# Genericity in the Enumeration Degrees

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The candidate confirms that the work submitted is her own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below.

“An application of 1-genericity in the $\Pi^0_2$ enumeration degrees”. Liliana Badillo and Charles M. Harris. Published in 9th Annual Conference on Theory and Applications of Models of Computation, TAMC 2012 Beijing, LNCS Volume 7287, pages 604-620, 2012. The problem and the basic modules for the main Theorem were suggested by the second author. The first author then completed the development of the construction and worked on the proof. The collaborative work on Section 4 benefited from substantial input from the second author. The paper was revised several times by both authors. This work is presented in chapters 3 and 5.

“Enumeration 1-genericity in the local enumeration degrees”. Liliana Badillo, Charles M. Harris and Mariya I. Soskova. Submitted to Mathematical Structures in Computer Science. An earlier version of Lemma 3.4 and Lemma 3.5 were developed by the first author and the final versions were prepared in collaboration with the second author. The proofs of Proposition 3.12 and Lemma 4.10 were developed jointly by the first and third author while they were visiting the Isaac Newton Institute. The proof of Theorem 5.3 and the basic technique of the construction were developed by the second author in collaboration with the first author. The section on Forcing and Enumeration 1-Genericity, Lemma 3.9, Corollary 3.10 and Proposition 4.3 were developed and written by the third author. Proposition 4.8, Theorem 4.9, Proposition 4.11 and Theorem 5.20 were developed and written by the second author. The paper was revised several times by the three authors. The section on Forcing and

Enumeration 1-genericity, Proposition 4.3, Proposition 4.8, Theorem 4.9, Proposition 4.11 and Theorem 5.20 were the work of the second and third authors and does not appear in this thesis. This work is presented in chapters 5 and 6.

"Avoiding uniformity in the $\Delta^0_2$ enumeration degrees". Liliana Badillo and Charles M. Harris. Accepted in Annals of Pure and Applied Logic. The main result presented in this paper is collaborative work of the authors, arising from a suggestion by the second author. The paper was revised several times by both authors. Section 5 of this work is presented in chapter 4.

The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

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# Abstract


In this thesis we study the notion of enumeration 1-genericity, various basic properties of it and its relationship with 1-genericity. We also study the problem of avoiding uniformity in the $\Delta^0_2$ enumeration degrees. In Chapter 2 we give a brief background survey of the notion of genericity in the context of the Turing degrees as well as in the enumeration degrees.

Chapter 3 presents a brief overview of the relationship between noncupping and genericity in the enumeration degrees. We give a result that will be useful in proving the existence of prime ideals of $\Pi^0_2$ enumeration degrees in Chapter 5, namely, we show the existence of a 1-generic enumeration degree $\mathbf{0}_e < \boldsymbol{a} < \mathbf{0}'_e$ which is noncuppable and low$_2$.

In Chapter 4 we investigate the property of incomparability relative to a class of degrees of a specific level of the Arithmetical Hierarchy. We show that for every uniform $\Delta^0_2$ class of enumeration degrees $\mathcal{C}$, there exists a high $\Delta^0_2$ enumeration degree $\boldsymbol{c}$ which is incomparable with any degree $\boldsymbol{b} \in \mathcal{C}$ such that $\boldsymbol{b} \notin \{\mathbf{0}_e, \mathbf{0}'_e\}$.

Chapter 5 is devoted to the introduction of the notions of "enumeration 1-genericity" and "symmetric enumeration 1-genericity". We study the distribution of the enumeration 1-generic degrees and show that it resembles to some extent the distribution of the class of 1-generic degrees. We also present an application of enumeration 1-genericity to show the existence of prime ideals of $\Pi^0_2$ enumeration degrees. We then look at the relationship between enumeration 1-genericity and highness.

Finally, in Chapter 6 we present two different approaches to the problem of separating the class of the enumeration 1-generic degrees from the class of 1-generic degrees. One of them is by showing the existence of a non trivial enumeration 1-generic set which is not 1-generic and the other is by proving that there exists a property that both classes do not share, namely, nonsplitting.

# Contents

# Chapter 1

# Introduction

In this Chapter we give a brief historical background of Computability Theory. The choice of results surveyed is restricted to those related to the main topic of this thesis. We thus refer the reader to [Odi92, Odi99, Soa99, Coo04, ASF06] for a thorough account of the development of Computability Theory as one of the main research areas of Mathematical Logic. In the second section we give a basic introduction to enumeration degrees along with a brief historical survey (we refer the reader to [Coo90, Sor97] for a fuller discussion of enumeration degrees). The last section is devoted to an introduction to the priority method. We illustrate the finite injury priority method with a simple example. We also discuss very briefly the tree method and the infinite injury priority method.

## 1.1 General overview and outline

In his historical speech at the International Congress of Mathematicians in Paris in 1900 the famous mathematician David Hilbert posed the Entscheidungsproblem, which asks for for an algorithm that takes as input a statement of a first-order logic and as an output gives "Yes" or "No" according to whether the statement is universally valid.

It was a young British mathematician, Alan Turing, who gave a negative solution to the Entscheidungsproblem in [Tur36]. In a later paper [Tur39], Turing introduced the notion of relativized computation using oracle Turing machines (i.e. Turing reducibility). This notion laid the foundations of what is now known as Computability Theory.

Turing's idea of relativized computation attracted other mathematicians such as Kleene and Post who developed further this concept in [Pos44, Kle52, KP54]; Kleene gave a formal definition in [Kle43]. Post defined the concept of degree of unsolvability in [Pos48]. Accordingly, two problems are of the same degree of unsolvability if each is reducible to the other. Kleene and Post introduced the notion of computable enumerable set (c.e. set) in [Kle36, Dav65]. Post showed in [Pos44] that the Halting set[1] (represented by $\mathcal{K}$) is complete, that is, every c.e. set is 1-reducible to $\mathcal{K}$.

Up until then, all computable enumerable problems known at the time were either computable or of the same degree as $\mathcal{K}$. Post asked if one could find an incomplete c.e. degree (this came to be known as Post's problem). Friedberg [Fri57] and Muchnik [Muc56] independently solved Post's problem using a technique which is now a fundamental tool in Computability Theory, namely, the priority method. In fact, the priority method had a finitary nature and was further developed to solve other problems. As the complexity of these problems grew, the need for more sophisticated techniques became evident. This gave rise to the infinite injury priority method invented independently by Shoenfield [Sho61] and Sacks [Sac63b]. Yates [Yat66a, Yat66b] and Lachlan [Lac66] extended the infinite injury priority method to continue solving complex problems but then this technique became quite complicated and difficult to follow. The use of priority trees was introduced in order to

[1]The Halting problem asks for an algorithm that decides whether an algorithm halts with a given input.

give a better understanding of the infinite injury priority method. Lachlan introduced the tree method in [Lac75] and further developed it in [Lac79]; Harrington gave useful remarks for the use of trees in [Har82].

Enumeration reducibility was introduced by Friedberg and Rogers in [FR59] as the notion of relative enumerability between sets. Intuitively a set $A$ is enumeration reducible to a set $B$, if there is an effective method (i.e. an algorithm) which given any enumeration of $B$, outputs an enumeration of $A$. We notice that a set $A$ is Turing reducible to a set $B$, if we can compute the characteristic function of $A$ using the characteristic function of $B$ as the input of an oracle Turing machine, whereas in enumeration reducibility we are only given an enumeration of $B$ as an input.

Myhill [Myh61] defined a computable embedding of the Turing degrees into the enumeration degrees, thus proving that the enumeration degrees are an extension of the Turing degrees. The enumeration degrees form a degree structure similar to that of the Turing degrees. Indeed, we say that two sets, $A$ and $B$, are of the same enumeration degree if $A$ is enumeration reducible to $B$ and vice versa.

The notion of genericity was invented by Cohen [Coh63] to prove the independence of the Axiom of Choice and the Continuum Hypothesis from Zermelo Fraenkel set theory. Genericity turned out to be a useful concept in Computability Theory to prove results related to the existence of Turing degrees with certain properties. We can transfer these results to the enumeration degrees via the computable embedding mentioned above, but we still need a notion of genericity which is appropriate to the definition of enumeration reducibility (in which only positive information can be used).

In this thesis we study the notions of enumeration 1-genericity and symmetric enumeration 1-genericity. We investigate various basic properties of enumeration 1-genericity in the enumeration degrees along with its relationship with the usual notion of 1-genericity. We also present an application of enumeration 1-genericity to show the existence of prime ideals of $\Pi^0_2$ enumeration degrees. As a sideline to the main focus of this thesis, we study the problem of avoiding uniformity in the $\Delta^0_2$ enumeration degrees.

In Chapter 2 we give a brief background survey of the notion of genericity in the context of the Turing degrees as well as in the enumeration degrees. This brief introduction to genericity will serve as a basis for the problems that will be investigated in subsequent chapters.

Chapter 3 presents a brief overview of the relationship between noncupping and genericity in the enumeration degrees. We give a result that will be useful in proving the existence of prime ideals of $\Pi^0_2$ enumeration degrees in Chapter 5, namely, we show the existence of a 1-generic enumeration degree $\mathbf{0}_{\mathrm{e}} < \boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ which is noncuppable and low$_2$.

In Chapter 4 we investigate the property of incomparability relative to a class of degrees of a specific level of the Arithmetical Hierarchy. We show that for every uniform $\Delta^0_2$ class of enumeration degrees $\mathcal{C}$, there exists a high $\Delta^0_2$ enumeration degree $\boldsymbol{c}$ which is incomparable with any degree $\boldsymbol{b} \in \mathcal{C}$ such that $\boldsymbol{b} \notin \{\mathbf{0}_{\mathrm{e}}, \mathbf{0}'_{\mathrm{e}}\}$. As a corollary, we get that such $\boldsymbol{c}$ caps with both a high and a low nonzero $\Delta^0_2$ enumeration degree.

Chapter 5 is devoted to the introduction of the notions of "enumeration 1-genericity" and "symmetric enumeration 1-genericity". We study the distribution of the enumeration 1-generic degrees and show that it resembles to some extent the distribution of the class of 1-generic degrees. We also present an application of enumeration 1-genericity to show the existence of prime ideals of $\Pi^0_2$ enumeration degrees. Finally, we look at the relationship between enumeration 1-genericity and highness.

In Chapter 6 we present two different approaches to the problem of separating the class of the enumeration 1-generic degrees from the class of 1-generic degrees. One of them is by showing the existence of a non trivial enumeration 1-generic set which is not 1-generic and the other is by proving that there exists a property that both classes do not share, namely, nonsplitting.

## 1.2 Enumeration Degrees

The notion of enumeration reducibility was introduced by Friedberg and Rogers in [FR59, Rog67] as a relation between sets of natural numbers. Intuitively, we say a set $A \subseteq \omega$ is enumeration reducible to a set $B \subseteq \omega$, if given an enumeration of $B$, we can effectively get an enumeration of $A$. More formally, we can define this notion by:

**Definition 1.2.1.** *A set $A$ is* enumeration reducible *to a set $B$ ($A \leq_{\mathrm{e}} B$) if there is a c.e. set $W$ such that*

$$x \in A \Leftrightarrow (\exists u)[\langle x, D_u \rangle \in W \ \& \ D_u \subseteq B],$$

*where $D_u$ is the finite set with canonical index $u$.*

Every c.e. set $W$ can be seen as corresponding to an "enumeration operator" $\Phi$ (*e-operator*) defined by $\Phi = \{\langle x, D_u \rangle \mid \exists s[\langle x, D_u \rangle \in W_s]\}$. Given an effective listing $\{W_e\}_{e \in \omega}$ of the c.e. sets we get a corresponding listing $\{\Phi_e\}_{e \in \omega}$ of the e-operators. Accordingly, we can also define enumeration reducibility as follows.

**Definition 1.2.2.** *A set $A$ is* enumeration reducible *to a set $B$ if and only if there exists an e-operator $\Phi$ such that:*

$$A = \Phi^B = \{x \mid (\exists u)[\langle x, D_u \rangle \in W \ \& \ D_u \subseteq B]\}.$$

Enumeration reducibility ($\leq_e$) is reflexive and transitive, i.e. is a pre-order on the powerset of $\omega$, namely $\mathcal{P}(\omega)$. This pre-order generates an equivalence relation ($\equiv_e$) defined by:

$$A \equiv_e B \Leftrightarrow A \leq_e B \ \&\ B \leq_e A.$$

Let $\boldsymbol{a} = deg_e(A)$ denote the enumeration degree of $A$ which is defined by $deg_e(A) = \{X \mid X \equiv_e A\}$. The degree structure $\langle \mathcal{D}_e, \leq \rangle$ is the class consisting of the enumeration degrees. If $A \in \boldsymbol{a}$ and $B \in \boldsymbol{b}$, then the relation $\leq$ is defined by

$$\boldsymbol{a} \leq \boldsymbol{b} \Leftrightarrow A \leq_e B.$$

The structure $\mathcal{D}_e$ is an upper semilattice with least element $\mathbf{0}_e$ and join operation. Indeed, given any set $B$ and a c.e. set $A$, if $B \leq_e A$ then, $B$ is c.e. On the other hand, given any c.e. set $A$ and any set $B$, we have $A \leq_e B$ via the e-operator $\Phi = \{\langle x, \emptyset \rangle \mid x \in A\}$. Moreover, we define the join operation as $deg_e(A) \vee deg_e(B) = deg_e(A \oplus B)$ where $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

In [Cas71] Case constructed an "exact pair" (see Definition below) for a countable ideal[2] $\mathcal{I}$ of $\mathcal{D}_e$. Case obtained as a corollary the existence of two degrees which do not have a greatest lower bound and hence the enumeration degrees do not form a lattice.

**Definition 1.2.3.** *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two enumeration degrees. We say $\boldsymbol{a}$ and $\boldsymbol{b}$ form an* exact pair *for a set of degrees $\mathcal{C}$ if,*

*i. For all $\boldsymbol{c} \in \mathcal{C}$ we have that $\boldsymbol{c} \leq \boldsymbol{a}$ and $\boldsymbol{c} \leq \boldsymbol{b}$, and*

*ii. For every degree $\boldsymbol{x}$ such that $\boldsymbol{x} \leq \boldsymbol{a}$ and $\boldsymbol{x} \leq \boldsymbol{b}$ then there exists $\boldsymbol{c} \in \mathcal{C}$ such that $\boldsymbol{x} \leq \boldsymbol{c}$.*

[2]If $\mathcal{D}$ is an upper semilattice then, the set $\mathcal{I} \subseteq \mathcal{D}$ is an ideal if $\mathcal{I}$ is closed under joins, and if $\boldsymbol{x} \in \mathcal{I}$ and $\boldsymbol{y} \leq \boldsymbol{x}$ then $\boldsymbol{y} \in \mathcal{I}$.

Enumeration reducibility can be seen as a reducibility between partial functions. If $\varphi$ is a partial function, then we define the graph of $\varphi$ by $graph(\varphi) = \{\langle x, y\rangle \mid \varphi(x) \downarrow = y\}$. Then for partial functions $\varphi$ and $\psi$,

$$\varphi \leq_{\mathrm{e}} \psi \Leftrightarrow graph(\varphi) \leq_{\mathrm{e}} graph(\psi).$$

The reducibility explained above coincides with the reducibility between partial functions that Myhill introduced in [Myh61]. Accordingly, $\mathcal{P}$ denotes the set of all partial degrees, where $\boldsymbol{\varphi} = deg_{\mathrm{e}}(\varphi) = \{\psi \mid \psi \equiv_{\mathrm{e}} \varphi\}$ with least degree $\mathbf{0}$ consisting of all partial computable functions. Let $\boldsymbol{a}$ be any enumeration degree and take $A \in \boldsymbol{a}$. We then define $graph(\varphi) = \{\langle x, 0\rangle \mid x \in A\}$. Clearly $A \equiv_{\mathrm{e}} graph(\varphi)$. Hence, $\mathcal{P}$ is isomorphic to $\mathcal{D}_{\mathrm{e}}$ since any enumeration degree contains the graph of a partial function.

Another interesting class of enumeration degrees is that of the "the total enumeration degrees", which is obtained by considering the enumeration degrees that contain the graph of a total function.

**Definition 1.2.4.** *An enumeration degree* $\boldsymbol{a}$ *is called* total *if there is a total function* $f$ *such that* $graph(f) \in \boldsymbol{a}$.

The necessary conditions for an enumeration degree to be total are the following[3].

**Lemma 1.2.5.** *For any enumeration degree* $\boldsymbol{a}$ *the following are equivalent:*

*i.* $\boldsymbol{a} = deg_{\mathrm{e}}(A)$ *is total;*

*ii.* $\overline{A} \leq_{\mathrm{e}} A$*;*

*iii.* $\chi_A \equiv_{\mathrm{e}} A \oplus \overline{A} \equiv_{\mathrm{e}} A$.

[3] We note that $\chi_A$ denotes the graph of the characteristic function of A and $\overline{A}$ denotes the set $A \setminus \omega$.

From the definition of enumeration reducibility $A \leq_{\mathrm{e}} B$ implies that $A$ is c.e. in $B$ but the reverse implication does not hold in general. For example, consider the halting set $\mathcal{K}$ (i.e. $\mathcal{K} = \{x \mid x \in W_x\}$). Then[4] $\overline{\mathcal{K}}$ is c.e. in $\mathcal{K}$ but it is not the case that $\overline{\mathcal{K}} \leq_{\mathrm{e}} \mathcal{K}$ (since $\overline{\mathcal{K}}$ is not c.e.). On the other hand, whenever $B$ is a total function we have the following.

**Lemma 1.2.6.** *$A$ is c.e. in $B$ if and only if $A \leq_{\mathrm{e}} B \oplus \overline{B}$.*

Myhill [Myh61] defined a computable embedding $\iota : \mathcal{D} \to \mathcal{D}_{\mathrm{e}}$ (i.e. an embedding of the Turing degrees into the enumeration degrees) by

$$\iota(deg_{\mathrm{T}}(A)) = deg_{\mathrm{e}}(A \oplus \overline{A}),$$

where $deg_{\mathrm{T}}(A)$ stands for the Turing degree of the characteristic function $\chi_A$ of $A$. In the Turing degrees if $A \leq_T B$, then we compute the characteristic function $\chi_A$ of $A$ using the characteristic function $\chi_B$ of $B$, i.e. this is a reduction between total functions.

**Lemma 1.2.7.** *For any two sets $A$ and $B$,*

$$A \leq_T B \Leftrightarrow \chi_A \leq_{\mathrm{e}} \chi_B.$$

Thus the image of the Turing degrees under $\iota$ is the class of the total enumeration degrees. However, in a more general sense we have that neither $A \leq_T B$ implies $A \leq_{\mathrm{e}} B$ nor the reverse implication. Moreover, consider the following result.

**Lemma 1.2.8.** *If $A$ is a c.e. set then,*

$$\chi_A \equiv_{\mathrm{e}} A \oplus \overline{A} \equiv_{\mathrm{e}} \overline{A}.$$

[4]We say $A$ is c.e. in $B$ if $A = W_e^B$ for some $e \in \omega$ (where $W_e^B$ denotes the domain of the $e$-th Turing machine using the characteristic function $\chi_B$ of $B$ as an oracle).

Therefore, the c.e. Turing degrees are isomorphic to the $\Pi^0_1$ enumeration degrees under $\iota$. Furthermore, $\iota$ preserves least upper bounds and least element. Notice that $\iota$ also preserves the greatest element since the enumeration degree of $\mathcal{K}$ is $deg_{\mathrm{e}}(\overline{\mathcal{K}})$ and so is the greatest $\Pi^0_1$ degree.

Medvedev showed in [Med55] that there exist enumeration degrees which are not total[5] by constructing a set $A$ which is not itself total and has no total predecessors other than $\mathbf{0}_{\mathrm{e}}$ (we say that such a set $A$ is "quasiminimal").

McEvoy [McE85] and Cooper [Coo84] defined a jump operation on the enumeration degrees in the following way.

**Definition 1.2.9.** *For any set $A$ let $K_A = \{x \mid x \in \Phi^A_x\}$. The* enumeration jump *$J_{\mathrm{e}}(A)$ (e-jump) of $A$ is defined by*

$$J_{\mathrm{e}}(A) = K_A \oplus \overline{K_A}.$$

Similarly, McEvoy and Cooper defined the jump of an e-degree $\boldsymbol{a}$ as $\boldsymbol{a}' = deg_{\mathrm{e}}(K_A \oplus \overline{K_A}) = deg_{\mathrm{e}}(J_{\mathrm{e}}(A))$. The e-jump can be iterated in the usual way to obtain the $n$-th jump $\boldsymbol{a}^{(n)}$ of $\boldsymbol{a}$. The enumeration jump has the same properties as the Turing jump (see below).

**Lemma 1.2.10.** *Let $A$ and $B$ be any two sets, then the enumeration jump of $A$ and $B$ have the following properties:*

*i. If $A \leq_{\mathrm{e}} B$ then $J_{\mathrm{e}}(A) \leq_{\mathrm{e}} J_{\mathrm{e}}(B)$, and*

*ii. $A <_{\mathrm{e}} J_{\mathrm{e}}(A)$.*

Moreover, McEvoy noted in [McE85] other interesting properties of the e-jump.

**Proposition 1.2.11** ([McE85]). *For any two sets $A$ and $B$, the following are equivalent*

[5] An enumeration degree $\boldsymbol{a}$ is total if $\boldsymbol{a}$ contains the graph of a total function.

*i.* $A \leq_{\mathrm{e}} B$,

*ii.* $A \leq_1 K_B$[6],

*iii.* $K_A \leq_1 K_B$.

**Lemma 1.2.12** ([McE85])**.** *For any two sets $A$ and $B$,*

$$A \in \Sigma^B_{n+1} \Leftrightarrow A \leq_{\mathrm{e}} J^{(n)}_{\mathrm{e}}(\chi_B).$$

**Corollary 1.2.13** ([McE85])**.** *For any set $A \subseteq \omega$,*

$$deg_{\mathrm{e}}(A) \leq_{\mathrm{e}} \mathbf{0}^{(n)}_{\mathrm{e}} \Leftrightarrow A \in \Sigma^0_{n+1}.$$

Under the embedding $\iota$ we have that $\iota(deg_{\mathrm{T}}(A)') = \iota(deg_{\mathrm{T}}(A))'$, that is, the Turing jump agrees with the e-jump.

Cooper showed in [Coo84] that $\mathcal{D}_{\mathrm{e}}[\leq \mathbf{0}'_{\mathrm{e}}]$ coincides with the set of all $\Sigma^0_2$ enumeration degrees. Indeed, given any $\Sigma^0_2$ enumeration degree $\boldsymbol{a}$ we can define a $\Sigma^0_2$ approximation (see Definition 1.2.14) to $A \in \boldsymbol{a}$ in the following way.

**Definition 1.2.14.** *A $\Sigma^0_2$* approximation *to a set $A$ is a computable sequence $\{A_s\}_{s\in\omega}$ of finite sets such that*

$$x \in A \Leftrightarrow \exists s \forall t \geq s[x \in A_t].$$

Then it follows that $x \in A \Leftrightarrow \lim_s A_s(x) = 1$ (whereas in a $\Delta^0_2$ approximation $\{A_s\}_{s\in\omega}$ to a set $A$ we have, $x \in A \Leftrightarrow \lim_s A_s(x)$ exists). Recall that the e-jump

[6] A set $A$ is one-one reducible to a set $B$ ($A \leq_1 B$) if there exists a one-one computable function $f$ such that

$$\forall x[x \in A \Leftrightarrow f(x) \in B].$$

agrees with the Turing jump and hence $\mathbf{0}'_\mathrm{e}$ is the e-degree of $\overline{\mathcal{K}}$. We then define a $\Sigma^0_2$ approximation to $A$ as

$$x \in A \Leftrightarrow \exists s \forall t \geq s[x \in \Phi^{\overline{\mathcal{K}}_t}_{e,t}].$$

Every total e-degree $\boldsymbol{a}$ below $\mathbf{0}'_\mathrm{e}$ is $\Delta^0_2$. On the other hand, there exist $\Delta^0_2$ enumeration degrees which are not total. Take as an example the construction of the existence of a quasiminimal set $A$ given in [Med55], in which $A$ is $\Delta^0_2$ but not total. In fact, Cooper and Copestake [CC88] showed that there exist properly $\Sigma^0_2$ e-degrees, that is, e-degrees which do not contain $\Delta^0_2$ sets. We illustrate the local structure of the e-degrees (consisting of the $\Sigma^0_2$ e-degrees) in Figure 1.1. Sorbi gives in [Sor97] a thorough discussion on the local structure of the enumeration degrees.

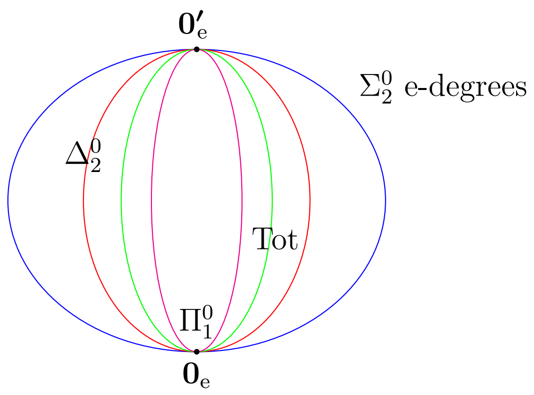


Figure 1.1: The local structure of the enumeration degrees.

Gutteridge showed in [Gut71] that there are no minimal[7] e-degrees and hence the enumeration degrees are downwards dense. Cooper [Coo84] extended this result and proved that the e-degrees below $\mathbf{0}'_\mathrm{e}$ are dense and form an ideal. Afterwards, in [Coo90] Cooper showed that the e-degrees below $\mathbf{0}^{(6)}_\mathrm{e}$ are not dense and Slaman and Woodin [SW97] proved that the e-degrees below $\mathbf{0}''_\mathrm{e}$ are not dense.

[7] A nonzero e-degree $\boldsymbol{a}$ is *minimal* if no nonzero e-degree is strictly below $\boldsymbol{a}$.

Cooper asked if the c.e. Turing degrees and $\mathcal{D}_{\mathrm{e}}[\leq \mathbf{0}'_{\mathrm{e}}]$ are elementary equivalent. Ahmad gave a negative answer in [Ahm91] by proving that the diamond lattice embeds into the $\Sigma^0_1$ enumeration degrees and in [AL98] where it is shown that there exists a nonsplittable[8] enumeration degree. In fact, these two results contrast with the Non-Diamond theorem by Lachlan [Lac66] and the Splitting theorem for the c.e. Turing degrees by Sacks [Sac63a].

As in the Turing degrees, an e-degree $\boldsymbol{a}$ is **low** if $\boldsymbol{a}' = \mathbf{0}'_{\mathrm{e}}$ and a set $A$ is low if $J_{\mathrm{e}}(A) \in \mathbf{0}'_{\mathrm{e}}$. Similarly, an e-degree is **high** if $\boldsymbol{a}' = \mathbf{0}''_{\mathrm{e}}$. More generally, for each $n > 0$, we define an e-degree $\boldsymbol{a}$ to be low$_n$ (high$_n$) if $\boldsymbol{a}^{(n)} = \mathbf{0}^{(n)}_{\mathrm{e}}$ ($\boldsymbol{a}^{(n)} = \mathbf{0}^{(n+1)}_{\mathrm{e}}$) and if $A \in \boldsymbol{a}$ then $A$ is low$_n$ (high$_n$). In fact, McEvoy gives in [MC85] a useful characterisation of the low e-degrees.

**Lemma 1.2.15** ([MC85])**.** *The following are equivalent, for any set A:*

*i. A is low.*

*ii. There is a $\Delta^0_2$ approximation $\{A_s\}_{s\in\omega}$ to A, in which, for every $e \in \omega$, there is a $\Delta^0_2$ approximation $\{\Phi^{A_s}_{e,s}\}_{s\in\omega}$ to $\Phi^A_e$.*

*iii. There is a $\Sigma^0_2$ approximation $\{A_s\}_{s\in\omega}$ to A, in which, for every $e \in \omega$, $lim_s \Phi^{A_s}_{e,s}(e)$ exists.*

### Approximations to sets below $\mathbf{0}'_{\mathrm{e}}$

As we have seen before, the sets below $\mathbf{0}'_{\mathrm{e}}$ are those which belong to the $\Sigma^0_2$ class and can be approximated by a $\Sigma^0_2$ approximation. Cooper mentions in [Coo84] that every $\Sigma^0_2$ set $A$ has a $\Sigma^0_2$ approximation $\{A_s\}_{s\in\omega}$ with infinitely many "thin" stages, namely, stages $s$ of the form $A_s \subseteq A$.

[8] We say an enumeration degree $\boldsymbol{a}$ is splittable if there exist incomparable degrees $\boldsymbol{a_0}$ and $\boldsymbol{a_1}$ such that $\boldsymbol{a} = \boldsymbol{a_0} \cup \boldsymbol{a_1}$. Otherwise we say $\boldsymbol{a}$ is nonsplittable.

**Lemma 1.2.16** ([Joc68],[Coo84]). *If $A \in \Sigma_2^0$ then, $A$ has a $\Sigma_2^0$ approximation $\{A_s\}_{s\in\omega}$ with infinitely many thin stages.*

*Proof.* Suppose that $A \in \Sigma_2^0$ and so $A$ is c.e. in $\mathcal{K}$. Thus $A = W_e^{\mathcal{K}}$ for some $e \in \omega$ (namely, $A$ is the domain of the e-th Turing machine using $\mathcal{K}$ as an oracle). Let $\{W_{e,s}\}_{s\in\omega}$ and $\{\mathcal{K}_s\}_{s\in\omega}$ be c.e. approximations to $W_e$ and $\mathcal{K}$.

First, we define a computable approximation $\{A_s\}_{s\in\omega}$ to $A$ as follows. We define a computable function $f$ by

$$f(s) = \begin{cases} \mu y[\mathcal{K}_s(y) \neq \mathcal{K}_{s-1}(y)] & \text{if } s > 0, \\ 0 & \text{otherwise,} \end{cases}$$

Then set $A_s = \{x \mid x \in W_{e,s}^{\mathcal{K}_s \upharpoonright f(s)}\}$. For every $x \in A$, we define $u(x,s) =$ the maximum element used of $\mathcal{K}_s$ by $x$ at stage $s$. It follows that $u(x,s) < f(s)$.

Now, we prove that $\{A_s\}_{s\in\omega}$ is a $\Sigma_2^0$ approximation to $A$. For any $x \in A$, choose a stage $s_x$ large enough such that for all $s \geq s_x$, $u(x,s_x) = u(x,s) = u(x)$. Moreover, choose a stage $s' \geq s_x$ such that for all $t \geq s'$, $f(t) \geq f(s')$. We then define,

$$x \in A \Leftrightarrow \exists s' \forall t \geq s' [x \in W_{e,t}^{\mathcal{K}_t \upharpoonright f(t)}].$$

Finally, we prove that $\{A_s\}_{s\in\omega}$ has indeed infinitely many thin stages. For any $n$, choose a stage $s$ large enough such that for all $t \geq s$ we have

$$\forall y < n \, [\mathcal{K} \upharpoonright n(y) = \mathcal{K}_t \upharpoonright n(y)].$$

Fix such a stage $s$, then it follows that

$$W_{e,s}^{\mathcal{K}_s} \subseteq W_e^{\mathcal{K}}.$$

□

Lemma 1.2.16 was originally proved by Jockusch in [Joc68]. Lachlan and Shore [LS92] and Harris [Har10] generalized it to the notion of "good approximation".

**Definition 1.2.17.** *A uniformly computable enumeration of finite sets $\{A_s\}_{s\in\omega}$ is said to be a* good approximation *to the set $A$ if*

*i.* $\forall s\,(\exists t \geq s)[\,A_t \subseteq A\,]$, *and*

*ii.* $\forall x\,[\,x \in A \quad\Leftrightarrow\quad \exists t\,(\forall s \geq t)[\,A_s \subseteq A \Rightarrow x \in A_s\,]\,]$.

In this case we say that $A$ is "good approximable". If we replace *ii.* by the condition $\forall x\,[\,x \in A \quad\Leftrightarrow\quad \exists t(\forall s \geq t)[\,x \in A_s\,]]$ then $\{A_s\}_{s\in\omega}$ is said to be a "good $\Sigma^0_2$ approximation". Stages $s$ of the form $A_s \subseteq A$ are called *good stages* (similar to thin stages in Lemma 1.2.16).

**Definition 1.2.18.** *We define an enumeration degree $\boldsymbol{a}$ to be* good *if $\boldsymbol{a}$ contains a good approximable set. Otherwise we say that $\boldsymbol{a}$ is* bad*.*

Lachlan and Shore proved in [LS92] that good approximations have the following interesting property.

**Lemma 1.2.19** ([LS92])**.** *Let $G(A) = \{s \mid A_s \subset A\}$ i.e. the set of all good stages. If $\{A_s\}_{s\in\omega}$ is a good approximation to $A$, and $\Phi$ an e-operator, then $lim_{s\in G(A)}\, \Phi^A_s = \Phi^A$.*

*Proof.* Consider $x \in \Phi^A$, then there exists an axiom $\langle x, D\rangle \in \Phi$ such that $D \subseteq A$. Set $m = \max\{D\} + 1$. Choose a stage $s \in G(A)$ such that for all good stages $t \geq s$, $A \restriction m \subseteq A_t \subseteq A$ and so $x \in \Phi^{A_t}_t \subseteq \Phi^{A_t} \subseteq \Phi^A$. On the other hand, if $x \notin \Phi^A$, then for no $s \in G(A)$ do we have $x \in \Phi^{A_s}_s$. □

We can define good approximations to sets that are members of the classes $\Sigma^0_2$, $\Delta^0_2$ and $\Pi^0_1$. However, there are some $\Pi^0_2$ sets that have no such good approximation [LS92, Har12].

Now, a natural question to ask is whether we can define a $\Sigma^0_2$ approximation to a set $B \leq_{\mathrm{e}} A$ (with $A \in \Sigma^0_2$). Let $B = \Phi^A_e$ for some $e \in \omega$. If we simply let $B_s = \Phi^{A_s}_{e,s}$ then, it might be the case that $y \in \Phi^{A_s}_{e,s}$ for cofinitely many stages $s$ but yet $y \notin \Phi^A_e$. McEvoy and Cooper presented in [McE85] a way of defining a $\Sigma^0_2$ approximation $\{\Phi^{A_s}_{e,s}\}_{s\in\omega}$ to $\Phi^A_e$.

**Proposition 1.2.20** ([McE85]). *If $A \leq_{\mathrm{e}} J_{\mathrm{e}}(\emptyset)$ then, there is a definition of $y \in \Phi^{A_s}_{e,s}$ such that for any $e \in \omega$, $\Phi^{A_s}_{e,s}$ is a $\Sigma^0_2$ approximation to $\Phi^A_e$.*

*Proof.* Let $A$ be a $\Sigma^0_2$ set with a $\Sigma^0_2$ approximation $\{A_s\}_{s\in\omega}$ and for some $e \in \omega$, we have an e-operator $\Phi_e$ with the usual c.e. approximation $\{\Phi_{e,s}\}_{s\in\omega}$.
First, at any stage $s$ for some $y$ such that $\langle y, D_u\rangle \in \Phi_{e,s}$ and $D_u \subset A_s$, we define the "use" function $u^A(e, y, s)$ so that $u^A$ "picks" the finite set $D_u$ that has been a subset of the current approximation $A_s$ to $A$ for the longest. Accordingly, we need to measure how long a given finite set has been a subset of the current approximation $A_s$ to $A$. Thus, we define

$$\lambda(u, s) = \mu t \leq s[\forall k\ (t \leq k \leq s) D_u \subset A].$$

We then define the use function $u^A(e, y, s)$ by

$$u^A(e, y, s) = \mu u \leq s[(\forall v)[(\langle y, D_v\rangle \in \Phi_{e,s}) \Rightarrow [\lambda(v, s) \geq \lambda(u, s)]]].$$

Finally, we define a $\Sigma^0_2$ approximation to $\Phi^A_e$.

<u>*Stage $s = 0$.*</u> Set $\Phi^{A_0}_{e,0} = \emptyset$.

<u>*Stage $s + 1$.*</u> Set $\Phi^{A_s}_{e,s} = \{y \mid u^A(e, y, s + 1) = u^A(e, y, s)\}$. □

We note that in Proposition 1.2.20, if $\{A_s\}_{s\in\omega}$ is a good $\Sigma^0_2$ approximation to $A$, then $\{\Phi^{A_s}_{e,s}\}_{s\in\omega}$ is a good approximation to $\Phi^A_e$ for any $e$.

# 1.3 Priority Method

## 1.3.1 Finite injury priority method

Occasionally, in Computability Theory we are required to prove that a set with certain properties exists. We then prove the existence of such set by constructing it by stages. There are several techniques for constructing sets according to the set we want to approximate. Throughout this thesis, when approximating sets, we will be using techniques such as the finite injury priority method, the tree method and the infinite injury priority method (we refer the reader to [Soa99, Coo04] for a thorough introduction to these techniques). In this section, we will illustrate the finite injury priority method by a very simple example and discuss the modifications needed to adapt this example to the tree method. Finally, we briefly discuss the difference between the finite injury priority method and the infinite injury method.

The simplest instance of the priority method in the Turing degrees is the finite injury priority method. The first result using this technique was by Friedberg [Fri57] and Muchnik [Muc56] in which two c.e. sets, $A$ and $B$, are constructed in such a way that they are incomparable ($A \not\leq_T B$ and $B \not\leq_T A$).

The general idea of the finite injury priority method is explained as follows. Suppose that we want to construct a set with a certain property and we express it as requirement $\mathcal{R}$. We break this requirement $\mathcal{R}$ into a list of smaller requirements, say $R_e$ (for all $e \in \omega$), and define a priority ordering as

$$R_0 < R_1 < R_2 < \cdots .$$

Note that requirement $R_0$ has higher priority than requirement $R_1$, requirement $R_1$ has higher priority than $R_2$ and etc. There are several ways (strategies) to meet each requirement $R_e$. The way each requirement will be met (satisfied) depends on the information available to us at stage $s$ during the construction. Consider any requirement $R_e$. In order to satisfy it we would have to wait until higher priority requirements $R_i$ ($i < e$) are satisfied. But this could cause the construction to wait forever for this to happen. Hence $R_e$ will have to act (receive attention), and take the risk of having its work destroyed (injured) at a later stage of the construction by higher priority requirements (whenever they take some action in order to be satisfied). Actually, the construction can be defined in such a way that after $R_e$ is injured finitely often, $R_e$ still has a chance to be satisfied at, say, stage $s$. Requirement $R_e$ might act depending on the information available to the construction at stage $s$ and remain satisfied for all stages $t \geq s$. Then we argue by induction that after a sufficiently large stage $s$ of the construction, such that all $R_i$ have finished acting, $R_e$ still has an opportunity to be satisfied. To illustrate this process in the context of enumeration degrees we give the following example.

**Theorem 1.3.1.** *There exists an enumeration degree* $\boldsymbol{a}$ *such that* $\boldsymbol{a}$ *is nonzero (not c.e.) and low (i.e.* $\boldsymbol{a}' = \boldsymbol{0}'_{\mathrm{e}}$*).*

*Proof.* We construct a set $A$ such that, for all $e \in \omega$, the following requirements are satisfied

$$
\begin{aligned}
N_e \quad &: \quad A \neq W_e, \\
L_e \quad &: \quad \lim_s \Phi_{e,s}^{A_s}(e) \text{ exists}.
\end{aligned}
$$

Let $\{W_e, \Phi_e\}_{e\in\omega}$ be a computable listing of all c.e. sets and enumeration operators with associated finite c.e. approximations $\{W_{e,s}\}_{s\in\omega}$ and $\{\Phi_{e,s}\}_{s\in\omega}$ for each $e \in \omega$.

Let $\boldsymbol{a}$ be the enumeration degree of $A$. Satisfaction of $L_e$ for all $e \in \omega$ ensures that $\boldsymbol{a}' = \boldsymbol{0}'_{\mathrm{e}}$ since it implies that $\lim_s \Phi_{e,s}^{A_s}(e)$ exists for all $e \in \omega$. Note that satisfaction of $N_e$ for all $e \in \omega$ entails that $A$ is not c.e. The set $A$ will be approximated by stages $s$. By $A_s$ we denote the finite set of numbers enumerated into $A$ by the end of stage $s$.

First, we define the priority of the requirements for $R \in \{N, L\}$. Accordingly, requirements $R_e$ (for all $e \in \omega$) are ordered in terms of priority so that

$$N_0 < L_0 < N_1 < L_1 < \cdots$$

We explain the strategies for satisfying each requirement $R_e$ as follows. The role of the $N$ strategy (module) working at index $e$ is to find a witness $x$ such that either $x \in A \backslash W_e$ or $x \in W_e \backslash A$. Initially, at stage $s$, $N$ chooses a witness $x$ and enumerates $x$ into $A$. If at a later stage, say $t > s$, $x$ enters $W_{e,t}$ then $N$ extracts $x$ from $A$ (this module is known as the "Friedberg-Muchnik" strategy). The $L$ module working at index $e$ attempts to satisfy that $\lim_s \Phi_{e,s}^{A_s}(e)$ exists. In doing so, it enumerates a finite set $D$ (with $\langle e, D \rangle \in \Phi_{e,})$ into $A$ such that $D$ does not contain any $x$ that some $N_i$ $(i \leq e)$ wants to keep out of $A$.

For each requirement $R_e$ we define a number of parameters for the sake of its satisfaction. The information contained in the parameters is useful when deciding what action the construction will take. Parameters are defined according to the type of requirement. We define the parameters in the following way.

