A STUDY OF STOCHASTIC LANDAU-LIFSCHITZ EQUATIONS

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Abstract. This Thesis studied two forms of Stochastic Landau-Lifschitz equations and proved the existence of the weak solution and some regularity properties.
The first form is given by

$$\begin{cases} \mathrm{d}u(t) & = \Big\{\lambda_1 u(t)\times[\Delta u(t)-\nabla\phi(u(t))] \\ & -\lambda_2 u(t)\times\Big(u(t)\times[\Delta u(t) \quad -\nabla\phi(u(t))]\Big)\Big\}\,\mathrm{d}t \\ & \qquad\qquad\qquad +\Big\{u(t)\times h\Big\}\circ\,\mathrm{d}W(t) \\ \frac{\partial u}{\partial\nu}\Big|_{\partial D} & = 0 \\ u(0) & = u_0 \end{cases}$$

which is a similar form as in Brzeźniak and Goldys and Jegaraj's paper [13] but with relatively more general energy.
The second form is given by

$$\begin{cases} \mathrm{d}M(t) = & [\lambda_1 M(t)\times\rho(t)-\lambda_2 M(t)\times(M(t)\times\rho(t))]\,\mathrm{d}t \\ & +\sum_{j=1}^{\infty}[\alpha M(t)\times h_j+\beta M(t)\times(M(t)\times h_j)]\circ\,\mathrm{d}W_j(t) \\ \mathrm{d}B(t) = & -\nabla\times E(t)\,\mathrm{d}t \\ \mathrm{d}E(t) = & [\nabla\times H(t)-1_D E(t)-\tilde{f}(t)]\,\mathrm{d}t, \quad t\in[0,T]. \end{cases}$$

which is the full version of the stochastic Landau Lifschitz equation coupled with the Maxwell's equations.

To my family,

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1. Introduction

The ferromagnetism theory was first studied by Weiß in 1907 and then further developed by Landau and Lifshitz [33] and Gilbert [24]. By their theory: there is a characteristic of the material called the Curie's temperature, whence below this critical temperature, ferromagnetic bodies which are large enough would break up into small uniformly magnetized regions and separated by thin transition layers. The small uniformly magnetized regions are called Weiß domains and the transition layers are called Bloch walls. Moreover the magnetization in a domain $D \subset \mathbb{R}^3$ at time $t > 0$ given by $M(t,x) \in \mathbb{R}^3$ satisfies the following Landau-Lifschitz equation:

$$\frac{\mathrm{d}M(t,x)}{\mathrm{d}t} = \lambda_1 M(t,x) \times \rho(t,x) - \lambda_2 M(t,x) \times (M(t,x) \times \rho(t,x)). \tag{1.1}$$

The ρ in the equation (1.1) is called the effective magnetic field and defined by

$$\rho = -\nabla_M \mathcal{E}, \tag{1.2}$$

where the $\mathcal{E}$ is the so called total electro-magnetic energy which composed by anisotropy energy, exchange energy and electronic energy. As explained in [13], in order to describe phase transitions between different equilibrium states induced by thermal fluctuations of the effective magnetic field ρ, we introduce the Gaussian noise into the Landau-Lifschitz equation to perturb ρ and which should have the following form:

$$\mathrm{d}M(t) = \left[\lambda_1 M(t) \times \rho(t) - \lambda_2 M(t) \times (M(t) \times \rho(t))\right] \mathrm{d}t + (M(t) \times h) \circ \mathrm{d}W(t).$$

This is the form of stochastic Landau-Lifschitz which will be studied in this thesis (but with different forms of ρ, h and $W(t)$).

The structure of this thesis is as following:

Section 2 contains some basic knowledge for reading this thesis and the Lemmas which will be referred in the following sections.

Section 3 gives details of explanation of the evolution situation in the Visintin's paper [51], which studies the deterministic Landau-Lifshitz Equation coupled by Maxwell's equations with the following form:

$$\begin{cases} \mathrm{d}M = \lambda_1 M \times \rho \,\mathrm{d}t - \lambda_2 M \times (M \times \rho) \,\mathrm{d}t \\ \mathrm{d}B = -\nabla \times E \,\mathrm{d}t \\ \mathrm{d}E = \nabla \times H \,\mathrm{d}t - (1_D E + \tilde{f}) \,\mathrm{d}t \end{cases} \tag{1.3}$$

where M is the magnetization, ρ is the effective magnetic field, E is the electric field, H is the magnetic field and B is defined by $B := H + \tilde{M}$, the tilde here means extend a function defined on $D \subset \mathbb{R}^3$ to $\mathbb{R}^3$ with value 0. This section will help us to understand the next two main sections of this thesis.

Brzeźniak and Goldys and Jegaraj [13] studied the Stochastic Landau-Lifschitz Equation with only the exchange energy taken into account:

(1.4)
$$\begin{cases} \mathrm{d}u(t) = (\lambda_1 u(t)\times\Delta u(t) - \lambda_2 u(t)\times(u(t)\times\Delta u(t)))\,\mathrm{d}t + (u(t)\times h)\circ\,\mathrm{d}W(t), \\ \frac{\partial u}{\partial n}(t,x) = 0, \qquad t>0, x\in\partial D, \\ u(0,x) = u_0(x), \qquad x\in D. \end{cases}$$

where Δu stands for the exchange energy. In section 4, a stochastic Landau-Lifshitz equation with a more general exchange energy has been studied, which has been formulated by:

$$\begin{cases} \mathrm{d}u(t) & = \Big\{\lambda_1 u(t)\times[\Delta u(t) - \nabla\phi(u(t))] \\ & \quad -\lambda_2 u(t)\times\Big(u(t)\times[\Delta u(t) \quad - \nabla\phi(u(t))]\Big)\Big\}\,\mathrm{d}t \\ & \qquad +\Big\{u(t)\times h\Big\}\circ\,\mathrm{d}W(t) \\ \frac{\partial u}{\partial \nu}\Big|_{\partial D} & = 0 \\ u(0) & = u_0 \end{cases} \tag{1.5}$$

where $\phi:\mathbb{R}^3\longrightarrow\mathbb{R}^+$ defined as a C^4 function which satisfies: ϕ' ϕ'' and $\phi^{(3)}$ are bounded. The existence of the weak solution has been concluded and some similar regularity properties as in Brzeźniak and Goldys and Jegaraj's paper [13] has been obtained.

In Section 5, a full stochastic Landau-Lifthitz equation coupled by the Maxwell's equations has been studied which formulated by:

$$\begin{cases} \mathrm{d}M(t) = & [\lambda_1 M(t)\times\rho(t) - \lambda_2 M(t)\times(M(t)\times\rho(t))]\,\mathrm{d}t \\ & + \sum_{j=1}^{\infty}[\alpha M(t)\times h_j + \beta M(t)\times(M(t)\times h_j)]\circ\,\mathrm{d}W_j(t) \\ \mathrm{d}B(t) = & -\nabla\times E(t)\,\mathrm{d}t \\ \mathrm{d}E(t) = & [\nabla\times H(t) - 1_D E(t) - \tilde{f}(t)]\,\mathrm{d}t, \quad t\in[0,T]. \end{cases} \tag{1.6}$$

The M,ρ,E,H,B have the same meaning as in Section 3. As previous section, the existence of the weak solution as well as some regularity have been obtained.

The Sections 3, 4 and 5 were written separately with independent notations. The proof of the existence of the weak solutions are all followed by two steps:

1st Step: Using the Faedo-Galerkin approximation to get a series of SDEs on finite dimensional spaces which have unique solution. Then prove some uniform (with respect to n) bounds in various norms of the solutions.

2nd Step: Using some compactness results and Skorohod's Theorem to show that there is another probability space in which there are some processes can be identified as a weak solution of the original equation.

2. Preliminaries

2.1. Linear operators.

Definition 2.1 (relation)**.** Let X, Y be nonempty sets, a *relation* on $X \times Y$ is a nonempty subset of $X \times Y$.

Example 2.2.(a). Identity relation:

$$R = \{(x, x) : x \in X\}.$$

(b). Let $X = \mathbb{R}$, $R = \{(x, y) : x \leq y\}$.

Definition 2.3 (function)**.** Let X, Y be nonempty sets, a *function* f on $X \times Y$ is a relation, such that

$$(x, y_1), (x, y_2) \in f \Longrightarrow y_1 = y_2.$$

We also define

$$D(f) := \{x \in X : \exists y \in Y, (x, y) \in f\},$$
$$R(f) := \{y \in Y : \exists x \in X, (x, y) \in f\}.$$

And we denote $f : D(f) \longrightarrow Y$ and $f(x) = y$ means that $(x, y) \in f$.

Definition 2.4 (linear operator)**.** Let X, Y be two Banach spaces, a *linear operator* A from X to Y is a function $X \supset D(A) \longrightarrow Y$ which is linear and bounded.

Definition 2.5 (symmetric)**.** Let H be a Hilbert space, a densely defined linear operator $A : H \supset D(A) \longrightarrow H$, is called *symmetric* iff for any $x, y \in D(A)$,

$$(Ax, y) = (x, Ay).$$

Lemma 2.6. *Let H be a Hilbert space, then the functions,*

$$(\cdot, y) : H \longrightarrow \mathbb{C}, \text{ and } (y, \cdot) : H \longrightarrow \mathbb{C}, \qquad y \in H,$$

are continuous on H.

Lemma 2.7. *Let H be a Hilbert space, if for some $y \in H$,*

$$(x, y) = 0, \qquad \forall x \in H, \tag{2.1}$$

then $y = 0$.

Proof. From (2.1), $(y, y) = 0$, hence $y = 0$. □

2.1.1. *Adjoint operator.*

Definition 2.8. Let H be a Hilbert space and A be a densely defined linear operator on H, then we define:

$$D(A^*) := \{y \in H : \exists z \in H, \forall x \in D(A), (Ax, y) = (x, z)\}. \tag{2.2}$$

Definition 2.9 (adjoint operator on Hilbert space)**.** Let H be a Hilbert space and A be a densely defined linear operator on H, then we define the *adjoint operator* A^* of A by $A^* : D(A^*) \longrightarrow H$, and for $x \in D(A)$, $y \in D(A^*)$,

$$(Ax, y) = (x, A^* y).$$

Remark 2.10. The A^* defined in Definition 2.9 is really a function. And this is why we need $D(A)$ is dense in H.

Proof. Suppose that $(Ax, y) = (x, z_1) = (x, z_2), \forall x \in D(A)$ in (2.2), then $(x, z_1 - z_2) = 0, \forall x \in D(A)$.
Actually $(x, z_1 - z_2) = 0, \forall x \in H$. Because $D(A)$ is dense in H, for $x \in H$, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} x_n = x$, and from Lemma 2.6,

$$(x, z_1 - z_2) = \lim_{n\to\infty} (x_n, z_1 - z_2) = 0.$$

Then from Lemma 2.7, $z_1 = z_2$, so by Definition 2.3, A^* is a function, this completes the proof. □

Definition 2.11 (adjoint operator on Banach space)**.** If A is a linear operator on Banach space X, and $D(A)$ is dense in X, X^* is the dual space of X then we define:

$$D(A^*) := \{x^* \in X^* : \exists y^* \in X^*, \forall x \in D(A), x^*(Ax) = \langle Ax, x^* \rangle = \langle x, y^* \rangle\}.$$

Then we define

$$A^* x^* = y^*.$$

Proposition 2.12. *Let H be a Hilbert space, if A is symmetric, then $A \subset A^*$.*

Proof. To prove this, we just need to show that

- $D(A) \subset D(A^*)$;
- $\forall y \in D(A), Ay = A^* y$.

Firstly, if $y \in D(A)$, then because A is symmetric, for $x \in D(A)$, $(Ax, y) = (x, Ay)$ which implies that $y \in D(A^*)$, hence $D(A) \subset D(A^*)$.
Next, if $y \in D(A)$, then for $x \in D(A)$, $(Ax, y) = (x, Ay) = (x, A^* y)$, and because $D(A)$ is dense in H, $(x, Ay) = (x, A^* y)$ for all $x \in H$, hence $Ay = A^* y, \forall y \in D(A)$.
This completes the proof. □

Definition 2.13 (self-adjoint)**.** Let H be a Hilbert space, a density defined operator A on H is called *self-adjoint* iff

$$A = A^*.$$

Example 2.14. $H = L^2(D, \mathbb{R})$, where D is open in $\mathbb{R}^d$. Let $D(A) = C_0^{\infty}(D)$, and $Au = \Delta u$. It is easily seen that A is symmetric, but $A \neq A^*$, because $D(A^*) \supset C_0^2 \supsetneq C_0^{\infty} = D(A)$. But there exist operators B, such that $B \supset A$, $B^* = B$.

Proposition 2.15. *Let H be a Hilbert space, then if the linear operators A_1, A_2 are densely defined and $A_1 \subset A_2$, then $A_2^* \subset A_1^*$. In particular, $D(A_2^*) \subset D(A_1^*)$.*

Proof. Let us take and fix $y \in D(A_2*)$. Then

$$(A_2 x, y) = (x, A_2^* y), \qquad \forall x \in D(A_2).$$

Since $D(A_1) \subset D(A_2)$, it follows that

$$(A_2 x, y) = (x, A_2^* y), \qquad \forall x \in D(A_1).$$

Since $A_1 \subset A_2$, we infer that

$$(A_1 x, y) = (x, A_2^* y), \qquad \forall x \in D(A_1).$$

Thus by the definition of adjoint operator, $y \in D(A_1^*)$, and $A_1^* y = A_2^* y$. So we proved that $A_2^* \subset A_1^*$. □

Proposition 2.16. *Let H be a Hilbert space and $A : D(A) \longrightarrow H$ be a symmetric operator on H, $\lambda \in \mathbb{R}$, then $\lambda I - A$ is a symmetric operator on H.*

Proof. Let us take $x, y \in D(A)$, then for $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} ((\lambda I - A)x, y) &= (\lambda x, y) - (Ax, y) \\ &= \lambda(x, y) - (x, Ay) \\ &= (x, \lambda y) - (x, Ay) \\ &= (x, (\lambda I - A)y) \end{aligned}$$

So, $\lambda I - A$ is also symmetric, which completes the proof. □

Proposition 2.17. *Let H be a Hilbert space and $A : D(A) \longrightarrow H$ be a linear operator on H, if $R(A) = H$, then $ker(A^*) = \{0\}$.*

Proof. Let us consider $y \in D(A^*)$ such that $A^* y = 0$. Then for any $x \in D(A)$, $(x, A^* y) = 0$, but $(x, A^* y) = (Ax, y)$, for $x \in D(A)$, hence

$$(Ax, y) = 0, \qquad \forall x \in D(A),$$

and because $R(A) = H$,

$$(x, y) = 0, \qquad \forall x \in H.$$

Therefore $y = 0$.
So $ker(A^*) = \{0\}$, which completes the proof. □

Proposition 2.18. *Let H be a Hilbert space and $A : D(A) \longrightarrow H$ be a symmetric operator on H that satisfies $R(A) = H$, then A is self-adjoint.*

Proof. Suppose that $y \in D(A^*)$, since $R(A) = H$, there exists $x \in D(A)$ such that, $Ax = A^* y$. From Proposition 2.12, $D(A) \subset D(A^*)$, so $A^* x = A^* y$, that is $A^*(x - y) = 0$. And from Proposition 2.17, because $R(A) = H$, $ker(A^*) = \{0\}$, so $y = x \in D(A)$, hence we proved $D(A) \supset D(A^*)$. Therefore $A = A^*$, which completes the proof. □

Corollary 2.19. *Let H be a Hilbert space, $A : D(A) \longrightarrow H$ be a symmetric operator on H and for a certain $\lambda \in \mathbb{R}$, $R(\lambda I - A) = H$, then A is self-adjoint.*

Proof. From Proposition 2.16, $\lambda I - A$ is also symmetric and from Proposition 2.18, $\lambda I - A$ is self-adjoint. Notice that

$$D((\lambda I - A)^*) = \{y \in H : \exists z \in H, \forall x \in D(\lambda I - A) = D(A), ((\lambda I - A)x, y) = (x, z)\}.$$

Suppose that $y \in D(A^*)$, then $\exists z \in H$, $\forall x \in D(A)$, $(Ax, y) = (x, z)$, then

$$\begin{aligned} ((\lambda I - A)x, y) &= \lambda(x, y) - (Ax, y) \\ &= \lambda(x, y) - (x, z) \\ &= (x, \lambda y - z) \end{aligned}$$

Hence $y \in D((\lambda I - A)^*)$, so

$$D(A) \subset D(A^*) \subset D((\lambda I - A)^*) = D(\lambda I - A) = D(A).$$

Therefore $D(A) = D(A^*)$, which means that A is self-adjoint, and this completes the proof. □

2.1.2. *Closed operator.*

Definition 2.20 (closed)**.** Let X be a Banach space, then a linear operator $A : D(A) \longrightarrow X$ is *closed* iff A is a closed subset of X^2.

Proposition 2.21. *Let X be a Banach space, then a linear operator $A : D(A) \longrightarrow X$ is closed iff: if $\{x_n\} \subset D(A)$ and $x_n \longrightarrow x$, $Ax_n \longrightarrow y$, then $x \in D(A)$, and $Ax = y$.*

Proposition 2.22. *Let X be a Banach space, if A is a bounded linear operator and $D(A) = X$, then A is closed.*

Proof. If $\{x_n\} \subset X$, and $x_n \to x$, $Ax_n \to y \in X$, then because $D(A) = X$, so $x \in D(A)$ and because A is bounded, so as $n \to \infty$,

$$\begin{aligned} \|A(x) - y\| &\le \|A(x) - A(x_n)\| + \|A(x_n) - y\| \\ &\le \|A\| \cdot \|x - x_n\| + \|A(x_n) - y\| \\ &\to 0 \end{aligned}$$

Hence $A(x) = y$, and therefore A is closed. □

Definition 2.23 (closable)**.** Let X be a Banach space, then the linear operator $A : D(A) \longrightarrow X$ is *closable* iff the closure of A in X^2 is some linear operator $S : D(S) \longrightarrow X$.

Definition 2.24 (smallest closed extension)**.** Let X be a Banach space, $A : D(A) \longrightarrow X$ is a linear operator. We define S as the *smallest closed extension* of A by

$$D(S) := \left\{ x \in X : \exists \{x_n\} \subset D(A), x_n \longrightarrow x \text{ and } \lim_{n\to\infty} Ax_n = y \text{ exists} \right\}, \tag{2.3}$$

and for such x, y in (2.3), we set

$$Sx = y.$$

Remark 2.25. (2.3) is equivalent to

$$D(S) = \left\{ x \in X : \exists \{x_n\} \subset D(A), y \in X, \text{ such that } \lim_{n\to\infty} (x_n, Ax_n) = (x, y) \right\}.$$

Proposition 2.26. *Let X be a Banach space, $A : D(A) \longrightarrow X$ be a linear operator. Then the smallest closed extension of A is closed.*

Proof. Let $w_n \in D(S)$, $w_n \longrightarrow w$ and $Sw_n \longrightarrow u$. Then by Proposition 2.21, we only need to prove $w \in D(S)$ and $Sw = u$.
Notice that $w_n \in D(S)$ means that there exists $\{x_{n,m}\} \subset D(A)$, $\lim_{m\to\infty} x_{n,m} = w_n$ and $\lim_{m\to\infty} Ax_{n,m} = Sw_n$. So $\forall w_n \in D(S)$, $\exists x_n \in D(A)$, such that

$$\|x_n - w_n\| < \frac{1}{n} \quad \text{and} \quad \|Ax_n - Sw_n\| < \frac{1}{n}.$$

Hence

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} w_n = w \quad \text{and} \quad \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sw_n = u.$$

So S is closed. This ends the proof. □

Proposition 2.27. *Let X be a Banach space, then the linear operator $A : D(A) \longrightarrow X$ is closable iff: if $\{x_n\} \subset D(A)$ and $x_n \longrightarrow 0$, $Ax_n \longrightarrow y$ then $y = 0$.*

Proof. We need to prove: $\overline{A} = S$ for some $S : D(S) \longrightarrow X \Longleftrightarrow$ if $D(A) \ni x_n \longrightarrow 0$ and $Ax_n \longrightarrow y$ exists, then $y = 0$.

"$\Longrightarrow$": If $(x_n, Ax_n) \longrightarrow (0, y)$, then $(0, y) \in \overline{A} = S$, so $S(0) = y$, because S is linear, so $y = 0$.
"$\Longleftarrow$": We prove that is satisfied for S is the smallest closed extension of A.
If $x \in D(A)$, then let $x_n = x, \forall n$, then $x_n \longrightarrow x$ and $Ax_n \longrightarrow Ax$, so $x \in D(S)$ and $Sx = Ax$. Hence $A \subset S$ and from Proposition 2.26, S is closed, so $\overline{A} \subset S$.
On the other hand, $\forall (x, y) \in S$, $\exists \{x_n\} \subset D(A)$, such that $(x_n, Ax_n) \longrightarrow (x, y)$. Hence $(x, y) \in \overline{A}$, so $\overline{A} \supset S$.
Therefore $\overline{A} = S$, this completes the proof.

□

Lemma 2.28. *Let X be a Banach space and $A : D(A) \longrightarrow X$ be a linear operator such that $D(A)$ is a subspace of X, A is invertible and A^{-1} is bounded. Then there exists $\delta > 0$ such that if a linear operator $B : X \longrightarrow X$ is bounded and $\|B\| < \delta$, then $A - B$ with $D(A - B) = D(A)$ is also invertible and $(A - B)^{-1}$ is also bounded.*

Proof. Let $\delta = \frac{1}{\|A^{-1}\|}$, then for $\|B\| < \delta$, $\|A^{-1}B\| < 1$. We will prove this Lemma by the following steps:

Step 1: $\sum_{k=0}^{\infty}(A^{-1}B)^k$ is converge to some element in H^*.
For any $m \in \mathbb{N}$,

$$\|\sum_{k=N+1}^{N+m}(A^{-1}B)^k\| \leq \sum_{k=N+1}^{N+m}\|A^{-1}B\|^k \longrightarrow 0, \text{ as } N \longrightarrow \infty,$$

so $\left(\sum_{k=0}^{N}(A^{-1}B)^k\right)_{N=0}^{\infty}$ is a Cauchy sequence in H^*, but H^* is complete, hence $\sum_{k=0}^{\infty}(A^{-1}B)^k$ is convergence.
Step 2: $(I - A^{-1}B)^{-1} = \sum_{k=0}^{\infty}(A^{-1}B)^k$.
That is because:

$$\begin{aligned}
&(I - A^{-1}B)\left(\sum_{k=0}^{N}(A^{-1}B)^k\right)\\
&= I + \left(\sum_{k=1}^{N}(A^{-1}B)^k\right) - \left(\sum_{k=1}^{N}(A^{-1}B)^k\right)\\
&= I
\end{aligned}$$

Step 3: $(A - B)^{-1} = (I - A^{-1}B)^{-1}A^{-1}$ is bounded. Because $(I - A^{-1}B)^{-1} \in H^*$, it is bounded, and A^{-1} is bounded, so $(A - B)^{-1}$ is bounded.

This completes the proof. □

2.1.3. *Compact operator.*

Definition 2.29 (precompact)**.** Let X be a Banach space, $A \subset X$ is called *precompact* iff $\overline{A}$ is compact.

Definition 2.30 (Compact operator)**.** Let X, Y be Banach spaces, $T : X \longrightarrow Y$ be a linear operator, T is called *compact* iff $T(B(0,1))$ is precompact in Y, where $B(0,1) = \{x \in X : \|x\| \leq 1\}$.

Proposition 2.31. *Let X, Y be Banach spaces, if $T : X \longrightarrow Y$ is compact, then T is bounded.*

Proof. We prove this by contradiction. Suppose that T is not bounded, then there exist $\{x_n\}_{n=1}^{\infty} \subset B(0,1)$, such that $T(x_n) \geq n$. $T(x_n) \in \overline{T(B(0,1))}$ and $\overline{T(B(0,1))}$ is compact, so there is a convergent subsequence $\{T(x_{n_k})\}_{k=1}^{\infty} \subset \{T(x_n)\}_{n=1}^{\infty}$. But $T(x_{n_k}) \geq n_k$, so it is not possible to converge. Hence we have got the contradiction, which completes the proof. □

Lemma 2.32. *Let H be a Hilbert space and $\dim H < \infty$, then $B(0,1)$ is compact; Conversely, if $B(0,1)$ is compact, then $\dim H < \infty$.*

Proposition 2.33. *Let H be a Hilbert space and $\dim H = n$ and $T : H \longrightarrow H$ is linear, then T is compact.*

Proof. Because H has only finite dimensional, T is bounded. Hence $\overline{T(B(0,1))}$ is a closed and bounded subset of a finite dimensional Hilbert space H. So $\overline{T(B(0,1))}$ is compact. Therefore T is compact, which completes the proof. □

Proposition 2.34. *Let H be a Hilbert space and $\dim H = \infty$ and $T = \lambda I$, $\lambda \in C$, then T is not compact.*

Proof. Notice that

$$\overline{T(B(0,1))} = B(0, |\lambda|),$$

and because $\dim H = \infty$, $B(0,|\lambda|)$ is not compact. So T is not compact. This completes the proof. □

Proposition 2.35. *A linear operator with a finite dimensional range is compact. If X, Y are Hilbert spaces, $T : X \longrightarrow Y$ is a linear bounded operator such that $dim T(X)$ is finite, then A is compact.*

Proof. Suppose that $A \subset X$ is bounded. Since T is bounded, $T(A)$ is bounded in Y, therefore $\overline{T(A)}$ is bounded too.
We can also prove that $\overline{T(A)} \subset T(X)$. Assume that $T(x) = lin\{e_1, \ldots, e_n\}$, $Y = lin\{e_1, \ldots, e_n, \ldots\}$, $y = \sum_{i=1}^{\infty} y_i e_i$. If $y \in \overline{T(A)}$, but $y \notin T(x)$, then $\exists k \in \mathbb{N}, k > n$, such that $y_k \neq 0$. Hence

$$|T(x) - y| \geq y_k, \qquad \forall x \in X.$$

This is contradict to $y \in \overline{T(A)}$. Therefore $\overline{T(A)} \subset T(X)$.
$\overline{T(A)}$ is a closed and bounded subset of a finite dimensional space, hence it is compact. Therefore T is compact. □

Proposition 2.36. *If $T_n : X \longrightarrow Y$ is a compact operator for each n, and $T : X \longrightarrow Y$ such that $\lim_{n\to\infty} \|T_n - T\| = 0$, then T is compact.*

Proof. Let B be the unit ball in X. For $\varepsilon > 0$, $\exists N > 0, N \in \mathbb{N}$, such that if $k > N$,

$$|T_k x - T x|_Y \leq \frac{1}{2}\varepsilon, \qquad \forall x \in B. \tag{2.4}$$

And T_k is compact, so $T_k(B)$ is precompact in Y, hence there exist $\{y_i\}_{i=1}^n \subset Y$, such that $T_k(B) \subset \bigcup_{i=1}^n B(y_i, \frac{1}{2}\varepsilon)$.
Hence for $x \in B$, $T_k(x) \in B(y_i, \frac{1}{2}\varepsilon)$ for some $i \in \mathbb{N}$, and by (2.4),

$$|T x - T_k x| < \frac{1}{2}\varepsilon, \qquad \forall k > N.$$

So $T x \in B(y_i, \varepsilon)$, therefore $T(B) \subset \bigcup_{i=1}^n B(y_i, \varepsilon)$, which means that $T(B)$ is precompact, hence T is compact. □

Example 2.37. Let

$$H = \left\{ x = \sum_{i=1}^{\infty} x_i e_i : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\} = l^2,$$

where $\{e_i\}$ is the ONB in H, so $\{e_i\}$ doesn't contain a convergent subsequence.

$$Y = \left\{ x = \sum_{i=1}^{\infty} x_i e_i : \sum_{i=1}^{\infty} \frac{1}{i} |x_i|^2 < \infty \right\},$$

Y is also a Hilbert space and

$$\|x\|_Y^2 = \sum_{i=1}^{\infty} \frac{1}{i} |x_i|^2 \leq \sum_{i=1}^{\infty} |x_i|^2 = \|x\|_H.$$

Hence $H \hookrightarrow Y$ is continuous. And also we can prove:
(a) : The imbedding $T : H \hookrightarrow Y$ is compact;
(b) : $e_n \longrightarrow 0$ in Y.

Proof.(a) : Let us define $T_n := \pi_n \circ T$. Then T_n is a bounded operator with a finite dimensional range, so from Proposition 2.35, T_n is a compact operator. And

$$\lim_{n\to\infty} \|T_n - T\| = \lim_{n\to\infty} \sup_{\{x:\sum |x_i|^2=1\}} \sum_{i=n+1}^{\infty} \frac{1}{i} |x_i|^2 \leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=n+1}^{\infty} |x_i|^2 = 0,$$

hence from Proposition 2.36, T is compact.
(b) :

$$\|e_n\|_Y^2 = \frac{1}{n} 1^2 = \frac{1}{n} \longrightarrow 0.$$

This completes the proof.

□

Lemma 2.38. *Suppose that X and Y are Banach spaces such that the embedding $X \hookrightarrow Y$ is compact. Assume that $\{x_n\}_{n=1}^{\infty} \subset X$ is a sequence such that $\|x_n\|_X \leq R$ for all $n \in \mathbb{N}$ and $R > 0$. Then there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ and a $y \in Y$, such that $x_{n_k} \longrightarrow y$ in Y.*

Proof. By the Definition 2.30 of a compact operator, $\overline{\{x \in Y : \|x\|_X \leq R\}}$ is compact in Y, but $\{x_n\}_{n=1}^{\infty} \subset \overline{\{x \in Y : \|x\|_X \leq R\}} \subset Y$ and for metric space, sequentially compact is equivalent to compact. Hence there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ and a $y \in Y$, such that $x_{n_k} \longrightarrow y$ in Y, which completes the proof. □

Lemma 2.39 (Schauder, p.282 in [53])**.** *Let X, Y be Banach spaces, an operator $T \in \mathcal{L}(X, Y)$ is compact iff its dual operator T' is compact.*

Lemma 2.40 (Lemma 2.5, p.99 in [28])**.** *For every separable Hilbert space H, there is another Hilbert space $U \subset H$, such that the embedding $U \hookrightarrow H$ is compact and dense.*

2.1.4. *Resolvent.*

Definition 2.41. Let X be a complex Banach space and $A : D(A) \longrightarrow X$ be a linear operator, $D(A)$ be a subspaces of X, we define the *resolvent set* $\rho(A)$ of A by

$$\rho(A) = \Big\{\lambda \in \mathbb{C} : R(\lambda I - A) \text{ is dense in } X \text{ and } \lambda I - A \text{ is invertible}$$
$$\text{and } (\lambda I - A)^{-1} \text{ is bounded}\Big\}.$$

And we define the *resolvent* by

$$R(\lambda; A) = (\lambda I - A)^{-1}.$$

Proposition 2.42. *Let X be a complex Banach space and $A : D(A) \longrightarrow X$ be a closed operator, $D(A)$ is a subspaces of X. If $\lambda \in \rho(A)$, then the resolvent $(\lambda I - A)^{-1}$ is defined on X, i.e. $R(\lambda I - A) = X$.*

Proof. Because the operator $(\lambda I - A)^{-1}$ is bounded, there exists $c > 0$, such that

$$|(\lambda I - A)^{-1} y| \leq c|y|, \qquad \forall y \in R(\lambda I - A)$$

, then

$$\|(\lambda I - A)^{-1}(\lambda I - A)x\| \leq c\|(\lambda I - A)x\|, \qquad \forall x \in D(A).$$

so

$$\|x\| \leq c\|(\lambda I - A)x\|. \tag{2.5}$$

Let us take $y \in X$, because $R(\lambda I - A)$ is dense in X, $\exists \{x_n\} \subset D(A)$, such that $\lim_{n\to\infty}(\lambda I - A)x_n = y$. Hence $\{(\lambda I - A)x_n\}$ is a Cauchy sequence, and by (2.5), we infer that $\{x_n\}$ is also Cauchy. X is complete, $\lim_{n\to\infty} x_n = x$ for some $x \in X$. Hence because A is closed, $x \in D(A)$ and $(\lambda I - A)x = y$. This means that $y \in R(\lambda I - A)$. So we proved that $R(\lambda I - A) = X$, which completes the proof. □

Proposition 2.43. *Let X be a complex Banach space and $A : D(A) \longrightarrow X$ is a closed operator, $D(A)$ is a subspaces of X. Then the resolvent set is an open set of the complex plane.*

Proof. From Proposition 2.42, $R(\lambda; A)$ is an everywhere defined continuous operator. Let $\lambda_0 \in \rho(A)$, and consider

$$S(\lambda) = R(\lambda_0; A)\left\{I + \sum_{n=1}^{\infty}(\lambda_0 - \lambda)^n R(\lambda_0; A)^n\right\}.$$

$S(\lambda)$ convergence and so bounded iff

$$|\lambda_0 - \lambda| \cdot \|R(\lambda_0; A)\| < 1,$$

and notice that multiplication by $(\lambda_0 I - A)(I - (\lambda_0 - \lambda)R(\lambda_0; A)) = \lambda I - A$ on the left or right of $S(\lambda)$ gives I. So $S(\lambda) = R(\lambda; A)$.
And because $R(\lambda; A)$ is densely defined, from the construction of $S(\lambda)$, it is also densely defined. Hence we have got: if $\lambda_0 \in \rho(A)$ and

$$|\lambda_0 - \lambda| < \frac{1}{\|R(\lambda_0; A)\|},$$

then $\lambda \in \rho(A)$. So $\rho(A)$ is open. This ends the proof. □

2.1.5. *Spectrum.*

Definition 2.44. Let X be a Banach space, and $T : X \longrightarrow X$ is linear, $\rho(T)$ is the resolvent of T, then we define:

(a) $\sigma(T) = \rho(T)^c$ as the *spectrum* of T.
(b) An $x \neq 0$ which satisfies $\lambda x = Tx$ for some $\lambda \in \mathbb{C}$ is called an *eigenvector* of T and λ is called the corresponding *eigenvalue*. The set of all the eigenvalues is called the *point spectrum* of T, which denoted by $\sigma_p(T)$.
(c) If λ is not an eigenvalue and if $R(\lambda I - T)$ is not dense, then λ is said to be in the *residual spectrum*.

Lemma 2.45. *Let H be a Hilbert space and $A : D(A) \longrightarrow H$ is a compact operator, then $\sigma(A)$ is a discrete set having no limit points except perhaps $\lambda = 0$. Further, any nozero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity.(i.e. the corresponding space of eigenvectors is finite dimensional).*

Corollary 2.46. *Let H be a Hilbert space and $T : H \longrightarrow H$ is a linear operator, if T is compact and $\lambda \in \sigma_p(T)\backslash\{0\}$, then* $\dim ker(\lambda I - T) < \infty$.

Proof. On $ker(\lambda I - T)$ which is also a Hilbert space, $T = \lambda I$, then from Proposition 2.34, $\dim ker(\lambda I - T) < \infty$. Or T should not compact. This completes the proof. □

Proposition 2.47. *Let H be a Hilbert space and $T : H \longrightarrow H$ is a linear operator, if T is bounded, then $\sigma(T) \subset B(0, \|T\|)$.*

Proof. To prove this, we only need to prove that if $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$.
If $|\lambda| > \|T\|$, then the series:

$$\frac{1}{\lambda}\left\{I + \sum_{n=1}^{\infty}\left(\frac{T}{\lambda}\right)^n\right\},$$

converges to a bounded operator in norm. Then we can deduce that

$$(\lambda I - T)\frac{1}{\lambda}\left\{I + \sum_{n=1}^{\infty}\left(\frac{T}{\lambda}\right)^n\right\} = \frac{1}{\lambda}\left\{I + \sum_{n=1}^{\infty}\left(\frac{T}{\lambda}\right)^n\right\}(\lambda I - T) = I,$$

hence $(\lambda I - T)$ is invertible and $(\lambda I - T)^{-1}$ is bounded. This means that $\lambda \in \rho(T)$, so the proof has been complete. □

Lemma 2.48. *Let X be a Banach space and $T : X \longrightarrow X$ is linear. Then,*

(a) *If λ is in the residual spectrum of T, then λ is in the point spectrum of T^*;*
(b) *If λ is in the point spectrum of T, then λ is in either the point spectrum or the residual spectrum of T^*.*

Lemma 2.49. [42] *Let H be a Hilbert space and $T : H \longrightarrow H$ is a self-adjoint operator, then*

(a) *T has no residual spectrum;*
(b) $\sigma(T) \subset \mathbb{R}$.
(c) *Eigenvectors corresponding to distinct eigenvalues of T are orthogonal.*

Remark 2.50. If we define the resolvent as

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ bijective and } (\lambda I - T)^{-1} \text{ is bounded}\}.$$

If $\lambda I - T$ is injective and $R(\lambda I - T)$ is dense which means that λ neither in the point spectrum nor the residual spectrum, then from Proposition 2.22, $\lambda I - T$ is closed, and from Proposition 2.42, $\lambda I - T$ is bijective.

Definition 2.51 (projection)**.** Let X be a Banach space, then the linear operator $A : D(A) \longrightarrow X$ is a *projection* iff $A^2 = A$.

Definition 2.52 (orthogonal projection)**.** Let X be a Banach space, then the linear operator $A : D(A) \longrightarrow X$ is a *orthogonal projection* iff A is a projection and $A = A^*$.

Theorem 2.53. [42]*[continuous functional calculus] Let A be a self-adjoint operator on a Hilbert space H. Then there is a unique map*

$$\phi : C(\sigma(A)) \longrightarrow \mathscr{L}(H),$$

with the following properties:

(a) *ϕ is an algebraic *-homomorphism, that is*

$$\phi(fg) = \phi(f)\phi(g) \qquad \phi(\lambda f) = \lambda\phi(f)$$

$$\phi(1) = I \qquad \phi(\overline{f}) = \phi(f)^*$$

(b) *ϕ is continuous, that is, $\|\phi(f)\|_{\mathscr{L}(H)} \leq C\|f\|_\infty$.*
(c) *Let f be the function $f(x) = x$; then $\phi(f) = A$.*

Moreover, ϕ has the additional properties:

(d) *If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.*
(e) *$\sigma[\phi(f)] = \{f(\lambda) : \lambda \in \sigma(A)\}$[spectral mapping theorem].*
(f) *If $f \geq 0$, then $\phi(f) \geq 0$.*
(g) *$\|\phi(f)\|_{\mathscr{L}(H)} = \|f\|_\infty$. [This strengthens (b)].*

Definition 2.54. Let A be a self-adjoint operator on a Hilbert space H, $f \in C(\sigma(A))$. Then we define

$$f(A) := \phi(f),$$

where $\phi : C(\sigma(A)) \longrightarrow \mathscr{L}(H)$ is the one in Theorem 2.53.

Lemma 2.55 (Riesz-Markov theorem)**.** [42] *Let X be a compact Hausdorff space. For any positive linear functional l on $C(X)$ there is a unique Baire measure μ on X with*

$$l(f) = \int_X f \,\mathrm{d}\mu.$$

2.1.6. *Spectral Theorem: bounded operator.*

Definition 2.56 (spectral measure)**.** Let A be a bounded self-adjoint operator on Hilbert space H, $\psi \in H$, then

$$f \longmapsto (\psi, f(A)\psi),$$

is a positive linear functional on $C(\sigma(A))$. Thus by the Riesz-Markov theorem 2.55, there is a unique measure μ_ψ on the compact set $\sigma(A)$ with

$$(\psi, f(A)\psi) = \int_{\sigma(A)} f(\lambda) \,\mathrm{d}\mu_\psi.$$

Then the measure μ_ψ is called the *spectral measure associated with the vector ψ.*

Definition 2.57. Let A be a bounded self-adjoint operator on Hilbert space H, $g \in \mathscr{B}(\mathbb{R})$. It is natural to define $g(A)$ such that

$$(\psi, g(A)\psi) = \int_{\sigma(A)} g(\lambda) \,\mathrm{d}\mu_\psi(\lambda), \qquad \psi \in H.$$

Theorem 2.58 (spectral theorem-functional calculus form)**.** [42] *Let A be a bounded self-adjoint operator on Hilbert space H. There is a unique map*

$$\phi : \mathscr{B}(\mathbb{R}) \longrightarrow \mathscr{L}(H),$$

such that

(a) *ϕ is an algebraic *-homomorphism, that is*

$$\phi(fg) = \phi(f)\phi(g) \qquad \phi(\lambda f) = \lambda\phi(f)$$

$$\phi(1) = I \qquad \phi(\overline{f}) = \phi(f)^*$$

(b) *ϕ is continuous, that is, $\|\phi(f)\|_{\mathscr{L}(H)} \le \|f\|_\infty$.*
(c) *Let f be the function $f(x) = x$; then $\phi(f) = A$.*
(d) *Suppose $f_n(x) \longrightarrow f(x)$ for each x and $\|f_n\|_\infty$ is bounded. Then $\phi(f_n) \longrightarrow \phi(f)$ strongly.*

Moreover, ϕ has the additional properties:

(e) *If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.*
(f) *If $f \ge 0$, then $\phi(f) \ge 0$.*
(g) *If $BA = AB$, then $\phi(f)B = B\phi(f)$.*

Definition 2.59 (cyclic vector)**.** A vector $\psi \in H$ is called a *cyclic vector* for A if finite linear combinations of the elements $\{A^n\psi\}_{n=0}^\infty$ are dense in H.

Theorem 2.60 (spectral theorem-multiplication operator form)**.** [42] *Let A be a bounded self-adjoint operator on a separable Hilbert space H. Then there exist measures $\{\mu_n\}_{n=1}^N$ ($N = 1, 2, \ldots, \infty$) on $\sigma(A)$ and a unitary operator*

$$U : H \longrightarrow \bigoplus_{n=1}^{N} L^2(\mathbb{R}, \, \mathrm{d}\mu_n),$$

such that

$$(UAU^{-1}\psi)_n(\lambda) = \lambda\psi_n(\lambda), \qquad \lambda \in \mathbb{R},$$

where we write an element $\psi \in \oplus_{n=1}^{N} L^2(\mathbb{R}, \, \mathrm{d}\mu_n)$ as an N-tuple $\langle \psi_1(\lambda), \ldots, \psi_N(\lambda)\rangle$. This realization of A is called a spectral representation.

Definition 2.61 (spectral measures)**.** The measure μ_n as in Theorem 2.60 are called *spectral measures*; they are just μ_ψ for suitable ψ.

Corollary 2.62. [42] *Let A be a bounded self-adjoint operator on a separable Hilbert space H. Then there exists a finite measure space (M, μ), a bounded function F on M, and a unitary map:*

$$U : H \longrightarrow L^2(M, \, \mathrm{d}\mu),$$

such that

$$(UAU^{-1}f)(m) = F(m)f(m), \qquad m \in M.$$

Definition 2.63 (spectral projection)**.** Let H be a Hilbert space and $A : H \supset D(A) \longrightarrow H$ is a bounded self-adjoint operator and D a Borel set of $\mathbb{R}$. $P_D := 1_D(A)$ is called a *spectral projection* of A.

Definition 2.64 (projection valued measure)**.** A family of projections obeying

(a) Each P_D is an orthogonal projection;
(b) $P_\emptyset = 0$, $P_{(-a,a)} = I$ for some a;
(c) If $D = \bigcup_{n=1}^{\infty} D_n$, with $D_n \cap D_m = \emptyset$ for all $n \neq m$, then

$$P_D = s - \lim_{N\to\infty} \left(\sum_{n=1}^{N} P_{D_n} \right).$$

is called a *projection valued measure*.

Remark 2.65. If $\{P_D\}$ is a projection-valued measure, then for $\phi \in H$, $(\phi, P_D\phi)$ is a Borel measure on $\mathbb{R}$, which we denote by $\mathrm{d}(\phi, P_\lambda\phi)$.

Theorem 2.66. [42] *If $\{P_D\}$ is a projection-valued measure, and f a bounded Borel function on* $\operatorname{supp} P_D$*, then there is an unique operator B which we denote $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}P_\lambda$ such that,*

$$(\phi, B\phi) = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}(\phi, P_\lambda\phi), \qquad \phi \in H.$$

Theorem 2.67 (spectral theorem-p.v.m. form)**.** [42] *There is a one-one correspondence between (bounded) self-adjoint operators A and (bounded) projection valued measures $\{P_D\}$ given by:*

$$A \longmapsto \{P_D\} = \{1_D(A)\}$$

$$\{P_D\} \longmapsto A = \int_{\mathbb{R}} \lambda \,\mathrm{d}P_\lambda$$

2.1.7. *Spectral Theorem: unbounded operator.*

Theorem 2.68. [42]*[spectral theorem-multiplication operator form] Let A be a self-adjoint operator on a separable Hilbert space H with domain $D(A)$. Then there is a measure space (M,μ) with μ a finite measure, a unitary operator*

$$U : H \longrightarrow L^2(M,\, \mathrm{d}\mu),$$

and a real-valued function f on M which is finite a.e. such that

(a) $\psi \in D(A)$ *iff* $f(\cdot)(U\psi)(\cdot) \in L^2(M,\, \mathrm{d}\mu)$.
(b) *If* $\phi \in U[D(A)]$, *then* $(UAU^{-1}\phi)(m) = f(m)\phi(m)$.

Theorem 2.69. *Let A be a self-adjoint operator on a separable Hilbert space H with domain $D(A)$. Then there is a measure space (M,μ) with μ a finite measure, an isometric isomorphism*

$$U : H \longrightarrow L^2(M,\, \mathrm{d}\mu),$$

and a measurable function $f : M \longrightarrow \mathbb{R}$, such that there is an operator B on $L^2(M,\mu)$, defined by

$$D(B) = \left\{ g \in L^2(M,\mu) : \int_M |g(x)|^2 f^2(x)\,\mathrm{d}\mu < \infty \right\},$$

$$Bg = fg, \qquad g \in D(B).$$

And

$$U\Big|_{D(A)} : D(A) \longrightarrow D(B),$$

is bijection. And

$$Au = U^{-1}BU.$$

Example 2.70. Let $H = L^2(\mathbb{R})$, $A = -\Delta$, $D(A) = H^2(\mathbb{R})$. U is the Fourier transform, $f(x) = x^2$.

$$\begin{aligned} -\Delta u &= \mathcal{F}^{-1}(|x|^2 \hat{u}) \\ &= \mathcal{F}^{-1}(|x|^2 \mathcal{F} u). \end{aligned}$$

Theorem 2.71 (spectral theorem-functional calculus form)**.** [42] *Let A be a self-adjoint operator on Hilbert space H. Then there is a unique map*

$$\phi : \mathscr{B}_b(\mathbb{R}) \longrightarrow \mathscr{L}(H),$$

such that

(a) *ϕ is an algebraic *-homomorphism, that is*

$$\phi(fg) = \phi(f)\phi(g) \qquad \phi(\lambda f) = \lambda\phi(f)$$
$$\phi(1) = I \qquad \phi(\overline{f}) = \phi(f)^*$$

(b) *ϕ is continuous, that is, $\|\phi(f)\|_{\mathscr{L}(H)} \leq \|f\|_\infty$.*
(c) *Let f_n be the sequence of bounded Borel functions with $\lim_{n\to\infty} f_n(x) = x$ for each $x \in \mathbb{R}$ and $|h_n(x) \leq |x|$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for any $\psi \in D(A)$, $\lim_{n\to\infty} \phi(f_n)\psi = A\psi$.*
(d) *Suppose $f_n(x) \longrightarrow f(x)$ for each x and $\|f_n\|_\infty$ is bounded. Then $\phi(f_n) \longrightarrow \phi(f)$ strongly.*

Moreover, ϕ has the additional properties:

(e) *If $A\psi = \lambda\psi$, then $\phi(f)\psi = f(\lambda)\psi$.*
(f) *If $f \geq 0$, then $\phi(f) \geq 0$.*

Theorem 2.72 (spectral theorem-projection valued measure form)**.** [42] *Let H be a Hilbert space, $A : D(A) \longrightarrow H$ is a self-adjoint operator, $\{P_D\}$ are projection-valued measures on H, then*

$$A = \int_{\mathbb{R}} \lambda \, \mathrm{d}P_\lambda.$$

And if $g(\cdot)$ is a real-valued Borel function on $\mathbb{R}$, then

$$g(A) = \int_{\mathbb{R}} g(\lambda) \, \mathrm{d}P_\lambda,$$

is self-adjoint on

$$D(g) = \left\{ \phi : \int_{-\infty}^{\infty} |g(\lambda)|^2 \, \mathrm{d}P_\lambda < \infty \right\}.$$

And if g is bounded,

$$g(A) = \phi(g),$$

where ϕ is as in Theorem 2.71.

2.2. **Sobolev space.**

2.2.1. *Definitions.*

Definition 2.73 (Weak derivative)**.** Let D be a nonempty open subset of $\mathbb{R}^n$. $f \in L^1_{\text{loc}}(D)$, if there exists a function $g \in L^1_{\text{loc}}(D)$ such that

$$\int_D f(x) D^\alpha \varphi(x)\,\mathrm{d}x = (-1)^\alpha \int_D g(x)\varphi(x)\,\mathrm{d}x, \quad \varphi \in C_0^\infty(D),$$

then we say that $D^\alpha f = g$ *in the weak sense.*

Definition 2.74 (Sobolev space)**.** Let us suppose that $D \subset \mathbb{R}^n$ open, $p \geq 1$ and k a non-negative integer, we define the *Soblev space* as

$$W^{k,p}(D) = \{ f \in L^p(D) : \mathrm{D}^\alpha f \text{ exitsts in weak sense and } \mathrm{D}^\alpha f \in L^p(D), |\alpha| \leq k \}.$$

The space $W^{k,p}(D)$ is equipped with the norm

$$\|f\|_{k,p;D} = \left(\int_D \sum_{|\alpha|\leq k} |\mathrm{D}^\alpha f|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Definition 2.75. For $k = 0, 1, 2, \ldots$, we define:

$$H^k(D) = \{u \in L^2 : \mathrm{D}^\alpha u \in L^2 \text{ in weak sense}, \forall |\alpha| \leq k\},$$
$$H_0^k(D) = \overline{C_0^\infty(D)} \subset H^k(D).$$

Informal interpretation:

$$H_0^k(D) = \{u \in H^k : \mathrm{D}^\alpha u|_{\partial D} = 0, |\alpha| \leq k-1\}.$$

Theorem 2.76. *If $f \in W^{1,2}(a,b)$ then f is a.e. equals to a function*

$$\tilde{f}(x) = f(x_0) + \int_{x_0}^x (\mathrm{D}f)(y)\,\mathrm{d}y, \qquad x_0 \in (a,b),$$

which is continuous on $[a,b]$.

Remark 2.77. So $f \in W^{1,2}(a,b)$ can be identified with this continuous version of it. And we can define a map:

$$\begin{array}{ccc} W^{1,2}(a,b) & \longrightarrow & C([a,b]) \\ f & \longmapsto & \tilde{f}, \end{array}$$

which is linear and continuous.

Proposition 2.78 ([1] Th. 3.3)**.** *$W^{m,p}(D)$ is a Banach space.*

Proposition 2.79 ([1] Th. 3.6)**.** *$W^{m,p}(D)$ is separable if $1 \leq p < \infty$. And $W^{m,p}(D)$ is uniformly convex and reflexive if $1 < p < \infty$.*

Proposition 2.80 ([1] Th. 3.6)**.** *$W^{k,2}$ is a separable Hilbert space.*

Definition 2.81. For $\alpha < 1$, we define

$$C^\alpha[a,b] := \left\{ f \in C[a,b] : \sup_{a \leq x_1 < x_2 \leq b} \frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|^\alpha} < \infty \right\}.$$

Definition 2.82. If $p = 2$ and $u, v \in C^k(D)$, we define:

$$(u,v)_k := \int_D \sum_{|\alpha|\le k} \mathrm{D}^\alpha u \overline{\mathrm{D}^\alpha v}\,\mathrm{d}x.$$

Theorem 2.83.

$$W^{1,2}(a,b) \hookrightarrow C^{\frac{1}{2}}[a,b].$$

Proof. Suppose $f \in W^{1,2}(a,b)$. Take f to be continuous, then

$$\begin{aligned} |f(x_2) - f(x_1)| &= \left| \int_{x_1}^{x_2} \mathrm{D}f(y)\,\mathrm{d}y \right| \\ &\le \left(\int_{x_1}^{x_2} |\mathrm{D}f(y)|^2\,\mathrm{d}y \right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} 1^2\,\mathrm{d}y \right)^{\frac{1}{2}} \\ &\le \left(\int_a^b |\mathrm{D}f(y)|^2\,\mathrm{d}y \right)^{\frac{1}{2}} |x_2 - x_1|^{\frac{1}{2}} \\ &\le \|f\|_{W^{1,2}} |x_2 - x_1|^{\frac{1}{2}} \end{aligned}$$

Thus,

$$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|^{\frac{1}{2}}} \le \|f\|_{W^{1,2}} < \infty.$$

This ends the proof. □

Definition 2.84. Let $D \subset \mathbb{R}^n$ be a bounded domain, for $0 < \theta < 1$, we define $u \in C^\theta(D)$, iff

$$\sup_x |u(x)| + \sup_{x,y} \frac{|u(x) - u(y)|}{|x-y|^\theta} < \infty.$$

Theorem 2.85. [20]*[Characterization of H^k by Fourier transform] Let $k \in \mathbb{N}$, then a function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ iff*

$$(1 + |\xi|^k)\hat{u} \in L^2(\mathbb{R}^n). \tag{2.6}$$

Remark 2.86. Equation (2.6) also make sense for $k \in \mathbb{R}$, so we have the following definition.

Definition 2.87. For $\theta \in \mathbb{R}$, we define $u \in H^\theta(\mathbb{R}^n)$ iff $u \in L^2(\mathbb{R}^n)$ and

$$(1 + |\xi|^\theta)\hat{u}(\xi) \in L^2(\mathbb{R}^n). \tag{2.7}$$

Theorem 2.88. [36] *If $0 < \theta < 1$, then $u \in H^\theta(\mathbb{R}^n)$ iff $u \in L^2(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{1+2\theta}}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Definition 2.89. [36] If $0 < \theta < 1$ and $D \subset \mathbb{R}^n$, we define $u \in W^{\theta,p}(D)$ iff $u \in L^p(D)$ and

$$\int_D\int_D \frac{|u(x) - u(y)|^p}{|x-y|^{1+p\theta}}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Lemma 2.90. [10] *If u is a function from the interval $[0,T]$ to $\mathbb{R}$, (if $T=\infty$, then we consider $[0,T)$,) and*

$$0 = t_0 < t_1 < \cdots < t_m = T,$$

$$u_j := u\Big|_{[t_{j-1},t_j]} \in W^{1,2}(t_{j-1},t_j), \quad j=1,\dots,m,$$

and $u_j(t_j) = u_{j+1}(t_j)$, $j=1,\dots,m-1$, $u_1(0)=u_0\in\mathbb{R}$. Then

$$u \in W^{1,2}(0,T).$$

Proof. We must show that $u\in L^2(0,T;\mathbb{R})$ and $u'\in L^2(0,T;\mathbb{R})$, where u' is the weak derivative of u. If we define

$$w(t) = u_j'(t), \qquad t\in(t_{j-1},t_j),$$

where $u_j'(t)$ is the weak derivative of $u_j(t)$. And

$$\int_0^T |w(t)|\,\mathrm{d}t \le 2\sum_{j=1}^m \int_{t_{j-1}}^{t_j} |u_j'(t)|^2\,\mathrm{d}t < \infty.$$

So now we have two facts: $u\in L^2(0,T;\mathbb{R})$ and $w\in L^2(0,T;\mathbb{R})$. Thus it suffices to prove that $w=u'$ in $(0,T)$.
In other words, we have to show that

$$-\int_0^T \phi'(t)u(t)\,\mathrm{d}t = \int_0^T \phi(t)w(t)\,\mathrm{d}t, \qquad \phi\in C_0^\infty(0,T). \tag{2.8}$$

If $\operatorname{supp}\phi\subset(t_{j-1},t_j)$, then from the definition of u, we see that (2.8) is true. If $\phi=\psi_1+\psi_2$ and (2.8) is valid for ψ_1 and ψ_2, then it is valid for ϕ too.
Therefore it is enough to show that (2.8) is true for the case $\operatorname{supp}\phi\subset(t_{j-1},t_{j+1})$. For simplicity we take $j=1$. Then

$$\int_0^T \phi'(t)u(t)\,\mathrm{d}t = \int_0^{t_2} \phi'(t)u(t)\,\mathrm{d}t,$$

$$\int_0^T \phi(t)w(t)\,\mathrm{d}t = \int_0^{t_2} \phi(t)w(t)\,\mathrm{d}t.$$

Let us take $\psi\in C_0^\infty(\mathbb{R})$, such that $0\le\psi(t)\le 1$ for every $t\in\mathbb{R}$ and $\psi(t)=1$ if $|t|\le\frac12$ and $\psi(t)=0$ if $|t|\ge 1$.
For $\varepsilon>0$, we define $\psi_\varepsilon(t) := \psi\left(\frac{1}{\varepsilon}(t-t_1)\right)$, then we have

$$\phi = \phi\cdot\psi_\varepsilon + (1-\psi_\varepsilon)\cdot\phi,$$

$$\phi' = \phi'\cdot\psi_\varepsilon + \phi\cdot\psi_\varepsilon' + \phi'(1-\psi_\varepsilon) + \phi(-\psi_\varepsilon').$$

If $\varepsilon<\frac12\min\{t_2-t_1,t_1\}$, then $\phi(1-\psi_\varepsilon)(t_1)=0$, so $\operatorname{supp}\phi(1-\psi_\varepsilon)\subset(0,t_1)\cup(t_1,t_2)$ and by previous part

$$-\int_0^{t_2} [\phi(t)(1-\psi_\varepsilon(t))]'u(t)\,\mathrm{d}t = \int_0^{t_2} \phi(t)(1-\psi_\varepsilon(t))w(t)\,\mathrm{d}t.$$

Let us observe that

$$\int \phi'(t)u(t)\,\mathrm{d}t = \int [\phi(t)(1-\psi_\varepsilon(t))]'u(t)\,\mathrm{d}t + \int [\phi(t)\psi_\varepsilon(t)]'u(t)\,\mathrm{d}t,$$

$$-\int \phi(t)w(t)\,\mathrm{d}t = -\int \phi(t)(1-\psi_\varepsilon(t))w(t)\,\mathrm{d}t - \int \phi(t)\psi_\varepsilon(t)w(t)\,\mathrm{d}t,$$

and the first summants of right hand side of both equations are equal. Since $\phi \cdot \psi_\varepsilon$ is bounded, $\mathrm{supp}\,(\phi \cdot \psi_\varepsilon) \subset (t_1 - \varepsilon, t_1 + \varepsilon)$ and $w \in L^2(0, T; \mathbb{R}) \subset L^1(0, T; \mathbb{R})$, we get

$$\int \phi(t)\psi_\varepsilon(t)w(t)\,\mathrm{d}t \longrightarrow 0, \qquad \text{as } \varepsilon \longrightarrow 0.$$

Similarly, we get

$$\int \phi'(t)\psi_\varepsilon(t)u(t)\,\mathrm{d}t \longrightarrow 0, \qquad \text{as } \varepsilon \longrightarrow 0.$$

$$\int \phi(t)\psi'_\varepsilon(t)u(t)\,\mathrm{d}t \longrightarrow 0, \qquad \text{as } \varepsilon \longrightarrow 0.$$

Hence

$$\int [\phi(t)\psi_\varepsilon(t)]'u(t)\,\mathrm{d}t \longrightarrow 0, \qquad \text{as } \varepsilon \longrightarrow 0.$$

Then the proof is complete. □

Lemma 2.91. *[[1], p.67, Thm 3.17] Let D be an nonempty open (unbounded) subset set of $\mathbb{R}^n$, $1 \le p < \infty$. Then $C^m(D) \cap W^{m,p}(D)$ is dense in $W^{m,p}(D)$.*

2.2.2. *Embedding Theorems.*

Definition 2.92. [23][cone property] A bounded domain D is said to have the *cone property* iff there exist positive constants α, h such that for any $x \in D$, one can construct a right spherical cone V_x with vertex x, opening α and height h such that $V_x \subset D$.

Proposition 2.93. [23] *Let D be a bounded domain, then D has*

(i) *If ∂D is of class C^1, then D satisfies the cone condition.*
(ii) *If D is a convex domain has the cone property.*

Theorem 2.94 ([23] Th. 11.1)**.** *Let $D \subset \mathbb{R}^n$ be a bounded domain satisfies the cone condition. If a function $u \in W^{j,p}$ with $j > m + \frac{n}{p}$ for some nonegative integer m, then $u \in C^m(D)$.*

Theorem 2.95. *$H^j(\mathbb{R}) \hookrightarrow C^m(\mathbb{R})$, if $j - m > \frac{1}{2}$.*

Theorem 2.96 ([23] Th. 10.2)**.** *Let $D \subset \mathbb{R}^n$ be a bounded domain with $\partial D \in C^1$, and let u be any function in $W^{m,r}(D)$, $1 \le r \le \infty$. Then for any integer j with $0 \le j < m$,*

$$|u|_{W^{j,p}(D)} \le C|u|_{W^{m,r}(D)}$$

where $m - \frac{n}{r} = j - \frac{n}{q}$, provided $p > 0$. The constant C depends only on Ω, m, j, r.

Theorem 2.97 ([23] Th. 11.2)**.** *Let $D \subset \mathbb{R}^n$ be a bounded domain with $\partial D \in C^1$. Let r be a positive number, $1 \le r < \infty$, and let j, m be integers, $0 \le j < m$. If $q \ge 1$ is any positive number satisfying*

$$m - \frac{n}{r} > j - \frac{n}{q},$$

then the imbedding $W^{m,r}(D) \hookrightarrow W^{j,p}(D)$ is compact.

Corollary 2.98. [23] *Let $D \subset \mathbb{R}^n$ be a bounded domain with $\partial D \in C^1$. Let r be a positive number, $1 \leq r < \infty$, and let j, m be integers, $0 \leq j < m$. If $q \geq 1$ is any positive number satisfying*

$$m - \frac{n}{r} \geq j - \frac{n}{q},$$

then $W^{m,r}(D) \hookrightarrow W^{j,q}(D)$ continuously.

Proof. By Theorem 2.97, if

$$m - \frac{n}{r} > j - \frac{n}{q},$$

$W^{m,r}(D) \hookrightarrow W^{j,q}(D)$ continuously. And by Theorem 2.96, if

$$m - \frac{n}{r} = j - \frac{n}{q},$$

$W^{m,r}(D) \hookrightarrow W^{j,q}(D)$ continuously. Hence the proof has been complete. □

Theorem 2.99 ([26], Th. 1.6.1)**.** *Let $D \subset \mathbb{R}^n$ is an open set having the C^m extension property. $1 \leq p < \infty$ and A is a sectinial operator in $X = L^p(D)$ with $D(A) = X^1 \subset W^{m,p}(D)$ for some $M \geq 1$. Then for $0 \leq \alpha \leq 1$,*

$$X^\alpha \subset W^{k,q}(D), \quad \text{if } k - \frac{n}{q} < m\alpha - \frac{n}{p}, \quad q \geq p,$$

$$X^\alpha \subset C^\nu(D), \quad \text{if } 0 \leq \nu < m\alpha - \frac{n}{p}.$$

Lemma 2.100 (Sobolev-Gagliardo inequality)**.** [23] *Assume that $q, r \in [1, \infty]$ and $j, m \in \mathbb{Z}$ satisfy $0 \leq j < m$. Then for any $u \in C_0^m(\mathbb{R}^n)$,*

$$\|D^j u\|_{L^p(\mathbb{R}^n)} \leq C\|D^m u\|^a_{L^r(\mathbb{R}^n)}\|u\|^{1-a}_{L^q(\mathbb{R}^n)}, \tag{2.9}$$

where $\frac{1}{p} = \frac{j}{n} + a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}$ for all $a \in [\frac{j}{m}, 1]$ and C is a constant depending only on n, m, j, q, r, a with the following exception. If $m - j - \frac{n}{r}$ is a nonegative integer, then the equality (2.9) *holds only for $a \in [\frac{j}{m}, 1)$.*

Lemma 2.101. [23] *Let D be an open bounded subset of $\mathbb{R}^n$ and $u \in W^{k,p}(D)$. If $k > \frac{n}{p}$, then there exists $\tilde{u} \in C(\overline{D}) \subset L^\infty$, such that $u = \tilde{u}$ almost everywhere.*

2.3. **Existence Theory for PDE.**

Theorem 2.102 (Lax-Milgram)**.** *Let H be a Hilbert space, $B[x,y]$ be a bilinear form in H. And we assume that*

$$\Big|B[x,y]\Big| \le C_1\|x\|\|y\|, \qquad \forall x, y \in H,$$

and

$$\Big|B[x,x]\Big| \ge C_2\|x\|^2, \qquad \forall x \in H,$$

for some constants $C_1, C_2 > 0$. Then every bounded linear functional $F(x)$ in H can be represented in the form

$$F(x) = B[x,v]$$

for some $v \in H$ uniquely determined by F.

Proof. Fix $v \in H$, then $B[x,v]$ is a bounded linear function on H, so by Riesz Lemma, there is a unique $y \in H$ such that $B[x,v] = \langle x, y\rangle$, for all $x \in H$. Then we can define a linear operator

$$\begin{aligned} A : H &\longrightarrow H \\ v &\mapsto y \end{aligned}$$

We claim that A is a bijection from H to H. That is because: If $Av = 0$, that is $B[x,v] = \langle x, 0\rangle = 0$, so $B[v,v] = 0$. But $B[v,v] \ge C_2|v|^2$, so $v = 0$. Hence A is $1-1$. If $R(A) \ne H$, then there exists $z \notin R(A)$, such that $\langle z, y\rangle = 0$ for all $y \in R(A)$. Since

$$\langle z, y\rangle = \langle z, Av\rangle = B[z,v],$$

for some $v \in H$, we have $B[z,v] = 0$ for all $v \in H$. Take $v = z$, then $B[z,z] = 0$, so $z = 0$. This contradict to $z \notin R(A)$. Hence A is a surjection. Therefore A is a bijection.

Since $F(x)$ is bounded and linear, by Riesz Lemma, there is a unique $y \in H$ such that $F(x) = \langle x, y\rangle$ for all $x \in H$. For A is bijection we have

$$F(x) = \langle x, y\rangle = \langle x, Av\rangle = B[x,v], \qquad x \in H.$$

If v' satisfies $F(x) = B[x,v']$ for all $x \in H$, then take $x = v-v'$, we have $B[v-v',v] = B[v-v',v'] = F(v-v')$, so $B[v-v',v-v'] = 0$. But $B[v-v',v-v'] \ge C_2\Big|v-v'|^2$, so $v = v'$. Hence such v is unique. This completes the proof. □

2.4. **Probability.**

2.4.1. *Uniform Integrability.*

Definition 2.103. Let $\{\xi_n\}$ be a series of random variables. They are said to be *uniformly integrable* iff

$$\lim_{r\to\infty}\sup_n \int_{\{|\xi_n|\geq r\}} |\xi_n| \, \mathrm{d}\mathbb{P} = 0.$$

Theorem 2.104. *If $p > 1$, $\sup_n \mathbb{E}|\xi_n|^p < \infty$, then $\{\xi_n\}_n$ is uniformly integrable.*

Proof. By the Cauchy-Schwartz inequality, for q satisfies $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}\int_{\{|\xi_n|\geq r\}} |\xi_n| \, \mathrm{d}\mathbb{P} &\leq \left(\int_{\{|\xi_n|\geq r\}} |\xi_n|^p \, \mathrm{d}\mathbb{P}\right)^{\frac{1}{p}} \left(\int_{\{|\xi_n|\geq r\}} 1^q \, \mathrm{d}\mathbb{P}\right)^{\frac{1}{q}} \\ &\leq \left(\mathbb{E}|\xi_n|^p\right)^{\frac{1}{p}} \mathbb{P}\left(\{|\xi_n| \geq r\}\right)^{\frac{1}{q}}.\end{aligned}$$

By the Chebyshev's inequality, we have

$$\mathbb{P}(\{|\xi_n| \geq r\}) \leq \frac{1}{r^p}\mathbb{E}(|\xi_n|^p).$$

Then let us assume that $\sup_n \mathbb{E}|\xi_n|^p < M < \infty$, then

$$\begin{aligned}\sup_n \int_{\{|\xi_n|\geq r\}} |\xi_n| \, \mathrm{d}\mathbb{P} &\leq \sup_n \left(\mathbb{E}|\xi_n|^p\right)^{\frac{1}{p}} \sup_n \mathbb{P}\left(\{|\xi_n| \geq r\}\right)^{\frac{1}{q}} \\ &\leq \frac{M}{r^{\frac{p}{q}}} \longrightarrow 0, \quad \text{as } r \longrightarrow \infty.\end{aligned}$$

This concludes the proof. □

Theorem 2.105. *[[5] Th. 6.5.4] Let $\{\xi_n\}_n$ be a series of random variables. If $\{\xi_n\}_n$ is uniformly integrable, and $\lim_{n\to\infty} \xi_n = \xi$, $\mathbb{P}$-a.s., then*

$$\lim_{n\to\infty} \mathbb{E}(\xi_n) = \mathbb{E}(\xi).$$

2.4.2. *Tightness.*

Definition 2.106 (Tight)**.** Let X be a Polish space, a family Λ of probability measures on $(X, \mathscr{B}(X))$ is *tight* iff for arbitrary $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset X$, such that

$$\mu(K_\varepsilon) > 1 - \varepsilon, \qquad \forall \mu \in \Lambda.$$

Theorem 2.107. *Let X, Y be separable Banach spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, we assume that $i : X \hookrightarrow Y$ is compact and the random variables $u_n : \Omega \longrightarrow X$, $n \in \mathbb{N}$, satisfy the following condition: there is a constant $C > 0$, such that*

$$\mathbb{E}(|u_n|_X) \leq C, \qquad \forall n.$$

Then the laws $\{\mathcal{L}(i \circ u_n)\}_{n\in\mathbb{N}}$ is tight on Y.

Proof. Let $\mu_n = \mathcal{L}(u_n)$, $\nu_n = \mathcal{L}(i \circ u_n)$.
For $\varepsilon > 0$, there exists $R > 0$, such that $\frac{C}{R} < \varepsilon$. By the Markov inequality Lemma 2.133,

$$\mu_n(|x|_X > R) = \mathbb{P}(|u_n| > R) \leq \frac{1}{R}\mathbb{E}(|u_n|) \leq \frac{C}{R} < \varepsilon, \quad \forall n.$$

Let $K_\varepsilon = \{x \in X : |x|_X \leq R\}$, it is a compact subset of Y. K_ε is bounded in X which is compactly embedded in Y, so K_ε is pre-compact in Y, but it is also closed, hence it is compact in Y.

$$\begin{aligned}\nu_n(K_\varepsilon) &= \mu_n(i^{-1}(K_\varepsilon)) = \mu_n(K_\varepsilon)\\ &= 1 - \mu_n(|x|_X > R) > 1 - \varepsilon, \qquad \forall n\end{aligned}$$

Therefore $\{\nu_n\} = \{\mathcal{L}(i \circ u_n)\}$ is tight on Y. This completes the proof. □

Lemma 2.108. *Let X, Y be separable Banach spaces and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, we assume that $f : X \hookrightarrow Y$ is continuous and the laws $\{\mathcal{L}(u_n)\}$ of the random variables $u_n : \Omega \longrightarrow X$, $n \in \mathbb{N}$, is tight on X. Then the laws $\{\mathcal{L}(f \circ u_n)\}$ is tight on Y.*

Proof. For $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset X$ such that

$$\mathbb{P}(u_n^{-1}(K_\varepsilon)) > 1 - \varepsilon, \qquad \forall n,$$

hence

$$\mathbb{P}\Big((f \circ u_n)^{-1}(f(K_\varepsilon))\Big) = \mathbb{P}(u_n^{-1}(K_\varepsilon)) > 1 - \varepsilon, \qquad \forall n.$$

And since f is continuous, $f(K_\varepsilon)$ is compact in Y. So $\{\mathcal{L}(f \circ u_n)\}$ is tight on Y. This completes the proof. □

Theorem 2.109. *Let X be a separable Banach space. Λ is a collection of probability measures on X. A, B are subspaces of X. If Λ is tight on both A and B, then Λ is tight on $A \cap B$.*

Proof. For $\varepsilon > 0$, there exist compact sets $K^1 \subset A$ and $K_2 \subset B$, such that

$$\mu(K_1) > 1 - \frac{\varepsilon}{2} \quad \text{and} \quad \mu(K_2) > 1 - \frac{\varepsilon}{2}, \qquad \forall \mu \in \Lambda.$$

Then $K_1 \cap K_2 \subset A \cap B$ is compact and for $\mu \in \Lambda$,

$$\begin{aligned}\mu(K_1 \cap K_2) &= 1 - \mu((K_1 \cap K_2)^c)\\ &= 1 - \mu(K_1^c \cup K_2^c)\\ &\geq 1 - \mu(K_1^c) - \mu(K_2^c)\\ &\geq 1 - \varepsilon\end{aligned}$$

Hence Λ is also tight on $A \cap B$. This completes the proof. □

Theorem 2.110. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X and Y be two Polish spaces, $\xi_n : \Omega \longrightarrow X \cap Y$, $n = 1, 2, \ldots$ be a series of random variables. Then if the laws of ξ_n, $(\mathcal{L}(\xi_n))_n$ is tight on $X \cap Y$, it is also tight on X.*

Proof. Suppose that $(\mathcal{L}(\xi_n))_n$ is tight on $X \cap Y$, we will show it is tight on X. Let us fix $\varepsilon > 0$, we need to find a compact $K \subset X$, such that $\mathcal{L}(\xi_n)(K) > 1 - \varepsilon$, for all n. But $(\mathcal{L}(\xi_n))_n$ is tight on $X \cap Y$, so there exists a compact $K' \subset X \cap Y$, such that $\mathcal{L}(\xi_n)(K') > 1 - \varepsilon$, for all n. Since K' is compact in $X \cap Y$, K' is also compact in X. Therefore we can take $K = K'$. This completes the proof of Theorem 2.110. ◻

Definition 2.111. [18][convergence weakly] Let X be a metric space and $\{\mu_n\}$ is a sequence of Borel probability measures on X, we say $\mu_n \longrightarrow \mu$ *weakly* (written as $\mu_n \xrightarrow{w} \mu$) iff

$$\int_X f \, \mathrm{d}\mu_n \longrightarrow \int_X f \, \mathrm{d}\mu, \qquad \forall f \in C_b(X, \mathbb{R}).$$

Definition 2.112 (converges in distribution)**.** Let X be a metric space and $\xi, \xi_1, \xi_2, \ldots$ be random elements with values in X, we say that ξ_n *converges in distribution* to ξ (written as $\xi_n \xrightarrow{d} \xi$) iff $\mathcal{L}(\xi_n) \xrightarrow{w} \mathcal{L}(\xi)$.

Definition 2.113 (relatively compact)**.** Let X be a separable Banach space, a family of measures Λ is called *relatively compact* iff an arbitrary sequence $\{\mu_n\} \subset \Lambda$ contains a subsequence which convergence weakly to a measure on $(X, \mathscr{B}(X))$.

Theorem 2.114 (Prokhorov)**.** [18] *Let X be a separable Banach space, a set of probability measures Λ on $(X, \mathscr{B}(X))$ is relatively compact iff it is tight.*

Lemma 2.115. [49] *Let X and Y be two (not necessary reflexive) Banach spaces with $Y \hookrightarrow X$ compactly. Assume that $p > 1$, let $\mathscr{G}$ be a set of functions in $L^1(\mathbb{R}; Y) \cap L^p(\mathbb{R}, X)$, with*

(a) *$\mathscr{G}$ is bounded in $L^1(\mathbb{R}; Y) \cap L^p(\mathbb{R}, X)$;*
(b) *$\int_{\mathbb{R}} \left| g(t+s) - g(s) \right|_X^p \mathrm{d}s \longrightarrow 0$, as $t \longrightarrow 0$, uniformly for $g \in \mathscr{G}$;*
(c) *The support of the functions $g \in \mathscr{G}$ is included in a fixed compact subset of $\mathbb{R}$.*

Then the set $\mathscr{G}$ is relatively compact in $L^p(\mathbb{R}; X)$.

Lemma 2.116. [22] *Assume that $B_0 \subset B \subset B_1$ are Banach spaces, B_0 and B_1 being reflexive. Assume that the embedding $B_0 \subset B$ is compact. Let $p \in (1, \infty)$ and $\alpha \in (0,1)$ be given. Then the embedding*

$$L^p(0, T; B_0) \cap W^{\alpha,p}(0, T; B_1) \hookrightarrow L^p(0, T; B)$$

is compact.

Lemma 2.117. [22] *Assume that $B_1 \subset B_2$ are two Banach spaces with compact embedding, and $\alpha \in (0,1)$, $p > 1$ satisfy $a - \frac{1}{p} > 0$. Then the space $W^{\alpha,p}(0, T; B_1)$ is compactly embedded into $C([0,T]; B_2)$.*

2.4.3. *Itô formula and SPDE.*

Definition 2.118 (progressively measurable process)**.** A stochastic process $(X(t))_{t\geq 0}$ is called *progressively measurable* with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if and only if:$\Omega \times [0,T] \ni (\omega, t) \longmapsto X_t(\omega) \in \mathbb{R}$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$ measurable.

Proposition 2.119. *If X_t is $\mathcal{F}_t$-measurable and the trajectories of X_t are a.s. continuous (or left\right continuous) then $(X_t)_{t\geq 0}$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$.*

Definition 2.120 (The σ-algebra $\mathcal{BF}$)**.**

$$\mathcal{BF} := \{A \subset [0,T] \times \Omega : \forall t \leq T, A \cap ([0,t] \times \Omega) \in \mathcal{B}[0,t] \otimes \mathcal{F}_t\}.$$

Proposition 2.121. *For $T > 0$, $(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration on Ω, the random process $X = (X_t)_{t\geq 0}$ is progressively measurable iff $X : [0,T] \times \Omega \ni (t,\omega) \mapsto X_t(\omega) \in \mathbb{R}$ is $\mathcal{BF}$ measurable.*

Proof. First we can show that: for any Borel set $B \subset \mathbb{R}$,

$$X^{-1}(B) \cap ([0,t] \times \Omega) = X|^{-1}_{[0,t]\times\Omega}(B).$$

That is because

$$X^{-1}(B) = \{(s,\omega) : s \in [0,T], \omega \in \Omega, X_s(\omega) \in B\},$$

$$X^{-1}(B) \cap ([0,t] \times \Omega) = \{(s,\omega) : s \in [0,t], \omega \in \Omega, X_s(\omega) \in B\} = X|^{-1}_{[0,t]\times\Omega}(B).$$

Then X is progressively measurable iff $X|_{[0,t]\times\Omega}$ is $\mathcal{B}[0,t] \otimes \mathcal{F}_t$-measurable iff for any Borel set $B \subset \mathbb{R}$, $X|^{-1}_{[0,t]\times\Omega}(B) \in \mathcal{B}[0,t] \otimes \mathcal{F}_t$ iff for any Borel set $B \subset \mathbb{R}$, $X^{-1}(B) \cap ([0,t] \times \Omega) \in \mathcal{B}[0,t] \otimes \mathcal{F}_t$ iff X is $\mathcal{BF}$-measurable. □

Definition 2.122 (Itô process)**.** A stochastic process $\{\xi(t)\}_{t\geq 0}$ is called an *Itô process* iff it has a.s. continuous paths and can be represented as

$$\xi(T) = \xi(0) + \int_0^T a(t)\,\mathrm{d}t + \int_0^T b(t)\,\mathrm{d}W(t), \qquad a.s.,$$

where $b(t) \in M_T^2$ for all $T > 0$, that is

$$\mathbb{E}\left(\int_0^T |b(t)|^2\,\mathrm{d}t\right) < \infty, \qquad \forall T > 0,$$

and $a(t) \in L_T^1$ for all $T > 0$, that is

$$\int_0^T |a(t)|\,\mathrm{d}t < \infty, \quad a.s. \quad \forall T > 0.$$

Definition 2.123. Let H be a Banach space and let $M^2(0,T;H)$ denote the space of H-valued measurable process with the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ which satisfy: $\phi \in M^2(0,T;H)$ if and only if

(i) $\phi(t)$ is $\mathcal{F}_t$ measurable for almost every t;
(ii) $\mathbb{E}\int_0^t |\phi(t)|^2\,\mathrm{d}t < \infty$.

