigma_1^2\sigma_2^2 \neq 0 \tag{1.13}$$

The intuition behind this condition is that it eliminates the opportunity for individuals to earn a risk free profit without any cost, which is called an Arbitrage Opportunity. In another words, there is not an *optimal solution* as an individual would want to invest an unlimited amount in the risky assets if such an opportunity exists. For example, if

$$\mu = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$$

and

$$\Omega = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$$

then the mean of the portfolio payoff is $0.1c_1 + 0.3c_2$ and the variance of the payoff is $0.2c_1^2 + 0.8c_1c_2 + 0.8c_2^2$. If an individual constructs the portfolio so that $c_1 = -2c_2$ then the portfolio variance is 0. And the utility function for

[4]It is frequent mistake that people think quadratic utility functions leads to mean variance preferences. See Robert and Edward (1991).

a mean variance preference individual is

$$U = 0.1c_2$$

which is monotonically increasing in c_2. As the risk parameter does not appear in this equation, an individual with any level of risk aversion would be willing to buy an unlimited amount of asset 2 and sell half that amount in asset 1. If

$$\mu = \begin{bmatrix} 0.1 \\ 0.15 \end{bmatrix}$$

hence the utility function is

$$U = -0.05c_2.$$

then an individual would be willing to sell an unlimited amount of asset 2 and buy twice that amount of asset 1 to earn an unlimited amount of money. Notice here this opportunity can only exist when there are no constraints on the individual's freedom to trade.

1.2.4 Expected Utility (EU) Preferences

According to Expected Utility theory, individuals choose the portfolio that maximises the sum of the von Neumann-Morgenstern utility function u of portfolio payoff w_j weighted by the probability of the corresponding state p_j, that is

$$U = \sum_j u(w_j)p_j \tag{1.14}$$

I assume that individuals are never satiated by money which means that the first derivative of the utility function is positive $u^{'} > 0$. One can measure individuals' attitudes to risk by two measures[5]:

[5]Generally the measure for a function's concavity is the second derivative $u^{''}$, but they are adjusted by $u^{'}$ because an EU utility function is unique only up to a linear transformation.

Absolute Risk Aversion Measure (ARA)

$$ARA(w) = -\frac{u(w)''}{u(w)'} \tag{1.15}$$

Relative Risk Aversion Measure (RRA)

$$RRA(w) = -\frac{wu(w)''}{u(w)'} \tag{1.16}$$

By assuming absolute risk aversion (1.15) and relative risk aversion (1.15) are constant respectively, we can derive two special, and often used, functions - belonging to the exponential utility family (Constant Absolute Risk Aversion (CARA)) and to the power utility family (Constant Relative Risk Aversion (CRRA)). I am going to assume these two kinds of functions in this portfolio choice problem, and later will use the experimental data to test which one is a better explanation of behaviour. Now I first present the problem using the general Expected Utility model, and then derive the optimal allocation for both the CARA and the CRRA utility functions.

Individuals who have Expected Utility preferences maximise the following function

$$\begin{aligned} U &= \sum_{j\in\{1,2,3\}} p_j \times u(w_j) \\ &= \sum_{j\in\{1,2,3\}} p_j \times u(c_1 \times d_{1j} + c_2 \times d_{2j} + e) \end{aligned} \tag{1.17}$$

The first order conditions are

$$\begin{cases} \dfrac{dEU}{dc_1} = & \sum_{j\in\{1,2,3\}} p_j d_{1j} u'(w_j) = 0 \\ \dfrac{dEU}{dc_2} = & \sum_{j\in\{1,2,3\}} p_j d_{2j} u'(w_j) = 0 \end{cases}$$

Here we assume $p_j \neq 0$, $d_{ij} \neq 0$, $i \in \{1,2\}$, $j \in \{1,2,3\}$. Dividing the two equations by the first element we get

$$\begin{cases} 1 + \dfrac{p_2 d_{12}}{p_1 d_{11}} \dfrac{u'(w_2)}{u'(w_1)} + \dfrac{p_3 d_{13}}{p_1 d_{11}} \dfrac{u'(w_3)}{u'(w_1)} & = 0 \\ 1 + \dfrac{p_2 d_{22}}{p_1 d_{21}} \dfrac{u'(w_2)}{u'(w_1)} + \dfrac{p_3 d_{23}}{p_1 d_{21}} \dfrac{u'(w_3)}{u'(w_1)} & = 0 \end{cases}$$

Denote x and y as follows:

$$\begin{cases} x = \dfrac{u'(w_2)}{u'(w_1)} \\ y = \dfrac{u'(w_3)}{u'(w_1)} \end{cases} \tag{1.18}$$

Then we get

$$\begin{cases} 1 + \dfrac{p_2 d_{12}}{p_1 d_{11}} x + \dfrac{p_3 d_{13}}{p_1 d_{11}} y & = 0 \\ 1 + \dfrac{p_2 d_{22}}{p_1 d_{21}} x + \dfrac{p_3 d_{23}}{p_1 d_{21}} y & = 0 \end{cases}$$

or written in matrix form:

$$\begin{bmatrix} \dfrac{p_2 d_{12}}{p_1 d_{11}} & \dfrac{p_3 d_{13}}{p_1 d_{11}} \\ \dfrac{p_2 d_{22}}{p_1 d_{21}} & \dfrac{p_3 d_{23}}{p_1 d_{21}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

The the first order condition for Expected Utility preferences becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{p_2 d_{12}}{p_1 d_{11}} & \dfrac{p_3 d_{13}}{p_1 d_{11}} \\ \dfrac{p_2 d_{22}}{p_1 d_{21}} & \dfrac{p_3 d_{23}}{p_1 d_{21}} \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \tag{1.19}$$

Condition for the Existence of an Optimal allocation with general EU preferences First, from the first order condition (1.19), we know that the matrix

$$\begin{bmatrix} \dfrac{p_2 d_{12}}{p_1 d_{11}} & \dfrac{p_3 d_{13}}{p_1 d_{11}} \\ \dfrac{p_2 d_{22}}{p_1 d_{21}} & \dfrac{p_3 d_{23}}{p_1 d_{21}} \end{bmatrix}$$

should not be singular which means that

$$(\frac{p_2}{p_1}\frac{d_{12}}{d_{11}})(\frac{p_3}{p_1}\frac{d_{23}}{d_{21}}) - (\frac{p_3}{p_1}\frac{d_{13}}{d_{11}})(\frac{p_2}{p_1}\frac{d_{22}}{d_{21}}) \neq 0.$$

Assume for $\forall j \in \{1,2,3\}$ that $p_j \neq 0$ and $d_{11},\, d_{21} \neq 0$, we get

$$d_{12}d_{23} - d_{13}d_{22} \neq 0 \tag{1.20}$$

If this condition does not hold, individuals would have an opportunity to increase their utility unboundedly. The proof follows.

Proof If the condition is not satisfied then it implies

$$d_{12}d_{23} - d_{13}d_{22} = 0$$

which can be written as

$$\frac{d_{12}}{d_{22}} = \frac{d_{13}}{d_{23}} = \delta$$

Individuals could construct a portfolio in which $c_2 = -\delta c_1$ for which

$$\begin{cases} w_1 & = d_{11}c_1 - \delta c_1 d_{21} + e = (d_{11} - \delta d_{21})c_1 + e \\ w_2 & = d_{12}c_1 - \delta c_1 d_{22} + e = (d_{12} - \delta d_{22})c_1 + e = e \\ w_3 & = d_{13}c_1 - \delta c_1 d_{23} + e = (d_{13} - \delta d_{23})c_1 + e = e \end{cases}$$

Then the Expected Utility (1.17) becomes

$$\begin{aligned} U &= p_1 u(w_2) + p_2 u(w_2) + p_3 u(w_3) \\ &= p_1 u((d_{11} - \delta c_1 d_{21})c_1 + e) + p_2 u(e) + p_3 u(e) \end{aligned}$$

Taking the first derivative of the utility function with respect to c_1 we get

$$\frac{dU}{dc_1} = p_1(d_{11} - \delta d_{21})u^{'}(w_1)$$

Then the Expected Utility is monotonically increasing/decreasing with respect to c_1. It is easy to understand since the portfolio payoffs for both state 2 and state 3 become constant by constructing such a portfolio. In this case:

- $d_{11} - \delta d_{21} > 0$ then $\dfrac{dU}{dc_1} > 0$ and individuals would like to buy infinite amount of asset 1 and sell δc_1 of asset 2 to increase utility;
- $d_{11} - \delta d_{21} < 0$ then $\dfrac{dU}{dc_1} < 0$ and individuals would like to sell infinite amount of asset 1 and buy δc_1 of asset 2 to increase utility;
- $d_{11} - \delta d_{21} = 0$ then $\dfrac{dU}{dc_1} = 0$ and individuals are indifferent to any allocation as long as $c_2 = -\delta c_1$

An example may help to illustrate this problem. Suppose that the return table is

$$D = \begin{bmatrix} 0.1 & -0.2 & 0.3 \\ 0.3 & 0.4 & -0.6 \end{bmatrix}$$

We can calculate that $\delta = \frac{-0.2}{0.4} = \frac{0.3}{-0.6} = 0.5$. Denoting by $c_2 = -\delta c_1 = 0.5c_1$ and assuming that $e = 100$ then the Expected Utility becomes

$$\begin{aligned} U &= p_1 u(w_2) + p_2 u(w_2) + p_3 u(w_3) \\ &= p_1 u(0.25c_1 + 100) + p_2 u(100) + p_3 u(100) \end{aligned}$$

The first derivative

$$\frac{dU}{dc_1} = 0.25 p_1 u^{'}(0.25c_1 + 100) > 0$$

and so utility increases as c_1 increases.

CARA utility function

Assume an individual who has constant absolute risk aversion and also assume the following particular functional form[6]:

$$u(w) = -\frac{1}{r} e^{-rw}. \tag{1.21}$$

[6] This function belongs to the exponential family. It assumes ARA is constant as

$$ARA(w) = -\frac{-wre^{-rw-1}}{e^{-rw}} = r$$

$r > 0$ represents risk-aversion and $r < 0$ represents risk-loving and $r = 0$ represents risk-neutrality. The degree of risk aversion increases as r increases.

Equation (1.18) becomes

$$\begin{cases} x & = e^{-r\times(w_2-w_1)} \\ y & = e^{-r\times(w_3-w_1)}. \end{cases}$$

Taking the logarithm of each side of the equation we get

$$\begin{cases} w_2 - w_1 & = -\frac{\ln x}{r} \\ w_3 - w_1 & = -\frac{\ln y}{r} \end{cases}$$

Inserting these into equation (1.3) we get

$$\begin{cases} c_1(d_{12} - d_{11}) + c_2\,(d_{22} - d_{21}) & = -\frac{\ln x}{r} \\ c_1(d_{13} - d_{11}) + c_2\,(d_{23} - d_{21}) & = -\frac{\ln y}{r} \end{cases}$$

and the matrix form is

$$\begin{bmatrix} d_{12} - d_{11} & d_{22} - d_{21} \\ d_{13} - d_{11} & d_{23} - d_{21} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{r} \begin{bmatrix} \ln x \\ \ln y \end{bmatrix}$$

Then we get that the optimal allocation is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{r} \begin{bmatrix} d_{12} - d_{11} & d_{22} - d_{21} \\ d_{13} - d_{11} & d_{23} - d_{21} \end{bmatrix}^{-1} \ln \begin{bmatrix} x \\ y \end{bmatrix} \tag{1.22}$$

here x and y are calculated by the first order condition (1.19).

Conditions for the existence of an Optimal Allocation with a CARA utility function First, as the logarithm is taken for x and y in equation (1.22), both of them should be positive. By expanding the matrix form x and y as defined in equation (1.19), we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \dfrac{p_1}{p_2} \dfrac{d_{13}d_{21} - d_{23}d_{11}}{d_{12}d_{23} - d_{13}d_{22}} \\ \dfrac{p_1}{p_3} \dfrac{d_{11}d_{22} - d_{12}d_{21}}{d_{12}d_{23} - d_{13}d_{22}} \end{bmatrix}$$

As we assume that for $\forall j \in \{1,2,3\}$ that $p_j \neq 0$, the first conditions are the following:

$$\begin{cases} \dfrac{d_{13}d_{21} - d_{23}d_{11}}{d_{12}d_{23} - d_{13}d_{22}} & > 0 \\ \dfrac{d_{11}d_{22} - d_{12}d_{21}}{d_{12}d_{23} - d_{13}d_{22}} & > 0 \end{cases} \tag{1.23}$$

Second, as the inverse of the matrix

$$\begin{bmatrix} d_{12} - d_{11} & d_{22} - d_{21} \\ d_{13} - d_{11} & d_{23} - d_{21} \end{bmatrix}$$

needs to be taken, it cannot be singular. So we get a further condition:

$$(d_{12} - d_{11})(d_{23} - d_{21}) - (d_{13} - d_{11})(d_{22} - d_{21}) \neq 0 \tag{1.24}$$

CRRA utility function

Assume an individual who has constant relative risk aversion and also assume the following particular functional form[7]:

$$u(w) = \begin{cases} \frac{w^{1-r}}{1-r}, & r \neq 1 \\ ln(w) & r = 1 \end{cases} \tag{1.25}$$

Then equation (1.18) becomes

$$\begin{cases} x & = (\frac{w_2}{w_1})^{-r} \\ y & = (\frac{w_3}{w_1})^{-r} \end{cases}$$

[7]The function belongs to the power family. It assumes RRA is constant as

$$RRA(w) = -\frac{-wrw^{-r-1}}{w^{-r}} = r$$

$r > 0$ represents risk aversion, $r < 0$ represents risk loving and $r = 0$ represents risk neutrality. The degree of risk aversion increases as r increases.

from which it follows that

$$\begin{cases} \dfrac{w_2}{w_1} & = x^{-\frac{1}{r}} \\ \dfrac{w_3}{w_1} & = y^{-\frac{1}{r}} \end{cases}$$

Inserting these into equation (1.3) we get

$$\begin{cases} \dfrac{c_1 d_{12} + c_2 d_{22} + e}{c_1 d_{11} + c_2 d_{21} + e} & = x^{-\frac{1}{r}} \\ \dfrac{c_1 d_{13} + c_2 d_{23} + e}{c_1 d_{11} + c_2 d_{21} + e} & = y^{-\frac{1}{r}} \end{cases}$$

which also can be written as

$$\begin{cases} (d_{12} - x^{-\frac{1}{r}} d_{11}) c_1 + (d_{22} - x^{-\frac{1}{r}} d_{21}) c_2 + (1 - x^{-\frac{1}{r}}) e & = 0 \\ (d_{13} - y^{-\frac{1}{r}} d_{11}) c_1 + (d_{23} - y^{-\frac{1}{r}} d_{21}) c_2 + (1 - y^{-\frac{1}{r}}) e & = 0 \end{cases}$$

and the matrix form is

$$\begin{bmatrix} d_{12} - x^{-\frac{1}{r}} d_{11} & d_{22} - x^{-\frac{1}{r}} d_{21} \\ d_{13} - y^{-\frac{1}{r}} d_{11} & d_{23} - y^{-\frac{1}{r}} d_{21} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e \begin{bmatrix} x^{-\frac{1}{r}} - 1 \\ y^{-\frac{1}{r}} - 1 \end{bmatrix}$$

Then we get the optimal allocation as

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = e \begin{bmatrix} d_{12} - x^{-\frac{1}{r}} d_{11} & d_{22} - x^{-\frac{1}{r}} d_{21} \\ d_{13} - y^{-\frac{1}{r}} d_{11} & d_{23} - y^{-\frac{1}{r}} d_{21} \end{bmatrix}^{-1} \begin{bmatrix} x^{-\frac{1}{r}} - 1 \\ y^{-\frac{1}{r}} - 1 \end{bmatrix} \tag{1.26}$$

here x and y are derived through the first order conditions (1.19).

Conditions for the existence of an Optimal Allocation using the CRRA utility function In equation (1.26), we have assumed the existence of the inverse of the matrix on the right hand side. Denoting by $||..||$ the determinant of a matrix we get

$$\left\| \begin{matrix} d_{12} - x^{-\frac{1}{r}} d_{11} & d_{22} - x^{-\frac{1}{r}} d_{21} \\ d_{13} - y^{-\frac{1}{r}} d_{11} & d_{23} - y^{-\frac{1}{r}} d_{21} \end{matrix} \right\| \neq 0$$

which leads us to the condition

$$\begin{bmatrix} d_{21}d_{13} - d_{11}d_{23} & d_{11}d_{22} - d_{21}d_{12} \end{bmatrix} \begin{bmatrix} x^{-\frac{1}{r}} \\ y^{-\frac{1}{r}} \end{bmatrix} \neq d_{13}d_{22} - d_{12}d_{23} \tag{1.27}$$

By the definition of x, y in equation (1.18), they are related to the probabilities. Unlike the case in CARA utility function, the condition (1.27) is also related to probabilities since x, y are related to the probabilities as they are calculated by the first order conditions (1.19).

1.2.5 Conclusion

In this section, we have derived the explicit solution to the optimal portfolio choice problem in an unconstrained setting for EU preferences (specifically for the CARA utility function, which is in equation (1.22), and the CRRA utility function, which is in equation (1.26) and for Mean-Variance preferences, which is in equation (1.12). We also have discussed the conditions for the existence of the corresponding solutions.

1.3 Constrained Portfolio Choice problem in the context of risk

1.3.1 Introduction

In the preceding section we studied the portfolio choice problem when individuals are free to buy or sell the assets without constraints. However in the real world, individuals are always subject to different levels of constraints as the supplies of the assets are not unlimited and also the trading volumes are usually restricted for regulation reasons. Also in the experiment, it is difficult to implement an environment in which subjects are free to buy/sell any amount. If they were free to do so, then it is possible that they would end up with negative cash. But we can not actually ask the subjects to pay the experimenter money. So there has to be some constraints about trading.

In this section, we are going to study the portfolio choice problem with constraints. Specifically, the constraint is set as 0 because the technical details are the same for any other values. It means that individuals cannot sell a particular amount of one asset when they actually do not hold enough of that asset. Moreover as individuals are endowed only with cash at the start of a problem, the constraints mean that they can not borrow money and also they can not sell any risky assets. Then the No-short-selling constraints for the allocation $\mathbf{C} = [c_1\ c_2]$ can be written as

$$\begin{cases} c_1 & \geq 0 \\ c_2 & \geq 0 \\ c_1 + c_2 & \leq e. \end{cases} \tag{1.28}$$

Here we define $\mathbb{C}$ as the complete set of $\mathbf{C}$ that satisfy equation (1.28). We also call the area which all allocations satisfy the No-short-selling constraints as the *Allocation Triangle*. In the preceding section we derived the analytical solution for the unconstrained optimal allocation $\mathbf{C}^*$, so we can now use that in what follows. When

$$\begin{cases} c_1^* & \geq 0 \\ c_2^* & \geq 0 \\ c_1^* + c_2^* & \leq e. \end{cases}$$

we have $\mathbf{C}^* \in \mathbb{C}$ so $\mathbf{C}^*$ is also the constrained optimal allocation, which denote by $\mathbf{C}^{**}$. When

$$\begin{cases} c_1^* & \geq 0 \\ c_2^* & < 0 \end{cases}$$

and

$$\begin{cases} c_2^* & \geq 0 \\ c_1^* & < 0 \end{cases}$$

and

$$\begin{cases} c_1^* & \geq 0 \\ c_2^* & \geq 0 \\ c_1^* + c_2^* & > e \end{cases}$$

and

$$\begin{cases} c_1^* & < 0 \\ c_2^* & < 0 \end{cases}$$

we have $\mathbf{C}^* \notin \mathbb{C}$ so $\mathbf{C}^*$ is *not* the constrained optimal allocation. Denoting by $\mathbf{C}^{**}$ the constrained optimal allocation, I show now how to find the $\mathbf{C}^{**}$ corresponding to these 5 scenarios. In each area, there are various ways of finding $\mathbf{C}^{**}$. Figure 1.1 to Figure 1.13 on Pages 22- 28 are illustration of the various cases for each area. In these figures, the horizontal axis represents c_1 and the vertical axis represents c_2. The contours are the indifference curves of utility, which are plotted, for illustration, using the utility function of Mean-Variance Preferences [8]. Notice that the area *within* (including the margins) the triangle represents all the allocations satisfying the No-short-selling constraints (1.28). There are 3 special allocations involved in this

[8]These figures also could represent the case for the Expected Utility Preferences as the utility functions for these are concave too.

analysis. Let me define them first. Denote by

$$^1\mathbf{C}^* = [^1c_1^* \;\; 0]$$

as the unconstrained optimal allocation with the constraint that $c_2 = 0$. Denote by

$$^2\mathbf{C}^* = [0 \;\; ^2c_2^*]$$

as the unconstrained optimal allocation with the constraint that $c_1 = 0$. Denote by

$$^0\mathbf{C}^* = [^0c_1^* \;\; ^0c_2^*]$$

as the unconstrained optimal allocation with the constraint that $c_1 + c_2 = e$.

- Area 0 ($\mathbb{A}_0$): $c_1^* \geq 0$ and $c_2^* \geq 0$ and $c_1^* + c_2^* \leq e$

  As shown in Figure 1.1 on Page 22, $\mathbf{C}^*$ lies in the constrained area $\mathbb{A}_0$. So the unconstrained optimal allocation is also the constrained optimal allocation, we have $\mathbf{C}^{**} = \mathbf{C}^*$.

- Area 1 ($\mathbb{A}_1$): $c_1^* \geq 0$ and $c_2^* < 0$

  As $c_2^* < 0$, $\mathbf{C}^*$ is not within $\mathbb{A}_0$. Then a reasonable guess is $\mathbf{C}^{**} =^1 \mathbf{C}^*$ as the latter is the tangency point of the indifference curves to the axis $c_2 = 0$. But we need to look through the following 3 cases.

  - Case 1: As shown in Figure 1.2 on Page 22, $0 <^1 c_1^* < e$. So $^1\mathbf{C}^*$ is a valid allocation. We have $\mathbf{C}^{**} = \mathbf{C}_1^*$
  - Case 2: As shown in Figure 1.3 on Page 23, $^1c_1^* > e$. So $^1\mathbf{C}^*$ is not a valid allocation. Then the optimal allocation is the point $[e \;\; 0]$, where is the nearest indifference touches $\mathbb{A}_0$. So we have $\mathbf{C}^{**} = [e \;\; 0]$
  - Case 3: As shown in Figure 1.4 on Page 23, $^1c_1^* < 0$. Again $^1\mathbf{C}^*$ is not a valid allocation. Then the optimal allocation is the point $[0 \;\; 0]$, where is the nearest indifference touches $\mathbb{A}_0$. So we have $\mathbf{C}^{**} = [0 \;\; 0]$

- Area 2 ($\mathbb{A}_2$): $c_2^* \geq 0$ and $c_1^* < 0$

$\mathbf{C}^*$ is not a valid allocation. Being similar with the scenarios when it is in $\mathbb{A}_1$, there are also 3 cases as follows:

- Case 1: As shown in Figure 1.5 on Page 24, ${}^2C^*$ is a valid allocation. So we have $\mathbf{C}^{**} =^2 C^*$
- Case 2: As shown in Figure 1.6 on Page 24, we have $\mathbf{C}^{**} = [e \;\; 0]$
- Case 3: As showing in Figure 1.7 on Page 25, we have $\mathbf{C}^{**} = [0 \;\; 0]$

- Area 3 ($\mathbb{A}_3$): $c_1^* \geq 0$ and $c_2^* \geq 0$ and $c_1^* + c_2^* > e$

  $\mathbf{C}^*$ is not a valid allocation as $c_1^* + c_2^* > e$. The position of $\mathbf{C}^{**}$ depends on 3 different locations of ${}^0\mathbf{C}^*$, where the indifference curves are tangential to the line $c_1 + c_2 = e$.

  - Case 1: As shown in Figure 1.8 on Page 25, ${}^0\mathbf{C}^*$ is a valid allocation because that ${}^0c_1^* \geq 0$ and ${}^0c_2^* \geq 0$. So we have $\mathbf{C}^{**} =^0 \mathbf{C}^*$.
  - Case 2: As shown in Figure 1.9 on Page 26, ${}^0c_2^* < 0$. ${}^0C^*$ is not a valid allocation, so we have $\mathbf{C}^{**} = [e \; 0]$
  - Case 3: As shown in Figure 1.10 on Page 26, ${}^0c_1^* < 0$. ${}^0C^*$ is not a valid allocation, so we have $\mathbf{C}^{**} = [0 \; e]$ Notice that we do not need to consider the case when both ${}^0c_1^* < e$ and ${}^0c_2^* < e$ since it is implicitly excluded by the condition ${}^0c_1^* +^0 c_2^* = e$

- Area 4 ($\mathbb{A}_4$): $c_1^* <$ and $c_2^* < 0$ This is a rather interesting scenario. We expected $\mathbf{C}^{**} = [0 \;\; 0]$ but it turns out that it is not always correct. Here still are 3 cases for $\mathbf{C}^{**}$.

  - Case 1: As shown in Figure 1.11 on Page 27, neither ${}^0C_1^*$ or ${}^0C_2^*$ is valid allocation. We have $\mathbf{C}^{**} = [0 \;\; 0]$ as expected.
  - Case 2: As shown in Figure 1.12 on Page 27, ${}^1c_1^* > 0$ and ${}^2c_2^* < 0$. So we have $\mathbf{C}^{**} = [{}^1c_1^* \; 0]$.

- Case 3: As shown in Figure 1.13 on Page 28, ${}^1c_1^* < 0$ and ${}^2c_2^* > 0$. So we have $\mathbf{C}^{**} = [0\ {}^2c_2^*]$.[9]

  We have not considered the case in which both ${}^1c_1^* > 0$ and ${}^2c_2^* > 0$ because it is not possible. The proof is following. Assume that ${}^1c_1^* > 0$ and ${}^1c_2^* > 0$ and $U({}^1c_1^*, 0) \leq U(0, {}^2c_2^*)$. As $C^* = [c_1^*\ \ c_2^*]$ is the unconstrained optimal allocation, we have

  $$U([{}^1c_1^*\ \ 0]) \leq U([0\ \ {}^2c_2^*] < U([c_1^*\ \ c_2^*])$$

  By the concavity of utility function, there exists a variable $\lambda \in [0\ 1]$ that makes

  $$[0\ \ {}^0c_2^*] = \lambda[{}^1c_1^*\ \ 0] + (1-\lambda)[c_1^*\ \ c_2^*]$$

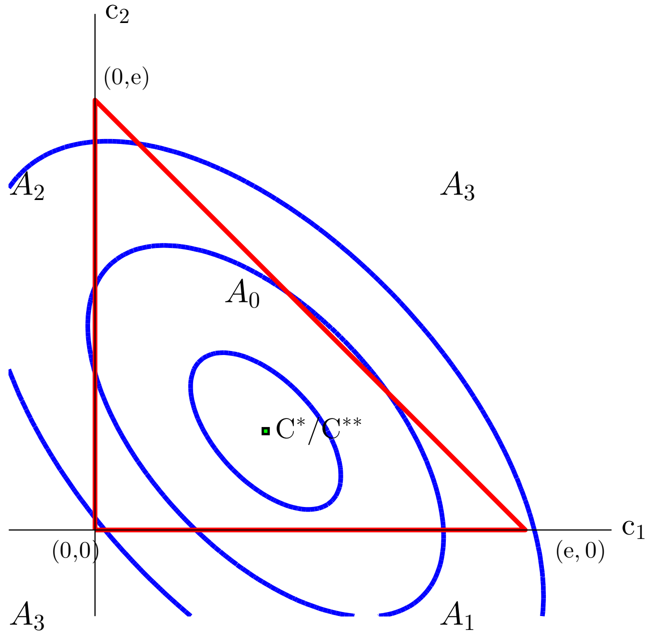
  which implies that

  $${}^2c_2^* = (1-\lambda)\ c_2^* \leq 0$$

  This is a contradiction with ${}^2c_2^* > 0$.

[9]If ${}^1c_1^* < 0$, we let $\mathbf{C}^{**} = [0\ \ 0]$ and if ${}^1c_1^* > e$, we let $\mathbf{C}^{**} = [e\ \ 0]$. Similar technique is applied to ${}^2c_2^*$. Note the constrained optima is relatively simple in the case that there is only dimension. That is there is only one variable. Denote by c the one variable and by c^* the unconstrained optima. Then the constrained optimal

$$c^{**} = \begin{cases} 0 & c^* < 0 \\ c^* & 0 \leq c^* \leq e \\ e & c^* > e. \end{cases}$$



[c]

FIGURE 1.1: $\mathbb{A}_0$

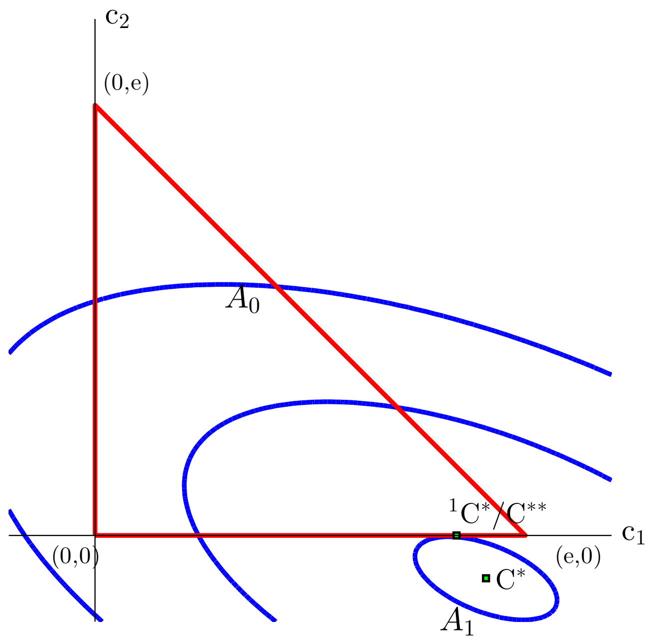


FIGURE 1.2: $\mathbb{A}_1$(Case 1)

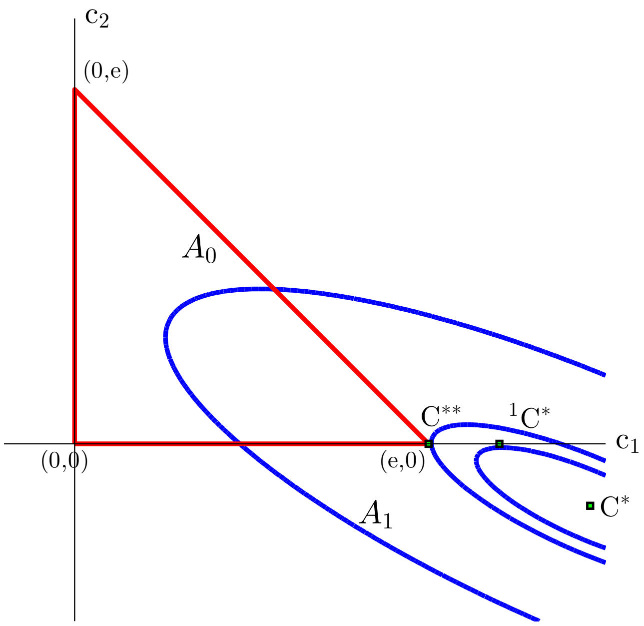


FIGURE 1.3: $\mathbb{A}_1$(Case 2)

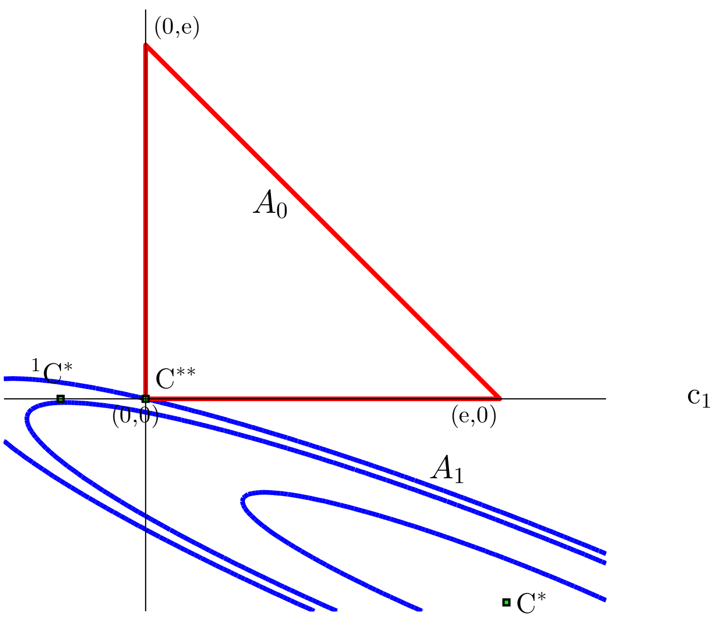


FIGURE 1.4: $\mathbb{A}_1$(Case 3)

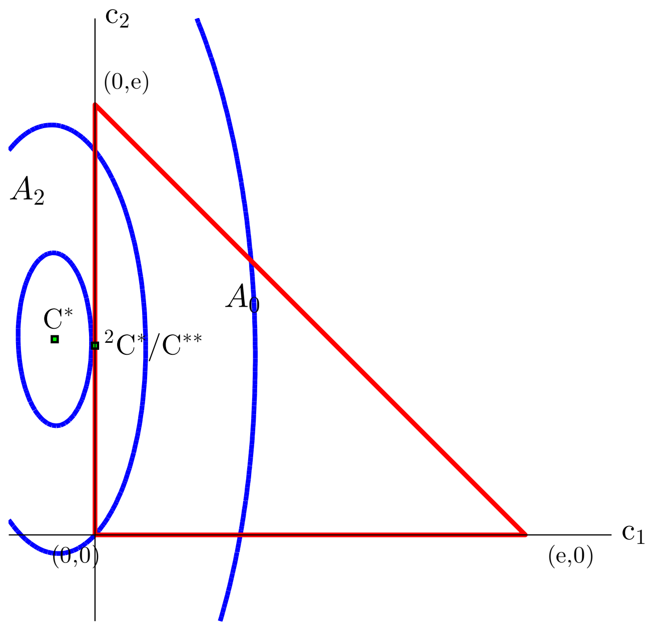


FIGURE 1.5: $\mathbb{A}_2$(Case 1)

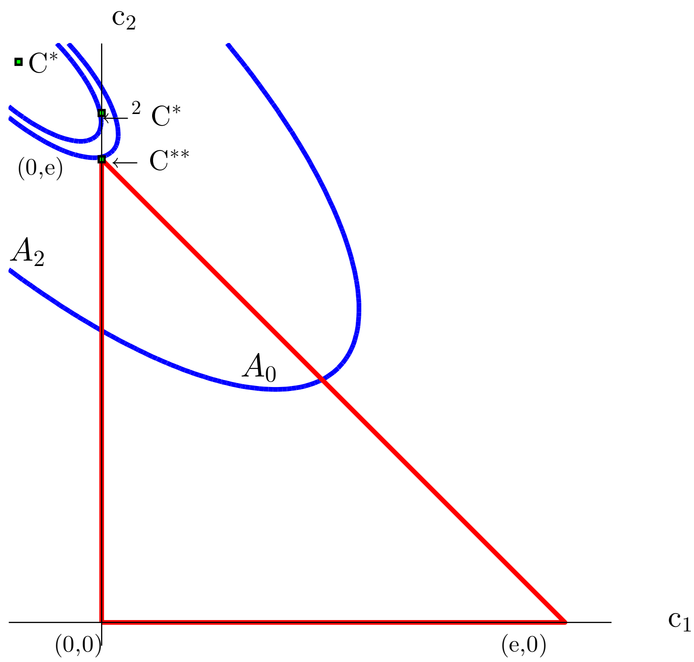


FIGURE 1.6: $\mathbb{A}_2$(Case 2)

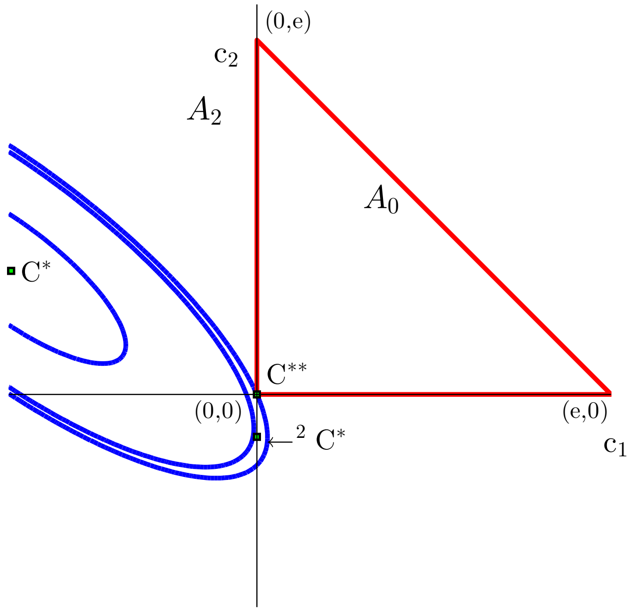


FIGURE 1.7: $\mathbb{A}_2$(Case 3)