- *Parameters for the $N_e$ requirements.* The outcome function $N(e, s) \in \{0, 1\}$ and the witness parameter $x(e, s) \in \omega \cup \{-1\}$.
- *Parameters for the $L_e$ requirements.* The outcome parameter $L(e, s) \in \{0, 1\}$, the finite set parameter $D(e, s) \in \mathcal{P}(\omega)$ and the avoidance parameter $\Omega(e, s) \in \mathcal{P}(\omega)$. We define $\Omega(e, s + 1)$ by

$$\Omega(e, s+1) \;=\; \bigcup_{i<e} \{x(i,s) \mid N(i,s) = 1\}.$$

Accordingly, $\Omega(e, s+1)$ records the finite set of elements that the construction wants to keep out of $A$ for the sake of higher priority $N$ requirements and cannot be enumerated into $A$ at stage $s+1$ for the sake of $L_e$.

Now, we establish under what conditions a requirement requires attention.

Case $N_e$. We say that $N_e$ *requires attention* at stage $s+1$ if, either $N(e,s) = 0$ and $x(e,s) = -1$ (i.e. $N_e$ does not have an associated witness), or $N(e,s) = 0$ and $x(e,s) \in W_{e,s}$.

Case $L_e$. We say that $L_e$ *requires attention* at stage $s+1$ if $L(e,s) = 0$ (i.e. $L_e$ has not received attention) and there exists a finite set $D$ such that $\langle e, D\rangle \in \Phi_{e,s}$ and $D \cap \Omega(e, s+1) = \emptyset$.

When a requirement $R_e$ acts, it can ignore lower priority requirements $R_i$ (i.e. for all $i \in \omega$ such that $i > e$) and destroy the work done so far in the construction for the satisfaction of all $R_i$. Whenever the construction encounters this situation, lower requirements $R_i$ have to be reset (i.e. cancel their parameters) and are forced to start all over again their work towards their satisfaction

*Resetting $N_e$.* When we say that the construction *resets* $N_e$ at stage $s+1$ we mean the following. If $x(e,s) = -1$ then, the construction does nothing (and in this case $x(e, s+1) = x(e,s) = -1$ and $N(e, s+1) = N(e,s) = 0$). Otherwise we set $x(e, s+1) = -1$ and $N(e, s+1) = 0$.

*Resetting $L_e$.* When we say that the construction *resets* $L_e$ at stage $s+1$ we mean the following. If $L(e,s) = 0$ the construction does nothing (and in this case $D(e,s+1) = D(e,s) = \emptyset$ and $L(e,s+1) = L(e,s) = 0$). On the other hand, if $L(e,s) = 1$ then we set $L(e,s+1) = 0$ and $D(e,s+1) = \emptyset$.

**The Construction.** $A$ is constructed in stages $s$ so that

$$x \in A \Leftrightarrow \exists s\, \forall t \geq s\, [x \in A_s].$$

Hence $A_s$ is finite for all $s$ and so the approximation to $A$ is $\Sigma^0_2$. Now, we give the formal definition of the construction. At every stage $s > 0$, if not otherwise specified, all parameters retain their values.

*Stage $s = 0$.* Define $A_0 = \emptyset$ and, for all $e \in \omega$, $N(e,0) = L(e,0) = 0$, $D(e,0) = \Omega(e,0) = \emptyset$ and $x(e,0) = -1$.

*Stage $s+1$.* Look for the least $e \leq s$ such that $R \in \{N_e, L_e\}$ is the highest priority requirement that requires attention and proceed as follows. Otherwise, if there does not exist such $e$ then go to stage $s+2$ (and all parameters retain their values from the preceding stage).

*Case a)* $Q = N_e$.

- If $x(e,s) = -1$ and $N(e,s) = 0$ then choose a new witness $x$ that has not appeared in the construction (fresh). Define $x(e,s+1) =$ the least such $x$, enumerate $x(e,s+1)$ into $A$, set outcome $N(e,s+1) = 0$. Reset lower priority requirements $R_i$. We say that $N_e$ receives attention.

- If $N(e,s) = 0$ and $x(e,s) \in W_{e,s}$ then extract $x(e,s)$ from $A$. Set outcome

$N(e, s+1) = 1$. Reset lower priority requirements $R_i$. We say that $N_e$ receives attention.

Case b) $Q = L_e$.

- If $L(e, s) = 0$ and there exists a finite set $D$ such that

$$\langle e, D \rangle \in \Phi_{e,s} \ \& \ D \cap \Omega(e, s+1) = \emptyset$$

then choose the least such $D$, set $D(e, s+1) = D$, enumerate $D(e, s+1)$ into $A$ and set outcome $L(e, s+1) = 1$. Reset lower priority requirements $R_i$. We say that $L_e$ receives attention.

Go to stage $s + 2$.

**Verification.** Consider any $e \in \omega$. As *Induction Hypothesis* we suppose that every requirement $R \in \{N_i, L_i \mid i < e\}$ only receives attention at most finitely often. Let $s$ be the least stage such that every such requirement $R$ does not receive attention at any stage $t > s$. We now check that $N_e$ and $L_e$ are satisfied and that the Induction Hypothesis is justified in each case. We proceed according to descending priority, noting that $N_e < L_e$ in the priority ordering.

*Case $N_e$.* By the definition of $s$ for all $t \geq s$, $x(e, t) = x(e, s)$. We write this limiting value as $x(e)$. If $N_e$ never receives attention after stage $s$, then $x(e) \in A \backslash W_e$ and outcome $N(e) = N(e, t) = 0$. Otherwise, if $N_e$ receives attention at some stage $u \geq t$, then $x(e) \in W_e \setminus A$ and outcome $N(e) = N(e, u) = 1$. Thus $N_e$ never receives attention at any stage $v > u$.

*Case $L_e$.* Consider $s$ as defined above. If $\Phi_{e,s}^{A_s}(e)$ reaches a limit then for all $t \geq s$, $D(e,t) = D(e,s)$. We write this limiting value as $D(e)$. Therefore, $\Phi_{e,t}^{A_t}(e) = \Phi_{e,s}^{A_s}(e)$ for all $t \geq s$ and outcome $L(e) = L(e,t) = 1$. Hence $D(e,t) \subseteq A_t \subseteq A$ and so $\lim_t \Phi_{e,t}^{A_t}(e)$ exists.

$\square$

### 1.3.2 Tree method

As we saw in the proof of Theorem 1.3.1, each requirement $R_e$ had a single strategy for its satisfaction which is reset every time a higher priority requirement $R_j$ acts. Instead of having a single strategy for every requirement, we can have multiple strategies for each $R_e$ if we use a "priority tree" $\mathcal{T}$. The use of priority trees was introduced by Lachlan in [Lac75] and further developed in [Lac79]; Harrington gave useful remarks for the use of trees in [Har82]. We will now explain briefly how this method works using requirements $N_e$ from Theorem 1.3.1 as an example.

We define a set of outcomes $\Sigma$ with a linear ordering $<_\Sigma$ and a priority tree $\mathcal{T}$, namely, the tree $\Sigma^{<\omega}$. Accordingly, we introduce the following notation. Lower case greek letters $\gamma, \beta, \sigma, \ldots$ range over $\Sigma^{<\omega}$. By $|\sigma|$ we denote the length of $\sigma$ which is defined as $|\sigma| = \mu\, x[x \notin \operatorname{dom} \sigma]$. Let $\lambda$ be the empty string. Let $\sigma \subseteq \tau$ denote that $\sigma$ is an initial segment of $\tau$ and $\sigma \subset \tau$ emphasises that $\sigma \subseteq \tau$ but $\sigma \neq \tau$. The concatenation $\sigma^\frown\tau$ of $\sigma$ and $\tau$ consists of $\sigma$ followed by $\tau$. The relation $<_L$ is the lexicographical ordering over strings given as follows.

**Definition 1.3.2.** *Let $\sigma, \tau$ be members of the priority tree $\mathcal{T}$.*

*i. We say $\sigma$* is to the left of *$\tau$ ($\sigma <_L \tau$) if, for the first $n \in \omega$ such that $\sigma(n) \neq \tau(n)$ we have that $\sigma(n) < \tau(n)$.*

*ii. We say $\sigma \leq \tau$ if $\sigma <_L \tau$ or $\sigma \subseteq \tau$.*

*iii. We say $\sigma < \tau$ if $\sigma \leq \tau$ and $\sigma \neq \tau$.*

Each node $\sigma \in \mathcal{T}$ attempts to satisfy requirement $R_e$ if $|\sigma| = e$, namely, $\sigma$ is a version of the strategy for satisfying $R_e$ and has its own "local parameters".

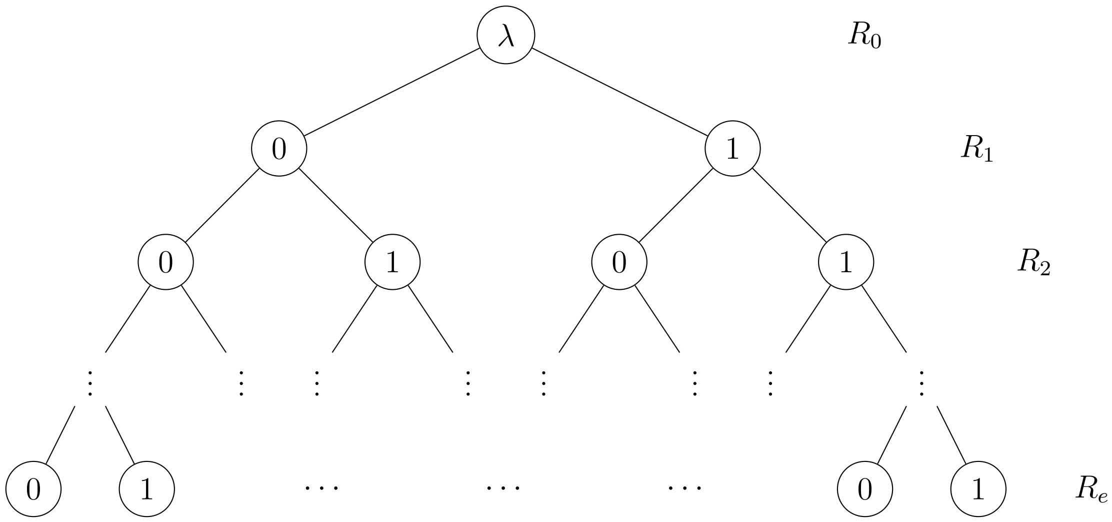


Figure 1.2: Priority Tree $\mathcal{T}$.

Figure 1.2 illustrates the setting explained above. For example, suppose that we want to construct a set $A$ satisfying requirements $N_e$ from Theorem 1.3.1 in isolation. Namely, we have requirements

$$N_0 < N_1 < N_2 < \dots$$

Then each $\sigma \in \mathcal{T}$ has local parameters $x(\sigma, s)$ and $N(\sigma, s)$.

The advantage of using a priority tree is that during the construction, each $\sigma \in \mathcal{T}$ has information about the outcome of the attempts to satisfy higher priority requirements. Accordingly, for every $\sigma \in \mathcal{T}$ and each $n < |\sigma|$, if $\sigma(n) = k$ then, we say that $\sigma$ "believes" the outcome of the strategy for satisfying $\alpha = \sigma \restriction n$ is $k$. We say $\sigma$ is allowed to act whenever its belief seems correct, i.e. if $\sigma = \alpha^\frown k$ and $k$ is the outcome of $\alpha$. We then reset every node $\tau$ such that $\tau > \sigma$.

Now we go back to our example (Theorem 1.3.1). Recall that requirement $N_e$ has two outcomes, namely, $x \in A \setminus W_e$ or $x \in W_e \setminus A$. We then have $2^e$ strategies for $N_e$. Consider requirement $N_1$. There are two strategies, say $\sigma_0$ and $\sigma_1$, for the satisfaction of $N_1$. We assume $\sigma_0 <_L \sigma_1$. Strategy $\sigma_1$ believes that $\lambda$ (the strategy for satisfying $N_0$) chooses a witness $x$ at a stage $s+1$, sets $x(\lambda, s+1) = x$, enumerates $x(\lambda, s+1)$ into $A$, and never extracts $x(\lambda, s+1)$ from $A$. On the other hand, strategy $\sigma_0$ believes that at a later stage $t > s+1$, $x(\lambda, s+1)$ enters the c.e. set $W_e$ ($x(\lambda, s+1) \in W_{e,t}$) and so $\lambda$ extracts $x(\lambda, s+1)$ from $A$ at stage $t+1$ (since $N_0$ has the highest priority in the construction). Only when $\sigma_1$'s belief is correct, i.e. $\lambda$ chooses a witness and enumerates it into $A$, does the construction allow $\sigma_1$ to receive attention and proceed as $N_1$ would have done in the original proof of Theorem 1.3.1. If at a later stage $\lambda$ indeed decides to extract its witness from $A$ then $\sigma_1$ is reset and all its work for satisfaction of $N_1$ is destroyed. Hence, $\sigma_1$ is never again allowed to receive attention and so the construction turns its attention to $\sigma_0$. Eventually, $\lambda$ will have a "true outcome" which is 1 if its witness $x(\lambda) \in A \setminus W_e$ or is 0 if $x(\lambda) \in W_e \setminus A$ (note that in this type of construction the ordering of the outcomes corresponds to the usual linear ordering of $\omega$).

The process explained above is generalized inductively on the tree $\mathcal{T}$ in the usual way. Each requirement $R_e$ will be satisfied by a unique $\sigma$ (such that $|\sigma| = e$) whose belief is correct. Hence $\sigma = f \upharpoonright e$ where $f \in 2^\omega$ is the "true path" (see Definition below) i.e. $f(e) \in \Sigma$ is the "true outcome". The true path $f$ of the construction is the unique infinite path[9] such that $f(e)$ is the outcome of $N_{f \upharpoonright e}$.

**Definition 1.3.3.** *Let $\sigma = f \upharpoonright e$. We define the true path $f \in 2^\omega$ through $\mathcal{T}$ by induction on $e$ as follows:*

[9] In general, we say $g$ is an infinite path through $\mathcal{T}$ if for all $e$, $g \upharpoonright e \in \mathcal{T}$.

$$f(e) = \begin{cases} 0 & \textit{if } (\exists s)[\sigma \textit{ receives attention at stage } s], \\ 1 & \textit{otherwise.} \end{cases}$$

We notice that $f \leq_T \mathcal{K}$. Accordingly, we define a computable approximation $\{\delta_s\}_{s\in\omega}$ to $f$ such that $f(e) = \lim_s \delta_s(e)$. In more complicated tree constructions (see below), we can only guarantee that $f(e) = \lim\inf_s \delta_s(e)$, that is, $f$ is the "leftmost path" visited infinitely often by $\{\delta_s\}_{s\in\omega}$.

### 1.3.3 Infinite injury priority method

The infinite injury priority method was introduced independently by Shoenfield in [Sho61] and Sacks in [Sac63b]. In a finite injury priority construction each requirement $R_e$ receives attention finitely often whereas in an infinite injury priority construction $R_e$ might receive attention infinitely often. In the case of an infinite injury priority requirement, if we reset lower priority requirements $R_i$ every time $R_e$ receives attention, then all such $R_i$ would never have a chance to receive attention. In order to give the opportunity to each $R_i$ to be satisfied we define multiple strategies for $R_i$, including a strategy for the case when $R_e$ acts infinitely often. The priority tree $\mathcal{T}$ is then defined as follows. The left outcome of requirement $R_e$ represents the belief that $R_e$ receives attention infinitely often and is allowed to receive attention every time $R_e$ does. The approximation $\{\delta_s\}_{s\in\omega}$ to the true path $f$ can move both left and right. Hence, $f$ is the leftmost path visited infinitely often by $\{\delta_s\}_{s\in\omega}$, that is, $f(e) = \lim\inf_s \delta_s(e)$. If $\sigma = f \restriction e$ then[10]

$$\exists^{<\infty}[\delta_s <_L \sigma] \text{ and } \exists^{\infty}[\sigma \subseteq \delta_s],$$

and so $f \leq_T \mathcal{K}'$.

[10]By $\exists^{\infty}$ we denote "there exists infinitely many".

# Chapter 2

# Genericity in the Enumeration Degrees

In this chapter we give a brief background survey of the notion of genericity in the context of the Turing degrees as well as in the enumeration degrees. The choice of the results surveyed in this chapter is dependent on the problems we will investigate in subsequent chapters. We refer the reader to [Odi99, Coo04, Cop87] for a fuller introduction to the notion of genericity in Computability Theory.

## 2.1 Genericity in the Turing Degrees

In [Coh63] Cohen presented a technique known as forcing, which is used in the context of set theory to prove the independence of the Axiom of Choice and the Continuum Hypothesis from Zermelo Fraenkel set theory. Forcing turned out to be a useful and power technique to prove other results in set theory and also in Computability Theory.

Feferman noted in [Fef64] that the notions of forcing and genericity can be used in arithmetic. Hinman continued in [Hin69] the study of forcing in arithmetic and found some useful applications. It was Posner [Pos77] who formulated an equivalent definition of 1-genericity replacing arithmetical relations by c.e. sets of strings (i.e. elements of $2^{<\omega}$).

**Definition 2.1.1.** *A set $A \subseteq \omega$ is* 1-generic *if for every $\Sigma^0_1$ set $X$ of strings, either*

*i.* $(\exists\sigma \subset A)\,[\sigma \in X]$, *or*

*ii.* $(\exists\sigma \subset A)\,(\forall\tau \supseteq \sigma)\,[\tau \notin X]$.

*Then we say $A$ forces $X\,(A \Vdash X)$ and $\boldsymbol{a} = deg_{\mathrm{T}}(A)$ is a 1-generic degree.*

In the above definition, the notion of 1-genericity can be generalized to $n$-genericity by replacing $\Sigma^0_n$ instead of $\Sigma^0_1$.

We can easily construct a 1-generic set which is Turing reducible to the halting set $\mathcal{K}$ by a finite extension argument(we refer the reader to [Soa99] for an introduction to this technique). We now list some of the properties that 1-generic sets have in the Turing degrees.

**Proposition 2.1.2.** *If $A$ is 1-generic, then $A$ is not computable.*

*Proof.* Let $A$ be a 1-generic set. Consider the following set $X_i$ of strings

$$X_i = \{\sigma \mid\ \exists\, x < |\sigma|\ [\varphi_i(x) \downarrow\neq \sigma(x)]\},$$

where $\varphi_i$ is the $i$-th computable function. We notice that $X_i$ is a c.e. set since it is in $\Sigma^0_1$ form. Then $A$ forces $X_i$ and one of the two following cases holds

- $\exists\sigma \subset A$ such that $\sigma \in X_i$, in which case $A$ cannot be computed by $\varphi_i$ since $\varphi_i$ and the characteristic function of $A$ disagree on such argument $x$.

- $\exists\sigma \subset A$ such that for any extension $\tau \supseteq \sigma$, $\tau \notin X_i$, but we cannot have $\varphi_i(x)\downarrow$ (where $x = |\sigma|$). Indeed suppose we could define a string $\tau \supset \sigma$ by $\tau = \sigma^\frown 1$ if $\varphi_i(x) = 0$ and $\tau = \sigma^\frown 0$ otherwise. This gives us $\tau \in X_i$, which is a contradiction. Hence $\varphi_i$ is not total.

□

**Proposition 2.1.3.** *A set $A$ is 1-generic if and only if $\overline{A}$ is 1-generic.*

Given any string $\sigma$, we define $\overline{\sigma}$ for every $x < |\sigma|$ as

$$\overline{\sigma}(x) = \begin{cases} 0 & \text{if } \sigma(x) = 1, \\ 1 & \text{if } \sigma(x) = 0. \end{cases}$$

We notice that Proposition 2.1.3 follows directly from the 1-genericity of $A$. Furthermore 1-generic sets cannot be c.e.

**Proposition 2.1.4.** *If $A$ is 1-generic, then $A$ is immune*[1].

*Proof.* Let $A$ be a 1-generic set. Consider the following set $X_i$ of strings

$$X_i = \{\sigma \mid \ \exists\, x < |\sigma|\ [\sigma(x) = 0\ \&\ x \in W_i\ ]\},$$

where $W_i$ is the $i$-th c.e. set. If we assume $(\exists\sigma \subset A)\,(\forall\tau \supseteq \sigma)\ [\tau \notin X_i]$ then $W_i$ is finite. Fix such $\sigma$. Indeed suppose for a contradiction that $W_i$ is infinite. Then we can find some $y > |\sigma|$ such that $y \in W_i$. We define a string $\rho \supset \sigma$ such that $\rho(y) = 0$ and so $\rho \in X_i$, a contradiction. It follows that there exists $\sigma \subset A$ such that $\sigma \in X_i$, and consequently $x \in \overline{A} \setminus W_i$.

□

An interesting property of a 1-generic set is that we can characterise it in terms of Turing reducibility.

[1]A set $A$ is immune if $A$ is infinite but does not contain any infinite c.e. set.

**Definition 2.1.5.** *A set $A$* forces its jump *if for all $e$, there is a $\sigma \subset A$ such that either*[2],

*i.* $\Phi_e^\sigma(e)\downarrow$, *or*

*ii.* $(\forall \tau \supseteq \sigma)\Phi_e^\sigma(e)\uparrow$.

If we consider the c.e. set of strings $X_{\langle x,y\rangle} = \{\,\sigma \mid \exists s\ \Phi_{x,s}^\sigma(y)\downarrow\}$ and a 1-generic set $A$, then $A \Vdash X_i$ and clearly $A$ forces its jump. It can also be shown that if $A$ forces its jump, then $A$ is 1-generic.

**Proposition 2.1.6.** *If $A$ is 1-generic, then*[3] $A \oplus \mathcal{K} \equiv_T A'$.

*Proof.* Consider again the set $X_{\langle x,y\rangle} = \{\,\sigma \mid \exists s\ \Phi_{x,s}^\sigma(y)\downarrow\}$ and let $A$ be a 1-generic set. Then $A$ forces its jump and so we can test all $\sigma \subset A$ to see which of the two parts of Definition 2.1.5 holds for $\sigma$. Whenever we find that $\Phi_e^\sigma(e)\downarrow$ then clearly $e \in A'$ and so $A' \leq_T A \oplus \mathcal{K}$. □

Notice that if we let $A \leq_T \mathcal{K}$, then the above implies that $A$ is low. Moreover, even if a 1-generic set is not c.e. we can still define a "splitting" of it.

**Definition 2.1.7.** *A set $B$ is* splittable *if there exist incomparable sets $B_0$ and $B_1$ such that $B = B_0 \oplus B_1$ and we say $\boldsymbol{b} = deg_{\mathrm{e}}(B)$ is splittable. Otherwise we say $B$ and $\boldsymbol{b} = deg_{\mathrm{e}}(B)$ are* nonsplittable.

**Proposition 2.1.8.** *If $A$ is 1-generic, then $A$ is splittable.*

*Proof.* Let $A$ be a 1-generic set. Then we define two sets $A_0$,$A_1$ such that $A = A_0 \oplus A_1$ by

$$A_0 = \{x \mid 2x \in A\} \text{ and } A_1 = \{x \mid 2x+1 \in A\}.$$

[2] $\Phi_e$ denotes the *eth* oracle Turing machine.
[3] $A'$ denotes the Turing jump of $A$, namely, $A' = \{e \mid \Phi_e^A(e)\downarrow\}$

Let $\sigma = \sigma_0 \oplus \sigma_1$, where $\sigma_0(x) = \sigma(2x)$ and $\sigma_1(x) = \sigma(2x+1)$. Consider the following c.e. set $X_i$ of strings

$$X_i = \{\sigma \mid \exists x < |\sigma|, s\, [\Phi_{i,s}^{\sigma_1}(x) \downarrow \neq \sigma_0(x)]\}.$$

Now, if $(\exists \sigma \subset A)\,(\forall \tau \supseteq \sigma)\ [\tau \notin X_i]$ then $\Phi_i^{A_1}$ is not total. Fix such $\sigma$. Indeed, for a contradiction assume that $\Phi_i^{A_1}$ is total and so $\Phi_i^{\tau_1}(x)\downarrow$ for some $\tau \supset \sigma$. Since $\sigma_0(|\sigma_0|)\uparrow$ we can find $\rho \supset \sigma$ such that $\rho_0(|\sigma_0|)\downarrow$ and $\rho_0(|\sigma_0|) \neq \Phi_i^{\tau_1}(|\sigma_0|)$ and hence $\rho \in X_i$. Likewise, we can also show that $A_1 \neq \Phi_i^{A_0}$. Therefore, whichever of the two cases of the definition of 1-genericity holds, we have that $A_0$ and $A_1$ are incomparable (i.e. $A_0 \mid_T A_1$). □

Furthermore, we can generalize Proposition 2.1.8 to the case where we split a 1-generic set into $n$ parts and every resulting part, say $A_n$, turns out to be strongly independent in the following sense.

**Definition 2.1.9.** *Let $A^{[i]} = \{y \mid \langle i, y\rangle \in A\}$, then for no $i$, $A^{[i]}$ is computable in $\{\langle x, y\rangle \mid \langle x, y\rangle \in A\ \&\ x \neq i\}$ i.e. every $A^{[i]}$ is* strongly independent.

We denote by $\mathcal{D}[\leq \boldsymbol{a}]$ the set $\{\boldsymbol{x} \mid \boldsymbol{x} \leq \boldsymbol{a}\}$. If $\boldsymbol{a}$ is 1-generic, then $\mathcal{D}[\leq \boldsymbol{a}]$ is a very rich structure, one example of this is the next result given by Jockusch in [Joc80].

**Proposition 2.1.10** ([Joc80]). *If $A$ is 1-generic and $\boldsymbol{a} = deg_{\mathrm{T}}(A)$, then $\mathcal{D}[\leq \boldsymbol{a}]$ is not a lattice.*

A natural question to ask is which sets are bounded by or bound a 1-generic set. It is known that any 1-generic degree does not bound any nonzero c.e. degree[4]. On the other hand, Shore noted that every nonzero c.e. degree bounds a 1-generic degree (this is mentioned in [CJ84]). With the motivation of finding a degree that does not bound a minimal degree, Chong and Jockusch [CJ84] showed the Theorem below (where $\mathbf{0'} = deg_{\mathrm{T}}(\mathcal{K})$).

[4] Folklore.

**Theorem 2.1.11** ([CJ84]). *If $\boldsymbol{a}$ is a 1-generic degree and $\mathbf{0} < \boldsymbol{b} \leq \boldsymbol{a} < \mathbf{0'}$, then there is a 1-generic degree $\boldsymbol{c} \leq \boldsymbol{b}$.*

In fact, Theorem 2.1.11 is a nice application of 1-genericity to prove the existence of a nonzero degree below $\mathcal{K}$ with no minimal predecessors. Moreover, Chong and Jockusch conjectured that the following is false: if $\boldsymbol{a}$ is 1-generic and $\mathbf{0} < \boldsymbol{b} \leq \boldsymbol{a} < \mathbf{0'}$, then $\boldsymbol{b}$ is 1-generic.

In [Hau86], Haught refuted Chong and Jockusch's conjecture by proving that indeed there exists a 1-generic set below a 1-generic degree which is Turing reducible to $\mathcal{K}$.

**Theorem 2.1.12** ([Hau86]). *If $\mathbf{0} < \boldsymbol{b} \leq \boldsymbol{a} \leq \mathbf{0'}$ and $\boldsymbol{a}$ is 1-generic, then $\boldsymbol{b}$ is 1-generic.*

The proof of Theorem 2.1.12 relies on an interesting relationship between 1-genericity and computable approximations, known as "$\Sigma_1$-correctness" (noted by Shore).

**Proposition 2.1.13** ([Hau86]). *If $B \leq_T G <_T \mathbf{0'}$ and $G$ is 1-generic then, $B$ has a computable approximation $\{\beta_s\}_{s\in\omega}$ (called a $\Sigma_1$-*correct approximation*) such that for any infinite c.e. set of stages $T \subseteq \omega$, there exists $t \in T$ such that $\beta_t \subset B$.*

*Proof.* Let $G$ be a 1-generic set such that $G <_T \mathbf{0'}$. Hence, there exists a computable approximation $\{\sigma_s\}_{s\in\omega}$ to $G$ such that $G = \lim_s \sigma_s$. Set[5] $B = \Phi_e^G$ for some $e \in \omega$ (the $e$-th Turing machine). For every $s \in \omega$, let $\beta_s = \Phi_e^{\sigma_s}$ and so $\{\beta_s\}_{s\in\omega}$ is a computable approximation to $B$. Let $T \subseteq \omega$ be any infinite c.e. set. Define $S = \{\sigma_t \mid t \in T\}$ and consequently $S$ is infinite and c.e. For a contradiction, assume that $\exists \sigma \subset G\ \forall \tau \supseteq \sigma\ [\tau \notin S]$. Since $T$ is infinite, if $\sigma \subset G$, then $\sigma_t \supset \sigma$ for some $t \in T$. A contradiction. Hence $\exists \sigma_t \subset G$ such that $\sigma_t \in S$ and so $\beta_t \subset B$. $\square$

[5] $\Phi_e$ denotes the *eth* oracle Turing machine.

Haught proved that if $B \in \boldsymbol{b}$ has a $\Sigma_1$-correct approximation, then there is a $C \equiv_T B$ such that $\boldsymbol{c} = deg_{\mathrm{T}}(C)$ is 1-generic. To prove this, she modified a construction of a 1-generic set $C$ using the finite injury priority method by adding permitting and coding to achieve $C \leq_T B$ and $B \leq_T C$.

## 2.2 Genericity in the Enumeration degrees

From the embedding $\iota$ of the Turing degrees into the enumeration degrees we can deduce the existence of $n$-generic sets below $\mathbf{0}_{\mathrm{e}}^{(n)}$. As we have seen before, the structure of the enumeration degrees is richer than that of the Turing degrees. This leads to the search for a notion of genericity which is appropriate for the definition of enumeration reducibility.

Case [Cas71] and Moore [Moo74] initiated the study of genericity in the enumeration degrees. In [Cas71] Case studied the notion of genericity for partial functions and used a forcing technique to prove some basic structural results. Analogously to the notion of an $n$-generic set, which is defined using 2-valued strings[6], a generic function is defined in terms of $\omega^*$-valued strings.

**Definition 2.2.1.** *Let $\omega^* = \omega \cup \{\omega\}$. For some $n \in \omega$, if "$\varphi(n) = \omega$", then we say $\varphi$ is undefined at $n$.*

By abuse of notation, we let $\sigma$ and $\tau$ stand for $\omega^*$-valued strings, that is, mappings from an initial segment of $\omega$ into $\omega^*$. Let $\tau \succeq \sigma$ denote $\tau$ *strongly extends* $\sigma$ in the sense that $\forall x < |\sigma|[\tau(x) \simeq \sigma(x)]$.

**Definition 2.2.2.** *A function $\varphi$ is generic if for every arithmetical set[7] $S$ of $\omega^*$-valued strings either*

[6] Members of $2^{<\omega}$.

[7] A set $A$ is arithmetical if $A \in \Sigma_n \cup \Pi_n$ for some $n \in \omega$.

*i.* $(\exists \sigma \prec \varphi)\,[\sigma \in S]$, *or*

*ii.* $(\exists \sigma \prec \varphi)\,(\forall \tau \succeq \sigma)\,[\tau \notin S]$.

*If* $\boldsymbol{a} = deg_{\mathrm{e}}(graph(\varphi))$, *then we say* $\boldsymbol{a}$ *is generic.*

We can also define what is meant by a generic set $A \subseteq \omega$ by replacing $\varphi$ by the characteristic function $\chi_A$ of $A$ in the above definition. Cooper [Coo90] noted that the advantage of studying generic functions rather than generic sets is that all results also hold for the structure of partial degrees $\mathcal{P}$. Furthermore, Case gave interesting properties of generic functions.

**Proposition 2.2.3** ([Cas71])**.** *i. If* $\varphi$ *is a generic function, then* $graph(\varphi)$ *is immune.*

*ii. If* $\varphi$ *is a generic function, then* $\varphi$ *has no partial recursive extension.*

*iii. If* $\varphi$ *is a generic function, then* $deg_{\mathrm{e}}(\varphi)$ *is quasiminimal.*

*iv. If* $\varphi$ *is a generic function and we define* $\varphi = \varphi_0 \oplus \varphi_1$, *then* $\varphi_0$ *and* $\varphi_1$ *are incomparable, generic and form a minimal pair*[8].

Copestake introduced in [Cop88] the notion of an *n-generic partial function* and studied its characteristics. In fact, Copestake explains that an $n$-generic partial function is a restriction of a generic function which expresses statements about partial functions in a direct way.

**Definition 2.2.4.** *A partial function* $\psi$ *is n-generic if for every* $\Sigma^0_n$ *set* $S$ *of* $\omega^*$*-valued strings either,*

[8]Nonzero functions $\varphi_0$ and $\varphi_1$ form a *minimal pair*, if

$$\text{For any partial function } \psi\ [\psi \leq_{\mathrm{e}} \varphi_0\ \&\ \psi \leq_{\mathrm{e}} \varphi_1 \Rightarrow \psi \in \mathbf{0}_{\mathrm{e}}].$$

*i.* $(\exists\sigma \prec \psi)\ [\sigma \in S]$, *or*

*ii.* $(\exists\sigma \prec \psi)\,(\forall\tau \succeq \sigma)\ [\tau \notin S]$.

*If* $\boldsymbol{a} = deg_{\mathrm{e}}(graph(\psi))$, *then we say* $\boldsymbol{a}$ *is* $n$*-generic.*

In particular, Copestake studied 1-generic functions and established the following characteristics (similar to those studied by Case).

**Proposition 2.2.5** ([Cop88])**.** *i. If* $\psi$ *is a* 1*-generic partial function, then* $graph(\psi)$ *is immune.*

*ii. If* $\psi$ *is a* 1*-generic partial function, then* $\psi$ *has no partial recursive extension.*

*iii. If* $\psi$ *is a* 1*-generic partial function, then* $deg_{\mathrm{e}}(\psi)$ *is quasiminimal.*

*iv. If we define* $\psi = \psi_0 \oplus \psi_1$ *for any* 1*-generic function* $\psi$*, then* $\psi_0$ *and* $\psi_1$ *are e-incomparable. We can then generalise to the nth case where the resulting functions* $\psi_i$ *(for* $i < n$*) are also e-independent (as defined in [McE84], see Definition below).*

**Definition 2.2.6.** *Let* $\{X_i\}_{i\in\omega}$ *be a sequence of subsets of* $\omega$ *and* $\bigoplus\{X_i\}_{i\in\omega} = \{\langle i, x\rangle \mid x \in X_i\}$*. The sequence of sets* $\{X_i\}_{i\in\omega}$ *is* e-independent *if for any* $i$*,* $X_i \not\leq_{\mathrm{e}} \bigoplus\{X_j \mid j \neq i\}$*.*

Moreover, Copestake studied the relationship between 1-generic functions and 1-generic sets. Copestake proved that there is no $n+1$-generic function below $\mathbf{0}_{\mathrm{e}}^{(n)}$. Furthermore, she mentions that, similarly to the construction of a 1-generic set, an $n$-generic function $\leq_{\mathrm{e}} \mathbf{0}_{\mathrm{e}}^{(n)}$ can be constructed in a straightforward way. Using a similar approach to that of the Shoenfield-Spector construction of a minimal Turing degree[9], she constructed an enumeration degree $\boldsymbol{a}$ known as "minimal-like" (see

[9] We refer the reader to [Soa99] for an introduction to this technique.

Definition below). The resulting minimal-like e-degree $\boldsymbol{a}$ is incomparable with all 1-generic function degrees.

**Definition 2.2.7.** *A nonzero enumeration degree* $\boldsymbol{a}$ *is* minimal-like *if* $\boldsymbol{a}$ *contains a function* $\psi$ *such that for any* $e$*, if* $\Phi_e^{graph(\psi)}$ *is a function, then either*

*i.* $\Phi_e^{graph(\psi)} \in \boldsymbol{a}$*, or*

*ii.* $\Phi_e^{graph(\psi)}$ *has a partial computable extension.*

**Theorem 2.2.8** ([Cop88])**.** *There is a total minimal-like e-degree* $\boldsymbol{a}$ *such that* $\boldsymbol{a} \leq_{\mathrm{e}} \mathbf{0}_{\mathrm{e}}''$ *.*

Copestake noted that even though there are some similarities between 1-generic functions and 1-generic sets (i.e. immunity, splittable, etc.) they cannot have the same e-degree. In fact, a 1-generic set is not the graph of a 1-generic function. Copestake deduces her observation from the the following results.

**Theorem 2.2.9** ([Cop88])**.** *A set* $A$ *is an* $n$*-generic set if and only if for some* $n$*-generic function* $\psi$*, we have* $A = dom(\psi)$*.*

**Theorem 2.2.10** ([Cop88])**.** *If* $\psi$ *is a* 1*-generic function, then* $dom(\psi) <_{\mathrm{e}} \psi$*.*

**Theorem 2.2.11** ([Cop88])**.** *If* $A$ *is a* 1*-generic set and there is a function* $\psi \leq_{\mathrm{e}} A$*, then* $\psi$ *has a partial computable extension.*

From Theorems 2.2.9, 2.2.10 and 2.2.11 it follows that, by contrast with Haught's main result [Hau86], if $\boldsymbol{a}$ is a 1-generic function e-degree, then there exists an e-degree $\boldsymbol{b} <_{\mathrm{e}} \boldsymbol{a}$ such that $\boldsymbol{b}$ does not contain the graph of any 1-generic function.

We saw in Proposition 2.1.6 that every 1-generic set $\leq_T \mathcal{K}$ is $\Delta_2^0$ and low. Copestake showed in [Cop90] that $\Delta_2^0$ 1-generic sets are also low in the enumeration degrees. On the other hand, she proved that there exists a 1-generic set below $\mathbf{0}_{\mathrm{e}}'$

which is not low [Cop90]. In fact she proved a stronger result by making the degree of a 1-generic set below $\mathbf{0}'_{\mathrm{e}}$ properly $\Sigma_2$ (see Definition below).

**Definition 2.2.12.** *An enumeration degree $\boldsymbol{a} \leq \mathbf{0}'_{\mathrm{e}}$ is properly $\Sigma^0_2$ if it contains no $\Delta^0_2$ sets.*

In his thesis [Cas71] Case mentions that splitting a generic function produces a minimal pair. Following this result, Copestake mentioned in [Cop88] that one can get a minimal pair out of the splitting of a 2-generic function and that this does not appear to be possible for the 1-generic function case.

Cooper, Li, Sorbi and Yang [CLSY05] showed that every $\Delta^0_2$ set bounds a minimal pair and constructed a $\Sigma^0_2$ set which does not bound a minimal pair. They conjectured that, following their proof, one could construct a 1-generic set that does not bound a minimal pair. Indeed, after making some modifications to the proof presented in [CLSY05], Soskova gave in [Sos07] the actual construction of a 1-generic set which does not bound a minimal e-degree. Moreover, in her thesis [Bia00] Bianchini gave the construction of a 1-generic set below every nonzero $\Delta^0_2$ set.

**Theorem 2.2.13** ([Bia00])**.** *If $B$ is a nonzero $\Delta^0_2$ set then, there is a 1-generic set $A \leq_{\mathrm{e}} B$.*

# Chapter 3

# Genericity and noncupping in the enumeration degrees

In this chapter we present a brief overview of the relationship between noncupping and genericity in the enumeration degrees. We then give a result that will be useful in proving the existence of prime ideals of $\Pi^0_2$ enumeration degrees in Chapter 5, namely, we show the existence of a 1-generic enumeration degree $\mathbf{0}_{\mathrm{e}} < \boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ which is noncuppable and low$_2$.

## 3.1 Introduction

We have discussed two important subclasses of the $\Sigma^0_2$ enumeration degrees. One of them is the $\Pi^0_1$ e-degrees, obtained under the embedding $\iota$ of the the c.e. Turing degrees and the second one is the $\Delta^0_2$ e-degrees. Cooper and Copestake started in [CC88] the study of another interesting subclass of the $\Sigma^0_2$ enumeration degrees, namely, the properly $\Sigma^0_2$ e-degrees. They noted that from the characterisation of the low enumeration degrees we can deduce that no properly $\Sigma^0_2$ e-degree can be low and constructed a high properly $\Sigma^0_2$ e-degree. Moreover, they showed that below any $\Sigma^0_2$

high enumeration degree $\boldsymbol{a}$ there exists an e-degree $\boldsymbol{b}$ incomparable with all the $\Delta^0_2$ enumeration degrees below $\boldsymbol{a}$ other than $\mathbf{0}_\mathrm{e}$ and (possibly) $\boldsymbol{a}$ itself.

In their paper [CSY96] Cooper, Sorbi and Yi proved that the noncuppable enumeration degrees (defined below) form a subclass of the properly $\Sigma^0_2$ enumeration degrees. This motivated the study of noncupping e-degrees since it gives insight on the distribution of the properly $\Sigma^0_2$ e-degrees.

**Definition 3.1.1.** *An enumeration degree* $\boldsymbol{x} < \mathbf{0}'_\mathrm{e}$ *is* cuppable *if there exists* $\boldsymbol{y} < \mathbf{0}'_\mathrm{e}$ *such that* $\boldsymbol{x} \cup \boldsymbol{y} = \mathbf{0}'_\mathrm{e}$*. We then say* $\boldsymbol{x}$ *is cuppable to* $\mathbf{0}'_\mathrm{e}$ *by* $\boldsymbol{y}$*.*

Whenever an e-degree $\boldsymbol{x}$ does not satisfy Definition 3.1.1 we say $\boldsymbol{x}$ is "noncuppable". We have figures 3.1 and 3.2 illustrating when an e-degree $\boldsymbol{x}$ is cuppable or noncuppable.