Theorem 2.124 (Itô Lemma)**.** [16] *Let $\xi(t)$ be an Itô process satisfies*

$$\mathrm{d}\xi(t) = a(t)\,\mathrm{d}t + b(t)\,\mathrm{d}W(t),$$

where $a \in L_t^1$ and $b \in M_t^2$ for all $t \geq 0$. Suppose that $F(t,x)$ is a real-valued function with continuous partial derivatives F'_t, F'_x and F''_{xx} for all $t \geq 0$ and $x \in \mathbb{R}$. We also assume that the process $b(t)F'_x(t,\xi(t)) \in M_T^2$, for all $T \geq 0$. Then $F(t,\xi(t))$ is an Itô process such that

$$\mathrm{d}F(t,\xi(t)) = \left(F'_t(t,\xi(t)) + F'_x(t,\xi(t))a(t) + \frac{1}{2}F''_{xx}(t,\xi(t))b(t)^2\right)\mathrm{d}t$$
$$+F'_x(t,\xi(t))b(t)\,\mathrm{d}W(t).$$

Theorem 2.125 (high dimensional Itô Lemma)**.** [52] *Let $\xi(t) := (\xi_i)_{i=1}^n$ be an Itô process satisfies*

$$\mathrm{d}\xi_i(t) = a_i(t)\,\mathrm{d}t + \sum_{j=1}^{m} b_{ij}(t)\,\mathrm{d}W_j(t), \qquad i = 1,\dots,n,$$

where $a_i \in L_t^1$ and $b_{ij} \in M_t^2$ for all $t \geq 0$. Suppose that $F(t,x)$, $t \geq 0$, $x \in \mathbb{R}^n$ is a real-valued function with continuous partial derivatives F'_t, $\frac{\partial F}{\partial x_i}$ and $\frac{\partial^2 F}{\partial x_i \partial x_j}$ for all $t \geq 0$ and $x_i, x_j \in \mathbb{R}$. We also assume that $\frac{\partial F}{\partial x_i}$ are bounded. Then $F(t,\xi(t))$ is an Itô process such that

$$\begin{aligned}\mathrm{d}F(t,\xi(t)) \;=\; & \sum_{j=1}^{m}\left[\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(t,\xi(t))b_{ij}(t)\right]\mathrm{d}W_j(t)\\ & +\left[\frac{\partial F}{\partial t}(t,\xi(t)) + \sum_{i=1}^{n}\frac{\partial F}{\partial x_i}(t,\xi(t))a_i(t)\right.\\ & \left.+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 F}{\partial x_i\partial x_j}(t,\xi(t))\sum_{k=1}^{m} b_{ik}(t)b_{jk}(t)\right]\mathrm{d}t.\end{aligned}$$

Theorem 2.126. [52] *Suppose that there exists a constant L such that for any $x, y \in \mathbb{R}^n$, $i, j = 1,\dots,n$,*

$$\left|g_{ij}(x) - g_{ij}(y)\right| \leq L\left|x - y\right|,$$

$$\left|f_i(x) - f_i(y)\right| \leq L\left|x - y\right|$$

and a random variable η with values in $(\mathbb{R}^n, \mathcal{B}^n)$ is $\mathcal{F}_s$-measurable and square integrable: $\mathbb{E}\left|\eta\right|^2 < \infty$. Then there exists a solution of stochastic equation with initial value problem:

$$\begin{cases} \mathrm{d}\xi_i(t) = \sum_{j=1}^m g_{ij}(\xi(t))\,\mathrm{d}W_j(t) + f_i(\xi(t))\,\mathrm{d}t, \quad 1 \leq i \leq n, t \geq s,\\ \xi(s) = \eta. \end{cases}$$

And $\mathbb{E}\left|\xi(t)\right|^2$ is bounded on any finite segment of variation of f.

Lemma 2.127 (Burkholder-Davis-Gundy Inequality)**.** [30] *Let H be a Hilbert space, $0 \leq T < \infty$, $(\eta_j(t))_{t\in[0,T]}$, $j = 1,2,\cdots$ are a series of H-valued random process, $(W_j(t))_{t\in[0,T]}$, $j = 1,2,\cdots$ are Brownian motions. Let us denote $\xi(t) = \sum_{j=1}^\infty \int_0^t \eta_j(s)\,\mathrm{d}W_j(s)$. Then for $p \geq 1$, there exists some K_p only depend on p such that:*

$$\mathbb{E}\sup_{0\leq t\leq T}|\xi(t)|_H^p \leq K_p\mathbb{E}\left[\left(\int_0^t\sum_{j=1}^{\infty}|\eta_j(s)|_H^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right].$$

2.4.4. *Other Relevant.*

Lemma 2.128 (Chebyshev inequality)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* X *is an random variable. Then*

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}, \qquad a > 0.$$

Theorem 2.129. *[*[5] *Th. 6.5.4] Let* $\{\xi_n\}_n$ *be a series of random variables. If* $\{\xi_n\}_n$ *is uniformly integrable, and* $\lim_{n\to\infty} \xi_n = \xi$, $\mathbb{P}$*-a.s., then*

$$\lim_{n\to\infty} \mathbb{E}(\xi_n) = \mathbb{E}(\xi).$$

Theorem 2.130. [19]*[Th 4.1.1] Let* $(\Omega, \mathcal{F}, \mu)$ *be a finite measure space,* X *be a Banach space,* $1 \leq p < \infty$*. Then the dual space of* $L^p(\Omega, X)$ *is* $L^q(\Omega; X^*)$ *for* $\frac{1}{p} + \frac{1}{q} = 1$ *if and only if for each* μ*- continuous vector measure* $G : \mathcal{F} \longrightarrow X^*$ *of bounded variation, there exists* $g \in L^1(\Omega; X^*)$*, such that* $G(E) = \int_E g \, \mathrm{d}\mu$*, for all* $E \in \mathcal{F}$*.*

Lemma 2.131 (Łomnick and Ulam)**.** *[*[29]*, Thm 3.19, page 55] For any probability measures* $\mu_1, \mu_2, \ldots$ *on some Borel spaces* $X_1, X_2, \ldots$*, there exist some independent random elements* $\xi_1, \xi_2, \ldots$ *on* $([0,1], \mathcal{B}([0,1]), \lambda)$ *(where* λ *is the Lebesgue's measure) with distributions* $\mu_1, \mu_2, \ldots$*.*

Lemma 2.132 (Skorohod)**.** *[*[29]*, Thm 4.30, page 79] Let* $\mu, \mu_1, \mu_2, \ldots$ *be probability measures on a separable metric space with the Borel* σ*-field* $(X, \mathcal{B}(X))$ *such that* $\mu_n \xrightarrow{w} \mu$*. Then there exist some random elements* $\eta, \eta_1, \eta_2, \ldots$ *on the probability space* $([0,1], \mathcal{B}([0,1]), \lambda)$ *with values in* X *such that* $\mathcal{L}(\eta)=\mu$ *and* $\mathcal{L}(\eta_n)=\mu_n$*,* $n \in \mathbb{N}$*, such that* $\eta_n \longrightarrow \eta$ *almost surely.*

Lemma 2.133 (Chebyshev inequality)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* X *is an random variable. Then*

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}, \qquad a > 0.$$

Lemma 2.134 (Kolmogorov test)**.** [18] *Let* $\{u(t)\}_{t\in[0,T]}$ *be a stochastic process with values in a separable Banach space* X*, such that for some* $C > 0$*,* $\varepsilon > 0$*,* $\delta > 1$ *and all* $t, s \in [0, T]$*,*

$$\mathbb{E}\Big|u(t) - u(s)\Big|_X^\delta \leq C|t - s|^{1+\varepsilon}.$$

Then there exists a version of u *with* $\mathbb{P}$ *almost surely trajectories being Hölder continuous functions with an arbitrary exponent smaller than* $\frac{\varepsilon}{\delta}$*.*

Lemma 2.135 ([22], Lem 2.1)**.** *Assume that* E *is a separable Hilbert space,* $p \in [2, \infty)$ *and* $a \in (0, \frac{1}{2})$*. Then there exists a constant* C *depending on* T *and* a*, such that for any progressively measurable process* $\xi = (\xi_j)_{j=1}^\infty$*,*

$$\mathbb{E}\, \|I(\xi)\|^p_{W^{a,p}(0,T;E)} \leq C\mathbb{E} \int_0^T \left(\sum_{j=1}^\infty |\xi_j(r)|_E^2 \right)^{\frac{p}{2}} \mathrm{d}t,$$

where $I(\xi_j)$ is defined by

$$I(\xi) := \sum_{j=1}^{\infty} \int_0^t \xi_j(s)\,\mathrm{d}W_j(s), \quad t \geq 0.$$

In particular, $\mathbb{P}$-a.s. the trajectories of the process $I(\xi_j)$ belong to $W^{a,2}(0,T;E)$.

2.5. **Other Lemmata.**

Definition 2.136 (principal part of differential operator)**.** Let $D \subset \mathbb{R}^n$ be a open bounded domain of $\mathbb{R}^n$, with smooth boundary ∂D. Consider the differential operator of order $2k$,

$$A(x, D) = \sum_{|\alpha|\leq 2k} a_\alpha(x)\mathrm{D}^\alpha,$$

where the coefficients $a_\alpha(x)$ are sufficiently smooth complex-valued functions of $x \in \overline{D}$. Then we define the *principal part* $A'(x, D)$ of $A(x, D)$ by

$$A'(x, D) = \sum_{|\alpha|=2k} a_\alpha(x)\mathrm{D}^\alpha.$$

Definition 2.137. Let $D \subset \mathbb{R}^n$ be a open bounded domain of $\mathbb{R}^n$, with smooth boundary ∂D. The differential operator $A(x, D)$ is said to be *strongly elliptic* iff there exists a constant $c > 0$ such that

$$Re(-1)^k A'(x, \xi) \geq c|\xi|^{2k}, \qquad x \in \overline{D}, \xi \in \mathbb{R}^n. \tag{2.10}$$

Lemma 2.138. [39] *Let $D \subset \mathbb{R}^n$ be a open bounded domain of $\mathbb{R}^n$, with smooth boundary ∂D. If $A(x, D)$ is a strongly elliptic operator of order $2k$, then there exist constant $c_0 > 0$ and $\lambda_0 \geq 0$ such that for every $u \in H^{2k}(D) \cap H_0^k(D)$ we have the Gårding's inequality:*

$$Re(Au, u)_0 \geq c_0\|u\|_{k,2}^2 - \lambda_0\|u\|_{0,2}^2. \tag{2.11}$$

And for every λ satisfying $Re\lambda > \lambda_0$ and every $f \in L^2(D)$, there exists a unique $u \in H^{2k}(D) \cap H_0^k(D)$ such that

$$(\lambda I + A(x, D))u = f.$$

Example 2.139. Let $D \subset \mathbb{R}^n$ be a open bounded domain of $\mathbb{R}^n$, with smooth boundary ∂D. We consider the operator A given by (in weak sense)

$$D(A) = H^2(D) \cap H_0^1(D),$$

$$Au = \Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}, \qquad \forall u \in D(A).$$

Then A is self-adjoint.

Proof. Because $-A(x, \xi)$ satisfies (2.10) for $c = 1$, $-A$ is strongly elliptic. And we notice that

$$Re(-Au, u)_0 = (-\Delta u, u)_0 = -\int_D u\Delta u \, \mathrm{d}x$$

$$= \int_D \nabla u \cdot \nabla u \, \mathrm{d}x = \|u\|_{1,2}^2 - \|u\|_{0,2}^2, \qquad \forall u \in C_0^\infty(D)$$

where the third equality is from the integration by parts and the forth equality is from the definition of the norms. $C_0^\infty(D)$ is dense in $H_0^1(D)$ as a subspace of $H^1(D)$, so $-A$ satisfies (2.11) for all $u \in D(A)$ and with $c_0 = \lambda_0 = 1$. Hence from Lemma 2.138, $(\lambda I - A)$ is surjective if $\lambda > 1$. Then from Corollary 2.19, A is self-adjoint. □

Lemma 2.140 (Poincaré's Inequality)**.** *([20], P265, Thm 3) Let us assume that D is an open bounded domain of $\mathbb{R}^n$. Suppose that $u \in W_0^{1,p}(D)$ for some $1 \le p < n$. Then we have the estimate*

$$\|u\|_{L^q(D)} \le C\|\nabla u\|_{L^2(D)},$$

for each $q \in [1, p^*]$, the constant C depending only on p, q, n and D.

Lemma 2.141 (Hölder's inequality)**.** *Let X, μ be a measurable space, we assume that $u \in L^p(\mu)$, $v \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then $uv \in L^r(\mu)$ and the following inequality holds:*

$$\|uv\|_{L^r} \le \|u\|_{L^p}\|v\|_{L^q},$$

the equality occur iff

$$\frac{|u(x)|^p}{\|u\|_{L^p}} = \frac{|v(x)|^q}{\|v\|_{L^q}}, \qquad \mu - a.e.$$

Theorem 2.142 (Kuratowski Theorem)**.** [41] *Let X_1, X_2 be Polish spaces equipped with their Borel σ-field $\mathscr{B}(X_1), \mathscr{B}(X_2)$, and $\phi : X_1 \longrightarrow X_2$ be a one to one Borel measurable map, then for any $E_1 \in \mathscr{B}(X)$, $E_2 := \phi(E_1) \in \mathscr{B}(X_2)$.*

Lemma 2.143 (Banach-Alaoglu [45] Th3.15)**.** *Let X be a topological vector space, K is the closed unit ball in X^*, i.e.*

$$K = \{\Lambda \in X^* : \|\Lambda\|_{X^*} \le 1\}.$$

Then K is compact with respect to the weak-topology.*

Remark 2.144. Hence, if a sequence in a reflexive Banach space is bounded, we can assume the sequence is convergent weakly.

Lemma 2.145. *Let D be a bounded domain in $\mathbb{R}^n$, $\{u_n\} \subset W^{j,p}(D)$, $1 < p < \infty$, and $u_n \longrightarrow u$ weakly in $L^p(D)$. Then $u \in W^{j,p}(D)$ and for any α satisfies $0 \le |\alpha| \le j$, $\mathrm{D}^\alpha u_n \longrightarrow D^\alpha u$ weakly in $L^p(D)$.*

Proof. Since $\{u_n\}$ is bounded in $W^{j,p}(D)$, and for $0 \le |\alpha| \le j$, $|\mathrm{D}^\alpha u_n|_{L^p} \le |u_n|_{W^{j,p}}$, $\{\mathrm{D}^\alpha u_n\}$ is bounded in $L^p(D)$ which is reflexive. Then By the Banach Alaoglu's Theorem 2.143, we can assume that $\{\mathrm{D}^\alpha u_n\}$ is weakly convergent to some $u^\alpha \in L^p(D)$. Then for any $\phi \in C_0^\infty(D)$,

$$\int_D u^\alpha \phi \,\mathrm{d}x = \lim_{n\to\infty} \int_D \mathrm{D}^\alpha u_n \phi \,\mathrm{d}x = \lim_{k\to\infty} (-1)^{|\alpha|} \int_D u_n \mathrm{D}^\alpha \phi \,\mathrm{d}x = (-1)^{|\alpha|} \int_D u \mathrm{D}^\alpha \phi \,\mathrm{d}x.$$

Thus u has weak derivatives $\mathrm{D}^\alpha u = u^\alpha = \lim_{n\to\infty} \mathrm{D}^\alpha u_n$, for all $|\alpha| \le j$. And it follows that $u \in W^{j,p}(D)$. This completes the proof of Lemma 2.145. □

Lemma 2.146 (Mazur Theorem [37] Th2.5.16)**.** *The closure and weak closure of a convex subset of norm space are the same. In particular, a convex subset of a norm space is closed iff it is weakly closed.*

Theorem 2.147 ([1] Th. 3.6)**.** *Let $O \subset \mathbb{R}^n$ be an arbitrary domain. $W^{m,p}(O)$ is separable if $1 \le p < \infty$, and is uniformly convex and reflexive if $1 < p < \infty$.*

Definition 2.148. A subset of a topological space X is called *nowhere dense* if it's closure has empty interior.

Definition 2.149. A subset of a topological space X is called:

(a) of *first category* in X if it is a union of countably many nowhere dense subsets;
(b) of *second category* in X if it is not of first category in X.

Lemma 2.150 (Baire's Theorem)**.** *[*[45]*, P43, 2.2] If X is either*

(a) *a complete metric space, or*
(b) *a locally compact Hausdorff space,*

then the intersection of every countable collection of dense open subsets of X is dense in X.

Corollary 2.151. *Complete metric spaces as well as locally compact Hausdorff spaces are of second category of themselves.*

Proof. Let X be a complete metric space or a locally compact Hausdorff space, $\{E_n\}$ be a countable collection of nowhere dense subsets of X, V_n be the complement of $\bar{E}_n$. Then V_n is dense and open in X. Hence by the Baire's Theorem 2.150, $\bigcap_n V_n$ still dense in X. Therefore $\bigcup_n E_n = (\bigcap_n V_n)^c \neq X$. So X can not be union of countable collection of nowhere dense subsets, hence it is of second category in itself. This completes the proof. □

Theorem 2.152 (Banach-Steinhaus)**.** *[*[45]*, P44, 2.5] Let X and Y be two topological vector spaces, Γ be a collection of bounded linear mappings from X to Y. We define*

$$\Gamma(x) := \{\lambda(x) : \lambda \in \Gamma\}, \qquad x \in X.$$
$$B := \{x \in X : \Gamma(x) \text{ is bounded in } Y\}.$$

Then if B is of second category of X, then $B = X$ and Γ is equi-continuous.

Proposition 2.153. *Let (X, μ) be a measurable space, $u_n : X \longrightarrow \mathbb{R}$, $n = 1, 2, \ldots$. Then if $\lim_{n\to\infty} \int_X |u_n(x)|\,\mathrm{d}x = 0$, then there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that*

$$\lim_{k\to\infty} |u_{n_k}(x)| = 0,$$

for almost every $x \in X$.

Proof. Since $\lim_{n\to\infty} \int_X |u_n(x)|\,\mathrm{d}t = 0$, the subsequence $\{u_{n_k}\}$ can be chosen by the following way: we choose n_1 such that if $n \geq n_1$, $\int_X |u_n(x)|\,\mathrm{d}x < \frac{1}{2}$, and for $k > 1$, we choose $n_k > n_{k-1}$ such that if $n \geq n_k$, $\int_X |u_n(x)|\,\mathrm{d}x < \frac{1}{k2^k}$. Then let $A_k := \bigcup_{i=k}^{\infty} \{x : |u_{n_i}(x)| \geq \frac{1}{i}\}$. Then since

$$\frac{1}{i2^i} > \int_X |u_{n_i}(x)|\,\mathrm{d}x \geq \frac{1}{i}\mu\left(\left\{|u_{n_i}(x)| \geq \frac{1}{i}\right\}\right),$$

we have $\mu\left(\left\{|u_{n_i}(x)| \geq \frac{1}{i}\right\}\right) < \frac{1}{2^i}$ and $\mu(A_k) < \frac{1}{2^{k-1}}$. Let $A = \bigcap_{k=1}^{\infty} A_k$, then $\mu(A) = 0$. Then for $x \notin A$, there exists k_1 such that $x \notin A_k$ for $k \geq k_1$, so $|u_{n_k}(x)| < \frac{1}{k}$ for all $k \geq k_1$. Hence $\lim_{n\to\infty} |u_{n_k}(x)| = 0$, for all $x \notin A$. This completes the proof. □

Definition 2.154. If X and Y are Hausdorff spaces and if $f : X \longrightarrow Y$, then f is said to be *sequentially continuous* provided that $\lim_{n\to\infty} f(x_n) = f(x)$ for every sequence $\{x_n\}$ in X that satisfies $\lim_{n\to\infty} x_n = x$.

Lemma 2.155 (p.395, A6 in [45])**.** *Let X and Y be Hausdorff spaces and $f : X \longrightarrow Y$ is sequentially continuous. If every point of X has a countably local base (in particular, if X is metrizable), then f is continuous.*

Lemma 2.156 (Jensen's inequality)**.** [6] *Let f be a measurable function on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}\big(f \in (a,b)\big) = 1$ for some interval (a,b), $-\infty \leq a < b \leq \infty$ and let $\phi : (a,b) \longrightarrow \mathbb{R}$ be convex, then*

$$\phi\left(\int_A f \,\mathrm{d}\mathbb{P}\right) \leq \int_A \phi(f) \,\mathrm{d}\mathbb{P}, \qquad A \in \mathcal{F},$$

provided $\int_\Omega |f| \,\mathrm{d}\mathbb{P} < \infty$ and $\int_\Omega \big|\phi(f)\big| \,\mathrm{d}\mathbb{P} < \infty$.

Lemma 2.157 (Gronwall inequality)**.** [43] *If ϕ is a positive locally bounded Borel function on $\mathbb{R}^+$ such that*

$$\phi(t) \leq a + b \int_0^t \phi(s) \,\mathrm{d}s,$$

for every t and two constants a and b with $b \geq 0$, then

$$\phi(t) \leq a e^{bt}.$$

If in particular $a = 0$, then $\phi \equiv 0$.

Lemma 2.158. *If (M,d) is an incomplete metric space, then we can find a complete metric space (M', d') such that M is isometric to a dense subset of M'.*

Proof. Let us consider the Cauchy sequences of elements of M. We say that two Cauchy sequences $\{x_n\}$, $\{y_n\}$ are equivalent iff

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Let M' be the family of equivalence classes of Cauchy sequences. Since for any two Cauchy sequences $\{x_n\}$, $\{y_n\}$, $\lim_{n\to\infty} d(x_n, y_n)$ exists and depends only on the equivalent classes of $\{x_n\}, \{y_n\}$. This limit defines a metric on M' and M' is complete. Finally, the map

$$\begin{aligned} i : M &\hookrightarrow M' \\ x &\mapsto \{x_n \equiv x\} \end{aligned}$$

is isometric. And $i(M)$ is dense in M'. □

Remark. The proof of the above Lemma is similar to the construction of the set $\mathbb{R}$ of real numbers from the set $\mathbb{Q}$ of rational numbers.

2.6. **Definition and Properties of −Laplacian with Neumann Boundary Condition.** The −Laplacian operator with the Neumann boundary conditions acting on $\mathbb{R}^3$ valued functions, denoted A, will be used a lot in next few main sections, so we will list the definition with notations and properties here for convenience.
In order to have the definition of A, we need some preparation:

Notation 2.159. For $O = D$ or $O = \mathbb{R}^3$, let us denote

$$\mathbb{L}^p(O) = L^p(O;\mathbb{R}^3), \qquad L^p(O) := L^p(O;\mathbb{R}).$$
$$\mathbb{W}^{k,p}(O) = W^{k,p}(O;\mathbb{R}^3), \qquad W^{k,p}(O) := W^{k,p}(O;\mathbb{R}).$$
$$\mathbb{H}^k(O) = H^k(O;\mathbb{R}^3), \qquad H^k(O) := H^k(O;\mathbb{R}).$$
$$H := \mathbb{L}^2(D), \qquad V := \mathbb{W}^{1,2}(D).$$

Definition 2.160. [48] Let us define a space $E(D)$ by

$$E(D) := \{u \in \mathbb{L}^2(D) : \nabla \cdot u \in L^2(D)\}.$$

Proposition 2.161. [48] *The $E(D)$ defined in Definition 2.160 is a Hilbert space with the inner product:*

$$(u, v)_{E(D)} := (u, v)_{\mathbb{L}^2(D)} + (\nabla \cdot u, \nabla \cdot v)_{L^2(D)}.$$

Lemma 2.162 ([48], p.6, 1.3)**.** *Let D be an open bounded domain in $\mathbb{R}^3$ with C^2 boundary Γ. Then there exists a linear continuous operator $\gamma_0 \in \mathcal{L}(H^1(D); L^2(\Gamma))$ (the trace operator) such that*

(i)
$$\gamma_0 u = u|_\Gamma, \qquad u \in H^1(D) \cap C^2(\bar{D});$$

(ii)
$$\ker \gamma_0 = H^1_0(D);$$

(iii) *We denote*
$$H^{\frac{1}{2}}(\Gamma) := \gamma_0(H^1(D)),$$
which is dense in $L^2(\Gamma)$;

(iv) *The space $H^{\frac{1}{2}}(\Gamma)$ can be equipped with the norm carried from $H^1(D)$ by γ_0;*

(v) *There exists a linear bounded operator $l_D \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma); H^1(D))$ (which is called a lifting operator), such that $\gamma_0 \circ l_D = \mathrm{id}$ on $H^{\frac{1}{2}}(\Gamma)$.*

Lemma 2.163 (Stokes theorem in the weak sense)**.** *[*[48]*, p.7, Th1.2] Let D be an open bounded domain in $\mathbb{R}^3$ with C^2 boundary Γ. n is the outward normal vector on Γ. Then there exists a linear continuous operator $\gamma_n \in \mathcal{L}(E(D), H^{-\frac{1}{2}}(\Gamma))$ such that*

$$\gamma_n u = (u, n)|_\Gamma, \qquad u \in C^\infty_0(\bar{D}).$$

The following generalized Stokes formula is true for all $u \in E(D)$ and $g \in H^1(D)$:

$$(u, \nabla g)_{\mathbb{R}^3} + (\nabla \cdot u, g)_{\mathbb{R}} = \langle \gamma_n u, \gamma_0 g \rangle.$$

Now we are ready to define A.

Definition 2.164. Let D be an open bounded domain of $\mathbb{R}^3$ with C^2 boundary, we define a linear operator A in the Hilbert space H by

$$D(A) = \left\{ u = (u_i)_{i=1,2,3} \in H^{2,2}(D;\mathbb{R}^3) : \frac{\partial u_i}{\partial n}\Big|_{\partial D} = 0 \right\} \subset L^2(D,\mathbb{R}^3),$$

$$Au = -(\Delta u_1, \Delta u_2, \Delta u_3), \qquad \forall u \in D(A).$$

Remark 2.165. In Definition 2.164, $\frac{\partial u_i}{\partial n}$ is an element of the dual space of $H^1(D)$ such that

$$\left\langle \frac{\partial u_i}{\partial n}, v \right\rangle = \langle \gamma_n \nabla u_i, \gamma_0 v \rangle, \qquad v \in H^1(D).$$

Proposition 2.166. *Let D be a bounded open domain in $\mathbb{R}^3$ with C^2 boundary, $u \in H^2(D;\mathbb{R}^3)$, $v \in H^1(D;\mathbb{R}^3)$, and $\frac{\partial u}{\partial n}\big|_{\partial D} = 0$ then we have*

$$(Au, v)_{L^2} = \int_D \Big(\nabla u(x), \nabla v(x)\Big)_{\mathbb{R}^{3\times 3}} \, \mathrm{d}x.$$

Proof. Let $u \in D(A)$, $v \in H^1(D,\mathbb{R}^3)$. Then

$$(Au, v)_{L^2} = \int_D \Big((Au)(x), v(x)\Big)_{\mathbb{R}^3} \, \mathrm{d}x = -\sum_{i=1}^3 \int_D \Delta u_i(x) v_i(x) \, \mathrm{d}x.$$

Thus by the Stokes Theorem, see Lemma 2.163, we get, for each $i = 1, 2, 3$.

$$\begin{aligned}
\int_D \Delta u_i v_i \, \mathrm{d}x &= \int_D \mathrm{div} \nabla u_i v_i \, \mathrm{d}x \\
&= -\int_D (\nabla u_i(x), \nabla v_i(x))_{\mathbb{R}^3} \, \mathrm{d}x + \langle \gamma_n \nabla u_i, \gamma_0 v_i \rangle \\
&= -\int_D (\nabla u_i(x), \nabla v_i(x))_{\mathbb{R}^3} \, \mathrm{d}x + \langle \frac{\partial u_i}{\partial n}, v_i \rangle \\
&= -\int_D (\nabla u_i, \nabla v_i)_{\mathbb{R}^3} \, \mathrm{d}x
\end{aligned}$$

The last equality above follows from the Neumann boundary condition satisfied by u:

$$\frac{\partial u_i}{\partial n} = 0, \qquad \forall i = 1, 2, 3.$$

Hence,

$$(Au, v)_{L^2} = \sum_{i=1}^3 \int_D (\nabla u_i, \nabla v_i) \, \mathrm{d}x = \int_D \sum_{i=1}^3 (\nabla u_i, \nabla v_i) \, \mathrm{d}x = \int_D (\nabla u, \nabla v) \, \mathrm{d}x,$$

and this completes the proof. ☐

Proposition 2.167. *If $v \in V$ and $u \in D(A)$, then*

$$\int_D \langle u(x) \times Au(x), v(x) \rangle_{\mathbb{R}^3} \, \mathrm{d}x = \sum_{i=1}^3 \int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial (v \times u)}{\partial x_i}(x) \right\rangle_{\mathbb{R}^3} \mathrm{d}x. \tag{2.12}$$

Proof. First of all, we need to check wether the integrals on both sides of equality (2.12) make sense. To do this let us fix $v \in V$ and $u \in D(A)$.
$u \in H^2 = W^{2,2}$ and $2 > \frac{3}{2}$, so by Lemma 2.101, we infer that $u \in L^\infty(D, \mathbb{R}^3)$. And since $u \in H^2$, $Au \in L^2(D, \mathbb{R}^3)$. Notice that

$$\int_D |u \times Au|^2_{\mathbb{R}^3} \, \mathrm{d}x \le \|u\|^2_{L^\infty} \cdot \|Au\|^2_{L^2} < \infty,$$

so $u \times Au \in L^2(D, \mathbb{R}^3)$. Moreover since $V \subset L^2(D, \mathbb{R}^3)$ and $v \in V$, the left hand side of the equality (2.12) is well defined. In respect to the right hand side of the equality (2.12), since $u \in H^2(D, \mathbb{R}^3)$, $\frac{\partial u}{\partial x_i}(x) \in L^2(D, \mathbb{R}^3)$. Moreover in the weak sense,

$$\frac{\partial (v \times u)}{\partial x_i} = \frac{\partial v}{\partial x_i} \times u + v \times \frac{\partial u}{\partial x_i}. \tag{2.13}$$

Now we prove the equality (2.13).
Since $\mathrm{D}_i v \in L^2(D, \mathbb{R}^3)$ and $u \in L^\infty$ we infer that $\mathrm{D}_i v \times u \in L^2(D, \mathbb{R}^3)$. From Sobolev-Gagliardo inequality 2.100, let $p = 6, j = 0, m = 1, r = q = 2$ in (2.9), we get $v \in L^6(D, \mathbb{R}^3)$ and with $p = 6, j = 1, m = 2, r = q = 2$ in (2.9), we get $\mathrm{D}_i u \in L^6(D, \mathbb{R}^3)$, which implies that $v \times \mathrm{D}_i u \in L^3(D, \mathbb{R}^3) \subset L^2(D, \mathbb{R}^3)$. By Lemma 2.91, $C^1(D; \mathbb{R}^3) \cap H^1(D; \mathbb{R}^3)$ is dense in $H^1(D; \mathbb{R}^3)$, so we can find sequences $u_n, v_n \in C^1(D, \mathbb{R}^3)$, such that u_n and v_n converges to u and v in V. It follows that

$$\lim_{n \to \infty} \int_D (v_n \times u_n) \mathrm{D}_i \phi \, \mathrm{d}x = \int_D (v \times u) \mathrm{D}_i \phi \, \mathrm{d}x, \qquad \phi \in C_0^\infty(D, \mathbb{R}^3).$$

Indeed we have,

$$\begin{aligned}
&\int_D |v_n \times u_n - v \times u| \, \mathrm{d}x \\
\le\ &\int_D |v_n \times u_n - v \times u_n| + |v \times u_n - v \times u| \, \mathrm{d}x \\
\le\ &\int_D |v_n - v| \cdot |u_n| \, \mathrm{d}x + \int_D |v| \cdot |u_n - u| \, \mathrm{d}x \\
\le\ &\left(\int_D |v_n - v|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_D |u_n|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + \left(\int_D |v|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_D |u_n - u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
\to\ &0, \qquad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\left| \int_D (v_n \times u_n) \mathrm{D}_i \phi - (v \times u) \mathrm{D}_i \phi \, \mathrm{d}x \right| \\
= &\left| \int_D [(v_n \times u_n) - (v \times u)] \mathrm{D}_i \phi \, \mathrm{d}x \right| \le M \int_D |v_n \times u_n - v \times u| \, \mathrm{d}x \longrightarrow 0,
\end{aligned}$$

where $M = \sup_{x \in D} |\mathrm{D}_i \phi(x)|$. So we have proved that

$$\lim_{n \to 0} \int_D (v_n \times u_n) \mathrm{D}_i \phi \, \mathrm{d}x = \int_D (v \times u) \mathrm{D}_i \phi \, \mathrm{d}x, \qquad \phi \in C_0^\infty(D).$$

Similarly we can prove that

$$\lim_{n\to\infty}\int_D (\mathrm{D}_i v_n \times u_n)\phi \, \mathrm{d}x = \int_D (\mathrm{D}_i v \times u)\phi \, \mathrm{d}x,$$

and

$$\lim_{n\to\infty}\int_D (v_n \times \mathrm{D}_i u_n)\phi \, \mathrm{d}x = \int_D (v \times \mathrm{D}_i u)\phi \, \mathrm{d}x.$$

Hence we infer that

$$\int_D (v\times u)\mathrm{D}_i\phi \, \mathrm{d}x = -\int_D (\mathrm{D}_i v \times u + v \times \mathrm{D}_i u)\phi \, \mathrm{d}x, \qquad \phi \in C_0^\infty(D,\mathbb{R}^3).$$

Hence we proved identity (2.13) in the weak sense. Notice that $\left(\frac{\partial u}{\partial x_i}, v \times \frac{\partial u}{\partial x_i}\right) = 0$, so we only need to check wether $\frac{\partial v}{\partial x_i} \times u \in L^2(D,\mathbb{R}^3)$. Since $v \in H^1(D,\mathbb{R}^3)$, we have $\frac{\partial v}{\partial x_i} \in L^2(D,\mathbb{R}^3)$. Moreover as observed earlier $u \in L^\infty(D,\mathbb{R}^3)$, so $\frac{\partial v}{\partial x_i} \times u \in L^2(D,\mathbb{R}^3)$, hence by the equality (2.13), $v \times u \in H^1(D,\mathbb{R}^3)$. Therefore the right hand side of equation (2.12) is also well defined. Since by an elementary property of inner product in $\mathbb{R}^3$,

$$(a\times b, c)_{\mathbb{R}^3} = (b, c\times a)_{\mathbb{R}^3}, \qquad a,b,c \in \mathbb{R}^3,$$

we infer that

$$\Big(u\times Au, v\Big)_{L^2(D,\mathbb{R}^3)} = \Big(Au, v\times u\Big)_{L^2(D,\mathbb{R}^3)} = \int_D \Big(Au(x), v(x)\times u(x)\Big)\,\mathrm{d}x.$$

Next because $v\times u \in H^1(D,\mathbb{R}^3)$ as just proved, by the Proposition 2.166,

$$\int_D \Big(Au(x), v(x)\times u(x)\Big)\,\mathrm{d}x = \Big(\nabla u, \nabla(v\times u)\Big)_{L^2(D,\mathbb{R}^{3\times 3})}.$$

Thus we have

$$(u\times Au, v)_{L^2(D,\mathbb{R}^3)} = \Big(\nabla u, \nabla(v\times u)\Big)_{L^2(D,\mathbb{R}^{3\times 3})}.$$

This completes the proof. □

Corollary 2.168. *If $v\in V$ and $u \in D(A)$, then*

$$\int_D \langle u(x)\times Au(x), v(x)\rangle\,\mathrm{d}x = \sum_{i=1}^{3}\int_D \left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial v}{\partial x_i}(x)\times u(x)\right\rangle \mathrm{d}x. \tag{2.14}$$

Definition 2.169 (Fractional power space of $A_1 := I+A$)**.** For any non-negative real number β we define the space $X^\beta := D(A_1^\beta)$, which is the domain of the fractional power operator A_1^b with the graph norm $|\cdot|_{X^\beta} := |A_1^\beta \cdot|_H$. For positive real β, the dual of X^β is denoted by $X^{-\beta}$ and the norm $|\cdot|_{X^{-\beta}}$ of $X^{-\beta}$ satisfies $|x|_{X^{-\beta}} = |A_1^{-\beta}x|_H$ when x is in H. And $X_\beta \hookrightarrow H \cong H^* \hookrightarrow X^{-\beta}$ is a Gelfand triple.

Lemma 2.170. [20] *The A defined in 2.164 has the following properties:*

0. *$D(A)$ is dense in $H := L^2(D;\mathbb{R}^3)$.*
1. *A is symmetric;*
2. *$R(I+A) = H = L^2(D;\mathbb{R}^3)$;*
3. *A is self-adjoint;*

4. $(I + A)^{-1}$ *is compact.*

*Proof.*0. This will not be proved here.

1. Let us assume that $u, v \in D(A)$, then

$$\langle Au, v\rangle_H = \left\langle -\sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}, v \right\rangle_H = \left\langle u, -\sum_{i=1}^{3} \frac{\partial^2 v}{\partial x_i^2} \right\rangle_H = \langle u, Av\rangle_H.$$

Hence A is symmetric.

2. By the Lax-Milgram Theorem 2.102, we have: for any $f \in H$, there exists a unique $u \in \mathbb{H}_0^2(D) \subset D(A)$ such that

$$\langle u + \nabla u, v + \nabla v\rangle_H = \langle f, v\rangle_H, \qquad v \in \mathbb{H}_0^2(D).$$

Hence

$$\langle (I + A)u, v\rangle_H = \langle f, v\rangle_H, \qquad v \in \mathbb{H}_0^2(D),$$

since $\mathbb{H}_0^2(D)$ is dense in H, we have

$$\langle (I + A)u, v\rangle_H = \langle f, v\rangle_H, \qquad v \in H.$$

Therefore $(I + A)u = f$ in H. Hence $R(I + A) = H$.

3. By Lemma 2.140 and the Lax-Milgram Theorem 2.102, in the same way of the proof of 2, we can see that $R(A) = H$. Then by Proposition 2.18, we see that A is self-adjoint.

4. If $(I + A)u = 0$, for some $u \in D(A)$, then $u = \Delta u$. Hence

$$0 \leq \langle u, u\rangle_H = \langle \Delta u, u\rangle_H = -\langle \nabla u, \nabla u\rangle_H \leq 0.$$

So $u = 0$. Hence $(I + A) : D(A) \longrightarrow H$ is a bijection. Therefore $(I + A)^{-1}$ exists. By Theorem 2.97, $\mathbb{H}^2(D)$ is a compact subspace of H. So any closed subset of $\mathbb{H}^2(D)$ is compact in H. But

$$(I + A)^{-1} : H \longrightarrow D(A) \subset \mathbb{H}^2(D),$$

so $(I + A)^{-1}$ is compact.

□

Lemma 2.171 (Eigenvalues of Laplace operators)**.** *[[20], p.335] Let A as be defined in 2.164. Then the following properties hold:*

(i) *There exists an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $L^2(D; \mathbb{R}^3)$, such that $e_k \in C^\infty(\bar{D})$ for all $k = 1, 2, \ldots,$.*

(ii) *There exists a sequence $\{\lambda_k\}_{k=1}^{\infty}$ in $\mathbb{R}^+$, such that*

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots,$$

and

$$\lim_{k\to\infty} \lambda_k = \infty.$$

(iii)

$$Ae_k = \lambda_k e_k, \qquad k = 1, 2, \ldots.$$

Proposition 2.172. *[[50],4.3.3] Let A as be defined in 2.164, then*

$$X^\gamma = D(A_1^\gamma) = \begin{cases} \left\{ u \in H^{2\gamma} : \frac{\partial u}{\partial n}\Big|_{\partial D} = 0 \right\}, & 2\gamma > \frac{3}{2} \\ H^{2\gamma}, & 2\gamma < \frac{3}{2} \end{cases}$$

Proposition 2.173. *Let $A_1 = I + A$. For a Borel function $g : \mathbb{R} \longrightarrow \mathbb{R}$,*

$$D(g(A_1)) = \left\{ h = \sum_{i=1}^{\infty} h_i e_i \in H : \sum_{i=1}^{\infty} |g(\lambda_i)|^2 h_i^2 < \infty \right\}.$$

Proof. By Theorem 2.72,

$$D(g(A_1)) = \left\{ h \in H : \int_{\mathbb{R}} |g(\lambda)|^2 \, \mathrm{d}(h, P_\lambda h) < \infty \right\}.$$

For $h \in H$, by Theorem 2.71 and Theorem 2.72,

$$(h, P_D y) = (h, 1_D(A_1) y) = 1_D(A_1) \sum_{i=1}^{\infty} h_i e_i = \sum_{i=1}^{\infty} 1_D(\lambda_i) h_i e_i.$$

Hence

$$D(g(A_1)) = \left\{ h \in H : \sum_{i=1}^{\infty} |g(\lambda_i)|^2 h_i^2 < \infty \right\}.$$

This completes the proof. □

Proposition 2.174. *For $\alpha > \beta > 0$, the embedding $\mathscr{I} : D(A_1^\alpha) \hookrightarrow D(A_1^\beta)$ is compact.*

Proof. By Proposition 2.173, we have

$$D(A_1^\alpha) = \left\{ h = \sum_{i=1}^{\infty} h_i e_i \in H : \sum_{i=1}^{\infty} \lambda_i^{2\alpha} h_i^2 < \infty \right\},$$

and

$$|h|_{D(A_1^\alpha)} = \left(\sum_{i=1}^{\infty} \lambda_i^{2\alpha} h_i^2 \right)^{\frac{1}{2}}.$$

Since $\lambda_n \nearrow \infty$ as $n \to \infty$, $|h|_{D(A_1^\alpha)} \geq |h|_{D(A_1^\beta)}$, so $D(A_1^\alpha) \hookrightarrow D(A_1^\beta)$. Let

$$\mathscr{I}_n := \pi_n \circ \mathscr{I} : D(A_1^\alpha) \longrightarrow D(A_1^\beta),$$

where

$$\pi_n : H \longrightarrow H_n,$$
$$\pi_n(x) = \sum_{i=1}^{n} (x, e_i)_H e_i, \quad x \in H,$$

is defined as the orthogonal projection. Since the dimension of the range of $\mathscr{I}_n$, *dim* $\mathscr{I}_n(D(A_1^u)) < \infty$, by Proposition 2.35, $\mathscr{I}_n$ is compact. Moreover,

$$(\mathscr{I} - \mathscr{I}_n) h = \sum_{i=n+1}^{\infty} h_i e_i, \qquad h \in H.$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\|\mathscr{I}_n-\mathscr{I}\|_{\mathscr{L}(D(A_1^{\alpha}),D(A_1^{\beta}))} &= \lim_{n\to\infty}\sup_{\{h:\sum\lambda_i^{2\alpha}h_i^2=1\}}\sum_{i=n+1}^{\infty}\lambda_i^{2\beta}h_i^2\\ &\le \lim_{n\to\infty}\sup_{\{h:\sum\lambda_i^{2\alpha}h_i^2=1\}}\sum_{i=n+1}^{\infty}\lambda_i^{2\alpha}h_i^2\frac{1}{\lambda_i^{2(\alpha-\beta)}}\\ &\le \lim_{n\to\infty}\frac{1}{\lambda_n^{2(\alpha-\beta)}}=0.\end{aligned}$$

Thus by Proposition 2.36, $\mathscr{I}$ is compact. This completes the proof. □

Lemma 2.175. *For $\beta>0$, let $|\cdot|_{-\beta}$ be a norm on H defined by*

$$|\cdot|_{-\beta}^2=\sum_{j=1}^{\infty}\lambda_j^{-2\beta}x_j^2.$$

Then $(H,|\cdot|_{-\beta})$ is not complete and the completion is $D(A^{-\beta})=X^{-\beta}$.

Proof. $H\subsetneq D(A^{-\beta})$, hence there exists a $x\in D(A^{-\beta})$ but $x\notin H$. So

$$|x|_{-\beta}^2=\sum_{j=1}^{\infty}\lambda_j^{-2\beta}x_j^2<\infty,$$

but

$$\sum_{j=1}^{\infty}\lambda_j^2x_j^2=\infty.$$

So $\{x^{(n)}=\pi_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $(H,|\cdot|_{-\beta})$, but do not converge to any element in H.
For any $x\in D(A^{-\beta})$, $|\pi_n(x)|_H^2=\sum_{j=1}^n\lambda_j^2x_j^2<\infty$, so $\pi_n(x)\in H$. And

$$|x-\pi_n(x)|_{-\beta}^2=\sum_{j=n+1}^{\infty}\lambda_j^{-2\beta}x_j^2\longrightarrow 0,\quad \text{as } n\longrightarrow\infty.$$

Hence $D(A^{-\beta})$ is the completion of $(H,|\cdot|_{-\beta})$. □

3. Landau-Lifshitz Equation

This section is a detailed explanations of the Visintin's paper [51], which is about the deterministic Landau-Lifshitz Equation and it will help us to understand the following main sections of this thesis.

3.1. Statement of the Problem.

Definition 3.1. Let $D \subset \mathbb{R}^3$ be an open and bounded domain with C^1 boundary.

(i) Suppose that $\phi \in C_0^2(\mathbb{R}^3;\mathbb{R}^+)$. For a magnetization field $M \in \mathbb{H}^1(D)$, we define the anisotropy energy by:

$$\mathcal{E}_{an}(M) := \int_D \phi(M(x))\,\mathrm{d}x.$$

(ii) We define the exchange energy by:

$$\mathcal{E}_{ex}(M) := \frac{1}{2}\int_D |\nabla M(x)|^2\,\mathrm{d}x = \frac{1}{2}\|\nabla M\|^2_{\mathbb{L}^2(D)}. \tag{3.1}$$

(iii) For a magnetic field $H \in \mathbb{L}^2(\mathbb{R}^3)$, we define the energy due to the magnetic field H by:

$$\mathcal{E}_{fi}(H) := \frac{1}{2}\int_{\mathbb{R}^3} |H(x)|^2\,\mathrm{d}x = \frac{1}{2}\|H\|^2_{\mathbb{L}^2(\mathbb{R}^3)}. \tag{3.2}$$

Definition 3.2. Given vector fields $M : D \longrightarrow \mathbb{R}^3$ and $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, we define a vector field $B : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$B := H + \tilde{M}, \tag{3.3}$$

where

$$\tilde{M}(x) := \begin{cases} M(x), & x \in D; \\ 0, & x \notin D. \end{cases}$$

Definition 3.3.(iv) We define the total magnetic energy as:

$$\begin{aligned}\mathcal{E}_{mag}(M,B) &:= \mathcal{E}_{an}(M) + \mathcal{E}_{ex}(M) + \mathcal{E}_{fi}(B-\tilde{M}) \\ &= \int_D \phi(M(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M\|^2_{\mathbb{L}^2(D)} + \frac{1}{2}\|B-\tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}.\end{aligned}$$

(v) Finally, for an electric field $E \in \mathbb{L}^2(\mathbb{R}^3)$, $M \in \mathbb{H}^1(D)$, $B \in \mathbb{L}^2(\mathbb{R}^3)$, we define the total electro-magnetic energy by

$$\begin{aligned}&\mathcal{E}_{el.mag}(M,B,E) \\ &:= \mathcal{E}_{mag}(M,B) + \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\ &= \int_D \phi(M(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M\|^2_{\mathbb{L}^2(D)} + \frac{1}{2}\|B-\tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)}.\end{aligned} \tag{3.4}$$

Notation 3.4. For simplicity, we define $V := \mathbb{H}^1(D)$, $H := \mathbb{L}^2(D)$, $\mathcal{E} := \mathcal{E}_{el.mag}$, $\phi' := \nabla\phi$. And $Q := [0,T]\times D$, $Q_\infty := [0,T]\times\mathbb{R}^3$.

Proposition 3.5. *For $M \in V$, if we define $\Delta M \in V'$ by*

$$ {}_{V'}\langle \Delta M, u\rangle_V := -\langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})}, \qquad \forall u \in V. \tag{3.5}$$

Then the total energy $\mathcal{E} : V \times \mathbb{L}^2(\mathbb{R}^3) \times \mathbb{L}^2(\mathbb{R}^3) \longrightarrow \mathbb{R}$ defined in (3.4) *has partial derivative with respect to M and satisfies*

$$\frac{\partial \mathcal{E}}{\partial M}(M,B,E) = \phi'(M) - (1_D B - M) - \Delta M, \qquad \text{in } V'. \tag{3.6}$$

Proof. For $M, u \in V$, $B, E \in \mathbb{L}^2(\mathbb{R}^3)$.

$$\begin{aligned} &\mathcal{E}(M+u,B,E) - \mathcal{E}(M,B,E)\\ &= \int_D \phi(M(x)+u(x)) - \phi(M(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M + \nabla u\|_H^2 - \frac{1}{2}\|\nabla M\|_H^2\\ &\quad + \frac{1}{2}\|B - \tilde{M} - \tilde{u}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|B - \tilde{M}\|_H^2, \end{aligned}$$

where

$$\int_D \phi(M(x)+u(x)) - \phi(M(x))\,\mathrm{d}x = \int_D \phi'(M(x))(u(x)) + \frac{1}{2}\phi''(M(x)+\theta(x)u(x))(u(x),u(x))\,\mathrm{d}x,$$

$\theta(x) \in [0,1]$ for $x \in D$. We assumed that ϕ'' is bounded, so there exists some constant $C > 0$ such that

$$\int_D \left|\frac{1}{2}\phi''(M(x)+\theta(x)u(x))(u(x),u(x))\right|\,\mathrm{d}x \le C\int_D |u(x)|^2\,\mathrm{d}x = C\|u\|_H^2 = o(\|u\|_V).$$

Hence

$$\begin{aligned} &\mathcal{E}(M+u,B,E) - \mathcal{E}(M,B,E)\\ &= \int_D \langle \phi'(M(x)), u(x)\rangle\,\mathrm{d}x + o(\|u\|_V) + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + \frac{1}{2}\|\nabla u\|_H^2\\ &\quad - \langle 1_D B - M, u\rangle_H + \frac{1}{2}\|u\|_H^2\\ &= \langle \phi'(M) - (1_D B - M), u\rangle_H + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + o(\|u\|_V) \end{aligned}$$

This implies that $\frac{\partial \mathcal{E}}{\partial M}(M,B,E)$ exists.
Hence as an element in V',

$${}_{V'}\left\langle \frac{\partial \mathcal{E}}{\partial M}(M,B,E), u\right\rangle_V = \langle \phi'(M) - (1_D B - M), u\rangle_H + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})}.$$

We have defined $\Delta M \in V'$ by

$${}_{V'}\langle \Delta M, u\rangle_V := -\langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})}, \qquad \forall u \in V.$$

So

$$\frac{\partial \mathcal{E}}{\partial M}(M,B,E) = \phi'(M) - (1_D B - M) - \Delta M, \qquad \text{in } V'.$$

□

Proposition 3.6. *For $u, v \in V$,*

$$\frac{\partial^2 \mathcal{E}}{\partial M^2}(M, B, E)(u, v) = \int_D \phi''(M(x))(u(x), v(x))\, \mathrm{d}x + \langle u, v\rangle_V. \tag{3.7}$$

Proof. By equality (3.6), we have

$$\frac{\partial \mathcal{E}}{\partial M}(M+u, B, E)(v) - \frac{\partial \mathcal{E}}{\partial M}(M, B, E)(v) = \langle \phi'(M+u) - \phi'(M), v\rangle_H + \langle u, v\rangle_V.$$

And by

$$\begin{aligned}
&\langle \phi'(M+u) - \phi'(M), v\rangle_H \\
&= \int_D [\phi'(M(x)+u(x)) - \phi'(M(x))](v(x))\, \mathrm{d}x \\
&= \int_D \phi''(M(x))(u(x), v(x))\, \mathrm{d}x + o(\|u\|_V),
\end{aligned}$$

The proof is complete. □

Proposition 3.7. *For the total energy $\mathcal{E} : V \times \mathbb{L}^2(\mathbb{R}^3) \times \mathbb{L}^2(\mathbb{R}^3) \longrightarrow \mathbb{R}$ defined in* (3.4)*, we have:*

(i)

$$\frac{\partial \mathcal{E}}{\partial B}(M, B, E) = B - \tilde{M}, \qquad \textit{in } \mathbb{L}^2(\mathbb{R}^3), \tag{3.8}$$

(ii)

$$\frac{\partial \mathcal{E}}{\partial E}(M, B, E) = E, \qquad \textit{in } \mathbb{L}^2(\mathbb{R}^3). \tag{3.9}$$

Proof.(i) For $v \in \mathbb{L}^2(\mathbb{R}^3)$,

$$\begin{aligned}
&\mathcal{E}(M, B+v, E) - \mathcal{E}(M, B, E) \\
&= \frac{1}{2}\|B + v - \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|B + v - \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \left\langle B - \tilde{M}, v\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|v\|^2_{\mathbb{L}^2(\mathbb{R}^3)}
\end{aligned}$$

Hence

$$\frac{\partial \mathcal{E}}{\partial B}(M, B, E) = B - \tilde{M}, \qquad \text{in } \mathbb{L}^2(\mathbb{R}^3).$$

(ii)

$$\begin{aligned}
&\mathcal{E}(M, B, E) - \mathcal{E}(M, B, E+v) \\
&= \frac{1}{2}\|E + v\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \langle E, v\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|v\|^2_{\mathbb{L}^2(\mathbb{R}^3)}
\end{aligned}$$

Hence

$$\frac{\partial \mathcal{E}}{\partial E}(M, B, E) = E, \qquad \text{in } \mathbb{L}^2(\mathbb{R}^3).$$

□

Now we state the main problem which we are going to study in this section:

Problem 3.8. Given the following objects:

$$M_0 \in \mathbb{L}^\infty(D) \cap V;$$
$$B_0 \in \mathbb{L}^2(\mathbb{R}^3): \quad \nabla \cdot B_0 = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R});$$
$$E_0 \in \mathbb{L}^2(\mathbb{R}^3);$$
$$f \in L^\infty(0,T;\mathbb{L}^2(D));$$
$$\phi \in C_0^2(\mathbb{R}^3;\mathbb{R}^+);$$
$$\lambda_1 \in \mathbb{R}, \quad \lambda_2 > 0.$$

Find $M:[0,T]\times D \longrightarrow \mathbb{R}^3$, $B:[0,T]\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $E:[0,T]\times\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $M \in L^2(0,T;V)\cap L^\infty(0,T;\mathbb{L}^\infty(D))$, $B \in L^2(0,T;\mathbb{L}^2(\mathbb{R}^3))$ and $E \in L^2(0,T;\mathbb{L}^2(\mathbb{R}^3))$, satisfying the following system of integral equations: for every $t \in [0,T]$,

$$\begin{aligned}
&\int_D \langle M(t) - M_0, u\rangle \, \mathrm{d}x \\
(3.10) \quad = &\int_0^t \int_D \Big\{ \langle B - M - \phi'(M), \lambda_1 u \times M - \lambda_2 (u\times M)\times M\rangle \\
&- \langle \nabla M, \nabla[\lambda_1 u \times M - \lambda_2(u\times M)\times M]\rangle \Big\} \, \mathrm{d}x\,\mathrm{d}s, \quad u \in C_0^\infty(Q;\mathbb{R}^3);
\end{aligned}$$

$$(3.11) \qquad \int_{\mathbb{R}^3} \langle B(t) - B_0, u\rangle \, \mathrm{d}x = -\int_0^t \int_{\mathbb{R}^3} \langle E, \nabla\times u\rangle \, \mathrm{d}x\,\mathrm{d}s, \quad u \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3);$$

(3.12)
$$\int_{\mathbb{R}^3} \langle E(t) - E_0, u\rangle \, \mathrm{d}x = \int_0^t \int_{\mathbb{R}^3} \left\langle B - \tilde{M}, \nabla\times u\right\rangle \mathrm{d}x\,\mathrm{d}s - \int_0^t \int_D \langle E+f, u\rangle \, \mathrm{d}x\,\mathrm{d}s, \quad u \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3),$$

and such that

$$(3.13) \qquad |M(x,t)| = |M_0(x)|, \qquad \text{a.e. in } Q.$$

Remark 3.9. Suppose that the functions M, B and E are sufficiently regular , then for $t \in [0,T]$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{E}(M(t),B(t),E(t))\Big] = \left\langle \frac{\partial\mathcal{E}}{\partial M}, \frac{\mathrm{d}M}{\mathrm{d}t}\right\rangle_H + \left\langle \frac{\partial\mathcal{E}}{\partial B}, \frac{\mathrm{d}B}{\mathrm{d}t}\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \left\langle \frac{\partial\mathcal{E}}{\partial E}, \frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \langle -\rho, \lambda_1 M\times\rho - \lambda_2 M\times(M\times\rho)\rangle_H + \left\langle B - \tilde{M}, -\nabla\times E\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
&\quad + \left\langle E, \nabla\times(B-\tilde{M}) - 1_D(E+\tilde{f})\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= -\lambda_2\|M\times\rho\|_H^2 - \|E\|_H^2 - \langle E, f\rangle_H.
\end{aligned}$$

Hence we infer that for $t \in [0,T]$,

$$\mathcal{E}(M(t),B(t),E(t)) - \mathcal{E}(M(0),B(0),E(0)) = -\int_0^t \lambda_2\|M(s)\times\rho\|_H^2 + \|E(s)\|_H^2 + \langle E(s), f(s)\rangle_H \, \mathrm{d}s.$$

Therefore, taking into the definition of $\mathcal{E}$, we get

$$\begin{aligned}&\int_D \phi(M(t))\,\mathrm{d}x + \frac{1}{2}\|\nabla M(t)\|_H^2 + \frac{1}{2}\|B(t) - \tilde{M}(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}\\ &+ \int_0^t \lambda_2\|M(s)\times\rho(s)\|_H^2 + \|E(s)\|_H^2 + \langle E(s), f(s)\rangle_H\,\mathrm{d}s\\ =& \int_D \phi(M(0))\,\mathrm{d}x + \frac{1}{2}\|\nabla M(0)\|_H^2 + \frac{1}{2}\|B(0) - \tilde{M}(0)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E(0)\|_{\mathbb{L}^2(\mathbb{R}^3)}.\end{aligned}$$

In the following section we will show the existence of a solution to Problem 3.8. To do this, we need the Galerkin Approximation:

3.2. **Galerkin Approximation and A'priori Estimates.**

Definition 3.10. We define

$$W := \left\{u \in \mathbb{L}^2(\mathbb{R}^3) : \nabla\times u \in \mathbb{L}^2(\mathbb{R}^3)\right\}.$$

Proposition 3.11. *W is a Hilbert space with the inner product:*

$$\langle u, v\rangle_W = \int_{\mathbb{R}^3} \left(\langle u(x), v(x)\rangle + \langle\nabla\times u(x), \nabla\times v(x)\rangle\right)\mathrm{d}x.$$

Proof. We need to prove W is complete. Suppose that $\{u_n\}$ is a Cauchy sequence in W. By definition of W,

$$\|u_n - u_m\|_W^2 = \|u_n - u_m\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \|\nabla\times u_n - \nabla\times u_m\|_{\mathbb{L}^2(\mathbb{R}^3)}^2.$$

Hence $\{u_n\}$ is a Cauchy sequence in $\mathbb{L}^2(\mathbb{R}^3)$ and $\{\nabla\times u_n\}$ is also a Cauchy sequence in $\mathbb{L}^2(\mathbb{R}^3)$. $\mathbb{L}^2(\mathbb{R}^3)$ is complete, so there are $u \in \mathbb{L}^2(\mathbb{R}^3)$ and $v \in \mathbb{L}^2(\mathbb{R}^3)$ such that $u_n \longrightarrow u$ in $\mathbb{L}^2(\mathbb{R}^3)$ and $\nabla\times u_n \longrightarrow v$ in $\mathbb{L}^2(\mathbb{R}^3)$. Hence for $\phi \in C_0^\infty(\mathbb{R}^3)$,

$$\langle v, \phi\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = \lim_{n\to\infty}\langle\nabla\times u_n, \phi\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = \lim_{n\to\infty}\langle u_n, \nabla\times\phi\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = \langle u, \nabla\times\phi\rangle_{\mathbb{L}^2(\mathbb{R}^3)}.$$

Hence $\nabla\times u$ exists and $\nabla\times u = v$. Therefore $u_n \longrightarrow u$ in W. This completes the proof. □

Let A as be define in Definition 2.164, by Lemma 2.171, we can define $H_n :=$ linspan$\{e_1, \dots, e_n\}$, where $\{e_n\}_{n=1}^\infty$ are eigenvectors of A. And since W is a separable Hilbert space, we can find $\{w_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$ such that $\{w_n\}$ is an orthogonal basis of $\mathbb{L}^2(\mathbb{R}^3)$. We define $W_n :=$ linspan$\{w_1, \dots, w_n\}$. Let us define the orthogonal projections

$$\pi_n : H \longrightarrow H_n,$$
$$\pi_n^W : \mathbb{L}^2(\mathbb{R}^3) \longrightarrow W_n.$$

Let us denote by $\mathcal{E}_n$ the restriction of the total energy function $\mathcal{E}$ to the finite dimensional space $H_n\times W_n\times W_n$, i.e.

$$\mathcal{E}_n : H_n\times W_n\times W_n \longrightarrow \mathbb{R},$$
$$\mathcal{E}_n(M, B, E) = \int_D \phi(M(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M\|_{\mathbb{L}^2(D)}^2 + \frac{1}{2}\|B - \pi_n^W\tilde{M}\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E\|_{\mathbb{L}^2(\mathbb{R}^3)}^2.$$

Proposition 3.12. *The function $\mathcal{E}_n$ is of class C^1 and for $M \in H_n$, $B, E \in W_n$ we have:*

(i)

$$(\nabla_M \mathcal{E}_n)(M, B, E) = \phi'(M) - (1_D B - \pi_n^W \tilde{M}) - \Delta M, \qquad \in H_n. \tag{3.14}$$

(ii)

$$(\nabla_B \mathcal{E}_n)(M, B, E) = B - \tilde{M}, \qquad \textit{in } W_n. \tag{3.15}$$

(iii)

$$(\nabla_E \mathcal{E}_n)(M, B, E) = E, \qquad \textit{in } W_n. \tag{3.16}$$

Proof.(i) For $M, u \in H_n$, $B, E \in W_n$. H_n is a finite dimensional space, so $\|\cdot\|_H \cong \|\cdot\|_V$ in H_n, so

$$\begin{aligned}
&\mathcal{E}_n(M+u, B, E) - \mathcal{E}_n(M, B, E)\\
= &\int_D \phi(M+u) - \phi(M)\,\mathrm{d}x + \frac{1}{2}\|\nabla M + \nabla u\|_H^2 - \frac{1}{2}\|\nabla M\|_H^2\\
&+\frac{1}{2}\|B - \tilde{M} - \tilde{u}\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 - \frac{1}{2}\|B - \tilde{M}\|_H^2\\
= &\int_D \langle \phi'(M), u\rangle\,\mathrm{d}x + o(\|u\|_H) + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + \frac{1}{2}\|\nabla u\|_H^2\\
&-\langle 1_D B - M, u\rangle_H + \frac{1}{2}\|u\|_H^2\\
= &\ \langle \phi'(M) - (1_D B - \pi_n^W M), u\rangle_H + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + o(\|u\|_H)\\
= &\ \langle \phi'(M) - (1_D B - \pi_n^W M) - \Delta M, u\rangle_H + o(\|u\|_H).
\end{aligned}$$

Hence by the definition of the gradient,

$$(\nabla_M \mathcal{E}_n)(M, B, E) = \phi'(M) - (1_D B - \pi_n^W M) - \Delta M, \qquad \in H_n.$$

(ii) For $v \in W_n$,

$$\begin{aligned}
&\mathcal{E}_n(M, B+v, E) - \mathcal{E}_n(M, B, E)\\
= &\ \frac{1}{2}\|B + v - \tilde{M}\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 - \frac{1}{2}\|B + v - \tilde{M}\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\\
= &\ \left\langle B - \tilde{M}, v\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|v\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\\
= &\ \left\langle B - \pi_n^W \tilde{M}, v\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + o(\|v\|_{\mathbb{L}^2(\mathbb{R}^3)}).
\end{aligned}$$

So

$$(\nabla_B \mathcal{E}_n)(M, B, E) = B - \pi_n^W \tilde{M}, \qquad \text{in } W_n.$$

(iii)

$$\begin{aligned}&\mathcal{E}_n(M,B,E+v)-\mathcal{E}_n(M,B,E)\\&=\frac{1}{2}\|E+v\|^2_{\mathbb{L}^2(\mathbb{R}^3)}-\frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\\&=\langle E,v\rangle_{\mathbb{L}^2(\mathbb{R}^3)}+\frac{1}{2}\|v\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\\&=\langle E,v\rangle_{\mathbb{L}^2(\mathbb{R}^3)}+o(\|v\|_{\mathbb{L}^2(\mathbb{R}^3)}).\end{aligned}$$

So

$$(\nabla_E\mathcal{E}_n)(M,B,E)=E,\qquad \text{in } W_n.$$

□

To solve the Problem 3.8, we first consider the following problem in finite dimensional spaces:

Problem 3.13. Given are the following objects:

$$\begin{gathered}M_0\in\mathbb{L}^\infty(D)\cap V;\\ B_0\in\mathbb{L}^2(\mathbb{R}^3);\qquad \nabla\cdot B_0=0,\quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R});\\ E_0\in\mathbb{L}^2(\mathbb{R}^3);\\ M_{0,n}=\pi_nM_0;\\ B_{0,n}=\pi_n^W B_0;\\ E_{0,n}=\pi_n^W E_0;\\ \lambda_1\in\mathbb{R},\quad \lambda_2>0;\\ f\in L^\infty(0,T;H);\\ \phi\in C^2(\mathbb{R}^3;\mathbb{R}^+);\end{gathered}$$

Find $M_n:[0,\infty)\longrightarrow H_n$, $B_n:[0,\infty)\longrightarrow W_n$, $E_n:[0,\infty)\longrightarrow W_n$, for $T\in[0,\infty]$, satisfying the following system of differential equations:

$$\begin{aligned}\frac{\mathrm{d}M_n(t)}{\mathrm{d}t}&=\lambda_1\pi_n(M_n(t)\times\rho_n(M_n,B_n)(t))\\&\quad-\lambda_2\pi_n(M_n(t)\times[M_n(t)\times\rho_n(M_n,B_n)(t)]),\quad \in H_n,\end{aligned}\tag{3.17}$$

where $\rho_n:H_n\times W_n\longrightarrow H_n$ is a map defined by

$$\rho_n(M_n,B_n):=-(\nabla_{M_n}\mathcal{E}_n)(M_n,B_n,E_n)=-\phi'(M_n)+\Delta M_n+1_D(B_n-\pi_n^W\tilde{M}_n)\in H_n.$$

$$\frac{\mathrm{d}B_n(t)}{\mathrm{d}t}=-\pi_n^W(\nabla\times E_n(t)),\quad \in W_n,\tag{3.18}$$

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t}=\pi_n^W[\nabla\times(B_n(t)-\pi_n^W(\tilde{M}_n(t)))]-\pi_n^W[1_D(E_n(t)+\tilde{f}_n(t))],\quad \in W_n.\tag{3.19}$$

And

$$M_n(0)=M_{0,n},\tag{3.20}$$

$$B_n(0)=B_{0,n},\tag{3.21}$$

$$E_n(0)=E_{0,n}.\tag{3.22}$$

Lemma 3.14. *Let X, Y be Banach spaces, $\psi_k : X^k \longrightarrow Y$ be a separately continuous k-linear functional. Then ψ_k is continuous.*

Proof. X is a Banach space so by Lemma 2.155, we only need to prove ψ_k is sequentially continuous. And we will prove it by induction.

When $k = 1$, the continuity of ψ_1 followed by the separate continuity.

Assume the conclusion is true for $k = m$, we prove it is also true for $k = m + 1$. Let us assume that $x_n^{m+1} \longrightarrow x_0^{m+1}$ in X^{m+1}, and we denote $x_n^{m+1} = (x_n, x_n^m)$, $x_0^{m+1} = (x_0, x_0^m)$, then $x_n^m \longrightarrow x_0^m$ in X^m and $x_n \longrightarrow x_0$ in X. We define $g_n : X \longrightarrow Y$ by

$$g_n(x) = \psi_{m+1}(x, x_n^m), \qquad x \in X.$$

Then g_n is linear and bounded by the separate continuity of ψ_{m+1}. For fixed $x \in X$, $\psi_{m+1}(x, \cdot) : X^m \longrightarrow Y$ is a separately continuous m-linear functional, which is continuous by our assumption. So for fixed $x \in X$, $\lim_{n\to\infty} g_n(x) = \psi_{m+1}(x, x_0^m) \in Y$. Hence for fixed $x \in X$, $\{g_n(x)\}_n$ is bounded in Y. Moreover by Corollary 2.151, X is of second category in itself. Hence by the Banach-Steinhaus Theorem 2.152, $\{g_n\}$ is equi-continuous. Then we prove $\lim_{n\to\infty} \psi_{m+1}(x_n^{m+1}) - \psi_{m+1}(x_0^{m+1}) = 0$.

$$\begin{aligned}\psi_{m+1}(x_n^{m+1}) - \psi_{m+1}(x_0^{m+1}) &= \psi_{m+1}(x_n, x_n^m) - \psi_{m+1}(x_0, x_0^m) \\ &= g_n(x_n - x_0) + \psi_{m+1}(x_0, x_n^m) - \psi_{m+1}(x_0, x_0^m).\end{aligned}$$

By the equi-continunity of g_n, $g_n(x_n - x_0) \longrightarrow 0$ as $n \longrightarrow \infty$. And if we define

$$\psi_m(y) := \psi_{m+1}(x_0, y), \qquad y \in X^m,$$

then

$$\psi_{m+1}(x_0, x_n^m) - \psi_{m+1}(x_0, x_0^m) = \psi_m(x_n^m) - \psi_m(x_0^m),$$

which goes to 0 as $n \longrightarrow \infty$ by the assumption. So when $k = m + 1$, ψ_{m+1} is continuous. This ends the proof.

□

Lemma 3.15. *Let X, Y be Banach spaces, $k < \infty$, $\psi_k : X^k \longrightarrow Y$ be a separately continuous k-linear functional. Then there exists $C > 0$, such that*

$$|\psi_k(x_1, \ldots, x_k)| \le C|x_1| \cdots |x_k|,$$

where $|\cdot| := \|\cdot\|_X$.

Proof. We prove by contradiction. Let us denote $|\cdot|_k := \|\cdot\|_{X^k}$. Suppose that for any $n \in \mathbb{N}$, there exists some $x_n \in X^k$, such that $|x_n|_k = 1$ and $|\psi_k(x_n)| > n$. So $|\psi_k(x_n)| \longrightarrow \infty$ as $n \longrightarrow \infty$. But as $k < \infty$, $\{x \in X^k : |x|_k = 1\}$ is compact. Hence there exists $x_0 \in X^k$ such that $|x_0|_k = 1$ and there is a subsequence of $\{x_n\}$ which we can still denote by $\{x_n\}$ which satisfies $x_n \longrightarrow x_0$ in X^k as $n \longrightarrow \infty$. By Lemma 3.14, ψ_k is continuous. So $\psi_k(x_n) \longrightarrow \psi_k(x_0) \in Y$, this is contradict to $|\psi_k(x_n)| \longrightarrow \infty$ as $n \longrightarrow \infty$. Hence there exists a $C > 0$, such that for $x \in X^k$,

$|x|_k \le 1$, $|\psi_k(x)| \le C$. Hence for $x = (x_1, \dots, x_k)$,

$$|\psi_k(x)| = \left|\psi_k(\frac{x_1}{|x_1|}, \dots, \frac{x_k}{|x_k|})\right| \prod_{i=1}^{k} |x_i| \le C \prod_{i=1}^{k} |x_i|.$$

This completes the proof. □

Lemma 3.16. *Let X, Y be Banach spaces, X has finite dimensional, $\psi_k : X^k \longrightarrow Y$ be a bounded k-linear functional. If we define $f(x) := \psi_k(x, \dots, x)$, then f is a C^1 function, so it is Lipschitz on balls in X.*

Proof.(i) Proof of $f \in C^1$.
For $x, y \in X$,

$$f(x+y) - f(x) = \sum_{i=1}^{k} \psi_k(x, \dots, x, y, x, \dots, x) + o(|y|).$$

Hence

$$f'(x)(y) = \sum_{i=1}^{k} \psi_k(x, \dots, x, y, x, \dots, x).$$

And for $x, z \in X$,

$$\begin{aligned} &\left\|f'(x+z) - f'(x)\right\| \\ =\ &\sup_{y \in X} \left|\sum_{i=1}^{k} \psi_k(x, \dots, x+z, y, x+z, \dots, x) - \sum_{i=1}^{k} \psi_k(x, \dots, x, y, x, \dots, x)\right| \\ =\ &O(|z|). \end{aligned}$$

Hence $\lim_{z \to 0} \|f'(x+z) - f'(x)\| = 0$, so f' is continuous.
(ii) Proof of f is Lipschitz on balls.
For $0 < R < \infty$, suppose that $a, b \in B(0, R) \subset X$, then by Lemma 3.15,

$$\begin{aligned} &|f(a) - f(b)| \\ =\ &\left|\int_0^1 f'(a + \theta(b-a))(b-a)\, d\theta\right| \\ =\ &\left|\int_0^1 \sum_{i=1}^{k} \psi(a + \theta(b-a), \cdots, b-a, \cdots, a + \theta(b-a))\, d\theta\right| \\ \le\ &CkR^{k-1}|b-a|, \end{aligned}$$

where $\psi(a + \theta(b-a), \cdots, b-a, \cdots, a+\theta(b-a))$ means $b-a$ at the ith position, and all the other positions are $a + \theta(b-a)$.

This completes the proof. □

Proposition 3.17. *Define the maps*

(3.23)
$$F_n^1 : H_n \times W_n \times W_n \ni (M, B, E) \mapsto \lambda_1 \pi_n(M \times \rho_n(M, B)) - \lambda_2 \pi_n(M \times [M \times \rho_n(M, B)]) \in H_n,$$

$$F_n^2 : H_n \times W_n \times W_n \ni (M, B, E) \mapsto -\pi_n^W(\nabla \times E) \in W_n, \tag{3.24}$$

(3.25)
$$F_n^3 : H_n\times W_n\times W_n \ni (M,B,E)\mapsto \pi_n^W[\nabla\times(B-\pi_n^W(\tilde{M}))]-\pi_n^W[1_D(E+\tilde{f}_n(t))]\in W_n.$$

The maps F_n^1, F_n^2 and F_n^3 are Lipschitz on balls.

Proof. H_n and W_n are finite dimensional spaces, so all the norms on them are equivalent. Hence there are some constants $C_1, C_2 > 0$, such that

$$C_1\|M\|_{\mathbb{L}^\infty(D)} \le \|M\|_{H_n} \le C_2\|M\|_{\mathbb{L}^\infty(D)},\quad C_1\|B\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \le \|B\|_{W_n} \le C_2\|B\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)},$$

$$C_1\|(M,B,E)\|_{H_n\times W_n\times W_n} \le \|M\|_{\mathbb{L}^\infty(D)}+\|B\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}+\|E\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \le C_2\|(M,B,E)\|_{H_n\times W_n\times W_n},$$

for all $(M,B,E)\in H_n\times W_n\times W_n$.

Assume that $\|(M,B,E)\|_{H_n\times W_n\times W_n}\le R$ for some $R>0$.

(i) :

Let us denote:

$$F_n^{11}(M,B,E) := M\times\rho_n(M,B),$$

$$F_n^{12}(M,B,E) := M\times[M\times\rho_n(M,B)].$$

Then

$$\begin{aligned}
&\left\|F_n^{11}(M_1,B_1,E_1)-F_n^{11}(M_2,B_2,E_2)\right\|_{H_n}\\
&=\left\|M_1\times[-\phi'(M_1)+\Delta M_1+(1_DB_1-M_1)]-M_2\times[-\phi'(M_2)+\Delta M_2+(1_DB_2-M_2)]\right\|_{H_n}\\
&\le\left\|M_1\times[-\phi'(M_1)+\Delta M_1-M_1]-M_2\times[-\phi'(M_2)+\Delta M_2-M_2]\right\|_{H_n}\\
&\quad+\|M_1\times 1_DB_1-M_2\times 1_DB_2\|_{H_n}.
\end{aligned}$$

In finite dimensional case, all the linear maps are bounded, so by Lemma 3.16 and since linear combination of Lipschitz continuous functions is Lipschitz, there exists $L_1>0$, such that

$$\begin{aligned}
&\left\|M_1\times[\Delta M_1-M_1]-M_2\times[\Delta M_2-M_2]\right\|_{H_n}\\
&\le L_1\|M_1-M_2\|_{H_n}\\
&\le L_1C_2\left\|(M_1,B_1,E_1)-(M_2,B_2,E_2)\right\|_{H_n\times W_n\times W_n}.
\end{aligned}$$

$$\begin{aligned}
&\left\|M_1\times\phi'(M_1)-M_2\times\phi'(M_2)\right\|_{H_n}\\
&\le\left\|M_1\times[\phi'(M_1)-\phi'(M_2)]\right\|_{H_n}+\left\|(M_1-M_2)\times\phi'(M_2)\right\|_{H_n}\\
&\le R\|\phi''\|_{L^\infty}\|M_1-M_2\|_{H_n}+\|\phi'\|_{L^\infty}\|M_1-M_2\|_{L^\infty}\\
&\le C_2\left(R\|\phi''\|_{L^\infty}+\|\phi'\|_{L^\infty}\right)\left\|(M_1,B_1,E_1)-(M_2,B_2,E_2)\right\|_{H_n\times W_n\times W_n}.
\end{aligned}$$

And

$$\begin{aligned}
&\|M_1 \times 1_D B_1 - M_2 \times 1_D B_2\|_{H_n} \\
&= \|M_1 \times (1_D B_1 - 1_D B_2) + (M_1 - M_2) \times 1_D B_2\|_{H_n} \\
&\le R\|1_D B_1 - 1_D B_2\|_{H_n} + R\|M_1 - M_2\|_{H_n} \\
&\le C_2 R\Big(\|B_1 - B_2\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} + \|M_1 - M_2\|_{\mathbb{L}^\infty(D)}\Big) \\
&= C_2 R\Big\|(M_1, B_1, E_1) - (M_2, B_2, E_2)\Big\|_{H_n\times W_n\times W_n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big\|F_n^{11}(M_1, B_1, E_1) - F_n^{11}(M_2, B_2, E_2)\Big\|_{H_n} \\
&\le C_2\left(L_1 + R + R\|\phi''\|_{L^\infty} + \|\phi'\|_{L^\infty}\right)\Big\|(M_1, B_1, E_1) - (M_2, B_2, E_2)\Big\|_{H_n\times W_n\times W_n}.
\end{aligned}$$

And

$$\begin{aligned}
&\Big\|F_n^{12}(M_1, B_1) - F_n^{12}(M_2, B_2)\Big\|_{H_n} \\
&= \Big\|M_1 \times \{M_1 \times [-\phi'(M_1) + \Delta M_1 + (1_D B_1 - M_1)]\} \\
&\quad -M_2 \times \{M_2 \times [-\phi'(M_2) + \Delta M_2 + (1_D B_2 - M_2)]\}\Big\|_{H_n} \\
&\le \Big\|M_1 \times \{M_1 \times [-\phi'(M_1) + \Delta M_1 - M_1]\} - M_2 \times \{M_2 \times [-\phi'(M_2) + \Delta M_2 - M_2]\}\Big\|_{H_n} \\
&\quad + \|M_1 \times \{M_1 \times 1_D B_1\} - M_2 \times \{M_2 \times 1_D B_2\}\|_{H_n}.
\end{aligned}$$

By Lemma 3.16 and since linear combination of Lipschitz continuous functions is Lipschitz, there exists $L_2 > 0$, such that

$$\begin{aligned}
&\Big\|M_1 \times \{M_1 \times [-\phi'(M_1) + \Delta M_1 - M_1]\} - M_2 \times \{M_2 \times [-\phi'(M_2) + \Delta M_2 - M_2]\}\Big\|_{H_n} \\
&\le L_2\|M_1 - M_2\|_{H_n} \\
&\le C_2^2 L_2\Big\|(M_1, B_1, E_1) - (M_2, B_2, E_2)\Big\|_{H_n\times W_n\times W_n}.
\end{aligned}$$

And

$$\begin{aligned}
&\|M_1 \times \{M_1 \times 1_D B_1\} - M_2 \times \{M_2 \times 1_D B_2\}\|_{H_n} \\
&\le \|M_1 \times \{M_1 \times (1_D B_1 - 1_D B_2)\}\|_{H_n} + \|M_1 \times \{(M_1 - M_2) \times 1_D B_2\}\|_{H_n} \\
&\quad + \|(M_1 - M_2) \times \{M_2 \times 1_D B_2\}\|_{H_n} \\
&\le R^2\|1_D B_1 - 1_D B_2\|_{H_n} + 2R^2\|M_1 - M_2\|_{H_n} \\
&\le 3C_2^2 R^2\Big\|(M_1, B_1, E_1) - (M_2, B_2, E_2)\Big\|_{H_n\times W_n\times W_n}.
\end{aligned}$$

Hence

$$\Big\|F_n^{12}(M_1, B_1, E_1) - F_n^{12}(M_2, B_2, E_2)\Big\|_{H_n} \le C_2^2(L_2 + 3R^2)\Big\|(M_1, B_1, E_1) - (M_2, B_2, E_2)\Big\|_{H_n\times W_n\times W_n}.$$

F_n^1 is a linear combination of F_n^{11} and F_n^{12} and both F_n^{11} and F_n^{12} are Lipschitz on balls, therefore F_n^1 is Lipschitz on balls.

