Essays in International Macroeconomics

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Abstract

This thesis is composed of two parts. Part I contains two essays on capital flows and, in particular, on the phenomenon of global imbalances. Part II includes an essay on the determination of exchange rates.

In the first part, I provide a framework to analyse the trade imbalances between the United States and East Asian countries. In a two-country OLG model with production, I investigate the relationship between East Asian economies' high propensity to save and global imbalances. It is suggested that the absence of pay-as-you-go pension systems can rationalize the saving behaviour of emerging economies and capital outflows to the United States. The model supports the view that there is a "global saving glut" in the world economy. The analysis implies that the introduction of a pay-as-you-go system in China would have the effect of reducing the imbalances.

In Chapter 2, I propose a two-country model to capture output per capita inequalities across countries. My motivation is that global imbalances involve countries at different stages of development. Consistently with empirical evidence, I assume that the East Asian country has a higher capital share in the aggregate production function. The analysis shows that technological differences provide incentives for capital to flow to the developing country. Given that the net foreign assets position of the United States is negative, I conclude that differences in social security systems is the most important basis for trade between the two countries.

In the second part, I develop a theory of nominal exchange rate determination. The model under study is a stochastic OLG economy with multiple currencies and goods. Currencies serve as stores of value and are also required to buy the country-specific good. Portfolios and nominal exchange rates can be pinned down at the stochastic steady state. The model makes a first step towards understanding changes in countries' net foreign assets positions as due to both portfolio adjustments and valuation effects driven by fluctuations of nominal exchange rates.

Contents

List of Tables

List of Figures

Preface

One of the main concerns of international macroeconomists is understanding countries' external positions. The external position of a country can be summarized by two variables. The first one is the *balance of trade*, which we can refer to as "excess demand" from the point of view of general equilibrium theory. The balance of trade is the synthesis of the flow of goods to and from a country expressed in a common numéraire. The second variable is *net foreign assets*. Its sign is informative of whether a country is net debtor or creditor towards the rest of the world. These two variables are deeply interrelated. For example, if a country runs a trade deficit, then intuition suggests that the country is able to consume more than what it produces because it is borrowing from abroad.

After the Second World War till the early 1980s, capital flows among countries were negligible. The celebrated Feldstein-Horioka paper [27] showed that domestic savings were highly correlated with domestic investments for a sample of developed countries. Following the rise in financial integration over the last two decades, it is now more frequent to observe trade imbalances. Another aspect of the same phenomenon is that cross-border holdings of assets has increased exponentially both for emerging and developed countries. In this context, no episode has gained more attention than "global imbalances", i.e. the trade imbalances between the United States and East Asian countries. The reasons are several. Firstly, the size of the trade imbalances is unprecedented as it concerns one of the richest countries in the world and a large area of growing, emerging economies. Second, these imbalances are not temporary. Therefore, global imbalances has now become a stylized fact in international macroeconomics.

The aim of Chapter 1 is to propose a theory of global imbalances. I choose an overlapping-generations framework as opposed to a model with infinitely-lived agents, with the purpose of investigating the relationship between excess savings and global imbalances. In fact, one of the common views on global imbalances is that they are the consequence of a "global saving glut", for which the high saving rates in East Asian countries are responsible [7]. In this thesis, it is pointed out that these economies do not have a welfare system as developed as the United States'. In particular, the absence of pay-as-you go social security systems implies that the working population need to save more in order to finance old age consumption. It is plausible that the divergence of the saving rates can be partially explained by the heterogeneity of pension systems across countries. In fact, this difference is structural as it can be observed in the data long before global imbalances emerged. As such, there is no immediate relationship between the high saving rates in East Asia and global imbalances.

In a two-country OLG economy with production in which only one country has a pay-as-you-go system, I show that capital flows to the country with the pay-as-you-go system. This is consistent with the fact that the United States' net foreign asset position has been negative since the early 1980s, which is when East Asian countries opened their financial markets. On the other hand, global imbalances are the long-run outcome of the financial integration between these two countries. Global imbalances arise during the transition to the world steady state, as soon as the interest rate falls below the growth rate of the economy. Once at steady state, the country with the pay-as-you-go system runs a trade deficit forever. Our analytical results indicate that the relationship between the balance of trade and net foreign assets is not as clear cut as it is commonly held[1]. If the borrower country is below the golden rule, the high interest rate paid on foreign assets (capital outflows) more than compensate the growth in net foreign liabilities (capital inflows). Therefore, the country is in trade surplus. Only in the capital over-accumulation case, the borrower country is in trade deficit. As far as the phenomenon of global imbalances is concerned, the model supports the hypothesis that there is a global saving glut in the world economy and makes the case for a pension reform in China in the direction of introducing a pay-as-you-go system.

The model of Chapter 1 predicts that, upon financial integration, the two countries have the same path of capital stock per capita

[1] Yet, these results are reminiscent of early findings of David Gale [28], [29] for the Solow model and the OLG model with inside money.

after the initial adjustment period. I consider this as a weakness of the theory, since global imbalances involve countries at different stages of development. The equalization of capital stocks per capita is an implication of the assumption that countries produce the consumption good with an identical technology[2]. In Chapter 2, we relax this assumption and allow for heterogenous technologies. In a constant returns to scale and perfect competition environment, this implies that income shares are different across countries. As a matter of principle, it is hard to justify why capital shares should vary across countries. On the other hand, empirical evidence suggests that the capital share of the main US trading partner, i.e. China, is higher than the US'.

As a matter of fact, the country with the highest capital share has a lower output per capita in autarky, and therefore it can be thought of as a developing country. Since its marginal product of capital is higher, capital flows to the developing country when the two countries open to trade. The model shows that output per capita differences persist although there are no frictions in capital markets. However, if a pay-as-you-go system is then introduced in country 1, strong predictions on the pattern of capital flows cannot be made. On the one hand, the higher capital share induces capital inflows to the developing country. On the other hand, the country saves more in the absence of the welfare system. The analysis concludes with the observation that the "saving channel" must be more important than the "technology channel", otherwise the net foreign assets' position of the US should be positive in the data.

Finally, I examine the effect of introducing a pay-as-you-go system in the developing country as far as global imbalances are concerned. Since there are technological differences inducing capital flows to the developing country, the pattern of capital flows would be reversed. Under the assumption that the autarkic interest rate of the US is the golden rule, the world economy would converge to a steady state interest rate higher than the growth rate of the economy. Therefore, the net pattern of trade prevailing in the long-run

[2] Another view is that GDP per capita differences are due to frictions in international financial markets. However, Caselli et al. [12] found that marginal products of capita are equalized across countries, therefore providing evidence in support of a frictionless view of capital markets.

would be the same as the one that we observe today. The United States would run a trade deficit, although in the position of international lenders.

In the last decade, there have been major advancements in the open economy literature from an empirical point of view. Lane and Milesi-Ferretti [41], [43] provided a database on cross-country holdings of assets and liabilities, which is unique in terms of asset disaggregation and country coverage. Following this work, international macroeconomists are no longer exclusively focused on explaining the determinants of countries' net foreign assets positions. One of the important topics on the agenda has become understanding of how countries allocate national savings across domestic and foreign assets. Very few theoretical results are available on this front, and the literature has mainly studied models in which real assets are traded. However, assets are actually currency-denominated and the nominal exchange rate matters when agents face a problem of portfolio choice in open economy.

In Part II (Chapter 3), I propose a new framework aimed at studying the role of the nominal exchange rate in countries' portfolio choices. The model under study is a stochastic OLG economy with multiple currencies and goods. Agents gain utility from consuming L goods but they are only endowed with the country-specific good (full specialization). Currencies serve as stores of value and no other assets are available for risk sharing purposes. Therefore, markets are sequentially incomplete. Money has also a transaction role, since currencies are needed to buy the country-specific good. This assumption allows to pin down portfolios and nominal exchange rates at the stochastic steady state. The fact that we find existence of stationary equilibrium in a number of examples is relevant to the literature on the stochastic OLG model, since existence is not generic in this class of models when $L \geq 2$.

First, I compute the analytical solution of an example with log utility and zero endowment in the second period of life. While useful to gain some intuition, this example is very special as the exchange rate and portfolios happen to be constant in equilibrium. We then generalize to isoelastic utility functions. Under this specification, the model does not have closed-form solutions, but it is still possible to compute the demand functions.

The model is able to generate state-dependent portfolios, even under identical homothetic preferences. In particular, I show that the distribution of money holdings is related to the distribution of wealth among countries. Trade imbalances also arise as a consequence of wealth, and therefore portfolios, being state dependent. For instance, a country is in surplus whenever its share of aggregate wealth is higher than in the immediate past. The nominal exchange rate is shown to be a function of the expected relative purchasing power of the two currencies, and fluctuates unless the stochastic process is i.i.d..

Thanks to Lane and Milesi-Ferretti's work, we are now aware that the dynamics of the net foreign assets of a country is not exclusively driven by trade imbalances. Movements in the value of assets and liabilities, which are not incorporated in national accounts and are known as "valuation effects", are quantitatively important.

The model presented in Chapter 3, albeit stylized, makes a first step towards understanding changes in countries' net foreign assets positions as due to both portfolio adjustments and valuation effects linked to fluctuations of nominal exchange rates. In the model, countries are in fact hit by positive (negative) valuation effects on their net foreign assets' position when the domestic currency depreciates (appreciates). Our numerical results indicate that the balance of trade comove negatively with valuation effects as long as the Markov process is persistent. The intuition can be briefly explained as follows. If a country experiences a positive shock, it runs a trade surplus since it holds more money than in the past. Since agents expect a high value of the endowment tomorrow, they desire more domestic currency as they wish to substitute the domestic for the foreign good. On the other hand, money supply is fixed. Therefore, the relative price of the domestic currency has to increase to counterbalance agents' high demand. Since the domestic currency appreciates, the country then experiences negative valuation effects.

Finally, for reasonable ranges of parameter values, valuation effects are sizable as they reduce the impact of current account positions on changes in net foreign assets by more than a half. Other papers find instead extremely small valuation effects[3].

[3]See, for example, Devereux et al. [21]. However, it is important to stress that valuation effects are of a different nature in [21]. They are driven by changes in equity prices and not by exchange rate fluctuations.

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Author’s declaration

The first chapter of this thesis has been presented at various conferences and workshops and benefited from comments of many participants: the General Equilibrium Days 2011 in York, the XVII Workshop in Dynamic Macroeconomics in Vigo, the RES Autumn School 2012, the Warwick Economics PhD Conference 2013, the RES Annual Conference 2013 and the EEA Annual Congress 2013.

All the work contained in this thesis is my own and it has not been submitted for examination at this or any other institution for another award.

Part I

A theory of global imbalances

Chapter 1

An OLG model of global imbalances

1.1 Introduction

Not only too little capital flows from rich to poor countries - as Lucas [45] pointed out - but we have observed the reverse pattern of net capital flows for over a decade. The trade imbalances between the United States and East Asian economies, or global imbalances, do not appear to be a temporary phenomenon. Figure 1.1 shows that the United States' current account deficit has steadily deteriorated since the late 1990s. The recent adjustment has involved trade with Europe and oil-producing countries, but not other emerging economies. In particular, the US deficit towards China, which mirrors very closely the Chinese surplus, did not shrink with the recession.

One of the most common views on global imbalances is the "global saving glut hypothesis", due to Bernanke [6]. The core of the argument is that the high saving rates in East Asia have created an excess of savings in the world economy, which has resulted in capital flows towards the US and low real interest rates. Bernanke [6] also claimed that the understanding of global imbalances requires a "global perspective" and that they do not "primarily reflect economic policies and other economic developments within the United States itself". In other words, current account imbalances must be thought of as an equilibrium phenomenon.

In this chapter, we provide a general equilibrium framework to discuss the global saving glut hypothesis and therefore investigate the relationship between emerging countries' high propensity to save and global imbalances. An interesting - and key, to us - aspect of the data is that while global imbalances emerged in the late 1990s, that East Asian countries save more than the United States is certainly not a new fact. Figure 1.2 depicts the saving rates of the US and a few East Asian countries over the last 30 years.

The heterogeneity in the pension systems is one of the plausible candidates to explain the structural difference in the countries' saving rates. In fact, pay-as-you-go social security systems are nearly absent in many emerging economies. Reforms aimed at introducing state pensions are still underway in China and other East Asian economies[1]. On the other hand, the pay-as-you go system was introduced in the United States during the Great Depression. There is a substantial body of evidence - summarized in [27] - which indicates that the pay-as-you-go system had the effect of crowding out private saving in the US. More generally, cross-sectional evidence [58] supports the idea that countries with pay-as-you-go systems tend to have lower saving rates, especially the more extensive is the coverage. Yet, the implications for global imbalances of the fact that East Asian countries need to save more to finance old age consumption are still unexplored. One of the contributions of this work is to fill this gap in the literature.

The model that we study is a two-country OLG model with production along the lines of Diamond [26], in which the two countries are identical except that only one country has a pay-as-you-go social security system. The Diamond model is a natural framework to address the question of excess savings in an economy. In fact, the model admits the possibility that, in a perfectly competitive economy, there is capital overaccumulation. The concept of "excess savings" has a precise meaning in the OLG model as it corresponds to the notion of dynamic inefficiency, and this motivates our modeling choice.

In section 2 and 3, we present the model and characterize the

[1] On the Chinese case, see "Social Security Reform in China: Issues and Options" at Peter A. Diamond's webpage: http://econ-www.mit.edu/files/691. Diamond was one of the leading economists who participated at this study on social security reforms in China. On pension systems in Asia, see e.g. [8].

direction of capital flows and trade at and outside steady states.

First, we show that the emerging country always lends to the developed country, as the young of the former country save relatively more in the absence of the pay-as-you-go system[2]. Yet, the pattern of trade in the consumption good does depend on the long-run efficiency of the world economy. We prove that the direction of trade depends on how the population growth rate compares with the interest rate, and this is also the case outside steady states. The emerging country runs a trade surplus only as long as the world economy is beyond the golden rule level of capital (capital overaccumulation). Otherwise, the emerging country runs a trade deficit despite the fact that it's the lender country. Only in the coincidental case of the golden rule, trade happens to be balanced.

The main implication of these results is that we would not observe the current pattern of trade if there was not an excess of savings in the world economy. In this sense, our work provides a formal argument in favor of the "global saving glut hypothesis". Caballero et al. [10] argue that the saving glut story can be interpreted within their framework, by positively shocking the emerging country's saving parameter. Here, a global excess of savings arise endogenously, as a long-term consequence of the financial integration between the United States and East Asian countries.

Another interesting aspect of the trade balance result is that the developed country runs a trade deficit in the capital overaccumulation case because aggregate consumption is higher than in the other country. The reason is that pensions' growth is high enough to compensate interest payments to the emerging country. It is often claimed that global imbalances are due to the fact that emerging countries are consuming too little. This model shows that this is nothing but equilibrium behavior.

Our findings are related to two seminal papers of David Gale [28], [29]. Gale made the important point that countries can run permanent trade imbalances in general equilibrium models. His intuition was that this is especially possible in OLG economies. Gale had discovered that the sign of the balance of trade depends

[2] Geide-Stevenson [33] found the same result in a two-country Diamond model in which the pay-as-you-go tax is proportional rather than lump-sum. Her analysis is limited to steady states, here we also look at the dynamics of capital flows.

on efficiency properties in a Solow model with heterogenous agents and in a pure exchange OLG economy with inside money. The paper is also related to Polemarchakis and Salto [53], which found that trade is balanced at the golden rule in a pure exchange OLG economy with outside money.

Previous work on international capital mobility that use the Diamond model as a framework include Buiter [9] and Geide-Stevenson [33]. Buiter [9] studies a two-country Diamond model in which the countries are heterogeneous in the discount factors, and finds that the most patient country always runs a current account surplus at the steady state, but not necessarily outside it. Following Buiter's paper, Geide-Stevenson [33] established that the social security country always runs a current account deficit at the steady state. On the contrary, we claim that the current account of the social security country can be in surplus both at and outside steady states, when interest rates are higher than the population growth rate. The reason for this divergence in the results is that the balance of trade equation, as stemming from the good market clearing equation, already includes net income from abroad. In fact, interest payments are done in the consumption good. The balance of trade must coincide with the current account in the Diamond model. This is why Buiter [9] and Geide-Stevenson [33] downplayed the importance of efficiency properties in their assessment of countries' net external position.

In section 4, we study the dynamics of capital flows and global imbalances for plausible initial conditions of the autarkic economies. It turns out that the model is able to account for the dynamics and the timing of global imbalances, as well as the dynamics of real interest rates and net foreign asset positions. First, the model can rationalize the fact that the US current account and real interest rates deteriorated *gradually* (Figure 1.1 and 1.4). Second, the model can explain why the accumulation of net foreign liabilities started in the early 1980s (Figure 1.3), well *before* the emergence of global imbalances[3].

The model provides intuitive explanations for these facts. Because of their higher saving rates, emerging countries started to lend abroad soon after they opened to trade with the US. The decline

[3]See section 4 for a comparison with the literature on these stylized facts.

of real interest rates can be read as a consequence of capital accumulation in the world economy (Figure 1.4). Global imbalances arose as soon as interest rates fell below the long-run growth rate, implying that the world economy is saving too much.

Finally, we ask whether it is plausible that the economy is experiencing a global saving glut. According to the model, this requires that the long-run growth rate of the economy is higher than the real interest rate. We find evidence of this in the data. This is hardly surprising, since US real interest rates have hit a historic low in the past decade.

This paper is mainly related to the body of literature which puts emphasis on differences in institutions as the main determinant of global imbalances, e.g. Caballero et al. [10], Mendoza et al. [49] and Angeletos et al. [2]. These papers' focus is on financial markets' different stages of development, and yet the sense of our analysis is similar as the type of pension system enforced in a country surely affects saving and investment possibilities. Caballero et al. [10] explain global imbalances as the result of a negative shock to emerging countries' level of financial development, while our view is that global imbalances arose as the outcome of the financial integration between the US and emerging economies. In this respect, this paper is closer to Mendoza et al. [49] and Angeletos et al. [2].

The novel element of this model is that global imbalances are neither a temporary phenomenon, meant to disappear in the long-run (in [49] and [2]), nor a benign aspect of the world economy (as in [10]). Moreover, the presence of excess savings in an OLG economy means that there is room for policy interventions.

Hence, this paper contributes to the debate on whether and how the imbalances should be addressed from a policy point of view. While there is widespread agreement that global imbalances must be reduced, this is advocated on the basis of a variety of arguments[4].

It is often claimed that East Asian countries should introduce policies to boost domestic demand, in view of correcting the imbalances. If we accept that the world economy is overaccumulating capital, long-term policies in this direction are clearly desirable. For instance, the introduction of a pay-as-you-go system in China would not only be Pareto-improving but also have the effect of reducing

[4]See a recent collection of papers written by central bankers on the topic [34].

the imbalances.

1.2 The world economy

In this section, we describe the two-country model, which maintains the basic structure of Diamond (1965). We will refer to country 1 (2) as the developed (emerging) country.

Agents live for two periods and a new generation is born in each country for all t. The size of the population follows $L_{i,t} = L_{i,0}(1+n)^t$, where $L_{i,0}$ are the young born in country i at date 0 and n is the (common) population growth rate[5]. The only source of growth in the model comes from population[6].

The two countries only differ in the pension systems. Country 1 has a pay-as-you-go social security system, while the system in country 2 is fully-funded. Country 1's government levies a time-invariant lump-sum tax τ_1 on the young, which is used to finance the old's pension b_1 at each t. The policy is balanced so that taxes are equal to transfers at each t: $\tau_1 L_{1,t} = b_1 L_{1,t-1}$. It follows that the transfer which the current old receive is equal to $b_1 = (1+n)\tau_1$, i.e. each generation receives a transfer which is bigger than the tax if population is increasing.

Finally, we need to specify which markets are open for international trade. We assume that the consumption good can be costlessly traded between the countries. As our focus is to analyze the pattern of trade in the good, we impose that labor is immobile.

1.2.1 Firms

Competitive firms use capital and labor to produce the consumption good by means of an identical, constant returns technology: $Y_{i,t} = F(K_{i,t}, L_{i,t})$.

As anticipated above, firms located in country i can only hire workers in the domestic labor market. We consider the production function in its intensive form as the number of workers is given at each t: $y_{i,t} = f(k_{i,t})$. The function f is strictly increasing and

[5]This is realistic as the population of both China and the United States have grown at an average rate of 1% for the last 30 years (World Bank data). However, we allow for the countries' size to be different.

[6]See Appendix A for an extension of the model with labour augmenting technological progress.

concave in $k_{i,t}$. Capital depreciates at the constant rate $0 \leqslant \delta \leqslant 1$ in both countries and we assume that the following boundary conditions hold:

$$\lim_{k_{i,t} \to +\infty} f'(k_{i,t}) = 0 \qquad\qquad \lim_{k_{i,t} \to 0} f'(k_{i,t}) = +\infty$$

At time 0, the two autarkic economies open to trade after production has taken place. Their "initial" level of capital will respectively be $k_{1,0}$ and $k_{2,0}$. Starting from period 1, firms' demand for capital is met in the world market and therefore they will face the same path of interest rates $\{r_t\}$. Firms solve the following maximization problem:

$$\max_{k_{i,t}} \pi_{i,t} = f(k_{i,t}) - (r_t + \delta)k_{i,t} - w_{i,t} \qquad \forall\ i, t \geqslant 1 \tag{1.1}$$

The necessary and sufficient conditions for a maximum are:

$$r_t = f'(k_{i,t}) - \delta \tag{1.2}$$

$$w_{i,t} = f(k_{i,t}) - f'(k_{i,t})k_{i,t} \tag{1.3}$$

Because the countries have access to the same technology, it is immediate that capital stocks per capita are equalized: because $k_{1,t} = k_{2,t} = k_t$, it is also true that $w_{1,t} = w_{2,t} = w_t$ for all t. While the two countries might start with different initial conditions, potential income differences vanish once the two countries open to trade.

This assumption is somewhat strong, but it is convenient to abstract from other potential bases for trade to study how differences in pension systems have an impact on capital accumulation and trade[7].

1.2.2 Consumers

Agents get utility from consuming in the two periods of life. Preferences are stationary and identical both within generation and across countries. The utility function is C^2, strictly increasing, strictly concave and additively separable:

$$U(c^t_{i,t}, c^t_{i,t+1}) = u(c^t_{i,t}) + \beta v(c^t_{i,t+1}) \tag{1.4}$$

[7] In Chapter 2, we will allow for technologies to be different across countries.

where $c_{i,t}^t$ denotes consumption when young and $c_{i,t+1}^t$ is consumption when old of the generation (born at time) t in country i. Also:

$$\lim_{c_{i,t}^t \to 0} u'(c_{i,t}^t) = +\infty \qquad \lim_{c_{i,t+1}^t \to 0} u'(c_{i,t+1}^t) = +\infty$$

The budget constraints are:

$$c_{i,t}^t = w_t - \tau_i - s_{i,t} \tag{1.5}$$

$$c_{i,t+1}^t = s_{i,t}(1 + r_{t+1}) + \tau_i(1 + n) \tag{1.6}$$

where $\tau_2 = 0$ as there is no pay-as-you-go system in country 2. In our two-country world, the young are allowed to lend both to domestic and foreign firms. Which country is going to be the borrower (lender) will be established in equilibrium.

The maximization problems of the two consumers are the following:

$$\max_{s_{1,t}} \quad u(w_t - \tau_1 - s_{1,t}) + \beta v(s_{1,t}(1 + r_{t+1}) + \tau_1(1 + n)) \tag{1.7}$$

$$\max_{s_{2,t}} \quad u(w_t - s_{2,t}) + \beta v(s_{2,t}(1 + r_{t+1})) \tag{1.8}$$

The necessary and sufficient conditions for a maximum are:

$$u'(w_t - \tau_1 - s_{1,t}) = \beta(1 + r_{t+1})v'(s_{1,t}(1 + r_{t+1}) + \tau_1(1 + n)) \tag{1.9}$$

$$u'(w_t - s_{2,t}) = \beta(1 + r_{t+1})v'(s_{2,t}(1 + r_{t+1})) \tag{1.10}$$

Agents' optimal savings are then a function of the wage and the interest rate. In country 1, they also depend upon the taxes and transfers related to the pension system.

In the OLG model, it is well known that savings are lower in presence of a pay-as-you-go system (see e.g. [3], or [57]):

$$\frac{ds_{1,t}}{d\tau_1} = -\frac{u''(c_{1,t}^t) + \beta(1 + n)(1 + r_{t+1})v''(c_{1,t+1}^t)}{u''(c_{1,t}^t) + \beta(1 + r_{t+1})^2 v''(c_{1,t+1}^t)} < 0 \tag{1.11}$$

Given that agents face the same factor prices (w_t, r_{t+1}), we can claim that the young in country 1 save less than in country 2. It is also known that the extent of the fall in saving will depend on how n and r_{t+1} compares. In particular, if $n > r_{t+1} (< r_{t+1})$ then the drop in saving is larger since $\frac{ds_{1,t}}{d\tau_1} < -1 (> -1)$. In fact, the income of country 1's consumers is higher (lower) when the rate of return on the pension system is higher (lower) than the interest rate. This can be seen from the consolidated budget constraint:

$$c^t_{1,t} + \frac{c^t_{1,t+1}}{1+r_{t+1}} = w_t - \tau_1 \frac{r_{t+1}-n}{1+r_{t+1}} \tag{1.12}$$

When $n > r_{t+1}$, $c^t_{1,t}$ increases, since consumption is a normal good[8]. Therefore, savings will be even lower. Only when $n = r_{t+1}$, savings decrease one for one with the tax as (1.11) shows.

We also characterize the saving functions by the following assumption.

Assumption 1 *Consumption when young and when old are gross substitutes:*

$$s_r > 0$$

where s_r is the partial derivative of the saving function with respect to the interest rate.

1.2.3 Equilibrium

Given $(\tau_1, k_{1,0}, k_{2,0})$, a competitive equilibrium is a sequence of capital stocks $\{k^*_t\}_{t\geqslant 1}$ and factor prices $\{r^*_t, w^*_t\}_{t\geqslant 1}$ such that:

(i) $\{c^{t*}_{i,t}, c^{t*}_{i,t+1}\}_{t\geqslant 0}$ maximize the agents' utility function (1.4) subject to the budget constraints (1.5),(1.6) for all i;

(ii) $\{k^*_t\}_{t\geqslant 1}$ maximize the firms' profit function (1.1);

(iii) the (world) capital market clears for $t \geqslant 0$:

$$\sum_i L_{i,t} s^*_{i,t} = \sum_i K^*_{i,t+1}$$

If the capital market clears at each t, the (world) market for the good will clear by Walras' Law. The good market is in equilibrium when the total resources available (after production) are equal to the consumption of the current young and old, and next period's capital stocks of the two countries.

$$\begin{aligned} \sum_i F(K^*_{i,t}, L_{i,t}) \quad &+ \quad (1-\delta)\sum_i K^*_{i,t} = \qquad (1.13) \\ &= \sum_i L_{i,t} c^{t*}_{i,t} + \sum_i L_{i,t-1} c^{t-1*}_{i,t} + \sum_i K^*_{i,t+1} \end{aligned}$$

[8]That savings are increasing in the wage can be derived from the first-order conditions. It can be checked that $\frac{ds_{i,t}}{dw_t} = \frac{1}{1+\beta(1+r_{t+1})^2 \frac{v''(c^t_{i,t+1})}{u''(c^t_{i,t})}}$, therefore $0 < s_w < 1$.

Equation (1.14) will be extensively used in the next section to study the pattern of trade between the two countries.

1.3 The pattern of trade

1.3.1 Dynamics in the capital market and capital flows

In this section, we analyze the direction of capital flows in the model described above. The first step is to study how capital accumulates. The capital market is equilibrium in as long as the world demand for capital is equal to the world supply (savings):

$$\begin{aligned} K_{t+1}^* &\equiv \sum_i K_{i,t+1}^* = L_{1,t}s_1(f(k_{1,t}^*) - f'(k_{1,t}^*)k_{1,t}^*, f'(k_{t+1}^*), \tau_1) + \\ &+ \; L_{2,t}s_2(f(k_{2,t}^*) - f'(k_{2,t}^*)k_{2,t}^*, f'(k_{t+1}^*)) \qquad (1.14) \end{aligned}$$

where K_{t+1}^* denotes the world capital stock at time $t+1$. We have already established that $k_{1,t} = k_{2,t} = k_t$ for $t \geqslant 1$, while at $t = 0$ countries might start with different levels of capital.

Before proceeding, it is convenient to introduce the following definition:

Definition 1 *Country i's size is:* $\rho_i \equiv \frac{L_{0,i}}{L_0}$.

Because the countries grow at a common rate, ρ_i is constant over time and depends on the countries' initial labor forces. We can now divide (1.14) by the world labor supply L_t and get:

$$\begin{aligned} (1+n)k_{t+1}^* &= \rho_1 s_1(f(k_{1,t}^*) - f'(k_{1,t}^*)k_{1,t}^*, f'(k_{t+1}^*), \tau_1) + \\ &+ \; \rho_2 s_2(f(k_{2,t}^*) - f'(k_{2,t}^*)k_{2,t}^*, f'(k_{t+1}^*)) \qquad (1.15) \end{aligned}$$

At each t, the world capital stock per capita (which is equivalent to the domestic capital stocks) is determined by the savings of country 1 and 2. Equation (1.15) shows that each country will contribute to the supply side of the market according to its size.

Hereafter, we study the above difference equation in the capital stock. The world economy is in steady state when $k_t^* = k_{t+1}^* = k^*$:

$$\begin{aligned} (1+n)k^* &= \rho_1 s_1[f(k^*) - f'(k^*)k^*, f'(k^*), \tau_1] + \\ &+ \; \rho_2 s_2[f(k^*) - f'(k^*)k^*, f'(k^*)] \qquad (1.16) \end{aligned}$$

Lemma 1 (i) *Given* $k_{1,0} > 0$ *and* $k_{2,0} > 0$*, there exists a unique intertemporal equilibrium as long as* $\tau_1 < \bar{\tau}_1(k_{1,0})$.