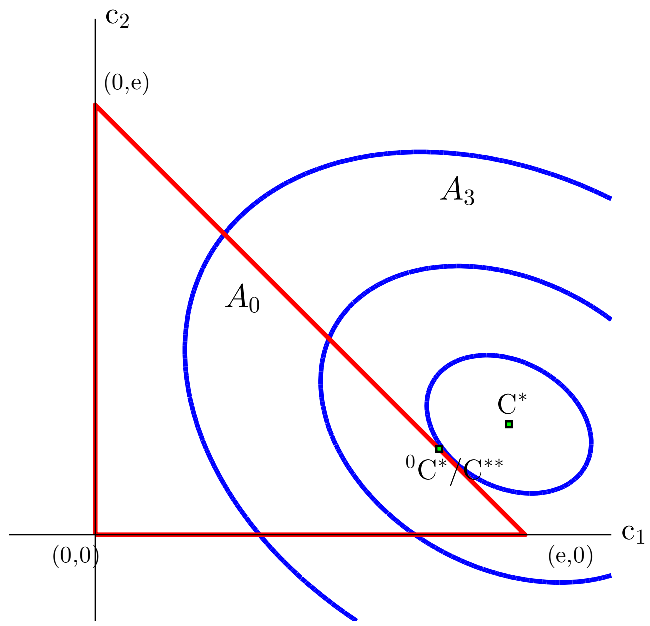


FIGURE 1.8: $\mathbb{A}_3$(Case 1)

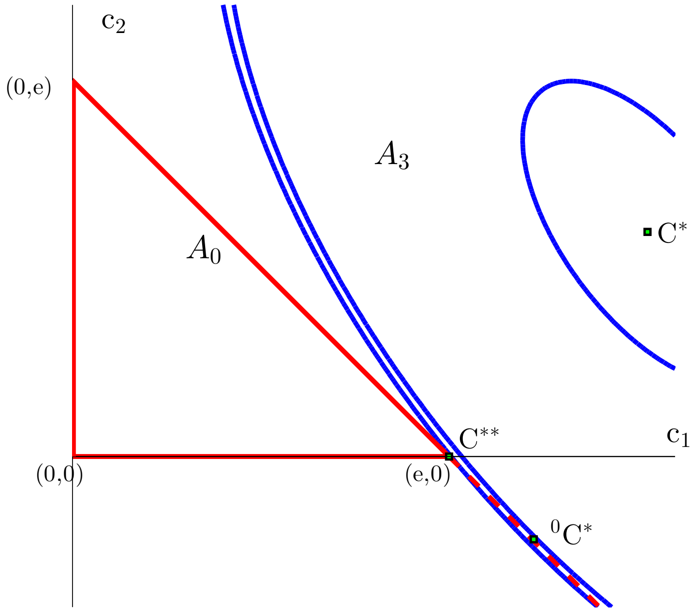


FIGURE 1.9: $\mathbb{A}_3$(Case 2)

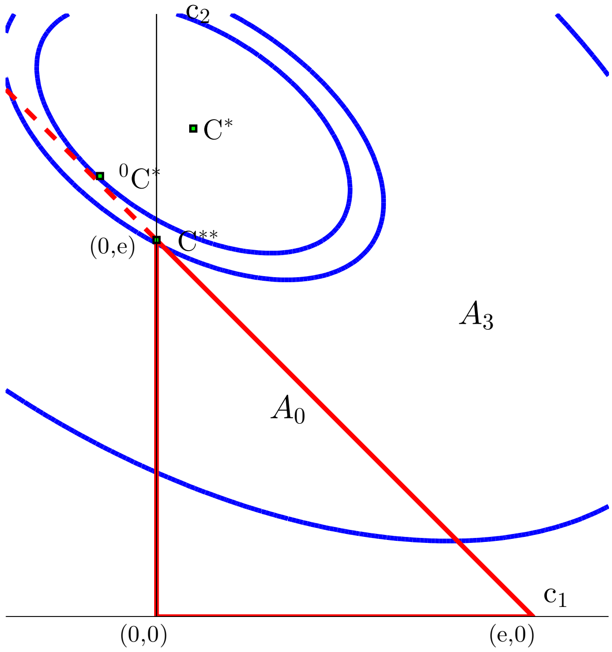


FIGURE 1.10: $\mathbb{A}_3$(Case 3)

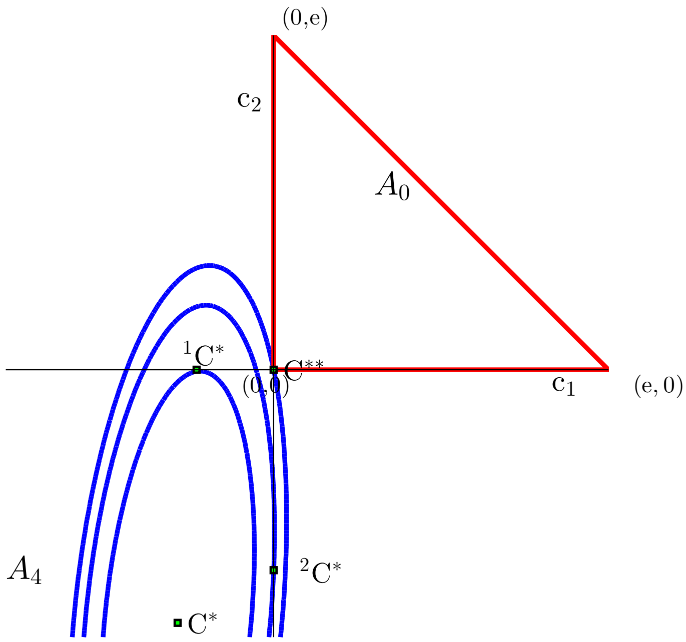


FIGURE 1.11: $\mathbb{A}_4$(Case 1)

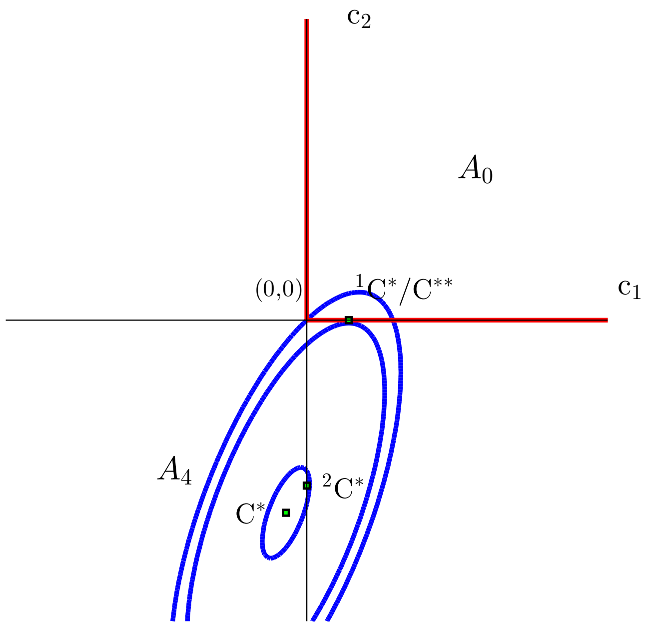


FIGURE 1.12: $\mathbb{A}_4$(Case 2)

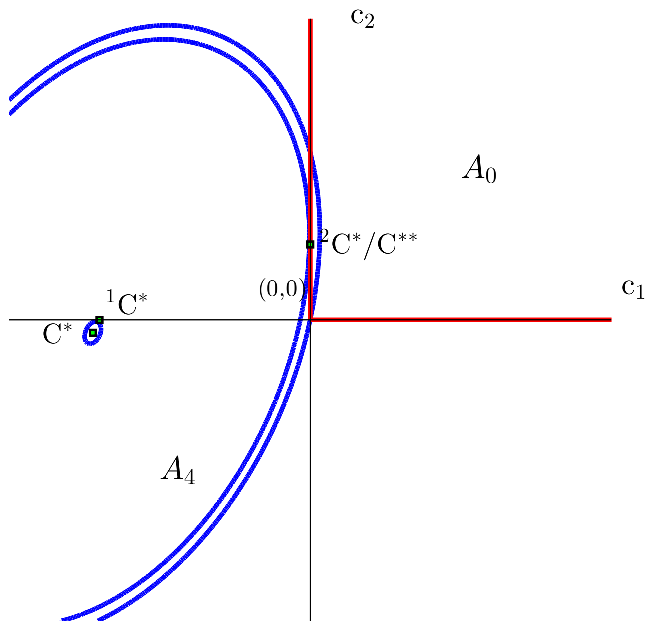


FIGURE 1.13: $\mathbb{A}_4$(Case 3)

1.3.2 Algorithm for the Constrained Optimal allocation

We have already described the various cases for the position of $\mathbf{C}^{**}$ in all the 5 areas. What we can do is to calculate $\mathbf{C}^{*}$ first and then check which area it is in. And then go through all the cases in that area. This method is straightforward but tedious and time-consuming. So we are going to adopt an alternative algorithm which is as follows. As analysed before, no matter which area $\mathbf{C}^{**}$ is in, there are only 7 different values that $\mathbf{C}^{**}$ can be assigned. They are C^{*}, ${}^{1}\mathbf{C}^{*}$, ${}^{2}\mathbf{C}^{*}$, ${}^{0}\mathbf{C}^{*}$, $[0 \quad e]$, $[e \quad 0]$ and $[0 \quad 0]$ and are denoted as $\mathbf{C}_0$, $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_5$ and $\mathbf{C}_6$ respectively. These 7 allocations cover all possible values that $\mathbf{C}^{**}$ could take. So we can calculate all of them and also the corresponding maximised utilities first. Then we can compare these maximised utilities and find the maximum one. The maximum utility of all maximised utilities is the constrained maximised utility and the corresponding optimal allocation is $\mathbf{C}^{**}$, the constrained optimal allocation. Notice that for $C_k \quad k \in \{1, 2, 3\}$, they could be invalid if they do not satisfy the No-short-selling constraints (1.28). In such scenarios, we specify the corresponding maximised utility to $-Inf$ to make sure they are not chosen. The details of the algorithm are in Algorithm 1 on Page 30. Algorithm 1 is a general approach for finding the constrained optimal allocation for different preferences. It needs us to provide the expression of $C_k \quad k \in \{0, 1, 2, 3, 4, 5, 6\}$, which are relevant to the objective functions. Next I provide these solutions and explain how to apply Algorithm 1 to MV preferences and EU preferences.

1.3.3 Constrained Optimal Allocation for MV Preferences

The objective function for Mean-Variance Preferences is function $(1.10$

$$U = e + \mathbf{C}\mu - \frac{1}{2} r \mathbf{C}\Omega\mathbf{C}'. \tag{1.29}$$

here μ and Ω are defined in equation (1.5) and (1.7) respectively.

Algorithm 1: Constrained Optimal allocation

Input : r, p_1, p_2, p_3, e
Output: $\mathbf{C}^*$

Calculate $\mathbf{C}^*$ using explicit solution ;
if $\mathbf{C}^* \in \mathbb{C}$ **then**
  $\mathbf{C}^{**}{=}\mathbf{C}_0$; $U^{**}{=}U(\mathbf{C}_0)$;
else
  $\tilde{U}_0 = -Inf$;
end
for $k = 1$ **to** 6 **do**
  Calculate C_k;
  if $C_k \in \mathbb{C}$ **then**
    $U_k{=}U(\mathbf{C}_k)$;
  else
    $\tilde{U}_k = -Inf$;
  end
end
Then we have

$$\mathbf{C}^{**} = \begin{cases} \mathbf{C}_0 & \text{if } C_0 \text{ is valid} \\ \mathbf{C}_k & \text{if for } k \text{ there is } U_k = max\{U_1, U_2, U_3, U_4, U_5, U_6\} \end{cases}$$

Step 1 As derived in equation (1.12), the unconstrained optimal allocation is

$$\mathbf{C}^* = \frac{1}{r}(\Omega^{-1}\mu)'. \tag{1.30}$$

We then check that whether $\mathbf{C}^* \in \mathbb{C}$, that is, if it satisfes equation (1.28). If not we then proceed to step 2.

Step 2

- For calculating ${}^1\mathbf{C}^*$, let $c_2 = 0$ so the objective function becomes

$$\begin{aligned} U &= e + \mathbf{C}\mu - \frac{1}{2}r\mathbf{C}\Omega C' \\ &= e + c_1\mu_1 - \frac{1}{2}rc_1^2\sigma_1^2 \end{aligned}$$

Using the first order condition, the optimal allocation is

$$c_1^* = \frac{\mu_1}{\sigma_1^2 r}.$$

The corresponding maximised utility is

$$U([c_1^*,\ 0]) = e + \frac{1}{2}\frac{\mu_1^2}{\sigma_1^2 r}.$$

Then we have

$$\mathbf{C}_1 = \begin{bmatrix} \frac{\mu_1}{\sigma_1^2 r} & 0 \end{bmatrix}$$

and

$$U_1 = \begin{cases} e + \frac{1}{2}\frac{\mu_1^2}{\sigma_1^2 r} & \text{if } 0 \leq \frac{\mu_1}{\sigma_1^2 r} \leq e \\ -Inf & \text{if } \frac{\mu_1}{\sigma_1^2 r} < 0 \text{ or } > e \end{cases}$$

- By the symmetry of c_1 and c_2 we have

$$C_2 = \begin{bmatrix} \frac{\mu_2}{\sigma_2^2 r} & 0 \end{bmatrix}$$

and

$$U_2 = \begin{cases} e + \dfrac{1}{2}\dfrac{\mu_2^2}{\sigma_2^2 r} & \text{if } 0 \leq \dfrac{\mu_2}{\sigma_2^2 r} \leq e \\ -Inf & \text{if } \dfrac{\mu_2}{\sigma_2^2 r} < 0 \text{ or } > e \end{cases}$$

- For calculating ${}^0C^*$ we impose the constraint that $c_1 + c_2 = e$ to maximise the function ((1.10)). It becomes an equality constrained optimisation problem and we can write the Lagrangian function as

$$\begin{aligned} L &= U - \lambda(c_1 + c_2 - e) \\ &= e + (c_1\mu_1 + c_2\mu_2) - \frac{1}{2}r(c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + c_1c_2\sigma_{12}) - \lambda(c_1 + c_2 - e) \end{aligned}$$

and the first order condition is

$$\begin{cases} L_{c_1} &= \mu_1 - r(c_1\sigma_1^2 + c_2\sigma_{12}) - \lambda \\ L_{c_1} &= \mu_2 - r(c_2\sigma_2^2 + c_1\sigma_{12}) - \lambda \\ L_\lambda &= c_1 + c_2 - e = 0. \end{cases}$$

Then we have that $\mu_1 - r(c_1\sigma_1^2 + c_2\sigma_{12}) = \mu_2 - r(c_2\sigma_2^2 + c_1\sigma_{12})$ and also $c_1 + c_2 - e = 0$. By solving these two equations we get the solution as

$$C_3 = \begin{bmatrix} c_1^* & c_2^* \end{bmatrix}$$

here we have that

$$\begin{cases} c_1^* &= \dfrac{\mu_1 - \mu_2 - r(\sigma_{12}e + er{\sigma_2}^2}{r{\sigma_1}^2 - 2\,\sigma_{12} + {\sigma_2}^2)} \\ c_2^* &= \dfrac{-\mu_1 + r{\sigma_1}^2 e + \mu_2 - r\sigma_{12}e}{r({\sigma_1}^2 - 2\,\sigma_{12} + {\sigma_2}^2)}. \end{cases}$$

The corresponding maximum is

$$U_3 = e + C_3\mu - \frac{1}{2}rC_3\Omega C_3'.$$

Step 3 This step does not involve a check as to whether any result satisfies the constraints, so we can directly get:

- $\mathbf{C}_4 = [e \ \ 0]$ and $U_4 = U([100 \ \ 0]) = e + e\mu_1 - \frac{1}{2}re^2\sigma_1^2$
- $\mathbf{C}_5 = [0 \ \ e]$ and $U_5 = U([0 \ \ 100]) = e + e\mu_2 - \frac{1}{2}re^2\sigma_1^2$
- $\mathbf{C}_6 = [0 \ \ 0]$ and $U_6 = U([0 \ \ 0]) = e$

1.3.4 Constrained Optimal Allocation for EU Preferences

CARA utility function

For an individual who has Expected Utility Preferences and has a CARA utility function, the objective function is

$$EU = -\sum_{j\in[1,2,3]} p_j \frac{1}{r} e^{-r(c_1 d_{1j} + c_2 d_{2j} + e)}.$$

Step 1: Calculating C_0 The unconstrained optimal allocation $\mathbf{C}^*$ is stated in equation (1.22). We then check that if $\mathbf{C}^*$ satisfies the equation (1.28). If not we then proceed to the step 2.

Step 2: Calculating $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$

- $\mathbf{C}_1$ In this case, $c_2 = 0$ so the objective function

$$EU = -\sum_{j\in[1,2,3]} p_j \frac{1}{r} e^{-r(c_1 d_{1j}+e)}.$$

The first order condition is

$$\frac{dEU}{c_1} = p_1 d_{11} e^{-rc_1 d_{11}} + p_2 d_{12} e^{-rc_1 d_{12}} + p_3 d_{13} e^{-rc_1 d_{13}} = 0. \quad (1.31)$$

There is not an explicit solution for this equation but the optimum can be found numerically. [10].Denoting by ${}^1c_1^*$ the answer then

$$\mathbf{C}_1 = [{}^1c_1^* \ \ 0]$$

and the corresponding maximum is

$$U_1 = \begin{cases} -\sum_j p_j \frac{1}{r} e^{-r({}^1c_1^* d_{1j}+e)} & \text{if } 0 \leq^1 c_1^* \leq e \\ -Inf & \text{if } {}^1c_1^* < 0 \text{ or } > e \end{cases}$$

- $\mathbf{C}_2$ By symmetry we have

$$\mathbf{C}_2 = [{}^1c_2^* \ \ 0]$$

here ${}^1c_2^*$ is the numerical solution to the equation

$$\frac{dEU}{c_2} = p_1 d_{21} e^{-rc_2 d_{11}} + p_2 d_{22} e^{-rc_2 d_{12}} + p_3 d_{23} e^{-rc_1 d_{23}} = 0. \quad (1.32)$$

[10] Most mathematical software can solve an equation of this form. In matlab, *fzero* can calculate the solution for an equation in a defined area.

The corresponding maximised utility is

$$U_2 = \begin{cases} -\sum^j \frac{1}{r} e^{-r({}^2c_2^* d_{1j}+e)} & \text{if } 0 \leq^2 c_2^* \leq e \\ -Inf & \text{if } {}^2c_2^* < 0 \text{ or } > e \end{cases}$$

- $\mathbf{C}_3$ This is an equality constrained optimisation problem which could be resolved by the Lagrangian method. But since there are only two variables here it might be easier just replace $c_2 = e - c_2$ in the objective function; then it becomes

$$EU = -\sum_{j\in[1,2,3]}^{j} p_j \frac{1}{r} e^{-r(c_1(d_{1j}-d_{2j})+ed_{2j}+e)}.$$

The first order condition is

$$\frac{dEU}{c_1} = \sum^j p_j (d_{1j} - d_{2j}) e^{-r(c_1(d_{1j}-d_{2j})} = 0.$$

There is also no explicit solution for this equation but the optimum can be found numerically. Denoting by ${}^0c_1^*$ the answer then

$$\mathbf{C}_3 = [{}^0c_1^* \quad e -^0 c_1^*]$$

and the corresponding maximum is

$$U_3 = \begin{cases} -\sum^j p_j \frac{1}{r} e^{-r({}^0c_1^*(d_{1j}-d_{2j})+ed_{2j}+e)} & \text{if } e \geq^0 c_1^* \geq 0 \\ -Inf & \text{if } {}^2c_2^* < 0 \text{ or } > e \end{cases}$$

Step 3 This step does not involve a check as to whether any result satisfies the constraints, so we can directly get:

- $C_4 = [e \ \ 0]$ and $U_4 = U([100 \ \ 0]) = -\sum_j p_j \frac{1}{r} e^{-r(ed_{1j}+e)}$
- $C_5 = [0 \ \ e]$ and $U_5 = U([0 \ \ 100]) = -\sum_j p_j \frac{1}{r} e^{-r(ed_{2j}+e)}$
- $C_6 = [0 \ \ 0]$ and $U_6 = U([0 \ \ 0]) = -\sum_j p_j \frac{1}{r} e^{-re}$

CRRA utility function

For an individual who has Expected Utility Preferences and also has a CRRA utility function, the objective function is

$$EU = \sum_{j\in[1,2,3]}^{j} (c_1 d_{1j} + c_2 d_{2j} + e)^r$$

Step 1: Calculating C_0 The expression for the unconstrained optimal allocation C is equation (1.26). We then check that if $\mathbf{C}^* \in \mathbb{C}$. If not we then proceed to the step 2.

Step 2: Calculating C_1, C_2, C_3

- $\mathbf{C}_1$ In this case, $c_2 = 0$ so the objective function becomes

$$EU = \sum_{j\in[1,2,3]} p_j (c_1 d_{1j} + e)^r$$

  The first order condition is

$$\frac{dEU}{c_1} = p_1 d_{11}(c_1 d_{11} + e)^{r-1} + p_2 d_{12}(c_1 d_{12} + e)^{r-1} + p_3 d_{13})^(c_1 d_{13} + e)^{r-1} = 0.$$

  There is no explicit solution for this equation but the optimum can be found numerically.Denoting by ${}^1c_1^*$ the answer then

$$\mathbf{C}_1 = [{}^1c_1^* \;\; 0]$$

  and the corresponding maximum is

$$U_1 = \begin{cases} \sum_j p_j ({}^1c_1^* d_{1j} + e)^r & \text{if } e \geq^1 c_1^* \geq 0 \\ -Inf & \text{if } {}^1c_1^* < 0 \text{ or } > e \end{cases}$$

- $\mathbf{C}_2$ By symmetry we have

$$\mathbf{C}_2 = [0 \;\; {}^1c_2^*]$$

here ${}^2c_2^*$ is the numerical solution to the equation

$$\frac{dEU}{c_2} = p_1 d_{21}(c_1 d_{21}+e)^{r-1}+p_2 d_{22}(c_1 d_{22}+e)^{r-1}+p_3 d_{23})^(c_1 d_{23}+e)^{r-1} = 0. \tag{1.33}$$

The corresponding maximised utility is

$$U_2 = \begin{cases} \sum^j p_j({}^2c_2^* d_{2j} + e)^r & \text{if } 0 \leq^2 c_2^* \leq e \\ -Inf & \text{if } {}^2c_2^* < 0 \text{ or } > e \end{cases}$$

- $\mathbf{C}_3$ This is an equality constrained optimisation problem which could be resolved by the Lagrangian method. But since there are only two variables here it might be easier just to replace $c_2 = e - c_2$ in the objective function; then it becomes

$$EU = -\sum_{j\in[1,2,3]}^{j} p_j(c_1(d_{1j} - d_{2j}) + ed_{2j} + e)^r$$

The first order condition is

$$\frac{dEU}{c_1} = \sum^j p_j(d_{1j} - d_{2j})(c_1(d_{1j} - d_{2j}) + ed_{2j} + e)^{r-1} = 0.$$

There is also no explicit solution for this equation but the optimum can be found numerically. Denote ${}^0c_1^*$ as the calculated optimal allocation.

$$\mathbf{C}_3 = [{}^0c_1^* \quad e -^0 c_1^*]$$

and the corresponding maximum is

$$U_3 = \begin{cases} -\sum_{j\in[1,2,3]}^j p_j({}^0c_1^*(d_{1j} - d_{2j}) + ed_{2j} + e)^r & \text{if } e \geq^0 c_1^* \geq 0 \\ -Inf & \text{if } {}^0c_1^* < 0 \text{ or } > e \end{cases}$$

Step 3 This step does not involve a check as to whether any result satisfies the constraints, so we can directly get:

- $\mathbf{C}_4 = [e \;\; 0]$ and $U_4 = U([100 \;\; 0]) = -\sum^j p_j(ed_{1j} + e)^r$

- $\mathbf{C}_5 = [0 \;\; e]$ and $U_5 = U([0 \;\; 100]) = -\sum^j p_j(ed_{2j} + e)^r$
- $\mathbf{C}_6 = [0 \;\; 0]$ and $U_6 = U([0 \;\; 0]) = e^r$

1.4 Portfolio Choices in the context of ambiguity

1.4.1 Introduction

I begin by applying the Maxmin Expected Utility (MEU) model to our portfolio choice problem. I provide two algorithms to find the MEU optimal allocation with the No-short-selling constraints. The first algorithm is purely numerical, but it takes a lot of time, and does not provide reliable solutions. The second algorithm is much more efficient but there is a lot of analysis to prove the validity of the algorithm. We are going to use the second algorithm for estimation when we fit this model to the experimental data. Then I introduce α-Maxmin Expected Utility (α-MEU) preferences, which is a generalisation of MEU. I also provide an algorithm to calculate the optimal allocation based on the previous analysis for MEU. The reader should be warned that there is a lot of technical detail in this section - concerning the derivation and application of algorithms to find the optimal portfolio allocations in the context of ambiguity. As is always the case, there is a trade-off between elegance and efficiency. While it would be nice to be able to find explicit analytical solutions to the determination of the optimal allocation, often, particularly when there are constraints on the allocation, explicit solutions cannot be obtained. In these cases we need to develop algorithms to find the solutions. Once again there is a trade-off between elegance and efficiency: it is not always the case that the most elegant algorithms are the computationally most efficient. One problem is that it is natural to explore the explicit route as far as possible before moving to computationally more efficient ones, though the latter in turn are driven by analytical considerations. The reader should try and keep this in mind throughout this section.

1.4.2 Maxmin Expected Utility (MEU) Preferences

Expected Utility theory assumes that individuals act as if they are maximising the probability weighted utilities over all the states. It assumes there is only one prior, that is individuals believe that the probability of each state is fixed. While in the MEU theory, it is assumed that there are multiple priors. Individuals are unware about the actual probabilities but believe that they lie in a set of possible probabilities. For each possible allocation, individuals work out the expected utility using the set of priors and consider the minimum expected utility as the utility of this allocation. By applying the same procedure to all the possible allocations, individuals then choose the allocation which produces the maximised utility. Notice that the major difference here is, whereas in the EU preference, the utility obtained from an allocation is just the expected utility, but in the MEU preferences, the utility over an allocation is the minimum expected utility among all the priors.

Denoting by $\mathbb{P}$ the set of all possible priors, and $\mathbb{C}$ as the set of all possible allocations, then the objective function of MEU model can be written as

$$\mathcal{U} = \underset{C \in \mathbb{C}}{max} \; \underset{P \in \mathbb{P}}{min} \{p_j \sum_{j \in (1,2,3)} u(w_j)\} \tag{1.34}$$

here w_j is the payoff from allocation. In Figure 1.14, the horizontal axis

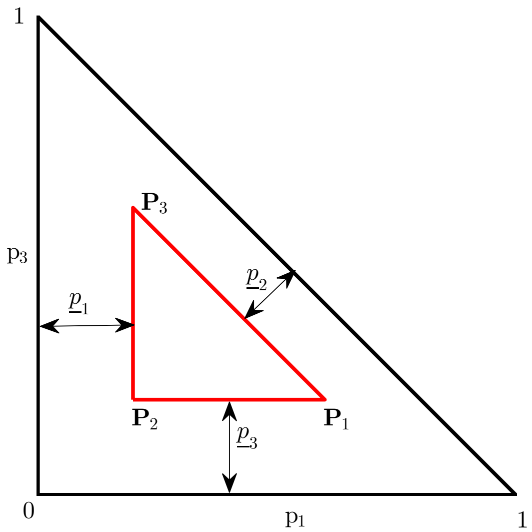


FIGURE 1.14: Marschk-Machina Triangle

represents p_1 and the vertical axis represents p_3 and we must have $0 \leq p_1, p_3 \leq 1$ as the sum of probabilities is equal to 1. This constraint makes a triangle area called the Marschak-Machina Triangle (MMT) and any point inside the triangle represents a probability vector for a three states gamble. The MEU preferences model does not tell us how the individuals specify $\mathbb{P}$, which is the set of possible priors. We are going to characterise this set in the following way. We are going to assume that for all individuals believe there is a probability lower bound $\underline{p_j}$ for each state j:

$$p_j \geq \underline{p}_j \quad \text{for} j = 1, 2, 3$$

There is $\underline{p}_1 + \underline{p}_2 + \underline{p}_3 \leq 1$. The small triangle [11] inside the MMT then represents the set $\mathbb{P}$.

Basically when we are finding the MEU optimal allocation, there are two steps involved. The first step is to calculate the minimum expected utility in terms of all the elements $\mathbf{P} \in \mathbb{P}$ for each allocation $\mathbf{C} \in \mathbb{C}$. We call this step Min-EU. The second step is to repeat the first step for all the $\mathbf{C} \in \mathbb{C}$ to find $\mathbf{C}^*$ which produces the maximized minimum expected utility. We call this step Max-Min-EU. With the No-short-selling constraints (1.28), $\mathbb{C}$ is the *Allocation Triangle*. Assume K and H as the size (the number of elements) of $\mathbb{C}$ and $\mathbb{P}$ respectively. [12]

Take a very simple example by assuming that individuals have an endowment of 3 units of cash and can only allocate integer amounts. Then $K = 10$ as all the possible allocations are $[0\ 0]$, $[0\ 1]$, $[0\ 2]$, $[0\ 3]$, $[1\ 0]$, $[1\ 1]$, $[1\ 2]$, $[2\ 0]$, $[2\ 1]$, $[3\ 0]$. Algorithm 2 on Page 41 provides a straightforward way to find the optimal allocation for MEU utility preferences (1.34).

[11] Another possible way to specify $\mathbb{P}$ is assuming it is a circle area inside MMT. The center of the circle is a subjective prior and the radius is also a subjective parameter. But it assumes symmetry for the three states. We think it is more flexible to use the lower bound assumption.

[12] Theoretically the numbers of elements of $\mathbb{C}$ and $\mathbb{P}$ infinity. But if we let computer to this computation continuously, it usually calculates using a small precision, so K and H can be very large.

Algorithm 2: Pure numerical method to find MEU optimal allocation

Input : $r, \underline{p_1}, \underline{p_2}, \underline{p_3}, e$
Output: $\mathbf{C}^*$

```
// The loop below is for step Max-Min-EU. We repeat K times
   to find the Maxmin EU in ℂ, which is the set of all
   allocations
```

for $k = 1$ **to** K **do**

```
    // The loop below is for step Min-EU. For each C_k we
       repeat H times to find the minimum EU in ℙ, which is
       the set of all priors
```

  for $h = 1$ **to** H **do**

    $EU_h = U(\mathbf{C}_k, P_h)$; `// calculate eu`

  end

  $\mathbf{EU}^H = \{EU_h | h \in [1, H]\}$;

  $EU_k = min\{\mathbf{EU}\}$; `// find the Min eu`

end

$\mathbf{EU}^K = \{EU_k | k \in [1, K]\}$;

$\mathbf{C}^* = \{\mathbf{C}_k | EU_K = max\{\mathbf{EU}^K\}$; `// find the Max-Min eu`

1.4.3 Algorithm with analytic results to find the optimal allocation with MEU preferences

Notice that in Algorithm 2, we need to calculate the utility $H * K$ times. This takes an a lot of time, especially when H and K are big. And the results are not reliable when using most in-built optimisation functions in mathematical software; since those optimisation functions are built on first and second order conditions. It requires the function to be smooth in the specified optimisation area. But our objective function (1.34) is a kinked function. If the optimal allocation happens to be at the kink, those optimisation routines may not be able to find it. That is why we have developed the following algorithm. We have tested it against the purely numerical algorithm and the former always returns the optimal allocation while the later fails (usually when the optimal allocation is at the kink).

A Complicated but Reliable and Efficient Algorithm

First, for any arbitrary allocation $\mathbf{C}$ and a given *return table* $\mathbf{D}$ there is a corresponding portfolio payoff vector $\mathbf{W} = [w_1\ w_2\ w_3]'$, where $\mathbf{W} = \mathbf{CD}$ is the product of the two matrices[13]. The *j-th* element of $\mathbf{W}$ represents the portfolio payoff in state j, $j = 1, 2, 3$. If *Minimum Portfolio Payoff* (MinPP) $w_j = min\{\mathbf{W}\}$, then $u(w_j) = min\{u(w_1), u(w_2), u(w_3)\}$ since utility function u is assumed to be monotonically increasing with respect to the outcome w. By assigning the maximum probability (1 minus the probabilities lower bounds of the other two states) to MinPP state j and the minimum probabilities (the probability lower bounds) to the other two states, we get the minimum expected utility. The corresponding probability vectors for the portfolio payoff to be then MinPP at state 1,2 and 3 are $\mathbf{P_1}$, $\mathbf{P_3}$ and $\mathbf{P_3}$ respectively. They are defined by equation (1.35) and also illustrated in Figure 1.14 on Page 39. As they are the 3 corners of the MMT, we call them corner probabilities.

$$\begin{cases} \mathbf{P_1} &= [1 - \underline{p}_2 - \underline{p}_3, \underline{p}_2, \underline{p}_3]'; \\ \mathbf{P_2} &= [\underline{p}_1, 1 - \underline{p}_1 - \underline{p}_3, \underline{p}_3]'; \\ \mathbf{P_3} &= [\underline{p}_1, \underline{p}_2, 1 - \underline{p}_1 - \underline{p}_2]'; \end{cases} \tag{1.35}$$

Given any allocation $\mathbf{C}$,

$$w_j = min\{\mathbf{W}\}^{14}$$

is both necessary and sufficient for the follow equation to be true.

$$\underset{P \in \mathbb{P}}{min} \left\{ \sum_{j \in (1,2,3)} p_j u(w_j) \right\} = U(\mathbf{P}_j, \mathbf{W}) \tag{1.36}$$

because the utility function *u* is monotonically increasing. Here *u* takes the form of a von Neumann-Morgenstern utility function and $U(\mathbf{P}_j, \mathbf{W})$ refers to the expected utility given the probability vector $\mathbf{P}_j$ and the payoff vector

[13] $\mathbf{C}$ is a 1*2 matrix and $\mathbf{D}$ is a 2*3 matrix so we have $\mathbf{W}$ is a 1*3 vector.

[14] Notice here we define the term minimum as strictly minimum. For example, $w_1 = min\{\mathbf{W}\}$ means $w_1 < w_2$ and $w_1 < w_3$.

Possibility	$subset$	MinPP state
1	$\mathbf{C}_0$	$w_1 = w_2 = w_3$
2	$\mathbb{L}_1$	$w_2 = w_3, w_2 < w_1, w_3 < w_1$
3	$\mathbb{L}_2$	$w_3 = w_1, w_3 < w_2, w_1 < w_2$
4	$\mathbb{L}_3$	$w_1 = w_2, w_1 < w_3, w_2 < w_3$
5	$\mathbb{A}_1$	$w_1 < w_2, w_1 < w_3$
6	$\mathbb{A}_2$	$w_2 < w_1, w_2 < w_3$
7	$\mathbb{A}_3$	$w_3 < w_1, w_3 < w_2$

TABLE 1.1: The seven possibilities for MinPP

$\mathbf{W}$. For example,

$$U(\mathbf{P}_1, \mathbf{W}) = (1 - \underline{p}_2 - \underline{p}_3)u(w_1) + \underline{p}_2 u(w_2) + \underline{p}_3 u(w_3).$$

The formal proof for equation (1.36) is as follows We let $\hat{U} = (\hat{\mathbf{P}}, \mathbf{W})$ with $\hat{\mathbf{P}} = [\hat{p}_1, \hat{p}_2, \hat{p}_3]$ to be any point in $\mathbb{P} - \mathbf{P}_1$. Next we use U_1 to refer to $U(\mathbf{P}_1, \mathbf{W})$.Then we have

$$\begin{aligned}
\hat{U} - U_1 &= [\hat{p}_1 u(w_1) + \hat{p}_2 u(w_2) + \hat{p}_3 u(w_3)] \\
&\quad - [(1 - \underline{p}_2 - \underline{p}_3)u(w_1) + \underline{p}_2 u(w_2) + \underline{p}_3 u(w_3)] \\
&= [(1 - \hat{p}_2 - \hat{p}_3)u(w_1) + \hat{p}_2 u(w_2) + \hat{p}_3 u(w_3)] \\
&\quad - [(1 - \underline{p}_2 - \underline{p}_3)u(w_1) + \underline{p}_2 u(w_2) + \underline{p}_3 u(w_3)] \\
&= (\hat{p}_2 - \underline{p}_2)[u(w_2) - u(w_1)] + (\hat{p}_3 - \underline{p}_3)[u(w_3) - u(w_1)] > 0.
\end{aligned}$$

The last inequality follows from $u(w_2) - u(w_1) > 0$, $u(w_3) - u(w_1) > 0$ and $\hat{p}_2 - \underline{p}_2 > 0$, $\hat{p}_3 - \underline{p}_3 > 0$. As we have proved that equation (1.36) is valid, the MEU objective function (1.34) can be written as

$$\mathcal{U} = \underset{\mathbf{C} \in \mathbb{C}}{max}\, U(\mathbf{P}_j, \mathbf{W}) \text{ s.t. } w_j = min\{\mathbf{W}\},\ j \in \{1, 2, 3\} \tag{1.37}$$

Notice there are 7 possibilities for $w_j = min\{\mathbf{W}\},\ j \in \{1, 2, 3\}$. Table 1.1 lists all the 7 possibilities. As w_j is calculated from the allocations, the 7 possibilities of MinPP actually imply 7 constraints on the allocations. Hence we divide the *Allocation Triangle* $\mathbb{C}$ into 7 corresponding subsets according to the constraints as shown in Table 1.1. Next I demonstrate how to define the subsets by imposing the constraints on the allocations. Figure 1.16 on Page 57 demonstrates the 7 subsets visually as the different areas in the *Allocation Triangle*. Notice that it is plotted with a particular *return table* so it does not cover the generality of various possible positions for these subsets. But at least it helps us to understand how do we divide these subsets. Next we first point out their positions in the figure, then derive the mathematical implications from the constraints.