(ii) F_n^2 is a bounded linear map, so it is Lipschitz on balls.

(iii) Except the term $\pi_n^W f_n(t)$, F_n^3 is a linear combination of bounded maps on M, B and E. So there is some constant $L_3 > 0$ such that,

$$\begin{aligned}
&\left\|F_n^3(M_1,B_1,E_1) - F_n^3(M_2,B_2,E_2)\right\|_{W_n}\\
=\ &\left\|\pi_n^W\left\{\nabla\times\left[(B_1-B_2)-\pi_n^W\left(\tilde{M}_1-\tilde{M}_2\right)\right]\right\} - \pi_n^W\left[1_D(E_1-E_2)\right]\right\|_{W_n}\\
\leq\ & L_3\left(\|M_1-M_2\|_{H_n} + \|B_1-B_2\|_{W_n} + \|E_1-E_2\|_{W_n}\right)\\
\leq\ & C_2^2 L_3\|(M_1,B_1,E_1)-(M_2,B_2,E_2)\|_{H_n\times W_n\times W_n}.
\end{aligned}$$

Hence F_n^3 is Lipschitz on balls.

This completes the proof. □

Definition 3.18 (Definition of solution of Problem 3.13)**.** We say that a function $(M_n, B_n, E_n) : [0,\infty) \longrightarrow H_n \times W_n \times W_n$, is *the solution of Problem 3.13* iff

$$\begin{aligned}
M_n(t) &= M_{0,n} + \int_0^t F_n^1(M_n(s),B_n(s),E_n(s))\,\mathrm{d}s,\\
B_n(t) &= B_{0,n} + \int_0^t F_n^2(M_n(s),B_n(s),E_n(s))\,\mathrm{d}s,\\
E_n(t) &= E_{0,n} + \int_0^t F_n^3(M_n(s),B_n(s),E_n(s))\,\mathrm{d}s,
\end{aligned}$$

for $t \in [0,\infty)$.

Lemma 3.19. *The problem 3.13 has a unique solution.*

Remark 3.20. The result of Lemma 3.19 is well known, see for example [3].

Theorem 3.21. *For all $n \in \mathbb{N}$,*

$$\|M_n(t)\|_H = \|M_n(0)\|_H. \tag{3.26}$$

Proof. By (3.17), we have

$$\frac{\mathrm{d}\|M_n(t)\|_H^2}{\mathrm{d}t} = 2\left\langle\frac{\mathrm{d}M_n(t)}{\mathrm{d}t}, M_n(t)\right\rangle_H.$$

And

$$\begin{aligned}
\left\langle\frac{\mathrm{d}M_n(t)}{\mathrm{d}t}, M_n(t)\right\rangle_H =\ & \lambda_1\langle\pi_n(M_n(t)\times\rho_n(M_n,B_n)(t)), M_n(t)\rangle_H\\
&-\lambda_2\langle\pi_n(M_n(t)\times[M_n(t)\times\rho_n(M_n,B_n)(t)]), M_n(t)\rangle_H
\end{aligned}$$

Since $\pi_n : H \longrightarrow H$ is self-adjoint and by the fact $\langle a\times b, a\rangle = 0$, we get

$$\langle\pi_n(M_n(t)\times\rho_n(M_n,B_n)(t)), M_n(t)\rangle_H = \langle\pi_n(M_n(t)\times[M_n(t)\times\rho_n(M_n,B_n)(t)]), M_n(t)\rangle_H = 0.$$

Therefore

$$\frac{\mathrm{d}\|M_n(t)\|_H^2}{\mathrm{d}t} = 2\left\langle \frac{\mathrm{d}M_n(t)}{\mathrm{d}t}, M_n(t)\right\rangle_H = 0.$$

Hence $\|M_n(t)\|_H = \|M_n(0)\|_H$. □

Theorem 3.22. *There is a constant $C > 0$ such that for every $n \in \mathbb{N}$,*

$$\|M_n\|_{L^\infty(0,T;V)} \le C; \tag{3.27}$$

$$\|B_n - \pi_n^W \tilde{M}_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \le C; \tag{3.28}$$

$$\|E_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \le C; \tag{3.29}$$

$$\|M_n \times \rho_n\|_{L^2(0,T;H)} \le C. \tag{3.30}$$

Proof. By the equations (3.17)-(3.22), we have

$$\begin{aligned}
\frac{\mathrm{d}\mathcal{E}_n}{\mathrm{d}t} &= \left\langle \frac{\partial \mathcal{E}_n}{\partial M_n}, \frac{\mathrm{d}M_n}{\partial t}\right\rangle_H + \left\langle \frac{\partial \mathcal{E}_n}{\partial B_n}, \frac{\mathrm{d}B_n}{\mathrm{d}t}\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \left\langle \frac{\partial \mathcal{E}_n}{\partial E_n}, \frac{\mathrm{d}E_n}{\mathrm{d}t}\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \langle -\rho_n, \lambda_1 \pi_n(M_n \times \rho_n) - \lambda_2 \pi_n(M_n \times (M_n \times \rho_n))\rangle_H + \left\langle B_n - \pi_n^W \tilde{M}_n, -\pi_n^W(\nabla \times E_n)\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
&\quad + \left\langle E_n, \pi_n^W(\nabla \times (B_n - \pi_n^W(\tilde{M}_n)) - 1_D(E_n + \tilde{f}_n))\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= -\lambda_2 \|M_n \times \rho_n\|_H^2 - \|E_n\|_H^2 - \langle E_n, f_n\rangle_H.
\end{aligned}$$

Hence we get the energy estimate:

$$\mathcal{E}_n(t) - \mathcal{E}_n(0) = -\int_0^t \lambda_2 \|M_n(s) \times \rho_n(s)\|_H^2 + \|E_n(s)\|_H^2 + \langle E_n(s), f_n(s)\rangle_H \,\mathrm{d}s, \quad t \in [0,T].$$

Therefore

$$\begin{aligned}
&\int_D \phi(M_n(t))\,\mathrm{d}x + \frac{1}{2}\|\nabla M_n(t)\|_H^2 \\
&+ \frac{1}{2}\|B_n(t) - \pi_n^W \tilde{M}_n(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E_n(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 \\
&+ \int_0^t \|M_n(s) \times \rho_n(s)\|_H^2 + \|E_n(s)\|_H^2 + \langle E_n(s), f_n(s)\rangle_H \,\mathrm{d}s \\
&= \int_D \phi(M_{0,n})\,\mathrm{d}x + \frac{1}{2}\|\nabla M_{0,n}\|_H^2 + \frac{1}{2}\|B_{0,n} - \pi_n^W \tilde{M}_{0,n}\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E_{0,n}\|_{\mathbb{L}^2(\mathbb{R}^3)}^2,
\end{aligned} \tag{3.31}$$

for all $t \in [0,T]$.

With the fact that

$$\left|\int_0^t \langle E_n(s), f_n(s)\rangle_H \,\mathrm{d}s\right| \le \frac{1}{2}\int_0^t \|E_n(s)\|_H^2 + \|f_n(s)\|_H^2 \,\mathrm{d}s,$$

we have

$$\begin{aligned}&\int_D \phi(M_n(t))\,\mathrm{d}x+\frac{1}{2}\|\nabla M_n(t)\|_H^2+\frac{1}{2}\|B_n(t)-\pi_n^W\tilde{M}_n(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\\&+\frac{1}{2}\|E_n(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\int_0^t\|M_n(s)\times\rho_n(s)\|_H^2+\frac{1}{2}\|E_n(s)\|_H^2\,\mathrm{d}s\\&\leq\int_D\phi(M_{0,n})\,\mathrm{d}x+\frac{1}{2}\|\nabla M_{0,n}\|_H^2+\frac{1}{2}\|B_{0,n}-\pi_n^W\tilde{M}_{0,n}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\\&+\frac{1}{2}\|E_{0,n}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\frac{1}{2}\int_0^T\|f_n(s)\|_H^2\,\mathrm{d}s,\quad t\in[0,T].\end{aligned}\tag{3.32}$$

Since we assumed that ϕ is bounded, so there exists $C_1>0$, such that $|\phi(M_{0,n}(x))|_{\mathbb{R}^3}\leq C_2$, hence

$$\int_D\phi(M_{0,n})\,\mathrm{d}x\leq C_1\mu(D).$$

And

$$\|\nabla M_{0,n}\|_H\leq\|M_0\|_V;$$

$$\|B_{0,n}-\pi_n^W\tilde{M}_{0,n}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\leq 2\left(\|B_{0,n}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|M_{0,n}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\right)\leq 2\left(\|B_0\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|M_0\|_H^2\right);$$

$$\|E_{0,n}\|_{\mathbb{L}^2(\mathbb{R}^3)}\leq\|E_0\|_{\mathbb{L}^2(\mathbb{R}^3)};$$

$$\int_0^T\|f_n(s)\|_H^2\,\mathrm{d}s\leq\|f\|^2_{L^2(0,T;H)},$$

Take $C^2:=C_1\mu(D)+\|M_0\|_V^2+2\left(\|B_0\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|M_0\|_H^2\right)+\|E_0\|_{\mathbb{L}^2(\mathbb{R}^3)}+\|f\|^2_{L^2(0,T;H)}$, then by (3.32), we get the inequalities (3.27)-(3.30). □

Theorem 3.23. *There is a constant $C>0$ such that for all $n\in\mathbb{N}$,*

$$\|M_n\|_{H^1(0,T;\mathbb{L}^{\frac{3}{2}}(D))}\leq C;\tag{3.33}$$

$$\|B_n\|_{W^{1,\infty}(0,T;W')}\leq C;\tag{3.34}$$

$$\|E_n\|_{W^{1,\infty}(0,T;W')}\leq C.\tag{3.35}$$

Proof of (3.33). By Theorem 2.96, there is a constant C_1 such that

$$\|u\|_{\mathbb{L}^6(D)}\leq C_1\|u\|_V,\quad u\in V.$$

So

$$\begin{aligned}\|M_n\times(M_n\times\rho_n)\|^2_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))}&=\int_0^T\|M_n\times(M_n\times\rho_n)\|^2_{\mathbb{L}^{\frac{3}{2}}(D)}\,\mathrm{d}t\\&\leq\int_0^T\|M_n\|^2_{\mathbb{L}^6(D)}\|M_n\times\rho_n\|_H^2\,\mathrm{d}t\\&\leq\sup_{t\in(0,T)}\|M_n\|^2_{\mathbb{L}^6(D)}\int_0^T\|M_n\times\rho_n\|_H^2\,\mathrm{d}t\\&\leq C_1\|M_n\|^2_{L^\infty(0,T;V)}\|M_n\times\rho_n\|^2_{L^2(0,T;H)}=:C_2^2.\end{aligned}$$

By theorem 2.98, there exists $C_3 > 0$ such that

$$\|u\|_{\mathbb{L}^{\frac{3}{2}}} \le C_3 \|u\|_H, \qquad u \in H.$$

Hence

$$\begin{aligned}
&\left\|\pi_n[M_n \times (M_n \times \rho_n)] - M_n \times (M_n \times \rho_n)\right\|_{\mathbb{L}^{\frac{3}{2}}(D)} \\
&\le C_3 \left\|\pi_n[M_n \times (M_n \times \rho_n)] - M_n \times (M_n \times \rho_n)\right\|_H \longrightarrow 0, \quad \text{as } n \to \infty.
\end{aligned}$$

So there exists a constant $C_2 > 0$, such that there is a $N \in \mathbb{N}$ for $n > N$, we have

$$\begin{aligned}
&\left\|\pi_n[M_n \times (M_n \times \rho_n)]\right\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} \\
&\le \left\|\pi_n[M_n \times (M_n \times \rho_n)] - M_n \times (M_n \times \rho_n)\right\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} + \|M_n \times (M_n \times \rho_n)\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} \\
&\le 2C_2.
\end{aligned}$$

Similarly, by (3.30) there is some $C_4 > 0$ such that for all $n \in \mathbb{N}$,

$$\|\pi_n(M_n \times \rho_n)\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} \le C_4.$$

Therefore by (3.17), we have

$$\begin{aligned}
&\left\|\frac{\mathrm{d}M_n}{\mathrm{d}t}\right\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} \\
&\le |\lambda_1| \, \|\pi_n(M_n \times \rho_n)\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} + \lambda_2 \left\|\pi_n[M_n \times (M_n \times \rho_n)]\right\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} \\
&\le |\lambda_1| 2C_2 + \lambda_2 C_4.
\end{aligned}$$

Together with (3.27), we get that there is some $C > 0$ such that

$$\begin{aligned}
&\|M_n\|_{H^1(0,T;\mathbb{L}^{\frac{3}{2}}(D))} \\
&\le \|M_n\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} + \left\|\frac{\mathrm{d}M_n}{\mathrm{d}t}\right\|_{L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D))} \\
&\le C.
\end{aligned}$$

This completes the proof. □

proof of (3.34). By (3.18),

$$\begin{aligned}
\left\|\frac{\mathrm{d}B_n}{\mathrm{d}t}\right\|_{L^\infty(0,T;W')} &= \sup_{t\in[0,T]} \left\|\pi_n^W(\nabla \times E_n(t))\right\|_{W'} \\
&= \sup_{t\in[0,T]} \sup_{w\in W} \frac{\left|{}_{W'}\left\langle \pi_n^W(\nabla \times E_n(t)), w\right\rangle_W\right|}{\|w\|_W} \\
&= \sup_{t\in[0,T]} \sup_{w\in W} \frac{\left|\left\langle E_n(t), \nabla \times \pi_n^W w\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\right|}{\|w\|_W} \\
&\le \|E_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}.
\end{aligned}$$

Then by (3.29), we get: there exists some $C_1 > 0$ independent of n, such that

$$\left\|\frac{\mathrm{d}B_n}{\mathrm{d}t}\right\|_{L^\infty(0,T;W')} \le C_1.$$

And

$$\begin{aligned}\|B_n\|_{L^\infty(0,T;W')} &= \sup_{t\in[0,T]}\sup_{w\in W}\frac{|_{W'}\langle B_n(t),w\rangle_W|}{\|w\|_W}\\ &\le \|B_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}\\ &\le \|B_n-\tilde{M}_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}+\|M_n\|_{L^\infty(0,T;\mathbb{L}^2(D))}.\end{aligned}$$

Then by (3.28) and (3.27), we have: there exists some $C_2 > 0$ independent of n, such that

$$\|B_n\|_{L^\infty(0,T;W')} \le C_2.$$

Hence let $C := C_1 + C_2$, we have got,

$$\|B_n\|_{W^{1,\infty}(0,T;W')} \le C.$$

This completes the proof. □

proof of (3.35).

$$\begin{aligned}&\left\|\frac{\partial E_n}{\partial t}\right\|_{L^\infty(0,T;W')}\\ &= \left\|\pi_n^W\left[\nabla\times(B_n-\pi_n^W(\tilde{M}_n))\right]-\pi_n^W\left[1_D E_n+\tilde{f}_n\right]\right\|_{L^\infty(0,T;W')}\\ &\le \sup_{t\in(0,T)}\left\|\pi_n^W\left[\nabla\times(B_n(t)-\pi_n^W(\tilde{M}_n(t)))\right]\right\|_{W'}+\sup_{t\in(0,T)}\left\|\pi_n^W\left[1_D E_n(t)+\tilde{f}_n(t)\right]\right\|_{W'}\\ &\le \sup_{t\in(0,T)}\sup_{w\in W}\left(\left|\frac{\langle B_n(t)-\pi_n^W(\tilde{M}_n(t)),\nabla\times\pi_n^W w\rangle_{\mathbb{L}^2(\mathbb{R}^3)}}{\|w\|_W}\right|\right.\\ &\quad\left.+\left|\frac{\langle 1_D E_n(t)+\tilde{f}_n(t),\pi_n^W w\rangle_{\mathbb{L}^2(\mathbb{R}^3)}}{\|w\|_W}\right|\right)\\ &\le \left\|B_n-\pi_n^W\tilde{M}_n\right\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}+\|E_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}+\|f_n\|_{L^\infty(0,T;\mathbb{L}^2(D))}\end{aligned}$$

In the proof of (3.34), we have proved that $\exists C_1 > 0$ independent of n, such that

$$\|B_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \le C_1.$$

$\pi_n^W\tilde{M}_n \longrightarrow \tilde{M}_n$ in $\mathbb{L}^2(\mathbb{R}^3)$ and by (3.27), we get: for n large enough, there exists some $C_2 > 0$, such that

$$\left\|\pi_n^W\tilde{M}_n\right\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \le C_2.$$

so

$$\left\|B_n-\pi_n^W\tilde{M}_n\right\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \le C_1 + C_2.$$

By (3.29), $\exists C_3 > 0$ independent of n, such that

$$\|E_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \le C_3$$

, by our assumption, $f \in L^\infty(0,T;\mathbb{L}^2(D))$, hence there exists $C_4 > 0$ independent of n such that

$$\|f_n\|_{L^\infty(0,T;\mathbb{L}^2(D))} \le C_4.$$

And

$$\|E_n\|_{L^\infty(0,T;W')} = \sup_{t\in(0,T)} \sup_{w\neq 0} \frac{|\langle E_n(t), w\rangle|}{|w|_W} \leq \|E_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C_3,$$

Hence

$$\|E_n\|_{W^{1,\infty}(0,T;W')} \leq [(C_1 + C_2 + C_3 + C_4)^2 + C_3^2]^{\frac{1}{2}} =: C.$$

This completes the proof. □

3.3. **Proof of Limit is a Weak Solution.** $L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))$ is the dual space of $L^1(0,T;\mathbb{L}^2(\mathbb{R}^3))$ and $L^\infty(0,T;W')$ is a dual space of $L^1(0,T;W)$. By the Banach-Alaoglu Theorem (Lemma 2.143), and (3.27)-(3.30) and (3.33)-(3.35), we have: There exist H, E, M, P such that for taking some subsequence,

$$B_n - \pi_n^W \tilde{M}_n \longrightarrow H \text{ weakly star in } L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3)) \cap L^\infty(0,T;W'), \tag{3.36}$$

and

$$\frac{\mathrm{d}(B_n - \pi_n^W \tilde{M}_n)}{\mathrm{d}t} \longrightarrow \frac{\mathrm{d}H}{\mathrm{d}t} \text{ weakly star in } L^\infty(0,T;W'). \tag{3.37}$$

$$E_n \longrightarrow E \text{ weakly star in } L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3)) \cap L^\infty(0,T;W'), \tag{3.38}$$

and

$$\frac{\mathrm{d}E_n}{\mathrm{d}t} \longrightarrow \frac{\mathrm{d}E}{\mathrm{d}t} \text{ weakly star in } L^\infty(0,T;W'). \tag{3.39}$$

$$M_n \longrightarrow M \text{ weakly star in } L^\infty(0,T;V), \text{ weakly in } H^1(0,T;\mathbb{L}^{\frac{3}{2}}(D)). \tag{3.40}$$

$$M_n \times \rho_n \longrightarrow P \text{ weakly in } L^2(0,T;H). \tag{3.41}$$

The proof of (3.37) and (3.39) are the same, so we will only prove (3.39).

Proof of (3.39). By (3.35),

$$\left\| \frac{\mathrm{d}E_n}{\mathrm{d}t} \right\|_{L^\infty(0,T;W')} < C.$$

By the Banach-Alaoglu Theorem 2.143 and $L^\infty(0,T;W')$ is a dual space of $L^1(0,T;W)$, there is some $F \in L^\infty(0,T;W')$ such that

$$\frac{\mathrm{d}E_n}{\mathrm{d}t} \longrightarrow F \text{ weakly star in } L^\infty(0,T;W').$$

Then we only need to prove $F = \frac{\mathrm{d}E}{\mathrm{d}t}$ in $L^\infty(0,T;W')$.
For any $u \in C_0^\infty(0,T;W)$, we have

$$\begin{aligned}\int_0^T {}_{W'}\langle E(t), u'(t)\rangle_W \,\mathrm{d}t &= \lim_{n\to\infty}\int_0^T {}_{W'}\langle E_n(t), u'(t)\rangle_W \,\mathrm{d}t\\ &= -\lim_{n\to\infty}\int_0^T {}_{W'}\left\langle \frac{\mathrm{d}E_n(t)}{\mathrm{d}t}, u(t)\right\rangle_W \mathrm{d}t\\ &= -\int_0^T {}_{W'}\langle F(t), u(t)\rangle_W \,\mathrm{d}t.\end{aligned}$$

Hence $F = \frac{\mathrm{d}E}{\mathrm{d}t}$ in $L^\infty(0,T;W')$. And the proof has been complete. □

We define $B := \tilde{M} + H$, then

$$B_n \longrightarrow B \text{ weakly star in } L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3)). \tag{3.42}$$

Proof of (3.42). For $u \in L^1(0,T;\mathbb{L}^2(\mathbb{R}^3))$, by (3.36) and (3.40),

$$\begin{aligned}\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\langle B, u\rangle\,\mathrm{d}x &= \int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\langle \tilde{M}+H, u\rangle\,\mathrm{d}x\\ &= \int_0^T\int_{\mathbb{R}^3}\langle \tilde{M}, u\rangle\,\mathrm{d}x + \int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\langle H, u\rangle\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\langle \pi_n^W\tilde{M}_n, u\rangle + \langle B_n - \pi_n^W\tilde{M}_n, u\rangle\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\langle B_n, u\rangle\,\mathrm{d}x.\end{aligned}$$

This completes the proof. □

Theorem 3.24. (M,B,E) *in* (3.36)-(3.40) *is a solution to Problem 3.8.*

Proof of (5.16). By (3.36) and (3.19), for $u \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$, $t\in[0,T]$, we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3}\langle B-\tilde{M}, \nabla\times u\rangle\,\mathrm{d}x\,\mathrm{d}s\\ &= \lim_{n\to\infty}\int_0^t\int_{\mathbb{R}^3}\langle B_n - \pi_n^W\tilde{M}_n, \nabla\times u\rangle\,\mathrm{d}x\,\mathrm{d}s\\ &= \lim_{n\to\infty}\int_0^t\int_{\mathbb{R}^3}\langle \nabla\times(B_n - \pi_n^W\tilde{M}_n), u\rangle\,\mathrm{d}x\,\mathrm{d}s\\ &= \lim_{n\to\infty}\int_0^t\int_{\mathbb{R}^3}\left\langle \frac{\mathrm{d}E_n}{\mathrm{d}t} + \pi_n^W(1_D E_n + \tilde{f}_n), u\right\rangle\mathrm{d}x\,\mathrm{d}s\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^3}\langle E_n(t) - E_{0,n}, u\rangle\,\mathrm{d}x + \lim_{n\to\infty}\int_0^t\int_{\mathbb{R}^3}\langle \pi_n^W(1_D E_n + \tilde{f}_n), u\rangle\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_{\mathbb{R}^3}\langle E(t) - E_0, u\rangle\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^3}\left\langle 1_D E + \tilde{f}, u\right\rangle\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

This completes the proof. □

Proof of (5.15). By (3.38), (3.18) and (3.42), we have: for $u \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \langle E, \nabla \times u\rangle \,\mathrm{d}x \\
&= \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^3} \langle E_n, \nabla \times u\rangle \,\mathrm{d}x\,\mathrm{d}s \\
&= \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^3} \langle \pi_n^W(\nabla \times E_n), u\rangle \,\mathrm{d}x\,\mathrm{d}s \\
&= \lim_{n\to\infty} \int_0^t \int_{\mathbb{R}^3} \left\langle -\frac{\mathrm{d}B_n}{\mathrm{d}t}, u\right\rangle \mathrm{d}x\,\mathrm{d}s \\
&= \lim_{n\to\infty} -\int_{\mathbb{R}^3} \langle B_n(t) - B_{0,n}, u\rangle \,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \langle B(t) - B_0, u\rangle \,\mathrm{d}x.
\end{aligned}$$

This completes the proof. □

Lemma 3.25 ([48],Th 3.2.1)**.** *Let X_0, X, X_1 be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, where the embeddings are continuous. And X_0, X_1 are reflexive and the embedding $X_0 \hookrightarrow X$ is compact. Let $T > 0$ be a fixed finite number, and let α_0, α_1 be two finite numbers such that $\alpha_i > 1$, $i = 0, 1$. We consider the space*

$$Y = \left\{ v \in L^{\alpha_0}(0,T;X_0),\ v' = \frac{\mathrm{d}v}{\mathrm{d}t} \in L^{\alpha_1}(0,T;X_1) \right\},$$

with the norm

$$\|v\|_Y = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v'\|_{L^{\alpha_1}(0,T;X_1)}.$$

Then $Y \subset L^{\alpha_0}(0,T;X)$ and the embedding $Y \hookrightarrow L^{\alpha_0}(0,T;X)$ is compact.

Proposition 3.26.

$$\lim_{n\to\infty} \|M - M_n\|_{\mathbb{L}^4(Q)} = 0. \tag{3.43}$$

Proof of (3.43). By Theorem 2.97, $V \hookrightarrow \mathbb{L}^4(D)$ is compact. Let $X_0 = V$, $X = \mathbb{L}^4(D)$, $X_1 = \mathbb{L}^{\frac{3}{2}}(D)$. Then $X_0 \hookrightarrow X \hookrightarrow X_1$. By Theorem 2.147, X_0 and X_1 are reflexive. Let $\alpha_0 = 4$, $\alpha_1 = 2$ and

$$Y = \left\{ v \in L^4(0,T;V), \frac{\mathrm{d}v}{\mathrm{d}t} \in L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D)) \right\}.$$

Then by Lemma 3.25, for $T < \infty$, the embedding $Y \hookrightarrow \mathbb{L}^4(Q)$ is compact.
By Theorem 2.98, $V \hookrightarrow \mathbb{L}^4(O)$ continuously, so by (3.27), we get

$$\|M_n\|_{L^\infty(0,T;\mathbb{L}^4(O))} \le C, \quad \forall n. \tag{3.44}$$

By (3.44) and (3.33), $\|M_n\|_Y \le C$. So $\{M_n\}$ has a subsequence converges (still denoted by $\{M_n\}$) in $\mathbb{L}^4(Q)$. Let us assume this limit is M'. Now we need to show

$M = M'$. $\mathbb{L}^4(Q) \hookrightarrow \mathbb{L}^2(Q)$ continuously, so M_n also converges to M' in $\mathbb{L}^2(Q)$. Hence by (3.40),

$$\begin{aligned}
\|M - M'\|^2_{\mathbb{L}^2(Q)} &= (M - M', M - M')_{\mathbb{L}^2(Q)} \\
&= \lim_{n\to\infty} (M - M_n, M - M')_{\mathbb{L}^2(Q)} \\
&= \lim_{n\to\infty} {}_{L^\infty(0,T;V)}\langle M - M_n, M - M'\rangle_{L^1(0,T;V')} \\
&= 0.
\end{aligned}$$

Therefore $M = M'$ a.e. and both in $\mathbb{L}^4(Q)$, so $M = M'$ in $\mathbb{L}^4(Q)$. This completes the proof. □

Proposition 3.27. *For almost every $t \in [0, T]$, $u \in C_0^\infty(D)$,*

$$\int_D \langle M(t) - M(0), u\rangle \,\mathrm{d}x = \lambda_1 \int_0^t \int_D \langle P, u\rangle \,\mathrm{d}x\,\mathrm{d}s - \lambda_2 \int_0^t \int_D \langle P, u \times M\rangle \,\mathrm{d}x\,\mathrm{d}s. \tag{3.45}$$

Proof of (3.45). By the Hölder's inequality (Lemma 2.141), for $M \in \mathbb{L}^4(D)$ and $P \in H$ we have

$$\|M \times P\|_{\mathbb{L}^{\frac{4}{3}}(D)} \le \|M\|_{\mathbb{L}^4(D)} \|M \times P\|_H.$$

Hence

$$\begin{aligned}
\|M \times P\|^{\frac{4}{3}}_{\mathbb{L}^{\frac{4}{3}}(Q)} &= \int_0^T \|M \times P\|^{\frac{4}{3}}_{\mathbb{L}^{\frac{4}{3}}(D)} \,\mathrm{d}t \\
&\le \int_0^T \|M(t)\|^{\frac{4}{3}}_{\mathbb{L}^4(D)} \|M(t) \times P(t)\|^{\frac{4}{3}}_H \,\mathrm{d}t \\
&\le \sup_{t\in[0,T]} \|M(t)\|^{\frac{4}{3}}_{\mathbb{L}^4(D)} \int_0^T \|M(t) \times P(t)\|^{\frac{4}{3}}_H \,\mathrm{d}t \\
&\le \|M\|^{\frac{4}{3}}_{L^\infty(0,T;\mathbb{L}^4(D))} \left(\int_0^T \left(\|P(t)\|^{\frac{4}{3}}_H \right)^{\frac{3}{2}} \mathrm{d}t \right)^{\frac{2}{3}} \cdot T^{\frac{1}{3}} \qquad \text{(again by Hölder's inequality)} \\
&= \|M\|^{\frac{4}{3}}_{L^\infty(0,T;\mathbb{L}^4(D))} \|P\|^{\frac{4}{3}}_{\mathbb{L}^2(Q)} \cdot T^{\frac{1}{3}}
\end{aligned}$$

By (3.40) $M \in L^\infty(0, T; V) \subset L^\infty(0, T; \mathbb{L}^4(D))$ and by (3.41) $P \in L^2(0, T; H) = \mathbb{L}^2(Q)$. Hence $M \times P \in \mathbb{L}^{\frac{4}{3}}(Q)$. Similarly $M_n \times (M_n \times \rho_n) \in \mathbb{L}^{\frac{4}{3}}(Q)$ too.

For any $u \in \mathbb{L}^4(Q)$, by (3.43) and (3.41),

$$\begin{aligned}
&\left| {}_{\mathbb{L}^{\frac{4}{3}}(Q)}\langle M_n \times (M_n \times \rho_n) - M \times P, u\rangle_{\mathbb{L}^4(Q)} \right| \\
= &\left| {}_{\mathbb{L}^{\frac{4}{3}}(Q)}\langle (M_n - M) \times (M_n \times \rho_n) + M \times (M_n \times \rho_n - P), u\rangle_{\mathbb{L}^4(Q)} \right| \\
\le &\int_0^T \mathrm{d}t \int_D \left|\langle M_n \times \rho_n, u \times (M_n - M)\rangle\right| \mathrm{d}x + \left| \int_0^T \mathrm{d}t \int_D \langle M_n \times \rho_n - P, u \times M\rangle \,\mathrm{d}x \right| \\
\le &\|M_n \times \rho_n\|_{\mathbb{L}^2(Q)} \|u\|^{\frac{1}{2}}_{\mathbb{L}^4(Q)} \|M_n - M\|^{\frac{1}{2}}_{\mathbb{L}^4(Q)} + \left|(M_n \times \rho_n - P, u \times M)_{\mathbb{L}^2(Q)}\right| \\
\longrightarrow &\ 0, \qquad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Hence we have

$$M_n \times (M_n \times \rho_n) \longrightarrow M \times P \text{ weakly in } \mathbb{L}^{\frac{4}{3}}(Q). \tag{3.46}$$

By (3.43), we have $\lim_{n\to\infty} \|M_n(t) - M(t)\|_{\mathbb{L}^4(D)} = 0$ for almost every $t \in [0, T]$. Hence for $u \in C_0^\infty(D)$, a.e. $t \in [0, T]$,

$$\begin{aligned}&\left|\int_D \langle M_n(t) - M(t), u\rangle \,\mathrm{d}x\right| \\ &\leq \|M_n(t) - M(t)\|_{\mathbb{L}^4(D)}\|u\|_{\mathbb{L}^{\frac{4}{3}}(D)} \longrightarrow \infty.\end{aligned}$$

Therefore by (3.17), (3.41) and (3.46),

$$\begin{aligned}&\int_D \langle M(t) - M(0), u\rangle \,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_D \langle M_n(t) - M_n(0), u\rangle \,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_0^t \mathrm{d}s \int_D \langle \lambda_1 \pi_n(M_n \times \rho_n) - \lambda_2 \pi_n(M_n \times (M_n \times \rho_n)), u\rangle \,\mathrm{d}x \\ &= \lambda_1 \int_0^t \int_D \langle P, u\rangle \,\mathrm{d}x \,\mathrm{d}s - \lambda_2 \int_0^t \int_D \langle P, u \times M\rangle \,\mathrm{d}x \,\mathrm{d}s.\end{aligned}$$

This completes the proof. □

Proof of (5.14). By (3.43), $M_n \longrightarrow M$ a.e. in Q, and by our assumption ϕ' is continuous, so $\phi'(M_n) \longrightarrow \phi'(M)$ a.e. in Q. And we assumed that ϕ' is bounded, so for $p \in [1, \infty)$, $\phi'(M_n) \in \mathbb{L}^p_{\mathrm{loc}}(Q)$ and $\phi'(M) \in \mathbb{L}^p_{\mathrm{loc}}(Q)$. Hence

$$\phi'(M_n) \longrightarrow \phi'(M) \text{ strongly in } \mathbb{L}^p_{\mathrm{loc}}(Q), \quad \forall 0 < p < \infty. \tag{3.47}$$

Hence similarly as before, we have

(3.48)

$$M_n \times [1_D(B_n - \pi_n^W \tilde{M}_n) - \phi'(M_n)] \longrightarrow M \times [1_D(B - \tilde{M}) - \phi'(M)] \text{ weakly in } \mathbb{L}^{\frac{4}{3}}(Q).$$

Then since $M_n \times \Delta M_n = M_n \times \rho_n - M_n \times [1_D(B_n - \pi_n^W \tilde{M}_n) - \phi'(M_n)]$, (3.41) and (3.48) yield

$$M_n \times \Delta M_n \longrightarrow P - M \times [1_D(B - \tilde{M}) - \phi'(M)] \text{ weakly in } \mathbb{L}^{\frac{4}{3}}(Q). \tag{3.49}$$

So for $t \in [0, T]$, $u \in C_0^\infty(Q)$, we have

$$\lim_{n\to\infty} \int_0^t \int_D \langle M_n \times \nabla M_n, \nabla u\rangle - \langle P - M \times \left[1_D(B - \tilde{M}) - \phi'(M)\right], u\rangle \,\mathrm{d}x \,\mathrm{d}s = 0. \tag{3.50}$$

For $u \in L^4(0,T;\mathbb{W}^{1,4}(D)$, let $X := \mathbb{W}^{1,4}(D)$, by (3.40), (3.43), we have: for $u \in L^4(0,T;X)$,

$$
\begin{aligned}
&\left|{}_{L^{\frac{4}{3}}(0,T;X')}\langle \nabla M_n \times M_n - \nabla M \times M, \nabla u\rangle_{L^4(0,T;X)}\right| \\
= &\left|\int_0^T \mathrm{d}t \int_D \langle \nabla M_n \times M_n - \nabla M \times M, \nabla u\rangle \ \mathrm{d}x\right| \\
\le &\int_0^T \mathrm{d}t \int_D |\langle \nabla M_n \times (M_n - M), \nabla u\rangle| \ \mathrm{d}x + \left|\int_0^T \mathrm{d}t \int_D \langle (\nabla M_n - \nabla M) \times M, \nabla u\rangle \ \mathrm{d}x\right| \\
\le &\left(\|\nabla M_n\|_{\mathbb{L}^2(Q)} \|M_n - M\|_{\mathbb{L}^4(Q)} \|\nabla u\|_{\mathbb{L}^4(Q)} + {}_{L^\infty(0,T;\mathbb{L}^2(D))}\langle \nabla M_n - \nabla M, M \times \nabla u\rangle_{L^1(0,T;\mathbb{L}^2(D))}\right) \\
\to &\ 0, \quad \text{as } n \to \infty.
\end{aligned}
$$

So for $t \in [0,T]$, $u \in C_0^\infty(Q)$, we have

$$
\lim_{n\to\infty} \int_0^t \int_D \langle \nabla M_n \times M_n - \nabla M \times M, \nabla u\rangle \ \mathrm{d}x \, \mathrm{d}s = 0. \tag{3.51}
$$

Comparing (3.50) and (3.51), we get

$$
\int_0^t \int_D \left\langle P - M \times \left[1_D(B - \tilde{M}) - \phi'(M)\right], u\right\rangle - \langle M \times \nabla M, \nabla u\rangle \ \mathrm{d}x \, \mathrm{d}s = 0.
$$

Hence for $t \in [0,T]$ and $u \in L^4(0,T;\mathbb{W}^{1,4}(D))$ we have

$$
\int_0^t \int_D \langle P, u\rangle \, \mathrm{d}x \, \mathrm{d}s = \int_0^t \int_D \langle M \times \nabla M, \nabla u\rangle + \langle 1_D(B - \tilde{M}) - \phi'(M), u \times M\rangle \, \mathrm{d}x \, \mathrm{d}s. \tag{3.52}
$$

We can define $\Delta M \in L^\infty(0,T;H)$ by

$$
\int_0^t \mathrm{d}s \int_D \langle M \times \Delta M, u\rangle \, \mathrm{d}x := \int_0^t \mathrm{d}s \int_D \langle \nabla M \times M, \nabla u\rangle \, \mathrm{d}x, \quad u \in \mathbb{W}^{1,4}(D),
$$

then

$$
\int_0^t \mathrm{d}s \int_D \langle P, u\rangle \, \mathrm{d}x = \int_0^t \mathrm{d}s \int_D \left\langle M \times [1_D(B - \tilde{M}) - \phi'(M) - \Delta M], u\right\rangle \, \mathrm{d}x, \quad u \in \mathbb{W}^{1,4}(D).
$$

Hence for $u \in C_0^\infty(Q)$,

$$
\int_0^t \int_D \langle P, u\times M\rangle \, \mathrm{d}x \, \mathrm{d}s = \int_0^t \int_D \langle M\times\nabla M, \nabla(u\times M)\rangle + \langle 1_D(B-\tilde{M}) - \phi'(M), (u\times M)\times M\rangle \, \mathrm{d}x \, \mathrm{d}s. \tag{3.53}
$$

Therefore by (3.52), (3.53) and (3.45), we get

$$
\begin{aligned}
&\int_D \langle M(t) - M_0, u\rangle \ \mathrm{d}x \\
= &\int_0^t \int_D \Big\{ \langle B - M - \phi'(M), \lambda_1 u \times M - \lambda_2 (u \times M) \times M\rangle \\
&- \langle \nabla M, \nabla[\lambda_1 u \times M - \lambda_2 (u \times M) \times M]\rangle \Big\} \mathrm{d}x \, \mathrm{d}s, \quad u \in C_0^\infty(Q;\mathbb{R}^3);
\end{aligned}
$$

This is (5.14). □

Proof of (3.13). Let $u \in C_0^\infty(D, \mathbb{R})$. Then we consider

$$\psi : H \ni M \longmapsto \langle M, uM \rangle_H \in \mathbb{R}.$$

For $v \in H$,

$$\begin{aligned} & \psi(M+v) - \psi(M) \\ = & \ \phi\langle M+v, M+v\rangle_H - \phi\langle M, M\rangle_H \\ = & \ \langle 2\phi M, v\rangle_H + \phi\langle v, v\rangle_H. \end{aligned}$$

Since $H^* = H$, we can see that $\psi'(M) = 2uM$.
Hence by (3.17),

$$\begin{aligned} & \langle M_n(t), uM_n(t)\rangle_H - \langle M_{0,n}, uM_{0,n}\rangle_H \\ = & \ \psi(M_n(t)) - \psi(M_{0,n}) \\ = & \int_0^t \left\langle \psi'(M_n(s)), \frac{\mathrm{d}M_n}{\mathrm{d}s} \right\rangle_H \mathrm{d}s \\ = & \int_0^t \left\langle 2uM(s), \frac{\mathrm{d}M_n}{\mathrm{d}s} \right\rangle_H \mathrm{d}s \\ = & \ 0, \qquad t \in [0, T]. \end{aligned}$$

Hence

$$\begin{aligned} & \int_D u(x)\left(\|M_n(t,x)\|^2 - \|M_{0,n}(x)\|^2\right) \mathrm{d}x \\ = & \int_D \langle M_n(t,x), u(x)M_n(t,x)\rangle \,\mathrm{d}x - \int_D \langle M_{0,n}(t,x), u(x)M_{0,n}(t,x)\rangle \,\mathrm{d}x \\ = & \ \langle M_n(t), uM_n(t)\rangle_H - \langle M_{0,n}, uM_{0,n}\rangle_H \\ = & \ 0, \qquad t \in [0, T]. \end{aligned}$$

$u \in C_0^\infty(D; \mathbb{R})$ is arbitrary, so $|M_n(t,x)| = |M_{0,n}(x)|$ a.e. in Q. And by (3.43), we have $\lim_{n\to\infty} M_n = M$ a.e. in Q. So

$$|M(t,x)| = \lim_{n\to\infty} |M_n(t,x)| = \lim_{n\to\infty} |M_{0,n}(t,x)| = |M_0(x)|, \quad \text{a.e. in } Q.$$

This completes the proof. □

Theorem 3.28.

$$E(0) = E_0. \tag{3.54}$$

Proof. Let $\psi \in C^1([0,T]; \mathbb{R}^3)$ such that $\psi(0) = 1$ and $\psi(T) = 0$. Then

$$\begin{aligned} \int_0^T \left\langle \frac{\partial E(t)}{\partial t}, \psi(t)v \right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \mathrm{d}t &= \langle E(t), \psi(t)v\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\Big|_0^T - \int_0^T \langle E(t), \psi'(t)v\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\ &= -\langle E(0), v\rangle_{\mathbb{L}^2(\mathbb{R}^3)} - \int_0^T \langle E(t), \psi'(t)v\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \,\mathrm{d}t. \end{aligned}$$

Hence by (3.38), we have:

$$
\begin{aligned}
&\langle E(0), v\rangle \\
&= \int_0^T \langle E(t), \psi'(t)v\rangle \,\mathrm{d}t - \int_0^T \left\langle \frac{\partial E(t)}{\partial t}, \psi(t)v \right\rangle \mathrm{d}t \\
&= \lim_{n\to\infty} \left(\int_0^T \langle E_n(t), \psi'(t)v\rangle \,\mathrm{d}t - \int_0^T \left\langle \frac{\partial E_n(t)}{\partial t}, \psi(t)v \right\rangle \mathrm{d}t \right) \\
&= \lim_{n\to\infty} \langle E_n(0), v\rangle \\
&= \lim_{n\to\infty} \langle \pi_n^W E^0, v\rangle \\
&= \langle E_0, v\rangle, \qquad \forall v \in W.
\end{aligned}
$$

Hence $E(0) = E_0$. □

Definition 3.29 (Lower semicontinuous)**.** A function f from a topological space X to $\mathbb{R}$ is called *lower semicontinuous* iff for $a \in \mathbb{R}$, the set $\{f > a\}$ is open in X.

Lemma 3.30. *Let X be a Banach space, then the norm on X^* is a lower semicontinuous function with respect to the weak star topology.*

Proof. For $a \in \mathbb{R}$, we have

$$
\begin{aligned}
\{y \in X^* : \|y\|_{X^*} > a\} &= \{y \in X^* : \sup_{\|x\|_X = 1} |\langle y, x\rangle| > a\} \\
&= \bigcup_{\|x\|_X=1} \{y \in X^* : |\langle y, x\rangle| > a\}.
\end{aligned}
$$

For $x \in X$, $\{y \in X^* : |\langle y, x\rangle| > a\}$ is open in the weak star topology of X^*, so $\{y \in X^* : \|y\|_{X^*} > a\}$ is also open. Hence $\|\cdot\|_{X^*}$ is lower semicontinuous. □

Lemma 3.31. *Let X be a Banach space, if $f : X \longrightarrow \mathbb{R}$ is lower semiconinuous, then if $x_n \longrightarrow x_0$ in X, we have*

$$
f(x_0) \le \liminf_{n\to\infty} f(x_n).
$$

Theorem 3.32.
(3.55)

$$
\mathcal{E}(t) + \int_0^t (\|1_D E(s)\|_H^2 + \|M(s)\times\rho(s)\|_H^2 + \langle f(s), 1_D E(s)\rangle_H)\,\mathrm{d}s \le \mathcal{E}(0), \quad 0 < t < T.
$$

Proof. For $r \in C([0,T];\mathbb{R})$, by (3.31) we have

$$\begin{aligned}
&\int_0^T \int_D \phi(M_n(t)) r^2(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\nabla M_n(t) r(t)\|_H^2\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^T \|[B_n(t) - \pi_n^W \tilde{M}_n(t)] r(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}t + \frac{1}{2}\int_0^T \|E_n(t) r(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}t \\
&+\int_0^T \int_0^t r^2(t)\|M_n(s)\times\rho_n(s)\|_H^2 + r^2(t)\|E_n(s)\|_H^2 + r^2(t)\langle E_n(s), f_n(s)\rangle_H\,\mathrm{d}s\,\mathrm{d}t \\
=&\int_0^T \int_D \phi(M_{0,n}) r^2(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\nabla M_{0,n} r(t)\|_H^2\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^T \|[B_{0,n} - \pi_n^W \tilde{M}_{0,n}] r(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}t + \frac{1}{2}\int_0^T \|E_{0,n} r(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}t, \quad t \in [0,T].
\end{aligned}$$

By (3.36)-(3.41), we have

$$r[B_n - \pi_n^W \tilde{M}_n] \longrightarrow rH \text{ weakly star in } L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3)) \cap W^{1,\infty}(0,T;W'). \tag{3.56}$$

$$rE_n \longrightarrow rE \text{ weakly star in } L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3)) \cap W^{1,\infty}(0,T;W'). \tag{3.57}$$

(3.58)

$$\left(\int_\cdot^T r^2(t)\,\mathrm{d}t\right)^{\frac{1}{2}} E_n \longrightarrow \left(\int_\cdot^T r^2(t)\,\mathrm{d}t\right)^{\frac{1}{2}} E \text{ weakly star in } L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))\cap W^{1,\infty}(0,T;W').$$

$$rM_n \longrightarrow rM \text{ weakly star in } L^\infty(0,T;V),\ \text{weakly in } H^1(0,T;\mathbb{L}^{\frac{3}{2}}(D)). \tag{3.59}$$

$$rM_n\times\rho_n \longrightarrow rP \text{ weakly in } L^2(0,T;H). \tag{3.60}$$

And by (3.43), we have $M_n(t,x) \longrightarrow M(t,x)$ almost everywhere in Q together with the continuity of ϕ, we have

$$\liminf_{n\to\infty} \int_0^T \int_D \phi(M_n(t)) r^2(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_D \phi(M(t)) r^2(t)\,\mathrm{d}x\,\mathrm{d}t.$$

Then with Lemma 3.30 and Lemma 3.31, we have

$$\int_0^T \|\nabla M(t) r(t)\|_H^2\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^T \|\nabla M_n(t) r(t)\|_H^2\,\mathrm{d}t.$$

$$\int_0^T \|[B(t) - \tilde{M}(t)] r(t)\|_H^2\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^T \|[B_n(t) - \pi_n^W \tilde{M}_n(t)] r(t)\|_H^2\,\mathrm{d}t.$$

$$\int_0^T \|E(t) r(t)\|_H^2\,\mathrm{d}t \le \liminf_{n\to\infty} \int_0^T \|E_n(t) r(t)\|_H^2\,\mathrm{d}t.$$

$$\int_0^T \int_0^t r^2(t)\|M(s)\times\rho(s)\|_H^2 + r^2(t)\|E(s)\|_H^2 \,\mathrm{d}s\,\mathrm{d}t$$
$$\leq \liminf_{n\to\infty}\int_0^T \int_0^t r^2(t)\|M_n(s)\times\rho_n(s)\|_H^2 + r^2(t)\|E_n(s)\|_H^2 \,\mathrm{d}s\,\mathrm{d}t.$$

And

$$\liminf_{n\to\infty}\int_0^T \int_0^t r^2(t)\langle E_n(s), f_n(s)\rangle_H \,\mathrm{d}s\,\mathrm{d}t = \int_0^T \int_0^t r^2(t)\langle E(s), f(s)\rangle_H \,\mathrm{d}s\,\mathrm{d}t.$$

Hence

$$\begin{aligned}
&\int_0^T \int_D \phi(M(t))r^2(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\nabla M(t)r(t)\|_H^2\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^T \|[B(t)-\tilde{M}(t)]r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t + \frac{1}{2}\int_0^T \|E(t)r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t \\
&+\int_0^T \int_0^t r^2(t)\|M(s)\times\rho(s)\|_H^2 + r^2(t)\|E(s)\|_H^2 + r^2(t)\langle E(s), f(s)\rangle_H\,\mathrm{d}s\,\mathrm{d}t \\
\leq\ &\liminf_{n\to\infty}\left\{\int_0^T \int_D \phi(M_n(t))r^2(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\nabla M_n(t)r(t)\|_H^2\,\mathrm{d}t\right. \\
&+\frac{1}{2}\int_0^T \|[B_n(t)-\pi_n^W\tilde{M}_n(t)]r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t + \frac{1}{2}\int_0^T \|E_n(t)r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t \\
&\left.+\int_0^T \int_0^t r^2(t)\|M_n(s)\times\rho_n(s)\|_H^2 + r^2(t)\|E_n(s)\|_H^2 + r^2(t)\langle E_n(s), f_n(s)\rangle_H\,\mathrm{d}s\,\mathrm{d}t\right\} \\
=\ &\liminf_{n\to\infty}\left\{\int_0^T \int_D \phi(M_{0,n})r^2(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\nabla M_{0,n}r(t)\|_H^2\,\mathrm{d}t\right. \\
&\left.+\frac{1}{2}\int_0^T \|[B_{0,n}-\pi_n^W\tilde{M}_{0,n}]r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t + \frac{1}{2}\int_0^T \|E_{0,n}r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t\right\} \\
=\ &\int_0^T \int_D \phi(M_0)r^2(t)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T \|\nabla M_0 r(t)\|_H^2\,\mathrm{d}t \\
&+\frac{1}{2}\int_0^T \|[B_0-\tilde{M}_0]r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t + \frac{1}{2}\int_0^T \|E_0 r(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}t.
\end{aligned}$$

This holds for all the $r\in C([0,T];\mathbb{R})$ hence we get

$$\begin{aligned}
&\int_D \phi(M(t))\,\mathrm{d}x + \frac{1}{2}\|\nabla M(t)\|_H^2 + \frac{1}{2}\|B(t)-\tilde{M}(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} \\
&+\int_0^t \lambda_2\|M(s)\times\rho(s)\|_H^2 + \|E(s)\|_H^2 + \langle E(s), f(s)\rangle_H\,\mathrm{d}s \\
\leq\ &\int_D \phi(M(0))\,\mathrm{d}x + \frac{1}{2}\|\nabla M(0)\|_H^2 + \frac{1}{2}\|B(0)-\tilde{M}(0)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E(0)\|_{\mathbb{L}^2(\mathbb{R}^3)}.
\end{aligned}$$

This completes the proof. □

4. Stochastic Landau-Lifschitz' Equation with General Energy

Brzeźniak and Goldys and Jegaraj [13] studied the Stochastic Landau-Lifschitz-Gilbert Equation with the following version:

$$\begin{cases} \mathrm{d}u(t) = (\lambda_1 u(t)\times\Delta u(t) - \lambda_2 u(t)\times(u(t)\times\Delta u(t)))\,\mathrm{d}t + (u(t)\times h)\circ\,\mathrm{d}W(t), \\ \frac{\partial u}{\partial n}(t,x) = 0, \qquad t>0, x\in\partial D, \\ u(0,x) = u_0(x), \qquad x\in D. \end{cases} \tag{4.1}$$

where Δu in the Equation (4.1) stands for the exchange energy. In this section we will consider a similar version of Stochastic Landau-Lifschitz-Gilbert Equation as (4.1) but with a more general exchange energy: $\Delta u - \nabla\phi(u)$.

4.1. Statement of the problem.

Notation 4.1. For $O = D$ or $O = \mathbb{R}^3$, let us denote

$$\mathbb{L}^p(O) = L^p(O;\mathbb{R}^3), \qquad L^p(O) := L^p(O;\mathbb{R}).$$

$$\mathbb{W}^{k,p}(O) = W^{k,p}(O;\mathbb{R}^3), \qquad W^{k,p}(O) := W^{k,p}(O;\mathbb{R}).$$

$$\mathbb{H}^k(O) = H^k(O;\mathbb{R}^3), \qquad H^k(O) := H^k(O;\mathbb{R}).$$

$$H := \mathbb{L}^2(D), \qquad V := \mathbb{W}^{1,2}(D).$$

Assumption 4.2. *Let D be an open and bounded domain in $\mathbb{R}^3$ with C^2 boundary $\Gamma := \partial D$. n is the outward normal vector on Γ. $\lambda_1\in\mathbb{R}$, $\lambda_2>0$, $h\in\mathbb{L}^\infty(D)\cap\mathbb{W}^{1,3}(D)$, $u_0\in V$. $\phi:\mathbb{R}^3\longrightarrow\mathbb{R}^+\cup\{0\}$ is in C^4 and ϕ, ϕ', ϕ'' and $\phi^{(3)}$ are bounded. And we also assume ϕ' is globally Lipschitz. Moreover, we also assume that we have a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$, and this probability space satisfies the so called usual conditions:*

(i) *$\mathbb{P}$ is complete on $(\Omega,\mathcal{F})$,*
(ii) *for each $t\geq 0$, $\mathcal{F}_t$ contains all $(\mathcal{F},\mathbb{P})$-null sets,*
(iii) *the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous.*

We also assume that $(W(t))_{t\geq 0}$ is a real-valued, $(\mathcal{F}_t)_{t\geq 0}$-adapted Wiener process defined on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$.

The equation we are going to study in this section is:

$$\begin{cases} \mathrm{d}u(t) &= \Big\{\lambda_1 u(t)\times[\Delta u(t) - \nabla\phi(u(t))] \\ & \quad -\lambda_2 u(t)\times\Big(u(t)\times[\Delta u(t) - \nabla\phi(u(t))]\Big)\Big\}\,\mathrm{d}t \\ & \quad +\Big\{u(t)\times h\Big\}\circ\,\mathrm{d}W(t) \\ \frac{\partial u}{\partial n}\Big|_\Gamma &= 0 \\ u(0) &= u_0 \end{cases} \tag{4.2}$$

The solution u of the Equation (4.2) will be an H-valued process.

Remark 4.3. In the Equations (4.2) we use the Stratonovich differential and in the Equation (4.10) we use the Itô differential, the following equality relates the two differentials: for the map $G : H \ni u \mapsto u \times h \in H$,

$$(Gu) \circ \mathrm{d}W(t) = \frac{1}{2} G'(u)[G(u)]\,\mathrm{d}t + G(u)\,\mathrm{d}W(t), \qquad u \in H.$$

Remark 4.4. Since $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, for every $x \in \mathbb{R}^3$ the Frechet derivative $\mathrm{d}_x\phi = \phi'(x) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is linear, and hence by the Riesz Lemma, there exists a vector $\nabla\phi(x) \in \mathbb{R}^3$ such that

$$\langle \nabla\phi(x), y\rangle = \mathrm{d}_x\phi(y), \quad y \in \mathbb{R}^3.$$

Definition 4.5 (Solution of (4.2))**.** Let $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$ be a filtered probability space, W is an $\mathbb{F}'$-adapted Wiener process. We say that an $\mathbb{F}'$-progressively measurable process $u = (u_i)_{i=1}^3 : \Omega' \times [0,T] \longrightarrow V \cap \mathbb{L}^\infty(D)$ is a weak solution (in both probabilistic sense and partial differential equation sense) of (4.2) if and only if for all the $\psi \in C_0^\infty(D;\mathbb{R}^3)$, $t \in [0,T]$, we have:

$$\begin{aligned}
\langle u(t), \psi\rangle_H &= \langle u_0, \psi\rangle_H - \lambda_1 \int_0^t \langle \nabla u(s), \nabla\psi \times u(s)\rangle_{L^2(D;\mathbb{R}^{3\times 3})}\,\mathrm{d}s \\
&+\lambda_1 \int_0^t \langle u(s) \times \nabla\phi(u(s)), \psi\rangle_H\,\mathrm{d}s \\
&-\lambda_2 \int_0^t \langle \nabla u(s), \nabla(u \times \psi)(s) \times u(s)\rangle_{L^2(D;\mathbb{R}^{3\times 3})}\,\mathrm{d}s \\
&+\lambda_2 \int_0^t \langle u(s) \times (u(s) \times \nabla\phi(u(s))), \psi\rangle_H\,\mathrm{d}s \\
&+ \int_0^t \langle u(s) \times h, \psi\rangle_H \circ \mathrm{d}W(s).
\end{aligned} \tag{4.3}$$

4.2. **Galerkin approximation.** Let $A := -\Delta$ be a linear operator as defined in Definition 2.164. As in Lemma 2.171, we set $H_n = \mathrm{linspan}\{e_1, e_2, \ldots, e_n\}$, where $e_1, \ldots, e_n, \ldots$ are eigenvectors of A. Let π_n denote the orthogonal projection from H to H_n. Then we consider the following equation in H_n ($H_n \subset D(A)$):

$$\begin{cases}
\mathrm{d}u_n(t) &= \pi_n\Big\{ - \lambda_1 u_n(t) \times \Big[Au_n(t) + \pi_n(\nabla\phi(u_n(t)))\Big] \\
&\qquad +\lambda_2 u_n(t) \times \Big(u_n(t) \times \Big[Au_n(t) + \pi_n(\nabla\phi(u_n(t)))\Big]\Big)\Big\}\,\mathrm{d}t \\
&\quad +\pi_n\Big[u_n(t) \times h\Big] \circ \mathrm{d}W(t) \\
u_n(0) &= \pi_n u_0
\end{cases} \tag{4.4}$$

In what follows, in order to simplify notation, instead of $\nabla\phi(u)$ we will write $\phi'(u)$.

Let us define the following maps:

$$F_n^1 : H_n \ni u \longmapsto -\pi_n(u \times Au) \in H_n, \tag{4.5}$$

$$F_n^2 : H_n \ni u \longmapsto -\pi_n(u \times (u \times Au)) \in H_n, \tag{4.6}$$

$$F_n^3 : H_n \ni u \longmapsto -\pi_n[u \times \pi_n(\phi'(u))] \in H_n, \tag{4.7}$$

$$F_n^4 : H_n \ni u \longmapsto -\pi_n\Big(u \times [u \times \pi_n(\phi'(u))]\Big) \in H_n, \tag{4.8}$$

$$G_n : H_n \ni u \longmapsto \pi_n(u \times h) \in H_n, \ h \in \mathbb{L}^\infty(D) \cap \mathbb{W}^{1,3}(D). \tag{4.9}$$

Since A restrict to H_n is linear and bounded (with values in H_n) and since $H_n \subset D(A) \subset \mathbb{L}^\infty(D)$, we infer that G_{jn} and $F_n^1, F_n^2, F_n^3, F_n^4$ are well defined maps from H_n to H_n.

The problem (4.4) can be written in a more compact way

$$\begin{cases} \mathrm{d}u_n(t) = \lambda_1[F_n^1(u_n(t)) + F_n^3(u_n(t))]\,\mathrm{d}t - \lambda_2\,[F_n^2(u_n(t)) + F_n^4(u_n(t))]\,\mathrm{d}t \\ \qquad\qquad +\frac{1}{2}G_n^2(u_n(t))\,\mathrm{d}t + G_n(u_n(t))\,\mathrm{d}W(t) \\ u_n(0) \ \ = \pi_n u_0 \end{cases} \tag{4.10}$$

Remark 4.6. As the equality (1.2), we have

$$-\nabla_{H_n}\mathcal{E}(u_n) = Au_n + \pi_n\phi'(u_n),$$

so with the "π_n"s in the equation (4.4), our approximation keeps as much as possible the structure of the equation (4.2).

Now we start to solve the Equation (4.10).

Lemma 4.7. *The maps F_n^i, $i = 1, 2, 3, 4$ are Lipschitz on balls, that is, for every $R > 0$ there exists a constant $C = C(n, R) > 0$ such that whenever $x, y \in H_n$ and $|x|_H \le R$, $|y|_H \le R$, we have*

$$|F_n^i(x) - F_n^i(y)|_H \le C|x - y|_H.$$

The map G_n is linear and

$$|G_n u|_{H_n} \le |h|_{L^\infty}|u|_H, \qquad u \in H_n. \tag{4.11}$$

*Proof.*Step 1: We will show that F_n^1 and F_n^2 are Lipschitz on balls.
For $u \in H_n = linspan\{e_1, \dots, e_n\}$, there exist $k_1, \dots, k_n \in \mathbb{R}$, such that

$$u = \sum_{i=1}^n k_i e_i,$$

and by Lemma 2.171, there exist $\mu_1, \dots, \mu_n$, such that

$$\Delta e_i = \mu_i e_i.$$

So we have

$$\Delta u = \Delta \sum_{i=1}^n k_i e_i = \sum_{i=1}^n k_i \mu_i e_i.$$

Put $\overline{\mu_n} := \max_{1\leq i\leq n} |\mu_i|$. Then we have

$$|\Delta u|_{H_n}^2 = \sum_{i=1}^n k_i^2\mu_i^2 \leq \overline{\mu_n}^2 \sum_{i=1}^n k_i^2 = \overline{\mu_n}^2 |u|_{H_n}^2.$$

Hence we proved

$$|\Delta u|_{H_n} \leq \overline{\mu_n}|u|_{H_n}, \qquad u \in H_n.$$

Next if $R > 0$, for $u_1, u_2 \in H_n$ satisfying $|u_1|_{H_n}, |u_2|_{H_n} \leq R$, we have

$$\begin{aligned}
&|F_n^1(u_1) - F_n^1(u_2)|_{H_n} = |\pi_n(u_1 \times \Delta u_1) - \pi_n(u_2 \times \Delta u_2)|_{H_n} \\
&\leq |u_1 \times \Delta u_1 - u_2 \times \Delta u_2|_H = |u_1 \times \Delta u_1 - u_1 \times \Delta u_2 + u_1 \times \Delta u_2 - u_2 \times \Delta u_2|_H \\
&\leq |u_1 \times (\Delta u_1 - \Delta u_2)|_H + |(u_1 - u_2) \times \Delta u_2|_H \\
&\leq |u_1|_{\mathbb{L}^\infty} \cdot |\Delta u_1 - \Delta u_2|_H + |\Delta u_2|_{\mathbb{L}^\infty} \cdot |u_1 - u_2|_H \\
&\leq \lambda R|u_1 - u_2|_{H_n} + \lambda R|u_1 - u_2|_{H_n} = 2\lambda R|u_1 - u_2|_{H_n}
\end{aligned}$$

Similarly, we get

$$|F_n^2(u_1) - F_n^2(u_2)|_{H_n} \leq 3\lambda R^2|u_1 - u_2|_{H_n}, \qquad u_1, u_2 \in H_n.$$

Therefore F_n^1 and F_n^2 are Lipschitz on balls.

Step 2: We will show that F_n^3 and F_n^4 are Lipschitz on balls.
Since π_n is a linear contraction in H it is enough to consider the functions without the external $-\pi_n$ in the definition of F_n^3 and F_n^4. Then we have

$$\begin{aligned}
\left|F_n^3(u_1) - F_n^3(u_2)\right|_{H_n} &\leq \left|u_1 \times \pi_n(\phi'(u_1)) - y \times \pi_n(\phi'(u_2))\right|_H \\
&\leq \left|(u_1 - u_2) \times \pi_n(\phi'(u_1))\right|_{L^2} + \left|y \times (\pi_n(\phi'(u_1)) - \pi_n(\phi'(u_2)))\right|_{L^2} \\
&\leq \left|\pi_n(\phi'(u_1))\right|_{L^2} |u_1 - u_2|_{L^\infty} + |y|_{L^\infty} \left|\pi_n(\phi'(u_1)) - \pi_n(\phi'(u_2))\right|_{L^2} \\
&\leq \left|\phi'(u_1)\right|_{L^2} |u_1 - u_2|_{L^\infty} + |y|_{L^\infty} \left|\phi'(u_1) - \phi'(u_2)\right|_{L^2}
\end{aligned}$$

Since by assumptions the function $\phi' : \mathbb{R}^3 \to \mathbb{R}^3$ is globally Lipschitz and bounded, we infer that there exists $C > 0$, such that

$$|\phi'(u)|_H^2 = \int_D |\phi'(x)|_{\mathbb{R}^3}^2 \, dx \leq C, \quad u \in H$$

and similarly,

$$|\phi'(u_1) - \phi'(u_2)|_H \leq C|u_1 - u_2|_H, \quad u_1, u_2 \in H.$$

And since $H_n = \text{linspan}\{e_1, \ldots, e_n\}$ and $e_j \in D(A) \subset H^2$, and by Theorem 2.94, $H^2 \subset C(D; \mathbb{R}^3)$. In particular, $H_n \subset L^\infty(D)$. Since H_n is a finite dimensional space, all norms on it are equivalent. In particular, there exists $C = C(n) > 0$ such that

$$|u|_{L^\infty} \leq C|u|_H, \quad u \in H_n.$$

Therefore, we infer that for $R > 0$, there exists $C_R > 0$: if $|u_1|_H, |u_2|_H \leq R$, then

$$\begin{aligned}
\left|F_n^3(u_1) - F_n^3(u_2)\right|_H &\leq \left|\phi'(x)\right|_{L^2} |u_1 - u_2|_{L^\infty} + |u_2|_{L^\infty} \left|\phi'(u_1) - \phi'(u_2)\right|_{L^2} \\
&\leq C_R|u_1 - u_2|_H.
\end{aligned}$$

And similarly we get, for $R > 0$, there exists $C_R > 0$: if $|u_1|_H, |u_2|_H \leq R$, then

$$\begin{aligned}
&\left|F_n^4(u_1) - F_n^4(y)\right|_H \\
\leq &\left|u_1 \times [u_1 \times \pi_n(\phi'(u_1))] - y \times [y \times \pi_n(\phi'(u_1))]\right|_H \\
\leq &\left|(u_1 - u_2) \times [u_1 \times \pi_n(\phi'(u_1))] + u_2 \times [u_1 \times \pi_n(\phi'(u_1)) - u_2 \times \pi_n(\phi'(u_2))]\right|_H \\
\leq &|u_1 - u_2|_{L^\infty}|u_1|_{L^\infty} \left|\phi'(u_1)\right|_H + |u_2|_{L^\infty} \left(|u_1 - u_2|_{L^\infty} \left|\phi'(u_1)\right|_H + |u_2|_{L^\infty} \left|\phi'(u_1) - \phi'(u_2)\right|_H\right) \\
\leq &C_R|u_1 - u_2|_H.
\end{aligned}$$

Step 3: We will show that G_n is linear and satisfies the inequality (4.11).
We have

$$\begin{aligned}
|G_n(u)|^2_{H_n} &\leq |u \times h|^2_H \\
&= \int_D |u \times h|^2 \,\mathrm{d}x \\
&\leq \int_D |u|^2|h|^2 \,\mathrm{d}x \\
&\leq |h|^2_{L^\infty}|u|^2_H
\end{aligned}$$

Hence

$$|G_n u|_{H_n} \leq |h|_{L^\infty}|u|_H, \qquad u \in H_n.$$

This completes the proof. □

Let us define functions F_n and $\hat{F}_n : H_n \longrightarrow H_n$ by

$$F_n = \lambda_1(F_n^1 + F_n^3) - \lambda_2(F_n^2 + F_n^4), \text{ and } \hat{F}_n = F_n + \frac{1}{2}G_n^2.$$

Then the problem (4.4) (or (4.10)) becomes

$$\mathrm{d}u_n(t) = \hat{F}_n(u_n(t))\,\mathrm{d}t + G_n(u_n(t))\,\mathrm{d}W(t). \tag{4.12}$$

Lemma 4.8. *Assume that $h \in \mathbb{L}^\infty$. Then*

$$G_n^* = -G_n,$$

and in particular for all $u \in H_n$

$$\langle G_n u, u\rangle_H = 0.$$

Moreover for $i = 1, 2, 3, 4$, we have

$$\left\langle F_n^i(u), u\right\rangle_H = 0.$$

Proof. Let's assume that $u, v \in H_n$. Then we have

$$\begin{aligned}
(G_n u, v)_{H_n} &= (G_n u, v)_H = (\pi_n(u \times h), v)_H \\
&= (u \times h, \pi_n v)_H = -(u, v \times h)_H \\
&= -(\pi_n u, v \times h)_H = -(u, \pi_n(v \times h))_H \\
&= -(u, G_n v)_H = -(u, G_n v)_{H_n}.
\end{aligned}$$

Thus $G_n^* = -G_n$. Therefore,

$$\begin{aligned}(G_n(u), u)_H &= (u, G_n^* u)_H \\ &= -(u, G_n u)_H = -(G_n u, u)_H = 0.\end{aligned}$$

Moreover, for $u \in H_n$, we have

$$\begin{aligned}(F_n^1(u), u)_H &= (\pi_n(u \times \Delta u), u)_H \\ &= (u \times \Delta u, \pi_n u)_H = (u \times \Delta u, u)_H = 0,\end{aligned}$$

$$\begin{aligned}(F_n^2(u), u)_H &= (\pi_n(u \times (u \times \Delta u)), u)_H \\ &= (u \times (u \times \Delta u), \pi_n u)_H = (u \times (u \times \Delta u), u)_H = 0,\end{aligned}$$

Finally, using the fact $\langle a \times b, a\rangle_H = 0$, we have:

$$\begin{aligned}\langle F_n^3(u), u\rangle_H &= \langle -\pi_n[u \times \pi_n(\phi'(u))], u\rangle_H = -\langle u \times \pi_n(\phi'(u)), \pi_n u\rangle_H \\ &= -\langle u \times \pi_n(\phi'(u)), u\rangle_H = 0,\end{aligned}$$

and

$$\begin{aligned}\langle F_n^4(u), u\rangle_H &= \langle -\pi_n(u \times [u \times \pi_n(\phi'(u))]), u\rangle_H = -\langle u \times [u \times \pi_n(\phi'(u))], \pi_n u\rangle_H \\ &= -\langle u \times [u \times \pi_n(\phi'(u))], u\rangle_H = 0.\end{aligned}$$

This completes the proof. □

The following existence and uniqueness Theorem is followed by Lemma 4.7 and Lemma 4.8.

Theorem 4.9. [3] *The Equation* (4.4) *has a unique global solution* $u_n : [0, T] \longrightarrow H_n$.

Proof. By the Lemma 4.7 and Lemma 4.8, the coefficients F_n^i, $i = 1, 2, 3, 4$ and G_n are locally Lipschitz and one side linear growth. Hence by Theorem 3.1 in [3], the Equation (4.4) has a unique global solution $u_n : [0, T] \longrightarrow H_n$. □

4.3. **a'priori estimates.** In this subsection we will get some properties of the solution of Equation (4.4) especially some a'priori estimates.

Theorem 4.10. *Assume that* $n \in \mathbb{N}$. *Let* u_n *be the solution of the Equation* (4.4) *which is constructed earlier. Then for every* $t \in [0, T]$,

$$|u_n(t)|_H = |u_n(0)|_H, \qquad a.s.. \tag{4.13}$$

Proof. Let us define $\psi : H_n \ni u \mapsto \frac{1}{2}|u|_H^2 \in \mathbb{R}$. Since

$$\begin{aligned}|u + g|_H^2 - |u|_H^2 &= \langle u_n + g, u + g\rangle_H - \langle u, u\rangle_H \\ &= \langle u + g, g\rangle_H + \langle g, u\rangle_H = 2\langle u, g\rangle_H + |g|_H^2, \quad u, g \in H_n.\end{aligned}$$

and

$$\langle u + k, g\rangle_H - \langle u, g\rangle_H = \langle k, g\rangle_H, \quad u, g, k \in H_n.$$

We infer that

$$\psi'(u)(g) = \langle u, g\rangle_H, \quad \text{and} \quad \psi''(u)(g,k) = \langle k, g\rangle_H.$$

By the Itô Lemma 2.124, we get

$$\begin{aligned}\frac{1}{2}\,\mathrm{d}|u_n(t)|_H^2 &= \left(\left\langle u_n(t), \hat{F}(u_n(t))\right\rangle_H + \frac{1}{2}\left\langle G_n(u_n(t)), G_n(u_n(t))\right\rangle_H\right)\mathrm{d}t \\ &\quad + \left\langle u_n(t), G_n(u_n(t))\right\rangle_H \,\mathrm{d}W_t \\ &= \frac{1}{2}\left(\left\langle u_n(t), G_n^2(u_n(t))\right\rangle + \left|G_n(u_n(t))\right|_H^2\right)\mathrm{d}t + 0\,\mathrm{d}W_t \\ &= 0\end{aligned}$$

Hence for $t \in [0,T]$,

$$|u_n(t)|_H = |u_n(0)|_H, \qquad a.s..$$

□

Lemma 4.11. *Let us define a function* $\Phi : H_n \longrightarrow \mathbb{R}$ *by*

$$\Phi(u) := \frac{1}{2}\int_D |\nabla u(x)|^2\,\mathrm{d}x + \int_D \phi(u(x))\,\mathrm{d}x, \qquad u \in H_n. \tag{4.14}$$

Then $\Phi \in C^2(H_n)$ *and for* $u, g, k \in H_n$,

$$\begin{aligned}\mathrm{d}_u\Phi(g) &= \Phi'(u)(g) = \langle \nabla u, \nabla g\rangle_{L^2(D,\mathbb{R}^{3\times 3})} + \int_D \langle \nabla\phi(u(x)), g(x)\rangle\,\mathrm{d}x \\ &= -\langle \Delta u, g\rangle_{L^2(D,\mathbb{R}^{3\times 3})} + \int_D \langle \nabla\phi(u(x)), g(x)\rangle\,\mathrm{d}x,\end{aligned} \tag{4.15}$$

$$\Phi''(u)(g,k) = \langle \nabla g, \nabla k\rangle_{L^2(D,\mathbb{R}^{3\times 3})} + \int_D \phi''(u(x))(g(x),k(x))\,\mathrm{d}x. \tag{4.16}$$

Proof. Let us introduce auxiliary functions Φ_0 and Φ_1 by:

$$\Phi_0(u) := \int_D \phi(u(x))\,\mathrm{d}x, \qquad u \in H_n.$$

$$\Phi_1(u) := \frac{1}{2}|\nabla u|^2_{L^2(D,\mathbb{R}^{3\times 3})}, \qquad u \in H_n.$$

It is enough to prove the results of Φ_0 and Φ_1.
Both Φ_0 and Φ_1 are of C^2-class and

$$\Phi_1'(u)g = (\nabla u, \nabla g)_{L^2(D,\mathbb{R}^{3\times 3})} \text{ and } \Phi_1''(u)(g,k) = (\nabla g, \nabla k)_{L^2(D,\mathbb{R}^{3\times 3})}, \quad \forall u, g, k \in H_n.$$

That is because

$$\begin{aligned}\Phi_1(u+g) - \Phi_1(u) &= \frac{1}{2}|\nabla u + \nabla g|^2_{L^2} - \frac{1}{2}|\nabla u|^2_{L^2} \\ &= (\nabla u, \nabla g)_{L^2} + \frac{1}{2}(\nabla g, \nabla g)_{L^2}\end{aligned}$$

and by Proposition 2.166,

$$(\nabla g, \nabla g)_{L^2(D,\mathbb{R}^{3\times 3})} = (Ag, g)_H.$$

By the Cauchy-Schwartz inequality

$$(g, Ag)_H \le |g|_H |Ag|_H,$$

and since $g \in H_n$, by Lemma 2.171, we have $g = \sum_{i=1}^n g_i e_i$, $g_i \in \mathbb{R}$, then $\Delta g = \sum_{i=1}^n \lambda_i g_i e_i$ where λ_i are eigenvalues of A at e_i. Let $K = \max_{1\le i\le n} |\lambda_i|$, then

$$|Ag|_H^2 = \sum_{i=1}^n |\lambda_i g_i|^2 \le K^2 \sum_{i=1}^n |g_i|^2 = K^2 |g|_H^2,$$

Therefore $|Ag|_H \le K|g|_H$, hence

$$(\nabla g, \nabla g)_{L^2(D,\mathbb{R}^{3\times 3})} \le K|g|_H^2 = o(|g|_H). \tag{4.17}$$

Therefore $\Phi_1'(u)g = (\nabla u, \nabla g)_{L^2(D,\mathbb{R}^{3\times 3})}$.
Moreover,

$$\Phi_1'(u+k)(g) - \Phi_1(u)(g) = (\nabla g, \nabla k)_{L^2(D,\mathbb{R}^{3\times 3})}.$$

The right hand side of the above equality is a linear map with respect to k, so $\Phi_1''(u)(g)(k) = (\nabla g, \nabla k)_{L^2(D,\mathbb{R}^{3\times 3})}$. Next we consider the parts related to Φ_0. For $u, g \in H_n$, by the mean value theorem, we have

$$\begin{aligned}
&\Phi_0(u+g) - \Phi_0(u) \\
&= \int_D \phi(u(x)+g(x))\,\mathrm{d}x - \int_D \phi(u(x))\,\mathrm{d}x = \int_D \left[\phi(u(x)+g(x)) - \phi(u(x))\right]\,\mathrm{d}x \\
&= \int_D \left[\phi'(u(x))(g(x)) + \int_0^1 (1-s)\phi''(u(x)+sg(x))(g(x),g(x))\,\mathrm{d}s\right]\mathrm{d}x \\
&= \int_D \phi'(u(x))(g(x))\,\mathrm{d}x + \int_D\int_0^1 (1-s)\phi''(u(x)+sg(x))(g(x),g(x))\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}$$

Since we assumed that ϕ'' is bounded, there is some $C_{\phi''} > 0$ such that

$$|\phi''(x)(h,h)| \le C_{\phi''}|h|^2, \qquad x, h \in \mathbb{R}^3.$$

Thus we have

$$\begin{aligned}
&\left|\int_D\int_0^1 (1-s)\phi''(u(x)+sg(x))(g(x),g(x))\,\mathrm{d}s\,\mathrm{d}x\right| \\
&\le \int_D\int_0^1 (1-s)\left|\phi''(u(x)+sg(x))(g(x),g(x))\right|\,\mathrm{d}s\,\mathrm{d}x \\
&\le \frac{1}{2}C_{\phi''}|g|_H^2 = o(|g|_H).
\end{aligned}$$

Hence we infer that

$$d_u\Phi_0(g) = \int_D \phi'(u(x))(g(x))\,\mathrm{d}x = \int_D \langle \nabla\phi(u(x)), g(x)\rangle\,\mathrm{d}x, \qquad u, g \in H_n.$$

Next we compute $\Phi_0''(u)$. We have the following inequalities:

$$
\begin{aligned}
&\int_D \phi'(u(x)+k(x))(g(x))\,\mathrm{d}x - \int_D \phi'(u(x))(g(x))\,\mathrm{d}x \\
= &\int_D [\phi'(u(x)+k(x)) - \phi'(u(x))](g(x))\,\mathrm{d}x \\
= &\int_D \phi''(u(x))(g(x),k(x))\,\mathrm{d}x + \int_D\int_0^1 (1-s)\phi^{(3)}(u(x)+sk(x))(k(x),k(x))\,\mathrm{d}s g(x)\,\mathrm{d}x,
\end{aligned}
$$

And since we assume that $\phi^{(3)}$ is bounded, so there is $C_{\phi^3} > 0$ such that

$$\left|\phi^{(3)}(x)(y,y)(z)\right| \leq C_{\phi^3}|y|^2|z|, \qquad x,y,z \in \mathbb{R}^3.$$

Then we infer that

$$
\begin{aligned}
&\left|\int_D \phi'(u(x)+k(x))g(x)\,\mathrm{d}x - \int_D \phi'(u(x))g(x)\,\mathrm{d}x - \int_D \phi''(u(x))(g(x),k(x))\,\mathrm{d}x\right| \\
= &\left|\int_D\int_0^1 (1-s)\phi^{(3)}(u(x)+sk(x))(k(x),k(x))\,\mathrm{d}s g(x)\,\mathrm{d}x\right| \\
\leq &\int_D\int_0^1 (1-s)\left|\phi^{(3)}(u(x)+sk(x))(k(x),k(x))g(x)\right|\,\mathrm{d}s\,\mathrm{d}x \\
\leq &\frac{1}{2}C_{\phi^{(3)}}\int_D |k(x)|^2_{\mathbb{R}^3}|g(x)|_{\mathbb{R}^3}\,\mathrm{d}x.
\end{aligned}
$$

Summarising we proved that

$$d_u^2\Phi_0(g,k) = \int_D \phi''(u(x))(g(x),k(x))\,\mathrm{d}x \qquad u,g,k \in H_n.$$

This ends the proof of Lemma 4.11. □

Proposition 4.12. *Let u_n be the solution of the Equation* (4.4)*. Then there exist constants $a,b,a_1,b_1 > 0$ such that for all $n \in \mathbb{N}$, $s \in [0,T]$,*

$$\left|\nabla G_n u_n(s)\right|^2_{L^2} \leq a\left|\nabla u_n(s)\right|^2_{L^2} + b, \tag{4.18}$$

and

$$\left|\nabla G_n^2 u_n(s)\right|^2_{L^2} \leq a_1\left|\nabla u_n(s)\right|^2_{L^2} + b_1. \tag{4.19}$$

Proof. By the Proposition 2.166,

$$\left|\nabla G_n u_n(s)\right|^2_{L^2} = \Big(AG_n u_n(s), G_n u_n(s)\Big)_H \leq \Big(A_1 G_n u_n(s), G_n u_n(s)\Big)_H.$$

Since $A_1 = I + A$ is self-adjoint,

$$\Big(A_1 G_n u_n(s), G_n u_n(s)\Big)_H = \left|A_1^{\frac{1}{2}} G_n u_n(s)\right|^2_{L^2} = \left|A_1^{\frac{1}{2}}\pi_n(u_n(s)\times h)\right|^2_{L^2}.$$

Thus

$$\left|\nabla G_n u_n(s)\right|^2_{L^2} \leq \left|A_1^{\frac{1}{2}}\pi_n(u_n(s)\times h)\right|^2_{L^2}.$$

Since $A_1^{\frac{1}{2}}$ and π_n commute, we have

$$\left|\nabla G_n u_n(s)\right|^2_{L^2} \leq \left|A_1^{\frac{1}{2}}\pi_n(u_n(s)\times h)\right|^2_H = \left|\pi_n A_1^{\frac{1}{2}}(u_n(s)\times h)\right|^2_H \leq \left|A_1^{\frac{1}{2}}(u_n(s)\times h)\right|^2_H.$$

Moreover, $\left|A_1^{\frac{1}{2}}(u_n(s)\times h)\right|_H^2 = \left|u_n(s)\times h\right|_V^2$, thus we get

$$\left|\nabla G_n u_n(s)\right|_{L^2}^2 \le \left|u_n(s)\times h\right|_V^2.$$

Hence

$$\begin{aligned}\left|\nabla G_n u_n(s)\right|_{L^2}^2 &\le \left|u_n(s)\times h\right|_V^2\\ &= \left|\nabla(u_n(s)\times h)\right|_{L^2}^2 + \left|u_n(s)\times h\right|_H^2\\ &\le 2\Big[\left|\nabla u_n(s)\times h\right|_{L^2}^2 + \left|u_n(s)\times \nabla h\right|_{L^2}^2\Big] + \left|u_n(s)\times h\right|_H^2\end{aligned}$$

By the Hölder's inequality,

$$\left|u_n(s)\times\nabla h\right|_{L^2}^2 \le \Big(u_n(s),\nabla h\Big)_{L^2}^2 \le \left|u_n(s)\right|_{L^6}^2\left|\nabla h\right|_{L^3}^2.$$

Since $0-\frac{2}{6}\ge 1-\frac{3}{2}$, by Theorem 2.98, $H^1\hookrightarrow L^6$ continuously. Hence there exists $c>0$, such that

$$\left|u_n(s)\right|_{L^6}^2 \le c(\left|\nabla u_n(s)\right|_{L^2}^2 + \left|u_n(s)\right|_{L^2}^2).$$

Together with $\left|u_n(s)\times h\right|_{L^2}^2 \le \left|u_n(s)\right|_{L^2}^2\left|h\right|_{L^\infty}^2$ and $\left|\nabla u_n(s)\times h\right|_{L^2}^2 \le \left|\nabla u_n(s)\right|_{L^2}^2\left|h\right|_{L^\infty}^2$, we can get

$$\begin{aligned}\left|\nabla G_n u_n(s)\right|_{L^2}^2 &\le 2\Big[\left|h\right|_{L^\infty}^2\left|\nabla u_n(s)\right|_{L^2}^2 + c(\left|\nabla u_n(s)\right|_{L^2}^2 + \left|u_n(s)\right|_{L^2}^2)\left|\nabla h\right|_{L^3}^2\Big]\\ &= 2\Big(\left|h\right|_{L^\infty}^2 + c\left|\nabla h\right|_{L^3}^2\Big)\left|\nabla u_n(s)\right|_{L^2}^2 + \Big(2c\left|\nabla h\right|_{L^3}^2 + \left|h\right|_{L^\infty}^2\Big)\left|u_n(s)\right|_{L^2}^2\end{aligned}$$

By (4.13), it is

$$\begin{aligned}\left|\nabla G_n u_n(s)\right|_{L^2}^2 &\le 2\Big(\left|h\right|_{L^\infty}^2 + c\left|\nabla h\right|_{L^3}^2\Big)\left|\nabla u_n(s)\right|_{L^2}^2 + \Big(2c\left|\nabla h\right|_{L^3}^2 + \left|h\right|_{L^\infty}^2\Big)\left|u_0\right|_{L^2}^2\\ &= a\left|\nabla u_n(s)\right|_{L^2}^2 + b\end{aligned}$$

where

$$a = 2\Big(\left|h\right|_{L^\infty}^2 + c\left|\nabla h\right|_{L^3}^2\Big),$$
$$b = \Big(2c\left|\nabla h\right|_{L^3}^2 + \left|h\right|_{L^\infty}^2\Big)\left|u_0\right|_{L^2}^2.$$

We can see that a and b depend only on h and u_0, but do not depend on n.
By (4.18), we can get

$$\begin{aligned}\left|\nabla G_n^2 u_n(s)\right|_{L^2}^2 &\le a\left|\nabla(G_n u_n(s))\right|_{L^2}^2 + \Big(2c\left|\nabla h\right|_{L^3}^2 + \left|h\right|_{L^\infty}^2\Big)\left|G_n u_0\right|_{L^2}^2\\ &\le a\Big(a\left|\nabla u_n(s)\right|_{L^2}^2 + b\Big) + \Big(2c\left|\nabla h\right|_{L^3}^2 + \left|h\right|_{L^\infty}^2\Big)\left|G_n u_0\right|_{L^2}^2\\ &= a_1\left|\nabla u_n(s)\right|_{L^2}^2 + b_1\end{aligned}$$

where

$$a_1 = a^2,$$
$$b_1 = b + \Big(2c\left|\nabla h\right|_{L^3}^2 + \left|h\right|_{L^\infty}^2\Big)\left|G_n u_0\right|_{L^2}^2.$$

a_1 and b_1 also depend only on h and u_0, but do not depend on n.
This completes the proof of the inequalities (4.18) and (4.19). □

Remark 4.13. The previous results will be used to prove the following fundamental a'prior estimates on the sequence $\{u_n\}$.