(ii) If $\lim_{k_t \to 0} \frac{\phi(k_t;\tau_1,\rho_1,\rho_2)}{k_t} > 1$, *there exists at least a stable steady state.*

Proof. The proof is in Appendix A. The function ϕ is defined there. ■

Part (i) of Lemma 1 establishes that there exists an equilibrium path only if each country's savings are positive at $t = 0$. It is intuitive that we need a condition on the tax level to avoid circumstances under which income is either zero or negative in the initial period. In other words, a perfect foresight equilibrium will exist only if the level of the tax is compatible with having positive savings in the economy[9]. Part (ii) shows that there exist paths converging to a stable steady state. This is important as the focus of the next section will be on the behavior of the economy near a stable steady state.

We can now analyze the pattern of trade between the countries[10]. We start with trade in the capital market. Given the capital market equilibrium equation, it is immediate to show which of the two countries has positive excess demand for capital.

Definition 2 *The (per capita) excess demand function of country* i *is:*

$$z_{i,t} \equiv (1+n)k_{t+1} - s_{i,t} \tag{1.17}$$

By dividing the capital market clearing equation by world population, we get that $\rho_1 z^*_{1,t} + \rho_2 z^*_{2,t} = 0$ using the above definition.

Proposition 1 (Borrowing and lending) *Country 1 (2) is the borrower (lender) country for all* $t \geqslant 1$.

Proof. First, substitute equation (1.15) into the excess demand function of country i. Equilibrium excess demands are:

$$z^*_{1,t} = \rho_2(s^*_{2,t} - s^*_{1,t}) \qquad z^*_{2,t} = -\rho_1(s^*_{2,t} - s^*_{1,t}) \tag{1.18}$$

From equation (1.11), we know that country 1 saves less than country 2 given factor prices. Therefore, it must be true that $s^*_{2,t} > s^*_{1,t}$

[9]See [25] for a detailed analysis of the (closed economy) Diamond model with lump-sum transfers.

[10]We postpone the discussion of the pattern of trade at the openness to section 4, where we study the dynamics of capital flows and global imbalances for realistic initial conditions of the autarkic economies.

for all k_t^*, k_{t+1}^*. The sign of excess demand for the two countries follows:

$$z_{1,t}^* > 0 \qquad\qquad z_{2,t}^* < 0 \qquad \forall\ t \geqslant 1 \tag{1.19}$$

■

Proposition 1 shows that country 2 (the emerging country) will always lend to country 1, it does not matter whether the economy is in a steady state or not. The intuition behind this result is simple. We know that the equilibrium capital stock is combination of savings in the two countries and the developed country saves less than the emerging economy. Therefore, while country 1 has to borrow to sustain k_{t+1}^*, country 2's savings (partly) find an outlet in country 1.

It might be noted that the extent of trade will depend on how large is the difference between the two countries' savings. For instance, countries trade more the bigger is the size of the pay-as-you-go system in country 1. It is worth stressing that the direction of trade in the capital market does not depend on whether we are in the capital overaccumulation case or not. However, this becomes relevant once we consider the countries' net trade.

1.3.2 The balance of trade and efficiency

We can now study the pattern of trade in the consumption good. First, we define the balance of trade of country i as the country's excess supply for the consumption good.

Definition 3 *The (per capita) trade balance of country i is:*

$$tb_{i,t}^* \equiv f(k_{i,t}^*) + (1-\delta)k_{i,t}^* - c_{i,t}^{t*} - \frac{c_{i,t}^{t-1*}}{1+n} - k_{i,t+1}^*(1+n) \tag{1.20}$$

If $tb_{i,t}^* > 0$ in equilibrium, then country i is net exporter as output is higher than "domestic absorption".

A few words are due to explain the above definition, as it is of fundamental importance for the results of the paper. Definition 3 stems from the per capita version of (1.14), the consumption good's market clearing equation. Equation (1.14) states that the sum of the countries' balances must be zero at each t. Once we divide it by the world population L_t, we obtain that $\rho_1 tb_{1,t} + \rho_2 tb_{2,t} = 0$. While this equation must hold, that $tb^*_{i,t} \neq 0$ for every i is still possible in equilibrium.

Another way to look at the balance of trade is in terms of net capital flows. Use the fact that $f(k^*_{i,t}) = w^*_t + (r^*_t + \delta)k^*_{i,t}$ to get:

$$tb^*_{i,t} = w^*_t + (r^*_t + \delta)k^*_{i,t} + (1-\delta)k^*_{i,t} - c^{t*}_{i,t} - \frac{c^{t-1*}_{i,t}}{1+n} - k^*_{i,t+1}(1+n)$$

Using the budget constraints of the young and the old living at time t, we obtain:

$$tb^*_{i,t} \equiv [s^*_{i,t} - k^*_{i,t+1}(1+n)] - \left(\frac{1+r^*_t}{1+n}\right)[s^*_{i,t-1} - k^*_{i,t}(1+n)] \quad (1.21)$$

Next, using Definition 2 rewrite (1.21) as follows:

$$tb^*_{i,t} = -z^*_{i,t} + \left(\frac{1+r^*_t}{1+n}\right) z^*_{i,t-1} \quad (1.22)$$

The above characterization shows that the balance of trade reflects trade in the capital market in period t and $t-1$.

Proposition 2 (Balance of trade and steady states) *At the golden rule allocation ($r^* = n$), trade is balanced.*

If the steady state is inefficient ($r^* < n$), country 2 (the emerging country) is in surplus while country 1 (the developed country) is in deficit.

If the steady state is efficient ($r^* > n$), the opposite is true.

Proof. Consider equation (1.22). Imposing $z^*_{i,t} = z^*_{i,t-1} = z^*_i$ and $r^*_t = r^*$, the trade balance of country i in the steady state is:

$$tb^*_i = -z^*_i \left(\frac{n - r^*}{1+n}\right) \quad (1.23)$$

It immediately follows that at the golden rule allocation $tb_i = 0 \; \forall \; i$. The other statements are a direct implication of our hypotheses and the sign of z^*_i (Proposition 1). ■

If the world economy converges to a steady state such that $r^* = n$, not only steady state consumption will be maximized but trade will be balanced in the long-run. Yet, that trade is balanced does not imply that the two countries do not trade at all. In fact, trade in the capital market still takes place at the golden rule (by Proposition 1) but each country's capital outflows are completely offset by capital inflows.

However, this can only happen by coincidence. In all other cases, there will be trade imbalances between the two countries. To comment on the result, let us consider the balance of trade of country 2:

$$tb_2^* = \underbrace{-z_2^*}_{\text{capital outflow}} + \underbrace{\left(\frac{1+r^*}{1+n}\right) z_2^*}_{\text{capital inflow}}$$

We have seen that the young in country 2 lend to firms located in country 1 as they save relatively more (capital outflow). At the same time, the old of country 1 pay the loan back, along with interest payments, to the old of country 2 (capital inflow).

The proposition states that the sign of net capital flows (or the balance of trade) will depend on how n and r^* compares. Indeed, notice that while z_i^* is constant at the steady state, $Z_{i,t}^*$ will grow at the population growth rate. Proposition 2 then says that the lender country will have a surplus as long as the net income from abroad is not enough to compensate the increase in capital outflows induced by population growth. Instead, if the interest rate was higher than the population growth rate, country 2 should be in deficit.

Therefore, the model implies that the reason why the US run a trade deficit is that there is a saving glut in the world economy. We postpone to section 4 the discussion of whether it is plausible that the world economy is on an inefficient path, with the support of some empirical evidence.

The fact that the sign of the balance of trade of a country depends on whether the world economy happens to be below or beyond the golden rule allocation is not just true at the steady state of the model. Next, we show that this holds outside stationary states too.

To this purpose, it is more convenient to work with equation (1.20). As technologies are identical, it is intuitive that all the action has to come from aggregate consumption. Because pension systems are different, the countries' consumption possibilities are not the same and this will explain the direction of trade in the consumption good.

Lemma 2 (Consumption) *For any generation $t \geqslant 1$, the agent born in country 1 consume relatively more (less) when $n > r_{t+1}$*

$(< r_{t+1})$.

Proof. The proof is in Appendix A. ■

In Lemma 2, we show that agents born in country 1 consume more in the capital overaccumulation case. It is interesting to note that this result supports the idea that East Asian countries are consuming too little relatively to the United States, and this has something to do with global imbalances. The reason is that country 1's generations have a higher income, despite that the United States have to pay interest rates to China. When $n > r^*_{t+1}$, there is enough growth in the economy for the pension to compensate interest payments to the foreign country. In other words, the net present value of the pay-as-you-go system is positive in the capital overaccumulation case. An examination of the two agents' budget constraints should convince the reader of this fact.

Given Lemma 2, we can analyze the pattern of trade in the consumption good outside steady states:

Proposition 3 (Balance of trade outside steady states) *Country 1 (the developed country) is in deficit at a given t when $n > r^*_t$ and $n > r^*_{t+1}$, while in surplus when $r^*_t > n$ and $r^*_{t+1} > n$. If $r^*_t > n$ and $r^*_{t+1} < n$, the sign is ambiguous.*

Proof. We consider the developed country, the opposite is obviously true for the emerging economy. If country 1 imports, then $tb_{1,t} < tb_{2,t}$. Given Definition 3 and because $k_{1,t} = k_{2,t}\ \forall\ t \geqslant 1$, the following must hold for country 1 to be in deficit:

$$c^{t*}_{1,t} + \frac{c^{t-1*}_{1,t}}{1+n} > c^{t*}_{2,t} + \frac{c^{t-1*}_{2,t}}{1+n}$$

Indeed, Lemma 2 showed that consumption is higher for generations in country 1 as long as next period's interest rate is lower than the population growth rate. Therefore, for $tb_{1,t} < 0$ it is sufficient that $n > r^*_t$ and $n > r^*_{t+1}$. Instead, when $r^*_t > n$ and $r^*_{t+1} > n$ generations of country 2 consume more and $tb_{2,t} < 0$.

Suppose that at a given t, we have that $r^*_t > n$ but next period's interest rate falls below the population growth rate. While $c^{t*}_{1,t-1} < c^{t*}_{2,t-1}$ by $r^*_t > n$, $c^{t*}_{1,t} > c^{t*}_{2,t}$ by $r^*_{t+1} < n$. The net effect will depend on other parameters of the economy (see Appendix A.2 for an illustration in the Cobb-Douglas case). ■

The proposition establishes that the deficit (surplus) country is the country which consumes relatively more (less) at a given t.

At the golden rule, it is worth noting that the consumption allocation of the two representative generations is identical despite the different pension systems (see the proof of Lemma 2 in Appendix A). This gives a different angle to the balanced trade result. Because savings decrease one for one with τ_1 and consumers' wealth is not affected by the pension system when $r^* = n$, consumption choices in the two countries are the same at the golden rule. Indeed, the planner would choose such allocation if giving the same weights to the agents (in fact, we did not allow for heterogeneity in preferences).

In this section, we have looked at the sign of the balance of trade as the trade balance is what drives the behaviour of the current account of the US and China. From equation (1.22), we can also derive an expression that relates the change in net foreign assets of a country (or current account) to the balance of trade and net income from abroad.

First, the net foreign asset position of country i at the beginning of period $t+1$ is equivalent to the amount of good which has been "exported" or lent to the other country in the previous period: $A_{i,t+1} \equiv -Z_{i,t}$. Therefore, equation (1.22) can be rewritten as:

$$tb^*_{i,t} = (1+n)a^*_{i,t+1} - (1+r^*_t)a^*_{i,t}$$

Finally, the current account of country i is the sum of the balance of trade and net income from abroad and it is also equivalent to the change in the net foreign asset position of the country:

$$ca^*_{i,t} \equiv (1+n)a^*_{i,t+1} - a^*_{i,t} = tb^*_{i,t} + r^*_t a^*_{i,t}$$

At the steady state of the economy, the current account of the lender (borrower) country will always be in surplus (deficit) as the country is accumulating net foreign assets at the growth rate of the world economy:

$$na^*_i = tb^*_i + r^* a^*_i$$

This result can be interpreted as follows. The current account of a country is usually defined as domestic savings minus domestic investment. As capital stocks per capita are equalized across countries, it is not investments per capita that drive the behaviour of the current account in the model. The lender country is always in surplus due to the fact that aggregate savings are relatively higher in the other country to the absence of the pay-as-you-go system.

Finally, it is worth noting that the sign of the balance of trade will closely follow the sign of the current account only if we are in the capital overaccumulation case.

1.4 The dynamics of net foreign assets and global imbalances

The results of section 3 imply that the dynamics of the countries' balance of trade are strongly related to the efficiency of the world economy's capital accumulation path. In particular, we have found that the lender country (the country with no pay-as-you-go pension system) runs a trade surplus only as long as the population growth rate is higher than the interest rate. Therefore, our theoretical results suggest that global imbalances are a signal that the world economy is overaccumulating capital.

In this section, we demonstrate that the model is able to qualitatively replicate the evolution of the US current account and net foreign assets' position since the early 1980s (the time of China's integration into the world economy). Second, we provide some evidence to support the claim that there is a "global saving glut" in the world economy. If we can say that the long-run growth rate of the world economy is higher than the real interest rate, it is then plausible that the world economy is on an equilibrium path characterized by an excess of savings.

To start with, we need to address the following questions. What are the conditions under which the world economy converges to an inefficient steady state? And are these reasonable enough? To make progress on these issues, we introduce some assumptions on the characteristics of the two countries in autarky. Moreover, we make a conjecture on the two countries' initial conditions at time 0, which would correspond to the financial openness of emerging countries[11].

[11] Lemma 1 established that there exists at least a stable steady state for the world economy. In this section, we restrict attention to those paths converging to a stable steady state.

Hypothesis 1 (Autarkic steady states) *Suppose country* 1 *has a locally stable steady state such that* $r_1^{autss} = n$*. For country* 2*, the locally stable steady state satisfies* $n > r_2^{aut}$*.*

Hypothesis 2 (Initial conditions) *At the time of financial integration* $t = 0$*, country* 1 *is at the autarkic steady state* $k_{1,0} \equiv k_1^{autss}$*.*

Country 2*'s initial capital stock satisfies* $k_{2,0} < k_2^{aut}$*. Moreover, it is low enough that* $k_{2,0} < k_{1,0}$ *and* $s_1(k_{1,0}, k_1^*, \tau_1) > s_2(k_{2,0}, k_1^*)$*, where* k_1^* *is the equilibrium capital stock at* $t = 1$*.*

Our main hypothesis is that the pay-as-you-go system, which has been introduced during the Great Depression, "fixed" the long-run inefficiency of the US economy. This assumption is also consistent with the fact that the US current account was balanced before 1980. We then assume that the autarkic steady state of the emerging economy is inefficient in the absence of social security. This is coherent with our previous analysis, as we treated the two countries as identical (except for the pension systems).

That country 2 opened to trade with a relatively low capital stock and along its transition path, while country 1 was already at the autarkic steady state, should not be controversial. We will explain Hypothesis 2 in more detail in the context of Proposition 5.

We are now ready to characterize the long-run equilibrium of the world economy.

Proposition 4 (World steady state) *Under Hypothesis* 1*, the world economy has a locally stable steady state such that* $n > r^*$*.*

Proof. It suffices to show that the (world) interest rate is between the autarkic interest rates: $r_1^{autss} > r^* > r_2^{aut}$, because we assumed that $r_1^{autss} = n$ (see Appendix A for a proof). ■

From Hypothesis 2, it can be inferred that the initial conditions of the world economy are such that the world economy starts to the left of the steady state. Our next step is to study trade dynamics in this context. First, we analyze trade at the time of China's financial integration. For instance, $t = 0$ could roughly correspond to 1980. That the world capital market is open means that the young can lend both to domestic and foreign firms. As it might be expected, the pattern of trade at the openness will depend on the two countries' initial conditions.

Proposition 5 (Financial integration) *Under Hypothesis* 2, (i) *the developed country is the lender and runs a trade surplus at* $t = 0$*;* (ii) *the developed country runs a trade deficit at* $t = 1$.

Proof. The proof is in Appendix A. ■

The proof shows that k_1^* is pinned down by total savings at $t = 0$, which depend on the two countries' initial conditions. At the outset of financial integration, a realistic scenario is one in which capital flows to the capital scarce, emerging country. To impose that $k_{2,0} < k_{1,0}$ is not enough because while country 1 has a higher wage, there is the negative partial equilibrium effect of the pay-as-you-go on country 1's savings to take into account. Therefore, we need more stringent conditions for country 1 to save more and therefore lend to country 2 (Hypothesis 2).

At $t = 1$, the developed country's current account position turns into deficit: the old in country 2 pay off their debt and country 1 now starts to borrow.

In the previous section, we established that capital flows to the developed country for $t \geq 1$ and that the sign of the balance of trade depends on whether the interest rate is higher or lower than the population growth rate. Here, we study the dynamics of net foreign assets and the balance of trade in more detail. We restrict our analysis to the case in which both the utility and the production functions are Cobb-Douglas, since it is analytically tractable:

Assumption 2 *The utility and the production functions are Cobb-Douglas:*

$$\begin{aligned} U(c_{i,t}^t, c_{i,t+1}^t) &= \beta \log c_{i,t}^t + (1-\beta) \log c_{i,t+1}^t \\ F(K_{i,t}, L_{i,t}) &= K_{i,t}^{\alpha} L_{i,t}^{1-\alpha} \end{aligned}$$

We fully derive the model under Assumption 2 in Appendix A. For our purposes, it is important to stress that the capital stock evolves over time as follows:

$$\begin{aligned} (1+n)k_{t+1}^* &= (1-\beta)(1-\alpha)k_t^{*\alpha} - \\ &- \rho_1 \tau_1 \left[(1-\beta) + \frac{\beta(1+n)}{1+\alpha k_{t+1}^{*\alpha-1} - \delta} \right] \end{aligned} \tag{1.24}$$

In particular, capital stock convergence is monotonic as the saving locus is increasing:

$$\frac{dk_{t+1}}{dk_t} = \frac{(1-\beta)\alpha(1-\alpha)k_t^{\alpha-1}}{(1+n) - \frac{\rho_1\tau_1\beta(1+n)\alpha(\alpha-1)k_{t+1}^{\alpha-2}}{(1+\alpha k_{t+1}^{\alpha-1}-\delta)^2}} > 0 \qquad (1.25)$$

Figure 1.3 illustrates the path of capital accumulation in the world economy under such assumptions on technology and preferences for an initial value of the world capital stock k_1 to the left of the world steady state.

Let us recall that we have defined the stock of net foreign assets held by residents of country i at the beginning of period $t+1$ as $A_{i,t+1} \equiv -Z_{i,t}$. We have already established that the United States is the borrower country for any $t \geq 1$.

Proposition 6 (The dynamics of net foreign assets) *Under Assumption 2, country* 1 (*the developed country*) *accumulates net foreign liabilities as the world economy converges to the steady state.*

Proof. In Appendix A, we show that equation (1.18) for country 1 becomes under Cobb-Douglas preferences:

$$z_{1,t}^* = \rho_2\tau_1 \left(1 - \beta\frac{\alpha k_{t+1}^{*\alpha-1} - \delta - n}{1+\alpha k_{t+1}^{*\alpha-1} - \delta}\right)$$

Therefore, using our definition of net foreign assets:

$$a_{1,t+1}^* = -\frac{\rho_2\tau_1}{1+n}\left(1 - \beta\frac{\alpha k_{t+1}^{*\alpha-1} - \delta - n}{1+\alpha k_{t+1}^{*\alpha-1} - \delta}\right)$$

Our goal is to study how $a_{1,t+1}$ changes with k_{t+1}:

$$\frac{\partial a_{1,t+1}}{\partial k_{t+1}} = \frac{\rho_2\tau_1\beta\alpha(\alpha-1)k_{t+1}^{\alpha-2}}{(1+\alpha k_{t+1}^{\alpha-1} - \delta)^2}$$

Hypotheses 1 and 2 imply that the initial conditions of the autarkic economies are such that the initial capital stock of the world economy (k_1^*) is to the left of the world steady state. Given that $\frac{\partial a_{1,t+1}}{\partial k_{t+1}} < 0$, country 1's net foreign liabilities increase as the capital stock accumulates.

■

Proposition 3 established that the sign of country 1's balance of trade at a given t depends on whether the current and next period's interest rates are lower or bigger than n. It should now be evident that trade dynamics depends both on the initial conditions and the long-run properties of the autarkic economies. By Proposition 2 and 4, we know already that country 1 will run a deficit in the long-run. In the next proposition, we study the dynamics of trade imbalances.

Proposition 7 (The dynamics of global imbalances) *Under Assumption 2, the balance of trade of country* 1 *deteriorates over time.*

Proof. The balance of trade of country 1 under log preferences is (see Appendix A.2):

$$tb_{1,t}^* = \rho_2\tau_1\left[(1-\beta)\frac{\alpha k_t^{*\alpha-1}-\delta-n}{1+n}+\beta\frac{\alpha k_{t+1}^{*\alpha-1}-\delta-n}{1+\alpha k_{t+1}^{*\alpha-1}-\delta}\right]$$

The balance of trade of country 1 decreases with the current and the future capital stock:

$$\begin{aligned}\frac{\partial tb_{1,t}}{\partial k_t} &= \frac{\rho_2\tau_1(1-\beta)\alpha(\alpha-1)k_t^{\alpha-2}}{1+n}<0\\ \frac{\partial tb_{1,t}}{\partial k_{t+1}} &= \frac{\rho_2\tau_1\beta\alpha(\alpha-1)k_{t+1}^{\alpha-2}(1+n)}{(1+\alpha k_{t+1}^{\alpha-1}-\delta)^2}<0\end{aligned}$$

As $k_1^* < k^*$, the balance of trade of country 1 decreases as the capital stock converges to the world steady state. ■

We can now compare the time-series of the US current account and net international position with the predictions of the model. Figure 1.1 shows that the sign of the US current account varied until the early 1990s, that is before the building up of global imbalances. For this period, we cannot say anything more specific as disaggregated data are not available before 1999. It is possible that China might have imported from the United States in the early stage of financial integration, as Proposition 5 suggests.

More importantly, Proposition 7 explains the widening of the United States' current account deficit versus China. Our model seems to be more successful in capturing the dynamics of global imbalances than other models, e.g. [2], [10], [49]. In these papers, the United States run a trade deficit immediately after China's financial integration (or a shock), and then the deficit gradually improves. Our framework is more consistent with the data as it predicts the *gradual* deterioration of the US deficit.

Another aspect of interest is the dynamics of US foreign assets. Proposition 6 establishes that US net foreign liabilities accumulate over time, starting from $t \geqslant 1$. Figure 1.4 shows this kind of pattern. In this respect, the contribution of this chapter is to explain why the US net foreign assets position turned negative *before* the emergence of global imbalances.

Finally, we show that the data validate the hypothesis that there is an excess of savings in the world economy. Let us focus on the key equation of the model (equation 1.23). The model requires that the interest rate is below the growth rate of the economy for the developed country to run a trade deficit.

The first variable of interest, the real interest rate, is the most controversial because the marginal product of capital and the interest rate in the international bond market are indistinguishable in the model. Figure 1.4 shows that the negative investment position of the US is due to net external debt (private and public), which has steadily increased and reached 40% of GDP in 2007. As it is known, the difference between NFA and net external debt is due to FDI and equity holdings, which tend to be positive for the US.

Because foreign lenders accumulate safe US assets, we take the rate of interest on the US government bonds at different maturities as a proxy for the real interest rate. Figure 1.4 indicates that while interest rates were quite high in the early 1980s, they have embarked on a negative trend since then.

As far as the growth part is concerned, we only allowed for population growth so far. Let us consider labor-augmenting technological progress and assume that technology grows at a common rate g in the two countries[12]. We show in the Appendix that equation (1.23) becomes:

$$\hat{tb}_i^* \approx -\hat{z}_i^* \frac{n+g-r^*}{1+n+g} \tag{1.26}$$

where the hat denotes variables per effective worker. We now take $g = 0.03$ as the (conservative) growth rate of technological progress for the world economy (similarly to Caballero et al.) and $n = 0.01$

[12] As Gourinchas and Jeanne [36] observe, "that countries have the same growth rate in the long run is a standard assumption, often justified by the fact that no country should have a share of world GDP converging to 0 or 100 percent.". The same would occur in this model in the long-run.

as the population growth rate (see footnote 3). Figure 1.5 reveals that real interest rates have been far below the combined growth rate of 4% since the 1990s. The gap between the two has particularly widened during the last decade, which saw the emergence of global imbalances.

We can conclude that there is evidence that the United States have accumulated a trade deficit because a higher saving rate in China (due to the absence of a pay-as-you-go system) has been pushing the real interest rate below the long-run growth rate of the world economy.

A final word is due about dynamic inefficiency. In our setup, we assume that the US economy was at the golden rule before integrating with "inefficient" countries. We have shown that the consequence is that the integrated economy is overaccumulating capital. Part of the literature is of the view that the capital overaccumulation case is only of theoretical interest because actual economies are not dynamically inefficient (see [25] for a discussion, p. 84). These statements are often based on early tests on the dynamic efficiency of stochastic OLG economies. However, Chattopadhyay [16] has recently shown that a widely used criterion to test dynamic efficiency, the net dividend criterion, does not actually give sufficient conditions for optimality. While we are far from having an empirically implementable test, the results of this chapter emphasize that the capital overaccumulation case cannot be ignored since it has something to tell us on relevant stylized facts such as global imbalances.

1.4.1 Country size

In this section, we show that country size has an impact on capital flows and current account dynamics.

First, we establish that the steady state capital stock of the world economy is increasing in country 2's size.

Proposition 8 *Let $k^*_{\rho_2,\rho_1}$ and $k^*_{\tilde{\rho}_2,\rho_1}$ be the steady state capital stocks of two economies, for which $\tilde{\rho}_2 > \rho_2$. Then, $k^*_{\rho_2,\rho_1} < k^*_{\tilde{\rho}_2,\rho_1}$.*

Proof. The logic of the proof is the same as for Proposition 4. Consider equation (A.4) in the Appendix. In Lemma 1(ii), we have proved that there exists a stable $k^*_{\rho_2,\rho_1}$ such that $g(k^*_{\rho_2,\rho_1}) = 0$. Now consider another economy such that $\tilde{\rho}_2 > \rho_2$. It is straightforward that if $k_{\tilde{\rho}_2,\rho_1} = k_{\rho_2,\rho_1}$, then $g(k_{\rho_2,\rho_1}) < 0$. Proposition 4 already showed that the function g is increasing in k if the steady state is stable. Hence, it must be true that $k^*_{\tilde{\rho}_2,\rho_1} > k_{\rho_2,\rho_1}$ for $g(k^*_{\tilde{\rho}_2,\rho_1}) = 0$. ■

This result shows that capital overaccumulation in the world economy is intensified if country 2 has a bigger size. The implications for trade are the following. First, the higher is ρ_2 the larger is $z^*_{1,t}$ or country 1's net foreign assets per capita (equation 1.19). Together with the fact that $\frac{n-r^*}{1+n}$ is also bigger, global imbalances are also larger (equation 1.23).

It might be argued that $\hat{\rho}_2$ is a better measure for country size (see Appendix A.3). Under technological progress, country i's share of world savings depends on country i's share of total labour productivity, as well as on population size. While China has a bigger population, the technological level of the US is higher[13]. Using the fact that $\frac{Y^*_t}{L_t A_t} \equiv \hat{y}^*_t = \hat{y}^*_{i,t} \equiv \frac{Y^*_{i,t}}{L_{i,t} A_{i,t}}$, we can rewrite $\hat{\rho}_i$ as follows:

$$\hat{\rho}_i = \frac{L_{i,t} A_{i,t}}{L_t A_t} = \frac{Y^*_{i,t}}{Y^*_t}$$

We compute East Asian countries' share of total GDP, where total GDP is computed as the sum of the US and East Asian countries GDP[14]. As expected, the US have a bigger size since the "productivity gap" compensates for China's bigger population: the size of emerging countries varies between 21% in 1980 up to 50% in 2010[15]. The model cannot account for the fact that East Asia's share has increased over time due to its spectacular economic growth, because $\hat{\rho}_2$ is constant in the model. A constant $\hat{\rho}_i$ is in fact the consequence of assuming identical growth rates for the two countries[16]. Yet, Proposition 8 can explain why capital flows and current account imbalances towards East Asian countries have a huge impact

[13] As a matter of fact, a simple way to account for the fact that China is poorer than the US is to assume that $A_{2,0} < A_{1,0}$. We thank Antonia Díaz and Timothy Kehoe for having raised this point.

[14] In particular, East Asian countries include China, Taiwan, South Korea, Hong Kong and Singapore. We take the countries' PPP-converted GDP, at current prices from Heston A., Summers R., Aten B., Penn World Table Version 7.1, Center for International Comparisons of Production, Income and Prices at the University of Pennsylvania, July 2012.

[15] In the Penn World Tables, there are two sets of data for China due to measurement problems. The above numbers are for China's version 2. For China version 1, the shares would be 15% in 1980 and 48% in 2010.

[16] See footnote 12 for a comment on this.

on the US economy: if China was a small country, the US current account deficit and net foreign asset liabilities would be negligible.

1.5 Conclusions and policy implications

This chapter takes seriously Bernanke's hypothesis that global imbalances might be due to a global saving glut. We have constructed a model in which a global excess of savings arises because of the financial integration between the United States and dynamically inefficient economies, which have a higher propensity to save than the US because they do not have a pay-as-you-go pension system. The increase in world savings had as long-run effects the drop of real interest rates and the emergence of global imbalances. These and other empirical evidences can be read through the lens of this model.

The model indicates that both the current direction of trade and the low real interest rates are signals that the world economy is on an inefficient path. If that was not the case, United States' current account should be zero or in surplus and we should also observe much higher interest rates. Pension reforms in China in the direction of introducing a pay-as-you-go system would increase domestic demand and therefore reduce world savings. The US deficit towards China would shrink, which is the outcome that many politicians and economists seem to hope for.