Subset $\mathbf{C}_0$ This refers to the origin in Figure 1.16. For $w_1 = w_2 = w_3$ we have

$$\begin{cases} d_{12}c_1 + d_{22}c_2 + e = d_{13}c_1 + d_{23}c_2 + e & \text{when } w_2 = w_3 \\ d_{11}c_1 + d_{21}c_2 + e = d_{13}c_1 + d_{23}c_2 + e & \text{when } w_3 = w_1 \\ d_{11}c_1 + d_{21}c_2 + e = d_{12}c_1 + d_{22}c_2 + e & \text{when } w_1 = w_2 \end{cases}$$

Rearranging the equation system we get

$$\begin{cases} c_2 = s_1 c_1 & \text{when } w_2 = w_3 \\ c_2 = s_2 c_1 & \text{when } w_3 = w_1 \\ c_2 = s_3 c_1 & \text{when } w_1 = w_2 \end{cases} \tag{1.38}$$

where s_1, s_2 and s_3 are given by equation (1.39).

$$\begin{cases} s_1 & = -\dfrac{d_{12} - d_{13}}{d_{22} - d_{23}} \\ s_2 & = -\dfrac{d_{11} - d_{13}}{d_{21} - d_{23}} \\ s_3 & = -\dfrac{d_{11} - d_{12}}{d_{21} - d_{22}} \end{cases} \tag{1.39}$$

As equation (1.38) implies $s_1c_1 = s_2c_1 = s_3c_1$ and we do not have $s_1 = s_2 = s_3$ [15] then the solution for equation (1.38) is $c_1 = c_2 = 0$. So the subset $\mathbf{C}_0 = [0\ 0]$ refers to the origin in Figure **??**.

[15] Please refer to equation (1.24)

Subset $\mathbb{L}_i,\ i \in \{1, 2, 3\}$ These refer to the three boundary lines inside the triangle in Figure 1.16. Notice that one of them is a dashed line. We will give the reason for that shortly. Without loss of generality, we adopt the convention as shown in equation (1.40)

$$\begin{cases} \text{if } i = 1 \text{ then } j = 2 \text{ and } k = 3 \\ \text{if } i = 2 \text{ then } j = 3 \text{ and } k = 1 \\ \text{if } i = 3 \text{ then } j = 1 \text{ and } k = 2 \end{cases} \tag{1.40}$$

For $\mathbb{L}_i$, the equality constraint is $w_j = w_k$ and the inequality constraints are $w_j < w_i$ and $w_k < w_i$. For $w_j = w_k$, the corresponding constraint is

$$c_2 = s_i c_1 \tag{1.41}$$

where s_i is its slope and is defined in equation (1.42)

$$s_i = -\frac{d_{1j} - d_{1k}}{d_{2j} - d_{2k}} \quad i \in \{1, 2, 3\} \tag{1.42}$$

For $w_j < w_i$, we have

$$\begin{aligned} d_{1j}c_1 + d_{2j}c_2 &< d_{1i}c_1 + d_{2i}c_2 \\ (d_{1j} - d_{1i})c_1 &< -(d_{2j} - d_{2i})c_2 \end{aligned}$$

The solution is

$$\begin{cases} c_2 > s_k c_1 & \text{if } d_{2i} > d_{2j} \\ c_2 < s_k c_1 & \text{if } d_{2i} < d_{2j} \end{cases} \tag{1.43}$$

Similarly the solution constraint is for $w_k < w_i$, we have

$$\begin{cases} c_2 > s_j c_1 & \text{if } d_{2i} > d_{2k} \\ c_2 < s_j c_1 & \text{if } d_{2i} < d_{2k} \end{cases} \tag{1.44}$$

As shown in the derivation, the different relations between d_{2i}, d_{2j} make the definition of the subsets $\mathbb{L}_i$ and $\mathbb{A}_i$ complicated while giving us very little insight. So we re-order the return table such that $d_{21} < d_{22} < d_{23}$ which can always be done without loss of generality. For example, consider a return table

$$D = \begin{bmatrix} -0.4 & -0.1 & 0.7 \\ 2.1 & -0.9 & -1 \end{bmatrix} \tag{1.45}$$

Since the order of the states does not influence the decisions we can write $\mathbf{D}$ as

$$D = \begin{bmatrix} 0.7 & -0.1 & -0.4 \\ -1 & -0.9 & 2.1 \end{bmatrix} \tag{1.46}$$

With the convention (1.40), the inequality constraints $w_j < w_i$ and $w_k < w_i$ for $\mathbb{L}_i$ become

$$\begin{cases} c_2 < s_3c_1,\ c_2 < s_2c_1 & \text{if } i = 1 \\ c_2 < s_1c_1,\ c_2 > s_3c_1 & \text{if } i = 2 \\ c_2 > s_2c_1,\ c_2 > s_1c_1 & \text{if } i = 3 \end{cases}$$

So in conclusion we have

$$\begin{cases} \mathbb{L}_1 & = \{\mathbf{C} \in \mathbb{C}\ | c_2 = s_1c_1, c_2 < min\{s_2, s_3\}c_1\} \\ \mathbb{L}_2 & = \{\mathbf{C} \in \mathbb{C}\ | c_2 = s_2c_1, s_3c_1 < c_2 < s_1c_1\} \\ \mathbb{L}_3 & = \{\mathbf{C} \in \mathbb{C}\ | c_2 = s_3c_1, c_2 > max\{s_1, s_2\}c_1\} \end{cases} \tag{1.47}$$

We can further explore equation (1.47) by listing all possible orderings for s_1, s_2 and s_3. Denote the ordering as follows

$$\begin{aligned} I:&\quad s_1 < s_2 < s_3 \\ II:&\quad s_1 < s_3 < s_2 \\ III:&\quad s_2 < s_1 < s_3 \\ IV:&\quad s_2 < s_3 < s_1 \\ V:&\quad s_3 < s_1 < s_2 \\ VI:&\quad s_3 < s_2 < s_1 \end{aligned}$$

Notice that the *Allocation Triangle* is in the positive quadrant. So it means that if the slope s_i is negative then there is no intersection of the line $c_2 = s_1c_1$ with the *Allocation Triangle*. In this case, we have $\mathbb{L}_i = \emptyset$ for sure. For the time being we assume that all $s_i > 0$. Take order I for example. First consider $\mathbb{L}_1$. As $s_2 < s_3$ implies $min\{s_2, s_3\} = s_2$ we have $\mathbb{L}_1 = \{\mathbf{C} \in$

$\mathbb{C}\ |c_2 = s_1c_1, c_2 < s_2c_1\}$. As $s_1 < s_2$, the line $c_2 = s_1c_1$ is inside the area $c_2 < s_2c_1$, we then have $\mathbb{L}_1 = \{\mathbf{C} \in \mathbb{C}\ |c_2 = s_1c_1\}$. There is part of the line $c_2 = s_1c_1$ inside the *Allocation Triangle*. Denote $\mathbf{I}$ as the intersection point of it with the line $c_1 + c_2 = e$. By solving the equations system

$$\begin{cases} c_1 + c_2 = e \\ c_2 = s_1c_1 \end{cases}$$

we have

$$\mathbf{I} = [\frac{e}{1+s_1}\ ,\ \frac{es_1}{1+s_1}]$$

So we have

$$\mathbb{L}_1 = \{c_2 = s_1c_1, 0 \leq c_1 \leq \frac{e}{1+s_1}\}$$

Now consider $\mathbb{L}_2$. As $s_3 > s_1$ it implies that $s_3c_1 < c_2 < s_1c_1 = \emptyset$ so we have $\mathbb{L}_2 = \emptyset$. The analysis for $\mathbb{L}_3$ is quite similar for $\mathbb{L}_1$, we get

$$\mathbb{L}_3 = \{c_2 = s_3c_1, 0 \leq c_1 \leq \frac{e}{1+s_3}\}$$

Mutatis mutandis, we can define $\mathbb{L}_i$ in Table 1.2 for all 6 orderings.

$\forall i \in \{1,2,3\},\ \mathbb{L}_i = \emptyset$ if $s_i < 0$			
Ordering	$\mathbb{L}_1$	$\mathbb{L}_2$	$\mathbb{L}_3$
I	$c_2 = s_1c_1$ $0 \leq c_1 \leq \frac{e}{1+s_1}$	$\emptyset$	$c_2 = s_3c_1$ $0 \leq c_1 \leq \frac{e}{1+s_3}$
II	$c_2 = s_1c_1$ $0 \leq c_1 \leq \frac{e}{1+s_1}$	$\emptyset$	$\emptyset$
III	$\emptyset$	$\emptyset$	$c_2 = s_3c_1$ $0 \leq c_1 \leq \frac{e}{1+s_3}$
IV	$\emptyset$	$\emptyset$	$\emptyset$
V	$\emptyset$	$\emptyset$	$\emptyset$
VI	$\emptyset$	$c_2 = s_2c_1$ $0 \leq c_1 \leq \frac{e}{1+s_2}$	$\emptyset$

TABLE 1.2: Definition of $\mathbb{L}_i, i \in \{1,2,3\}$

Figure 1.15 demonstrates the 3 possible positions for each $\mathbb{L}_i$ in the *Allocation Triangle*.

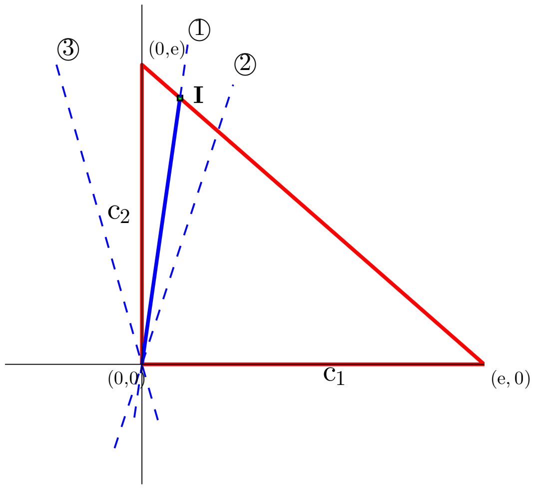


FIGURE 1.15: Boundary Lines

1. Position ①

   For this position, on the line $c_2 = s_i c_1$ there is $w_j = w_k$, which makes is the equality constraint. And $s_i > 0$ so the line crosses $\mathbb{C}$. The inequality constraints $w_j < w_i$ and $w_k < w_i$ are also satisfied on it. So $\mathbb{L}_i$ is OI as shown in the figure. Here O is the original point of $\mathbb{C}$.

2. Position ②

   For this position, it is similar to position 1 except that the inequality constraints $w_j < w_i$ and $w_k < w_i$ are not satisfied so $\mathbb{L}_i = \emptyset$.

3. Position ③

   Here we have $s_i < 0$ the line does not cross $\mathbb{C}$ so $\mathbb{L}_i = \emptyset$.

Subset $\mathbb{A}_i,\ i \in \{1, 2, 3\}$ They refer to the triangle areas separated by $\mathbb{L}_i$ Figure 1.16. For $\mathbb{A}_i$ we have two inequality constraints $w_i < w_j$ and $w_i < w_k$

as $w_i = min\{w_i, w_j, w_k\}$. We have for $\mathbb{A}_i$

$$\begin{cases} c_2 > s_k c_1, c_2 > s_j c_1 & \text{if } d_{2i} < d_{2j} \text{ and } d_{2i} < d_{2k} \\ c_2 < s_k c_1, c_2 > s_j c_1 & \text{if } d_{2i} > d_{2j} \text{ and } d_{2i} < d_{2k} \\ c_2 > s_k c_1, c_2 < s_j c_1 & \text{if } d_{2i} < d_{2j} \text{ and } d_{2i} > d_{2k} \\ c_2 < s_k c_1, c_2 < s_j c_1 & \text{if } d_{2i} > d_{2j} \text{ and } d_{2i} > d_{2k} \end{cases} \tag{1.48}$$

By re-ordering the return table such that $d_{21} < d_{22} < d_{23}$ we have the constraints to define the $\mathbb{A}_1$,$\mathbb{A}_2$ and $\mathbb{A}_3$ as

$$\begin{cases} \mathbb{A}_1 & = \{\mathbf{C} \in \mathbb{C} \mid c_2 > s_3 c_1\} \\ \mathbb{A}_2 & = \begin{cases} \{\mathbf{C} \in \mathbb{C} \mid s_1 c_1 < c_2 < s_3 c_1\} & if\ s_1 < s_3 \\ \emptyset & if\ s_1 > s_3 \end{cases} \\ \mathbb{A}_3 & = \{\mathbf{C} \in \mathbb{C} | c_2 < s_1 c_1\} \end{cases} \tag{1.49}$$

At this point we stop further discussion of the relation between s_1 and s_3 because it is unnecessary for developing the Algorithm to find the MEU optimal allocation. As we will see next, we adopt a different strategy which can be used to find the boundary lines and determine the area of the local optimal allocations. Now I will discuss how to calculate the optimal allocation for each subset in detail.

- Origin $\mathbf{C}_0$

  For $\mathbf{C}_0 = [0\ 0]$ the MEU objective function (1.37) becomes

$$\begin{aligned} \tilde{U}_0 &= \max_{\mathbf{C} \in \{[0,\ 0]\}} min\{U(\mathbf{P_1}),\ U(\mathbf{P_2}),\ U(\mathbf{P_3})\} \\ &= min\{u(e),\ u(e),\ u(e)\} \\ &= u(e) \end{aligned} \tag{1.50}$$

- Areas $\mathbb{A}_1$, $\mathbb{A}_2$ and $\mathbb{A}_3$

  From equation (1.49) we know that $\mathbb{A}_2 = \emptyset$ when $s_1 > s_3$. So we can always check this first, if $s_1 > s_3$ then we do not need to calculate the local optimal allocation for $\mathbb{A}_2$. We can just let $\tilde{U}_2^* = -Inf$

to let exclude this case. If $s_1 < s_3$, then we adopt the following strategy. Ignoring the constraints for area $\mathbb{A}_i,\ i \in [1, 2, 3]$, the MEU objective function becomes a normal Expected Utility function. So we can get the optimal allocation $\mathbf{C}^*_{\mathbb{A}_i}$ and the corresponding local maximised expected utility $U^*_{\mathbb{A}_i}$ for the function $U(\mathbf{P_i}\}$ by Algorithm 1. Though Algorithm 1 guarantees that the local optimal allocations are inside $\mathbb{C}$, it does not guarantee that they satisfy their corresponding constraints of their areas. If $\mathbf{C}^*_{\mathbb{A}_i}$ is outside the area of $\mathbb{C}_i$, it means it is not a valid local optimal allocation. Then the valid local optimal allocation must lie on the boundary lines of area $\mathbb{A}_i$, so will be captured when calculating the local optimal allocation for the boundary lines. Hence we can let $\tilde{U}^*_i = -Inf$ to exclude the possibility that the local optimal allocation is the global optimal allocation in this case. In conclusion we have

$$\tilde{U}^*_{\mathbb{A}_i \in \{1,2,3\}} = \begin{cases} U_{\mathbf{C}^*_{\mathbb{A}_i}} & \text{if } \mathbf{C}^*_{\mathbb{A}_i} \in \mathbb{A}_i \\ -Inf & \text{if } \mathbf{C}^*_{\mathbb{A}_i} \notin \mathbb{A}_i \end{cases}$$

- Boundary Lines $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{L}_3$

  In the *Allocation Triangle*, these subsets are actually the boundary lines of the areas $\mathbb{A}_i$, where there is a strictly minimum portfolio payoff. According to Table (1.2), there are various cases for which $\mathbb{L}_i = \emptyset$. We can calculate s_1, s_2, s_3 first and then check if any such cases exist. If so, then we let $\tilde{U}_{L_i} = -Inf$. When $\mathbb{L}_i \neq \emptyset$, the following procedure is adopted. As we have $w_j = w_k = min\{w_1,\ w_2,\ w_3\}$, the MEU objective function for $\mathbb{L}_i$ is

$$\tilde{U}_{\mathbb{L}_i} = \underset{\mathbf{C} \in \mathbb{L}_j}{max} U(\mathbf{P}_j\} = \underset{\mathbf{C} \in \mathbb{L}_k}{max} U(\mathbf{P}_k\} \tag{1.51}$$

  But we cannot find the local optimal allocation here in the same way as we have done for area $\mathbb{A}_i$, that is ignoring the constraints to find the optimal allocation first and then excluding the point it if it does not satisfy the constraints. since a local optimal allocation lying on the boundaries will be missed. As the constraints are incorporated in the process of finding the optimal allocation and they are implying $w_j = w_k$, we can just calculate either $\underset{\mathbf{C}\in\mathbb{L}_j}{max}U(\mathbf{P}_j\}$ or $\underset{\mathbf{C}\in\mathbb{L}_k}{max}U(\mathbf{P}_k\}$. As shown in equation (1.47), there are various cases for which $\mathbb{L}_i = \emptyset$ depending on s_i. We will check them first. If $\mathbb{L}_i = \emptyset$ we let $\tilde{U}_{\mathbb{L}_i} = -Inf$. If not then we use the following strategy. As when $\mathbb{L}_i \neq \emptyset$ the constraints for $\mathbb{L}_i$ are $c_2 = s_i c_i$ and $0 \leq c_1 \leq \frac{e}{1+s_i}$. We denote ${}^1\mathbf{C}_i$ as the optimal allocation for the function $U(\mathbf{P}_j\}$ with the equality constraint $c_2 = s_i c_1$. Notice here we put a superscript on the left of $\mathbf{C}$ to emphasize that we are solving for c_1. We can find the optimal allocation with the equality constraint and then modify it to satisfy the inequality constraint. Next I will demonstrate how to get ${}^1\mathbf{C}_i$ for two types of utility functions - CARA and CRRA, which are the two functions used in this thesis.

CARA Assume individuals have the particular form of CARA utility function defined in equation (1.21)

$$u(w) = -\frac{1}{r}e^{-rw}$$

Then the expected utility function is

$$U(w_j) = -\sum_j p_j \frac{1}{r} e^{-rw_j}$$

By inserting the constraint that $c_2 = s_i c_1$, we get the first order condition

$$\sum_j p_j(d_{1j} + s_i d_{2j})e^{-r(d_{1j}c_1 + d_{2j}s_i c_1 + e)} = 0 \qquad (1.52)$$

Denote c_1^* as the solution for equation (1.52). So we have ${}^1\mathbf{C} = [c_1^*\ s_i c_1^*]$.

CRRA Assume individuals have the particular form of CRRA utility function defined in equation (1.25)

$$u(w) = \frac{w^{1-r}}{1-r}$$

Then the expected utility function is

$$U(w_j) = \sum_j p_j \frac{w_j^{1-r}}{1-r} \tag{1.53}$$

By inserting the constraint that $c_2 = s_i c_1$, we get the first order condition is

$$\sum_j p_j (d_{1j} + s_i d_{2j})(d_{1j} c_1 + d_{2j} s_i c_1 + e)^{-r} = 0 \tag{1.54}$$

Denote c_1^* as the solution for equation (1.54). So we have ${}^1\mathbf{C} = [c_1^* \;\; s c_1^*]$. Notice that ${}^1\mathbf{C}_i = [c_1^* \;\; s_i c_1^*]$ may not be inside $\mathbb{C}$. Using the constraints for $\mathbb{L}_i$ in Table 1.2, we get that for $\mathbb{L}_i$, the optimal allocation is

$$\mathbf{C}^*_{\mathbb{L}_i} = \begin{cases} \mathbf{C}^1 & 0 \le c_1^* \le \frac{e}{1+s_i} \\ [0, \;\; 0] & c_1^* < 0 \\ [\frac{e}{1+s_i}, \;\; \frac{e s_i}{1+s_i}] & c_1^* > \frac{e}{1+s_i} \end{cases} \tag{1.55}$$

and the corresponding maximized utility is $\tilde{U}^*_{\mathbb{L}_i} = U(\mathbf{C}^*_{\mathbb{L}_i})$.

We have discussed how to find the local optimal allocation for the various situations in each the 7 subsets in $\mathbb{C}$ in detail. The global optimal allocation $\mathbf{C}^{**}$ is the one that generates the highest local utility. Based on the analysis, Algorithm 3 shows how to find the MEU optimal allocation combining numerical methods with analytical results.

Algorithm 3: Numerical method employing analytical results to find MEU optimal allocation (Part 1)

Input : $r, \underline{p_1}, \underline{p_2}, \underline{p_3}, \mathbf{D}, e$
Output: $\mathbf{C}^*$

```
// Find s_i, i ∈ {1,2,3} according to equation (1.39)
s_1 = −(d_12 − d_13)/(d_22 − d_23);
s_2 = −(d_11 − d_13)/(d_21 − d_23);
s_3 = −(d_11 − d_12)/(d_21 − d_22);

// Find the corner probabilities P_i, i ∈ {1,2,3} according to
   equation (1.35)
P_1 = [1 − p_2 − p_3, p_2, p_3]' ;
P_2 = [p_1, 1 − p_1 − p_3, p_3]' ;
P_3 = [p_1, p_2, 1 − p_1 − p_2]' ;

// Find C_{A_i}, i ∈ {1,2,3} according to equation (1.49)
for i = 1 to 3 do
    [C_i, Ũ_i] = f_1(U(P_i)) ;                    // C_i = C_{A_i}, Ũ_i = Ũ_{A_i}
    // f_1 refers to Algorithm 1
end

// Exclude the temporary local optimal allocation of A_i if
   it is not in the area of A_i
for i = 1 to 3 do
    W = C_i D;
    if W(i) ≠ min{W} then
        Ũ_i = −Inf
    end
end
```

Algorithm 3: Numerical method employing analytical results to find the MEU optimal allocation (Part 2)

```
// Find C_{L_i}, i ∈ {1,2,3} according to equation (1.55)
// C_4 = C_{L_1} C_5 = C_{L_2} C_6 = C_{L_3}
C_4 = f_2(U(P_2)) // f_2 refers to equation (1.55)
C_5 = f_2(U(P_3));
C_6 = f_2(U(P_1));
// Ũ_4 = Ũ_{L_1} Ũ_5 = Ũ_{L_2} Ũ_6 = Ũ_{L_3}
U_4 = U*(P_2)) ;
U_5 = U*(P_3));
U_6 = U*(P_1));

// Exclude local optimal allocations according to Table
   (1.2)
if s_1 < s_2 < s_3 then
|  Ũ_5 = −Inf;
else if s_1 < s_3 < s_2 then
|  Ũ_5 = Ũ_6 = −Inf;
else if s_2 < s_1 < s_3 then
|  Ũ_4 = Ũ_5 = −Inf;
else if s_2 < s_3 < s_1 or s_3 < s_1 < s_2 then
|  Ũ_4 = Ũ_5 = Ũ_6 = −Inf;
else
|  Ũ_5 = −Inf;
end
for i = 4 to 6 do
|  if s_{i−3} < 0 then
|  |  Ũ_i = −Inf;
|  end
end
C** = {C_k | Û_k = max{Û_1, Û_2, Û_3, Û_4, Û_5, Û_6, Û_7}}
```

Illustration of Algorithm 3 to the find MEU optimal allocation

First I apply Algorithm 3 to find the MEU optimal allocation, then I illustrate the local optimal allocation in figure 1.16 to illustrate how we get the global optimal allocation. Consider a return table

$$D = \begin{bmatrix} 0.7 & -0.1 & -0.4 \\ -1 & -0.9 & 2.1 \end{bmatrix} \tag{1.56}$$

and a probability lower bound vector

$$\mathbf{P}^{-} = \begin{bmatrix} 0.2 & 0.1 & 0.4 \end{bmatrix}^{'}$$

Consider an individual who has MEU Preferences with a CARA utility function and risk parameter $r = 0.002$. Then we can write his/her objective function as

$$U = -\sum_{j} p_j \frac{1}{0.002} e^{-0.002 w_j}$$

Applying Algorithm 3

Find $s_i, i \in \{1, 2, 3\}$ **according to equation (1.39)**

$$\begin{aligned} s_1 &= -\frac{d_{12} - d_{13}}{d_{23} - d_{23}} = 0.1 \\ s_2 &= -\frac{d_{11} - d_{13}}{d_{21} - d_{23}} = 0.35 \\ s_3 &= -\frac{d_{11} - d_{12}}{d_{21} - d_{22}} = 8 \end{aligned}$$

Find the corner probabilities $\mathbf{P}_i, i \in \{1, 2, 3\}$ **according to equation (1.35)**

$$\begin{aligned} \mathbf{P_1} &= [0.5, 0.1, 0.4]^{'} \\ \mathbf{P_2} &= [0.2, 0.4, 0.4]^{'} \\ \mathbf{P_3} &= [0.2, 0.1, 0.7]^{'} \end{aligned}$$

Find $\mathbf{C}_{\mathbb{A}_i}, i \in \{1, 2, 3\}$ **according to equation (**1.49**)** For $\mathbf{C}_{\mathbb{A}_i}$ the corresponding probability vector is $\mathbf{P}_i$. Then by Algorithm 1 we have $\mathbf{C}_1 =$

$[66.2\ 33.8], \tilde{U}_1 = -393.5$, $\mathbf{C}_2 = [0,\ 66.8], \tilde{U}_2 = -401.9$ and $\mathbf{C}_3 = [0,\ 100], \tilde{U}_3 = -337.3$.

Exclude the temporary local optimal allocation of $\mathbb{A}_i$ if it is not in the area of $\mathbb{A}_i$ For $\mathbb{A}_1$, $\mathbf{W} = \mathbf{C}_1\mathbf{D} = [12.6\ \ -37.1\ 44.4]'$. $\mathbf{W}(1) = 12.6 \neq min\{\mathbf{W}\}$, so we update $\tilde{U}_1 = -Inf$. For $\mathbb{A}_2$, $\mathbf{W} = \mathbf{C}_2\mathbf{D} = [-66.8\ \ -60.1\ 140.2]'$. $\mathbf{W}(2) = -60.1 \neq min\{\mathbf{W}\}$, so we update $\tilde{U}_2 = -Inf$. For $\mathbb{A}_3$, $\mathbf{W} = \mathbf{C}_3\mathbf{D} = [-100\ \ -90.1\ 210.0]'$. $\mathbf{W}(3) = 210.0 \neq min\{\mathbf{W}\}$, so we update $\tilde{U}_3 = -Inf$.

Find $\mathbf{C}_{\mathbb{L}_i},\ i \in \{1,2,3\}$ according to equation (1.55) By Algorithm 1 and assigning probability vectors as $\mathbf{P}_2$ and $\mathbf{P}_3$ and $\mathbf{P}_1$, we get $\mathbf{C}_4 = [8.5\ \ 68.2], \tilde{U}_4 = -402.0$, $\mathbf{C}_5 = [73.8\ \ 26.2], \tilde{U}_5 = -407.6$ and $\mathbf{C}_6 = [0\ 0], \tilde{U}_6 = -409.4$.

Exclude local optimal allocations according to Table (1.2) As there is $s_1 < s_2 < s_3$ we update $\tilde{U}_5 = -Inf$

Find MEU utility of C_0 according to equation (1.50) $\mathbf{C}_0 = [0\ 0]$ and $\tilde{U}_0 = u(100) = -409.4$

Find the global maximum $\mathbb{C}^*$ As $\tilde{U}_4 = max\{\tilde{U}_0, \tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \tilde{U}_4, \tilde{U}_5, \tilde{U}_6\}$ we have $\mathbf{C}^* = \mathbf{C}_4 = [8.5\ 68.2]$.

Illustration for finding the global optimal allocation

Table 1.3 lists all seven local optimal allocations and their corresponding MEU utility for the purpose of illustration. In Figure 1.16, seven subsets and their corresponding seven local optimal allocations are marked in *Allocation Triangle*. $\mathbb{A}_1$ is the area above $\mathbb{L}_3$. $\mathbb{A}_2$ is the area between $\mathbb{L}_1$ and $\mathbb{L}_3$. $\mathbb{A}_3$ is the area below $\mathbb{L}_1$. As shown in the figure, $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ are outside their corresponding areas so are excluded.

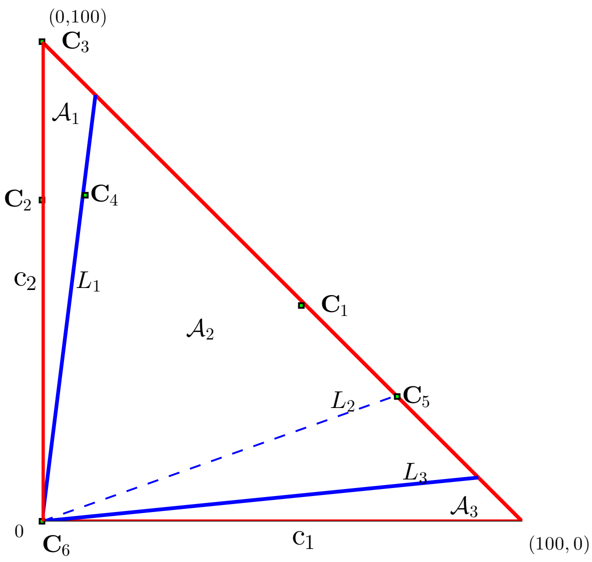


FIGURE 1.16: Example : The seven Local Optimal Allocations

$\mathbb{C}_i$	Local Optimal Allocation	Valid	$U(\mathbf{C}_i)$	$\tilde{U}_i$
$\mathbf{C}_0$	$\mathbf{C}_0 = [0,\ 0]$	YES	-409.4	-409.4
$\mathbb{A}_1$	$\mathbf{C}_1 = [66.2\ 33.8]$	NO	-393.5	$-Inf$
$\mathbb{A}_2$	$\mathbf{C}_2 = [0,\ 66.8]$	NO	-401.9	$-Inf$
$\mathbb{A}_3$	$\mathbf{C}_3 = [0,\ 100]$	NO	-337.3	$-Inf$
$\mathbb{L}_1$	$\mathbf{C}_4 = [8.5,\ 68.2]$	YES	-402.0	-402.0
$\mathbb{L}_2$	$\mathbf{C}_5 = [73.8,\ 26.2]$	NO	-407.6	$-Inf$
$\mathbb{L}_3$	$\mathbf{C}_6 = [0,\ 0]$	YES	-409.4	-409.4

TABLE 1.3: Example : 7 Local Optimal Allocations

For $\mathbb{L}_1$, its local optimal allocation $\mathbf{C}_4$ is a valid one. For $\mathbb{L}_2$, the line $c_2 = s_2 s_1 = 0.35c_1$ is lying in the area $\mathbb{A}_2$ where $w_2 = min\{\mathbf{W}\}$. But the definition for $\mathbb{L}_2$ is $w_1 = w_3 = min\{\mathbf{W}\}$. So $\mathbb{L}_2 = \emptyset$ and its local optimal allocation $\mathbf{C}_5$ indicated by the dashed line in this figure. For $\mathbb{L}_3$, its local optimal allocation $\mathbf{C}_6$ is a valid one though it coincides with $\mathbf{C}_0$. By assigning $-Inf$ to the utility of invalid optimal allocations, we compare all utilities and are able to conclude that $\mathbf{C}_4$ is the global optimal allocation.

1.4.4 α-Maxmin Expected Utility (α-MEU) Preferences

α-MEU is a generalisation of MEU. Instead of assuming individuals only look at the worst case that would happen, α-MEU assumes individuals both look at the worst case and also the best case. α-MEU introduces a parameter α which weights the two cases in the objective utility function. Denote $\mathbb{P}$ as the set of all possible priors, and $\mathbb{C}$ as the set of all possible allocations, then the objective function of α-MEU can be written as

$$U = \underset{C\in\mathbb{C}}{max}\left(\alpha \underset{P\in\mathbb{P}}{min} \sum_{j\in\{1,2,3\}} \{p_j u(w_j)\} + (1-\alpha)\underset{P\in\mathbb{P}}{max} \sum_{j\in(1,2,3)} \{p_j u(w_j)\}\right) \tag{1.57}$$

We can see, when $\alpha = 1$, α-MEU becomes MEU, which means extreme ambiguity aversion. When $\alpha = 0$, the individual only considers the best case. Just like we describe risk attitude, we can consider this individual as ambiguity loving. To work out the optimal allocation, we can use a similar method. Similar to the derivation of equation (1.36), we have

$$\underset{P\in\mathbb{P}}{max}\left\{\sum_{j\in(1,2,3)} \{p_j u(w_j)\right\} = U(\mathbf{P}_j, \mathbf{W}) \ \ s.t \ \ w_j = max\{W\} \tag{1.58}$$

The meaning of equation (1.58) is, the maximum utility is calculated by assigning the biggest probability to the state that generates the biggest payoff. For any given allocation $\mathbf{C}$, we can calculate $\mathbf{W}$ and then check the minimum and maximum payoff state. Then we assign the biggest probability to the minimum payoff state and assign the biggest probability to the maximum payoff state to calculate the minimum Expected Utility and maximum Expected Utility accordingly. The α-MEU objective function (1.57) can thus be written as

$$\mathcal{U} = \underset{\mathbf{C}\in\mathbb{C}}{max}\,(\,\alpha U(\mathbf{P}_i, \mathbf{W}) + (1-\alpha)U(\mathbf{P}_j, \mathbf{W})) \tag{1.59}$$
$$s.t. \ w_i = min\{\mathbf{W},\ i \in \{1,2,3\}\},\ w_j = max\{\mathbf{W},\ j \in \{1,2,3\}\}$$

Now we are not going to develop a similar algorithm as Algorithm 3. But at least we can improve the totally numerical Algorithm 2 to Algorithm 4 using equation (1.59). First we denote K as the total number of $\mathbf{C} \in \mathbb{C}$.

Algorithm 4: Improved numerical method for finding α-MEU optimal allocation

Input : $r, \underline{p_1}, \underline{p_2}, \underline{p_3}, e, \mathbf{D}, \alpha$
Output: $\mathbf{C}^*$

Calculate $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$;
for $k = 1$ **to** K **do**

```
    // The loop below is for calculating the α-MEU objective
       utility for each C_k
```
$\quad\mathbf{W} = \mathbf{C}_k\mathbf{D}$;
$\quad w_i = min\mathbf{W}$;
$\quad w_j = max\mathbf{W}$;
$\quad\mathcal{U}_k = \alpha\mathbf{U}(\mathbf{P}_i, W_i) + (1-\alpha)\mathbf{U}(\mathbf{P}_j, W_j)$; `// calculate the objective function (1.59) for any given` $\mathbf{C}$

end
$\mathbb{U}^K = \{\mathcal{U}_k | k \in [1, K]\}$;
$\mathbf{C}^* = \{\mathbf{C}_k | \hat{U}_k = max\{\mathbb{U}^K\}$;

Notice that Algorithm 4 is plausible when we have a relatively small size for $\mathbb{C}$.

1.4.5 Conclusion

In this section, we have developed Algorithm 3 for calculating the optimal allocation for MEU, which is more efficient and reliable than the purely numerical Algorithm 2. We have developed Algorithm 4 to calculate the optimal allocation for α-MEU. It is plausible when we have a relatively small size of $\mathbb{C}$.[16]

[16] These two algorithms will be called for estimation when we fit these two preferences theories in the experimental data.

Chapter 2

Portfolio Choice under Ambiguity - Experiment

[1]

2.1 Experimental Design

2.1.1 Introduction

In Chapter 1, we provided Algorithms for finding the optimal allocation if individuals have Expected Utility (EU), Mean Variance (MV), Maxmin Expected Utility (MEU) and α-MEU preferences. Now we carry out an experiment to test how well these models work on real data. The problem setup in the experiment is exactly the same as in the theory. There is one safe asset/cash, for which the relative return is 0 and two ambiguous assets, for which the absolute returns are contingent on the three states. Subjects are endowed with a certain amount of money e and can allocate the money to the safe asset and the two ambiguous assets which are both priced at 1 unit of cash. Subjects can not short sell any asset and they do not need to spend

[1] This experiment was financed with funds from MIUR.