Theorem 4.14. *Assume that $p \geq 1$, $\beta > \frac{1}{4}$. Then there exists a constant $C > 0$, such that for all $n \in \mathbb{N}$,*

$$\mathbb{E} \sup_{r\in[0,t]} \left\{ \left|\nabla u_n(r)\right|^2_{L^2} + \int_D \phi(u_n(r,x))\, \mathrm{d}x \right\}^p \leq C, \qquad t \in [0,T], \tag{4.20}$$

$$\mathbb{E}\left[\left(\int_0^T \left|u_n(t) \times [\Delta u_n(t) - \nabla\phi(u_n(t))]\right|^2_H \mathrm{d}t\right)^p\right] \leq C, \tag{4.21}$$

$$\mathbb{E}\left[\left(\int_0^T \left|u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \nabla\phi(u_n(t))])\right|^2_{\mathbb{L}^{\frac{3}{2}}} \mathrm{d}t\right)^p\right] \leq C, \tag{4.22}$$

$$\mathbb{E}\int_0^T \left|\pi_n\left(u_n(t) \times (u_n(t) \times [\Delta u_n(t) - \nabla\phi(u_n(t))])\right)\right|^2_{X^{-\beta}} \mathrm{d}t \leq C. \tag{4.23}$$

We will prove the inequalities (4.20) and (4.21) in Theorem 4.14 together and prove (4.22) and (4.23) in Theorem 4.14 separately. In the argument below we will frequently use, without referring to this, that π_n is an orthogonal projection from H onto H_n.

Proof of (4.20) *and* (4.21). Let us define a function Φ same as in the Equation (4.14). Then by the Itô Lemma 2.124,

$$\begin{aligned} &\Phi(u_n(t)) - \Phi(u_n(0)) \\ &= \int_0^t \left(\Phi'(u_n(s))\hat{F}_n(u_n(s)) + \frac{1}{2}\Phi''(u_n(s))G_n(u_n(s))^2\right) \mathrm{d}s \\ &\quad + \int_0^t \Phi'(u_n(s))G_n(u_n(s))\, \mathrm{d}W(s), \qquad t \in [0,T]. \end{aligned} \tag{4.24}$$

Then we consider each terms of the Equation (4.24),

$$\begin{aligned} \Phi'(u)\hat{F}_n(u) = &- \lambda_2\left|u \times (\Delta u - \pi_n(\phi'(u))\right|^2_H \\ &- \frac{1}{2}\langle \Delta u - \pi_n(\phi'(u)), \pi_n(u \times h) \times h\rangle_H \end{aligned} \tag{4.25}$$

Let us now prove the equality (4.25). Since
$\hat{F}_n = \lambda_1(F_n^1 + F_n^3) - \lambda_2(F_n^2 + F_n^4) + \frac{1}{2}G_n^2$, for $u \in H_n$ we have,

$$\begin{aligned}
\Phi'(u)\hat{F}_n(u) &= \lambda_1\Phi'(u)\hat{F}_n^1(u) - \lambda_2\Phi'(u)\hat{F}_n^2(u) \\
&+ \lambda_1\Phi'(u)\hat{F}_n^3(u) - \lambda_2\Phi'(u)\hat{F}_n^4(u) + \frac{1}{2}\Phi'(u)G_n^2(u) \\
&= \lambda_1\Phi'(u)[\pi_n(u\times\Delta u)] - \lambda_2\Phi'(u)[\pi_n(u\times(u\times\Delta u))] \\
&+ \lambda_1\Phi'(u)[\pi_n(u\times\pi_n(-\phi'(u)))] - \lambda_2\Phi'(u)[\pi_n\big(u\times(u\times\pi_n(-\phi'(u)))\big)] \\
&+ \frac{1}{2}\Phi'(u)[\pi_n(\pi_n(u\times h)\times h)] \\
&= -\lambda_1\langle\Delta u, \pi_n(u\times\Delta u)\rangle + \lambda_2\langle\Delta u, \pi_n(u\times(u\times\Delta u))\rangle \\
&+ -\lambda_1\langle\Delta u, \pi_n(u\times\pi_n(-\phi'(u)))\rangle + \lambda_2\langle\Delta u, \pi_n\big(u\times(u\times\pi_n(-\phi'(u)))\big)\rangle \\
&- \frac{1}{2}\langle\Delta u, \pi_n(\pi_n(u\times h)\times h)\rangle \\
&+ \lambda_1\int_D\langle\nabla\phi(u(x)), \pi_n(u\times\Delta u)\rangle\,\mathrm{d}x \\
&- \lambda_2\int_D\langle\nabla\phi(u(x)), \pi_n(u\times(u\times\Delta u))\rangle\,\mathrm{d}x \\
&+ \lambda_1\int_D\langle\nabla\phi(u(x)), \pi_n(u\times\pi_n(-\phi'(u)))\rangle\,\mathrm{d}x \\
&- \lambda_2\int_D\langle\nabla\phi(u(x)), \pi_n\big(u\times(u\times\pi_n(-\phi'(u)))\big)\rangle\,\mathrm{d}x \\
&+ \frac{1}{2}\int_D\langle\nabla\phi(u(x)), \pi_n(\pi_n(u\times h)\times h)\rangle\,\mathrm{d}x
\end{aligned}$$

In above $\langle\cdot,\cdot\rangle$ is either the inner product in $H = L^2(D,\mathbb{R}^3)$ or in $\mathbb{R}^3$.

Since $\pi_n(\Delta u) = \Delta u$, for $u \in H_n$, we infer that

$$I = \langle\Delta u, \pi_n(u\times\Delta u)\rangle = \langle\Delta u, u\times\Delta u\rangle = 0,$$

Next, since $\langle a, b\times c\rangle = -\langle c, b\times a\rangle = -\langle b\times a, c\rangle$ and $\langle a, b\times(b\times a)\rangle = -|a\times b|^2$ for $a,b,c\in\mathbb{R}^3$, we get

$$\begin{aligned}
II &= \Big\langle\Delta u, \pi_n\big(u\times(u\times\Delta u)\big)\Big\rangle = \langle\Delta u, u\times(u\times\Delta u)\rangle \\
&= -\langle u\times\Delta u, u(x)\times\Delta u\rangle = -|u\times\Delta u|^2,
\end{aligned}$$

$$\begin{aligned}
III = \Big\langle\Delta u, \pi_n\big(u\times\pi_n(-\phi'(u))\big)\Big\rangle_H &= \langle\Delta u, u\times\pi_n(-\phi'(u))\rangle_H \\
&= -\langle\pi_n(-\phi'(u)), u\times\Delta u\rangle,
\end{aligned}$$

$$\begin{aligned}
IV &= \Big\langle\Delta u, \pi_n\big(u\times(u\times\pi_n(-\phi'(u)))\big)\Big\rangle = \langle\Delta u, u\times(u\times\pi_n(-\phi'(u)))\rangle \\
&= -\langle u\times\Delta u, u\times\pi_n(-\phi'(u))\rangle_H, \\
V &= \langle\Delta u, \pi_n(\pi_n(u\times h)\times h)\rangle = \langle\Delta u, \pi_n(u\times h)\times h\rangle
\end{aligned}$$

Since all the integral terms are simply appropriate scalar products in the space L^2 we infer that

$$\begin{aligned}
\Phi'(u)\hat{F}_n(u) = & -\lambda_2 |u\times\Delta u|^2 + \lambda_1\langle \pi_n(-\phi'(u)), u\times\Delta u\rangle - \lambda_2 \langle u\times\Delta u, u\times(-\phi'(u))\rangle \\
& - \frac{1}{2}\langle \Delta u, \pi_n(u\times h)\times h\rangle \\
& + \lambda_1\langle \pi_n[\phi'(u)], u\times\Delta u\rangle + \lambda_2\langle \pi_n(-\phi'(u)), u\times(u\times\Delta u)\rangle \\
& - \lambda_1\langle \pi_n(-\phi'(u)), u\times[-\phi'(u)]\rangle + \lambda_2\langle \pi_n(-\phi'(u)), u\times(u\times\pi_n(-\phi'(u)))\rangle \\
& + \frac{1}{2}\langle \pi_n(\phi'(u)), \pi_n(u\times h)\times h\rangle \\
= & -\lambda_2\Big[|u\times\Delta u|^2 + \langle u\times\Delta u, u\times\pi_n(-\phi'(u))\rangle \\
& - \langle \pi_n(-\phi'(u)), u\times(u\times\Delta u)\rangle - \langle \pi_n(-\phi'(u)), u\times(u\times\pi_n(-\phi'(u)))\rangle\Big] \\
& + \lambda_1\Big[\langle \pi_n(-\phi'(u)), u\times\Delta u\rangle - \langle \pi_n(-\phi'(u)), u\times\Delta u\rangle \\
& \quad + \langle \pi_n(-\phi'(u)), u\times\pi_n(-\phi'(u))\rangle\Big] \\
& - \frac{1}{2}\Big[\langle \Delta u, \pi_n(u\times h)\times h\rangle + \langle \pi_n(-\phi'(u), \pi_n(u\times h)\times h)\rangle\Big]
\end{aligned}$$

Using again the classical identities

$$\langle a, b\times c\rangle = -\langle b\times a, c\rangle, \text{ for } a,b,c\in\mathbb{R}^3,$$

$$\langle a, b\times a\rangle = 0, \text{ for } a,b\in\mathbb{R}^3,$$

the equality (4.25) has been obtained.

Similarly,

$$\begin{aligned}
\Phi'(u)[G_n(u)] = & \ \Phi'(u)\pi_n(u\times h) = -\langle \Delta u, \pi_n(u\times h)\rangle + \int_D \langle \nabla\phi(u(x)), [\pi_n(u\times h)](x)\rangle \, dx \\
= & -\langle \Delta u, u\times h\rangle + \langle \phi'(u), \pi_n(u\times h)\rangle
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
\Phi''(u)[G_n(u)^2] = & \ \Phi''(u)(G_n(u), G_n(u)) = \Phi''(u)(\pi_n(u\times h), \pi_n(u\times h)) \\
= & \ \langle \nabla\pi_n(u\times h), \nabla\pi_n(u\times h)\rangle_{L^2} + \int_D \phi''(u(x))(\pi_n(u\times h)(x), \pi_n(u\times h)(x)) \, dx \\
= & \ |\nabla\pi_n(u\times h)|_{L^2}^2 + \int_D \phi''(u(x))(\pi_n(u\times h)(x), \pi_n(u\times h)(x)) \, dx
\end{aligned} \tag{4.27}$$

Therefore by the Equations (4.14), (4.25), (4.26) and (4.27), the Equation (4.24) becomes:

$$
\begin{aligned}
&\frac{1}{2}|\nabla u_n(t)|_{L^2}^2 + \frac{1}{2}\int_D \phi(u_n(t,x))\,\mathrm{d}x \\
&+\lambda_2 \int_0^t |u_n(s)\times(\Delta u_n(s)-\pi_n\phi'(u_n(s)))|_H^2\,\mathrm{d}s \\
=\;& \frac{1}{2}|\nabla u_n(0)|_{L^2}^2 + \frac{1}{2}\int_D \phi(u_n(0,x))\,\mathrm{d}x - \frac{1}{2}\int_0^t \langle \Delta u_n(s), \pi_n(u_n(s)\times h)\times h\rangle_H\,\mathrm{d}s \\
&+\frac{1}{2}\int_0^t \langle \phi'(u_n(s)), \pi_n(u_n(s)\times h)\times h\rangle_H\,\mathrm{d}s + \frac{1}{2}\int_0^t |\nabla \pi_n(u_n(s)\times h)|_{L^2}^2\,\mathrm{d}s \\
&+\frac{1}{2}\int_0^t\int_D \phi''(\pi_n(s,x))(\pi_n(u_n(s)\times h)(x), \pi_n(u_n(s)\times h)(x))\,\mathrm{d}x\,\mathrm{d}s \\
&-\int_0^t \langle \Delta u_n(s), u_n(s)\times h\rangle_H\,\mathrm{d}W(s) + \int_0^t \langle \phi'(u_n(s)), \pi_n(u_n(s)\times h)\rangle_H\,\mathrm{d}W(s).
\end{aligned} \tag{4.28}
$$

Next we will get estimates for some terms on the right hand side of Equation (4.28).

For the first term on the right hand side of Equation (4.28), we have

(4.29)

$$
|\nabla u_n(0)|_{L^2}^2 = |\nabla \pi_n u_0|_{L^2}^2 \le |\pi_n u_0|_V^2 = |A_1^{\frac{1}{2}}\pi_n u_0|_H^2 = |\pi_n A_1^{\frac{1}{2}} u_0|_H^2 \le |A_1^{\frac{1}{2}} u_0|_H^2 = |u_0|_V^2.
$$

We assumed that ϕ is bounded, so there is a constant $C_\phi > 0$, such that $|\phi(y)| \le C_\phi$, for $y \in \mathbb{R}^3$. So for the second term on the right hand side of Equation (4.28), we have

$$
\left|\int_D \phi(u_n(0,x))\,\mathrm{d}x\right| \le C_\phi \mu(D). \tag{4.30}
$$

We assume that ϕ' and ϕ'' are bounded, so there exist constants $C_{\phi'} > 0$ and $C_{\phi''} > 0$, such that

$$
|\phi'(y)| \le C_{\phi'}, \qquad y \in \mathbb{R}^3, \tag{4.31}
$$

and

$$
|\phi''(x)(y,y)| \le C_{\phi''}|y|^2, \qquad x,y \in \mathbb{R}^3. \tag{4.32}
$$

For the third term on the right hand side of Equation (4.28), by (4.19) and Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
|\langle \Delta u_n(s), \pi_n(u_n(s)\times h)\times h\rangle_H| &= \left|\langle \nabla u_n(s), \nabla G_n^2 u_n(s)\rangle_{L^2}\right| \\
&\le |\nabla u_n(s)|_{L^2}\sqrt{a_1|\nabla u_n(s)|_{L^2}^2 + b_1} \le \sqrt{a_1}|\nabla u_n(s)|_{L^2}^2 + \frac{b_1}{2\sqrt{a_1}}.
\end{aligned} \tag{4.33}
$$

For the fourth term on the right hand side of Equation (4.28), by the equalities (4.31), (4.13) and Cauchy-Schwartz inequality, we have

(4.34)

$$
\langle \phi'(u_n(s)), \pi_n(u_n(s)\times h)\times h\rangle_H \le C_{\phi'}\mu(D)|u_n(s)\times h\times h|_H \le C_{\phi'}\mu(D)|u_0|_H|h|_{L^\infty}^2.
$$

For the fifth term on the right hand side of Equation (4.28), by (4.18), we have

$$|\nabla \pi_n(u_n(s) \times h)(s)|^2_{L^2} = |\nabla G_n(u_n(s))|^2_{L^2} \leq a|\nabla u_n(s)|^2_{L^2} + b. \tag{4.35}$$

For the sixth term on the right hand side of Equation (4.28), we have

$$\begin{aligned}
&\int_D \left|[\phi''(u_n(s,x))]\,[\pi_n(u_n(s) \times h)(x), \pi_n(u_n(s) \times h)(x)]\right| \mathrm{d}x \\
&\leq C_{\phi''} \int_D |\pi_n(u_n(s) \times h)(x))|^2 \,\mathrm{d}x = C_{\phi''}\,|\pi_n(u_n(s) \times h)|^2_H \\
&\leq C_{\phi''}\,|u_n(s) \times h|^2_H \leq C_{\phi''}|h|^2_{L^\infty}|u_0|^2_H.
\end{aligned} \tag{4.36}$$

Then by the equalities (4.28)-(4.36), there exists a constant $C_2 > 0$ such that for all $n \in \mathbb{N}$, $t \in [0, T]$ and $\mathbb{P}$-almost surely:

$$\begin{aligned}
&\frac{1}{2}\,|\nabla u_n(t)|^2_{L^2} + \frac{1}{2}\int_D \phi\big(u_n(t,x)\big)\,\mathrm{d}x \\
&+\lambda_2 \int_0^t |u_n(s) \times (\Delta u_n(s) - \pi_n\phi'(u_n(s)))|^2_H \,\mathrm{d}s \\
\leq\ & \frac{1}{2}|u_0|^2_V + \frac{1}{2}C_\phi\mu(D) + \frac{T}{2}C_{\phi'}|u_0|_H|h|^2_{L^\infty} + \frac{T}{2}C_{\phi''}|u_0|^2_H|h|^2_{L^\infty} \\
&+\frac{1}{2}\sqrt{a_1}\int_0^t |\nabla u_n(s)|^2\,\mathrm{d}s + \frac{b_1}{4\sqrt{a_1}}T + \frac{1}{2}a\int_0^t |\nabla u_n(s)|^2_{L^2}\,\mathrm{d}s + \frac{Tb}{2} \\
&+\int_0^t \big\langle \nabla u_n(s), \nabla G_n\big(u_n(s)\big)\big\rangle_{L^2}\,\mathrm{d}W_s + \frac{1}{2}\int_0^t \big\langle \phi'\big(u_n(s)\big), G_n\big(u_n(s)\big)\big\rangle_H\,\mathrm{d}W_s \\
=\ & \frac{1}{2}(\sqrt{a_1} + a)\int_0^t |\nabla u_n(s)|^2_H\,\mathrm{d}s + \int_0^t \big\langle \nabla u_n(s), \nabla G_n\big(u_n(s)\big)\big\rangle_{L^2}\,\mathrm{d}W_s \\
&+\frac{1}{2}\int_0^t \big\langle \phi'\big(u_n(s)\big), G_n\big(u_n(s)\big)\big\rangle_H\,\mathrm{d}W_s + C_2.
\end{aligned} \tag{4.37}$$

Therefore for all $n \in \mathbb{N}$, $t \in [0, T]$ and $\mathbb{P}$-almost surely:

$$\begin{aligned}
&|\nabla u_n(t)|^2_{L^2} + \int_D \phi\big(u_n(t,x)\big)\,\mathrm{d}x \\
&+2\lambda_2\int_0^t |u_n(s) \times (\Delta u_n(s) - \pi_n\phi'(u_n(s)))|^2_H\,\mathrm{d}s \\
\leq\ & (\sqrt{a_1} + a)\int_0^t |\nabla u_n(s)|^2_{L^2}\,\mathrm{d}s + C_2 \\
&+2\int_0^t \big\langle \nabla u_n(s), \nabla G_n\big(u_n(s)\big)\big\rangle_{L^2}\,\mathrm{d}W(s) \\
&+\int_0^t \big\langle \phi'\big(u_n(s)\big), G_n\big(u_n(s)\big)\big\rangle_H\,\mathrm{d}W(s).
\end{aligned} \tag{4.38}$$

Hence for $p \geq 1$,

$$
\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{|\nabla u_n(r)|^2_{L^2}+\int_D \phi(u_n(r,x))\,\mathrm{d}x+2\lambda_2\int_0^t |u_n(s)\times(\Delta u_n(s)-\pi_n\phi'(u_n(s)))|^2_H\,\mathrm{d}s\right\}^p\\
&\leq \mathbb{E}\sup_{r\in[0,t]}\left\{(\sqrt{a_1}+a)\int_0^r |\nabla u_n(s)|^2_{L^2}\,\mathrm{d}s+C_2\right.\\
&\quad\left.+2\int_0^r \left\langle \nabla u_n(s),\nabla G_n(u_n(s))\right\rangle_{L^2}\,\mathrm{d}W_s+\int_0^r \left\langle \phi'(u_n(s)),G_n(u_n(s))\right\rangle_H\,\mathrm{d}W_s\right\}^p\\
&\leq 4^{p-1}(\sqrt{a_1}+a)^p t^{p-1}\mathbb{E}\left(\int_0^t |\nabla u_n(s)|^{2p}_{L^2}\,\mathrm{d}s\right)\\
&\quad+4^{p-1}2\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \left\langle \nabla u_n(s),\nabla G_n(u_n(s))\right\rangle_{L^2}\,\mathrm{d}W_s\right|^p\\
&\quad+4^{p-1}\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \left\langle \phi'(u_n(s)),G_n(u_n(s))\right\rangle_{L^2}\,\mathrm{d}W_s\right|^p+4^{p-1}C_2^p
\end{aligned}
$$

By the Burkholder-Davis-Gundy inequality (Lemma 2.127), there exists a constant $K>0$ such that for all $n \in \mathbb{N}$,

$$
\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \left\langle \nabla u_n(s),\nabla G_n(u_n(s))\right\rangle_{L^2}\,\mathrm{d}W_s\right|^p \leq K\mathbb{E}\left|\int_0^t \left\langle \nabla u_n(s),\nabla G_n(u_n(s))\right\rangle^2_{L^2}\,\mathrm{d}s\right|^{\frac{p}{2}},
$$

$$
\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \left\langle \phi'(u_n(s)),G_n(u_n(s))\right\rangle_{L^2}\,\mathrm{d}W_s\right|^p \leq K\mathbb{E}\left|\int_0^t \left\langle \phi'(u_n(s)),G_n(u_n(s))\right\rangle^2_{L^2}\,\mathrm{d}s\right|^{\frac{p}{2}}.
$$

By the inequality (4.18) we get, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\mathbb{E}\left|\int_0^t \left\langle \nabla u_n(s),\nabla G_n(u_n(s))\right\rangle^2_{L^2}\,\mathrm{d}s\right|^{\frac{p}{2}} \leq \mathbb{E}\left[\sup_{r\in[0,t]}|\nabla u_n(r)|^p_{L^2}\left(\int_0^t |\nabla G_n(u_n(s))|^2_{L^2}\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\leq \mathbb{E}\left[\varepsilon\sup_{r\in[0,t]}|\nabla u_n(r)|^{2p}_{L^2}+\frac{4}{\varepsilon}\left(\int_0^t |\nabla G_n|^2_{L^2}\,\mathrm{d}s\right)^p\right]\\
&\leq \varepsilon\mathbb{E}\left(\sup_{r\in[0,t]}|\nabla u_n(r)|^{2p}_{L^2}\right)+\frac{4}{\varepsilon}(2t)^{p-1}a^p\mathbb{E}\left(\int_0^t |\nabla u_n(s)|^{2p}_{L^2}\,\mathrm{d}s\right)+\frac{4}{\varepsilon}2^{p-1}(bt)^p.
\end{aligned}
$$

And

$$
\begin{aligned}
&\mathbb{E}\left|\int_0^t \left\langle \phi'(u_n(s)),G_n(u_n(s))\right\rangle^2_{L^2}\,\mathrm{d}s\right|^{\frac{p}{2}} \leq \mathbb{E}\left[\sup_{r\in[0,t]}|\phi'(u_n(r))|^p_{L^2}\left(\int_0^t |\nabla G_n(u_n(s))|^2_{L^2}\,\mathrm{d}s\right)^{\frac{p}{2}}\right]\\
&\leq \varepsilon\left[C_{\phi'}\mu(D)\right]^{2p}+\frac{4}{\varepsilon}(2t)^{p-1}a^p\mathbb{E}\left(\int_0^t |\nabla u_n(s)|^{2p}_{L^2}\,\mathrm{d}s\right)+\frac{4}{\varepsilon}2^{p-1}(bt)^p.
\end{aligned}
$$

Hence we infer that for $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \langle \nabla u_n(s), \nabla G_n(u_n(s))\rangle_{L^2}\, \mathrm{d}W_s\right|^p \\
&\le K\varepsilon\mathbb{E}\left(\sup_{r\in[0,t]}|\nabla u_n(r)|_{L^2}^{2p}\right) + \frac{4K}{\varepsilon}(2t)^{p-1}a^p\mathbb{E}\left(\int_0^t |\nabla u_n(s)|_{L^2}^{2p}\,\mathrm{d}s\right) \\
&+\frac{4K}{\varepsilon}2^{p-1}(bt)^p.
\end{aligned}\tag{4.39}$$

and similarly for $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \langle \phi'(u_n(s)), G_n(u_n(s))\rangle_{L^2}\, \mathrm{d}W_s\right|^p \\
&\le K\varepsilon\left[C_{\phi'}\mu(D)\right]^{2p} + \frac{4K}{\varepsilon}(2t)^{p-1}a^p\mathbb{E}\left(\int_0^t |\nabla u_n(s)|_{L^2}^{2p}\,\mathrm{d}s\right) + \frac{4K}{\varepsilon}2^{p-1}(bt)^p.
\end{aligned}\tag{4.40}$$

Hence for every $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{|\nabla u_n(r)|_{L^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x + 2\lambda_2\int_0^t |u_n(s)\times(\Delta u_n(s) - \pi_n\phi'(u_n(s)))|_H^2\,\mathrm{d}s\right\}^p \\
&\le 4^{p-1}(\sqrt{a_1}+a)^p t^{p-1}\mathbb{E}\left(\int_0^t |\nabla u_n(s)|_{L^2}^{2p}\,\mathrm{d}s\right) + K\varepsilon\mathbb{E}\left(\sup_{r\in[0,t]}|\nabla u_n(r)|_{L^2}^{2p}\right) + K\varepsilon\left[C_{\phi'}\mu(D)\right]^{2p} \\
&+\frac{8K}{\varepsilon}(2t)^{p-1}a^p\mathbb{E}\left(\int_0^t |\nabla u_n(s)|_{L^2}^{2p}\,\mathrm{d}s\right) + \frac{8K}{\varepsilon}2^{p-1}(bt)^p
\end{aligned}$$

Set $\varepsilon = \frac{1}{2K}$ in the above inequality, we have:

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{|\nabla u_n(r)|_{L^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x \right.\\
&\left.+2\lambda_2\int_0^t |u_n(s)\times(\Delta u_n(s) - \pi_n\phi'(u_n(s)))|_H^2\,\mathrm{d}s\right\}^p \\
&\le \left[2\cdot 4^{p-1}(\sqrt{a_1}+a)^p t^{p-1} + 32K^2(2t)^{p-1}a^p\right]\mathbb{E}\left(\int_0^t |\nabla u_n(s)|_{L^2}^{2p}\,\mathrm{d}s\right) \\
&+\left[C_{\phi'}\mu(D)\right]^{2p} + 32K^2 2^{p-1}(bt)^p \\
&= C_3\mathbb{E}\left(\int_0^t |\nabla u_n(s)|_{L^2}^{2p}\,\mathrm{d}s\right) + C_4.
\end{aligned}\tag{4.41}$$

where the constants C_3 and C_4 are defined by:

$$C_3 = 2\cdot 4^{p-1}(\sqrt{a_1}+a)^p t^{p-1} + 32K^2(2t)^{p-1}a^p,$$

$$C_4 = \left[C_{\phi'}\mu(D)\right]^{2p} + 32K^2 2^{p-1}(bt)^p,$$

note that they not depend on n.

And since $\int_D \phi(u_n(r,x))\,\mathrm{d}x$ and $\lambda_2 \int_0^t |u_n(s)\times(\Delta u_n(s)-\pi_n\phi'(u_n(s)))|_H^2\,\mathrm{d}s$ are non-negative, so by the inequality (4.41), we have

$$
\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{|\nabla u_n(r)|_{L^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x\right.\\
&\left.+2\lambda_2\int_0^t |u_n(s)\times(\Delta u_n(s)-\pi_n\phi'(u_n(s)))|_H^2\,\mathrm{d}s\right\}^p\\
&\leq C_3\int_0^t \mathbb{E}\sup_{r\in[0,s]}\left\{|\nabla u_n(r)|_{L^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x\right.\\
&\left.+2\lambda_2\int_0^r |u_n(\tau)\times(\Delta u_n(\tau)-\pi_n\phi'(u_n(\tau)))|_H^2\,\mathrm{d}\tau\right\}^p\mathrm{d}s + C_4.
\end{aligned}
\tag{4.42}
$$

Let us define a function ψ by:

$$
\begin{aligned}
\psi(s) = \mathbb{E}\sup_{r\in[0,s]}&\left\{\left|\nabla u_n(r)\right|_{L^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x\right.\\
&\left.+2\lambda_2\int_0^r |u_n(\tau)\times(\Delta u_n(\tau)-\pi_n\phi'(u_n(\tau)))|_H^2\,\mathrm{d}\tau\right\}^p, \qquad s\in[0,T].
\end{aligned}
$$

Then by the inequality (4.42), we deduce that:

$$
\psi(t)\leq C_3\int_0^t \psi(s)\,\mathrm{d}s + C_4.
$$

Observe that ψ is a bounded Borel function. The boundedness is because

$$
|\nabla u_n(r)|_{L^2}\leq |u_n(r)|_V\leq C_n|u_n(r)|_H < C_n|u_0|_H,
$$

and

$$
\begin{aligned}
&\left|u_n(s)\times(\Delta u_n(s)-\pi_n\phi'(u_n(s)))\right|_{H_n}\\
&\leq |u_n(s)|_{\mathbb{L}^\infty(D)}\left(|\Delta u_n(s)|_H + |\pi_n\phi'(u_n(s))|_H\right)\\
&\leq C_n|u_n(s)|_H\left(C_n|u_n(s)|_H + C_{\phi'}\mu(D)\right)\\
&\leq C_n|u_0|_H\left(C_n|u_0|_H + 2C_{\phi'}\mu(D)\right).
\end{aligned}
$$

where C_n is from the norm equivalent in n-dimensional space. Therefore

$$
|\psi(s)|\leq\left(C_n^2|u_0|_H^2 + C_\phi\mu(D) + 2\lambda_2 T C_n^2|u_0|_H^2\left(C_n|u_0|_H + 2C_{\phi'}\mu(D)\right)^2\right)^p.
$$

Therefore by the Gronwall inequality Lemma 2.157, we have

$$
\psi(t)\leq C_3 e^{C_4 t}, \qquad t\in[0,T].
$$

Since C_3 and C_4 are independent of n, we have proved that for $T\in(0,\infty)$,

$$
\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left\{\left|\nabla u_n(r)\right|_{L^2}^2 + \int_D \phi(u_n(r,x))\,\mathrm{d}x + 2\lambda_2\int_0^r |u_n(\tau)\times(\Delta u_n(\tau)-\pi_n\phi'(u_n(\tau)))|_H^2\,\mathrm{d}\tau\right\}^p\\
&\leq C_3 e^{C_4 T} = C_T, \qquad t\in[0,T]
\end{aligned}
$$

where C_T independent of n. Therefore we infer that

$$\mathbb{E}\sup_{r\in[0,t]}\left\{\left|\nabla u_n(r)\right|^2_{L^2}+\int_D \phi(u_n(r,x))\,\mathrm{d}x\right\}^p < C_T,$$

and

$$\mathbb{E}\left(\int_0^T |u_n(\tau)\times(\Delta u_n(\tau)-\pi_n\phi'(u_n(\tau)))|^2_H\,\mathrm{d}\tau\right)^p < C_T.$$

This completes the proof of the inequalities (4.20) and (4.21). □

Proof of (4.22). By the Hölder inequality and the Sobolev imbedding Theorem 2.94 of $\mathbb{H}^1 \hookrightarrow \mathbb{L}^6$, we have that for some constant $c > 0$

$$\left|u_n(t)\times(u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))])\right|_{\mathbb{L}^{\frac{3}{2}}} \le \left|u_n(t)\right|_{\mathbb{L}^6}\left|u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))]\right|_{\mathbb{L}^2}$$
$$\le c\left|u_n(t)\right|_{\mathbb{H}^1}\left|u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))]\right|_{\mathbb{L}^2}.$$

Then by (4.13), (4.20) and (4.21), there exists some constant $c_1 > 0$, such that

$$\mathbb{E}\left[\left(\int_0^T \left|u_n(t)\times(u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))])\right|^2_{\mathbb{L}^{\frac{3}{2}}}\ \mathrm{d}t\right)^p\right]$$
$$\le c_1\mathbb{E}\left[\sup_{r\in[0,T]}\left|u_n(r)\right|^{2p}_{\mathbb{H}^1}\left(\int_0^T \left|u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))]\right|_{\mathbb{L}^2}\mathrm{d}t\right)^p\right]$$
$$\le c_1\left(\mathbb{E}\left[\sup_{r\in[0,T]}\left|u_n(r)\right|^{4p}_{\mathbb{H}^1}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left(\int_0^T \left|u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))]\right|_{\mathbb{L}^2}\mathrm{d}t\right)^{2p}\right]\right)^{\frac{1}{2}} \le C,$$

Note that C is independent of n. This completes the proof of (4.22). □

Proof of (4.23). By Proposition 2.172, if $\beta > \frac{1}{4}$, $X^\beta \hookrightarrow \mathbb{H}^{2\beta}(D)$ continuously. And by Theorem 2.98, if $\beta > \frac{1}{4}$, $\mathbb{H}^{2\beta}(D)$ is continuously imbedded in $\mathbb{L}^3(D)$. Therefore $\mathbb{L}^{\frac{3}{2}}(D)$ is continuously imbedded in $X^{-\beta}$. And since for $\xi \in H$,

$$|\pi_n\xi|_{X^{-\beta}} = \sup_{|\varphi|_{X^\beta}\le 1}\left|{}_{X^{-\beta}}\langle\pi_n\xi,\varphi\rangle_{X^\beta}\right| = \sup_{|\varphi|_{X^\beta}\le 1}|\langle\pi_n\xi,\varphi\rangle_H|$$
$$= \sup_{|\varphi|_{X^\beta}\le 1}|\langle\xi,\pi_n\varphi\rangle_H| \le \sup_{|\pi_n\varphi|_{X^\beta}\le 1}\left|{}_{X^{-\beta}}\langle\xi,\pi_n\varphi\rangle_{X^\beta}\right| = |\xi|_{X^{-\beta}}.$$

Therefore we infer that

$$\mathbb{E}\int_0^T \left|\pi_n(u_n(t)\times(u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))]))\right|_{X^{-\beta}}\mathrm{d}t$$
$$\le \mathbb{E}\int_0^T \left|u_n(t)\times(u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))])\right|_{X^{-\beta}}\mathrm{d}t$$
$$\le c\mathbb{E}\int_0^T \left|u_n(t)\times(u_n(t)\times[\Delta u_n(t)-\phi'(u_n(t))])\right|^2_{\mathbb{L}^{\frac{3}{2}}}\ \mathrm{d}t.$$

Then (4.23) follows from (4.22). □

Proposition 4.15. *Let u_n, for $n \in \mathbb{N}$, be the solution of the equation* (4.4) *and assume that $\alpha \in (0, \frac{1}{2})$, $\beta > \frac{1}{4}$, $p \geq 2$. Then the following estimates holds:*

$$\sup_{n\in\mathbb{N}} \mathbb{E}(|u_n|^2_{W^{\alpha,p}(0,T;X^{-\beta})}) < \infty. \tag{4.43}$$

Proof. Let us fix $\alpha \in (0, \frac{1}{2})$, $\beta > \frac{1}{4}$, $p \geq 2$. By the equation (4.10), we get

$$\begin{aligned} u_n(t) &= u_{0,n} + \lambda_1 \int_0^t [F_n^1(u_n(s)) + F_n^3(u_n(s))] \, ds - \lambda_2 \int_0^t [F_n^2(u_n(s)) + F_n^4(u_n(s))] \, ds \\ &\quad + \frac{1}{2} \int_0^t G_n^2(u_n(s)) \, ds + \int_0^t G_n(u_n(s)) \, dW(s) \\ &=: u_{0,n} + \sum_{i=1}^4 u_n^i(t), \qquad t \in [0, T]. \end{aligned}$$

By Theorem 4.14, we have the following results:

(1) There exists $C > 0$ such that for all $n \in \mathbb{N}$,
$$\mathbb{E}\left[|u_n^1|^2_{W^{1,2}(0,T;H)}\right] \leq C.$$
(2) There exists $C > 0$ such that for all $n \in \mathbb{N}$,
$$\mathbb{E}\left[|u_n^2|^2_{W^{1,2}(0,T;X^{-\beta})}\right] \leq C.$$
(3) There exists $C > 0$ such that for all $n \in \mathbb{N}$,
$$|u_n^3|^2_{W^{1,2}(0,T;H)} \leq C, \qquad \mathbb{P} - a.s..$$

Moreover, by the equality (4.13), there exists $C > 0$ such that for all $n \in \mathbb{N}$,
$$\mathbb{E}\left[\sup_{t\in[0,T]} |u_n(t)|^p_H\right] = \mathbb{E}\left[|u_n(0)|^p_H\right] \leq C.$$
By the inequality (4.11) and Lemma 2.135, we have: there exists $C > 0$ such that for all $n \in \mathbb{N}$
$$\mathbb{E}\left[|u_n^4|^p_{W^{\alpha,p}(0,T;X^{-\beta})}\right] \leq C.$$
Therefore since $H \hookrightarrow X^{-\beta}$ and by Theorem 2.98, $H^1(0, T; X^{-\beta}) \hookrightarrow W^{\alpha,p}(0, T; X^{-\beta})$ continuously, we get
$$\sup_{n\in\mathbb{N}} \mathbb{E}(|u_n|^2_{W^{\alpha,p}(0,T;X^{-\beta})}) < \infty.$$
This completes the proof of the inequality (4.43). □

4.4. **Tightness results.** In this subsection we will use the a'priori estimates (4.13)-(4.23) to show that the laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ are tight on a suitable path space.

Now we are going to prove our tightness result.

Lemma 4.16. *For any $p \geq 2$, $q \in [2, 6)$ and $\beta > \frac{1}{4}$ the set of laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ on the Banach space*
$$L^p(0, T; \mathbb{L}^q(D)) \cap C([0, T]; X^{-\beta})$$
is tight.

Proof. Let us choose and fix $p \geq 2$, $q \in [2,6)$ and $\beta > \frac{1}{4}$. Since $q < 6$ we can choose $\gamma \in (\frac{3}{4} - \frac{3}{2q}, \frac{1}{2})$, $\beta' \in (\frac{1}{4}, b)$, $\alpha \in (\frac{1}{p}, 1)$. Then by Proposition 2.174, $H^1 = D(A^{\frac{1}{2}}) \hookrightarrow X^\gamma = D(A^\gamma)$ is compact, hence by Lemma 2.116, the embedding

$$L^p(0,T;H^1) \cap W^{\alpha,p}(0,T;X^{-\beta'}) \hookrightarrow L^p(0,T;X^\gamma)$$

is compact. We note that for any positive real number r and random variables ξ and η, since

$$\Big\{\omega : \xi(\omega) > \frac{r}{2}\Big\} \cup \Big\{\omega : \eta(\omega) > \frac{r}{2}\Big\} \supset \Big\{\omega : \xi(\omega) + \eta(\omega) > r\Big\},$$

we have

$$\begin{aligned}
&\mathbb{P}\Big(|u_n|_{L^p(0,T;H^1)\cap W^{\alpha,p}(0,T;X^{-\beta'})} > r\Big)\\
&= \mathbb{P}\Big(|u_n|_{L^p(0,T;H^1)} + |u_n|_{W^{\alpha,p}(0,T;X^{-\beta'})} > r\Big)\\
&\leq \mathbb{P}\left(|u_n|_{L^p(0,T;H^1)} > \frac{r}{2}\right) + \mathbb{P}\left(|u_n|_{W^{\alpha,p}(0,T;X^{-b'})} > \frac{r}{2}\right)\\
&\leq \dots
\end{aligned}$$

then by the Chebyshev inequality in Lemma 2.133,

$$\dots \leq \frac{4}{r^2}\mathbb{E}\left(|u_n|^2_{L^p(0,T;V)} + |u_n|^2_{W^{\alpha,p}(0,T;X^{-\beta'})}\right).$$

By the estimates in (4.43), (4.13) and (4.20), the expected value on the right hand side of the last inequality is uniformly bounded in n. Let $X_T := L^p(0,T;V) \cap W^{\alpha,p}(0,T;X^{-\beta'})$. There is a constant C, such that

$$\mathbb{P}(\|u_n\|_{X_T} > r) \leq \frac{C}{r^2}, \qquad \forall r, n.$$

Since

$$\mathbb{E}(\|u_n\|_X) = \int_0^\infty \mathbb{P}(\|M_n\| > r)\,\mathrm{d}r,$$

we can infer that

$$\mathbb{E}(\|u_n\|_{X_T}) \leq 1 + \int_1^\infty \frac{C}{r^2}\,\mathrm{d}r = 1 + C < \infty, \quad \forall n \in \mathbb{N}.$$

Therefore by Theorem 2.107 the family of laws $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;X^\gamma)$. By Proposition 2.172, $X^\gamma = \mathbb{H}^{2\gamma}(D)$. Therefore since by the assumption $\gamma > \frac{3}{4} - \frac{3}{2q}$, i.e.

$$2\gamma - \frac{3}{2} > 0 - \frac{3}{q},$$

by Theorem 2.97 we deduce that $X^\gamma \hookrightarrow \mathbb{L}^q(D)$ continuously. Hence $L^p(0,T;X^\gamma) \hookrightarrow L^p(0,T;\mathbb{L}^q(D))$ continuously. By Lemma 2.108, $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is also tight on $L^p(0,T;\mathbb{L}^q(D))$.
Since $\beta' < \beta$, by Lemma 2.117, $W^{\alpha,p}(0,T;X^{-\beta'}) \hookrightarrow C([0,T];X^{-\beta})$ compactly. Therefore by the estimates in (4.43) and Lemma 2.107, we can conclude that $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $C([0,T];X^{-\beta})$.
Therefore by Theorem 2.109, $\{\mathcal{L}(u_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;L^q) \cap C([0,T];X^{-\beta})$. Hence the proof of Lemma 4.16 is complete. □

4.5. **Construction of new probability space and processes.** In this section we will use Skorohod's theorem to obtain another probability space and an almost surely convergent sequence defined on this space whose limit is a weak martingale solution of the equation (4.2).
By Lemma 4.16 and Prokhorov's Theorem, we have the following property.

Proposition 4.17. *Let us assume that W is the Wiener process and $p \in [2,\infty)$, $q \in [2,6)$ and $\beta > \frac{1}{4}$. Then there is a subsequence of $\{u_n\}$ which we will denote it in the same way as the full sequence, such that the laws $\mathcal{L}(u_n, W)$ converge weakly to a certain probability measure μ on $[L^p(0,T;\mathbb{L}^q(D)) \cap C([0,T];X^{-\beta})] \times C([0,T];\mathbb{R})$.*

Proof. By Lemma 4.16 and Theorem 2.114, the laws $\mathcal{L}(u_n)$ converge weakly to a certain probability measure μ_1 on $L^p(0,T;L^q) \cap C([0,T];X^{-\beta})$. Thus the laws $\mathcal{L}(u_n, W)$ converge weakly to a certain probability measure μ on $[L^p(0,T;L^q) \cap C([0,T];X^{-\beta})] \times C([0,T];\mathbb{R})$. □

Next by the Skorohod's theorem, we have the following proposition.

Proposition 4.18. *There exists a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and there exists a sequence (u_n', W_n') of $[L^4(0,T;\mathbb{L}^4(D)) \cap C([0,T];X^{-\beta})] \times C([0,T];\mathbb{R})$-valued random variables defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that*

(a) *On $[L^4(0,T;\mathbb{L}^4(D)) \cap C([0,T];X^{-\beta})] \times C([0,T];\mathbb{R})$,*

$$\mathcal{L}(u_n, W) = \mathcal{L}(u_n', W_n'), \qquad \forall n \in \mathbb{N}$$

(b) *There exists a random variable*

$$(u', W') : (\Omega', \mathcal{F}', \mathbb{P}') \longrightarrow [L^4(0,T;\mathbb{L}^4(D)) \cap C([0,T];X^{-\beta})] \times C([0,T];\mathbb{R})$$

such that

(i) *On $[L^4(0,T;\mathbb{L}^4(D)) \cap C([0,T];X^{-\beta})] \times C([0,T];\mathbb{R})$,*

$$\mathcal{L}(u', W') = \mu,$$

where μ is same as in Proposition 4.17.

(ii) *$u_n' \longrightarrow u'$ in $L^4(0,T;\mathbb{L}^4(D)) \cap C([0,T];X^{-\beta})$ almost surely*

(iii) *$W_n' \longrightarrow W'$ in $C([0,T];\mathbb{R})$ almost surely.*

Proof. Since $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-\beta}) \times C([0,T];\mathbb{R})$ is a separable metric space , this proposition is a direct result from the Skorohod Theorem (Lemma 2.132). □

Notation 4.19. We will use $\mathbb{F}'$ to denote the filtration generated by u' and W' in the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$.

From now on we will prove that u' is the weak solution of the equation (4.2). And we begin with showing that $\{u_n'\}$ satisfies the same a'priori estimates as the original sequence $\{u_n\}$.

Proposition 4.20. *The Borel subsets of $C([0,T];H_n)$ are Borel subsets of $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-\frac{1}{2}})$.*

Proof. Since $H_n \subset \mathbb{H}^1$, and by the Sobolev imbedding Theorem 2.98, $\mathbb{H}^1 \subset \mathbb{L}^4$, we infer that $H_n \subset \mathbb{L}^4$. Hence $C([0,T];H_n) \subset L^4(0,T;\mathbb{L}^4)$. Moreover $D(A^{\frac{1}{2}}) = \mathbb{H}^1 \subset \mathbb{L}^2$, so that $H_n \subset \mathbb{L}^2 \subset X^{-\frac{1}{2}}$. Hence $C([0,T];H_n) \subset C([0,T];X^{-\frac{1}{2}})$. Therefore $C([0,T];H_n) \subset L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-\frac{1}{2}})$.
By the Kuratowski Theorem 2.142, the Borel subsets of $C([0,T];H_n)$ are Borel subsets of $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-\frac{1}{2}})$. This concludes the proof. □

Corollary 4.21. *u_n' takes values in H_n and the laws on $C([0,T];H_n)$ of u_n and u_n' are equal.*

Proof. Since u_n is the solution of the equation (4.4), we infer that $\mathbb{P}\{u_n \in C([0,T];H_n)\} = 1$. Hence by Propositions 4.20 and 4.18 (a), $\mathbb{P}'\{u_n' \in C([0,T];H_n)\} = 1$. So we can assume that u_n' takes values in H_n and the laws on $C([0,T];H_n)$ of u_n and u_n' are equal. □

Lemma 4.22. *The $\{u_n'\}$ defined in Proposition 4.18 satisfies the following estimates:*

$$\sup_{t\in[0,T]} \left|u_n'(t)\right|_H \le \left|u_0\right|_H, \qquad \mathbb{P}' - a.s., \tag{4.44}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\sup_{t\in[0,T]} \left|u_n'(t)\right|_{\mathbb{H}^1}^{2r}\right] < \infty, \qquad \forall r \ge 1, \tag{4.45}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}'\left[\left(\int_0^T |u_n'(t) \times [\Delta u_n'(t) - \phi'(u_n'(t))]|_{\mathbb{L}^2}^2\, dt\right)^r\right] < \infty, \qquad \forall r \ge 1, \tag{4.46}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}' \int_0^T \left|u_n'(t) \times (u_n'(t) \times [\Delta u_n'(t) - \phi'(u_n'(t))])\right|_{\mathbb{L}^{\frac{3}{2}}}^2 \, dt < \infty, \tag{4.47}$$

$$\sup_{n\in\mathbb{N}} \mathbb{E}' \int_0^T \left|\pi_n \left[u_n'(t) \times (u_n'(t) \times [\Delta u_n'(t) - \phi'(u_n'(t))])\right]\right|_{X^{-\beta}}^2 \, dt < \infty. \tag{4.48}$$

Proof of (4.44). By (4.13), $\left|u_n(t)\right|_H = \left|u_n(0)\right|_H$, $\mathbb{P}-a.s.$ and together with the Corollary 4.21, u_n and u_n' have the same distribution on $C([0,T];H_n)$, we have

$$\mathbb{P}'(|u_n'|_{C([0,T];H_n)} \le |u_0|_H) = \mathbb{P}(|u_n|_{C([0,T];H_n)} \le |u_0|_H) = 1.$$

Hence

$$\sup_{t\in[0,T]} \left|u_n'(t)\right|_H \le \left|u_0\right|_H, \qquad \mathbb{P}' - a.s..$$

□

Proof of (4.45)*,* (4.46)*,* (4.47) *and* (4.48). The maps:

$$u \in C([0,T];H_n) \longrightarrow L^\infty(0,T;V) \ni u,$$

$$u \in C([0,T];H_n) \longrightarrow L^2(0,T;H) \ni u \times (\Delta u + \phi'(u)),$$

$$u \in C([0,T];H_n) \longrightarrow L^2(0,T;\mathbb{L}^{\frac{3}{2}}) \ni u \times \{u \times (\Delta u + \phi'(u))\},$$

$$u \in C([0,T];H_n) \longrightarrow L^2(0,T;X^{-\beta}) \ni \pi_n(u \times \{u \times (\Delta u + \phi'(u))\}),$$

are continuous so they are measurable. Hence by Corollary 4.21, we have

$$\mathcal{L}(u_n) = \mathcal{L}(u'_n) \text{ on } L^\infty(0,T;V),$$
$$\mathcal{L}(u_n \times (\Delta u_n + \phi'(u_n))) = \mathcal{L}(u'_n \times (\Delta u'_n + \phi'(u'_n))) \text{ on } L^2(0,T;H),$$
$$\mathcal{L}(u_n \times [u_n \times (\Delta u_n + \phi'(u_n))]) = \mathcal{L}(u'_n \times [u'_n \times (\Delta u'_n + \phi'(u'_n))]) \text{ on } L^2(0,T;\mathbb{L}^{\frac{3}{2}}),$$
$$\mathcal{L}(\pi_n(u_n \times [u_n \times (\Delta u_n + \phi'(u_n))])) = \mathcal{L}(\pi_n(u'_n \times [u'_n \times (\Delta u'_n + \phi'(u'_n))])) \text{ on } L^2(0,T;X^{-\beta}).$$

Therefore we get the estimates (4.45), (4.46), (4.47) and (4.48). □

Now we will study some inequalities satisfied by the limiting process u'.

Proposition 4.23. *Let u' be the process which defined in Proposition 4.18. Then we have*

$$\operatorname*{ess\,sup}_{t\in[0,T]} |u'(t)|_{\mathbb{L}^2} \le |u_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s. \tag{4.49}$$

$$\sup_{t\in[0,T]} |u'(t)|_{X^{-\beta}} \le c|u_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s. \tag{4.50}$$

Proof of (4.49). Since u'_n converges to u' in $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-\beta})$ $\mathbb{P}'$ almost surely,

$$\lim_{n\to\infty} \int_0^T |u'_n(t) - u'(t)|^4_{\mathbb{L}^4}\, dt = 0, \quad \mathbb{P}' - a.s.$$

Since $\mathbb{L}^4(D) \hookrightarrow \mathbb{L}^2(D)$, we infer that

$$\lim_{n\to\infty} \int_0^T |u'_n(t) - u'(t)|^2_{\mathbb{L}^2}\, dt = 0.$$

Hence u'_n converges to u' in $L^2(0,T;\mathbb{L}^2)$ $\mathbb{P}'$ almost surely. Therefore by (4.44),

$$\operatorname*{ess\,sup}_{t\in[0,T]} |u'(t)|_{\mathbb{L}^2} \le |u_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s.$$

□

Proof of (4.50). Since $\mathbb{L}^2(D) \hookrightarrow X^{-\beta}$, there exists some constant $c > 0$, such that $|u'_n(t)|_{X^{-\beta}} \le c|u'_n(t)|_{\mathbb{L}^2}$ for all $n \in \mathbb{N}$. By (4.44), we have

$$\sup_{t\in[0,T]} |u'_n(t)|_{X^{-\beta}} \le c \sup_{t\in[0,T]} |u'_n(t)|_{\mathbb{L}^2} \le |u_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s.$$

And by Proposition 4.18 (ii) u'_n converges to u' in $C([0,T];X^{-\beta})$, we infer that

$$\sup_{t\in[0,T]} |u'(t)|_{X^{-\beta}} \le c|u_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s.$$

□

We continue with investigating properties of the process u'. The next result and it's proof are related to the estimate (4.45).

Proposition 4.24. *Let u' be the process which defined in Proposition 4.18. Then we have*

$$\mathbb{E}'[\operatorname*{ess\,sup}_{t\in[0,T]} |u'(t)|^{2r}_V] < \infty, \quad r \ge 2. \tag{4.51}$$

Proof. Since $L^{2r}(\Omega';L^\infty(0,T;V))$ is isomorphic to $\left[L^{\frac{2r}{2r-1}}(\Omega';L^1(0,T;X^{-\frac12}))\right]^*$, by the Banach-Alaoglu Theorem (Lemma 2.143), we infer that the sequence $\{u_n'\}$ contains a subsequence, denoted in the same way as the full sequence, and there exists an element $v\in L^{2r}(\Omega';L^\infty(0,T;V))$ such that $u_n'\to v$ weakly* in $L^{2r}(\Omega';L^\infty(0,T;V))$. In particular, we have

$$\langle u_n',\varphi\rangle\to\langle v,\varphi\rangle,\qquad \varphi\in L^{\frac{2r}{2r-1}}(\Omega';(L^1(0,T;X^{-\frac12}))).$$

This means that

$$\int_{\Omega'}\int_0^T\langle u_n'(t,\omega),\varphi(t,\omega)\rangle\,dt\,d\mathbb{P}'(\omega)\to\int_{\Omega'}\int_0^T\langle v(t,\omega),\phi(t,\omega)\rangle\,dt\,d\mathbb{P}'(\omega).$$

On the other hand, if we fix $\varphi\in L^4(\Omega';L^{\frac43}(0,T;\mathbb{L}^{\frac43}))$, by the inequality (4.45) we have

$$\sup_n\int_{\Omega'}\left|\int_0^T{}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac43}}\,dt\right|^2 d\mathbb{P}'(\omega)\le\sup_n\int_{\Omega'}\left|\int_0^T|u_n'|_{\mathbb{L}^4}|\varphi|_{\mathbb{L}^{\frac43}}\,dt\right|^2 d\mathbb{P}'(\omega)$$
$$\le\sup_n\int_{\Omega'}|u_n'|^2_{L^\infty(0,T;\mathbb{L}^4)}|\varphi|^2_{L^1(0,T;\mathbb{L}^{\frac43})}\,d\mathbb{P}'(\omega)\le\sup_n|u_n'|^2_{L^4(\Omega';L^\infty(0,T;\mathbb{L}^4))}|\varphi|^2_{L^4(\Omega';L^1(0,T;\mathbb{L}^{\frac43}))}<\infty.$$

So by Lemma 2.104 the sequence $\int_0^T{}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac43}}\,dt$ is uniformly integrable on Ω'. Moreover, by the $\mathbb{P}'$ almost surely convergence of u_n' to u' in $L^4(0,T;\mathbb{L}^4)$, we get $\mathbb{P}'$-a.s.

$$\left|\int_0^T{}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac43}}\,dt-\int_0^T{}_{\mathbb{L}^4}\langle u'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac43}}\,dt\right|$$
$$\le\int_0^T\left|{}_{\mathbb{L}^4}\langle u_n'(t)-u'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac43}}\right|dt\le\int_0^T|u_n'(t)-u'(t)|_{\mathbb{L}^4}|\varphi(t)|_{\mathbb{L}^{\frac43}}\,dt$$
$$\le|u_n'(t)-u'(t)|_{L^4(0,T;\mathbb{L}^4)}|\varphi|_{L^{\frac43}(0,T;\mathbb{L}^{\frac43})}\to0.$$

Therefore we infer that $\int_0^T{}_{\mathbb{L}^4}\langle u_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac43}}\,dt$ converges to $\int_0^T{}_{\mathbb{L}^4}\langle u'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac43}}\,dt$ $\mathbb{P}'$ almost surely. Thus by Lemma 2.129,

$$\int_{\Omega'}\int_0^T{}_{\mathbb{L}^4}\langle u_n'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac43}}\,dt\,d\mathbb{P}'(\omega)\to\int_{\Omega'}\int_0^T{}_{\mathbb{L}^4}\langle u'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac43}}\,dt\,d\mathbb{P}'(\omega).$$

Hence we deduce that

$$\int_{\Omega'}\int_0^T{}_{\mathbb{L}^4}\langle v(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac43}}\,dt\,d\mathbb{P}'(\omega)=\int_{\Omega'}\int_0^T{}_{\mathbb{L}^4}\langle u'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac43}}\,dt\,d\mathbb{P}'(\omega)$$

By the arbitrariness of φ and density of $L^4(\Omega';L^{\frac43}(0,T;\mathbb{L}^{\frac43}))$ in $L^{\frac{2r}{2r-1}}(\Omega';L^1(0,T;X^{-\frac12}))$, we infer that $u'=v$ and since v satisfies (4.51) we infer that u' also satisfies (4.51). In this way the proof (4.51) is complete. □

Now we will strength part (ii) of Proposition 4.18 about the convergence of u_n' to u'.

Proposition 4.25.

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T |u_n'(t) - u'(t)|_{\mathbb{L}^4}^4 \, \mathrm{d}t = 0. \tag{4.52}$$

Proof. Since $u_n' \longrightarrow u'$ in $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-\beta})$ $\mathbb{P}'$-almost surely, $u_n' \longrightarrow u'$ in $L^4(0,T;\mathbb{L}^4)$ $\mathbb{P}'$-almost surely, i.e.

$$\lim_{n\to\infty} \int_0^T |u_n'(t) - u'(t)|_{\mathbb{L}^4}^4 \, \mathrm{d}t = 0, \qquad \mathbb{P}' - a.s.,$$

and by (4.45) and (4.51),

$$\sup_n \mathbb{E}' \left(\int_0^T |u_n'(t) - u'(t)|_{\mathbb{L}^4}^4 \, \mathrm{d}t \right)^2 \le 2^7 \sup_n \left(|u_n'|_{L^4(0,T;\mathbb{L}^4(D))}^8 + |u'|_{L^4(0,T;\mathbb{L}^4(D))}^8 \right) < \infty,$$

by Theorem 2.129 we infer that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T |u_n'(t) - u'(t)|_{\mathbb{L}^4}^4 \, \mathrm{d}t = 0.$$

This completes the proof. □

By (4.45), $\{u_n'\}_{n=1}^\infty$ is bounded in $L^2(\Omega';L^2(0,T;\mathbb{H}^1))$. And since $u_n' \longrightarrow u'$ in $L^2(\Omega';L^2(0,T;\mathbb{L}^2))$, by Lemma 2.145,

$$\frac{\partial u_n'}{\partial x_i} \longrightarrow \frac{\partial u'}{\partial x_i} \text{ weakly in } L^2(\Omega';L^2(0,T;\mathbb{L}^2)),\ i = 1,2,3. \tag{4.53}$$

Lemma 4.26. *There exists a unique $\Lambda \in L^2(\Omega';L^2(0,T;H))$ such that for $v \in L^2(\Omega';L^2(0,T;\mathbb{W}^{1,4}(D)))$,*

$$\mathbb{E}' \int_0^T \langle \Lambda(t), v(t) \rangle_H \, \mathrm{d}t = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u'(t), u'(t) \times D_i v(t) \rangle_H \, \mathrm{d}t. \tag{4.54}$$

Proof. We will omit"(t)" in this proof. Let us denote $\Lambda_n := u_n' \times A u_n'$. By the estimate (4.46), there exists a constant C such that

$$\|\Lambda_n\|_{L^2(\Omega';L^2(0,T;H))} \le C, \quad n \in \mathbb{N}.$$

Hence by the Banach-Alaoglu Theorem, there exists $\Lambda \in L^2(\Omega';L^2(0,T;H))$ such that $\Lambda_n \to \Lambda$ weakly in $L^2(\Omega';L^2(0,T;H))$.
Let us fix $v \in L^2(\Omega';L^2(0,T;\mathbb{W}^{1,4}(D)))$. Since $u_n'(t) \in D(A)$ for almost every $t \in [0,T]$ and $\mathbb{P}'$-almost surely, by the Proposition 2.167 and estimate (4.46) again, we have

$$\mathbb{E}' \int_0^T \langle \Lambda_n, v \rangle_H \, \mathrm{d}t = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u_n', u_n' \times D_i v \rangle_H \, \mathrm{d}t.$$

Moreover, by the results: (5.80), (4.45) and (4.52), we have for $i = 1, 2, 3$,

$$\begin{aligned}
&\left|\mathbb{E}' \int_0^T \langle D_i u', u' \times D_i v\rangle_H \, dt - \mathbb{E}' \int_0^T \langle D_i u'_n, u'_n \times D_i v\rangle_H \, dt\right| \\
&\leq \left|\mathbb{E}' \int_0^T \langle D_i u' - D_i u'_n, u' \times D_i v\rangle_H \, dt\right| + \left|\mathbb{E}' \int_0^T \langle D_i u'_n, (u' - u'_n) \times D_i v\rangle_H \, dt\right| \\
&\leq \left|\mathbb{E}' \int_0^T \langle D_i u' - D_i u'_n, u' \times D_i v\rangle_H \, dt\right| \\
&\quad + \left(\mathbb{E}' \int_0^T \|D_i u'_n\|_H^2 \, dt\right)^{\frac{1}{2}} \left(\mathbb{E}' \int_0^T \|u' - u'_n\|^4_{\mathbb{L}^4(D)} \, dt\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \|D_i v\|^4_{\mathbb{L}^4(D)} \, dt\right)^{\frac{1}{4}} \to 0.
\end{aligned}$$

Therefore we infer that

$$\lim_{n\to\infty} \mathbb{E} \int_0^T \langle \Lambda_n, v\rangle_H \, dt = \sum_{i=1}^3 \mathbb{E} \int_0^T \langle D_i u', u' \times D_i v\rangle \, dt.$$

Since on the other hand we have proved $\Lambda_n \to \Lambda$ weakly in $L^2(\Omega'; L^2(0, T; H))$ the equality (5.81) follows.
It remains to prove the uniqueness of Λ, but this, because $L^2(\Omega'; L^2(0, T; \mathbb{W}^{1,4}(D)))$ is dense in $L^2(\Omega'; L^2(0, T; H))$, follows from (5.81). This complete the proof of Lemma 5.53. □

Notation 4.27. We will use $u' \times \Delta u'$ to denote Λ in Lemma 5.53 which is an element of $L^2(\Omega'; L^2(0, T; H))$ such that the following identity is satisfied: for all v in a class of test functions includes $L^2(\Omega'; L^2(0, T; \mathbb{W}^{1,4}(D)))$:

$$\mathbb{E}' \int_0^T \langle (u' \times \Delta u')(t), v(t)\rangle_H \, dt = \sum_{i=1}^3 \mathbb{E}' \int_0^T \langle D_i u'(t), u'(t) \times D_i v(t)\rangle_H \, dt.$$

Next we will show that the limits of $\{u'_n \times [\Delta u'_n - \pi_n \phi'(u'_n)]\}_n$, $\{u'_n \times (u'_n \times [\Delta u'_n - \pi_n \phi'(u'_n)])\}_n$ and $\{\pi_n(u'_n \times (u'_n \times [\Delta u'_n - \pi_n \phi'(u'_n)]))\}_n$ are equal to $u' \times [\Delta u' - \phi'(u')]$, $u' \times (u' \times [\Delta u' - \phi'(u')])$ and $u' \times (u' \times [\Delta u' - \phi'(u')])$ respectively.

By (4.46)-(4.48), the sequence $\{u'_n \times [\Delta u'_n - \phi'(u'_n(t))]\}_n$ is bounded in $L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2))$ for $r \geq 1$, $\{u'_n \times (u'_n \times [\Delta u'_n - \phi'(u'_n(t))])\}_n$ is bounded in $L^2(\Omega'; L^2(0, T; \mathbb{L}^{\frac{3}{2}}))$ and $\{\pi_n(u'_n \times (u'_n \times [\Delta u'_n - \phi'(u'_n(t))]))\}_n$ is bounded in $L^2(\Omega'; L^2(0, T; X^{-\beta}))$. And since $L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2))$, $L^2(\Omega'; L^2(0, T; \mathbb{L}^{\frac{3}{2}}))$ and $L^2(\Omega'; L^2(0, T; X^{-\beta}))$ are all reflexive, by the Banach-Alaoglu theorem 2.143, there exist subsequences weakly convergent. So we can assume that there exist $Y \in L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2))$, $Z \in L^2(\Omega'; L^2(0, T; \mathbb{L}^{\frac{3}{2}}))$ and $Z_1 \in L^2(\Omega'; L^2(0, T; X^{-\beta}))$, such that

$$u'_n \times [\Delta u'_n - \phi'(u'_n)] \longrightarrow Y \quad \text{weakly in } L^{2r}(\Omega'; L^2(0, T; \mathbb{L}^2)), \tag{4.55}$$

$$u'_n \times (u'_n \times [\Delta u'_n - \phi'(u'_n)]) \longrightarrow Z \quad \text{weakly in } L^2(\Omega'; L^2(0, T; \mathbb{L}^{\frac{3}{2}})), \tag{4.56}$$

$$\pi_n \left(u'_n \times (u'_n \times [\Delta u'_n - \phi'(u'_n)])\right) \longrightarrow Z_1 \quad \text{weakly in } L^2(\Omega'; L^2(0, T; X^{-\beta})). \tag{4.57}$$

Remark. Similar argument (but with less details) has been done in [13] for terms not involving ϕ'. Our main contribution here is to show the validity of such an argument for term containing ϕ' (and to be more precise). This works because earlier have been able to prove generalized estimates as in [13] as in Lemma 4.22.

Proposition 4.28. *If Z and Z_1 defined as above, then $Z = Z_1 \in L^2(\Omega'; L^2(0,T; X^{-\beta}))$.*

Proof. Notice that $(L^{\frac{3}{2}})^* = L^3$, and by Proposition 2.172, $X^\beta = \mathbb{H}^{2\beta}$. By Theorem 2.98, $X^\beta \subset L^3$ for $\beta > \frac{1}{4}$, hence $L^{\frac{3}{2}} \subset X^{-\beta}$, so

$$L^2(\Omega'; L^2(0,T; \mathbb{L}^{\frac{3}{2}})) \subset L^2(\Omega'; L^2(0,T; X^{-\beta})).$$

Therefore $Z \in L^2(\Omega'; L^2(0,T; X^{-\beta}))$ and also $Z_1 \in L^2(\Omega'; L^2(0,T; X^{-\beta}))$.
Since $X^\beta = D(A_1^\beta)$ and A_1 is self-adjoint, we can define

$$X_n^\beta = \left\{ \pi_n x = \sum_{j=1}^n x_j e_j : \sum_{j=1}^\infty \lambda_j^{2\beta} x_j^2 < \infty \right\}.$$

Then $X^\beta = \bigcup_{n=1}^\infty X_n^\beta$, $L^2(\Omega'; L^2(0,T; X^\beta)) = \bigcup_{n=1}^\infty L^2(\Omega'; L^2(0,T; X_n^\beta))$. We have for $\psi_n \in L^2(\Omega'; L^2(0,T; X_n^\beta))$,

$$\begin{aligned}
& {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle \pi_n(u_n' \times (u_n' \times [\Delta u_n' - \phi'(u_n'(t))])), \psi_n \rangle_{L^2(\Omega';L^2(0,T;X^\beta))} \\
&= \mathbb{E}' \int_0^T {}_{X^{-\beta}}\langle \pi_n(u_n'(t) \times (u_n'(t) \times [\Delta u_n' - \phi'(u_n'(t))])), \psi_n(t) \rangle_{X^\beta} \, dt \\
&= \mathbb{E}' \int_0^T {}_{H}\langle \pi_n(u_n'(t) \times (u_n'(t) \times [\Delta u_n' - \phi'(u_n'(t))])), \psi_n(t) \rangle_{H} \, dt \\
&= \mathbb{E}' \int_0^T {}_{H}\langle u_n'(t) \times (u_n'(t) \times [\Delta u_n' - \phi'(u_n'(t))]), \psi_n(t) \rangle_{H} \, dt \\
&= \mathbb{E}' \int_0^T {}_{X^{-\beta}}\langle u_n'(t) \times (u_n'(t) \times [\Delta u_n' - \phi'(u_n'(t))]), \psi_n(t) \rangle_{X^\beta} \, dt \\
&= {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle u_n' \times (u_n' \times [\Delta u_n' - \phi'(u_n'(t))]), \psi_n \rangle_{L^2(\Omega';L^2(0,T;X^\beta))}.
\end{aligned}$$

Hence

$${}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z_1, \psi_n \rangle_{L^2(\Omega';L^2(0,T;X^\beta))} = {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z, \psi_n \rangle_{L^2(\Omega';L^2(0,T;X^\beta))},$$

$\forall \psi_n \in L^2(\Omega'; L^2(0,T; X_n^\beta))$. For any $\psi \in L^2(\Omega'; L^2(0,T; X^\beta)$, there exists $L^2(\Omega'; L^2(0,T; X_n^\beta)) \ni \psi_n \longrightarrow \psi$ as $n \longrightarrow \infty$, hence for all $\psi \in L^2(\Omega'; L^2(0,T; X^\beta)$,

$$\begin{aligned}
{}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z_1, \psi \rangle_{L^2(\Omega';L^2(0,T;X^\beta))} &= \lim_{n\to\infty} {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z_1, \psi_n \rangle_{L^2(\Omega';L^2(0,T;X^\beta))} \\
&= \lim_{n\to\infty} {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z, \psi_n \rangle_{L^2(\Omega';L^2(0,T;X^\beta))} \\
&= {}_{L^2(\Omega';L^2(0,T;X^{-\beta}))}\langle Z, \psi \rangle_{L^2(\Omega';L^2(0,T;X^\beta))}
\end{aligned}$$

Therefore $Z = Z_1 \in L^2(\Omega'; L^2(0,T; X^{-\beta}))$ and this concludes the proof of Proposition 4.28. □

Lemma 4.29. *For any measurable process $\psi \in L^4(\Omega'; L^4(0,T;\mathbb{W}^{1,4}))$, we have the equality*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u_n'(s) \times [\Delta u_n'(s) - \phi'(u_n'(t))], \psi(s)\rangle_H \, ds \\
&= \mathbb{E}' \int_0^T \langle Y(s), \psi(s)\rangle_H \, ds \\
&= \mathbb{E}' \int_0^T \sum_{i=1}^3 \langle \frac{\partial u'(s)}{\partial x_i}, u'(s) \times \frac{\partial \psi(s)}{\partial x_i}\rangle_H \, ds - \mathbb{E}' \int_0^T \langle u'(t) \times \phi'(u'(t)), \psi\rangle_H \, dt.
\end{aligned}$$

Proof. Let us fix $\psi \in L^4(\Omega'; L^4(0,T;\mathbb{W}^{1,4}))$. Firstly, we will prove that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle u_n'(t) \times \Delta u_n'(t), \psi(t)\rangle_{\mathbb{L}^2} \, dt = \mathbb{E}' \int_0^T \sum_{i=1}^3 \left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t) \times \frac{\partial \psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2} \, dt.$$

For each $n \in \mathbb{N}$ we have

$$\langle u_n'(t) \times \Delta u_n'(t), \psi\rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial u_n'(t)}{\partial x_i}, u_n'(t) \times \frac{\partial \psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2} \tag{4.58}$$

for almost every $t \in [0,T]$ and $\mathbb{P}'$ almost surely. By Corollary 4.21, $\mathbb{P}(u_n' \in C([0,T];H_n)) = 1$. For each $i \in \{1,2,3\}$ we may write

$$\begin{aligned}
&\left\langle \frac{\partial u_n'}{\partial x_i}, u_n' \times \frac{\partial \psi}{\partial x_i}\right\rangle_{\mathbb{L}^2} - \left\langle \frac{\partial u'}{\partial x_i}, u' \times \frac{\partial \psi}{\partial x_i}\right\rangle_{\mathbb{L}^2} \\
&= \left\langle \frac{\partial u_n'}{\partial x_i} - \frac{\partial u'}{\partial x_i}, u' \times \frac{\partial \psi}{\partial x_i}\right\rangle_{\mathbb{L}^2} + \left\langle \frac{\partial u_n'}{\partial x_i}, (u_n' - u') \times \frac{\partial \psi}{\partial x_i}\right\rangle_{\mathbb{L}^2}
\end{aligned} \tag{4.59}$$

Since $\mathbb{L}^4 \hookrightarrow \mathbb{L}^2$ and $\mathbb{W}^{1,4} \hookrightarrow \mathbb{L}^2$, so there are constants C_1 and $C_2 < \infty$ such that

$$\begin{aligned}
&\left\langle \frac{\partial u_n'(t)}{\partial x_i}, (u_n'(t) - u'(t)) \times \frac{\partial \psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2} \le \left\|\frac{\partial u_n'(t)}{\partial x_i}\right\|_{\mathbb{L}^2} \left\|(u_n'(t) - u'(t)) \times \frac{\partial \psi(t)}{\partial x_i}\right\|_{\mathbb{L}^2} \\
&\le \left\|u_n'(t)\right\|_{\mathbb{H}^1} C_1 \|u_n'(t) - u'(t)\|_{\mathbb{L}^4} C_2 \|\psi(t)\|_{\mathbb{W}^{1,4}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}' \int_0^T \left|\left\langle \frac{\partial u_n'(t)}{\partial x_i}, (u_n'(t) - u'(t)) \times \frac{\partial \psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\right| \, dt \\
&\le C_1 C_2 \mathbb{E}' \int_0^T \|u_n'(t)\|_{\mathbb{H}^1} \|u_n'(t) - u'(t)\|_{\mathbb{L}^4} \|\psi(t)\|_{\mathbb{W}^{1,4}} \, dt.
\end{aligned}$$

Moreover by the Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}'\int_0^T \|u_n'(t)\|_{\mathbb{H}^1}\|u_n'(t)-u'(t)\|_{\mathbb{L}^4}\|\psi(t)\|_{\mathbb{W}^{1,4}}\,\mathrm{d}t\\
&\le \left(\mathbb{E}'\int_0^T \|u_n'(t)\|^2_{\mathbb{H}^1}\,\mathrm{d}t\right)^{\frac12}\left(\mathbb{E}'\int_0^T \|u_n'(t)-u'(t)\|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\psi(t)\|^4_{\mathbb{W}^{1,4}}\,\mathrm{d}t\right)^{\frac14}\\
&\le T^{\frac12}\left(\mathbb{E}'\sup_{t\in[0,T]}\|u_n'(t)\|^2_{\mathbb{H}^1}\right)^{\frac12}\left(\mathbb{E}'\int_0^T \|u_n'(t)-u'(t)\|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\phi(t)\|^4_{\mathbb{W}^{1,4}}\,\mathrm{d}t\right)^{\frac14}
\end{aligned}$$

By (4.45), (4.52) and since $\psi\in L^4(\Omega';L^4(0,T;\mathbb{W}^{1,4}))$, we have

$$\lim_{n\to\infty}\left(\mathbb{E}'\sup_{t\in[0,T]}\|u_n'(t)\|^2_{\mathbb{H}^1}\right)^{\frac12}\left(\mathbb{E}'\int_0^T \|u_n'(t)-u'(t)\|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\psi(t)\|^4_{\mathbb{W}^{1,4}}\,\mathrm{d}t\right)^{\frac14}=0$$

Hence

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\left|\left\langle\frac{\partial u_n'(t)}{\partial x_i},(u_n'(t)-u'(t))\times\frac{\partial\psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\right|\,\mathrm{d}t=0 \tag{4.60}$$

Both u' and $\frac{\partial\psi}{\partial x_i}$ are in $L^2(\Omega';L^2(0,T;\mathbb{L}^2))$, so $u'\times\frac{\partial\psi}{\partial x_i}\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$. Hence by (5.80), we have

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\left\langle\frac{\partial u_n'(t)}{\partial x_i}-\frac{\partial u'(t)}{\partial x_i},u'(t)\times\frac{\partial\psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\,\mathrm{d}t=0. \tag{4.61}$$

Therefore by (4.59), (4.60), (4.61),

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\left\langle\frac{\partial u_n'(t)}{\partial x_i},u_n'(t)\times\frac{\partial\psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\,\mathrm{d}t=\mathbb{E}'\int_0^T\left\langle\frac{\partial u'(t)}{\partial x_i},u'(t)\times\frac{\partial\psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\,\mathrm{d}t \tag{4.62}$$

Then by (4.58), we have

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle u_n'(t)\times\Delta u_n'(t),\psi(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t=\mathbb{E}'\int_0^T\sum_{i=1}^3\left\langle\frac{\partial u'(t)}{\partial x_i},u'(t)\times\frac{\partial\psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\,\mathrm{d}t \tag{4.63}$$

Secondly, we will show that

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle u_n'(t)\times\pi_n\phi'(u_n'(t)),\psi\rangle_H\,\mathrm{d}t=\mathbb{E}'\int_0^T\langle u'(t)\times\phi'(u'(t)),\psi\rangle_H\,\mathrm{d}t.$$

Since

$$\begin{aligned}
&\left|\langle u_n'(t)\times\pi_n\phi'(u_n'(t)),\psi\rangle_H-\langle u'(t)\times\phi'(u'(t)),\psi\rangle_H\right|\\
&\le\left|\langle[u_n'(t)-u'(t)]\times\pi_n\phi'(u_n'(t)),\psi\rangle_H\right|+\left|\langle u'(t)\times[\pi_n\phi'(u_n'(t))-\phi'(u'(t))],\psi\rangle_H\right|\\
&\le\|\psi\|_H\|u_n'(t)-u'(t)\|_H\|\phi'(u_n'(t))\|_H+\|\psi\|_H\|u'(t)\|_H\|\pi_n\phi'(u_n'(t))-\phi'(u'(t))\|_H,
\end{aligned}$$

we have

$$\begin{aligned}
&\left|\mathbb{E}'\int_0^T \langle u_n'(t)\times\pi_n\phi'(u_n'(t)),\psi\rangle_H \,\mathrm{d}t - \mathbb{E}'\int_0^T \langle u'(t)\times\phi'(u'(t)),\psi\rangle_H \,\mathrm{d}t\right| \\
&\le \mathbb{E}'\int_0^T \Big(\|\psi\|_H\|u_n'(t)-u'(t)\|_H\|\phi'(u_n'(t))\|_H + \|\psi\|_H\|u'(t)\|_H\|\pi_n\phi'(u_n'(t))-\phi'(u'(t))\|_H\Big)\,\mathrm{d}t \\
&\le \left(\mathbb{E}'\int_0^T \|\psi\|^4_{\mathbb{L}^{1,4}}\,\mathrm{d}t\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|u_n'(t)-u'(t)\|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\phi'(u_n'(t))\|^2_{\mathbb{L}^2}\,\mathrm{d}t\right)^{\frac12} \\
&\quad + \left(\mathbb{E}'\int_0^T \|\psi\|^4_{\mathbb{W}^{1,4}}\,\mathrm{d}t\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|u'(t)\|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\pi_n\phi'(u_n'(t))-\phi'(u'(t))\|^2_H\,\mathrm{d}t\right)^{\frac14} \to 0.
\end{aligned}$$

We need to prove why

$$\mathbb{E}'\int_0^T \|\pi_n\phi'(u_n'(t))-\phi'(u'(t))\|^2_H\,\mathrm{d}t \to 0 \tag{4.64}$$

This is because

$$\begin{aligned}
&\left(\mathbb{E}'\int_0^T \|\pi_n\phi'(u_n'(t))-\phi'(u'(t))\|^2_H\,\mathrm{d}t\right)^{\frac12} \\
&\le \left(\mathbb{E}'\int_0^T \|\pi_n\phi'(u_n'(t))-\pi_n\phi'(u'(t))\|^2_H\,\mathrm{d}t\right)^{\frac12} + \left(\mathbb{E}'\int_0^T \|\pi_n\phi'(u'(t))-\phi'(u'(t))\|^2_H\,\mathrm{d}t\right)^{\frac12} \le \cdots
\end{aligned}$$

Since ϕ' is global Lipschitz, there exists a constant C such that

$$\cdots \le C\left(\mathbb{E}'\int_0^T \|u_n'(t)-u'(t)\|^2_H\,\mathrm{d}t\right)^{\frac12} + \left(\mathbb{E}'\int_0^T \|\pi_n\phi'(u'(t))-\phi'(u'(t))\|^2_H\,\mathrm{d}t\right)^{\frac12}.$$

By (4.52), the first term on the right hand side of above inequality converges to 0. And since $\|\pi_n\phi'(u'(t))-\phi'(u'(t))\|^2_H \to 0$ for almost every $(t,\omega)\in[0,T]\times\Omega$, and since ϕ' is bounded, $\|\pi_n\phi'(u'(t))-\phi'(u'(t))\|^2_H$ is uniformly integrable, by Lemma 2.129 the second term of right hand side also converges to 0 as $n\to\infty$. Therefore we have proved (4.64).