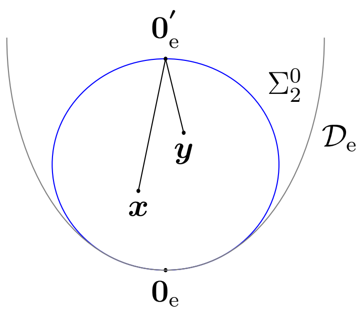


Figure 3.1: An e-degree $\boldsymbol{x}$ which is cuppable

**Theorem 3.1.2** ([CSY96])**.** *If* $\mathbf{0}_\mathrm{e} < \boldsymbol{x} < \mathbf{0}'_\mathrm{e}$ *is* $\Delta^0_2$ *then* $\boldsymbol{x}$ *is cuppable.*

In addition, Cooper, Sorbi and Yi proved that noncuppable enumeration degrees do exist.

**Theorem 3.1.3** ([CSY96])**.** *There exists a nonzero noncuppable enumeration degree* $\boldsymbol{y}$*.*

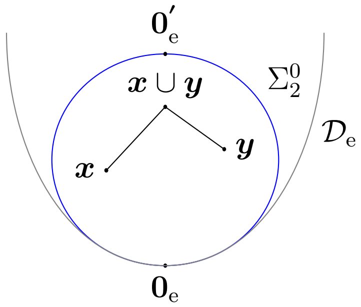


Figure 3.2: An e-degree $\boldsymbol{x}$ which is noncuppable

As a consequence, they deduced that below any noncuppable enumeration degree there is no $\Delta^0_2$ e-degree. Before stating this result, we need the following definition.

**Definition 3.1.4.** *An enumeration degree $\boldsymbol{x}$ is downwards properly $\Sigma^0_2$ if every $\boldsymbol{y} \in \{\boldsymbol{z} \mid \mathbf{0}_\mathrm{e} < \boldsymbol{z} \leq \boldsymbol{x}\}$ is properly $\Sigma^0_2$.*

**Corollary 3.1.5** ([CSY96])**.** *Every noncuppable $\mathbf{0}_\mathrm{e} < \boldsymbol{x} < \mathbf{0}_\mathrm{e}'$ is downwards properly $\Sigma^0_2$ (see Definition 2.2.12).*

Soskova and Wu sharpened Theorem 3.1.2 in [SW07] by proving that every nonzero $\Delta^0_2$ e-degree is cuppable to $\mathbf{0}_\mathrm{e}'$ by a 1-generic degree. They noted that, since a 1-generic set is quasi-minimal and low, every nonzero $\Delta^0_2$ e-degree is cuppable to $\mathbf{0}_\mathrm{e}'$ by a low nontotal e-degree.

Giorgi, Sorbi and Yang showed in [GSY06] that every total nonlow $\Sigma^0_2$ enumeration degree bounds a noncuppable degree. Whereas Giorgi proved in [Gio08] that there exists a high noncuppable enumeration degree. Both results shed light on the distribution of the noncuppable properly $\Sigma^0_2$ e-degrees (these proofs involve infinite priority arguments).

In [Har11] Harris deduced, from results in the context of the $\Sigma^0_1$ Turing degrees and the jump preserving properties of the embedding $\iota$, the existence of a $\Sigma^0_2$ noncuppable e-degree below $\mathbf{0}_\mathrm{e}'$ which is low$_2$. Bearing Lemma 1.2.6 in mind, $A \leq_\mathrm{e} \overline{\mathcal{K}}$ if and only if A is c.e. in $\mathcal{K}$. Hence, there exists a computable approximation $\{A_s\}_{s\in\omega}$ to $A$ which is c.e. in $\mathcal{K}$ such that $A = \bigcup_{s\in\omega} A_s$. Using this relationship between enumeration reducibility and relative computability enumerability Harris obtained a method for constructing a high properly $\Sigma^0_2$ and low$_2$ noncuppable enumeration degree by a finite injury priority method. In fact Harris [Har11] showed that there are high noncuppable e-degrees that are also upwards properly $\Sigma^0_2$.

Following the proof given in [Har11], that there exists a nonzero noncuppable $\Sigma^0_2$ e-degree $\boldsymbol{a}$ such that $\boldsymbol{a}'' = \mathbf{0}''_{\mathrm{e}}$ (i.e. $\boldsymbol{a}$ is low$_2$), we strengthen Soskova's result [Sos07] of the existence of a 1-generic enumeration degree that does not bound a minimal pair, by showing the existence of a noncuppable (and hence downwards properly $\Sigma^0_2$) enumeration degree containing a 1-generic set $A$. The set constructed in Theorem 3.2.1 will be useful in proving the existence of prime ideals of $\Pi^0_2$ enumeration degrees in Chapter 5.

## 3.2 A noncuppable enumeration degree containing a 1-generic set

**Theorem 3.2.1.** *There exists a* 1*-generic enumeration degree* $\boldsymbol{a}$ *such that* $\mathbf{0}_{\mathrm{e}} < \boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ *which is noncuppable and low*$_2$ *(i.e.* $\boldsymbol{a}'' = \mathbf{0}''_{\mathrm{e}}$*).*

Before giving the proof of Theorem 3.2.1, we mention some properties of the e-jump of sets with good approximations that we will be using. In [Gri03] Griffith gave the following characterisation.

**Lemma 3.2.2** ([Gri03])**.** *If* $A = \{e \mid C^{[e]}$ *is finite*$\}$[1] *for some set* $C \leq_{\mathrm{e}} X$*, then* $A \leq_{\mathrm{e}} J_{\mathrm{e}}(X)$*.*

[1]For any set $X$, we define $X^{[e]} = \{\langle e, x\rangle \mid \langle e, x\rangle \in X\}$.

Harris proved in [Har10] that Lemma 3.2.2 holds whenever $X$ has a good approximation.

**Lemma 3.2.3** ([Har10])**.** *If $X$ has a good approximation then, for any set $A$, $A \leq_{\mathrm{e}} J_{\mathrm{e}}(X)$ if and only if there exists a set $C \leq_{\mathrm{e}} X$ such that $A = \{e \mid C^{[e]}$ is finite$\}$.*

Moreover, Harris gave the following characterisation of the double e-jump of a set with a good approximation.

**Lemma 3.2.4** ([Har10])**.** *If $A$ has a good approximation then $\{\, e \mid \Phi_e^A$ is infinite$\,\} \equiv_{\mathrm{e}} J_{\mathrm{e}}^{(2)}(A)$.*

We now give the main construction of Theorem 3.2.1.

### 3.2.1 Requirements

We construct sets $A$ and $C$ c.e. in $\mathcal{K}$ such that (for all $e \in \omega$) the following requirements are satisfied:

$$R_e \quad : \quad \exists \alpha \subseteq \chi_A \,[\, \alpha \in W_e \;\vee\; \forall \beta (\, \alpha \subseteq \beta \;\Rightarrow\; \beta \notin W_e \,)\,]\,. \tag{3.2.1}$$

$$L_e \quad : \quad \Phi_e^A \text{ is infinite } \;\Leftrightarrow\; C^{[e]} \text{ is finite,} \tag{3.2.2}$$

$$P_e \quad : \quad \overline{\mathcal{K}} \;=\; \Phi_e^{B_e \oplus A} \quad\Rightarrow\quad \overline{\mathcal{K}} \leq_{\mathrm{e}} B_e\,, \tag{3.2.3}$$

where $\{W_e, \Phi_e, B_e\}_{e\in\omega}$ is a computable listing of all c.e. sets, enumeration operators and $\Sigma^0_2$ sets with associated finite c.e. approximations $\{W_{e,s}\}_{s\in\omega}$, $\{\Phi_{e,s}\}_{s\in\omega}$ and c.e. in $\mathcal{K}$ approximations $\{B_{e,s}\}_{s\in\omega}$ for each $e \in \omega$.

Supposing $\boldsymbol{a}$ to be the enumeration degree of $A$, satisfaction of $L_e$ for all $e \in \omega$ ensures that $\boldsymbol{a}'' = \mathbf{0}''_{\mathrm{e}}$ since it entails that $\{\, e \mid \Phi_e^A \text{ infinite} \,\} = \{\, e \mid C^{[e]} \text{ finite} \,\}$. By Lemma 3.2.4 we have $J_{\mathrm{e}}^{(2)}(A) \equiv_{\mathrm{e}} \{\, e \mid \Phi_e^A \text{ is infinite} \,\}$. Now, from the construction it follows that $C \leq_{\mathrm{e}} \overline{\mathcal{K}}$ and so by Lemma 3.2.3 we have $\{\, e \mid \Phi_e^A \text{ is infinite} \,\} \leq_{\mathrm{e}} J_{\mathrm{e}}(\overline{\mathcal{K}})$.

Thus $J_{\mathrm{e}}^{(2)}(A) \leq_{\mathrm{e}} \mathbf{0}''_{\mathrm{e}}$ since $J_{\mathrm{e}}(\overline{\mathcal{K}}) \equiv_{\mathrm{e}} \mathbf{0}''_{\mathrm{e}}$ . Note that satisfaction of $\{P_e\}_{e\in\omega}$ implies that $\boldsymbol{a} = deg_{\mathrm{e}}(A)$ is noncuppable whereas satisfaction of $\{R_e\}_{e\in\omega}$ entails that $A$ is 1-generic.

**Definitions and Notation.**

The construction will proceed by stages $s$, each stage being computable in $\mathcal{K}$. We use $A_s$ to denote the finite set of numbers enumerated into $A$ by the end of stage $s$.

1) *The Priority of Requirements.*

For $S \in \{R, L, P\}$, the requirements $S_e$ are ordered in terms of priority by $R_e < L_e < P_e < R_{e+1}$ for all $e \in \omega$.

2) *Environment Parameters.*

We define a number of parameters used by the construction for the satisfaction of individual requirements. Firstly, we use the string parameter $\alpha_s \in 2^{<\omega}$ for the stage $s$ approximation (in the form of an initial segment) and the associated parameters $\alpha_s^+ = \{\, n \mid \alpha_s(n) = 1\,\}$ and $\alpha_s^- = \{\, n \mid \alpha_s(n) = 0\,\}$. Also, for clarity and notational convenience, we define the enumerating parameter $W(s) \in \mathcal{F}$ (the class of finite sets) and the parameter $\mathcal{I}(e, s)$ which is a finite set of numbers that the construction at stage $s$ already knows to be in $\Phi_e^A$ (i.e. $\mathcal{I}(e, s) \subseteq \Phi_e^A$).

- *Parameters for the $R_e$ requirements.* The outcome function $R(e, s) \in \{0, 1, 2\}$ and the restraint parameter $\varepsilon(e, s) \in \mathcal{F}$ (the class of finite sets).

- *Parameters for the $L_e$ requirements.* The outcome parameter $L(e, s) \in \{0, 1\}$, the restraint parameter $\delta(e, s) \in \mathcal{F}$, the individual axiom parameter $v(e, s) \in \omega \cup \{-1\}$ and the enumerating parameter $V(s) \in \mathcal{F}$.

- *Parameters for the $P_e$ requirements.* The outcome parameter $P(e, s) \in \{1, 2\}$, and the avoidance parameter $\Omega(e, s) \in \mathcal{F}$. The definition of $\Omega(e, s+1)$ is:

$$\Omega(e, s+1) \; = \; \bigcup_{i \leq e} (\varepsilon(i, s) \, \cup \, \delta(i, s)) \, . \tag{3.2.4}$$

Accordingly, $\Omega(e, s+1)$ records the finite set of elements that the construction wants to keep out of $A$ for the sake of higher priority $R$ and $L$ requirements, and that it thus cannot enumerate into $A$ at stage $s+1$ for the sake of $P_e$.

3) *Requiring attention.*

<u>Case $R_e$.</u> We say that $R_e$ *requires attention* at stage $s+1$ if $R(e, s) = 0$.

<u>Case $L_e$.</u> We say that $L_e$ *requires attention* at stage $s+1$ if $L(e, s) = 0$ and for all $x \in \omega$ and $D \in \mathcal{F}$,

$$x \notin \mathcal{I}(e, s) \;\; \& \;\; \langle x, D \rangle \in \Phi_e \quad \Rightarrow \quad D \cap \alpha_s^- \; \neq \; \emptyset \, . \tag{3.2.5}$$

<u>Case $P_e$.</u> We say that $P_e$ *requires attention* at stage $s+1$ if $P(e, s) = 1$ and there exists $x \leq s$ and a pair of finite sets $(D, E)$ such that

$$x \in \mathcal{K} \;\; \& \;\; \langle x, D \oplus E \rangle \in \Phi_e[s] \;\; \& \;\; D \subseteq B_e[s] \;\; \& \;\; E \cap \Omega(e, s+1) = \emptyset \tag{3.2.6}$$

where we note that $\Omega(e, s+1)$ is a finite set[2].

4) *Resetting.*

*Resetting $R_e$.* When we say that the construction *resets* $R_e$ at stage $s+1$ we mean the following. If $R(e, s) = 0$ the construction does nothing (and in this case $\varepsilon(e, s+1) = \varepsilon(e, s) = \emptyset$ and $R(e, s+1) = R(e, s)$). On the other hand, if $R(e, s) \in \{1, 2\}$ then we set $\varepsilon(e, s) = \emptyset$ and $R(e, s) = 0$.

*Resetting $L_e$.* When we say that the construction *resets* $L_e$ at stage $s+1$ we mean the following. If $L(e, s) = 0$ we do nothing (and in this case $\delta(e, s+1) = \delta(e, s) = \emptyset$

[2]Notice also that, since the construction uses oracle $\mathcal{K}$ the conditions in (3.2.6) could be defined so that the search for an axiom $\langle x, D \oplus E \rangle$ is unbounded (i.e. in the whole of $\Phi_e$). However this is unnecessary as $\lim_{s \to \infty} \Omega(e, s+1)$ exists.

and $L(e, s+1) = L(e, s) = 0$). On the other hand, if $L(e, s) = 1$ then we set $\delta(e, s+1) = \emptyset$ and $L(e, s) = 0$.

### 3.2.2 Basic Idea of the Construction

We can think of the construction as comprising a module for each type of requirement. In anticipation of the formal proof a brief description of these modules follows below.

The role of the $R$ module working at index $e$ is to find an initial segment $\alpha \subseteq \chi_A$ witnessing satisfaction of $R_e$ as stated in (3.2.1). As the construction uses $\mathcal{K}$ as oracle, the $R$ module is able to test at any even stage $s+1$ whether, for $\alpha_s$ (i.e. the current approximation to $\chi_A$), there exists $\beta \supseteq \alpha_s$ such that $\beta \in W_e$. Accordingly it carries out this test at any stage $s+1 > e+1$ if $R_e$ still appears not to be satisfied (in which case $R(e, s) = 0$) and if for all $i < e$, no requirement $R_i$ requires attention at stage $s+1$. It will thus pick some $\alpha \supseteq \alpha_s$ such that $\alpha$ satisfies (3.2.1)—where $\alpha = \alpha_s$ if there exists no $\beta \supseteq \alpha_s$ such that $\beta \in W_e$, and $\alpha =$ one such $\beta$ otherwise. Moreover, for stages $t \geq s+1$, the $R$ module will try to restrain $\alpha \subseteq \alpha_t$ and it will follow that, if assumption (3.2.7) below is correct, then $\alpha \subseteq \chi_A$ and $R_e$ will be satisfied.

"No higher priority $P$ requirement receives attention at a later stage." (3.2.7)

The $L$ module working at index e tries to make $\Phi_e^A$ infinite. In doing this it uses at stage $s+1 > e+1$ the finite set of numbers being restrained out of $A$ by $R$ and other $L$ requirements at the end of stage $s$. Accordingly at stage $s+1$ (provided that $L(e, s) = 1$, i.e. that $L_e$ does not appear to be already satisfied) the $L$ module will try to put some finite set[3] $D \subseteq A_s \cup \{z \mid z \geq |\alpha_s|\}$ into $A_{s+1} \subseteq A$ to ensure that some $x \notin \mathcal{I}(e,s)$ enters $\mathcal{I}(e,s+1) \subseteq \Phi_e^A$. This is the role of $V(s+1)$ which is simply the union of all those sets that the $L$ module enumerates into $A_{s+1}$ for the sake of requirements $L_i$ such that $i \leq s$. Note that, due to the definition of $\alpha_s$, this action will cause no injury to any $R$ and (other) $L$ requirements. This is important since the $L$ module may carry out this action infinitely often for the sake of $e$ in order to make $\Phi_e^A$ infinite. If the $L$ module does succeed in putting some[4] $x$ into $\Phi_e^A - \mathcal{I}(e,s)$ it enumerates no numbers into $C^{[e]}$ at stage $s+1$. If on the other hand it cannot achieve this, it knows that for every axiom $\langle x, D\rangle \in \Phi_e$ such that $x \notin \mathcal{I}(e,s)$, $\alpha_s^- \cap D \neq \emptyset$. Accordingly it restrains $\delta(e,s+1) = \alpha_s^-$ out of $A_{s+1}$. It also enumerates all of $\omega^{[e]} \upharpoonright s$ into $C_{s+1}$. Now, if assumption (3.2.7) is correct, this restraint will stay in place forcing $\Phi_e^A = \mathcal{I}(e,s)$ and the module will enumerate all of $\omega^{[e]} \upharpoonright t$ into $C$ at every subsequent stage $t+1 > s+1$ thus making $C^{[e]} = \omega^{[e]}$.

The $P$ module working at index $e$ tries to diagonalise $\overline{\mathcal{K}} = \Phi_e^{B_e \oplus A}$. Its strategy is to search for $x \notin \overline{\mathcal{K}}$ such that $x \in \Phi_e^{B_e \oplus (\omega - \Omega)}$ for some finite set $\Omega \subseteq \overline{A}$. This search starts from stage $s+1 > e+1$ onwards with $\Omega$ being the set of elements restrained out of $A$ by higher priority $R$ and $L$ requirements at stage $s$—i.e. the set $\Omega(e,s+1)$ in the notation of the proof. If the module finds such an $x$ it will at some stage $s+1$ enumerate a requisite finite set $E \subseteq \omega - \Omega(e,s+1)$—i.e. where, for some $D \subseteq B_e[s]$, $\langle x, D \oplus E\rangle$ is an axiom in $\Phi_e[s]$—into $A$ thus ensuring that $x \in \Phi_e^{B_e \oplus A} - \overline{\mathcal{K}}$. On the other hand if this search fails then, under the assumption that $\Omega(e,s+1)$ converges in the limit (over stages $s \in \omega$) to a finite set $\Omega(e) \subseteq \overline{A}$, it will follow that $\overline{\mathcal{K}} = \Phi_e^{B_e \oplus A}$ implies that $\overline{\mathcal{K}} = \Phi_e^{B_e \oplus (\omega - \Omega(e))}$, i.e. that $\overline{\mathcal{K}} \leq_{\mathrm{e}} B_e$.

Note that the action of enumerating some finite set $E$ (for the sake of $P_e$ ) into A might injure lower priority $R$ and $L$ requirements. For example, suppose that

[3] Note that $A_s = \alpha_s^+$.

[4] In which case $\mathcal{I}(e,s+1) = \mathcal{I}(e,s) \cup \{x\}$.

$i > e$ is such that $R(i, s) \in \{1, 2\}$. Then this means that the $R$ module working at index $i$ is trying to restrain the set $\varepsilon(i, s)$ out of $A$. Hence if $E \cap \varepsilon(i, s) \neq \emptyset$, and $E$ is enumerated into $A$ at stage $s + 1$, then for $t \geq s + 1$ it is not the case that $\varepsilon(i, s) \subseteq \overline{A}[t]$. A similar observation holds if we replace $R(i, s)$ by $L(i, s)$ and $\varepsilon(i, s)$ by $\delta(i, s)$. Accordingly, all requirements $R_i$ and $L_i$ such that $i > e$ are reset. Now since each $P$ requirement receives attention at most once and, for any $R$ or $L$ requirement there are only finitely many $P$ requirements of higher priority, and the latter can only be reset (i.e. injured) finitely often. Accordingly, for any index $e$, at every stage $s + 1 > e + 1$ the $R$ and $L$ modules can safely be set to work with index $e$ under assumption (3.2.7) since, from some stage $r_e$ onwards this assumption will indeed be correct.

Before proceeding to the formal construction note the difference in roles of $V(s+1)$ and $W(s + 1)$ at stage $s + 1$. The former as described above is enumerated into $A_{s+1}$ for the sake of forcing $\Phi_e^A - \mathcal{I}(e, s) \neq \emptyset$ for each $e \leq s$, where this turns out to be possible respecting the above conditions. $W(s + 1)$ on the other hand is a finite set (perhaps $= \emptyset$) to be enumerated into $A_{s+1}$ if a $P$ requirement receives attention at stage $s + 1$.

### 3.2.3 The Construction

The sets $A$ and $C$ are enumerated in stages so that, for $X \in \{A, C\}$, $X = \bigcup_{s \in \omega} X_s$ and $X_s$ is finite for all $s$.

*Stage $s = 0$.* Define $\alpha_0 = \lambda$, $A_0 = C_0 = \emptyset$ and, for all $e \in \omega$, $v(e, 0) = -1$, $\varepsilon(e, 0) = \delta(e, 0) = \emptyset$, $R(e, 0) = L(e, 0) = 0$ and $\mathcal{I}(e, 0) = \emptyset$. Note that accordingly $\Omega(e, 0) = \emptyset$ for all $e \in \omega$ by definition. Also define $V(0) = W(0) = \emptyset$.

*Stage $s + 1$.* Using $\mathcal{K}$ as Turing oracle proceed as follows according as to whether

$s$ is even or odd.

*Case I. $s$ is even.* Search for the least $e \leq s$ such that $R_e$ requires attention, set $e_{s+1} = e$ and test whether there exists $\beta \supset \alpha_s$ such that $\beta \in W_{e_{s+1}}$.

- If there exists such a $\beta$, define $\alpha_{s+1}$ to be the (lexicographically) least such string. Set $R(e_{s+1}, s+1) = 2$
- Otherwise define $\alpha_{s+1} = \alpha_s$ and set $R(e_{s+1}, s+1) = 1$.

In both of these sub cases set $\varepsilon(e_{s+1}, s+1) = \alpha_{s+1}^{-}$ and $A_{s+1} = \alpha_{s+1}^{+}$. (Notice that $A_s \subseteq A_{s+1}$.) In this case we say that $R_{e_{s+1}}$ *receives attention.*

*Remark.* Note that for all $t \geq s+1$ such that $\varepsilon(e_{s+1}, s+1)$ is not destroyed by the resetting activity of higher priority $P$ requirements at any stage $r$ such that $s+1 \leq r \leq t$, $\alpha_{s+1} \subseteq \alpha_t$. Thus, if no higher $P$ requirement receives attention after stage $s+1$, $\alpha_{s+1} \subseteq \chi_A$.

*Case II. $s$ is odd.* There are three steps in this case.

*Step A.* For all $e \leq s$, define $v(e, s+1)$ as follows. If $L(e, s) = 1$ (i.e. $L_e$ is satisfied for the moment) or $L(e, s) = 0$ and $L_e$ requires attention at stage $s+1$ then set $v(e, s+1) = -1$. Otherwise—i.e. if $L(e, s) = 0$ and $L_e$ does not require attention at stage $s+1$—choose in a consistent manner[5] some $\langle x, D\rangle$ such that

$$x \notin \mathcal{I}(e, s),\ \langle x, D\rangle \in \Phi_e \ \&\ D \subseteq A_s \cup \{z \mid z \geq |\alpha_s|\}$$

and set $v(e, s+1) = \langle x, D\rangle$. Now define the finite set

$$V(s+1) \;=\; \bigcup_{\substack{e \leq s\ ,\ x \in \omega, \\ v(e,s+1) = \langle x, D\rangle}} D\,,$$

and, for all $e \leq s$ set

[5] I.e. via a uniformly computable search using the construction's oracle $\mathcal{K}$.

$$\mathcal{I}(e,s+1) \quad = \quad \begin{cases} \mathcal{I}(e,s) & \text{if } v(e,s+1) = -1, \\ \mathcal{I}(e,s) \cup \{(v(e,s+1))_0\} & \text{if}^{6} v(e,s+1) \neq -1. \end{cases}$$

Step B. Look for the least $e \leq s$ such that $S \in \{L_e, P_e\}$ is the highest priority requirement that requires attention. If there exists such an $e$ then set $e_{s+1} = e$ and proceed according to case (a) or case (b) below. Otherwise set $e_{s+1} = s$, $W(s+1) = \emptyset$ and go to Step C.

**a)** $e_{s+1} < s$ and $S = L_{e_{s+1}}$. In this case, for any axiom $\langle x, D\rangle$ such that $x \notin \mathcal{I}(e,s)$ and $\langle x, D\rangle \in \Phi_e$ it holds that $D \cap \alpha_s^- \neq \emptyset$. Accordingly set $\delta(e_{s+1}, s+1) = \alpha_s^-$ and define $L(e_{s+1}, s+1) = 1$. Also set $W(s+1) = \emptyset$. We say that $L_{e_{s+1}}$ *receives attention* in this case.

**b)** $e_{s+1} < s$ and $S = P_{e_{s+1}}$. In this case choose the least axiom $\langle x, D \oplus E\rangle$ satisfying (3.2.6). Set $W(s+1) = E$ and define $P(e,s) = 2$ (permanently satisfied). Reset—as defined on page 45—all $R_i$ and $L_i$ such that $i > e_{s+1}$. We say that $P_{e_{s+1}}$ *receives attention* in this case.

Step C. Let

$$l \quad = \quad \max\left(\{|\alpha_s|\} \cup V(s+1) \cup W(s+1)\right) + 1\,.$$

and

$$A_{s+1} \quad = \quad A_s \cup V(s+1) \cup W(s+1)\,,$$

and define $\alpha_{s+1}$ to be the least string of length $l$ such that $\alpha_{s+1}(x) = A_{s+1}(x)$ for all $x < l$.

[6] I.e. if $v(e,s+1) = \langle x, D\rangle$ then $\mathcal{I}(e,s+1) = \mathcal{I}(e,s) \cup \{x\}$.

*To end stage $s+1$.* After both case I and II, for all requirement parameters $\gamma(j,s)$ not mentioned during stage $s+1$ reset $\gamma(j,s+1)=\gamma(j,s)$. Define

$$C_{s+1}=C_s\cup\{\langle e,z\rangle \mid e\le s\ \&\ z\le s\ \&\ L(e,s+1)=1\} \tag{3.2.8}$$

and Proceed to stage $s+2$.

### 3.2.4 Verification.

Consider any $e\in\omega$. As Induction Hypothesis we suppose that every requirement $S\in\{\,R_i,L_i,P_i \mid i<e\,\}$ only receives attention at most finitely often. (Notice that it is obvious by construction that each $P$ requirement receives attention at most once.) Accordingly, let $s_e\ge e$ be the least (even) stage such that no such requirement $S$ receives attention at any stage $t>s_e$. Note that this means that, for every $i<e$ and $\gamma\in\{\varepsilon,\delta,R,L,P\}$, $\gamma(i,t)=\gamma(i,s_e)$ for all $t\ge s_e$. We write this limiting value as $\gamma(i)$. We now check that $R_e$, $L_e$ and $P_e$ are satisfied, and that the Induction Hypothesis is preserved in each case. We proceed according to descending priority, noting that $R_e<L_e<P_e$ in the priority ordering.

*Case $R_e$.* By definition of $s_e$, $R_e$ receives attention at stage $s_e+1$ and is not reset at any stage $t\ge s_e+1$. It follows that either $\alpha_{s_e+1}\in W_e$, or else that, for all $\beta\supseteq\alpha_{s_e+1}$, $\beta\notin W_e$ and, moreover that $R(e,t)\in\{1,2\}$ and $\alpha_{e_{s+1}}\subseteq\alpha_t$ for all $t\ge s_e+1$. Thus $R_e$ never again receives attention and $R(e)=\lim_{t\to\infty}R(e,t)=R(e,s_e+1)$ is the final outcome of $R_e$, whereas $\varepsilon(e)=\lim_{t\to\infty}\varepsilon(e,t)=\varepsilon(e,s_e+1)$. (Note that the latter is precisely the set of numbers restrained out of $A$ for the sake of $R_e$.)

*Case $L_e$.* We firstly show that

$$\Phi_e^A \text{ infinite} \quad\Leftrightarrow\quad C^{[e]} \text{ finite}. \tag{3.2.9}$$

Set $\tilde{s}_e=s_e+1$. (Thus $\tilde{s}_e$ is such that $R_e$ does not receive attention at any stage $t\ge\tilde{s}_e$.)

$\Rightarrow$ Consider any $t \geq \tilde{s}_e$ and suppose that $L(e,t) = 1$. Then there exists some (odd) $r < t$ such that $L_e$ received attention at stage $r+1$ and $L_e$ has not been reset since stage $r+1$. But this means that $\delta(e, r+1) = \alpha_r^-$ and that, by (3.2.5), for all $\langle x, D\rangle$,

$$x \notin \mathcal{I}(e,r) \;\;\&\;\; \langle x, D\rangle \in \Phi_e \quad \Rightarrow \quad D \cap \delta(e, r+1) \;\neq\; \emptyset\,.$$

Moreover, since by definition of $\tilde{s}_e$ it is also the case that no requirement of higher priority receives attention—and so as a result that $L_e$ cannot be reset at any stage $s \geq t$—it follows that $\delta(e, r+1) = \lim_{s\to\infty}\delta(e,s) = \delta(e)$. On the other hand, for the same reasons, we know, by an easy induction over stages $s$, that $\delta(e) \subseteq \overline{A}$. So we can see, by inspection of the construction, that $\Phi_e^A = \mathcal{I}(e,r)$. In other words $\Phi_e^A$ is finite, contradicting the hypothesis. Therefore $L(e,s) = 0$ for all $s \geq \tilde{s}_e$ and so $C^{[e]} \subseteq \omega^{[e]}\!\upharpoonright\!\langle e, \tilde{s}_e\rangle$. I.e. $C^{[e]}$ is finite.

$\Leftarrow$ Now suppose that $\Phi_e^A$ is finite, and note that by construction

$$\mathcal{I}(e,t) \subseteq \mathcal{I}(e,t+1) \subseteq \Phi_e^A$$

for all $t \in \omega$. Also, as $\Phi_e^A$ is finite there is a least (odd) stage $r \geq \tilde{s}$ such that $\mathcal{I}(e,r) = \Phi_e^A$ (this again follows by inspection of the construction). Hence

$$\mathcal{I}(e,s) \quad = \quad \mathcal{I}(e,r) \quad \text{for all } s \geq r\,. \tag{3.2.10}$$

Then, if $L(e,r) \neq 1$ it is clear that $L_e$ will require—and hence receive—attention at stage $r+1$ since otherwise the construction would ensure that $\mathcal{I}(e,r+1) - \mathcal{I}(e,r) \neq \emptyset$, due to action taken during step A of stage $r+1$. Hence $L(e,r+1) = 1$. Furthermore, as $L_e$ cannot be reset after this stage (by definition of $\tilde{s}_e$), it follows that $L(e,t) = 1$ for all $t \geq r+1$. So by construction (see (3.2.8)), $C^{[e]} = \omega^{[e]}$. I.e. $C^{[e]}$ is infinite.

Finally, notice that the above implies that $L_e$ only receives attention at most once after stage $\tilde{s}_e$ and is satisfied. Moreover, letting stage $r$ be as above, $L(e) = \lim_{t\to\infty} L(e,t) = L(e,r+1)$ is the final outcome of $L_e$, and $\delta(e) = \lim_{t\to\infty}\delta(e,t) =$

$\delta(e, r+1)$. (The latter is precisely the set of numbers restrained out of $A$ for the sake of $L_e$.)

*Case $P_e$.* Let $\hat{s}_e \geq \tilde{s}_e + 1$ be a stage at or after which $L_e$ does not receive attention at any stage $t > \hat{s}_e$. Thus, by definition of $\hat{s}_e$, for all such $t$, $\Omega(e,t) = \Omega(e, \hat{s}_e)$. Accordingly we define $\Omega(e)$ to be this set.

Now suppose that $\overline{\mathcal{K}} = \Phi_e^{B_e \oplus A}$. We show that, in this case, $\overline{\mathcal{K}} = \Phi_e^{B_e \oplus (\omega - \Omega(e))}$.

- If $x \in \overline{\mathcal{K}}$ then, since $\Omega(e) \subseteq \overline{A}$—as is easily proved by a simple induction over $s$—it is clear that $x \in \Phi_e^{B_e \oplus (\omega - \Omega(e))}$ follows from our supposition that $\overline{\mathcal{K}} = \Phi_e^{B_e \oplus A}$.

- If $x \notin \overline{\mathcal{K}}$ and $x \in \Phi_e^{B_e \oplus (\omega - \Omega(e))}$ then we know that there exists (odd) $s \geq \hat{s}_e$ and a least axiom $\langle x, D \oplus E \rangle \in \Phi_e[s]$, $D \subseteq B_e[s]$ and $E \cap \Omega(e) = \emptyset$. There are 2 cases.

  1) $P(e,s) = 2$. Then there exists (odd) $t < s$, $z \leq t$ and a pair of finite sets $(F, G)$ such that $z \notin \overline{\mathcal{K}}$, $\langle z, F \oplus G \rangle \in \Phi_e[t]$, $F \subseteq B_e[t]$, and $G \cap \Omega(e, t+1) = \emptyset$ and such that $G$ was enumerated into $A$ at stage $t+1$. But then $z \in \Phi_e^{B_e \oplus A}$ (since $B_e[t] \subseteq B_e$ and $A_{t+1} \subseteq A$) whereas $z \notin \overline{\mathcal{K}}$. Contradiction.

  2) Otherwise $P(e,s) = 1$. In this case the construction enumerates $E$ into $A$ at stage $s+1$, so obtaining $x \in \Phi_e^{B_e \oplus A}$ and $x \notin \overline{\mathcal{K}}$, once again a contradiction.

  This proves that if $x \notin \overline{\mathcal{K}}$ then $x \notin \Phi_e^{B_e \oplus (\omega - \Omega(e))}$.

We thus conclude that $\overline{\mathcal{K}} = \Phi_e^{B_e \oplus (\omega - \Omega(e))}$, i.e. that $\overline{\mathcal{K}} \leq_{\mathrm{e}} B_e$, since $\Omega(e)$ is finite.

Notice that $P_e$ only receives attention once and that there thus exists a stage $\hat{t}_e \geq \hat{s}_e$ such that for all $s \geq \hat{t}_e$ $P(e,s) = P(e, \hat{t}_e)$. I.e. $P(e) = P(e, \hat{t}_e)$

We see from the above that, assuming the Induction Hypothesis for $e$, the requirements $R_e$, $L_e$ and $P_e$ are satisfied and that the Induction Hypothesis is preserved for requirement $e+1$. This concludes the proof.

# Chapter 4

# Avoiding Uniformity in the $\Delta_2^0$ Enumeration Degrees

In this chapter we investigate the property of incomparability relative to a class of degrees of a specific level of the Arithmetical Hierarchy. Indeed, we show that for every uniform $\Delta_2^0$ class of enumeration degrees $\mathcal{C}$ (see Definition 4.1.5), there exists a high $\Delta_2^0$ enumeration degree $\boldsymbol{c}$ which is incomparable with any degree $\boldsymbol{b} \in \mathcal{C}$ such that $\boldsymbol{b} \notin \{\mathbf{0}_{\mathrm{e}}, \mathbf{0}'_{\mathrm{e}}\}$. As a corollary, we get that such $\boldsymbol{c}$ caps with both a high and a low nonzero $\Delta_2^0$ enumeration degree.

## 4.1 Introduction

Yates proved in [Yat67] the existence of a $\Delta_2^0$ Turing degree incomparable with all c.e. Turing degrees $\notin \{\mathbf{0}_{\mathrm{T}}, \mathbf{0}'_{\mathrm{T}}\}$ (intermediate c.e. Turing degrees). In fact, an immediate corollary of Yates' result in the enumeration degrees, is the existence of a $\Delta_2^0$ degree incomparable with all intermediate $\Pi_1^0$ degrees (i.e. $\notin \{\mathbf{0}_{\mathrm{e}}, \mathbf{0}'_{\mathrm{e}}\}$)[CC88].

In the context of the $\Sigma_2^0$ enumeration degrees, Cooper and Copestake [CC88] showed that given any high $\Sigma_2^0$ enumeration degree $\boldsymbol{h}$, there exists a degree $\boldsymbol{c}$ such that $\mathbf{0}_{\mathrm{e}} < \boldsymbol{c} < \boldsymbol{h}$ and $\boldsymbol{c}$ is incomparable with all $\Delta^0_2$ degrees intermediate between $\mathbf{0}_{\mathrm{e}}$ and $\boldsymbol{h}$. Recently, Harris showed in [Har11] the special case when $\boldsymbol{c}$ can be constructed to be high and $\boldsymbol{h} = \mathbf{0}'_{\mathrm{e}}$ .

We now turn our attention to the question of whether a similar result holds in the $\Delta^0_2$ enumeration degrees. The $\Sigma^0_2$ enumeration degrees can be divided via the lower levels of the Arithmetical Hierarchy, that is, the class of the $\Pi^0_1$ degrees (the degrees of the characteristic functions of c.e. sets) and the class of $\Delta^0_2$ degrees (which extends the $\Pi^0_1$ class). In [Lee11, LHC12] it is explained that the notion of "uniform $\Delta^0_2$ class" plays a central role in the classification of the $\Delta^0_2$ degrees. We will only give the definition of the notion of "uniform $\Delta^0_2$ class" (originally introduced in [Lee11, LHC12]) and refer the reader to [Lee11, LHC12] for a thorough explanation and further details of such notion.

The main result of this chapter (Theorem 4.2.1) can be seen as an extension of Kalimullin's work presented in [Kal00], in which given any uniform $\Delta^0_2$ class of enumeration degrees $\mathcal{C}$, there exists a nonzero $\Delta^0_2$ enumeration degree $\boldsymbol{a}$ such that $\{\, \boldsymbol{x} \mid \mathbf{0}_{\mathrm{e}} < \boldsymbol{x} < \boldsymbol{a} \,\} \cap \mathcal{C} = \emptyset$. From this result, Kalimullin gets as a corollary that every nonzero low enumeration degree $\boldsymbol{c}$ caps with a nonzero $\Delta^0_2$ enumeration degree. In the same way, Theorem 4.2.1 allows us to deduce that such $\boldsymbol{c}$ caps with both a high and a low nonzero $\Delta^0_2$ enumeration degree.

In the definitions below if $f$ is a binary (ternary) function then $f_e$ ($f_{e,s}$) is shorthand of $\lambda n f(e,n)$ ($\lambda n f(e,s,n)$).

**Definition 4.1.1.** *If $\mathcal{F}$ is a class of unary functions (mapping $\omega \to \omega$), $\mathcal{F}$ is defined to be* uniform $\Delta^0_2$ *(*subuniform $\Delta^0_2$*) if there is a binary function $f \leq_{\mathrm{T}} \mathcal{K}$ such that*

$$\mathcal{F} \;=\; \{\, f_e \mid e \in \omega \,\} \qquad (\, \mathcal{F} \subseteq \{\, f_e \mid e \in \omega \,\}\,)\,.$$

*A class of sets $\mathcal{C} \subseteq \mathcal{P}(\omega)$ is defined to be* uniform $\Delta^0_2$ *(*subuniform $\Delta^0_2$*) if the class of characteristic functions of $\mathcal{C}$ is uniform $\Delta^0_2$ (subuniform $\Delta^0_2$).*

We note here that the notion "uniform $\Delta_2^0$" corresponds to the notion "$\mathbf{0'}$-uniform" derived from Jockusch's notation in [Joc72]. The motivation for the present terminology is our use of Definition 4.1.2 below.

**Definition 4.1.2.** *We say that a computable function $f : \omega \times \omega \times \omega \to \omega$ is* uniform $\Delta_2^0$ approximating *if $lim_{s\to\infty} f_{e,s}(n)$ exists for all $n \in \omega$ and, in this case, we say that $\{f_{e,s}\}_{e,s\in\omega}$ is a* uniform $\Delta_2^0$ approximation. *Accordingly $f$ defines a class $\{f_e\}_{e\in\omega}$ such that $f_e(n) = lim_{s\to\infty} f_{e,s}(n)$ for all $e, n \in \omega$.*

By application of Lemma 4.1.3 (known as the Limit Lemma) we know that Definition 4.1.1 can be derived from this notion.

**Lemma 4.1.3** (Shoenfield)**.** *Let $A$ be a set. The following are equivalent.*

1) $A \leq_{\mathrm{T}} \mathcal{K}$.

2) *$A$ is $\Delta_2^0$.*

3) *There is a computable function $f : \omega \times \omega \to \{0, 1\}$ such that*

   (a) *For all $n \in \omega$, $f(n, 0) = 0$.*

   (b) *For all $n \in \omega$, $lim_{s\to\infty} f(n, s) = A(n)$.*

**Lemma 4.1.4.** *A class of functions $\mathcal{F}$ is uniform $\Delta_2^0$ if and only if there exists a uniform $\Delta_2^0$ approximation function $f$ such that $\mathcal{F} = \{f_e\}_{e\in\omega}$. In particular, a class of sets $\mathcal{C}$ is uniform $\Delta_2^0$ if and only if there exists a uniform $\Delta_2^0$ approximation $\{A_{e,s}\}_{e,s\in\omega}$ such that $\mathcal{C} = \{A_e\}_{e\in\omega}$ (using the standard shorthand identification of a set predicate with its characteristic function.)*

**Definition 4.1.5.** *A class $\mathcal{A} \subseteq \mathcal{D}_{\mathrm{e}}$ is said to be* uniform $\Delta_2^0$ *(*subuniform $\Delta_2^0$*) if there exists a uniform $\Delta_2^0$ class $\{A_e\}_{e\in\omega} \subseteq \mathcal{P}(\omega)$ such that, for all $\boldsymbol{a} \in \mathcal{D}_{\mathrm{e}}$,*

$$\boldsymbol{a} \in \mathcal{A} \Leftrightarrow \exists e[\, A_e \in \boldsymbol{a}\,] \qquad (\, \boldsymbol{a} \in \mathcal{A} \Rightarrow \exists e[\, A_e \in \boldsymbol{a}\,]\,)\,.$$

A uniform $\Delta^0_2$ class relevant to Theorem 4.2.1 is the following.