Figure 1.1: Current accounts of the United States and China

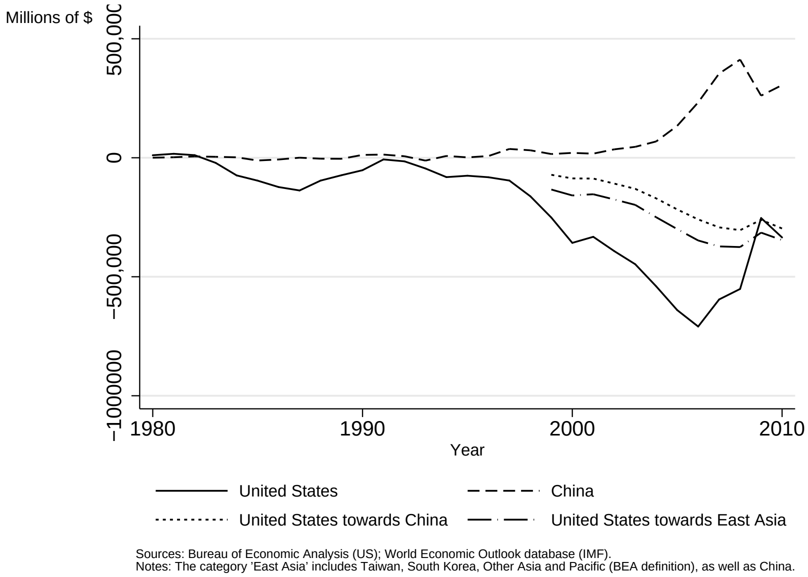


Figure 1.2: Gross national savings

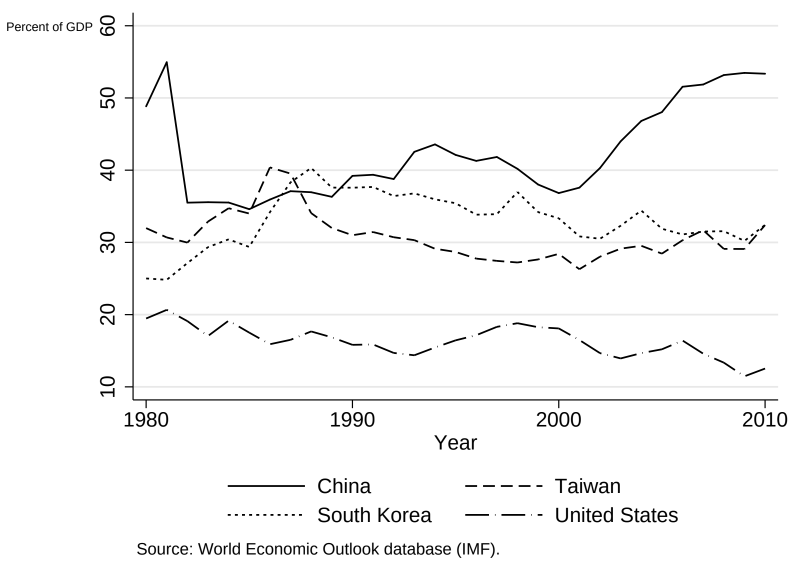


Figure 1.3: Capital accumulation in the world economy under Cobb-Douglas utility and production functions

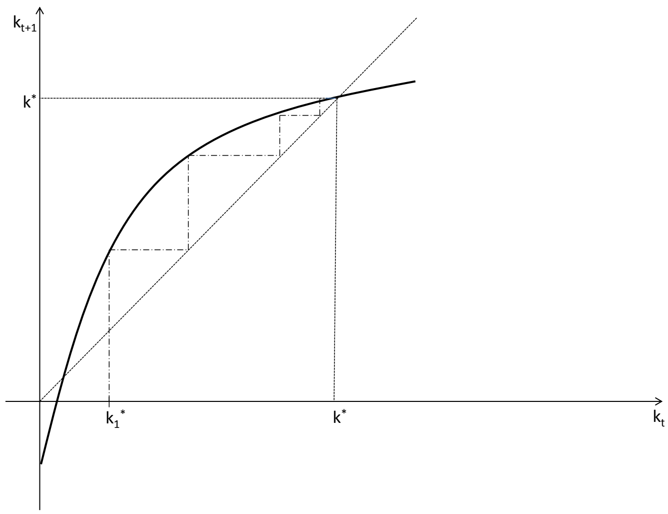


Figure 1.4: United States' net international position

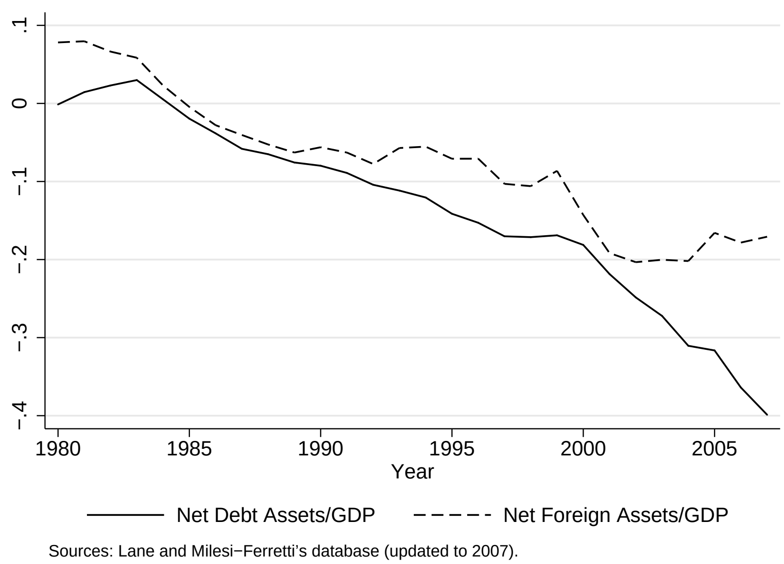
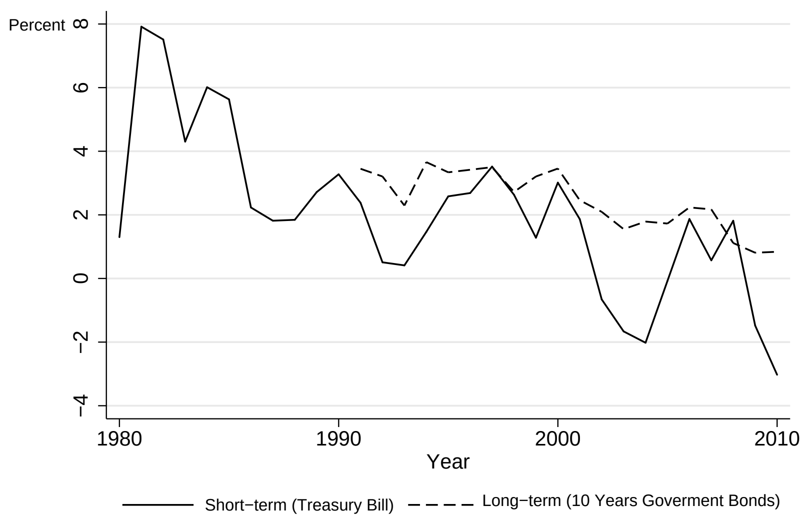


Figure 1.5: Real interest rates in the United States



Notes: IFS data, yearly rates. The Treasury Bill rate is adjusted for the actual CPI.
The long−term yield is adjusted for expected inflation from the Survey of Professional Forecasters as in Caballero et al. (2008).

Chapter 2

Technological differences and trade dynamics in a two-country world

2.1 Introduction

In neoclassical growth models, capital flows to the country whose marginal product of capital is higher. Since developing countries have a relatively low capital stock per capita, the marginal product of capital is higher than in developed countries. Therefore, capital should flow to those countries by the law of diminishing marginal returns. In the model studied in Chapter 1, when financial markets integrate, capital flows to the emerging country provided that the country's initial capital stock is sufficiently low. In the following periods, as the emerging country does not have a pay-as-you-go system, its young agents save relatively more to finance consumption when old and therefore capital flows to the developed country.

After this initial adjustment, the model predicts that both countries have the same path of capital stock per capita. If the theory was right in this respect, there should be no output differences between the United States and East Asian countries. So why do these differences persist, despite the increasing integration of financial markets? Explaining cross-country income inequality remains an open and very challenging question[1].

[1]The literature often refers to income differences, since the problem is thought of in a closed economy framework where output and income are identical (see [11]). In open economy, income and output do not coincide if there is asset trade among countries. Our

In Chapter 1, we have allowed for initial levels of the labour-augmenting technology to be different across countries[2]. Under that specification, output per effective worker is equalized but not output per capita. This is a convenient way to introduce output differences, but it is only a scale effect.

In a one-good economy with constant returns to scale and perfect competition, capital stocks per capita are equalized across countries as an implication of assuming that aggregate production functions are identical. The main objective of this Chapter is to relax this assumption and study capital accumulation and trade when production functions are heterogenous. In such framework, this means allowing for income shares to be different across countries.

Earlier studies had suggested that capital shares decrease at higher stages of development (see Bernanke et al. [7]). Gollin [24] challenged this view and argued that the share of capital income tends to be overestimated in developing countries, since income from self-employment is wrongly considered as capital income. According to Gollin, in those countries most of GDP is generated by self-employed workers in the agricultural sector, whose income should rather be treated as labour income. Gollin then estimated that adjusted capital shares vary between 0.20 and 0.35. He concluded that although there is some variation of income shares across countries, this is not as huge as it was previously thought. However, Young [64] found that in China the average capital share across sectors is 0.40, while it varies between 0.46 and 0.50 in the work of Bai et al. [4][3]. Therefore, these papers suggest that the capital share in the aggregate production function of China is significantly higher than the United States', which is estimated to be around 1/3.

In principle, it is hard to explain why capital shares should be different between these two countries and more generally across countries. On the other hand, the empirical literature suggests that the heterogeneity in income shares is potentially relevant for our case of interest. Our approach is to take this fact as exogenous and explore whether it can explain why the integration of financial

focus will be on differences in output per capita.

[2]See Appendix A.3.

[3]Young argued that there are no accountancy problems in China for at least two reasons. First, self-employment in China is quite rare. Secondly, the Chinese national accounts directly assume that the income of the self-employed is labour income.

markets has not meant the disappearance of GDP inequality between the United States and China. Our purpose is not to draw general conclusions on the relationship between the share of capital and income per capita, since no robust correlation has been found between capital shares and GDP per capita (e.g. [24], [7] and [12]). The alternative, traditional view is that income inequality is due to frictions in international capital markets. While it is true that capital controls in some emerging countries are still in place, there is also evidence that marginal products of capital are equalized across countries (see Caselli et al. [12]). These recent findings are supportive of the hypothesis that capital markets are substantially integrated and therefore calls for other causes to be found. The objectives of this Chapter are twofold. The first is to contribute to the theory of capital flows, since a two-country model with heterogeneous income shares has not been studied before. The second aim is to reassess the conclusions of the previous Chapter when this additional element of heterogeneity is introduced.

To start with, we abstract from the heterogeneity in the pension systems and we analyze the pattern of trade between two countries whose production functions have different capital shares. In a Cobb-Douglas setting, the country with the lowest capital share (i.e. the developed country) is more productive in the sense that it can produce more units of output given the same input. In equilibrium, the developed country's output per capita is always higher than in the developing country, so are wages. Yet, the young in the developed country lend to firms in the developing country. They do not find convenient to invest all their savings in the domestic firms, because of the decreasing returns to scale. Therefore, capital flows to the developing country, but output differences persist despite the absence of frictions in the world capital market. Yet, output differences at the world steady state are reduced as compared to autarky.

Second, we introduce the pension system in the developed country and ask what is the pattern of trade when countries are heterogenous both in their saving rates and technologies. We find that we cannot make strong predictions on the pattern of capital flows. The lower capital share induces capital outflows from the developed country, while the pension system has the opposite effect. Going back to our case study, the United States' net foreign asset position is negative in the data. Therefore, our conclusion is that the "saving channel" must be more important than the "technology channel".

Third, we ask what would happen to global imbalances if emerging countries introduced a pension system. For simplicity, suppose that the lump-sum tax is of the same size as in the US. If capital shares were the same, there would be no trade at impact. In fact, countries start with the same initial conditions and their savings become identical. The world economy would converge to a steady state with a lower capital stock per capita[4], but there would be no trade both along the transition path and at steady state. Global imbalances would basically disappear[5].

Since the capital shares are different, the pattern of trade would be reversed since capital would now flow to the developing country. Under Hypothesis 1 of Chapter 1, the world economy converges to a steady state at which the interest rate is higher than the population growth rate. As far as trade imbalances are concerned, the developed country would still run a trade deficit in the long-run, despite that the country becomes the international lender. Our conclusion is that global imbalances would not vanish if China introduced a pension system, because of the heterogeneity in the income shares.

2.2 A two-country model with heterogenous capital shares

The set up is a two-country Diamond model in which the two countries have access to different CRS technologies for the production of the consumption good. First, we need the following definition:

Definition 4 (Productivity) *Given two technologies* 1 *and* 2*, we say that technology* 1 *is more productive than technology* 2 *if* $f_1(k) > f_2(k)$ *for any* k*.*

We then assume:

[4]Under Hypothesis 1, the new world steady state would be the golden rule.

[5]If the pension system was smaller in the emerging country, global imbalances would be reduced.

Assumption 3 *Country 1 & 2 have Cobb-Douglas production functions and $\alpha_1 < \alpha_2$.*

Under Assumption 3, country 1's technology produces more output than country 2's technology as long as $k < 1$ (see Figure 2.1). When both the production functions and the utility functions are Cobb-Douglas, the steady state capital stock is less than one (see equation (2.5) below). Therefore, we concentrate our analysis in the region $0 \leqslant k_i < 1$ for every i.

We assume that the utility function is Cobb-Douglas to avoid multiplicity of steady states. For simplicity, we also assume that $\delta = 1$. For the moment, we abstract from other bases of trade, namely differences in social security systems, to isolate the effect of heterogenous technologies on capital flows. In the next section, we shall reintroduce the pay-as-you-go system in country 1.

2.2.1 Autarky

First, consider the two economies in autarky as this will be useful to analyze the open economy. For the moment, suppose that both capital and labour are immobile.

From profit maximization, factor prices in country i are:

$$r_{i,t} = \alpha_i k_{i,t}^{\alpha_i - 1} \tag{2.1}$$

$$w_{i,t} = (1 - \alpha_i) k_{i,t}^{\alpha_i} \tag{2.2}$$

With Cobb-Douglas utility function and $\tau_1 = \tau_2 = 0$, the saving function of country i is[6]:

$$s_{i,t} = (1 - \beta) w_{i,t} \tag{2.3}$$

where $1 - \beta$ is the weight attached to consumption when old. The dynamics of the capital stock per capita of country i is characterized by the following equation:

$$(1 + n) k_{i,t+1}^{aut} = (1 - \beta)(1 - \alpha_i) k_{i,t}^{aut\,\alpha_i} \qquad i = \{1, 2\} \tag{2.4}$$

It is known that a unique and stable steady state exists under these assumptions on preferences and technology. The steady state cap-

[6] See section A.2 in the Appendix A.2.

ital stocks and interest rates in autarky are:

$$k_i^{aut} = \left[\frac{(1-\beta)(1-\alpha_i)}{1+n}\right]^{\frac{1}{1-\alpha_i}} \tag{2.5}$$

$$r_i^{aut} = \frac{\alpha_i(1+n)}{(1-\beta)(1-\alpha_i)} \tag{2.6}$$

We can now prove that the first result:

Proposition 9 (Autarkic steady states) *(i) The capital stock per capita of the country with the lower share of capital is higher at the autarkic steady state: $k_1^{aut} > k_2^{aut}$. (ii) The interest rate of the country with the lower share of capital is lower at the autarkic steady state: $r_1^{aut} < r_2^{aut}$.*

Proof. (i) To show this, we compute the derivative of k with respect to α using equation (2.5):

$$\frac{\partial k}{\partial \alpha} = \left[\frac{(1-\beta)(1-\alpha)}{1+n}\right]^{\frac{1}{1-\alpha}} \cdot \frac{1}{(1-\alpha)^2} \cdot \left[\log\left(\frac{(1-\beta)(1-\alpha)}{1+n}\right) - 1\right]$$

Since $(1-\beta)(1-\alpha)/(1+n) < 1$, then $\frac{\partial k}{\partial \alpha} < 0$.

(ii) The derivative of r with respect to α is (see equation (2.6)):

$$\frac{\partial r}{\partial \alpha} = \frac{1+n}{(1-\beta)(1-\alpha)^2} > 0$$

■

Since country 1 produces more than country 2 given the same input, $k_1^{aut} > k_2^{aut}$ imply that $y_1^{aut} > y_2^{aut}$. If the two countries do not open to trade, country 1 has a higher output per capita than country 2 at the autarkic steady state. Therefore, country 1 (2) can be thought of as the developed (developing) country.

2.2.2 Open economy

In this section, there is a common capital market and countries are allowed to trade. Since there are no frictions, all firms take the same interest rate as given. The demands for inputs in the two countries now satisfy:

$$r_t = \alpha_1 k_{1,t}^{\alpha_1 - 1} = \alpha_2 k_{2,t}^{\alpha_2 - 1} \tag{2.7}$$

$$w_{i,t} = (1-\alpha_i) k_{i,t}^{\alpha_i} \tag{2.8}$$

For any r_t, we must have that $k_{1,t} \neq k_{2,t}$ and $y_{1,t} \neq y_{2,t}$ as the capital shares are not identical. Differently from Chapter 1, there is no wage equalization.

The saving function of country i is the same as in autarky (equation (2.3)).

We can now write the equilibrium equation and analyze capital accumulation as well as the pattern of trade. The (world) capital market is in equilibrium as long as (world) savings are equal to the sum of the domestic capital stocks:

$$\sum_i K^*_{i,t+1} = \sum_i s^*_{i,t} = (1-\beta)\sum_i L_{i,t}(1-\alpha_i)k^{*\alpha_i}_{i,t} \qquad t \geqslant 0 \quad (2.9)$$

Dividing by the world population L_t, the equation becomes:

$$\rho_1 k^*_{1,t+1} + \rho_2 k^*_{2,t+1} = \frac{1-\beta}{1+n}\left[\rho_1(1-\alpha_1)k^{*\alpha_1}_{1,t} + \rho_2(1-\alpha_2)k^{*\alpha_2}_{2,t}\right] \quad (2.10)$$

where ρ_i is country i's share of the world population. We are at a steady state of the world economy when the capital stocks of the two countries do not change over time:

$$\rho_1 k^*_1 + \rho_2 k^*_2 = \frac{1-\beta}{1+n}\left[\rho_1(1-\alpha_1)k^{*\alpha_1}_1 + \rho_2(1-\alpha_2)k^{*\alpha_2}_2\right] \qquad (2.11)$$

To prove the existence of a steady state, we rewrite the difference equation using the first-order conditions of the firms and show the existence of a steady state interest rate.

Proposition 10 (Existence of steady states) *For any given r_0, the (world) interest rate converges to a unique and stable steady state r^*.*

Proof. First, let us rewrite (2.10) using $r_t = \alpha_i k^{\alpha_i - 1}_{i,t}$:

$$\sum_i \rho_i \left(\frac{r^*_{t+1}}{\alpha_i}\right)^{\frac{1}{\alpha_i - 1}} = \frac{1-\beta}{1+n}\left[\sum_i \rho_i(1-\alpha_i)\left(\frac{r^*_t}{\alpha_i}\right)^{\frac{\alpha_i}{\alpha_i - 1}}\right]$$

Using the implicit function theorem, we find that:

$$\frac{dr_{t+1}}{dr_t}(r_t) = \frac{\frac{1-\beta}{1+n}\left[\sum_i \rho_i \left(\frac{r_t}{\alpha_i}\right)^{\frac{1}{\alpha_i - 1}}\right]}{\sum_i \frac{\rho_i}{\alpha_i(1-\alpha_i)}\left(\frac{r_{t+1}}{\alpha_i}\right)^{\frac{2-\alpha_i}{\alpha_i - 1}}} > 0 \qquad \forall\ r_t \qquad (2.12)$$

Since the derivative exists, we can write $r_{t+1} = \phi(r_t)$. The difference equation is an increasing function, with $\phi(0) = 0$. It can also be checked that $\lim_{r_t \to 0} \phi'(r_t) = \infty$ and $\lim_{r_t \to \infty} \phi'(r_t) = 0$. These conditions imply that a stable steady state exists. Moreover, we have that $\phi''(r_t) < 0$. Concavity of the ϕ function means that the steady state is unique (see Galor et al. [30]). ■

Corollary 1 *The capital stock per capita of country i accumulates according to the following equation:* $k_{i,t+1} = \left(\frac{\phi(\alpha_i k_{i,t}^{\alpha-1})}{\alpha_i}\right)^{\frac{1}{\alpha_i - 1}}$. *Moreover,* $\frac{\partial k_{i,t+1}}{\partial k_{i,t}} > 0$ *and* $\frac{\partial^2 k_{i,t+1}}{(\partial k_{i,t})^2} < 0$.

Proof. See Appendix B.

■

Suppose that $r_0 > r^*$. By the function ϕ, the world interest rate falls until it reaches the steady state interest rate. Since capital stocks per capita decrease with the interest rate, the domestic capital stocks accumulate over time and converge to the steady state capital stocks k_1^* and k_2^*. We can now study the pattern of trade in the capital market at the steady state:

Proposition 11 (Borrowing and lending at steady state) *Country 1 (2) is the lender (borrower) country at the world interest rate r^*, which lies between the two autarkic steady states: $r_1^{aut} < r^* < r_2^{aut}$.*

Proof. First, let us consider country i's excess demand per capita:

$$z_i(r) \equiv (1+n)\left(\frac{r}{\alpha_i}\right)^{\frac{1}{\alpha_i - 1}} - (1-\beta)(1-\alpha_i)\left(\frac{r}{\alpha_i}\right)^{\frac{\alpha_i}{\alpha_i - 1}} \quad (2.13)$$

Suppose that $z_i(r^*) > 0$ for some i. By directly manipulating the above equation, it turns out that is true only as long as $r_i^{aut} > r^*$. Similarly, $z_i(r^*) < 0$ if $r^* > r_i^{aut}$.

The world interest rate cannot satisfy $r_i^{aut} \geqslant r^*$ for every i. Otherwise, we would have that $z_i(r^*) \geqslant 0$ for every i and $\sum_i z_i(r^*) > 0$, since $r_1^{aut} \neq r_2^{aut}$ by assumption. For the same reason, $r^* \geqslant r_i^{aut}$ for every i is impossible. In order for r^* to clear the world capital market, the only possibility is that $r_1^{aut} < r^* < r_2^{aut}$. Hence, at the world steady state, $z_2(r^*) > 0$ and $z_1(r^*) < 0$. ■

Corollary 2 *The steady state output per capita of country 1 (2) in open economy is lower (higher) than in autarky.*

Financial openness allows the developing country to reach a higher output per capita and a higher wage as compared to the autarkic steady state.

Nonetheless, the model implies that there is no actual convergence of GDP per capita across countries, even though inequality is reduced with financial integration.

Proposition 12 (GDP inequality) *If* $\frac{1+n}{(1-\alpha_1)(1-\beta)} > \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_2(1-\alpha_1)}{\alpha_2-\alpha_1}}$, *then* $y_{1,t} > y_{2,t}$ *for all* $r_t \geq r^*$.

Proof. See Appendix B. ∎

The proposition applies if the initial condition of the world economy satisfies $r_0 \geq r^*$, i.e. the capital stocks per capita of the two countries are below their steady state values. In this case, country 1 is richer than country 2 both in the transition path to the world steady state and at steady state.

Note the above condition is not very stringent. On the one hand, it is known that $\alpha_1 = 1/3$. Young [64] and Bai et al. [4] estimated that the capital share for China is between 0.40 and 0.50. Since we do not want this condition to be sensitive on other parameters, we assume no discounting and no population growth: $1-\beta = 0.5$ and $n = 0$. For $\alpha_2 = 0.40$, the inequality is satisfied since $3 > 2.0736$. Similarly, $3 > 2.25$ for $\alpha_2 = 0.50$.

Corollary 3 *In open economy, wages and savings in country 1 are higher than in country 2*[7].

Proof. Recalling that $w_{i,t} = (1-\alpha_i)k_{i,t}^{\alpha_i}$, we must have that $w_{1,t} > w_{2,t}$ given that $y_{1,t} > y_{2,t}$ and the share of labour in country 1 is higher. Since β is identical across countries, the second part follows. ∎

We can now study trade dynamics in detail. First, we need to make some assumptions on the two countries' initial conditions when they open to trade at $t = 0$.

Assumption 4 *At* $t = 0$, $k_{1,0} = k_1^{aut}$ *and* $k_{2,0} < k_2^{aut}$.

As in Chapter 1, we look at the case in which the marginal product of capital in the developing country (country 2) is higher than

[7] In this chapter, wages are not equalized when countries open to trade because of the heterogenous technologies. Therefore, the assumption that labour markets are closed matters as far as the world equilibrium is concerned. If labour migration was allowed, there would be incentives for workers to move to country 1 until the wage differential disappears. We abstract from this issue as our focus is to study the pattern of trade in the capital market.

in the developed country (country 1) at the openness. We assume that the developed country is at its autarkic steady state level of capital, while the developing country is still along its transition path to the autarkic steady state.

It follows from these assumptions that $s_{1,0} > s_{2,0}$. We must then find the interest rate r_1^* which clears the world capital market in the initial period. After that, we can analyze the direction of capital flows at $t = 0$.

To start with, we need the following definitions. Given $k_{1,0}$, $r_{1,1}^{aut}$ is the interest rate which country 1 would face if it did not open to trade with country 2. Since $k_{1,0} = k_1^{aut}$, then $r_{1,1}^{aut} = r_1^{aut}$. Similarly, given $k_{2,0}$, $r_{2,1}^{aut}$ would be the interest rate prevailing in country 2 if still in autarky.

Lemma 3 (World interest rate at $t = 1$) *Given Assumption 4,* $r_{2,1}^{aut} > r_1^* > r_1^{aut}$.

Proof. First, let us prove by contradiction that $r_1^* \neq r_{2,1}^{aut}$, where $r_{2,1}^{aut}$ is the interest rate which would clear the (domestic) capital market of country 2 if the country remained in autarky. If $r_1^* = r_{2,1}^{aut}$, then the young in country 2 would not actually trade even if they are allowed to do so. Since savings are given, the demand for capital would be the same in autarky: $s_{2,0} = k_{2,1}(1+n)$, where $k_{2,1} = k_{2,1}^{aut}$. In country 1, the demand for capital would be lower than in autarky as $r_{2,1}^{aut} > r_2^{aut} > r_1^{aut}$. Therefore, $k_{1,1}(1+n) < s_{1,0}$. But then, the world capital market does not clear since $(1+n)\sum_i k_{i,1} < \sum_i s_{i,0}$.

Similarly, it can be proved that $r_1^* \neq r_1^{aut}$ as the aggregate demand for capital would exceed aggregate savings. In the initial period, each country's savings are given. Since the demand for capital is decreasing in r, it must be that $r_{2,1}^{aut} > r_1^* > r_1^{aut}$ in order for $\sum_i z_i(r_1^*) = 0$ to be satisfied. ■

Proposition 13 (Financial integration) *At $t = 0$, the developing country borrows from the developed country.*

Proof. Lemma 3 established that $r_1^* < r_{2,1}^{aut}$. Then, $k_{2,1}^*(1+n) > s_{2,0}$ and therefore $z_{2,0}^* > 0$. ■

The next step is to analyse the pattern of trade along the transition to the steady state.

Intuition suggests that capital should flow to the developing country even during the transition. The developing country, because of its lower savings, should have a higher marginal product of capital. While it is true that country 1's technology is more productive, decreasing returns to scale should provide an incentive to export capital to the developing country. Our strategy for the proof is similar to Lemma 3. In each period, we can take the interest rate r_t^* as given and ask which country has a higher marginal product of capital, i.e. the interest rate which would prevail if the countries were in autarky in period $t+1$.

Lemma 4 *Given $r_1^* > r^*$, then $r_{2,t+1}^{aut} > r_{t+1}^* > r_{1,t+1}^{aut}$ for every r_t^*.*

Proof. See Appendix B.

■

Proposition 14 (Borrowing and lending during the transition) *The developed country lends for any $t \geqslant 1$.*

Proof. The proof follows from the previous lemma. Let us define the excess demand function of country i along the transition:

$$z_i(r_t^*, r_{t+1}^*) \equiv \left(\frac{r_{t+1}^*}{\alpha_i}\right)^{\frac{1}{\alpha_i-1}} - \frac{(1-\beta)(1-\alpha_i)}{1+n}\left(\frac{r_t^*}{\alpha_i}\right)^{\frac{\alpha_i}{\alpha_i-1}}$$

For country 1, we have that $r_{t+1}^* > r_{1,t+1}^{aut}$. Then, $(1+n)k_{1,t+1}^* < s_{1,t}^*$ or $z_1(r_t^*, r_{t+1}^*) < 0$. On the other hand, country 2 will have excess demand for capital, i.e. $z_2(r_t^*, r_{t+1}^*) > 0$. ■

2.3 The pattern of trade with heterogenous technologies and pension systems

In section 2, we found that capital flows to the country with the least productive technology, i.e. the developing country.

We now reintroduce the pay-as-you-go system in country 1. The purpose is to check whether or under which conditions the results of Chapter 1 are robust when we relax the assumption of identical technologies. Given the analysis above, it should be evident that there are two basis for trade which force capital flows in opposite directions. On the one hand, the emerging country saves more than the other country because of the absence of the pay-as-you-go system and therefore it has incentives to lend to the developed country. On the other hand, capital should flow to the emerging country since its marginal product of capital is higher. As a result, the theoretical model cannot give a clean prediction on the direction of capital flows. Since capital actually flows to the United States, it must be that there are some restrictions in the model such that the "saving motive" prevails. In particular, this is true under the condition that the autarkic interest rate of the developing country is higher than the interest rate of the other country.

Assumption 5 *The autarkic steady states satisfy* $r_1^{autss} > r_2^{aut}$.

This assumption implies that the size of pay-as-you-go system in country 1 is big enough to neutralize the difference in the production technologies, or that technological differences are small (see Figure 2.2)[8].