Hence we have

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T \langle u_n'(t)\times\pi_n\phi'(u_n'(t)),\psi\rangle_H\,\mathrm{d}t = \mathbb{E}'\int_0^T \langle u'(t)\times\phi'(u'(t)),\psi\rangle_H\,\mathrm{d}t. \tag{4.65}$$

Therefore by the equalities (4.63) and (4.65), we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}'\int_0^T \langle u_n'(t)\times[\Delta u_n'(t)-\pi_n\phi'(u_n'(t))],\psi(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t \\
&= \mathbb{E}'\int_0^T \sum_{i=1}^3\left\langle \frac{\partial u'(t)}{\partial x_i}, u'(t)\times\frac{\partial\psi(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\mathrm{d}t + \mathbb{E}'\int_0^T \langle u'(t)\times\phi'(u'(t)),\psi\rangle_H\,\mathrm{d}t.
\end{aligned}\tag{4.66}$$

Moreover, by (4.55),

(4.67)
$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle u_n'(t)\times[\Delta u_n'(t)-\pi_n\phi'(u_n'(t))],\psi\rangle_{\mathbb{L}^2}\,\mathrm{d}t=\mathbb{E}'\int_0^T\langle Y(t),\psi\rangle_{\mathbb{L}^2}\,\mathrm{d}t,\quad \psi\in L^2(\Omega';L^2(0,T;\mathbb{L}^2)).$$

Hence by (4.66) and (4.67),

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle u_n'(t)\times[\Delta u_n'(t)-\pi_n\phi'(u_n'(t))],\psi(s)\rangle_H\,\mathrm{d}t\\&=\mathbb{E}'\int_0^T\langle Y(t),\psi(t)\rangle_H\,\mathrm{d}t\\&=\mathbb{E}'\int_0^T\sum_{i=1}^3\langle\frac{\partial u'(t)}{\partial x_i},u'(t)\times\frac{\partial\psi(t)}{\partial x_i}\rangle_H\,\mathrm{d}t+\mathbb{E}'\int_0^T\big\langle u'(t)\times\phi'(u'(t)),\psi(t)\big\rangle_H\,\mathrm{d}t.\end{aligned}$$

This completes the proof of Lemma 4.29. □

Lemma 4.30. *For any process $\psi\in L^4(\Omega';L^4(0,T;\mathbb{L}^4))$ we have*

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T{}_{\mathbb{L}^{\frac{3}{2}}}\langle u_n'(s)\times(u_n'(s)\times[\Delta u_n'-\phi'(u_n'(t))]),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s$$

$$=\mathbb{E}'\int_0^T{}_{\mathbb{L}^{\frac{3}{2}}}\langle Z(s),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s \tag{4.68}$$

$$=\mathbb{E}'\int_0^T{}_{\mathbb{L}^{\frac{3}{2}}}\langle u'(s)\times Y(s),\psi(s)\rangle_{\mathbb{L}^3}\,\mathrm{d}s. \tag{4.69}$$

Proof. Let us take $\psi\in L^4(\Omega';L^4(0,T;\mathbb{L}^4))$. For $n\in\mathbb{N}$, put $Y_n:=u_n'\times[\Delta u_n'+\phi'(u_n')]$. $L^4(\Omega';L^4(0,T;\mathbb{L}^4))\subset L^2(\Omega';L^2(0,T;\mathbb{L}^3))=\left[L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}}))\right]'$. Hence (4.56) implies that (4.68) holds.
So it remains to prove equality (4.69). Since by the Hölder's inequality

$$\|\psi\times u'\|_{\mathbb{L}^2}^2=\int_D|\psi(x)\times u'(x)|^2\,\mathrm{d}x\le\int_D|\psi(x)|^2|u'(x)|^2\,\mathrm{d}x\le\|\psi\|_{\mathbb{L}^4}^2\|u'\|_{\mathbb{L}^4}^2\le\|\psi\|_{\mathbb{L}^4}^4+\|u'\|_{\mathbb{L}^4}^4.$$

And since by (4.52), $u'\in L^4(\Omega';L^4(0,T;\mathbb{L}^4))$, we infer that

$$\mathbb{E}'\int_0^T\|\psi\times u'\|_{\mathbb{L}^2}^2\,\mathrm{d}t\le\mathbb{E}'\int_0^T\|\psi\|_{\mathbb{L}^4}^4\,\mathrm{d}t+\mathbb{E}'\int_0^T\|u'\|_{\mathbb{L}^4}^4\,\mathrm{d}t<\infty.$$

This proves that $\psi\times u'\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$ and similarly $\psi\times u_n'\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$. Thus since by (4.55), $Y_n\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$, we infer that

$$\begin{aligned}{}_{\mathbb{L}^{\frac{3}{2}}}\langle u_n'\times Y_n,\psi\rangle_{\mathbb{L}^3}&=\int_D\langle u_n'(x)\times Y_n(x),\psi(x)\rangle\,\mathrm{d}x\\&=\int_D\langle Y_n(x),\psi(x)\times u_n'(x)\rangle\,\mathrm{d}x=\langle Y_n,\psi\times u_n'\rangle_{\mathbb{L}^2}.\end{aligned} \tag{4.70}$$

Similarly, since by (4.55), $Y \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$, we have

$$
\begin{aligned}
{}_{\mathbb{L}^{\frac{3}{2}}}\langle u' \times Y, \psi\rangle_{\mathbb{L}^3} &= \int_D \langle u'(x) \times Y(x), \psi(x)\rangle \, \mathrm{d}x \\
&= \int_D \langle Y(x), \psi(x) \times u'(x)\rangle \, \mathrm{d}x = \langle Y, \psi \times u'\rangle_{\mathbb{L}^2}. \qquad (4.71)
\end{aligned}
$$

Thus by (4.70) and (4.71), we get

$$
\begin{aligned}
{}_{\mathbb{L}^{\frac{3}{2}}}\langle u'_n \times Y_n, \psi\rangle_{\mathbb{L}^3} - {}_{\mathbb{L}^{\frac{3}{2}}}\langle u' \times Y, \psi\rangle_{\mathbb{L}^3} &= \langle Y_n, \psi \times u'_n\rangle_{\mathbb{L}^2} - \langle Y, \psi \times u'\rangle_{\mathbb{L}^2} \\
&= \langle Y_n - Y, \psi \times u'\rangle_{\mathbb{L}^2} + \langle Y_n, \psi \times (u'_n - u')\rangle_{\mathbb{L}^2}.
\end{aligned}
$$

In order to prove (4.69), we are aiming to prove that the expectation of the left hand side of the above equality goes to 0 as $n \to \infty$. By (4.55), since $\psi \times u' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$,

$$
\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle Y_n(s) - Y(s), \psi(s) \times u'(s)\rangle_{\mathbb{L}^2} \, \mathrm{d}s = 0.
$$

By the Cauchy-Schwartz inequality and the equation (4.52), we have

$$
\begin{aligned}
&\mathbb{E}'\left(\langle Y_n, \psi \times (u'_n - u')\rangle^2_{\mathbb{L}^2}\right) \le \mathbb{E}'\left(\|Y_n\|^2_{\mathbb{L}^2}\|\psi \times (u'_n - u')\|^2_{\mathbb{L}^2}\right) \le \mathbb{E}'\left(\|Y_n\|^2_{\mathbb{L}^2}(\|\psi\|^2_{\mathbb{L}^4} \cdot \|u'_n - u'\|^2_{\mathbb{L}^4})\right) \\
&\le \mathbb{E}' \int_0^T \left\|Y(s)\right\|_{\mathbb{L}^2}\left\|\psi(s)\right\|_{\mathbb{L}^4}\left\|u'_n(s) - u'(s)\right\|_{\mathbb{L}^4} \mathrm{d}s \\
&\le \left(\mathbb{E}' \int_0^T \left\|Y_n(s)\right\|^2_{\mathbb{L}^2} \mathrm{d}s\right)^{\frac{1}{2}} \left(\mathbb{E}' \int_0^T \left\|\psi(s)\right\|^4_{\mathbb{L}^4} \mathrm{d}s\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T \left\|u'_n(s) - u'(s)\right\|^4_{\mathbb{L}^4} \mathrm{d}s\right)^{\frac{1}{4}} \to 0.
\end{aligned}
$$

Therefore, we infer that

$$
\lim_{n\to\infty} \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle u'_n(s) \times (u'_n(s) \times \Delta u'_n(s)), \psi(s)\rangle_{\mathbb{L}^3} \, \mathrm{d}s = \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle u'(s) \times Y(s), \psi(s)\rangle_{\mathbb{L}^3} \, \mathrm{d}s.
$$

This completes the proof of the Lemma 4.30. □

The next result will be used to show that the process u' satisfies the condition $|u'(t,x)|_{\mathbb{R}^3} = 1$ for all $t \in [0,T]$, $x \in D$ and $\mathbb{P}'$-almost surely.

Lemma 4.31. *For any bounded measurable function $\psi : D \longrightarrow \mathbb{R}$ we have*

$$
\langle Y(s,\omega), \psi u'(s,\omega)\rangle_H = 0,
$$

for almost every $(s,\omega) \in [0,T] \times \Omega'$.

Proof. Let $B \subset [0,T] \times \Omega'$ be a measurable set.

$$
\begin{aligned}
&\left|\mathbb{E}' \int_0^T 1_B(s) \left\langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))], \psi u'_n(s)\right\rangle_H \, \mathrm{d}s - \mathbb{E}' \int_0^T 1_B(s) \langle Y(s), \psi u'(s)\rangle_H \, \mathrm{d}s\right| \\
&= \left|\mathbb{E}' \int_0^T \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))] - Y(s), \psi u'_n(s)\rangle_H + \langle Y(s), \psi u'_n(s) - \psi u'(s)\rangle_H \, \mathrm{d}s\right| \\
&\le \left|\mathbb{E}' \int_0^T \langle u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))] - Y(s), \psi u'_n(s)\rangle_H \, \mathrm{d}s\right| + \left|\mathbb{E} \int_0^T \langle Y(s), \psi u'_n(s) - \psi u'(s)\rangle_H \, \mathrm{d}s\right|.
\end{aligned}
$$

$\psi u'(s) \in H$, by (4.55) and (4.46) and (ii) of Proposition 4.18, we have

$$\begin{aligned}
&\left|\mathbb{E}' \int_0^T \langle u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))] - Y(s), \psi u_n'(s)\rangle_H \, \mathrm{d}s\right| \\
\leq &\left|\mathbb{E}' \int_0^T \langle u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))] - Y(s), \psi u'(s)\rangle_H \, \mathrm{d}s\right| \\
&+ \left|\mathbb{E}' \int_0^T \langle u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))] - Y(s), \psi [u'(s) - u_n'(s)]\rangle_H \, \mathrm{d}s\right| \longrightarrow 0.
\end{aligned}$$

And since ψ is bounded and $\mathbb{L}^4 \hookrightarrow \mathbb{L}^2$, by (4.52), we have

$$\begin{aligned}
&\left|\mathbb{E} \int_0^T \langle Y(s), \psi u_n'(s) - \psi u'(s)\rangle_H \, \mathrm{d}s\right| \leq \mathbb{E}' \int_0^T \left|\langle Y(s), \psi u_n'(s) - \psi u'(s)\rangle_H\right| \mathrm{d}s \\
\leq &\left(\mathbb{E}' \int_0^T \left|Y(s)\right|_H^2 \mathrm{d}s\right)^{\frac{1}{2}} \left(\mathbb{E}' \int_0^T \left|\psi u_n'(s) - \psi u'(s)\right|_H^2 \mathrm{d}s\right)^{\frac{1}{2}} \\
\leq &\left(\mathbb{E}' \int_0^T \left|Y(s)\right|_H^2 \mathrm{d}s\right)^{\frac{1}{2}} C_1 \left(\mathbb{E}' \int_0^T \left|u_n'(s) - u'(s)\right|_{\mathbb{L}^4}^4 \mathrm{d}s\right)^{\frac{1}{4}} \left(\mathbb{E}' \int_0^T 1^4 \, \mathrm{d}s\right)^{\frac{1}{4}} \\
\leq &\; C\left(\mathbb{E}' \int_0^T \left|Y(s)\right|_H^2 \mathrm{d}s\right)^{\frac{1}{2}} \left(\mathbb{E}' \int_0^T \left|u_n'(s) - u'(s)\right|_{\mathbb{L}^4}^4 \mathrm{d}s\right)^{\frac{1}{4}} \\
\to &\; 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &= \lim_{n\to\infty} \mathbb{E}' \int_0^T 1_B(s) \langle u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))], \psi u_n'(s)\rangle_H \, \mathrm{d}s \\
&= \mathbb{E}' \int_0^T 1_B(s) \langle Y(s), \psi u'(s)\rangle_H \, \mathrm{d}s.
\end{aligned}$$

This concludes the proof of Lemma 4.31. □

4.6. **Conclusion of the proof of the existence of a weak solution.** Our aim in this subsection is to prove that the process u' from Proposition 4.18 is a weak solution of the equation (4.2).

We define a sequence of H-valued process $(M_n(t))_{t\in[0,T]}$ on the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\begin{aligned}
M_n(t) := &\; u_n(t) - u_n(0) - \lambda_1 \int_0^t \pi_n \left(u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))]\right) \mathrm{d}s \\
&+ \lambda_2 \int_0^t \pi_n \left(u_n(s) \times (u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))])\right) \mathrm{d}s \\
&- \frac{1}{2} \int_0^t \pi_n [(\pi_n(u_n(s) \times h)) \times h] \, \mathrm{d}s.
\end{aligned} \tag{4.72}$$

By (4.4), we have

$$\begin{aligned}u_n(t) \;=\; & u_n(0) + \lambda_1 \int_0^t \pi_n \left(u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))]\right) \,\mathrm{d}s \\ & -\lambda_2 \int_0^t \pi_n \left(u_n(s) \times (u_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u_n(s))])\right) \,\mathrm{d}s \\ & +\frac{1}{2} \int_0^t \pi_n(\pi_n(u_n(s) \times h) \times h) \,\mathrm{d}s + \int_0^t \pi_n(u_n(s) \times h) \,\mathrm{d}W(s).\end{aligned}$$

Hence we have

$$M_n(t) = \sum_{j=1}^{N} \int_0^t \pi_n(u_n(s) \times h_j) \,\mathrm{d}W_j(s), \quad t \in [0, T]. \tag{4.73}$$

It will be 2 steps to prove u' is a weak solution of the Equation (4.2):

Step 1 : we are going to find some $M'(t)$ defined similar as in (4.72), but with u' instead of u_n.

Step 2 : We will show the similar result as in (4.73) but with u' instead of u_n and W'_j instead of W_j.

4.6.1. *Step 1.* We also define a sequence of H-valued process $(M'_n(t))_{t\in[0,T]}$ on the new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ by a formula similar as (4.72)

$$\begin{aligned}M'_n(t) \;:=\; & u'_n(t) - u'_n(0) - \lambda_1 \int_0^t \pi_n \left(u'_n(s) \times [\Delta u_n(s) - \pi_n \phi'(u'_n(s))]\right) \,\mathrm{d}s \\ & +\lambda_2 \int_0^t \pi_n \left(u'_n(s) \times (u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))])\right) \,\mathrm{d}s \\ & -\frac{1}{2} \int_0^t \pi_n[(\pi_n(u'_n(s) \times h)) \times h] \,\mathrm{d}s.\end{aligned} \tag{4.74}$$

It will be natural to ask if $\{M'_n\}$ has limit and if yes, what is the limit. The next result answers this question.

Lemma 4.32. *For each $t \in [0, T]$ the sequence of random variables $M'_n(t)$ converges weakly in $L^2(\Omega'; X^{-\beta})$ to the limit*

$$\begin{aligned}M'(t) \;:=\; & u'(t) - u_0 - \lambda_1 \int_0^t \left(u'(s) \times [\Delta u'(s) - \phi'(u'(s))]\right) \,\mathrm{d}s \\ & +\lambda_2 \int_0^t \left(u'_n(s) \times (u'_n(s) \times [\Delta u'_n(s) - \pi_n \phi'(u'_n(s))])\right) \,\mathrm{d}s \\ & -\frac{1}{2} \int_0^t (u'(s) \times h) \times h \,\mathrm{d}s\end{aligned}$$

as n goes to infinity.

Proof. By Theorem 2.130, the dual space of $L^2(\Omega'; X^{-\beta})$ is $L^2(\Omega'; X^{\beta})$. Let $t \in (0, T]$ and $U \in L^2(\Omega'; X^{\beta})$. We have

$$\begin{aligned}
&{}_{L^2(\Omega';X^{-\beta})}\langle M_n'(t), U\rangle_{L^2(\Omega';X^{\beta})} = \mathbb{E}'\left[{}_{X^{-\beta}}\langle M_n'(t), U\rangle_{X^{\beta}}\right]\\
= {}& \mathbb{E}'\Big[{}_{X^{-\beta}}\langle u_n'(t), U\rangle_{X^{\beta}} - {}_{X^{-\beta}}\langle u_n(0), U\rangle_{X^{\beta}}\\
&-\lambda_1 \int_0^t \langle u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))], \pi_n U\rangle_{\mathbb{L}^2}\, \mathrm{d}s\\
&+\lambda_2 \int_0^t {}_{X^{-\beta}}\left\langle (u_n'(s) \times (u_n'(s) \times [\Delta u_n'(s) - \pi_n \phi'(u_n'(s))])), U\right\rangle_{X^{\beta}}\, \mathrm{d}s\\
&-\frac{1}{2}\int_0^t \langle \pi_n(u_n'(s) \times h) \times h, \pi_n U\rangle_{\mathbb{L}^2}\, \mathrm{d}s\Big].
\end{aligned}$$

We know that $u_n' \longrightarrow u'$ in $C([0, T]; X^{-\beta})$ $\mathbb{P}'$-a.s., so

$$\sup_{t\in[0,T]} |u_n(t) - u(t)|_{X^{-\beta}} \longrightarrow 0, \quad \mathbb{P}' - a.s.$$

so $u_n'(t) \longrightarrow u'(t)$ in $X^{-\beta}$ $\mathbb{P}'$-almost surely for any $t \in [0, T]$. And ${}_{X^{-\beta}}\langle \cdot, U\rangle_{X^{\beta}}$ is a continuous function on $X^{-\beta}$, hence

$$\lim_{n\to\infty} {}_{X^{-\beta}}\langle u_n'(t), U\rangle_{X^{\beta}} = {}_{X^{-\beta}}\langle u'(t), U\rangle_{X^{\beta}}, \quad \mathbb{P}' - a.s.$$

By (4.44), $\sup_{t\in[0,T]} |u_n'(t)|_H \le |u_0|_H$, since $H \hookrightarrow X^{-\beta}$ continuously, we can find a constant C such that

$$\begin{aligned}
&\sup_n \mathbb{E}'\left[\left|{}_{X^{-\beta}}\langle u_n'(t), U\rangle_{X^{\beta}}\right|^2\right] \le \sup_n \mathbb{E}'|U|^2_{X^{\beta}} \mathbb{E}'|u_n'(t)|^2_{X^{-\beta}}\\
\le {}& C\mathbb{E}'|U|^2_{X^{\beta}}\mathbb{E}' \sup_n |u_n'(t)|^2_H \le C\mathbb{E}'|U|^2_{X^{\beta}}\mathbb{E}'|u_0|^2_H < \infty.
\end{aligned}$$

Hence the sequence ${}_{X^{-\beta}}\langle u_n'(t), U\rangle_{X^{\beta}}$ is uniformly integrable. So the almost surely convergence and uniform integrability implies that

$$\lim_{n\to\infty} \mathbb{E}'[{}_{X^{-\beta}}\langle u_n'(t), U\rangle_{X^{\beta}}] = \mathbb{E}'[{}_{X^{-\beta}}\langle u'(t), U\rangle_{X^{\beta}}].$$

By (4.55),

$$\lim_{n\to\infty} \mathbb{E}'\int_0^t \langle u_n'(s) \times [\Delta u'(s) - \phi'(u'(s))], \pi_n U\rangle_{\mathbb{L}^2}\, \mathrm{d}s = \mathbb{E}'\int_0^t \langle Y(s), U\rangle_{\mathbb{L}^2}.$$

By (4.57)

$$\lim_{n\to\infty} \mathbb{E}'\int_0^t {}_{X^{-\beta}}\left\langle \pi_n\left(u_n'(s) \times (u_n'(s) \times [\Delta u_n'(s) - \pi_n\phi'(u_n'(s))])\right), U\right\rangle_{X^{\beta}}\, \mathrm{d}s = \mathbb{E}'\int_0^t \langle Z(s), U\rangle_{X^{\beta}}\, \mathrm{d}s.$$

By the Hölder's inequality,

$$|u_n'(t) - u'(t)|^2_{\mathbb{L}^2} \le |u_n'(t) - u'(t)|^2_{\mathbb{L}^4}.$$

Hence by (4.52),

$$\begin{aligned}
&\mathbb{E}'\int_0^t \langle \pi_n((u_n'(s)-u'(s))\times h)\times h, \pi_n U\rangle_{\mathbb{L}^2}\,\mathrm{d}s \\
\leq\ & |U|_{L^2(\Omega';L^2(0,T;\mathbb{L}^2))}\left(\mathbb{E}'\int_0^t (\pi_n|(u_n'(s)-u'(s))\times h)\times h|^2_{\mathbb{L}^2}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
\leq\ & |h|^2_{\mathbb{L}^\infty}|U|_{L^2(\Omega';L^2(0,T;\mathbb{L}^2))}\left(\mathbb{E}'\int_0^t |u_n'(s)-u'(s)|^2_{\mathbb{L}^2}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
\leq\ & |h|^2_{\mathbb{L}^\infty}|U|_{L^2(\Omega';L^2(0,T;\mathbb{L}^2))}\left(\mathbb{E}'\int_0^t |u_n'(s)-u'(s)|^2_{\mathbb{L}^4}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
\leq\ & |h|^2_{\mathbb{L}^\infty}|U|_{L^2(\Omega';L^2(0,T;\mathbb{L}^2))}\left(\mathbb{E}'t\int_0^t |u_n'(s)-u'(s)|^4_{\mathbb{L}^4}\,\mathrm{d}s\right)^{\frac{1}{4}} \\
\longrightarrow\ & 0.
\end{aligned}$$

The last "$\leq$" is from the Jensen's inequality.
Hence

$$\begin{aligned}
&\lim_{n\to\infty} {}_{L^2(\Omega';X^{-\beta})}\langle M_n'(t), U\rangle_{L^2(\Omega';X^\beta)} \\
=\ & \mathbb{E}'\Big[{}_{X^{-\beta}}\langle u'(t), U\rangle_{X^\beta} - {}_{X^{-\beta}}\langle u_0, U\rangle_{X^\beta} - \lambda_1\int_0^t \langle Y(s), U\rangle_{\mathbb{L}^2}\,\mathrm{d}s \\
&+\lambda_2\int_0^t {}_{X^{-\beta}}\langle Z(s), U\rangle_{X^\beta}\,\mathrm{d}s - \frac{1}{2}\int_0^t \langle (u'(s)\times h)\times h, U\rangle_{\mathbb{L}^2}\,\mathrm{d}s\Big].
\end{aligned}$$

Since by Lemma 4.29 and Lemma 4.30, we have $Y = u'\times\Delta u'$ and $Z = u'\times(u'\times\Delta u')$. Therefore for any $U \in L^2(\Omega'; X^\beta)$,

$$\lim_{n\to\infty} {}_{L^2(\Omega';X^{-\beta})}\langle M_n'(t), U\rangle_{L^2(\Omega';X^\beta)} = {}_{L^2(\Omega';X^{-\beta})}\langle M'(t), U\rangle_{L^2(\Omega';X^\beta)}.$$

This concludes the proof of Lemma 4.32. □

Before we can continue to prove u' is the weak solution of equation (4.2), we need to show that the W' and W_n' in Proposition 4.18 are Brownian motions. And that will be done in Lemma 4.34 and Lemma 4.35. And we need Lemma 4.33 to prove Lemma 4.34.

Lemma 4.33. *The Borel σ-field $\mathcal{B}$ and the cylindrical σ-field C on $C([0,T];\mathbb{R})$ are identical.*

Proof.(i) We will show that $C \subset \mathcal{B}$.
We claim that all the cylindrical sets are Borel sets. For some $n \in \mathbb{N}$, let

$$C = \{x : (x(t_1), \dots, x(t_n)) \in A\},$$

for some open set $A \in \mathcal{B}(\mathbb{R}^n)$. For any $y \in C$, $(y(t_1), \dots, y(t_n)) \in A$, and since A is open, $\exists\varepsilon > 0$, such that if x satisfies

$$|(x(t_1), \dots, x(t_n)) - (y(t_1), \dots, y(t_n))| < \varepsilon,$$

then $(x(t_1), \dots, x(t_n)) \in A$, so $x \in C$. But

$$|(x(t_1), \dots, x(t_n)) - (y(t_1), \dots, y(t_n))| \leq n\|x - y\|_{C([0,T];\mathbb{R})},$$

hence if $\|x - y\| < \frac{\varepsilon}{n}$, then $x \in C$. So we proved that if A is open, then $C = \{x : (x(t_1), \dots, x(t_n)) \in A\}$ is also open. And notice that

$$\{x : (x(t_1), \dots, x(t_n)) \in A\}^c = \{x : (x(t_1), \dots, x(t_n)) \in A^c\},$$

$$\bigcup_{k=1}^{\infty}\{x : (x(t_1), \dots, x(t_n)) \in A_k\} = \{x : (x(t_1), \dots, x(t_n)) \in \bigcup_{k=1}^{\infty} A_k\},$$

therefore if A is a Borel set, then C is also a Borel set, which means that all the cylindrical sets are Borel set. Hence $\mathcal{C} \subset \mathcal{B}$.

(ii) We will show that $\mathcal{B} \subset \mathcal{C}$.

We only need to prove all the open sets are in $\mathcal{C}$. And since $C([0,T];\mathbb{R})$ is a separable metric space, any open set is a countable union of open balls. Hence we only need to show all the open balls belong to $\mathcal{C}$. Let us first consider

$$B_\varepsilon = \left\{x : \|x\| = \sup_{t\in[0,T]} |x(t)| < \varepsilon\right\},$$

for some $\varepsilon > 0$. Suppose that $\{t_i\}$ is countable and dense in $[0, T]$, then we have

$$\sup_{t\in[0,T]} |x(t)| = \sup_i |x(t_i)|.$$

By the definition of cylindrical set, the set $\{x : x(t_i) \in A\}$ for some Borel set A is cylindrical. So the map

$$f_i : C([0,T];\mathbb{R}) \longrightarrow \mathbb{R}$$
$$x \longmapsto x(t_i)$$

is $(\mathcal{C}, \mathcal{B}(\mathbb{R}))$-measurable. And by the propositions of measurable maps, the map

$$f : C([0,T];\mathbb{R}) \ni x \longrightarrow \sup_i |f_i(x)| \in \mathbb{R}$$

is also $(\mathcal{C}, \mathcal{B}(\mathbb{R}))$-measurable. Since $B_\varepsilon = f^{-1}([0,\varepsilon))$, $B_\varepsilon \in \mathcal{C}$. So $\mathcal{B} \subset \mathcal{C}$.

This concludes the proof if Lemma 4.33. □

Lemma 4.34. *Suppose the W'_n defined in $(\Omega', \mathcal{F}', \mathbb{P}')$ has the same distribution as the Brownian Motion W defined in $(\Omega, \mathcal{F}, \mathbb{P})$ as in Proposition 4.18. Then W'_n is also a Brownian Motion.*

Proof. We prove W'_n is a Brownian Motion. By Lemma 4.33, we can use the cylinder subsets as the Borel subsets in $C([0,T];\mathbb{R})$.

(i) $W'_n(0) = 0$ $\mathbb{P}'$-almost surely.

$$\begin{aligned} &\mathbb{P}'(W'_n(0) = 0) \\ =\ &\mathbb{P}'\{\omega' : W'_n(\cdot, \omega') \in \{x \in C([0,T];\mathbb{R}) : x_0 = 0\}\} \\ =\ &\mathbb{P}\{\omega : W(\cdot, \omega) \in \{x \in C([0,T];\mathbb{R}) : x_0 = 0\}\} \\ =\ &\mathbb{P}(W(0) = 0) = 1. \end{aligned}$$

(ii) W_n' has independent increment.
For $0 \le t_1 < t_2 \le t_3 < t_4 \le T$, any $A, B \in \mathscr{B}(\mathbb{R})$.

$$\begin{aligned}&\{W_n'(t_2) - W_n'(t_1) \in A\} \cap \{W_n'(t_4) - W_n'(t_3) \in B\}\\ = &\{\omega' \in \Omega' : W_n'(\cdot, \omega') \in \{x : x_{t_2} - x_{t_1} \in A\} \cap \{x : x_{t_4} - x_{4_3} \in B\}\}.\end{aligned}$$

$A, B \in \mathscr{B}(\mathbb{R})$, so $\{x : x_{t_2} - x_{t_1} \in A\}$, $\{x : x_{t_4} - x_{4_3} \in B\}$ are cylindrical sets, so $\{x : x_{t_2} - x_{t_1} \in A\} \cap \{x : x_{t_4} - x_{4_3} \in B\} \in \mathscr{B}^T$. Since W_n' and W have the same law,

$$\begin{aligned}&\mathbb{P}'\Big(\{\omega' \in \Omega' : W_n'(\cdot, \omega') \in \{x : x_{t_2} - x_{t_1} \in A\} \cap \{x : x_{t_4} - x_{t_3} \in B\}\}\Big)\\ = &\mathbb{P}\Big(\{\omega \in \Omega : W(\cdot, \omega) \in \{x : x_{t_2} - x_{t_1} \in A\} \cap \{x : x_{t_4} - x_{4_3} \in B\}\}\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\mathbb{P}'(\{W_n'(t_2) - W_n'(t_1) \in A\} \cap \{W_n'(t_4) - W_n'(t_3) \in B\})\\ = &\mathbb{P}\Big(\{\omega \in \Omega : W(\cdot, \omega) \in \{x : x_{t_2} - x_{t_1} \in A\} \cap \{x : x_{t_4} - x_{4_3} \in B\}\}\Big)\\ = &\mathbb{P}(\{W(t_2) - W(t_1) \in A\} \cap \{W(t_4) - W(t_3) \in B\})\\ = &\mathbb{P}(\{W(t_2) - W(t_1) \in A\})\mathbb{P}(\{W(t_4) - W(t_3) \in B\})\\ = &\mathbb{P}(W \in \{x \in C([0,T];\mathbb{R}) : x_{t_2} - x_{t_1} \in A\})\mathbb{P}(W \in \{x \in C([0,T];\mathbb{R}) : x_{t_4} - x_{t_3} \in B\})\\ = &\mathbb{P}'(W' \in \{x \in C([0,T];\mathbb{R}) : x_{t_2} - x_{t_1} \in A\})\mathbb{P}'(W' \in \{x \in C([0,T];\mathbb{R}) : x_{t_4} - x_{t_3} \in B\})\\ = &\mathbb{P}'(\{W_n'(t_2) - W_n'(t_1) \in A\})\mathbb{P}(\{W_n'(t_4) - W_n'(t_3) \in B\})\end{aligned}$$

Hence $W_n'(t_2) - W_n'(t_1)$ and $W_n'(t_4) - W_n'(t_3)$ are independent.
(iii) $W_n'(t) \sim N(0, t)$ for $t \in [0, T]$. Similarly as before, we have

$$\mathbb{P}'(W_n'(t) \in A) = \mathbb{P}(W(t) \in A) = \frac{1}{\sqrt{2\pi t}} \int_A e^{-\frac{x^2}{2t}}\, \mathrm{d}x,$$

for $A \in \mathscr{B}(\mathbb{R})$ and $t \in [0, T]$.

Hence $W_n'(t)$ is a Brownian Motion. This concludes the proof of Lemma 4.34. □

Lemma 4.35. *The process $(W'(t))_{t\in[0,T]}$ is a real-valued Brownian motion on $(\Omega', \mathcal{F}', \mathbb{P}')$ and if $0 \le s < t \le T$ then the increament $W'(t) - W'(s)$ is independent of the σ-algebra generated by $u'(r)$ and $W'(r)$ for $r \in [0, s]$.*

Proof. We consider the characteristic functions of W'. Let $k \in \mathbb{N}$ and $0 = s_0 < s_1 < \cdots < s_k \le T$. For $(t_1, \ldots, t_k) \in \mathbb{R}^k$, we have for each $n \in \mathbb{N}$:

$$\mathbb{E}'\left[e^{i\sum_{j=1}^k t_j(W_n'(s_j) - W_n'(s_{j-1}))}\right] = e^{-\frac{1}{2}\sum_{j=1}^k t_j^2(s_j - s_{j-1})}.$$

Notice that $\left|e^{i\sum_{j=1}^k t_j(W_n'(s_j) - W_n'(s_{j-1}))}\right| \le 1$, by the Lebesgue's dominated convergence theorem,

$$\mathbb{E}'\left[e^{i\sum_{j=1}^k t_j(W'(s_j) - W'(s_{j-1}))}\right] = \lim_{n\to\infty} \mathbb{E}'\left[e^{i\sum_{j=1}^k t_j(W_n'(s_j) - W_n'(s_{j-1}))}\right] = e^{-\frac{1}{2}\sum_{j=1}^k t_j^2(s_j - s_{j-1})}.$$

Hence $W'(t)$ has the same distribution with $W_n'(t)$ for $t \in [0, T]$. Since random variables are independent if and only if the characteristic function of the sum of them equals to the multiplication of their characteristic functions, here

$$\mathbb{E}'\left[e^{i\sum_{j=1}^{k} t_j(W'(s_j)-W'(s_{j-1}))}\right] = \prod_{j=1}^{k}\mathbb{E}'\left[e^{it_j(W'(s_j)-W'(s_{j-1}))}\right].$$

Hence W' has independent increments.
And

$$W'(0) = \lim_{n\to\infty} W_n'(0) = 0, \quad \mathbb{P}' - a.s.,$$

so $(W'(t))_{t\in[0,T]}$ is a real-valued Brownian motion on $(\Omega', \mathcal{F}', \mathbb{P}')$.
The law of (u_n, W) is same as (u_n', W_n') and if $t > s \geq r$, $W(t) - W(s)$ is independent with $u_n(r)$, so $W_n'(t) - W_n'(s)$ is independent with $u_n'(r)$ for all n. By Proposition 4.18, $\lim_{n\to\infty} \|u_n'(r)\|_{V'} = \|u'(r)\|_{V'}$ and $\lim_{n\to\infty}(W_n'(t) - W_n'(s)) = W'(t) - W'(s)$, hence by the Lebesgue's dominated convergence theorem we have

$$\begin{aligned}
&\mathbb{E}'\left(e^{it\left(\|u'(r)\|_{V'}+W'(t)-W'(s)\right)}\right) = \lim_{n\to\infty}\mathbb{E}'\left(e^{it\left(\|u_n'(r)\|_{V'}+W_n'(t)-W_n'(s)\right)}\right)\\
&= \lim_{n\to\infty}\mathbb{E}'\left(e^{it\left(\|u_n'(r)\|_{V'}\right)}\right)\mathbb{E}'\left(e^{it(W_n'(t)-W_n'(s))}\right) = \mathbb{E}'\left(e^{it\left(\|u'(r)\|_{V'}\right)}\right)\mathbb{E}'\left(e^{it(W'(t)-W'(s))}\right).
\end{aligned}$$

So $W'(t)-W'(s)$ is independent of $u'(r)$. Hence this completes the proof of Lemma 4.35. □

Remark 4.36. We will denote $\mathbb{F}'$ the filtration generated by (u', W') and $\mathbb{F}_n'$ the filtration generated by (u_n', W_n'). Then by Lemma 4.35, u' is progressively measurable with respect to $\mathbb{F}'$ and by Lemma 4.34, u_n' is progressively measurable with respect to $\mathbb{F}_n'$.

4.6.2. *Step 2.* Let us summarize what we have achieved so far. We have got our process M' and have shown W' is a Wiener process. Next we will show a similar result as in equation (4.73) to prove u' is a weak solution of the Equation (4.2).
But before that we still need some preparation. The following estimate will be used to prove Lemma 4.38.

Proposition 4.37. *For every $h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$, there exists $c = c(h,\beta) > 0$ such that for every $u \in \mathbb{L}^2$, we have*

$$|u\times h|_{X^{-\beta}} \leq c|u|_{X^{-\beta}} < \infty. \tag{4.75}$$

Proof. Let $z \in \mathbb{H}^1$, $h \in \mathbb{L}^\infty \cap \mathbb{W}^{1,3}$. Then

$$\begin{aligned}
&|z\times h|_{\mathbb{H}^1}^2 = |\nabla(z\times h)|_{\mathbb{L}^2}^2 + |z\times h|_{\mathbb{L}^2}^2\\
&\leq 2(|\nabla z\times h|_{\mathbb{L}^2}^2 + |z\times\nabla h|_{\mathbb{L}^2}^2) + |z\times h|_{\mathbb{L}^2}^2 \leq 2(|h|_{\mathbb{L}^\infty}^2|\nabla z|_{\mathbb{L}^2}^2 + |\nabla h|_{\mathbb{L}^3}^2|z|_{\mathbb{L}^6}^2) + |h|_{\mathbb{L}^\infty}^2|z|_{\mathbb{L}^2}^2\\
&\leq 2(|h|_{\mathbb{L}^\infty}^2 + c^2|\nabla h|_{\mathbb{L}^3}^2)|z|_{\mathbb{H}^1}^2,
\end{aligned}$$

so the map

$$\mathbb{H}^1 \ni z \longmapsto z\times h \in \mathbb{H}^1$$

is linear and bounded. And so for $u \in \mathbb{L}^2$, $z \in X^\beta$.

$$|{}_{X^{-\beta}}\langle u\times h, z\rangle_{X^\beta}| = |{}_{X^{-\beta}}\langle u, z\times h\rangle_{X^\beta}| \leq \sqrt{2(|h|_{\mathbb{L}^\infty}^2 + c^2|\nabla h|_{\mathbb{L}^3}^2)}|z|_{X^\beta}|u|_{X^{-\beta}}.$$

Let $c_h = \sqrt{2(|h|^2_{\mathbb{L}^\infty} + c^2|\nabla h|^2_{\mathbb{L}^3})}$, we have get

$$|u \times h|_{X^{-\beta}} \leq c_h |u|_{X^{-\beta}} < \infty, \quad u \in \mathbb{L}^2.$$

This completes the proof of Proposition 4.37. □

Lemma 4.38. *For each $m \in \mathbb{N}$, we define the partition $\{s^m_j := \frac{jT}{m},\ j = 0, \dots, m\}$ of $[0, T]$. Then for any $\varepsilon > 0$, we have the following results:*

(i) *We begin with the proof of part (i).*
We can choose $m \in \mathbb{N}$ large enough such that

$$\lim_{n\to\infty} |\left(\mathbb{E}'\left[\left|\int_0^t (\pi_n(u'_n(s) \times h) - \sum_{j=0}^{m-1} \pi_n(u'_n(s^m_j) \times h) 1_{(s^m_j, s^m_{j+1}]}(s))\,\mathrm{d}W'_n(s)\right|^2_{X^{-\beta}}\right]\right)^{\frac{1}{2}} < \frac{\varepsilon}{2};$$

(ii)

$$\lim_{n\to\infty} \mathbb{E}'\left[\left|\sum_{j=0}^{m-1} \pi_n(u'_n(s^m_j) \times h)(W'_n(t \wedge s^m_{j+1}) - W'_n(t \wedge s^m_j)) - \sum_{j=0}^{m-1} \pi_n(u'(s^m_j) \times h)(W'(t \wedge s^m_{j+1}) - W'(t \wedge s^m_j))\right|^2_{X^{-\beta}}\right] = 0;$$

(iii)

$$\lim_{n\to\infty} \left(\mathbb{E}'\left[\left|\int_0^t (\pi_n(u'(s) \times h) - \sum_{j=0}^{m-1} \pi_n(u'(s^m_j) \times h) 1_{(s^m_j, s^m_{j+1}]}(s))\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right]\right)^{\frac{1}{2}} < \frac{\varepsilon}{2};$$

(iv)

$$\lim_{n\to\infty} \mathbb{E}'\left[\left|\int_0^t (\pi_n(u'(s) \times h) - (u'(s) \times h))\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right] = 0.$$

Proof.(i) By the Itô isometry,

$$\begin{aligned}
&\left(\mathbb{E}'\left[\left|\int_0^t (\pi_n(u'_n(s) \times h) - \sum_{j=0}^{m-1} \pi_n(u'_n(s^m_j) \times h) 1_{(s^m_j, s^m_{j+1}]}(s))\,\mathrm{d}W'_n(s)\right|^2_{X^{-\beta}}\right]\right)^{\frac{1}{2}} \\
&= \left(\mathbb{E}'\left[\int_0^t \left|(\pi_n(u'_n(s) \times h) - \sum_{j=0}^{m-1} \pi_n(u'_n(s^m_j) \times h) 1_{(s^m_j, s^m_{j+1}]}(s))\right|^2_{X^{-\beta}} \mathrm{d}s\right]\right)^{\frac{1}{2}} \\
&\leq \left(\mathbb{E}' \int_0^t |u'_n(s) \times h - u'(s) \times h|^2_{X^{-\beta}}\,\mathrm{d}s\right)^{\frac{1}{2}} + \left(\mathbb{E}' \int_0^t |u'(s) \times h - \sum_{j=0}^{m-1} (u'(s^m_j) \times h) 1_{(s^m_j, s^m_{j+1}]}(s)|^2_{X^{-\beta}}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad + \left(\mathbb{E}' \int_0^t |\sum_{j=0}^{m-1} (u'(s^m_j) - u'_n(s^m_j)) \times h 1(s^m_j, s^m_{j+1}](s)|^2_{X^{-\beta}}\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

Note that by Jensen's inequality,

$$\int_0^t |u_n'(s)\times h - u'(s)\times h|^2_{X^{-\beta}}\,\mathrm{d}s \le C^2|h|^2_{\mathbb{L}^\infty}\int_0^t |u_n'(s)-u'(s)|^2_{\mathbb{L}^4}\,\mathrm{d}s$$

$$\le C^2|h|^2_{\mathbb{L}^\infty}\sqrt{t}\left(\int_0^t |u_n'(s)-u'(s)|^4_{\mathbb{L}^4}\,\mathrm{d}s\right)^{\frac{1}{2}} \longrightarrow 0, \quad \mathbb{P}'-a.s.$$

and by Sobolev embedding, (4.44) and (4.50),

$$\sup_n \mathbb{E}'\left(\int_0^t |u_n'(s)\times h - u'(s)\times h|^2_{X^{-\beta}}\,\mathrm{d}s\right)^2 \le \sup_n C^4|h|^4_{\mathbb{L}^\infty}\mathbb{E}'\left(\int_0^t |u_n'(s)-u'(s)|^2_{\mathbb{L}^2}\,\mathrm{d}s\right)^2$$

$$\le C^4|h|^4_{\mathbb{L}^\infty}T\sup_n\sup_t \mathbb{E}'|u_n'(t)-u'(t)|^2_{\mathbb{L}^2} \le C^4|h|^4_{\mathbb{L}^\infty}T\sup_n\sup_t\mathbb{E}'\left(|u_n'(t)|^2_{\mathbb{L}^2}+|u'(t)|^2_{\mathbb{L}^2}\right)$$

$$\le C^4|h|^4_{\mathbb{L}^\infty}T\sup_n\left(\sup_t\mathbb{E}'|u_n'(t)|^2_{\mathbb{L}^2}+\sup_t\mathbb{E}'|u'(t)|^2_{\mathbb{L}^2}\right) \le C^4|h|^4_{\mathbb{L}^\infty}T\left(\mathbb{E}'|u_0|^2_{\mathbb{L}^2}+\sup_t\mathbb{E}'|u'(t)|^2_{\mathbb{L}^2}\right)$$

$$\le C^4|h|^4_{\mathbb{L}^\infty}T(1+c^2)|u_0|^2_{\mathbb{L}^2} < \infty.$$

Hence $\int_0^t |u_n'(s)\times h - u'(s)\times h|^2_{X^{-\beta}}\,\mathrm{d}s$ tends to 0 almost surely as $n \longrightarrow \infty$ and the sequence is uniformly integrable. Thus

$$\lim_{n\to\infty}\mathbb{E}'\int_0^t |u_n'(s)\times h - u'(s)\times h|^2_{X^{-\beta}}\,\mathrm{d}s = 0.$$

$u'\times h \in C([0,T];X^{-\beta})$ $\mathbb{P}'-a.s.$, next since

$$\lim_{m\to\infty}\sup_{s\in[0,T]}\left|u'(s)\times h - \sum_{j=0}^{m-1}(u'(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s)\right|_{X^{-\beta}} = 0, \quad \mathbb{P}'-a.s.$$

Hence

$$\int_0^t \left|u'(s)\times h - \sum_{j=0}^{m-1}(u'(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s)\right|^2_{X^{-\beta}}\,\mathrm{d}s \longrightarrow 0, \quad \mathbb{P}'-a.s.$$

And by Sobolev embedding,

$$\sup_m \mathbb{E}'\left(\int_0^t \left|u'(s)\times h - \sum_{j=0}^{m-1}(u'(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s)\right|^2_{X^{-\beta}}\,\mathrm{d}s\right)^2$$

$$\le c_h^4\sup_m\mathbb{E}'\left(\int_0^t \left|u'(s) - \sum_{j=0}^{m-1}u'(s_j^m)1_{(s_j^m,s_{j+1}^m]}(s)\right|^2_{X^{-\beta}}\,\mathrm{d}s\right)^2$$

$$\le c_h^4\sup_m\mathbb{E}'\left(\int_0^T 2|u'(s)|^2_{X^{-\beta}} + 2\left|\sum_{j=0}^{m-1}u'(s_j^m)1_{(s_j^m,s_{j+1}^m]}(s)\right|^2_{X^{-\beta}}\,\mathrm{d}s\right)^2 \le \cdots$$

Then by (4.50), $\sup_{t\in[0,T]}|u'(t)|_{X^{-\beta}} \le c|u_0|_{\mathbb{L}^2}$, $\mathbb{P}'$-almost surely,

$$\cdots \le 16T^2 c_h^4 \mathbb{E}'(|u_0|^4_{\mathbb{L}^2}) = 16T^2 c_h^4 |u_0|^4_{\mathbb{L}^2} < \infty.$$

Hence $\int_0^t \left| u'(s) \times h - \sum_{j=0}^{m-1} (u'(s_j^m) \times h) 1_{(s_j^m, s_{j+1}^m]}(s) \right|^2_{X^{-\beta}} \mathrm{d}s$ is uniform integrable. Therefore

$$\lim_{m\to\infty} \mathbb{E}' \left(\int_0^t \left| u'(s) \times h - \sum_{j=0}^{m-1} (u'(s_j^m) \times h) 1_{(s_j^m, s_{j+1}^m]}(s) \right|^2_{X^{-\beta}} \mathrm{d}s \right) = 0.$$

Hence for $\varepsilon > 0$ we can choose $m \in \mathbb{N}$ such that

$$\left(\mathbb{E}' \int_0^t |u'(t) \times h - \sum_{j=0}^{m-1} (u'(s_j^m) \times h) 1_{(s_j^m, s_{j+1}^m]}(s)|^2_{X^{-\beta}} \mathrm{d}s \right)^{\frac{1}{2}} < \frac{\varepsilon}{2}.$$

Again since $u'_n \longrightarrow u'$ in $C([0,T]; X^{-\beta})$, we have

$$\begin{aligned}
&\int_0^t \left| \sum_{j=0}^{m-1} (u'(s_j^m) - u'_n(s_j^m)) \times h 1_{(s_j^m, s_{j+1}^m]}(s) \right|^2_{X^{-\beta}} \mathrm{d}s \\
&\le c_h^2 \int_0^T \left| \sum_{j=0}^{m-1} (u'(s_j^m) - u'_n(s_j^m)) 1_{(s_j^m, s_{j+1}^m]}(s) \right|^2_{X^{-\beta}} \mathrm{d}s \\
&= c_h^2 \sum_{j=0}^{m-1} |u'(s_j^m) - u'_n(s_j^m)|^2_{X^{-\beta}} \frac{T}{m} \le c_h^2 \sup_{s\in[0,T]} |u'(s) - u'_n(s)|^2_{X^{-\beta}} \longrightarrow 0, \quad \mathbb{P}' - a.s.
\end{aligned}$$

Then by (4.50), $\sup_{t\in[0,T]} |u'(t)|_{X^{-\beta}} \le c|u_0|_{\mathbb{L}^2}$, $\mathbb{P}'$-almost surely, we have

$$\begin{aligned}
&\sup_n \mathbb{E}' \left(\sup_{s\in[0,T]} |u'(s) - u'_n(s)|^2_{X^{-\beta}} \right)^2 \\
&\le \sup_n \mathbb{E}' \left(\sup_{s\in[0,T]} |u'(s)|_{X^{-\beta}} + \sup_{s\in[0,T]} |u'_n(s)|^2_{X^{-\beta}} \right)^2 \le 4c^4 |u_0|^4_{\mathbb{L}^2}.
\end{aligned}$$

Hence

$$\sup_n \mathbb{E}' \left(\int_0^t \left| \sum_{j=0}^{m-1} (u'(s_j^m) - u'_n(s_j^m)) \times h 1_{(s_j^m, s_{j+1}^m]}(s) \right|^2_{X^{-\beta}} \mathrm{d}s \right)^2 < \infty.$$

So $\int_0^t \left| \sum_{j=0}^{m-1} (u'(s_j^m) - u'_n(s_j^m)) \times h 1_{(s_j^m, s_{j+1}^m]}(s) \right|^2_{X^{-\beta}} \mathrm{d}s$ is uniformly integrable. Hence

$$\lim_{n\to\infty} \mathbb{E}' \left(\int_0^t \left| \sum_{j=0}^{m-1} (u'(s_j^m) - u'_n(s_j^m)) \times h 1_{(s_j^m, s_{j+1}^m]}(s) \right|^2_{X^{-\beta}} \mathrm{d}s \right) = 0.$$

Therefore we have get for any $\varepsilon > 0$ we can choose large enough m, such that

$$\lim_{n\to\infty}\left(\mathbb{E}'\left[\left|\int_0^t(\pi_n(u_n'(s)\times h)-\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s))\,\mathrm{d}W_n'(s)\right|^2\right]\right)^{\frac{1}{2}} < \frac{\varepsilon}{2};$$

(ii) Next we will deal with the proof of part (ii).
Since $u_n' \longrightarrow u'$ in $C([0,T];\mathbb{R})$ $\mathbb{P}'$ almost surely and $W_n' \longrightarrow W'$ in $C([0,T];\mathbb{R})$ $\mathbb{P}'$ almost surely,

$$\begin{aligned}
&\Big|\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)(W_n'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m))\\
&-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\Big|_{X^{-\beta}}\\
\le&\Big|\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)(W_n'(t\wedge s_{j+1}^m)-W'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m)+W'(t\wedge s_j^m))\Big|_{X^{-\beta}}\\
&+\Big|\sum_{j=0}^{m-1}\pi_n((u_n'(s_j^m)-u'(s_j^m))\times h)(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\Big|_{X^{-\beta}}\\
\le&\ c_h\Big|\sum_{j=0}^{m-1}\pi_n u_n'(s_j^m)(W_n'(t\wedge s_{j+1}^m)-W'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m)+W'(t\wedge s_j^m))\Big|_{X^{-\beta}}\\
&+c_h\Big|\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)-u'(s_j^m))(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\Big|_{X^{-\beta}}\longrightarrow 0,\quad \mathbb{P}'-a.s.
\end{aligned}$$

And since W_n' are Brownian Motions and we have prove W' is also Brownian Motion, together with (4.50), we have

$$\begin{aligned}
&\sup_n\mathbb{E}'\Big[\Big|\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)(W_n'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m))\\
&-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\Big|_{X^{-\beta}}\Big]^4\\
\le&\ c_h\sup_n\mathbb{E}'\Big[\Big|\sum_{j=0}^{m-1}\pi_n u_n'(s_j^m)(W_n'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m))\\
&-\sum_{j=0}^{m-1}\pi_n u'(s_j^m)(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\Big|_{X^{-\beta}}\Big]^4\\
\le&\ c_h c^4|u_0|^4_{\mathbb{L}^2}\sup_n\mathbb{E}'\Big[\sum_{j=0}^{m-1}(|W_n'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m)|+|W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m)|)\Big]^4<\infty.
\end{aligned}$$

So

$$\Big|\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)(W_n'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m))$$
$$-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\Big|^2_{X^{-\beta}}$$

is uniform integrable. Hence we have

$$\lim_{n\to\infty}\mathbb{E}'\Big[\Big|\sum_{j=0}^{m-1}\pi_n(u_n'(s_j^m)\times h)(W_n'(t\wedge s_{j+1}^m)-W_n'(t\wedge s_j^m))$$
$$-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\Big|^2_{X^{-\beta}}\Big]=0.$$

(iii) Next we move to the proof of part (iii).
By the Itô isometry and the result in (i):

$$\left(\mathbb{E}'\left[\left|\int_0^t(\pi_n(u'(s)\times h)-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s))\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right]\right)^{\frac{1}{2}}$$
$$=\left(\mathbb{E}'\left[\int_0^t\left|(\pi_n(u'(s)\times h)-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s))\right|^2_{X^{-\beta}}\mathrm{d}s\right]\right)^{\frac{1}{2}}<\frac{\varepsilon}{2}$$

(iv) Finally, we will prove part (iv).
By Itô isometry,

$$\mathbb{E}'\left[\left|\int_0^t(\pi_n(u'(s)\times h)-(u'(s)\times h))\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right]$$
$$=\mathbb{E}'\left[\int_0^t\left|\pi_n(u'(s)\times h)-(u'(s)\times h)\right|^2_{X^{-\beta}}\mathrm{d}s\right]$$

By the Sobolev embedding $\mathbb{L}^2\hookrightarrow X^{-\beta}$,

$$\int_0^t|\pi_n(u'(s)\times h)-u'(s)\times h|^2_{X^{-\beta}}\,\mathrm{d}s\le C\int_0^t|\pi_n(u'(s)\times h)-u'(s)\times h|^2_{\mathbb{L}^2}\,\mathrm{d}s.$$

Since $\pi_n(u'(s)\times h)\longrightarrow u'(s)\times h$ in $\mathbb{L}^2$ $\mathbb{P}'$ almost surely,

$$\int_0^t|\pi_n(u'(s)\times h)-u'(s)\times h|^2_{X^{-\beta}}\,\mathrm{d}s\longrightarrow 0,\quad \mathbb{P}'-a.s.$$

And by (4.49), (4.50) and (4.75),

$$\begin{aligned}
&\sup_n \mathbb{E}'\left[\int_0^t \left|\pi_n(u'(s)\times h) - (u'(s)\times h)\right|^2_{X^{-\beta}}\, \mathrm{d}s\right]^2 \\
&\leq 2\sup_n \mathbb{E}'\left[\int_0^t \left|\pi_n(u'(s)\times h)\right|^2_{X^{-\beta}} + \left|(u'(s)\times h)\right|^2_{X^{-\beta}}\, \mathrm{d}s\right]^2 \\
&\leq 2\sup_n \mathbb{E}'\left[\int_0^t C^2|h|^2_{\mathbb{L}^\infty}\,|u_0|^2_{\mathbb{L}^2} + c_h^2\,|u_0|^2_{\mathbb{L}^2}\, \mathrm{d}s\right]^2 = (C^2|h|^2_{\mathbb{L}^\infty} + c_h)^2|u_0|^4_{\mathbb{L}^2}T^2 < \infty.
\end{aligned}$$

So $\int_0^t |\pi_n(u'(s)\times h) - (u'(s)\times h)|^2_{X^{-\beta}}\, \mathrm{d}s \longrightarrow 0$ $\mathbb{P}'$ almost surely and uniform integrable, hence

$$\lim_{n\to\infty} \mathbb{E}'\left[\left|\int_0^t (\pi_n(u'(s)\times h) - (u'(s)\times h))\, \mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right] = 0.$$

This completes the proof. □

Now we are ready to state the Theorem which means that u' is the weak solution of the equation (4.2).

Theorem 4.39. *For each $t\in[0,T]$ we have $M'(t) = \int_0^t (u'(s)\times h)\,\mathrm{d}W'(s)$.*

Proof. Firstly, we show that $M'_n(t) = \sum_{j=1}^N \int_0^t \pi_n(u'_n(s)\times h_j)\,\mathrm{d}W'_{jn}(s)$ $\mathbb{P}'$ almost surely for each $t\in[0,T]$ and $n\in\mathbb{N}$.
Let us fix that $t\in[0,T]$ and $n\in\mathbb{N}$. Let us also fix $m\in\mathbb{N}$ and define the partition $\left\{s_i^m := \frac{iT}{m}, i = 0,\ldots,m\right\}$ of $[0,T]$. Let us recall that (u'_n, W'_n) and (u_n, W) have the same laws on the separable Banach space $C([0,T];H_n)\times C([0,T];\mathbb{R}^N)$. Since the map

$$\begin{aligned}
\Psi:\ & C([0,T];H_n)\times C([0,T];\mathbb{R}^N) \longrightarrow H_n \\
& (u_n, W) \longmapsto M_n(t) - \sum_{i=0}^{m-1}\sum_{j=1}^N \pi_n(u_n(s_i^m)\times h_j)(W_j(t\wedge s_{i+1}^m) - W_j(t\wedge s_i^m))
\end{aligned}$$

is continuous so measurable. By involving the Kuratowski Theorem we infer that the H-valued random variables:

$$M_n(t) - \sum_{i=0}^{m-1}\sum_{j=1}^N \pi_n(u_n(s_i^m)\times h_j)(W_j(t\wedge s_{i+1}^m) - W_j(t\wedge s_i^m))$$

and

$$M'_n(t) - \sum_{j=0}^{m-1}\sum_{j=1}^N \pi_n(u'_n(s_i^m)\times h_j)(W'_{jn}(t\wedge s_{i+1}^m) - W'_{jn}(t\wedge s_j^m))$$

have the same laws. Let us denote $\tilde{u}_n := \sum_{i=0}^{m-1} u_n(s_i^m)1_{[s_i^m, s_{i+1}^m)}$. By the Itô isometry, we have

$$\left\|\sum_{i=0}^{m-1}\pi_n(u_n(s_i^m)\times h_j)(W_j(t\wedge s_{i+1}^m)-W_j(t\wedge s_i^m))-\int_0^t\pi_n(u_n(s)\times h_j)\,\mathrm{d}W_j(s)\right\|^2_{L^2(\Omega;H)}$$
$$=\mathbb{E}\left\|\int_0^t\left[\pi_n(\tilde{u}_n\times h_j)-\pi_n(u_n(s)\times h_j)\right]\mathrm{d}W_j(s)\right\|_H^2\leq\|h_j\|^2_{\mathbb{L}^\infty(D)}\mathbb{E}\int_0^t\|\tilde{u}_n(s)-u_n(s)\|_H^2\,\mathrm{d}s.$$

Since $u_n \in C([0,T];H_n)$ $\mathbb{P}$-almost surely, we have

$$\lim_{n\to\infty}\int_0^t\|\tilde{u}_n(s)-u_n(s)\|_H^2\,\mathrm{d}s=0,\quad \mathbb{P}-a.s..$$

Moreover by the equality (4.13), we infer that

$$\sup_n\mathbb{E}\left|\int_0^t\|\tilde{u}_n(s)-u_n(s)\|_H^2\,\mathrm{d}s\right|^2\leq\sup_n\mathbb{E}\left|\int_0^t\left[2\|\tilde{u}_n(s)\|_H^2+2\|u_n(s)\|_H^2\right]\mathrm{d}s\right|^2$$
$$\leq\mathbb{E}\left|4\|u_0\|_H^2T\right|^2=16\|u_0\|_H^4T^2<\infty.$$

Therefore by the almost surely convergence and uniformly integrability, we have

$$\lim_{m\to\infty}\left\|\sum_{i=0}^{m-1}\pi_n(u_n(s_i^m)\times h_j)(W_j(t\wedge s_{i+1}^m)-W_j(t\wedge s_i^m))-\int_0^t\pi_n(u_n(s)\times h_j)\,\mathrm{d}W_j(s)\right\|^2_{L^2(\Omega;H)}=0.$$

Similarly, because u_n' satisfies the same conditions as u_n, we also get

$$\lim_{m\to\infty}\left\|\sum_{i=0}^{m-1}\pi_n(u_n'(s_i^m)\times h)(W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m))-\int_0^t\pi_n(u_n'(s)\times h_j)\,\mathrm{d}W_{jn}'(s)\right\|^2_{L^2(\Omega;H)}=0.$$

Hence, since L^2 convergence implies weak convergence, we infer that the random variables $M_n(t)-\sum_{j=1}^N\int_0^t\pi_n(u_n(s)\times h_j)\,\mathrm{d}W_j(s)$ and $M_n'(t)-\sum_{j=1}^N\int_0^t\pi_n(u_n'(s)\times h_j)\,\mathrm{d}W_{jn}'(s)$ have same laws. But $M_n(t)-\sum_{j=1}^N\int_0^t\pi_n(u_n(s)\times h_j)\,\mathrm{d}W_j(s)=0$ $\mathbb{P}$-almost surely, so

$$M_n'(t)=\sum_{j=1}^N\int_0^t\pi_n(u_n'(s)\times h_j)\,\mathrm{d}W_j'(s),\quad \mathbb{P}'-a.s.$$

Secondly, we show that $M'_n(t)$ converges in $L^2(\Omega'; X^{-\beta})$ to $\int_0^t (u'(s)\times h)\,\mathrm{d}W'(s)$ as $n\to\infty$. Notice that

$$\begin{aligned}
&\left(\mathbb{E}'\left|M'_n(t)-\int_0^t (u'(s)\times h)\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right)^{\frac{1}{2}}\\
=&\left(\mathbb{E}'\left|\int_0^t \pi_n(u'_n(s)\times h)\,\mathrm{d}W'_n(s)-\int_0^t (u'(s)\times h)\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right)^{\frac{1}{2}}\\
\leq&\left(\mathbb{E}'\left|\int_0^t \pi_n(u'_n(s)\times h)\,\mathrm{d}W'_n(s)-\sum_{j=0}^{m-1}\pi_n(u'_n(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s)\,\mathrm{d}W'_n(s)\right|_{X^{-\beta}}\right)^{\frac{1}{2}}\\
&+\left(\mathbb{E}'\left|\sum_{j=0}^{m-1}\pi_n(u'_n(s_j^m)\times h)(W'_n(t\wedge s_{j+1}^m)-W'_n(t\wedge s_j^m))\right.\right.\\
&\left.\left.-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)(W'(t\wedge s_{j+1}^m)-W'(t\wedge s_j^m))\right|^2_{X^{-\beta}}\right)^{\frac{1}{2}}\\
&+\left(\mathbb{E}'\left|\int_0^t (\pi_n(u'(s)\times h)-\sum_{j=0}^{m-1}\pi_n(u'(s_j^m)\times h)1_{(s_j^m,s_{j+1}^m]}(s))\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right)^{\frac{1}{2}}\\
&+\left(\mathbb{E}'\left|\int_0^t (\pi_n(u'(s)\times h)-(u'(s)\times h))\,\mathrm{d}W'(s)\right|^2_{X^{-\beta}}\right)^{\frac{1}{2}},
\end{aligned}$$

And then by Lemma 4.38, we complete the proof of Theorem 4.39. □

Summarizing, it follows from Theorem 4.39 that the process u' satisfies the following equation in $L^2(\Omega'; X^{-\beta})$ for $t\in[0,T]$:

$$\begin{aligned}
u'(t) \;=\; & u_0+\lambda_1\int_0^t (u'\times[\Delta u'-\phi'(u')])(s)\,\mathrm{d}s\\
&-\lambda_2\int_0^t u'(s)\times(u'\times[\Delta u'-\phi'(u')])(s)\,\mathrm{d}s\\
&+\int_0^t (u'(s)\times h)\circ\,\mathrm{d}W'(s)
\end{aligned}\tag{4.76}$$

4.7. **Regularities of the weak solution.** Now we will start to show some regularity of u'.

Theorem 4.40. *The process u' from Proposition 4.18 satisfies:*

$$|u'(t,x)|_{\mathbb{R}^3}=1,\quad \textit{for Lebesgue a.e. } (t,x)\in[0,T]\times D \textit{ and } \mathbb{P}'-a.s.. \tag{4.77}$$

To prove Theorem 4.40, we need the following Lemma:

Lemma 4.41. [40]*(Th. 1.2) Let $(\Omega, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space and let V and H be two separable Hilbert spaces, such that $V \hookrightarrow H$ continuously and densely. We identify H with it's dual space and have a Gelfand triple: $V \hookrightarrow H \cong H' \hookrightarrow V'$. We assume that*

$$u \in M^2(0,T;V),\ u_0 \in H,\ v \in M^2(0,T;V'),\ z \in M^2(0,T;H),$$

for every $t \in [0,T]$,

$$u(t) = u_0 + \int_0^t v(s)\,\mathrm{d}s + \int_0^t z(s)\,\mathrm{d}W_s, \quad \mathbb{P}-a.s..$$

Let ψ be a twice differentiable functional on H, which satisfies:

(i) *ψ, ψ' and ψ'' are locally bounded.*
(ii) *ψ and ψ' are continuous on H.*
(iii) *Let $\mathscr{L}^1(H)$ be the Banach space of all the trace class operators on H. Then $\forall Q \in \mathscr{L}^1(H)$, $Tr[Q \circ \psi'']$ is a continuous functional on H.*
(iv) *If $u \in V$, $\psi'(u) \in V$; $u \mapsto \psi'(u)$ is continuous from V (with the strong topology) into V endowed with the weak topology.*
(v) *$\exists k$ such that $\|\psi'(u)\|_V \le k(1 + \|u\|_V)$, $\forall u \in V$.*

Then for every $t \in [0,T]$,

$$\begin{aligned}\psi(u(t)) = \psi(u_0) &+ \int_0^t {}_{V^*}\langle v(s), \psi'(u(s))\rangle_V\,\mathrm{d}s + \int_0^t {}_H\langle \psi'(u(s)), z(s)\rangle_H\,\mathrm{d}W_s \\ &+\frac{1}{2}\int_0^t {}_H\langle \psi''(u(s))z(s), z(s)\rangle_H\,\mathrm{d}s, \quad \mathbb{P}-a.s..\end{aligned}$$

Proof of Theorem 4.40. Let $\xi \in C_0^\infty(D,\mathbb{R})$. Then we consider a function

$$\psi : H \ni u \longmapsto \langle u, \xi u\rangle_H \in \mathbb{R}.$$

It's easy to see that ψ is of C^2 class and $\psi'(u) = 2\xi u$, $\psi''(u)(v) = 2\xi v$, $u, v \in H$. Next we will check the assumptions of Lemma 4.41. By previous work (see details below), u' satisfies:

$$\mathbb{E}'\int_0^T \|u'(t)\|_V^2\,\mathrm{d}t < \infty, \quad \text{by (4.51)},$$

$$\mathbb{E}'\int_0^T \|\left(u' \times [\Delta u' + \phi'(u')]\right)(t)\|_{X^{-\beta}}^2\,\mathrm{d}t < \infty, \quad \text{by (4.55)},$$

$$\mathbb{E}'\int_0^T \|u'(t) \times \left(u' \times [\Delta u' + \phi'(u')]\right)(t)\|_{X^{-\beta}}^2\,\mathrm{d}t < \infty, \quad \text{by (4.57)},$$

$$\mathbb{E}'\int_0^T \|(u'(s) \times h) \times h\|_{X^{-\beta}}^2\,\mathrm{d}t < \infty, \quad \text{by (4.49)},$$

$$\mathbb{E}'\int_0^T \|u'(s) \times h\|_H^2\,\mathrm{d}t < \infty, \quad \text{by (4.49)}.$$

And ψ satisfies:

(i) ψ, ψ', ψ'' are locally bounded.
(ii) Since ψ', ψ'' exist, ψ, ψ' are continuous on H.

(iii) $\forall Q \in \mathscr{L}^1(H)$,

$$Tr[Q \circ \psi''(a)] = \sum_{j=1}^{\infty} \langle Q \circ \psi''(a)e_j, e_j \rangle_H = 2 \sum_{j=1}^{\infty} \langle Q(\eta e_j), e_j \rangle_H,$$

which is a constant in $\mathbb{R}$, so the map $H \ni a \longmapsto Tr[Q \circ \psi''(a)] \in H$ is a continuous functional on H.

(iv) If $u \in V$, $\psi'(u) \in V$; $u \mapsto \psi'(u)$ is continuous from V (with the strong topology) into V endowed with the weak topology.
This is because: For any $v^* \in X^{-\beta}$, we have

$$\begin{aligned} {}_{X^\beta}\langle \psi'(u+v) - \psi'(u), v^* \rangle_{X^{-\beta}} &= {}_{X^\beta}\langle 2\phi v, v^* \rangle_{X^{-\beta}} \\ &\le 2|\xi|_{C(D,\mathbb{R})} {}_{X^\beta}\langle v, v^* \rangle_{X^{-\beta}}, \end{aligned}$$

hence ψ' is weakly continuous. Let us denote τ as the strong topology of V and τ_w the weak topology of V. Take $B \in \tau_w$, by the weak continuity $(\psi')^{-1}(B) \in \tau_w$, but $\tau_w \subset \tau$. Hence $(\psi')^{-1}(B) \in \tau$, which implies (iv).

(v) $\exists k$ such that $\|\psi'(u)\|_V \le k(1 + \|u\|_V)$, $\forall u \in V$.

Hence by Lemma 4.41, we have that for $t \in [0, T]$, $\mathbb{P}'$ almost surely,

$$\begin{aligned} &\langle u'(t), \xi u'(t) \rangle_H - \langle u_0, \xi u_0 \rangle_H \\ = &\int_0^t {}_{X^{-\beta}}\langle \lambda_1 \left(u' \times [\Delta u' - \phi'(u')]\right)(s) - \lambda_2 u'(s) \times (u' \times [\Delta u' \\ &-\phi'(u')])(s) + \frac{1}{2}(u'(s) \times h) \times h, 2\xi u'(s) \rangle_{X^\beta} \, \mathrm{d}s \\ &+ \int_0^t \langle 2\xi u'(s), u'(s) \times h \rangle_H \, \mathrm{d}W'(s) + \int_0^t \langle \xi u'(s) \times h, u'(s) \times h \rangle_H \, \mathrm{d}s. \end{aligned}$$

By Lemma 4.31,

$${}_{X^{-\beta}}\langle \lambda_1 \left(u' \times [\Delta u' + \phi'(u')]\right)(s), 2\xi u'(s) \rangle_{X^\beta} = 0.$$

And since

$${}_{X^{-\beta}}\langle \lambda_2 u'(s) \times \left(u' \times [\Delta u' + \phi'(u')]\right)(s), 2\xi u'(s) \rangle_{X^\beta} = 0,$$

$${}_{X^{-\beta}}\langle (u'(s) \times h) \times h, \xi u'(s) \rangle_{X^\beta} = -{}_{X^{-\beta}}\langle u'(s) \times h, \xi u'(s) \times h \rangle_{X^\beta},$$

$$\langle 2\xi u'(s), u'(s) \times h \rangle_H = 0,$$

we have

$$\langle u'(t), \xi u'(t) \rangle_H - \langle u_0, \xi u_0 \rangle_H = 0, \quad \mathbb{P}' - a.s.$$

Since ϕ is arbitrary and $|u_0(x)| = 1$ for almost every $x \in D$, we infer that $|u'(t,x)| = 1$ for almost every $x \in D$ as well. This completes the proof of Theorem 4.40. □

By Theorem 4.40, we can show that:

Theorem 4.42. *The process u' from Proposition 4.18 satisfies: for every $t \in [0,T]$,*

$$u'(t) = u_0 + \lambda_1 \int_0^t (u' \times [\Delta u' - \phi'(u')])(s)\,\mathrm{d}s - \lambda_2 \int_0^t u'(s) \times (u' \times [\Delta u' - \phi'(u')])(s)\,\mathrm{d}s + \int_0^t (u'(s) \times h) \circ \mathrm{d}W'(s) \tag{4.78}$$

in $L^2(\Omega'; H)$.

Proof. By (4.55) and Lemma 4.29,

$$\mathbb{E}'\left(\int_0^T \left|(u' \times [\Delta u' - \phi'(u')])(t)\right|_H^2 \mathrm{d}t\right)^r < \infty, \quad r \geq 1. \tag{4.79}$$

And then by (4.77), we see that

$$|u'(t,\omega) \times ((u' \times [\Delta u' - \phi'(u')])(t,\omega))|_H \leq \left|(u' \times [\Delta u' - \phi'(u')])(t,\omega)\right|_H \tag{4.80}$$

for almost every $(t,\omega) \in [0,T] \times \Omega'$. And so

$$\mathbb{E}' \int_0^T \left|u'(t) \times (u' \times [\Delta u' + \phi'(u')])(t)\right|_H^2 \mathrm{d}t < \infty.$$

Therefore all the terms in the equation (4.78) are in the space $L^2(\Omega'; H)$. This completes the proof the Theorem 4.42. □

Theorem 4.43. *The process u' defined in Proposition 4.18 satisfies: for every $\alpha \in (0, \frac{1}{2})$,*

$$u' \in C^\alpha([0,T]; H), \qquad \mathbb{P}' - a.s.. \tag{4.81}$$

We need the following Lemma to prove Theorem 4.43.

Lemma 4.44 (Kolmogorov test)**.** [18] *Let $\{u(t)\}_{t\in[0,T]}$ be a stochastic process with values in a separable Banach space X, such that for some $C > 0$, $\varepsilon > 0$, $\delta > 1$ and all $t, s \in [0,T]$,*

$$\mathbb{E}\left|u(t) - u(s)\right|_X^\delta \leq C|t-s|^{1+\varepsilon}.$$

Then there exists a version of u with $\mathbb{P}$ almost surely trajectories being Hölder continuous functions with an arbitrary exponent smaller than $\frac{\varepsilon}{\delta}$.