Table 2.1: *Payoff Table*

	Pink	Green	Blue
Asset 1	1.7	0.9	0.6
Asset 2	0	0.1	3.1

all the cash endowment. They can see *Payoff Table* which gives information about the end-of-period total payoff in all three states for each of the two assets. Also for implementing the experiment, we name the three states as Pink, Green and Blue respectively (the reason for this terminology will become clearer later). Thus in the experiment, subjects are given a *Payoff Table* with the format as in Table 2.1. Notice that in the theoretical part, we always employ the asset relative *return table* for concise mathematical analysis. But we think *Payoff Table* is more obvious for subjects so it is used in the experiment. A *Payoff Table* implies a unique *return table*. Consider a *Payoff Table*. This means if the Pink state occurs, for every unit of Asset 1 invested the total payoff is 1.7 unit of cash and for Asset 2 is 0. As both assets are priced as 1 unit of cash, the relative returns for Asset 1 and 2 are $(1.7-1)/1 = 0.7$ and $(0-1)/1 = -1$ respectively. The relative return can be calculated as the payoff minus 1 as the assets are priced at 1. So This *Payoff Table* implies the following *return table*.

$$\mathbf{D} = \begin{bmatrix} 0.7 & -0.1 & -0.4 \\ -1 & -0.9 & 2.1 \end{bmatrix}$$

Each subject is given a *Payoff Table* and endowment e, then they are asked to choose a portfolio. We call this a *Problem*. Because there are two allocations (the cash allocation is automatically decided by the two allocations to the ambiguous assets), it is more informative to let the subjects do an allocation problem rather than a pairwise problem. So in the experiment, we let the subjects actually choose the amounts they wish to buy of the two assets. As we need to specify a particular number of decimal places for the allocation amounts, we restricted subjects to choosing integer amount (e was always 100). When subjects make the allocations for the two ambiguous assets, the program calculates the remaining cash and subjects can read the *Portfolio*

Table 2.2: *Portfolio*

Asset 1	c_1
Asset 2	c_2
Cash	$e - c_1 - c_2$

Table 2.3: *Portfolio Payoff*

	Pink	Green	Blue
Portfolio	w_1	w_2	w_3

table as shown in Table 2.2. They are also shown a *Portfolio Payoff* table which shows the total payoffs in each state for the portfolio they choose. The availability of the *Portfolio Payoff* table saves the subjects from complicated mathematical calculations and lets them focus on choosing their desired portfolios. The format of the *Portfolio Payoff* is shown in Table 2.3.

We have

$$w_j = c_1 d_{1j} + c_2 d_{2j} + e - c_1 - c_2$$

Each subject was given 65 problems. These 65 problems were chosen by us after extensive simulations [2]. In each problem, there is different *Payoff Table* but the same cash endowment (100 units of experimental money). As long as the *Payoff Tables* are different from problem to problem, each problem is different from the others. We keep the cash endowment to be the same for all problems to make the experiment as simple as possible.

The payment is also decided by the colour. Here is how we implemented ambiguity. We placed a Bingo Blower [3] in the clear view of the subjects and in continuous motion throughout the experiment. A camera projected the image of the Bingo Blower onto two screens in the laboratory. The subjects

[2] We used Matlab to do the simulations; the program is available on request. The simulation was designed to produce a set of problems that would enable us to distinguish between subjects in terms of their preference functional and beliefs. Clearly what are 'good' problems depends upon what these preferences and beliefs are, so we chose a range of problems to distinguish different subjects.

[3] A Bingo Blower is a rectanglular machine with glass sides in which a number of coloured balls were in continuous motion - being driven by a fan of air.

were able to see there were balls of three different colours (Pink, Green and Blue) but they could not count the balls of different colours (because they are in continuous motion). It is a good way to implement ambiguity in an experiment. Subjects are able to formulate some belief about the numbers of each colour, but not form precise probabilities. We actually put 10 pink balls and 20 green balls and 10 blue balls in the bingo blower so the true probabilities of the three colours were 0.25, 0.5 and 0.25. It is likely that subjects realised that the number of green balls were more than pink and blue. [4]

At the end of the experiment, each subject individually drew a ticket from a bag containing 65 tickets numbering from 1 to 65; this number determined which problem was to be played out for that subject. Then the subject was asked expel one ball out of the Bingo Blower - they could not control the colour to be expelled. The problem to be played out having already been decided, the colour of the expelled ball determined their payoff (w^*) in experimental money, taken from the appropriate Payoff Table. Their final payment in money was that portfolio payoff divided by the exchange rate, 12 units of experimental money equivalent to $\pounds 1.00$[5], plus a $\pounds 2.50$ show up fee.

$$Payment = \pounds(\frac{w^*}{12} + 2.5)$$

Taking a particular *Payoff Table* as an example

	Pink	Green	Blue
Asset 1	1.2	0.6	1.6
Asset 2	0.5	1.4	1.4

.

Suppose a subject chose the portfolio $[40\ \ 30]$ for the problem that was randomly selected, then it means his/her cash remaining is $100 - 30 - 40 =$

[4] This property suggested an interesting heuristic rule which we formulated later. More details about this heuristic are given in Section 2.3.

[5] which they had been told in the Instructions.

30. So his/her portfolio payoff would be

$$\begin{cases} w_P = & 40 * 1.2 + 30 * 0.5 + 30 = 93 \\ w_G = & 40 * 0.6 + 30 * 1.4 + 30 = 93 \\ w_B = & 40 * 1.6 + 30 * 1.4 + 30 = 136 \end{cases}$$

and the corresponding payment for each state would be

$$\begin{cases} P_P = & 93/12 + 2.5 = 10.3 \\ P_G = & 93/12 + 2.5 = 10.3 \\ P_B = & 136/12 + 2.5 = 13.8 \end{cases}$$

2.1.2 Experimental Details

Subjects were asked to arrive at EXEC laboratory at a specific time and then they were asked to read the Instructions at their seats. After 10 minutes, they were given a PowerPoint presentation of the experimental instructions, and then given the chance to ask any questions they may have had. A short demonstration was given about how the Bingo Blower works. Then they were asked to read Instructions again. Subjects individually drew the attention of the experimenter when they were ready to start. As mentioned in the previsou section when they finished all the questions in the experiment they needed to draw a ticket and then expel a ball from Bingo Blower as mentioned in the previous section. Their payment then was calculated and they were paid. The total time for the experiment was from one hour to 2 hours; most subjects finished in around one-and-a-half hours. The average payment was £13, including the show-up fee.

Software

The experimental software was written in Visual Basic. It is an executable file called *portfoliochoice.exe* which runs under any Windows system. The *return table* and endowment for each problem were stored in an Access file which was put in the same directory as the executable file. The *portfoliochoice.exe* and *parameters.mdb* were stored on the lab server so every machine in the lab could run the program and read the input data. When each subject began the experiment, another Access file under the subject's number is created; his/her allocations were stored in that file.

The main experimental interface contains an *Allocation Triangle* (AT), which is an important design feature of this experiment. Also on the screen there were three tables which give information about the assets and the current portfolio choice. AT is consistent with the definition of the Allocation Triangle in the theory section of this thesis, that is, a triangular area in which the allocations satisfy the No-short-selling constraints (1.28) on Page 17. Then any point in AT represents a possible allocation. Notice subjects can only make integer portfolio choice as any point inside the Triangle will be automatically rounded to the nearest integer. The three tables are *Payoff Table*, *Portfolio table* and *Portfolio Payoff table* . They are defined in Table 2.1, 2.2 and 2.3 respectively.

When the main interface was opened, subjects saw the AT with red lines on the screen. When they moved the cursor inside the AT they saw that each particular point in the triangle representing an allocation of their 100-unit endowment of experimental money to the two assets and to residual experimental money. The horizontal distance from the left hand side indicated the number of units allocated to Asset 1; the vertical distance from the bottom indicated the number of units allocated to Asset 2; the residual amount of experimental money was then automatically calculated by the software and shown in *Portfolio table*. For example, if they put the cursor at the bottom corner of the AT (actually the mouse cursor always started at this point which represented their initial situation), this represented buying zero of both assets and hence the residual cash is 100: the bottom-right hand corner represented spending all the endowment on Asset 1; the top-left hand corner represented spending all the endowment on Asset 2; and the middle of the triangle represented spending one-third of their endowment on Asset 1, one-third on Asset 2 and keeping one-third in the form of experimental money. Subjects could see this information in the *Portfolio* table. When they moved the cursor inside the AT, the information in the *Portfolio table* changed dynamically. Given a *Payoff Table*, subjects could move the cursor around in the AT; the *Payoff Table* changed dynamically according to the *Portfolio table*.

Once subjects had decided on their desired allocation, they needed to double click with their mouse to register the choice. When the cursor was outside the AT, *Portfolio Table* and the *Payoff Table* table became blank; similalry when subjects double clicked outside the AT, they saw a warning message in a pop-out window telling them the allocations were invalid. After double clicking the cursor, subjects were asked to confirm their decision, and they could change it if they wished (by clicking on 'No'). To stop subjects just clicking rapidly through the experiment, we put a minimum time for each problem of 30 seconds. If they tried to make a decision before it, they would get a warning box popping out. There was also a maximum time for each problem of 120 seconds. If a subject did not make a decision in the maximum time, the progam would record his/her porfolio choice as [0 0]. Subjects see a timer showing the count-down at the upper-right corner. Figure 2.1 shows the main screen used in the experiment. The Instructions given to subjects are attached in Appendix A.

2.1.3 The Design of the Problems

Why did we choose 3 States?

This experiment is to study individuals' portfolio choices. On the one hand, there has been at least two states, otherwise the assets are not ambiguous. On the other hand, we do not want the problems to be too complicated for subjects. We initially considered having two states cases but decided it was not informative enough. The reason is as follows. Remember that we use the *return table* for the theoretical analysis, so from now we use the term *return table*. If there are only two possible states for the ambiguous assets.

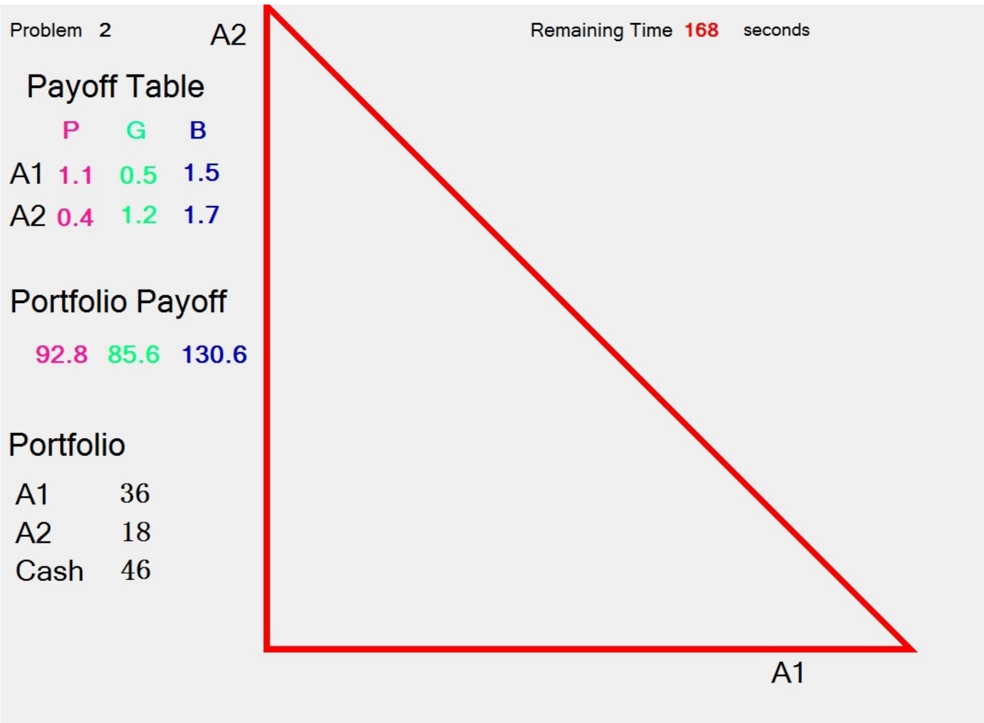


FIGURE 2.1: Experiment Main Screen

Then the *return table* is

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

and the probability vector is

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 \end{bmatrix}'$$

and the portfolio payoff vector is

$$\mathbf{W} = \mathbf{CD} = \begin{bmatrix} c_1 d_{11} + c_2 d_{21} + e \\ c_1 d_{12} + c_2 d_{22} + e \end{bmatrix}$$

The objective function for EU preferences is

$$\begin{aligned} U &= p_1 u(w_1) + p_2 u(w_2) \\ &= p_1 u(c_1 d_{11} + c_2 d_{21} + e) + p_2 u(c_1 d_{12} + c_2 d_{22} + e) \end{aligned}$$

If we use a CARA utility function $u = -\frac{1}{r}e^{-rw}$, the derivative in terms of c_1 is

$$\begin{aligned}\frac{\partial U}{\partial c_1} &= d_{11}p_1 e^{-r(c_1 d_{11}+c_2 d_{21}+e)} + d_{12}p_2 e^{-r(c_1 d_{12}+c_2 d_{22}+e)} \\ &= -rd_{11}p_1 u(w_1) - rd_{12}p_2 u(w_2)\end{aligned}$$

Similarly we have

$$\frac{\partial U}{\partial c_2} = -rd_{21}p_1 u(w_1) - rd_{22}p_2 u(w_2)$$

so the the first order conditions become

$$\begin{cases} p_1 d_{11} u(w_1) + p_2 d_{12} u(w_2) & = 0 \\ p_1 d_{21} u(w_1) + p_2 d_{22} u(w_2) & = 0 \end{cases}$$

For this equation system, the solution only exists when the following equation stands.

$$(p_1 d_{11})(p_2 d_{22}) = (p_1 d_{21})(p_2 d_{22}) \tag{2.1}$$

For a CRRA utility function $u = \frac{w^{1-r}}{1-r}$, we have that the first order conditions are

$$\begin{cases} \frac{\partial U}{\partial c_1} & = d_{11}p_1 w_1^{-r} + d_{12}p_2 w_2^{-r} \\ \frac{\partial U}{\partial c_2} & = d_{12}p_1 w_1^{-r} + d_{12}p_2 w_2^{-r} \end{cases}$$

For this equation system, the solution also only exists when equation (2.1) is satisfied. The intuition here is that when equation (2.1) is satisfied, individuals can always construct a portfolio that makes the maximised expected utility infinite. As in the experiment the allocations are bounded, it is likely subjects will make the same allocations and it would be difficult to estimate their subjective parameters. So two states are not enough. So we decided on three states, to keep the experiment as simple as possible for the subjects, while also being informative to us.

What are, and do we want, Sure-Win return tables?

A Sure-Win *return table* is the one that gives subjects the opportunity to earn from investing in the ambiguous assets irrespective of which state occurs. To demonstrate such a table we use the terminology of Chapter 1, though we still do the theoretical analysis with the relative *return table*. A Sure-Win *return table* would not be informative as all the subjects are going to spend all the cash on the two ambiguous assets in some proportion as long as they are non-satiated with money. Before I formally define a Sure-Win *return table*, I would like to give an example. Consider the *return table*

$$\mathbf{D} = \begin{bmatrix} -0.4 & 0.4 & 0.3 \\ 0.5 & -0.4 & 0.1 \end{bmatrix}$$

Assume that the individuals spend *x* units of cash in buying the two assets and let π be the portion invested in asset 1 and $1-\pi$ in asset 2. Since both assets are priced at 1, the portfolio is $\mathbf{C} = [x\pi\, x(1-\pi)]$. Then the portfolio payoff vector is

$$\mathbf{W} = \mathbf{CD} = x \begin{bmatrix} 0.5 - 0.9\pi + 1 \\ -0.4 + 0.8\pi + 1 \\ 0.1 + 0.2\pi + 1 \end{bmatrix}$$

Let $w_j > 0$ for $j \in 1, 2, 3$ and the solution is

$$\frac{1}{2} < \pi < \frac{5}{9}$$

It means as long as investors make the allocations in the two assets as a proportion between $\frac{1}{2}$ and $\frac{5}{9}$, they will earn more than *x* irrespective of the state which occurs. So subjects may just spend all their cash irrespective of their risk attitudes and their beliefs about the probabilities.[6] Sure-Win *return tables* should not be included in the experiment as they are not informative in the sense of revealing information about the subjects preferences and beliefs.

[6] Notice this is different from the concept of arbitrage. Arbitrage means the opportunity to construct a portfolio with variance equal to 0. But in this case, the portfolio variance could still be positive.

Formal definition A sure win *return table* means there exists a π that makes

$$w_j = \pi d_{1j} + (1-\pi)d_{2j} > 0 \quad \text{for } \forall j \in \{1,2,3\}$$

which can be written as

$$\begin{cases} \pi(d_{21} - d_{11}) < d_{21} \\ \pi(d_{22} - d_{12}) < d_{22} \\ \pi(d_{23} - d_{13}) < d_{23} \end{cases} \tag{2.2}$$

If we want to exclude Sure-Win *return tables*, then we need to make sure there is no solution to equation (2.2). First, if

$$d_{2j} - d_{1j} > 0 \quad \forall j \in \{1,2,3\} \tag{2.3}$$

then there is always a solution which is

$$\pi < min\{\frac{d_{21}}{d_{21}-d_{11}}, \frac{d_{22}}{d_{22}-d_{12}}, \frac{d_{23}}{d_{23}-d_{13}}\} \tag{2.4}$$

So we should check the *return table* to make sure that equation (2.3) cannot be satisfied. By symmetry,

$$d_{1j} - d_{2j} > 0 \quad \forall j \in \{1,2,3\} \tag{2.5}$$

cannot be satisfied either. The intuition for equation (2.3) and (2.5) is that there is one asset which dominates the other. That means the return for one asset is higher than the other in all three states. When they are excluded, then it means for a *return table*, the following scenarios exist - one asset has lower payoff in one state and higher payoffs in the other two states. Assume it is asset 2. As the order of the states is not important, the *return table* can always be arranged to

$$\begin{cases} d_{21} - d_{11} < 0 \\ d_{22} - d_{12} > 0 \\ d_{23} - d_{13} > 0 \end{cases} \tag{2.6}$$

Then equation (2.2) becomes

$$\pi > \frac{d_{21}}{d_{21} - d_{11}}$$
$$\pi < \frac{d_{22}}{d_{22} - d_{12}}$$
$$\pi < \frac{d_{23}}{d_{23} - d_{13}}$$

And then the condition for there to be no solution is

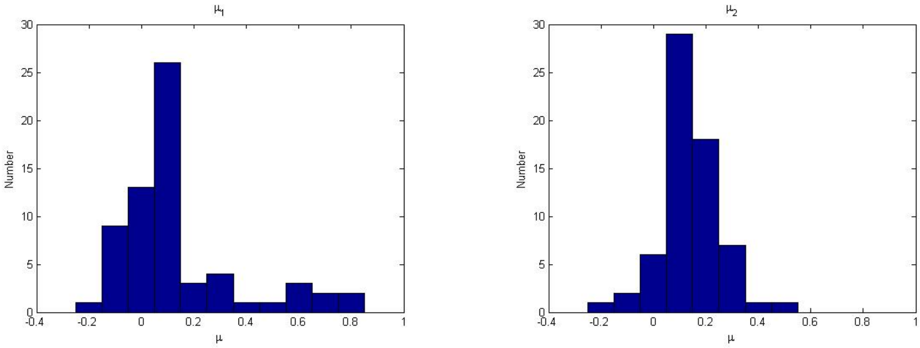
$$\frac{d_{21}}{d_{21} - d_{11}} > \min\{\frac{d_{22}}{d_{22} - d_{12}}, \frac{d_{23}}{d_{23} - d_{13}}\} \tag{2.7}$$

In conclusion, we can exclude Sure-Win *return tables* by creating *return tables* which satisfy the following conditions.

1. Equation (2.3) and (2.5) are not satisfied.
2. Arrange the *return table* to let equation (2.6) to be satisfied and hence let equation (2.7) be satisfied.

Choosing the return tables

As we analyse the experimental data individual by individual, it is important to give return tables which could be informative enough for us to estimate subjects' parameters. We expect heterogeneity in risk aversion and in beliefs about the priors. Since the subjects' decisions are determined by the *return table* as well as their preferences and beliefs, it is crucial to design a good set of *return tables* to be as informative as possible, with respect to the revelation of subject's risk and ambiguity attitudes. For example, if all the *return tables* are chosen in the sense that the two assets are really ambiguous, then the majority of subjects would just hold all cash to be safe. Then we would be unable to distinguish between different attitudes. When we apply the data to different ambiguity models, we may need to estimate different parameters. It is hard to design a set of *return tables* that will be "good" for every model.



Asset 1 Asset 2

FIGURE 2.2: Histogram of expected return

We are going to estimate the risk and ambiguity parameters of our subjects. We expected that most of the subjects would be risk-averse. They could be either extremely risk-averse or approaching risk-neutrality. If subjects are really risk-averse then it is likely that they do not invest any cash into the ambiguous assets at all, so we need to give some really attractive *return tables*-with high payoff and low risk. If a subject is approaching risk-neutrality, then he/she is likely to invest all the cash in the ambiguous assets so we need to include some less attractive *return tables*. To do this, we calculate the mean and variance for the two ambiguous assets using the true probabilities $p = [0.25, 0.5, 0.25]$. Figure 2.2 shows the histograms of the expected return for both assets. Table 2.4 summarises the mean (expected return) and variance information for the two assets. Note that for Asset 1, for 18 of the 65 *return tables* the expected return are negative, and 6 of the 65 *return tables* the expected return negative for Asset 2. There are 50 *return tables* in which the covariance of the two assets are positive. Because if the covariance is positive, it is more likely subjects want to construct a portfolio of buying one asset and selling another. But they cannot sell assets since this is not allowed under the experimental rules. As discussed in Section 1.3, when the Unconstrained optimal allocation is positive for one asset and negative for another asset, the Constrained optimal allocation is always putting 0 in the one which is negative. And it is the same for either Mean Variance or Expected Utility preferences. So we choose more *return tables* with positive

TABLE 2.4: Mean Variance Information for the 65 *return tables*

	Mean		Covariance	
	≥ 0	< 0	≥ 0	< 0
Asset1	47	18	15	50
Asset2	59	6		

covariances. We still want some *return tables* with negative covariances because we are not sure about subjects' preferences. The negative ones may lead us to find some interesting results.

As we assume subjects' beliefs are described by lower bounds on the probabilities, the lower bounds $\underline{\mathbf{P}} = [\underline{p}_1\ \underline{p}_2\ \underline{p}_3]'$ have to be estimated. So the *return tables* need to be designed in such a way that we can estimate the lower bounds precisely. As stated in equation (1.36), for one *return table*, only two lower bounds are actually involved in the optimal allocation. So it means the loss of information about the lower bound in one state. And it is not clear which one would be omitted unless we know the subjects' parameters, since we do not have an explicit expression of the optimal allocation in terms of the risk parameter. But what we want to do is to provide to the subjects a set of problems to reveal all the information about the lower bounds. In order to realise this we need to do some 'manipulation' of the *return tables*. The strategy is to add some Least-Payoff *return tables*, for which the portfolio payoff in a particular state is always the smallest, and hence the lower bounds of the other two states will be captured in the optimal solution. We call such states Least-Payoff states. A least Payoff-State can be created just by making the portfolio payoff to be the smallest in one state for any portfolio choice. For example, if we choose state 1 as the least portfolio payoff and we let $d_{11} = min\{d_{11}, d_{12}, d_{12}\}$ and $d_{21} = min\{d_{21}, d_{22}, d_{22}\}$ hence

$$\begin{aligned} w_2 - w_1 &= (d_{12} - d_{11})c_1 + (d_{22} - d_{21})c_2 \geq 0 \\ w_3 - w_1 &= (d_{13} - d_{11})c_1 + (d_{23} - d_{21})c_2 \geq 0 \end{aligned}$$

as we have $c_1 \geq 0$ and $c_2 \geq 0$. We have included 31 Least-Payoff *return* *tables* in total[7], among them there are 15 for state 1, 7 for state 2 and 9 for state 3.

Conditions for the Optimal Solution

As discussed in the section 1.2, there are explicit solutions for the unconstrained portfolio choice problem, though there are some conditions for the existence for solutions. Because these solutions are to be used in the algorithms to get the Constrained optimal allocation and furthermore the algorithms are called in the estimation program, it is better that we let these conditions be satisfied. The conditions are different for different preferences and different utility functions. For Expected Utility preferences with a CARA utility function the conditions are only related to the *return table*, but for Mean-Variance Preferences and EU preferences with CRRA utility function the conditions also concern the probabilities. As the subjective probabilities are unknown *ex ante*, what we can do is to let the *return table* satisfy the conditions for EU preferences with CARA utility functions, which are equations (1.20) and (1.23) and (1.24).

2.2 Error Specification

2.2.1 Introduction

Experimental Literature (e.g. Kroll *(*et al) 1998) shows Modern Portfolio choice (MPC) theory does not work well (the allocations are far away from optimal) in the laboratory, but surprisingly the Capital Asset Pricing Model, which is built on MPC, works well at the market level. Bossaerts *(*et al) (2007) propose a CAPM+ϵ model which is not rejected by their experimental data. In their method they add a utility perturbation to individuals' demand functions.

[7] We do not want to let all 65 *return tables* to have the property because we do not want subjects to notice the pattern.

Most experimental result shows that people are inconsistent even in simple binary choices experiments when they are asked to repeat the same problem (Camerer 1989, Starmer and Sugden 1989, Hey and Orme 1994, and Ballinger and Wilcox 1997). Research results (Hey 1995, Buschena and Zilberman 2000) suggest that the standard deviation of error tends to be higher when subjects are facing problem with more outcomes. So it is not hard to imagine that the degree of error that subjects would make when they are facing a portfolio choice problem, if we assume they do have deterministic preferences. Because a portfolio choice problem basically means that subjects are facing an infinite, or a relatively huge amount of multiple choice problems. Their brains have to deal with a huge number of the possible outcomes of their possible choices. Then it strikes me to try to answer the following question: Is it that Modern Portfolio choice theory really does not work in the lab, or it does work, but we just have not found the correct model of the error term?

It appears that the mainstream of experimental study of error stories is in the environment of pairwise choice problems. When studying portfolio choice, pairwise choices are less informative than allocation problems. So we hope our study could be valuable in suggesting the proper stochastic specification for allocation problems, especially in the scenario where the allocation is bounded. In most experimental finance studies, the allocations have to be bounded as we do not want subjects to go bankrupt in an experiment as in such cases subjects would have to pay money to the experimenter. Besides in real life there does not exist a situation where people can allocate an unlimited amount. We emphasise this point here because the stochastic specification for boundaries raises many interesting issues. Furthermore, as one portfolio choice involves two allocation, the dependence of the two stochastic variables makes the problem more complicated, and interesting.

In this chapter, we report on a Portfolio Choice under Ambiguity (PCA) experiment. The intention of conducting this experiment is to shed light on individual portfolio choice when the assets are ambiguous, rather than risky. In order to achieve this, first we need to find a good stochastic specification for our portfolio choice problem. In this section, I discuss the plausibility of error stories falling into three categories. The first two categories assuming there is a deterministic preference and error is coming from the implementation and the measurement of utility respectively. The third category is rather different from the first two by assuming the preferences are stochastic and hence the decisions are stochastic.

2.2.2 Implementation Error

Introduction

In this section, we discuss errors are coming from *implementation*. For the time being, we assume that the two allocations are independent. So next I use *c* to refer to the two allocations in general. We assume subject have deterministic preferences which give them an optimal allocation c^*. This optimal allocation c^* is decided by the problem itself and their subjective parameters of preferences.

$$c^* = \mathcal{OPT}(U(\mathbf{D}, \mathbf{S})). \tag{2.8}$$

here $\mathbf{S}$ is a vector that contains the subjects' parameters. The elements of $\mathbf{S}$ may be different depending on which preference theory we are considering. For example, if we use Subjective Expected Utility, then $\mathbf{S}$ contains the risk parameter and 3 probabilities for the 3 states. If we use Maxmin Expected Utility, then $\mathbf{S}$ contains risk parameter and 3 lower bounds on probabilities for the 3 states. Remember in the experiment, subjects can only make integer allocations. We think this leads to two different mind processing ways when subjects are making their choices. The first is they think about the allocations in a continuous sense and hence c^* is continuous. And when implementing their *intended* allocation, they 'add on' an error ϵ, which is also continuous. The experimental interface then rounds up $c^* + \epsilon$, which is a continuous number, to the actual allocation $\hat{c}$, which is an integer number. It means for any observed $\hat{c}$, the likelihood is calculated by the cumulative probability at the point $\hat{c} + 0.5$ minus probability at the point $\hat{c} - 0.5$.

TABLE 2.5: Error Specifications

Continuous	Discrete
-	Binomial
Beta	Beta Binomial
Biased Beta (BB)	Biased Beta Binomial (BBB)
Two Beta[1]	Two Beta Binomial
Beta Exponential[2]	-

[1] We use a Beta distribution for the open area from 0 to 1 and another Beta distribution only for boundaries. Similar for the Two Beta Binomial.

[2] We use a Beta distribution for the open area from 0 to 1 and a truncated Exponential distribution only for boundaries.

The second approach is to assume that they discretise the problem so c^* are integers and the error ϵ is also an integer. So we need continuous and discrete distributions to model *c* respectively. Which approach is a realistic description of the actual processes of the subjects is difficult to say *ex ante*.

We need to use bounded distribution as *c* is bounded in the area between 0 and 100 (the cash endowment in each problem being 100). We have to treat the error stories for the boundaries very carefully, as we will see. In Table 2.5 we summarise the various error stories that we discuss in detail in this section.

Continuous Stochastic Specification - BB specification

As $c \in [0, 100]$ and is continuous, we assume

$$x \sim Beta(\alpha, \beta), \quad 0 \leq x = \frac{c}{100} \leq 1$$

Here we scale the allocation *c* by the initial endowment *e* to make it possible to be represented by a beta distribution, which is defined on the range $[0,\ 1]$. We will use the same notation for the scaled allocation *x* as *c*. For example, $\hat{x} = \dfrac{c}{100}$ means the scaled actual allocation and $x^* = \dfrac{c^*}{100}$ means the scaled optimal allocation.

We want to specify the parameters α and β. Notice that they need to satisfy $\alpha > 0$ and $\beta > 0$. For a beta distribution, the relation between the mean $\bar{x}$, and variance V of its parameters α and β is as follows. The mean and variance can be calculated from α and β as

$$\begin{cases} \bar{x} &= \dfrac{\alpha}{\alpha+\beta} \\ V &= \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \end{cases} \tag{2.9}$$

and the α and β can be derived from the mean and the variance as

$$\begin{cases} \alpha &= (\frac{1-\bar{x}}{V} - \frac{1}{\bar{x}})\bar{x}^2 \\ \beta &= \alpha(\frac{1}{\bar{x}} - 1) \end{cases} \tag{2.10}$$

A first possibility is to follow others in assuming that the actual allocations are *centred* on their *optimal* allocation and that the variance of errors made by subjects is the same everywhere - that is for all $x^* = [0,\ 1]$. But a constant variance is inappropriate as we think subjects make big error in the middle and less error when approach to the boundaries. Hey and Pace (2014) suggest a way of specifying α and β as follows.

$$\begin{cases} \alpha &= x^*(s-1) \\ \beta &= (1-x^*)(s-1). \end{cases} \tag{2.11}$$

hence the mean and variance are as follows

$$\begin{cases} \bar{x} &= x^* \\ V &= \frac{(1-x^*)x^*}{s} \end{cases} \tag{2.12}$$

where *s* is defined as *precision* parameter. They assume the distribution is centred on the optimal allocation and the variance is specified such that it is smallest at the boundaries and reaches at the maximum in the middle. In their case they have $0 < x^* < 1$ while we have $0 \leq x^* \leq 1$. According to equation (2.12), the variance V is equal to 0 when $x^* = 0$ or $x^* = 1$. It implies subjects make no mistake at the boundaries. In the experiment, since subjects are not allowed to short sell, it is likely that they are going to the hit the boundaries often[8] If we do not want the variance to be equal to zero at the boundaries, then the distribution of x has to be biased. If we let $\bar{x} = 0$ for $x^* = 0$, then the distribution has to be spread both in the part of $x > 0$ and $x < 0$ to make $\bar{x} = 0$. As x can not be negative, $\bar{x}$ has to be positive if the variance is not zero. The similar analysis applies to the situation when $x^* = 1$. $\bar{x}$ has to be less than one at $x^* = 1$. Hence our solution is to add a bias parameter to $\bar{x}$. Differently from Hey and Pace (2014) who assumed that the distribution is unbiased, we assume it is centred on the *biased optimal allocation* which is defined as follows.

$$x^{'} = \frac{b}{2} + (1-b)x^* \quad 0 < b < 1 \tag{2.13}$$

As we want $0 < x^{'} < 1$, we let $0 < b < 2$. And we also want the $x^{'}$ to be positively related to x^*, we let $b < 1$. So we assume $0 < b < 1$. And the degree of bias is increasing when x is away from 0.5. When $b = 0$ we have $x^{'} = x^*$ then there is no bias. When $b = 1$ we have $x^{'} = \frac{b}{2}$ which is not related to x^*, which means subjects' allocations are total random. As $x \in [0,1]$ we have $x^{'} \in [\frac{b}{2}, 1 - \frac{b}{2}]$. There is $x^{'} - x^* = b(\frac{1}{2} - x^*)$ which can be seen as a indication of the bias. When $x^* < \frac{1}{2}$, we have $x^{'} - x^* > 0$, so x is positively biased. When $x^* = \frac{1}{2}$, x is not biased. When $x^* > \frac{1}{2}$, x is negatively biased. Notice as $x^* \in [0,1]$ we have $x^{'} \in [\frac{b}{2}, 1 - \frac{b}{2}]$ and hence $0 < \bar{x} < 1$. We can call $x^{'}$ as *biased scaled optimal allocation*. Replacing x^* by x' in equation (2.12), we have a new specification for x as follows.

$$\begin{cases} \bar{x} & = x^{'} \\ V & = \dfrac{(1-x^{'})x^{'}}{s} \end{cases} \tag{2.14}$$

where α and β are as follows

$$\begin{cases} \alpha & = x^{'}(s-1) \\ \beta & = (1-x^{'})(s-1). \end{cases} \tag{2.15}$$

Figure 2.3 on Page 80 shows the relation of V to x^*. We have $x^{'} = \frac{b}{2} +$

[8] The experimental data also proves this point.

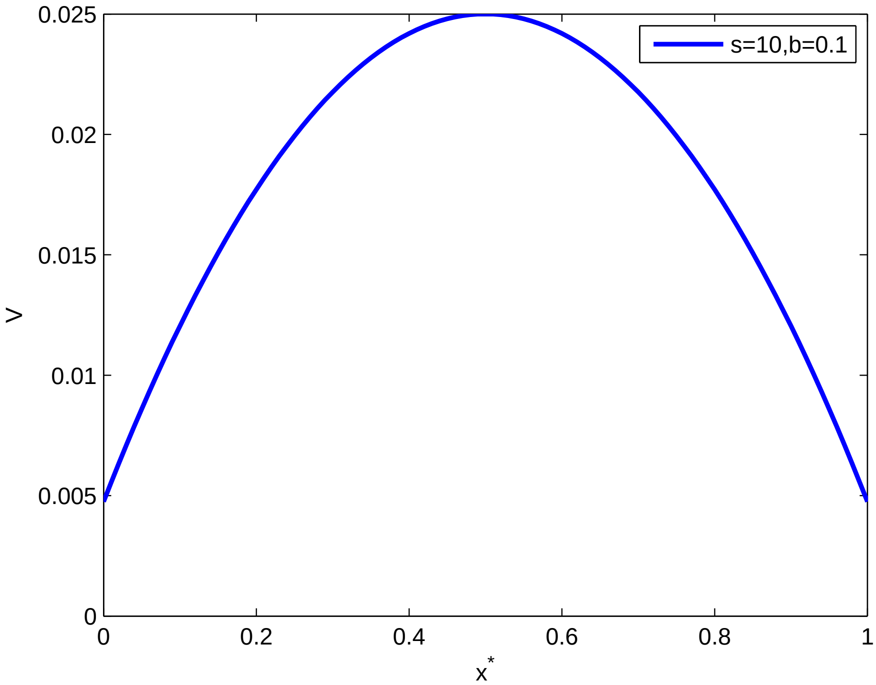


FIGURE 2.3: Concave Variance under Biased Beta Distribution

$(1-b)x^*$, so $V(x^*=0) = V(x^*=1) = \frac{1}{4}\frac{(2-b)b}{s} \neq 0$ which are the minima and $V(x^*=0.5) = \frac{1}{4s}$ which is the maxima. So far we have constructed a satisfactory biased beta distribution specification, which we call Biased Beta specification. Equation (2.16) shows the complete Biased Beta specification.

$$x \sim Beta(\alpha, \beta), \quad 0 \leq x \leq 1 \tag{2.16}$$

where

$$\begin{cases} x' &= \frac{b}{2} + (1-b)x^* \\ \alpha &= x'(s-1) \\ \beta &= (1-x')(s-1). \end{cases}$$

and

$$0 < b < 1, \qquad s > 1$$

Denote $F(x;\alpha,\beta)$ [9]as the cumulative distribution function for a beta distribution. Now we can write the sum of the log-likelihood function for the 65

[9]When $x > 1$ we have $F = 1$, and when $x < 0$ we have $F = 0$.

problems $\mathbf{C}_j = [\hat{c}_1,\ \hat{c}_2], j \in \mathbb{N}^+_{65}$ as

$$L = \sum_j^{65} log \left(\prod_i^2 L_{ij} \right),\ i \in \mathbb{N}^+_2,\ j \in \mathbb{N}^+_{65} \tag{2.17}$$

where

$$L_{ij} = F(\frac{\hat{c}_{ij} + 0.5}{100}, \alpha_i, \beta_i) - F(\frac{\hat{c}_{ij} - 0.5}{100}, \alpha_i, \beta_i),\quad c^*_{ij} \in \mathbb{N}_{100}$$

where

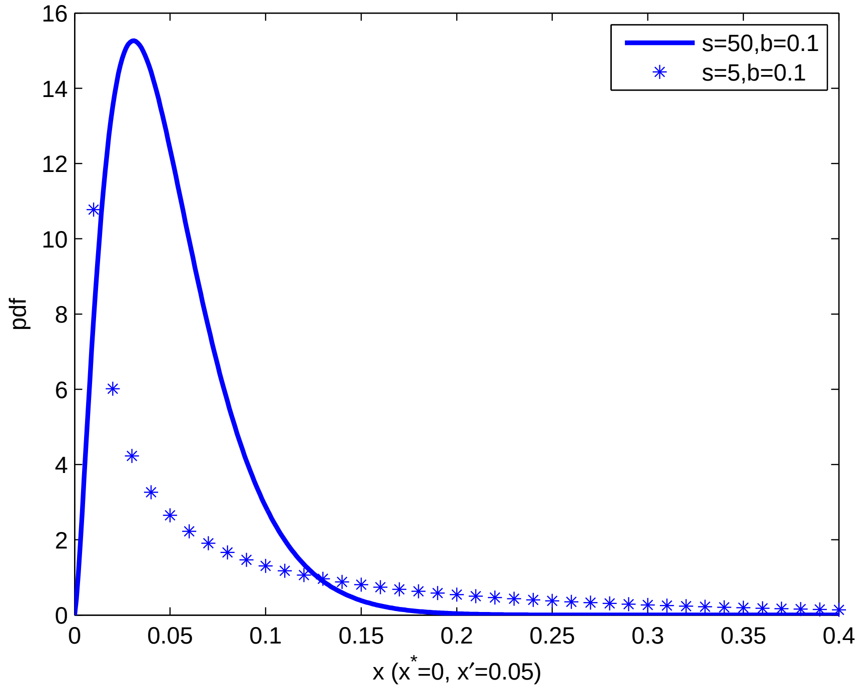
$$\begin{cases} \alpha_i = \left(\frac{b_i}{2} + (1 - b_i)\frac{c^*_{ij}}{100} \right)(s_i - 1) \\ \beta_i = \left(1 - \frac{b_i}{2} - (1 - b_i)\frac{c^*_{ij}}{100} \right)(s_i - 1) \\ 0 < b_i < 1, \quad s_i > 1. \end{cases}$$

Now let us explore a little more the implications about this specification, especially the implication of the precision and bias parameters, as they have not been used before with this particular specification.