The world capital market clearing condition is now the following:

$$\begin{aligned}(1+n)\sum_i \rho_i k^*_{i,t+1} &= \rho_1(1-\beta)[(1-\alpha_1)k_{1,t}^{*\alpha_1} - \tau_1] - \\ &- \rho_1\tau_1 \frac{\beta(1+n)}{1+\alpha_1 k_{1,t+1}^{*\alpha_1-1}} + \rho_2(1-\beta)(1-\alpha_2)k_{2,t}^{*\alpha_2}\end{aligned} \tag{2.14}$$

or:

$$\begin{aligned}(1+n)\sum_i \rho_i \left(\frac{r^*_{t+1}}{\alpha_i}\right)^{\frac{1}{\alpha_i-1}} &= (1-\beta)\sum_i \rho_i(1-\alpha_i)\left(\frac{r^*_t}{\alpha_i}\right)^{\frac{\alpha_i}{\alpha_i-1}} - \\ &- \rho_1\tau_1\left[(1-\beta) + \frac{\beta(1+n)}{1+r^*_{t+1}}\right]\end{aligned} \tag{2.15}$$

Lemma 5 *Under Assumption 5, a stable steady state exists.*

Proof. A discussion on the conditions under which a world stable steady state exists when $\tau_1 > 0$ and $\alpha_i = \alpha$ are provided in section A.2. The crucial point was that the size of the pension system in country 1 has to be small enough in order for savings to be positive. Changing the α parameter in country 2 does not pose any additional problems, so the same arguments can be used here. ■

[8] We refer to section A.2 of Chapter 1 for an analysis of country 1's autarkic dynamical system with $\tau_1 > 0$ (it is enough to impose that $\rho_1 = 1$). Note that since there is no closed form solution for r_1^{autss}, we cannot derive a condition on the parameters of the economy such that Assumption 5 holds.

Let's refer to the stable steady state of this economy as $\bar{r}^*$. Since the autarkic interest rate of country 1 does not have a closed form solution, we cannot use the arguments of section 2.2 to study the pattern of trade. Moreover, we cannot follow the logic of the proof of Proposition 1.4, since it relied on the equalization of the capital stocks. The following proofs are only sketched.

Lemma 6 *Under Assumption 5, the steady state interest rate lie between the two autarkic interest rates.*

Proof. Consider the economy of Chapter 1, where $\tau_1 > 0$ and $\alpha_i = \alpha$. In that case, we showed that $r_1^{autss} > r_2^{aut}$. Next, allow for $\alpha_2 > \alpha_1$. Remember that Assumption 5 is satisfied only as long as α_2 is sufficiently close to α_1. See Figure 2.2 for an illustration. Since the autarkic interest rate of country 2 has increased, the world interest rate should be higher than in the case with identical capital shares. However, it is still the case that $r_1^{autss} > \bar{r}^* > r_2^{aut}$. ■

Proposition 15 *Under Assumption 5, capital flows to the developed country at the steady state.*

Proof. The excess demand function of country 2 is the same as in section 2.2. From Proposition 11, we know that $z_2(\bar{r}^*) < 0$ if $r_2^{aut} < \bar{r}^*$ and viceversa. Given Lemma 6, we can establish that $z_2(\bar{r}^*) < 0$. Therefore, capital flows to the developed country under Assumption 5. ■

Figures 2.3 illustrates the dynamics of the capital stocks in the two countries when there is no social security in country 1 (see Corollary 1). The steady state capital stocks satisfy $k_1^* < k_1^{aut}$ and $k_2^* > k_2^{aut}$ since $r_2^{aut} > r^* > r_1^{aut}$. If we introduce the social security system in country 1, its autarkic difference equation shifts below the autarkic difference equation of country 2 provided that τ_1 is big enough (see Figure 2.4). In that case, we have that $r_1^{autss} > \bar{r}^* > r_2^{aut}$ and, therefore, $\bar{k}_1^* > k_1^{aut}$ and $\bar{k}_2^* < k_2^{aut}$.

2.4 The introduction of a pay-as-you-go system in emerging countries and global imbalances

Finally, we want to ask the following question: "Would global imbalances disappear if emerging countries introduced a pay-as-you-go system?"

To start with, it is important to stress that the basic equation of the balance of trade does not change across different versions of the model, as it can be obtained by manipulating the market clearing equation for the good (see Chapter 1):

$$tb^*_{i,t} = -z^*_{i,t} + \frac{1+r^*_t}{1+n} z^*_{i,t-1}$$

so that at the steady state:

$$tb^*_i = -z^*_i \left(\frac{n-r^*}{1+n} \right)$$

We approach this question in two steps. Firstly, we analyse the case in which countries have identical capital shares and then introduce technological differences.

2.4.1 Identical capital shares

We still work under the hypothesis that country 1's autarkic steady state is the golden rule (Hypothesis 1). In Chapter 1, we have shown that since $r_1^{autss} > r^* > r_2^{aut}$, then the world economy's interest rate is lower than the population growth rate.

Suppose that country 2 introduces a pay-as-you-go pension system at t^*. For simplicity, assume that $\tau_1 = \tau_2 = \tau$. The world capital market clearing equation now becomes:

$$(1+n)k_{t+1} = (1-\beta)[(1-\alpha)k_t^\alpha - \tau] - \tau \frac{\beta(1+n)}{1+\alpha k_{t+1}^{\alpha-1}} \qquad t \geqslant t^*$$

Note also that both countries have the same "initial condition" at t^* because of the identical technologies.

The steady state interest rate is precisely equal to the autarkic interest rate of country 1, which is at the golden rule. We denote this new steady state as r^{**}. If the world economy is at the old steady state at t^*, since $k^* > k^{**}$ the capital stock per capita is outside steady state and will converge to the new steady state from above. If the world economy is not at the old steady state and $k_{t^*} > k^{**}$, there will be a similar adjustment process. If $k_{t^*} < k^{**}$ capital would instead converge from below.

Independently from the initial condition, for $t \geqslant t^*$ the two countries' savings would be identical and therefore there would be no trade in the capital market. In the first period, that $z_{i,t^*} = 0$ implies that $tb_{1,t^*} > 0$ (since $z_{1,t^*-1} > 0$). Country 1 has a trade surplus since it repays the previous period's loan to country 2. However, for every $t > t^*$ we have $tb_{i,t} = 0$. The introduction of a pension system of the same size would mean the permanent disappearance of global imbalances.

Instead, if the emerging country introduced a smaller pay-as-you-go system, so that $\tau_2 < \tau_1$, it is intuitive that capital would keep flowing to the developed country and that the latter would still run a trade deficit in the long run. However, the size of global imbalances would be smaller.

2.4.2 Heterogenous capital shares

Let's now allow for country 2's capital share to be higher than in country 1. Since we claimed that the difference in the capital shares is sufficiently small (Assumption 5), at the outset we still have that $n > \bar{r}^*$ under Hypothesis 1.

If the emerging country introduces a pension system of the same size of country 1 at t^*, the world capital market clearing equation becomes:

$$
\begin{aligned}
(1+n)\sum_i \rho_i k_{i,t+1} &= (1-\beta)\sum_i \rho_i[(1-\alpha_i)k_{i,t}^{\alpha_i} - \tau] - \\
&- \beta\tau(1+n)\sum_i \rho_i \frac{1}{1+\alpha_i k_{i,t+1}^{\alpha_i - 1}} \qquad t \geqslant t^*
\end{aligned}
$$

The first observation is that countries start with different "initial conditions" at t^*, since $y_{1,t^*} > y_{2,t^*}$ (see Proposition 12). As in the case with identical α, the effect of the pension system in country 2 is that $\bar{r}^{**} > \bar{r}^*$. Therefore, the economy will converge to a steady state with a higher interest rate.

However, there would still be trade since technologies are different. We have shown that capital flows to the developing country in the long-run, so the pattern of trade would be reversed. At the new world steady state, we would have that $\bar{z}_1^{**} < 0$. Moreover, $\bar{r}^{**}$ would not be at the golden rule as in the case with identical capital shares, but to its left: $\bar{r}^{**} > n$. Therefore, country 1 would still run a trade deficit in the long-run. However, since we argued that the difference in technologies is small, these imbalances would not be as large as the ones induced by the heterogeneity in the pension systems.

2.5 Conclusions and future research

In this Chapter, we have studied a model in which countries are heterogeneous in their aggregate production functions. We have shown that the integration of financial markets does not imply the equalization of GDP per capita across countries, even though capital flows to the developing country. The model is a way to think about GDP inequalities in a setting of frictionless capital markets.

We have also showed that the pattern of capital flows would be reversed if China introduced a pay-as-you-go system, because of technological differences between China and the United States. Capital would flow to the country where the marginal product of capital is higher, which is the emerging economy because of its higher capital share. The United States would run a surplus or a deficit depending on where the new steady state interest rate would be as compared to the golden rule allocation. Therefore, we argued that global imbalances would not vanish if China pursed some pension reforms.

Finally, it is true that the United States are net borrowers but they also have a positive FDI position against the rest of the world (Figure 1.5). The data might reflect the fact that while emerging countries are net lenders because of a saving motive, the US invest more in emerging countries than viceversa since there are investment opportunities in emerging countries offering higher returns. This can explain the ambiguity in the pattern of trade when both sources of heterogeneity are present.

As a matter of fact, there is only one asset in the model and it is not possible to distinguish and explain the pattern of capital flows at a more disaggregated level. The next step towards a better understanding of gross capital flows requires the introduction of a menu of assets yielding different rates of return, which requires a model with uncertainty. We leave this task for future research.

Figure 2.1: Cobb-Douglas production functions with $\alpha_2 > \alpha_1$.

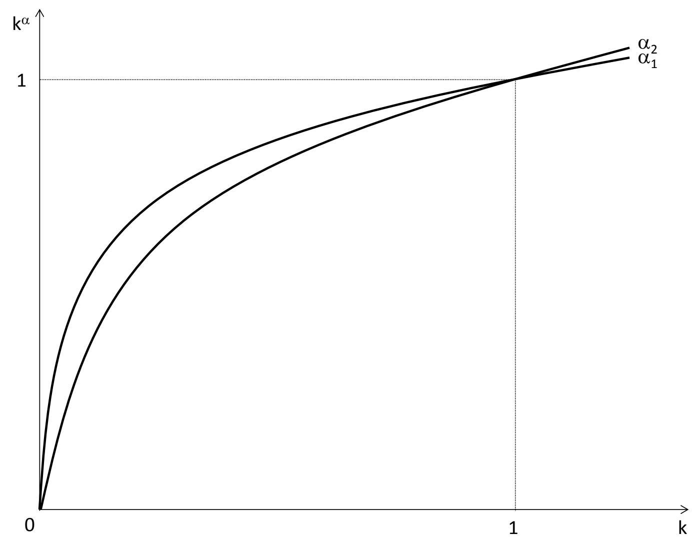


Figure 2.2: Autarkic steady states with social security in country 1

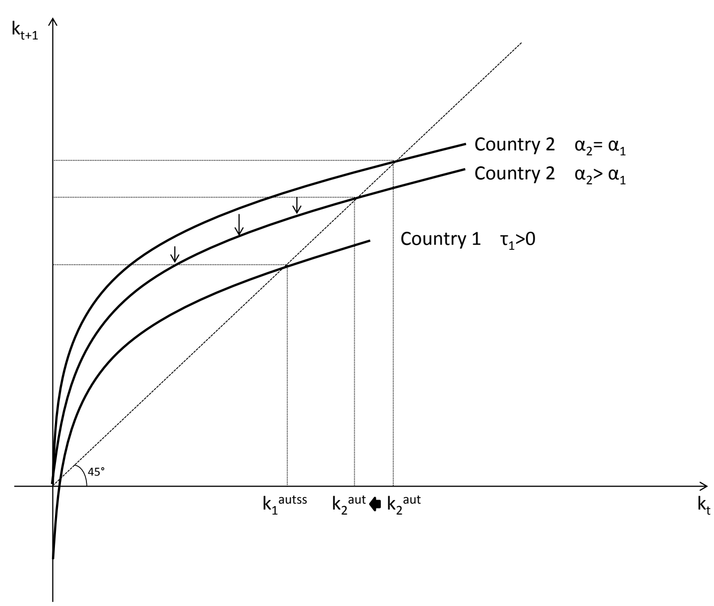


Figure 2.3: Capital accumulation with $\alpha_2 > \alpha_1$ and no social security in country 1

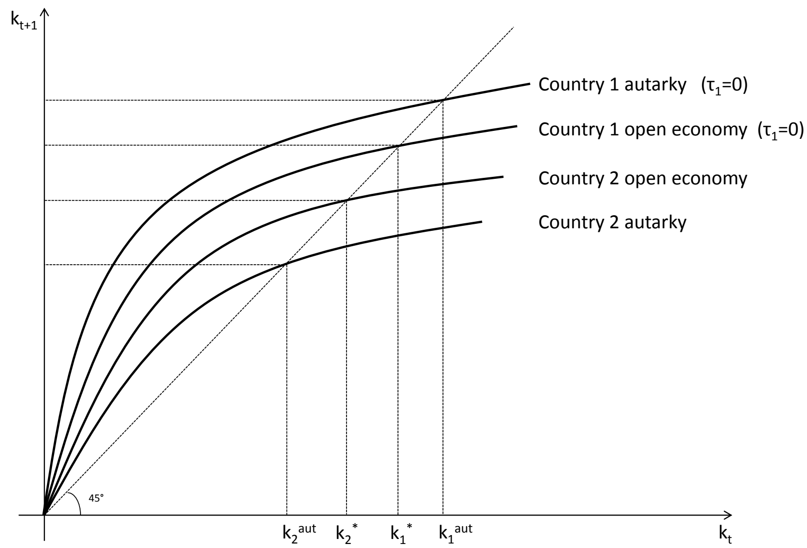
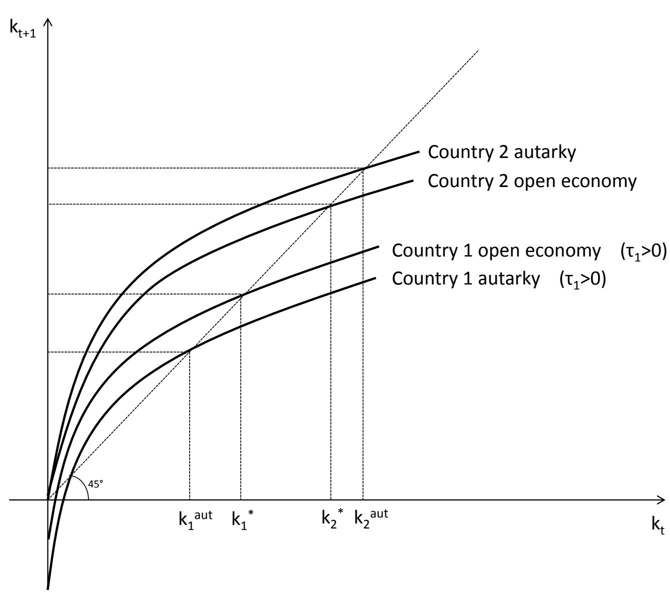


Figure 2.4: Capital accumulation with $\alpha_2 > \alpha_1$ and social security in country 1



Part II

A theory of exchange rate determination

Chapter 3

Portfolio choice and nominal exchange rate determination in a stochastic OLG economy

3.1 Introduction

Since data on countries' gross foreign assets positions have been made available in the last decade, one of the main issues on the agenda of international macroeconomists is the understanding of countries' portfolio choices[1]. However, as Gourinchas and Rey [38] put it, "the open economy literature has so far not managed to come up with a new generation of portfolio balance models microfounded and embedded in a general equilibrium set up.". The purpose of this chapter is to develop a general equilibrium framework with incomplete financial markets and nominal assets in which countries' portfolios choices can be studied.

The dynamics of the net foreign assets position of a country is driven by two main forces, i.e. portfolio adjustments and changes in the value of assets and liabilities. For us to be able to interpret the data, it is therefore essential to come up with a model in which portfolio rebalancing takes place. In models with complete markets, i.e. the Lucas asset pricing model, it is always optimal not

[1]Lane and Milesi-Ferretti [41], [43] pioneered the empirical work on the topic and provided a unique database in terms of country coverage and level of asset disaggregation.

to retrade assets following a new realization of uncertainty[2]. This framework was extended by Lucas [44] to incorporate money. However, the flexible exchange rates' equilibrium allocation is shown to be equivalent to the barter allocation and is supported by constant holdings of money balances, as well as of Lucas trees. While it is reasonable to conjecture that relaxing the assumption of complete markets might be the crucial step to improve our understanding of countries' portfolio choices, the lack of tractability is a major issue when markets are allowed to be incomplete.

Recent work has proposed local solution methods to analyze incomplete markets' models (e.g. Devereux et al. [20] and Tille et al. [63]). The advantage of these methods is that they can deal with any state space. On the other hand, Rabitsch et al. [54] have questioned their accuracy and showed that the global solution does not always coincide with the local one[3]. Pavlova and Rigobon [51] were able to derive closed form solutions under log utility, but uncertainty in the endowments is not enough to generate time-varying portfolios. The common element of this strand of literature is the focus on incomplete markets' models with real assets.

The novelty of this work is that we investigate the role of the nominal exchange rate in countries' saving and portfolio decisions. In real life, assets are currency-denominated and intuition suggests that the nominal exchange rate matters when agents face the problem of allocating their savings across domestic and foreign issued assets.

If we hope to understand the behaviour of the exchange rate as well as of countries' portfolios as seen in the data, it is important to have a model in which these variables can be pinned down. However, the incomplete markets' literature has pointed out that this task is particularly daunting when nominal assets are traded. In a two-period economy, Balasko and Cass [5] and Geanakoplos and Mas-Colell [31] have shown that the equilibrium allocation is indeterminate when markets are incomplete.

The OLG model is a suitable approach to study portfolio choices with nominal assets, since stochastic stationary equilibria are known to be determinate when markets are incomplete as long as money is

[2]See Lucas [44] and Judd et al. [40]. for a more general version of the model.
[3]See also Courdacier et al. [18] for a critical assessment of local solution methods.

held as one of the assets (Cass et al. [13], Gottardi [35]). Our asset structure is going to be very simple: currencies are the only assets available for trade. This allows to derive some analytical results where possible, which facilitates the understanding of the model.

The seminal paper on the OLG model with multiple currencies is Kareken and Wallace [39]. They analyze a one-good economy in which there are no legal restrictions in currency trading, in the sense that each currency can be used to buy the consumption good in any location[4]. One of their key findings is that the equilibrium exchange rate is indeterminate. The indeterminacy is related to the perfect substitutability of currencies from the point of view of the agents, since their rates of return are equalized. For the same reason, portfolios cannot be pinned down since agents are indifferent as to how allocate their saving across the currencies. Another important implication of the rate of returns' equalization is that the exchange rate is constant over time. These results are due to the assumption of no legal restrictions and are not related to the OLG structure of the economy[5].

Our solution to these problems is to work under the assumption that each currency can only buy the country specific good. The idea of distinguishing commodities by their location goes back to Debreu [19]: "A good at a certain location and the same good at another location are *different* economic objects, and the specification of the location at which it will be available is essential."..."A *commodity* is therefore defined by a specification of all its physical characteristics, of its availability date, and of its availability location. As soon as one of these three factors changes, a *different* commodity results.". The assumption that the endowment of a country can only be bought by the local currency has been adopted by the cash-in-advance literature started by Lucas [44].

This assumption turns out to be crucial in a number of respects. Firstly, it allows to pin down portfolios and the exchange rate. Second, a stationary equilibrium would not exist otherwise. This is related to earlier work of Spear [62], who showed that steady state

[4]Their approach has been followed by Fischer [23] and Manuelli and Peck [48]. Fischer has extended Kareken and Wallance's results with L goods. Manuelli and Peck have looked at the issue of exchange rate volatility under uncertainty.

[5]Sargent [59] showed that the same indeterminacy result holds when agents are infinitely lived.

equilibrium does not generically exist in pure exchange stochastic OLG economies with multiple goods. By introducing "cash-in-advance" constraints, we are able to get existence for functional forms of the utility functions widely used in macroeconomics[6].

Under log utility and zero endowment in the second period of life, the equilibrium system of equations is linear and it is possible to compute the equilibrium prices and the allocation analytically. While extremely useful, this example is very special as the exchange rate and portfolios happen to be constant in equilibrium. As soon as we generalize to isoelastic utility functions, the model does not have closed-form solutions but demand functions can be computed. When utility functions are identical, some analytical results can be derived but equilibrium has to be computed numerically.

The first result of the chapter is that the model is able to generate state-varying portfolios, even under identical homothetic preferences. We show that countries' portfolio holdings in a given state depend on the current distribution of wealth among countries. Except for degenerate values of the endowments, the equilibrium distribution of wealth varies across states and, as a consequence, agents born in different states of nature choose a different portfolio of currencies. Since portfolios vary across states, current account imbalances arise among countries. For instance, suppose that country h at time t is wealthier than at time $t-1$. Then, the young born at t will hold more currency than the young born at $t-1$, since they now hold a higher share of aggregate wealth. As a result, we show that country h runs a trade surplus at time t. To sum up, the sign of the balance of trade depends on the past as well as on the current state. It is important to stress that a country might be the richest country in all states, but still run a trade deficit in some state of nature. What is really important is how the current wealth distribution compares with the past one.

Secondly, this model is able to give some insights on the relationship between countries' external positions and valuation effects. Traditionally, the net foreign asset position of a country was calculated based on the assumption that the change in net foreign assets of a country in a given period is equivalent to the current account

[6]It should be possible to prove generic existence under fairly standard assumptions. We leave this task for future research.

position. However, recent empirical literature has pointed out that the cumulated current account position of a country between any two periods does not always correspond to the overall change in net foreign assets[7]. Those gaps are sizable for many countries and it has been suggested that they are induced by changes in the value of foreign assets and liabilities, which are known in the literature as valuation effects. Valuations effects are not accounted for in national statistics and can be generated by capital gains or losses on equity positions or nominal exchange rate variations. In the case of the United States, it has been shown that they have "benefited" from positive valuation effects over the last 30 years, i.e. they have gained more on foreign assets than they lost on foreign liabilities. As a result, net foreign liabilities are much lower than they would be if we only took into account the current account deficits and ignored valuation effects. It has been suggested that the exchange rate might have played an important role in generating positive valuation effects in the US. Since US foreign assets tend to be denominated in foreign currencies while liabilities in dollars, a depreciation of the dollar against other currencies might have "stabilized" US net foreign liabilities. Many politicians and economists seem to think at these positive valuation effects as an "exorbitant privilege" due to their centrality of the dollar in the international monetary system.

Our numerical finding is that valuation effects, as driven by exchange rate fluctuations, are negatively related to the current account position of a country. Therefore, valuation effects have a stabilizing effect on net foreign assets. This result obtains only if the Markov process is persistent, but it is robust to changes in other parameters of the economy. The intuition can be explained as follows. Suppose that a country receives a high value of the endowment in some state. This country is wealthier than in the past and therefore runs a trade surplus (see above). Since the Markov process is persistent, agents expect that the domestic country's endowment is high tomorrow with higher probability. Therefore, they substitute the domestic good for the foreign good as they expect the domestic good to be relatively cheaper. In order to do so, all agents demand

[7]See Gourinchas et al. [38] for a review of the literature on valuation effects and an assessment of the US net external position in particular.

more domestic currency. However, the money supply is fixed and the price of the domestic currency today must rise in order to offset this increase in demand. As a consequence, the value of the foreign currencies' holdings (foreign assets) decrease, while the value of foreign agents' holdings of the domestic currency (foreign liabilities) increases, hence the negative valuation effects. The negative relationship between the balance of trade and valuation effects is also consistent with the empirical findings of Lane and Milesi-Ferretti [42], who showed that excess returns on net foreign assets and the balance of trade comove negatively for a cross-section of developed countries.

Another result is that the model is able to generate valuation effects that are quantitatively large. In particular, valuation effects can reduce the impact of a trade deficit or surplus on the change in net foreign assets by more than a half for reasonable parameter values. Valuations effects are instead quantitatively small in Devereux et al. [21]. The reason for these large effects is that we compute the solution of the non-linear model, as opposed to Devereux et al. [21] which uses local methods[8].

The mechanism is different from what Gourinchas et al. [37] suggest in their empirical work. In Gourinchas et al. [37], deteriorations in the external account must imply *future* excess returns or/and trade surpluses. Their empirical results are based on the assumption that a no-Ponzi game condition is satisfied[9]. While it is true that this assumption holds in a large class of models with infinitely lived agents, it does not necessarily in an OLG framework. In this model, a deterioration in the external account means instead that the country is experiencing *current* excess returns. As far as expectations are concerned, let us consider an example with two states of nature. Since tomorrow's state is the same as today's with high probability, the exchange rate does not change with high probability (since prices are only state dependent) and appreciates with low probability. Therefore, agents expect an appreciation of

[8]Notice that Devereux et al. [21] focus on valuation effects as induced by capital gains or losses in equity positions, not by exchange rate changes.

[9]Gourinchas and Rey model the world economy as a stochastic economy deviating from a deterministic steady state. One of the stark implications of the no-Ponzi game condition is that the lender (borrower) country is the deficit (surplus) country at the deterministic steady state.

the domestic currency and therefore negative valuation effects. For the same reason, the current young will expect a trade surplus if today's balance of trade is in deficit.

Finally, our work offers a new perspective on the theoretical underpinnings of the nominal exchange rate. In the cash-in-advance literature (see e.g. Lucas [44], Svensson [61] and Alvarez et al. [1]), the nominal exchange rate is a function of *current* endowments, money supplies and relative prices. The structure of the economy is such that money is not used to transfer wealth across periods and agents are indifferent to exchange rate fluctuations. Here, we abstract from the presence of other assets in the economy and, in a sense, gives money a much stronger role. Firstly, it functions as store of value, as in a standard pure exchange OLG setting. Moreover, it is necessary for transactions, since agents are required to hold foreign currencies in order to purchase foreign goods. As a result, the spot nominal exchange rate between any two currencies is a forward looking variable which depends on the expected purchasing power of the two currencies weighted by the old's marginal utilities. The demand for current and future goods, as well as the demand for assets, are explicit functions of the spot and the future nominal exchange rate.

Another closely related paper is Neumeyer [50], since we share the focus on incomplete markets models with nominal assets. The paper is an open economy extension of Magill and Quinzii's [46] two-period model of money, in which the indeterminacy problem is solved by injecting money via an institution called "the Central Exchange". Agents cannot consume their own endowment but they are required to sell it to the domestic Central Exchange in return for domestic currency. The implication of this device is that the nominal price level of each country is pinned down by a quantity equation[10]. The law of one price is assumed to hold and therefore the exchange rate directly depends on relative endowments and money supplies. Differently from the cash-in-advance literature, Neumeyer's is a one-good economy and therefore relative prices do

[10]In Neumeyer, agents are not allowed to store the currencies by assumption. Magill and Quinzii allow for this possibility and show that money will be stored when the interest rate is zero. In that case, money and a risk-free bond are perfect substitutes. If the equilibrium interest rate is instead positive, money is not held intertemporally.

not contribute to the determination of the exchange rate. In this respect, the exchange rate is not fully endogenous.

3.2 The model

We consider a pure exchange OLG economy with L countries. In each period, an agent with a two-period lifetime is born in country h and a state of nature s is realized, where $s = \{1, ..., S\}$.

The basic set up has a few similarities to Lucas [44]. Endowments are assumed to follow a first-order stationary Markov process. Agents gain utility from the consumption of L goods but they are only endowed with a country-specific good. Therefore, there are as many goods as countries. We will use the superscript ℓ to indicate goods and currencies, while we will refer to agents with the subscript h. As in [44], we study the stationary equilibria of the model. Hence, we assume that prices are only state, not time dependent.

The utility function is time-separable and preferences are not state dependent. u_h (v_h) is the utility function in the first (second) period of life. We introduce the same set of assumptions as the stochastic OLG literature on existence (e.g. Cass et al. [13] and Gottardi [35]):

Assumption 6 *The utility functions are continuous, strictly monotone and strictly quasi-concave.*

Assumption 7 *The utility functions are twice continuously differentiable on $\mathbb{R}^L_{++}$ and the closure of the indifference curves are strictly contained in $\mathbb{R}^L_{++}$.*

At time 0, the initial old are endowed with some units of L currencies. M^ℓ is the stock of money issued in country ℓ. Monetary authorities are inactive.

The timing is organized as follows. For descriptive purposes, suppose that endowments and preferences are such that currencies have a positive value. In the first period of life, young agents consume both the domestic and foreign goods. In order to consume foreign goods, they engage in trade with the foreign young. To finance future consumption, they sell part of their endowment to the current old in exchange for money. We now state the key assumptions of the model.

Assumption 8 *Currency M^ℓ can only buy good ℓ.*

Assumption 9 *To buy foreign goods when old, agents must purchase foreign currencies when young.*

The first restriction that we impose is that agents need the local currency to buy the local good. Therefore, Assumption 8 introduces what we will refer to as "cash-in-advance constraints".

Under this Assumption, agents could buy the exact amount of foreign currencies which allow them to consume foreign goods when young, while hold all their savings in the domestic currency. In order to consume foreign goods when old, they could buy foreign currencies after uncertainty has been realized. In that case, there is no actual portfolio choice to be made when young. The rationale behind Assumption 9 is that we want uncertainty to be realized *after* the currencies are chosen, with the purpose of introducing an element of exchange rate risk in agents' decision problem. Assumption 8 alone does not restrict them to buy foreign currencies one period in advance.

These Assumptions are a crucial aspect of the model in many respects. First of all, they allow to pin down the equilibrium exchange rate and countries' portfolios. Currencies are not perfect substitutes in the sense that each of them has a specific role, that is to allow agents to consume a particular good. On the contrary, in a world of no legal restrictions in which portfolios and exchange rates are indeterminate, only total money holdings matter and not the currency composition. Moreover, the legal restrictions implicit in Assumptions 8 and 9 are of fundamental importance, since a stationary equilibrium would not exist otherwise (see below).

In principle, agents should face cash-in-advance constraints in both periods of life. Only for simplicity, we will assume that the young can engage in barter with other agents in their cohort and face cash-in-advance constraints only when old. In the Appendix, we show that the maximization problem does not change if we introduced the cash-in-advance constraints when young.

Taking as given the vector of transition probabilities $\rho(ss')$ and goods' and currencies' prices $(p(s), q(s))$, agent h born in state s

solves the following maximization problem :

$$\max_{c_{1h}(s),c_{2h}(ss')} u_h\left(c_{1h}(s)\right) + \beta_h \sum_{s'} \rho(ss') v_h\left(c_{2h}(ss')\right) \tag{3.1}$$

subject to its budget set:

$$B_h^s(p(s), q(s)) = \left\{(c_{1h}(s), c_{2h}(ss'), m_h(s)) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^{LS} \times \mathbb{R}_+^L : \begin{array}{r} p(s)\cdot[c_{1h}(s) - \omega_{1h}(s)] \leqslant -q(s)\cdot m_h(s) \\ p^\ell(s')[c_{2h}^\ell(ss') - \omega_{2h}^\ell(s')] \leqslant q^\ell(s')m_h^\ell(s) \;\; \forall\, \ell, s' \\ m_h^\ell(s) \geqslant 0 \qquad \forall\, \ell \end{array}\right\} \tag{3.2}$$

Prices are all expressed in abstract units of account. Since each agent is only endowed with the domestic good, $\omega_{1h}(s)$ is a ℓ-dimensional vector where all but one element are zeros. Moreover, $\omega_{2h}^\ell(s') = 0$ whenever $h \neq \ell$. The second set of budget constraints can also be thought of as cash-in advance constraints.