**Lemma 4.1.6** ([MC85]). *Let $\boldsymbol{a}$ be a low enumeration degree and suppose that $A \in \boldsymbol{a}$. Then the class $\{ X \mid X \leq_{\mathrm{e}} A \}$ is uniform $\Delta^0_2$. Thus $\{ \boldsymbol{x} \mid \boldsymbol{x} \leq \boldsymbol{a} \}$ is uniform $\Delta^0_2$.*

# 4.2 Main Construction

**Theorem 4.2.1.** *For every uniform $\Delta^0_2$ class of enumeration degrees $\mathcal{C}$ there exists a high $\Delta^0_2$ enumeration degree $\boldsymbol{c}$ such that for all $\boldsymbol{b} \in \mathcal{C} \cap \{ \boldsymbol{x} \mid \boldsymbol{0}_{\mathrm{e}} < \boldsymbol{x} < \boldsymbol{0}'_{\mathrm{e}} \}$, $\boldsymbol{c} \perp \boldsymbol{b}$*[1][2].

*Proof.* Let $\mathcal{B} = \{B_e\}_{e\in\omega}$ be a uniform $\Delta^0_2$ class of sets with uniform $\Delta^0_2$ approximation $\{\widehat{B}_{e,s}\}_{e,s\in\omega}$ such that, for each $\boldsymbol{c} \in \mathcal{C}$ there exists $C \in \boldsymbol{c}$ such that $C \in \mathcal{B}$ (and also such that $\mathcal{B} \subseteq \bigcup \{ \boldsymbol{x} \mid \boldsymbol{x} \in \mathcal{C} \}$). We note that the proof of Theorem 4.2.1 ends in page 92.

## 4.2.1 Requirements

The overall strategy is to construct a set $C$ such that the following requirements are satisfied (for all $e \in \omega$).

$$\begin{aligned} R &: \; C \text{ is } \Delta^0_2, \\ H_e &: \; W_e \text{ is infinite } \Leftrightarrow C^{[2e]} \text{ is finite}, \\ P_e &: \; C = \Phi_e^{B_e} \Rightarrow \overline{\mathcal{K}} \leq_{\mathrm{e}} B_e, \\ N_e &: \; B_e = \Phi_e^{C} \Rightarrow B_e \text{ is computably enumerable}, \end{aligned}$$

where $\{(W_e, \Phi_e, B_e)\}_{e\in\omega}$ is a computable listing of all triples of c.e. sets, c.e. operators and $\mathcal{B}$ with associated uniform c.e. approximations $\{W_{e,s}\}_{e,s\in\omega}$ and $\{\Phi_{e,s}\}_{e,s\in\omega}$ for

[1]Notice that this Theorem obviously implies the same statement with "uniform $\Delta^0_2$" replaced by "subuniform $\Delta^0_2$".

[2]$\boldsymbol{c} \perp \boldsymbol{b}$ denotes that degrees $\boldsymbol{c}$ and $\boldsymbol{b}$ are incomparable.

the former and uniform $\Delta^0_2$ approximation $\{B_{e,s}\}_{e,s\in\omega}$ for the latter—e.g. for all $e$, let $B_e = \widehat{B}_{(e)_2}$ etc.

By Lemma 3.2.4 we have that $J^{(2)}_{\mathrm{e}}(\emptyset) \equiv_{\mathrm{e}} \{\, e \mid W_e \text{ is infinite}\,\}$. Now by Lemma 3.2.3, satisfaction of $\{H_e\}_{e\in\omega}$ implies that $\{\, e \mid W_e \text{ is infinite}\,\} \leq_{\mathrm{e}} J_{\mathrm{e}}(C)$. Hence $J^{(2)}_{\mathrm{e}}(\emptyset) \leq_{\mathrm{e}} J_{\mathrm{e}}(C)$ and so $J^{(2)}_{\mathrm{e}}(\emptyset) \equiv_{\mathrm{e}} J^{(2)}_{\mathrm{e}}(C)$.

### Definitions and notation

During the construction a number is said to be *new* if it is larger than any number mentioned in $C$ up to that point in the construction.

Cooper-McEvoy Enumeration Reduction Approximations.

Throughout this thesis we will be assuming that if $X \in \Sigma^0_2$ then, for a given e-operator $\Phi_e$ with c.e. approximation $\{\Phi_{e,s}\}_{s\in\omega}$, $\{\Phi^X_{e,s}\}_{s\in\omega}$ is a $\Sigma^0_2$ approximation to $\Phi^X_e$ in the sense of 1.2.20.

The Priority Ordering.

The $H$, $P$ and $N$ requirements are ordered as follows: $H_i < P_i < N_i < H_{i+1}$ for all $i \in \omega$.

The Tree of Outcomes $\mathcal{T}$.

A tree of outcomes $\mathcal{T} \subseteq 3^{<\omega}$ is defined during the construction. For each $e$, the nodes of length $3e$, $3e+1$ and $3e+2$ are allocated respectively to requirements $H_e$, $P_e$ and $N_e$. At every stage $s$ of the construction a path $\alpha_s \in 3^{<\omega}$ of length $s$ is defined. For any given node $\sigma$ we say that a stage $s$ is $\sigma$-true if $\sigma \subset \alpha_s$ (i.e. $\sigma\,\hat{}\,i \subseteq \alpha_s$ for some $i \in \{0,1,2\}$). The full definition of $\mathcal{T}$ is given in Definition 4.2.2 below. Note that in the informal discussion below we we will refer to the *true path* $\delta$ as specified in Definition 4.2.2. For $S \in \{H, P, N\}$ node $\sigma$ *is* $S_e$ (shorthand $\sigma \in S_e$) if $\sigma$ is allocated to $S_e$ and we say that $\sigma$ *is* $S$ (shorthand $\sigma \in S$) if $\sigma \in S_e$ for some index $e$.

### 4.2.2 Basic idea of the construction

We describe the basic modules $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{N}$ for (respectively) the $H$, $P$ and $N$ requirements by considering their action relative to a given index $e$. For $\mathcal{J} \in \{\mathcal{H}, \mathcal{P}, \mathcal{N}\}$ the *outcome* of basic module $\mathcal{J}$ is the least $i \in \{1, 2, 3\}$ such that $\mathcal{J}.i$ is visited infinitely often.

**Note.** *In the description of the basic modules, "go to $\mathcal{J}.i$" indicates going to $\mathcal{J}.i$ at the* next stage. *Also, an action specified during a* wait *(for example adding a new number to $C^{[e]}$ in $\mathcal{H}.2$) is understood to be performed at every stage during the wait. The notation for the sets involved in the basic modules stands for their finite approximation at the stage under consideration with the exception of the notation*[3] *$\mathcal{K}[\max D]$ in (4.2.1) which stands for the approximation to $\mathcal{K}$ at stage $s = \max D$. We say that an outcome is* finitary *if it stabilises after a finite number of stages. Otherwise we say that it is* infinitary.

Basic Module $\mathcal{H}$. The basic strategy involves directly processing numbers in $\omega^{[2e]}$ as follows.

$\mathcal{H}.1$. Remove all numbers from $C^{[2e]}$ and go to $\mathcal{H}.2$.

$\mathcal{H}.2$. Wait for a number to enter $W_e$. Add a new number to $C^{[2e]}$.

$\mathcal{H}.3$. (Some number enters $W_e$). Go back to $\mathcal{H}.1$.

*The outcomes.* There are two outcomes: $\mathcal{H}.1$ (infinitary) and $\mathcal{H}.2$ (finitary). Outcome $\mathcal{H}.1$ occurs if $W_e$ is infinite, in which case $C^{[2e]} = \emptyset$. Outcome $\mathcal{H}.2$ occurs if $W_e$ is finite, in which case $C^{[2e]}$ is infinite.

Basic Module $\mathcal{P}$. The basic strategy involves directly processing numbers in $\omega^{[2e+1]}$

[3] We sometimes use the shorthand $X[s]$ instead of $X_s$.

as follows.

$\mathcal{P}$.1. Wait for some number $n$ such that $\langle 2e+1, n\rangle \in C$ and

$$n \in \mathcal{K} \ \ \& \ \ \langle\langle 2e+1, n\rangle, D\rangle \in \Phi_e \ \ \& \ \ D \subseteq B_e \ \ \& \ \ n \notin \mathcal{K}[\max D]\,. \tag{4.2.1}$$

Add the least $m$ such that $\langle 2e+1, m\rangle \notin C$ into $C$.

$\mathcal{P}$.2. (4.2.1) applies for some $\langle\langle 2e+1, n\rangle, D\rangle \in \Phi_e$. Choose the least such $\langle 2e+1, n\rangle$ and go to $\mathcal{P}$.3.

$\mathcal{P}$.3. Wait for $D \nsubseteq B_e$. Remove[4] the number $\langle 2e+1, n\rangle$ chosen in $\mathcal{P}$.2 from $C$.

$\mathcal{P}$.4. ($D \nsubseteq B_e$). Go back to $\mathcal{P}$.1.

*The outcomes.* There are two outcomes: $\mathcal{P}$.1 (infinitary or finitary) and $\mathcal{P}$.3 (finitary). If outcome $\mathcal{P}$.3 occurs and $\langle\langle 2e+1, n\rangle, D\rangle$ is the axiom that causes this, then $\langle 2e+1, n\rangle \notin C$ whereas $\langle\langle 2e+1, n\rangle, D\rangle \in \Phi_e$ and $D \subseteq B_e$. Hence $\langle 2e+1, n\rangle \in \Phi_e^{B_e} - C$. If outcome $\mathcal{P}$.1 occurs then note firstly that, for any given $n$, $\mathcal{P}$.3 (and $\mathcal{P}$.2) can only apply to $\langle 2e+1, n\rangle$ finitely often. This is because $\mathcal{P}$.3 only applies to $\langle 2e+1, n\rangle$ if $n \in \mathcal{K}$. In this case it also only applies relative to some axiom $\langle\langle 2e+1, n\rangle, D\rangle$ if $D \subseteq B_e$ and $\max D \leq t_n$ where $t_n$ is the stage such that $n \in \mathcal{K}[t_n+1] - \mathcal{K}[t_n]$. Since there are only finitely many such axioms and $\{B_{e,s}\}_{s\in\omega}$ is a $\Delta^0_2$ approximation, it follows from the fact that $\mathcal{P}$.3 is not the outcome that there exists a stage $\hat{s}$ after which no such axiom for $\langle 2e+1, n\rangle$ can cause an instance of $\mathcal{P}$.3. Indeed let $E \subseteq \overline{B_e}$ be a finite set such that $D \cap E \neq \emptyset$ for each such axiom $\langle\langle 2e+1, n\rangle, D\rangle$. Then, using the fact that $\{B_{e,s}\}_{s\in\omega}$ is a $\Delta^0_2$ approximation we can let $\hat{s}$ be a stage such that $E \cap B_e[s] = \emptyset$ for all $s > \hat{s}$. We deduce from this that $\omega^{[2e+1]} \subseteq C$ when outcome $\mathcal{P}$.1 occurs. Suppose now that $C = \Phi_e^{B_e}$. Consider any

[4] Of course, when the $\mathcal{P}$ basic module is looked at in isolation after the first removal of $\langle 2e+1, n\rangle$ from $C$ this removal activity is redundant since $\langle 2e+1, n\rangle$ is then not in $C$ in any case. However it is necessary in the context of the interaction of basic modules $\mathcal{P}$ and $\mathcal{N}$. Note also that in the construction, for reasons of bookkeeping every $\langle 2e+1, m\rangle \in C$ such that $m \geq n$ is removed from $C$.

$n \in \omega$. If $n \in \overline{\mathcal{K}}$ then, as $\langle 2e+1, n\rangle \in C = \Phi_e^{B_e}$ there exists a finite set $D$ such that $\langle\langle 2e+1, n\rangle, D\rangle \in \Phi_e$ and $D \subseteq B_e$ and (trivially) $n \notin \mathcal{K}[\max D]$. On the other hand if $n \in \mathcal{K}$ then there is no such axiom $\langle\langle 2e+1, n\rangle, D\rangle$ since this would cause outcome $\mathcal{P}.3$ to happen. Hence for all $n \in \omega$, $n \in \overline{\mathcal{K}}$ if and only if there exists $D$ such that $\langle\langle 2e+1, n\rangle, D\rangle \in \Phi_e$, $D \subseteq B_e$, and $n \notin \mathcal{K}[\max D]$. In other words $\overline{\mathcal{K}} \leq_{\mathrm{e}} B_e$.

Basic Module $\mathcal{N}$. The basic strategy involves enumerating a c.e. set $W(e)$ and auxiliary set $I(e)$ of instigator candidates.

$\mathcal{N}.1$. Wait for some $n \in W(e)$ such that $n \notin B_e$. For any $m \in B_e \cap \Phi_e^C$ such that $m \notin W(e)$ enumerate $m$ into $W(e)$ and enumerate a single axiom $\langle m, D\rangle$ witnessing $m \in \Phi_e^C$ into $I(e)$.

$\mathcal{N}.2$. (For some $n \in W(e)$, $n \notin B_e$.) Wait for $n \in B_e$. Restrain in $C$ the unique finite set $D$ such that $\langle n, D\rangle \in I(e)$.

$\mathcal{N}.3$. ($n \in B_e$) Remove the restraint over $D$. Return to $\mathcal{N}.1$.

*The outcomes.* There are two outcomes: $\mathcal{N}.1$ (infinitary or finitary) and $\mathcal{N}.2$ (finitary[5]). If the outcome is $\mathcal{N}.2$ then for some fixed axiom $\langle n, D\rangle \in \Phi_e$, $n \notin B_e$ whereas $n \in \Phi_e^C$ since $D$ is restrained in $C$. Thus $B_e \neq \Phi_e^C$. On the other hand, if the outcome is $\mathcal{N}.1$ and $B_e = \Phi_e^C$ then, for every $n \in B_e$, $n$ will eventually be enumerated into $W(e)$. Hence $B_e \subseteq W(e)$. Conversely $n \in W(e) - B_e$ is impossible since this would entail outcome $\mathcal{N}.2$. Thus $W(e) = B_e$ and so $B_e$ is c.e.

Coherence between the Basic Modules. There is no direct interference between the $\mathcal{H}$ and $\mathcal{P}$ basic modules. However $\mathcal{H}.1$ and $\mathcal{P}.3$ clearly conflict with the restraining activity of $\mathcal{N}.2$. Accordingly, in order for the overall strategy to cohere the construction distributes the requirements over a ternary priority tree $\mathcal{T}$—where reinitialisation of nodes to the right of $\alpha_s$ at the end of stage $s$ plays an important

[5]Note that outcome $\mathcal{N}.2$ is finitary as a consequence of the fact that $\{B_{e,s}\}_{s\in\omega}$ is a $\Delta^0_2$ approximation.

role—so that injury to the activity of any node $\sigma$ such that $\sigma \subset \delta$ (the true path) can only originate from the finite number of nodes to the left or above $\sigma$ on $\mathcal{T}$ (i.e. from nodes of higher priority on $\mathcal{T}$).

Indeed an $H$ node keeps a record of restraints of higher priority $N$ nodes and only removes numbers from $C$ that do not occur within the union of this finite number of restraints. Thus outcome $\mathcal{H}.1$ at index $e$ corresponds to the action of $H_e$ node $\sigma \subset \delta$ removing all numbers from $C^{[2e]}$ except for a finite set.

A node $\sigma \in P_e$ on the other hand only processes numbers in $C^{[2e+1]}$ above a certain threshold bounding those numbers mentioned in $C$ up to the least point in the construction at which $\sigma$ was visited and since when $\sigma$ has not been reinitialised. In this way the *main* extracting activity corresponding to $\mathcal{P}.3$ never affects the restraints of higher priority $N$ nodes. However $\sigma$ may also extract numbers from $C^{[2e+1]}$ below its threshold in order to preserve extractions by higher priority $P_e$ nodes. Nevertheless this extracting activity is defined so as to respect restraints belonging to $N$ nodes of higher priority than $\sigma$. (This *auxiliary* extracting activity is necessary in order to ensure that $\{C_s\}_{s\in\omega}$ is a $\Delta^0_2$ approximation.) Note also that any $P$ node $\sigma \subset \delta$ has outcomes $\sigma\widehat{\ }0$, $\sigma\widehat{\ }1$ or $\sigma\widehat{\ }2$ corresponding (respectively) to the infinitary case of the $\mathcal{P}.1$ outcome, the $\mathcal{P}.3$ outcome, and the finitary case of the $\mathcal{P}.1$ outcome. This means that, for any stage $s$, the extracting activity undertaken due to $\sigma\widehat{\ }1 \subseteq \alpha_s$ may remove numbers lying within restraints or involved in instigator candidates[6] belonging to $N$ nodes $\tau$ on the subtree below $\sigma\widehat{\ }0$. However, if $\sigma\widehat{\ }0 \subset \delta$ then there exists a stage $t$, such that for all stages $s \geq t$, *any* $\langle 2e+1, n\rangle$ extracted due to $\sigma\widehat{\ }1 \subseteq \alpha_s$ will be (permanently) enumerated back into $C$ at some later stage. Hence, even though the activity of $\sigma$ ($\subset \delta$) may cause infinite injury along the true path, the injury itself tends to infinity (in terms of the numbers involved).

[6] I.e. in the formalism of the proof such numbers lie within some restraint $E(\tau, s)$ or some $D$ such that, for some $n$, $\langle n, D\rangle \in I(\tau, s)$, with $\tau$ being an $N$ node such that $\sigma\widehat{\ }0 \subseteq \tau$.

A node $\sigma \in N$ only restrains a finite set $D$ in $C$ (corresponding to $\mathcal{N}.2$) if, for some $n$, $\langle n, D\rangle$ is an instigator candidate for $\sigma$ and if no $H$ or $P$ node $\beta \subset \sigma$ extracts numbers belonging to $D$. Hence the restraining activity of $\sigma$ can only injure the activity of lower priority $H$ or $P$ nodes. However the activity (of collecting valid instigator candidates and maintaining a restraint) of $\sigma$ may be injured by $\beta$ if $\beta$ is a $P$ node and $\beta\frown 0 \subseteq \sigma$. Nevertheless, as explained in the previous paragraph, if $\sigma$ is on the true path this injury does not affect the final outcome of the activity of $\sigma$.

<u>A sketch of how requirement $R$ is satisfied.</u>

If a number $x$ is extracted from $C$ at infinitely many stages this is due to the extraction activity of some $H$ or $P$ node on the true path. Consider index $e$ and suppose, without loss of generality, that for $P_e$ node $\sigma$, $\sigma\frown 1$ is on the true path, and note here that the case when $\sigma$ is an $H_e$ node and $\sigma\frown 0 \subset \delta$ is similar. Then there exists a least $\sigma\frown 1$-true stage $s_\sigma$, after which the construction never visits nodes to the left of $\sigma\frown 1$, such that at $s_\sigma$, and all subsequent $\sigma$ true stages, $x = \langle 2e+1, z\rangle$ is always extracted.

In order to see this, consider firstly the case in which, for every $N$ node $\beta \subseteq \sigma$ it is the case that $\beta\frown 1 \subseteq \sigma$. Then by construction, at every subsequent stage $s$ the $P_e$ node $\sigma_s$ on the $s$-stage path $\alpha_s$ is free to extract $x$ from[7] $C$ (although in this particular case—with $\beta\frown 1 \subseteq \sigma$ for all $N$ nodes $\beta \subseteq \sigma$—$x$ does not recur in $C$ in any case.)

Now consider the case in which there exists precisely one $N$ node $\beta$ such that $\beta\frown 0 \subseteq \sigma$. Then $x$ may lie in some $D$ such that $\langle n, D\rangle$ has already been chosen as

[7] This is because all restraints belonging to $N$ nodes above $\sigma$ along the true path are already fixed. Also, all $N$ nodes $\tau$ to the right of the true path are reinitialised at stage $s_\sigma$ and so the set of instigator candidates belonging to $\tau$ is set to $\emptyset$ at stage $s_\sigma$. Thus, by construction, for any such $\tau$ and instigation candidate $\langle n, D\rangle$ belonging to $\tau$ at stages subsequent to $s_\sigma$, $x \notin D$. Thus $x$ will appear in no active $N$ restraint (the restraints for $N$ nodes on the subtree below $\sigma\frown 0$ having no effect relative to the extracting activity of $\sigma$ after stage $s_\sigma$) subsequent to stage $s_\sigma$. Thus there is no obstacle to $P_e$ node $\alpha_s$ extracting $x$ from $C$.

an instigator candidate for $\beta$ at stage $s_\sigma$. Hence there may be stages $s > s_\sigma$ such that $\beta^\frown 1 \subseteq \alpha_s$ and $\beta$ restrains $D$, and thus also $x$, in $C$. However, as $\beta^\frown 0 \subseteq \sigma$ is on the true path and since $\{B_{e,s}\}_{s\in\omega}$ is a $\Delta^0_2$ approximation, there will be a stage such that—supposing that $i$ is such that $\beta$ is an $N_i$ node—$n$ permanently enters $B_i$. Hence there exists a stage $r_\sigma$ such that, for all $s \geq r_\sigma$ and for all such instigator candidates $\langle n, D\rangle$, $D$ is never restrained by $\beta$ at stage $s$. Moreover, instigation candidates $\langle n, D\rangle$ for $\beta$ can only be chosen at $\beta^\frown 0$-true stages $s$ if $D \subseteq C_t$ where $t$ is the last $\beta^\frown 0$-true stage[8]. However, for $t \geq s_\sigma$, by construction at each such stage the $P_e$ node $\sigma_t \subseteq \alpha_t$ removes (or is free to remove) $x$ from $C$ (the first such stage being $t = s_\sigma$) so no instigator candidate $\langle m, D\rangle$ chosen by $\beta$ at a stage subsequent to $s_\sigma$ is such that $x \in D$. Therefore $x \notin C_s$ for all $s \geq r_\sigma$.

The case of more than one node $\beta$ such that $\beta^\frown 0 \subseteq \sigma$ is a straightforward generalisation of the above case and is discussed in the proof of Lemma 4.2.14. Accordingly it can be deduced that the approximation $\{C_s\}_{s\in\omega}$ defined during the construction is $\Delta^0_2$.

<u>Parameters.</u>

There is one timing parameter $t(\sigma, s+1) \in \omega\restriction s+1$ used in the construction to record the last stage $t$ at which $\sigma \subset \alpha_t$.

Parameters associated with each requirement and each node are defined during the construction.

We use bracket notation (e.g. $V(\sigma, s)$) for parameters associated with nodes and subscript notation (e.g. $V_{e,s}$) for parameters associated with requirements. Upper case letters are used for parameters ranging over (finite) sets and states and lower case letters are used for parameters ranging over numbers and singleton sets. $\mathcal{F}$

[8]And $\beta$ has not been reinitialised since stage $t$—however this is the case for $s > s_\sigma$ by definition of $s_\sigma$.

denotes the class of finite subsets of $\omega$ and $\mathcal{F}_e$ ($\mathcal{S}_e$) the class of finite (singleton) subsets of $\omega^{[e]}$.

*Remark.* Notes on the meanings of the parameters and the action taken can be found after each separate case of the description of the construction at stage $s+1$.

### 4.2.3 The Construction.

The construction proceeds by stages $s \in \omega$. At each stage a finite approximation $C_s$ to the set $C$ is defined where $C =_{\mathrm{def}} \{\, x \mid \exists t \forall s \geq t [\, x \in C_s \,] \,\}$.

*Stage* 0. Initially $C_0 = \emptyset$ whereas all parameters (defined below) are set to their initial values as follows. For all indices $e$ and nodes $\sigma$, $t(\sigma, 0) = 0$; if $|\sigma| = 0 \bmod 3$ then $H(\sigma, 0) = 0$, and $U_{e,0} = V_{e,0} = \Omega(\sigma, 0) = \emptyset$; if $|\sigma| = 1 \bmod 3$ then $P(\sigma, 0) = -1$, $b(\sigma, 0) = h(\sigma, 0) = w(\sigma, 0) = \uparrow$ and $T_{e,0} = V(\sigma, 0) = \emptyset$; if $|\sigma| = 2 \bmod 3$ then $N(\sigma, 0) = -1$, $x(\sigma, 0) = \uparrow$ and $E(\sigma, 0) = W(\sigma, 0) = I(\sigma, 0) = \emptyset$.

*Stage* $s+1$. The $s+1$ stage path $\alpha_{s+1}$ of length $s+1$ is constructed at this stage. There are $s+1$ substages: at each substage $n$ (such that $1 \leq n \leq s+1$) a node of length $n-1$ is processed. We thus assume for $n$ such that the node $\sigma \subseteq \alpha_{s+1}$ of length $n-1$ has been defined (so that if $n = 1$, $\sigma = \lambda$) and define below how the node $\sigma$ is processed. The parameter $t(\sigma, s+1)$ is defined at the beginning of this stage as follows.

$$t(\sigma, s+1) \;=\; \begin{cases} \max\{\, t \mid t \leq s \quad \& \;\; \sigma \subset \alpha_t \,\} & \\ & \text{if there exists such } t, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.2}$$

The node $\sigma$ is now processed according as to whether $n-1 = |\sigma| = 0$, 1, or 2 modulo 3.

*Case* 1: $|\sigma| = 0 \bmod 3$. Thus $\sigma$ is an $H$ node. Suppose that $|\sigma| = 3e$, i.e. that

$\sigma \in H_e$.

*Parameters.* $H(\sigma, s) \in \{0, 1\}$ is the *state*, $u(\sigma, s) \in \mathcal{S}_{2e} \cup \{\emptyset\}$ the *enumerator*, $U_{e,s} \in \mathcal{F}_{2e}$ the *inclusion*, $V_{e,s} \in \mathcal{F}_{2e}$ the *exclusion*, and $\Omega(\sigma, s) \in \mathcal{F}_{2e}$ the *avoidance* parameter. Note that $u(\sigma, s)$ is a temporary parameter in the sense that it only has meaning—and is non trivially defined—when $s$ is $\sigma$-true.

The parameter $\Omega(\sigma, s)$ is redefined as follows.

$$\Omega(\sigma, s+1) \quad = \bigcup_{\substack{\tau \in N \\ \& \ \tau <_L \sigma}} E(\tau, s)^{[2e]} \quad \cup \bigcup_{\substack{\tau \in N \\ \& \ \tau \subset \sigma}} E(\tau, s+1)^{[2e]} \,. \tag{4.2.3}$$

There are two subcases.

<u>*Case 1.A:*</u> $W_{e,s+1} - W_{e,t(\sigma,s+1)} \neq \emptyset$.

Set the state parameter $H(\sigma, s+1) = 0$ and redefine the set ($\subseteq \omega^{[2e]}$) that $H_e$ wants to keep out of $C$: $V_{e,s+1} = V_{e,s} \cup U_{e,s}$. Now reinitialise the set that $H_e$ wants to put in $C$: $U_{e,s+1} = \emptyset$ (and set $u(\sigma, s+1) = \emptyset$).

<u>*Notes.*</u> *$H(\sigma, s+1) = 0$ means that the present guess of the construction is that $W_e$ is infinite. Note that the construction wants to extract $V_{e,s+1}$ from $C$ with the (present) aim of making $C^{[2e]}$ finite. $U_{e,s}$ is the set of numbers in $\omega^{[2e]}$ present in $C$ at stage $s$ due to previous action taken at $H_e$ nodes.*

<u>*Case 1.B:*</u> $W_{e,s+1} - W_{e,t(\sigma,s+1)} = \emptyset$.

Set $H(\sigma, s+1) = 1$, choose a new number $\langle 2e, z\rangle$, and set $u(\sigma, s+1) = \{\langle 2e, z\rangle\}$ (to be enumerated into $C$ at the end of stage $s+1$). Also set $U_{e,s+1} = U_{e,s} \cup u(\sigma, s+1)$ and $V_{e,s+1} = V_{e,s}$.

<u>*Notes.*</u> *$H(\sigma, s+1) = 1$ means that the present guess of the construction is that $W_e$ is finite so that the construction wants $C^{[2e]}$ to be infinite. Note that, in this case, for $t \geq s$, $U_{e,t} \subseteq U_{e,t+1}$ and $|U_{e,t+1} \setminus U_{e,t}| = 1$ for as long as $H(\sigma_{t+1}, t+1) = 1$—where $\sigma_{t+1}$ is the $H_e$ node $\subseteq \alpha_{t+1}$.*

**To end this substage.** After both case 1.A and case 1.B define the set to be removed from $C$ at the end of stage $s+1$: $V(\sigma, s+1) = V_{e,s+1} - \Omega(\sigma, s+1)$. Go to the *Final Step of Substage n* (below).

*Notes. The parameter $\Omega(\sigma, s+1)$ contains the set of numbers restrained by $N$ nodes $\tau$ of higher priority in $\mathcal{T}$ (i.e. $\tau <_L \sigma$ or $\tau \subset \sigma$). $V(\sigma, s+1)$ is the actual set of numbers that the construction extracts from $C^{[2e]}$ at stage $s+1$.*

*Case* 2: $|\sigma| = 1 \bmod 3$. Thus $\sigma$ is a $P$ node. Suppose that $|\sigma| = 3e+1$, i.e. that $\sigma \in P_e$.

*Parameters.* $P(\sigma, s) \in \{-1, 0, 1, 2\}$ is the *state*, $b(\sigma, s) \in \omega \cup \{\uparrow\}$ the *threshold*, $h(\sigma, s) \in \omega \cup \{\uparrow\}$ the *height*, $w(\sigma, s) \in \omega \cup \{\uparrow\}$ the *witness*, $u(\sigma, s) \in \mathcal{S}_{2e+1} \cup \{\emptyset\}$ the *enumerator*, $T_{e,s} \in \mathcal{F}_{2e+1}$ the *inclusion*, $V^-(\sigma, s) \in \mathcal{F}_{2e+1}$ the *lower exclusion*, $V^+(\sigma, s) \in \mathcal{F}_{2e+1}$ the *upper exclusion* and $V(\sigma, s)$ the *combined exclusion* parameter. Note that $u(\sigma, s)$, $V^-(\sigma, s)$, and $V^+(\sigma, s)$ are temporary parameters, i.e. they only have meaning when $s$ is $\sigma$-true.

Define the temporary parameter $V^-(\sigma, s+1)$ as follows.

$$V^-(\sigma, s+1) \quad = \quad \{\, z \mid (\exists \gamma \in P_e)[\, \gamma <_L \sigma \;\;\&\;\; z \in V(\gamma, s) \;\;\&\;\; z \notin \varepsilon(\gamma, \sigma, s+1)\,]\,\}$$

with

$$\varepsilon(\gamma, \sigma, s+1) \quad = \quad \varepsilon^{<_L}(\gamma, \sigma, s+1) \;\cup\; \varepsilon^{\subset}(\gamma, \sigma, s+1)$$

and

$$\varepsilon^{<_L}(\gamma, \sigma, s+1) \quad = \quad \bigcup \{\, E(\beta, s) \mid \beta \in N \;\;\&\;\; \gamma <_L \beta^\frown 1 <_L \sigma \,\}$$

whereas

$$\varepsilon^{\subset}(\gamma, \sigma, s+1) \quad = \quad \bigcup \{\, E(\beta, s+1) \mid \beta \in N \;\;\&\;\; \beta^\frown 1 \subseteq \sigma \,\}.$$

*Remark.* The following definition of $V^-(\sigma, s+1)$ is equivalent to the above. Define

$$S^{<_L}(\sigma, s+1) = \{\, \gamma \mid \gamma \in P_e \cap \mathcal{T}_s \;\;\&\;\; \gamma <_L \sigma \,\}$$

and let $\Lambda(\sigma, s+1)$ be the $<_L$-rightmost $\tau \in S^{<_L}(\sigma, s+1)$ if $S^{<_L}(\sigma, s+1) \neq \emptyset$ (and be undefined otherwise). Now set $V^-(\sigma, s+1)$

$$= \begin{cases} \emptyset & \text{if } S^{<_L}(\sigma, s+1) = \emptyset, \\ \{\, z \mid z \in V(\Lambda(\sigma, s+1), s) \;\;\&\;\; z \notin \varepsilon(\Lambda(\sigma, s+1), \sigma, s+1) \,\} & \text{otherwise.} \end{cases}$$

*Notes.* $V^-(\sigma, s+1) \subseteq \{\, \langle 2e+1, n\rangle \mid n < b(\sigma, s+1) \,\}$*—with* $b(\sigma, s+1)$ *defined below—is the set that the construction extracts from* $C$ *at the end of stage* $s+1$ *for the sake of higher priority* $P_e$ *nodes, i.e. nodes in the set* $\{\, \gamma \mid |\gamma| = 3e+1 \;\;\&\;\; \gamma <_L \sigma \,\}$.

There are two subcases.

*Case 2.A:* $P(\sigma, s) = -1$. Then choose $b(\sigma, s+1) \in \omega$ so that $\langle 2e+1, b(\sigma, s+1)\rangle$ is new. Also set $h(\sigma, s+1) = b(\sigma, s+1)$. Define $P(\sigma, s+1) = 0, w(\sigma, s+1) = \uparrow$ and $V^+(\sigma, s+1) = u(\sigma, s+1) = \emptyset$ and $T_{e,s+1} = T_{e,s}$.

*Notes.* $P(\sigma, s) = -1$ *means that* $\sigma$ *is in its initial state so that* $b(\sigma, s) = h(\sigma, s) = w(\sigma, s) = \uparrow$ *and* $V(\sigma, s) = \emptyset$. *The parameter* $b(\sigma, s+1)$ *gives a lower bound for numbers* $n$ *such that* $\langle 2e+1, n\rangle$ *is added or removed from* $C$ *for the sake of* $\sigma$ *at stage* $s+1$ *whereas* $\{\, \langle 2e+1, n\rangle \mid b(\sigma, s+1) \leq n < h(\sigma, s+1) \,\}$ *is the set put into* $C$ *for the sake of* $\sigma$ *at this stage (i.e. the empty set in the present case).*

*Case 2.B:* $P(\sigma, s) \in \{0, 1, 2\}$. Then test whether there exists $n$ such that $b(\sigma, s) \leq n < h(\sigma, s)$ and for some finite set $D$,

$$n \in \mathcal{K}[s] \;\;\&\;\; \langle \langle 2e+1, n\rangle, D\rangle \in \Phi_e[s] \;\;\&\;\; D \subseteq B_e[s] \;\;\&\;\; n \notin \mathcal{K}[\max D]\,. \tag{4.2.4}$$

There are now 3 possible cases. Apply the first case that holds.

*Case 2.B.1:* There is such an $n$. Then for the least such $n$ set $h(\sigma, s+1) = n$.

Also define[9] $P(\sigma, s+1) = 0$, $V^+(\sigma, s+1) = u(\sigma, s+1) = \emptyset$, $w(\sigma, s+1) = \uparrow$, $b(\sigma, s+1) = b(\sigma, s)$ and $T_{e,s+1} = T_{e,s}$.

Notes. $\langle 2e+1, h(\sigma, s+1)\rangle$ *is a new (better) prediction of a diagonalisation via* (4.2.4) *(with* $n = h(\sigma, s+1)$*) i.e. the prediction that* $\langle 2e+1, h(\sigma, s+1)\rangle \in \Phi_e^{B_e} \backslash C$. $P(\sigma, s+1) = 0$ *indicates that the construction does not yet have enough evidence that this is a true diagonalisation—since the diagonalisation* candidate $\langle 2e+1, h(\sigma, s+1)\rangle$ *has changed. More generally* $P(\sigma, s+1) \in \{0, 2\}$ *indicates the construction's prediction that*

$$C \cap \omega^{[2e+1]} = \{ \langle 2e+1, n\rangle \mid n \geq b(\sigma, s+1) \}$$

*and that* $C = \Phi_e^{B_e}$ *and* $\overline{\mathcal{K}} \leq_{\mathrm{e}} B_e$. $T_{e,s+1}$ *(*$\supseteq T_{e,s}$*) contains every number* $x \in \omega^{[2e+1]}$ *mentioned in* $C$ *up to this point in the construction.* $V^+(\sigma, s+1) \subseteq \{ \langle 2e+1, n\rangle \mid n \geq h(\sigma, s+1) \}$ *is the set that the construction extracts from* $C$ *for the sake of* $\sigma$ *at (the end of) this stage and is only non empty when* $P(\sigma, s+1) = 1$.

Case 2.B.2: $n = h(\sigma, s)$ satisfies (4.2.4) (but no $b(\sigma, s) \leq \hat{n} < h(\sigma, s)$ does). Then set $h(\sigma, s+1) = w(\sigma, s+1) = n$, $P(\sigma, s+1) = 1$ and

$$V^+(\sigma, s+1) = T_{e,s} \cap \{ \langle 2e+1, m\rangle \mid m \geq h(\sigma, s+1) \}. \tag{4.2.5}$$

Also set $u(\sigma, s+1) = \emptyset$, $b(\sigma, s+1) = b(\sigma, s)$, and $T_{e,s+1} = T_{e,s}$.

Notes. *This is the only case in which* $w(\sigma, s+1)$ *is non trivially defined. Indeed* $P(\sigma, s+1) = 1$ *indicates the construction's prediction that* $\langle 2e+1, w(\sigma, s+1)\rangle$ *witnesses* $\Phi_e^{B_e} \setminus C \neq \emptyset$ *via* (4.2.4) *(with* $n = w(\sigma, s+1)$*). Notice that* (4.2.5) *means that, for every* $m \geq h(\sigma, s+1)$ *such that* $\langle 2e+1, m\rangle$ *has been mentioned thus far in the construction,* $\langle 2e+1, m\rangle$ *is extracted from* $C$ *at this stage. (The ultimate objective being that* $\langle 2e+1, w(\sigma, s+1)\rangle \notin C$

[9] We could choose to define $V^+(\sigma, s+1)$ as in (4.2.5) in order to try to avoid injury to $N$ nodes in the subtree below $\sigma\frown 0$ when $\sigma\frown 1 \subseteq \alpha_{t+1}$ at some later stage $t+1$. However, even if we do this, if $h(\sigma, r+1) < h(\sigma, s+1)$ at some later $\sigma$-true stage $r+1$ (and so $\sigma\frown 0 \subseteq \alpha_{r+1}$) the $N$ nodes in the subtree below $\sigma\frown 0$ will nevertheless sustain injury at stage $r+1$. (Note that the option of reinitialising all such $N$ nodes in this case—e.g. at stage $r+1$—would cause fatal injury to $N$ nodes along the true path.)

*if the prediction* $P(\sigma, s+1) = 1$ *is correct.)*

<u>*Case 2.B.3:*</u> (Neither of the above 2 cases applies.)

There are two subcases.

<u>*Case 2.B.3.i:*</u> $P(\sigma, s) \in \{0, 2\}$. Then set $u(\sigma, s+1) = \{\langle 2e+1, h(\sigma, s)\rangle\}$, $h(\sigma, s+1) = h(\sigma, s) + 1$, and define $P(\sigma, s+1) = 2$, $V^+(\sigma, s+1) = \emptyset$, $w(\sigma, s+1) = \uparrow$ and $T_{e,s+1} = T_{e,s} \cup u(\sigma, s+1)$.

<u>Notes.</u> *The construction predicts that* $C = \Phi_e^{B_e}$ *and* $\overline{\mathcal{K}} \leq B_e$ *in this case, i.e. when* $P(\sigma, s+1) = 2$. *This is the only case when the set* $u(\sigma, s+1)$*—to be enumerated into* $C$ *at the end of this stage—is non trivially defined. Note that if the present prediction is indeed true (and* $s$ *is a late enough stage and* $\sigma$ *is on the true path) the construction will eventually enumerate all of the set* $\{\, \langle 2e+1, m\rangle \mid m \geq b(\sigma, s)\,\}$ *into* $C$.

<u>*Case 2.B.3.ii:*</u> $P(\sigma, s) = 1$. Then set $h(\sigma, s+1) = h(\sigma, s)$ and define $P(\sigma, s+1) = 0$, $V^+(\sigma, s+1) = u(\sigma, s+1) = \emptyset$, $w(\sigma, s+1) = \uparrow$, $b(\sigma, s+1) = b(\sigma, s)$ and $T_{e,s+1} = T_{e,s}$.

<u>Notes.</u> *The construction's previous prediction that* $\langle 2e+1, w(\sigma, s)\rangle$ *(with* $w(\sigma, s) = h(\sigma, s)$*) witnesses* $\Phi_e^{B_e} \setminus C \neq \emptyset$ *has collapsed in this case and* $P(\sigma, s+1) = 0$ *indicates that the construction now considers the opposite outcome (as described above) more likely. Notice also that, as* $\sigma\hat{\ }0 \subseteq \alpha_{s+1}$ *in this case, all the nodes* $\gamma$ *such that* $\sigma\hat{\ }1 \subseteq \gamma$ *are reinitialised at the end of stage* $s+1$.

**To end this substage.** After each of the cases 2.A and 2.B.1-2.B.3, define the set to be removed from $C$ at the end of stage $s{+}1$: $V(\sigma, s{+}1) = V^-(\sigma, s{+}1) \cup V^+(\sigma, s{+}1)$. Go to the *Final Step of Substage* $n$.

<u>*Case 3:* $|\sigma| = 2$ *mod* 3.</u> Thus $\sigma$ is an $N$ node. Suppose that $|\sigma| = 3e+1$, i.e. that $\sigma \in N_e$.