Figure 2.4 [10] on Page 82 shows the *pdf* for different values of *s* and *b* when $x^* = 0$ and $x^* = 0.5$.The mean of the both distributions are equal to $x' = x^* + \frac{b}{2} = 0.05$. Assume the same bias parameter. The individual with lower precision, say *s=5*, is likely to make bigger errors overall. But he/she seems to better express his intention when x^* approaches the boundaries. The likelihood of choosing 0 (also the ones quite near 0) when $x^* = 0$ is much bigger for *s=5* than for *s=50*. This does not seem to be quite sensible. So we ask the question: Is it sensible to add a bias to a whole range when we only want to tackle the issue concerning the boundaries?

So let us now focus on how to specify the biased distribution only at the boundaries. We would like to think the mode of the distribution is at the boundaries, it means we assume subjects are most likely to choose the boundaries when the actual optimal allocation is at the boundaries. The relation

[10] For better demonstration for the *pdf* functions, the range for *x* is set from 0 to 0.4 instead of 0 to 1 when in the case $x^* = 0$.



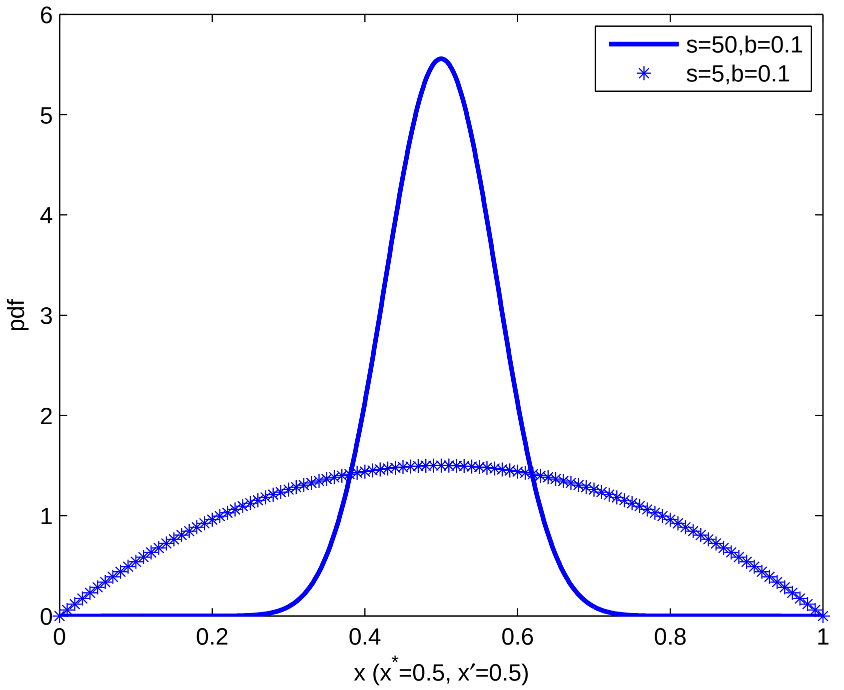


FIGURE 2.4: Biased Beta pdf for $x^* = 0$ and $x^* = 0$

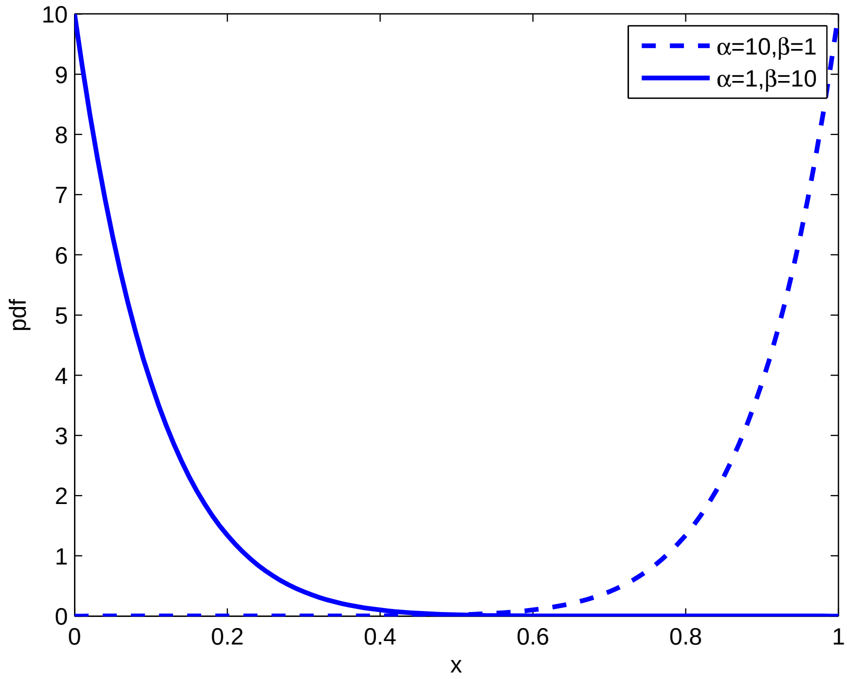


FIGURE 2.5: Beta distribution for boundaries

of the mode to the parameters for a beta distribution is as follows.

$$Mo = \frac{\alpha - 1}{\alpha + \beta - 2}, \ \alpha > 1, \ \beta > 1$$

It implies

$$\begin{cases} Mo = 0 & \text{when } \alpha = 1, \beta > 1 \\ Mo = 1 & \text{when } \alpha > 1, \beta = 1 \end{cases}$$

When $\alpha = 1, \beta > 1$, we get a distribution whose mode is equal to 0 and the *pdf* is strictly decreasing. We can use it to model the distribution when $x^* = 0$. It seems sensible to assume that the most likely choice for *x* is equal to 0 when $x^* = 0$. When $\alpha > 1, \beta = 1$, we have $Mo = 1$ and the *pdf* is strictly increasing, which could be used to specify the distribution when $x^* = 1$. Figure 2.5 on Page 83 shows the *pdf* for these two special cases. We would like to assume that the distributions at the two boundaries are symmetric by letting $x \sim Beta(1, b)$ when $x^* = 0$, and $x \sim Beta(b, 1)$ when $x^* = 1$. When $0 < x^* < 1$, we assume the unbiased Beta distribution which centred on the x^*. We call this specification as the Two Betas specification as it involves two different types of Beta distributions. Equation (2.18) shows the complete Two Betas specification.

$$\begin{cases} x \sim Beta(1,b), & x^* = 0 \\ x \sim Beta(\alpha,\beta) & 0 < x^* < 1 \\ x \sim Beta(b,1), & x^* = 1 \end{cases} \tag{2.18}$$

where

$$\begin{cases} \alpha & = x^*(s-1) \\ \beta & = (1-x^*)(s-1). \end{cases}$$

and

$$b > 1, \qquad s > 1$$

Now we can write the sum of likelihood function for the 65 portfolio choice $\mathbf{C}_j = [\hat{c}_1,\ \hat{c}_2], j \in \mathbb{N}^+_{65}$ as

$$L = \sum_{j}^{65} log \left(\prod_{i}^{2} L_{ij} \right),\ i \in \mathbb{N}^+_2,\ j \in \mathbb{N}^+_{65} \tag{2.19}$$

where

$$L_{ij} = \begin{cases} F(\frac{0.5}{100}, 1, b_i) & \text{if } c^*_{ij} = 0 \\ \\ F(\frac{\hat{c}_{ij}+0.5}{100}, \alpha_i, \beta_i) - F(\frac{\hat{c}_{ij}-0.5}{100}, \alpha_i, \beta_i) & \text{if } c^*_{ij} \in \mathbb{N}_{99^+} \\ \\ 1 - F(\frac{\hat{c}_{ij}}{100}, b_i, 1) & \text{if } c^*_{ij} = 100 \end{cases}$$

where

$$\begin{cases} \alpha_i = \frac{c^*_{ij}}{100}(s_1 - 1) \\ \beta_i = (1 - \frac{c^*_{ij}}{100})(s_2 - 1) \\ b_i > 1, \qquad s_i > 1. \end{cases}$$

Beta Exponential

We have put a lot of effort into modelling the error at the boundaries. One might think, why not just use a truncated distribution for the boundaries? We have explored this possibility too. We still keep using the scaled allocation *x* for analysis here. Let us assume that x has an (unbiased) Beta distribution, when $0 < x^* < 1$, but let us assume that it has an *exponential* distribution when $x^* = 0$ or 1. Next I go through the details about this specification.

Suppose we use a exponential distribution to model the distribution when $x^* = 0$. The *pdf* for an exponential distribution is

$$f(x;\lambda) = \begin{cases} \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and we have $\bar{x} = \lambda$ and $V(x) = \lambda$ and the mode $Mo = 0$. It is good in the way that the *pdf* is monotonically decreasing from 0. As the exponential distribution is defined in $[0,\ \infty]$, we need to truncate the part that $x > 1$. Denote $F(x;\lambda)$ as the probability mass function. We cannot just do the following.

$$L(1) = F(\infty;\lambda) - (0.995;\lambda)$$

[11] as it gives a big likelihood for $x = 1$. Figure 2.6 on Page 86 shows the *pdf* for $\lambda = 0.3$. The cumulative probability from 0.995 to ∞ is equal to 0.0363. We have $L(0.99) = F(0.995; 0.3) - F(0.985; 0.3) = 0.0012 < L(1)$. It means the subject is more likely to choose 1 than 0.99. This is not sensible. We need to truncate the exponential distribution and scale the *pdf* as follows.

$$f(x;\lambda) = \frac{\frac{1}{\lambda} e^{-\frac{1}{\lambda}x}}{1 - (F(\infty;\lambda) - F(1;\lambda))}, \quad 0 \leq x \leq 1$$

For $x^* = 1$, we can use the rescaled distribution of x^*. The *pdf* is as follows.

$$f(x;\lambda) = 1 - \frac{\frac{1}{\lambda} e^{-\frac{1}{\lambda}x}}{1 - (F(\infty;\lambda) - F(1;\lambda))}, \quad 0 \leq x \leq 1$$

[11] Remember that in the experiment, an observed allocation 100 is assumed to be rounded, so the likelihood is calculated from 99.5 to 100.

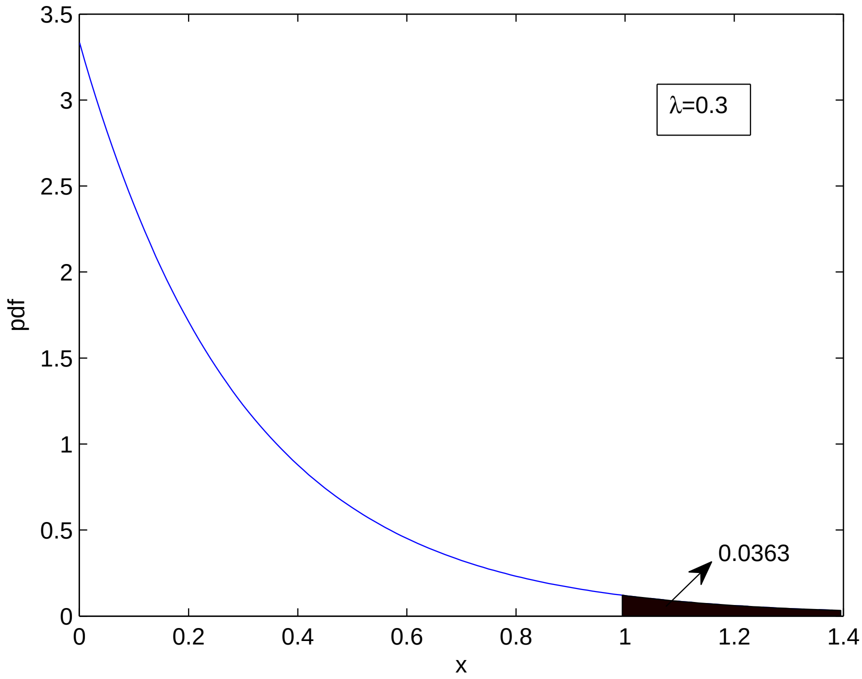


FIGURE 2.6: Truncated Exponential Distribution for $x^* = 0$

The sum of the log-likelihood function is similar to that in equation (2.19) so we do not give details here.

Discrete Stochastic Specification - the BBB specification

Hareless and Camerer (1994) proposed the use of a *tremble* to model the stochastic process involved in the actual decision. Their story is based on a pairwise-choice problem, either choose lottery l_1 or lottery l_2 in a problem. They think individuals' behaviour rule could be explained by a deterministic preference theory plus a *tremble*. Denote $\mathbf{S}$ as a unique set of parameters represents an individual's preferences. Determined by his/her preferences, an individual has an optimal decision l^* which also can be called the individual's *intended* decision. They further claim that a subject will make a mistake in implementation with a constant probability λ by choosing otherwise instead of l^* across all problems.

$$\hat{l} = \begin{cases} l^* & p = 1 - \lambda \\ \{l_1, l_2\} - \{l^*\} & p = \lambda \end{cases}$$

This model can be adapted to our portfolio choice problem, by discretisation of the allocations. Remember in the experiment, there are two allocations in one portfolio choice problem, which we denote by $\mathbf{C} = [c_1, c_2]$. The allocation can only be made in integers and they satisfy

$$c_1 \geq 0,\ c_2 \geq 0,\ c_1 + c_2 \leq 100$$

In a pairwise-choice experiment, the choice is between two options while we have 5151 choices in the experiment.[12] We can think in this way. Subjects have a deterministic decision $\mathbf{C}^*$, which is in integer, based on their preferences, but when they implement their decisions they make mistakes by choosing another. It might because that they tremble away from $\mathbf{C}^*$ when they are clicking the mouse. And the further away a choice from $\mathbf{C}^*$, the less likely subjects choose it by mistake. The stochastic specification of c is written as follows.

$$\hat{c} = \begin{cases} \vdots \\ c^* - 1 & p_{c^*-1} \\ c^* & p_{c^*} \\ c^* + 1 & p_{c^*+1} \\ \vdots \end{cases}$$

[12]Assuming that c_1 and c_2 are independent, so for each of them the total number is 101. But all possible portfolio choices are not simply 101*101 as $c_1 + c_2 \leq 100$. For example, if the total endowment is 5 then for each of them the total number of possible allocation is 36. But the possible portfolios are $[0,0]$, $[0,1]$, $[0,2]$, $[0,3]$,$[0,4]$, $[0,5]$, $[1,0]$, $[1,1]$, $[1,2]$, $[1,3]$, $[1,4]$, $[2,0]$, $[2,1]$, $[2,2]$, $[2,3]$, $[3,0]$, $[3,1]$, $[3,2]$, $[4,0]$, $[4,1]$, $[5,0]$. The total number pf possible portfolio choice is only 21. Similarly, when the endowment in 100, the total number of portfolio choice is 5151. More details about the independence of the two allocations will be discussed later in this section.

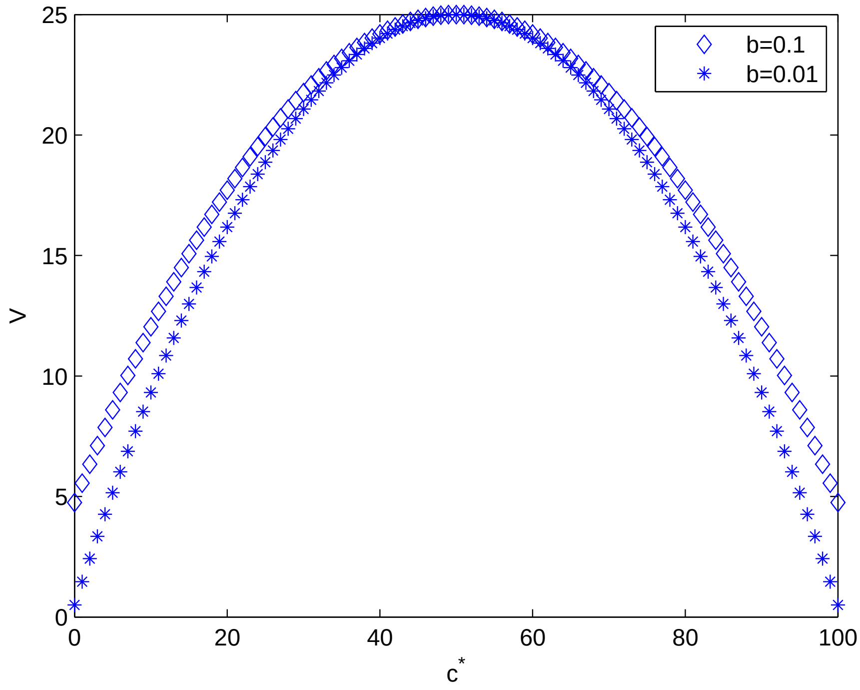


FIGURE 2.7: Variance for different bias

The Binomial distribution seems to be a possible specification of the tremble. Denote by $Bin(p, n)$ a binomial distribution with probability parameter $p \geq 0$ and $n \in \mathbb{N}^+$. The variable has a discrete distributed over the set of integers between 0 and n, with mean $\bar{c} = np$ and $V = np(1-p)$. Let us assume that

$$c \sim Bin(p^*, 100) \;\text{ where }\; p^* = \frac{c^*}{100}$$

hence that the mean of c is c^*. There is no parameter to be estimated, that is we are assuming all subjects are making the same errors. Given that we think that heterogeneity (over subjects) in error is important in our problem, this Binomial distribution does not seem to be a good specification.

A way round this problem is to use a *Beta-Binomial* distribution. Such a distribution can be thought of as a binomial distribution

$$c \sim Bin(p, n)$$

where the probability parameter

$$p \sim Beta(\alpha, \beta).$$

This Beta-Binomial involves two steps of stochastic processes. The first step is in generating the probability parameter p from the beta distribution. The second is generating the actual allocation from a Binomial Distribution using that probability parameter, which we denote as $\hat{p}$. The probability mass function for a Beta-Binomial distribution is

$$f^{BB}(\hat{c}|\alpha, \beta, n) = \binom{n}{k} \frac{f^{Beta}(k+\alpha, n-k+\beta)}{f^{Beta}(\alpha, \beta)} \tag{2.20}$$

here f^{Beta} is the probability density function for Beta distribution.

We can let the mean of p to be equal p^* and the variance to be related to p^* and another parameter *s*, which we call the *precision*. This extra parameter allows us to estimate subjects' heterogeneity in errors. Before we go further, we should warn ourselves about the implications of assuming that the mean of p is equal to p^*. Remember in the continuous case, we found that there is no variance for a Beta distribution at the boundaries 0 and 1. And we do not want to assume that subjects do not make mistakes at the boundaries. We have the same problems here. If we let p be centred on p^*, then $\hat{p}$ equal to 0 for sure when $p^* = 0$ instead of stochastically generated from a Beta Distribution. Furthermore, the variance is equal to zero again for a Binomial Distribution $Bin(0, n)$. We fail to incorporate variance for $p^* = 0$ in both steps. The same analysis applies to the case when $p^* = 1$. Luckily we have solved a similar problem in the continuous case by adding bias, which makes sure that the Beta distribution is not centred on either 0 or 1. The bias can be added over the whole range of $p^* = [0,\ 1]$ and also can only be added at the boundaries. If we add the bias over the whole range, we let p be centred on p', which is defined as following

$$p^{'} = \frac{c^*}{100}(1-b) + \frac{b}{2}, \quad 0 < b < 1, \tag{2.21}$$

Here we obtain equation (2.21) by the inspiration of equation (2.13). Suppose $x^* = 0$, then we have $p^* = 0$ and $p^{'} = \frac{b}{2} > 0$. So $\hat{p}$ is stochastically generated from a Beta distribution with mean equal to $\frac{b}{2} > 0$. Notice the actual allocation $c \sim Bin(\hat{p}, 100)$ and $\hat{p}$ could still be equal to 0. But we do not need to worry about it, as we have already incorporated the variance from the first step. We still need to specify the variance for p. In the continuous case, we decided that a constant variance does not seem to be a good specification for bounded distribution as in our setting. So again, we use similar specification as we use in the continuous case. We specify α and β as follows.

$$\begin{cases} \alpha & = p^{'}(s-1) \\ \beta & = (1-p^{'})(s-1). \end{cases} \tag{2.22}$$

hence

$$\begin{cases} \bar{p} & = p^{'} \\ V & = \dfrac{(1-p^{'})p^{'}}{s}. \end{cases} \tag{2.23}$$

here $s > 1$ is a precision parameter *s*, which allows us to specify subjects' heterogeneity in errors. Here we obtain equation (2.22) by replacing x' with $p^{'}$ in equation (2.15). By specifying α and β as in equation (2.22) for equation (2.20), we completed the specification for Biased Beta Binomial distribution.

Remember that Biased Beta-Binomial distribution is the case in which we add the bias over the whole range. We can also add bias only for the boundaries. We call this specification as Two Beta-Binomial distribution and its specification is as follows.

$$P(\hat{c})^{13} \begin{cases} f^{BB}(\hat{c}|b, 1, 100), & \hat{c} = 0 \\ f^{BB}(\hat{c}|\alpha, \beta, 100), & \hat{c} \in \mathbb{N}_{99} \\ f^{BB}(\hat{c}|1, b, 100), & \hat{c} = 0^{14} \end{cases} \tag{2.24}$$

[13]When $c^* = 0$ and hence $p^* = 0$, we assume $p \sim Beta(b, 1)$ which guarantees the mode of the distribution is at 0 and is monotonically decreasing from 0 to 1. The mean is equal to $\frac{b}{1+b} > 0$, so the distribution of p is positively biased. When $c^* = 1$ and hence $p^* = 1$, we assume $pis \sim Beta(1, b)$ which guarantees the mode of the distribution is at 1 and the *pdf* is monotonically decreasing from 1 to 0. The mean is equal to $\frac{1}{1+b} < 1$,

where

$$\begin{cases} \alpha = p^*(s-1) \\ \beta = (1-p^*)(s-1) \\ 0 < b < 1, \quad s > 1 \end{cases}$$

Remark : Independence Issues about Two allocations

So far we have assumed c_1 and c_2 are independent, in terms of both the optimal allocation c_1^*, c_2^* and the errors. Though the optimal allocations satisfy the constraint $c_1^* + c_2^* \leq 100$, the actual allocation implied by these stochastic modelling does not necessarily satisfy $\hat{c}_1 + \hat{c}_2 \leq 100$. For example, $c_1^* = 40$ and $c_2^* = 40$. Suppose they are independently distributed in $[0,\ 100]$, then it is possible that $\hat{c}_1 = 56$ and $\hat{c}_2 = 44$ and hence $\hat{c}_1 + \hat{c}_2 = 110 \leq 100$. Such violations do not happen in the experimental data as subjects can not implement such a portfolio. So we may have a divergence between the error story model and its application when we are estimating data. The subjects might choose one allocation first and then another allocation, whose error has to be conditioned on the first allocation. Or they choose the two allocations simultaneously and the errors are formulated in a specific way. We have only considerred the first possibility in our specifications. Take Biased Beta-Binomial distribution for an example. When $c_1^* = 40, b_1 = 0.2$ and $c_2^* = 40, b_2 = 0.2$, we have $p_1^{'} = p_2^{'} = 0.42$ by equation (2.21), then

$$c_1 \sim Bin(0.42, 100), \quad c_2 \sim Bin(0.42, 100) \tag{2.25}$$

The distributions of c_1 and c_2 are shown in Figure 2.8 on Page 93. The two distributions are exactly the same as they have the same parameters. But if

so the distribution is negatively biased. When $\hat{c} \in \mathbb{N}_{99}$, the specification of α and β guarantees the mean and variance to be follows.

$$\begin{cases} \bar{p} & = p^* \\ V & = \dfrac{(1-p^*)p^*}{s}. \end{cases}$$

As $\bar{p} = p^*$, the distribution of p over $0 < p^* < 1$is unbiased. We have discussed this in detail when we specified the Biased Beta specification in the continuous case in equation (2.18).

$\hat{c}_1 = 30$, the area from 70 to 100 are actual not valid as $\hat{c}_2$ can not exceeding 70. This is the situation when $\hat{c}_1 < c_1^*$. When $\hat{c}_1 = 60 > c_1^*$ the area is relatively bigger in the shadowed area, which is around 0.04.

Define c_2 on $100 - c_1^*$ Instead of assuming both the two allocations are distributed on $[0,\ 100]$, an alternative way is assuming

$$c_1 \sim Bin(p_1^{''}, 100), \quad c_2 \sim Bin(p_2^{''}, 100(1 - p_1^{''})) \tag{2.26}$$

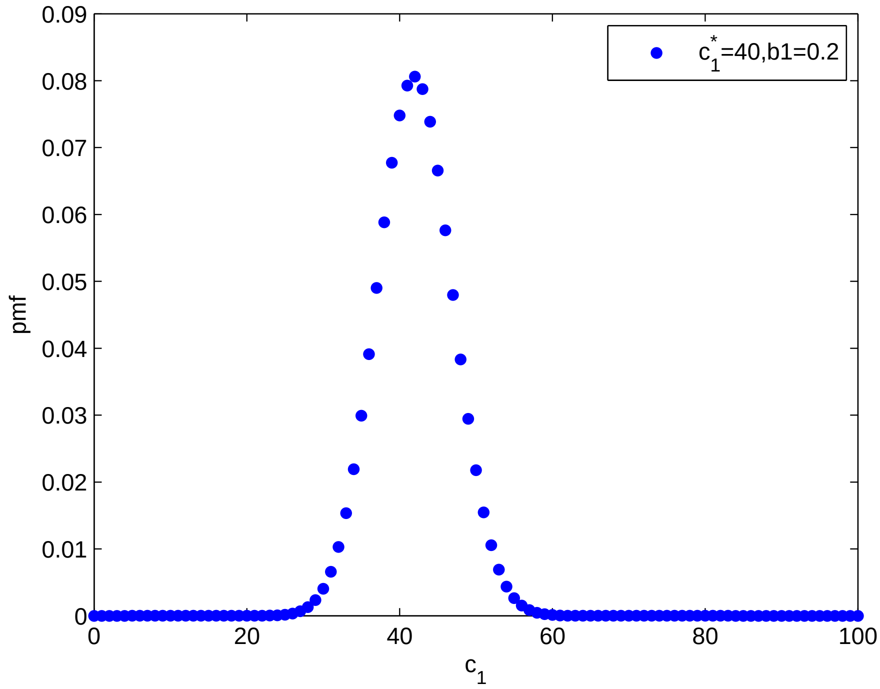
where

$$\begin{cases} p_1^{''} &= \frac{c_1^*}{100}(1 - b_1) + \frac{b_1}{2} \\ p_2^{''} &= \dfrac{\frac{c_2^*}{100}(1 - b_2) + \frac{b_2}{2}}{1 - p_1^{''}} \end{cases}$$

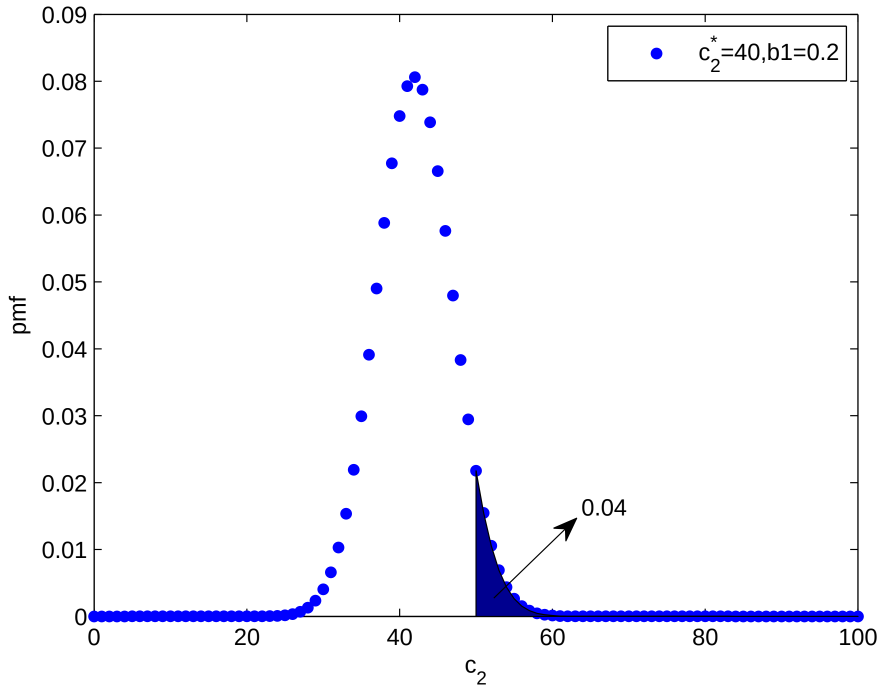
such that c_1 and c_2 are still centred on their own biased optimal allocation. It seems that the specification in equation (2.26) is more sensible; after all it incorporates the constraints that $c_1^* + c_2^* \leq 100$. Take $c_1^* = 40, b1 = 0.2$ and $c_2^* = 40, b1 = 0.2$ as an example. By equation (2.27), we have $p_1^{''} = \frac{40}{100}(1 - 0.2) + \frac{0.2}{2} = 0.42$ and $p_2^{''} = \frac{40}{100*(1-0.42)}(1 - 0.2) + \frac{0.2}{2} = 0.47$

$$c_1 \sim Bin(0.42, 100), \quad c_2 \sim Bin(0.72, 58)$$

The distribution of c_1 is exactly the same as in Figure 2.8. Figure 2.9 on Page 94 shows the distribution of the c_2 in this case. Their means are the same while the variance of c_2 is less than c_1 though they share the the same parameters. That is because the range of c_2 is reduced from 0 to 100 to 0 to $100 - c_2$. Again it does not totally eliminate the possibility of $\hat{c}_1 + \hat{c}_2 > 100$, but at least it is less likely to happen as many times as the specification defined in equation (2.25). But it does not cover a lot of possibilities for c_2. For example, if $c_1 = 30$, c_2 can be between 0 to 70. But in this specification, the likelihood of any number more than 58 is equal to 0 as c_2 is defined on 0 to 58.



$\bar{c}_1 = 42,\ V(c_1) = 24.4$



$\bar{c}_2 = 42,\ V(c_2) = 24.4$

FIGURE 2.8: Biased Beta Binomial distributions of c_1 and c_2

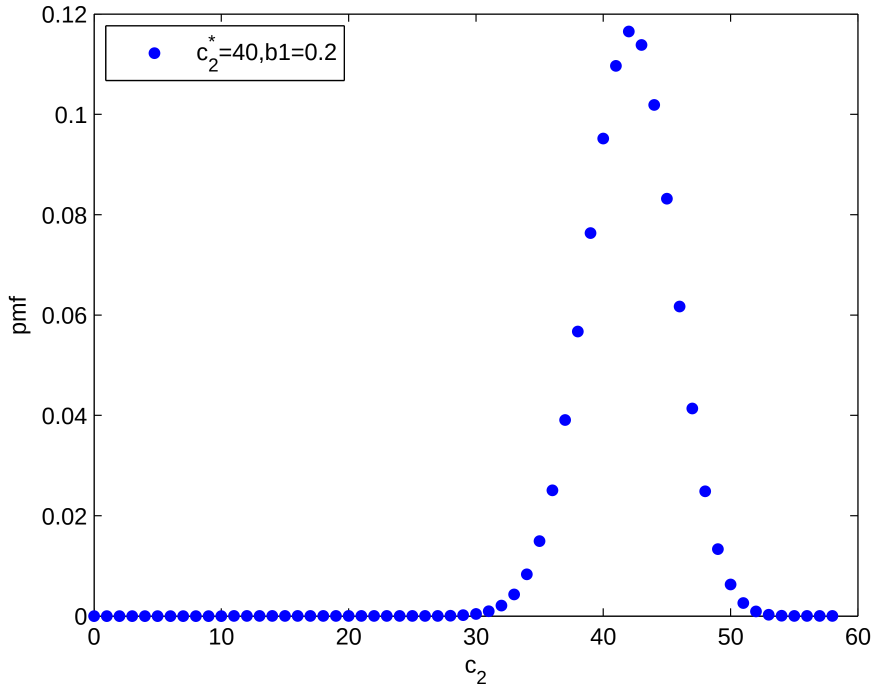


FIGURE 2.9: Beta Binomial Distribution of $c_2 \sim Bin(p_2', 100 - c_1^*)$

Define c_2 on $100 - \hat{c}_1$ Another way of is to assume

$$c_1 \sim Bin(p_1''', 100), \quad c_2 \sim Bin(p_2''', 100 - \hat{c}_1) \tag{2.27}$$

where

$$\begin{cases} p_1''' &= \frac{c_1^*}{100}(1 - b_1) + \frac{b_1}{2} \\ p_2''' &= \dfrac{\frac{c_2^*}{100}(1 - b_2) + \frac{b_2}{2}}{1 - \frac{\hat{c}_1}{100}} \end{cases}$$

This specification guarantees that $\hat{c}_1 + \hat{c}_2 \leq 100$. But p_2''' can be bigger than 1 when $\hat{c}_1 < c^*$. For example, $\hat{c}_2 = 60$ we have $p_2''' = \dfrac{0.42}{1 - 0.6} > 1$.

2.2.3 Stochastic Utility

We have been assuming the $\hat{c}_1$ $\hat{c}_2$ are independent. Figure 2.10 on Page 95 shows the Indifference Curves in the *Allocation Triangle*. In the direction of OO_1 the expected utility decreases at the largest speed, while in the direction OO_2 it decreases at the minimum speed. If two points in these two directions

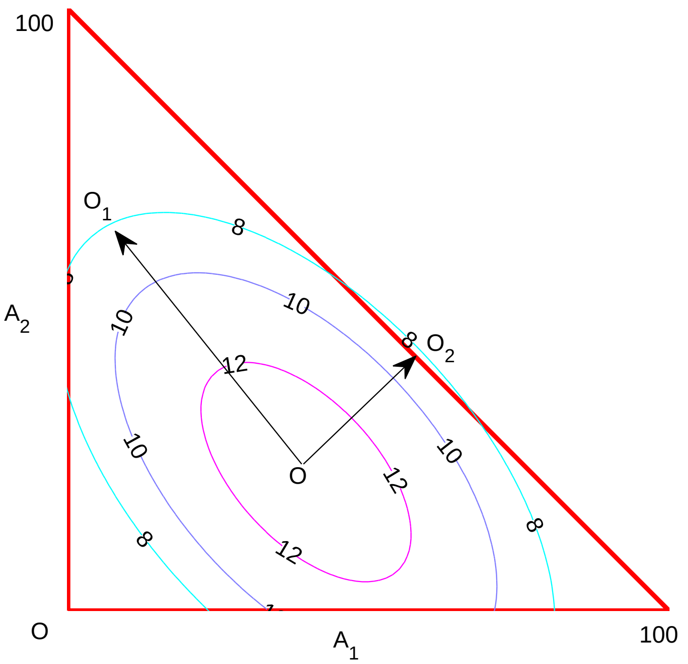


FIGURE 2.10: Indifference Curve in Allocation Triangle

have the same distance to the optimal allocation, the likelihoods calculated are the same which seems inplausible on the direction of OO_1 as it generates lower expected utility.