Let $\lambda_h(s)$ be the multiplier associated to the young's budget constraint, $\lambda_h^\ell(ss')$ the multiplier of cash-in-advance constraint related to good ℓ in state s' and $\mu_h(s)$ the vector of multipliers of money holdings' non-negativity constraints. The necessary and sufficient conditions for a maximum are the following first-order conditions:

$$\begin{aligned} c_{1h}^\ell(s) &: \; u_h^\ell(c_{1h}(s)) = \lambda_h(s)p^\ell(s) && (3.3)\\ c_{2h}^\ell(ss') &: \; \beta_h\rho(ss')v_h^\ell(c_{2h}(ss')) = \lambda_h^\ell(ss')p^\ell(s') \qquad \forall\, \ell, s' && (3.4)\\ m_h^\ell(s) &: \; -\lambda_h(s)q^\ell(s) + \sum_{s'} \lambda_h^\ell(ss')q^\ell(s') \leqslant 0 \\ & \quad = 0 \text{ if } m_h^\ell(s) > 0 \qquad \forall\, \ell && (3.5) \end{aligned}$$

and the complementary slackness conditions:

$$\begin{aligned} \lambda_h(s)\{-p(s)\cdot[c_{1h}(s) - \omega_{1h}(s)] - q(s)m_h(s)\} &= 0 && (3.6)\\ \lambda_h(s) &\geqslant 0 \\ \lambda_h^\ell(ss')\{q^\ell(s')m_h^\ell(s) - p^\ell(s')[c_{2h}^\ell(ss') - \omega_{2h}(s')]\} &= 0 && (3.7)\\ \lambda_h^\ell(ss') &\geqslant 0 \;\forall\, \ell, s' \end{aligned}$$

where $u_h^\ell(\cdot)$ and $v_h^\ell(\cdot)$ are partial derivatives with respect to good ℓ. All budget constraints will hold with equality by Assumption 1 (monotonicity), therefore $\lambda_h(s) > 0$ and $\lambda_h^\ell(ss') > 0$.

Definition 5 *A stationary equilibrium is a system of prices* $(p, q) \in \mathbb{R}_{++}^{LS} \times \mathbb{R}_+^{LS}$*, consumption allocations and portfolios* $(c_{1h}(s), c_{2h}(ss'),$

$m_h(s)) \in \mathbb{R}^L_{++} \times \mathbb{R}^{LS}_{++} \times \mathbb{R}^L_+$ for every $h = 1, ..., L$ and $s = 1, ..., S$ such that:

(i) agent h maximizes (1.4) on his budget set (3.2) for every s;

(ii) $z_1^\ell(s) + z_2^\ell(s's) = 0 \qquad \forall\ s', s$ and ℓ;

(iii) $\sum_h m_h^\ell(s) = M^\ell \qquad \forall\ \ell, s$

where $z_1^\ell(s) \equiv c_1^\ell(s) - \omega_1^\ell(s)$, $z_2^\ell(s's) \equiv c_2^\ell(s's) - \omega_2^\ell(s)$, $c_1^\ell(s) \equiv \sum_h c_{1h}^\ell(s)$ and $c_2^\ell(s's) \equiv \sum_h c_{2h}^\ell(s's)$.

3.2.1 The issue of existence of a stationary (monetary) equilibrium

Spear [62] proved that a steady state equilibrium does not generically exist in a stochastic OLG economy with multiple goods. Heuristically speaking, the non existence result is due to the fact that there are too many equations with respect to the number of unknowns.

However, his generic result does not rule out the possibility that a stationary equilibrium may exist under some restrictions. For example, he showed that additively time-separable utility functions and one type of agent per generation are sufficient conditions.

The economy studied in this chapter is another case in which a stationary equilibrium exists with multiple goods. In a standard OLG economy, the old face as many budget constraints as states of nature. If there are heterogenous agents, as we require in open economy, separable preferences are not enough to guarantee existence. The key element of this economy is that the old face L budget constraints in each state because of legal restrictions in currency trading. The next proposition shows that the presence of these constraints has the important consequence that many market clearing equations become redundant. This explains why we find existence later in our examples.

Proposition 16 *Under Assumption 6 (strict monotonicity), we have a system of $(L-1)S + LS$ equations and unknowns.*

Proof. Under Assumption 6, all budget constraints are binding. Let us plug the second period budget constraints into the second period utility function and write the maximization problem as one in which agent h choose L consumption goods when young and L money balances:

$$\max_{c_{1h}(s),m_h(s)} u_h\left(c_{1h}(s)\right)+\beta_h \sum_{s'} \rho(ss') v_h(\underbrace{\omega^\ell_{2h}(s') + \frac{q^\ell(s')}{p^\ell(s')} m^\ell_h(s)}_{c^\ell_{2h}(ss')}) \quad (3.8)$$

subject to:

$$p(s) \cdot [c_{1h}(s) - \omega_{1h}(s)] = -q(s) \cdot m_h(s) \quad (3.9)$$

Let's recall the goods' market clearing equations:

$$\sum_h c^\ell_{1h}(s) + \sum_h c^\ell_{2h}(s's) = \omega^\ell_1(s) + \omega^\ell_2(s) \qquad (LS^2 \text{ equations})$$

However, the aggregate consumption of the old does not depend on the previous state: $\sum_h c^\ell_{2h}(s's) = \omega^\ell_2(s) + \frac{q^\ell(s)}{p^\ell(s)} M^\ell$. Therefore, it is enough that the following LS goods' market clearing equations clear:

$$c^\ell_1(s) + c^\ell_2(ss) = \omega^\ell_1(s) + \omega^\ell_2(s) \qquad \text{(LS equations)}$$

We are basically left with LS equations plus LS money market clearing equations. Given (3.9), Walras Law will take the following form:

$$p(s) \cdot [c_1(s) - \omega_1(s)] = -q(s) \cdot M(s)$$

Therefore, further S market clearing equations becomes redundant. Our economy can be reduced to a system of $(L-1)S + LS$ equations. We can also adopt the normalization that $q^1(s) = 1$ for every s[11]. The number of equations and the number of unknowns is identical. ∎

Remark 1 *The introduction of more currencies* per se *does not solve the issue of non-existence of a stationary equilibrium.*

In a one currency economy, a stationary equilibrium does not exist generically since there are $(L-1)S^2 + S$ independent equations and only LS unknowns (see Spear [62]). Now introduce more currencies but assume that there are no legal restrictions, i.e. that each

[11] We follow this normalization later in the chapter.

currency can buy all goods. In the absence of the cash-in-advance constraints, the introduction of more currencies has the effect of increasing the number of independent equations to $(L-1)S^2+LS$, because L currency markets must now clear in each state of nature. On the other hand, the number of unknowns increases to $LS+(L-1)S$, where $(L-1)S$ are the exchange rates. This would even aggravate the non existence problem as it would add relatively more equations than unknowns.

Proposition 17 *Under Assumptions* 6*,* 8 *and* 9*, if a stationary equilibrium exists it is a monetary equilibrium.*

Proof. We must prove that an autarkic equilibrium in which $q(s)=0$ cannot exist. By Assumption 6, the domain of v_h is $\mathbb{R}^L_{++}$. Therefore, old agents consume L goods in strictly positive quantities. Since they are not endowed with foreign goods, the only option for them to consume them is to hold foreign currency. In fact, agents are not allowed to trade the domestic endowment when old by Assumptions 8 and 9. Hence, foreign currencies have always a positive value so at least $L-1$ conditions in (3.5) are binding for each h. But each currency is a foreign currency to at least one agent. Therefore, if a stationary equilibrium exists it must be an equilibrium in which all currencies have positive value. ■

3.2.2 Nominal exchange rate determination

Define exchange rates as the relative price of currency ℓ with respect to currency 1 in the state s: $e^\ell(s) \equiv \frac{q^\ell(s)}{q^1(s)}$. If $e^\ell(s) > e^\ell(s')$, currency ℓ is worth more in state s than in state s'. If s' was yesterday's realization and s is today's, we can say that currency ℓ has appreciated with respect to currency 1.

Using equations (3.4), (3.5) and our definition, we can write exchange rates as follows:

$$e^\ell(s) = \frac{\sum_{s'} \rho(ss') v_h^\ell(c_{2h}(ss')) \frac{q^\ell(s')}{p^\ell(s')}}{\sum_{s'} \rho(ss') v_h^1(c_{2h}(ss')) \frac{q^1(s')}{p^1(s')}} \qquad \ell = 2, ..., L \qquad s = 1, ..., S \tag{3.10}$$

$\frac{q^\ell(s')}{p^\ell(s')}$ gives how many units of good ℓ we can afford in state s' per unit of currency ℓ held. Therefore, the relative price of currency ℓ is the ratio of the expected purchasing power of currency ℓ over the expected purchasing power of currency 1, weighted by agent h's marginal utilities. Given that the currencies are used to transfer wealth across periods, it is expected that exchange rates follow some sort of asset pricing equations.

The nominal exchange rate is a key price in this model as it affects agents' saving and portfolio problems. In the cash-in-advance literature, the spot exchange rate simply depends on the *current* realization of the stochastic variables and not on future ones (see Lucas [44]). This is due to the different role that it is attributed to money. Money is only used to carry out exchange in a given period. Money is a "veil" and the exchange rate does not ultimately affect the allocation, which is the same as in the barter economy. We will provide a more detailed discussion on the role of the exchange rate in the context of our examples.

3.2.3 Portfolio rebalancing generates trade imbalances

We now explore the relationship between the behaviour of portfolios over time and the balance of trade.

Lemma 7 (Balance of trade) *If portfolios are constant across states, trade is always balanced.*

Proof. For any pair of states s' and s, premultiply each market clearing equation for the respective prices $p^{\ell}(s)$ and then sum across ℓ:

$$p(s) \cdot [z_1(s) + z_2(s's)] = 0$$

Define the balance of trade of country h as[12]:

$$TB_h(s's) \equiv -p(s) \cdot [z_{1h}(s) + z_{2h}(s's)] \qquad (3.11)$$

Using the budget constraints, it should be immediate that the balance of trade can be rewritten as:

$$TB_h(s's) = q(s) \cdot [m_h(s) - m_h(s')] \qquad (3.12)$$

It now follows that $TB_h(s's) = 0$ if $m_h(s') = m_h(s)$ for every s'. ■

If portfolios were to be constant across states of nature in equilibrium, then the change in the net foreign assets position of a

[12]The balance of trade is defined as minus excess demand, so that it is positive when there is excess supply in the country.

country over time would always be equal to zero. In this case, we have shown that the balance of trade would be in equilibrium as we would expect from national accounting identities[13]. However, the net foreign assets position of a country is not always equal to the balance of trade. In fact, there can also be valuation effects deriving from fluctuations of nominal exchange rates[14]. Our examples will show that valuation effects are absent only in very special cases. Therefore, the Lemma stresses that we cannot have an equilibrium in which the net foreign assets position of a country does not change over time and yet there are trade imbalances and the nominal exchange rate fluctuates. This is a scenario which is indeed plausible in the Lucas model [44]. In our model, if portfolios are constant across states then *both* the balance of trade and valuation effects will be zero.

In general, it is reasonable to expect that the set of parameters of the economy under which portfolios are state invariant has a very small measure. If portfolios are constant across states, then the consumption of the old does not depend on his state of birth[15]. If endowments are random, agents born in different states of nature do not have the same endowment and are therefore likely to have different demands for the goods. In order for portfolios to be constant, the demand function must be very special. The example that we present in the next section has this property that portfolios are constant in equilibrium. Therefore, the balance of trade is zero as well as valuation effects by Lemma 7. If the utility function is generalized to isoelastic utility (see sections 3.4 and 3.5), constant portfolios can only occur for degenerate values of the endowments.

Note that the above condition does not depend on the fact that we restrict to stationary equilibria. Instead, the following corollary requires that prices are state dependent:

Corollary 4 *If today's realized state is the same as yesterday's, then trade is balanced.*

The corollary is related to Polemarchakis and Salto's result for deterministic OLG economies [53]. In a one-currency economy,

[13]In fact, net income from abroad is equal to zero as there are no interest payments in the model.

[14]See equation (3.18) in the next section.

[15]See the budget constraints of the agents when old.

which can be thought of as a monetary union, they showed that the balance of trade is in equilibrium at the monetary steady state. Here, the monetary steady state is stochastic and trade imbalances are possible when $s \neq s'$.

3.2.4 Net foreign assets and valuation effects: the role of exchange rates

In this section, we explore the relationship between net foreign assets, the balance of trade and valuation effects. Consider the balance of trade of country 1 in state $s's$ (see equation (3.12)):

$$TB_1(s's) = q^1(s)[m_1^1(s) - m_1^1(s')] + \sum_{\ell=2}^{L} q^\ell(s)[m_1^\ell(s) - m_1^\ell(s')] \quad (3.13)$$

Now, adopt the normalization that $q^1(s) = 1$ and use the definition of exchange rates at page 72:

$$tb_1(s's) = [m_1^1(s) - m_1^1(s')] + \sum_{\ell=2}^{L} e^\ell(s)[m_1^\ell(s) - m_1^\ell(s')] \quad (3.14)$$

where $tb_1(s's) \equiv \frac{TB_1(s's)}{q^1(s)}$. By currency 1's market clearing equations, we can rewrite the first two terms on the right hand side:

$$tb_1(s's) = \underbrace{\sum_{h=2}^{L} m_h^1(s')}_{FL_1(s')} - \underbrace{\sum_{h=2}^{L} m_h^1(s)}_{FL_1(s)} + \underbrace{\sum_{\ell=2}^{L} e^\ell(s) m_1^\ell(s)}_{FA_1(s)} - \underbrace{\sum_{\ell=2}^{L} e^\ell(s) m_1^\ell(s')}_{\text{current value } FA_1(s')} \quad (3.15)$$

$FA(s)$ are holdings of foreign assets in state s and $FL(s)$ are foreign holdings of the domestic currency, i.e. foreign liabilities. Now define net foreign assets as $NFA(s) \equiv FA(s) - FL(s)$, and rewrite the above as follows:

$$NFA_1(s) = \text{current value } NFA_1(s') + tb_1(s's) \quad (3.16)$$

Equation (3.16) states that the end of period net foreign assets in country 1 is equal to the current value of the net foreign assets accumulated in the previous period and the balance of trade[16].

Now, we rewrite equation (3.15) in order to highlight valuation effects. In the right hand side, sum and subtract $FA_1(s')$ and use

[16] This equation is equivalent to equation (1) of Gourinchas and Rey [37] (see in particular footnote 2).

the definition of net foreign assets to obtain:

$$tb_1(s's) = NFA_1(s) - NFA_1(s') + \underbrace{\sum_{\ell=2}^{L} e^{\ell}(s')m_1^{\ell}(s')}_{FA_1(s')} - \underbrace{\sum_{\ell=2}^{L} e^{\ell}(s)m_1^{\ell}(s')}_{\text{current value } FA_1(s')} \tag{3.17}$$

This equation can be rewritten as:

$$\Delta NFA_1(s's) = tb_1(s's) + \underbrace{\sum_{\ell=2}^{L} r^{\ell}(s's)e^{\ell}(s')m_1^{\ell}(s')}_{\text{valuation effects}} \tag{3.18}$$

where

$$r^{\ell}(s's) = R^{\ell}(s's) - 1 \equiv \frac{e^{\ell}(s)}{e^{\ell}(s')} - 1$$

In this model, valuation effects are entirely determined by exchange rate movements. If foreign currencies have appreciated with respect to the past, the "rate of return" on foreign assets is positive and therefore we say that the country experiences positive valuation effects. Conversely, a country experiences negative valuation effects if foreign currencies have depreciated.

Valuation effects are thought to be very important in explaining the dynamics of net foreign assets in the US and other countries. This chapter provides a framework in which the exchange rate can be state dependent and therefore valuation effects can arise. Therefore, the change in net foreign assets will be a combination of the balance of trade position and valuation effects. We will discuss in more detail the interaction between exchange rates, net foreign assets and the balance of trade in section 3.5.

3.3 A special case

In this section, we illustrate an example which can be fully solved analytically. We specialize to a two-country setting in which consumers have logarithmic preferences[17]. Agents born in country 1 are endowed with good 1 and agents born in country 2 are endowed with good 2. In order to be able to derive the demand functions in closed-form solutions, we also assume that endowments in the second period of life are always zero:

[17]The example can be extended to L countries but computation becomes more cumbersome.

Assumption 10 $\omega_{2h}^{\ell}(s) = 0$ *for every* h, s.

From now onwards, we drop the subscript related to age in endowments. We also use the normalization that $q^1(s) = 1$ for every s. The budget constraints are then expressed in terms of the numéraire currency. $e(s)$ denotes the relative price of currency 2 with respect to currency 1 in state s. To facilitate interpretation, we adopt the convention that $\pi^2(s) \equiv \frac{p^2(s)}{q^2(s)}$ is the nominal price of good 2, so that $\pi^2(s)e(s)$ is the price of good 2 expressed in units of the numéraire currency. Similarly, $\pi^1(s) \equiv \frac{p^1(s)}{q^1(s)}$ is the nominal price of good 1.

Agent h born in state s solves the following maximization problem:

$$\max \sum_{\ell} \log c_{1h}^{\ell}(s) + \beta_h \sum_{s'} \rho(ss') \sum_{\ell} \log c_{2h}^{\ell}(ss')$$

subject to:

$$\begin{aligned}
\pi^1(s)[c_{1h}^1(s) - \omega_h^1(s)] &+ \pi^2(s)e(s)[c_{1h}^2(s) - \omega_h^2(s)] = \\
&= -m_h^1(s) - e(s)m_h^2(s) \\
\pi^1(s')c_{2h}^1(ss') &= m_h^1(s) \qquad \forall\ s' \\
\pi^2(s')c_{2h}^2(ss') &= m_h^2(s) \qquad \forall\ s'
\end{aligned}$$

where m_h^1 (m_h^2) is the demand of agent h for currency 1 (2).

The first-order conditions are:

$$c_{1h}^1(s) \quad : \quad \frac{1}{c_{1h}^1(s)} = \lambda_h(s)\pi^1(s) \tag{3.19}$$

$$c_{1h}^2(s) \quad : \quad \frac{1}{c_{1h}^2(s)} = \lambda_h(s)\pi^2(s)e(s) \tag{3.20}$$

$$c_{2h}^{\ell}(ss') \quad : \quad \frac{\beta_h\rho(ss')}{c_{2h}^{\ell}(ss')} = \lambda_h^{\ell}(ss')\pi^{\ell}(s') \qquad \forall\ \ell, s' \tag{3.21}$$

$$m_h^1(s) \quad : \quad -\lambda_h(s) + \sum_{s'} \lambda_h^1(ss') \leqslant 0 \qquad = 0 \text{ if } m_h^1(s) > 0 \tag{3.22}$$

$$m_h^2(s) \quad : \quad -\lambda_h(s)e(s) + \sum_{s'} \lambda_h^2(ss') \leqslant 0 \quad = 0 \text{ if } m_h^2(s) > 0 \tag{3.23}$$

$$\begin{aligned}
\lambda_h(s) \quad &: \quad \pi^1(s)[c_{1h}^1(s) - \omega_h^1(s)] + \pi^2(s)e(s)[c_{1h}^2(s) - \omega_h^2(s)] + \\
&+ \quad m_h^1(s) + e(s)m_h^2(s) = 0
\end{aligned} \tag{3.24}$$

$$\lambda_h^1(ss') \quad : \quad \pi^1(s')c_{2h}^1(ss') - m_h^1(s) = 0 \qquad \forall\ s' \tag{3.25}$$

$$\lambda_h^2(ss') \quad : \quad \pi^2(s')c_{2h}^2(ss') - m_h^2(s) = 0 \qquad \forall\ s' \tag{3.26}$$

Since the endowment in the old age is zero for all agents, the first-order conditions for the currencies must be binding at the solution.

In Appendix C, we show how to derive the demand for the two currencies of agent h:

$$\begin{aligned} m_h^1(s) &= \frac{1}{2}\frac{\beta_h}{1+\beta_h}w_h(s) \\ m_h^2(s) &= \frac{1}{2}\frac{\beta_h}{1+\beta_h}\frac{w_h(s)}{e(s)} \end{aligned}$$

$w_h(s)$ denotes wealth, therefore $w_1(s) = \pi^1(s)\omega^1(s)$ and $w_2(s) = \pi^2(s)e(s)\omega^2(s)$. Money holdings are linear functions of wealth, which is to be expected under log utility. Keeping nominal prices constant, as $e(s)$ appreciates the demand of agent 1 (2) for the foreign currency decreases (increases).

Savings are also a constant fraction of wealth:

$$s_h(s) \equiv m_h^1(s) + e(s)m_h^2(s) = \frac{\beta_h}{1+\beta_h}w_h(s)$$

The demand functions are:

$$\begin{aligned} c_{1h}^1(s) &= \frac{1}{2}\frac{1}{1+\beta_h}\frac{w_h(s)}{\pi^1(s)} \\ c_{1h}^2(s) &= \frac{1}{2}\frac{1}{1+\beta_h}\frac{w_h(s)}{\pi^2(s)e(s)} \\ c_{2h}^1(ss') &= \frac{1}{2}\frac{\beta_h}{1+\beta_h}\frac{w_h(s)}{\pi^1(s')} \qquad \forall\ s' \\ c_{2h}^2(ss') &= \frac{1}{2}\frac{\beta_h}{1+\beta_h}\frac{w_h(s)}{\pi^2(s')e(s)} \qquad \forall\ s' \end{aligned}$$

For agent 1, a high $e(s)$ means that the "exchange value" of the domestic endowment is lower and so is the demand for the foreign good in the first period of life. The demand for good 2 when old decreases as well since he acquires less foreign currency if it is relatively expensive. For agent 2, a higher $e(s)$ results instead in more purchasing power with respect to the foreign good (good 1).

Definition 6 *A stationary equilibrium is a system of prices* $(\pi, e) \in \mathbb{R}^{2S}_{++} \times \mathbb{R}^{S}_{++}$, *consumption allocations and portfolios* $(c_{1h}(s), c_{2h}(ss'), m_h(s)) \in \mathbb{R}^2_{++} \times \mathbb{R}^{2S}_{++} \times \mathbb{R}^2_{++}$ *for every* $h = 1, ..., L$ *and* $s = 1, ..., S$ *such that:*

(i) *agent* h *solves the above maximization problem in every* s*;*

(ii) $z_1^1(s) + z_1^\ell(ss) = 0 \qquad s = 1, ..., S$

(iii) $\sum_h m_h^\ell(s) = M^\ell \qquad s = 1, ..., S \qquad \ell = 1, 2$

In the previous section, we have shown that we have a system of $(L-1)S$ market clearing equations in the goods' markets and LS monetary equations. Since $L=2$, we have to solve for $3S$ prices in a system of $3S$ equations.

Now, substitute the demand functions for good 1 into the market clearing equation:

$$\frac{1}{2}\sum_h \frac{1}{1+\beta_h}\frac{w_h(s)}{\pi^1(s)} + \frac{1}{2}\sum_h \frac{\beta_h}{1+\beta_h}\frac{w_h(s)}{\pi^1(s)} = \omega^1(s)$$

Substituting back wealth in its original form and rearranging, we can pin down relative prices:

$$\frac{\pi^2(s)e(s)}{\pi^1(s)} = \frac{\omega^1(s)}{\omega^2(s)} \qquad s=1,...,S \tag{3.27}$$

Under this specification, the terms of trade simply depend on relative endowments. Note that purchasing power parity does not hold unless $\omega^1(s)=\omega^2(s)$.

Using (3.27), the money market clearing equations pin down nominal prices:

$$\pi^1(s) = \frac{2M^1}{\omega^1(s)\sum_h \frac{\beta_h}{1+\beta_h}} \qquad s=1,...,S \tag{3.28}$$

$$\pi^2(s) = \frac{2M^2}{\omega^2(s)\sum_h \frac{\beta_h}{1+\beta_h}} \qquad s=1,...,S \tag{3.29}$$

Finally, the exchange rate can be computed:

$$e(s) = e = \frac{M^1}{M^2} \tag{3.30}$$

Under log utility, it is remarkable that the exchange rate is constant even though endowments follow a stochastic process and discount factors differ across agents.

Plugging the equilibrium prices into the demand functions for the currencies, we find that portfolios are constant across states:

$$m_h^\ell(s) = m_h^\ell = \frac{\frac{\beta_h}{1+\beta_h}}{\sum_h \frac{\beta_h}{1+\beta_h}} M^\ell \qquad \forall\ h,\ell$$

The reason for this result is very simple. Equations (3.28), (3.29) and (3.30) show that wealth and the exchange rate are constant in equilibrium and therefore the demand for the currencies is no longer state dependent.

Remark 2 *Since portfolios are state-invariant, the balance of trade is always in equilibrium (by Remark 7):*

$$tb_h(s's) = 0 \qquad \forall\ s', s$$

The fact that the spatial distribution of portfolios is constant across states is reflected by the fact that the consumption of the old does not depend on the state of birth (see the discussion of the previous section). The consumption allocation is in fact:

$$\begin{array}{rlrl}
c_{11}^1(s) = \frac{1}{2}\frac{1}{1+\beta_1}\omega^1(s) & & c_{12}^1(s) = \frac{1}{2}\frac{1}{1+\beta_2}\omega^1(s) \\
c_{11}^2(s) = \frac{1}{2}\frac{1}{1+\beta_1}\omega^2(s) & & c_{12}^2(s) = \frac{1}{2}\frac{1}{1+\beta_2}\omega^2(s) \\
c_{21}^1(s's) = \frac{1}{2}\frac{\beta_1}{1+\beta_1}\omega^1(s) & & c_{22}^1(s's) = \frac{1}{2}\frac{\beta_2}{1+\beta_2}\omega^1(s) \\
c_{21}^2(s's) = \frac{1}{2}\frac{\beta_1}{1+\beta_1}\omega^2(s) & & c_{22}^1(s's) = \frac{1}{2}\frac{\beta_2}{1+\beta_2}\omega^2(s)
\end{array}$$

Remark 3 *If discount factors are heterogenous, the country with the most patient agents has a positive net foreign assets position in all states:*

$$NFA_1 \equiv e \cdot m_1^2 - m_2^1 = \frac{M^1}{\frac{\beta_1}{1+\beta_1} + \frac{\beta_2}{1+\beta_2}} \left[\frac{\beta_1}{1+\beta_1} - \frac{\beta_2}{1+\beta_2} \right]$$

Yet, trade is always balanced and the exchange rate is constant, therefore the change in net foreign assets is obviously equal to zero.

Our findings in the log case are related to Cass and Pavlova [14], who have shown that logarithmic utility yields very special results even when markets are incomplete. In a two-period economy with N Lucas trees, they found that the matrix of portfolio returns is degenerate and that the equilibrium allocation is Pareto optimal. In the same model but with infinitely-lived agents, Pavlova and Rigobon [51] need to introduce demand shocks in order to have a model with time-varying portfolios. Here, a similar outcome could be achieved by introducing state-dependent discount factors.

3.4 The leading example

In the previous section, we have presented a two-country example in which the equilibrium allocation can be computed analytically. However, the model loses many of its interesting features with logarithmic utility, since portfolios and the exchange rate are constant. In a sense, the predictions of the model are not "typical".

Yet, the analytical solution shows why things "go wrong". Since the system of equations is linear, in equilibrium country h's wealth is constant across states of nature (see equations (3.28) and (3.29)) and therefore the demand for the currencies does not vary across states. To break this result, we now work with isoelastic utility functions:

$$\max \sum_{\ell} \frac{c^{\ell}_{1h}(s)^{1-\frac{1}{\varepsilon_h}}}{1-\frac{1}{\varepsilon_h}} + \beta_h \sum_{s'} \rho(ss') \sum_{\ell} \frac{c^{\ell}_{2h}(ss')^{1-\frac{1}{\varepsilon_h}}}{1-\frac{1}{\varepsilon_h}} \; \varepsilon_h > 0, \; \varepsilon_h \neq 1$$

This is going to be the only difference with respect to the previous section. Although we are not able to compute equilibrium prices and the allocation by hand, we find closed form solutions for the demand functions. This allows us to show analytically that portfolios vary across states even when $\varepsilon_h = \varepsilon$.