Given stages $t \leq s$ we define $x \in \Phi_e^C[t, s]$ if and only if $x \in \Phi_e^C[q]$ for all stages $q$

such that $t \leq q \leq s$. Note that, with reference to the timing parameter $t(\sigma, s)$, if $t(\sigma, s) = 0$ (with $s > 0$), then $\Phi^C_e[t(\sigma, s), s-1] = \emptyset$ since $\Phi^C_e[0] = \emptyset$.

*Parameters.* $N(\sigma, s) \in \{-1, 0, 1\}$ is the *state*, $x(\sigma, s) \in \omega \cup \{\uparrow\}$ the *witness*, $E(\sigma, s) \in \mathcal{F}$ the *restraint*, $W^*(\sigma, s) \in \mathcal{F}$ the *enumerator set*, $W(\sigma, s) \in \mathcal{F}$ the *c.e. set approximation*, $I(\sigma, s) \in \omega \times \mathcal{F}$ the *overall set of instigator candidates*, $I^-(\sigma, s) \subseteq I(\sigma, s)$ the *set of invalid instigator candidates*, $I^+(\sigma, s) \subseteq I(\sigma, s)$ the *set of valid instigator candidates*, and $I^*(\sigma, s)$ the *enumerator set of instigator candidates*. Note that $W^*(\sigma, s)$, $I^-(\sigma, s)$, $I^+(\sigma, s)$, and $I^*(\sigma, s)$ are temporary parameters, i.e. they only have meaning when $s$ is $\sigma$-true.

Define the auxiliary parameter.

$$\chi(\sigma, s+1) \quad = \quad \bigcup \{\, V(\beta, s+1) \mid \beta \in N \cup H \ \ \& \ \ \beta \subset \sigma \,\} \tag{4.2.6}$$

and set

$$\begin{aligned} I^+(\sigma, s+1) \quad = \quad \{\, & \langle x, D \rangle \mid \langle x, D \rangle \in I(\sigma, s) \\ & \& \ \ D^{[\leq 2e+1]} \subseteq C[t(\sigma, s+1), s] \ \cap \ \overline{\chi(\sigma, s+1)} \,\} \end{aligned} \tag{4.2.7}$$

whereas

$$I^-(\sigma, s+1) \quad = \quad I(\sigma, s+1) \ - \ I^+(\sigma, s+1)\,. \tag{4.2.8}$$

There are three subcases.

<u>*Case 3.A.*</u> $\sigma$ is in its initial or reinitialised state $N(\sigma, s) = -1$.

Notice that for $X \in \{I, W, E\}$, $X(\sigma, s) = \emptyset$ whereas $x(\sigma, s) = \uparrow$. Then set $N(\sigma, s+1) = 0$ and reset $E(\sigma, s+1) = \emptyset$ and $x(\sigma, s+1) = \uparrow$. Note that for all other $N$ nodes $\sigma'$ such that $\sigma \subset \sigma' \subset \alpha_{s+1}$, $N(\sigma', s) = -1$ and so $\sigma'$ will receive the same treatment.

<u>Notes.</u> $N(\sigma, s+1) = 0$ *is a default prediction at this stage—i.e. the construction does not*

*have enough information to make an* informed *choice for* $N(\sigma, s+1) \in \{0,1\}$.

<u>*Case* 3.*B*.</u> $N(\sigma, s) = 0$.

Note that this means that $t(\sigma, s+1) > 0$, that $N(\sigma, t(\sigma, s+1)) = 0$ and that $\sigma$ has not been reinitialised since the last time that $\sigma$ was visited (i.e. at stage $t(\sigma, s+1)$). There are 3 cases that ensue. Process the first case that applies.

<u>*Case* 3.*B*.1</u> There exists some axiom $\langle x, D\rangle \in I^+(\sigma, s+1)$ such that $x \notin \mathcal{B}_e[s]$. (Note that $x \in W(\sigma, s)$ by definition in this case.) Choose $\langle x, D\rangle$ such that for all such axioms $\langle y, D\rangle$, $x \leq y$. (For every $z$ there is at most one axiom $\langle z, D\rangle$ in $I(\sigma, s)$ and hence also at most one such axiom in $I^+(\sigma, s+1)$). Set $N(\sigma, s+1) = 1$, $E(\sigma, s+1) = D$, $x(\sigma, s+1) = x$.

*Notation.* When Case 2.B.1 holds we say that the axiom $\langle x, D\rangle$ *instigates* the outcomes $N(\sigma, s+1) = 1$, $E(\sigma, s+1) = D$ and $x(\sigma, s+1) = x$.

<u>Notes.</u> $N(\sigma, s+1) = 1$ *means that the construction predicts that* $x(\sigma, s+1) \in \Phi_e^C \setminus B_e$. *Note that* $I^+(\sigma, s+1)$ *contains every* $\langle n, D\rangle \in I(\sigma, s)$ *such that* $D^{[\leq 2e+1]}$ *appeared to be in* $C$ *at stage* $s$ *and such that no number in* $D^{[\leq 2e+1]}$ *is extracted by higher priority* $H$ *and* $P$ *nodes (i.e. nodes* $\beta$ *such that* $\beta \subset \sigma$*) at stage* $s+1$ *so that restraining* $D$ *(and in particular* $E(\sigma, s+1)$*) does not interfere with previous action taken by higher priority* $H$ *and* $P$ *nodes.* $E(\sigma, s+1)$ *is the finite set of numbers that the construction wants to restrain in* $C$ *in order to force* $x(\sigma, s+1) \in \Phi_e^C$ *if the prediction* $N(\sigma, s+1) = 1$ *is correct.*

<u>*Case* 3.*B*.2</u> $I^-(\sigma, s+1) \neq \emptyset$. (And case 3.B.1 does not apply.)

Then set $N(\sigma, s+1) = 0$, $E(\sigma, s+1) = \emptyset$ and $x(\sigma, s+1) = \uparrow$.

<u>Notes.</u> $N(\sigma, s+1) = 0$ *again corresponds to a default prediction. (See the Notes after case* 3.*C*.2 *for further explanation.)*

<u>*Case* 3.*B*.3</u> $I^-(\sigma, s+1) = \emptyset$. Then define

$$\begin{aligned} I^*(\sigma, s+1) \quad = \quad \{\, \langle x, D\rangle \mid x \in B_e[s] \ \ \& \ \ & x \notin W(\sigma, s) \\ \& \ \ & D^{[\leq 2e+1]} \subseteq \overline{\chi(\sigma, s+1)} \\ \& \ \ & \langle x, D\rangle \text{ witnesses } x \in \Phi^C_e[t(\sigma, s+1), s]\,\} \end{aligned} \tag{4.2.9}$$

and set

$$W^*(\sigma, s+1) \quad = \quad \{\, x \mid \exists D[\,\langle x, D\rangle \in I^*(\sigma, s+1)\,]\,\} \tag{4.2.10}$$

Set $N(\sigma, s+1) = 0$, $E(\sigma, s+1) = \emptyset$ and $x(\sigma, s+1) = \uparrow$.

<u>Notes.</u> *In this case* $N(\sigma, s+1) = 0$ *is an informed guess that* $B_e = \Phi^C_e$ *and* $B_e$ *is c.e. (and* $= \bigcup_{t \geq s^*} W(\sigma, t)$ *for some fixed stage* $s^* \leq s+1$ *if* $\sigma \subset \delta$ *and* $s+1$ *is a large enough stage).* $I(\sigma, s+1)$ *contains all possible instigator candidates and* $W(\sigma, s+1) = \{\, n \mid \exists D[\,\langle n, D\rangle \in I(\sigma, s+1)\,]\,\}$ *is a set that the construction guesses to be a finite stage of a c.e. approximation to* $B_e$ *if indeed* $B_e = \Phi^C_e$. *Note that* $|I(\sigma, s+1)| = |W(\sigma, s+1)|$. *Notice also that* $I^+(\sigma, s+1) = I(\sigma, s)$ *in this case (i.e. the whole of* $I(\sigma, s)$ *appears to be* valid*).* $W^*(\sigma, s+1)$ *contains every number* $n \notin W(\sigma, s)$ *that appears to be in* $B_e \cap \Phi^C_e$ *whereas* $I^*(\sigma, s+1)$ *contains, for each such* $n$*, the axiom that appears to witness* $n \in \Phi^C_e$.

<u>*Case* 3.*C*.</u> $N(\sigma, s) = 1$.

Note that this means that $x(\sigma, s) \in \omega$ and moreover that $\langle x(\sigma, s), E(\sigma, s)\rangle \in I^+(\sigma, t(\sigma, s+1))$ (and that $I^+(\sigma, t(\sigma, s+1)) \subseteq I(\sigma, s)$). There are 2 cases that ensue.

<u>*Case* 3.*C*.1</u> $\langle x(\sigma, s), E(\sigma, s)\rangle \in I^+(\sigma, s+1)$ & $x(\sigma, s) \notin \mathcal{B}_e[s]$.

Then reset $N(\sigma, s+1) = 1$, $E(\sigma, s+1) = E(\sigma, s)$ and $x(\sigma, s+1) = x(\sigma, s)$.

<u>Notes.</u> *The construction maintains its prediction that* $x(\sigma, s) \in \Phi^C_e \setminus B_e$ *in this case.*

<u>*Case* 3.*C*.2</u> $\langle x(\sigma, s), E(\sigma, s)\rangle \in I^-(\sigma, s+1)$ $\vee$ $x(\sigma, s) \in \mathcal{B}_e[s]$.

Then set $N(\sigma, s+1) = 0$, $E(\sigma, s+1) = \emptyset$, and $x(\sigma, s+1) = \uparrow$.

Notes. *The construction now no longer believes that $x(\sigma, s+1) \in \Phi_e^C \setminus B_e$ and falls back on its default prediction that $B_e = \Phi_e^C$ and $B_e$ is c.e. Note that, if $\sigma$ is on the true path, then infinitely many default predictions corresponding to $N(\sigma, s+1) = 0$, as in this case and cases 3.A and 3.B.2, in fact entail that case 3.B.3 applies infinitely often and consequently that the default prediction is in fact correct (see Lemma 4.2.17 of the verification).*

**To end this substage.** For every one of the outcomes covered by cases 3.A-3.C except case 3.B.3 set $I^*(\sigma, s+1) = W^*(\sigma, s+1) = \emptyset$. Define $I(\sigma, s+1) = I(\sigma, s) \cup I^*(\sigma, s+1)$ and $W(\sigma, s+1) = W(\sigma, s) \cup W^*(\sigma, s+1)$. Go to the *Final Step of Substage n.*

*Final Step of Substage n.*

In order to finalise substage $n$—and so after each of the above cases—the following action is taken according to $Q \in \{H, P, N\}$ corresponding to $n - 1 \in \{0, 1, 2\}$ modulo 3.

*Case I.* $n < s + 1$.

Go to substage $n + 1$ with the node $\sigma\widehat{\ }Q(\sigma, s+1)$ being eligible to be processed at substage $n + 1$.

*Case II.* $n = s + 1$.

Define the $s + 1$ stage path $\alpha_{s+1} = \sigma\widehat{\ }Q(\sigma, s+1)$. Reinitialise all $N$ nodes $\tau$, and $P$ nodes $\gamma$ such that $\alpha_{s+1} <_L \tau, \gamma$—i.e. set $N(\tau, s+1) = -1$, $x(\tau, s+1) = \uparrow$ and $E(\tau, s+1) = W(\tau, s+1) = I(\tau, s+1) = \emptyset$ and set $P(\gamma, s+1) = -1$, $V(\gamma, s+1) = \emptyset$, and $v(\gamma, s+1) = \uparrow$ for $v \in \{b, h, w\}$. Define

$$C_{s+1} = C_s \cup \bigcup_{\substack{\beta \in H \cup P \\ \& \ \beta \subset \alpha_{s+1}}} u(\beta, s+1) \cup \bigcup_{\substack{\tau \in N \\ \& \ \tau \subset \alpha_{s+1}}} E(\tau, s+1) - \bigcup_{\substack{\beta \in H \cup P \\ \& \ \beta \subset \alpha_{s+1}}} V(\beta, s+1)\,. \tag{4.2.11}$$

*Remark.* The reader should note here that, for any nodes $\beta, \beta' \in H \cup P$ such that $\beta, \beta' \subset \alpha_{s+1}$, and $N$ node $\tau \subset \alpha_{s+1}$, it is clear that $V(\beta, s+1) \cap u(\beta', s+1) = \emptyset$. Moreover if $\beta \subset \tau$ it follows from cases 3.B.1 and 3.C.1 and the definition of $I^+(\sigma, s+1)$ that $V(\beta, s+1) \cap E(\tau, s+1) = \emptyset$. On the other hand, if $\tau \subset \beta$ and $\beta \in H$, then $E(\tau, s+1)^{[2e]} \subseteq \Omega(\beta, s+1)$—if (say) $\beta \in H_e$—whereas if $\beta \in P$, then max $E(\tau, s+1) <$ min $V^+(\sigma, s+1)$ and also $V^-(\sigma, s+1) \cap E(\tau, s+1) = \emptyset$ by definition of $V^-(\sigma, s+1)$. Hence $E(\tau, s+1) \cap V(\beta, s+1) = \emptyset$ in this case also.

Reset any (nontemporary) node parameter $p(\gamma, s+1)$ and any index parameter $q(i, s+1)$ not mentioned above to its previous value, i.e. $p(\gamma, s+1) = p(\gamma, s)$ and $q(i, s+1) = q(i, s)$. We call this *automatic resetting.*

Go to stage $s+2$.

### 4.2.4 Verification

We verify the correctness of the construction in the proofs of Lemmas 4.2.9-4.2.17 below.

**Definition 4.2.2.** *We define the approximation* $\{\mathcal{T}_s\}_{s\in\omega}$ *of finite trees* $\mathcal{T}_s \subseteq 3^{<\omega}$ *as follows.*

$$\mathcal{T}_0 = \emptyset$$

$$\mathcal{T}_{s+1} = \{\, \sigma \mid \sigma \subseteq \alpha_{s+1} \,\} \quad \cup \quad \{\, \sigma \mid \sigma \in \mathcal{T}_s \;\&\; \sigma <_L \alpha_{s+1} \,\},$$

*Accordingly we define the tree*

$$\mathcal{T} = \bigcup_{s\in\omega} \mathcal{T}_s$$

*and we let* $|\mathcal{T}|$ *denote the set of infinite paths through* $\mathcal{T}$*. I.e.*

$$|\mathcal{T}| = \{\, \mu \mid \mu \in 3^{\omega} \;\&\; \forall n \exists s [\, \mu \restriction n \in \mathcal{T}_s \,] \,\}$$

*where $\mu\restriction n$ is the finite initial segment of $\mu$ of length $n$.*

**Note 4.2.3.** For all $s \in \omega$, $\mathcal{T}_s \subseteq 3^{<\omega}$ ($\alpha_s \in 3^{<\omega}$) and $|\alpha_s| = s$. Therefore it follows that $|\mathcal{T}|$ is non empty and that there exists a (lexicographically) leftmost member of $|\mathcal{T}|$. We call the latter the *true path* through $\mathcal{T}$ and we use the symbol $\delta$ to denote the true path. We also use $\delta_n$ to denote $\delta\restriction n$, i.e. the initial segment of $\delta$ of length $n$.

**Note 4.2.4.** If $\beta$ is an $H_e$ node and $r < s+1$ are $\beta$-true stages such that $\alpha_t \not<_L \beta$ for all $t$ such that $r < t < s+1$, then $\Omega(\beta, t) = \Omega(\beta, r)$ for all $t$ such that $r \leq t \leq s+1$. Indeed firstly consider any $N$ node $\tau \subset \beta$. There are two cases. If $\tau\hat{\ }0 \subseteq \beta$, then $N(\tau, r) = 0$ and $E(\tau, r) = \emptyset$. So if $E(\tau, t) \neq \emptyset$ for some $t$ such that $r < t \leq s+1$ then, supposing this to be the least such stage, $N(\tau, t) = 1$ and $\tau\hat{\ }1 \subseteq \alpha_t$. Thus $\beta <_L \alpha_t$, a contradiction if $t = s+1$, and otherwise implying that $\Omega(\beta, t) = \Omega(\beta, t-1)$ by automatic resetting. On the other hand, if $\tau\hat{\ }1 \subseteq \beta$, then $N(\tau, t) = 1$ and $E(\tau, t) = E(\tau, r)$ for all $t$ such that $r < t \leq s+1$ since $N(\tau, t) = 0$ would imply that $\alpha_t <_L \beta$. Secondly consider any $N$ node $\tau <_L \beta$. Then $E(\tau, t-1) = E(\tau, r-1)$ for all $t$ such that $r < t \leq s+1$ (by automatic resetting since $\tau <_L \alpha_t$ for all such $t$). Using these remarks we can deduce by straightforward induction that $\Omega(\tau, t) = \Omega(\tau, r)$ for all $t$ such that $r \leq t \leq s+1$. Notice also that, for all $t$ such that $r < t \leq s+1$ and $\tau <_L \alpha_t$, not only is $E(\tau, t-1) = E(\tau, r-1)$ but also $E(\tau, t-1)^{[2e]} \subseteq C_{t-1}$ since only $H_e$ nodes can extract numbers from $C^{[2e]}$—even though the extraction activity of $P$ nodes might mean that $E(\tau, t-1) \not\subseteq C_{t-1}$—thus we can also deduce that $\Omega(\tau, t) \subseteq C_t$ for all $t$ such that $r \leq t \leq s+1$.

**Note 4.2.5.** Application of a similar argument to that of Note 4.2.4 shows that, if $\beta$ is a $P$ node and $r < s+1$ are $\beta$-true stages such that $\alpha_t \not<_L \beta$ for all $t$ such that $r < t < s+1$ then $b(\beta, t) = b(\beta, r)$ for all stages $t$ such that $r \leq t \leq s+1$ whereas $V^-(\beta, t) = V^-(\beta, r)$ for all $\beta$-true stages $t$ such that $r \leq t \leq s+1$.

**Note 4.2.6.** As already mentioned in the remark on page 4.2.3, it follows from the definition of the construction that, for any stage $s+1$ and nodes $\beta, \tau \subseteq \alpha_{s+1}$ such that $\beta \in H \cup P$ and $\tau \in N$, it is the case that

$$E(\tau, s+1) \cap V(\beta, s+1) = \emptyset\,. \tag{4.2.12}$$

Moreover (4.2.12) also holds if $\beta \subseteq \alpha_{s+1}$ and $\tau <_L \beta$ by definition of $\Omega(\beta, s+1)$ if $\beta \in H$, and of $V^-(\beta, s+1)$ if $\beta \in P$.

**Note 4.2.7.** If $\sigma$ is a $P_e$ node and $P(\sigma, s+1) \in \{0, 1, 2\}$, then

$$\{\, \langle 2e+1, z\rangle \;\mid\; b(\sigma, s+1) \leq z < h(\sigma, s+1)\,\} \quad \subseteq \quad C_{s+1}\,. \tag{4.2.13}$$

This can be seen, for any such $\langle 2e+1, z\rangle$, by a straightforward induction on the stages following the stage $s_z$ at which $\langle 2e+1, z\rangle$ is enumerated into $C$ (via $u(\sigma, s_z) = \{\langle 2e+1, z\rangle\}$). This is because only $P_e$ nodes can remove members of $\omega^{[2e+1]}$ from $C$. Moreover $P(\sigma, s+1) = -1$ if $\alpha_{s+1} <_L \sigma$, whereas if $\sigma <_L \alpha_{s+1}$ and $\sigma'$ is the $P_e$ node $\subseteq \alpha_{s+1}$, then $b(\sigma', s+1)$ ($= b(\sigma', s)$ if $b(\sigma', s) \neq \uparrow$) is greater than $h(\sigma, s) = h(\sigma, s+1)$ and so action taken for the sake of $\sigma'$ at stage $s+1$ does not involve $\langle 2e+1, z\rangle$.

**Note 4.2.8.** Suppose that $\sigma \in P \cup H$ and that $\sigma \subset \delta$. Accordingly let $s_\sigma$ be the least $\sigma$-true stage such that $\alpha_t \not<_L \sigma$ for all $t \geq s_\sigma$. Then it follows from Notes 4.2.4 and 4.2.5 that, if $\sigma \in H$, then $\Omega(\sigma, s) = \Omega(\sigma, s_\sigma) \subseteq C_s$ for all $s \geq s_\sigma$ whereas, if $\sigma \in P$, then $b(\sigma, s) = b(\sigma, s_\sigma)$ for all $s \geq s_\sigma$ and $V^-(\sigma, s) = V^-(\sigma, s_\sigma)$ for all $\sigma$-true stages $s \geq s_\sigma$. Thus for any $H$ node $\sigma$ on the true path we let $\Omega(\sigma) = \lim_{s\to\infty}\Omega(\sigma, s)$ ($= \Omega(\sigma, s_\sigma)$ for $s_\sigma$ defined as above). Similarly, for any $P$ node $\sigma$ on the true path we let $b(\sigma) = \lim_{s\to\infty} b(\sigma, s)$. Likewise if $\sigma^\frown 1 \subset \delta$ then $\lim_{s\to\infty} v(\sigma, s)$ exists for $v \in \{h, w\}$ and we use $v(\sigma)$ to denote this value.

**Lemma 4.2.9.** *Suppose that $\sigma$ is a $P$ node such that $\sigma^\frown i \subseteq \delta$ for some $i \in \{0, 2\}$. Then lim inf$_{s\to\infty} h(\sigma, s) = \infty$.*

*Proof.* Suppose that $\sigma\hat{\ }2 \subseteq \delta$. Accordingly let $t_\sigma$ be a $\sigma\hat{\ }2$-true stage such that $\alpha_t \not<_L \sigma\hat{\ }2$ for all $s \geq t_\sigma$. Then, at every $\sigma$-true stage $s+1 > t_\sigma$, $h(\sigma, s+1) = h(\sigma, t(\sigma, s+1)) + 1$. It follows therefore (in the case $\sigma\hat{\ }2 \subseteq \delta$) that $\lim_{s\to\infty} h(\sigma, s) = \infty$.

Suppose now that $\sigma\hat{\ }0 \subseteq \delta$. Let $s_\sigma$ be a $\sigma$-true stage such that $\alpha_t \not<_L \sigma$ for all $t \geq s_\sigma$. Then $b(\sigma, s) = b(\sigma, s_\sigma) =_{\text{def}} b(\sigma)$ for all $s \geq s_\sigma$ whereas $N(\sigma, s+1) = 0$ at infinitely many $\sigma$-true stages $s+1 \geq s_\sigma$. Suppose that there exists $x \geq b(\sigma)$ such that $h(\sigma, s+1) = x$ at infinitely many ($\sigma$-true) stages. Let $x_0$ be the least such number. Then this means that $x_0 \in \mathcal{K}$. Accordingly let $t_0$ be the stage such that $x_0 \in \mathcal{K}[t_0+1] - \mathcal{K}[t_0]$. Then, at each $\sigma$-true stage $s+1$ such that $h(\sigma, s+1) = x_0$ the construction verifies that there exists some finite set $D$ in the set

$$J \quad = \quad \{\, D \mid \langle x_0, D\rangle \in \Phi_e \ \ \&\ \ D \subseteq \{0, \ldots, t_0\}\,\}$$

and also that $D \subseteq \mathcal{B}_e[s]$. However $J$ is a finite set whereas $\{B_{e,s}\}_{s\in\omega}$ is a $\Delta^0_2$ approximation. Hence there is a $\sigma$-true stage $s^*+1$ at which $N(\sigma, s^*+1)$ is permanently set to 1, i.e. $\sigma\hat{\ }1 \subset \delta$. A contradiction. Hence (in the case $\sigma\hat{\ }0 \subseteq \delta$) it follows that $\liminf_{s\to\infty} h(\sigma, s) = \infty$. □

**Lemma 4.2.10.** *Suppose that $\sigma$ is a $P$ node such that $\sigma\hat{\ }1 \subseteq \delta$. Then there exists a stage $s^*$ such that $\sigma\hat{\ }1 \subseteq \alpha_s$ for every $\sigma$-true stage $s \geq s^*$.*

*Proof.* This observation follows from the fact that, for all $s$, $P(\sigma, s+1) = 2$ only if $P(\sigma, s) \in \{0, 2\}$ (case 2.B.3.$i$) so that, if $P(\sigma, s) = 1$, then $P(\sigma, s+1) \in \{0, 1\}$. □

**Lemma 4.2.11.** *Let $\sigma$ be the $N_e$ node on the true path $\delta$. Then there exists a stage $s_e$ such that $I(\sigma, s) \subseteq I(\sigma, s+1)$ and $W(\sigma, s) \subseteq W(\sigma, s+1)$ for all stages $s \geq s_e$.*

*Moreover, for every $s \geq s_e$ and axiom $\langle x, D\rangle \in I(\sigma, s)$ there exists a stage $s_D \geq s$ such that $D^{[\leq 2e+1]} \subseteq \mathcal{C}[t]$ for all $t \geq s_D$.*

*Proof.* Define $s_e > 0$ to be the least $\sigma$-true stage such that $\alpha_t \not<_L \sigma$ for all $t \geq s_e$. Then clearly $I(\sigma, s) \subseteq I(\sigma, s+1)$ and $W(\sigma, s) \subseteq W(\sigma, s+1)$ for all $s \geq s_e$.

Now, by definition of $s_e$, for every axiom $\langle x, D\rangle \in \bigcup_{s \geq s_e} I(\sigma, s)$ there exists a stage $s+1 \geq s_e$ such that $\langle x, D\rangle$ enters $I(\sigma, s+1)$. Choose some such axiom $\langle x, D\rangle$ and let $s^*$ be the least stage $s$ such that $\langle x, D\rangle \in I(\sigma, s+1)$. Note that this means that $D^{[\leq 2e+1]} \subseteq \overline{\chi(\sigma, s^*+1)}$ and in fact that $D^{[\leq 2e+1]} \subseteq C_{s^*+1}$ by definition of $I^*(\sigma, s^*+1)$—since $D^{[\leq 2e+1]} \subseteq C_{s^*}$ and $D \cap V(\beta, s^*+1) = \emptyset$ for all $\beta \in H \cup P$ such that $\beta \subset \sigma$.

Now suppose that $j \leq 2e+1$ and consider $D^{[j]}$. There are 2 cases to consider.

1) $j = 0 \bmod 3$. Thus for some $d$, $j = 3d$. Let $\tau$ be the $H_d$ node on the true path. In other words $\tau \subset \sigma \subset \delta$. There are 2 subcases.

   a) $\tau\widehat{\ }0 \subseteq \sigma$. Then, at stage $s^*+1$, $D^{[2d]} \subseteq \Omega(\tau, s^*+1)$ since $U_{e,s^*+1} = \emptyset$ and $V(\tau, s^*+1) = V_{e,s^*+1} - \Omega(\tau, s^*+1)$—i.e. any $z \notin \Omega(\tau, s^*+1)$ mentioned in $C^{[2d]}$ up to the end of stage $s^*+1$ is in $V(\tau, s^*+1)$. Moreover, by Note 4.2.8 we see that $\Omega(\tau, s) = \Omega(\tau, s_e) \subseteq C_s$ for all $s \geq s_e$. Thus, in this case $D^{[2d]} \subseteq \mathcal{C}[s]$ for all $s \geq s^*+1$.

   b) $\tau\widehat{\ }1 \subseteq \sigma$. Then we can see that $D^{[2d]} \cap \Omega(\tau, s^*+1) \subseteq \mathcal{C}[s]$ for all $s \geq s^*+1$ by the same argument as that for the case $\tau\widehat{\ }0 \subseteq \sigma$ above. On the other hand, for any $\langle 2d, z\rangle \in D$ such that $\langle 2d, z\rangle \notin \Omega(\tau, s^*+1)$, $\langle 2d, z\rangle$ is not extracted at any stage $s^* \geq s+1$ since this would imply that $\tau$ is processed via case 1.A. and entail $\tau\widehat{\ }0 \subseteq \alpha_t$ at the next $\tau$-true stage $t$ in contradiction with the definition of $s_e$.

   We deduce therefore that $D^{[2d]} \subseteq \mathcal{C}[s]$ for all $s \geq s^*+1$.

2) $j = 1 \bmod 3$. Thus for some $d$, $j = 3d+1$. Let $\tau$ be the $P_d$ node on the true path. In other words $\tau \subset \sigma \subset \delta$. There are 2 subcases.

a) $\tau^\frown i \subseteq \sigma$ for some $i \in \{0,2\}$. Note firstly that $b(\tau,s) = b(\tau,s_e) =_{\mathrm{def}} b(\tau)$ for all $s \geq s_e$ by definition of the latter. By Lemma 4.2.9, $\liminf_{s\to\infty} h(\tau,s) = \infty$ in both these cases. So it follows from Note 4.2.7 that, for any $z \geq b(\tau)$ such that $\langle 2d+1, z\rangle \in D$ there exists a stage $s_z$ such that $\langle 2d+1, z\rangle \in \mathcal{C}[s]$ for all $s \geq s_z$. So now suppose that $z < b(\tau)$. Then as $\langle 2d+1, z\rangle \in D^{[2d+1]} \subseteq C_{s^*+1}$ we know that $\langle 2d+1, z\rangle \notin V^-(\tau, s^*+1)$. Moreover, by Note 4.2.8 and by definition of $s_e$, $V^-(\tau,s) = V^-(\tau,s^*+1)$ for all $\tau$-true stages $s \geq s^*+1$. On the other hand, for any $\gamma \in P_d$ such that $\tau <_L \gamma$, and any $\gamma$-true stages $s+1 \geq s^*+1$ it is the case that $\langle 2d+1, z\rangle \notin V^+(\gamma, s+1)$ (see the last sentence of Note 4.2.7). It follows that $\langle 2d+1, z\rangle$ is not extracted from $\mathcal{C}[s]$ and hence remains in $\mathcal{C}[s]$ for all $s \geq s^*+1$ (as $\langle 2d+1, z\rangle \in C_{s^*+1}$). Thus (in the case $\tau^\frown i \subseteq \sigma$ for $i \in \{0,2\}$) there exists a stage $t^* \geq s^*+1$ such that $D^{[2d+1]} \subseteq \mathcal{C}[s]$ for all $s \geq t^*$.

b) $\tau^\frown 1 \subseteq \sigma$. This implies that $h(\tau,s) = h(\tau,s_e) =_{\mathrm{def}} h(\tau)$ for all $s \geq s_e$. In particular at stage $s^*+1$

$$V^+(\tau, s^*+1) \quad = \quad T_{e,s^*} \;\cap\; \{\, \langle 2d+1, m\rangle \mid m \geq h(\tau, s^*+1) \,\}$$

by definition. This implies that

$$D \;\cap\; \{\, \langle 2d+1, m\rangle \mid m \geq h(\tau) \,\} \quad = \quad \emptyset\,.$$

On the other hand by Note 4.2.7

$$\{\, \langle 2d+1, m\rangle \mid b(\tau) \leq m < h(\tau) \,\} \quad \subseteq \quad \mathcal{C}[s]$$

for all $s \geq s_e$. Moreover, as in the case $\tau^\frown i \subseteq \sigma$ for $i \in \{0,2\}$,

$$D^{[2d+1]} \;\cap\; \{\, \langle 2d+1, z\rangle \mid z < b(\tau) \,\} \quad \subseteq \quad \mathcal{C}[s]$$

for all $s \geq s^*+1$. Thus (in the case $\tau^\frown 1 \subseteq \sigma$) we know that $D^{[2d+1]} \subseteq \mathcal{C}[s]$ for all $s \geq s^*+1$.

We deduce therefore that in both cases there exists a stage $t^* \geq s^* + 1$ such that $D^{[2d+1]} \subseteq \mathcal{C}[s]$ for all $s \geq t^*$.

□

**Lemma 4.2.12.** *Suppose that $\sigma$ is an $N_e$ node such that $\sigma\hat{\ }1 \subset \delta$. Let $s_\sigma$ be the least $\sigma$-true stage such that $\alpha_s \not<_L \sigma$ for all $s \geq s_\sigma$. Then $N(\sigma, s) = N(\sigma, s_\sigma) = 1$, $x(\sigma, s) = x(\sigma, s_\sigma) \in \omega$ and $E(\sigma, s) = E(\sigma, s_\sigma) \subseteq C_s$ for all $s \geq s_\sigma$.*

*Proof.* The fact that $y(\sigma, s) = y(\sigma, s_\sigma)$ for $y \in \{N, x, E\}$ follows by a straightforward induction over $s \geq s_\sigma$.

*Notation.* Accordingly we use the terminology $y(\sigma) =_{\text{def}} \lim_{s\to\infty} y(\sigma, s)$ for $y \in \{N, x, E\}$ in the case of any $N$ node $\sigma$ such that $\sigma\hat{\ }1 \subset \delta$ throughout the rest of the proof of Theorem 4.2.1. In particular, $E(\sigma) = E(\sigma, s_\sigma)$ in the present case.

Consider $E(\sigma)$ and note that $E(\sigma) \subseteq \mathcal{C}[s_\sigma]$ by (4.2.11) applied to stage $s+1 = s_\sigma$. Let $s^*+1$ be the next $\sigma$-true stage. Then, by definition of $s_\sigma$, $\sigma\hat{\ }1 \subseteq \alpha_{s^*+1}$. Now, this implies that $E(\sigma)^{[\leq 2e+1]} \subseteq C[s_\sigma, s^*] \cap \overline{\chi(\sigma, s^*+1)}$ by definition of $I^+(\sigma, s^*+1)$ (see (4.2.7)) and the fact that case 3.C.1 must apply (to get $N(\sigma, s^*+1) = 1$). It follows that $E(\sigma)^{[\leq 2e+1]} \subseteq \mathcal{C}[t]$ for all $t$ such that $s_\sigma \leq t \leq s^* + 1$. Now $E(\sigma)^{[>2e+1]} \subseteq \mathcal{C}[s_\sigma]$ since $E(\sigma) = E(\sigma, s_\sigma) \subseteq \mathcal{C}[s_\sigma]$ by definition (see (4.2.11)) whereas an easy induction on $t$ shows that $E(\sigma, t)^{[>2e+1]} \subseteq \mathcal{C}[t]$ for all $t$ such that $s_\sigma \leq t \leq s^*$ due to the definitions of the restraint $\Omega(\tau, t)$ and parameter $V^-(\gamma, t)$ for any $H$ node $\tau$ and $P$ node $\gamma$ visited at stage $t$. Moreover $E(\sigma, s^*+1)^{[>2e+1]} \subseteq E(\sigma, s^*+1) \subseteq \mathcal{C}[s^*+1]$ by (4.2.11) with $s = s^*$ (since case 3.C.1 applies at stage $s^*+1$). We conclude therefore that $E(\sigma)^{[>2e+1]} \subseteq \mathcal{C}[t]$ for all $t$ such that $s_\sigma \leq t \leq s^* + 1$. Hence $E(\sigma) \subseteq \mathcal{C}[t]$ for all such $t$ and thus by induction (on the $\sigma$-true stages $t_0 < t_1 < t_2 \ldots$ where $t_0 = s_\sigma$) we see that $E(\sigma) \subseteq \mathcal{C}[t]$ for all $t \geq s_\sigma$. □

**Lemma 4.2.13.** *Suppose that $\sigma$ is an $N_e$ node such that $\sigma\widehat{\ }1 \subset \delta$. Then $x(\sigma) \in \Phi_e^C - B_e$.*

*Proof.* Let $s_\sigma$ be defined as in Lemma 4.2.12, so that $x(\sigma) = x(\sigma, s_\sigma)$. Then obviously $x(\sigma) \notin \mathcal{B}_e[s]$ at every $\sigma$-true stage $s + 1 > s_\sigma$ since case 3.C.1 applies at each such stage. Therefore $x(\sigma) \notin B_e$. Moreover $\langle x(\sigma), E(\sigma)\rangle \in \Phi_e$ whereas, by Lemma 4.2.12, $E(\sigma) \subseteq C$ (since $C =_{\text{def}} \{\, n \mid \exists t(\forall s \geq t)[\, n \in \mathcal{C}[s]\,]\,\}$). Thus $x(\sigma) \in \Phi_e^C - B_e$. □

**Lemma 4.2.14.** *For all $x$, $lim_{s\to\infty}\mathcal{C}[s](x)$ exists. Hence $C$ is $\Delta_2^0$.*

*Proof.* Consider any $x$. There are two cases to consider.

1) $x = \langle 2e + 1, z\rangle$ for some $e, z \in \omega$. Suppose that $\sigma$ is the $P_e$ node on the true path $\delta$ and let $s_\sigma$ be the least $\sigma$-true stage such that $\alpha_t \not<_L \sigma$ for all $t \geq s_\sigma$. Note that this means that $b(\sigma, s) = b(\sigma, s_\sigma) =_{\text{def}} b(\sigma)$ for all $s \geq s_\sigma$. There are two subcases.

   a) $z \geq b(\sigma)$. If $\sigma\widehat{\ }i \subset \delta$ for some $i \in \{0, 2\}$ then we know that

$$\liminf_{s\to\infty} h(\sigma, s) = \infty$$

   by Lemma 4.2.9. Hence, by Note 4.2.7 there exists a stage $t^*$ such that $x \in C_s$ for all $s \geq t^*$. Therefore we need only consider the case $\sigma\widehat{\ }1 \subset \delta$. Accordingly let $t_\sigma \geq s_\sigma$ be the least $\sigma\widehat{\ }1$-true stage such that $\alpha_t \not<_L \sigma\widehat{\ }1$ for all $t \geq t_\sigma$. Note that $y(\sigma, s) = y(\sigma, t_\sigma) =_{\text{def}} y(\sigma)$ for all $s \geq t_\sigma$ and $y \in \{h, w\}$ in this case. Now, if $b(\sigma) \leq z < h(\sigma)$ it follows once again by Note 4.2.7 that $x \in C_s$ for all $s \geq t_\sigma$. Hence we can suppose that $z \geq h(\sigma)$. Notice that case 2.B.2 applies at every $\sigma$-true stage $s + 1 \geq s_\sigma$. Thus, by the fact that $x \in T_{e,s} \cap \{\, \langle 2e + 1, m\rangle \mid m \geq h(\sigma)\,\} = V^+(\sigma, s + 1)$ and that

$$V^+(\sigma, s + 1) \;\subseteq\; V(\sigma, s + 1) \;\subseteq\; \overline{\mathcal{C}[s + 1]}$$

in conjunction with (4.2.11) we know that $x \notin \mathcal{C}[s+1]$ at every such stage. Hence, if $x \in C_t$ for some $t > t_\sigma$ it must be the case that $\sigma <_L \alpha_t$. Consider all $N$ nodes $\beta \subset \sigma$. If $\beta\widehat{\ }1 \subseteq \sigma$ then $x \notin E(\beta)$ by definition of $b(\sigma)$—since if $s_\beta + 1$ is the least $\beta\widehat{\ }1$-true stage such that $\alpha_t \not<_L \beta\widehat{\ }1$ for all $t \geq s_\beta + 1$, then $P(\sigma, s_\beta) = -1$ and $b(\sigma, s_\beta) = \uparrow$ by construction and so $b(\sigma) > \max E(\beta)$ by definition of $b(\sigma) = b(\sigma, s_\sigma)$ (as $s_\sigma > s_\beta$). Accordingly let

$$\mathcal{B}(\sigma) \quad = \quad \{\, \beta \mid \beta \in N \;\;\&\;\; \beta\widehat{\ }0 \subseteq \sigma \,\}.$$

- If $\mathcal{B}(\sigma) = \emptyset$, then we show by induction that, at every stage $s \geq t_\sigma$, $x \in V(\sigma_s, s) \subseteq \overline{C_s}$ (see the Remark below) where we define $\sigma_s$ to be the $P_e$ node $\sigma' \subseteq \alpha_s$. Indeed, the case $s = t_\sigma$ being true by definition, consider $s{+}1 = t_\sigma{+}1$. Then, for all $N$ nodes $\beta \subseteq \alpha_{s+1}$ such that $\beta \subset \sigma$, $x \notin E(\beta, s{+}1)$ since in this case we know that it is also the case that $\beta\widehat{\ }1 \subseteq \sigma$ (since $\mathcal{B}(\sigma) = \emptyset$). Also, for any $N$ node $\sigma <_L \beta \subset \sigma_{s+1}$, $N(\beta, s) = -1$ and so $I(\beta, s+1) = \emptyset$. Thus $x \in V^-(\sigma_{s+1}, s+1) \subseteq V(\sigma_{s+1}, s+1) \subseteq \overline{\mathcal{C}[s+1]}$.

So now consider any stage $s + 1 = t_\sigma + n$ for $n > 1$. By the induction hypothesis $x \in V(\sigma_t, t) \subseteq \overline{\mathcal{C}[t]}$ and $x \notin \bigcup\{\, D \mid \exists y[\,\langle y, D\rangle \in I(\beta, t)\,]\,\}$ for every $t_\sigma \leq t \leq s$ and $N$ node $\sigma <_L \beta \subset \sigma_t$. Note once again that for all $N$ nodes $\beta \subset \sigma_{s+1}$ such that $\beta \subset \sigma$, $x \notin E(\beta, s+1)$ as explained above. Moreover, if $\sigma <_L \beta \subset \sigma_{s+1}$ then $x \notin \bigcup\{\, D \mid \exists y[\,\langle y, D\rangle \in I(\beta, s)\,]\,\}$, by the induction hypothesis, and so again $x \notin E(\beta, s+1)$ (since if $E(\beta, s+1) \neq \emptyset$ then $\langle x(\beta, s+1), E(\beta, s+1)\rangle \in I(\beta, s)$). On the other hand, for any $\langle z, D\rangle \in I(\beta, s{+}1) - I(\beta, s)$ we know that $\langle z, D\rangle \in I^*(\beta, s{+}1)$ due to the application of case 3.B.3 to $\beta$ at stage $s + 1$. However this implies that $N(\beta, s) = 0$ and so $t(\beta, s{+}1) > t_\sigma$. Moreover, by (4.2.9), $D \subseteq \mathcal{C}[t(\beta, s{+}1)]$. Hence $x \notin D$ by the induction hypothesis. Therefore $x \notin \bigcup\{\, D \mid \exists y[\,\langle y, D\rangle \in I(\beta, s+1)\,]\,\}$ and so the induction hypothesis is validated. Thus $x \in V(\sigma_s, s) \subseteq \overline{C_s}$ for all

$s \geq t_\sigma$.