Proof of Theorem 4.43. By (4.76), we have

$$
\begin{aligned}
&u'(t)-u'(s)\\
&=\lambda_1\int_s^t (u'\times[\Delta u'-\phi'(u')])(\tau)\,\mathrm{d}\tau-\lambda_2\int_s^t u'(\tau)\times(u'\times[\Delta u'-\phi'(u')])(\tau)\,\mathrm{d}\tau\\
&\quad+\int_s^t (u'(\tau)\times h\circ\,\mathrm{d}W'(\tau)\\
&=\lambda_1\int_s^t (u'\times[\Delta u'-\phi'(u')])(\tau)\,\mathrm{d}\tau-\lambda_2\int_s^t u'(\tau)\times(u'\times[\Delta u'-\phi'(u')])(\tau)\,\mathrm{d}\tau\\
&\quad+\frac12\int_s^t (u'(\tau)\times h)\times h\,\mathrm{d}\tau+\int_s^t u'(\tau)\times h\,\mathrm{d}W'(\tau),\quad 0\le s<t\le T.
\end{aligned}
$$

Hence by Jensen's inequality, for $q>1$,

$$
\begin{aligned}
&\mathbb{E}'\left[\left|u'(t)-u'(s)\right|_H^{2q}\right]\\
&\le\mathbb{E}'\Bigg(|\lambda_1|\int_s^t\left|(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H\,\mathrm{d}\tau+|\lambda_2|\int_s^t\left|u'(\tau)\times(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H\,\mathrm{d}\tau\\
&\quad+\frac12\int_s^t\left|u'(\tau)\times h\times h\right|_H\mathrm{d}\tau+\left|\int_s^t u'(\tau)\times h\,\mathrm{d}W'(\tau)\right|_H\Bigg)^{2q}\\
&\le 4^q\mathbb{E}'\Bigg(|\lambda_1|^{2q}\left(\int_s^t\left|(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H\mathrm{d}\tau\right)^{2q}\\
&\quad+|\lambda_2|^{2q}\left(\int_s^t\left|u'(\tau)\times(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H\mathrm{d}\tau\right)^{2q}\\
&\quad+\frac{1}{4^q}\left(\int_s^t\left|u'(\tau)\times h\times h\right|_H\mathrm{d}\tau\right)^{2q}+\left|\int_s^t u'(\tau)\times h\,\mathrm{d}W'(\tau)\right|_H^{2q}\Bigg).
\end{aligned}
$$

By (4.79), there exists $C^1>0$, such that

$$
\begin{aligned}
\mathbb{E}'\left(\int_s^t\left|(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H\mathrm{d}\tau\right)^{2q}&\le(t-s)^q\mathbb{E}'\left(\int_s^t\left|(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H^2\mathrm{d}\tau\right)^{q}\\
&\le C_1^q(t-s)^q.
\end{aligned}
$$

By (4.80)

$$
\begin{aligned}
&\mathbb{E}'\left(\int_s^t\left|u'(\tau)\times(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H\mathrm{d}\tau\right)^{2q}\le\mathbb{E}'\left(\int_s^t\left|(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H\mathrm{d}\tau\right)^{2q}\\
&\le(t-s)^q\mathbb{E}'\left(\int_s^t\left|(u'\times[\Delta u'-\phi'(u')])(\tau)\right|_H^2\mathrm{d}\tau\right)^{q}\le C_1^q(t-s)^q.
\end{aligned}
$$

And by (4.49),

$$
\mathbb{E}'\left(\int_s^t\left|u'(\tau)\times h\times h\right|_H\mathrm{d}\tau\right)^{2q}\le(t-s)^q\mathbb{E}'\left(\int_s^t\left|u'(\tau)\times h\times h\right|_H^2\mathrm{d}\tau\right)^{q}\le|u_0|_H^{2q}T|h|_{\mathbb{L}^\infty}^{4q}(t-s)^q.
$$

By the Burkholder-Davis-Gundy Inequality (Lemma 2.127),

$$\mathbb{E}'\left|\int_s^t u'(\tau)\times h\,\mathrm{d}W'(\tau)\right|_H^{2q} \le K_q\mathbb{E}'\left(\int_s^t \left|u'(\tau)\times h\right|_H^2 \mathrm{d}\tau\right)^q \le K_q|u_0|_H^{2q}|h|_{\mathbb{L}^\infty}^{2q}(t-s)^q.$$

Therefore, let $C = 2C_1^q + |u_0|_H^{2q}T|h|_{\mathbb{L}^\infty}^{4q} + K_q|u_0|_H^{2q}|h|_{\mathbb{L}^\infty}^{2q}$, we have

$$\mathbb{E}'\left[\left|u'(t)-u'(s)\right|_H^{2q}\right] \le C(t-s)^q, \quad q \ge 1.$$

Then by Lemma 4.44,

$$u \in C^\alpha([0,T];H), \quad \alpha \in \left(0, \frac{1}{2}\right).$$

This completes the proof of Theorem 4.43. □

4.8. **Main theorem.** Summarizing, we state all the results of this section of the thesis in one Theorem:

Theorem 4.45. *There exists a probability space* $(\Omega', \mathcal{F}', \mathbb{P}')$ *and there exists a process* u' *in the probability space such that:*

(i) u' *is a weak solution of* (4.2)

(ii) *For every* $t \in [0,T]$*,*

$$\begin{aligned} u'(t) = {} & u_0 + \lambda_1 \int_0^t \left(u' \times [\Delta u' - \phi'(u')]\right)(s)\,\mathrm{d}s \\ & - \lambda_2 \int_0^t u'(s) \times \left(u' \times [\Delta u' - \phi'(u')]\right)(s)\,\mathrm{d}s \\ & + \int_0^t (u'(s)\times h) \circ \mathrm{d}W'(s) \end{aligned}$$

in $L^2(\Omega';H)$*. And this implies that* u' *is a weak solution of the equation* (4.2)*;*

(iii)

$$|u'(t,x)|_{\mathbb{R}^3} = 1, \quad \textit{for Lebesgue a.e. } (t,x) \in [0,T]\times D \textit{ and } \mathbb{P}' - a.s..$$

(iv) *For every* $\alpha \in (0, \frac{1}{2})$*,*

$$u' \in C^\alpha([0,T];H), \qquad \mathbb{P}' - a.s..$$

Proof. The three results in Theorem 4.45 are from Theorem 4.42, Theorem 4.40 and Theorem 4.43. □

5. Full Stochastic Landau-Lifshitz' Equation Coupled with Max-Well Equations

A stochastic Landau-Lifthitz equation with full energy coupled by the Maxwell's equations will be studied in this section.

5.1. **Statement of the problem.**

Definition 5.1. Let $D \subset \mathbb{R}^3$ be an open and bounded domain with C^2 boundary.

(i) Suppose that $\phi \in C_0^2(\mathbb{R}^3; \mathbb{R}^+)$. For a magnetization field $M \in \mathbb{H}^1(D)$, we define the anisotropy energy of M by:

$$\mathcal{E}_{an}(M) := \int_D \phi(M(x))\, \mathrm{d}x.$$

(ii) We define the exchange energy of M by:

$$\mathcal{E}_{ex}(M) := \frac{1}{2}\int_D |\nabla M(x)|^2\, \mathrm{d}x = \frac{1}{2}\|\nabla M\|^2_{\mathbb{L}^2(D)}. \tag{5.1}$$

(iii) For a magnetic field $H \in \mathbb{L}^2(\mathbb{R}^3)$, we define the energy due to the magnetic field H by:

$$\mathcal{E}_{fi}(H) := \frac{1}{2}\int_{\mathbb{R}^3} |H(x)|^2\, \mathrm{d}x = \frac{1}{2}\|H\|^2_{\mathbb{L}^2(\mathbb{R}^3)}. \tag{5.2}$$

Definition 5.2. Given a magnetization field $M : D \longrightarrow \mathbb{R}^3$ and a magnetic field $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, we define a vector field $B : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$B := H + \tilde{M}, \tag{5.3}$$

where

$$\tilde{M}(x) := \begin{cases} M(x), & x \in D; \\ 0, & x \notin D. \end{cases}$$

Definition 5.3.(iv) We define the total magnetic energy as:

$$\begin{aligned} \mathcal{E}_{mag}(M, B) &:= \mathcal{E}_{an}(M) + \mathcal{E}_{ex}(M) + \mathcal{E}_{fi}(B - \tilde{M}) \\ &= \int_D \phi(M(x))\, \mathrm{d}x + \frac{1}{2}\|\nabla M\|^2_{\mathbb{L}^2(D)} + \frac{1}{2}\|B - \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)}. \end{aligned}$$

(v) Finally, for an electric field $E \in \mathbb{L}^2(\mathbb{R}^3)$, a magnetization field $M \in \mathbb{H}^1(D)$ and a magnetic field $H \in \mathbb{L}^2(\mathbb{R}^3)$, so the vector field $B \in \mathbb{L}^2(\mathbb{R}^3)$, we define the total electro-magnetic energy by

$$\begin{aligned} \mathcal{E}_{el.mag}(M, B, E) &:= \mathcal{E}_{mag}(M, B) + \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\ &= \int_D \phi(M(x))\, \mathrm{d}x + \frac{1}{2}\|\nabla M\|^2_{\mathbb{L}^2(D)} + \frac{1}{2}\|B - \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)}. \end{aligned} \tag{5.4}$$

Notation 5.4. For simplicity, we denote $V := \mathbb{H}^1(D)$, $H := \mathbb{L}^2(D)$, the dual space of V by V' and the dual space of H by H', so $V \hookrightarrow H \simeq H' \hookrightarrow V'$. We also denote $\mathcal{E} := \mathcal{E}_{el.mag}$, $\phi' := \nabla\phi$, $Q := [0, T] \times D$, $Q_\infty := [0, T] \times \mathbb{R}^3$.

The total energy $\mathcal{E}$ is the generalization of the exchange energy which has the density: $\Delta u - \nabla\phi(u)$ used in Section 4. We begin with investigation of some properties of $\mathcal{E}$.

Proposition 5.5. *For $M \in V$, if we define $\Delta M \in V'$ by*

$$ {}_{V'}\langle \Delta M, u\rangle_V := -\langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})}, \qquad \forall u \in V. \tag{5.5}$$

Then the total energy $\mathcal{E} : V \times \mathbb{L}^2(\mathbb{R}^3) \times \mathbb{L}^2(\mathbb{R}^3) \longrightarrow \mathbb{R}$ defined in (5.4) *has partial derivative with respect to M which satisfies*

$$\frac{\partial \mathcal{E}}{\partial M}(M,B,E) = \phi'(M) - (1_D B - M) - \Delta M, \qquad \text{in } V'. \tag{5.6}$$

Proof. For $M, u \in V$, $B, E \in \mathbb{L}^2(\mathbb{R}^3)$.

$$\begin{aligned}
&\mathcal{E}(M+u,B,E) - \mathcal{E}(M,B,E)\\
&= \int_D \phi(M(x)+u(x)) - \phi(M(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M + \nabla u\|_H^2 - \frac{1}{2}\|\nabla M\|_H^2\\
&\quad + \frac{1}{2}\|B - \tilde{M} - \tilde{u}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|B - \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)},
\end{aligned}$$

where

$$\int_D \phi(M(x)+u(x)) - \phi(M(x))\,\mathrm{d}x = \int_D \phi'(M(x))(u(x)) + \frac{1}{2}\phi''(M(x)+\theta(x)u(x))(u(x),u(x))\,\mathrm{d}x,$$

$\theta(x) \in [0,1]$ for $x \in D$. We assumed that ϕ'' is bounded, so there exists some constant $C > 0$ such that

$$\int_D \left|\frac{1}{2}\phi''(M(x)+\theta(x)u(x))(u(x),u(x))\right| \mathrm{d}x \le C\int_D |u(x)|^2\,\mathrm{d}x = C\|u\|_H^2 = o(\|u\|_V).$$

Hence

$$\begin{aligned}
&\mathcal{E}(M+u,B,E) - \mathcal{E}(M,B,E)\\
&= \int_D \langle \phi'(M(x)), u(x)\rangle\,\mathrm{d}x + o(\|u\|_V) + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + \frac{1}{2}\|\nabla u\|_H^2\\
&\quad - \langle 1_D B - M, u\rangle_H + \frac{1}{2}\|u\|_H^2\\
&= \langle \phi'(M) - (1_D B - M), u\rangle_H + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + o(\|u\|_V)
\end{aligned}$$

This implies that $\frac{\partial\mathcal{E}}{\partial M}(M,B,E)$ exists.

Hence as an element in V',

$${}_{V'}\left\langle \frac{\partial \mathcal{E}}{\partial M}(M,B,E), u\right\rangle_V = \langle \phi'(M) - (1_D B - M), u\rangle_H + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})}.$$

We have defined $\Delta M \in V'$ by

$${}_{V'}\langle \Delta M, u\rangle_V := -\langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})}, \qquad \forall u \in V.$$

So

$$\frac{\partial \mathcal{E}}{\partial M}(M,B,E) = \phi'(M) - (1_D B - M) - \Delta M, \qquad \text{in } V'.$$

□

Notation 5.6. We will denote

$$\rho := \phi'(M) - (1_D B - M) - \Delta M, \qquad \text{in } V'. \tag{5.7}$$

Proposition 5.7. *For $u, v \in V$,*

$$\frac{\partial^2 \mathcal{E}}{\partial M^2}(M,B,E)(u,v) = \int_D \phi''(M(x))(u(x),v(x))\,\mathrm{d}x + \langle u,v\rangle_V. \tag{5.8}$$

Proof. By Proposition 5.5, we have

$$\frac{\partial \mathcal{E}}{\partial M}(M+u,B,E)(v) - \frac{\partial \mathcal{E}}{\partial M}(M,B,E)(v) = \langle \phi'(M+u) - \phi'(M), v\rangle_H + \langle u,v\rangle_V.$$

And by

$$\begin{aligned} &\langle \phi'(M+u) - \phi'(M), v\rangle_H \\ =& \int_D [\phi'(M(x)+u(x)) - \phi'(M(x))](v(x))\,\mathrm{d}x \\ =& \int_D \phi''(M(x))(u(x),v(x))\,\mathrm{d}x + o(\|u\|_V), \end{aligned}$$

The proof is complete. □

Proposition 5.8. *For the total energy $\mathcal{E} : V \times \mathbb{L}^2(\mathbb{R}^3) \times \mathbb{L}^2(\mathbb{R}^3) \longrightarrow \mathbb{R}$ defined in* (5.4)*, we have:*

(i)

$$\frac{\partial \mathcal{E}}{\partial B}(M,B,E) = B - \tilde{M}, \qquad \text{in } \mathbb{L}^2(\mathbb{R}^3). \tag{5.9}$$

(ii)

$$\frac{\partial \mathcal{E}}{\partial E}(M,B,E) = E, \qquad \text{in } \mathbb{L}^2(\mathbb{R}^3). \tag{5.10}$$

Proof.(i) For $v \in \mathbb{L}^2(\mathbb{R}^3)$,

$$\begin{aligned} &\mathcal{E}(M,B+v,E) - \mathcal{E}(M,B,E) \\ =& \frac{1}{2}\|B+v-\tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|B-\tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\ =& \left\langle B-\tilde{M}, v\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|v\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \end{aligned}$$

Hence

$$\frac{\partial \mathcal{E}}{\partial B}(M,B,E) = B - \tilde{M}, \qquad \text{in } \mathbb{L}^2(\mathbb{R}^3).$$

(ii)

$$\begin{aligned} &\mathcal{E}(M,B,E) - \mathcal{E}(M,B,E+v) \\ =& \frac{1}{2}\|E+v\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\ =& \langle E, v\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|v\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \end{aligned}$$

Hence

$$\frac{\partial \mathcal{E}}{\partial E}(M,B,E) = E, \qquad \text{in } \mathbb{L}^2(\mathbb{R}^3).$$

□

After having finished with studying the basic properties of the total energy $\mathcal{E}$, we are ready to state the main problem with assumptions which we are going to study in this section. But before that we need one more notation:

Notation 5.9. We also define

$$Y := \{u \in \mathbb{L}^2(\mathbb{R}^3) : \nabla \times u \in \mathbb{L}^2(\mathbb{R}^3)\},$$

with the inner product

$$(u,v)_Y := (u,v)_{\mathbb{L}^2(\mathbb{R}^3)} + (\nabla \times u, \nabla \times v)_{\mathbb{L}^2(\mathbb{R}^3)}.$$

Problem 5.10. Let D be an open bounded domain in $\mathbb{R}^3$ with C^2 boundary. Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, $(W_j)_{j=1}^{\infty}$ be pairwise independent, real valued, $\mathbb{F}$-adapted Wiener processes. Given $T \geq 0$ (we are interested in global solutions for all time $t \geq 0$, but we fix $T > 0$ for simplicity) and

$$M_0 \in \mathbb{L}^\infty(D);$$
$$B_0 \in \mathbb{L}^2(\mathbb{R}^3); \qquad \nabla \cdot B_0 = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3;\mathbb{R});$$
$$E_0 \in \mathbb{L}^2(\mathbb{R}^3);$$

$$h_j \in \mathbb{L}^\infty(D) \cap \mathbb{W}^{1,3}(D), \quad \text{for } j = 1, \dots, \infty, \text{ such that } c_h := \sum_{j=1}^{\infty} \|h_j\|_{\mathbb{L}^\infty(D)\cap\mathbb{W}^{1,3}(D)} < \infty,$$

$$f \in L^2(0,T;\mathbb{L}^2(D));$$
$$\phi \in C_0^2(\mathbb{R}^3;\mathbb{R}^+);$$
$$\lambda_1 \in \mathbb{R}, \quad \lambda_2 > 0, \quad \alpha, \beta \in \mathbb{R}.$$

Find $\mathbb{F}$-progressively measurable processes $M : [0,T]\times\Omega \longrightarrow V$, $B : [0,T]\times\Omega \longrightarrow \mathbb{L}^2(\mathbb{R}^3)$, $E : [0,T]\times\Omega \longrightarrow \mathbb{L}^2(\mathbb{R}^3)$ such that the following system is satisfied: for $t \in [0,T]$,

$$\begin{aligned} M(t) &= M_0 + \int_0^t [\lambda_1 M \times \rho - \lambda_2 M \times (M \times \rho)]\, \mathrm{d}s \\ &\quad + \sum_{j=1}^{\infty} \left\{ \int_0^t \left[\alpha M \times h_j + \beta M \times (M \times h_j) \right] \circ \mathrm{d}W_j(s) \right\}, \end{aligned} \tag{5.11}$$

in $L^2(\Omega; V')$, where V' is the dual space of V.

$$B(t) = B_0 - \int_0^t \nabla \times E(s)\, \mathrm{d}s, \quad \in Y', \ \mathbb{P}-a.s.. \tag{5.12}$$

$$E(t) = E_0 + \int_0^t \nabla\times[B(s) - \tilde{M}(s)]\, \mathrm{d}s - \int_0^t [1_D E(s) + \tilde{f}(s)]\, \mathrm{d}s, \quad \in Y', \ \mathbb{P}-a.s.. \tag{5.13}$$

Remark 5.11. The "$\circ\, \mathrm{d}W_j(s)$" in the equation (5.14) denotes the Stratonovich differential, in our case it relates to the Itô differential by the following formula:

$$\begin{aligned}
&\Big[\alpha M \times h_j + \beta M \times (M \times h_j)\Big] \circ\ \mathrm{d}W_j(s)\\
=\ & \frac{1}{2}\Big[\alpha^2 \Big[(M \times h_j) \times h_j\Big] + \alpha\beta \Big[[M \times (M \times h_j)] \times h_j\Big]\\
&+\beta^2 \Big[M \times [M \times (M \times h_j) \times h_j]\Big] + \alpha\beta \Big[M \times [(M \times h_j) \times h_j]\Big]\\
&+\beta^2 \Big[M \times (M \times h_j) \times (M \times h_j)\Big]\Big] \mathrm{d}s + \Big[\alpha \Big[M \times h_j\Big] + \beta \Big[M \times (M \times h_j)\Big]\Big]\ \mathrm{d}W_j(s).
\end{aligned}$$

Remark 5.12. (5.12) infers that $\nabla \cdot B(t) = 0$.

Remark 5.13. The Problem 5.10 is a generalised version of equation 4.2. Some methods of dealing with it are similar to the methods used in Section 4. In particular, we will use the Galerkin approximation and get some a'priori estimates.

Definition 5.14 (Solution of Problem 5.10)**.** A weak solution of Problem 5.10 is system consisting of a filtered probability space $(\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')$, an ∞-dimensional $\mathbb{F}'$-Wiener process $W' = (W_j')_{j=1}^{\infty}$ and an $\mathbb{F}'$-progressively measurable process

$$M' = (M_i')_{i=1}^{3} : \Omega' \times [0, T] \longrightarrow V \cap \mathbb{L}^{\infty}(D)$$

such that for all the $u \in C_0^{\infty}(D; \mathbb{R}^3)$, $t \in [0, T]$, we have, $\mathbb{P}'$-a.s.,

$$\begin{aligned}
(5.14)\qquad & \int_D \langle M'(t) - M_0, u\rangle\ \mathrm{d}x\\
&= \int_0^t \int_D \Big\{ \langle B' - M' - \phi'(M'), \lambda_1 u \times M' - \lambda_2 (u \times M') \times M'\rangle\\
&\quad - \sum_{i=1}^{3} \Big\langle \frac{\partial M'}{\partial x_i}, \lambda_1 \frac{\partial u}{\partial x_i} \times M' - \lambda_2 \Big(\frac{\partial u}{\partial x_i} \times M' + u \times \frac{\partial M'}{\partial x_i}\Big) \times M' \Big\rangle \Big\} \mathrm{d}x\, \mathrm{d}s\\
&\quad + \sum_{j=1}^{\infty} \int_0^t \Big\langle \alpha M \times h_j + \beta M \times (M \times h_j), u \Big\rangle \circ\ \mathrm{d}W_j(s);
\end{aligned}$$

$$(5.15)\qquad \int_{\mathbb{R}^3} \langle B'(t) - B_0, u\rangle\ \mathrm{d}x = -\int_0^t \int_{\mathbb{R}^3} \langle E', \nabla \times u\rangle\ \mathrm{d}x\, \mathrm{d}s;$$

(5.16)

$$\int_{\mathbb{R}^3} \langle E'(t) - E_0, u\rangle\ \mathrm{d}x = \int_0^t \int_{\mathbb{R}^3} \Big\langle B' - \tilde{M}, \nabla \times u \Big\rangle\ \mathrm{d}x\, \mathrm{d}s - \int_0^t \int_D \langle E' + f, u\rangle\, \mathrm{d}x\, \mathrm{d}s.$$

5.2. **Galerkin Approximation.** Let $A := -\Delta$ be a linear operator as defined in Definition 2.164. As in Lemma 2.171, we can define $H_n := \mathrm{linspan}\{e_1, \dots, e_n\}$, where $\{e_n\}_{n=1}^{\infty}$ are eigenvectors of A. Since $\mathbb{L}^2(\mathbb{R}^3)$ is a separable Hilbert space, we can find $\{y_n\}_{n=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^3; \mathbb{R}^3)$ such that $\{y_n\}$ is an orthogonal basis of $\mathbb{L}^2(\mathbb{R}^3)$. We define $Y_n := \mathrm{linspan}\{y_1, \dots, y_n\}$ and the orthogonal projections

$$\pi_n : H \longrightarrow H_n,$$

$$\pi_n^Y : \mathbb{L}^2(\mathbb{R}^3) \longrightarrow Y_n.$$

Remark 5.15. The processes B and E will take values in $\mathbb{L}^2(\mathbb{R}^3)$, so $\nabla \times B \in Y'$ and $\nabla \times E \in Y'$.

$$\pi_n^Y|_Y : Y \longrightarrow Y_n.$$

On H_n and Y_n we consider the scalar product inheritted from H and Y. Let us denote by $\mathcal{E}_n$ the restriction of the total energy function $\mathcal{E}$ to the finite dimensional space $H_n \times Y_n \times Y_n$, i.e.

$$\mathcal{E}_n : H_n \times Y_n \times Y_n \longrightarrow \mathbb{R},$$

$$\mathcal{E}_n(M, B, E) = \int_D \phi(M(x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M\|^2_{\mathbb{L}^2(D)} + \frac{1}{2}\|B - \pi_n^Y \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)}.$$

Proposition 5.16. *The function $\mathcal{E}_n$ is of class C^2 and for $M \in H_n$, $B, E \in Y_n$ we have:*

(i)

$$(\nabla_M \mathcal{E}_n)(M, B, E) = \pi_n[\phi'(M) - (1_D B - \pi_n^Y \tilde{M})] - \Delta M, \qquad \in H_n. \tag{5.17}$$

(ii)

$$(\nabla_B \mathcal{E}_n)(M, B, E) = B - \pi_n^Y \tilde{M}, \qquad \textit{in } Y_n. \tag{5.18}$$

(iii)

$$(\nabla_E \mathcal{E}_n)(M, B, E) = E, \qquad \textit{in } Y_n. \tag{5.19}$$

(iv)

$$\frac{\partial^2 \mathcal{E}_n}{\partial M^2}(M, B, E)(u, v) = \int_D \phi''(M(x))(u(x), v(x))\,\mathrm{d}x + \langle u, v\rangle_V. \tag{5.20}$$

Proof.(i) For $M, u \in H_n$, $B, E \in Y_n$. H_n is a finite dimensional space, so $\|\cdot\|_H \simeq \|\cdot\|_V$ in H_n, so

$$\begin{aligned}
&\mathcal{E}_n(M + u, B, E) - \mathcal{E}_n(M, B, E)\\
&= \int_D \phi(M + u) - \phi(M)\,\mathrm{d}x + \frac{1}{2}\|\nabla M + \nabla u\|^2_H - \frac{1}{2}\|\nabla M\|^2_H\\
&\quad + \frac{1}{2}\|B - \tilde{M} - \tilde{u}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|B - \tilde{M}\|^2_H\\
&= \int_D \langle \phi'(M), u\rangle\,\mathrm{d}x + o(\|u\|_H) + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + \frac{1}{2}\|\nabla u\|^2_H\\
&\quad - \langle 1_D B - M, u\rangle_H + \frac{1}{2}\|u\|^2_H\\
&= \langle \phi'(M) - (1_D B - \pi_n^Y M), u\rangle_H + \langle \nabla M, \nabla u\rangle_{L^2(D;\mathbb{R}^{3\times 3})} + o(\|u\|_H)\\
&= \langle \pi_n[\phi'(M) - (1_D B - \pi_n^Y M)] - \Delta M, u\rangle_H + o(\|u\|_H).
\end{aligned}$$

Hence by the definition of the gradient,

$$(\nabla_M \mathcal{E}_n)(M, B, E) = \pi_n[\phi'(M) - (1_D B - \pi_n^Y M)] - \Delta M, \qquad \in H_n.$$

(ii) For $v \in Y_n$,

$$\begin{aligned}
&\mathcal{E}_n(M, B+v, E) - \mathcal{E}_n(M, B, E) \\
&= \frac{1}{2}\|B + v - \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|B - \tilde{M}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \left\langle B - \tilde{M}, v \right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|v\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \left\langle B - \pi_n^Y \tilde{M}, v \right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + o(\|v\|_{\mathbb{L}^2(\mathbb{R}^3)}).
\end{aligned}$$

So

$$(\nabla_B \mathcal{E}_n)(M, B, E) = B - \pi_n^Y \tilde{M}, \qquad \text{in } Y_n.$$

(iii)

$$\begin{aligned}
&\mathcal{E}_n(M, B, E+v) - \mathcal{E}_n(M, B, E) \\
&= \frac{1}{2}\|E + v\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - \frac{1}{2}\|E\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \langle E, v \rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|v\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\
&= \langle E, v \rangle_{\mathbb{L}^2(\mathbb{R}^3)} + o(\|v\|_{\mathbb{L}^2(\mathbb{R}^3)}).
\end{aligned}$$

So

$$(\nabla_E \mathcal{E}_n)(M, B, E) = E, \qquad \text{in } Y_n.$$

□

Notation 5.17. There exists a function $\psi : \mathbb{R}^3 \longrightarrow [0, 1]$ such that:
(i) $\psi \in C^1(\mathbb{R}^3)$.
(ii)

$$\psi(x) = \begin{cases} 1, & |x| \le 3, \\ 0, & |x| \ge 5, \end{cases}$$

(iii) $|\nabla \psi| \le 1$.

Remark 5.18. The ψ defined here is to make sure we can get the estimates in Proposition 5.23 below. By Theorem 5.70, we will prove that $|M(t, x)| = 1$ for almost every $x \in D$, therefore we can remove this ψ at the end.

Let us define the function $\rho_n : H_n \times Y_n \times Y_n \longrightarrow H_n$ which corresponds to ρ by:

(5.21) $\rho_n := -(\nabla_M \mathcal{E}_n)(M_n, B_n, E_n) = \pi_n[-\phi'(M_n) + 1_D(B_n - \pi_n^Y \tilde{M}_n)] + \Delta M_n \in H_n.$

To solve Problem 5.10, we first consider the following problem with values in finite dimensional space:

Problem 5.19. Let D be an open bounded domain in $\mathbb{R}^3$ with C^2 boundary. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0})$ be a filtered probability space, $(W_j)_{j=1}^\infty$ are pairwise independent, real valued, $(\mathcal{F}_t)$ adapted Wiener processes. Given

$$M_0 \in \mathbb{L}^\infty(D);$$

$$B_0 \in \mathbb{L}^2(\mathbb{R}^3); \qquad \nabla \cdot B_0 = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3; \mathbb{R});$$

$$E_0 \in \mathbb{L}^2(\mathbb{R}^3);$$

$$h_j \in \mathbb{L}^\infty(D) \cap \mathbb{W}^{1,3}(D), \quad \text{for } j = 1, \dots, \infty, \text{ and } c_h := \sum_{j=1}^{\infty} \|h_j\|_{\mathbb{L}^\infty(D)\cap\mathbb{W}^{1,3}(D)} < \infty,$$

$$f \in L^2(0, T; \mathbb{L}^2(D));$$

$$\phi \in C_0^2(\mathbb{R}^3; \mathbb{R}^+);$$

$$\lambda_1 \in \mathbb{R}, \quad \lambda_2 > 0, \quad \alpha, \beta \in \mathbb{R}.$$

For $T \in [0, \infty]$, find $M_n : [0, T] \times \Omega \longrightarrow H_n$, $B_n : [0, T] \times \Omega \longrightarrow Y_n$, $E_n : [0, T] \times \Omega \longrightarrow Y_n$ such that the following system has been satisfied: For $t \in [0, T]$, $\mathbb{P}$ almost surely on Ω,

$$\begin{aligned}
\mathrm{d}M_n = & \Big\{\pi_n [\lambda_1 M_n \times \rho_n] - \lambda_2 \pi_n [M_n \times (M_n \times \rho_n)]\ \mathrm{d}t \\
& + \sum_{j=1}^{\infty} \Big\{\Big[\alpha\pi_n \Big[M_n \times h_j\Big] + \beta\pi_n \Big[\psi(M_n) M_n \times (M_n \times h_j)\Big]\Big] \circ \mathrm{d}W_j\Big\} \\
= & \Big\{\pi_n [\lambda_1 M_n \times \rho_n] - \lambda_2 \pi_n [M_n \times (M_n \times \rho_n)] \\
& + \frac{1}{2} \sum_{j=1}^{\infty} \Big[\alpha^2 \pi_n \Big[(M_n \times h_j) \times h_j\Big] \\
& + \alpha\beta\psi(M_n)\pi_n \Big[[M_n \times (M_n \times h_j)] \times h_j\Big] \\
& + \beta^2 \psi(M_n)\pi_n \Big[M_n \times [M_n \times (M_n \times h_j) \times h_j]\Big] \\
& + \alpha\beta\psi(M_n)\pi_n \Big[M_n \times [(M_n \times h_j) \times h_j]\Big] \\
& + \beta^2 \psi(M_n)\pi_n \Big[M_n \times (M_n \times h_j) \times (M_n \times h_j)\Big]\Big]\Big\}\ \mathrm{d}t \\
& + \sum_{j=1}^{\infty} \Big\{\Big[\alpha\pi_n \Big[M_n \times h_j\Big] + \beta\pi_n \Big[\psi(M_n) M_n \times (M_n \times h_j)\Big]\Big]\ \mathrm{d}W_j\Big\}.
\end{aligned} \tag{5.22}$$

$$\mathrm{d}E_n(t, \omega) = -\pi_n^Y [1_D (E_n(t, \omega) + f)]\,\mathrm{d}t + \pi_n^Y [\nabla \times (B_n(t, \omega) - \pi_n^Y (\tilde{M}_n(t, \omega))]\,\mathrm{d}t, \tag{5.23}$$

$$\mathrm{d}B_n(t, \omega) = -\pi_n^Y [\nabla \times E_n(t, \omega)]\,\mathrm{d}t, \tag{5.24}$$

$$M_n(0) = \pi_n M_0, \tag{5.25}$$

$$E_n(0) = \pi_n^Y E_0, \tag{5.26}$$

$$B_n(0) = \pi_n^Y B_0, \tag{5.27}$$

Remark 5.20. The Equations (5.22), (5.23) and (5.24) should be understood in the integral form.

Notation 5.21. Let us define vector fields $F_n, G_{nj} : H_n \longrightarrow H_n$, $j = 1, 2, \ldots$, by

$$\begin{aligned}F_n := \pi_n\left[\lambda_1 M_n \times \rho_n\right] - \lambda_2 \pi_n\left[M_n \times (M_n \times \rho_n)\right] \\ +\frac{1}{2}\sum_{j=1}^{\infty}\Big[\alpha^2\pi_n\left[(M_n \times h_j) \times h_j\right] \\ +\alpha\beta\psi(M_n)\pi_n\left[\left[M_n \times (M_n \times h_j)\right] \times h_j\right] \\ +\beta^2\psi(M_n)\pi_n\left[M_n \times \left[M_n \times (M_n \times h_j) \times h_j\right]\right] \\ +\alpha\beta\psi(M_n)\pi_n\left[M_n \times \left[(M_n \times h_j) \times h_j\right]\right] \\ +\beta^2\psi(M_n)\pi_n\left[M_n \times (M_n \times h_j) \times (M_n \times h_j)\right]\Big],\end{aligned} \tag{5.28}$$

$$G_{n,j} := \alpha\pi_n\left[M_n \times h_j\right] + \beta\pi_n\left[\psi(M_n)M_n \times (M_n \times h_j)\right]. \tag{5.29}$$

Then the equation (5.22) becomes

$$\mathrm{d}M_n(t) = F_n\,\mathrm{d}t + \sum_{j=1}^{\infty} G_{n,j}\,\mathrm{d}W_j(t), \tag{5.30}$$

which as always has to be understood in the integral form.

Hence the system (5.22), (5.23) and (5.24) becomes:

$$\begin{aligned}\mathrm{d}\begin{pmatrix} M_n(t) \\ E_n(t) \\ B_n(t)\end{pmatrix} = \sum_{j=1}^{\infty}\begin{pmatrix} G_{nj} \\ 0 \\ 0\end{pmatrix}\mathrm{d}W_j(t) \\ +\begin{pmatrix} F_n \\ -\pi_n^Y[1_D(E_n(t) + f(t))]\,\mathrm{d}t + \pi_n^Y[\nabla \times (B_n(t) - \pi_n^Y(\tilde{M}_n(t)))] \\ -\pi_n^Y[\nabla \times E_n(t)]\end{pmatrix}\mathrm{d}t.\end{aligned} \tag{5.31}$$

Finally we define the following vector fields:

$$\begin{aligned}\hat{F}_n : H_n \times Y_n \times Y_n \longrightarrow H_n \times Y_n \times Y_n \\ \begin{pmatrix} M_n \\ E_n \\ B_n\end{pmatrix} \longmapsto \begin{pmatrix} F_n \\ -\pi_n^Y[1_D(E_n + f)]\,\mathrm{d}t + \pi_n^Y[\nabla \times (B_n - \pi_n^Y(\tilde{M}_n))] \\ -\pi_n^Y[\nabla \times E_n]\end{pmatrix},\end{aligned} \tag{5.32}$$

$$\begin{aligned}\hat{G}_{nj} : H_n \times Y_n \times Y_n \longrightarrow H_n \times Y_n \times Y_n \\ \begin{pmatrix} M_n \\ E_n \\ B_n\end{pmatrix} \longmapsto \begin{pmatrix} G_{nj} \\ 0 \\ 0\end{pmatrix}.\end{aligned} \tag{5.33}$$

Use (5.32) and (5.33), the system (5.22), (5.23) and (5.24) becomes:

$$\mathrm{d}\begin{pmatrix} M_n(t) \\ E_n(t) \\ B_n(t) \end{pmatrix} = \hat{F}_n \begin{pmatrix} M_n(t) \\ E_n(t) \\ B_n(t) \end{pmatrix} \mathrm{d}t + \hat{G}_{nj} \begin{pmatrix} M_n(t) \\ E_n(t) \\ B_n(t) \end{pmatrix} \mathrm{d}W_j(t). \tag{5.34}$$

The next result corresponds to Corollary 4.9 from Section 4.

Proposition 5.22. [3] *$\hat{F}_n$, $\hat{G}_{nj}$ defined in* (5.32) *and* (5.33) *are Lipschitz on balls and one side linear growth. And hence there exists a unique global solution* (M_n, B_n, E_n) *of the Problem 5.19. Moreover,* $(M_n, B_n, E_n) \in C^1([0,T]; H_n \times Y_n \times Y_n)$*,* $\mathbb{P}$*-almost surely.*

5.3. a'priori estimates.

Proposition 5.23. *For $p \geq 1$, $b > \frac{1}{4}$, there is constant $C = C(p,b) > 0$ independent of n such that:*

$$\|M_n\|_{L^\infty(0,T;H)} \leq \|M_0\|_H, \qquad \mathbb{P} - a.s., \tag{5.35}$$

$$\mathbb{E}\|B_n - \pi_n^Y \tilde{M}_n\|^p_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C, \tag{5.36}$$

$$\mathbb{E}\|E_n\|^p_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C, \tag{5.37}$$

$$\mathbb{E}\|M_n\|^p_{L^\infty(0,T;\mathbb{H}^1(D))} \leq C. \tag{5.38}$$

$$\mathbb{E}\|M_n \times \rho_n\|^p_{L^2(0,T;\mathbb{L}^2(D))} \leq C, \tag{5.39}$$

$$\mathbb{E}\|B_n\|^p_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C, \tag{5.40}$$

$$\mathbb{E}\left(\int_0^T \|M_n(t) \times (M_n(t) \times \rho_n(t))\|^2_{\mathbb{L}^{\frac{3}{2}}(D)} \,\mathrm{d}t\right)^{\frac{p}{2}} \leq C, \tag{5.41}$$

$$\mathbb{E}\int_0^T \left\|\pi_n\left[M_n(t) \times (M_n(t) \times \rho_n(t))\right]\right\|^2_{X^{-b}} \,\mathrm{d}t \leq C, \tag{5.42}$$

$$\mathbb{E}\left\|\frac{\mathrm{d}E_n}{\mathrm{d}t}\right\|^p_{L^\infty(0,T;Y')} \leq C, \tag{5.43}$$

$$\mathbb{E}\left\|\frac{\mathrm{d}B_n}{\mathrm{d}t}\right\|^p_{L^\infty(0,T;Y')} \leq C, \tag{5.44}$$

where X^{-b} is the dual space of $X^b = D(A^b)$.

Proof of (5.35). By the Itô formula from Lemma 2.125, we have

$$\begin{aligned}
\mathrm{d}\|M_n\|_H^2 &= \sum_{j=1}^{\infty} 2\langle M_n, G_{nj}\rangle_H \,\mathrm{d}W_j + \left[2\langle M_n, F_n\rangle_H + \sum_{j=1}^{\infty}\|G_{nj}\|_H^2\right]\mathrm{d}t \\
&= \sum_{j=1}^{\infty}\left\|\alpha\pi_n\left[M_n\times h_j\right] + \beta\pi_n\left[\psi(M_n)M_n\times(M_n\times h_j)\right]\right\|_H^2 \\
&\quad + \sum_{j=1}^{\infty}\Big\{\langle\alpha^2\pi_n[(M_n\times h_j)\times h_j], M_n\rangle_H \\
&\quad + \langle\beta^2\psi(M_n)\pi_n[M_n\times(M_n\times h_j)\times(M_n\times h_j)], M_n\rangle_H\Big\} \\
&= 0.
\end{aligned}$$

Therefore

$$\|M_n(t)\|_H^2 = \|M_n(0)\|_H^2 = \|\pi_n M_0\|_H^2 \le \|M_0\|_H^2, \qquad t \ge 0.$$

The proof of (5.35) has been complete. □

Proof of (5.36), (5.37), (5.38), (5.39). By the Itô Lemma 2.125 and using (5.30), (5.23), (5.24) we get:

$$\begin{aligned}
&\mathrm{d}\mathcal{E}_n(M_n(t), B_n(t), E_n(t)) \\
&= \Bigg[\frac{\partial\mathcal{E}_n}{\partial M_n}(F_n(t)) + \frac{1}{2}\sum_{j=1}^{\infty}\frac{\partial^2\mathcal{E}_n}{\partial M_n^2}(G_{n,j}(t), G_{n,j}(t)) - \frac{\partial\mathcal{E}_n}{\partial B_n}(\pi_n^Y(\nabla\times E_n(t))) \\
&\quad + \frac{\partial\mathcal{E}_n}{\partial E_n}(\pi_n^Y[\nabla\times(B_n(t) - \pi_n^Y(\tilde{M}_n(t)))] - \pi_n^Y[1_D(E_n(t) + \tilde{f}(t))])\Bigg]\mathrm{d}t \\
&\quad + \sum_{j=1}^{\infty}\frac{\partial\mathcal{E}_n}{\partial M_n}(G_{n,j}(t))\,\mathrm{d}W_j(t).
\end{aligned}$$

Then by (5.17)-(5.20) and (5.21), we have

$$
\begin{aligned}
&\mathcal{E}_n(t) - \mathcal{E}_n(0) \\
= &\int_0^t \Big\{ - \int_D \langle \rho_n(s,x), F_n(s,x)\rangle \,\mathrm{d}x \\
&+\frac{1}{2}\sum_{j=1}^{\infty} \int_D \phi''(M_n(s,x))(G_{n,j}(s,x), G_{n,j}(s,x))\,\mathrm{d}x \\
&+\frac{1}{2}\sum_{j=1}^{\infty} \|G_{n,j}(s)\|_V^2 - \Big\langle B_n(s) - \pi_n^Y \tilde{M}_n(s), \pi_n^Y(\nabla \times E_n(s))\Big\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
&+\Big\langle E_n(s), \pi_n^Y[\nabla \times (B_n(s) - \pi_n^Y(\tilde{M}_n(s)))] - \pi_n^Y[1_D(E_n(s) + \tilde{f}(s))]\Big\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \Big\}\,\mathrm{d}s \\
&-\sum_{j=1}^{\infty} \int_0^t \int_D \langle \rho_n(s,x), G_{n,j}(s,x)\rangle \,\mathrm{d}x\,\mathrm{d}W_j(s).
\end{aligned}
\tag{5.45}
$$

Now let's consider each term in the equality (5.45).
For the term on the left hand side of (5.45),

$$
\begin{aligned}
&\mathcal{E}_n(t) - \mathcal{E}_n(0) \\
= &\int_D \phi(M_n(t,x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M_n(t)\|_H^2 \\
&+\frac{1}{2}\left\|B_n(t) - \pi_n^Y(\tilde{M}_n(t))\right\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E_n(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 \\
&-\Big(\int_D \phi(M_n(0,x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M_n(0)\|_H^2 \\
&+\frac{1}{2}\left\|B_n(0) - \pi_n^Y(\tilde{M}_n(0))\right\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{1}{2}\|E_n(0)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\Big).
\end{aligned}
\tag{5.46}
$$

For the 1st term on the right hand side of (5.45), by (5.28),

$$
\begin{aligned}
&-\langle \rho_n, F_n\rangle_H = -\alpha\langle \rho_n, \pi_n[M_n \times \rho_n]\rangle_H - \beta\langle \rho_n, \pi_n[M_n \times (M_n \times \rho_n)]\rangle_H \\
&-\frac{1}{2}\sum_{j=1}^{\infty}\Big\{\alpha^2\langle \rho_n, \pi_n[(M_n \times h_j)\times h_j]\rangle_H \\
&+\alpha\beta\psi(M_n)\langle \rho_n, \pi_n[[M_n \times (M_n \times h_j)] \times h_j] + \pi_n[M_n \times [(M_n \times h_j)\times h_j]]\rangle_H \\
&+\beta^2\psi(M_n)\langle \rho_n, \pi_n[M_n \times [M_n \times (M_n \times h_j)\times h_j]] \\
&+\pi_n[[M_n \times (M_n \times h_j)] \times (M_n \times h_j)]\rangle_H\Big\}.
\end{aligned}
$$

Since $\pi_n : H \longrightarrow H$ is a self-adjoint operator,

$$\langle \rho_n, \pi_n[M_n \times \rho_n]\rangle_H = \langle \rho_n, M_n \times \rho_n\rangle_H = 0.$$

$$\langle \rho_n, \pi_n[M_n \times (M_n \times \rho_n)]\rangle_H = \langle \rho_n, M_n \times (M_n \times \rho_n)\rangle_H = -\|M_n \times \rho_n\|_H^2.$$

So

$$
\begin{aligned}
&-\langle \rho_n, F_n\rangle_H \\
(5.47)= \ &\beta\|M_n\times\rho_n\|_H^2 - \frac{1}{2}\sum_{j=1}^{\infty}\Big\{\alpha^2\left\langle \rho_n, \pi_n[(M_n\times h_j)\times h_j]\right\rangle_H \\
&+\alpha\beta\psi(M_n)\left\langle \rho_n, \pi_n[[M_n\times(M_n\times h_j)]\times h_j] + \pi_n[M_n\times[(M_n\times h_j)\times h_j]]\right\rangle_H \\
&+\beta^2\psi(M_n)\Big\langle \rho_n, \pi_n[M_n\times[M_n\times(M_n\times h_j)\times h_j]] \\
&+\pi_n[[M_n\times(M_n\times h_j)]\times(M_n\times h_j)]\Big\rangle_H\Big\}.
\end{aligned}
$$

For the 2nd term on the right hand side of (5.45), by (5.29),

$$
\begin{aligned}
(5.48)\qquad &\frac{1}{2}\sum_{j=1}^{\infty}\int_D \phi''(M_n(s,x))(G_{n,j}(s,x), G_{n,j}(s,x))\,\mathrm{d}x \\
= \ &\frac{1}{2}\sum_{j=1}^{\infty}\int_D \phi''(M_n(s,x))\Big(\alpha\pi_n(M_n\times h_j) + \beta\pi_n[\psi(M_n)M_n\times(M_n\times h_j)], \\
&\qquad \alpha\pi_n(M_n\times h_j) + \beta\pi_n[\psi(M_n)M_n\times(M_n\times h_j)]\Big)\,\mathrm{d}x.
\end{aligned}
$$

For the 3rd term on the right hand side of (5.45),

$$
\begin{aligned}
(5.49)\qquad &\frac{1}{2}\sum_{j=1}^{\infty}\|G_{n,j}(s)\|_V^2 \\
= \ &\frac{1}{2}\sum_{j=1}^{\infty}\int_D\left(\alpha^2\left|M_n\times h_j\right|^2 + \beta^2\left|\psi(M_n)M_n\times(M_n\times h_j)\right|^2\right)\mathrm{d}x \\
&+\frac{1}{2}\sum_{j=1}^{\infty}\int_D\left|\nabla\left(\alpha M_n\times h_j + \beta\psi(M_n)M_n\times(M_n\times h_j)\right)\right|^2\,\mathrm{d}x.
\end{aligned}
$$

For the 4th and 5th terms on the right hand side of (5.45), let us notice that

$$
\begin{aligned}
&-\left\langle B_n(s) - \pi_n^Y\tilde{M}_n(s), \pi_n^Y(\nabla\times E_n(s))\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \left\langle E_n(s), \pi_n^Y[\nabla\times(B_n(s) - \pi_n^Y(\tilde{M}_n(s)))]\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
= \ &-\left\langle B_n(s) - \pi_n^Y\tilde{M}_n(s), \nabla\times E_n(s)\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} + \left\langle E_n(s), \nabla\times(B_n(s) - \pi_n^Y(\tilde{M}_n(s)))\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = 0.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&-\left\langle B_n(s) - \pi_n^Y\tilde{M}_n(s), \pi_n^Y(\nabla\times E_n(s))\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
(5.50)\quad &+\left\langle E_n(s), \pi_n^Y[\nabla\times(B_n(s) - \pi_n^Y(\tilde{M}_n(s)))] - \pi_n^Y[1_D(E_n(s) + \tilde{f}(s))]\right\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\
= \ &-\langle E_n(s), 1_D(E_n(s)) + \tilde{f}(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} = -\|1_D E_n\|_H^2 - \langle f, 1_D E_n\rangle_H.
\end{aligned}
$$

For the 6th terms on the right hand side of (5.45),

$$
\begin{aligned}
&-\sum_{j=1}^{\infty}\int_D \langle \rho_n(s,x), G_{n,j}(s,x)\rangle \,\mathrm{d}x \\
&= -\sum_{j=1}^{\infty}\int_D \alpha\langle \rho_n, M_n\times h_j\rangle + \beta\langle \rho_n, \psi(M_n)M_n\times(M_n\times h_j)\rangle \,\mathrm{d}x.
\end{aligned}
\tag{5.51}
$$

By (5.46)-(5.51), the equality (5.45) becomes

$$
\begin{aligned}
&\int_D \phi(M_n(t,x))\,\mathrm{d}x + \frac{1}{2}\|\nabla M_n(t)\|_H^2 \\
&+\frac{1}{2}\left\|B_n(t)-\pi_n^Y(\tilde{M}_n(t))\right\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{1}{2}\|E_n(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\
&+\beta\int_0^t\int_D |M_n\times\rho_n|^2\,\mathrm{d}x\,\mathrm{d}s \\
&+\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_D \Big\{\alpha^2\left\langle \rho_n, M_n\times h_j\times h_j\right\rangle \\
&+\alpha\beta\left\langle \rho_n, \psi(M_n)\Big([M_n\times(M_n\times h_j)]\times h_j + M_n\times[(M_n\times h_j)\times h_j]\Big)\right\rangle \\
&+\beta^2\Big\langle \rho_n, \psi(M_n)\Big(M_n\times[M_n\times(M_n\times h_j)\times h_j]\Big\rangle \\
&+M_n\times(M_n\times h_j)\times(M_n\times h_j)\Big)\Big\}\,\mathrm{d}x\,\mathrm{d}s \\
&-\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_D\left(\alpha^2\left|M_n\times h_j\right|^2+\beta^2\left|\psi(M_n)M_n\times(M_n\times h_j)\right|^2\right)\mathrm{d}x\,\mathrm{d}s \\
&-\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_D\left[\nabla\left(\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\right)\right]^2\mathrm{d}x\,\mathrm{d}s \\
&-\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_D \phi''(M_n)\Big(\alpha\pi_n(M_n\times h_j)+\beta\pi_n[\psi(M_n)M_n\times(M_n\times h_j)], \\
&\qquad\qquad \alpha\pi_n(M_n\times h_j)+\beta\pi_n[\psi(M_n)M_n\times(M_n\times h_j)]\Big)\,\mathrm{d}x\,\mathrm{d}s \\
&+\int_0^t\left(\|1_D E_n\|_H^2+\langle f, E_n\rangle_H\right)\mathrm{d}s \\
&+\sum_{j=1}^{\infty}\int_0^t\int_D\Big[\alpha\left\langle \rho_n, M_n\times h_j\right\rangle+\beta\left\langle \rho_n,\psi(M_n)M_n\times(M_n\times h_j)\right\rangle\Big]\,\mathrm{d}x\,\mathrm{d}W_j(s) \\
&=\int_D\phi(M_n(0,x))\,\mathrm{d}x+\frac{1}{2}\|\nabla M_n(0)\|_H^2 \\
&+\frac{1}{2}\left\|B_n(0)-\pi_n^Y(\tilde{M}_n(0))\right\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\frac{1}{2}\|E_n(0)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}, \qquad \forall t\in(0,T).
\end{aligned}
\tag{5.52}
$$

Now let us consider some terms in the equality (5.52).
By the definition of ρ_n in (5.21), we have

$$\begin{aligned}&\int_D \left\langle \rho_n, (M_n \times h_j) \times h_j \right\rangle \mathrm{d}x\\ = &-\int_D \langle \pi_n[\phi'(M_n)], (M_n \times h_j) \times h_j \rangle \mathrm{d}x + \int_D \langle \Delta M_n, (M_n \times h_j) \times h_j \rangle \mathrm{d}x\\ &+ \int_D \langle \pi_n[B_n - \pi_n^Y \tilde{M}_n], (M_n \times h_j) \times h_j \rangle \mathrm{d}x.\end{aligned}$$

By the Proposition 2.166, we have

$$\begin{aligned}&\left|\int_D \left\langle \Delta M_n, (M_n \times h_j) \times h_j \right\rangle \mathrm{d}x\right| = \left|\langle \nabla M_n, \nabla[(M_n \times h_j) \times h_j] \rangle_{L^2}\right|\\ &\leq \|\nabla M_n\|_{L^2}^2 \|h_j\|_{\mathbb{L}^\infty(D)}^2 + 2\|\nabla M_n\|_{L^2} \|M_n\|_{\mathbb{L}^6(D)} \|h_j\|_{\mathbb{L}^\infty(D)} \|\nabla h_j\|_{\mathbb{L}^3(D)}\\ &\leq \|M_n\|_V^2 \|h_j\|_{\mathbb{L}^\infty(D)}^2 + 2\|M_n\|_V^2 \|h_j\|_{\mathbb{L}^\infty(D)} \|\nabla h_j\|_{\mathbb{L}^3(D)}.\end{aligned}$$

Next we have

$$\begin{aligned}&\left|\int_D \langle \pi_n[B_n - \pi_n^Y(\tilde{M}_n)], (M_n \times h_j) \times h_j \rangle \mathrm{d}x\right|\\ &\qquad \leq \frac{1}{2}\|h_j\|_{\mathbb{L}^\infty(D)}^2 \left(\left\|1_D \left[B_n - \pi_n^Y(\tilde{M}_n)\right]\right\|_H^2 + \|M_n\|_H^2\right).\end{aligned}$$

Since we assume that ϕ' is bounded, there exists some constant $C_1 > 0$ independent of n such that

$$\left|\int_D \langle \pi_n[\phi'(M_n)], (M_n \times h_j) \times h_j \rangle \mathrm{d}x\right| \leq C_1 m(D) \|h_j\|_{\mathbb{L}^\infty(D)}^2 \|M_n\|_H.$$

Hence by the last four inequalities, there exists constant $C_2 > 0$ independent of n such that

$$\begin{aligned}&\left|\frac{1}{2}\alpha^2 \sum_{j=1}^\infty \int_0^t \int_D \left\langle \rho_n, M_n \times h_j \times h_j \right\rangle \mathrm{d}x\, \mathrm{d}s\right|\\ &\qquad \leq \frac{1}{2}\alpha^2 c_h C_2 \int_0^t \left(\|M_n\|_V^2 + \left\|1_D \left[B_n - \pi_n^Y(\tilde{M}_n)\right]\right\|_H^2\right) \mathrm{d}s. \qquad (5.53)\end{aligned}$$

Similarly as before, we can find a constant $C_3 > 0$ independent of n, such that

$$\begin{aligned}&\left|\frac{1}{2}\alpha\beta \sum_{j=1}^\infty \int_0^t \int_D \left\langle \rho_n, \psi(M_n)\Big\{[M_n \times (M_n \times h_j)] \times h_j \right.\right.\\ &\qquad \left.\left. + M_n \times [(M_n \times h_j) \times h_j]\Big\} \right\rangle \mathrm{d}x\, \mathrm{d}s\right|\\ &\leq C_3 c_h \alpha\beta \int_0^t \left(\|M_n\|_V^2 + \left\|1_D \left[B_n - \pi_n^Y(\tilde{M}_n)\right]\right\|_H^2\right) \mathrm{d}s + C_3 c_h \alpha\beta. \qquad (5.54)\end{aligned}$$

We continue and find $C_4 > 0$ such that

$$\frac{\beta^2}{2}\Big|\sum_{j=1}^{\infty}\int_0^t\int_D\Big\langle\rho_n,\psi(M_n)\Big\{M_n\times[[M_n\times(M_n\times h_j)]\times h_j] \\ +[M_n\times(M_n\times h_j)]\times(M_n\times h_j)\Big\}\Big\rangle\,\mathrm{d}x\,\mathrm{d}s\Big|$$

$$\leq C_4c_h\frac{\beta^2}{2}\int_0^t\left(\|M_n\|_V^2+\left\|1_D\left[B_n-\pi_n^Y(\tilde{M}_n)\right]\right\|_H^2\right)\mathrm{d}s+C_4c_h\frac{\beta^2}{2} \tag{5.55}$$

Next, we can find a constant $C_5 > 0$ independent of n such that

$$\Big|\sum_{j=1}^{\infty}\int_0^t\int_D\left(\alpha^2\big|M_n\times h_j\big|^2+\beta^2\big|\psi(M_n)M_n\times(M_n\times h_j)\big|^2\right)\mathrm{d}x\,\mathrm{d}s \\ +\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_D\left[\nabla\left(\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\right)\right]^2\,\mathrm{d}x\,\mathrm{d}s \\ +\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_D\phi'(M_n)\Big[\alpha^2\big|M_n\times h_j\big|^2+\beta^2\big|\psi(M_n)M_n\times(M_n\times h_j)\big|^2\Big]\,\mathrm{d}x\,\mathrm{d}s\Big|$$

$$\leq C_5c_h\int_0^t\|M_n\|_V^2\,\mathrm{d}s+C_5c_h.. \tag{5.56}$$

Remark 5.24. Please notice that we need to use the boundness property of ψ to get the inequalities (5.54), (5.55) and (5.56). And this is the role ψ played.

Next, let us notice that by the Cauchy-Schwartz inequality,

$$\left|\int_0^t\int_D\langle f,E_n\rangle\,\mathrm{d}x\,\mathrm{d}s\right|\leq\frac{1}{2}\int_0^t\int_D\left(|f|^2+|E_n|^2\right)\mathrm{d}x\,\mathrm{d}s. \tag{5.57}$$

By (5.52) and (5.53)-(5.57) we infer that there exists a constant $C_6 > 0$ independent of n such that

$$\frac{1}{2}\left(\|B_n(t)-\pi_n^Y(\tilde{M}_n(t))\|_{\mathbb{L}^2(\mathbb{R}^3)}^2+\|E_n(t)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\right)+\frac{1}{2}\int_0^t\|E_n\|_{\mathbb{L}^2(\mathbb{R}^3)}^2\,\mathrm{d}s \\ +\beta\int_0^t\|M_n\times\rho_n\|_H^2\,\mathrm{d}s+\int_D\phi(M_n(t))\,\mathrm{d}x+\frac{1}{2}\|M_n(t)\|_V^2 \\ \leq\frac{1}{2}\int_{\mathbb{R}^3}(|[B_n-\pi_n^Y(\tilde{M}_n)](0)|^2+|E_n(0)|^2)\,\mathrm{d}x+\frac{1}{2}\int_0^t\int_D|f|^2\,\mathrm{d}x\,\mathrm{d}s \\ +\int_D\left(\phi(M_n(0))+\frac{1}{2}|\nabla M_n(0)|^2\right)\mathrm{d}x+C_6c_h\int_0^t\left(\|M_n\|_V^2+\left\|1_D\left[B_n-\pi_n^Y(\tilde{M}_n)\right]\right\|_H^2\right)\mathrm{d}s+C_6c_h \\ +\left|\sum_{j=1}^{\infty}\int_0^t\int_{\mathbb{R}^3}\alpha\left\langle\rho_n,M_n\times h_j\right\rangle+\beta\left\langle\rho_n,\psi(M_n)M_n\times(M_n\times h_j)\right\rangle\,\mathrm{d}x\,\mathrm{d}W_j(s)\right|,\qquad\forall t\in(0,T).$$

We are going to estimate the stochastic integral term in the above inequality, but before that let us take the supreme for $r \in (0,t)$ on both sides of the above inequality where $t \in [0,T]$ is fixed. Then we get

$$\begin{aligned}
&\frac{1}{2}\sup_{r\in(0,t)}\left(\|B_n(r)-\pi_n^Y(\tilde{M}_n(r))\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|E_n(r)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\right)+\frac{1}{2}\int_0^t\|E_n\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s\\
&+\beta\int_0^t\|M_n\times\rho_n\|^2_H\,\mathrm{d}s+\sup_{r\in(0,t)}\int_D\phi(M_n(r))\,\mathrm{d}x+\frac{1}{2}\|M_n(r)\|^2_V\\
\leq\ &\frac{1}{2}\int_{\mathbb{R}^3}(|[B_n-\pi_n^Y(\tilde{M}_n)](0)|^2+|E_n(0)|^2)\,\mathrm{d}x+\frac{1}{2}\int_0^t\int_D|f|^2\,\mathrm{d}x\,\mathrm{d}s\\
&+\int_D\left(\phi(M_n(0))+\frac{1}{2}|\nabla M_n(0)|^2\right)\mathrm{d}x\\
&+C_6c_h\int_0^t\sup_{r\in(0,s)}\left(\|M_n\|^2_V+\left\|1_D\left[B_n-\pi_n^Y(\tilde{M}_n)\right]\right\|^2_H\right)\mathrm{d}s+C_6c_h\\
&+\sup_{r\in(0,t)}\left|\sum_{j=1}^{\infty}\int_0^r\int_{\mathbb{R}^3}\left\langle\rho_n,\left[\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\right]\right\rangle\mathrm{d}x\,\mathrm{d}W_j(s)\right|.
\end{aligned}$$

Now let us fix $p \geq 1$ and rise both sides of the above equation to power p for $p \geq 1$ and then take expectation on both sides, so for some constant C_7, we have

(5.58)

$$\begin{aligned}
&\mathbb{E}\sup_{r\in(0,t)}\left(\|[B_n-\pi_n^Y(\tilde{M}_n)](r)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|E_n(r)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|M_n(r)\|^2_V+2\beta\int_0^t\|M_n\times\rho_n\|^2_{L^2(D)}\,\mathrm{d}s\right)^p\\
\leq\ &C_7c_h\left[\int_0^t\sup_{r\in(0,s)}\left(\|M_n(r)\|^2_V+\|[B_n-\pi_n^Y(\tilde{M}_n)](r)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\right)\mathrm{d}s\right]^p\\
&+\mathbb{E}\left(\sup_{r\in(0,t)}\left|\sum_{j=1}^{\infty}\int_0^r\int_{\mathbb{R}^3}\left\langle\rho_n,\left[\alpha M_n\times h_j+\beta\psi(M_n)M_n^R\times(M_n\times h_j)\right]\right\rangle\mathrm{d}x\,\mathrm{d}W_j(s)\right|\right)^p+C_7c_h.
\end{aligned}$$

Then by the Burkholder-Davis-Gundy inequality (Theorem 2.127), there exists an constant $K = K(p) > 0$ independent of n such that:

$$\mathbb{E}\left(\sup_{r\in(0,t)}\left|\sum_{j=1}^{\infty}\int_0^r\int_{\mathbb{R}^3}\left\langle\rho_n,\left[\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\right]\right\rangle\,\mathrm{d}x\,\mathrm{d}W_j\right|\right)^p$$
$$\le K\mathbb{E}\left|\int_0^t\sum_{j=1}^{\infty}\left\langle\rho_n,\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\right\rangle^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s\right|^{\frac{p}{2}}$$
$$\le K\mathbb{E}\left|\int_0^t\sum_{j=1}^{\infty}\left\langle\rho_n,\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\right\rangle^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s\right|^{p}+K$$

Hence we can find $C_8>0$ independent of n such that

(5.59)
$$\mathbb{E}\left(\sup_{r\in(0,t)}\left|\sum_{j=1}^{\infty}\int_0^r\int_{\mathbb{R}^3}\left\langle\rho_n,\left[\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\right]\right\rangle\,\mathrm{d}x\,\mathrm{d}W_j\right|\right)^p$$
$$\le C_8c_h\left(\int_0^t\mathbb{E}\sup_{r\in(0,s)}\left(\|M_n(r)\|_V^2+\left\|1_D\left[B_n-\pi_n^Y(\tilde{M}_n)\right](r)\right\|_H^2\right)\mathrm{d}s\right)^p+C_8c_h$$

Hence by (5.58) and (5.59) there exists $C_9>0$ independent of n such that,

$$\mathbb{E}\sup_{r\in(0,t)}\left(\|[B_n-\pi_n^Y(\tilde{M}_n)](r)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|E_n(r)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\|M_n(r)\|_V^2+2\beta\int_0^t\|M_n\times\rho_n\|^2_{L^2(D)}\,\mathrm{d}\tau\right)^p$$
$$\le C_9c_h\int_0^t\sup_{r\in(0,s)}\left(\|M_n(r)\|_V^2+\left\|1_D\left[B_n-\pi_n^Y(\tilde{M}_n)\right](r)\right\|_H^2\right)^p\,\mathrm{d}s+C_9c_h$$

Hence by the Gronwall inequality 2.157, with $C=C_9c_he^{C_9c_hT}$, we get the following four a'priori estimates,

$$\mathbb{E}\|B_n-\pi_n^Y\tilde{M}_n\|^{2p}_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}\le C,$$

$$\mathbb{E}\|E_n\|^{2p}_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}\le C,$$

$$\mathbb{E}\|M_n\|^{2p}_{L^\infty(0,T;\mathbb{H}^1(D))}\le C,$$

$$\mathbb{E}\|M_n\times\rho_n\|^{2p}_{L^2(0,T;\mathbb{L}^2(D))}\le C.$$

And since $L^{2p}(\Omega)\hookrightarrow L^p(\Omega)$ continuously, these four inequalities imply the inequalities: (5.36), (5.37), (5.38), (5.39). □

We continue with the proof of Proposition 5.23.

Proof of (5.40). As before we fix $p\ge 1$.

$$\mathbb{E}\|B_n\|^{2p}_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}\le 2^p\left(\mathbb{E}\|[B_n-\pi_n^Y(\tilde{M}_n)]\|^{2p}_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}+\mathbb{E}\|M_n\|^{2p}_{L^\infty(0,T;\mathbb{L}^2(D))}\right).$$

By the above inequality (5.36) and (5.38), there exists some $C > 0$ independent of n such that

$$\mathbb{E}\|B_n\|^{2p}_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C.$$

Together with the fact $L^{2p}(\Omega) \hookrightarrow L^p(\Omega)$ continuously, we complete the proof of (5.40). □

Proof of (5.41). Applying Theorem 2.96, with $n = 3$, $m = 1$, $r = 2$, so $p = 6$. We infer that there is a constant C, such that

$$\left\|M_n\right\|_{\mathbb{L}^6} \leq C\left\|M_n\right\|_{\mathbb{H}^1}, \qquad \forall u_n \in H^1(D, \mathbb{R}^3).$$

Therefore by the Hölder inequality:

$$\left\|M_n(t) \times (M_n(t) \times \rho_n(t))\right\|_{\mathbb{L}^{\frac{3}{2}}} \leq \left\|M_n(t)\right\|_{\mathbb{L}^6}\left\|M_n(t) \times \rho_n(t)\right\|_{\mathbb{L}^2} \leq C\left\|M_n(t)\right\|_{\mathbb{H}^1}\left\|M_n(t) \times \rho_n(t)\right\|_{\mathbb{L}^2}.$$

Hence, by Cauchy-Schwartz inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left\|M_n(t) \times (M_n(t) \times \rho_n(t))\right\|^2_{\mathbb{L}^{\frac{3}{2}}} \,\mathrm{d}t\right)^{\frac{p}{2}}\right] \\
&\leq C^p\mathbb{E}\left[\sup_{r\in[0,T]} \left\|M_n(r)\right\|^p_{H^1}\left(\int_0^T \left\|M_n(t) \times \rho_n(t)\right\|^2_{\mathbb{L}^2} \,\mathrm{d}t\right)^{\frac{p}{2}}\right] \\
&\leq C^p\left(\mathbb{E}\left[\sup_{t\in[0,T]} \left\|M_n(t)\right\|^{2p}_{\mathbb{H}^1}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left(\int_0^T \left\|M_n(t) \times \rho_n(t)\right\|^2_{\mathbb{L}^2} \,\mathrm{d}t\right)^p\right]\right)^{\frac{1}{2}} \leq \dots
\end{aligned}$$

The last inequality above is from the Cauchy-Schwartz inequality.
By Jensen's inequality 2.156,

$$\left\|M_n(t)\right\|^{2p}_{\mathbb{H}^1} = \left(\left\|M_n(t)\right\|^2_{\mathbb{L}^2} + \left\|\nabla M_n(t)\right\|^2_{\mathbb{L}^2}\right)^p \leq 2^{p-1}\left(\left\|M_n(t)\right\|^{2p}_{\mathbb{L}^2} + \left\|\nabla M_n(t)\right\|^{2p}_{\mathbb{L}^2}\right)$$

Moreover, since

$$\left\|M_n(t)\right\|^{2p}_{\mathbb{L}^2} = \left\|M_n(0)\right\|^{2p}_{\mathbb{L}^2} = \left\|\pi_n M_0\right\|^{2p}_{\mathbb{L}^2} \leq \left\|M_0\right\|^{2p}_{\mathbb{L}^2}.$$

We infer that,

$$\dots \leq C^p\left(\mathbb{E}\left\|M_0\right\|^{2p}_{\mathbb{L}^2} + \mathbb{E}\sup_{t\in[0,T]}\left\|\nabla M_n(t)\right\|^{2p}_{\mathbb{L}^2}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T \left\|M_n(t) \times \rho_n(t)\right\|^2_{\mathbb{L}^2} \,\mathrm{d}t\right]^p\right)^{\frac{1}{2}}.$$

Then by (5.38) and (5.39), we get (5.41). □

Proof of (5.42). By Theorem 2.99, $X^b \hookrightarrow \mathbb{L}^3(D)$ continuously if $b > \frac{1}{4}$. Hence $\mathbb{L}^{\frac{3}{2}}$ is continuously embedded in X^{-b}. Thus there is a constant C_1 independent of n

such that

$$
\begin{aligned}
&\mathbb{E}\int_0^T \left\|\pi_n\left[M_n(t)\times(M_n(t)\times\rho_n(t))\right]\right\|_{X^{-b}}^2\,\mathrm{d}t\\
&\qquad\le \mathbb{E}\int_0^T \left\|\left[M_n(t)\times(M_n(t)\times\rho_n(t))\right]\right\|_{X^{-b}}^2\,\mathrm{d}t\\
&\qquad\qquad\le C_1\mathbb{E}\int_0^T \left\|\left[M_n(t)\times(M_n(t)\times\rho_n(t))\right]\right\|_{\mathbb{L}^{\frac{3}{2}}}^2\,\mathrm{d}t.
\end{aligned}
$$

Then by (5.41), we get (5.42). □

Remark 5.25. From now on we need to assume that $b > \frac{1}{4}$.

Proof of (5.43) *and* (5.44). By (5.23)

$$
\begin{aligned}
\mathbb{E}\left\|\frac{\mathrm{d}E_n}{\mathrm{d}t}\right\|_{L^\infty(0,T;Y')}^p &= \mathbb{E}\|\pi_n^Y(\nabla\times[B_n-\pi_n^Y(\tilde{M}_n)])-\pi_n^Y[1_D(E_n+f)]\|_{L^\infty(0,T;Y')}^p\\
&\le C_p\mathbb{E}\sup_{t\in(0,T)}\|\nabla\times B_n(t)-\pi_n^Y(\tilde{M}_n(t))\|_{Y'}^p + C_p\mathbb{E}\sup_{t\in(0,T)}\|1_D(E_n(t)+f(t))\|_{Y'}^p\\
&\le C_p\mathbb{E}\sup_{t\in(0,T)}\sup_{y\neq 0}\left(\left|\frac{\langle B_n(t)-\pi_n^Y(\tilde{M}_n(t)),\nabla\times y\rangle_{\mathbb{L}^2(\mathbb{R}^3)}}{\|y\|_Y}\right|^p + \left|\frac{\langle 1_D(E_n+f),y\rangle_{\mathbb{L}^2(\mathbb{R}^3)}}{\|y\|_Y}\right|^p\right)\\
&\le C_p\mathbb{E}\|[B_n-\pi_n^Y(\tilde{M}_n)]\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}^p + C_p\mathbb{E}\|E_n\|_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))}^p + C_p\|f\|_{\mathbb{L}^2(Q)}^p.
\end{aligned}
$$

Hence, since $f\in L^2(0,T;\mathbb{L}^2(D))$, by (5.36) and (5.37), we get (5.43) and similarly (5.44). □

After so many pages of long calculation, the proof of Proposition 5.23 has been finished.

Lemma 5.26. *If $a\in(0,\frac{1}{2})$ and $p\ge 2$, there exists a constant $C\ge 0$ such that for all $n\in\mathbb{N}$,*

$$
\begin{aligned}
\text{(5.60)}\quad &\mathbb{E}\Bigg\|\sum_{j=1}^{\infty}\int_0^t\Big\{\beta\psi(M_n(s))\pi_n\Big[M_n(s)\times(M_n(s)\times h_j)\Big]\\
&\qquad +\alpha\pi_n\Big[M_n(s)\times h_j\Big]\Big\}\,\mathrm{d}W_j(s)\Bigg\|_{W^{a,p}(0,T;\mathbb{L}^2(D))}^p \le C.
\end{aligned}
$$

Proof. By Lemma 2.135, there exists constant $C_1 > 0$, such that

$$\begin{aligned}
&\mathbb{E}\Bigg\| \sum_{j=1}^{\infty} \int_0^t \Big\{\alpha\pi_n\Big[M_n(s)\times h_j\pi_n\Big] \\
&+\beta\psi(M_n(s))\pi_n\Big[M_n(s)\times(M_n(s)\times h_j)\Big]\Big\}\,\mathrm{d}W_j(s)\Bigg\|^p_{W^{a,p}(0,T;\mathbb{L}^2(D))} \\
\le{}& C_1\mathbb{E}\int_0^T\left(\sum_{j=1}^{\infty}\Big\|\alpha\pi_n(M_n\times h_j)+\beta\psi(M_n)\pi_n\Big[M_n\times(M_n\times h_j)\Big]\Big\|^2_{\mathbb{L}^2(D)}\right)^{\frac{p}{2}}\,\mathrm{d}t \\
\le{}& 2^{p-1}C_1\mathbb{E}\left(\sum_{j=1}^{\infty}\|h_j\|^p_{\mathbb{L}^\infty(D)}\int_0^T \alpha^p\|M_n\|^p_{\mathbb{L}^2(D)}+\beta^p\,\mathrm{d}t\right)\le C,
\end{aligned}$$

where the last inequality followed by (5.38). This completes the proof of the estimate (5.60). ◻

Remark 5.27. From now on we will always assume $a \in (0, \frac{1}{2})$, $b > \frac{1}{4}$ and $p \ge 2$.

Lemma 5.28. *For $a \in (0, \frac{1}{2})$, $b > \frac{1}{4}$, $p \ge 2$, there exists $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{E}\,\|M_n\|^2_{W^{a,p}(0,T;X^{-b})} \le C. \tag{5.61}$$

Proof. By (5.22),

$$\begin{aligned}
\mathbb{E}\,\|M_n\|^2_{W^{a,p}(0,T;X^{-b})} ={}& \mathbb{E}\Bigg\|\int_0^t \pi_n\Big\{\alpha M_n\times\rho_n-\beta M_n\times(M_n\times\rho_n)+\frac{1}{2}\sum_{j=1}^{\infty}\Big[\alpha^2(M_n\times h_j)\times h_j \\
&+\alpha\beta\psi(M_n)[M_n\times(M_n\times h_j)]\times h_j+\beta^2\psi(M_n)M_n\times[M_n\times(M_n\times h_j)\times h_j] \\
&+\alpha\beta\psi(M_n)M_n\times[(M_n\times h_j)\times h_j]+\beta^2\psi(M_n)M_n\times(M_n\times h_j)\times(M_n\times h_j)\Big]\Big\}\,\mathrm{d}s \\
&+\sum_{j=1}^{\infty}\int_0^t\pi_n\Big\{\Big[\alpha M_n\times h_j+\beta\psi(M_n)M_n\times(M_n\times h_j)\Big]\,\mathrm{d}W_j\Big\}\Bigg\|^2_{W^{a,p}(0,T;X^{-b})}.
\end{aligned}$$

We assumed that $\alpha \in (0, \frac{1}{2})$, $p \ge 2 \ge 0$, so by Theorem 2.98, $H^1(0,T;X^{-b}) \hookrightarrow W^{\alpha,p}(0,T;X^{-b})$ continuously. And since $\mathbb{L}^2(D) \hookrightarrow X^{-b}$ continuously, there is a constant C independent of n such that

$$\mathbb{E}\,\|M_n\|^2_{W^{a,p}(0,T;X^{-b})} \le C\mathbb{E}\bigg\| \int_0^t \pi_n \Big\{ \alpha M_n \times \rho_n + \frac{1}{2}\sum_{j=1}^\infty \Big[\alpha^2 (M_n \times h_j)\times h_j$$
$$+\alpha\beta\psi(M_n)[M_n \times (M_n \times h_j)]\times h_j + \beta^2\psi(M_n) M_n \times [M_n \times (M_n\times h_j)\times h_j]$$
$$+\alpha\beta\psi(M_n) M_n \times [(M_n \times h_j)\times h_j] + \beta^2 \psi(M_n) M_n \times (M_n \times h_j)\times (M_n \times h_j)\Big]\Big\}\,\mathrm{d}s\bigg\|^2_{H^1(0,T;\mathbb{L}^2(D))}$$
$$+C\mathbb{E}\bigg\|\int_0^t \beta\pi_n\,[M_n\times(M_n\times\rho_n)]\,\mathrm{d}s\bigg\|^2_{H^1(0,T;X^{-b})}$$
$$+C\mathbb{E}\bigg\|\sum_{j=1}^\infty \pi_n\int_0^t \Big\{\Big[\alpha M_n\times h_j + \beta\psi(M_n)M_n\times(M_n\times h_j)\Big]\,\mathrm{d}W_j\Big\}\bigg\|^2_{W^{a,p}(0,T;\mathbb{L}^2(D))}$$

To prove (5.61), it is enough to consider each term on the right hand side of the above inequality. By (5.39), (5.42) and (5.60), we can conclude (5.61). ◻

5.4. **Tightness results.** In this subsection we will use the a'priori estimates (5.35)-(5.44) to show that the laws $\{\mathcal{L}(M_n, B_n, E_n) : n \in \mathbb{N}\}$ are tight on a suitable path space. Then we will use Skorohod's theorem to obtain another probability space and an almost surely convergent sequence defined on this space whose limit is a weak martingale solution of the Problem 5.10. Now let's state and prove our tightness Lemma.

Lemma 5.29. *For any $p \ge 2$, $q \in [2, 6)$ and $b > \frac{1}{4}$ the set of laws $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ on the Banach space*

$$L^p(0,T;\mathbb{L}^q(D)) \cap C([0,T];X^{-b})$$

is tight.

Proof. Let us choose and fix $p \ge 2$, $q \in [2, 6)$ and $b > \frac{1}{4}$. Since $q < 6$ we can choose $\gamma \in (\frac{3}{4} - \frac{3}{2q}, \frac{1}{2})$, $b' \in (\frac{1}{4}, b)$, $a \in (\frac{1}{p}, 1)$. Then by Proposition 2.174, $H^1 = D(A^{\frac{1}{2}}) \hookrightarrow X^\gamma = D(A^\gamma)$ is compact, hence by Lemma 2.116, the embedding

$$L^p(0,T;H^1) \cap W^{a,p}(0,T;X^{-b'}) \hookrightarrow L^p(0,T;X^\gamma)$$

is compact. We note that for any positive real number r and random variables ξ and η, since

$$\Big\{\omega : \xi(\omega) > \frac{r}{2}\Big\} \cup \Big\{\omega : \eta(\omega) > \frac{r}{2}\Big\} \supset \Big\{\omega : \xi(\omega) + \eta(\omega) > r\Big\},$$

we have

$$\mathbb{P}\Big(|M_n|_{L^p(0,T;H^1)\cap W^{a,p}(0,T;X^{-b'})} > r\Big)$$
$$= \mathbb{P}\Big(|M_n|_{L^p(0,T;H^1)} + |M_n|_{W^{a,p}(0,T;X^{-b'})} > r\Big)$$
$$\le \mathbb{P}\left(|M_n|_{L^p(0,T;H^1)} > \frac{r}{2}\right) + \mathbb{P}\left(|M_n|_{W^{a,p}(0,T;X^{-b'})} > \frac{r}{2}\right) \le \dots$$

then by the Chebyshev inequality in Lemma 2.133,

$$\dots \le \frac{4}{r^2}\mathbb{E}\left(|M_n|^2_{L^p(0,T;H^1)} + |M_n|^2_{W^{a,p}(0,T;X^{-b'})}\right).$$

By the estimates in (5.61) and (5.38), the expected value on the right hand side of the last inequality is uniformly bounded in n. Let $X_T := L^p(0,T;H^1)\cap W^{a,p}(0,T;X^{-b'})$. There is a constant C, such that

$$\mathbb{P}(\|M_n\|_{X_T} > r) \le \frac{C}{r^2}, \qquad \forall r, n.$$

Since

$$\mathbb{E}(\|M_n\|_X) = \int_0^\infty \mathbb{P}(\|M_n\| > r)\,\mathrm{d}r,$$

we can infer that

$$\mathbb{E}(\|M_n\|_{X_T}) \le 1 + \int_1^\infty \frac{M}{r^2}\,\mathrm{d}r = 1 + M < \infty, \quad \forall n \in \mathbb{N}.$$

Therefore by Theorem 2.107 the family of laws $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;X^\gamma)$. By Proposition 2.172, $X^\gamma = \mathbb{H}^{2\gamma}(D)$. Therefore since by the assumption $\gamma > \frac{3}{4} - \frac{3}{2q}$, i.e.

$$2\gamma - \frac{3}{2} > 0 - \frac{3}{q},$$

by Theorem 2.97 we deduce that $X^\gamma \hookrightarrow \mathbb{L}^q(D)$ continuously. Hence $L^p(0,T;X^\gamma) \hookrightarrow L^p(0,T;L^q)$ continuously. By Lemma 2.108, $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is also tight on $L^p(0,T;L^q)$.

Since $b' < b$, by Lemma 2.117, $W^{a,p}(0,T;X^{-b'}) \hookrightarrow C([0,T];X^{-b})$ compactly. Therefore by the estimates in (5.61) and Lemma 2.107, we can conclude that $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is tight on $C([0,T];X^{-b})$.