The first-order conditions are:

$$\begin{aligned}
c^1_{1h}(s) &: \quad c^1_{1h}(s)^{-\frac{1}{\varepsilon_h}} = \lambda_h(s)\pi^1(s) && (3.31)\\
c^2_{1h}(s) &: \quad c^2_{1h}(s)^{-\frac{1}{\varepsilon_h}} = \lambda_h(s)\pi^2(s)e(s) && (3.32)\\
c^{\ell}_{2h}(ss') &: \quad \beta_h \rho(ss') c^{\ell}_{2h}(ss')^{-\frac{1}{\varepsilon_h}} = \lambda^{\ell}_h(ss')\pi^{\ell}(s') \qquad \forall\ \ell, s' && (3.33)\\
m^1_h(s) &: \quad -\lambda_h(s) + \sum_{s'} \lambda^1_h(ss') \leqslant 0 && (3.34)\\
m^2_h(s) &: \quad -\lambda_h(s)e(s) + \sum_{s'} \lambda^2_h(ss') \leqslant 0 && (3.35)\\
\lambda_h(s) &: \quad \pi^1(s)[c^1_{1h}(s) - \omega^1_h(s)] + \pi^2(s)e(s)[c^2_{1h}(s) - \omega^2_h(s)] + \\
&+ \quad m^1_h(s) + e(s)m^2_h(s) = 0 && (3.36)\\
\lambda^1_h(ss') &: \quad \pi^1(s')c^1_{2h}(ss') - m^1_h(s) = 0 \qquad \forall\ s' && (3.37)\\
\lambda^2_h(ss') &: \quad \pi^2(s')c^2_{2h}(ss') - m^2_h(s) = 0 \qquad \forall\ s' && (3.38)
\end{aligned}$$

We show in the Appendix how to find closed-form solutions for the agents' portfolios:

$$\begin{aligned}
m^1_h(s) &= \frac{\beta_h^{\varepsilon_h} \left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}}{A_h(s)} w_h(s) && (3.39)\\
m^2_h(s) &= \frac{\beta_h^{\varepsilon_h} e(s)^{1-\varepsilon_h} \left[\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}}{A_h(s)} \frac{w_h(s)}{e(s)} && (3.40)
\end{aligned}$$

where

$$A_h(s) \equiv \pi^1(s)^{1-\varepsilon_h} + [\pi^2(s)e(s)]^{1-\varepsilon_h} + \beta_h^{\varepsilon_h}\left[\sum_{s'}\rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h} + \\ + \beta_h^{\varepsilon_h}e(s)^{1-\varepsilon_h}\left[\sum_{s'}\rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}$$

Agent h's demand functions can be derived using (3.39), (3.40) and the budget constraints (calculations of the demand functions when young are provided in the Appendix):

$$c_{1h}^1(s) = \frac{\pi^1(s)^{-\varepsilon_h}}{A_h(s)}w_h(s) \qquad \forall\ \ell \tag{3.41}$$

$$c_{1h}^2(s) = \frac{[\pi^2(s)e(s)]^{-\varepsilon_h}}{A_h(s)}w_h(s) \qquad \forall\ \ell \tag{3.42}$$

$$c_{2h}^1(ss') = \frac{\beta_h^{\varepsilon_h}\left[\sum_{s'}\rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}}{A_h(s)}\frac{w_h(s)}{\pi^1(s')} \tag{3.43}$$

$$c_{2h}^2(ss') = \frac{\beta_h^{\varepsilon_h}e(s)^{-\varepsilon_h}\left[\sum_{s'}\rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}}{A_h(s)}\frac{w_h(s)}{\pi^2(s')} \tag{3.44}$$

$\pi(s)$ and $e(s)$ are equilibrium prices if they solve the following system of equations (see the discussion in the log case):

$$\begin{aligned}\sum_h c_{1h}^1(s) + \sum_h c_{2h}^1(ss) &= \omega^1(s) \qquad \forall\ s\\ \sum_h m_h^1(s) &= M^1 \qquad \forall\ s\\ \sum_h m_h^2(s) &= M^2 \qquad \forall\ s\end{aligned}$$

In the case of identical preferences ($\varepsilon_h = \varepsilon$), this system is:

$$\frac{\pi^2(s)e(s)}{\pi^1(s)} = \frac{\omega^1(s)}{\omega^2(s)}\frac{[\pi^2(s)e(s)]^{1-\varepsilon} + \beta^\varepsilon e(s)^{1-\varepsilon}\left[\sum_{s'}\rho(ss')\pi^2(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon}{\pi^1(s)^{1-\varepsilon} + \beta^\varepsilon\left[\sum_{s'}\rho(ss')\pi^1(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon} \tag{3.45}$$

$$M^1 = \frac{\beta^\varepsilon\left[\sum_{s'}\rho(ss')\pi^1(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon}{A(s)}\sum_h w_h(s) \tag{3.46}$$

$$e(s)M^2 = \frac{\beta^\varepsilon e(s)^{1-\varepsilon}\left[\sum_{s'}\rho(ss')\pi^2(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon}{A(s)}\sum_h w_h(s) \tag{3.47}$$

for $s = \{1, ..., S\}$, where the first equation is obtained by plugging the demand functions into the market clearing equations for good 1 and rearranging (as in the log case). The system is highly nonlinear so equilibrium prices cannot be solved for analytically.

This case is of particular interest for comparison purposes, since e.g. Lucas [44] assumes identical preferences. It can be observed that marginal rates of substitutions are equalized. In fact, the intertemporal marginal rates of substitution for goods 1 and 2 are the following:

$$
\begin{aligned}
\frac{\lambda_h^1(ss')}{\lambda_h(s)} &= \frac{\pi^1(s)}{c_{1h}^1(s)^{-\frac{1}{\varepsilon_h}}} \frac{\beta_h \rho(ss') c_{2h}^1(ss')^{-\frac{1}{\varepsilon_h}}}{\pi^1(s')} \qquad \forall\ s, s' \\
\frac{\lambda_h^2(ss')}{\lambda_h(s)} &= \frac{\pi^2(s) e(s)}{c_{1h}^2(s)^{-\frac{1}{\varepsilon_h}}} \frac{\beta_h \rho(ss') c_{2h}^2(ss')^{-\frac{1}{\varepsilon_h}}}{\pi^2(s')} \qquad \forall\ s, s'
\end{aligned}
$$

Plugging the demand functions, we get:

$$
\begin{aligned}
\frac{\lambda_h^1(ss')}{\lambda_h(s)} &= \frac{\rho(ss') \pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}}{\sum_{s'} \rho(ss') \pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}} \qquad \forall\ s, s' \\
\frac{\lambda_h^2(ss')}{\lambda_h(s)} &= \frac{\rho(ss') \pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}} e(s)}{\sum_{s'} \rho(ss') \pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}} \qquad \forall\ s, s'
\end{aligned}
$$

It is now immediate we have equalization of the normalized price vectors when $\varepsilon_h = \varepsilon$. That marginal rates of substitutions might be equalized when preferences are identical and homothetic, despite the incompleteness of the markets, is known in the literature[18]. On the other hand, the equalization of marginal rates of substitution is not a sufficient condition to achieve conditional Pareto optimality, the criterion used to assess optimality of stationary OLG economies, even when markets are sequentially complete[19].

In Lucas, homotheticity is not necessary for Pareto optimality to obtain since markets are complete. The optimal allocation is achieved by keeping asset holdings constant across states of nature. In this work, the OLG structure implies that the uncertainty faced when young cannot be insured. Therefore, agents born in different states are likely to choose a different portfolio of assets. However,

[18]For instance, Geanakoplos and Polemarchakis [32] found that the equilibrium allocation of a two-period incomplete markets economy with numéraire assets is constrained optimal under such preferences. Chapter 3 of Magill and Quinzii's book [47] is another useful reference.

[19]See Chattopadhyay et al. [15] for some results on optimality of stationary OLG economies with sequentially incomplete markets.

the marginal rates of substitution of agents born in the same state are equalized since preferences are identical and homothetic.

3.4.1 Portfolio holdings and the distribution of wealth

The following proposition establishes that there is a strong relationship between the distribution of wealth across countries, portfolio holdings and trade imbalances.

Proposition 18 *Under identical isoelastic preferences:* (*i*) *country h's portfolio holdings depends on its current share of aggregate wealth;* (*ii*) *if country h has a higher* (*lower*) *share of aggregate wealth with respect to the past, it runs a trade surplus* (*deficit*)*.*

Proof. (i) Under identical isoelastic preferences, the demand of agent h for the two currencies has the following form (see equations (3.39) and (3.40)):

$$m_h^\ell(s) = k^\ell(s)\omega_h(s)$$

where $k^\ell(s)$ is identical across agents. Summing across h, we get the following equation:

$$M^\ell = k^\ell(s)\sum_h \omega_h(s)$$

Combining the two equations, we obtain the desired result:

$$\frac{m_h^\ell(s)}{M^\ell} = \frac{w_h(s)}{w(s)} \qquad \ell = 1, 2$$

where $w(s) = \sum_h w_h(s)$.

(ii) Suppose that today's realized state is s and yesterday's was s'. By hypothesis, $\frac{w_h(s)}{w(s)} > \frac{w_h(s')}{w(s')}$. But then the above equation implies that:

$$\frac{m_h^\ell(s)}{M^\ell} > \frac{m_h^\ell(s')}{M^\ell} \qquad \ell = 1, 2$$

Finally, equation (3.14) implies country h has a trade surplus in state s. The other case can be worked out in a similar way. ■

It is reasonable to expect that each country's share of aggregate wealth - expressed in the numéraire currency - varies across states for most parameter values. The equilibrium system of equations is highly non linear and constant wealth can only happen by coincidence[20].

The numerical examples in section 3.5 will show that this model can generate state-dependent portfolios even under identical homothetic preferences, which is a common assumption in the international macroeconomics' literature. In the Lucas model, that portfolios are constant across states is true with any additively time-separable utility function. In a more general Lucas-tree economy, Judd et al. [40] find the same result even with heterogenous utility functions. When markets are complete, wealth is identical across agents and therefore agents do not adjust their portfolio holdings following to new shocks. As such, cash-in-advance models with complete markets are not suitable to study the dynamics of countries' net foreign assets. This model makes a first step in this direction.

It is also interesting to compare the behaviour of the balance of trade between this model and Lucas'. In the latter, each country always exports half of the domestic endowment and imports half of the foreign endowment. The sign of the balance of trade depends on how prices respond to changes in endowments. Under isoelastic preferences, a country is in surplus as long as the domestic endowment is higher than the foreign in the current state[21]. The stark implication is that a country whose endowment is always lower than the foreign country in all states (e.g. a developing country) is always in deficit. This result is in contrast with the observation that while many developing and emerging countries run current account surpluses, the United States have run current account deficits for more than a decade. In this model, both current and past variables are important to establish the sign of the balance of trade. If a country is in surplus it is because it is relatively wealthier with respect to the past, even though it can be poorer than the other country in all states. This can give some intuition as to why emerging countries run a trade surplus against the United States: higher growth in their domestic economies has meant a higher share of world GDP and hence the trade surpluses.

[20]We find numerically that a degenerate case in which wealth is constant is when countries have the same output in all states of nature.

[21]See the Appendix for a derivation of the balance of trade in the Lucas model under isoelastic preferences.

As far as the allocation of savings across assets is concerned, Proposition 19 reveals that if preferences are identical countries do not have very sophisticated portfolio strategies. Since wealth is the only aspect that matters, agents hold the same share of both currencies.

3.4.2 Exchange rate determination

Combining equations (3.46) and (3.47), we obtain the following expression for the exchange rate:

$$e(s) = \left(\frac{M^1}{M^2}\right)^{\frac{1}{\varepsilon}} \frac{\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon}{\varepsilon}}}{\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon}{\varepsilon}}} \qquad s = 1, ..., S \qquad (3.48)$$

The exchange rate is a non linear function of relative money supplies and of expected nominal prices in the two countries, where the non linearity comes from the elasticity of substitution being different than one. It is also interesting to note that if the stochastic process is i.i.d., the exchange rate is constant under isoelastic utility. This is because not only prices are stationary, but agents will have the same expectations in every state.

Equation (3.45) combined with (3.46) and (3.47) yields an expression for the nominal price levels in the two countries:

$$\begin{aligned}
\pi^1(s) &= \frac{M^1}{\omega^1(s)} \frac{\pi^1(s)^{1-\varepsilon} + \beta^\varepsilon \left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon}{\beta^\varepsilon \left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon} \\
\pi^2(s) &= \frac{M^2}{\omega^2(s)} \frac{[\pi^2(s)e(s)]^{1-\varepsilon} + \beta^\varepsilon e(s)^{1-\varepsilon} \left[\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon}{\beta^\varepsilon e(s)^{1-\varepsilon} \left[\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon}{\varepsilon}}\right]^\varepsilon}
\end{aligned}$$

Differently from the log case, we do not have closed-form solutions for the exchange rate and the price levels. As for the exchange rate, nominal price levels are related to expected price levels.

Exchange rate movements are very important, since they convey information about the sign of valuation effects and they also have an impact on net foreign assets. Unfortunately, we cannot make further progress on the relationship between the exchange rate, the distribution of wealth and net foreign assets from an analytical point of view. Above, we have shown that a country that is wealthier with respect to the past experiences a trade surplus. In the next section, numerical examples will show that such country does also experience an appreciation of the exchange rate and hold a positive net foreign asset position.

3.5 Countries' external positions and valuation effects

The aim of this section is to gain some insights on the behaviour of open economy variables such as the exchange rate, the balance of trade and net foreign asset positions. For this purpose, we study in detail two examples. In the first example, both countries are developed in the sense that the mean endowment across states is the same. In the second, one country is developed and the other is developing since the former country has a higher endowment in all states of nature. For simplicity, we assume that there are only two states.

The first result is that current account deficits (surpluses) are associated with positive (negative) valuation effects as long as the Markov process is persistent. This is consistent with Lane and Milesi-Ferretti's [42] empirical finding that "adjusted returns" are negatively correlated with the trade balance in a cross-section of developed countries[22]. In section 3.3, we have shown that a country experiences positive valuation effects on foreign assets when the domestic currency depreciates. In general, exchange rates movements are not the only source of valuation effects, but the literature has stressed that they are key to the understanding of the dynamics of countries' net external positions (e.g. [38]). We will explain the relationship between the current account and the exchange rate in the context of the examples. For the moment, we wish to emphasize that these two variables behave in a very different way. The sign of the balance of trade depends on how the current distribution of wealth compares to the past one. On the other hand, the exchange rate is driven by agents' expectations of nominal price levels in the two countries.

Moreover, we find that valuation effects can reduce the extent of changes in net foreign assets due to trade imbalances by more

[22] Lane and Milesi-Ferretti do not distinguish across different sources of valuation effects. "Adjusted returns" include valuation effects stemming from exchange rate movements, but it is more broadly defined. For instance, it also incorporates capital gains and losses.

than 50% for reasonable parameter values, and therefore they significantly contribute to the dynamics of net foreign assets. This is important since valuation effects are a crucial component of countries' net external positions. For instance, a stylized fact for the US is that there is a sizable gap between the United States' net external position and their cumulated current account deficits due to positive valuation effects on net foreign assets (see the discussion in Gourinchas and Rey [38]). Other papers, such as Devereux and Sutherland [21], have focused on valuation effects stemming from movements in equity prices and dividend payments. However, they find valuation effects that are quantitatively small.

3.5.1 Example 1: developed countries

In this example, the two countries are developed since the mean endowment is the same across countries. We assume that country 1's output is more volatile than country 2[23]:

Symmetric shocks		Asymmetric shocks	
$\omega^1(1) = 8$	$\omega^1(2) = 12$	$\omega^1(1) = 8$	$\omega^1(2) = 12$
$\omega^2(1) = 9$	$\omega^2(2) = 11$	$\omega^2(1) = 11$	$\omega^2(2) = 9$

We say that shocks are symmetric when a state is good (or bad) for both countries. In this example, state 1 is the bad state and state 2 is the good state. In the asymmetric case, state 1 (2) is the good state for country 2 (1). The other parameter values are chosen as follows[24]:

$$\begin{aligned} M^1 = M^2 = M &= 10 \\ \varepsilon_1 = \varepsilon_2 = \varepsilon &= 2 \\ \beta_1 = \beta_2 = \beta &= 0.9 \\ \rho(ss) &= 0.8 \end{aligned}$$

When the elasticity of substitution between tradable goods is greater than 1, consumption goods are gross substitutes. Such parametrization rules out episodes of "immiserizing growth". In fact, when $0 < \varepsilon < 1$, a country that experiences a positive shock (everything

[23] If countries had identical, but state-dependent endowments, nominal price levels would be identical across countries and the exchange rate would be constant.

[24] The money supplies are chosen to be identical since their level does not affect the allocation.

else equal) is poorer in value terms since the price of the domestic good falls too much. In other words, the terms of trade effect dominates changes in endowments[25]. Empirical work based on low-frequency data found elasticities between 4 and 15, while estimates at higher frequency suggest that the elasticity is much lower and in the range of 0.2 to 3.5 (see Ruhl [55]). Our parametrization is more in line with the international real business cycle literature. We have also assumed that the Markov process is persistent[26].

We report in brackets the equilibrium prices related to the asymmetric shocks case:

$$
\begin{array}{rlrl}
\pi^1(1) &= 2.6875\ [2.6875] & \qquad \pi^1(2) &= 1.9281\ [1.9281] \\
\pi^2(1) &= 2.4370\ [2.0676] & \qquad \pi^2(2) &= 2.0677\ [2.4369] \\
e(1) &= 1.0309\ [1.0830] & \qquad e(2) &= 0.9804\ [0.9333]
\end{array}
$$

The first observation is that the nominal price level of each country decreases with the domestic endowment. Relative prices, expressed in the numéraire currency, are defined as:

$$\pi(s) \equiv \frac{\pi^2(s)e(s)}{\pi^1(s)}$$

Therefore:

$$
\begin{aligned}
\pi(1) &= 0.9348\ [0.8332] \\
\pi(2) &= 1.0514\ [1.1796]
\end{aligned}
$$

From country 1's perspective, when $\pi(s)$ increases the terms of trade worsens since the price of imports rises relatively to the price of exports. In the asymmetric case, good 1 is relatively cheaper in state 2 since the country experiences a positive shock. When shocks are symmetric, this is still the case because the shock in country 1 is bigger than in country 2.

It can also be observed that the country that experiences a positive shock (or a larger positive shock) has its currency appreciated. For instance, let us explain the mechanics of why currency 1 appreciates in state 2 ($e(s)$ falls). From the exchange rate equation (3.48), it can be checked that the exchange rate decreases (increases) with

[25] A similar issue arises in the simplest possible setting, i.e. a static GE model with isoelastic utility and corner endowments. See also Cole and Obstfeld [17] and Lucas [44].

[26] We provide results for different values of the elasticity and the persistence parameter in the Appendix.

future price levels in country 2 (1):

$$\frac{\partial e(s)}{\partial \pi^2(s')} < 0 \qquad \frac{\partial e(s)}{\partial \pi^1(s')} > 0 \qquad\qquad \text{for } \varepsilon > 1$$

The intuition is the following. Country 1 (2) has a low (high) price level in state 2. Since the Markov process is persistent, agents will expect a lower (higher) price level in country 1 (2) than if they were born in the other state. As goods are gross substitutes, the young substitute good 1 for good 2 and therefore demand relatively more currency 1 (see equation (C.14) in the Appendix). However, the money supply is fixed, therefore the relative price of currency 1 has to increase. When shocks are symmetric, both countries experience a positive shock in state 2. However, the shock in country 1 is bigger and therefore currency 1 appreciates.

On the left (right) column, we report the money holdings of agents born in country 1 (2):

$$\begin{array}{ll}
m_1^1(1) = 4.8741 \ [4.6606] & \qquad m_2^1(1) = 5.1259 \ [5.3394] \\
m_1^2(1) = 4.8741 \ [4.6606] & \qquad m_2^2(1) = 5.1259 \ [5.3394] \\
& \\
m_1^1(2) = 5.0922 \ [5.3059] & \qquad m_2^1(2) = 4.9078 \ [4.6941] \\
m_1^2(2) = 5.0922 \ [5.3059] & \qquad m_2^2(2) = 4.9078 \ [4.6941]
\end{array}$$

To interpret these results, let us define country 1's share of aggregate wealth in state s as $\theta_1(s)$, measured in the numéraire currency:

$$\theta_1(s) \equiv \frac{\pi^1(s)\omega^1(s)}{\pi^1(s)\omega^1(s) + \pi^2(s)e(s)\omega^2(s)}$$

Since preferences are identical, the share of the country's aggregate wealth dictates the share of currencies held by its young agents. Country 1 is always more endowed than country 2 in state 2 and viceversa. Therefore, country 1 is wealthier in state 2 but poorer in state 1. In fact, the share held by agents born in country 1 in the two states are: $\theta_1(1) = 0.4874$ [0.4661] and $\theta_1(2) = 0.5092$ [0.5306].

In the previous section, we have shown that we can infer the sign of the balance of trade from countries' portfolio holdings. The balance of trade of country h in state $(s's)$ is:

$$tb_h(s's) \equiv [m_h^1(s) - m_h^1(s')] + e(s)[m_h^2(s) - m_h^2(s')]$$

Therefore:

$$\begin{aligned} tb_1(21) &= -0.4429\ [-1.3442] \\ tb_1(12) &= 0.4319\ [1.2476] \end{aligned}$$

Obviously, $tb_2(s's) = -tb_1(s's)$. As expected, the country that experiences a positive shock is in surplus since it is the wealthiest country, while the poorer country is in deficit. Note that a current account surplus is associated with an appreciation of the domestic currency and viceversa. When shocks are asymmetric, portfolios change more across states and therefore trade imbalances are even larger.

A more traditional definition of the balance of trade is the difference between exports and imports:

$$tb_1(s's) \equiv \pi^1(s)[c^1_{12}(s) + c^1_{22}(s's)] - \pi^2(s)e(s)[c^2_{11}(s) + c^2_{21}(s's)]$$

The left (right) column reports the consumption allocation for agents born in country 1 (2):

$$\begin{aligned} c^1_{11}(1) &= 2.0856\ [1.9943] & c^1_{12}(1) &= 2.1934\ [2.2848] \\ c^2_{11}(1) &= 2.3866\ [2.8726] & c^2_{12}(1) &= 2.5100\ [3.2911] \\ c^1_{21}(11) &= 1.8137\ [1.7341] & c^1_{22}(11) &= 1.9073\ [1.9868] \\ c^2_{21}(11) &= 2.0000\ [2.2540] & c^2_{22}(11) &= 2.1034\ [2.5823] \\ c^1_{21}(12) &= 2.5278\ [2.4172] & c^1_{22}(12) &= 2.6585\ [2.7693] \\ c^2_{21}(12) &= 2.3573\ [1.9123] & c^2_{22}(12) &= 2.4790\ [2.1910] \end{aligned}$$

$$\begin{aligned} c^1_{11}(2) &= 3.4697\ [3.6152] & c^1_{12}(2) &= 3.3439\ [3.1984] \\ c^2_{11}(2) &= 3.1387\ [2.5981] & c^2_{12}(2) &= 3.0250\ [2.2986] \\ c^1_{21}(21) &= 1.8948\ [1.9743] & c^1_{22}(21) &= 1.8262\ [1.7466] \\ c^2_{21}(21) &= 2.0895\ [2.5661] & c^2_{22}(21) &= 2.0139\ [2.2702] \\ c^1_{21}(22) &= 2.6409\ [2.7519] & c^1_{22}(22) &= 2.5454\ [2.4346] \\ c^2_{21}(22) &= 2.4627\ [2.1772] & c^2_{22}(22) &= 2.3736\ [1.9262] \end{aligned}$$

Consider the balance of trade of country 1 in state 12. First, let us look at the asymmetric case. The terms of trade worsens for country 1 since the foreign good is relatively more expensive. As a matter of fact, country 1's consumption of the foreign good ($c^2_{11}(2)$ and $c^2_{21}(12)$) is lower than in state 21. Despite that the price of the foreign good is higher, the value of imports falls. On the other hand, the value of exports increase since good 1 is now cheaper.

In fact, country 2's consumption of the domestic good ($c^1_{12}(2)$ and $c^1_{22}(12)$) are higher than in state 21. The net effect is unambiguous: country 1 experiences a trade surplus in state 12.

When shocks are symmetric, both imports and exports increase since both goods are cheaper than in state 1. However, the relative price of good 2 is still higher since country 2 experiences a smaller positive shock. Therefore, exports increase more than imports and the net effect is that the balance of trade is in surplus.

In this context, net foreign assets can be defined as holdings of foreign currency (foreign assets) minus foreign holdings of domestic currency (foreign liabilities):

$$\begin{aligned} NFA_1(s) &\equiv m^2_1(s)e(s) - m^1_2(s) \\ NFA_2(s) &\equiv -NFA_1(s) \end{aligned}$$

Net foreign assets of country i are:

$$\begin{aligned} NFA_1(1) &= -0.1012\ [-0.2920] \\ NFA_1(2) &= 0.0846\ [0.2579] \end{aligned}$$

Country 1 has a positive net foreign asset position in state 2 for two reasons. Since the country is wealthier, it holds more money. Therefore, the country holds more foreign assets and also less liabilities. The foreign currency depreciates, but quantity effects are more important than exchange rate movements. In the asymmetric shock case, the net position is larger since the country holds even more assets and less liabilities.

It is worth noting that while the balance of trade's position depends on the previous state as well as on the current state, the net foreign assets' position only depends on the current state. In this example, country 1's positive (negative) net foreign asset position is associated with a trade surplus (deficit) in state 12 (21). In states 11 and 22, countries have non-zero net foreign assets but balanced trade.

We now examine the issue of valuation effects and how they affect countries' net external position. In this two-country two-states example, the change in net foreign assets can be written as follows:

$$\Delta NFA_1(s's) \equiv tb_1(s's) + \underbrace{r^2(s's)e(s')m_1^2(s')}_{\text{valuation effects}}$$

where $r^2(s's) \equiv \frac{e(s)}{e(s')} - 1$. We will refer to $r^2(s's)$ as to excess returns. Therefore:

$$\begin{aligned} r^2(21) &= 0.0515\ [0.1604] \\ r^2(12) &= -0.0490\ [-0.1382] \end{aligned}$$

and

$$\begin{aligned} VAL_1(21) &= 0.2571\ [0.7943] \\ VAL_1(12) &= -0.2462\ [-0.6976] \end{aligned}$$

The overall change in net foreign assets in the two states is:

$$\begin{aligned} \Delta NFA_1(21) &= -0.1858\ [-0.5499] \\ \Delta NFA_1(12) &= 0.1857\ [0.5499] \end{aligned}$$

In both states, it can be observed that the change in net foreign assets in country 1 is much lower than the actual trade imbalance. Valuation effects have the opposite sign of the balance of trade and they reduce surpluses or deficits by more than 50%. Since the exchange rate is more volatile in the asymmetric case, excess returns are larger and so are valuation effects.

To sum up, this is what happens to a country experiencing a positive shock:

$$\begin{aligned} \omega_h(s)\uparrow \quad &\Rightarrow \quad \text{TOT worsens} \Rightarrow tb_h(s's) > 0 \\ &\Rightarrow \quad \text{higher money holdings} \Rightarrow NFA_h(s) > 0 \\ &\Rightarrow \quad \text{domestic currency appreciates} \Rightarrow VAL_h(s) < 0 \end{aligned}$$

and

$$tb_h(s's) > 0 \ \ \& \ \ VAL_h(s) < 0 \qquad \Rightarrow \ \Delta NFA_h(s's) > 0$$

The country that experiences a positive shock has a higher wealth as long as the elasticity is greater than one. The young agents will hold more currency and therefore the country's net external position is positive. Since the domestic good is relatively cheaper, exports increase while imports fall. Therefore, the country runs a trade surplus. The domestic currency is more expensive (appreciates) to offset a high demand for the domestic good due to a low expected price level in the country. As a result, foreign assets accumulated in the previous period depreciates. The negative valuation effects counterbalance the trade surplus and therefore reduce the size of the positive change in net foreign assets.

Robustness checks In the Appendix, we vary elasticity of substitution and the persistence parameter and find the following:

Result 1 *Current account surpluses (deficits) are associated with negative (positive) valuation effects for* $0 < \varepsilon < \infty$*, but only as long as* $\rho(ss) > 0.5$.

Assuming that $\varepsilon < 1$ does not invalidate the negative relationship between the balance of trade and valuation effects. The only difference is that the effects of endowments shocks are the opposite.

For instance, a country with the positive shock is instead poorer in equilibrium. Therefore, that country holds less currency and has a negative net foreign asset position. The terms of trade falls too much and offset the increase in export, hence the country runs a current account deficit. The domestic currency now depreciates following a positive shock. Despite that agents expect a low price of good 1 tomorrow, the wealth effect dominates the substitution effects, so domestic agents will demand less currency 1. Foreign agents need less currency 1 to carry out their consumption plan, if they expect the price of good 1 to be low. However, the money supply is fixed and therefore currency 1 depreciates in order for both agents to demand the same share of both currencies.

Even in this case, the actual change in net foreign assets is lower than the trade imbalance. On the other hand, it is important to assume that the Markov process is persistent for such result to hold.

Now, suppose that $\varepsilon > 1$ but that the Markov process is not persistent ($\rho(ss) < 0.5$). Valuation effects now reinforce the country's current account balance. For example, if a country is in surplus it experiences positive valuation effects. The reason is that agents will not expect a low price level in the surplus country following a positive shock. Since the probability that tomorrow is a bad state is higher, agents will expect inflation and demand less currency instead. Therefore, the domestic currency depreciates and foreign assets are revaluated.

3.5.2 Example 2: developing vs. developed country

Now, country 1 is our developed country since it has a higher endowment than country 2 in both states. Throughout this example, we wish to highlight the fact that both the current and the past state of nature are important to establish the direction of trade among countries. In the Lucas model with isoelastic preferences, a country whose endowment is higher than the other in all states of nature (developed country) is always in surplus (see the Appendix). This result is at odds with the fact that the United States run current account deficits against emerging economies. Here, we show that the developed country runs a deficit if the other country experiences a positive shock (or a larger positive shock if shocks are symmetric).

In the symmetric scenario, state 1 is the bad state and state 2 is the good state for both countries. In the asymmetric one, the good state for a country is the bad state for the other. Output is also more volatile in the developing country.

Symmetric shocks		Asymmetric shocks	
$\omega^1(1) = 10$	$\omega^1(2) = 11$	$\omega^1(1) = 11$	$\omega^1(2) = 10$
$\omega^2(1) = 7$	$\omega^2(2) = 9$	$\omega^2(1) = 7$	$\omega^2(2) = 9$

Other parameters are chosen as in Example 1. We report in square brackets the equilibrium prices and allocation for the asymmetric shock case:

$$\begin{aligned} \pi^1(1) &= 2.2151 \ [2.0488] & \pi^1(2) &= 2.0488 \ [2.2151] \\ \pi^2(1) &= 3.1182 \ [3.1184] & \pi^2(2) &= 2.5380 \ [2.5381] \\ e(1) &= 0.8543 \ [0.8345] & e(2) &= 0.8877 \ [0.9087] \end{aligned}$$

Since endowments in country 2 are lower than in country 1 in all states, the nominal price level in country 2 is always higher. Money holdings are:

$$\begin{array}{ll} m_1^1(1) = 5.4294\ [5.5301] & m_2^1(1) = 4.5706\ [4.4699] \\ m_1^2(1) = 5.4294\ [5.5301] & m_2^2(1) = 4.5706\ [4.4699] \\ & \\ m_1^1(2) = 5.2639\ [5.1624] & m_2^1(2) = 4.7361\ [4.8376] \\ m_1^2(2) = 5.2639\ [5.1624] & m_2^2(2) = 4.7361\ [4.8376] \end{array}$$

Since country 1 has always a higher endowment in both states, it is always richer than country 2 and its agents hold a higher share of both currencies in every state. However, country 2's share of aggregate wealth ($\theta_2(s)$) is higher in state 2 and therefore its agents holds relatively more money in this state[27]. In fact, the share held by agents born in the developed country in the two states are: $\theta_1(1) = 0.5429$ [0.5530] and $\theta_1(2) = 0.5264$ [0.5162].