*Remark.* In fact for the case $\mathcal{B}(\sigma) = \emptyset$ the argument above can be simplified to just show that $x \notin C_s$ for all $s \geq t_\sigma$ without the need to show that $x \in V(\sigma_s, s)$ at each such stage. However the latter condition is essential for the adaptation of this argument to the more general case of $\mathcal{B}(\sigma) \neq \emptyset$ below since it shows that $\sigma_s$ is free to remove $x$ from $C_s$ if $x$ has reappeared in $C$.

- If $\mathcal{B}(\sigma) \neq \emptyset$ then, supposing that $|\mathcal{B}(\sigma)| = m+1$ for some $m \geq 0$, we label the members of $\mathcal{B}(\sigma)$ as $\beta_0, \ldots, \beta_m$ where

$$\beta_m{}^\frown 0 \subseteq \ldots \subseteq \beta_0{}^\frown 0 \subseteq \sigma \,.$$

Consider $\beta_0$. Note that $I(\beta_0, t_\sigma)$ is finite. Define

$$I^x(\beta_0, s) \quad = \quad \{\, \langle z, D\rangle \mid \langle z, D\rangle \in I(\beta_0, s) \;\;\&\;\; x \in D \,\} \,.$$

We firstly show that $I^x(\beta_0, s) = I^x(\beta_0, t_\sigma)$ for all $s \geq t_\sigma$. Consider the set of stages

$$S^* \quad = \quad \{\, s+1 \mid s+1 \geq t_\sigma \;\;\&\;\; \beta_0{}^\frown 0 \subseteq \alpha_{s+1} \,\} \,.$$

By the argument used in the case $\mathcal{B}(\sigma) = \emptyset$ with the set $S^*$ replacing the set $\{\, s+1 \mid s+1 \geq t_\sigma \,\}$ and using the fact that $\mathcal{C} = \emptyset$ where

$$\mathcal{C} \quad =_{\text{def}} \quad \{\, \beta \mid \beta \in N \;\;\&\;\; \beta_0 \subset \beta \;\;\&\;\; \beta{}^\frown 0 \subseteq \sigma \,\}$$

in the same way that we used the fact that $\mathcal{B}(\sigma) = \emptyset$ in the above argument, we can deduce that $x \in V(\sigma_{s+1}, s+1) \subseteq \overline{C_{s+1}}$ for all $s+1 \in S^*$. Therefore, since by construction any axiom $\langle z, D\rangle$ is picked as an instigator candidate for $\beta_0$ at a stage $s+1 > t_\sigma$ only when $s+1 \in S^*$ it follows that $x \notin D$ as in this case (see (4.2.9)) $D \subseteq C[t(\sigma, s+1)]$ whereas $t(\sigma, s+1) \in S^*$ by definition. Thus indeed $I^x(\beta_0, s+1) = I^x(\beta_0, t_\sigma)$ for all $s \geq t_\sigma$. Now consider any $\langle y, D\rangle \in I^x(\beta_0, t_\sigma)$. Supposing that $\beta$ is an $N_d$ node (say) it follows from Lemma 4.2.11 that there exists a stage $s_D$ such that $D^{[\leq 2d+1]} \subseteq \mathcal{C}[t]$ for all $s \geq s_D$. Suppose that $y \notin B_e$. Then there exists a stage $\hat{s} \geq s_D$ such that $y \notin \mathcal{B}_e[s]$ for all $s \geq \hat{s}$. However this implies that $N(\beta, s)$ switches permanently to 1 at some stage $s \geq \hat{s}$ so that $\beta_0{}^\frown 1 \subseteq \delta$ (either for the sake of this axiom $\langle y, D\rangle$ or for the sake of some other instigator candidate $\langle \hat{y}, \hat{D}\rangle$ such that $\hat{y} < y$ and $\hat{y} \notin B_e$). A contradiction. Therefore for each $\langle y, D\rangle \in I^x(\beta_0, t_\sigma)$ there exists a stage $s_{y,D}$ such that $\langle y, D\rangle$ does not instigate $N(\beta_0, s+1) = 1$ (i.e. instigating $E(\beta_0, s+1) = D$) at any stage $s \geq s_{y,D}$. Let $s_0 = \max\{\, s_{y,D} \mid \langle y, D\rangle \in I^x(\beta_0, t_\sigma)\,\}$. Then for all stages $s$ in the set

$$S_0 \quad =_{\text{def}} \quad \{\, s+1 \mid s \geq s_0 \;\;\&\;\; \beta_0 \subseteq \alpha_{s+1}\,\}$$

we can deduce that $x \in V(\sigma_{s+1}, s+1) \subseteq \overline{\mathcal{C}[s+1]}$ by once again applying the argument used in the case $B(\sigma) = \emptyset$ with (this time) the whole set $S_0$ replacing the set $\{\, s+1 \mid s+1 \geq t_\sigma\,\}$, and using the fact that $\mathcal{C} = \emptyset$ (in place of the fact that $B(\sigma) = \emptyset$). This argument shows that, for the case $m = 0$ there exists a stage $s_0$ such that $x \notin C_s$ for all $s \geq s_0$. On the other hand, if $m > 0$ we argue by induction for $0 \leq i \leq m$ relative to $\beta_i$ to obtain $s_i \geq s_{i-1}$ such that, for all $s$ in the set

$$S_i \quad =_{\text{def}} \quad \{\, s+1 \mid s \geq s_i \;\;\&\;\; \beta_i \subseteq \alpha_{s+1}\,\}$$

$x \in V(\sigma_{s+1}, s+1) \subseteq \overline{\mathcal{C}[s+1]}$. We thus obtain in the general case a stage $s_m$ such that $x \notin \mathcal{C}[s]$ for all stages $s \geq s_m$.

b) $z < b(\sigma)$. If $x = \langle 2e+1, z\rangle \notin V^-(\sigma, s_\sigma)$ then, by Note 4.2.8, $x \notin V^-(\sigma, s+1)$ at every $\sigma$-true stage $s+1 \geq s_\sigma$. Moreover, inspection of the construction shows that, for all $s+1 \geq s_\sigma$

$$V^-(\sigma_{s+1}, s+1) \cap \omega^{[2e+1]}\restriction\langle 2e+1, b(\sigma)\rangle \;\subseteq\; V^-(\sigma, s_\sigma) \cap \omega^{[2e+1]}\restriction\langle 2e+1, b(\sigma)\rangle$$

so that $x \notin V(\sigma_{t+1}, t+1)$ for all $t+1 \geq s_\sigma$. Thus $\mathcal{C}[t+1](x) \neq \mathcal{C}[t](x)$ for at most one stage $t \geq s_\sigma$ in this case (and this can only be due to $\mathcal{C}[t](x) = 0$ and $\mathcal{C}[t+1](x) = 1$).

On the other hand, if $x \in V^-(\sigma, s_\sigma)$, then $x \in V^-(\sigma, s+1) \subseteq V(\sigma, s+1) \subseteq \overline{\mathcal{C}[s+1]}$ at every $\sigma$-true stage $s+1 \geq s_\sigma$. But then we can apply the argument used for the case $z \geq h(\sigma)$ and $\sigma\frown 1 \subseteq \delta$ above to show that there exists a stage $s_{m'}$ (say) such that $x \notin \mathcal{C}[s]$ for all stages $s \geq s_{m'}$.

2) $x = \langle 2e, z\rangle$ for some $e, z \in \omega$. Consider the $H_e$ node $\sigma$ such that $\sigma \subset \delta$. Let $s_\sigma$ be defined as above and note that this means, by Note 4.2.8, that $\Omega(\sigma, s) = \Omega(\sigma, s_\sigma) =_{\text{def}} \Omega(\sigma) \subseteq C_s$ for all $s \geq s_\sigma$. So we can suppose that $x \notin \Omega(\sigma)$ but that $x \in \mathcal{C}[s]$ for some $s \geq s_\sigma$. Let $s_x$ be the least such stage. There are 2 subcases to to consider.

- $\sigma\frown 1 \subset \delta$. Then define $t_\sigma$ as above (i.e. $\alpha_t \not<_L \sigma\frown 1$ for all $t \geq t_\sigma$). Then no number is extracted from $\mathcal{C}[s]^{[2e]}$ at any stage $t \geq t_\sigma$. Hence either $s_x < t_\sigma$, in which case $\mathcal{C}[t+1](x) \neq \mathcal{C}[t](x)$ for at most one stage $t \geq t_\sigma$ or otherwise $s_x \geq t_\sigma$ and $x \in \mathcal{C}[s]$ for all $s \geq s_x$.
- $\sigma\frown 0 \subset \delta$. Then $x \in V(\sigma, s) \subseteq \overline{\mathcal{C}[s]}$ at every $\sigma\frown 0$-true stage $s > s_\sigma$. Moreover, $x$ can only be reinserted into $C$ at later stages by action taken for the sake of $N$ nodes. Hence we can apply a similar argument to that used in case 1 above to show that there exists a stage $s_{\hat{m}}$ (say) above such that $x \notin \mathcal{C}[s]$ for all $s \geq s_{\hat{m}}$.

□

**Lemma 4.2.15.** *For all $e$, $W_e$ is infinite if and only if $C^{[2e]}$ is finite.*

*Proof.* For all $s > 3e$ let $\sigma_s$ denote the $H_e$ node satisfying $\sigma_s \subseteq \alpha_s$. There are two possible cases.

- $W_e$ is finite. Then there exists a stage $s^* > 2e$ such that $W_{e,s} = W_{e,s^*}$ for all $s \geq s^*$. Hence there exists a stage $t^* \geq s^*$ such that, for every $s \geq t^*$, $W_{e,s} = W_{e,t(\sigma_s,s+1)}$. This means that $\sigma_s{}^\frown 1 \subseteq \alpha_s$, and that $u(\sigma_s, s) = \{\langle 2e+1, z_s\rangle\}$ where $\langle 2e+1, z_s\rangle$ is a new number, and that $u(\sigma_s, s) \subseteq C_s^{[2e]} - C_{s-1}^{[2e]}$. Now, $u(\sigma_s, s) \cap V_{e,s} = \emptyset$, whereas $V_{e,r} = V_{e,s}$ $(= V_{e,t^*})$ for all $r \geq s$ by definition of $t^*$, so that $u(\sigma_s, s) \cap V_{e,r} = \emptyset$ for all such $r$. Therefore $\bigcup\{\, u(\sigma_s, s) \mid s \geq t^* \,\} \subseteq C^{[2e]}$. I.e. $C^{[2e]}$ is infinite.

- Now suppose that $W_e$ is infinite. Let $\sigma$ be the $H_e$ node such that $\sigma \subseteq \delta$ (the true path). Then the set $\{\, s \mid \sigma \subseteq \alpha_s \;\&\; W_{e,s} \neq W_{e,t(\sigma,s)} \,\}$ is infinite. Hence at each such stage $U_{e,s}$ is included in $V_{e,s+1}$ and $U_{e,s+1}$ is set to $\emptyset$. Hence all numbers $x \in \omega^{[2e]}$ mentioned in $C$ at any stage $t \leq s$ and such that $x \notin \Omega(\sigma, s+1)$ are removed from $C$ at stage $s+1$. Moreover, by Note 4.2.8 we know that $\Omega(\sigma)\!\downarrow$. We thus see that, for all $\langle 2e, x\rangle \notin \Omega(\sigma)$ the set $\{\, s \mid \langle 2e, x\rangle \notin C_s \,\}$ is infinite—and in fact cofinite by Lemma 4.2.14. Hence $C^{[2e]} = \Omega(\sigma)$ in this case. In other words, $C^{[2e]}$ is finite. $\square$

**Lemma 4.2.16.** *For all $e$, $C = \Phi_e^{B_e} \Rightarrow \overline{\mathcal{K}} \leq_{\mathrm{e}} B_e$.*

*Proof.* Suppose that $C = \Phi_e^{B_e}$. Let $\sigma$ be the $P_e$ node on the true path $\delta$. Let $s_\sigma$ be the least $\sigma$-true stage such that $\alpha_s \not<_L \sigma$ for all $s \geq s_\sigma$. Note that $b(\sigma, s) = b(\sigma, s_\sigma) =_{\mathrm{def}} b(\sigma)$ for all $s \geq s_\sigma$. We begin by showing that it is not the case that $\sigma{}^\frown 2 \subset \delta$. Indeed, suppose otherwise. Then there exists a $\sigma$-true stage $t_\sigma \geq s_\sigma$ such that $P(\sigma, s) = 2$ for all $s \geq t_\sigma$ and so $h(\sigma, s+1) \geq h(\sigma, s)$ for all $s \geq t_\sigma$—and in fact $h(\sigma, s+1) > h(\sigma, t(\sigma, s+1))$ for all $\sigma$-true stages $s+1 > t_\sigma$. Let $h_\sigma = h(\sigma, s_\sigma)$. We show that (in this case) for all $x \geq h_\sigma$

$$\begin{aligned} x \in \overline{\mathcal{K}} \quad \Leftrightarrow \quad \exists D(\exists s \geq t_\sigma)\,[\,\langle\langle 2e+1, x\rangle, D\rangle \in \Phi_e[s] \;&\&\; D \subseteq B_e[s] \\ &\&\; x \notin \mathcal{K}[\max D] \\ &\&\; s+1 \text{ is a } \sigma\text{-true stage} \\ &\&\; h(\sigma, s) > x\,]\,. \qquad (4.2.14) \end{aligned}$$

Note that $\{\, \langle 2e+1, z\rangle \mid \geq b(\sigma)\,\} \subseteq C$ by Note 4.2.7 as $\liminf_{s\to\infty} h(\sigma,s) = \infty$.

$(\Rightarrow)$ Suppose that $x \in \overline{\mathcal{K}}$. Then, by the last sentence $\langle 2e+1, x\rangle \in C$ and so (as $C = \Phi_e^{B_e}$) there exists $\langle\langle 2e+1, x\rangle, D\rangle \in \Phi_e$ such that $D \subseteq B_e$. Let $s+1 > t_\sigma$ be a $\sigma$-true stage such that $D \subseteq B_e[s]$, $\langle\langle 2e+1, x\rangle, D\rangle \in \Phi_e[s]$ and $h(\sigma, s+1) > x$. Then it suffices to note that that $x \notin \mathcal{K}[\max D]$ since $x \notin \mathcal{K}$. This proves $\Rightarrow$ of (4.2.14).

$(\Leftarrow)$ Suppose that $x \in \mathcal{K}$ and, for a contradiction, that for some finite set $D$ and stage $s \geq t_\sigma$ the right hand side of (4.2.14) holds. Then this implies that $P(\sigma, s+1) = 0$ and $h(\sigma, s+1) < h(\sigma, s)$ in contradiction with the definition of $t_\sigma$. This proves $\Leftarrow$ of (4.2.14).

We know therefore that $\sigma\widehat{\ }2 \subset \delta$ implies that $x \in \overline{\mathcal{K}} \Leftrightarrow \exists D(\exists s \geq t_\sigma)R(D,s,x)$ where $R(D,s,x)$ is the computable condition on the right hand side of (4.2.14). This contradicts the fact that $\overline{\mathcal{K}}$ is not c.e.

- We now show that it is not the case that $\sigma\widehat{\ }1 \subset \delta$. Suppose otherwise and let $r_\sigma \geq s_\sigma$ be a $\sigma$-true stage such that $P(\sigma,s) = P(\sigma, r_\sigma)$ for all $s \geq r_\sigma$. Then, $w(\sigma,s) = w(\sigma, r_\sigma) =_{\mathrm{def}} w(\sigma)$ for all $s \geq r_\sigma$ and case 2.B.2 applies relative to $n = w(\sigma)$ at every $\sigma$-true stage $s+1 \geq r_\sigma$. However this implies that, at every such stage the construction verifies that

$$\begin{aligned} w(\sigma) \in \mathcal{K}[s] \ \&\ \langle\langle 2e+1, w(\sigma)\rangle, D\rangle \in \Phi_e[s] \ &\&\ D \subseteq \{0,\dots,t_{w(\sigma)}\} \\ &\&\ D \subseteq B_e[s] \end{aligned}$$

where $t_{w(\sigma)}$ is such that $w(\sigma) \in \mathcal{K}[t_{w(\sigma)+1}] - \mathcal{K}[t_{w(\sigma)}]$. Thus, for some finite set $D \subseteq \{0,\dots,t_{w(\sigma)}\}$, $\langle\langle 2e+1, w(\sigma)\rangle, D\rangle \in \Phi_e$, it must be the case that $D \subseteq B_e$ (since $\{B_{e,s}\}_{s\in\omega}$ is a $\Delta^0_2$ approximation and the set $\{\, D \mid D \subseteq \{0,\dots,t_{w(\sigma)}\}\,\}$ is finite). Thus $\langle 2e+1, w(\sigma)\rangle \in \Phi_e^{B_e}$. On the other hand, at every such stage $s+1$, we have that $\langle 2e+1, w(\sigma)\rangle \in V^+(\sigma, s+1) \subseteq V(\sigma, s+1) \subseteq \overline{\mathcal{C}[s+1]}$. Thus $\langle 2e+1, w(\sigma)\rangle \in \Phi_e^{B_e}$ whereas $\langle 2e+1, w(\sigma)\rangle \notin C$. A contradiction.

• So it must be the case that $\sigma\widehat{\ }0 \subseteq \delta$. Note that we showed in Lemma 4.2.9 that this implies that $\liminf_{s\to\infty} h(\sigma, s) = \infty$. So, by Note 4.2.7, $\{\, \langle 2e+1, z\rangle \mid z \geq b(\sigma)\,\} \subseteq C$. We show that (in this case), for all $x \geq b(\sigma)$,

$$x \in \overline{\mathcal{K}} \quad \Leftrightarrow \quad \exists D \left( \langle\langle 2e+1, x\rangle, D\rangle \in \Phi_e \;\;\&\;\; D \subseteq B_e \;\;\&\;\; x \notin \mathcal{K}[\max D] \right). \quad (4.2.15)$$

($\Rightarrow$) Suppose that $x \in \overline{\mathcal{K}}$. As $\langle 2e+1, x\rangle \in C$ (and $C = \Phi_e^{B_e}$) there exists $\langle\langle 2e+1, x\rangle, D\rangle \in \Phi_e$ with $D \subseteq B_e$ witnessing $\langle 2e+1, x\rangle \in \Phi_e^{B_e}$. Moreover $x \in \overline{\mathcal{K}}$ implies that $x \notin \mathcal{K}[\max D]$.

($\Leftarrow$) Now suppose that the right hand side of (4.2.15) holds. Then choose an axiom $\langle\langle 2e+1, x\rangle, D\rangle$ witnessing this and let $s^* \geq s_\sigma$ be a stage such that $\langle\langle 2e+1, x\rangle, D\rangle \in \Phi_e[s]$ and $D \subseteq B_e[s]$ for all $s \geq s^*$. Suppose also that $x \in \mathcal{K}$ and that $x$ is the least number satisfying these conditions. Then, since the set $S = \{\, s+1 \mid \sigma\widehat{\ }0 \subseteq \alpha_{s+1}\,\}$ is infinite there exists some stage $\hat{s}+1 \geq s^*$ such that $P(\sigma, \hat{s}+1)$ switches to 1 and $w(\sigma, \hat{s}+1) = x$. However this implies that $P(\sigma, s) = P(\sigma, \hat{s}+1) = 1$ for all $s \geq \hat{s}+1$. A contradiction since $\sigma\widehat{\ }0 \subset \delta$ (and $|S| = \infty$). Thus $x \in \overline{\mathcal{K}}$.

We conclude therefore that (4.2.15) applies for all $x \geq b(\sigma)$ and so $\overline{\mathcal{K}} \leq_{\mathrm{e}} B_e$. □

**Lemma 4.2.17.** *For all $e$, if $B_e = \Phi_e^C$, then $B_e$ is c.e.*

*Proof.* Let $\sigma$ be the $N_e$ node on the true path. Let $s_\sigma$ be the least $\sigma$-true stage such that $\alpha_t \not<_L \sigma$ for all $t \geq s_\sigma$. Note that, for all $s \geq s_\sigma$, $I(\sigma, s) \subseteq I(\sigma, s+1)$ and $W(\sigma, s) \subseteq W(\sigma, s+1)$. Accordingly we define $I(\sigma) = \bigcup_{s \geq s_\sigma} I(\sigma, s)$ and $W(\sigma) = \bigcup_{s \geq s_\sigma} W(\sigma, s)$. Notice that it follows that both $I(\sigma)$ and (in particular) $W(\sigma)$ are c.e. sets. There are two cases to consider.

• Suppose that $\sigma\widehat{\ }1 \subset \delta$. Then—letting $x(\sigma)$ be defined as on page 82—we know that $x(\sigma) \in \Phi_e^C - B_e$ by Lemma 4.2.13. A contradiction.

• Thus $\sigma\widehat{\ }0 \subset \delta$. We show that $B_e = W(\sigma)$ in this case. Consider any $x \in \omega$.

Suppose firstly that $x \in B_e$. Then, as $B_e = \Phi_e^C$, there exists $D$ such that $\langle x, D\rangle \in \Phi_e$ and $D \subseteq C$. As both $\{\mathcal{B}_e[s]\}_{s\in\omega}$ and $\{\mathcal{C}[s]\}_{s\in\omega}$ are $\Delta_2^0$ approximations there exists a $\sigma$-true stage $s^* \geq s_\sigma$ such that $x \in \mathcal{B}_e[s]$, $\langle x, D\rangle \in \Phi_e[s]$ and $D \subseteq \mathcal{C}[s]$ for all $s \geq s^*$. Note that it follows from the fact that $\sigma\hat{\ }0 \subset \delta$ and Lemma 4.2.11 that there exist infinitely many $\sigma$-true stages $s+1 \geq s_\sigma$ such that $I^-(\sigma, s+1) = \emptyset$. So, at each such stage, case 3.B.3 applies. Let $\hat{s}+1$ be the least such $\sigma$-true stage $> s^*$. Then, at stage $\hat{s}+1$, if $x \notin W(\sigma, \hat{s})$, $x$ will be enumerated into $W(\sigma, \hat{s}+1)$ (and some axiom $\langle x, E\rangle$ will be enumerated into $I(\sigma, \hat{s}+1)$). Hence $x \in W(\sigma)$.

Now suppose that $x \in W(\sigma)$. This means that at some $\sigma$-true stage $s+1 \geq s_\sigma$ some axiom $\langle x, D\rangle$ is (permanently) enumerated into $I(\sigma, s+1)$. By Lemma 4.2.11 there exists a stage $s_D \geq s+1$ such that $D^{[\leq 2e+1]} \subseteq \mathcal{C}[t]$ for all stages $t \geq s_D$. Suppose that $x \notin B_e$. Then there exists a stage $s' \geq s_D$ such that $x \notin \mathcal{B}_e[s]$ for all stages $s \geq s'$. Let $t+1$ be the least $\sigma$-true stage $\geq s'$. Then $N(\sigma, t+1)$ will be permanently set to 1 entailing that $\sigma\hat{\ }1 \subseteq \delta$. A contradiction. It follows that $x \in B_e$.

We conclude that $W(\sigma) = B_e$. In other words $B_e$ is c.e. □

On the strength of Lemmas 4.2.14-4.2.17 we conclude that requirement $R$ and, for all $e \in \omega$, the requirements $H_e$, $P_e$ and $N_e$ are satisfied. □

We now give a strengthened version of Kalimullin's result [Kal00] that every low enumeration degree $\boldsymbol{a}$ *caps* with some $\Delta_2^0$ enumeration degree $\boldsymbol{b}$.

**Corollary 4.2.18.** *Every low $\Delta_2^0$ enumeration degree $\boldsymbol{a}$ caps with both a high $\Delta_2^0$ enumeration degree $\mathbf{c}$ and a low enumeration degree $\boldsymbol{d}$.*

*Proof.* To obtain $\boldsymbol{c}$ apply Theorem 4.2.1 using the fact that the class $\{\, \boldsymbol{b} \mid \boldsymbol{b} \leq \boldsymbol{a} \,\}$ is uniform $\Delta_2^0$ by Lemma 4.1.6. Now, since every nonzero $\Delta_2^0$ enumeration degree bounds a nonzero low $\Delta_2^0$ enumeration degree, we obtain low $\boldsymbol{d}$ such that $\mathbf{0}_{\mathrm{e}} < \boldsymbol{d} < \boldsymbol{c}$. $\square$

*Remark.* Notice that Corollary 4.2.18 applies to the local structure (rather than just to its $\Delta_2^0$ substructure) due to the downwards $\Delta_2^0$ closure implied by Lemma 4.1.6.

# Chapter 5

# Enumeration 1-genericity

As mentioned earlier, the richness of the enumeration degrees leads to the search of a notion of genericity appropriate to this context. In this chapter, we introduce the notions of “enumeration 1-genericity” and “symmetric enumeration 1-genericity” which are defined by adapting the underlying definition of 1-genericity to the context in which only positive information can be used. We then study the distribution of the enumeration 1-generic degrees and show that it resembles to some extent the distribution of the class of 1-generic degrees. We also present an application of enumeration 1-genericity to show the existence of prime ideals of $\Pi^0_2$ enumeration degrees. Finally, we look at the relationship between enumeration 1-genericity and highness.

## 5.1 Enumeration 1-genericity

**Definition 5.1.1.** *A set* $A \subseteq \omega$ *is* enumeration 1-generic *if for all* $\Sigma^0_1$ *sets* $W$ *of finite subsets of* $\omega$ *either,*

*i.* $(\exists D \subseteq A)\ [D \in W]$, *or* *ii.* $(\exists E \subseteq \overline{A})$ *such that* $\forall D \in W\ [D \cap E \neq \emptyset]$.

We note that in Definition 5.1.1, $D$ and $E$ are finite sets. Similarly to the case of 1-genericity in the Turing degrees, it turns out that we can characterise Definition 5.1.1 in terms of e-operators.

**Proposition 5.1.2.** *A set $A$ is enumeration 1-generic if and only if, for any $e \in \omega$, either*

*i.* $e \in \Phi_e^A$; *or*

*ii.* $(\exists E \subseteq \overline{A})$ *such that* $e \notin \Phi_e^{\omega \setminus E}$.

*Proof.* $(\Rightarrow)$: Suppose that $A$ is enumeration 1-generic. For any e-operator $\Phi_e$, we consider the c.e. set $W_{g(e)} = \{D \mid \langle e, D\rangle \in \Phi_e\}$. From the enumeration 1-genericity of $A$ we have that if $\exists D \subseteq A$ such that $D \in W_{g(e)}$, then $e \in \Phi_e^A$. On the other hand, if $\exists E \subseteq \overline{A}$ such that $\forall D \in W_{g(e)}\ [D \cap E \neq \emptyset]$, then it follows that $e \notin \Phi_e^{\omega \setminus E}$[1].
$(\Leftarrow)$: For any c.e. set $W_e$ we define the c.e. set $\Phi_{f(e)} = \{\, \langle x, D\rangle \mid x \in \omega \ \&\ D \in W_e \,\}$. Take $f(e)$, then if $f(e) \in \Phi_e^A$ there exists some $\langle f(e), D\rangle \in \Phi_{f(e)}$ and so $D \in W_e$. Otherwise, if $f(e) \notin \Phi_e^{\omega\setminus E}$, then $\exists E \subseteq \overline{A}$ such that for all $\langle f(e), D\rangle \in \Phi_e$ we have that $D \cap E \neq \emptyset$. $\square$

**Lemma 5.1.3.** *If $A$ is enumeration 1-generic then $A$ is infinite.*

*Proof.* Let $A$ be an enumeration 1-generic set. For a contradiction suppose that $A$ is finite, that is, $A \subseteq \omega \restriction n$ for some $n \in \omega$. Now, consider the c.e. set $W = \{\{m\} \mid n \leq m\}$. Clearly, there exists some $E \subseteq \overline{A}$ such that $\forall D \in W$, $D \cap E \neq \emptyset$. However, this is a contradiction since $E$ is not a finite set. $\square$

[1]Note that $g$ and $f$ are suitable computable functions.

We note that the notion of enumeration 1-genericity is weak in the sense that there are c.e. sets which are enumeration 1-generic (i.e. the set of all natural numbers $\omega$). We consider how to strengthen enumeration 1-genericity in such a way that we do not have trivial enumeration 1-generic sets. Our first approach is to look at coinfiniteness.

**Lemma 5.1.4.** *If $A$ is enumeration 1-generic and coinfinite, then $\overline{A}$ is immune. Thus $A$ is not $\Pi^0_1$.*

*Proof.* Let $A$ be an enumeration 1-generic set. Suppose that $W$ is an infinite c.e. set such that $W \subseteq \overline{A}$. Let $\widehat{W} = \{\, \{n\} \mid n \in W \,\}$. By the enumeration 1-genericity of $A$, there exists a finite set $E \subseteq \overline{A}$ such that $W \subseteq E$. An obvious contradiction. Hence $\overline{A}$ is immune. $\square$

In fact we can generalize Lemma 5.1.4 to hyperimmunity(see Definition below).

**Definition 5.1.5.** *Let $\{F_n\}_{n\in\omega}$ be a computable list of all finite sets and let $g$ be a computable function.*

*i. We call a c.e. set $\{F_{g(n)}\}_{n\in\omega}$ of mutually disjoint finite sets $F_{g(n)}$ a c.e. array.*

*ii. We say a set $A$ avoids $\{X_n\}_{n\in\omega}$ if for some $n$, $A \cap X_n = \emptyset$.*

*iii. If $A$ avoids every c.e. array then we say $A$ is hyperimmune.*

**Lemma 5.1.6.** *If $A$ is enumeration 1-generic and coinfinite, then $\overline{A}$ is hyperimmune. Thus $A$ is not $\Pi^0_1$.*

*Proof.* Suppose that there exists a sequence of mutually disjoint finite sets $\{F_{g(n)}\}_{n\in\omega}$ with $g$ computable such that, for all $n$, $F_{g(n)} \cap \overline{A} \neq \emptyset$. Let $W = \{\, F_{g(n)} \mid n \in \omega \,\}$. By enumeration 1-genericity there exists a finite set $E \subseteq \overline{A}$ such that for all $F \in W$, $F \cap E \neq \emptyset$. This is an obvious contradiction since $W$ contains mutually disjoint finite sets. Hence $\overline{A}$ is hyperimmune. $\square$

However, requiring coinfiniteness does not prevent us from having trivial enumeration 1-generic sets.

**Lemma 5.1.7.** *There exists a coinfinite c.e. enumeration 1-generic set $A$.*

*Proof.* We enumerate a set $A \subseteq \omega$ in stages such that for all $e \in \omega$ the following requirements are satisfied:

$$P_e \ : \ \exists D \subset A[D \in W_e] \ \vee \ \exists E \subset \overline{A}\, \forall D \in W_e \, [E \cap D \neq \emptyset].$$

And the overall requirement:

$$R \ : \ |\overline{A}| = \infty.$$

We construct $A$ by the usual finite injury priority method and define $A = \bigcup_{s \in \omega} A_s$.

To satisfy requirement $P_e$ we enumerate into $A$ some finite set $D$ such that $D \in W_e$. For every $P_e$ we define an outcome parameter $P(e,s) \in \{0,1\}$ where 0 stands for "not satisfied" and 1 for "already satisfied". We satisfy the overall requirement $R$ by ensuring that we keep infinitely many numbers out of $A$. For every $e \in \omega$ we define a restrain function $R(e,s) \in \omega$ and require that at stage $s+1$, $P_e$ enumerates into $A$ a finite set $D$ only if $\forall z \in D, z > R(e,s)$.

**The Construction.** At any given stage $s+1$, if not otherwise specified, for every $e \in \omega$ the function $R(e,s)$ and outcome parameter $P(e,s)$ retain their value.

<u>*Stage $s = 0$.*</u> Set $A_0 = \emptyset$. For all $e \in \omega$, we define $R(e,0) = 2e$ and $P(e,s) = 0$.

*Stage $s+1$.* Choose the least $e \leq s$ such that

$$\exists D \; [D \in W_{e,s}] \;\; \& \;\; \min\{x|x \in D\} > R(e,s) \;\; \& \;\; P(e,s) = 0.$$

If such $e$ exists, pick the least such $D$ and let $P(e, s+1) = 1$. Set $d = \max\{x \mid x \in A_s \cup D\}$.

For all $i > e$, define $R(i,s) = d + (i - e)$ and $P(i, s+1) = 0$.

Define $A_{s+1} = A_s \cup D$.

Go to stage $s+2$.

□

**Lemma 5.1.8.** *Every requirement $P_e$ is satisfied.*

*Proof.* Fix $e$ and assume the lemma by induction for all $i < e$. Choose stage $s$ large enough so that no $P_i$ (for $i < e$) receives attention at any stage $t > s$. It is clear by the construction that $R(e,t) = R(e,v)$ for all stages $v \geq t$ and therefore $R(e) = lim_s R(e,s)$. If there exists a set $D \in W_{e,t}$ such that $\forall z \in D, z > R(e,t)$, we then enumerate $D$ into $A$ and requirement $P_e$ is forever satisfied. Otherwise, there exists some finite set $F \subseteq \{0, 1, ..., R(e,t)\}$ such that for all $D$ in $W_e$, $F \cap D \neq \emptyset$ and again requirement $P_e$ is forever satisfied. Moreover, for all stages $v \geq t$, $P(e,t) = P(e,v)$ and so $P(e) = lim_s P(e,s)$. □

**Lemma 5.1.9.** *$A$ is coinfinite.*

*Proof.* Fix $e$ and choose $s$ large enough so that no $P_i$ $(i < e)$ receives attention at any stage $t > s$. Let $P_j$ be the last requirement $(j < e)$ that received attention at stage $s' \leq s$ and so for all $t > s$, $R(e,t) = R(e,s') = \max\{x \mid x \in A_{s'}\} + (j - e)$ and so $R(e) = R(e,t)$. Otherwise, if no $P_i$ received attention then $R(e,t) = 2e = R(e)$. Finally, let $y = R(e)$ and hence $y \in \overline{A}$. $\square$

## 5.2 Symmetric enumeration 1-genericity

As we have seen, imposing coinfiniteness to the notion of enumeration 1-genericity does not confer nontriviality. Another way of strengthening enumeration 1-genericity is to impose symmetry of this notion over a set and its complement.

**Definition 5.2.1.** *A set A is* symmetric enumeration 1-generic (s.e. 1-generic) *if both A and $\overline{A}$ are enumeration 1-generic.*

**Lemma 5.2.2.** *If A is s.e.* 1*-generic then* $A \notin \Sigma^0_1 \cup \Pi^0_1$.

*Proof.* Suppose that $A$ is s.e. 1-generic. By Lemma 5.1.3 both $A$ and $\overline{A}$ are infinite. Since $\overline{A}$ is infinite, $\overline{A}$ is immune and $A$ is not $\Pi^0_1$. It follows that $A$ is not $\Sigma^0_1$ and so $A \notin \Sigma^0_1 \cup \Pi^0_1$. $\square$

Thus, by Lemma 5.2.2 we can conclude that s.e. 1-genericity indeed yields nontriviality.

**Lemma 5.2.3.** *If A is 1-generic, then A is s.e. 1-generic.*

*Proof.* Recall from Proposition 2.1.3 that A is 1-generic if and only if $\overline{A}$ is 1-generic. Hence, we only have to prove that if $A$ is 1-generic then $A$ is enumeration 1-generic.

Consider any c.e. set $W$ of finite sets. We will show that either there exists $D \subset A$ such that $D \in W$, or that there exists $E \subset \overline{A}$ such that for all $D \in W$, $D \cap E \neq \emptyset$.

We define the following set

$$\mu(D) \;=\; \{\, \sigma \mid \sigma \in 2^{<\omega} \;\&\; |\sigma| > \max D \;\&\; (\forall x \in D)[\, \sigma(x) = 1 \,]\, \}.$$

Accordingly, let

$$\widehat{W} = \bigcup_{D \in W} \mu(D) .$$

We notice that $\widehat{W}$ is a c.e. set. Since $A$ is 1-generic, either,

i. $(\exists \sigma \subset A)\,[\sigma \in \widehat{W}]$. Fix such $\sigma$. Then for some $D$, $\sigma \in \mu(D)$ and so $D \in W$; or

ii. $(\exists \sigma \subset A)\,(\forall \tau \supseteq \sigma)\,[\tau \notin \widehat{W}]$. Fix such $\sigma$. Define $E = \{x \mid x < |\sigma| \;\&\; \sigma(x) = 0\}$. Hence $E \subset \overline{A}$.

   For a contradiction, suppose that there exists $D \in W$ such that $D \cap E = \emptyset$. Define $\tau$ by $\tau = \max\{\max\{x \mid x \in D\} + 1, |\sigma|\}$, and for all $x < |\tau|$,

$$\tau(x) = \begin{cases} 1 & \text{if } x < |\sigma| \text{ and } \sigma(x) = 1, \text{ or } x \in D, \\ 0 & \text{otherwise.} \end{cases}$$

   Clearly, $\tau \in \mu(D)$, $\tau \supseteq \sigma$ and so $\tau \in \widehat{W}$. A contradiction. Hence, for all $D \in W$ we have that $D \cap E \neq \emptyset$.

By a similar argument we can prove that $\overline{A}$ is also enumeration 1-generic and so $A$ is s.e. 1-generic.

$\square$

*Remark.* We note that the class of 1-generic degrees is contained in the class of s.e. 1-generic degrees which is itself a subclass of the enumeration 1-generic degrees.

## 5.3 Distribution of the class of the enumeration 1-generic degrees

We now study the jump of enumeration 1-generic sets.

**Lemma 5.3.1.** *If $A$ is enumeration 1-generic, then $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} A \oplus \overline{A} \oplus J_{\mathrm{e}}(\emptyset)$.*

*Proof.* Recall that $K_A \equiv_{\mathrm{e}} A$. Let $A$ be an enumeration 1-generic set. Consider the set $\overline{\Phi_e^{\omega \setminus E}}$ (which is $\Pi^0_1$). Then $\overline{\Phi_e^{\omega \setminus E}}$ is e-reducible to $J_{\mathrm{e}}(\emptyset)$ uniformly in $e$ and $E$ via the e-operator $\Phi_{g(e,E)}$. By the enumeration 1-genericity of $A$,

$$\overline{K_A} = \{e \mid \exists E[e \in \Phi_{g(e,E)}^{J_{\mathrm{e}}(\emptyset)}] \ \&\ E \subset \overline{A}\}.$$

It follows that $\overline{K_A} \leq_{\mathrm{e}} \overline{A} \oplus J_{\mathrm{e}}(\emptyset)$. Now, since $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} K_A \oplus \overline{K_A}$, we have that $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} A \oplus \overline{A} \oplus J_{\mathrm{e}}(\emptyset)$. □

**Lemma 5.3.2.** *If $A$ is s.e. 1-generic, then $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} J_{\mathrm{e}}(\overline{A})$. In particular, if $A$ is 1-generic, then $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} J_{\mathrm{e}}(\overline{A}) \equiv_{\mathrm{e}} J_{\mathrm{e}}(A \oplus \overline{A})$.*

*Proof.* Let $A$ be an s.e. 1-generic set. Then $A$ and $\overline{A}$ are both enumeration 1-generic. By Lemma 5.3.1, $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} A \oplus \overline{A} \oplus J_{\mathrm{e}}(\emptyset)$. Similarly, $J_{\mathrm{e}}(\overline{A}) \equiv_{\mathrm{e}} A \oplus \overline{A} \oplus J_{\mathrm{e}}(\emptyset)$.

Now, suppose that $A$ is 1-generic. By Proposition 2.1.6, $A \oplus \mathcal{K} \equiv_T A'$. Since the embedding $\iota$ of the Turing degrees into the enumeration degrees preserves the jump operation, $J_{\mathrm{e}}(A \oplus \overline{A}) \equiv_{\mathrm{e}} A \oplus \overline{A} \oplus J_{\mathrm{e}}(\emptyset)$. □

**Corollary 5.3.3.** *If $A$ is $\Pi^0_2$ enumeration 1-generic then $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} A \oplus J_{\mathrm{e}}(\emptyset)$. In particular, if $A$ is $\Delta^0_2$ enumeration 1-generic then, $deg_{\mathrm{e}}(A)$ is low.*

Moreover, a straightforward argument shows the following.

**Corollary 5.3.4.** *If $A \in \Delta^0_2$ is enumeration 1-generic then there exists a $\Delta^0_2$ approximation $\{A_s\}_{s\in\omega}$ to $A$ which is good and low.*

*Proof.* Let $\{\widetilde{A}_s\}_{s\in\omega}$ be any $\Delta^0_2$ approximation to $A$, and let $\{\Phi_e\}_{e\in\omega}$ be the standard computable listing of enumeration operators with c.e. approximation $\{\Phi_{e,s}\}_{e,s\in\omega}$. We define a good $\Delta^0_2$ approximation $\{A_s\}_{s\in\omega}$ to $A$ in the usual way. For all $s \in \omega$ we define:

- Case $s = 0$. Then set $A_s = \emptyset$ and $n_0 = 0$.
- Case $s > 0$. If $\widetilde{A}_s \neq \widetilde{A}_{s+1}$ then, set $n_s = \mu x[\widetilde{A}_s(x) \neq \widetilde{A}_{s+1}(x)]$. Otherwise, set $n_s = s$. Define $A_s = \widetilde{A}_s \upharpoonright n_s$.