Hey and Orme (1994) proposed a model of adding noise to the valuation of the gambles. They assumed that subjects make mistakes when they measure the possible valuations in a problem. Denote $\mathbf{U}$ as subjects' utility function and $\mathbb{C}$ as all possible allocations. Instead of finding a optimum to maximise their utility function $\mathbf{U}$, they think subjects are maxmising the stochastic valuation.

$$\hat{\mathbf{C}}^* = \mathcal{OPT}(\hat{U}), \quad \hat{U} = U(\mathbf{D}, \mathbf{S}) + \epsilon \tag{2.28}$$

In the original Hey-Orme model paper, they considered pairwise-choice problems. It is relatively easy to understand this model in that setting. That is, subjects makes mistakes when measuring the vauation of the two lotteries and take the better one from the mistaken valuation. They assume the error is a so called white noise, which means a normally distribution with mean to be equal to 0 and and with a constant standard error. Later Hey (1995)

TABLE 2.6: Portfolio Choice with Stochastic Valuation

	U ($U^* = U_j$)	$\hat{U} = U^* + \epsilon$ ($\hat{U}^* = U_k$)
$\mathbf{C}_1$	U_1	$U_1 + \epsilon_1$
$\vdots$	$\vdots$	$\vdots$
$\mathbf{C}^* = \mathbf{C}_h$	$U^* = U_h$	$U_h + \epsilon_h$
$\vdots$	$\vdots$	$\vdots$
$\hat{\mathbf{C}}^* = \mathbf{C}_k$	U_j	$U_k + \epsilon_1$
$\vdots$	$\vdots$	$\vdots$
$\mathbf{C}_{5151}$	U_{5151}	$U_{5151} + \epsilon_{5151}$

generalised the model by assuming the error to be heteroskedastic, that is related to the each problem itself. This seems easy to understand. Assume an individual is asked to solve two problems. In the first one he could earn a maximum of £100 and minimum of £90 In the second one he could earn £1000 and could lose £1000. Then it is not hard to imagine that the he would try harder to make less mistake in the second one. So the error for the second problem is likely to be smaller. But for time being, let us explore the possibility of applying this model in our setting. Consider the discrete portfolio choice.[15] Remember that there are 5151 possible portfolio choices in $\mathbb{C}_j,\ j \in \mathbb{N}^+_{5151}$. As shown in Table 2.6 on page 96, for each portfolio choice $\mathbf{C}_j$, there is a corresponding utility denoted as U_j. But subjects evaluates U_j as $\hat{U}_j$ by mistake, here $\hat{U} = U + \epsilon$. If subjects make no errors, they would choose $\mathbf{C}_j$ as it produces the maximised utility. But since they make mistakes when evaluating prospects, they will choose $\mathbf{C}_k$ instead, as $\hat{U}_j$ is the maximised noised valuation.

[15]When applying the Hey-Orme story, I use the discrete set-up. Because we want to assume the error is heteroscedastic. So we need to find the magnititude of the difference between the best case and worst case for each problem. That is, among all 5151 possible choices, we need calculate the minimum utility and maxmimum utility. In the discrete optimisation routine it is easy to do so as we calculate the utilities for all choices. Before we just pick out the best one. Now we can just add another line to identify the minimum one. But in the the continuous optimisation routine, we have used the first order condition to help find the optimal allocation. We do not see a efficient way to find the allocation which minimises the utility function, which are all concave.

Suppose we can assume $\epsilon_j \sim N(U_j, \sigma^2)$ and for portfolio choice its ϵ_j are independent from each other. We assume subjects have constant variance for all problems. Notice here $\hat{U}$ can be more than U^*. Blavatskyy (2007) suggests that truncating the error such that the stochastic utility is inside the area of the minimum utility and maximum utility. For now let us just keep it simple. We think stochastic utility model would be really suitable for specifying portfolio choice errors. But the problem is, we do not have clear idea to construct the likelihood function for a observed portfolio choice $\hat{\mathbf{C}}$. This could be future work.

2.2.4 Random Preference Model

Random Preference theory (Loomes and Sugden 1995) might be a good way to specify the errors in the sense that it captures the correlation of the two allocations. Notice that Random Preference theory is very different from what we have done before where we assumed an *implementation* error. Instead of assuming the actual implemented choice are stochastic, they assume that preference function does not have determistic parameters but ones that are stochastic. In our experiment, if we fit the MEU model, we have the set of parameter $\mathbf{S} = [r, \underline{p}_1, \underline{p}_2, \underline{p}_3]$, which are the risk parameter and the three lower bounds on probabilities. We assume *r* is stochastic while $\underline{p}$ are non-stochastic. [16] Assuming the element $\hat{r}$ for each decision choice is randomly generated from a normal distribution $\mathcal{N}(\bar{r}, \sigma)$, each set of observations of choice $\hat{\mathbf{C}}_j$ can be rationalised as

$$\mathbf{S}_j = [\hat{r}_j, \underline{p}_1, \underline{p}_2, \underline{p}_3] = \mathcal{U}^{-1}(\hat{\mathbf{C}}) \tag{2.29}$$

here $\mathcal{U}^{-1}$ means the inverse function of $\mathcal{U}$ as defined in equation (2.8). For 65 problems, we calculate the mean and standard deviation of *r*, which we

[16] If we also assume the lower bounds are stochastic, then we have to assume all three are as they are indifferent from each other. But we have $0 \leq \underline{p}1$ and $0 \leq \underline{p}_1 + \underline{p}_2 + \underline{p}_3 \leq 1$. It would be difficult to implement as a two-allocation choice make the stochastic specification very complicated.

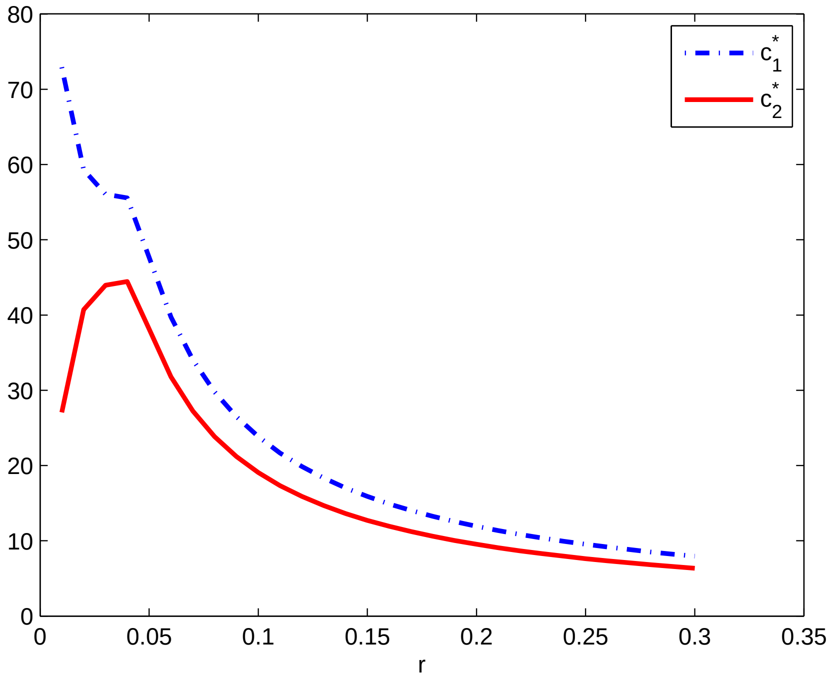


FIGURE 2.11: Optimal Allocations for different risk parameter

denote by $\bar{r}$ and σ_r. The sum of the log-likelihoods function is written as

$$L = \sum_{j}^{65} log(f(\hat{r}_j, \bar{r}, \sigma)) \tag{2.30}$$

here f is the probability density function of normal distribution

$$f(\hat{r}_j, \bar{r}, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\hat{r}-\bar{r})^2}{2\sigma^2}}$$

As we assume $r \in \mathcal{N}(\bar{r},\ \sigma^2)$, there is no constraint on it so the specification is less messy. But here is the problem. In order to use this model, the optimisation function $\mathcal{U}$ needs to be invertible, that is there always exists a risk parameter which can rationalise any given observed optimal choice. Unfortunately for the optimisation function, which is based on either MEU, EU or MV preference theory, cannot do this. Figure 2.11 on Page 98 shows an example of the optimal allocation of MEU preference given different risk parameters. We can see that in general, any point between the the lines cannot be rationalised by the any risk parameters.

2.2.5 Conclusion

In this chapter, we have discussed stochastic specifications for portfolio problems. There are two major possibilities of the stochastic specification. One is to assume errors come from implementation. Another is to assume errors come from evaluating the prospects. We have given a formal analysis for the first possibility, both in the discrete and the continuous cae. We have discussed the possibilities for the second possibility, and pointed out the their advantages and difficulties of applications.

2.3 Experimental Analysis and Results

2.3.1 Introduction

In the experiment, we had 77 subjects. Each subject made 65 portfolio choices $\mathbf{C}$, with each choice containing two allocations c_1 and c_2. We use four preference theories, α-Maxmin Expected Uitlity (α-MEU), Maxmin Expected Utility (MEU), Subjective Expected Utility (SEU), Mean-Variance, to explain the data. We also use Safety-First (SF), which is a heuristic rule to explain the data. We use two different error stories, which are the Biased Beta-Binomial (BBB) specification and the Biased Beta (BB) specification. We then combine different preference theories, including SF (which we describe below), combined with the two error stories to fit the data using Maximum Likelihood Estimation (MLE). BBB specification assumes the subjects are calculating in integers and make errors in integers, and BB specification assumes subjects are calculating continuously and also make continuous errors. Table 2.7 shows the preferences theories we have estimated with BBB or BB respectively. The reason why we do not combine α-MEU and SF with BB is that it is very difficult to calculate the continuous optimal allocation. In the experiment, as we have imposed the No-short-selling constraint, we have no explicit solution for any objective function for any preference theories other than EU. We have developed a reliable and precise algorithm for finding the MEU optimal allocation building on the EU solution. We could develop a

Table 2.7: Estimation Methods

	BBB(discrete)	BB (continuous)
Preference Theory	α-MEU, MEU, EU, MV, SF	MEU, EU

similar one for α-MEU and SF but it would be very difficult. We also do not trust the purely numerical method, that is optimising the objective function using the built-in optimisation routine in most mathematical software, because of the fact that the objectives functions for α-MEU and SF are kinked. So we only use the Grid search method to find the integer optimal allocation for α-MEU and SF. And so they are only combined with the discrete specification. The general idea for grid search is that we calculate the utility for all possible allowable portfolio choices which can be implemented in the experiment, which are 5151 sets, and pick out the one with the maximum utility.

This section is organised as follows. In Section 2.3.2 I introduce the SF rule. In Section 2.3.3 I demonstrate the algorithms for finding the optimal allocation for α-MEU, ME, SEU,MV and SF using the Grid Search Method. In Section 2.3.5, I compare the results of different preference theories and error models using statistical tests. In Section 2.3.6 I present other interesting findings in the estimation results. Section 2.3.7 concludes.

2.3.2 Heuristic Rule - Safety-First (SF)

At the end of each session of the experiment, we gave a short post-experimental questionnaire to the subjects. Many of the subjects claimed that they followed a two-step simple heuristic rule: first they make sure that the payoff in each state is above a threshold; then they look for an allocation that maximises the payoff in the Green state. The green balls are half of the total number in the Bingo Blower, so subjects are almost certain that the the probability for a Green ball to be chosen is the largest, though they do not know the exact probability. We think this rule may come from subjects' reaction to ambiguity. For the first step, we think it suggests ambiguity aversion. For the second step, it does not necessary to be seen as ambiguity averse. Next I present the formal construction of the SF rule.

The maximisation function is easy to explain with matrices. Remind ourselves that $\mathbf{C} = [c_1,\ c_2]$ is a 1*2 matrix, which represents the the two allocations. And the *return table* $\mathbf{D}$ is a 2 by 3 matrix, with each row giving the return for the assets in the 3 states. In the experiment, we used colour Pink, Green and Blue to present the three states. The colour Green corresponds to state 2, so its return is the second column in $\mathbf{D}$. The payoff for a portfolio in the three states is $\mathbf{W} = \mathbf{CD}$ which is a 1*3 vector. So the payoff of a portfolio for in state 2 is the second column in $\mathbf{W}$. And that is the number that subjects would like to maximise, with the constraints that each element in $\mathbf{W}$ is bigger than $\underline{w}$, which is the minimum payoff that subjects want to keep.

$$C^* = max\{\mathbf{W}(2)\} \quad s.t.\ \ W(i) \geq \underline{w},\ i = 1, 2, 3 \tag{2.31}$$

2.3.3 Grid Search Method of Finding Optimal Allocation

The general idea of the Grid Search Method is to calculate the utility of a specific preferences theory for all possible portfolio choices and then choose the one with the highest utility. It is reliable and efficient, when the size of the choices is not too large. In our experiment, we have 5151 possible choices, this method has proved to be more efficient than using the algorithms for finding continuous optima. The former takes just 5% of the time of the latter. We suspect subjects may also arrive at their portfolio choice also based on the integers. To implement this method, first we work out the set of portfolio choices $\mathbb{C}$. The procedure is first let $c_1 = 0$, then we have $c_2 \in \mathbb{N}_{100}$, which is any integer number from 0 to 100. We then repeat this step with $c_1 = 1$. Generally when $c_1 = i$, then $c_2 \in \mathbb{N}_{100-i}$. Then we save all choices in a 5151*2 matrix file under the name $\mathbb{C}$. Then we calculate the utility for $\forall \mathbf{C} \in \mathbb{C}$.[17] As all possible utility values are there, it is easy to find the the maximised one and the corresponding optimal allocation. We next give the details for SF Optimal Allocation and state briefly the algorithms for the remaining theories as they are similar to SF.

SF Optimal Allocation

Algorithm 5: Safety-First Optimal Choice

Input : $\underline{w}$, $\mathbb{C}$, $\mathbf{D}$

// $\underline{w}$: lower bound for payoff

// $\mathbb{C}$: (5151*2 matrix) The set of all possible portfolio choice in integers

// $\mathbf{D}$: return table

Output: $\mathbf{C}^*$: Optimal Allocation

// calculate the payoff of 3 states for all allocations $\mathbf{W}$: 5151*3 matrix

$\mathbf{W} = \mathbb{C}\mathbf{D}$;

// check each row of $\mathbf{W}$ if there is any number less than the lower bound. If so, assign the second element (the payoff in green state) of that row as minus Infinity. Doing so we eliminate this portfolio choice as it violates the lower bound condition

$\mathbf{W}(any(\mathbf{W} < \underline{w}, 2), 2) = -Inf$;

// in the updated $\mathbb{W}$ find the highest number in second column and read its index maxnum. We do not need to know the highest number itself, so we use $\sim$ means we do not need that value

$[\sim, maxnum] = max(W(:, 2))$;

// use the index to find the optimal allocation in $\mathbb{C}$

$\mathbf{C}^*$=$\mathbb{C}$(maxnum,:)

The objective function for SF preferences is defined in function (2.31). The general routine is as follows. First, for each portfolio choice $\mathbf{C}_i \in \mathbb{C}, i \in$

[17]We could also write a loop for doing this instead of work out the set of portfolio choices first. But in Matlab, it is always much efficient to use matrix calculation than writing a loop is Matlab is designed for matrix calculation.

$\mathbb{N}^{+}_{5151}$, we calculate the payoff for each 3 states. If for any $\mathbb{C}$, there is a state in which the payoff is less than the subjects' lower bound $\underline{w}$, then we eliminate its possibility to be the optimal allocation. Then in the remaining set, we look for the optimal allocation for which the payoff in state 2 is the highest among all remaining portfolio choices. Notice here for calculating the optimal allocation, the subject's only parameter is the lower bound $\underline{w}$. Algorithm 5 on Page 102 shows the pseudo code for finding the optimal allocation for SF rule in Matlab.

Notice SF is a rather simple heuristic rule. It differs radically from the rule followed by a expected payoff maximiser. First, it assumes that the individual has a required minimum portfolio payoff, no matter which state occurs. Second individuals do not care about the probability weighted expected payoff. They only maximise the payoff in the state that is most likely to happen. We use the BBB stochastic specification because we solved the maximisation problem in the discrete setting. So the parameters estimated are

$$S^{SF} = S^{MV} = [\underline{w}, s_1, b_1, s_1, s_2]$$

where $\underline{w}$ is the minimum payoff *s* and *b* are the precision and bias parameters for the two allocations. Notice that in both the BBB and BB specification, we have 4 parameters s_1, b_1, s_1, s_2 to be estimated no matter what preference theory we are assuming. We call these four parameters as error parameters and other parameters, which are related to individuals' preferences, as preference parameters. In the case of SF, there is only one preference parameter, $\underline{w}$.

EU, MEU, α-MEU

Grid Search in EU is straightforward. We calculate the expected utility $\forall \mathbf{C} \in \mathbb{C}$ given subjects' parameter set, the risk parameter r, and the subjective probability vector $\mathbf{P} = [p_1, p_2, p_3]'$. Then we locate the optimal allocation as that with the maximised utility.

For MEU, we have an extra step compared with EU. Given the risk parameter, and subjective probability lower bound vector $\underline{\mathbf{P}} = [\underline{p}_1, \underline{p}_2, \underline{p}_3]'$, we calculate three expected utilities $[U_1, U_2, U_3]$ for each allocation based on three probability sets [18] and choose the minimum one which we call it minEU. Then we repeat the step similar to EU, locating the optimal allocation with the maximised minEU.

For α-MEU, we need one extra subject's parameter α, which measures the subject's ambiguity aversion level. The difference from calculating utility from MEU is that we take into acccount both minEU and maxEU, which is the maximum in $[U_1, U_2, U_3]$ as follows

$$\mathcal{U} = \alpha minEU + (1 - \alpha) maxEU$$

The details are in Algorithm 4 on Page 59.

2.3.4 Estimation Method

The general procedure is as follows. We specify the parameter space $\mathbb{S}$. Then we search $\mathbf{S} \in \mathbb{S}$ and use the preference parameters of it to calculate its theoretical optimal allocations. Then we use the error parameters of it, along with the theoretical optimal allocations using the appropriate error specification to calculate the sum of the log-likelihoods. The optimal $\mathbf{S}^*$ is the one maximises the sum of the log-likelihoods. Table 2.8 summarises the preference parameters estimated for each preference theory. Notice that for EU and MV, though it seems we have to estimate p_1, p_2, p_3, there are only two free parameters as one of them is 1 minus the other two. Next I use MEU (BB) as an example to illustrate the method. For MEU, the estimated parameters is

$$S^{MEU} = [\underline{p}_1, \underline{p}_1, \underline{p}_1, r, s_1, b_1, s_1, b_2]$$

[18] They are $\mathbf{P}_1 = [1 - \underline{p}_2 - \underline{p}_3, \underline{p}_2, \underline{p}_3]'$, $\mathbf{P}_2 = [\underline{p}_1, 1 - \underline{p}_1 - \underline{p}_3, \underline{p}_3]'$, $\mathbf{P}_3 = [\underline{p}_1, \underline{p}_1, 1 - \underline{p}_2 - \underline{p}_3]'$.

Algorithm 6: Estimation for MEU preferences

Input : $\mathbf{D}^n$, $\mathbf{C}^n$ where $n \in [1, N]$,e
Output: $\mathbf{S}^* = [r, \underline{p_1}, \underline{p_2}, \underline{p_3}, s_1, b_1, s_2, b_2], L^*$

special treatment of the first line;
for $i = 1$ **to** S **do**
  choose a start point $\mathbf{S}_1 \in \mathbb{S}$;
  for $n = 1$ **to** N **do**
    $\mathbf{C}^* = f^M(r, \underline{p_1}, \underline{p_2}, \underline{p_3}, \mathbf{D}^n)$;
    $x_1^* = c_1^*/e$;
    $x_2^* = c_1^*/e$;
    $\alpha_1 = [\frac{b_i}{2} + (1 - b_i)x_1^*](s - 1)$;
    $\beta_1 = [1 - \frac{b_i}{2} - (1 - b_i)x_1^*](s - 1)$;
    $\alpha_2 = [\frac{b_i}{2} + (1 - b_i)x_2^*](s - 1)$;
    $\beta_2 = [1 - \frac{b_i}{2} - (1 - b_i)x_2^*](s - 1)$;
    $[c_1, c_2] = \mathbf{C}^n$;
    $l_{x_1} = f^b(\frac{c_1 - 0.5}{e}, \alpha_1, \beta_1) - f^b(\frac{c_1 - 0.5}{e}, \alpha_1, \beta_1)$;
    $l_{x_2} = f^b(\frac{c_2 - 0.5}{e}, \alpha_2, \beta_2) - f^b(\frac{c_2 - 0.5}{e}, \alpha_2, \beta_2)$;
    $l^n = l_{x_1} l_{x_2}$;
  end
  $L_i = \sum^N log(l_n)$;
  choose the next point that $\mathbf{S}_{i+1} \in \mathbb{S}$;
end
Find $L_i^* = max\{\mathbf{L}\}$;
$[r, \underline{p_1}, \underline{p_2}, \underline{p_3}, s_1, b_1, s_2, b_2] = S_i$;

TABLE 2.8: Estimated Preference Parameters

	Parameters	Total number
EU	p_1, p_2, r	3
MV	p_1, p_2, r	3
MEU	$\underline{p}_1, \underline{p}_2, \underline{p}_3, r$	4
α-MEU	$\underline{p}_1, \underline{p}_2, \underline{p}_3, r, \alpha$	5
SF	$\underline{w}$	1

For the lower bounds on the probabilities we have

$$\underline{p_1} + \underline{p_2} + \underline{p_3} \leq 1$$

For the risk parameter, we let $r > 0$ as we assume risk aversion. For the bias parameter we have $0 < b_1 < 1$, $0 < b_2 < 1$. And for the precision we have $s_1 > 1$ and $s_2 > 1$. The parameter space $\mathbb{S}$ is the set that satisfies all these constraints. For any $\mathbf{S}$, we calculate the theoretical optimal $\mathbf{C}^*$ first. By denoting f^M as Algorithm 3 on Page 53, we can calculate the MEU optimal allocation for problem $i \in N_{65}^+$.

$$\mathbf{C}^* = f^M(r, \underline{p_1}, \underline{p_2}, \underline{p_3}, \mathbf{D}, e) \tag{2.32}$$

here $\mathbf{D}$ is the *return table* and *e* is the cash endowment for the problem *i*. We repeat this step for $\forall i \in \mathbb{N}_{65}^+$. Then we read the actual portfolio choice of each problem for one subject and calculate the sum of the log-likelihoods use equation (2.19). We then search for $\mathbb{S}^*$ that maximises the sum of the log-likelihoods. Algorithm 6 on Page 105 summarises the method.

2.3.5 Preference Theory Comparison Results

Within one error specification, we compare the performance of nested preferences theories using the Likelihood Ratio Test (LRT), which is a statistical method for testing two nested models. Denote L_0 as the sum of the log-likelihoods of the null model and L_1 as the sum of log-likelihoods of the alternative model, we have the test statistic

$$T = -2((L_1) - (L_0))$$

The test statistic has a Chi-square distribution with degrees of freedom equal to the difference in the number of parameters in these two models. We have EU nested in MEU, and MEU nested in α-MEU. MEU has one more parameter than EU and α-MEU has one more parameter than MEU and α-MEU has two more parameter than EU. So the corresponding degrees of freedom are 1, 1 and 2. We have 77 subjects, so we compare each subject's

TABLE 2.9: Likelihood Ratio Test Result comparing EU, MEU and α-MEU

A : Five percentage significance level

	BBB	BB
MEU *v* EU	21 (27%)	14 (18%)
α-MEU *v* MEU	10 (12%)	$\sim$
α-MEU *v* EU	17 (22%)	$\sim$
α-MEU+MEU *v* EU	23 (32%)	$\sim$

B : One percentage significance level

	BBB	BB
MEU *v* EU	19 (25%)	13 (17%)
α-MEU *v* MEU	9 (12%)	$\sim$
α-MEU *v* EU	14 (18%)	$\sim$
α-MEU+MEU *v* EU	20 (26%)	$\sim$

sum of log-likelihoods and calculate the test statistic. For a 5% significance level test, the *p*-values for Chi square test at 1 and 2 degree of freedom are 3.84 and 5.99. For a 1% significance level test, the *p*-values for Chi square test at 1 and 2 degree of freedom are 6.63 and 9.21. In the BB specification, we have not fitted α-MEU using the data, so we only test MEU against EU. The results are summarised in Table 2.9 on Page 107. In each entry, the first number is the number of subjects for who the fit is significantly better, while the second number is its percentage out of the 77 subjects. It seems that neither MEU nor α-MEU are particularly better than EU if we consider the percentages of the subjects. But as we analyse the data individually, we may want to interpret that result as that most subjects seems to have EU preferences and a small proportion of subjects are sophisticated enough to have MEU and α-MEU preferences. At the five percentage significance level, 15 subjects are significantly better both in MEU and α-MEU, which means there are $21 + 17 - 15 = 23$ (32%) in total subjects are significantly better explained by ambiguity preference. This number at the one percentage level is 20 (26%). It means one third of the total subjects seem to have multiple priors.

We compare the performance of non-nested theories using the Clarke Test

(Clarke 2007) [19]. The Clarke test is a distribution-free test used for comparing non-nested models. For example, if we compare the EU with SF, the null hypothesis is

$$H_0 : P(L_1 - L_2 > 0) = 0.5$$

here L_1 denotes the individual log-likelihood (the log-likelihood of each 65 problems, which is calculated by the estimated parameters) of EU and L_2 denotes the log-likelihood of SF. The test statistic is the

$$T = \sum_{i}^{65} I_i(L_1 - L_2) \tag{2.33}$$

here

$$I_i = \begin{cases} 1, & L_1 - L_2 > 0 \\ 0, & L_1 - L_2 \leq 0 \end{cases}$$

The test statistic is $T \sim Bin(65,\ 0.5)$. Under a 5% significance level, the condition (for an upper tail test) of rejecting the null hypothesis is $T \geq 40$. For example, if for one subject, $T = 42 > 40$, then we can reject the null hypothesis. We say EU is significantly 'better" than SF at 5% significance level. Table 2.10 on Page 109 summarise the tests results. Notice we have not corrected the log-likelihood using the number of parameters. From Table 2.10, EU, MEU and α-MEU seem to fit better than SF for the majority of the subjects, though they do have more parameters than SF (See Table 2.8 on Page 105). EU and MEU seems to fit better than MV. It is not surprising to get this result considering we only have three states for the ambiguous assets. d'Acremont and Bossaerts (2008) suggest Mean Variance preferences are more likely to be adopted by the human brain when the number of states is increasing.

[19] In the paper, Clarke suggests correcting the test statistic by adding a factor of the number of parameters of the two competing models. We think it is vicious. If we do so, the individual corrected log-likelihoods of SF are always bigger than those of the other theories as it has only 1 preference parameter.

TABLE 2.10: Clarke Test Result

Comparison between SF and EU, MEU α-MEU

EU *v* SF		MEU *v* SF		α-MEU *v* SF	
EU $\succ$ SF	SF $\succ$ EU	MEU $\succ$ SF	SF $\succ$ MEU	α-MEU$\succ$SF	SF $\succ\alpha$-MEU
55%	8%	56%	10%	58%	4%

Comparison between MV and EU, MEU, α-MEU

EU *v* MV		MEU*v* MV		α-MEU *v* MV	
EU $\succ$ MV	MV $\succ$ EU	MEU $\succ$ MV	MV$\succ$ MEU	α-MEU$\succ$MV	MV$\succ\alpha$-MEU
42%	19%	49%	7%	$\sim$	$\sim$

2.3.6 Estimation Results concerning beliefs about probabilities

Even though the subjects do not know the exact number of balls of each colour, the results suggest that they *on average* do a good job of guessing the true values but there is considerable variation. We put 15 pink balls, 30 green balls and 15 blue balls into the Bingo Blower, so the corresponding probabilities are 0.25, 0.5 and 0.25. Figure B.1 on page 116 shows the histogram of the three estimated probabilities in the EU (BBB) estimation. On average, the estimated probabilities for each colour in the same order are 0.28, 0.49 and 0.23. Figure B.6 on page 121 shows the estimation results for the lower bounds of probabilities in the three states in the MEU(BBB) estimation. Figure B.7 on page 122 shows the estimation results for the sum of the lower bounds of probabilities in the MEU (BBB) estimation. Figure B.8 on page 123 shows the estimation results for probabilities in three states in the MV (BB). The average estimated probabilities for the three colours are 0.26 0.50 and 0.24, which are even better than EU (BB) in terms of the closeness to the true probabilities. Table 2.11 A on Page 110 summarises the estimation results on the (lower bounds of the) probabilities in the BBB specification. In the BB specification, the results are quite close as in the BBB specification except the there is a divergence in EU preferences. Table 2.11 B on Page 110 summarise the estimation results on the (lower bounds of the) probabilities in the BB specification. It is interesting to find that even

Table 2.11: Estimation Results of Priors

A : BBB specification

	Pink(0.25)	Green(0.5)	Blue(0.25)	sum
EU	0.29	0.52	0.19	1
MEU	0.24	0.46	0.18	0.88
α-MEU	0.22	0.45	0.10	0.77
MV	0.26	0.50	0.24	1

B : BB specification

	Pink(0.25)	Green(0.5)	Blue(0.25)	sum
EU	0.28	0.49	0.23	1
MEU	0.25	0.47	0.17	0.89
MV	0.26	0.50	0.24	1

though there is a big disparity among the subjects of the estimation of the lower bounds, on average they guess the probabilities quite well. This is also a big disparity among the subjects concerning the sum of the three lower bounds as shown in Figure $B.4$ on Page 119 and $B.7$ on Page 122.

2.3.7 Conclusion

In this section, we fit different preference theories to the experiment data. The statistics suggests that multiple priors preferences (MEU and α-MEU) fit better than EU for one third of the subjects. Subjects have heterogeneous beliefs about the (lower bounds of the) probabilities but on average do a job at guessing the probabilities. EU and MEU fit significantly better than MV in the majority of subjects. EU, MEU, α-MEU seems to fit better than SF if we do not consider the number of parameters.

Appendix

A Experimental Instruction

There are the Instructions given to the subjects in the experiment.

Instructions

Preamble

Welcome to this experiment. Thank you for coming. These instructions are to help you to understand what the experiment is about and what you are being asked to do during it. The experiment gives you the chance to earn money, which will be paid to you in cash after you have completed the experiment. The payments described below are *in addition* to a participation fee of £2.50 that you will be paid independently of your answers.

The Bingo Blower

At the back of this laboratory you will see a Bingo Blower. You can inspect the Blower at any time that you want. At the front you will see the projections of the Bingo Blower. In it, as you will see, there are balls of three different colours – Pink, Green and Blue – which are being blown around inside the Blower. After you have responded to the various problems in the experiment, you will go to the Bingo Blower and you will activate a mechanism which will expel **one** ball from the Bingo Blower. The colour of this one ball, combined with your answer to a randomly chosen one of the problems during the experiment, will determine your payment for taking part in this experiment – as we describe below.

Payment

At the end of the experiment, one of the 65 problems will be picked at random by your picking at random a lottery ticket from a set of such tickets numbered from 1 to 65. We will then look at what you did on that problem (your allocation to Asset 1, your allocation to Asset 2 and your experimental money remaining). You will then go to the Bingo Blower and you will eject one ball. The colour of that ball will determine your payment, because it will determine the payments on the two assets. You will be paid in cash and then we will ask you to sign a receipt and to confirm that you participated voluntarily in this experiment. Then you will be free to go.

The Problems

This experiment is a simulated investment game. The experiment consists of 65 different problems. In each problem, you will be endowed with **100** units of experimental money, and you can use it to buy two assets. The price of each asset is **1** unit of experimental money. You do not need to spend all of your endowment on the assets, and you cannot borrow any experimental money to buy the assets. Any experimental money not allocated to the two assets will remain as experimental money. We call an allocation of your 100-unit endowment of experimental money between the two assets and experimental money a *portfolio decision*. The computer software will tell you for any given portfolio decision the payoff you would get for each of the three colours.

There are three different possible states (Pink, Green, and Blue) that could happen and each of the states gives a payoff to each asset. For each problem, the possible payoffs for the two assets are different. You will get this information in a *payoff table* when you start each problem. An example is given here:

	Pink	Green	Blue
Asset 1	1.2	0.6	1.6
Asset 2	0.5	1.3	1.4

The colour will be decided by one draw from the Bingo Blower – as we have described. If the payoff table for the randomly selected problem is that above, then for every unit of Asset 1 that you bought in that problem you will be given 1.2 units of experimental money if the ball drawn is Pink, 0.6 units if the ball drawn is Green and 1.6 units if the ball drawn is Blue; similarly for every unit of Asset 2 that you bought in that problem you will be given 0.5 units of experimental money if the ball drawn is Pink, 1.3 units if the ball drawn is Green and 1.4 units if the ball drawn is Blue. As we have already said any units of experimental money that you kept in that problem will remain as experimental money.

Continuing with the example above, if your portfolio decision in this problem was to buy 40 units of Asset 1 and 30 units of Asset 2 (thus keeping 30 units of experimental money as experimental money) then you would end up with 93 (= 40*1.2 + 30*0.5 + 30) units of experimental money if a Pink ball is drawn, 93 (= 40*0.6 + 30*1.3 + 30) if a Green ball is drawn and 136 (= 40*1.6 + 30*1.4 + 30) if an Blue ball is drawn:

Pink	Green	Blue
93	93	136

If instead you had decided to buy 60 units of Asset 1 and 40 units of Asset 2 (thus having no units of experimental money left) then you would end up with 92 (= 60*1.2 + 40*0.5 + 0) units of experimental money if a Pink ball is drawn, 88 (= 60*0.6 + 40*1.3 + 0) if a Green ball is drawn and 152 (= 60*1.6 + 40*1.4 + 0) if an Blue ball is drawn. Notice that you do not need to calculate these payoffs as the computer will tell you the payoffs for each of the three states. The experimental money that you end up with will be converted into real money (which you will be paid in addition to the £2.50 show-up fee) at the rate of 12 units of experimental money equal to £1 in real money.

How you express your desired purchases

When you start each problem you will see a triangle with red sides on the screen. When you move the cursor into the screen you will see that each particular point in the triangle represents an allocation of your 100-unit endowment of experimental money to the two assets and to residual experimental money. This will be written alongside the triangle. The horizontal distance from the left hand side indicates the number of units of Asset 1; the vertical distance from the bottom indicates the number of units of Asset 2; the residual amount of experimental money is written beside the triangle. You can see a *Portfolio Table* on the screen that shows the amounts for two assets and experimental money when you move your cursor inside the triangle. For example, if you put the cursor at the bottom corner of the triangle (actually the mouse cursor is always starting at this point which represents your initial situation), this represents buying zero of both assets: the bottom-right hand corner represents spending all the endowment on Asset 1; the top-left hand corner represents spending all the endowment on Asset 2; and the middle of the triangle represents spending one-third of your endowment on Asset 1, one-third on Asset 2 and keeping one-third in the form of experimental money. Move the cursor around until you see your desired allocation. Then **double click** with your mouse to make this choice. You will be asked to confirm your decision, though you can change it if you wish by clicking on 'No'.

How Long the Experiment Will Last

Because it is in your interest to consider the problems carefully, we are imposing a *minimum* time of 30 seconds for your answer on any problem; you cannot go onto the next problem until these 30 seconds have elapsed. We are also imposing a *maximum* time of 180 seconds; if you do not answer and confirm you decision in these 180 seconds, we will assume that your allocation to each of the two assets is zero.

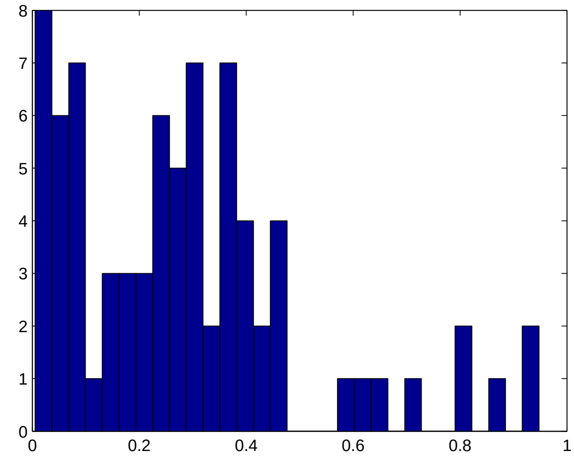
We estimate that the experiment will take at least 60 minutes of your time. You can take longer and it is clearly in your interests to be as careful as you can when you are answering the questions.