The balance of trade of country 1 when $s \neq s'$ is:

$$\begin{aligned} tb_1(21) &= 0.3069\ [0.6745] \\ tb_1(12) &= -0.3124\ [-0.7018] \end{aligned}$$

Despite that country 1 is richer in both states, it runs a trade deficit when the developing country experiences a positive shock. Even though both countries experience a positive shock in state 2, the shock is bigger in country 2 and therefore its young agents hold more currency than if they were born in state 1. This simple fact generates a trade surplus for the developing country. As in the previous example, a trade deficit (surplus) is associated to the depreciation (appreciation) of the domestic currency.

Let us now consider the balance of trade in terms of exports and

[27] When the shocks are asymmetric, country 1 obviously holds more in the good state and less in the bad state.

imports. The consumption allocation is:

$$\begin{aligned}
c_{11}^1(1) &= 2.9784\ [3.3840] & c_{12}^1(1) &= 2.5073\ [2.7351] \\
c_{11}^2(1) &= 2.0594\ [2.0976] & c_{12}^2(1) &= 1.7336\ [1.6954] \\
c_{21}^1(11) &= 2.4511\ [2.6992] & c_{22}^1(11) &= 2.0634\ [2.1817] \\
c_{21}^2(11) &= 1.7413\ [1.7735] & c_{22}^2(11) &= 1.4658\ [1.4335] \\
c_{21}^1(12) &= 2.6500\ [2.4965] & c_{22}^1(12) &= 2.2309\ [2.0179] \\
c_{21}^2(12) &= 2.1392\ [2.1789] & c_{22}^2(12) &= 1.8008\ [1.7612]
\end{aligned}$$

$$\begin{aligned}
c_{11}^1(2) &= 3.2210\ [2.8319] & c_{12}^1(2) &= 2.8981\ [2.6537] \\
c_{11}^2(2) &= 2.6636\ [2.6122] & c_{12}^2(2) &= 2.3965\ [2.4477] \\
c_{21}^1(21) &= 2.3763\ [2.5197] & c_{22}^1(21) &= 2.1381\ [2.3612] \\
c_{21}^2(21) &= 1.6882\ [1.6556] & c_{22}^2(21) &= 1.5189\ [1.5514] \\
c_{21}^1(22) &= 2.5692\ [2.3305] & c_{22}^1(22) &= 2.3117\ [2.1839] \\
c_{21}^2(22) &= 2.0740\ [2.0340] & c_{22}^2(22) &= 1.8660\ [1.9061]
\end{aligned}$$

Agents born in country 1 always consume more than agents in country 2 because they are relatively richer. The terms of trade for country 1 are:

$$\begin{aligned}
\pi(1) &= 1.2026\ [1.2706] \\
\pi(2) &= 1.0997\ [1.0412]
\end{aligned}$$

To start with, it is easier to look at the case of asymmetric shocks. Country 1 is in deficit in state 12 for two reasons. In state 2, the foreign country experiences a positive shock. First, domestic consumption of the foreign good is higher than in state 1 since good 2 is cheaper. Since goods are gross substitutes, changes in quantities are greater than changes in prices. Therefore, the value of imports increase. Second, foreign consumption of good 1 is lower precisely because good 1 is relatively more expensive. As a consequence, the value of exports drops. Higher imports and lower exports means that country 1 is in deficit. A similar argument can be made to explain why country 1 is in surplus in state 21.

In the symmetric shocks case, good 2 is still cheaper and therefore imports increase. The difference is now that exports increase as well, since the domestic price level falls due to a positive shock. However, the demand of good 2 increases more in relative terms since the terms of trade improves. Overall, imports rise more than exports and country 1 runs a deficit.

Net foreign assets are:

$$\begin{aligned} NFA_1(1) &= 0.0677\ [0.1450] \\ NFA_1(2) &= -0.0633\ [-0.1465] \end{aligned}$$

while excess returns are:

$$\begin{aligned} r^2(12) &= 0.0391\ [0.0889] \\ r^2(21) &= -0.0376\ [-0.0817] \end{aligned}$$

Therefore:

$$\begin{aligned} \Delta NFA_1(12) &= -0.1310\ [-0.2915] \\ VAL_1(12) &= 0.1814\ [0.4103] \\ \Delta NFA_1(21) &= 0.1310\ [0.2915] \\ VAL_1(21) &= -0.1758\ [-0.3830] \end{aligned}$$

As in the previous example, a current account deficit (surplus) is associated with positive (negative) valuation effect due to the depreciation (appreciation) of the domestic currency.

Robustness checks We do not comment the results of the sensitivity analysis since they are the same as for the previous example (see the Appendix).

3.6 Conclusions and future research

This chapter provides a stylized model of exchange rate determination where changes in the net foreign assets' position of a country are a consequence of trade imbalances among countries and valuation effects due to exchange rate movements. Under identical isoelastic preferences, agents born in a given state of nature hold a fraction of the total money stocks equivalent to their share of aggregate wealth. Since the distribution of wealth among countries varies across states, trade imbalances arise at the country level. We have also shown that the exchange rate fluctuates because of the Markov structure of uncertainty. The spot exchange rate between any two currencies is a function of expected purchasing power of the currencies with respect to the domestic good. Since agents' expectations depend on the state of birth, the exchange rate is state dependent. In this framework, exchange rate movements are big enough to generate relevant valuation effects.

An aspect of the model that needs further investigation is countries' portfolio choices when preferences are heterogenous. Under identical isoelastic preferences, agents choose all currencies in the same proportion. If the elasticity of substitution is allowed to vary across agents, wealth is no longer the only variable that pins down portfolios. Depending on the endowment structure, agents might prefer to hold more domestic currency in some state while more foreign currency in some other state, or even have a clear cut "preference" for a particular currency. For instance, it would be interesting to investigate under which conditions agents tend to have a bias towards domestic assets, as observed in the empirical literature[28]. An important direction of research would also be to provide a proof of generic existence and determinacy of equilibrium, in order to assess the robustness of the model.

Finally, a major challenge is to generalize the model by introducing further assets, in addition to the currencies in positive net supply. One option would be to look at the case of assets in zero net supply (e.g. bonds), to be denominated in different currencies. Past research on the topic has shown that this is no simple task. When there are no currencies in positive net supply, Polemarchakis [52] showed that the degree of indeterminacy in a two-period model with incomplete markets is $NS - A(N-1) - N$, where N is the number of currencies and A is the number of assets. Another extension would be to introduce other assets in positive net supply such as Lucas trees, in order to make our framework directly comparable with Lucas [44]. The incompleteness due to the OLG structure could be the key to break Lucas' [44] constant portfolios result. Both these extensions do not appear to be straightforward, given the particular structure of our model. However, this is the path to follow if we wish to achieve a better understanding of countries' portfolio choices and exchange rate behaviour.

[28] See Coeurdacier et al. [18] for a recent review of the literature on "home bias".

Appendix A

A.1 Proofs

Proof of Lemma 1

(*i*) Take equation (1.15) for any $t \geqslant 1$ and define the function g as follows:

$$\begin{aligned} g(k_{t+1}; k_t, \tau_1, \rho_1, \rho_2) &\equiv (1+n)k_{t+1} - \\ &- [\rho_1 s_1(f(k_t) - f'(k_t)k_t, f'(k_{t+1}), \tau_1) + \\ &+ \rho_2 s_2(f(k_t) - f'(k_t)k_t, f'(k_{t+1}))] \end{aligned}$$

We want to establish the existence of $k_{t+1} > 0$ given $k_t > 0$, such that $g(k_{t+1}; k_t, \tau_1, \rho_1, \rho_2) = 0$. To do that, we study the sign of g as k_{t+1} tends to infinity and zero. The first limit tells us that g is positive for k_{t+1} approaching infinity:

$$\lim_{k_{t+1} \to +\infty} g(k_{t+1}; k_t, \tau_1, \rho_1, \rho_2) = +\infty \tag{A.1}$$

(savings are always bounded above by w_t). Therefore, for at least a $k_{t+1} > 0$ to exist we need:

$$\lim_{k_{t+1} \to 0} g(k_{t+1}; k_t, \tau_1, \rho_1, \rho_2) < 0 \tag{A.2}$$

When $\rho_1 = 1$ (closed economy), De La Croix et al. [25] show that it is enough that the young's income after tax is strictly positive for savings to be positive, as savings are increasing in income. In particular, the following condition must hold : $w_t > \tau_1$. It turns out that the same condition is valid in a two-country economy. It is not sufficient that aggregate savings are positive, since we only allow for strictly positive consumption. Therefore, for an equilibrium to exist we need both countries' savings to be positive.

Now, define $\bar{\tau}_1(k_t)$ as the level of tax for which savings are zero in country 1 (it is obvious that $\bar{\tau}_1$ is increasing in k_t). Therefore, as long as $\tau_1 < \bar{\tau}_1(k_t)$, equation (A.2) is satisfied and therefore k_{t+1} exists.
We now prove that k_{t+1} is unique given k_t. By Assumption 1, g is increasing in k_{t+1}:

$$g'(k_{t+1}) = 1 + n - s_r f''(k_{t+1}) > 0 \qquad\qquad \forall\ k_{t+1}$$

This is enough to ensure uniqueness. We can then write

$$k_{t+1} = \phi(k_t; \tau_1, \rho_1, \rho_2)$$

which is a single-valued, strictly increasing function in k_t[1].

The above discussion is also valid at $t = 0$. It follows that if $\tau_1 < \bar{\tau}_1(k_{1,0})$ at time 0, $k_1 > 0$ exists given $(k_{1,0}, k_{2,0})$ and is unique. A unique intertemporal equilibrium will exist by induction.

(*ii*) We know already that the saving locus of the economy is increasing. Suppose that

$$\lim_{k_t \to 0} \frac{\phi(k_t; \tau_1, \rho_1, \rho_2)}{k_t} > 1$$

For the saving locus to cross the 45 degree line from above at least once, we need to show that the following is true:

$$\lim_{k_t \to +\infty} \frac{\phi(k_t; \tau_1, \rho_1, \rho_2)}{k_t} < 1 \tag{A.3}$$

The argument is the same as for closed economies and relies on the fact that savings can never exceed the wage (see Azariadis [3], p. 84). Since

$$(1 + n)k_{t+1} = \rho_1 s_{1,t} + \rho_2 s_{2,t} \leqslant w_t$$

that condition (A.3) is satisfied can be shown by dividing both sides of the inequality by k_t and then taking the limit:

$$\lim_{k_t \to +\infty} \left[\frac{\phi(k_t; \tau_1, \rho_1, \rho_2)}{k_t} \right] \leqslant \frac{1}{1+n} \lim_{k_t \to +\infty} \left[\frac{f(k_t)}{k_t} - f'(k_t) \right] = 0$$

This proves the existence of at least one locally stable steady state.

[1] See Galor et al. [30] for a throughout study of the function ϕ.

Proof of Lemma 2

Consider the budget constraints of the agents born at t at equilibrium:

$$\begin{aligned} c_{1,t}^{t*} + \frac{c_{1,t+1}^{t*}}{1+r_{t+1}^*} &= w_t^* - \tau_1 \frac{r_{t+1}^* - n}{1+r_{t+1}^*} \equiv I_{1,t}^* \\ c_{2,t}^{t*} + \frac{c_{2,t+1}^{t*}}{1+r_{t+1}^*} &= w_t^* \equiv I_{2,t}^* \end{aligned}$$

It is easy to see that the two agents will always have different budget sets, except in the case $r^* = n$ where $I_{1,t}^* = I_{2,t}^*$. Iff $n > r_{t+1}^*$, $I_{1,t}^* > I_{2,t}^*$. Because of that, note that the budget line of agent 1 is to the right of agent 2's budget line. It is parallel as they face the same interest rate r_{t+1}^*. Marginal rates of substitutions of the two agents are obviously equalized:

$$1 + r_{t+1}^* = \frac{u'(c_{1,t}^{t*})}{\beta v'(c_{1,t+1}^{t*})} = \frac{u'(c_{2,t}^{t*})}{\beta v'(c_{2,t+1}^{t*})}$$

Because utility functions are identical across agents and consumption goods are normal, we can conclude that $c_{1,t}^{t*} > c_{2,t}^{t*}$ and $c_{1,t+1}^{t*} > c_{2,t+1}^{t*}$. If $r_{t+1}^* > n$, the opposite is true.

Proof of Proposition 4

Let k_2^{aut} be the level of capital such that country 2 is at the autarkic steady state, and define the function g_2 as follows:

$$g_2(k_2^{aut}) \equiv (1+n)k_2^{aut} - s_2(f(k_2^{aut}) - f'(k_2^{aut})k_2^{aut}, f'(k_2^{aut})) = 0$$

where

$$g_2'(k_2^{aut}) = 1 + n + s_w f''(k_2^{aut})k_2^{aut} - s_r f''(k_2^{aut})$$

When the steady state is stable, $g_2'(k_2^{aut}) > 0$ as

$$\frac{dk_{2,t+1}}{dk_{2,t}}(k_2^{aut}) = \frac{-f''(k_2^{aut})k_2^{aut}s_w}{1+n-s_r f''(k_2^{aut})} < 1$$

Similarly, let k^* be the steady state world capital stock and define the function g for the world economy:

$$\begin{aligned} g(k^*;\tau_1,\rho_1,\rho_2) \;&\equiv\; (1+n)k^* - [\rho_1 s_1(f(k^*) - f'(k^*)k^*, f'(k^*), \tau_1) + \\ &+\; \rho_2 s_2(f(k^*) - f'(k^*)k^*, f'(k^*))] = 0 \qquad \text{(A.4)} \end{aligned}$$

Now suppose that $k^* = k_2^{aut}$. From equation (1.11), we know that country 1 saves less than country 2 for any k, then $g(k_2^{aut}; \tau_1, \rho_1, \rho_2) > 0$. Note that $g'(k_2^{aut}; \tau_1, \rho_1, \rho_2) = g_2'(k_2^{aut})$, and therefore for g to be zero k must fall. It follows that $k^* < k_2^{aut}$.

Similarly, it can be shown that $k_1^{autss} < k^*$. Diminishing returns to capital implies that $r_1^{autss} > r^* > r_2^{aut}$.

Proof of Proposition 5

(i) At $t = 0$, the world capital market clears if the following equation holds:

$$\begin{aligned}(1+n)k_1^* &= \rho_1 s_1(f(k_{1,0}) - f'(k_{1,0})k_{1,0}, f'(k_1^*), \tau_1) + \\ &+ \rho_2 s_2(f(k_{2,0}) - f'(k_{2,0})k_{2,0}, f'(k_1^*))\end{aligned}$$

Under Hypothesis 2, $s_{1,0}^* > s_{2,0}^*$. By Proposition 1, it follows that:

$$z_{1,0} < 0 \qquad\qquad z_{2,0} > 0$$

Because of no trade in the previous period, the countries' trade balances will only reflect the current trade in the capital market: $tb_{i,0} = -z_{i,0}$. Hence:

$$tb_{1,0} > 0 \qquad\qquad tb_{2,0} < 0$$

(ii) Let us write the balance of trade of country 1 at $t = 1$:

$$tb_{1,1}^* = -z_{1,1}^* + z_{1,0}^* \frac{1 + r_1^*}{1 + n}$$

Because $z_{1,1}^* > 0$ (Proposition 1) and we have shown that $z_{1,0}^* < 0$, then $tb_{1,1}^* < 0$.

A.2 A Cobb-Douglas Example

In this section, we derive the model for Cobb-Douglas utility and production functions:

$$U(c_{i,t}^t, c_{i,t+1}^t) = \beta \log c_{i,t}^t + (1-\beta) \log c_{i,t+1}^t \tag{A.5}$$

$$f(k_t) = k_t^\alpha \tag{A.6}$$

We can study this example in some detail as our variables of interest have a simpler dynamics with Cobb-Douglas functions.

From profit maximization, the factor prices are:

$$r_t = \alpha k_t^{\alpha-1} - \delta \tag{A.7}$$

$$w_t = (1-\alpha)k_t^{\alpha} \tag{A.8}$$

The saving functions in the two countries are:

$$s_{1,t} = (1-\beta)(w_t - \tau_1) - \beta\tau_1 \frac{1+n}{1+r_{t+1}} \tag{A.9}$$

$$s_{2,t} = (1-\beta)w_t \tag{A.10}$$

It is known that, with log-utility, savings are a constant fraction of the wage and do not depend on the rate of interest. In country 1, the young also consume a fraction of the discounted future transfer.

Overall, the impact of the pay-as-you-go system on country 1's savings is:

$$\frac{\partial s_{1,t}}{\partial \tau_1} = -(1-\beta) - \beta\frac{1+n}{1+r_{t+1}} = -1 + \beta\frac{r_{t+1}-n}{1+r_{t+1}} \tag{A.11}$$

The market clearing equation for capital is:

$$\begin{aligned} K^*_{t+1} &= L_{1,t}\left[(1-\beta)((1-\alpha)k_t^{*\alpha} - \tau_1) - \beta\tau_1\frac{1+n}{1+\alpha k_{t+1}^{*\alpha-1} - \delta}\right] + \\ &+ L_{2,t}(1-\beta)(1-\alpha)k_t^{*\alpha} \end{aligned} \tag{A.12}$$

The capital stock evolves over time as follows:

$$\begin{aligned} (1+n)k^*_{t+1} &= (1-\beta)(1-\alpha)k_t^{*\alpha} - \\ &- \rho_1\tau_1\left[(1-\beta) + \frac{\beta(1+n)}{1+\alpha k_{t+1}^{*\alpha-1} - \delta}\right] \end{aligned} \tag{A.13}$$

while the steady state capital stock satisfies:

$$(1+n)k^* = (1-\beta)(1-\alpha)k^{*\alpha} - \rho_1\tau_1\left[(1-\beta) + \frac{\beta(1+n)}{1+\alpha k^{*\alpha-1} - \delta}\right] \tag{A.14}$$

For any given $k_t > 0$, it can be verified that $k_{t+1} > 0$ exists as long as $(1-\alpha)k_t^{\alpha} - \tau_1 > 0$ (see Lemma 1) and that the higher is τ_1, the lower k_{t+1} will be given k_t. It can also be checked that the saving locus is increasing (here, $s_r = 0$):

$$\frac{dk_{t+1}}{dk_t} = \frac{(1-\beta)\alpha(1-\alpha)k_t^{\alpha-1}}{(1+n) - \frac{\rho_1\tau_1\beta(1+n)\alpha(\alpha-1)k_{t+1}^{\alpha-2}}{(1+\alpha k_{t+1}^{\alpha-1}-\delta)^2}} > 0 \tag{A.15}$$

The specific feature of this example is that the saving locus is concave as $\frac{d^2 k_{t+1}}{(dk_t)^2} < 0$. However, note that the saving locus of the economy does not start at the origin as in the case $\tau_1 = 0$. In fact, $(k_t, k_{t+1}) = (0,0)$ does not satisfy equation (A.13). When $k_t = 0$, k_{t+1} must be negative.

With $\tau_1 = 0$, it is known that there exists a globally unique steady state with Cobb Douglas utility and production function. With $\tau_1 > 0$, the number of steady states depends on how big is the tax. If the tax is small enough, then there are two steady states (one unstable and one stable). At a certain threshold for the tax, the steady state is not hyperbolic and above that we have non-existence of steady states. See [25] for a detailed discussion[2].

A.2.1 Trade and Consumption

We can now compute the excess demand. For instance, for country 1:

$$z^*_{1,t} = \rho_2 \tau_1 \left[(1-\beta) + \frac{\beta(1+n)}{1+\alpha k^{*\alpha-1}_{t+1} - \delta} \right] \tag{A.16}$$

It can be verified that $\frac{\partial z_{1,t}}{\partial k_{t+1}} > 0$.

At the golden rule k^{GR} and other stationary allocations, z_1 is respectively:

$$z_1^{GR} = \rho_2 \tau_1 \tag{A.17}$$

$$z_1^* = \rho_2 \tau_1 \left[(1-\beta) + \frac{\beta(1+n)}{1+\alpha k^{*\alpha-1} - \delta} \right] \tag{A.18}$$

Using the capital flows definition (24), we can plug equation (A.16) in and compute the balance of trade of country 1:

$$\begin{aligned} tb^*_{1,t} &= \rho_2 \tau_1 (1-\beta) \left[\frac{(\alpha k_t^{*\alpha-1} - \delta) - n}{1+n} \right] + \\ &+ \rho_2 \tau_1 \beta \left[\frac{(\alpha k^{*\alpha-1}_{t+1} - \delta) - n}{1+\alpha k^{*\alpha-1}_{t+1} - \delta} \right] \end{aligned} \tag{A.19}$$

$$\tag{A.20}$$

When both interest rates are bigger than the population growth rate, it is evident that $tb^*_{1,t} > 0$. Suppose now at a given $\bar{t}$, $k^*_{\bar{t}}$ and $k^*_{\bar{t}+1}$ are such that $r^*_{\bar{t}} > n$ and $r^*_{\bar{t}+1} < n$. The first part of the equation is positive and reflects the fact that the old in country 2 are consuming more (exports). But part two is negative as the

[2] They discuss a closed economy, but the substance of the argument does not change.

young in country 2 are now consuming less (imports). It is now clear that which of the two is bigger will also depend on β.

In the long-run, the balance of trade satisfies:

$$tb_1^* = \frac{(\alpha k^{*\alpha-1} - \delta) - n}{1+n} \rho_2 \tau_1 \left[(1-\beta) + \frac{\beta(1+n)}{1+\alpha k^{*\alpha-1} - \delta} \right] \quad \text{(A.21)}$$

The two representative agents' consumption obeys:

$$\begin{aligned}
c_{1,t}^{t*} &= \beta \left[(1-\alpha) k_t^{\alpha *} - \tau_1 \frac{(\alpha k_{t+1}^{\alpha-1*} - \delta) - n}{1+\alpha k_{t+1}^{\alpha-1*} - \delta} \right] \\
c_{1,t+1}^{t*} &= (1+\alpha k_{t+1}^{\alpha-1*} - \delta)(1-\beta) \left[(1-\alpha) k_t^{\alpha *} - \tau_1 \frac{(\alpha k_{t+1}^{\alpha-1*} - \delta) - n}{1+\alpha k_{t+1}^{\alpha-1*} - \delta} \right] \\
c_{2,t}^{t*} &= \beta(1-\alpha) k_t^{\alpha *} \\
c_{2,t+1}^{t*} &= (1+\alpha k_{t+1}^{\alpha-1*} - \delta)(1-\beta)(1-\alpha) k_t^{\alpha *}
\end{aligned}$$

As we established in Lemma 2, agents born in country 1 consumes more (less) when the world economy happens to be beyond (below) the golden rule allocation.

A.3 Labour-augmenting technological progress

The aim of this section is to show how to get the condition for country 1 to run a trade deficit in the long-run under labour-augmenting technological progress (equation (1.26)). Under this assumption, the production function is still homogeneous of degree one in the two arguments:

$$Y_{i,t} = F(K_{i,t}, A_{i,t} L_{i,t}) \qquad\qquad A_{i,t} = (1+g) A_{i,t-1}$$

where, in principle, $A_{1,0} \neq A_{2,0}$.

We define $\hat{k}_{i,t} \equiv \frac{K_{i,t}}{A_{i,t} L_{i,t}}$ as capital per effective worker. The first-order conditions of the firms now become:

$$\begin{aligned}
r_t &= f'(\hat{k}_{i,t}) - \delta \\
\hat{w}_t &= f(\hat{k}_{i,t}) - f'(\hat{k}_{i,t}) \hat{k}_{i,t}
\end{aligned}$$

where $\hat{w}_t \equiv \frac{w_{i,t}}{A_{i,t}}$.

Taxes must grow at the same rate of technological progress, for the tax to have an impact on savings in the long-run: $\tau_{1,t} = (1+g)\tau_{1,t-1}$. At each t, because $L_{1,t}\tau_{1,t} = L_{1,t-1} b_{1,t}$ must hold, $b_{1,t} = \tau_{1,t}(1+n)$. Therefore, the budget constraints become:

$$\begin{aligned}
c_{i,t}^t &= w_{i,t} - \tau_{i,t} - s_{i,t} \\
c_{i,t+1}^t &= s_{i,t}(1+r_{t+1}) + \tau_{i,t}(1+n)(1+g)
\end{aligned}$$

where $\tau_2 = 0$. The market clearing condition for capital expressed in capital per effective worker becomes:

$$\hat{k}^*_{t+1}(1+n)(1+g) = \hat{\rho}_1 \hat{s}^*_{1,t} + \hat{\rho}_2 \hat{s}^*_{2,t}$$

where $\hat{\rho}_i \equiv \frac{L_{i,t}A_{i,t}}{L_t A_t}$. Following the same steps as in section 2.3, we derive the balance of trade per effective worker for country 1:

$$\hat{tb}^*_{1,t} \equiv [\hat{s}^*_{1,t} - (1+n)(1+g)\hat{k}^*_{t+1}] - \frac{1+r^*_t}{(1+n)(1+g)}[\hat{s}^*_{1,t-1} - \hat{k}^*_t(1+n)(1+g)]$$

which at the steady state simplifies as follows:

$$\hat{tb}^*_1 = -\hat{z}^*_1 \frac{(1+n)(1+g) - (1+r^*)}{(1+n)(1+g)} \approx -\hat{z}^*_1 \frac{(n+g) - r^*}{1+n+g}$$

where $\hat{z}^*_1 \equiv \frac{Z^*_{1,t}}{A_{1,t}L_{1,t}}$.

Appendix B

B.1 Proofs

Proof of Corollary 1

Since $k_{i,t+1} = \left(\frac{r_{t+1}}{\alpha_i}\right)^{\frac{1}{\alpha_i - 1}}$, $r_{t+1} = \phi(r_t)$ and $r_t = \alpha_i {k_{i,t}}^{\alpha_i - 1}$, we can write the following accumulation equation for the capital stock per capita of country i:

$$k_{i,t+1} = \left(\frac{\phi(\alpha_i k_{i,t}^{\alpha_i - 1})}{\alpha_i}\right)^{\frac{1}{\alpha_i - 1}}$$

We now compute the first and the second derivative:

$$\begin{aligned}
\frac{\partial k_{i,t+1}}{\partial k_{i,t}} &= \left(\frac{\phi(\alpha_i k_{i,t}^{\alpha - 1})}{\alpha_i}\right)^{\frac{2-\alpha_i}{\alpha_i - 1}} \cdot \phi'(r_t) \cdot k_{i,t}^{\alpha_i - 2} \\
\frac{\partial^2 k_{i,t+1}}{(\partial k_{i,t})^2} &= \phi''(r_t) \cdot k_{i,t}^{\alpha_i - 2} \cdot \left(\frac{\phi(\alpha_i k_{i,t}^{\alpha_i - 1})}{\alpha_i}\right)^{\frac{2-\alpha_i}{\alpha_i - 1}} + \\
&+ \phi'(r_t) \cdot (\alpha_i - 2) k_{i,t}^{\alpha_i - 3} \cdot \left(\frac{\phi(\alpha_i k_{i,t}^{\alpha_i - 1})}{\alpha_i}\right)^{\frac{2-\alpha_i}{\alpha_i - 1}} + \\
&+ \phi'(r_t) \cdot k_{i,t}^{\alpha_i - 2} \cdot (2 - \alpha_i) \left(\frac{\phi(\alpha_i k_{i,t}^{\alpha_i - 1})}{\alpha_i}\right)^{\frac{3-\alpha_i}{\alpha_i - 1}} \phi'(r_t) k_{i,t}^{\alpha_i - 2}
\end{aligned}$$

The first derivative is positive since $\phi'(r_t) > 0$ by Lemma 10. The second derivative is negative since $\phi''(r_t) < 0$. Therefore, the above function is increasing and concave.

Proof of Proposition 12

We must show that $y_{1,t} > y_{2,t}$ for any $r_t \geq r^*$.

$$y_{1,t} > y_{2,t} \qquad \Leftrightarrow \qquad \left(\frac{r_t}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1 - 1}} > \left(\frac{r_t}{\alpha_2}\right)^{\frac{\alpha_2}{\alpha_2 - 1}}$$

After a few steps, the inequality can be rearranged as follows:

$$r_t > \frac{\alpha_2^{\frac{\alpha_2(1-\alpha_1)}{\alpha_2-\alpha_1}}}{\alpha_1^{\frac{\alpha_1(1-\alpha_2)}{\alpha_2-\alpha_1}}} \equiv \bar{r}$$

Our strategy is the following. We know that $r_2^{aut} > r^* > r_1^{aut}$ by Proposition 11. If we can show that $r_1^{aut} > \bar{r}$, then $r^* > \bar{r}$ and therefore the inequality holds for any $r_t \geqslant r^*$.

$$r_1^{aut} > \bar{r} \qquad \Leftrightarrow \qquad \frac{\alpha_1(1+n)}{(1-\alpha_1)(1-\beta)} > \frac{\alpha_2^{\frac{\alpha_2(1-\alpha_1)}{\alpha_2-\alpha_1}}}{\alpha_1^{\frac{\alpha_1(1-\alpha_2)}{\alpha_2-\alpha_1}}}$$

which, upon rearranging, becomes:

$$\frac{1+n}{(1-\alpha_1)(1-\beta)} > \left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{\alpha_2(1-\alpha_1)}{\alpha_2-\alpha_1}}$$

which is the condition stated in the Proposition.

Proof of Lemma 4

Given r_t^*, the autarkic interest rate of country i is derived by manipulating the domestic capital market clearing equation. We find that $r_{i,t+1}^{aut} \equiv \left[\frac{(1+n)\alpha_i}{(1-\alpha_i)(1-\beta)}\right]^{1-\alpha_i} r_t^{*\alpha_i}$.