It can be easily shown that $\{A_s\}_{s\in\omega}$ is indeed a good approximation. Now, for all $e \in \omega$, we define

$$\Phi^A_e[s] = \{x \mid \langle x, D\rangle \in \Phi_{e,s} \;\&\; D \subseteq A[s]\}.$$

Finally, we check that for any given $e$,

$$\forall x \; lim_s \Phi^A_{e,s}(x) \text{ exists}$$

Suppose that $x \in \Phi^A_e$. Then there exists an axiom $\langle x, D\rangle \in \Phi_e$. Consider $m = \max\{x \mid x \in D\} + 1$.Choose $s$ large enough such that for all $t \geq s$ we have $A \upharpoonright m \subseteq A_t$ and thus $D \subseteq A_t$. Therefore, $x \in \Phi^{A_t}_{e,t} \subseteq \Phi^A_e$.

Now, suppose that $x \notin \Phi^A_e$. Then since $A$ is enumeration 1-generic there exists some finite set $E \subseteq \overline{A}$ such that $x \notin \Phi^{\omega-E}_e$. Notice that $\Phi^A_e \subseteq \Phi^{\omega-E}_e$. Since $\{A_s\}_{s\in\omega}$ is a good $\Delta^0_2$ approximation to $A$, there exists a stage $s$ such that for all $t \geq s$, $E \subseteq \overline{A}_t$. Hence, $x \notin \Phi^{A_t}_{e,t} \subseteq \Phi^A_e$.

□

**Lemma 5.3.5.** *For any $A \in \Sigma^0_2$, if $A$ is s.e. 1-generic and $deg_{\mathrm{e}}(\overline{A})$ is good then $A$ and $\overline{A}$ are low.*

Before giving the proof of Lemma 5.3.5 we need the following results.

**Proposition 5.3.6** ([McE85]). *If $A \in \Sigma^0_2$ then $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} \overline{K_A}$.*

**Proposition 5.3.7** ([Har10]). *If $A$ has a good approximation then $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} \overline{K_A}$.*

**Lemma 5.3.8.** *Let $\boldsymbol{a}$ be a good enumeration degree. Then, for every $A \in \boldsymbol{a}$, $K_A \leq_{\mathrm{e}} \overline{K_A}$. Thus, $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} \overline{K_A}$.*

*Proof.* Let $\boldsymbol{a}$ be a good enumeration degree. Choose a good approximable set $B \in \boldsymbol{a}$ and let $A \in \boldsymbol{a}$ be any set. Note that $A \equiv_{\mathrm{e}} B$. From Lemma 1.2.11 it follows that $K_A \leq_1 K_B$ and $K_B \leq_1 K_A$. Hence, $K_A \equiv_1 K_B$ and so $\overline{K_A} \equiv_1 \overline{K_B}$. Moreover, Proposition 5.3.7 leads to $K_B \oplus \overline{K_B} \leq_{\mathrm{e}} \overline{K_B}$ and consequently $K_B \leq_{\mathrm{e}} \overline{K_B}$. Now, since $K_A \equiv_{\mathrm{e}} A$ and by assumption $A \leq_{\mathrm{e}} B$, it follows that

$$K_A \leq_{\mathrm{e}} B \leq_{\mathrm{e}} \overline{K_B} \leq_{\mathrm{e}} \overline{K_A}.$$

Thus, $K_A \leq_{\mathrm{e}} \overline{K_A}$ and so $J_{\mathrm{e}}(A) \equiv_{\mathrm{e}} \overline{K_A}$.

□

Now, we can finally give the proof of Lemma 5.3.5.

*Proof.* Let $A \in \Sigma^0_2$ be an s.e. 1-generic set such that $deg_{\mathrm{e}}(\overline{A})$ is good. Notice that from Lemma 5.3.8 it follows that $J_{\mathrm{e}}(\overline{A}) \leq_{\mathrm{e}} \overline{K_{\overline{A}}}$.

Let $R_A$ be a computable predicate such that for any finite set $E$,

$$E \subseteq A \Leftrightarrow \exists s \, \forall t \, R_A(E, s, t).$$

Now, from the definition of $\overline{K_{\overline{A}}}$,

$$
\begin{aligned}
e \in \overline{K_{\overline{A}}} \quad &\Leftrightarrow \quad e \notin \Phi_e^{\overline{A}}, \\
&\Leftrightarrow \quad (\exists E \subseteq A)[\, e \notin \Phi_e^{\omega \setminus E}\,], \quad \text{by enumeration 1-genericity of } \overline{A}, \\
&\Leftrightarrow \quad \exists E\, \exists s\, \forall t\, \forall s'\, \forall D[\, R_A(E,s,t) \;\&\; \big(\langle e, D\rangle \in \Phi_{e,s'} \Rightarrow D \cap E \neq \emptyset\big)\,].
\end{aligned}
$$

Clearly, $\overline{K_{\overline{A}}}$ is $\Sigma_2^0$. Hence, $J_{\mathrm{e}}(\overline{A})$ is $\Sigma_2^0$ and consequently $\overline{A}$ is low.

We proceed to show that $A$ is low. By assumption $A \in \Sigma_2^0$. Applying the same reasoning to $\overline{K_A}$ as we did to $\overline{K_{\overline{A}}}$, and using the fact that $\overline{A}$ is $\Sigma_2^0$, it follows that $\overline{K_A} \in \Sigma_2^0$. Thus, $A$ is low.

$\square$

**Lemma 5.3.9.** *If $A \in \Delta_2^0$ is enumeration 1-generic then $deg_{\mathrm{e}}(A)$ is low$_2$.*

*Proof.* Let $\boldsymbol{a} = deg_{\mathrm{e}}(A)$. By Lemma 3.2.4 we know that

$$\{\, e \mid \Phi_e^A \text{ is infinite}\,\} \;\in\; \boldsymbol{a}''.$$

Thus to show that $\boldsymbol{a}$ is low$_2$ it suffices to show that there exists a set $C \leq_{\mathrm{e}} \overline{\mathcal{K}}$ such that $\{\, e \mid C^{[e]} \text{ is finite}\,\}$ since this implies, by Lemma 3.2.3, that $\boldsymbol{a}'' = \mathbf{0}_{\mathrm{e}}''$ . We do this by enumerating $C$ using a construction with $\overline{\mathcal{K}}$ as oracle. Now, since $A$ is $\Delta_2^0$ we know that there is a function $f \leq_{\mathrm{T}} \mathcal{K}$ such that $\mathrm{Ran}\,(f) = A$ and such that for all $n$, $f(n) < f(n+1)$. Accordingly we let $a_0 < a_1 < a_2 \ldots$ be the resulting c.e. in $\mathcal{K}$ enumeration of $A$. At each stage $s+1$ of the construction $a_s$ is enumerated into $A$. Note that this means that the set

$$U_{s+1} =_{\mathrm{def}} \{\, z \mid z < a_s \;\&\; z \notin A_{s+1}\,\} \;\subseteq\; \overline{A}\,.$$

<u>*Stage* 0.</u> Set $C_0 = \emptyset$.

<u>*Stage* $s+1$.</u> For each $e > s$ do nothing (so that $C^{[e]}_{s+1} = \emptyset$). For each $e \leq s$ on the other hand, test whether, for all $x > s$,

$$\langle x, D\rangle \in \Phi_e \;\Rightarrow\; D \cap U_{s+1} \neq \emptyset\,.$$

- If so, then enumerate $\langle e, s\rangle$ into $C$.
- Otherwise do nothing for index $e$.

Having processed each $e \leq s$ proceed to stage $s+2$. This completes the description of the construction.

In order to verify the correctness of the construction there are two cases to consider.

<u>*Case A.*</u> $\Phi_e^A$ is infinite. Consider any stage $s+1$. If, for every $x > s$ and finite set $D$ such that $\langle x, D\rangle \in \Phi_e$, it is the case that $D \cap U_{s+1} \neq \emptyset$ then $\Phi_e^A \subseteq \{0, \ldots, s\}$ since $U_{s+1} \subseteq \overline{A}$. Hence $C^{[e]} = \emptyset$.

<u>*Case B.*</u> $\Phi_e^A$ is finite. Then, for some $s_A$, for all $x > s_A$, $x \notin \Phi_e^A$. Let

$$W =_{\rm def} \{\, D \mid \langle z, D\rangle \in \Phi_e \;\;\&\;\; z > s_A \,\}\,,$$

and note that $W$ is c.e. By the enumeration 1-genericity of $A$ (and since there is no $\langle z, D\rangle \in \Phi_e$ with $z > s_A$ such that $D \subseteq A$), there exists a finite set $E_A \subseteq \overline{A}$ such that $D \cap E_A \neq \emptyset$ for all $D \in W$. Let $z_A = \max E_A + 1$ and also let $t_a$ be a stage such that $a_{t_A} \geq z_A$. Then, by construction, we can see that $C^{[e]} \supseteq \{\, \langle e, s\rangle \mid s > t_A \,\}$. In other words $C^{[e]}$ is cofinite (and so infinite). $\square$

We now consider the question of what overall restrictions there are on the distribution of enumeration 1-generic degrees.

**Definition 5.3.10.** *If* $\boldsymbol{a}$ *is a* $\Pi^0_2$ *(*$\Sigma^0_2$*) e-degree then* $\boldsymbol{a}$ *is* downwards properly $\Pi^0_2$ ($\Sigma^0_2$) closed *if* $\mathcal{D}[\leq \boldsymbol{a}] = \{\boldsymbol{x} \mid \boldsymbol{x} \leq \boldsymbol{a}\}$ *contains only* $\Pi^0_2$ *(*$\Sigma^0_2$*) sets.*

**Definition 5.3.11.** *Let* $A$ *be any set and* $\{A_s\}_{s\in\omega}$ *be a computable approximation to* $A$*. We say* $\{A_s\}_{s\in\omega}$ *is a* $\Pi^0_2$ approximation *to* $A$ *if,*

$$A = \{x \mid \forall s\, \exists t \geq s\, [x \in A_t]\}.$$

**Definition 5.3.12.** *Let $\mathcal{A} = \{A_i\}_{i\in\omega}$ be a class of $\Pi^0_2$ sets (i.e. $\mathcal{A} \subseteq \Pi^0_2$). We say a computable approximation $\{A_{i,s}\}_{i,s\in\omega}$ is a* uniform $\Pi^0_2$ approximation *to $\mathcal{A}$ if for all $i \in \omega$,*

*i.* $\{A_{i,s}\}_{s\in\omega}$ *is a $\Pi^0_2$ approximation to $A_i$, and*

*ii.* $\mathcal{A} = \{A_{i,s}\}_{i,s\in\omega}$.

*In this case, we say $\mathcal{A}$ is* uniform $\Pi^0_2$.

Similarly, we can define a uniform $\Sigma^0_2$ approximation $\{B_{i,s}\}_{i,s\in\omega}$ to a $\Sigma^0_2$ class $\mathcal{B} = \{B_i\}_{i\in\omega}$ by letting $B_i = \{y \mid \exists s\, \forall t \geq s\, [y \in B_t]\}$.

**Lemma 5.3.13.** *If $A$ is a $\Pi^0_2$ enumeration 1-generic set, then the class $\mathcal{A} = \{X \mid X \leq_{\mathrm{e}} A\}$ is uniform $\Pi^0_2$.*

*Proof.* Let $A$ be a $\Pi^0_2$ enumeration 1-generic set with an associated $\Pi^0_2$ approximation $\{A_s\}_{s\in\omega}$. Let $\{\Phi_e\}_{e\in\omega}$ be the standard computable listing of enumeration operators with c.e. approximation $\{\Phi_{e,s}\}_{e,s\in\omega}$. We define a uniform $\Pi^0_2$ class $\{A_e\}_{e\in\omega}$ with a uniform $\Pi^0_2$ approximation $\{A_{e,s}\}_{e,s\in\omega}$ and set $\mathcal{A} = \{A_e\}_{e\in\omega}$. In order to define, for every $e \in \omega$, a $\Pi^0_2$ approximation $\{A_{e,s}\}_{s\in\omega}$ we need the following parameters. For every $y \in \omega$ and every finite set $D$, let

- $\mu(y, D, e, 0) = \emptyset$, and for $s + 1$,

$$\mu(y, D, e, s+1) = \begin{cases} \mu(y, D, e, s) \;\cup\; \{\, x \mid x \in D \;\&\; x \in A_{s+1} \,\} & \\ \qquad\qquad \text{if } \langle y, D\rangle \in \Phi_{e,s+1} \text{ and } \mu(y, D, e, s) \neq D, & \\ \emptyset \qquad\qquad \text{otherwise.} & \end{cases}$$

- $m(y,e,0)=0$, and for $s+1$,

$$m(y,e,s+1) \;=\; \begin{cases} 1 & \text{if } \mu(y,D,e,s)=D \text{ for some axiom } \langle y,D\rangle \in \Phi_{e,s}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for every $e,s \in \omega$, we define a uniform $\Pi^0_2$ approximation $\{A_{e,s}\}_{e,s\in\omega}$,

$$A_{e,s} \;=\; \begin{cases} \emptyset & \text{if } e \geq s, \\ \{\, x \mid m(y,e,s)=1 \,\} & \text{otherwise.} \end{cases}$$

□

**Lemma 5.3.14.** *For every $e \in \omega$, $\{A_{e,s}\}_{s\in\omega}$ is a $\Pi^0_2$ approximation to $A_e = \Phi_e^A$.*

*Proof.* First, supposing that $y \in \Phi_e^A$, there exists some axiom $\langle y,D\rangle \in \Phi_e$ such that $D \subseteq A$. We define the set $S_y = \{\, s \mid \mu(y,D,e,s)=D \text{ and } \langle y,D\rangle \in \Phi_{e,s} \,\}$. Since $\{A_s\}_{s\in\omega}$ is a $\Pi^0_2$ approximation to $A$ and by the approximation $\{A_{e,s}\}_{s\in\omega}$ to $A_e$ we have that $S_y$ is infinite. Hence, there exist infinitely many stages $s \in S_y$ and so $m(y,e,s)=1$. Hence, $y \in A_{e,s}$ for every $s \in S_y$. Finally, suppose that $y \notin \Phi_e^A$. Since $A$ is enumeration 1-generic, for some $E \subseteq \overline{A}$, we have $y \notin \Phi_e^{\omega\setminus E}$. Let $\{\overline{A}_s\}_{s\in\omega}$ be a $\Sigma^0_2$ approximation to $\overline{A}$. Then there exists some stage $s$ such that for all $t \geq s$, $E \subseteq \overline{A}_t$. Hence, $m(y,e,t)=0$ and so $y \notin \Phi_{e,t}^A$. □

**Corollary 5.3.15.** *If $A$ is a $\Sigma^0_2$ set such that $\overline{A}$ is enumeration 1-generic, then $\mathcal{B} = \{X \mid X \leq_{\mathrm{e}} \overline{A}\}$ is uniform $\Pi^0_2$.*

*Proof.* Notice that $\overline{A}$ is $\Pi^0_2$. Apply Lemma 5.3.13 to $\overline{A}$. □

In fact, if we let $\boldsymbol{b} = deg_{\mathrm{e}}(\overline{A})$, then Corollary 5.3.15 implies that for any $\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x}$ contains sets that are at most $\Pi^0_2$.

Using the notion of s.e. 1-genericity and the set constructed in Theorem 3.2.1 (Chapter 2) we are able to show that any 1-generic $\Pi^0_2$ degree is downwards $\Pi^0_2$ closed.

**Proposition 5.3.16.** *There exists a $\Pi^0_2$ enumeration degree $\boldsymbol{b}$ such that the following is true for $\boldsymbol{b}$, some $B \in \boldsymbol{b}$, $A = \overline{B}$ and $\boldsymbol{a} = deg_{\mathrm{e}}(A)$.*

*i. $A$ is 1-generic (and hence $\boldsymbol{a}$ is 1-generic).*

*ii. $\boldsymbol{a}$ is noncuppable (and hence downwards properly $\Sigma^0_2$).*

*iii. $\boldsymbol{a}$ is low$_2$ (i.e. $\boldsymbol{a}'' = \mathbf{0}''_{\mathrm{e}}$ ).*

*iv. The class $\mathcal{B} =_{\mathrm{def}} \{ X \mid X \leq_{\mathrm{e}} B \}$ is uniformly $\Pi^0_2$ so that, in particular, $\boldsymbol{b}$ (and any $\boldsymbol{x} \leq \boldsymbol{b}$) only contains $\Pi^0_2$ sets.*

*v. $\boldsymbol{b}$ is properly $\Pi^0_2$.*

*vi. $K_B \not\leq_{\mathrm{e}} \overline{K_B}$ (and hence $J_{\mathrm{e}}(B) \not\leq_{\mathrm{e}} \overline{K_B}$).*

*vii. $\boldsymbol{b}' \leq \mathbf{0}''_{\mathrm{e}}$ .*

*Proof. i.* Apply Theorem 3.2.1.
*ii.* Apply Theorem 3.2.1.
*iii.* Apply Theorem 3.2.1.
*iv.* Apply Theorem 3.2.1, then apply Lemma 5.2.3 and Lemma 5.3.13.
*v.* Follows from *iv.*
*vi.* From $B \in \Pi^0_2$ and $K_B \equiv_{\mathrm{e}} B$ it follows that $K_B$ is $\Pi^0_2$. Hence, $\overline{K_B}$ is $\Sigma^0_2$. For a contradiction, assume that $K_B \leq_{\mathrm{e}} \overline{K_B}$. Then $K_B \in \Sigma^0_2$. A contradiction. Thus $K_B \not\leq_{\mathrm{e}} \overline{K_B}$.
*vii.* Notice that $K_B$ is $\Pi^0_2$ and hence $\overline{K_B}$ is $\Sigma^0_2$. Hence, $K_B \oplus \overline{K_B}$ is $\Sigma^0_3$ and so $\boldsymbol{b}' \leq \mathbf{0}''_{\mathrm{e}}$ .

□

Now we turn our attention to the local structure of the enumeration degrees. A natural question to ask is whether $\mathbf{0}'_{\mathrm{e}}$ is enumeration 1-generic. The next property, which is also possessed by the notion of 1-genericity, settles this question.

**Proposition 5.3.17.** *Every enumeration* 1*-generic degree* $\boldsymbol{a} > \mathbf{0}_{\mathrm{e}}$ *is quasiminimal.*

*Proof.* Suppose that $A$ is an enumeration 1-generic set and that there exists some set $C$ set such that $C \oplus \overline{C} \leq_{\mathrm{e}} A$. Let $\Phi$ witness this reduction, that is, $C \oplus \overline{C} = \Phi^A$. Consider the c.e. set

$$S \;=\; \{\, D \mid \exists F\, \exists F' [\, \langle 2x, F\rangle \in \Phi \;\;\&\;\; \langle 2x+1, F'\rangle \in \Phi \;\;\&\;\; D = F \cup F' \,]\,\}.$$

Since $C \oplus \overline{C}$ is the characteristic function of $C$, it follows that $D \not\subseteq A$ for all $D \in S$. Hence, by enumeration 1-genericity of $A$, there exists a finite set $E \subseteq \overline{A}$ such that for all $D \in S$, $D \cap E \neq \emptyset$. However, this implies that $C \oplus \overline{C} = \Phi^{\omega - E}$.

Indeed, clearly $C \oplus \overline{C} \subseteq \Phi^{\omega-E}$ (as $A \subseteq \omega - E$). Suppose that there exists $y \in \Phi^{\omega-E} \setminus C \oplus \overline{C}$. Then, $y = 2x + i$ for some $i \in \{0, 1\}$. Without loss of generality, suppose that $i = 0$. Therefore there is a finite set $F \subseteq \omega - E$ such that $\langle 2x, F\rangle \in \Phi$. Since $C \oplus \overline{C}$ is the characteristic function of $C$, and $2x \notin C \oplus \overline{C}$, it follows that $2x + 1 \in C \oplus \overline{C} = \Phi^A$. Hence there exists a finite set $F'$ such that $\langle 2x + 1, F'\rangle \in \Phi$ and $F' \subseteq A \subseteq \omega - E$. Set $D = F \cup F'$. Clearly $D \in S$ whereas, by the above, $D \cap E = \emptyset$. This contradicts the definition of $E$. Thus $\Phi^{\omega-E} \subseteq C \oplus \overline{C}$ and so $C \oplus \overline{C} = \Phi^{\omega-E}$, i.e. $C \oplus \overline{C}$ is c.e. □

We now illustrate another characteristic of $\Delta^0_2$ enumeration 1-generic sets which resembles 1-genericity, namely, $\Sigma_1$-correctness (see chapter 2, Proposition 2.1.13).

**Proposition 5.3.18.** *If* $B \leq_{\mathrm{e}} A <_{\mathrm{e}} J_{\mathrm{e}}(\emptyset)$ *and* $A$ *is a* $\Delta^0_2$ *enumeration 1-generic set then* $B$ *has a* $\Delta^0_2$ *approximation* $\{B_s\}_{s\in\omega}$ *such that for any infinite c.e. set* $T \subseteq \omega$, $\exists t \in T (B_t \subset B)$.

*Proof.* Let $A \in \Delta_2^0$ be an enumeration 1-generic set with an associated $\Delta_2^0$ approximation $\{A_s\}_{s\in\omega}$. Since $B \leq_{\mathrm{e}} A$, for some $e \in \omega$ we have $B = \Phi_e^A$. We set $B_s = \Phi_e^{A_s}$ for every $s \in \omega$. Let $T \subseteq \omega$ be any infinite c.e. set. Define

$$D(A_t) = \{y \mid y \in A_t\},$$

and set

$$S = \{\ D(A_t)\ :\ t \in T\}.$$

We notice that $S$ is a c.e. set of finite sets. For a contradiction, assume that $\exists E \subset \overline{A}\, \forall D \in S\, [E \cap D \neq \emptyset]$. By assumption $\{A_s\}_{s\in\omega}$ is a $\Delta_2^0$ approximation so there exists $v$ such that for all $t \geq v$, $E \not\subseteq A_t$. Hence, $E \cap D(A_t) = \emptyset$ (whenever $t \in T$). Clearly a contradiction. Therefore, $\exists D(A_t) \subset A$ such that $D(A_t) \in S$ and so $B_t \subset B$.

□

We have seen that enumeration 1-genericity displays some form of lowness and this leads us to the question of the relationship between enumeration 1-genericity and $\Sigma_2^0$ highness (see Definition 5.3.20 below). For the final part of this section we investigate this question.

**Definition 5.3.19.** *If $\{A_s\}_{s\in\omega}$ is a $\Sigma_2^0$ approximation to a set $A$, then we define* the computation function relative to $A$ *by*

$$C_A(x) = \mu s\, [s > x\ \&\ A_s \restriction x \subset A].$$

**Definition 5.3.20.** *A set $A \leq_{\mathrm{e}} \mathbf{0}'_{\mathrm{e}}$ is* $\Sigma_2^0$ high *if it has a $\Sigma_2^0$ approximation for which $C_A$ is total and dominates every computable function. A degree is $\Sigma_2^0$ high if it contains a $\Sigma_2^0$ high set.*

**Lemma 5.3.21** ([SS99])**.** *A degree $\boldsymbol{a} \leq \mathbf{0}'_{\mathrm{e}}$ is high if and only if it is $\Sigma^0_2$ high.*

**Lemma 5.3.22.** *If $A$ is $\Sigma^0_2$ high then $A$ is not enumeration 1-generic.*

*Proof.* For a contradiction, assume that $A$ is a $\Sigma^0_2$ high enumeration 1-generic set. Let $\{A_s\}_{s\in\omega}$ be a high $\Sigma^0_2$ approximation to $A$ with associated computation function $c_A$. Let $s_A \in \omega$ be such that $c_A(s) > s+1$ (i.e. the successor function) for all $s > s_A$. Define the c.e. set

$$W \;=\; \{\, A_{s+1}{\restriction}s \;\mid\; s > s_A \,\}$$

and notice that, by definition of $s_A$, for all $D \in W$, $D \not\subseteq A$. Then there exists a finite set $E \subseteq \overline{A}$ such that $D \cap E \neq \emptyset$ for all $D \in W$. Let

$$m \;=\; \max\{E \cup \{s_A\}\} + 1$$

and let $s_m$ be such that $s_m + 1 = c_A(m)$ (and so $s_m \geq m$). By definition of $c_A$, $A_{s_m+1}{\restriction}m \subseteq A$ (whereas $E \subseteq \overline{A}{\restriction}m$). Thus, letting $D = A_{s_m+1}{\restriction}s_m$ we see that $D \in W$ and $D \cap E = \emptyset$, a contradiction. Thus $A$ is not enumeration 1-generic. $\square$

**Corollary 5.3.23.** *If $A$ is is $\Sigma^0_2$ high then $A$ is not 1-generic.*

*Proof.* Let $A$ be a $\Sigma^0_2$ high 1-generic set. Then by Lemma 5.2.3, $A$ is enumeration 1-generic. Apply Lemma 5.3.22 and deduce a contradiction. $\square$

# Chapter 6

# Separating enumeration 1-genericity and 1-genericity

In this final chapter we present two different approaches to the problem of separating the class of the enumeration 1-generic degrees from the class of 1-generic degrees. One of them is by showing the existence of a non trivial enumeration 1-generic set which is not 1-generic and the other is by proving that there exists a property that both classes do not share, namely, nonsplitting.

## 6.1 Introduction

As we saw in the previous chapter, every 1-generic set is s.e. 1-generic and hence enumeration 1-generic. Moreover, the class of nonzero enumeration 1-generic degrees shares at least two properties with the 1-generic degrees, namely, quasiminimality (Proposition 5.3.17) and $\Pi^0_2$ downwards closure (Proposition 5.3.16). We now investigate the question of how enumeration 1-genericity and 1-genericity may be separated within the enumeration degrees.

### 6.1.1 An enumeration 1-generic set which is not 1-generic

We start by taking a quite straightforward approach to our question i.e. we present a direct construction of a non trivial enumeration 1-generic set which is not 1-generic.

**Proposition 6.1.1.** [1] *There exists an nonzero enumeration 1-generic set $A$ which is not 1-generic.*

*Proof.* We construct a set $A$ such that, for all $e \in \omega$, the following requirements are satisfied

$$\begin{aligned} N_e &: \quad A \neq W_e, \\ P_e &: \quad (\exists D \in W_e)[\, D \subseteq A\,] \ \vee \ (\exists E \subseteq \overline{A})(\forall D \in W_e)[\, D \cap E \neq \emptyset\,]\,. \end{aligned}$$

We enumerate a c.e. set of strings $W$ which satisfies the overall non 1-genericity requirement

$$N_G \ : \ (\forall \sigma \subseteq A[\sigma \notin W \quad \wedge \quad \exists \tau \supset \sigma(\tau \in W)]).$$

And we say $W$ is a witness of $A$ being not 1-generic, i.e. $W$ is not forced by $A$. Let $\{W_e\}_{e\in\omega}$ be a computable listing of all c.e. sets with associated finite c.e. approximations $\{W_{e,s}\}_{s\in\omega}$ for each $e \in \omega$. The priority ordering of the requirements $R \in \{N, P\}$ is given by

$$N_0 < P_0 < N_1 < P_1 < \cdots$$

Requirements $N_e$ are satisfied using the basic Friedberg-Muchnik strategy whereas the basic module for $P_e$ is explained as follows. Indeed, to satisfy $P_e$, whenever we see some $D \in W_e$ such that $D \cap F = \emptyset$, for some finite set $F$ that the construction wants to restrain from $A$, we enumerate $D$ in $A$. The basic module for satisfying

[1] Thanks are due to Mariya Soskova for suggesting this problem.

the overall requirement $N_G$ enumerates a c.e. set of strings $W$ by stages $s$ in such a way that $W$ is not forced by $A$. Accordingly, at the end of every stage $s > 0$ we define a string $\tau$ such that $\tau \supset \sigma$ (with $\sigma \subseteq A_s$) and enumerate $\tau$ into $W$. We note that satisfaction of $N_G$ ensures that for every initial segment of $A$, say $\sigma \subset A$, there exists an extension $\tau \supset \sigma$ in $W$ and $\sigma \notin W$ (since we make sure that we never enumerate an initial segment of $A$ into $W$).

### Definitions and notation

• *Parameters for the $N_e$ requirements.* The outcome function $N(e, s) \in \{0, 1\}$ and the witness parameter $x(e, s) \in \omega \cup \{-1\}$.

• *Parameters for the $P_e$ requirements.* The outcome parameter $P(e, s) \in \{0, 1\}$, the finite set parameter $D(e, s) \in \mathcal{P}(\omega)$ and the avoidance parameter $\Omega(e, s) \in \mathcal{P}(\omega)$. We define $\Omega(e, s + 1)$ by

$$\Omega(e, s + 1) \ = \ \bigcup\{x(i, s) \mid i < e \ \& \ N(i, s) = 1\}.$$

Thus, $\Omega(e, s+1)$ records the finite set of elements that the construction wants to keep out of $A$ for the sake of higher priority $N$ requirements and cannot be enumerated into $A$ at stage $s + 1$ for the sake of $P_e$.

• *Global parameters.* The enumerating parameter $C(s+1) \in \mathcal{P}(\omega)$, the extracting parameter $F(s + 1) \in \mathcal{P}(\omega)$, the approximating string $\sigma_{s+1}$ and the witness string $\tau_{s+1}$.

Now, we establish under what conditions requirements $R \in \{N, P\}$ require attention.

<u>Case $N_e$.</u> We say that $N_e$ *requires attention* at stage $s + 1$ if, either $N(e, s) = 0$ and $x(e, s) = -1$ (i.e. $N_e$ does not have an associated witness), or $N(e, s) = 0$ and $x(e, s) \in W_{e,s}$.

Case $P_e$. We say that $P_e$ *requires attention* at stage $s+1$ if $P(e,s)=0$ (i.e. $P_e$ has not received attention) and there exists a finite set $D$ such that $D \in W_{e,s}$ and $D \cap \Omega(e,s+1)=\emptyset$.

*Resetting $N_e$.* When we say that the construction *resets* $N_e$ at stage $s+1$ we mean the following. If $x(e,s)=-1$ then, the construction does nothing (and in this case $x(e,s+1)=x(e,s)=-1$ and $N(e,s+1)=N(e,s)=0$). Otherwise we set $x(e,s+1)=-1$ and $N(e,s+1)=0$.

*Resetting $P_e$.* When we say that the construction *resets* $P_e$ at stage $s+1$ we mean the following. If $P(e,s)=0$ the construction does nothing (and in this case $D(e,s+1)=D(e,s)=\emptyset$ and $P(e,s+1)=P(e,s)=0$). On the other hand, if $P(e,s)=1$ then we set $\Omega(e,s+1)=\emptyset$, $P(e,s+1)=0$ and $D(e,s+1)=\emptyset$.

**The Construction.**

At every stage $s>0$, if not otherwise specified, all parameters retain their values.

*Stage $s=0$.* Define $A_s=C_s=F_s=\emptyset$ and, for all $e \in \omega$, $N(e,s)=P(e,s)=0$, $D(e,s)=\Omega(e,s)=\emptyset$ and $x(e,s)=-1$.

*Stage $s+1$.* Look for the least $e \leq s$ such that $Q \in \{N_e, P_e\}$ is the highest priority requirement that requires attention and proceed as follows. Otherwise, if there does not exist such $e$ then go to stage $s+2$ (and all parameters retain their values from the preceding stage).

*Case a)* $Q=N_e$.

- If $x(e,s)=-1$ and $N(e,s)=0$ then choose a new witness $x$ that has not appeared in the construction. Define $x(e, s+1)$ = the least such $x$, set the enumerating parameter $C_{s+1} = \{x(e, s+1)\}$ and set outcome $N(e, s+1) = 0$. Reset lower priority requirements $R_i$. We say that $N_e$ receives attention.

- If $N(e, s) = 0$ and $x(e, s) \in W_{e,s}$ then set the extracting parameter $F_{s+1} = \{x(e, s)\}$. Set outcome $N(e, s+1) = 1$. Reset lower priority requirements $R_i$. We say that $N_e$ receives attention.

*Case b)* $Q = P_e$.

- If $P(e, s) = 0$ and there exists a finite set $D$ such that

$$D \in W_{e,s} \ \& \ D \cap \Omega(e, s+1) = \emptyset$$

then choose the least such $D$, set $D(e, s+1) = D$, set the enumerating parameter $C_{s+1} = D(e, s+1)$ and set outcome $P(e, s+1) = 1$. Reset lower priority requirements $R_i$. We say that $P_e$ receives attention.

*End of Stage $s+1$.*

Define $A_{s+1} = \{A_s \cup C_{s+1}\} \setminus F_{s+1}$. Set $m_A = \max\{x \mid x \in A_{s+1}\}$.

For all $x \leq m_A$ define[2]

$$\sigma_{s+1}(x) = \begin{cases} 1 & \text{if } x \in A_{s+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Set $\tau_{s+1} = \sigma_{s+1}{}^\frown 0$. Enumerate $\tau_{s+1}$ into $W_{s+1}$.

Go to stage $s+2$.

[2] I.e. we set $\sigma_{s+1} = \chi_{A_{s+1}}$.

**Verification.**

Consider any $e \in \omega$. As *Induction Hypothesis* we suppose that every requirement $R \in \{N_i, P_i \mid i < e\}$ only receives attention at most finitely often. Let $s$ be the least stage such that every requirement $R$ does not receive attention at any stage $t > s$. We now check that $N_e$ and $P_e$ are satisfied and that the Induction Hypothesis is justified in each case. We proceed according to descending priority, noting that $N_e < P_e$ in the priority ordering.

*Case $N_e$.* By the definition of $s$, for all $t \geq s$, $x(e,t) = x(e,s)$. We write this limiting value as $x(e)$. If $N_e$ never receives attention after stage $t$, then $x(e) \in A \setminus W_e$ and outcome $N(e) = N(e,t) = 0$. Otherwise, if $N_e$ receives attention at some stage $u \geq t$, then $x(e) \in W_e \setminus A$ and outcome $N(e) = N(e,u) = 1$. Thus $N_e$ never receives attention at any stage $v > u$.

*Case $P_e$.* Consider $s$ as defined above. Then for all $t \geq s$, $D(e,t) = D(e,s)$ and $\Omega(e,t) = \Omega(e,s)$. We write these limiting values as $D(e)$ and $\Omega(e)$. If $P_e$ never receives attention after stage $t$ then for all $D \in W_e$, $D \cap \Omega(e,t) \neq \emptyset$ and outcome $P(e) = P(e,t) = 0$. Otherwise, if $P_e$ receives attention at some stage $u > t$ then, $D(e) \in W_e$ and outcome $P(e) = P(e,u) = 1$. Thus $P_e$ never receives attention at any stage $v > u$.

□

**Lemma 6.1.2.** *$A$ is $\Delta^0_2$*

*Proof.* Fix $x$. Then either $x = x(e)$ or $x \in D(e)$ for some $e \in \omega$. Choose stage $s$ large enough so that for all $t \geq s$, $x = x(e,t)$ or $x \in D(e,t)$. If $x = x(e,t)$ then either $N(e,t) = 0$ or $N(e,t) = 1$. Suppose that $N(e,t) = 0$, then $\forall t \geq s$ we have that $x \in A_t$. Otherwise, if $N(e,t) = 1$ then $\forall t \geq s$ we have that $x \notin A_t$. On the other hand, if $x \in D(e,t)$ then $\forall t \geq s$, $x \in A_t$. □

**Lemma 6.1.3.** *$A$ is not 1-generic.*

*Proof.* Fix $n$ and choose stage $s$ large enough such that $\forall t \geq s$, $A_t \upharpoonright n = A \upharpoonright n$. We then set $\sigma_t = \chi_{A_t}$. Inspection of the construction shows that $\tau_t = \sigma_t{}^\frown 0$, $\tau_t \in W$ and $\sigma_t(|\sigma_t|-1) = 1$. For a contradiction suppose that $\sigma_t \in W$. Then by the construction $\sigma_t(|\sigma_t| - 1) = 0$. Clearly a contradiction. Hence $\sigma_t \notin W$. □

## 6.2 An enumeration 1-generic set which is not splittable

### 6.2.1 Introduction

In the enumeration degrees we say a degree $\boldsymbol{a}$ is splittable if for some $A \in \boldsymbol{a}$ we have that $A = A_0 \oplus A_1$ with $A_0, A_1 <_{\mathrm{e}} A$. Otherwise we say $\boldsymbol{a}$ is nonsplittable. In her thesis [AL98] Ahmad showed the existence of a low nonsplittable $\Sigma^0_2$ enumeration degree (by contrast with the Turing degrees, in which every c.e. degree is splittable). Moreover, in his thesis [Ken05] Kent gave a direct construction of a nonzero nonsplittable enumeration degree using a tree of strategies. Kent and Sorbi [KS07] then adapted this direct construction of a nonsplittable enumeration degree on a tree to prove the following.

**Theorem 6.2.1** ([KS07])**.** *Every nonzero $\Sigma^0_2$ enumeration degree bounds a nonzero nonsplittable enumeration degree.*

We now turn our attention to 1-genericity and note that it is commonly known that, as in the Turing degrees, 1-generic sets are splittable in the enumeration degrees.

**Proposition 6.2.2.** *If $A$ is a 1-generic set then $\boldsymbol{a} = deg_{\mathrm{e}}(A)$ is splittable.*

Our proof starts with the following two lemmas, in which we assume $A$ is 1-generic, $A = A_0 \oplus A_1$ and $\sigma = \sigma_0 \oplus \sigma_1$, in a similar way to Proposition 2.1.8 (see chapter 2).

**Lemma 6.2.3.** *The sets $A_0$ and $A_1$ are infinite.*

*Proof.* For a contradiction assume that $A_0$ is finite (likewise we can prove $A_1$ is infinite). Define $m_{A_0} = \max\{x \mid x \in A_0\}$ and consider the following c.e. set of strings

$$S = \{\sigma \mid \exists x\ [\sigma(2x) = 1 \ \&\ x > m_{A_0}]\}.$$

We notice that no initial segment $\sigma$ of $A$ can be a member of $S$ since we assumed $A(2x) = A_0(x)$ and consequently $\forall x > m_{A_0}[A(2x) = A_0(x)]$. We thus get $\exists \sigma\ \forall\, \tau \supseteq \sigma\ [\tau \notin S]$. Fix such $\sigma$. Then we define a string $\tau$ by

$$\tau(x) = \begin{cases} \sigma(x) & \text{if } x < |\sigma|, \\ 1 & \text{if } |\sigma| \leq x < 2m_{A_0}, \\ 0 & \text{if } x = 2m_{A_0}. \end{cases}$$

This gives us $\tau \in S$, clearly a contradiction. □

**Lemma 6.2.4.** *The sets $A_0$ and $A_1$ are not c.e.*

*Proof.* For a contradiction assume that $A_0$ is c.e. (we can prove $A_1$ is not c.e. in a similar way) and consider the following c.e. set

$$S = \{\sigma \mid \exists x\ [x \in A_0 \ \&\ \sigma(2x) = 0]\}.$$

Clearly not $\exists \sigma \subset A[\sigma \in S]$ since we assumed $A(2x) = A_0(x)$. We thus get $\exists \sigma\ \forall\, \tau \supseteq \sigma\ [\tau \notin S]$. Fix such $\sigma$. By Lemma 6.2.3, $A_0$ is infinite and so we can find $2x > |\sigma|$

such that $A_0(x) = 1$. Then we define a string $\tau$ with $|\tau| \geq 2x+1$ and $\tau(2x) = 0$. This gives $\tau \in S$, a contradiction.

$\square$

We can now proceed to the proof of Proposition 6.2.2:

*Proof.* For a contradiction suppose that $A_0 \leq_{\mathrm{e}} A_1$. It follows that $A_0 = \Phi_e^{A_1}$ for some $e \in \omega$ and consequently

$$x \in A_0 \Leftrightarrow (\exists u)[\langle x, D_u \rangle \in \Phi_e \& D_u \subseteq A_1].$$

Thus we have

$$2x \in A \Leftrightarrow (\exists u)[\langle x, D_u \rangle \in \Phi_e \& \forall z \in D_u[z \in D_u \Rightarrow 2z+1 \ \in A]].$$

Now, consider the following c.e. set of strings

$$S = \{\sigma \mid \exists x \ [\sigma_0(x) = 0 \ \& \ x \ \in \ \Phi_e^{\sigma_1}]\},$$

where $\Phi_e^{\sigma_1} = \{x \mid (\exists u)[\langle x, D_u \rangle \in \Phi_e \ \& \ D_u \subseteq \{y \mid \sigma_1(y) = 1\}]\}$. We notice that no $\sigma \subset A$ can be a member of $S$ since by assumption $A_0 \leq_{\mathrm{e}} A_1$. We claim that if $\exists \sigma \ \forall \tau \supseteq \sigma[\tau \ \notin \ S]$, then $\Phi_e^{A_1}$ is finite. Indeed for a contradiction, assume $\Phi_e^{A_1}$ is infinite and fix such $\sigma$. By Lemma 6.2.3, $A_0$ is infinite and so is $\Phi_e^{A_1}$. Hence we can find $y \in \Phi_e^{A_1}$ with $y \geq |\sigma|$. But this is a contradiction since we could then define a string $\tau$ such that $\forall x < |\sigma| \ [\tau(x) = \sigma(x)]$ with $\tau(2y) = 0$ and, in consequence $\tau \in S$. $\square$

We adapt the methodology used in [AL98, KS07, Ken08] to prove the existence of a low nonzero nonsplittable enumeration 1-generic degree $\boldsymbol{a}$. The existence of this particular enumeration degree $\boldsymbol{a}$ taken in conjunction with Proposition 6.2.2, shows that the classes of enumeration 1-generic degrees and 1-generic degrees can be separated using nonsplitting.