Enrica Carbone
Xueqi Dong
John Hey
May 2013

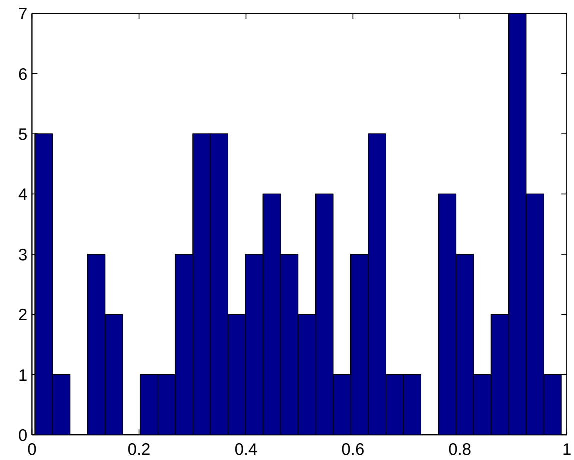
B Experimental Results

FIGURE B.1: Estimated probabilities for EU(BBB)

Pink ($p_1 = 0.25$, $\hat{p_1} = 0.29$)



Green ($p_2 = 0.5$, $\hat{p_3} = 0.52$)



Blue ($p_2 = 0.25$, $\hat{p_3} = 0.19$)

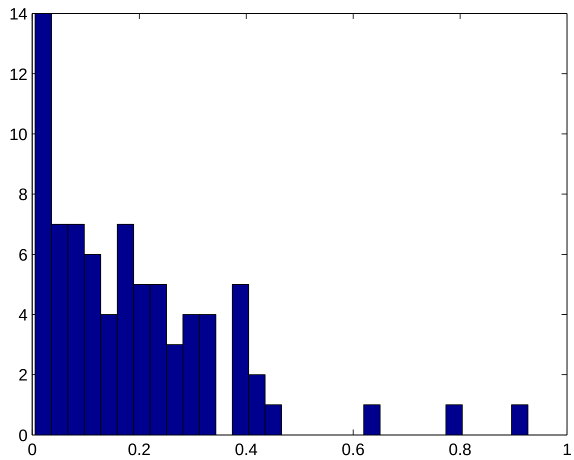
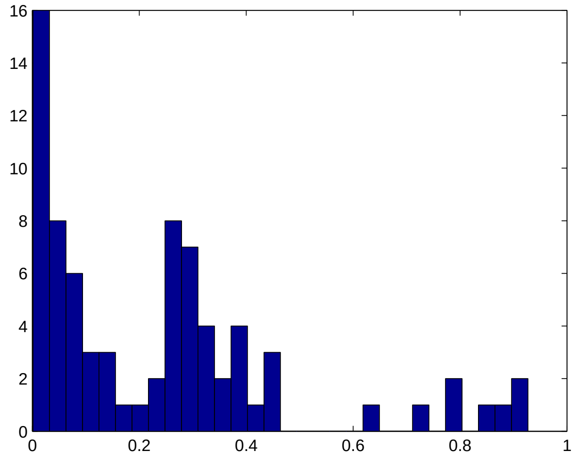
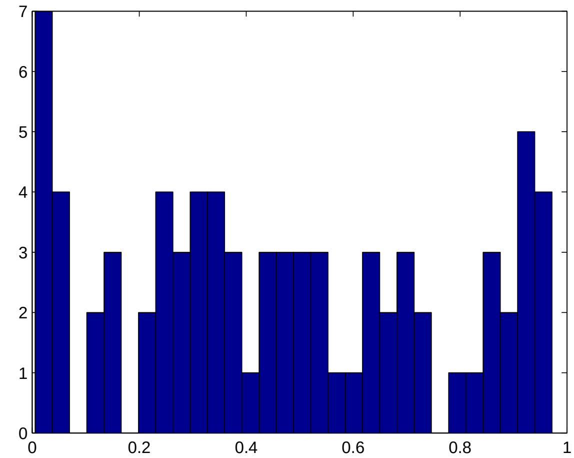


FIGURE B.2: Estimated lower bound of probabilities for MEU(BBB)

Pink ($p_1 = 0.25$)



Green ($p_2 = 0.5$)



Blue ($p_2 = 0.25$)

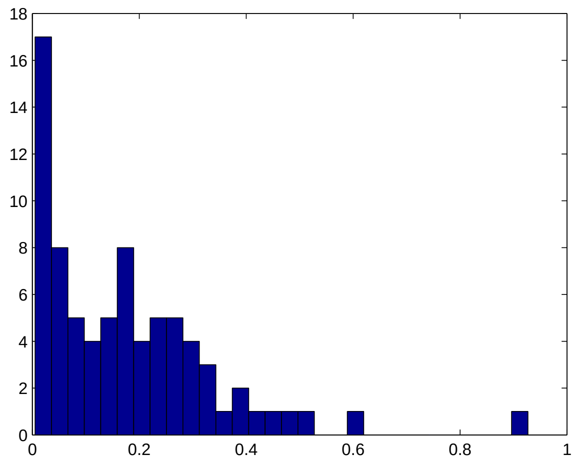
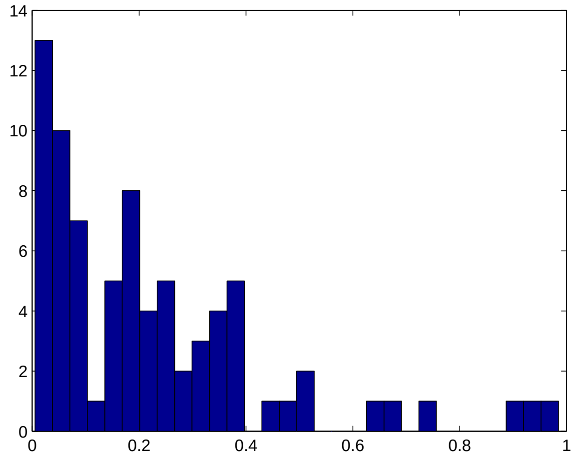
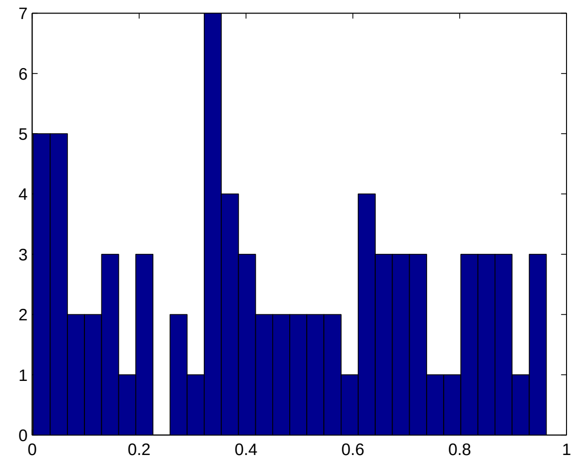


FIGURE B.3: Estimated lower bound of probabilities for α-MEU(BBB)

Pink ($p_1 = 0.25$)



Green ($p_2 = 0.5$)



Blue ($p_2 = 0.25$)

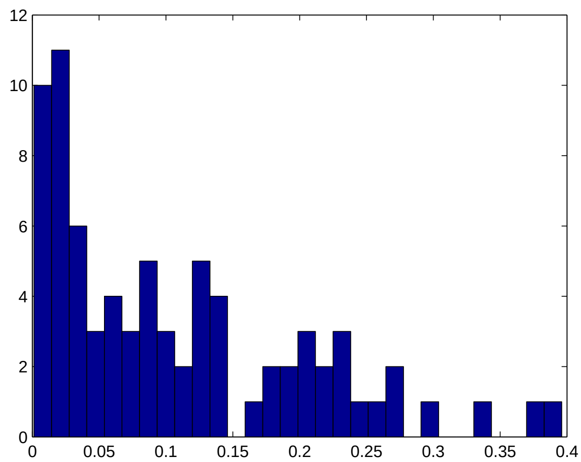
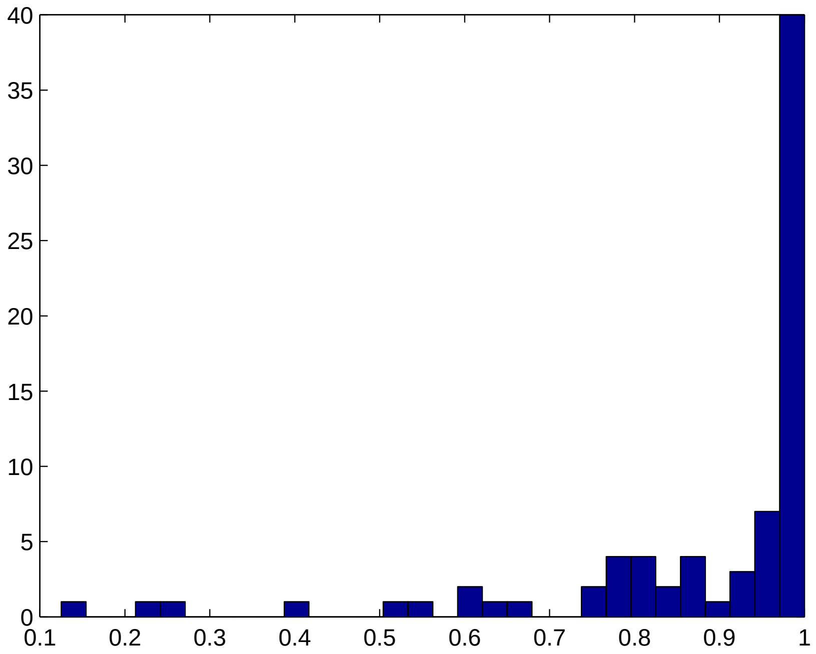
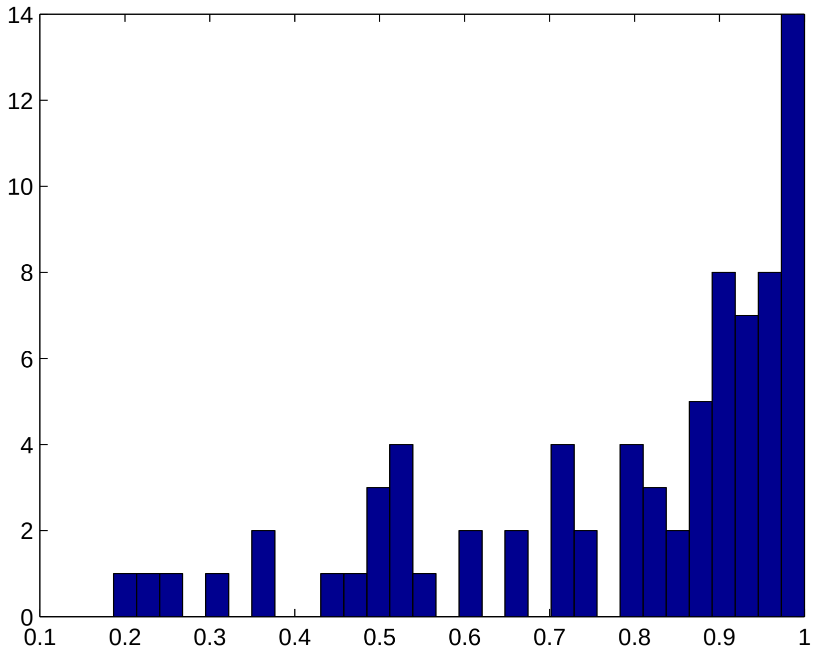


Figure B.4: Estimated sum of lower bound for MEU(BBB) and α-MEU(BBB)



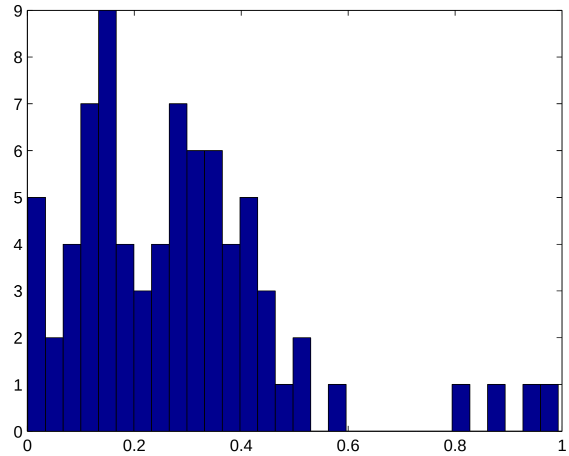
MEU



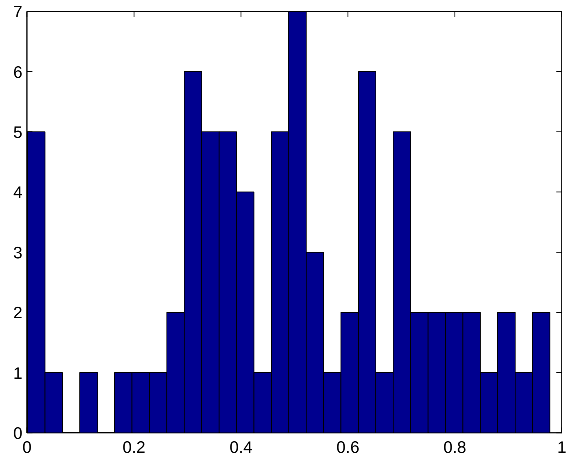
α-MEU

FIGURE B.5: Estimated probabilities for EU(BB)

Pink ($p_1 = 0.25$, $\hat{p_1} = 0.28$)



Green ($p_2 = 0.5$, $\hat{p_3} = 0.49$)



Blue ($p_2 = 0.25$, $\hat{p_3} = 0.23$)

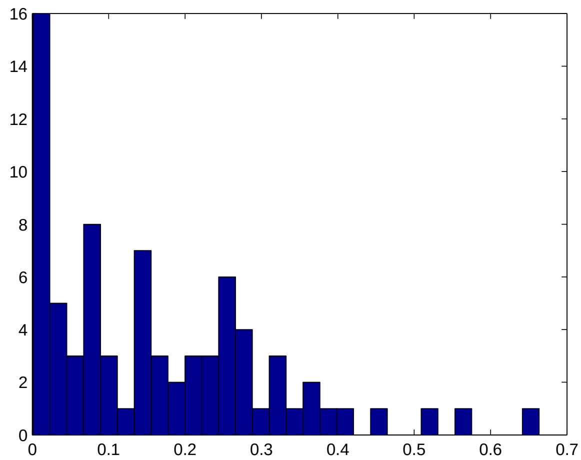
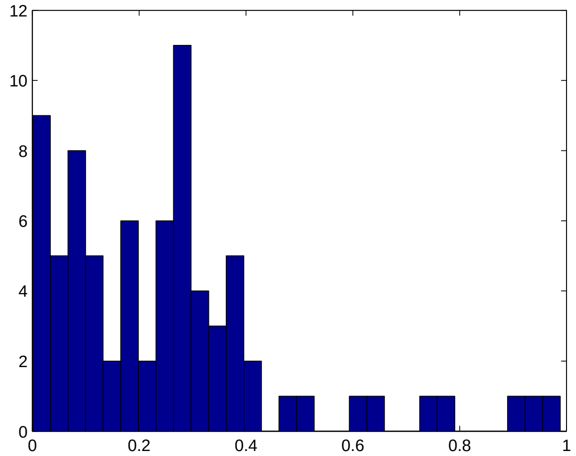
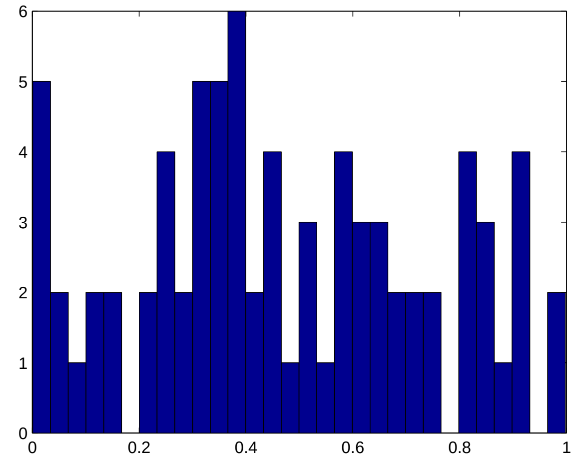


FIGURE B.6: Estimated lower bound of probabilities for MEU(BB)

Pink ($p_1 = 0.25$)



Green ($p_2 = 0.5$)



Blue ($p_2 = 0.25$)

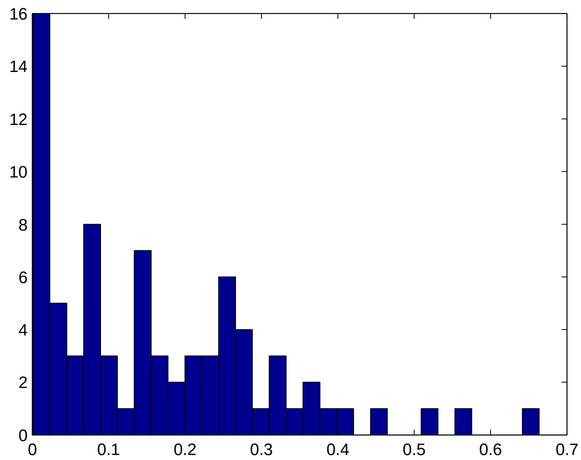


FIGURE B.7: Estimated sum of lower bound of probabilities for MEU(BB)

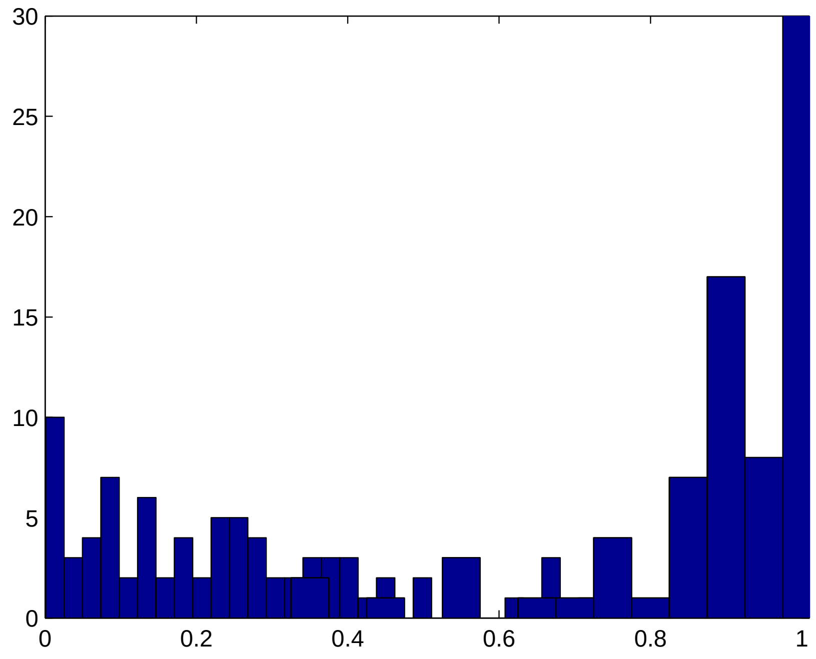
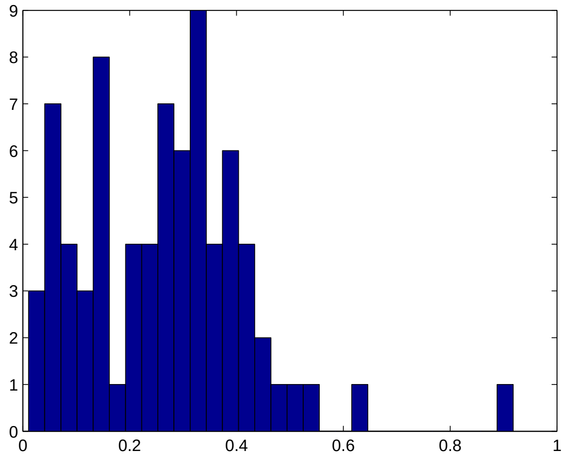
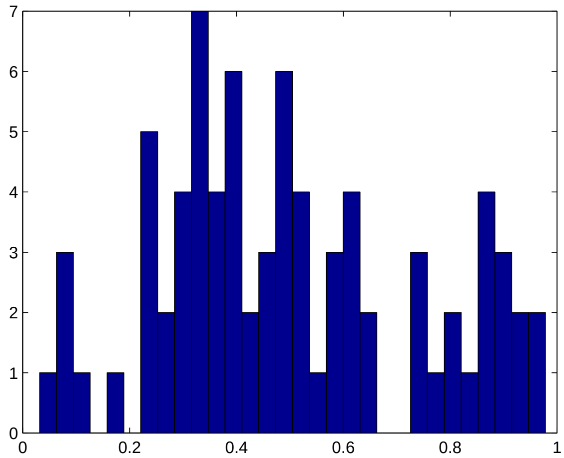


FIGURE B.8: Estimated lower bound of probabilities for MV(BB)

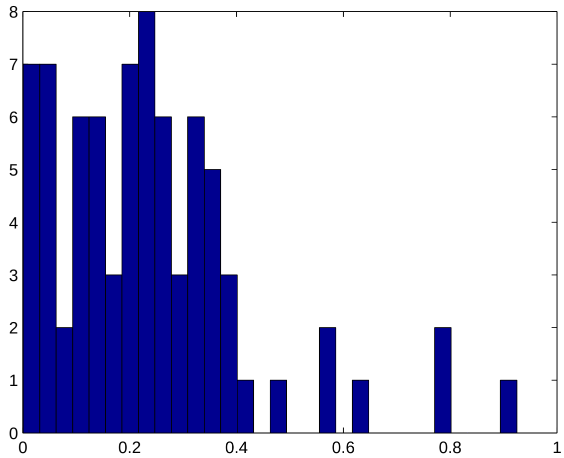
Pink ($p_1 = 0.25$, $\hat{p_1} = 0.26$)



Green ($p_2 = 0.5$, $\hat{p_1} = 0.50$)



Blue ($p_2 = 0.25$, $\hat{p_1} = 0.24$)



Chapter 3

Error Stories and the Estimation of Preference Functionals using 3-Way Allocation Data

[1]

3.1 Introduction

Experimentalists are increasingly using *allocation* problems to make inferences about subjects' preferences – the reason being that allocation problems appear more informative than other types of problems - such as pairwise choices, Holt-Laury price lists and the Becker-DeGroot-Marschak mechanism. At the same time some experimentalists are broadening the type of allocation problem, moving from allocations over just two events to allocations over more than two, again to get more information from experiments. Even with just two allocations, the issue of the error process is already interesting; going to allocations over more than two increases the interest as well as the

[1] This chapter is joint written with John Hey as a result of join research. It is an on going project.

complexity of the problem. This paper examines some of the various possibilities and solutions. It also carries out a simulation exercise to investigate the problems caused by using the wrong stochastic specification.

3.2 The decision Problem

Consider a 3-way allocation problem, in which subjects are asked, in a series of problems (one of which will be randomly selected at the end of the experiment to determine the subject's payment) to allocate a given quantity of tokens between three risky events, with given exchange rates between tokens and money for each state, and with given probabilities for each state. Denote by *m* the quantity of tokens to allocate, by e_1, e_2 and e_3 the three exchange rates, and by p_1, p_2 and p_3 the three probabilities. Let x_1, x_2 and x_3 denote the three allocations (where $x_1+x_2+x_3 = m$), then, assuming an Expected Utility maximiser, the subject's decision is to choose the allocations to maximise $p_1u(e_1x_1) + p_2u(e_2x_2) + p_3u(e_3x_3)$ subject to the constraint. The first-order conditions are $p_ie_iu'(e_ix_i^*) = \lambda$, for $i = 1, 2, 3$, where λ is the Lagrangian multiplier and the asterisk denotes the optimal allocation. After making all the allocations, one of the problems is chosen at random, a random device is invoked (with the specified probabilities for that problem) and the subject paid the money equivalent (given the exchange rates of that problem) of the number of tokens allocated to that state by that subject. From the experiment are obtained observations of the allocations that subjects actually made. These observations will be used to infer the preference functions of the subjects. Specifically it might be the case that is desired to infer whether these preferences are CARA (Constant Absolute Risk Averse) or CRRA (Constant Relative Risk Averse); it also desired to infer their degree of risk aversion. This latter is captured by the parameter *r* of the utility function. The two functions are as follows:

$$\text{CARA} \quad u(w) = \begin{cases} -e^{-rx} & r > 0 \\ x & r = 0 \\ e^{-rx} & r < 0 \end{cases}$$

here if r is positive, zero, negative, then the individual is risk-averse, risk-neutral, risk-loving.

$$\text{CRRA} \quad u(w) = \begin{cases} x^r & r > 0 \\ ln(x) & r = 0 \\ -x^r & r < 0 \end{cases}$$

here if *r* is less than, equal to or larger than 1, then the individual is risk-averse, risk-neutral, risk-loving. For either preference functional and for any value of the parameter *r* we can find the optimal unconstrained allocations. These are specified in Appendix C on Page 138. It will be seen there that with the CARA specification there is a problem in that the optimal unconstrained allocations may be negative or may exceed the amount to be allocated. Clearly in an experiment we cannot allow subjects to make allocations such that they may lose money as a consequence. So we have to tell the subjects that they cannot make negative allocations: they will be implementing their optimal constrained allocations. That is what will be observed. Again details are given in Appendix C. These problems do not arise with the CRRA specification as the optimal unconstrained allocations necessarily strictly satisfy the non-negativity constraints, as Appendix C makes clear.

3.3 Error Specifications

Now we come to the meat of the chapter – the stochastic specification. It is clear that subjects make mistakes when taking their decisions, and do not precisely implement the optimal (constrained or unconstrained as appropriate) allocations. Their actual allocations depart from these in some way. We need an error story for our estimation (which will be by means of Maximum Likelihood estimation). The stochastic assumptions underlying this estimation embody a story about the error process.

The reader should be warned that there is a lot of technical detail in what follows, but it is important. We want to somehow capture the process followed by the subjects in arriving at their actual allocations - to try and get inside their minds. What we presume, in keeping with all tests of economic theories, is that subjects do have some preference functional (which we are assuming is deterministic, so we are excluding random preference stories in what follows), and that they try and optimise it. We assume that subjects calculate their (constrained) optimal allocations with error - so that the error is 'added into' the actual (constrained) allocations. We note that in most experiments subjects are usually restricted to discrete allocations, so that the data we have is discrete, and it is an interesting question as to whether subjects do this discretisation *before* or *after* errors are 'added'. In what follows we assume that subjects first arrive at continuous allocations with error and then discretise them. Different error stories[2] would be needed if subjects first discretise their optimal (constrained) allocations before the error is 'added in'.

We presume that these errors are built 'on top of' the optimal constrained allocations, rather than on the optimal unconstrained allocations[3]. In this context one cannot do as many people do – just add on a normally distributed error term to the optimal allocations to get the actual allocations – because the implied actual allocations might then violate the non-negativity constraints. However, we can use the fact that the optimal (constrained if necessary) allocations must lie between 0 and *m* (the amount to be allocated). So the proportions x_i^*/m must lie between 0 and 1. This suggests one obvious error story: that the actual proportions x_i/m have beta distributions centred on the optimal allocations x_i^*/m. The way to achieve this has been used by Hey and Pace (2014). If we take x_i/m to have a beta

[2]Obvious contenders here are a beta-binomial distribution, a beta-binomial with bias (the meaning of which will become clearer later) and a two-beta-binomial (again the meaning of which will become clearer later).

[3]It is difficult to conceive of subjects making errors in their unconstrained allocations and then somehow adding error into them.

distribution[4] with parameters

$$\begin{cases} \alpha_i &= x_i^*(s_i - 1)/m \\ \beta_i &= (m - x_i^*)(s_i - 1)/m \end{cases} \tag{3.1}$$

then the mean of x_i/m is x_i^*/m and its variance is $x_i^*(m - x_i^*)/(m^2 s_i)$, and hence it follows that the mean of x_i is x_i^* and its variance is $x_i^*(m - x_i^*)/s_i$. There is no bias in the allocations. Here the parameter s_i is an indicator of the precision of the subject in taking decisions – the higher the more precise. Note also that this variance expression implies that the spread of the distribution of x_i is smaller the closer that x_i^* is to its bounds; this seems a natural behavioural assumption: subjects make smaller errors towards the bounds.

But we have three allocations. We can ignore one as they must add up to *m*. Let us concentrate on allocations 1 and 2. The above method works for each of the other two individually, but, if we apply it to two of them at the same time, there is no guarantee that $x_1 + x_2 \leq m$. If this is not true it would imply that $x_3 < 0$, which is not allowed by the rules of the experiment. There are two ways to proceed at this point: first, one can simply ignore the cases in which $x_1 + x_2 > m$; second, one can devise an error story for which it is guaranteed that $x_1 + x_2 \leq m$. This latter is what we do here. To do this, we use the fact given x_1, x_2 needs to lie between 0 and $m - x_1$. So let us first assume that x_1/m has a beta distribution with parameters

$$\begin{cases} \alpha_1 &= x_1^*(s_1 - 1)/m \\ \beta_1 &= (m - x_1^*)(s_1 - 1)/m \end{cases} \tag{3.2}$$

which guarantees that the mean of x_1 equals x_1^* and its variance equals $x_i^*(m - x_i^*)/s_1$; and then assume that $x_2/(m - x_1)$ has a beta distribution

[4] A beta distribution with parameters α and β lies in [0,1] and has mean $\dfrac{\alpha}{\alpha+\beta}$ and variance $\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

with parameters

$$\begin{cases} \alpha_2 & = x_2^*(s_2-1)/(m-x_1^*) \\ \beta_2 & = (m-x_1^*-x_2^*)(s_2-1)/(m-x_1^*) \end{cases} \tag{3.3}$$

This would imply that the mean of $x_2/(m-x_1)$ would be equal to $x_2^*/(m-x_1^*)$ and its variance would be equal to

$$\frac{x_2^*(m-x_1^*-x_2^*)(s_2-1)^2}{(m-x_1^*)^2(s_2-1)^2 s_2} = \frac{x_2^*(m-x_1^*-x_2^*)}{(m-x_1^*)^2 s_2};$$

and hence the mean of x_2 is equal to $x_2^*(m-x_1)/(m-x_1^*)$ while the variance of x_2 is $x_2^*(m-x_1^*-x_2^*)(m-x_1)^2/[s_2(m-x_1^*)^2]$. Note that the nice properties of the variance are still preserved and that s_2 remains a measure of the precision of the subject [5]. It might appear from the result on the mean of x_2 that it is biased – but this is conditional on the value of x_1 and it, in turn, is unbiased, so that the unconditional mean of x_2 is indeed equal to x_2^*. Behaviourally it seems reasonable to assume that the (distribution of the) error on x_2 depends on x_1, which is what the above specification implicitly assumes and implies. Note that this method guarantees that $x_1 + x_2 \leq m$ and hence that $x_3 \geq 0$.

This error specification is appropriate for when the preference functional is CRRA – since the optimal allocations are strictly between 0 and m, and so the error distributions are not degenerate. When we combine CRRA with this error specification we will refer to this combination as Specification 1.

However there are problems with this error specification with CARA since the optimal unconstrained allocations may lie outside this interval and therefore the optimal constrained allocations may lie on the bounds. It will be clear from the above that if x_i^* is equal to either 0 or *m* then the variance of x_i is zero and its distribution is degenerate at the bound. Implicitly this specification implies that subjects do not make mistakes at the bounds. So this model cannot rationalise any observation inside the bounds in the cases where the optimal constrained allocation is at a bound. We suspect that subjects may

[5] In what follows we will put $s_1 = s_2$ as there seems to be no reason why subjects should be more precise on one state than on the other.

still make errors even at the bounds, and, to cover such case, propose the following two specifications. We note that one important criterion is that the sum of the three actual allocations must be *m*.

Specification 2 'beta with bias'

Here we assume a beta distribution as above but with bias. We define variables $x_i' = b_i m/3 + (1 - b_i)x_i^*$ and replace x_i^* in the above with x_i'. Here the parameters b_i are bias parameters. If $b_i = 0$ then there is no bias; but if $b_i > 0$ there is bias, which depends upon the value of x_i^*. If $x_i^* = m/3$ once again there is no bias, so the bias increases away from the equal division. We note that here, even if the value of x_i^* is at a bound the distributions of the x_i are not degenerate, so we may observe a non-zero actual allocation even though the optimal allocation is zero.

Specification 3 'two betas'

In this we continue to use Specification 1 when the optimal allocation is within the bounds, but we add in a special case - when the x_i^* is 0 (or *m*). When x_i^* is 0 we assume that the actual allocation is beta with parameters 1 and *d* so that x_i has mean $1/(1 + d)$ and variance $d/[(1 + d)^2(d + 2)]$. When x_i^* is 1 we assume that the actual allocation is beta with parameters *d* and 1 so that x_i has mean $d/(1 + d)$ and variance $d/[(1 + d)^2(d + 2)]$. But we also need to take into account the fact that the actual allocations sum to *m*. There are various cases that we need to consider:

- If all of the x_i^* are positive, then we do as in Specification 1.
- If one of the x_i^* is *m* (and hence the other two are zero), then we generate the corresponding x_i as $Beta(1, d)$; this will be less than *m*, and we make the other two allocations equal to half of $(m - x_i)$.
- If one of the x_i^* is zero and the other two positive, we have three cases. Let us take just one of them – when $x_1^* = 0, x_2^* > 0$ and $x_3^* > 0$ –

the other two cases are treated symmetrically. In this case we make the actual allocation x_1 to be beta with parameters 1 and *d*, and then make the allocation $x_2/(m - x_1)$ to be beta with parameters

$$\begin{cases} \alpha_2 & = x_2^*(s_2 - 1)/(m - x_1^*) \\ \beta_2 & = (m - x_1^* - x_2^*)(s_2 - 1)/(m - x_1^*) \end{cases} \tag{3.4}$$

This guarantees that $x_1 + x_2 \leq m$.

3.4 A simulation study

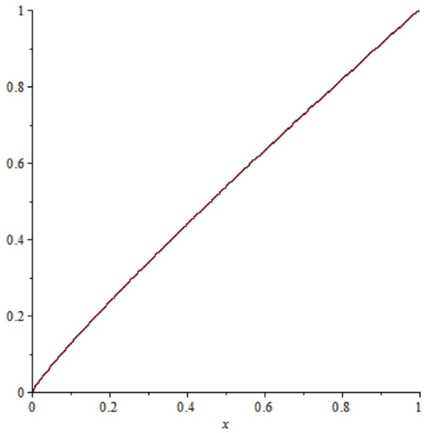
Some econometricians who analyse experimental data – particularly Wilcox (2008) – feel that the error specification may be more important than the preference functional. Wilcox has extensively investigated and compared different error stories in the context of an experiment involving pairwise choices. To see whether his results – which show that the error specification is a crucial decision variable for the experimenter – carry over to our (allocation) context, we report on simulation results that we have generated.

The simulation was conducted as follows We have the three stochastic specifications described above. We examine the estimation results obtained from the 9 pairwise combinations of the three true error specifications with the three assumed-true (estimated) specifications. This will enable us to see if the inferences drawn are very different if we use the wrong stochastic specification. Clearly the results from any such simulation may be sensitive to the underlying parameters, so the simulations have been carried out with a number of different parameter sets. These parameters are those relating to the preferences of the subjects and the precision and bias in their responses. The preferences we take to be either CRRA or CARA with risk aversion index denoted by *r* in both cases. The implied utility functions that we use are those stated above.

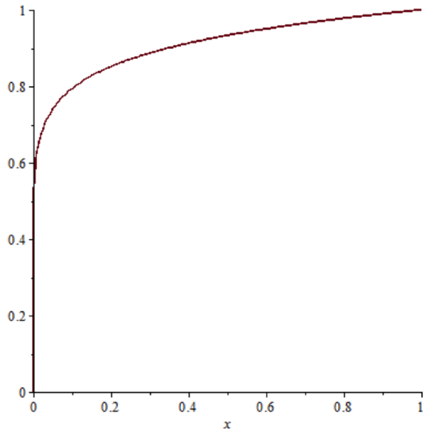
In the simulation we normalise by putting the value of *m* equal to 1, so that these functions span the range from 0 to 1. For a CRRA subject, a value of *r* greater or equal to 1 indicates a subject who is risk-loving or risk-neutral; for such subjects their optimal decision is simple – to allocate everything to the state *i* for which $p_i e_i$ is the greatest. Clearly we are not going to get any information about their value of *r* if it is greater than 1. For a CRRA person the value *r* can be zero or negative, in which cases the function takes different forms. We restrict attention to positive values of *r* (between 0 and 1) as this covers a range of reasonably risk-averse subjects. Those with negative values of *r* are very risk-averse. For CARA subjects, roughly the same range of risk attitudes is captured by letting r range from 1 to 5, though one cannot strictly compare the risk attitudes of the two types of subjects. For a CARA person the value of *r* can be zero or negative, in which cases once again the function takes different forms. If *r* is zero the subject is risk-neutral and if *r* is negative the subject is risk-loving; once again we would not be able to discriminate between different degrees of risk-loving. Note that with CRRA an increase in *r* implies less risk-aversion while for CARA an increase in r implies more risk-aversion. Our sets of parameters are listed in Table D.1 on Page 141 . There are 10 different parameter sets in two blocks of 5: the same set of risk-aversion parameters are in the two blocks and the blocks differ in their precision – the amount of noise in behaviour – the second block is more precise. Within a block risk-aversion increases throughout the block for both CRRA and CARA. Although, as we have already noted, there is no strict mapping between a value of *r* for a CRRA subject and the value of r for a CARA subject, we have chosen the parameter values so that the highest value of r for the CRRA subjects implies roughly the same (low) amount of risk-aversion (concavity) as the lowest value of *r* for the CARA subjects, and so that the lowest value of r for the CRRA subjects implies roughly the same (high) amount of risk-aversion (concavity) as the highest value of r for the CARA subjects. Figure 3.1 shows the implied utility functions for both the least risk-averse and the most risk-averse simulated subjects for both CRRA and CARA; Table 3.1 is a rough mapping between a value of *r* for a CRRA subject and the value of *r* for a CARA subject, though we should emphasise that the mapping is not precise.