To start with, we verify that $r_{2,t+1}^{aut} > r_{1,t+1}^{aut}$ for any $r_t^* \geqslant r^*$. Rearranging the inequality, we obtain:

$$r_t^* > \frac{1+n}{1-\beta}\left[\frac{\left(\frac{\alpha_1}{1-\alpha_1}\right)^{1-\alpha_1}}{\left(\frac{\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}}\right]^{\frac{1}{\alpha_2-\alpha_1}} \equiv \tilde{r}$$

Next, we prove that $r_1^{aut} > \tilde{r}$:

$$\frac{\alpha_1(1+n)}{(1-\alpha_1)(1-\beta)} > \frac{1+n}{1-\beta}\left[\frac{\left(\frac{\alpha_1}{1-\alpha_1}\right)^{1-\alpha_1}}{\left(\frac{\alpha_2}{1-\alpha_2}\right)^{1-\alpha_2}}\right]^{\frac{1}{\alpha_2-\alpha_1}}$$

After a few steps, we obtain $\alpha_1 < \alpha_2$. Hence, the inequality holds. Since we know that $r_2^{aut} > r^* > r_1^{aut}$, then $r^* > \tilde{r}$ and therefore the above inequality holds as long as $r_t^* \geqslant r^*$.

Finally, we prove that $r_{2,t+1}^{aut} > r_{t+1}^* > r_{1,t+1}^{aut}$. The argument is now routine. If $r_{i,t+1}^{aut} \geqslant r_{t+1}^*$ for every i, we would have that $\sum_i z_i(r_t^*, r_{t+1}^*) > 0$. The opposite is true for $r_{t+1}^* \geqslant r_{i,t+1}^{aut}$. Given r_t^*, the equilibrium interest rate at $t+1$ must lie between the corresponding autarkic steady states for the world capital market to clear.

Appendix C

C.1 The cash-in-advance constraints in the young age

Suppose that agents face cash-in-advance constraints in both periods of life. Consider the constraints of the agent born in country 1:

$$
\begin{aligned}
(\lambda_1^1(s))\ p^1(s)[c_{11}^1(s) - \omega_{11}^1(s)] &\leqslant -q^1(s)\bar{m}_1^1(s) && \text{(C.1)}\\
(\lambda_1^m(s))\ \sum_{\ell=2}^{L} q^\ell(s)\bar{m}_1^\ell(s) + q^1(s)m_1^1(s) &\leqslant q^1(s)\bar{m}_1^1(s) && \text{(C.2)}\\
(\lambda_1^\ell(s))\ p^\ell(s)c_{11}^\ell(s) + q^\ell(s)m_1^\ell(s) &\leqslant q^\ell(s)\bar{m}_1^\ell(s)\ \ \ell \neq 1 && \text{(C.3)}\\
(\lambda_1^\ell(ss'))\ p^\ell(s')[c_{21}^\ell(ss') - \omega_{21}^\ell(s')] &\leqslant q^\ell(s')m_1^\ell(s)\ \forall\ s' && \text{(C.4)}\\
\bar{m}_1^\ell \geqslant 0 \qquad m_1^\ell &\geqslant 0 \qquad \forall\ \ell && \text{(C.5)}
\end{aligned}
$$

Obviously, $\omega_{21}^\ell(s') = 0$ for $\ell \neq 1$. First, the agent would decide how much domestic endowment to consume and then would sell the rest in exchange for the domestic currency $(\bar{m}_1^1(s))$. Then, he would possibly keep part of the domestic currency to consume the domestic good when old $(m_1^1(s))$ and buy foreign currencies in order to be able to buy foreign goods $(\bar{m}_1^\ell(s))$. Part of the foreign currencies will be used to consume the foreign goods today and part will be stored in order to buy the goods tomorrow $(m_1^\ell(s))$.

The necessary conditions for a maximum are:

$$
\begin{aligned}
c_{11}^1(s) &: \; u_1^1(c_{11}(s)) = \lambda_1^1(s)p^1(s) \\
c_{11}^\ell(s) &: \; u_1^\ell(c_{11}(s)) = \lambda_1^\ell(s)p^\ell(s) \qquad \forall\ \ell \neq 1 \\
c_{21}^\ell(ss') &: \; \beta_1\rho(ss')v_1^\ell(c_{21}(ss')) = \lambda_1^\ell(ss')p^\ell(s') \qquad \forall\ s' \\
\bar{m}_1^1(s) &: \; -\lambda_1^1(s)q^1(s) + \lambda_1^m(s)q^1(s) \leqslant 0 \\
& \qquad\qquad\qquad\qquad = 0 \text{ if } \bar{m}_1^1(s) > 0 \\
\bar{m}_1^\ell(s) &: \; -\lambda_1^m(s)q^\ell(s) + \lambda_1^\ell(s)q^\ell(s) \leqslant 0 \\
& \qquad\qquad\qquad\qquad = 0 \text{ if } m_1^\ell(s) > 0 \ \ell \neq 1 \\
m_1^1(s) &: \; -\lambda_1^m(s)q^1(s) + \sum_{s'} \lambda_1^1(ss')q^1(s') \leqslant 0 \\
& \qquad\qquad\qquad\qquad = 0 \text{ if } m_1^1(s) > 0 \\
m_1^\ell(s) &: \; -\lambda_1^\ell(s)q^\ell(s) + \sum_{s'} \lambda_1^\ell(ss')q^\ell(s') \leqslant 0 \\
& \qquad\qquad\qquad\qquad = 0 \text{ if } m_1^\ell(s) > 0 \ \ell \neq 1
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_1^1(s)\{-q^1(s)\bar{m}_1^1(s) - p^1(s)[c_{11}^1(s) - \omega_{11}^1(s)]\} &= 0 \\
\lambda_1^1(s) &\geqslant 0 \\
\lambda_1^m(s)\{q^1(s)\bar{m}_1^1(s) - \sum_{\ell=2}^{L} q^\ell(s)\bar{m}_1^\ell(s) - q^1(s)m_1^1(s)\} &= 0 \\
\lambda_2^m(s) &\geqslant 0 \\
\lambda_1^\ell(s)\{q^\ell(s)\bar{m}_1^\ell(s) - p^\ell(s)c_{11}^\ell(s) - q^\ell(s)m_1^\ell(s)\} &= 0 \\
\lambda_2^\ell(s) &\geqslant 0 \ \ell \neq 1 \\
\lambda_1^\ell(ss')\{q^\ell(s')m_1^\ell(s) - p^\ell(s')[c_{21}^\ell(ss') - \omega_{21}(s')]\} &= 0 \\
\lambda_1^\ell(ss') &\geqslant 0 \ \forall\ \ell, s'
\end{aligned}
$$

By monotonicity, (C.1), (C.3) and (C.4) are binding. Then, (C.3) can be substituted into (C.2). As a consequence, (C.2) must be binding as well. Since all constraints are binding, the budget constraints can be rearranged as follows:

$$
\begin{aligned}
p(s)\cdot[c_{11}(s) - \omega_{11}(s)] &= -q(s)\cdot m_1(s) \\
p^\ell(s')[c_{21}^\ell(ss') - \omega_{21}^\ell(s')] &= q^\ell(s')m_1^\ell(s) \ \ \forall\ \ell, s' \\
m_1^\ell(s) &\geqslant 0 \qquad \ell = 1, ..., L
\end{aligned}
$$

which is the same problem stated in the main body of the chapter. It can also be checked that the necessary conditions of the above problem are identical to the necessary conditions of a problem in which the young do not face cash-in-advance constraints. Since agent 1 is not endowed with foreign goods, we must have that $m_1^\ell(s) > 0$ for $\ell \neq 1$ so that $c_{21}^\ell(ss') > 0$. But if $m_1^\ell(s) > 0$, then $\bar{m}_1^\ell(s) > 0$, which is possible only if $\bar{m}_1^1(s) > 0$. Therefore, at the solution $\lambda_1^1(s) = \lambda_1^m(s) = \lambda_1^\ell(s) \equiv \lambda_1(s)$.

C.2 Derivation of portfolios in the special case

Solving the maximization problem involves the following steps. First, combine (3.19) and (3.20):

$$\pi^1(s)c_{1h}^1(s) = \pi^2(s)e(s)c_{1h}^2(s) \tag{C.6}$$

Plug this equation into the young's budget constraint and get:

$$\pi^1(s)c_{1h}^1(s) = \frac{1}{2}[w_h(s) - m_h^1(s) - e(s)m_h^2(s)] \tag{C.7}$$

where $w_h(s)$ is wealth in state s. Take the first-order conditions for good 1 in all spots and plug them into (3.22):

$$\frac{1}{\pi^1(s)c_{1h}^1(s)} = \beta_h \sum_{s'} \frac{\rho(ss')}{\pi^1(s')c_{2h}^1(ss')} \tag{C.8}$$

Then, substitute (C.7) and (3.25) into (C.8) and obtain:

$$m_h^1(s)\left(1 + \frac{1}{2}\beta_h\right) = \frac{1}{2}\beta_h[w_h(s) - e(s)m_h^2(s)] \tag{C.9}$$

Now take (C.6) and this time rewrite the budget constraint when young getting rid of good 1. Follow the same steps as for good 1 in combining (3.23), (3.20) and (3.21) for good 2. Then, plug in the rewritten budget constraint in state s and (3.26):

$$e(s)m_h^2\left(1 + \frac{1}{2}\beta_h\right) = \frac{1}{2}\beta_h[w_h(s) - m_h^1(s)] \tag{C.10}$$

Equations (C.9) and (C.10) yield agent h's demand for the currencies:

$$\begin{aligned} m_h^1(s) &= \frac{1}{2}\frac{\beta_h}{1+\beta_h}w_h(s) \\ m_h^2(s) &= \frac{1}{2}\frac{\beta_h}{1+\beta_h}\frac{w_h(s)}{e(s)} \end{aligned}$$

C.3 Derivation of portfolios in the leading example

The derivation of portfolios with isoelastic utility functions is a bit more complicated than in the log case. First, combine (3.34), and (3.31) and (3.33) for $\ell = 1$:

$$\frac{c_{1h}^1(s)^{-\frac{1}{\varepsilon_h}}}{\pi^1(s)} = \beta_h \sum_{s'} \frac{\rho(ss')c_{1h}^1(ss')^{-\frac{1}{\varepsilon_h}}}{\pi^1(s')}$$

and rewrite it as follows:

$$\frac{\pi^1(s)^{\frac{1-\varepsilon_h}{\varepsilon_h}}}{[\pi^1(s)c_{1h}^1(s)]^{\frac{1}{\varepsilon_h}}} = \beta_h \sum_{s'} \frac{\rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}}{[\pi^1(s')c_{1h}^1(ss')]^{\frac{1}{\varepsilon_h}}}$$

Plugging $\pi^1(s')c_{2h}^1(ss') = m_h^1(s)$ for every s', we can sum up the numerators in the right hand side and elevate both sides of the equation to ε_h:

$$\frac{\pi^1(s)^{1-\varepsilon_h}}{\pi^1(s)c_{1h}^1(s)} = \beta_h^{\varepsilon_h} \frac{\left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}}{m_h^1(s)} \tag{C.11}$$

As in the log case, using the first-order conditions for the goods in state s we can write:

$$\frac{c_{1h}^1(s)^{-\frac{1}{\varepsilon_h}}}{\pi^1(s)} = \frac{c_{1h}^2(s)^{-\frac{1}{\varepsilon_h}}}{\pi^2(s)e(s)}$$

After some manipulations, the above equation can be rewritten as follows:

$$\pi^2(s)e(s)c_{1h}^2(s) = \frac{[\pi^2(s)e(s)]^{1-\varepsilon_h}}{\pi^1(s)^{1-\varepsilon_h}}\pi^1(s)c_{1h}^1(s) \tag{C.12}$$

Now, plug (C.12) into the budget constraint when young and obtain:

$$\pi^1(s)c_{1h}^1(s) = \frac{\pi^1(s)^{1-\varepsilon_h}}{\pi^1(s)^{1-\varepsilon_h} + [\pi^2(s)e(s)]^{1-\varepsilon_h}}[w_h(s) - m_h^1(s) - e(s)m_h^2(s)]$$

Plug it into (C.11) and rearrange:

$$m_h^1(s) = \frac{\beta_h{}^{\varepsilon_h}\left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}[w_h(s) - e(s)m_h^2(s)]}{\pi^1(s)^{1-\varepsilon_h} + [\pi^2(s)e(s)]^{1-\varepsilon_h} + \beta_h^{\varepsilon_h}\left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}} \tag{C.13}$$

Now, combine (3.35) with (3.33) for $\ell = 2$:

$$\lambda_h(s)e(s) = \beta_h \sum_{s'} \frac{\rho(ss')c_{2h}^2(ss')^{-\frac{1}{\varepsilon_h}}}{\pi^2(s')}$$

Multiplying and dividing each term of the right hand side by $\pi^2(s')^{\frac{1}{\varepsilon_h}}$ and then substituting $\pi^2(s')c_{2h}^2(ss') = m_h^2(s)$, we can sum the numerators on the right hand side and get the following equation:

$$\lambda_h(s) = \beta_h \frac{\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}}{m_h^2(s)^{\frac{1}{\varepsilon_h}} e(s)}$$

Because $\lambda_h(s) = \sum_{s'} \lambda_h^1(ss')$, we can write:

$$\underbrace{\beta_h \frac{\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}}{m_h^1(s)^{\frac{1}{\varepsilon_h}}}}_{\sum_{s'} \lambda_h^1(ss')} = \beta_h \frac{\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon}}}{m_h^2(s)^{\frac{1}{\varepsilon_h}} e(s)}$$

or

$$\frac{m_h^1(s)}{m_h^2(s)} = e(s)^{\varepsilon_h} \frac{\left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}}{\left[\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}} \tag{C.14}$$

The higher is the exchange rate (the price of currency 2), the higher is the relative demand for currency 1 $\left(\frac{m_h^1(s)}{m_h^2(s)}\right)$. The relative demand is also a function of expected (nominal) prices in the two countries, which reflect the purchasing power of the two currencies, in a way that depends on the agents' elasticity of substitution for the two goods.

Solving (C.13) and (C.14) simultaneously, we obtain the demand for the two currencies:

$$m_h^1(s) = \frac{\beta_h^{\varepsilon_h}\left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h}}{A_h(s)} w_h(s) \tag{C.15}$$

$$m_h^2(s) = \frac{\beta_h^{\varepsilon_h} e(s)^{-\varepsilon_h}\left[\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon}}\right]^{\varepsilon_h}}{A_h(s)} w_h(s) \tag{C.16}$$

where

$$\begin{aligned} A_h(s) \;&\equiv\; \pi^1(s)^{1-\varepsilon_h} + [\pi^2(s)e(s)]^{1-\varepsilon_h} + \beta_h^{\varepsilon_h}\left[\sum_{s'} \rho(ss')\pi^1(s')^{\frac{1-\varepsilon_h}{\varepsilon}}\right]^{\varepsilon_h} + \\ &+\; \beta_h^{\varepsilon_h} e(s)^{1-\varepsilon_h}\left[\sum_{s'} \rho(ss')\pi^2(s')^{\frac{1-\varepsilon_h}{\varepsilon_h}}\right]^{\varepsilon_h} \end{aligned}$$

C.4 Derivation of the demand functions when young

Let us recall the budget constraint of agent h born in state s:

$$\pi^1(s)c^1_{1h}(s) + \pi^2(s)e(s)c^2_{1h}(s) = w_h(s) - m^1_h(s) - e(s)m^2_h(s)$$

Firstly, we can obtain total expenditure by substituting the demand for the currencies (3.39) and (3.40):

$$\pi^1(s)c^1_{1h}(s) + \pi^2(s)e(s)c^2_{1h}(s) = \frac{p^1(s)^{1-\varepsilon_h} + [\pi^2(s)e(s)]^{1-\varepsilon_h}}{A_h(s)} w_h(s)$$

Combining the above equation with (C.12), we can derive the demand functions of the young agents.

C.5 The balance of trade in the Lucas model

We will use the same notation as above to facilitate the comparison between Lucas [44] and this chapter. Since in equilibrium $c^1_h(s) = \frac{1}{2}\omega^1(s)$ and $c^2_h(s) = \frac{1}{2}\omega^2(s)$ for every h, each agent exports (imports) half of the domestic (foreign) good in each period. Here is the balance of trade of country 1 under the normalization that $q^1(s) = 1$:

$$tb_1(s) \equiv \frac{1}{2}\pi^1(s)\omega^1(s) - \frac{1}{2}\pi^2(s)e(s)\omega^2(s) \tag{C.17}$$

The nominal price levels are:

$$\begin{aligned}\pi^1(s) &= \frac{M^1}{\omega^1(s)}\\ \pi^2(s) &= \frac{M^2}{\omega^2(s)}\end{aligned}$$

By arbitrage, the price of good 2 in the domestic currency must be equal to the the price of good 2 expressed in the numéraire currency divided by the exchange rate:

$$\pi^2(s) = \frac{\pi^1(s)\pi(s)}{e(s)} \tag{C.18}$$

where the price of good 2 expressed in the numéraire currency (the numerator at the right hand side) is given by the price of good 1 multiplied by relative prices ($\pi(s)$). Now, assume that the instantaneous utility function is isoelastic:

$$u(c^1_h, c^2_h) = \frac{{c^1_h}^{1-\frac{1}{\varepsilon}}}{1-\frac{1}{\varepsilon}} + \frac{{c^2_h}^{1-\frac{1}{\varepsilon}}}{1-\frac{1}{\varepsilon}}$$

It may be checked that relative prices are:

$$\pi(s) = \frac{u_2(\frac{1}{2}\omega^1, \frac{1}{2}\omega^2)}{u_1(\frac{1}{2}\omega^1, \frac{1}{2}\omega^2)} = \left[\frac{\omega^2(s)}{\omega^1(s)}\right]^{-\frac{1}{\varepsilon}}$$

Substituting the nominal and the relative prices into equation (C.18), we get:

$$e(s) = \frac{M^1}{M^2}\left(\frac{\omega^2(s)}{\omega^1(s)}\right)^{1-\frac{1}{\varepsilon}}$$

We can now plug all prices into (C.17) and obtain:

$$tb_1(s) = \frac{1}{2}M^1\left[1 - \left(\frac{\omega^2(s)}{\omega^1(s)}\right)^{1-\frac{1}{\varepsilon}}\right]$$

When $\varepsilon > 1$, the balance of trade is always in surplus (deficit) as long as the country's endowment is higher (lower) than the other. A positive (negative) shock in country 1 means that the domestic currency appreciates (depreciates). For $\varepsilon < 1$, the opposite occurs.

Note that trade is balanced only as long as $\omega^1(s) = \omega^2(s)$.

C.6 Sensitivity analysis

In this section of the Appendix, we report the robustness checks that we performed for the two examples. For both examples, we vary the elasticity of substitution from 0.5 to 8 and $\rho(ss)$ between 0.4 and 0.9.

Similar observations can be made for both examples. For high values of the elasticity, agents react more to changes in relative prices and therefore trade imbalances are larger. Since countries trade more, portfolios holdings are more volatile across states. As a consequence, net foreign asset positions are larger as well. Note that valuation effects increase at first and eventually decrease for high values of the elasticity. If the endowment process is i.i.d., the exchange rate is constant since agents' expectations of price levels are not conditional on the state of birth. The higher is persistence, the more different are the expectations of agents born in different states of nature. This makes the exchange rate more volatile, and therefore valuation effects increase with persistence. We have also seen that money demand depends on current wealth and prices, as well as price expectations. If the process is highly persistent, expectations vary a lot across states and portfolios are more volatile. Therefore, trade imbalances increase with persistence.

Table C.1: Varying the elasticity of substitution parameter (Example 1)

	$\theta(1)$ $\theta(2)$	$tb_i(21)$ $tb_i(12)$	e(1) e(2)	$NFA_i(1)$ $NFA_i(2)$	$r^2(21)$ $r^2(12)$	$VAL_i(21)$ $VAL_i(12)$	$\Delta NFA_i(21)$ $\Delta NFA_i(12)$
$\varepsilon = 0.5$	0.5253	0.7579	0.9120	0.0432	-0.1260	-0.6386	0.1193
	[0.5642]	[2.1130]	[0.7977]	[0.1417]	[-0.3313]	[-1.7650]	[0.3480]
	0.4856	-0.8100	1.0435	-0.0762	0.1442	0.6907	-0.1193
	[0.4467]	[-2.5776]	[1.1929]	[-0.2063]	[0.4954]	[2.2296]	[-0.3480]
$\varepsilon = 1$	0.5	0	1	0	0	0	0
	[0.5]	[0]	[1]	[0]	[0]	[0]	[0]
	0.5	0	1	0	0	0	0
	[0.5]	[0]	[1]	[0]	[0]	[0]	[0]
$\varepsilon = 1.5$	0.4917	-0.2876	1.0227	-0.0538	0.0375	0.1872	-0.1004
	[0.4778]	[-0.8668]	[1.0605]	[-0.1547]	[0.1156]	[0.5713]	[-0.2955]
	0.5060	0.2824	0.9857	0.0466	-0.0362	-0.1819	-0.1004
	[0.5200]	[0.8206]	[0.9506]	[0.1408]	[-0.1036]	[-0.5251]	[0.2955]
$\varepsilon = 2$	0.4874	-0.4429	1.0309	-0.1012	0.0515	0.2571	-0.1858
	[0.4661]	[-1.3442]	[1.0830]	[-0.2920]	[0.1604]	[0.7943]	[-0.5499]
	0.5092	0.4319	0.9804	0.0846	-0.0490	-0.2462	0.1858
	[0.5306]	[1.2476]	[0.9333]	[0.2579]	[-0.1382]	[-0.6976]	[0.5499]
$\varepsilon = 4$	0.4800	-0.7081	1.0349	-0.2321	0.0548	0.2770	-0.4311
	[0.4461]	[-2.1484]	[1.0905]	[-0.6741]	[0.1712]	[0.8748]	[-1.2736]
	0.5148	0.6894	0.9811	0.1991	-0.0520	-0.2583	0.4311
	[0.5488]	[1.9846]	[0.9311]	[0.5985]	[-0.1462]	[-0.7110]	[1.2736]
$\varepsilon = 8$	0.4753	-0.8709	1.0295	-0.3530	0.0371	0.1907	-0.6802
	[0.4336]	[-2.6140]	[1.0669]	[-1.0377]	[0.1137]	[0.6099]	[-2.0041]
	0.5183	0.8551	0.9927	0.3272	-0.0357	-0.1749	0.6802
	[0.5601]	[2.4763]	[0.9580]	[0.9664]	[-0.1021]	[-0.4722]	[2.0041]

Table C.2: Varying the persistence parameter (Example 1)

	$\theta(1)$	$tb_i(21)$	e(1)	$NFA_i(1)$	$r^2(21)$	$VAL_i(21)$	$\Delta NFA_i(21)$
	$\theta(2)$	$tb_i(12)$	e(2)	$NFA_i(2)$	$r^2(12)$	$VAL_i(12)$	$\Delta NFA_i(12)$
$\rho = 0.4$	0.4898	-0.3458	1.0004	-0.2014	-0.01224	-0.0629	-0.4087
	[0.4729]	[-1.0181]	[0.9884]	[-0.5973]	[-0.0358]	[-0.1923]	[-1.2104]
	0.5071	0.3480	1.0128	0.2073	0.0124	0.0607	0.4087
	[0.5241]	[1.0369]	[1.0251]	[0.6131]	[0.0371]	[0.1735]	[1.2104]
$\rho = 0.5$	0.4894	-0.3647	1.0060	-0.1832	0	0	-0.3647
	[0.4716]	[-1.0796]	[1.0060]	[-0.5407]	[0]	[0]	[-1.0796]
	0.5076	0.3647	1.0060	0.1815	0	0	0.3647
	[0.5254]	[1.0796]	[1.0060]	[0.5389]	[0]	[0]	[1.0796]
$\rho = 0.6$	0.4888	-0.3864	1.0127	-0.1613	0.0143	0.0726	-0.3138
	[0.4700]	[-1.1521]	[1.0269]	[-0.4734]	[0.0430]	[0.2229]	[-0.9292]
	0.5080	0.3837	0.9984	0.1525	-0.0141	-0.0699	0.3138
	[0.5269]	[1.1280]	[0.9846]	[0.4558]	[-0.0411]	[-0.1988]	[0.9292]
$\rho = 0.8$	0.4874	-0.4429	1.0309	-0.1012	0.0515	0.2571	-0.1858
	[0.4661]	[-1.3442]	[1.0830]	[-0.2920]	[0.1604]	[0.7943]	[-0.5499]
	0.5092	0.4319	0.9804	0.0846	-0.0490	-0.2462	0.1858
	[0.5306]	[1.2476]	[0.9333]	[0.2579]	[-0.1382]	[-0.6976]	[0.5499]
$\rho = 0.9$	0.4865	-0.4807	1.0437	-0.0582	0.0763	0.3774	-0.1033
	[0.4635]	[-1.4751]	[1.1219]	[-0.1656]	[0.2437]	[1.1715]	[-0.3036]
	0.5100	0.4633	0.9697	0.0451	-0.0709	-0.3600	0.1033
	[0.5330]	[1.3223]	[0.9021]	[0.1380]	[-0.1959]	[-1.0187]	[0.3036]

Table C.3: Varying the elasticity of substitution parameter (Example 2)

	$\theta(1)$ $\theta(2)$	$tb_i(21)$ $tb_i(12)$	e(1) e(2)	$NFA_i(1)$ $NFA_i(2)$	$r^2(21)$ $r^2(12)$	$VAL_i(21)$ $VAL_i(12)$	$\Delta NFA_i(21)$ $\Delta NFA_i(12)$
$\varepsilon = 0.5$	0.4153	-0.7149	1.3974	-0.0448	0.1093	0.6129	-0.1020
	[0.3973]	[-1.6507]	[1.4893]	[-0.1113]	[0.2600]	[1.4245]	[-0.2262]
	0.4451	0.6738	1.2597	0.0572	-0.0985	-0.5718	0.1020
	[0.4636]	[1.4469]	[1.1820]	[0.1149]	[-0.2063]	[-1.2207]	[0.2262]
$\varepsilon = 1$	0.5	0	1	0	0	0	0
	[0.5]	[0]	[1]	[0]	[0]	[0]	[0]
	0.5	0	1	0	0	0	0
	[0.5]	[0]	[1]	[0]	[0]	[0]	[0]
$\varepsilon = 1.5$	0.5286	0.2055	0.8990	0.0374	-0.0278	-0.1330	0.0725
	[0.5352]	[0.4525]	[0.8837]	[0.0807]	[-0.0606]	[-0.2913]	[0.1612]
	0.5177	-0.2083	0.9247	-0.0351	0.0286	0.1358	-0.0725
	[0.5111]	[-0.4662]	[0.9407]	[-0.0805]	[0.0645]	[0.3050]	[-0.1612]
$\varepsilon = 2$	0.5429	-0.3124	0.8543	0.0677	0.0391	0.1814	-0.1310
	[0.5530]	[-0.7018]	[0.8345]	[0.1450]	[0.0889]	[0.4103]	[-0.2915]
	0.5264	0.3069	0.8877	-0.0633	-0.0376	-0.1758	0.1310
	[0.5162]	[0.6745]	[0.9087]	[-0.1465]	[-0.0817]	[-0.3830]	[0.2915]
$\varepsilon = 4$	0.5650	0.4716	0.7964	0.1500	-0.0399	-0.1784	0.2932
	[0.5809]	[1.0361]	[0.7769]	[0.3222]	[-0.0864]	[-0.3841]	[0.6510]
	0.5388	-0.4802	0.8295	-0.1432	0.0416	0.1870	-0.2932
	[0.5226]	[-1.0790]	[0.8504]	[-0.3298]	[0.0946]	[0.4270]	[-0.6520]
$\varepsilon = 8$	0.5767	0.5718	0.7737	0.2280	-0.0273	-0.1181	0.4537
	[0.5961]	[1.2596]	[0.7607]	[0.4947]	[-0.0597]	[-0.2533]	[1.0063]
	0.5444	-0.5788	0.7954	-0.2257	0.0280	0.1251	-0.4537
	[0.5245]	[-1.2942]	[0.8090]	[-0.5116]	[0.0635]	[0.2879]	[-1.0063]

Table C.4: Varying the persistence parameter (Example 2)

	$\theta(1)$ $\theta(2)$	$tb_i(21)$ $tb_i(12)$	e(1) e(2)	$NFA_i(1)$ $NFA_i(2)$	$r^2(21)$ $r^2(12)$	$VAL_i(21)$ $VAL_i(12)$	$\Delta NFA_i(21)$ $\Delta NFA_i(12)$
$\rho = 0.4$	0.5412	0.2461	0.8744	0.1435	0.0094	0.0428	0.2889
	[0.5492]	[0.5483]	[0.8795]	[0.3215]	[0.0211]	[0.0946]	[0.6429]
	0.5280	-0.2450	0.8663	-0.1454	0.0093	-0.0439	-0.2889
	[0.5200]	[-0.5429]	[0.8613]	[-0.3214]	[-0.0207]	[-0.1000]	[0.6429]
$\rho = 0.5$	0.5415	0.2581	0.8706	0.1294	0	0	0.2581
	[0.5499]	[0.5737]	[0.8706]	[0.2798]	[0]	[0]	[0.5737]
	0.5277	-0.2581	0.8760	-0.1287	0	0	-0.2581
	[0.5193]	[-0.5737]	[0.8706]	[-0.2869]	[0]	[0]	[-0.5737]
$\rho = 0.6$	0.5419	0.2721	0.8661	0.1126	-0.0107	-0.0496	0.2225
	[0.5508]	[0.6028]	[0.8604]	[0.2469]	[-0.0237]	[-0.1084]	[0.4944]
	0.5273	-0.2734	0.8755	-0.1099	0.0109	0.0509	-0.2225
	[0.5184]	[-0.6095]	[0.8813]	[-0.2475]	[0.0243]	[0.1151]	[-0.4944]
$\rho = 0.8$	0.5429	-0.3124	0.8543	0.0677	0.0391	0.1814	-0.1310
	[0.5530]	[-0.7018]	[0.8345]	[0.1450]	[0.0889]	[0.4103]	[-0.2915]
	0.5264	0.3069	0.8877	-0.0633	-0.0376	-0.1758	0.1310
	[0.5162]	[0.6745]	[0.9087]	[-0.1465]	[-0.0817]	[-0.3830]	[0.2915]
$\rho = 0.9$	0.5436	0.3292	0.8465	0.0379	-0.0546	-0.2571	0.0721
	[0.5545]	[0.7203]	[0.8179]	[0.0797]	[-0.1173]	[-0.5597]	[0.1606]
	0.5258	-0.3379	0.8954	-0.0342	0.0578	0.2658	-0.0721
	[0.5149]	[-0.7633]	[0.9266]	[-0.0809]	[0.1329]	[0.6027]	[-0.1606]

Part III

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