Therefore by Theorem 2.109, $\{\mathcal{L}(M_n) : n \in \mathbb{N}\}$ is tight on $L^p(0,T;L^q)\cap C([0,T];X^{-b})$. Hence the proof is complete. □

Remark 5.30. From now on we will always assume that $p \ge 2$, and $q \in [2,6)$ and $b > \frac{1}{4}$. Here $q \ge 2$ is because we want $M_n(t) \in H = \mathbb{L}^2(D)$, $q < 6$ is because we need to find some $\gamma < \frac{1}{2}$ ($X^\gamma \hookrightarrow \mathbb{H}^1(D)$ compactly) and $X^\gamma \hookrightarrow \mathbb{L}^q$ continuously.

Definition 5.31 (Aldous condition)**.** [14] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F}$. Let (S,ρ) be a separable metric space, we say that the sequence $\{X_n(t)\}$, $t \in [0,T]$, of S-valued random variables satisfies the *Aldous condition* iff $\forall \varepsilon > 0$, $\forall \eta > 0$, $\exists \delta > 0$ such that for every sequence $\{\tau_n\}$ of $\mathbb{F}$-stopping times with $\tau_n \le T$ one has:

$$\sup_{n\in\mathbb{N}} \sup_{0\le\theta\le\delta} \mathbb{P}\{\rho(X_n(\tau_n+\theta), X_n(\tau_n)) \ge \eta\} \le \varepsilon. \tag{5.62}$$

Lemma 5.32 (Tightness Criterion)**.** [14] *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a filtration* $\mathbb{F}$*. Let* H *be a separable Hilbert space,* U *be another Hilbert space such that* $U \hookrightarrow H$ *compact and dense,* U' *be the dual space of* U*. Let* $\{X_n(t)\}_{n\in\mathbb{N}}$*,* $t \in [0,T]$ *be a sequence of continuous* $\mathbb{F}$*-adapted* U' *valued process such that*

(a) *there exists a positive constant C such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{s\in[0,T]} \|X_n(s)\|_H\right] \leq C.$$

(b) $\{X_n\}_{n\in\mathbb{N}}$ *satisfies the Aldous condition* (5.62) *in* U'.

Then if we denote the law of X_n *by* $\mathbb{P}_n$, *then for every* $\varepsilon > 0$ *there exists a compact subset* K_ε *of* $C([0,T];U') \cap L^2_w([0,T];H) \cap C([0,T];H_w)$ *such that*

$$\mathbb{P}_n(K_\varepsilon) \geq 1 - \varepsilon, \quad \forall n.$$

where L^2_w *and* H_w *means the spaces* L^2 *and* H *equipped with the weak topology.*

Lemma 5.33. *The sets of laws* $\{\mathcal{L}(E_n)\}$ *and* $\{\mathcal{L}(B_n)\}$ *on the space* $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$ *are tight.*

Proof. We will only prove the result about $\{\mathcal{L}(E_n)\}$, the proof about $\{\mathcal{L}(B_n)\}$ is exactly the same.

In order to use Lemma 5.32, we put $H = \mathbb{L}^2(\mathbb{R}^3)$. Let us recall that Y is introduced in Notation 5.9. Then by Lemma 2.40, we can choose an auxiliary Hilbert space U such that the embedding $U \hookrightarrow Y$ is compact. Since the embedding $Y \hookrightarrow \mathbb{L}^2(\mathbb{R}^3)$ is bounded, the embedding $U \hookrightarrow \mathbb{L}^2(\mathbb{R}^3)$ is compact.

Firstly let us observe that by the estimate (5.37), condition (a) of the Lemma 5.32 is satisfied.

Secondly we will check the Aldous condition in Definition 5.31. Let us fix $\varepsilon > 0$ and $\eta > 0$. The embedding $Y' \hookrightarrow U'$ is compact so bounded and thus there exists a constant $C_1 > 0$ such that $\|\cdot\|_{Y'} \geq C_1\|\cdot\|_{U'}$. Hence together with the Chebyshev inequality and estimate (5.43), we have,

$$\begin{aligned}
&\mathbb{P}\left(\|E_n(\tau_n+\theta) - E_n(\tau_n)\|_{U'} \geq \eta\right)\\
&\leq \mathbb{P}\left(\|E_n(\tau_n+\theta) - E_n(\tau_n)\|_{Y'} \geq C_1\eta\right)\\
&\leq \frac{1}{C_1\eta}\mathbb{E}\left(\|E_n(\tau_n+\theta) - E_n(\tau_n)\|_{Y'}\right)\\
&\leq \frac{1}{C_1\eta}\mathbb{E}\int_{\tau_n}^{\tau_n+\theta} \left\|\frac{\mathrm{d}E_n(s)}{\mathrm{d}s}\right\|_{Y'} \mathrm{d}s \leq \frac{C\theta}{C_1\eta}.
\end{aligned}$$

So if $\delta \leq \frac{C_1}{C}\varepsilon\eta$, then we have

$$\sup_{n\in\mathbb{N}} \sup_{0\leq\theta\leq\delta} \mathbb{P}\left(\|E_n(\tau_n+\theta) - E_n(\tau_n)\|_{U'} \geq \eta\right) \leq \varepsilon.$$

Hence the Aldous condition (5.62) has been verified.

Therefore by Lemma 5.32, the laws $\{\mathcal{L}(E_n)\}$ are tight on $C([0,T];U') \cap L^2_w(0,T;H) \cap C([0,T];H_w)$. And the result follows. □

Remark 5.34. If we define a map

$$i_w : L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \ni f \longrightarrow i_w(f) \in \mathbb{L}^2_w([0,T]\times\mathbb{R}^3),$$

by setting

$$i_w(f)(t,x) = f(t)(x), \quad (t,x) \in [0,T]\times\mathbb{R}^3.$$

Then i is a homeomorphism.

Proof. Actually the map

$$i : L^2(0,T;\mathbb{L}^2(\mathbb{R}^3)) \ni f \longrightarrow i(f) \in \mathbb{L}^2([0,T]\times\mathbb{R}^3),$$

with

$$i(f)(t,x) = f(t)(x), \quad (t,x) \in [0,T]\times\mathbb{R}^3.$$

is a homeomorphism. So i_w is a homeomorphism. □

Proposition 5.35. *There is a subsequences of* $\{(M_n, B_n, E_n)\}$*, which we will still denote them as same as the original sequence, such that the laws* $\mathcal{L}(M_n, B_n, E_n, W_j)$ *(*W_j *is the Wiener process) converge weakly to a certain probability measure* μ_j *on* $L^p(0,T;\mathbb{L}^q) \cap C([0,T];X^{-b}) \times L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \times L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \times C([0,T];\mathbb{R})$*, where* $p \in [2,\infty)$*,* $q \in [2,6)$ *and* $b > \frac{1}{4}$*.*

Proof. If $p \in [2,\infty)$, $q \in [2,6)$ and $b > \frac{1}{4}$, by Lemma 5.29, Lemma 5.33 and Theorem 2.114, there is a subsequence of $\mathcal{L}(M_n, B_n, E_n)$ and there exist certain probability measure μ_1 on $L^p(0,T;\mathbb{L}^q)\cap C([0,T];X^{-b})\times L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))\times L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$ such that:

$$\mathcal{L}(M_n, B_n, E_n) \xrightarrow{w} \mu_1,$$

Let $\mu_j := \mu_1 \times \mathcal{L}(W_j)$, we have

$$\mathcal{L}(M_n, B_n, E_n, W_j) \xrightarrow{w} \mu_j$$

on $L^p(0,T;\mathbb{L}^q)\cap C([0,T];X^{-b})\times L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))\times L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))\times C([0,T];\mathbb{R})$. This ends the proof of Proposition 5.35. □

5.5. **Construction of new probability space and processes.** Now we are going to use Skorohod Theorem to construct our new probability space and processes as the weak solution of Problem 5.10.

Theorem 5.36. *For* $p \in [2,\infty)$*,* $q \in [2,6)$ *and* $b > \frac{1}{4}$*, there exists a probability space* $(\Omega', \mathcal{F}', \mathbb{P}') = ([0,1], \mathcal{B}([0,1]), Leb.)$ *and there exists a sequence* $\{(M'_n, E'_n, B'_n, W'_{jn})\}$ *of*

$$\begin{aligned} &L^p(0,T;\mathbb{L}^q(D)) \cap C([0,T];X^{-b}) \\ \times\ &L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ &L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ &C([0,T];\mathbb{R}) \end{aligned}$$

-valued random variables defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ *such that*

(a) *On*

$$\begin{aligned} &L^p(0,T;\mathbb{L}^q) \cap C([0,T];X^{-b}) \\ \times\ &L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ &L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ &C([0,T];\mathbb{R}), \end{aligned}$$

$$\mathcal{L}(M_n, E_n, B_n, W_j) = \mathcal{L}(M'_n, E'_n, B'_n, W'_{jn}), \qquad \forall n \in \mathbb{N}$$

(b) *There exists a random variable* (M', E', B', W'_j) :

$$\begin{aligned}(\Omega', \mathcal{F}', \mathbb{P}') \longrightarrow\ & L^p(0,T;\mathbb{L}^q) \cap C([0,T];X^{-b}) \\ \times\ & L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ & L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ & C([0,T];\mathbb{R}),\end{aligned}$$

such that

(i) *On*

$$\begin{aligned}& L^p(0,T;\mathbb{L}^q) \cap C([0,T];X^{-b}) \\ \times\ & L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ & L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3)) \\ \times\ & C([0,T];\mathbb{R}),\end{aligned}$$

$$\mathcal{L}(M', E', B', W'_j) = \mu_j,$$

where μ_j *is same as in Proposition 5.35. And*

(ii) $M'_n \longrightarrow M'$ *in* $L^p(0,T;\mathbb{L}^q(D)) \cap C([0,T];X^{-b})$ $\mathbb{P}'$ *almost surely,*

(iii) $E'_n \longrightarrow E'$ *in* $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$ $\mathbb{P}'$ *almost surely,*

(iv) $B'_n \longrightarrow B'$ *in* $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$ $\mathbb{P}'$ *almost surely,*

(v) $W'_{jn} \longrightarrow W'_j$ *in* $C([0,T];\mathbb{R})$ $\mathbb{P}'$ *almost surely.*

To prove Theorem 5.36, we need the following Lemma.

Lemma 5.37 ([15], Thm A.1)**.** *Let X be a topological space such that there exists a sequence $\{f_m\}$ of continuous functions $f_m : X \longrightarrow \mathbb{R}$ that separates points of X. Let us denote by $\mathcal{S}$ the σ-algebra generated by the maps $\{f_m\}$. Then*

(i) *every compact subset of X is metrizable,*

(ii) *if μ_m is a tight sequence of probability measures on $(X, \mathcal{S})$, then there exists a subsequence (m_k), a probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda)$ with X valued Borel measurable variables ξ_k, ξ such that μ_{m_k} is the law of ξ_k and ξ_k converges to ξ almost surely on Ω. Moreover, the law of ξ is a random measure.*

Proof of Theorem 5.36. $L^p(0,T;\mathbb{L}^q(D)) \cap C([0,T];X^{-b})$ and $C([0,T];\mathbb{R})$ are separable metric spaces, so (ii) and (v) of the Proposition are followed by Lemma 2.132. To prove (iii) and (iv) we will use Proposition 5.35 and Lemma 5.37. For this aim we only need to prove that there exist a sequence $\{f_m\}$, $f_m : L^2_w(0,T;\mathbb{L}^2) \longrightarrow \mathbb{R}$ continuous, $\{f_m\}$ separates points of $L^2_w(0,T;\mathbb{L}^2)$ and generates the Borel σ-field on $L^2_w(0,T;\mathbb{L}^2)$.

Let X be a separable Hilbert space, so there is a sequence $\{u_m\}$ dense in X. Let us define $f_m := \langle u_m, \cdot\rangle_X$, then f_m, $m = 1, 2, \dots$ are continuous on X_w.

Suppose that $f_m(v) = \langle u_m, v\rangle = 0$ for all m. Let us fix $u \in X$, since $\{u_m\}$ dense in

X, there exists a subsequence $\{u_{m_k}\}$ such that $u_{m_k} \longrightarrow u$ in X, so $\langle u_{m_k}, v\rangle \longrightarrow \langle u, v\rangle$. Hence $\langle u, v\rangle = 0$, which implies that $v = 0$. Therefore $\{f_m\}$ separates points in X. For any U open in X_w, by the definition of the weak topology, U will be the combination of unions and finite intersections of the following sets:

$$\tilde{U} = \{x \in X : \langle u, x\rangle \in V\},$$

for some $u \in X$ and V is open in $\mathbb{R}$. Next we will prove $\tilde{U}$ can be generated by f_m.

Suppose that $u_{m_k} \longrightarrow u$ in X, so for any $i \geq 0$, $f_{m_k} \longrightarrow \langle u, \cdot\rangle$ uniformly for $\|x\|_X \leq i$. So there exists some $N_i \in \mathbb{N}$ such that if $k \geq N_i$, then $f_{m_k}(x) = \langle u_{m_k}, x\rangle \in V$ for $\|x\|_X \leq i$. So

$$\tilde{U} \subset \bigcup_{i=1}^{\infty} \bigcap_{k \geq N_i} \left[f_{m_k}^{-1}(V) \cap \{\|x\|_X \leq i\}\right].$$

If $x \in \bigcup_{i=1}^{\infty} \bigcap_{k \geq N_i} \left[f_{m_k}^{-1}(V) \cap \{\|x\|_X \leq i\}\right]$, then $\|x\|_X \leq i$ for some $i \geq 1$ and $f_{m_k}(x) = \langle u_{m_k}, x\rangle \in V$ for all $k \geq N_i$, so $\langle u, x\rangle \in V$, so $x \in U$. Hence

$$\tilde{U} \supset \bigcup_{i=1}^{\infty} \bigcap_{k \geq N_i} \left[f_{m_k}^{-1}(V) \cap \{\|x\|_X \leq i\}\right].$$

So

$$\tilde{U} = \bigcup_{i=1}^{\infty} \bigcap_{k \geq N_i} \left[f_{m_k}^{-1}(V) \cap \{\|x\|_X \leq i\}\right].$$

But

$$\bigcup_{i=1}^{\infty} \bigcap_{k \geq N_i} \left[f_{m_k}^{-1}(V) \cap \{\|x\|_X \leq i\}\right] = \bigcup_{i=1}^{\infty} \bigcap_{k \geq N_i} f_{m_k}^{-1}(V).$$

So $\tilde{U} = \bigcup_{i=1}^{\infty} \bigcap_{k \geq N_i} f_{m_k}^{-1}(V)$. Therefore the Borel σ-field $\mathcal{B}(X_w) \subset \sigma\{f_m\}$, but the Borel σ-field is generated by all the continuous functions and f_m are continuous, so $\mathcal{B}(X_w) \supset \sigma\{f_m\}$, hence $\mathcal{B}(X_w) = \sigma\{f_m\}$. And since $L^2(0, T; \mathbb{L}^2(\mathbb{R}^3))$ is a separable Hilbert space, the proof of Theorem 5.36 has been complete. □

Let M'_n, B'_n and E'_n be as in Theorem 5.36, we have the following result:

Proposition 5.38.(i) *$M'_n \in C([0, T]; H_n)$ almost surely and $\mathcal{L}(M'_n) = \mathcal{L}(M_n)$ on $C([0, T]; H_n)$;*
(ii) *$E'_n \in C([0, T]; Y_n)$ almost surely and $\mathcal{L}(E'_n) = \mathcal{L}(E_n)$ on $C([0, T]; Y_n)$;*
(iii) *$B'_n \in C([0, T]; Y_n)$ almost surely and $\mathcal{L}(B'_n) = \mathcal{L}(B_n)$ on $C([0, T]; Y_n)$.*

To prove Proposition 5.38, we need the following Lemma:

Lemma 5.39 ([45], Page 66, Thm 3.12)**.** *Suppose E is a convex subset of a locally convex space X. Then the weak closure $\bar{E}_w$ of E is equal to its original closure $\bar{E}$.*

Proof of 5.38.(i) By the Kuratowski Theorem 2.142, the Borel sets in $C([0, T]; H_n)$ are the Borel sets in $L^p(0, T; \mathbb{L}^1(D)) \cap C([0, T]; X^{-b})$. By Theorem 5.36, $\mathcal{L}(M'_n) = \mathcal{L}(M_n)$ on $L^p(0, T; \mathbb{L}^1(D)) \cap C([0, T]; X^{-b})$, so $\mathcal{L}(M'_n) = \mathcal{L}(M_n)$ on $C([0, T]; H_n)$. And by (5.22), $\mathbb{P}\{M_n \in C([0, T]; H_n)\} = 1$. Hence $\mathbb{P}'\{M'_n \in C([0, T]; H_n)\} = 1$. That is $M'_n \in C([0, T]; H_n)$ almost surely.

(ii) By the Kuratowski Theorem 2.142, the Borel sets in $C([0,T];Y_n)$ are Borel sets in $L^2(0,T;Y_n)$. And since $L^2(0,T;Y_n)$ is closed in $L^2(0,T;\mathbb{L}^2(\mathbb{R}^3))$, by the Lemma 5.39 $L^2(0,T;Y_n)$ is also closed in the space $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$. Hence the Borel sets in $L^2(0,T;Y_n)$ are also Borel sets in $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$. Therefore the Borel sets in $C([0,T];Y_n)$ are the Borel sets in $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$. By Theorem 5.36, $\mathcal{L}(E'_n) = \mathcal{L}(E_n)$ on $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$, so $\mathcal{L}(E'_n) = \mathcal{L}(E_n)$ on $C([0,T];Y_n)$.
And by (5.23), $\mathbb{P}\{E_n \in C([0,T];Y_n)\} = 1$. Hence $\mathbb{P}'\{E'_n \in C([0,T];Y_n)\} = 1$. That is $E'_n \in C([0,T];Y_n)$ almost surely.

(iii) Exactly the same as the proof of (ii).

This complete the proof of 5.38. □

Lemma 5.40. *Let X be a metric space, A is open in X. Let $A_t := \{f \in C([0,T];X) : f(t) \in A\}$, then A_t is open in $C([0,T];X)$.*

Proof. Let us fix $t \in [0,T]$. Then the map

$$i_t : C([0,T];X) \ni f \longmapsto f(t) \in X$$

is linear and bounded and so is continuous. Thus $i_t^{-1}(A)$ is open in $C([0,T];X)$. The equality

$$i_t^{-1}(A) = \{f \in C([0,T];X) : f(t) \in A\}$$

concludes the proof of Lemma 5.40. □

Proposition 5.41. *For $t \in [0,T]$, $\mathcal{L}(E'_n(t)) = \mathcal{L}(E_n(t))$ on Y_n and $\mathcal{L}(B'_n(t)) = \mathcal{L}(B_n(t))$ on Y_n.*

Proof. We will only prove $\mathcal{L}(E'_n(t)) = \mathcal{L}(E_n(t))$, the proof of the result corresponds B_n is the same.
For $t \in [0,T]$, we only need to show that for all $A \subset Y_n$ open, we have $\mathbb{P}(E_n(t) \in A) = \mathbb{P}'(E'_n(t) \in A)$. Let us fix such an open set A and let $A_t := \{f \in C([0,T];Y_n) : f(t) \in A\}$, then by the Lemma 5.40, A_t is open in $C([0,T];Y_n)$. We have

$$\begin{aligned}\{E_n(t) \in A\} &= \{\omega \in \Omega : E_n(t,\omega) \in A\}\\ &= \{\omega \in \Omega : E_n \in C([0,T];Y_n), E_n(t,\omega) \in A\}\\ &= \{\omega \in \Omega : E_n(\omega) \in A_t\}.\end{aligned}$$

Hence

$$\mathbb{P}(\{E_n(t) \in A\}) = \mathbb{P}(\{E_n \in A_t\}).$$

By Proposition 5.38, $\mathcal{L}(E_n) = \mathcal{L}(E'_n)$ on $C([0,T];Y_n)$, so

$$\mathbb{P}(\{E_n \in A_t\}) = \mathbb{P}'(\{E'_n \in A_t\}).$$

Therefore

$$\mathbb{P}\{E_n(t) \in A\} = \mathbb{P}'\{E'_n \in A_t\} = \mathbb{P}'\{E'_n(t) \in A\}.$$

This completes the proof of Proposition 5.41. □

The next result shows that the sequence (M'_n, B'_n, E'_n) satisfies the similar a'priori estimates as (M_n, B_n, E_n) in Proposition 5.23.

Theorem 5.42. *Let us define*

$$\rho_n' := \pi_n\big[-\phi'(M_n') + 1_D(B_n' - \pi_n^Y \tilde{M}_n')\big] + \Delta M_n',$$

Then for all $m \geq 1$, $b > \frac{1}{4}$, there exists $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|M_n'\|_{L^\infty(0,T;H)} \leq \|M_0\|_H, \qquad \mathbb{P}' - a.s., \tag{5.63}$$

$$\mathbb{E}'\|B_n' - \pi_n^Y \tilde{M}_n'\|^m_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C, \tag{5.64}$$

$$\mathbb{E}'\|E_n'\|^m_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C, \tag{5.65}$$

$$\mathbb{E}'\|M_n'\|^m_{L^\infty(0,T;\mathbb{H}^1(D))} \leq C, \tag{5.66}$$

$$\mathbb{E}'\|M_n' \times \rho_n'\|^m_{L^2(0,T;\mathbb{L}^2(D))} \leq C, \tag{5.67}$$

$$\mathbb{E}\|B_n'\|^m_{L^\infty(0,T;\mathbb{L}^2(\mathbb{R}^3))} \leq C, \tag{5.68}$$

$$\mathbb{E}'\left(\int_0^T \left\|M_n'(t) \times (M_n'(t) \times \rho_n'(t))\right\|^2_{\mathbb{L}^{\frac{3}{2}}(D)} \,\mathrm{d}t\right)^{\frac{m}{2}} \leq C, \tag{5.69}$$

$$\mathbb{E}' \int_0^T \left\|\pi_n\left[M_n'(t) \times (M_n'(t) \times \rho_n'(t))\right]\right\|^2_{X^{-b}} \,\mathrm{d}t \leq C, \tag{5.70}$$

$$\mathbb{E}'\left\|\frac{\mathrm{d}E_n'}{\mathrm{d}t}\right\|^m_{L^\infty(0,T;Y')} \leq C. \tag{5.71}$$

$$\mathbb{E}'\left\|\frac{\mathrm{d}B_n'}{\mathrm{d}t}\right\|^m_{L^\infty(0,T;Y')} \leq C. \tag{5.72}$$

Proof. H_n and Y_n are finite dimensional spaces, so the norms on them are all equivalent. Therefore by the Proposition 5.38 and Proposition 5.23, we got the estimates (5.63)-(5.72). □

Notation 5.43. We will use $\mathbb{F}'$ to denote the filtration generated by M' and W' in the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$.

Remark 5.44. From now on will set $p = q = 4$ and $b = \frac{1}{2}$. And that will be enough to show the existence of the solution of the Problem 5.10.

Proposition 5.45. *As defined in Theorem 5.36, the M' satisfies the following estimates:*

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|M'(t)\|_H \le \|M_0\|_H, \qquad \mathbb{P}' - a.s., \tag{5.73}$$

And for some constant $C > 0$,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|M'(t)\|_{X^{-b}} \le C\|M_0\|_H, \qquad \mathbb{P}' - a.s.. \tag{5.74}$$

Proof of (5.73). Since M'_n converges to M' in $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-b})$ $\mathbb{P}'$-almost surely,

$$\lim_{n\to\infty} \int_0^T |M'_n(t) - M'(t)|^4_{\mathbb{L}^4}\, \mathrm{d}t = 0, \quad \mathbb{P}' - a.s.$$

Since $\mathbb{L}^4 \hookrightarrow \mathbb{L}^2$, we infer that

$$\lim_{n\to\infty} \int_0^T |M'_n(t) - M'(t)|^2_H\, \mathrm{d}t = 0.$$

Hence M'_n converges to M' in $L^2(0,T;\mathbb{L}^2)$ $\mathbb{P}'$-almost surely. Therefore by (5.35),

$$\operatorname*{ess\,sup}_{t\in[0,T]} |M'(t)|_{\mathbb{L}^2} \le |M_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s.$$

□

Proof of (5.74). Since $\mathbb{L}^2 \hookrightarrow X^{-b}$ continuously, there exists some constant $C > 0$, such that $|M'_n(t)|_{X^{-b}} \le C|M'_n(t)|_{\mathbb{L}^2}$ for all $n \in \mathbb{N}$. By (5.35), we have

$$\sup_{t\in[0,T]} |M'_n(t)|_{X^{-b}} \le C \sup_{t\in[0,T]} |M'_n(t)|_{\mathbb{L}^2} \le |M_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s.$$

And by Theorem 5.36 (ii), $M'_n(t)$ converges to $M'(t)$ in $C([0,T];X^{-b})$ we infer that

$$\sup_{t\in[0,T]} |M'(t)|_{X^{-b}} \le C|M_0|_{\mathbb{L}^2}, \quad \mathbb{P}' - a.s.$$

□

We continue with investigating properties of the process M', the next result and it's proof are related to the estimate (5.66).

Proposition 5.46. *The process M' define in Theorem 5.36 satisfies:*

$$\mathbb{E}'\big[\operatorname*{ess\,sup}_{t\in[0,T]} \|M'(t)\|^{2r}_V\big] < \infty, \quad r \ge 2. \tag{5.75}$$

Proof. Since $L^{2r}(\Omega';L^\infty(0,T;V))$ is isomorphic to $\big[L^{\frac{2r}{2r-1}}(\Omega';L^1(0,T;X^{-\frac{1}{2}}))\big]^*$, by the estimate (5.66) and the Banach-Alaoglu Theorem we infer that the sequence $\{M'_n\}$ contains a subsequence, denoted in the same way as the full sequence, and there exists an element $v \in L^{2r}(\Omega';L^\infty(0,T;V))$ such that $M'_n \to v$ weakly* in $L^{2r}(\Omega';L^\infty(0,T;V))$. In particular, we have

$$\langle M'_n, \varphi\rangle \to \langle v, \varphi\rangle, \qquad \varphi \in L^{\frac{2r}{2r-1}}(\Omega';(L^1(0,T;X^{-\frac{1}{2}}))).$$

This means that

$$\int_{\Omega'}\int_0^T \langle M_n'(t,\omega),\varphi(t,\omega)\rangle\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega) \to \int_{\Omega'}\int_0^T \langle v(t,\omega),\phi(t,\omega)\rangle\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega).$$

On the other hand, if we fix $\varphi \in L^4(\Omega'; L^{\frac{4}{3}}(0,T;\mathbb{L}^{\frac{4}{3}}))$, by the inequality (4.45) we have

$$\sup_n \int_{\Omega'} \left|\int_0^T {}_{\mathbb{L}^4}\langle M_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\right|^2 \mathrm{d}\mathbb{P}'(\omega) \le \sup_n \int_{\Omega'} \left|\int_0^T |M_n'|_{\mathbb{L}^4}|\varphi|_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\right|^2 \mathrm{d}\mathbb{P}'(\omega)$$

$$\le \sup_n \int_{\Omega'} |M_n'|^2_{L^\infty(0,T;\mathbb{L}^4)}|\varphi|^2_{L^1(0,T;\mathbb{L}^{\frac{4}{3}})}\,\mathrm{d}\mathbb{P}'(\omega) \le \sup_n |M_n'|^2_{L^4(\Omega';L^\infty(0,T;\mathbb{L}^4))}|\varphi|^2_{L^4(\Omega';L^1(0,T;\mathbb{L}^{\frac{4}{3}}))} < \infty.$$

So by Lemma 2.104 the sequence $\int_0^T {}_{\mathbb{L}^4}\langle M_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t$ is uniformly integrable on Ω'. Moreover, by the $\mathbb{P}'$ almost surely convergence of M_n' to M' in $L^4(0,T;\mathbb{L}^4)$, we get $\mathbb{P}'$-a.s.

$$\left|\int_0^T {}_{\mathbb{L}^4}\langle M_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t - \int_0^T {}_{\mathbb{L}^4}\langle M'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\right|$$
$$\le \int_0^T \left|{}_{\mathbb{L}^4}\langle M_n'(t)-M'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\right|\mathrm{d}t \le \int_0^T |M_n'(t)-M'(t)|_{\mathbb{L}^4}|\varphi(t)|_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t$$
$$\le |M_n'(t)-M'(t)|_{L^4(0,T;\mathbb{L}^4)}|\varphi|_{L^{\frac{4}{3}}(0,T;\mathbb{L}^{\frac{4}{3}})} \to 0.$$

Therefore we infer that $\int_0^T {}_{\mathbb{L}^4}\langle M_n'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t$ converges to $\int_0^T {}_{\mathbb{L}^4}\langle M'(t),\varphi(t)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t$ $\mathbb{P}'$ almost surely. Thus by Lemma 2.129,

$$\int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle M_n'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega) \to \int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle M'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega).$$

Hence we deduce that

$$\int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle v(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega) = \int_{\Omega'}\int_0^T {}_{\mathbb{L}^4}\langle M'(t,\omega),\varphi(t,\omega)\rangle_{\mathbb{L}^{\frac{4}{3}}}\,\mathrm{d}t\,\mathrm{d}\mathbb{P}'(\omega)$$

By the arbitrariness of φ and density of $L^4(\Omega'; L^{\frac{4}{3}}(0,T;\mathbb{L}^{\frac{4}{3}}))$ in $L^{\frac{2r}{2r-1}}(\Omega'; L^1(0,T;X^{-\frac{1}{2}}))$, we infer that $M' = v$ and since v satisfies (5.75) we infer that M' also satisfies (5.75). In this way the proof (5.75) is complete. □

We also investigate the following property of B'.

Proposition 5.47.

$$\mathbb{E}'\int_0^T \|B'(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}t < \infty. \tag{5.76}$$

Proof. Since $L^2(\Omega'; L^2(0,T;\mathbb{L}^2(\mathbb{R}^3)))$ is isomorphic to the dual space of itself, by the Banach-Alaoglu Theorem we infer that the sequence $\{B_n'\}$ contains a subsequence, denoted in the same way as the full sequence, and there exists an element

$v \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2(\mathbb{R}^3)))$ such that $B_n' \to v$ weakly* in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2(\mathbb{R}^3)))$. In particular, we have

$$\langle B_n', \varphi \rangle \to \langle v, \varphi \rangle, \qquad \varphi \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2(\mathbb{R}^3))).$$

This means that

$$\int_{\Omega'} \int_0^T \langle B_n'(t,\omega), \varphi(t,\omega) \rangle \, \mathrm{d}t \, \mathrm{d}\mathbb{P}'(\omega) \to \int_{\Omega'} \int_0^T \langle v(t,\omega), \phi(t,\omega) \rangle \, \mathrm{d}t \, \mathrm{d}\mathbb{P}'(\omega).$$

On the other hand, if we fix $\varphi \in L^4(\Omega'; L^2(0,T;\mathbb{L}^2(\mathbb{R}^3)))$, by the inequality (5.68) we have

$$\sup_n \int_{\Omega'} \left| \int_0^T \langle B_n'(t), \varphi(t) \rangle_{\mathbb{L}^2(\mathbb{R}^3)} \, \mathrm{d}t \right|^2 \mathrm{d}\mathbb{P}'(\omega) \le \sup_n \int_{\Omega'} \left| \int_0^T |B_n'|_{\mathbb{L}^2} |\varphi|_{\mathbb{L}^2} \, \mathrm{d}t \right|^2 \mathrm{d}\mathbb{P}'(\omega)$$

$$\le \sup_n \int_{\Omega'} |B_n'|^2_{L^\infty(0,T;\mathbb{L}^2)} |\varphi|^2_{L^2(0,T;\mathbb{L}^2)} \, \mathrm{d}\mathbb{P}'(\omega) \le \sup_n |M_n'|^2_{L^4(\Omega'; L^\infty(0,T;\mathbb{L}^2))} |\varphi|^2_{L^4(\Omega'; L^2(0,T;\mathbb{L}^2))} < \infty.$$

So by Lemma 2.104 the sequence $\int_0^T \langle M_n'(t), \varphi(t) \rangle_{\mathbb{L}^2} \, \mathrm{d}t$ is uniformly integrable on Ω'. Moreover, by the $\mathbb{P}'$ almost surely convergence of B_n' to M' in $L^2_w(0,T;\mathbb{L}^2)$ and by Lemma 2.129,

$$\int_{\Omega'} \int_0^T \langle B_n'(t,\omega), \varphi(t,\omega) \rangle_{\mathbb{L}^2} \, \mathrm{d}t \, \mathrm{d}\mathbb{P}'(\omega) \to \int_{\Omega'} \int_0^T {}_{\mathbb{L}^2}\langle B'(t,\omega), \varphi(t,\omega) \rangle_{\mathbb{L}^2} \, \mathrm{d}t \, \mathrm{d}\mathbb{P}'(\omega).$$

Hence we deduce that

$$\int_{\Omega'} \int_0^T \langle v(t,\omega), \varphi(t,\omega) \rangle_{\mathbb{L}^2} \, \mathrm{d}t \, \mathrm{d}\mathbb{P}'(\omega) = \int_{\Omega'} \int_0^T \langle B'(t,\omega), \varphi(t,\omega) \rangle_{\mathbb{L}^2} \, \mathrm{d}t \, \mathrm{d}\mathbb{P}'(\omega)$$

By the arbitrariness of φ and density of $L^4(\Omega'; L^2(0,T;\mathbb{L}^2))$ in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2(\mathbb{R}^3)))$, we infer that $B' = v$ and since v satisfies (5.76) we infer that M' also satisfies (5.76). In this way the proof (5.76) is complete. □

Next we will strength part (ii) and (iv) of Theorem 5.36 about the convergence.

Proposition 5.48.

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \|M_n'(t) - M'(t)\|^4_{\mathbb{L}^4(D)} \, \mathrm{d}t = 0. \tag{5.77}$$

Proof of (5.77). By the Theorem 5.36, $M_n'(t) \longrightarrow M'(t)$ in $L^4(0,T;\mathbb{L}^4) \cap C([0,T];X^{-b})$ $\mathbb{P}'$-almost surely, $M_n'(t) \longrightarrow M'(t)$ in $L^4(0,T;\mathbb{L}^4)$ $\mathbb{P}'$-almost surely, that is

$$\lim_{n\to\infty} \int_0^T |M_n'(t) - M'(t)|^4_{\mathbb{L}^4} \, \mathrm{d}t = 0, \qquad \mathbb{P}' - a.s.,$$

and by (5.66) and (5.75),

$$\sup_n \mathbb{E}' \left(\int_0^T |M_n'(t) - M'(t)|^4_{\mathbb{L}^4} \, \mathrm{d}t \right)^2 \le 2^7 \sup_n \left(|M_n'|^8_{L^4(0,T;\mathbb{L}^4(D))} + |M'|^8_{L^4(0,T;\mathbb{L}^4(D))} \right) < \infty,$$

hence by Theorem 2.129,

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T |M_n'(t) - M'(t)|^4_{\mathbb{L}^4} \, \mathrm{d}t = \mathbb{E}' \left(\lim_{n\to\infty} \int_0^T |M_n'(t) - M'(t)|^4_{\mathbb{L}^4} \, \mathrm{d}t \right) = 0.$$

This completes the proof. □

Corollary 5.49. *There is a subsequence of* $\{M'_n\}$ *which we can still denote by* $\{M'_n\}$*, such that* $M'_n \longrightarrow M'$ *almost everywhere in* $\Omega' \times [0,T] \times D$*.*

Proof. By (5.77), we have

$$\int_{\Omega'\times[0,T]\times D} |M'_n(\omega,t,x) - M'(\omega,t,x)|^4 \,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\omega \longrightarrow 0.$$

Then by the Proposition 2.153, there is a subsequence of $\{M'_n\}$ which we can still denoted by $\{M'_n\}$, such that $M'_n \longrightarrow M'$ almost everywhere in $\Omega' \times [0,T] \times D$. □

Proposition 5.50.

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \|\pi_n \phi'(M'_n(s)) - \phi'(M'(s))\|^2_{\mathbb{L}^2} \,\mathrm{d}s = 0. \tag{5.78}$$

Proof of (5.78). By Corollary 5.49, $M'_n \longrightarrow M'$ almost everywhere in $\Omega' \times [0,T] \times D$. And since ϕ' is continuous,

$$\lim_{n\to\infty} \left|\phi'(M'_n) - \phi'(M')\right|^2 = 0,$$

almost everywhere in $\Omega' \times [0,T] \times D$. Moreover, ϕ' is bounded, so there exists some constant $C > 0$ such that $|\phi'(x)| \le C$ for all $x \in \mathbb{R}^3$. Therefore for almost every $(\omega,s) \in \Omega' \times [0,T]$,

$$\int_D \left|\phi'(M'_n(\omega,s,x)) - \phi'(M'(\omega,s,x))\right|^4 \,\mathrm{d}x \le 16C^4 m(D) < \infty.$$

Hence $\left|\phi'(M'_n(\omega,s)) - \phi'(M'(\omega,s))\right|^2$ is uniformly integrable on D, so

$$\lim_{n\to\infty} \left\|\phi'(M'_n(\omega,s)) - \phi'(M'(\omega,s))\right\|^2_H = 0, \quad \Omega' \times [0,T] - a.e..$$

Therefore for almost every $(\omega,s) \in \Omega' \times [0,T]$,

$$\begin{aligned} &\left\|\pi_n\phi'(M'_n(\omega,s)) - \phi'(M'(\omega,s))\right\|^2_H \\ &\le 2\left\|\phi'(M'_n(\omega,s)) - \phi'(M'(\omega,s))\right\|^2_H + 2\left\|\pi_n\phi'(M'(\omega,s)) - \phi'(M'(\omega,s))\right\|^2_H \to 0. \end{aligned}$$

Moreover since

$$\mathbb{E}' \int_0^T \left\|\pi_n\phi'(M'_n(\omega,s)) - \phi'(M'(\omega,s))\right\|^4_H \,\mathrm{d}s \le 16TC^4 m(D) < \infty,$$

$\left\|\pi_n\phi'(M'_n) - \phi'(M')\right\|^2_H$ is uniformly integrable on $\Omega' \times [0,T]$. Hence

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \|\pi_n\phi'(M'_n(s)) - \phi'(M'(s))\|^2_{\mathbb{L}^2} \,\mathrm{d}s = 0.$$

This completes the proof of (5.78). □

Proposition 5.51. *For any* $u \in L^2(0,T;H)$*, we have*

$$\lim_{n\to\infty} \mathbb{E}' \left|\int_0^T \langle u(s), \pi_n 1_D (B'_n - B')(s)\rangle_H \,\mathrm{d}s\right| = 0. \tag{5.79}$$

Proof of (5.79). By (iv) of Theorem 5.36, we have

$$\lim_{n\to\infty}\left|\int_0^T \left\langle u(s),\pi_n 1_D(B_n'-B')(s)\right\rangle_H \,\mathrm{d}s\right| = 0,\quad \mathbb{P}'-a.s..$$

Moreover, by (5.68) and (5.76) we have

$$\begin{aligned}&\mathbb{E}'\left|\int_0^T \left\langle u(s),\pi_n 1_D(B_n'-B')(s)\right\rangle_H \,\mathrm{d}s\right|^2\\ &\leq 2\|u\|^2_{L^2(0,T;H)}\mathbb{E}'\left(\int_0^T \|1_D B_n'(s)\|_H^2\,\mathrm{d}s+\int_0^T \|1_D B'(s)\|_H^2\,\mathrm{d}s\right)<\infty.\end{aligned}$$

Hence $\left|\int_0^T \left\langle u(s),\pi_n 1_D(B_n'-B')(s)\right\rangle_H \,\mathrm{d}s\right|$ is uniformly integrable on Ω', so

$$\lim_{n\to\infty}\mathbb{E}'\left|\int_0^T \left\langle u(s),\pi_n 1_D(B_n'-B')(s)\right\rangle_H \,\mathrm{d}s\right| = 0.$$

The proof of (5.79) has been complete. □

Proposition 5.52.

$$\frac{\partial M_n'}{\partial x_i}\longrightarrow\frac{\partial M'}{\partial x_i}\ \textit{weakly in}\ L^2(\Omega';L^2(0,T;\mathbb{L}^2)),\ i=1,2,3. \tag{5.80}$$

Proof. Let us fix $\varphi\in L^2(\Omega';L^2(0,T;V))$, then since $M_n'\longrightarrow M'$ in $L^2(\Omega';L^2(0,T;\mathbb{L}^2(D)))$, we have:

$$\mathbb{E}'\int_0^T\left\langle M',\frac{\partial\varphi}{\partial x_i}\right\rangle_H \mathrm{d}x=\lim_{n\to\infty}\mathbb{E}'\int_0^T\left\langle M_n',\frac{\partial\varphi}{\partial x_i}\right\rangle_H \mathrm{d}x=-\lim_{n\to\infty}\mathbb{E}'\int_0^T\left\langle\frac{\partial M_n'}{\partial x_i},\varphi\right\rangle_H \mathrm{d}x.$$

By the estimate (5.38), $\{M_n'\}_{n=1}^\infty$ is bounded in $L^2(\Omega';L^2(0,T;\mathbb{H}^1))$, so the limit of the right hand side of above equation exists. Hence the result follows. □

Lemma 5.53. *There exists a unique* $\Lambda\in L^2(\Omega';L^2(0,T;H))$ *such that for* $v\in L^2(\Omega';L^2(0,T;\mathbb{W}^{1,4}(D)))$,

$$\mathbb{E}'\int_0^T\langle\Lambda(t),v(t)\rangle_H\,\mathrm{d}t=\sum_{i=1}^3\mathbb{E}'\int_0^T\langle D_iu'(t),u'(t)\times D_iv(t)\rangle_H\,\mathrm{d}t. \tag{5.81}$$

Proof. We will omit"(t)" in this proof. Let us denote $\Lambda_n := M_n'\times AM_n'$. By the estimate (5.39), there exists a constant C such that

$$\|\Lambda_n\|_{L^2(\Omega';L^2(0,T;H))}\leq C,\quad n\in\mathbb{N}.$$

Hence by the Banach-Alaoglu Theorem (Lemma 2.143), there exists $\Lambda\in L^2(\Omega';L^2(0,T;H))$ such that $\Lambda_n\to\Lambda$ weakly in $L^2(\Omega';L^2(0,T;H))$.

Let us fix $v\in L^2(\Omega';L^2(0,T;\mathbb{W}^{1,4}(D)))$. Since $M_n'(t)\in D(A)$ for almost every $t\in[0,T]$ and $\mathbb{P}'$-almost surely, by the Proposition 2.167 and estimate (5.39) again, we have

$$\mathbb{E}'\int_0^T\langle\Lambda_n,v\rangle_H\,\mathrm{d}t=\sum_{i=1}^3\mathbb{E}'\int_0^T\langle D_iM_n',M_n'\times D_iv\rangle_H\,\mathrm{d}t.$$

Moreover, by the results: (5.80), (5.38) and (5.77), we have for $i = 1, 2, 3$,

$$\left|\mathbb{E}'\int_0^T \langle D_iM', M'\times D_iv\rangle_H\,\mathrm{d}t - \mathbb{E}'\int_0^T \langle D_iM'_n, M'_n\times D_iv\rangle_H\,\mathrm{d}t\right|$$

$$\leq \left|\mathbb{E}'\int_0^T \langle D_iM' - D_iM'_n, M'\times D_iv\rangle_H\,\mathrm{d}t\right| + \left|\mathbb{E}'\int_0^T \langle D_iM'_n, (M' - M'_n)\times D_iv\rangle_H\,\mathrm{d}t\right|$$

$$\leq \left|\mathbb{E}'\int_0^T \langle D_iM' - D_iM'_n, M'\times D_iv\rangle_H\,\mathrm{d}t\right| + \Big(\mathbb{E}'\int_0^T \|D_iM'_n\|_H^2\,\mathrm{d}t\Big)^{\frac{1}{2}}$$

$$\times\Big(\mathbb{E}'\int_0^T \|M' - M'_n\|^4_{\mathbb{L}^4(D)}\,\mathrm{d}t\Big)^{\frac{1}{4}}\left(\mathbb{E}'\int_0^T \|D_iv\|^4_{\mathbb{L}^4(D)}\,\mathrm{d}t\right)^{\frac{1}{4}} \to 0.$$

Therefore we infer that

$$\lim_{n\to\infty}\mathbb{E}\int_0^T \langle \Lambda_n, v\rangle_H\,\mathrm{d}t = \sum_{i=1}^3 \mathbb{E}\int_0^T \langle D_iM', M'\times D_iv\rangle\,\mathrm{d}t.$$

Since on the other hand we have proved $\Lambda_n \to \Lambda$ weakly in $L^2(\Omega'; L^2(0,T;H))$ the equality (5.81) follows.
It remains to prove the uniqueness of Λ, but this, because $L^2(\Omega'; L^2(0,T;\mathbb{W}^{1,4}(D)))$ is dense in $L^2(\Omega'; L^2(0,T;H))$, follows from (5.81). This complete the proof of Lemma 5.53. □

Notation 5.54. The process Λ introduced in Lemma 5.53 will be denoted by $M'\times\Delta M'$. Note that $M'\times\Delta M'$ is an element of $L^2(\Omega'; L^2(0,T;H))$ such that for all test functions $v \in L^2(\Omega'; L^2(0,T;\mathbb{W}^{1,4}(D)))$ the following identity holds

$$\mathbb{E}'\int_0^T \langle (M'\times\Delta M')(t), v(t)\rangle_H\,\mathrm{d}t = \sum_{i=1}^3 \mathbb{E}'\int_0^T \langle D_iM'(t), M'(t)\times D_iv(t)\rangle_H\,\mathrm{d}t.$$

Notation 5.55. Since by the estimate (5.38), $M' \in L^2(\Omega', L^\infty(0,T;V))$ and by Notation 5.54, $\Lambda \in L^2(\Omega'; L^2(0,T;H))$, the process $M'\times\Lambda \in L^{\frac{4}{3}}(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}}(D)))$. And $M'\times\Lambda$ will be denoted by $M'\times(M'\times\Delta M')$.

Let us denote:

$$\rho' := -\phi'(M') + 1_D(B') - M' + \Delta M'.$$

Next we will show that the limits of $\{M'_n\times\rho'_n\}_n$, $\{M'_n\times(M'_n\times\rho'_n)\}_n$ and $\{\pi_n(M'_n\times(M'_n\times\rho'_n))\}_n$ are actually $M'\times\rho'$, $M'\times(M'\times\rho')$ and $M'\times(M'\times\rho')$.

Proposition 5.56. *For $p \geq 1$, we can assume that there exist $Z_1 \in L^{2p}(\Omega'; L^2(0,T;H))$, $Z_2 \in L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$ and $Z_3 \in L^2(\Omega'; L^2(0,T;X^{-b}))$, such that*

$$M'_n\times\rho'_n \longrightarrow Z_1 \quad \textit{weakly in } L^{2p}(\Omega'; L^2(0,T;H)), \tag{5.82}$$

$$M'_n\times(M'_n\times\rho'_n) \longrightarrow Z_2 \quad \textit{weakly in } L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}})), \tag{5.83}$$

$$\pi_n(M'_n\times(M'_n\times\rho'_n)) \longrightarrow Z_3 \quad \textit{weakly in } L^2(\Omega'; L^2(0,T;X^{-b})). \tag{5.84}$$

Proof. By Theorem 2.147, the spaces $L^{2p}(\Omega';L^2(0,T;H))$, $L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$ and $L^2(\Omega';L^2(0,T;X^{-b}))$ are reflexive. Then by equations (5.67), (5.69), (5.70) and by the Banach-Alaoglu Theorem (Lemma 2.143), we get equations (5.82), (5.83) and (5.84). □

Proposition 5.57.

$$Z_2 = Z_3 \in L^2(\Omega';L^2(0,T;X^{-b})).$$

Proof. Notice that $(L^{\frac{3}{2}})^* = L^3$, and by Proposition 2.172, $X^b = \mathbb{H}^{2b}$. By Theorem 2.98, $X^b \subset L^3$ for $b > \frac{1}{4}$, hence $L^{\frac{3}{2}} \subset X^{-b}$, so

$$L^2(\Omega';L^2(0,T;\mathbb{L}^{\frac{3}{2}})) \subset L^2(\Omega';L^2(0,T;X^{-b})).$$

Therefore $Z_2 \in L^2(\Omega';L^2(0,T;X^{-b}))$ as well as $Z_3 \in L^2(\Omega';L^2(0,T;X^{-b}))$.
Since $X^b = D(A^b)$ and A is self-adjoint, we can define

$$X^b_n = \left\{ \pi_n x = \sum_{j=1}^{n} x_j e_j : \sum_{j=1}^{\infty} \lambda_j^{2b} x_j^2 < \infty \right\},$$

where e_j, $j = 1,2,\dots$ are eigenvectors of A, λ_j are eigenvalues of A and $x_j = \langle X, e_j\rangle_{\mathbb{L}^2(D)}$. Then $X^b = \bigcup_{n=1}^{\infty} X^b_n$, $L^2(\Omega';L^2(0,T;X^b)) = \bigcup_{n=1}^{\infty} L^2(\Omega';L^2(0,T;X^b_n))$. For $u_n \in L^2(\Omega';L^2(0,T;X^b_n))$, we have for $m \geq n$,

$$\begin{aligned}
&{}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle \pi_m(M'_m \times (M'_m \times \rho'_m)), u_n\rangle_{L^2(\Omega';L^2(0,T;X^b))}\\
&= \mathbb{E}' \int_0^T {}_{X^{-b}}\langle \pi_m(M'_m(t) \times (M'_m(t) \times \rho'_m(t))), u_n(t)\rangle_{X^b}\, \mathrm{d}t\\
&= \mathbb{E}' \int_0^T \langle \pi_m(M'_m(t) \times (M'_m(t) \times \rho'_m(t))), u_n(t)\rangle_{\mathbb{L}^2(D)}\, \mathrm{d}t\\
&= \mathbb{E}' \int_0^T \langle M'_m(t) \times (M'_m(t) \times \rho'_m(t)), u_n(t)\rangle_{\mathbb{L}^2(D)}\, \mathrm{d}t\\
&= \mathbb{E}' \int_0^T {}_{X^{-b}}\langle M'_m(t) \times (M'_m(t) \times \rho'_m(t)), u_n(t)\rangle_{X^b}\, \mathrm{d}t\\
&= {}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle M'_m \times (M'_m \times \rho'_m), u_n\rangle_{L^2(\Omega';L^2(0,T;X^b))}.
\end{aligned}$$

Hence

$${}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle Z_3, u_n\rangle_{L^2(\Omega';L^2(0,T;X^b))} = {}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle Z_2, u_n\rangle_{L^2(\Omega';L^2(0,T;X^b))},$$

$\forall u_n \in L^2(\Omega';L^2(0,T;X^b_n))$. For any $u \in L^2(\Omega';L^2(0,T;X^b)$, there exists $L^2(\Omega';L^2(0,T;X^b_n)) \ni u_n \longrightarrow u$ as $n \longrightarrow \infty$, hence for all $u \in L^2(\Omega';L^2(0,T;X^b)$,

$$\begin{aligned}
{}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle Z_3, \phi\rangle_{L^2(\Omega';L^2(0,T;X^b))} &= \lim_{n\to\infty} {}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle Z_3, u_n\rangle_{L^2(\Omega';L^2(0,T;X^b))}\\
&= \lim_{n\to\infty} {}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle Z_2, u_n\rangle_{L^2(\Omega';L^2(0,T;X^b))}\\
&= {}_{L^2(\Omega';L^2(0,T;X^{-b}))}\langle Z_2, u\rangle_{L^2(\Omega';L^2(0,T;X^b))}
\end{aligned}$$

Therefore $Z_2 = Z_3 \in L^2(\Omega';L^2(0,T;X^{-b}))$ and this concludes the proof. □

Lemma 5.58. *For any measurable process $u \in L^4(\Omega'; L^4(0,T; \mathbb{W}^{1,4}(D)))$, we have the equality*

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle M_n'(s) \times \rho_n'(s), u(s)\rangle_{\mathbb{L}^2(D)} \,\mathrm{d}s \\
&= \mathbb{E}' \int_0^T \langle Z_1(s), u(s)\rangle_{\mathbb{L}^2(D)} \,\mathrm{d}s \\
&= \mathbb{E}' \int_0^T \big\langle M'(s) \times \rho'(s), u(s)\big\rangle_{\mathbb{L}^2(D)} \,\mathrm{d}s.
\end{aligned}$$

Proof. Firstly we show that

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle M_n'(t) \times \Delta M_n'(t), u(t)\rangle_{\mathbb{L}^2} \,\mathrm{d}t = \mathbb{E}' \int_0^T \langle M'(t) \times \Delta M'(t), u(t)\rangle_{\mathbb{L}^2} \,\mathrm{d}t.$$

For each $n \in \mathbb{N}$ we have

$$\langle M_n'(t) \times \Delta M_n'(t), u\rangle_{\mathbb{L}^2} = \sum_{i=1}^3 \left\langle \frac{\partial M_n'(t)}{\partial x_i}, M_n'(t) \times \frac{\partial u(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \tag{5.85}$$

for almost every $t \in [0,T]$ and $\mathbb{P}'$ almost surely. By Propostion 5.38, $\mathbb{P}'(M_n' \in C([0,T]; H_n)) = 1$. For each $i \in \{1,2,3\}$ we may write

$$\begin{aligned}
&\left\langle \frac{\partial M_n'}{\partial x_i}, M_n' \times \frac{\partial u}{\partial x_i} \right\rangle_{\mathbb{L}^2} - \left\langle \frac{\partial M'}{\partial x_i}, M' \times \frac{\partial u}{\partial x_i} \right\rangle_{\mathbb{L}^2} \\
&= \left\langle \frac{\partial M_n'}{\partial x_i} - \frac{\partial M'}{\partial x_i}, M' \times \frac{\partial u}{\partial x_i} \right\rangle_{\mathbb{L}^2} + \left\langle \frac{\partial M_n'}{\partial x_i}, (M_n' - M') \times \frac{\partial u}{\partial x_i} \right\rangle_{\mathbb{L}^2}
\end{aligned} \tag{5.86}$$

Since $\mathbb{L}^4 \hookrightarrow \mathbb{L}^2$ and $\mathbb{W}^{1,4} \hookrightarrow \mathbb{L}^2$, so there are constants C_1 and $C_2 < \infty$ such that

$$\begin{aligned}
&\left\langle \frac{\partial M_n'(t)}{\partial x_i}, (M_n'(t) - M'(t)) \times \frac{\partial u(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \\
&\le \left| \frac{\partial M_n'(t)}{\partial x_i} \right|_{\mathbb{L}^2} \left| (M_n'(t) - M'(t)) \times \frac{\partial u(t)}{\partial x_i} \right|_{\mathbb{L}^2} \\
&\le \left| M_n'(t) \right|_{\mathbb{H}^1} C_1 |M_n'(t) - M'(t)|_{\mathbb{L}^4} C_2 |u(t)|_{\mathbb{W}^{1,4}}
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}' \int_0^T \left| \left\langle \frac{\partial M_n'(t)}{\partial x_i}, (M_n'(t) - M'(t)) \times \frac{\partial u(t)}{\partial x_i} \right\rangle_{\mathbb{L}^2} \right| \mathrm{d}t \\
&\le C_1 C_2 \mathbb{E}' \int_0^T |M_n'(t)|_{\mathbb{H}^1} |M_n'(t) - M'(t)|_{\mathbb{L}^4} |u(t)|_{\mathbb{W}^{1,4}}
\end{aligned}$$

And by the Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}'\int_0^T |M_n'(t)|_{\mathbb{H}^1}|M_n'(t)-M'(t)|_{\mathbb{L}^4}|u(t)|_{\mathbb{W}^{1,4}}\\
\leq &\left(\mathbb{E}'\int_0^T |M_n'(t)|^2_{\mathbb{H}^1}\,\mathrm{d}t\right)^{\frac{1}{2}}\left(\mathbb{E}'\int_0^T |M_n'(t)-M'(t)|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac{1}{4}}\left(\mathbb{E}'\int_0^T |u(t)|^4_{\mathbb{W}^{1,4}}\,\mathrm{d}t\right)^{\frac{1}{4}}\\
\leq &\, T^{\frac{1}{2}}\left(\mathbb{E}'\sup_{t\in[0,T]}|M_n'(t)|^2_{\mathbb{H}^1}\right)^{\frac{1}{2}}\left(\mathbb{E}'\int_0^T |M_n'(t)-M'(t)|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac{1}{4}}\left(\mathbb{E}'\int_0^T |u(t)|^4_{\mathbb{W}^{1,4}}\,\mathrm{d}t\right)^{\frac{1}{4}}
\end{aligned}$$

By (5.38), (5.77) and $u\in L^4(\Omega';L^4(0,T;\mathbb{W}^{1,4}))$, we have

$$\lim_{n\to\infty}\left(\mathbb{E}'\sup_{t\in[0,T]}|M_n'(t)|^2_{\mathbb{H}^1}\right)^{\frac{1}{2}}\left(\mathbb{E}'\int_0^T |M_n'(t)-M'(t)|^4_{\mathbb{L}^4}\,\mathrm{d}t\right)^{\frac{1}{4}}\left(\mathbb{E}'\int_0^T |u(t)|^4_{\mathbb{W}^{1,4}}\,\mathrm{d}t\right)^{\frac{1}{4}}=0$$

Hence

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\left|\left\langle\frac{\partial M_n'(t)}{\partial x_i},(M_n'(t)-M'(t))\times\frac{\partial u(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\right|\mathrm{d}t=0 \tag{5.87}$$

Both M' and $\frac{\partial u}{\partial x_i}$ are in $L^2(\Omega';L^2(0,T;\mathbb{L}^2))$, so $M'\times\frac{\partial u}{\partial x_i}\in L^2(\Omega';L^2(0,T;\mathbb{L}^2))$. Hence by Lemma 2.145, we have

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\left\langle\frac{\partial M_n'(t)}{\partial x_i}-\frac{\partial M'(t)}{\partial x_i},M'(t)\times\frac{\partial u(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\mathrm{d}t=0. \tag{5.88}$$

Therefore by (5.86), (5.87), (5.88),

(5.89)
$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\left\langle\frac{\partial M_n'(t)}{\partial x_i},M_n'(t)\times\frac{\partial u(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\mathrm{d}t=\mathbb{E}'\int_0^T\left\langle\frac{\partial M'(t)}{\partial x_i},M'(t)\times\frac{\partial u(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\mathrm{d}t$$

Then by (5.85), and Notation 5.54 we have

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle M_n'(t)\times\Delta M_n'(t),u(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t\\
=&\ \mathbb{E}'\int_0^T\sum_{i=1}^3\left\langle\frac{\partial M'(t)}{\partial x_i},M'(t)\times\frac{\partial u(t)}{\partial x_i}\right\rangle_{\mathbb{L}^2}\mathrm{d}t\\
=&\ \mathbb{E}'\int_0^T\langle M'(t)\times\Delta M'(t),u(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t.
\end{aligned} \tag{5.90}$$

Secondly we show that

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T\langle M_n'(t)\times\pi_n\phi'(M_n'(t)),u(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t=\mathbb{E}'\int_0^T\langle M'(t)\times\phi'(M'(t)),u(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t.$$

Proof of the above inequality: By (5.77) and (5.78), we have

$$\begin{aligned}
&\left|\mathbb{E}'\int_0^T \langle M_n'(s)\times\pi_n\phi'(M_n'(s)) - M'(s)\times\phi'(M'(s)), u(s)\rangle_H \,\mathrm{d}s\right| \\
&\leq \mathbb{E}'\int_0^T \left|\langle [M_n'(s)-M'(s)]\times u(s), \pi_n\phi'(M_n'(s))\rangle_H\right| \,\mathrm{d}s \\
&\quad + \mathbb{E}'\int_0^T \left|\langle M'(s)\times u(s), \pi_n\phi'(M_n'(s)) - \phi'(M'(s))\rangle_H\right| \,\mathrm{d}s \\
&\leq \left(\mathbb{E}'\int_0^T \|M_n'(s)-M'(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|u(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\phi'(M_n'(s))\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^{\frac12} \\
&\quad + \left(\mathbb{E}'\int_0^T \|M'(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|u(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\pi_n\phi'(M_n'(s)) - \phi'(M'(s))\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^{\frac12} \to 0.
\end{aligned}$$

Finally, we will show that

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T \langle M_n'(t)\times\pi_n 1_D(B_n' - \pi_n^Y\tilde{M}'_n)(t), u(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t = \mathbb{E}'\int_0^T \langle M'(t)\times 1_D(B' - \tilde{M}')(t), u(t)\rangle_{\mathbb{L}^2}\,\mathrm{d}t.$$

Proof of the above equality: By (5.77) and (5.79), we have

$$\begin{aligned}
&\left|\mathbb{E}'\int_0^T \langle M_n'(s)\times\pi_n 1_D(B_n' - \pi_n^Y\tilde{M}'_n)(s) - M'(s)\times 1_D(B' - \tilde{M}')(s), u(s)\rangle_H \,\mathrm{d}s\right| \\
&\leq \mathbb{E}'\int_0^T \left|\langle [M_n'(s)-M'(s)]\times u(s), \pi_n 1_D(B_n' - \pi_n^Y\tilde{M}'_n)(s)\rangle_H\right| \,\mathrm{d}s \\
&\quad + \mathbb{E}'\left|\int_0^T \langle M'(s)\times u(s), \pi_n 1_D(B_n' - \pi_n^Y\tilde{M}'_n)(s) - 1_D(B' - \tilde{M}')(s)\rangle_H \,\mathrm{d}s\right| \\
&\leq \left(\mathbb{E}'\int_0^T \|M_n'(s)-M'(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|u(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14} \\
&\quad \left(\mathbb{E}'\int_0^T \|1_D(B_n' - \pi_n^Y\tilde{M}'_n)(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^{\frac12} \\
&\quad + \left(\mathbb{E}'\int_0^T \|M'(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|u(s)\|_{\mathbb{L}^4}^4 \,\mathrm{d}s\right)^{\frac14}\left(\mathbb{E}'\int_0^T \|\pi_n 1_D\pi_n^Y\tilde{M}'_n(s) - 1_D\tilde{M}'(s)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^{\frac12} \\
&\quad + \mathbb{E}'\left|\int_0^T \langle M'(s)\times u(s), \pi_n 1_D(B_n' - B')(s)\rangle_H \,\mathrm{d}s\right| \longrightarrow 0.
\end{aligned}$$

Hence

$$\lim_{n\to\infty}\mathbb{E}'\int_0^T \langle M_n'(s)\times\rho_n'(s), u(s)\rangle_{\mathbb{L}^2(D)}\,\mathrm{d}s = \mathbb{E}'\int_0^T \langle M'(s)\times\rho'(s), u(s)\rangle_{\mathbb{L}^2(D)}\,\mathrm{d}s.$$

This completes the proof. □

Lemma 5.59. *For any process $\eta \in L^4(\Omega'; L^4(0,T;\mathbb{L}^4(D))$ we have*

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle M_n'(s) \times (M_n'(s) \times \rho_n'(s)), \eta(s)\rangle_{\mathbb{L}^3(D)}\, \mathrm{d}s$$

$$= \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle Z_2(s), \eta(s)\rangle_{\mathbb{L}^3(D)}\, \mathrm{d}s \tag{5.91}$$

$$= \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle M'(s) \times Z_1(s), \eta(s)\rangle_{\mathbb{L}^3(D)}\, \mathrm{d}s \tag{5.92}$$

Proof. Put $Z_{1n} := M_n' \times \rho_n'$ for each $n \in \mathbb{N}$. $L^4(\Omega'; L^4(0,T;\mathbb{L}^4)) \subset L^2(\Omega'; L^2(0,T;\mathbb{L}^3))$ which is the dual space of $L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$. Hence (5.83) implies that (5.91) holds.

$$\begin{aligned} |\eta \times M'|^2_{\mathbb{L}^2} &= \int_D |\eta(x) \times M'(x)|^2\, \mathrm{d}x \\ &\le \int_D |\eta(x)|^2 |M'(x)|^2\, \mathrm{d}x \\ &= \Big\| |\eta| \cdot |M'| \Big\|^2_{\mathbb{L}^2}. \end{aligned}$$

By the Hölder's inequality

$$\Big\| |\eta| \cdot |M'| \Big\|^2_{\mathbb{L}^2} \le |\eta|^2_{\mathbb{L}^4} |M'|^2_{\mathbb{L}^4}.$$

Therefore

$$|\eta \times M'|^2_{\mathbb{L}^2} \le |\eta|^2_{\mathbb{L}^4} |M'|^2_{\mathbb{L}^4} \le |\eta|^4_{\mathbb{L}^4} + |M'|^4_{\mathbb{L}^4}.$$

By (5.77) $M' \in L^4(\Omega'; L^4(0,T;\mathbb{L}^4))$. Then we have

$$\begin{aligned} &\mathbb{E}' \int_0^T |\eta \times M'|^2_{\mathbb{L}^2}\, \mathrm{d}t \\ &\le \mathbb{E}' \int_0^T |\eta|^4_{\mathbb{L}^4}\, \mathrm{d}t + \mathbb{E}' \int_0^T |M'|^4_{\mathbb{L}^4}\, \mathrm{d}t < \infty. \end{aligned}$$

So $\eta \times M' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ and similarly $\eta \times M_n' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$. By (5.82), $Z_{1n} \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$. And $\eta \times M_n' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$. Hence

$$\begin{aligned} {}_{\mathbb{L}^{\frac{3}{2}}}\langle M_n' \times Z_{1n}, \eta\rangle_{\mathbb{L}^3} &= \int_D \langle M_n'(x) \times Z_{1n}(x), \eta(x)\rangle\, \mathrm{d}x \\ &= \int_D \langle Z_{1n}(x), \eta(x) \times M_n'(x)\rangle\, \mathrm{d}x \\ &= \langle Z_{1n}, \eta \times M_n'\rangle_{\mathbb{L}^2} \end{aligned} \tag{5.93}$$

By (5.82), $Z_1 \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$. And $\eta \times M' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$. So

$$
\begin{aligned}
{}_{\mathbb{L}^{\frac{3}{2}}}\langle M' \times Z_1, \eta\rangle_{\mathbb{L}^3} &= \int_D \langle M'(x) \times Z_1(x), \eta(x)\rangle \,\mathrm{d}x \\
&= \int_D \langle Z_1(x), \eta(x) \times M'(x)\rangle \,\mathrm{d}x \qquad (5.94)\\
&= \langle Z_1, \eta \times M'\rangle_{\mathbb{L}^2}
\end{aligned}
$$

By (5.93) and (5.94),

$$
\begin{aligned}
{}_{\mathbb{L}^{\frac{3}{2}}}\langle M_n' \times Z_{1n}, \eta\rangle_{\mathbb{L}^3} - {}_{\mathbb{L}^{\frac{3}{2}}}\langle M' \times Z_1, \eta\rangle_{\mathbb{L}^3} &= \langle Z_{1n}, \eta \times M_n'\rangle_{\mathbb{L}^2} - \langle Z_1, \eta \times M'\rangle_{\mathbb{L}^2} \\
&= \langle Z_{1n} - Z_1, \eta \times M'\rangle_{\mathbb{L}^2} + \langle Z_{1n}, \eta \times (M_n' - M')\rangle_{\mathbb{L}^2}.
\end{aligned}
$$

By (5.82), and since $\eta \times M' \in L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$,

$$
\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle Z_{1n}(s) - Z_1(s), \eta(s) \times M'(s)\rangle_{\mathbb{L}^2} \,\mathrm{d}s = 0.
$$

By the Cauchy-Schwartz inequality

$$
\begin{aligned}
&\langle Z_{1n}, \eta \times (M_n' - M')\rangle_{\mathbb{L}^2}^2 \\
\leq\ & |Z_{1n}|_{\mathbb{L}^2}^2 |\eta \times (M_n' - M')|_{\mathbb{L}^2}^2 \\
\leq\ & |Z_{1n}|_{\mathbb{L}^2}^2 (|\eta|_{\mathbb{L}^4}^4 + |M_n' - M'|_{\mathbb{L}^4}^4) \\
\to\ & 0, \qquad \text{as } n \longrightarrow \infty.
\end{aligned}
$$

Hence

$$
\lim_{n\to\infty} \mathbb{E}' \int_0^T \langle Z_{1n}(s), \eta \times (M_n' - M')(s)\rangle_{\mathbb{L}^2} \,\mathrm{d}s = 0.
$$

Therefore,

$$
\lim_{n\to\infty} \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle M_n'(s) \times (M_n'(s) \times \rho_n'(s)), \eta(s)\rangle_{\mathbb{L}^3} \,\mathrm{d}s = \mathbb{E}' \int_0^T {}_{\mathbb{L}^{\frac{3}{2}}}\langle M'(s) \times Z_1(s), \eta(s)\rangle_{\mathbb{L}^3} \,\mathrm{d}s.
$$

This completes the proof of the Lemma. □

Remark 5.60. Lemma 5.59 has proved that

$$
Z_2 = M' \times (M' \times \rho')
$$

in $L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}}))$. So

$$
M_n' \times (M_n' \times \rho_n') \longrightarrow M' \times (M' \times \rho') \quad \text{weakly in } L^2(\Omega'; L^2(0,T;\mathbb{L}^{\frac{3}{2}})).
$$

The next result will be used to show that the process M' satisfies the condition $|M'(t,x)|_{\mathbb{R}^3} = 1$ for all $t \in [0,T]$, $x \in D$ and $\mathbb{P}'$-almost surely.

Lemma 5.61. *For any bounded measurable function $\varphi : D \longrightarrow \mathbb{R}$, we have*

$$
\langle Z_1(s,\omega), \varphi M'(s,\omega)\rangle_{\mathbb{L}^2(D)} = 0,
$$

for almost every $(s,\omega) \in [0,T] \times \Omega'$.

Proof. Let $B \subset [0,T] \times \Omega'$ be a measurable set and 1_B be the indicator function of B. Then

$$\begin{aligned} &\mathbb{E}' \int_0^T |1_B \varphi M_n'(t) - 1_B \varphi M'(t)|_{\mathbb{L}^2} \, \mathrm{d}t \\ &= \mathbb{E}' \int_0^T |1_B \varphi [M_n'(t) - M'(t)]|_{\mathbb{L}^2} \, \mathrm{d}t \\ &\le |\varphi|_{\mathbb{L}^\infty} \mathbb{E}' \int_0^T |M_n'(t) - M'(t)|_{\mathbb{L}^2} \, \mathrm{d}t \\ &\le C|\varphi|_{\mathbb{L}^\infty} \mathbb{E}' \int_0^T |M_n'(t) - M'(t)|_{\mathbb{L}^4} \, \mathrm{d}t, \end{aligned}$$

for some constant $C > 0$. Hence by (5.77), we have

$$\lim_{n\to\infty} \mathbb{E}' \int_0^T |1_B \varphi M_n'(t) - 1_B \varphi M'(t)|_{\mathbb{L}^2} \, \mathrm{d}t = 0.$$

Together with the fact that $M_n' \times \rho_n'$ converges to Z_1 weakly in $L^2(\Omega'; L^2(0,T;\mathbb{L}^2))$ we can infer that

$$0 = \lim_{n\to\infty} \mathbb{E}' \int_0^T 1_B(s) \langle M_n'(s) \times \rho_n'(s), \varphi M_n'(s) \rangle_{\mathbb{L}^2} \, \mathrm{d}s = \mathbb{E}' \int_0^T 1_B(s) \langle Z_1(s), \varphi M'(s) \rangle_{\mathbb{L}^2} \, \mathrm{d}s.$$

This proves the Lemma. □

5.6. **Prove that the limit process is a weak solution.** Our aim in this subsection is to prove that the process (M', B', E') from Theorem 5.36 is a weak solution of the Problem 5.10.

We define a sequence of H-valued process $(\xi_n(t))_{t\in[0,T]}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\begin{aligned} \xi_n(t) := \; & M_n(t) - M_n(0) - \int_0^t \Big\{ \pi_n [\lambda_1 M_n \times \rho_n] - \lambda_2 \pi_n [M_n \times (M_n \times \rho_n)] \\ & + \frac{1}{2} \sum_{j=1}^\infty \Big[\alpha^2 \pi_n \big[(M_n \times h_j) \times h_j \big] \\ & + \alpha\beta \psi(M_n) \pi_n \big[[M_n \times (M_n \times h_j)] \times h_j \big] \\ & + \beta^2 \psi(M_n) \pi_n \big[M_n \times [M_n \times (M_n \times h_j) \times h_j] \big] \\ & + \alpha\beta \psi(M_n) \pi_n \big[M_n \times [(M_n \times h_j) \times h_j] \big] \\ & + \beta^2 \psi(M_n) \pi_n \big[M_n \times (M_n \times h_j) \times (M_n \times h_j) \big] \Big] \Big\} \, \mathrm{d}s. \end{aligned}$$

By (5.22), we have

$$\xi_n(t) = \sum_{j=1}^\infty \left\{ \int_0^t \Big[\alpha \pi_n \big[M_n \times h_j \big] + \beta \pi_n \big[\psi(M_n) M_n \times (M_n \times h_j) \big] \Big] \, \mathrm{d}W_j(s) \right\}.$$

We also define a sequence of H-valued process $(\xi'_n(t))_{t\in[0,T]}$ on the probability space $(\Omega',\mathcal{F}',\mathbb{P}')$ by

$$
\begin{aligned}
\xi'_n(t) := \; & M'_n(t) - M'_n(0) - \int_0^t \Big\{ \pi_n\left[\lambda_1 M'_n \times \rho'_n\right] - \lambda_2 \pi_n \left[M'_n \times (M'_n \times \rho'_n)\right] \\
& + \frac{1}{2}\sum_{j=1}^{\infty} \Big[\alpha^2 \pi_n \left[(M'_n \times h_j) \times h_j\right] \\
& + \alpha\beta\psi(M'_n)\pi_n \left[[M'_n \times (M'_n \times h_j)] \times h_j\right] \\
& + \beta^2\psi(M'_n)\pi_n \left[M'_n \times [M'_n \times (M'_n \times h_j) \times h_j]\right] \\
& + \alpha\beta\psi(M'_n)\pi_n \left[M'_n \times [(M'_n \times h_j) \times h_j]\right] \\
& + \beta^2\psi(M'_n)\pi_n \left[M'_n \times (M'_n \times h_j) \times (M'_n \times h_j)\right] \Big] \Big\} \, \mathrm{d}s.
\end{aligned}
$$

Lemma 5.62. *For each $t \in [0,T]$ the sequence of random variables $\xi'_n(t)$ converges weakly in $L^2(\Omega'; X^{-b})$ to the limit*

$$
\begin{aligned}
\xi'(t) := \; & M'(t) - M_0 - \int_0^t \Big\{ \left[\lambda_1 M' \times \rho'\right] - \lambda_2 \left[M' \times (M' \times \rho')\right] \\
& + \frac{1}{2}\sum_{j=1}^{\infty} \Big[\alpha^2 \left[(M' \times h_j) \times h_j\right] \\
& + \alpha\beta\psi(M') \left[[M' \times (M' \times h_j)] \times h_j\right] \\
& + \beta^2\psi(M') \left[M' \times [M' \times (M' \times h_j) \times h_j]\right] \\
& + \alpha\beta\psi(M') \left[M' \times [(M' \times h_j) \times h_j]\right] \\
& + \beta^2\psi(M') \left[M' \times (M' \times h_j) \times (M' \times h_j)\right] \Big] \Big\} \, \mathrm{d}s.
\end{aligned}
$$

as $n \to \infty$.

Remark 5.63. There is term B' in ρ', so we can not simply repeat the argument in Section 4. However, the terms contains B'_n have already been dealt with in Lemma 5.58 and Lemma 5.59.