We ran the simulations on 72 different allocation problems (combinations of p_1, p_2, p_3, e_1, e_2 and e_3). These are listed in Table D.2 on Page 142. As

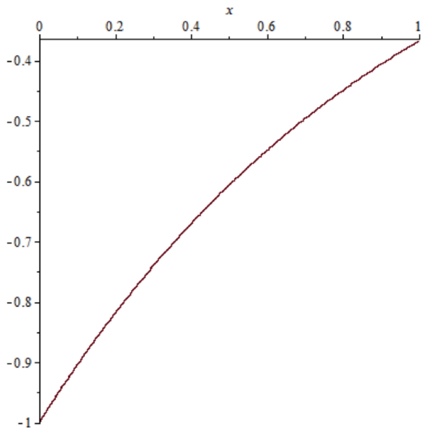
FIGURE 3.1: The utility functions implied by the lowest and highest risk aversion indices



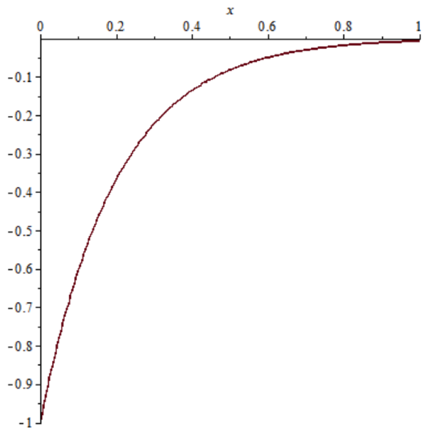
CRRA with $r = 0.9$ (low risk-aversion)



CRRA with $r = 0.1$ (high risk-aversion)



CARA with $r = 1$ (low risk-aversion)



CARA with r=5 (high risk-aversion)

CRRA *r*	CARA *r*
0.9	1
0.7	2
0.5	3
0.3	4
0.1	5

TABLE 3.1: Rough mapping from CRRA *r* to CARA *r*

will be seen there is a range of possible values for all of these. We assume that all states are treated equally, given their probabilities, and hence that there is no psychological bias towards or away from particular states[6]. The set of problems was chosen to span as much of the state space as possible, so we might expect a variety of behaviours from different subjects[7]. Mirroring what is done in experiments, we restricted the number of decimal places in the stated allocations to 2 – exactly as if the total number of tokens to allocate was 100 and subjects had to choose integer allocations. This has an effect, as we will see, on the inferences drawn. A total of 1000[8] simulations was implemented; the means and the standard deviations of the estimated parameters *r* and *s*, (for all three Specifications), *b* (for Specification 2) and *d* (for Specification 3) are presented in Table D.3 through Table D.6 from Page 143 to Page 146 with the standard deviation in italics under the corresponding mean. These tables are arranged in 10 blocks, each corresponding to a particular one of the 10 parameter sets. The means and standard deviations of the maximised log-likelihood are presented in Table D.7 on Page 147, with, again, the standard deviations are in italics. We should note that no specification is nested inside any other. It might look on the surface the Specification 2 is nested inside Specification 1 when $b = 0$, but the former specification involves CARA preferences while the latter involves CRRA preferences. Similarly it might look as if Specification 3 is nested inside Specification 1 when $d = \infty$, but, once again, the former specification involves CARA preferences while the latter involves CRRA preferences. We begin by looking at Table D.3, which reports the means and standard deviations of the risk-aversion parameter *r*. Looking down the main diagonal of each block we see that everywhere the mean estimated parameter is close to the true value – as one would expect: if one uses the correct specification one should recover the true preferences, though the CRRA estimated values seem to be closer to the true values than the CARA estimated values[9]. However it is the off-diagonal elements that are most interesting and informative as these tell

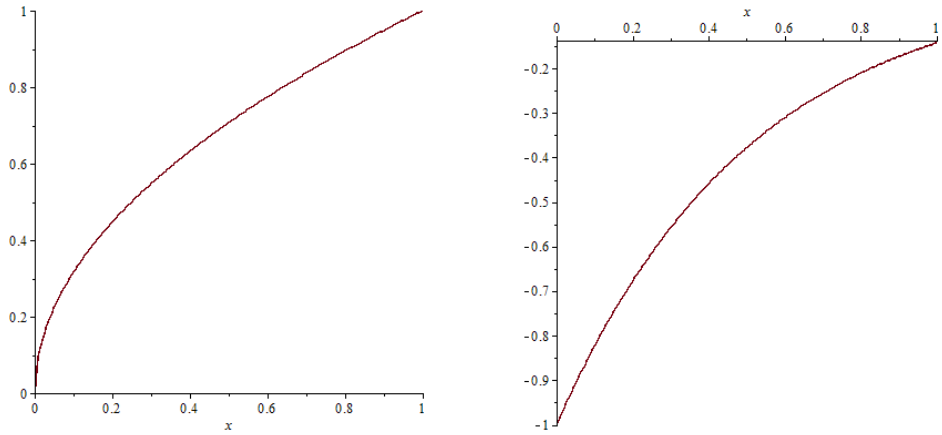
[6] Perhaps because of their representation.

[7] Actually one of the purposes of this simulation exercise was to enable us to choose a 'good' problem set for use in a future experiment.

[8] Matlab was used for the simulations; the program is available on request.

[9] Though it seems to be never the case that the mean *r* value are significantly different at 1% from their true values.

FIGURE 3.2: An example of mis-specification



CRRA with $r = 0.9$ (low risk-aversion)

CRRA with $r = 0.1$ (high risk-aversion)

us about the dangers of misspecification[10]. Care must be exercised however, since the r for CRRA means something different from an r for CARA. For example, look at Parameter set 1 (the first block) when the true specification is CRRA with $r = 0.9$ and when specification 2 is used for estimation, the mean estimated value of *r* is 0.634. Figure 3.2 on Page 135 illustrates (where the true model is CRRA with $r = 0.5$ and the estimated model is CARA under Specification 2 (where the mean estimated CARA is 1.948)); while one cannot say that the two functions are the same, they are similar. The same is true elsewhere. Precisions, however, are comparable; examine Table D.4 on Page 144. Again along the main diagonal of each block the mean estimated precision is close to the true precision, though the standard errors are quite large. This latter is a consequence of the likelihood function being rather flat along towards its minimum, indicating that differences in precision do not make a big difference to behaviour. The off-diagonal elements, however, do depart quite sharply from the true values. As a general rule, though it is not always the case, the estimated precision is less than the true precision. This is an interesting result which suggests that mis-specification might lead to an under-estimation of the precision of the subjects. The results for the

[10]Note that in some cases the mean estimate of r is 0.01 (and the standard error is 0); these are cases when the estimate hit its lowest bound (specified as 0.01) in all simulations; this lower bound was there to stop the software crashing if s reached zero. Perhaps a lower lower-bound might have been appropriate.

bias b parameter in Table D.5 on Page 145 are interesting. This is appropriate only when the estimated specification is Specification 2. When the true specification is Specification 1 the estimates of b are close to their true values but occasionally depart significantly from them. When the true value of b is zero, the estimated values are not significantly different from zero. The estimates of the d parameter in Specification 3 appear reasonable as Table D.6 on Page 146 shows. This is appropriate only when the estimated specification is Specification 3. It is it interesting to note, however, that for the parameter sets with higher risk aversion, the estimated values of d for true Specification 1 are 'well below' (if that means anything) their true value of ∞. Finally we examine Table D.7 which reports the means and standard deviations of the minimised negative log-likelihoods. We feel that these log-likelihoods are comparable across specifications. What we had expected was that the entries down the main diagonal of each block would be the smallest in each row (remember that these numbers are the negative of the minimised log-likelihood) – indicating that if one chooses between specifications on the basis of the maximised log-likelihoods, then one would always correctly identify the true specification. But that is not true. Take the rows where Specification 1 is the true specification. Everywhere Specification 3 has the lowest (negative) log-likelihood; it is almost the same when Specification 2 is the true specification. There appears to be a systematic bias: experimentalists could be wrongly led to believe that Specification 3 is the true specification even when it is not. This is rather worrying. But the reasons are clear and it is to do with our recurring issue about the bounds on the optimal allocations. It is also to do with the fact that actual allocations were rounded to two decimal places[11]. Examine Table D.8 on Page 148. This displays the optimal allocations for CRRA and CARA – in the first 6 columns to 7 decimal places and in the last 6 columns to just 2. Of particular importance are the '0's and '1's. It will be noted that, to 7 decimal places, where there is a '0' in the CARA columns, there is (but this is just to 7 decimal places) a positive number in the CRRA columns; similarly where there is a '1' in the CARA columns there is (but this is just to 7 decimal places)

[11]The reason for that, as we have mentioned before, is that in experiments subjects are not allowed to express allocations to any number of decimal places, and usually, for example, when the amount to allocate is 100, they are restricted to integer allocations.

a number less than 1 in the CRRA columns. However when the number of decimal places is cut to 2, there are usually '0's and '1's in the same place. As a consequence the estimation procedure gets confused as to whether the true preferences are CRRA or CARA. In addition, Specification 3 tells a nice story about what happens at the bounds – a story different from that told elsewhere.

3.5 Conclusions

The messages that this study suggest are negative in some ways and positive in others, though this depends on your perspective. If you are solely interested in getting an idea as to the magnitude of the risk-aversion of the subjects, then the message seems to be that the specification is relatively unimportant. If, however, you are interested in the precision of the subjects, then it would appear that the specification is important – getting it wrong can lead to systematic under-estimates of the precision. One might also get wrong the bias of the subjects. But the really negative, and surprising, result is that choosing the 'best' specification on the basis of the best log-likelihood can seriously lead you astray. But if you are only interested in estimates of risk-aversion, it might not matter. Perhaps there is a moral here – if experimentalists are really concerned about deciding whether preferences are CRRA or CARA, they ought to let subjects express their allocations to more decimal places.

Appendix

C The optimal allocations

CARA

Unconstrained

$$
\begin{aligned}
x_1^* &= \{rme_2e_3 - e_2ln(p_3e_3/p_1e_1) - e_3ln(p_2e_2/p_1e_1)\}/d_1 \\
x_2^* &= \{rme_3e_1 - e_2ln(p_3e_3/p_2e_2) - e_3ln(p_1e_1/p_2e_2)\}/d_1 \\
x_3^* &= \{rme_1e_2 - e_2ln(p_1e_1/p_3e_3) - e_1ln(p_2e_2/p_3e_3)\}/d_1
\end{aligned}
$$

where $d_1 = r(e_2e_3 + e_3e_1 + e_1e_2$.

Constrained 1

$$
\begin{aligned}
x_1^* &= 0 \\
x_2^* &= \{rme_3 - ln(p_3e_3/p_2e_2)\}/d_2 \\
x_3^* &= \{rme_2 - ln(p_2e_2/p_3e_3)\}/d_2
\end{aligned}
$$

where $d_2 = r(e_2 + e_3)$.

Constrained 2

$$\begin{aligned} x_1^* &= \{rme_3 - ln(p_3e_3/p_1e_1)\}/d_3 \\ x_2^* &= 0 \\ x_3^* &= \{rme_1 - ln(p_1e_1/p_3e_3)\}/d_3 \end{aligned}$$

where $d_3 = r(e_1 + e_3)$.

Constrained 3

$$\begin{aligned} x_1^* &= \{rme_2 - ln(p_2e_2/p_1e_1)\}/d_4 \\ x_2^* &= \{rme_1 - ln(p_1e_1/p_2e_2)\}/d_4 \\ x_3^* &= 0 \end{aligned}$$

where $d_4 = r(e_1 + e_2)$.

Constrained 4

$$\begin{aligned} x_1^* &= 0 \\ x_2^* &= 0 \\ x_3^* &= m \end{aligned}$$

Constrained 5

$$\begin{aligned} x_1^* &= 0 \\ x_2^* &= m \\ x_3^* &= 0 \end{aligned}$$

Constrained 6

$$\begin{aligned} x_1^* &= m \\ x_2^* &= 0 \\ x_3^* &= 0 \end{aligned}$$

If the unconstrained solution violates the non-negativity constraints we need to check the six constrained allocations and choose the optimal which is the one that satisfies the constraints and yields the maximum expected utility.

CRRA

$$\begin{aligned} x_1^* &= m(p_1e_1)^{1/(1+r)}e_2e_3/d \\ x_2^* &= m(p_2e_2)^{1/(1+r)}e_3e_1/d \\ x_3^* &= m(p_3e_3)^{1/(1+r)}e_1e_2/d \end{aligned}$$

where

$$d = (p_1e_1)^{1/(1+r)}e_2e_3 + (p_2e_2)^{1/(1+r)}e_3e_1 + (p_3e_3)^{1/(1+r)}e_1e_2$$

D Tables for Simulation

This appendix contains the tables of simulation parameters and simulation results.

TABLE D.1: The Parameters Sets

Parameter set number	Risk-aversion index *r*		Precision parameter *s*	Bias parameter *b*	Second beta parameter *d*
	CRRA	CARA			
1	0.9	1	25	0.01	20
2	0.7	2	25	0.01	20
3	0.5	3	25	0.01	20
4	0.3	4	25	0.01	20
5	0.1	5	25	0.01	20
6	0.9	1	50	0.02	40
7	0.7	2	50	0.02	40
8	0.5	3	50	0.02	40
9	0.3	4	50	0.02	40
10	0.1	5	50	0.02	40

Table D.2: The Problem Sets

p_1	p_2	p_3	e_1	e_2	e_3
0.40	0.30	0.30	0.75	0.75	0.75
0.40	0.40	0.20	0.75	0.75	0.75
0.50	0.30	0.20	0.75	0.75	0.75
0.50	0.40	0.10	0.75	0.75	0.75
0.60	0.20	0.20	0.75	0.75	0.75
0.60	0.30	0.10	0.75	0.75	0.75
0.70	0.20	0.10	0.75	0.75	0.75
0.80	0.10	0.10	0.75	0.75	0.75
0.40	0.30	0.30	0.75	0.75	1.00
0.40	0.40	0.20	0.75	0.75	1.00
0.50	0.30	0.20	0.75	0.75	1.00
0.50	0.40	0.10	0.75	0.75	1.00
0.60	0.20	0.20	0.75	0.75	1.00
0.60	0.30	0.10	0.75	0.75	1.00
0.70	0.20	0.10	0.75	0.75	1.00
0.80	0.10	0.10	0.75	0.75	1.00
0.40	0.30	0.30	0.75	0.75	1.25
0.40	0.40	0.20	0.75	0.75	1.25
0.50	0.30	0.20	0.75	0.75	1.25
0.50	0.40	0.10	0.75	0.75	1.25
0.60	0.20	0.20	0.75	0.75	1.25
0.60	0.30	0.10	0.75	0.75	1.25
0.70	0.20	0.10	0.75	0.75	1.25
0.80	0.10	0.10	0.75	0.75	1.25
0.40	0.30	0.30	0.75	1.00	0.75
0.40	0.40	0.20	0.75	1.00	0.75
0.50	0.30	0.20	0.75	1.00	0.75
0.50	0.40	0.10	0.75	1.00	0.75
0.60	0.20	0.20	0.75	1.00	0.75
0.60	0.30	0.10	0.75	1.00	0.75
0.70	0.20	0.10	0.75	1.00	0.75
0.80	0.10	0.10	0.75	1.00	0.75
0.40	0.30	0.30	0.75	1.00	1.00
0.40	0.40	0.20	0.75	1.00	1.00
0.50	0.30	0.20	0.75	1.00	1.00
0.50	0.40	0.10	0.75	1.00	1.00
0.60	0.20	0.20	0.75	1.00	1.00
0.60	0.30	0.10	0.75	1.00	1.00
0.70	0.20	0.10	0.75	1.00	1.00
0.80	0.10	0.10	0.75	1.00	1.00
0.40	0.30	0.30	0.75	1.00	1.25
0.40	0.40	0.20	0.75	1.00	1.25
0.50	0.30	0.20	0.75	1.00	1.25
0.50	0.40	0.10	0.75	1.00	1.25
0.60	0.20	0.20	0.75	1.00	1.25
0.60	0.30	0.10	0.75	1.00	1.25
0.70	0.20	0.10	0.75	1.00	1.25
0.80	0.10	0.10	0.75	1.00	1.25
0.40	0.30	0.30	0.75	1.25	0.75
0.40	0.40	0.20	0.75	1.25	0.75
0.50	0.30	0.20	0.75	1.25	0.75
0.50	0.40	0.10	0.75	1.25	0.75
0.60	0.20	0.20	0.75	1.25	0.75
0.60	0.30	0.10	0.75	1.25	0.75
0.70	0.20	0.10	0.75	1.25	0.75
0.80	0.10	0.10	0.75	1.25	0.75
0.40	0.30	0.30	0.75	1.25	1.00
0.40	0.40	0.20	0.75	1.25	1.00
0.50	0.30	0.20	0.75	1.25	1.00
0.50	0.40	0.10	0.75	1.25	1.00
0.60	0.20	0.20	0.75	1.25	1.00
0.60	0.30	0.10	0.75	1.25	1.00
0.70	0.20	0.10	0.75	1.25	1.00
0.80	0.10	0.10	0.75	1.25	1.00
0.40	0.30	0.30	0.75	1.25	1.25
0.40	0.40	0.20	0.75	1.25	1.25
0.50	0.30	0.20	0.75	1.25	1.25
0.50	0.40	0.10	0.75	1.25	1.25
0.60	0.20	0.20	0.75	1.25	1.25
0.60	0.30	0.10	0.75	1.25	1.25
0.70	0.20	0.10	0.75	1.25	1.25
0.80	0.10	0.10	0.75	1.25	1.25

TABLE D.3: Means and standard deviations of the estimated value of the risk-aversion prametr r

True spec	Parameter set 1				Parameter set 6			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	0.900	0.634	0.793	0.900	0.900	0.601	0.752	0.900
	0.004	*0.140*	*0.150*		*0.006*	*0.132*	*0.131*	
2	0.657	0.954	1.076	1	0.589	0.929	1.160	1
	0.013	*0.034*	*0.064*		*0.023*	*0.073*	*0.071*	
3	0.678	1.121	1.061	1	0.625	3.072	1.088	1
	0.011	*0.730*	*0.053*		*0.022*	*2.171*	*0.079*	
	Parameter set 2				Parameter set 5			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	0.699	1.724	1.389	0.700	0.700	1.636	1.497	0.700
	0.010	*0.544*	*0.211*		*0.014*	*0.550*	*0.237*	
2	0.434	1.910	2.003	2	0.355	1.830	1.969	2
	0.014	*0.091*	*0.216*		*0.031*	*0.119*	*0.303*	
3	0.461	2.035	1.822	2	0.414	2.265	1.761	2
	0.018	*0.242*	*0.132*		*0.030*	*0.362*	*0.071*	
	Parameter set 3				Parameter set 8			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	0.501	2.249	1.949	0.500	0.499	2.306	1.935	0.500
	0.015	*0.376*	*0.196*		*0.023*	*0.359*	*0.227*	
2	0.143	3.044	2.953	3	0.070	3.042	2.670	3
	0.030	*0.097*	*0.105*		*0.051*	*0.208*	*0.228*	
3	0.191	3.034	2.709	3	0.167	3.061	2.521	3
	0.032	*0.104*	*0.170*		*0.042*	*0.149*	*0.064*	
	Parameter set 4				Parameter set 9			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	0.300	2.803	2.455	0.300	0.294	2.824	2.417	0.300
	0.027	*0.185*	*0.102*		*0.040*	*0.273*	*0.141*	
2	0.010	4.036	4.246	4	0.010	5.841	5.972	4
	0.000	*0.310*	*0.199*		*0.000*	*0.899*	*0.668*	
3	0.010	3.935	3.510	4	0.010	5.102	4.933	4
	0.000	*0.183*	*0.213*		*0.000*	*0.684*	*0.784*	
	Parameter set 5				Parameter set 10			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	0.101	3.157	2.983	0.100	0.090	3.188	2.565	0.100
	0.038	*0.121*	*0.288*		*0.052*	*0.155*	*0.351*	
2	0.010	4.835	5.352	5	0.010	5.833	6.016	5
	0.000	*0.604*	*0.274*		*0.000*	*0.886*	*0.574*	
3	0.010	4.837	5.058	5	0.010	4.981	4.941	5
	0.000	*0.426*	*0.264*		*0.000*	*0.629*	*0.826*	

Table D.4: Means and standard deviations of the estimated value of the precision parameter s

True spec	Parameter set 1				Parameter set 6			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	54.69	6.61	6.48	50	29.11	6.11	5.64	25
	11.52	*2.75*	*4.76*		*7.76*	*2.34*	*3.30*	
2	27.67	53.62	58.90	50	17.63	26.17	32.48	25
	4.60	*9.23*	*15.10*		*3.21*	*4.27*	*11.40*	
3	25.07	47.75	56.22	50	16.42	9.84	29.89	25
	3.99	*13.94*	*12.91*		*3.20*	*9.85*	*9.91*	
	Parameter set 2				Parameter set 5			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	52.59	15.13	29.06	50	26.47	12.08	15.48	25
	9.00	*10.36*	*13.56*		*5.18*	*6.86*	*6.48*	
2	35.49	53.05	53.40	50	22.69	25.01	25.97	25
	6.59	*9.39*	*13.70*		*4.67*	*4.68*	*7.71*	
3	32.32	48.14	51.30	50	20.42	21.58	25.81	25
	5.88	*11.01*	*13.77*		*3.59*	*5.94*	*7.06*	
	Parameter set 3				Parameter set 8			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	51.82	26.56	43.01	50	25.89	16.16	24.20	25
	9.77	*9.66*	*13.78*		*4.95*	*4.82*	*7.95*	
2	43.91	51.64	49.19	50	21.86	25.32	19.40	25
	8.86	*8.87*	*10.56*		*4.32*	*5.14*	*5.57*	
3	42.70	51.37	46.95	50	22.29	25.32	22.45	25
	7.14	*9.09*	*12.91*		*4.74*	*5.36*	*5.94*	
	Parameter set 4				Parameter set 9			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	50.70	38.81	49.49	50	25.45	21.21	26.33	25
	8.16	*6.48*	*10.29*		*4.88*	*4.18*	*6.89*	
2	37.92	51.18	52.30	50	13.69	22.39	22.90	25
	5.85	*9.54*	*9.55*		*2.62*	*5.08*	*4.88*	
3	43.06	51.50	44.31	50	16.89	24.20	23.32	25
	7.54	*9.39*	*10.07*		*3.37*	*5.95*	*4.32*	
	Parameter set 5				Parameter set 10			
	Estimated specification			True value	Estimated specification			True value
	1	2	3		1	2	3	
1	50.22	45.89	47.13	50	24.00	23.42	21.79	25
	8.46	*8.12*	*10.56*		*4.88*	*4.54*	*5.84*	
2	22.22	51.12	49.80	50	13.83	22.79	22.11	25
	4.32	*8.15*	*8.91*		*2.98*	*5.66*	*5.06*	
3	25.60	49.86	50.23	50	17.33	24.11	23.91	25
	5.25	*9.47*	*9.31*		*4.04*	*4.97*	*5.56*	

TABLE D.5: Means and standard deviations of the estimated value of the bias parameter *b* (note that this is only estimated for specification 2)

True specification	Parameter set 1		Parameter set 6	
		True value		True value
1	0.0074	0	0.0085	0
	0.0082		*0.0075*	
2	0.0670	0.05	0.1276	0.1
	0.0100		*0.0229*	
3	0.0444	*	0.0330	*
	0.0136		*0.0445*	
	Parameter set 2		Parameter set 7	
		True value		True value
1	0.0121	0	0.0141	0
	0.0151		*0.0179*	
2	0.0628	0.05	0.1270	0.1
	0.0137		*0.0276*	
3	0.0343	*	0.0358	*
	0.0218		*0.0411*	
	Parameter set 3		Parameter set 8	
		True value		True value
1	0.0232	0	0.0189	0
	0.0300		*0.0315*	
2	0.0270	0.05	0.0781	0.1
	0.0322		*0.0656*	
3	0.0070	*	0.0126	*
	0.0138		*0.0171*	
	Parameter set 4		Parameter set 9	
		True value		True value
1	0.0150	0	0.0156	0
	0.0214		*0.0279*	
2	0.0449	0.05	0.0319	0.1
	0.0488		*0.0680*	
3	0.0170	*	0.0194	*
	0.0273		*0.0469*	
	Parameter set 5		Parameter set 10	
		True value		True value
1	0.0442	0	0.0344	0
	0.0310		*0.0292*	
2	0.0787	0.05	0.0337	0.1
	0.0829		*0.0688*	
3	0.0345	*	0.0247	*
	0.0604		*0.0602*	

*Indicates parameter b is irrelevant

TABLE D.6: Means and standard deviations of the estimated value of the second beta parameter *d* (note that his is only estimated for specification 3)

True specification	Parameter set 1		Parameter set 6	
		True value		True value
1	2559.90	∞	2637.72	∞
	2012.78		*2006.78*	
2	44.87	*	22.37	*
	5.68		*2.83*	
3	48.09	40	23.56	20
	9.31		*5.26*	
	Parameter set 2		Parameter set 7	
		True value		True value
1	147.44	∞	193.08	∞
	82.57		*110.16*	
2	50.41	*	25.01	*
	10.50		*5.07*	
3	50.91	40	26.68	20
	10.48		*5.91*	
	Parameter set 3		Parameter set 8	
		True value		True value
1	33.07	∞	35.09	∞
	7.56		*10.54*	
2	66.16	*	25.84	*
	12.36		*7.50*	
3	51.72	40	28.56	20
	18.66		*6.74*	
	Parameter set 4		Parameter set 9	
		True value		True value
1	16.19	∞	2.49	∞
	1.83		*0.01*	
2	7.00	*	0.00	*
	60.18		*18.71*	
3	53.40	40	3.40	20
	42.37		*2.49*	
	Parameter set 5		Parameter set 10	
		True value		True value
1	10.75	∞	10.03	∞
	1.20		*2.22*	
2	0.01	*	0.01	*
	0.00		*0.00*	
3	40.00	40	18.86	20
	0.00		*3.25*	

*Indicates parameter s is irrelevant

TABLE D.7: Means and standard deviations of the maximised log-likelihoods

True spec	Parameter set 1			Parameter set 6		
	Estimated specification			Estimated specification		
	1	2	3	1	2	3
1	105.0	141.0	41.0	111.5	137.6	38.4
	6.4	*6.7*	*6.9*	*6.9*	*7.3*	*5.9*
2	234.4	205.0	144.8	254.2	236.3	183.2
	6.9	*6.0*	*6.7*	*6.1*	*5.4*	*6.4*
3	230.4	204.6	146.5	249.9	261.2	183.6
	6.9	*17.4*	*7.0*	*6.5*	*20.0*	*7.5*
	Parameter set 2			*Parameter set 5*		
	Estimated specification			Estimated specification		
	1	2	3	1	2	3
1	192.9	227.4	139.4	208.2	229.1	141.3
	5.9	*14.3*	*6.3*	*6.3*	*13.8*	*8.6*
2	235.6	222.7	167.7	252.3	248.9	198.9
	5.2	*5.7*	*6.0*	*5.4*	*5.3*	*5.4*
3	236.1	223.0	165.4	252.1	250.1	194.7
	4.8	*7.5*	*6.1*	*4.7*	*7.5*	*5.9*
	Parameter set 3			Parameter set 8		
	Estimated specification			Estimated specification		
	1	2	3	1	2	3
1	222.2	242.0	183.8	241.5	251.4	190.9
	6.7	*9.3*	*5.1*	*5.6*	*6.9*	*5.7*
2	239.7	232.8	198.1	257.8	254.2	223.9
	5.9	*5.6*	*5.2*	*4.4*	*5.2*	*6.1*
3	239.7	231.2	196.8	256.8	252.2	215.1
	5.0	*5.8*	*6.1*	*4.8*	*5.9*	*6.3*
	Parameter set 4			Parameter set 9		
	Estimated specification			Estimated specification		
	1	2	3	1	2	3
1	231.9	239.1	206.9	251.9	255.0	217.7
	5.3	*4.8*	*4.7*	*5.4*	*5.2*	*5.6*
2	248.1	237.3	236.4	271.1	256.1	255.8
	4.4	*5.7*	*6.0*	*3.2*	*4.4*	*4.1*
3	243.7	237.2	228.4	267.3	256.3	255.9
	5.6	*5.6*	*5.0*	*3.9*	*4.9*	*3.7*
	Parameter set 5			Parameter set 10		
	Estimated specification			Estimated specification		
	1	2	3	1	2	3
1	235.9	237.8	225.5	256.3	256.1	236.5
	5.5	*5.2*	*5.4*	*5.0*	*4.8*	*6.0*
2	263.7	237.8	238.5	270.9	255.9	256.7
	4.0	*4.9*	*5.1*	*3.2*	*4.7*	*4.2*
3	260.3	238.7	238.4	267.0	256.4	255.6
	4.8	*5.5*	*5.6*	*4.3*	*4.4*	*4.4*

Table D.8: Allocations under CRRA and under CARA for parameter set 1

CRRA to 7 decimal places			CARA to 7 decimal places			CRRA to 2 decimal places			CARA to 2 decimal places		
x_1*	x_2*	x_3*	x_1*	x_2*	x_3*	x_1*	x_2*	x_3*	x_1*	x_2*	x_3*
0.8987738	0.0506131	0.0506131	0.5890507	0.2054746	0.2054746	0.90	0.05	0.05	0.59	0.21	0.21
0.4997560	0.4997560	0.0004880	0.5000000	0.5000000	0.0000000	0.50	0.50	0.00	0.50	0.50	0.00
0.9938861	0.0060096	0.0001042	0.8405504	0.1594496	0.0000000	0.99	0.01	0.00	0.84	0.16	0.00
0.9030370	0.0969629	0.0000001	0.6487624	0.3512376	0.0000000	0.90	0.10	0.00	0.65	0.35	0.00
0.9999661	0.0000169	0.0000169	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9990244	0.0009756	0.0000000	0.9620981	0.0379019	0.0000000	1.00	0.00	0.00	0.96	0.04	0.00
0.9999964	0.0000036	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
1.0000000	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.5536138	0.0311759	0.4152103	0.5031186	0.1195425	0.3773389	0.55	0.03	0.42	0.50	0.12	0.38
0.4967695	0.4967695	0.0064611	0.5000000	0.5000000	0.0000000	0.50	0.50	0.01	0.50	0.50	0.00
0.9926118	0.0060019	0.0013862	0.8398944	0.1587935	0.0013121	0.99	0.01	0.00	0.84	0.16	0.00
0.9030360	0.0969628	0.0000012	0.6487624	0.3512376	0.0000000	0.90	0.10	0.00	0.65	0.35	0.00
0.9997576	0.0000169	0.0002255	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9990242	0.0009756	0.0000002	0.9620981	0.0379019	0.0000000	1.00	0.00	0.00	0.96	0.04	0.00
0.9999963	0.0000036	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
1.0000000	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.1505061	0.0084755	0.8410184	0.4634851	0.0799090	0.4566059	0.15	0.01	0.84	0.46	0.08	0.46
0.4768937	0.4768937	0.0462126	0.4407143	0.4407143	0.1185713	0.48	0.48	0.05	0.44	0.44	0.12
0.9838147	0.0059488	0.0102365	0.7713357	0.0902349	0.1384294	0.98	0.01	0.01	0.77	0.09	0.14
0.9030288	0.0969620	0.0000092	0.6487624	0.3512376	0.0000000	0.90	0.10	0.00	0.65	0.35	0.00
0.9983055	0.0000169	0.0016776	0.9188933	0.0000000	0.0811067	1.00	0.00	0.00	0.92	0.00	0.08
0.9990228	0.0009756	0.0000016	0.9620981	0.0379019	0.0000000	1.00	0.00	0.00	0.96	0.04	0.00
0.9999960	0.0000036	0.0000004	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9999999	0.0000000	0.0000001	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.5536138	0.4152103	0.0311759	0.5031186	0.3773389	0.1195425	0.55	0.42	0.03	0.50	0.38	0.12
0.0698360	0.9300958	0.0000682	0.4070388	0.5929612	0.0000000	0.07	0.93	0.00	0.41	0.59	0.00
0.9253814	0.0745216	0.0000970	0.6989392	0.3010608	0.0000000	0.93	0.07	0.00	0.70	0.30	0.00
0.4115157	0.5884843	0.0000000	0.5345494	0.4654506	0.0000000	0.41	0.59	0.00	0.53	0.47	0.00
0.9997576	0.0002255	0.0000169	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9871608	0.0128392	0.0000000	0.8031229	0.1968771	0.0000000	0.99	0.01	0.00	0.80	0.20	0.00
0.9999517	0.0000483	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
1.0000000	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.4000000	0.3000000	0.3000000	0.4000000	0.3000000	0.3000000	0.40	0.30	0.30	0.40	0.30	0.30
0.0697773	0.9293151	0.0009075	0.4070388	0.5929612	0.0000000	0.07	0.93	0.00	0.41	0.59	0.00
0.9242766	0.0744326	0.0012908	0.6989392	0.3010608	0.0000000	0.92	0.07	0.00	0.70	0.30	0.00
0.4115155	0.5884840	0.0000006	0.5345494	0.4654506	0.0000000	0.41	0.59	0.00	0.53	0.47	0.00
0.9995491	0.0002254	0.0002254	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9871606	0.0128392	0.0000002	0.8031229	0.1968771	0.0000000	0.99	0.01	0.00	0.80	0.20	0.00
0.9999517	0.0000483	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
1.0000000	0.0000000	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.1362781	0.1022086	0.7615133	0.3495682	0.2621761	0.3882557	0.14	0.10	0.76	0.35	0.26	0.39
0.0693712	0.9239065	0.0067223	0.3651809	0.5615678	0.0732513	0.07	0.92	0.01	0.37	0.56	0.07
0.9166444	0.0738180	0.0095376	0.6585173	0.2707444	0.0707383	0.92	0.07	0.01	0.66	0.27	0.07
0.4115140	0.5884819	0.0000042	0.5345494	0.4654506	0.0000000	0.41	0.59	0.00	0.53	0.47	0.00
0.9980976	0.0002251	0.0016773	0.9188933	0.0000000	0.0811067	1.00	0.00	0.00	0.92	0.00	0.08
0.9871592	0.0128391	0.0000016	0.8031229	0.1968771	0.0000000	0.99	0.01	0.00	0.80	0.20	0.00
0.9999514	0.0000483	0.0000004	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9999999	0.0000000	0.0000001	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.1505061	0.8410184	0.0084755	0.4634851	0.4566059	0.0799090	0.15	0.84	0.01	0.46	0.46	0.08
0.0099771	0.9900132	0.0000097	0.3695872	0.6304128	0.0000000	0.01	0.99	0.00	0.37	0.63	0.00
0.6249590	0.3749754	0.0000655	0.6250000	0.3750000	0.0000000	0.62	0.37	0.00	0.63	0.38	0.00
0.0858028	0.9141972	0.0000000	0.4811590	0.5188410	0.0000000	0.09	0.91	0.00	0.48	0.52	0.00
0.9983055	0.0016776	0.0000169	0.9188933	0.0811067	0.0000000	1.00	0.00	0.00	0.92	0.08	0.00
0.9116573	0.0883426	0.0000000	0.7161608	0.2838392	0.0000000	0.91	0.09	0.00	0.72	0.28	0.00
0.9996404	0.0003596	0.0000000	0.9959687	0.0040313	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9999999	0.0000001	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.1362781	0.7615133	0.1022086	0.3495682	0.3882557	0.2621761	0.14	0.76	0.10	0.35	0.39	0.26
0.0099759	0.9898944	0.0001297	0.3695872	0.6304128	0.0000000	0.01	0.99	0.00	0.37	0.63	0.00
0.6244550	0.3746730	0.0008721	0.6250000	0.3750000	0.0000000	0.62	0.37	0.00	0.63	0.38	0.00
0.0858028	0.9141971	0.0000001	0.4811590	0.5188410	0.0000000	0.09	0.91	0.00	0.48	0.52	0.00
0.9980976	0.0016773	0.0002251	0.9188933	0.0811067	0.0000000	1.00	0.00	0.00	0.92	0.08	0.00
0.9116572	0.0883426	0.0000002	0.7161608	0.2838392	0.0000000	0.91	0.09	0.00	0.72	0.28	0.00
0.9996404	0.0003596	0.0000000	0.9959687	0.0040313	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9999999	0.0000001	0.0000000	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.0821296	0.4589352	0.4589352	0.2922592	0.3538704	0.3538704	0.08	0.46	0.46	0.29	0.35	0.35
0.0099675	0.9890666	0.0009659	0.3350894	0.6097142	0.0551964	0.01	0.99	0.00	0.34	0.61	0.06
0.6209618	0.3725771	0.0064611	0.6019873	0.3611924	0.0368203	0.62	0.37	0.01	0.60	0.36	0.04
0.0858027	0.9141964	0.0000009	0.4811590	0.5188410	0.0000000	0.09	0.91	0.00	0.48	0.52	0.00
0.9966504	0.0016748	0.0016748	0.8820267	0.0589867	0.0589867	1.00	0.00	0.00	0.88	0.06	0.06
0.9116560	0.0883425	0.0000015	0.7161608	0.2838392	0.0000000	0.91	0.09	0.00	0.72	0.28	0.00
0.9996401	0.0003596	0.0000004	0.9959687	0.0040313	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.9999998	0.0000001	0.0000001	1.0000000	0.0000000	0.0000000	1.00	0.00	0.00	1.00	0.00	0.00
0.8987738	0.0506131	0.0506131	0.5890507	0.2054746	0.2054746	0.90	0.05	0.05	0.59	0.21	0.21
0.4997560	0.4997560	0.0004880	0.5000000	0.5000000	0.0000000	0.50	0.50	0.00	0.50	0.50	0.00

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