Proof of Lemma 5.62. By Theorem 2.130, the dual space of $L^2(\Omega'; X^{-b})$ is $L^2(\Omega'; X^b)$. Let $t \in (0, T]$ and $U \in L^2(\Omega'; X^b)$. We have

$$
\begin{aligned}
&{}_{L^2(\Omega';X^{-b})}\langle \xi_n'(t), U\rangle_{L^2(\Omega';X^b)} \\
&= \mathbb{E}'\left[{}_{X^{-b}}\langle \xi_n'(t), U\rangle_{X^b}\right] \\
&= \mathbb{E}'\Bigg\{\langle M_n'(t), U\rangle_H - \langle M_n(0), U\rangle_H \\
&-\lambda_1 \int_0^t \langle M_n'(s) \times \rho_n'(s), \pi_n U\rangle_H \,\mathrm{d}s \\
&+\lambda_2 \int_0^t \langle (M_n'(s) \times (M_n'(s) \times \rho_n'(s))), \pi_n U\rangle_H \,\mathrm{d}s \\
&-\frac{1}{2}\sum_{j=1}^{\infty}\Bigg[\alpha^2 \int_0^t \langle (M_n'(s) \times h_j) \times h_j, \pi_n U\rangle_H \,\mathrm{d}s \\
&+\alpha\beta \int_0^t \langle \psi(M_n'(s))[M_n'(s) \times (M_n'(s) \times h_j)] \times h_j, \pi_n U\rangle_H \,\mathrm{d}s \\
&+\beta^2 \int_0^t \langle \psi(M_n'(s))[M_n'(s) \times [M_n'(s) \times (M_n'(s) \times h_j) \times h_j]], \pi_n U\rangle_H \,\mathrm{d}s \\
&+\alpha\beta \int_0^t \langle \psi(M_n'(s))[M_n'(s) \times [(M_n'(s) \times h_j) \times h_j]], \pi_n U\rangle_H \,\mathrm{d}s \\
&+\beta^2 \int_0^t \langle \psi(M_n'(s))[M_n'(s) \times (M_n'(s) \times h_j) \times (M_n'(s) \times h_j)], \pi_n U\rangle_H \,\mathrm{d}s\Bigg]\Bigg\}.
\end{aligned}
$$

By the Theorem 5.36, $M_n' \longrightarrow M'$ in $C([0,T]; X^{-b})$ $\mathbb{P}'$-a.s., so

$$\sup_{t\in[0,T]} |M_n(t) - M(t)|_{X^{-b}} \longrightarrow 0, \quad \mathbb{P}' - a.s.$$

so $M_n'(t) \longrightarrow M'(t)$ in X^{-b} $\mathbb{P}'$-almost surely for any $t \in [0, T]$. And ${}_{X^{-b}}\langle \cdot, U\rangle_{X^b}$ is a continuous function on X^{-b}, hence

$$\lim_{n\to\infty} {}_{X^{-b}}\langle M_n'(t), U\rangle_{X^b} = {}_{X^{-b}}\langle M'(t), U\rangle_{X^b}, \quad \mathbb{P}' - a.s.$$

By (5.63), $\sup_{t\in[0,T]} |M_n'(t)|_H \le |M_0|_H$, since $H \hookrightarrow X^{-\beta}$ continuously, we can find a constant C such that

$$
\begin{aligned}
\sup_n \mathbb{E}'\left[\left|{}_{X^{-b}}\langle M_n'(t), U\rangle_{X^b}\right|^2\right] &\le \sup_n \mathbb{E}'\|U\|_{X^b}^2 \mathbb{E}'\|M_n'(t)\|_{X^{-b}}^2 \\
\le C\mathbb{E}'\|U\|_{X^b}^2 \mathbb{E}' \sup_n \|M_n'(t)\|_H^2 &\le C\mathbb{E}'\|U\|_{X^b}^2 \mathbb{E}'\|M_0\|_H^2 < \infty.
\end{aligned}
$$

Hence the sequence ${}_{X^{-b}}\langle M_n'(t), U\rangle_{X^b}$ is uniformly integrable. So the almost surely convergence and uniform integrability implies that

$$\lim_{n\to\infty} \mathbb{E}'[{}_{X^{-b}}\langle M_n'(t), U\rangle_{X^b}] = \mathbb{E}'[{}_{X^{-b}}\langle M'(t), U\rangle_{X^b}].$$

By (5.82),

$$\lim_{n\to\infty} \mathbb{E}' \int_0^t \langle M_n'(s) \times \rho_n'(s), \pi_n U\rangle_H \,\mathrm{d}s = \mathbb{E}' \int_0^t \langle Z_1(s), U\rangle_H.$$

By (5.84)

$$\lim_{n\to\infty} \mathbb{E}' \int_0^t {}_{X^{-b}}\langle \pi_n(M_n'(s) \times (M_n'(s) \times \rho_n'(s))), U\rangle_{X^b} \,\mathrm{d}s = \mathbb{E}' \int_0^t \langle Z_2(s), U\rangle_{X^b} \,\mathrm{d}s.$$

By the Hölder's inequality 2.141,

$$|M_n'(t) - M'(t)|^2_{\mathbb{L}^2} \leq |M_n'(t) - M'(t)|^2_{\mathbb{L}^4}.$$

Hence by (5.77) and $\sum_{j=1}^{\infty} \|h_j\|^2_{\mathbb{L}^\infty} < \infty$,

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty} \mathbb{E}' \int_0^t \langle ((M_n'(s) - M'(s)) \times h_j) \times h_j, \pi_n U\rangle_H \,\mathrm{d}s\right| \\
&\leq \|U\|_{L^2(\Omega';L^2(0,T;H))} \sum_{j=1}^{\infty} \left(\mathbb{E}' \int_0^t (\|(M_n'(s) - M'(s)) \times h_j) \times h_j\|^2_H \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j=1}^{\infty} \|h_j\|^2_{\mathbb{L}^\infty}\right) \|U\|_{L^2(\Omega';L^2(0,T;H))} \left(\mathbb{E}' \int_0^t \|M_n'(s) - M'(s)\|^2_H \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j=1}^{\infty} \|h_j\|^2_{\mathbb{L}^\infty}\right) \|U\|_{L^2(\Omega';L^2(0,T;H))} \left(\mathbb{E}' \int_0^t \|M_n'(s) - M'(s)\|^2_{\mathbb{L}^4} \,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{j=1}^{\infty} \|h_j\|^2_{\mathbb{L}^\infty}\right) \|U\|_{L^2(\Omega';L^2(0,T;H))} \left(\mathbb{E}' t \int_0^t \|M_n'(s) - M'(s)|^4_{\mathbb{L}^4} \,\mathrm{d}s\right)^{\frac{1}{4}} \longrightarrow 0.
\end{aligned}$$

By Corollary 5.49, $M_n' \longrightarrow M'$ almost everywhere on $\Omega' \times [0,T] \times D$. $\psi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous, so

$$\mathbb{E}' \int_0^t |\psi(M_n'(s)) - \psi(M'(s))| \,\mathrm{d}s \longrightarrow 0, \qquad \text{as } n \longrightarrow \infty.$$

So

$$\Big|\sum_{j=1}^{\infty}\mathbb{E}'\int_0^t\langle\psi(M_n'(s))[M_n'(s)\times(M_n'(s)\times h_j)]\times h_j$$
$$-\psi(M'(s))[M'(s)\times(M'(s)\times h_j)]\times h_j,\pi_nU\rangle_H\,\mathrm{d}s\Big|$$
$$\leq \|U\|_{L^2(\Omega';L^2(0,T;H))}\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{L}^\infty(D)}\mathbb{E}'\int_0^t\Big\|\psi(M_n')[M_n'\times(M_n'\times h_j)]$$
$$-\psi(M')[M'\times(M'\times h_j)]\Big\|_H\,\mathrm{d}s$$
$$\leq \|U\|_{L^2(\Omega';L^2(0,T;H))}\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{L}^\infty(D)}\mathbb{E}'\int_0^t\Big|\psi(M_n')-\psi(M')\Big|\Big\|M_n'\times(M_n'\times(M_n'\times h_j)\Big\|_H$$
$$+\Big\|M_n'\times(M_n'\times h_j)-M'\times(M'\times h_j)\Big\|_H\,\mathrm{d}s$$
$$\leq \|U\|_{L^2(\Omega';L^2(0,T;H))}\mathbb{E}'\int_0^t\Big|\psi(M_n')-\psi(M')\Big|\|M_n'\|^2_{\mathbb{L}^4(D)}$$
$$+\|M_n'-M'\|^2_{\mathbb{L}^4(D)}(\|M_n'\|^2_{\mathbb{L}^4(D)}+\|M'\|^2_{\mathbb{L}^4(D)})\,\mathrm{d}s\sum_{j=1}^{\infty}\|h_j\|^2_{\mathbb{L}^\infty(D)}\longrightarrow 0.$$

Noticed that we can assume that $|M_n'|_{\mathbb{R}^3}<5$ and $|M'|_{\mathbb{R}^3}<5$, we also have:

$$\Big|\sum_{j=1}^{\infty}\mathbb{E}'\int_0^t\langle\psi(M_n'(s))M_n'(s)\times[M_n'(s)\times(M_n'(s)\times h_j)\times h_j]$$
$$-\psi(M'(s))M'\times[M'(s)\times(M'(s)\times h_j)\times h_j],\pi_nU\rangle_H\,\mathrm{d}s\longrightarrow 0,$$

$$\Big|\sum_{j=1}^{\infty}\mathbb{E}'\int_0^t\langle\psi(M_n'(s))M_n'(s)\times[(M_n'(s)\times h_j)\times h_j]$$
$$-\psi(M'(s))M'\times[(M'(s)\times h_j)\times h_j],\pi_nU\rangle_H\,\mathrm{d}s\longrightarrow 0,$$

and

$$\Big|\sum_{j=1}^{\infty}\mathbb{E}'\int_0^t\langle\psi(M_n'(s))[M_n'(s)\times(M_n'(s)\times h_j)]\times(M_n'\times h_j)$$
$$-\psi(M'(s))[M'(s)\times(M'(s)\times h_j)]\times(M'\times h_j),\pi_nU\rangle_H\,\mathrm{d}s\longrightarrow 0.$$

Hence

$$
\begin{aligned}
&\lim_{n\to\infty} {}_{L^2(\Omega';X^{-b})}\langle \xi'_n(t), U\rangle_{L^2(\Omega';X^b)}\\
= &\ \mathbb{E}'\Big[{}_{X^{-b}}\langle M'(t), U\rangle_{X^b} - {}_{X^{-\beta}}\langle M_0, U\rangle_{X^b}\\
&-\lambda_1 \int_0^t \langle Z_1(s), U\rangle_H \,\mathrm{d}s\\
&+\lambda_2 \int_0^t {}_{X^{-b}}\langle Z_2(s), U\rangle_{X^b} \,\mathrm{d}s\\
&-\frac{1}{2}\sum_{j=1}^{\infty}\Big[\alpha^2 \int_0^t \langle (M'(s)\times h_j)\times h_j, U\rangle_H \,\mathrm{d}s\\
&+\alpha\beta \int_0^t \langle \psi(M'(s))[M'(s)\times (M'(s)\times h_j)]\times h_j, U\rangle_H \,\mathrm{d}s\\
&+\beta^2 \int_0^t \langle \psi(M'(s))[M'(s)\times [M'(s)\times (M'(s)\times h_j)\times h_j]], U\rangle_H \,\mathrm{d}s\\
&+\alpha\beta \int_0^t \langle \psi(M'(s))[M'(s)\times [(M'(s)\times h_j)\times h_j]], U\rangle_H \,\mathrm{d}s\\
&+\beta^2 \int_0^t \langle \psi(M'(s))[M'(s)\times (M'(s)\times h_j)\times (M'(s)\times h_j)], U\rangle_H \,\mathrm{d}s\Big]\Big\}.
\end{aligned}
$$

Since by Lemma 5.58 and Lemma 5.59, we have $Z_1 = M'\times\rho'$ and $Z_2 = M'\times(M'\times\rho')$. Therefore for any $U\in L^2(\Omega';X^b)$,

$$
\lim_{n\to\infty} {}_{L^2(\Omega';X^{-b})}\langle \xi'_n(t), U\rangle_{L^2(\Omega';X^b)} = {}_{L^2(\Omega';X^{-b})}\langle \xi'(t), U\rangle_{L^2(\Omega';X^b)}.
$$

This concludes the proof. □

Lemma 5.64. *For $j = 1,2,\ldots$, suppose the W'_{jn} defined in $(\Omega',\mathcal{F}',\mathbb{P}')$ has the same distribution as the Brownian Motion W_j defined in $(\Omega,\mathcal{F},\mathbb{P})$ as in Propsition 5.36. Then W'_{jn} is also a Brownian Motion.*

Proof. This is same as the proof of Lemma 4.34. □

Lemma 5.65. *For $j = 1,2,\ldots$, the processes $(W'_j(t))_{t\in[0,T]}$ are real-valued Brownian Motion on $(\Omega',\mathcal{F}',\mathbb{P}')$ and if $0\le s<t\le T$ then the increment $W'_j(t)-W'_j(s)$ is independent of the σ-algebra generated by $M'(s_1)$ and $W'(s_1)$ for $s_1\in[0,s]$.*

Proof. This is same as the proof of Lemma 4.35. □

Lemma 5.66. *For each $m\in\mathbb{N}$, we define the partition $\{s_i^m := \frac{iT}{m},\ i = 0,\ldots,m\}$ of $[0,T]$. Then for any $\varepsilon>0$, We can choose $m\in\mathbb{N}$ large enough such that:*

(i)

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\int_0^t\Big[\alpha\pi_n\Big[M_n'(s)\times h_j\Big]+\beta\pi_n\Big[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\Big]\Big]$$
$$-\sum_{i=0}^{m-1}\Big[\alpha\pi_n\Big[M_n'(s_i^m)\times h_j\Big]+\beta\pi_n\Big[\psi(M_n'(s_i^m))M_n'(s_i^m)$$
$$\times(M_n'(s_i^m)\times h_j)\Big]\Big]1_{(s_i^m,s_{i+1}^m]}(s)\,\mathrm{d}W_{jn}'(s)\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}<\frac{\varepsilon}{2};$$

(ii)

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}[\alpha\pi_n\Big[M_n'(s_i^m)\times h_j\Big]+\beta\pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m)$$
$$\times(M_n'(s_i^m)\times h_j)]](W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m))$$
$$-\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\Big[\alpha\pi_n\Big[M_n'(s_i^m)\times h_j\Big]+\beta\pi_n\Big[\psi(M_n'(s_i^m))M_n'(s_i^m)$$
$$\times(M_n'(s_i^m)\times h_j)\Big]\Big](W_j'(t\wedge s_{i+1}^m)-W_j'(t\wedge s_i^m))\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}=0;$$

(iii)

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\int_0^t\sum_{j=1}^{\infty}\sum_{i=1}^{m-1}\Big[\alpha\pi_n\Big[M_n'(s_i^m)\times h_j\Big]$$
$$+\beta\pi_n\Big[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)\Big]\Big]1_{(s_i^m,s_{i+1}^m]}(s)$$
$$-\sum_{j=1}^{\infty}\Big[\alpha\pi_n\Big[M_n'(s)\times h_j\Big]+\beta\pi_n\Big[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\Big]\Big]\,\mathrm{d}W_j'(s)\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}<\frac{\varepsilon}{2};$$

(iv)

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\int_0^t\sum_{j=1}^{\infty}\Big[\alpha\pi_n\Big[M_n'(s)\times h_j\Big]+\beta\pi_n\Big[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\Big]\Big]$$
$$-\sum_{j=1}^{\infty}\Big[\alpha M'(s)\times h_j+\beta\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big]\,\mathrm{d}W_j'(s)\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}=0.$$

Proof.(i) Firstly let us consider:

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\int_0^t\alpha\pi_n\Big[M_n'(s_i^m)\times h_j\Big]-\sum_{i=0}^{m-1}\Big[\alpha\pi_n\Big[M_n'(s_i^m)\times h_j\Big]\,1_{(s_i^m,s_{i+1}^m]}(s)\,\mathrm{d}W_{jn}'(s)\Bigg\|_{X^{-b}}\Bigg)^{\frac{1}{2}}.$$

By the Itô isometry,

$$\left(\mathbb{E}'\left[\left\|\sum_{j=1}^{\infty}\int_0^t(\pi_n(M_n'(s)\times h_j)-\sum_{i=0}^{m-1}\pi_n(M_n'(s_j^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s))\,\mathrm{d}W_{jn}'(s)\right\|_{X^{-b}}^2\right]\right)^{\frac{1}{2}}$$

$$\leq \sum_{j=1}^{\infty}\left(\mathbb{E}'\left[\int_0^t\left\|(\pi_n(M_n'(s)\times h_j)-\sum_{j=0}^{m-1}\pi_n(M_n'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s))\right\|_{X^{-b}}^2\,\mathrm{d}s\right]\right)^{\frac{1}{2}}$$

$$\leq \sum_{j=1}^{\infty}\left(\mathbb{E}'\left[\int_0^t\left\|M_n'(s)\times h_j-\sum_{j=0}^{m-1}M_n'(s_i^m)\times h_j1_{(s_i^m,s_{i+1}^m]}(s)\right\|_{X^{-b}}^2\,\mathrm{d}s\right]\right)^{\frac{1}{2}}$$

$$\leq \sum_{j=1}^{\infty}\left(\mathbb{E}'\int_0^t\left\|M_n'(s)\times h_j-M'(s)\times h_j\right\|_{X^{-b}}^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$

$$+\sum_{j=1}^{\infty}\left(\mathbb{E}'\int_0^t\left\|M'(s)\times h_j-\sum_{i=0}^{m-1}(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\right\|_{X^{-b}}^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$

$$+\sum_{j=1}^{\infty}\left(\mathbb{E}'\int_0^t\left\|\sum_{i=0}^{m-1}(M'(s_i^m)-M_n'(s_i^m))\times h_j1(s_i^m,s_{i+1}^m](s)\right\|_{X^{-b}}^2\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

There exists some constant $C>0$ such that

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}\left(\mathbb{E}'\int_0^t\left\|M_n'(s)\times h_j-M'(s)\times h_j\right\|_{X^{-b}}^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$

$$\leq \sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\lim_{n\to\infty}\left(\mathbb{E}'\int_0^t\|M_n'(s)-M'(s)\|_{X^{-b}}^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$

$$\leq C\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\lim_{n\to\infty}\left(\mathbb{E}'\int_0^t\|M_n'(s)-M'(s)\|_{\mathbb{L}^4(D)}^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$

$$\leq C\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}T\lim_{n\to\infty}\left(\mathbb{E}'\int_0^t\|M_n'(s)-M'(s)\|_{\mathbb{L}^4(D)}^4\,\mathrm{d}s\right)^{\frac{1}{4}}$$

$$=0.$$

$M'\in C([0,T];X^{-b})$ $\mathbb{P}'-a.s.$, so

$$\lim_{m\to\infty}\sup_{s\in[0,T]}\left\|M'(s)-\sum_{i=0}^{m-1}M'(s_i^m)1_{(s_i^m,s_{i+1}^m]}(s)\right\|_{X^{-b}}=0,\quad \mathbb{P}'-a.s.$$

Hence

$$\int_0^t \left\| M'(s) - \sum_{i=0}^{m-1} M'(s_i^m) 1_{(s_i^m, s_{i+1}^m]}(s) \right\|_{X^{-b}}^2 \mathrm{d}s \longrightarrow 0, \quad \mathbb{P}' - a.s.$$

And by Sobolev embedding,

$$\begin{aligned}
&\sup_m \mathbb{E}' \left(\int_0^t \left\| M'(s) - \sum_{i=0}^{m-1} M'(s_i^m) 1_{(s_i^m, s_{i+1}^m]}(s) \right\|_{X^{-b}}^2 \mathrm{d}s \right)^2 \\
\leq\ &\sup_m \mathbb{E}' \left(\int_0^T 2\|M'(s)\|_{X^{-b}}^2 + 2 \left\| \sum_{i=0}^{m-1} M'(s_j^m) 1_{(s_i^m, s_{i+1}^m]}(s) \right\|_{X^{-b}}^2 \mathrm{d}s \right)^2 \\
\leq\ &\cdots
\end{aligned}$$

Then by (5.63), $\sup_{t\in[0,T]} \|M'(t)\|_{X^{-b}} \leq \|M_0\|_H$, $\mathbb{P}'$-almost surely,

$$\cdots \leq 16T^2 \mathbb{E}'(\|M_0\|_H^4) = 16T^2 \|M_0\|_H^4 < \infty.$$

Hence $\int_0^t \left\| M'(s) - \sum_{i=0}^{m-1} M'(s_i^m) 1_{(s_i^m, s_{i+1}^m]}(s) \right\|_{X^{-b}}^2 \mathrm{d}s$ is uniform integrable. Therefore

$$\begin{aligned}
&\lim_{m\to\infty} \sum_{j=1}^\infty \left(\mathbb{E}' \int_0^t \left\| M'(s) \times h_j - \sum_{i=0}^{m-1} (M'(s_i^m) \times h_j) 1_{(s_i^m, s_{i+1}^m]}(s) \right\|_{X^{-b}}^2 \mathrm{d}s \right)^{\frac{1}{2}} \\
\leq &\sum_{j=1}^\infty \|h_j\|_{\mathbb{W}^{1,\infty}} \lim_{m\to\infty} \left(\mathbb{E}' \int_0^t \left\| M'(s) - \sum_{i=0}^{m-1} M'(s_i^m) 1_{(s_i^m, s_{i+1}^m]}(s) \right\|_{X^{-b}}^2 \mathrm{d}s \right)^{\frac{1}{2}} = 0.
\end{aligned}$$

Hence for $\varepsilon > 0$ we can choose $m \in \mathbb{N}$ such that

$$\sum_{j=1}^\infty \left(\mathbb{E}' \int_0^t \left\| M'(t) \times h_j - \sum_{i=0}^{m-1} (M'(s_i^m) \times h_j) 1_{(s_i^m, s_{i+1}^m]}(s) \right\|_{X^{-b}}^2 \mathrm{d}s \right)^{\frac{1}{2}} < \frac{\varepsilon}{4\alpha}.$$

Again since $M_n' \longrightarrow M'$ in $C([0,T]; X^{-b})$, we have

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \|M'(s) - M_n'(s)\|_{X^{-b}}^2 = 0.$$

There exists a constant $C > 0$ such that $\sup_{t\in[0,T]} \|M'(t)\|_{X^{-b}} \leq C\|M_0\|_H$, $\mathbb{P}'$-almost surely, we have

$$\begin{aligned}
&\sup_n \mathbb{E}' \left(\sup_{s\in[0,T]} \|M'(s) - M_n'(s)\|_{X^{-b}}^2 \right)^2 \\
\leq\ &\sup_n \mathbb{E}' \left(\sup_{s\in[0,T]} \|M'(s)\|_{X^{-b}} + \sup_{s\in[0,T]} \|M_n'(s)\|_{X^{-b}}^2 \right)^2 \leq 4C^4 \|M_0\|_H^4 < \infty.
\end{aligned}$$

Hence by Theorem 2.104, $\sup_{s\in[0,T]}\|M'(s)-M'_n(s)\|^2_{X^{-b}}$ is uniformly integrable. Hence

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}\left(\mathbb{E}'\int_0^t\left\|\sum_{i=0}^{m-1}(M'(s_i^m)-M'_n(s_i^m))\times h_j 1_{(s_i^m,s_{i+1}^m]}(s)\right\|^2_{X_{-b}}\mathrm{d}s\right)^{\frac{1}{2}}$$
$$\leq\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\lim_{n\to\infty}\left(\mathbb{E}'\int_0^t\left\|\sum_{i=0}^{m-1}(M'(s_i^m)-M'_n(s_i^m))1_{(s_i^m,s_{i+1}^m]}(s)\right\|^2_{X_{-b}}\mathrm{d}s\right)^{\frac{1}{2}}$$
$$\leq\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\lim_{n\to\infty}\left(\sum_{i=0}^{m-1}(s_{i+1}^m-s_i^m)\mathbb{E}'\left\|M'(s_i^m)-M'_n(s_i^m)\right\|^2_{X^{-b}}\right)^{\frac{1}{2}}=0.$$

Therefore we have get for any $\varepsilon>0$ we can choose large enough m, such that

$$\lim_{n\to\infty}\left(\mathbb{E}'\left[\left\|\int_0^t(\pi_n(M'_n(s)\times h_j)-\sum_{i=0}^{m-1}\pi_n(M'_n(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s))\,\mathrm{d}W'_{jn}(s)\right\|^2\right]\right)^{\frac{1}{2}}<\frac{\varepsilon}{4\alpha};$$

Secondly, we consider

$$\lim_{n\to\infty}\left(\mathbb{E}'\left\|\sum_{j=1}^{\infty}\int_0^t\pi_n[\psi(M'_n(s))M'_n(s)\times(M'_n(s)\times h_j)]\right.\right.$$
$$\left.\left.-\sum_{i=0}^{m-1}\pi_n[\psi(M'_n(s_i^m))M'_n(s_i^m)\times(M'_n(s_i^m)\times h_j)]1_{(s_i^m,s_{i+1}^m]}(s)\,\mathrm{d}W'_{jn}(s)\right\|^2_{X^{-b}}\right)^{\frac{1}{2}}.$$

Similar as before, we have

$$\begin{aligned}
&\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\int_0^t \pi_n[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)]\\
&-\sum_{i=0}^{m-1}\pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)]1_{(s_i^m,s_{i+1}^m]}(s)\,\mathrm{d}W_{jn}'(s)\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}\\
\le{}& \sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\Bigg\|\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\\
&-\sum_{i=0}^{m-1}\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_{X^{-b}}^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\\
\le{}& \sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\left\|\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)-\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right\|_{X^{-b}}^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\\
&+\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\Bigg\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)\\
&-\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_{X^{-b}}^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\\
&+\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\Bigg\|\sum_{i=0}^{m-1}\Big[\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)\\
&-\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)\Big]1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_{X^{-b}}^2\mathrm{d}s\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

There exists a constant $C>0$, such that

$$\begin{aligned}
&\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\left\|\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)-\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right\|_{X^{-b}}^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\\
\le{}& C\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\left\|\psi(M_n'(s))(M_n'(s)-M'(s))\times(M_n'(s)\times h_j)\right\|_H^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\\
&+C\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\left\|\psi(M_n'(s))M'(s)\times((M_n'(s)-M'(s))\times h_j)\right\|_H^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\\
&+C\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\left\|[\psi(M_n'(s))-\psi(M'(s))]M'(s)\times(M'(s)\times h_j)\right\|_H^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\longrightarrow 0,\quad \text{as } n\to\infty.
\end{aligned}$$

For

$$\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\Bigg\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)$$
$$-\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_{X^{-b}}^2\mathrm{d}s\Bigg)^{\frac{1}{2}}$$

There exists a constant $C>0$ such that:

$$\sum_{j=1}^{\infty}\sup_m\Bigg(\mathbb{E}'\int_0^t\Bigg\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)$$
$$-\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_{X^{-b}}^2\mathrm{d}s\Bigg)^{\frac{1}{2}}$$
$$\leq C\sum_{j=1}^{\infty}\sup_m\Bigg\{\Bigg(\mathbb{E}'\int_0^t\Big\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big\|_H^2\mathrm{d}s\Bigg)^{\frac{1}{2}}$$
$$+\Bigg(\mathbb{E}'\int_0^t\Bigg\|\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_H^2\mathrm{d}s\Bigg)^{\frac{1}{2}}\Bigg\}$$

Since

$$\Bigg(\mathbb{E}'\int_0^t\Big\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big\|_H^2\mathrm{d}s\Bigg)^{\frac{1}{2}}$$
$$\leq\left(\mathbb{E}'\int_0^T\|\psi(M'(s))M'(s)\|_{\mathbb{L}^4}^4\,\mathrm{d}s\right)^{\frac{1}{4}}\left(\mathbb{E}'\int_0^T\|M'(s)\|_{\mathbb{L}^4}^4\|h_j\|_{\mathbb{L}^\infty}^4\,\mathrm{d}s\right)^{\frac{1}{4}}$$
$$\leq\left(\mathbb{E}'\int_0^T\|M'(s)\|_{\mathbb{L}^4}^4\,\mathrm{d}s\right)^{\frac{1}{2}}\|h_j\|_{\mathbb{L}^\infty}$$

and

$$\Bigg(\mathbb{E}'\int_0^t\Bigg\|\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_H^2\mathrm{d}s\Bigg)^{\frac{1}{2}}$$
$$\leq\left(\mathbb{E}'\sum_{i=0}^{m-1}\int_D|\psi(M'(s_i^m,x))|^2|M'(s_i^m,x)|^4|h_j(x)|^2\,\mathrm{d}x\frac{T}{m}\right)^{\frac{1}{2}}$$
$$\leq T^{\frac{1}{2}}R^2\|h_j\|_H,$$

we have

$$\sum_{j=1}^{\infty}\sup_m\left(\mathbb{E}'\int_0^t\left\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right.\right.$$
$$\left.\left.-\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\right\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}$$
$$\leq C\sum_{j=1}^{\infty}\left\{\left(\mathbb{E}'\int_0^T\|M'(s)\|_{\mathbb{L}^4}^4\,\mathrm{d}s\right)\frac{1}{2}\|h_j\|_{\mathbb{L}^\infty}+T^{\frac{1}{2}}R^2\|h_j\|_H\right\}<\infty.$$

Let us denote

$$\eta_{j,m}:=\left(\mathbb{E}'\int_0^t\left\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right.\right.$$
$$\left.\left.-\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\right\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}>0.$$

Then we have $\sum_{j=1}^{\infty}\sup_m\eta_{j,m}<\infty$. And since $\psi(M')M'\times(M'\times h_j):[0,T]\longrightarrow X^{-b}$ is continuous and $\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}$ is the approximation function of $\psi(M')M'\times(M'\times h_j)$, we have $\lim_{m\to\infty}\eta_{j,m}=0$. So by the Lebesgue's Dominated Convergence Theorem, we have $\lim_{m\to\infty}\sum_{j=1}^{\infty}\eta_{j,m}=0$. So we can choose m large enough such that

$$\left(\mathbb{E}'\int_0^t\left\|\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right.\right.$$
$$\left.\left.-\sum_{i=0}^{m-1}\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)1_{(s_i^m,s_{i+1}^m]}(s)\right\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}<\frac{\varepsilon}{4\beta}.$$

And there exists some constant $C > 0$ such that

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}\Bigg(\mathbb{E}'\int_0^t\Bigg\|\sum_{i=0}^{m-1}\Big[\psi(M'(s_i^m))M'(s_i^m)\times(M'(s_i^m)\times h_j)$$
$$-\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)\Big]1_{(s_i^m,s_{i+1}^m]}(s)\Bigg\|_{X^{-b}}^2\,\mathrm{d}s\Bigg)^{\frac{1}{2}}$$
$$\leq \lim_{n\to\infty}\sqrt{\frac{T}{m}}\sum_{j=1}^{\infty}\Bigg\{\Bigg(\mathbb{E}'\sum_{i=0}^{m-1}\Big\|\Big[\psi(M'(s_i^m))M'(s_i^m)-\psi(M_n'(s_i^m))M_n'(s_i^m)\Big]\times(M'(s_i^m)\times h_j)\Big\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}$$
$$+\Bigg(\mathbb{E}'\sum_{i=0}^{m-1}\Big\|\psi(M_n'(s_i^m))M_n'(s_i^m)\times\Big[(M_n'(s_i^m)-M_n'(s_i^m))\times h_j\Big]\Big\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}\Bigg\}$$
$$\leq C\sqrt{\frac{T}{m}}\lim_{n\to\infty}\Bigg(\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\Big(\mathbb{E}'\big\|\psi(M'(s_i^m))M'(s_i^m)-\psi(M_n'(s_i^m))M_n'(s_i^m)\big\|_{\mathbb{L}^4}^4\Big)^{\frac{1}{4}}\Big(\mathbb{E}'\big\|M'(s_i^m)\times h_j\big\|_{\mathbb{L}^4}^4\Big)^{\frac{1}{4}}$$
$$+R\sum_{i=0}^{m-1}\Bigg(\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{L}^\infty}\Bigg)\Big(\mathbb{E}'\big\|M'(s_i^m)-M_n'(s_i^m)\big\|_{\mathbb{L}^4}^4\Big)^{\frac{1}{4}}\Bigg)=0.$$

So, we can choose m large enough such that

$$\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\int_0^t\pi_n[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)]$$
$$-\sum_{i=0}^{m-1}\pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)]1_{(s_i^m,s_{i+1}^m]}(s)\,\mathrm{d}W_{jn}'(s)\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}<\frac{\varepsilon}{4\beta}.$$

Therefore, we can choose m large enough such that

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\int_0^t\Big[\alpha\pi_n\Big[M_n'(s)\times h_j\Big]+\beta\pi_n\Big[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\Big]\Big]$$
$$-\sum_{i=0}^{m-1}\Big[\alpha\pi_n\Big[M_n'(s_i^m)\times h_j\Big]+\beta\pi_n\Big[\psi(M_n'(s_i^m))M_n'(s_i^m)$$
$$\times(M_n'(s_i^m)\times h_j)\Big]\Big]1_{(s_i^m,s_{i+1}^m]}(s)\,\mathrm{d}W_{jn}'(s)\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}<\frac{\varepsilon}{2}.$$

(ii) We only need to prove:

(a)

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\pi_n[M_n'(s_i^m)\times h_j](W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m))$$
$$-\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\pi_n[M_n'(s_i^m)\times h_j](W_j'(t\wedge s_{i+1}^m)-W_j'(t\wedge s_i^m))\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}=0.$$

And
(b)

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)](W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m))$$
$$-\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)]](W_j'(t\wedge s_{i+1}^m)-W_j'(t\wedge s_i^m))\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}=0.$$

Proof of (a) :

$$\Bigg(\mathbb{E}'\Bigg\|\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\pi_n[M_n'(s_i^m)\times h_j](W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m))$$
$$-\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\pi_n[M_n'(s_i^m)\times h_j](W_j'(t\wedge s_{i+1}^m)-W_j'(t\wedge s_i^m))\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}}$$
$$\leq\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\Bigg(\mathbb{E}'\left\|\pi_n\left[M_n'(s_i^m)\times h_j\right]\right\|_{X^{-b}}^2\Big|W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m)$$
$$-W_j'(t\wedge s_{i+1}^m)+W_j'(t\wedge s_i^m)\Big|^2\Bigg)^{\frac{1}{2}}$$
$$\leq\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\Bigg(\mathbb{E}'\left\|M_n'(s_i^m)\right\|_{X^{-b}}^2\|h_j\|_{\mathbb{W}^{1,\infty}}^2\Big|W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m)$$
$$-W_j'(t\wedge s_{i+1}^m)+W_j'(t\wedge s_i^m)\Big|^2\Bigg)^{\frac{1}{2}}$$
$$\leq\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\sum_{i=0}^{m-1}\left(\mathbb{E}'\left\|M_n'(s_i^m)\right\|_{X^{-b}}^4\right)^{\frac{1}{4}}\Big(\mathbb{E}'\Big|W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m)$$
$$-W_j'(t\wedge s_{i+1}^m)+W_j'(t\wedge s_i^m)\Big|^4\Big)^{\frac{1}{4}}$$

By (5.66),

$$\sup_n\mathbb{E}'\left\|M_n'(s_i^m)\right\|_{X^{-b}}^4<\infty.$$

By Theorem 5.36, $W'_{jn} \longrightarrow W'_j$ in $C([0,T];R)$ $\mathbb{P}'$ almost surely, so

$$\lim_{n\to\infty} \left|W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i) - W'_j(t \wedge s^m_{i+1}) + W'_j(t \wedge s^m_i)\right|^4 = 0.$$

And since

$$\sup_n \mathbb{E}\left|W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i) - W'_j(t \wedge s^m_{i+1}) + W'_j(t \wedge s^m_i)\right|^8 < \infty,$$

$\left|W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i) - W'_j(t \wedge s^m_{i+1}) + W'_j(t \wedge s^m_i)\right|^4$ is uniformly integrable. So

$$\lim_{n\to\infty} \mathbb{E}\left|W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i) - W'_j(t \wedge s^m_{i+1}) + W'_j(t \wedge s^m_i)\right|^4 = 0.$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty}\Bigg(\mathbb{E}'\Bigg\| \sum_{j=1}^{\infty}\sum_{i=0}^{m-1} \pi_n[\psi(M'_n(s^m_i))M'_n(s^m_i) \times (M'_n(s^m_i) \times h_j)](W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i)) \\
&- \sum_{j=1}^{\infty}\sum_{i=0}^{m-1} \pi_n[\psi(M'_n(s^m_i))M'_n(s^m_i) \times (M'_n(s^m_i) \times h_j)]](W'_j(t \wedge s^m_{i+1}) - W'_j(t \wedge s^m_i))\Bigg\|^2_{X^{-b}}\Bigg)^{\frac{1}{2}} \\
&\le \lim_{n\to\infty} \sum_{j=1}^{\infty} \|h_j\|_{\mathbb{L}^\infty(D)} \sum_{i=0}^{m-1} \left(\mathbb{E}'\left\|M'_n(s^m_i)\right\|^4_{X^{-b}}\right)^{\frac{1}{4}} \Big(\mathbb{E}'\Big|W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i) \\
&-W'_j(t \wedge s^m_{i+1}) + W'_j(t \wedge s^m_i)\Big|^4\Big)^{\frac{1}{4}} = 0.
\end{aligned}$$

Proof of (b) :

$$\begin{aligned}
&\Bigg(\mathbb{E}'\Bigg\| \sum_{j=1}^{\infty}\sum_{i=0}^{m-1} \pi_n[\psi(M'_n(s^m_i))M'_n(s^m_i) \times (M'_n(s^m_i) \times h_j)](W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i)) \\
&- \sum_{j=1}^{\infty}\sum_{i=0}^{m-1} \pi_n[\psi(M'_n(s^m_i))M'_n(s^m_i) \times (M'_n(s^m_i) \times h_j)]](W'_j(t \wedge s^m_{i+1}) - W'_j(t \wedge s^m_i))\Bigg\|^2_{X^{-b}}\Bigg)^{\frac{1}{2}} \\
&\le \sum_{j=1}^{\infty}\sum_{i=0}^{m-1} \Bigg(\mathbb{E}'\left\|\pi_n[\psi(M'_n(s^m_i))M'_n(s^m_i) \times (M'_n(s^m_i) \times h_j)]\right\|^2_{X^{-b}} \\
&\left|W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i) - W'_j(t \wedge s^m_{i+1}) + W'_j(t \wedge s^m_i)\right|^2\Bigg)^{\frac{1}{2}} \\
&\le \sum_{j=1}^{\infty}\sum_{i=0}^{m-1} \left(\mathbb{E}'\left\|\psi(M'_n(s^m_i))M'_n(s^m_i) \times (M'_n(s^m_i) \times h_j)\right\|^4_{X^{-b}}\right)^{\frac{1}{4}} \\
&\left(\mathbb{E}'\left|W'_{jn}(t \wedge s^m_{i+1}) - W'_{jn}(t \wedge s^m_i) - W'_j(t \wedge s^m_{i+1}) + W'_j(t \wedge s^m_i)\right|^4\right)^{\frac{1}{4}}.
\end{aligned}$$

Because we can find some constant $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}' \left\|\psi(M_n'(s_i^m))M_n'(s_i^m) \times (M_n'(s_i^m) \times h_j)\right\|_{X^{-b}}^4 \\
&\leq C^4 \mathbb{E}' \left\|\psi(M_n'(s_i^m))M_n'(s_i^m) \times (M_n'(s_i^m) \times h_j)\right\|_H^4 \leq C^4 \|h_j\|_{\mathbb{L}^\infty}^4 R^4 \mu(D)^2 < \infty.
\end{aligned}$$

And since we have proved

$$\lim_{n\to\infty} \left(\mathbb{E}' \left|W_{jn}'(t \wedge s_{i+1}^m) - W_{jn}'(t \wedge s_i^m) - W_j'(t \wedge s_{i+1}^m) + W_j'(t \wedge s_i^m)\right|^4\right)^{\frac{1}{4}} = 0$$

in part (a), we have

$$\begin{aligned}
&\lim_{n\to\infty} \Bigg(\mathbb{E}' \Bigg\| \sum_{j=1}^\infty \sum_{i=0}^{m-1} \pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m) \times (M_n'(s_i^m) \times h_j)](W_{jn}'(t \wedge s_{i+1}^m) - W_{jn}'(t \wedge s_i^m)) \\
&- \sum_{j=1}^\infty \sum_{i=0}^{m-1} \pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m) \times (M_n'(s_i^m) \times h_j)]](W_j'(t \wedge s_{i+1}^m) - W_j'(t \wedge s_i^m))\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}} \\
&\leq CR\mu(D)^{\frac{1}{2}} \sum_{j=1}^\infty \|h_j\|_{\mathbb{L}^\infty} \lim_{n\to\infty} \left(\mathbb{E}' \left|W_{jn}'(t \wedge s_{i+1}^m) - W_{jn}'(t \wedge s_i^m) - W_j'(t \wedge s_{i+1}^m) + W_j'(t \wedge s_i^m)\right|^4\right)^{\frac{1}{4}} = 0.
\end{aligned}$$

Hence we have proved (ii).

(iii) The proof of (iii) is same as the proof of (i).

(iv) By the Itô isometry, we have

$$\begin{aligned}
&\Bigg(\mathbb{E}' \Bigg\| \int_0^t \sum_{j=1}^\infty \left[\alpha\pi_n\left[M_n'(s) \times h_j\right] + \beta\pi_n\left[\psi(M_n'(s))M_n'(s) \times (M_n'(s) \times h_j)\right]\right] \\
&- \sum_{j=1}^\infty \left[\alpha M'(s) \times h_j + \beta\psi(M'(s))M'(s) \times (M'(s) \times h_j)\right] \mathrm{d}W_j'(s)\Bigg\|_{X^{-b}}^2\Bigg)^{\frac{1}{2}} \\
&= \Bigg(\mathbb{E}' \int_0^t \Bigg\| \sum_{j=1}^\infty \left[\alpha\pi_n\left[M_n'(s) \times h_j\right] + \beta\pi_n\left[\psi(M_n'(s))M_n'(s) \times (M_n'(s) \times h_j)\right]\right] \\
&- \sum_{j=1}^\infty \left[\alpha M'(s) \times h_j + \beta\psi(M'(s))M'(s) \times (M'(s) \times h_j)\right]\Bigg\|_{X^{-b}}^2 \mathrm{d}(s)\Bigg)^{\frac{1}{2}} \\
&\leq \alpha \left(\mathbb{E}' \int_0^t \left\|\pi_n \sum_{j=1}^\infty [M_n'(s) \times h_j] - \sum_{j=1}^\infty M'(s) \times h_j\right\|_{X^{-b}}^2 \mathrm{d}(s)\right)^{\frac{1}{2}} \\
&+\beta\left(\mathbb{E}' \int_0^t \left\|\pi_n \sum_{j=1}^\infty [\psi(M_n'(s))M_n'(s) \times (M_n'(s) \times h_j)] - \sum_{j=1}^\infty \psi(M'(s))M'(s) \times (M'(s) \times h_j)\right\|_{X^{-b}}^2 \mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

Hence to prove (iv), we only need to prove

(a)

$$\lim_{n\to\infty}\left(\mathbb{E}'\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}[M_n'(s)\times h_j]-\sum_{j=1}^{\infty}M'(s)\times h_j\right\|_{X^{-b}}^2\mathrm{d}(s)\right)^{\frac{1}{2}}=0,$$

(b)

$$\lim_{n\to\infty}\left(\mathbb{E}'\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)]\right.\right.$$
$$\left.\left.-\sum_{j=1}^{\infty}\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}=0.$$

Proof of (a). Notice that

$$\lim_{n\to\infty}\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}[M_n'(s)\times h_j]-\sum_{j=1}^{\infty}M'(s)\times h_j\right\|_{X^{-b}}^2\mathrm{d}s=0.$$

This is because

$$\begin{aligned}
&\left(\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}[M_n'(s)\times h_j]-\sum_{j=1}^{\infty}M'(s)\times h_j\right\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}\\
&\le\left(\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}[M_n'(s)\times h_j]-\pi_n\sum_{j=1}^{\infty}M'(s)\times h_j\right\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+\left(\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}M'(s)\times h_j-\sum_{j=1}^{\infty}M'(s)\times h_j\right\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}\\
&\le\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\left(\int_0^t\|M_n'(s)-M'(s)\|_{X^{-b}}^2\mathrm{d}s\right)^{\frac{1}{2}}\\
&\quad+C\left(\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}M'(s)\times h_j-\sum_{j=1}^{\infty}M'(s)\times h_j\right\|_{H}^2\mathrm{d}s\right)^{\frac{1}{2}}\to0.
\end{aligned}$$

And

$$\sup_n\mathbb{E}'\left(\int_0^t\left\|\pi_n\sum_{j=1}^{\infty}[M_n'(s)\times h_j]-\sum_{j=1}^{\infty}M'(s)\times h_j\right\|_{X^{-b}}^2\mathrm{d}(s)\right)^2<\infty.$$

This is because: By (5.63) and (5.74), there exist some constant $C > 0$ such that

$$\begin{aligned}
&\sup_n \mathbb{E}'\left(\int_0^t \left\|\pi_n \sum_{j=1}^{\infty}[M_n'(s)\times h_j] - \sum_{j=1}^{\infty} M'(s)\times h_j\right\|_{X^{-b}}^2 \mathrm{d}(s)\right)^2 \\
&\leq 8\left(\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\right)^2 \sup_n \mathbb{E}'\left[\left(\int_0^T \|M_n'(s)\|_{X^{-b}}^2 \,\mathrm{d}s\right)^2 + \left(\int_0^T \|M'(s)\|_{X^{-b}}^2 \,\mathrm{d}s\right)^2\right] \\
&\leq 16\left(\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{W}^{1,\infty}}\right)^2 T^2C^2\|M_0\|_H^2 < \infty.
\end{aligned}$$

So $\int_0^t \left\|\pi_n \sum_{j=1}^{\infty}[M_n'(s)\times h_j] - \sum_{j=1}^{\infty} M'(s)\times h_j\right\|_{X^{-b}}^2 \mathrm{d}s$ is uniformly integrable. Hence we have

$$\lim_{n\to\infty}\left(\mathbb{E}'\int_0^t \left\|\pi_n \sum_{j=1}^{\infty}[M_n'(s)\times h_j] - \sum_{j=1}^{\infty} M'(s)\times h_j\right\|_{X^{-b}}^2 \mathrm{d}(s)\right)^{\frac{1}{2}} = 0.$$

□

Proof of (b). Similar as the previous proof. Notice that

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t \Big\|\pi_n \sum_{j=1}^{\infty}[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)] \\
&-\sum_{j=1}^{\infty}\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big\|_{X^{-b}}^2 \mathrm{d}s = 0.
\end{aligned}$$

And

$$\begin{aligned}
&\sup_n \mathbb{E}'\Big(\int_0^t \Big\|\pi_n \sum_{j=1}^{\infty}[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)] \\
&-\sum_{j=1}^{\infty}\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big\|_{X^{-b}}^2 \mathrm{d}s\Big)^2 < \infty.
\end{aligned}$$

So

$$\begin{aligned}
&\int_0^t \Big\|\pi_n \sum_{j=1}^{\infty}[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)] \\
&-\sum_{j=1}^{\infty}\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big\|_{X^{-b}}^2 \mathrm{d}s
\end{aligned}$$

is uniformly integrable. Hence we have

$$\lim_{n\to\infty}\Bigg(\mathbb{E}'\int_0^t\Big\|\pi_n\sum_{j=1}^{\infty}[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)]$$
$$-\sum_{j=1}^{\infty}\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big\|_{X^{-b}}^2\,\mathrm{d}s\Bigg)^{\frac{1}{2}}=0.$$

□

Therefore we have proved (iv).

□

Lemma 5.67. *For each $t\in[0,T]$, we have*

$$\xi'(t)=\sum_{j=1}^{\infty}\left\{\int_0^t\left[\alpha\left[M'\times h_j\right]+\beta\left[\psi(M')M'\times(M'\times h_j)\right]\right]\mathrm{d}W_j'(s)\right\},$$

in $L^2(\Omega';X^{-b})$.

Proof. Firstly, we show that

$$\xi_n'(t)=\sum_{j=1}^{\infty}\left\{\int_0^t\left[\alpha\pi_n\left[M_n'(s)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\right]\right]\mathrm{d}W_{nj}'(s)\right\}$$

$\mathbb{P}'$ almost surely for each $t\in[0,T]$ and $n\in\mathbb{N}$.
Let us fix that $t\in[0,T]$ and $n\in\mathbb{N}$. For each $m\in\mathbb{N}$ we define the partition $\{s_i^m:=\frac{iT}{m}, i=0,\dots,m\}$ of $[0,T]$. (M_n',W_{jn}') and (M_n,W_j) have same distribution on $L^4(0,T;\mathbb{L}^4)\cap C([0,T];X^{-b})\times C([0,T];\mathbb{R})$, so for each m the random variables in $H=\mathbb{L}^2(D)$:

$$\xi_n(t)-\sum_{j=1}^{\infty}\Bigg\{\sum_{i=0}^{m-1}\Big[\alpha\pi_n(M_n(s_i^m)\times h_j)+$$
$$\beta\pi_n\Big[\psi(M_n(s_i^m))M_n(s_i^m)\times(M_n(s_i^m)\times h_j)\Big]\Big](W_j(t\wedge s_{i+1}^m)-W_j(t\wedge s_i^m))\Bigg\}$$

and

$$\xi_n'(t)-\sum_{j=1}^{\infty}\Bigg\{\sum_{i=0}^{m-1}\Big[\alpha\pi_n(M_n'(s_i^m)\times h_j)+$$
$$\beta\pi_n\Big[\psi(M_n'(s_i^m))M_n(s_i^m)\times(M_n'(s_i^m)\times h_j)\Big]\Big](W_{nj}'(t\wedge s_{i+1}^m)-W_{nj}'(t\wedge s_i^m))\Bigg\}$$

have the same distribution. For each j, we have

$$\lim_{m\to\infty}\Bigg|\sum_{i=0}^{m-1}\Big[\alpha\pi_n(M_n(s_i^m)\times h_j)+\beta\pi_n\Big[\psi(M_n(s_i^m))M_n(s_i^m)\times(M_n(s_i^m)\times h_j)\Big]\Big]$$
$$(W_j(t\wedge s_{i+1}^m)-W_j(t\wedge s_i^m))$$
$$-\int_0^t\Big[\alpha\pi_n(M_n(s)\times h_j)+\beta\pi_n\Big[\psi(M_n(s))M_n(s)\times(M_n(s)\times h_j)\Big]\Big]\mathrm{d}W_j(s)\Bigg|_{L^2(\Omega;H)}=0$$

and

$$\lim_{m\to\infty}\Bigg|\sum_{i=0}^{m-1}\Big[\alpha\pi_n(M_n'(s_i^m)\times h_j)+\beta\pi_n\Big[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)\Big]\Big]$$
$$(W_{nj}'(t\wedge s_{i+1}^m)-W_{nj}'(t\wedge s_i^m))$$
$$-\int_0^t\Big[\alpha\pi_n(M_n'(s)\times h_j)+\beta\pi_n\Big[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\Big]\Big]\mathrm{d}W_{nj}'(s)\Bigg|_{L^2(\Omega';H)}=0,$$

$$\xi_n-\sum_{j=1}^{\infty}\int_0^t\Big[\alpha\pi_n(M_n(s)\times h_j)+\beta\pi_n\Big[\psi(M_n(s))M_n(s)\times(M_n(s)\times h_j)\Big]\Big]\mathrm{d}W_j(s)$$

and

$$\xi_n'-\sum_{j=1}^{\infty}\int_0^t\Big[\alpha\pi_n(M_n'(s)\times h_j)+\beta\pi_n\Big[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\Big]\Big]\mathrm{d}W_{nj}'(s)$$

have the same distribution. But

$$\xi_n=\sum_{j=1}^{\infty}\int_0^t\Big[\alpha\pi_n(M_n(s)\times h_j)+\beta\pi_n\Big[\psi(M_n(s))M_n(s)\times(M_n(s)\times h_j)\Big]\Big]\mathrm{d}W_j(s)$$

$\mathbb{P}$-almost surely, so we have

$$\xi_n'=\sum_{j=1}^{\infty}\int_0^t\Big[\alpha\pi_n(M_n'(s)\times h_j)+\beta\pi_n\Big[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\Big]\Big]\mathrm{d}W_{nj}'(s)$$

$\mathbb{P}'$-almost surely. Secondly, we show that $\xi_n'(t)$ converges in $L^2(\Omega';X^{-b})$ to

$$\sum_{j=1}^{\infty}\int_0^t\Big[\alpha M'(s)\times h_j+\beta\psi(M'(s))M'(s)\times(M'(s)\times h_j)\Big]\mathrm{d}W_j'(s)$$

as $n \to \infty$. Notice that

$$
\begin{aligned}
&\left(\mathbb{E}'\left\|\xi_n'(t)-\sum_{j=1}^{\infty}\int_0^t\left[\alpha M'(s)\times h_j+\beta\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right]\mathrm{d}W_j'(s)\right\|_{X^{-b}}^2\right)^{\frac{1}{2}}\\
&=\left(\mathbb{E}'\left\|\sum_{j=1}^{\infty}\int_0^t\left[\alpha\pi_n\left[M_n'(s)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\right]\right]\mathrm{d}W_{nj}'(s)\right.\right.\\
&\qquad\left.\left.-\sum_{j=1}^{\infty}\int_0^t\left[\alpha M'(s)\times h_j+\beta\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right]\mathrm{d}W_j'(s)\right\|_{X^{-b}}^2\right)^{\frac{1}{2}}\\
&\le\left(\mathbb{E}'\left\|\sum_{j=1}^{\infty}\int_0^t\left[\alpha\pi_n\left[M_n'(s)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\right]\right]\right.\right.\\
&\left.\left.-\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\left[\alpha\pi_n\left[M_n'(s_i^m)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)\right]\right]1_{(s_i^m,s_{i+1}^m]}(s)\,\mathrm{d}W_{jn}'(s)\right\|_{X^{-b}}\right)^{\frac{1}{2}}\\
&+\left(\mathbb{E}'\left\|\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\left[\alpha\pi_n\left[M_n'(s_i^m)\times h_j\right]+\beta\pi_n[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)]\right](W_{jn}'(t\wedge s_{i+1}^m)-W_{jn}'(t\wedge s_i^m))\right.\right.\\
&\left.\left.-\sum_{j=1}^{\infty}\sum_{i=0}^{m-1}\left[\alpha\pi_n\left[M_n'(s_i^m)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)\right]\right](W_j'(t\wedge s_{i+1}^m)-W_j'(t\wedge s_i^m))\right\|_{X^{-b}}^2\right)^{\frac{1}{2}}\\
&\qquad+\left(\mathbb{E}'\left\|\int_0^t\sum_{j=1}^{\infty}\left[\alpha\pi_n\left[M_n'(s_i^m)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s_i^m))M_n'(s_i^m)\times(M_n'(s_i^m)\times h_j)\right]\right]1_{(s_i^m,s_{i+1}^m]}(s)\right.\right.\\
&\qquad\left.\left.-\sum_{j=1}^{\infty}\left[\alpha\pi_n\left[M_n'(s)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\right]\right]\mathrm{d}W_j'(s)\right\|_{X^{-b}}^2\right)^{\frac{1}{2}}\\
&\qquad+\left(\mathbb{E}'\left\|\int_0^t\sum_{j=1}^{\infty}\left[\alpha\pi_n\left[M_n'(s)\times h_j\right]+\beta\pi_n\left[\psi(M_n'(s))M_n'(s)\times(M_n'(s)\times h_j)\right]\right]\right.\right.\\
&\qquad\left.\left.-\sum_{j=1}^{\infty}\left[\alpha M'(s)\times h_j+\beta\psi(M'(s))M'(s)\times(M'(s)\times h_j)\right]\mathrm{d}W_j'(s)\right\|_{X^{-b}}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

And then by Lemma 5.66, we conclude the result. □

Remark 5.68. Observe that the Lemma 5.67 means that:

$$
\begin{aligned}
M'(t) = M_0 + \int_0^t \Bigg\{ & [\lambda_1 M' \times \rho'] - \lambda_2 [M' \times (M' \times \rho')] \\
& + \frac{1}{2} \sum_{j=1}^{\infty} \Big[\alpha^2 \Big[(M' \times h_j) \times h_j \Big] \\
& + \alpha\beta\psi(M') \Big[[M' \times (M' \times h_j)] \times h_j \Big] \\
& + \beta^2 \psi(M') \Big[M' \times [M' \times (M' \times h_j) \times h_j] \Big] \\
& + \alpha\beta\psi(M') \Big[M' \times [(M' \times h_j) \times h_j] \Big] \\
& + \beta^2 \psi(M') \Big[M' \times (M' \times h_j) \times (M' \times h_j) \Big] \Big] \Bigg\} \, \mathrm{d}s \\
& + \sum_{j=1}^{\infty} \Bigg\{ \int_0^t \Big[\alpha \Big[M' \times h_j \Big] + \beta \Big[\psi(M') M' \times (M' \times h_j) \Big] \Big] \, \mathrm{d}W'_j(s) \Bigg\},
\end{aligned} \tag{5.95}
$$

in $L^2(\Omega'; X^{-b})$.

Lemma 5.69. [40]*(Th. 1.2) Let V and H be two separable Hilbert spaces, such that $V \hookrightarrow H$ continuously and densely. We identify H with it's dual space. And let $M^2(0,T;H)$ denote the space of H-valued measurable process with the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ which satisfy: $\phi \in M^2(0,T;H)$ if and only if*

(i) *$\phi(t)$ is $\mathcal{F}_t$ measurable for almost every t;*
(ii) $\mathbb{E} \int_0^t |\phi(t)|^2 \, \mathrm{d}t < \infty$.

We suppose that

$$
\begin{aligned}
& u \in M^2(0,T;V), \quad u_0 \in H, \quad v \in M^2(0,T;V'), \\
& \mathbb{E} \int_0^T \sum_{j=1}^{\infty} \|z_j(t)\|_H^2 \, \mathrm{d}t < \infty,
\end{aligned}
$$

with

$$
u(t) = u_0 + \int_0^t v(s) \, \mathrm{d}s + \sum_{j=1}^{\infty} \int_0^t z_j(s) \, \mathrm{d}W_j(s).
$$

Let γ be a twice differentiable functional on H, which satisfies:

(i) *γ, γ' and γ'' are locally bounded.*
(ii) *γ and γ' are continuous on H.*
(iii) *Let $\mathscr{L}^1(H)$ be the Banach space of all the trace class operators on H. Then $\forall Q \in \mathscr{L}^1(H)$, $Tr[Q \circ \gamma'']$ is a continuous functional on H.*
(iv) *If $u \in V$, $\psi'(u) \in V$; $u \mapsto \gamma'(u)$ is continuous from V (with the strong topology) into V endowed with the weak topology.*
(v) *$\exists k$ such that $\|\gamma'(u)\|_V \leq k(1 + \|u\|_V)$, $\forall u \in V$.*

Then $\mathbb{P}$ *almost surely,*

$$\gamma(u(t)) = \gamma(u_0) + \int_0^t {}_{V^*}\langle v(s), \gamma'(u(s))\rangle_V \,\mathrm{d}s + \sum_{j=1}^{\infty}\int_0^t {}_H\langle \gamma'(u(s)), z_j(s)\rangle_H \,\mathrm{d}W_j(s)$$
$$+\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t {}_H\langle \gamma''(u(s))z_j(s), z_j(s)\rangle_H \,\mathrm{d}s.$$

Theorem 5.70. *The* M' *defined in Theorem 5.36 satisfies: for each* $t \in [0,T]$*, we have* $\mathbb{P}'$*-almost surely*

$$|M'(t,x)|_{\mathbb{R}^3} = 1, \quad \text{for Lebesgue a.e. } x \in D. \tag{5.96}$$

Proof. Let $\eta \in C_0^\infty(D,\mathbb{R})$. Then we consider

$$\gamma : H \ni M \longmapsto \langle M, \eta M\rangle_H \in \mathbb{R}.$$

It's easy to see that γ is of C^2 class and $\gamma'(M) = 2\eta M$ and $\gamma''(M)(v) = 2\eta v$ for $M, v \in H$.
Next we check the assumptions of Lemma 5.69 with u, v, z_j in Lemma 5.69 from (5.95). By previous work (See details below), we can see M' satisfies:

$$\mathbb{E}'\int_0^T \left\|M'(t)\right\|_V^2 \,\mathrm{d}t < \infty, \quad \text{by (5.66)},$$

$$\mathbb{E}'\int_0^T \left\|(M'\times\rho')(t)\right\|_{X^{-\beta}}^2 \,\mathrm{d}t < \infty, \quad \text{by (5.67)},$$

$$\mathbb{E}'\int_0^T \left\|M'(t)\times(M'\times\rho')(t)\right\|_{X^{-\beta}}^2 \,\mathrm{d}t < \infty, \quad \text{by (5.70)},$$

and by the assumption $\sum_{j=1}^\infty \|h_j\|^2_{\mathbb{L}^\infty(D)} < \infty$ and the definition of ψ, we have

$$\mathbb{E}'\int_0^T \left\|\sum_{j=1}^\infty (M'(s)\times h_j)\times h_j\right\|_{X^{-\beta}}^2 \,\mathrm{d}t < \infty,$$

$$\mathbb{E}'\int_0^T \left\|\sum_{j=1}^\infty \psi(M'(s))[M'(s)\times(M'(s)\times h_j)]\times h_j\right\|_{X^{-\beta}}^2 \,\mathrm{d}t < \infty,$$

$$\mathbb{E}'\int_0^T \left\|\sum_{j=1}^\infty \psi(M'(s))M'(s)\times[M'(s)\times(M'(s)\times h_j)\times h_j]\right\|_{X^{-\beta}}^2 \,\mathrm{d}t < \infty,$$

$$\mathbb{E}'\int_0^T \left\|\sum_{j=1}^\infty \psi(M'(s))M'(s)\times[(M'(s)\times h_j)\times h_j]\right\|_{X^{-\beta}}^2 \,\mathrm{d}t < \infty,$$

$$\mathbb{E}'\int_0^T \left\|\sum_{j=1}^\infty \psi(M'(s))[M'(s)\times(M'(s)\times h_j)]\times M'(s)\times h_j\right\|_{X^{-\beta}}^2 \,\mathrm{d}t < \infty,$$

and

$$\mathbb{E}' \int_0^T \left\| \sum_{j=1}^{\infty} M'(s) \times h_j \right\|_H^2 \, \mathrm{d}t < \infty,$$

$$\mathbb{E}' \int_0^T \left\| \sum_{j=1}^{\infty} \psi(M'(s)) M'(s) \times [M'(s) \times h_j] \right\|_H^2 \, \mathrm{d}t < \infty.$$

And γ satisfies:

(i) γ, γ', γ' are locally bounded.
(ii) Since γ', γ'' exist, γ, γ' are continuous on H.
(iii) $\forall Q \in \mathscr{L}^1(H)$,

$$Tr[Q \circ \gamma''(a)] = \sum_{j=1}^{\infty} \langle Q \circ \gamma''(a) e_j, e_j \rangle_H = 2 \sum_{j=1}^{\infty} \langle Q(\eta e_j), e_j \rangle_H,$$

which is a constant in $\mathbb{R}$, so the map $H \ni a \longmapsto Tr[Q \circ \gamma''(a)] \in \mathbb{R}$ is a continuous functional on H.
(iv) If $u \in V$, $\gamma'(u) \in V$; $u \mapsto \gamma'(u)$ is continuous from V (with the strong topology) into V endowed with the weak topology.
This is because: For any $v^* \in X^{-b}$, we have

$${}_{X^b}\langle \gamma'(u+v) - \gamma'(u), v^* \rangle_{X^{-b}} = {}_{X^b}\langle 2\eta v, v^* \rangle_{X^{-b}} \leq 2|\eta|_{C(D,\mathbb{R})}\, {}_{X^b}\langle v, v^* \rangle_{X^{-b}},$$

hence γ' is weakly continuous. Let us denote τ as the strong topology of V and τ_w the weak topology of V. Take $B \in \tau_w$, by the weak continuity $(\gamma')^{-1}(B) \in \tau_w$, but $\tau_w \subset \tau$. Hence $(\gamma')^{-1}(B) \in \tau$, which implies (iv).
(v) $\exists k$ such that $\|\gamma'(u)\|_V \leq k(1 + \|u\|_V)$, $\forall u \in V$.

Hence by Lemma 5.69, we have that for $t \in [0, T]$ and $\mathbb{P}'$ almost surely,

$$\begin{aligned}
&\langle M'(t), \eta M'(t) \rangle_H - \langle M_0, \eta M_0 \rangle_H \\
= &\int_0^t {}_{X^{-b}}\Big\langle \lambda_1 M' \times \rho' - \lambda_2 M' \times (M' \times \rho') \\
&+ \frac{1}{2} \sum_{j=1}^{\infty} \Big[\alpha^2 (M' \times h_j) \times h_j + \alpha\beta\psi(M')[[M' \times (M' \times h_j)] \times h_j] \\
&+ \beta^2 \psi(M')[M' \times [M' \times (M' \times h_j) \times h_j]] + \alpha\beta\psi(M')[M' \times [(M' \times h_j) \times h_j]] \\
&+ \beta^2 \psi(M')[M' \times (M' \times h_j)] \times (M' \times h_j) \Big], 2\eta M'(s) \Big\rangle_{X^b} \mathrm{d}s \\
&+ \sum_{j=1}^{\infty} \int_0^t \langle 2\eta M'(s), \alpha M' \times h_j + \beta\psi(M') M' \times (M' \times h_j) \rangle_H \, \mathrm{d}W'_j(s) \\
&+ \sum_{j=1}^{\infty} \int_0^t \langle \eta\alpha M' \times h_j + \beta\psi(M') M' \times (M' \times h_j), \alpha M' \times h_j + \beta\psi(M') M' \times (M' \times h_j) \rangle_H \, \mathrm{d}s = 0.
\end{aligned}$$

Hence we have

$$\langle \eta, |M'(t)|^2_{\mathbb{R}^3} - |M'_0|^2_{\mathbb{R}^3} \rangle_{L^2(D;\mathbb{R})} = \langle M'(t), \eta M'(t)\rangle_H - \langle M'_0, \eta M'_0\rangle_H = 0.$$

Since η is arbitrary and $|M_0(x)| = 1$ for almost every $x \in D$, we infer that $|M'(t,x)| = 1$ for almost every $x \in D$ as well. □

Finally we are ready to give the proof of the main result.

Theorem 5.71. *The process* (M', E', B') *is a solution of Problem 5.10. That is,* (M', E', B') *satisfies the following equations:* (5.14)*,* (5.12) *and* (5.13).

Proof of (5.14). By Lemma 5.67 and Lemma 5.70, we have $\psi(M'(t)) \equiv 1$ for $t \in [0, T]$. Hence we deduce that for $t \in [0, T]$, the following equation holds in $L^2(\Omega'; X^{-b})$.

$$\begin{aligned} M'(t) = {} & M_0 + \int_0^t \Big\{ [\lambda_1 M' \times \rho'] - \lambda_2 [M' \times (M' \times \rho')] \\ & + \frac{1}{2}\sum_{j=1}^\infty \Big[\alpha^2 \big[(M' \times h_j) \times h_j \big] + \alpha\beta \big[[M' \times (M' \times h_j)] \times h_j \big] \\ & + \beta^2 \big[M' \times [M' \times (M' \times h_j) \times h_j] \big] + \alpha\beta \big[M' \times [(M' \times h_j) \times h_j] \big] \\ & + \beta^2 \big[M' \times (M' \times h_j) \times (M' \times h_j) \big] \Big] \Big\} \, \mathrm{d}s \\ & + \sum_{j=1}^\infty \Big\{ \int_0^t \Big[\alpha \big[M' \times h_j \big] + \beta \big[M' \times (M' \times h_j) \big] \Big] \, \mathrm{d}W'_j(s) \Big\} \\ = {} & M_0 + \int_0^t [\lambda_1 M' \times \rho' - \lambda_2 M' \times (M' \times \rho')] \, \mathrm{d}s \\ & + \sum_{j=1}^\infty \Big\{ \int_0^t \Big[\alpha M' \times h_j + \beta M' \times (M' \times h_j) \Big] \circ \mathrm{d}W'_j(s) \Big\}. \end{aligned}$$

□

Proof of (5.12). By Proposition 5.41 and the equation (5.24), we have

$$B'_n(t) - B'_n(0) = -\int_0^t \pi_n^Y [\nabla \times E'_n(s)] \, \mathrm{d}s, \qquad \mathbb{P}' - a.s. \tag{5.97}$$

By Theorem 5.36, we also have

(a) $E'_n \longrightarrow E'$ in $L^2_w(0, T; \mathbb{L}^2(\mathbb{R}^3))$ $\mathbb{P}'$ almost surely, and
(b) $B'_n \longrightarrow B'$ in $L^2_w(0, T; \mathbb{L}^2(\mathbb{R}^3))$ $\mathbb{P}'$ almost surely.

Hence for any $u \in H^1(0,T;Y)$,

$$\int_0^t \langle B'(s), \frac{\mathrm{d}u(s)}{\mathrm{d}s}\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s = \lim_{n\to\infty}\int_0^t \langle B_n'(s), \frac{\mathrm{d}u(s)}{\mathrm{d}s}\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s$$
$$= -\lim_{n\to\infty}\int_0^t \langle \frac{\mathrm{d}B_n'(s)}{\mathrm{d}s}, u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s = \lim_{n\to\infty}\int_0^t \langle \pi_n^Y[\nabla\times E_n'(s)], u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s$$
$$= \lim_{n\to\infty}\int_0^t \langle \nabla\times E_n'(s), \pi_n^Y u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s = \lim_{n\to\infty}\int_0^t \langle E_n'(s), \nabla\times \pi_n^Y u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s$$

Since

$$\lim_{n\to\infty}\left|\int_0^t \langle E_n'(s), \nabla\times\pi_n^Y u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} - \langle E'(s), \nabla\times u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s\right|$$
$$\le \lim_{n\to\infty}\int_0^t \left|\langle E_n'(s), \nabla\times\Big(\pi_n^Y u(s) - u(s)\Big)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\right|\,\mathrm{d}s$$
$$+ \lim_{n\to\infty}\left|\int_0^t \langle E_n'(s) - E'(s), \nabla\times u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s\right|$$
$$\le \lim_{n\to\infty}\left(\int_0^t \|E_n'(s)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s\right)^{\frac12}\left(\int_0^t \|\pi_n^Y u(s) - u(s)\|_Y^2\,\mathrm{d}s\right)^{\frac12} + 0 = 0, \qquad \mathbb{P}' - a.s.,$$

we have

$$\lim_{n\to\infty}\int_0^t \langle E_n'(s), \nabla\times\pi_n^Y u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s = \int_0^t \langle E'(s), \nabla\times u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s.$$

Therefore

$$\int_0^t \langle B'(s), \frac{\mathrm{d}u(s)}{\mathrm{d}s}\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s = \int_0^t \langle E'(s), \nabla\times u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)}\,\mathrm{d}s,$$

for all $u \in H^1(0,T;Y)$.

Hence for $t \in [0,T]$,

$$B'(t) = B_0 - \int_0^t \nabla\times E'(s)\,\mathrm{d}s, \quad \in Y', \ \mathbb{P}' - a.s..$$

□

Proof of (5.13). Similar as in the proof of (5.12). Let $p = q = 2$ in Theorem 5.36, we have

(a) $M_n' \longrightarrow M'$ in $L^2(0,T;\mathbb{L}^2(D))$ $\mathbb{P}'$ almost surely,
(b) $E_n' \longrightarrow E'$ in $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$ $\mathbb{P}'$ almost surely, and
(c) $B_n' \longrightarrow B'$ in $L^2_w(0,T;\mathbb{L}^2(\mathbb{R}^3))$ $\mathbb{P}'$ almost surely.

Hence by (5.23) we have for all $u \in H^1(0,T;Y)$,

$$\int_0^t \langle E'(s), \frac{\mathrm{d}u(s)}{\mathrm{d}s}\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \,\mathrm{d}s = \lim_{n\to\infty}\int_0^t \langle E'_n(s), \frac{\mathrm{d}u(s)}{\mathrm{d}s}\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \,\mathrm{d}s$$
$$= -\lim_{n\to\infty}\int_0^t \langle \pi_n^Y[1_D(E'_n(s)) + f(s)] - \pi_n^Y[\nabla\times(B'_n(s) - \pi_n^Y(\tilde{M}'_n(s)))], u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \,\mathrm{d}s$$
$$= \int_0^t \langle B'(s) - \tilde{M}'(s), \nabla\times u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} - \langle 1_D E'(s) + f(s), u(s)\rangle_{\mathbb{L}^2(\mathbb{R}^3)} \,\mathrm{d}s.$$

Hence for $t \in [0,T]$,

$$E'(t) = E_0 + \int_0^t \nabla\times[B'(s) - \tilde{M}'(s)]\,\mathrm{d}s - \int_0^t [1_D E'(s) + \tilde{f}(s)]\,\mathrm{d}s, \quad \in Y', \ \mathbb{P}' - a.s..$$

□

Next we will show some regularity of M'.

Theorem 5.72. *For $t \in [0,T]$ the following equation holds in $L^2(\Omega';H)$.*

$$\begin{aligned} M'(t) &= M_0 + \int_0^t \Big\{ [\lambda_1 M'\times\rho'] - \lambda_2 [M'\times(M'\times\rho')] \\ &\quad + \frac{1}{2}\sum_{j=1}^\infty \Big[\alpha^2\big[(M'\times h_j)\times h_j\big] + \alpha\beta\big[[M'\times(M'\times h_j)]\times h_j\big] \\ &\quad + \beta^2\big[M'\times[M'\times(M'\times h_j)\times h_j]\big] + \alpha\beta\big[M'\times[(M'\times h_j)\times h_j]\big] \\ &\quad + \beta^2\big[M'\times(M'\times h_j)\times(M'\times h_j)\big]\Big]\Big\}\,\mathrm{d}s \\ &\quad + \sum_{j=1}^\infty \Big\{\int_0^t \big[\alpha\big[M'\times h_j\big] + \beta\big[M'\times(M'\times h_j)\big]\big]\,\mathrm{d}W'_j(s)\Big\} \\ &= M_0 + \int_0^t \{\lambda_1 M'\times\rho' - \lambda_2 M'\times(M'\times\rho')\}\,\mathrm{d}s \\ &\quad + \sum_{j=1}^\infty \Big\{\int_0^t \big[\alpha M'\times h_j + \beta M'\times(M'\times h_j)\big]\circ\,\mathrm{d}W'_j(s)\Big\}. \end{aligned} \tag{5.98}$$

Proof. By (5.67) and (5.96), there exists some constant $C > 0$ such that

$$\mathbb{E}\int_0^T \|M'_n(t)\times\rho'_n(t)\|_H^2\,\mathrm{d}t \le C^2, \qquad \forall n.$$

By Lemma 5.58, for $u \in L^2(\Omega'; L^2(0,T;H))$ we have

$$\begin{aligned}\left(\mathbb{E}'\|M' \times \rho'\|^2_{L^2(0,T;H)}\right)^{\frac{1}{2}} &= \sup_{\|u\|_{L^2(\Omega';L^2(0,T;H))}\le 1} \mathbb{E}\int_0^T \langle M'(t)\times\rho'(t), u(t)\rangle_H \,\mathrm{d}t \\ &= \sup_{\|u\|_{L^2(\Omega';L^2(0,T;H))}\le 1} \lim_{n\to\infty}\mathbb{E}\int_0^T \langle M'_n(t)\times\rho'_n(t), u(t)\rangle_H \,\mathrm{d}t \\ &\le \sup_n \left(\mathbb{E}'\|M'_n\times\rho'_n\|^2_{L^2(0,T;H)}\right)^{\frac{1}{2}} \le C.\end{aligned}$$

Hence for almost every $t \in [0,T]$, $M'(t)\times\rho'(t) \in L^2(\Omega';H)$. Again by (5.96) we have

$$\begin{aligned}\|M'(t)\times(M'(t)\times\rho'(t))\|^2_{L^2(\Omega';H)} &= \mathbb{E}'\int_D |M'(t,x)\times(M'(t,x)\times\rho'(t,x))|^2\,\mathrm{d}x \\ \le \mathbb{E}'\int_D |M'(t,x)|^2|M'(t,x)\times\rho'(t,x)|^2\,\mathrm{d}x &\le \|M'(t)\times\rho'(t)\|^2_{L^2(\Omega';H)} < \infty.\end{aligned}$$

Therefore all the terms of (5.98) are in $L^2(\Omega';H)$, so the proof has been complete. □

Finally we will prove that M' has more regularity time-wise.

Theorem 5.73. *The process M' introduced in Theorem 5.36 satisfies the following condition: for $\theta \in (0, \frac{1}{2})$,*

$$M' \in C^\theta(0,T;H), \qquad \mathbb{P}' - a.s..$$

Proof. The proof is based on the Kolmogorov test (Lemma 2.134). By (5.98), we have

$$\begin{aligned}M'(t) - M'(s) = &\int_s^t \Big\{ [\lambda_1 M'\times\rho'] - \lambda_2\,[M'\times(M'\times\rho')] \\ &+\frac{1}{2}\sum_{j=1}^\infty \Big[\alpha^2\big[(M'\times h_j)\times h_j\big] + \alpha\beta\big[[M'\times(M'\times h_j)]\times h_j\big] \\ &+\beta^2\big[M'\times[M'\times(M'\times h_j)\times h_j]\big] + \alpha\beta\big[M'\times[(M'\times h_j)\times h_j]\big] \\ &+\beta^2\big[M'\times(M'\times h_j)\times(M'\times h_j)\big]\Big]\Big\}\,\mathrm{d}\tau \\ &+\sum_{j=1}^\infty\left\{\int_s^t \Big[\alpha\big[M'\times h_j\big] + \beta\big[M'\times(M'\times h_j)\big]\Big]\,\mathrm{d}W'_j(\tau)\right\},\end{aligned}$$

for $0 \leq s < t \leq T$, in $L^2(\Omega'; H)$. Hence for fixed $q \geq 1$, we have

$$\left\{\mathbb{E}' \left\|M'(t) - M'(s)\right\|_H^{2q}\right\}^{\frac{1}{2q}}$$
$$\leq |\lambda_1| \left\{\mathbb{E}' \left\|\int_s^t M'(\tau) \times \rho'(\tau)\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}} + |\lambda_2| \left\{\mathbb{E}' \left\|\int_s^t M'(\tau) \times [M'(\tau) \times \rho'(\tau)]\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}$$
$$+\frac{1}{2}\alpha^2 \sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t [M'(\tau) \times h_j] \times h_j\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}$$
$$+\frac{1}{2}|\alpha\beta| \sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t (M'(\tau) \times [M'(\tau) \times h_j]) \times h_j\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}$$
$$+\frac{1}{2}\beta^2 \sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t \{M'(\tau) \times (M'(\tau) \times [M'(\tau) \times h_j])\} \times h_j\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}$$
$$+\frac{1}{2}|\alpha\beta| \sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t M'(\tau) \times ([M'(\tau) \times h_j] \times h_j) \times h_j\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}$$
$$+\frac{1}{2}\beta^2 \sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t (M'(\tau) \times [M'(\tau) \times h_j]) \times [M'(\tau) \times h_j]\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}$$
$$+|\alpha| \sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t M'(\tau) \times h_j\, \mathrm{d}W_j'(\tau)\right\|_H^{2q}\right\}^{\frac{1}{2q}} + |\beta| \sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t M'(\tau) \times [M'(\tau) \times h_j]\, \mathrm{d}W_j'(\tau)\right\|_H^{2q}\right\}^{\frac{1}{2q}}.$$

We also have the following results: By (5.82), $\exists C > 0$ such that

$$\mathbb{E}' \left\|\int_s^t M'(\tau) \times \rho'(\tau)\, \mathrm{d}\tau\right\|_H^{2q} \leq (t-s)^q \mathbb{E}' \left(\int_s^t \|M'(\tau) \times \rho'(\tau)\|_H^2\, \mathrm{d}\tau\right)^q \leq C^q (t-s)^q.$$

By (5.96), $\|M'(t) \times (M'(t) \times \rho'(t))\|_H^2 \leq \|M'(t) \times \rho'(t)\|_H^2$. So

$$\mathbb{E}' \left\|\int_s^t M'(t) \times (M'(t) \times \rho'(t))\, \mathrm{d}\tau\right\|_H^{2q} \leq C^q (t-s)^q.$$

By (5.73),

$$\sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t [M'(\tau) \times h_j] \times h_j\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}} \leq \sum_{j=1}^{\infty} \|h_j\|_{\mathbb{L}^\infty(D)} \left(\mathbb{E}' \int_s^t \|M'(\tau)\|_H^2\, \mathrm{d}\tau\right)^{\frac{1}{2}} (t-s)^{\frac{1}{2}} \leq C^{\frac{1}{2}} (t-s)^{\frac{1}{2}}.$$

Using (5.96) and similarly as before we also have:

$$\sum_{j=1}^{\infty} \left\{\mathbb{E}' \left\|\int_s^t (M'(\tau) \times [M'(\tau) \times h_j]) \times h_j\, \mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}} \leq C^{\frac{1}{2}} (t-s)^{\frac{1}{2}},$$

$$\sum_{j=1}^{\infty}\left\{\mathbb{E}'\left\|\int_s^t \{M'(\tau)\times(M'(\tau)\times[M'(\tau)\times h_j])\}\times h_j\,\mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}\le C^{\frac{1}{2}}(t-s)^{\frac{1}{2}},$$

$$\sum_{j=1}^{\infty}\left\{\mathbb{E}'\left\|\int_s^t M'(\tau)\times([M'(\tau)\times h_j]\times h_j)\times h_j\,\mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}\le C^{\frac{1}{2}}(t-s)^{\frac{1}{2}},$$

$$\sum_{j=1}^{\infty}\left\{\mathbb{E}'\left\|\int_s^t (M'(\tau)\times[M'(\tau)\times h_j])\times[M'(\tau)\times h_j]\,\mathrm{d}\tau\right\|_H^{2q}\right\}^{\frac{1}{2q}}\le C^{\frac{1}{2}}(t-s)^{\frac{1}{2}}.$$

By Theorem 2.127,

$$\begin{aligned}&\sum_{j=1}^{\infty}\left\{\mathbb{E}'\left\|\int_s^t M'(\tau)\times h_j\,\mathrm{d}W_j'(\tau)\right\|_H^{2q}\right\}^{\frac{1}{2q}}\\ &\le (q(2q-1))^{\frac{1}{2}}\left(\frac{2q}{2q-1}\right)^{2q}\sum_{j=1}^{\infty}\mathbb{E}'\left(\int_s^t \left\|M'(\tau)\times h_j\right\|_H^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\\ &\le \|u_0\|_H\sum_{j=1}^{\infty}\|h_j\|_{\mathbb{L}^\infty}(q(2q-1))^{\frac{1}{2}}\left(\frac{2q}{2q-1}\right)^{2q}(t-s)\frac{1}{2}\le C^{\frac{1}{2}}(t-s)^{\frac{1}{2}}.\end{aligned}$$

And

$$\sum_{j=1}^{\infty}\left\{\mathbb{E}'\left\|\int_s^t M'(\tau)\times[M'(\tau)\times h_j]\,\mathrm{d}W_j'(\tau)\right\|_H^{2q}\right\}^{\frac{1}{2q}}\le C^{\frac{1}{2}}(t-s)^{\frac{1}{2}}.$$

Therefore there exists $C_1>0$ such that

$$\mathbb{E}'\left[\left\|M'(t)-M'(s)\right\|_H^{2q}\right]\le C_1(t-s)^q,\quad q\ge 1.$$

Then by the Kolmogorov test (Lemma 2.134),

$$M'\in C^{\alpha}([0,T];\mathbb{L}^2),\quad \alpha\in\left(0,\frac{1}{2}\right).$$

This completes the proof of Theorem 5.73. □

5.7. **Main theorem.** Summarizing, we state all the results in this section in the following Theorem:

Theorem 5.74. *There exists a filtered probability space* $(\Omega',\mathcal{F}',\mathbb{F}'=(\mathcal{F}_t')_{t\ge0},\mathbb{P}')$ *with pairwise independent, real valued,* $\mathbb{F}'$*-adapted Wiener process* $(W_j')_{j=1}^{\infty}$*. And there exist processes* $M':\Omega'\times[0,T]\longrightarrow V$*,* $B':\Omega'\times[0,T]\longrightarrow\mathbb{L}^2(\mathbb{R}^3)$ *and* $E':\Omega'\times[0,T]\longrightarrow\mathbb{L}^2(\mathbb{R}^3)$ *in this probability space such that:*

(i) *For every* $t \in [0,T]$,

$$M'(t) = M_0 + \int_0^t \{\lambda_1 M' \times \rho' - \lambda_2 M' \times (M' \times \rho')\} \, \mathrm{d}s$$
$$+ \sum_{j=1}^{\infty} \left\{ \int_0^t \left[\alpha M' \times h_j + \beta M' \times (M' \times h_j) \right] \circ \, \mathrm{d}W_j'(s) \right\}.$$

in $L^2(\Omega'; H)$.

$$B'(t) = B_0 - \int_0^t \nabla \times E'(s) \, \mathrm{d}s, \quad \in Y', \ \mathbb{P}' - a.s..$$

$$E'(t) = E_0 + \int_0^t \nabla \times [B'(s) - \tilde{M}'(s)] \, \mathrm{d}s - \int_0^t [1_D E'(s) + \tilde{f}(s)] \, \mathrm{d}s, \quad \in Y', \ \mathbb{P}' - a.s..$$

(ii)

$$|M'(t,x)|_{\mathbb{R}^3} = 1, \quad \textit{for Lebesgue a.e. } (t,x) \in [0,T] \times D \textit{ and } \mathbb{P}' - a.s..$$

(iii) *For every* $\theta \in (0, \frac{1}{2})$,

$$M' \in C^{\theta}([0,T]; H), \qquad \mathbb{P}' - a.s..$$

Proof. The claim (i) is from Theorem 5.71 and Theorem 5.72. The claim (ii) is from Theorem 5.70 and the claim (iii) is from Theorem 5.74. □

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