**Theorem 6.2.5.** *There exists a low enumeration* 1*-generic nonsplittable degree* $\boldsymbol{a} > \mathbf{0}_{\mathrm{e}}$.

### 6.2.2 Requirements

We will define a set $A$ with $\Delta^0_2$ approximation $\{A_s\}_{s\in\omega}$ satisfying the following requirements.

$$
\begin{aligned}
R_{\Psi,\Omega_0,\Omega_1} \quad &: \quad A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow A \leq_{\mathrm{e}} \Omega_i^A \text{ for some } i \in \{0,1\} \text{ or } A \text{ c.e.}, \\
N_W \quad &: \quad A \neq W, \\
P_W \quad &: \quad (\exists D \in W)[\, D \subseteq A\,] \ \vee\ (\exists E \subseteq \overline{A})(\forall D \in W)[\, D \cap E \neq \emptyset\,].
\end{aligned}
$$

Note our use of shorthand notation in the above (introduced to simplify the presentation) whereby we understand $(\Psi, \Omega_0, \Omega_1) \in \{(\Psi_e, \Omega_{e,0}, \Omega_{e,1})\}_{e\in\omega}$ where the latter is a standard effective listing of all triples of enumeration operators. Likewise $W$ ranges over a standard effective listing of c.e. sets $\{W_e\}_{e\in\omega}$. In each case we assume that the listing is associated with standard uniform c.e. approximations of the sets/operators involved.

### 6.2.3 Basic idea of the construction

Before giving the construction of $A$, we describe the basic modules for satisfying the requirements. In order to facilitate the processes described below, for every requirement $L \in \{R, N, P\}$ we define a stream $S$ which is basically the set of numbers inside $A$ which were enumerated by lower requirements and are available to $L$ for processing. Requirements $N_W$ are satisfied using the basic Friedberg-Muchnik strategy whereas the basic module for $P_W$ is quite straightforward. Indeed, to satisfy $P_W$, whenever we see some $D \in W$ such that $D \subseteq S \cup \{$ some number available to $P_W\}$ we then permanently place $D$ in $A$ (i.e. $D$ is "dumped" into $A$). Requirements $R_{\Psi,\Omega_0,\Omega_1}$ are in fact more complicated. Assume we are trying to satisfy a single requirement $R_{\Psi,\Omega_0,\Omega_1}$. Suppose that $x$ is enumerated by some lower requirement. Whenever we see $x \in A \cap \Psi^{\Omega_0^A \oplus \Omega_1^A}$ we then arrange either $A \leq_{\mathrm{e}} \Omega_0^A$ or $A \leq_{\mathrm{e}} \Omega_1^A$ by defining enumeration operators $\Gamma_0$ and $\Gamma_1$ (which witness $A = \Gamma_0^{\Omega_0^A}$ or $A = \Gamma_1^{\Omega_1^A}$). Hence at stage $s$ we add axioms $\langle x, \Omega_0^A[s]\rangle$ to $\Gamma_0$ and $\langle x, \Omega_1^A[s]\rangle$ to $\Gamma_1$ and prevent lower priority requirements from destroying our work by dumping all $y \in S$. Our aim is to arrange $A = \Gamma_1^{\Omega_1^A}$. If later on during the construction $x \notin A$ and $x \in \Psi^{\Omega_0^A \oplus \Omega_1^A}$ then we do nothing since the requirement is trivially satisfied (i.e. $A \neq \Psi^{\Omega_0^A \oplus \Omega_1^A}$). If $x \notin A$ and $x \in \Gamma_1^{\Omega_1^A}$ (i.e. $x \notin \Psi^{\Omega_0^A \oplus \Omega_1^A}$) then we change our aim to $A = \Gamma_0^{\Omega_0^A}$ and again we dump all $y \in S$ into $A$. Whichever requirement we are trying to satisfy, we note that for some $x$ if there is a stage $s+1$ such that $A[s+1] \upharpoonright x \neq A[s] \upharpoonright x$ then $x$ is dumped into $A$.

### Definitions and notation

*The Tree of Strategies.*

We define the overall set of *outcomes* to be $\Sigma = \{0,1,2\} \cup \{\text{void}\}$ and the set of *tree outcomes* to be $\{0,1,2\}$. We fix an arbitrary effective priority ordering $\{L_e\}_{e\in\omega}$ of all $R$, $N$ and $P$ requirements. We also define $\mathcal{T} \subseteq \{0,1,2\}^{<\omega}$ and we refer to it as the *tree of strategies*. Each node $\alpha \in \mathcal{T}$ will be associated, and so identified, with the strategy for the satisfaction of $L_{|\alpha|}$. We use the notation $\mathcal{R}_{\Psi,\Omega_0,\Omega_1}$ for the set of $R_{\Psi,\Omega_0,\Omega_1}$ strategies and $\mathcal{R}$ for the set of all $R$ strategies. Likewise, for $(\mathcal{Q}, Q) \in \{(\mathcal{N}, N), (\mathcal{P}, P)\}$ we will use the notation $\mathcal{Q}_W$ for the set of strategies associated with $Q_W$ and we let $\mathcal{Q}$ denote the set of all such strategies.

We assign requirements to nodes on $\mathcal{T}$ by induction as follows. Define $\emptyset \in \mathcal{T}$. Given $\alpha \in \mathcal{T}$ we distinguish three cases depending on the requirement $L$ associated with $\alpha$.

*Case 1.* $\alpha \in \mathcal{R}$: define $\alpha^\frown\langle n\rangle \in \mathcal{T}$ for $n \in \{0,1,2\}$.

*Case 2.* $\alpha \in \mathcal{N}$: define $\alpha^\frown\langle n\rangle \in \mathcal{T}$ for all $n \in \{0,1\}$.

*Case 3.* $\alpha \in \mathcal{P}$: define $\alpha^\frown\langle n\rangle \in \mathcal{T}$ for all $n \in \{0,1\}$.

*Environment Parameters*

*Local parameters for $\alpha \in \mathcal{R}_{\Psi,\Omega_0,\Omega_1}$.* $R(\alpha,s) \in \{0,1,2,\mathrm{void}\}$ is the *outcome* parameter, and $\Gamma_{\alpha,0}[s]$ and $\Gamma_{\alpha,1}[s]$ finite approximations to enumeration operators constructed so as to (possibly) witness $A \leq_{\mathrm{e}} \Omega_0^A$ or $A \leq_{\mathrm{e}} \Omega_1^A$. (Note that, for $i \in \{0,1\}$, we use $\Gamma_i$ as shorthand for $\Gamma_{\alpha,i}$ when there is no danger of ambiguity.) Outcome $R(\alpha,s) = j$ for $j \leq 1$ corresponds to $\alpha$'s belief that, if $A = \Psi^{\Omega_0^A \oplus \Omega_1^A}$, then $A \leq_{\mathrm{e}} \Omega_j^A$ (as witnessed by $\Gamma_j$ in the limit). Likewise, under the same assumption, $R(\alpha,s) = 2$ corresponds to $\alpha$'s belief that $A$ is c.e. (contradicting the definition of $\boldsymbol{a}$). For ease of description in the construction $\alpha$ also has a *dummy witness* parameter $x(\alpha,s) = -1$.

*Local parameters for $\alpha \in \mathcal{N}_W$.* $N(\alpha,s) \in \{0,1,\mathrm{void}\}$ is the *outcome* parameter, and $x(\alpha,s) \in \{-1\} \cup \omega$ is the *witness* parameter associated with $\alpha$. Outcome $R(\alpha,s) = 0$ corresponds to $\alpha$'s knowledge that $x(\alpha,s) \in W$ and belief that $x(\alpha,s) \notin A$ (which will be vindicated if $\alpha$ is not initialised at any stage $t > s$). $N(\alpha,s) = 1$, on the other hand, means that $\alpha$ believes that $x(\alpha,s) \in A \setminus W$.

*Local parameters for $\alpha \in \mathcal{P}_W$.* $P(\alpha,s) \in \{0,1,\mathrm{void}\}$ is the *outcome* parameter and $x(\alpha,s) = -1$ a *dummy witness* parameter for $\alpha$. $P(\alpha,s) = 0$ corresponds to $\alpha$'s belief that there is some $D \in W$ such that $D \subseteq A$ (which will be vindicated if $\alpha$ is on the true path and is not initialised at any stage $t > s$). $P(\alpha,s) = 1$, on the other hand, corresponds to $\alpha$'s belief that there is no such $D$ in $W$.

*The stream for any $\alpha \in \mathcal{T}$.* $S(\alpha, s) = \{ x(\beta, s) \mid x(\beta, s) \geq 0 \ \ \& \ \ \alpha \subseteq \beta \}$ is the (finite) *stream* associated with $\alpha$ at stage $s$ and corresponds to the set of numbers already processed by the construction at stage $s$ and which are (roughly speaking) available for processing by $\alpha$ at stage $s+1$. Note that by definition $x(\alpha, s) \notin S(\alpha^\frown\langle n\rangle, s)$ for any $n \in \{0, 1, 2\}$. (This observation is significant for the construction for the case $\alpha \in \mathcal{N}$ and trivial otherwise.)

*Global parameters for stage $s+1$.* Each stage $s+1$ has the following parameters.

(i) $z(s+1, t) \in \omega \cup \{\text{break}\}$ is a floating witness which is passed down the $s+1$ stage approximation to the true path. When $t = 0$, $z(s+1, t)$ starts life by denoting the number $s$. For $t \geq 0$, the witness $z(s+1, t)$ is passed to the strategy $\alpha$ of length $t$ eligible to act at substage $t+1$ provided that $z(s+1, t) \neq \text{break}$. The strategy $\alpha$ decides whether (a) to set $z(s+1, t+1) = \text{break}$, thus causing stage $s+1$ to terminate[3], or (b) to reallocate $z(s+1, t+1)$ to some number belonging to its $s$ stage *stream*, or (c) to reset $z(s+1, t+1) = z(s+1, t)$. In case (a) the strategy $\alpha$ either sets[4] $x(\alpha, s+1) = z(s+1, t)$ or *dumps* $z(s+1, t)$ into $A$, whereas in case (b) $\alpha$ always dumps $z(s+1, t)$ into $A$. Note that case (a) corresponds to $\alpha \in \mathcal{N} \cup \mathcal{P}$, case (b) to $\alpha \in \mathcal{R}$ whereas case (c) may apply to any strategy $\alpha$. Also notice that in cases (b) and (c) the new value of the floating witness $z(s+1, t+1)$ is passed to the strategy $\alpha^\frown\langle i\rangle$ of length $t+1$ eligible to act at stage $t+2$.

(ii) $D(s+1, t) \in \mathcal{F}$ is a record, established at substage $t$, that defines a set of numbers that will be *dumped* at the end of stage $s+1$. When $t = 0$, $D(s+1, t)$ starts life as $\emptyset$. $D(s+1, t+1)$ is defined provided that $z(s+1, t) \neq \text{break}$ (i.e. the stage

[3]Note that termination of a stage is determined by the value of $z(s+1, t)$ only, not by the length of the strategies eligible to act.

[4]This first case (i.e. $x(\alpha, s+1) = z(s+1, t)$) happens only if $\alpha \in \mathcal{N}$.

has not yet terminated) and in this case $D(s+1,t) \subseteq D(s+1,t+1)$.

(iii) $D(s+1)$ is the overall set of numbers *dumped* into $A$ at the end of stage $s+1$. Thus by definition $D(s+1) = D(s+1, |\beta_s|+1)$ where $\beta_s$ is the $s$ stage approximation to the true path.

*Initialisation.* For $(Q, \mathcal{Q}) \in \{(R, \mathcal{R}), (N, \mathcal{N}), (P, \mathcal{P})\}$ and any $\alpha \in \mathcal{Q}$ we say that 'void' is the *initial value* of $Q(\alpha, s)$ and that $-1$ is the initial value of $x(\alpha, s)$. For $\alpha \in \mathcal{R}$ we say that $\emptyset$ is the initial value of $\Gamma_{\alpha,i}$ for $i \in \{0,1\}$. *Initialisation* of a node $\alpha \in \mathcal{T}$ is the process of resetting its associated parameters to their initial values.

### 6.2.4 The Construction.

The construction proceeds in stages $s \in \omega$. At each stage $s$ the construction defines the following finite sets. $D_A[s]$ is the set of numbers already ***D**umped* into $A$ while $F_A[s]$ is the set of numbers already used by the construction (i.e. having visited $A$ during at least one stage) but still ***F**ree*, i.e. nondumped. $I_A[s]$ is the set of (free) numbers ***I**nside* $A$ and $O_A[s]$ is the set of (free) numbers ***O**utside* $A$. The intention here is that $I_A[s] \cap O_A[s] = \emptyset$, $F_A[s] = I_A[s] \cup O_A[s]$, $F_A[s] \cap D_A[s] = \emptyset$, and $F_A[s] \cup D_A[s] = \omega \restriction s$. The $s$ stage approximation to $A$ will be defined to be $A[s] = I_A[s] \cup D_A[s]$.

We say that a number $x \in \omega$ is *new* if it is greater than any number used in the construction so far.

To facilitate understanding of the construction we suggest that the reader also consult the informal observations relative to stage $s+1$ made on page 128.

*Stage $s = 0$.*

Set $A[s] = I_A[s] = O_A[s] = F_A[s] = D_A[s] = \emptyset$ and initialise all $\alpha \in \mathcal{T}$.

<u>*Stage* $s+1$.</u>

This stage consists of substages $t \geq 0$ such that some strategy $\alpha \in \mathcal{T}$ acts (i.e. is processed) at substage $t+1$ provided that $z(s+1,t) \neq$ break. If so, $\alpha$ decides the value of $z(s+1,t+1)$ and $D(s+1,t+1)$, the value of its local parameters and, if $z(s+1,t+1) \in \omega$, which strategy $\alpha^\frown\langle n\rangle$ is *eligible to act next.*

<u>*Substage* 0.</u>

Set $z(s+1,0) = s$ and $D(s+1,0) = \emptyset$.

<u>*Substage* $t+1$.</u> (Under the assumption that $z(s+1,t) \in \omega$.)

We suppose that $\alpha$ is the strategy of length $t$ which is eligible to act at this substage. We distinguish cases depending on the requirement $R$ assigned to $\alpha$.

<u>*Case* 1.</u> $\alpha \in \mathcal{R}_{\Psi,\Omega_0,\Omega_1}$. Process the first of the following cases which is applicable.

*Reminder.* We are using the notations $\Psi$ and $\Omega_i$ as shorthand for $\Psi_e$ and $\Omega_{e,i}$ for some index $e$, and $\Gamma_i$ as shorthand for $\Gamma_{\alpha,i}$.

<u>*Case* 1.1</u> There is a number $z \in S(\alpha^\frown\langle 1\rangle, s)$ such that $z \notin A[s]$ but $z \in \Gamma_1^{\Omega_1^A}[s]$.

Then set $z(s+1,t+1) = z$ for the least such $z$, define

$$D(s{+}1,t{+}1) \;=\; D(s{+}1,t) \cup \{z(s{+}1,t)\} \cup \left( \bigcup_{1\leq i\leq 2} S(\alpha^\frown\langle i\rangle, s) \setminus \{z(s+1,t+1)\} \right),$$

and $\Gamma_1[s+1] = \emptyset$. Also reset $\Gamma_0[s+1] = \Gamma_0[s]$. Set $R(\alpha, s+1) = 0$.

*Remark.* $R(\alpha, s+1) = 0$ indicates that $\alpha^\frown\langle 0\rangle$ will be eligible to act at substage $t+2$. (See *Ending substage* $t+1$ on page 127.) Note that the floating witness $z(s+1,t+1)$ will be passed to $\alpha^\frown\langle 0\rangle$.

<u>*Case* 1.2</u> There is a number $z \in S(\alpha^\frown\langle 2\rangle, s)$ such that $z \in A[s] \cap \Psi^{\Omega_0^A \oplus \Omega_1^A}[s]$.

Then set $z(s+1,t+1) = z$ for the least such $z$, define

$$D(s+1,t+1) \;=\; D(s+1,t) \;\cup\; \{z(s+1,t)\} \;\cup\; \Big(S(\alpha^\frown\langle 2\rangle, s) \setminus \{z(s+1,t+1)\}\Big),$$

and, for $0 \leq i \leq 1$ define $\Gamma_i[s+1] = \Gamma_i[s] \cup \{\langle z(s+1,t+1), \Omega_i^A[s]\rangle\}$.

Set $R(\alpha, s+1) = 1$.

<u>*Case* 1.3</u> Otherwise.

Then reset $z(s+1,t+1) = z(s+1,t)$, $D(s+1,t+1) = D(s+1,t)$, $\Gamma_i[s+1] = \Gamma_i[s]$ for $0 \leq i \leq 1$ and set $R(\alpha, s+1) = 2$.

<u>*Case* 2.</u> $\alpha \in \mathcal{N}_W$. Process the first of the following cases which is applicable.

<u>*Case* 2.1.</u> $N(\alpha, s) = 0$.

(Note that this means that $x(\alpha, s) \in O_A[s] \subseteq \omega \setminus A[s]$.) Set $z(s+1,t+1) = z(s+1,t)$, $D(s+1,t+1) = D(s+1,t)$ and reset $x(\alpha, s+1) = x(\alpha, s)$ and $N(\alpha, s+1) = 0$.

<u>*Case* 2.2.</u> $N(\alpha, s) = 1$ and $x(\alpha, s) \in W[s]$.

Set $z(s+1,t+1) = \text{break}$ and

$$D(s+1,t+1) \quad = \quad D(s+1,t) \ \cup \ \{z(s+1,t)\} \ \cup \ S(\alpha\widehat{\ }\langle 1\rangle, s)\,.$$

(Note that $S(\alpha\widehat{\ }\langle 1\rangle, s) = S(\alpha, s) \setminus \{x(\alpha, s)\}$ in this case.) Reset $x(\alpha, s+1) = x(\alpha, s)$ and set $N(\alpha, s+1) = 0$.

<u>*Case* 2.3.</u> $N(\alpha, s) = 1$ and $x(\alpha, s) \notin W[s]$.

Reset $z(s+1,t+1) = z(s+1,t)$ and $D(s+1,t+1) = D(s+1,t)$. Also reset $x(\alpha, s+1) = x(\alpha, s)$ and $N(\alpha, s+1) = 1$.

<u>*Case* 2.4.</u> $N(\alpha, s) = \text{void}$ and $z(s+1,t) \geq |\alpha|$.

Set $z(s+1,t+1) = \text{break}$ and $D(s+1,t+1) = D(s+1,t)$. Also set $x(\alpha, s+1) = z(s+1,t)$ and $N(\alpha, s+1) = 1$.

<u>*Case* 2.5.</u> Otherwise (i.e. $N(\alpha, s) = \text{void}$ and $z(s+1,t) < |\alpha|$).

Set $z(s+1,t+1) = \text{break}$ and $D(s+1,t+1) = D(s+1,t) \cup \{z(s+1,t)\}$. Also reset $x(\alpha, s+1) = -1$ and $N(\alpha, s+1) = \text{void}$.

<u>*Case* 3.</u> $\alpha \in \mathcal{P}_W$. Process the first of the following cases which is applicable.

*Notation.* For the sake of Cases 3.2 and 3.3 we use the notation

$$\begin{aligned}\Omega_{\alpha,s+1} \quad = \quad & \{\, x(\beta,s) \mid x(\beta,s) \geq 0 \;\;\&\;\; N(\beta,s) = 1 \;\;\&\;\; \beta <_L \alpha \,\} \\ \cup \;\; & \{\, x(\beta,s+1) \mid x(\beta,s+1) \geq 0 \;\;\&\;\; N(\beta,s+1) = 1 \;\;\&\;\; \beta \subset \alpha \,\}\,.\end{aligned}$$

(Note that $N(\beta, s+1) = 1$ and $\beta \subset \alpha$ implies that $\beta\widehat{\;}\langle 1\rangle \subseteq \alpha$.)

<u>*Case* 3.1.</u> $P(\alpha, s) = 0$.

(Note that the implication here is that there is some $D \in W[s]$ such that $D \subseteq A[s]$.) Reset $z(s+1, t+1) = z(s+1, t)$, $D(s+1, t+1) = D(s+1, t)$ and reset $P(\alpha, s+1) = 0$.

<u>*Case* 3.2.</u> $P(\alpha, s) = 1$ and for some $D \in W[s]$.

$$D \;\subseteq\; \Omega_{\alpha,s+1} \cup D_A[s] \cup D(s+1,t) \cup \{z(s+1,t)\} \cup S(\alpha,s)\,. \tag{6.2.1}$$

(Note that $S(\alpha, s) = S(\alpha\widehat{\;}\langle 1\rangle, s)$ in this case.) Set $z(s + 1, t + 1) =$ break,

$$D(s+1,t+1) \;=\; D(s+1,t) \;\cup\; \{z(s+1,t)\} \;\cup\; S(\alpha,s)\,,$$

and set $P(\alpha, s + 1) = 0$.

<u>*Case* 3.3.</u> Otherwise. (I.e. $P(\alpha, s) =$ void or $P(\alpha, s) = 1$ and (6.2.1) holds for no $D \in W[s]$.) Reset $z(s + 1, t + 1) = z(s + 1, t)$, $D(s + 1, t + 1) = D(s + 1, t)$ and set $P(\alpha, s + 1) = 1$.

*Ending substage* $t+1$. Supposing that $\alpha \in \mathcal{Q}$ with $\mathcal{Q} \in \{\mathcal{R}, \mathcal{N}, \mathcal{P}\}$, if $z(s+1, t+1) \in \omega$ then define $\alpha\widehat{\;}\langle Q(\alpha, s + 1)\rangle$ to be *eligible to act next* and go to substage $t + 2$. Otherwise (i.e. if $z(s + 1, t + 1) =$ break) go to *End of Stage* $s + 1$.

*Remark.* The last node eligible to act, and hence be processed, at stage $s + 1$ is either an $N$ node via Case 2.2, 2.4 or 2.5 or otherwise a $P$ node via Case 3.2.

*End of Stage $s+1$.* Supposing that $\alpha$ of length $t$ is the last strategy to be processed define $\beta_{s+1} = \alpha$. Set $D(s+1) = D(s+1, t+1)$ and initialise all nodes in the set $G = \{\, \beta \mid \alpha < \beta \,\}$ (i.e. all nodes $\beta$ such that $\alpha <_L \beta$ or $\alpha \subset \beta$). For every $\beta \in \mathcal{T}$ such that $\beta <_L \alpha$ reset $\beta$'s parameters for stage $s+1$ to their value at stage $s$. Before proceeding note that, by initialisation, for any $\beta \in \mathcal{N}$ such that $N(\beta, s+1) \in \{0, 1\}$, $\beta \leq \alpha$. Define

$$\begin{aligned} I_A[s+1] &= \{\, x(\beta, s+1) \mid x(\beta, s+1) \geq 0 \;\;\&\;\; N(\beta, s+1) = 1 \,\}\,, \\ O_A[s+1] &= \{\, x(\beta, s+1) \mid x(\beta, s+1) \geq 0 \;\;\&\;\; N(\beta, s+1) = 0 \,\}\,, \\ F_A[s+1] &= I_A[s+1] \cup O_A[s+1]\,, \\ D_A[s+1] &= D_A[s] \cup D(s+1)\,, \end{aligned}$$

and

$$A[s+1] = I_A[s+1] \cup D_A[s+1]\,.$$

(And note that $F_A[s+1] = \{\, x(\beta, s+1) \mid x(\beta, s+1) \geq 0 \,\}$.) For every $\gamma \in \mathcal{T}$ redefine the stream for $\gamma$ as follows.

$$S_A(\gamma, s+1) = \{\, x(\beta, s+1) \mid x(\beta, s+1) \in F_A[s+1] \;\;\&\;\; \gamma \subseteq \beta \,\}\,.$$

Note that by resetting, if $\gamma <_L \alpha$ then $S(\gamma, s+1) = S(\gamma, s)$ whereas, by initialisation, if $\alpha < \gamma$ then $S(\gamma, s+1) = \emptyset$.

Go to stage $s+2$.

### 6.2.5 Verification.

The following informal observations clarify the mechanics of the construction and underline its inherent simplicity.

*Some properties of stage $s+1$.*

(i) $F_A[s+1]$ comprises precisely the set of witnesses $x(\gamma, s+1) \geq 0$ such that $\gamma \leq \beta_{s+1}$.

(ii) At most one number is removed from $A$ at stage $s+1$. Indeed, this can only happen if Case 2.2 applies at substage $|\beta_{s+1}|+1$ and the witness[5] $x = x(\beta_{s+1}, s)$ is extracted from $A$.

(iii) $F_A[s+1] \setminus F_A[s] \subseteq \{s\}$. And if indeed $s \in F_A[s+1]$ then Case 2.4 applies at substage $|\beta_{s+1}|+1$ and $s = x(\beta_{s+1}, s+1)$. Also this means that the floating witness $z(s+1,t)$ never changes value. I.e. $z(s+1,t) = s$ for all $t$ such that $0 \leq t \leq |\beta_{s+1}|$.

(iv) If $z(s+1, |\beta_{s+1}|) \neq s$ (i.e. if the floating witness changes value at least once) then $z(s+1, |\beta_{s+1}|) = x(\gamma, s)$ for some $\gamma >_L \beta_{s+1}$. Likewise each intermediate value of the floating witness $z(s+1,t)$ is a witness $x(\gamma', s)$ for some $\gamma' >_L \beta_{s+1}$. Moreover, the only one of the values of the floating witness that (possibly) remains in $F_A[s+1]$ is $x(\gamma, s)$. Note that this happens if Case 2.4 applies at substage $|\beta_{s+1}|+1$ forcing $x(\beta_{s+1}, s+1) = x(\gamma, s)$. All other values (including $s$) of $z(s+1,t)$ are dumped into $A$.

(v) Nontrivial cases of (ii), (iii) and (iv) are mutually exclusive. In other words, extraction of a number from $A$ (see (ii)) forces $s$ and all $x(\gamma, s) \geq 0$, such that $\gamma > \beta_{s+1}$ to be dumped into $A$. On the other hand $s \in F_A[s+1]$ (see (iii)) precludes any extraction from $A$ and forces all $x(\gamma, s) \geq 0$ such that $\gamma > \beta_{s+1}$ to be dumped into $A$. Likewise $x(\gamma, s) \in F_A[s+1]$ for some $\gamma >_L \beta_{s+1}$ (see (iv)) precludes any extraction from $A$ and forces $s$ (as well as all other $x(\hat{\gamma}, s) \geq 0$ such that $\hat{\gamma} > \beta_{s+1}$) to be dumped into $A$.

[5] $x = x(\beta_{s+1}, s) = x(\beta_{s+1}, s+1)$ in this case.

(vi) $\beta_{s+1}$ is either in $\mathcal{N}$ and Case 2.2, 2.4 or 2.5 applies at substage $|\beta_{s+1}|+1$ or otherwise $\beta_{s+1}$ is in $\mathcal{P}$ and Case 3.2 applies at substage $|\beta_{s+1}|+1$.

We now verify the correctness of the construction via the Lemmas below. Note firstly that Lemmas 6.2.6-6.2.8 are proved by inspection (only for some of the statements involved) and straightforward induction arguments over the stages of the construction, using the observations above.

**Lemma 6.2.6.** *For all stages $s>0$ and $x \in F_A[s]$ both (1) and (2) are true.*

*(1) One of the three following (mutually exclusive) cases applies for $x$.*

*(a) $x = s-1$, $\beta_s \in \mathcal{N}$ and $x = x(\beta_s, s)$.*

*(b) There exists $\gamma \in \mathcal{N}$ such that $\gamma \leq \beta_{s-1}$, $\beta_s <_L \gamma$, $x = x(\gamma, s-1)$, $x(\gamma, s) =$ void and $x(\beta_s, s) = x$.*

*(c) There exists $\gamma \in \mathcal{N}$ such that $\gamma \leq \beta_{s-1}$, $\gamma \leq \beta_s$ and $x = x(\gamma, s-1) = x(\gamma, s)$.*

*(2) For all $\gamma_1, \gamma_2 \in \mathcal{N}$ such that $x = x(\gamma_1, s) = x(\gamma_2, s)$, $\gamma_1 = \gamma_2$.*

*Proof. (1)* Suppose that $s>0$ and $x \in F_A[s]$. Then we have the following two cases to consider:

- $x \in I_A[s]$. Then for some $\beta \in \mathcal{N}$, $x = x(\beta, s)$ and $N(\beta, s) = 1$. Now by inspection of the construction, if $N(\beta, s) = 1$ then either case 2.3 or case 2.4 was applicable. If case 2.3 was applied then, $x(\beta, s-1) \notin W[s-1]$ and hence $\beta \leq \beta_{s-2}$, $\beta \leq \beta_{s-1}$ and $x = x(\beta, s-1) = x(\beta, s)$. Thus we conclude *(c)*. On the other hand, if case 2.4 was applied then, $N(\beta, s-1) = void$ and $x = z(s,t) \geq |\beta|$. Hence $x(\beta, s) = x$. In the case that $x = s-1$ we conclude *(a)*. Otherwise, if $x \neq s-1$ then, $x \in S(\gamma^\frown\langle i\rangle, s)$ with $i \in \{1,2\}$ for some $\gamma \subseteq \beta$ and we then conclude *(b)*.

- $x \in O_A[s]$. Then for some $\beta \in \mathcal{N}$, $x = x(\beta, s)$ and $N(\beta, s) = 0$. By the construction we have that either case 2.1 or case 2.2 was applicable at stage $s$. If case 2.1 was applied at stage $s$ then $N(\beta, s-1) = 1$ and $x(\beta, s-1) \in W[s]$; and the construction defined $N(\beta, s) = 0$, $x(\beta, s) = x(\beta, s-1)$ and hence *(c)*. Otherwise, suppose that case 2.2 was applied, then $N(\beta, s-1)$, $x = x(\beta, s-1)$ and $x \notin W[s]$. Hence at stage $s$ the construction defined $x(\beta, s) = x(\beta, s-1)$, $N(\beta, s) = 0$ and thus *(c)*.

*(2)* For a contradiction, suppose that $\gamma_1 \neq \gamma_2$, $x(\gamma_1, s) = x(\gamma_2, s)$, $x(\gamma_1, s) \neq -1$ and $\gamma_1, \gamma_2 \in \mathcal{N}$. Then either $\gamma_1 < \gamma_2$, or $\gamma_1 > \gamma_2$ Without loss of generality suppose that $\gamma_1 < \gamma_2$. By definition, at stage $s$ the construction defines either $x(\gamma_1, s) = z(s, t)$ (with $|\gamma_1| = t$) or $x(\gamma_1, s) = x(\gamma_1, s-1)$. If $x(\gamma_1, s) = z(s, t)$ then the construction sets $z(s, t+1) = \mathit{break}$. Hence the construction ends stage $s$, initialises $\gamma_2$, and sets $x(\gamma_2) = -1$. A contradiction. On the other hand, if $x(\gamma_1, s) = x(\gamma_1, s-1)$ then $x(\gamma_1, s) \notin S(\gamma_1{}^\frown \langle n \rangle, s)$ for any $n \in \{0, 1\}$, and so $x(\gamma_1, s)$ is not available for processing to lower priority requirements $R_{\Psi, \Omega_0, \Omega_1}$. Thus there is no substage $v > t$ (with $t = |\gamma_1|$) such that $z(\gamma_2, v) = x(\gamma_1, s)$. Hence $x(\gamma_1, s) \neq x(\gamma_2), s$. A contradiction.

$\square$

*Remark.* By Lemma 6.2.6, and the definition of $F_A[s]$ we can now assume that $x \in F_A[s]$ if and only if there exists a unique ($N$ strategy) $\gamma \leq \beta_s$ such that $x = x(\gamma, s)$. Clearly also in this case for $(L, i) \in \{(I, 1), (O, 0)\}$, we have that $x \in L_A[s]$ if and only if $N(\gamma, s) = i$.

**Lemma 6.2.7.** *For all $s \geq 0$, the following statements are true.*

1. *$D(s) \subseteq D_A[s] \subseteq D_A[s+1]$.*

2. *$F_A[s] = I_A[s] \cup O_A[s]$ and $I_A[s] \cap O_A[s] = \emptyset$.*

*3.* $D_A[s] \cap F_A[s] = \emptyset$*.*

*4.* $\{\, n \mid 0 \leq n < s \,\} = F_A[s] \cup D_A[s]$*.*

*5. For any* $\alpha \in \mathcal{T}$ *such that* $\alpha \subseteq \beta_s$*,*

$$F_A[s] \;=\; S(\alpha, s) \,\cup\, \{\, x(\gamma, s) \mid x(\gamma, s) \geq 0 \;\&\; \gamma < \alpha \,\}\,.$$

*Proof.* Inspection of the construction show that if $s = 0$ then 1. to 5. are trivially true.

1. Consider the last stage $s > 0$ under the Induction Hypothesis (I.H.) that 1. holds for $s-1$. Then by the construction, at stage $s$ we have that $D(s) \subseteq D_A[s]$ and $D_A[s+1] = D_A[s] \cup D(s+1)$. Hence $D(s) \subseteq D_A[s] \subseteq D_A[s+1]$.

2. Clearly, by definition of the construction $F_A[s] = I_A[s] \cup O_A[s]$. For a contradiction, suppose that there exists some $x \in I_A[s] \cap O_A[s]$. Inspection of the stage $s$ shows that if $x \in F_A[s]$ then $x = x(\beta, s)$ for some $\beta \in \mathcal{N}$ and either $N(\beta, s) = 1$ or $N(\beta, s) = 1$. Thus by the construction either $x \in I_A[s]$ or $x \in O_A[s]$. A contradiction.

3. For a contradiction, suppose that $x \in D_A[s] \cap F_A[s]$. Then inspection of the construction shows that $x = x(\beta, s)$ for some $\beta \in \mathcal{N}$ and $N(\beta, s) \in \{0, 1\}$. Moreover, if $x \in D_A[s]$ then for no $\beta \in \mathcal{N}$ do we have $x = x(\beta, s)$. Clearly a contradiction.

4. Consider the last stage $s > 0$ under the Induction Hypothesis (I.H.) that 4. holds for $s-1$. Then inspection of stage $s+1$ shows that $s$ is enumerated into either $F_A[s+1]$ or $D_A[s+1]$. Hence $\{\, n \mid 0 \leq n < s+1 \,\} = F_A[s+1] \cup D_A[s+1]$.

5. Consider the last stage $s > 0$ under the Induction Hypothesis (I.H.) that 5. holds for $s-1$. Supposing $\alpha \leq \beta_{s+1}$. By definition, $S(\alpha, s+1) = \{x(\beta, s+1) \mid$

$x(\beta, s+1) \geq 0$ and $\alpha \subseteq \beta\}$. Now, supposing $\gamma < \alpha$, we have two cases to consider:

- If $\gamma \subset \alpha$ then, $\gamma \subset \beta_{s+1}$. If $x(\gamma, s+1) \geq 0$ then $x(\alpha, s+1) \in F_A[s+1]$.
- If $\gamma <_L \alpha$ then, for some $x$, $x = x(\gamma, s{+}1) \neq x(\gamma, s)$ and so $N(\gamma, s{+}1) = 1$. Hence $x \in F_A[s+1]$.

□

**Lemma 6.2.8.** *Suppose that $\beta \in \mathcal{T}$ is such that $x(\beta, s) \geq 0$. Then for all $\gamma \subseteq \beta$ such that $\gamma \in \mathcal{N}$, $N(\gamma, s) \in \{0, 1\}$ and $x(\gamma, s) \geq 0$.*

*Proof.* For a contradiction, suppose that $\beta \in \mathcal{T}$ is such that $x(\beta, s) \geq 0$ and there exists $\gamma \subseteq \beta$ such that $N(\gamma, s) \notin \{0, 1\}$ and $x(\gamma, s) = -1$. Inspection of the construction shows that at stage $s$ either Case 2.5 was applied or $\gamma$ was initialised. If Case 2.5 was applied then the construction defined $z(s, t{+}1) = \mathit{break}$ (with $t = |\gamma|$), thus causing the end of stage $s$. Hence $\beta$ was initialised and so $x(\beta, s) = -1$. A contradiction. On the other hand, if $\gamma$ was initialised then $\beta$ was initialised and so $x(\beta, s) = -1$. Clearly a contradiction. □

**Lemma 6.2.9.** *For any $\alpha \in \mathcal{T}$ and stage $s \geq 0$, $|S(\alpha, s+1) \setminus S(\alpha, s)| \leq 1$.*

*Proof.* This follows by inspection of the construction at stage $s+1$. Indeed, if $z \in S(\alpha, s+1) \setminus S(\alpha, s)$ then for some substage $t$ of stage $s+1$, $z(s+1, t) = z$. However at most one such $z$ survives without being dumped into $D(s+1)$. (And in this case $z = x(\beta_{s+1}, s+1)$.) □

**Lemma 6.2.10.** *For all stages $s \geq 0$ and any strategies $\alpha, \beta \in \mathcal{T}$ such that $S(\alpha, s) \neq \emptyset$ and $S(\beta, s) \neq \emptyset$, if $\alpha <_L \beta$, then $\max S(\alpha, s) < \min S(\beta, s)$.*

*Proof.* By induction over stages $s \geq 0$. The case $s = 0$ is trivially true. So consider case $s+1$. For the hypotheses of the Lemma to be true at stage $s+1$ it must be the case that $\beta \leq \beta_{s+1}$ (otherwise $S(\beta, s+1) = \emptyset$). If $\beta <_L \beta_{s+1}$ then $S(\alpha, s+1) = S(\alpha, s)$ and $S(\beta, s+1) = S(\beta, s)$ and the result follows by the induction hypothesis. Otherwise $\beta \subseteq \beta_{s+1}$. As seen in Lemma 6.2.9, if $D = S(\beta, s+1) \setminus S(\beta, s)$, then $|D| \leq 1$. If $|D| = 0$ then the result follows as above. Otherwise suppose that $z$ is the number contained in $D$. Then either $z = s$ and so $z > \max S(\alpha, s) \subseteq \{ n \mid n < s \}$ or $z \in S(\gamma, s)$ for some $\beta <_L \gamma$ (via Case 1.1 or 1.2 applied at some substage $1 \leq t \leq |\beta|$ of stage $s+1$) in which case $z > \max S(\beta, s) > \max S(\alpha, s)$, by application of the induction hypothesis. □

From inspection of Lemma 6.2.10 and its proof we have the following Corollary.

**Corollary 6.2.11.** *For any stage $s \geq 0$, strategy $\alpha \in \mathcal{T}$, and number $z$, if $z \in S(\alpha, s+1) \setminus S(\alpha, s)$ then $z > \max S(\alpha, s)$.*

**Lemma 6.2.12.** *For all $x, y \in \omega$, stages $0 \leq s < t$ and nodes $\alpha \in \mathcal{T}$, if $x \in S(\alpha, s) \cap I_A[s]$, $y \in S(\alpha, s+1) \setminus S(\alpha, s)$, and $\{x, y\} \subseteq S(\alpha, t)$, then $x \in I_A[t]$.*

*Remark* 1. Less formally Lemma 6.2.12, says that if $y$ enters[6] a stream to which $x$ already belongs as well as already belonging to $A$ (at this point in the construction) then, for as long as both $x$ and $y$ remain in the stream, $x$ remains in $A$.

*Remark* 2. Notice that, by Corollary 6.2.11, $x < y$.

*Proof.* We reason by induction over stages $t \geq s+1$.

*Case* $t = s+1$. By inspection of the construction we see that $y = x(\beta_{s+1}, s+1)$. Let $\beta \in \mathcal{N}$ be such that $x = x(\beta, s+1)$. From Lemma 6.2.8 and the definition of Case 2.4 of the construction we can deduce that it is not the case that $\beta_{s+1} \subseteq \beta$. Moreover $\beta_{s+1} \not<_L \beta$ since then $\beta$ would be initialised at stage $s+1$ forcing $x \in D(s+1) \subseteq D_A[s+1]$ and hence $x \notin S(\alpha, s+1) \subseteq F_A[s+1]$ by Lemma 6.2.7(3).

[6] By Lemma 6.2.9 $y$ is the unique such number.

Thus there are two subcases as follows.

*Subcase* $\beta <_L \beta_{s+1}$. Then $x = x(\beta, s+1)$ by Lemma 6.2.6(1)(c) and $N(\beta, s+1) = N(\beta, s)$ by resetting. Hence $x \in I_A[s+1]$ by definition.

*Subcase* $\beta \subset \beta_{s+1}$. Then, as above, $x = x(\beta, s+1)$. Moreover, notice that if $x \in O_A[s+1]$, then Case 2.2 applies at substage $|\beta|+1$ forcing $\beta_{s+1} = \beta$, a contradiction. Hence $x \in I_A[s+1]$.

*Case* $t > s+1$. We assume the *extended induction hypothesis* that, not only does the Lemma hold for stage $t-1$, but also that the nodes $\beta, \gamma \in \mathcal{N}$ such that $x(\beta, t-1) = x$ and $x(\gamma, t-1) = y$ satisfy $\beta < \gamma$. (Notice that we have already seen that the extended induction hypothesis is true when $t-1 = s+1$.) Again we reason by subcases.

*Subcase* $\beta_t < \beta$. Notice that $\beta_t \subset \beta$ can only happen via Case 3.2 of the construction in which case $\beta$ is initialised, forcing $x \in D(t)$. So we can suppose that $\beta_t <_L \beta$. However in this case there is at most one strategy $\beta_t <_L \mu$ such that[7] $x(\mu, t-1)$ is not forced into $D(t)$ by initialisation. However, $\beta_t <_L \beta < \gamma$ and we have $\{x, y\} \cap D(t) = \emptyset$ by hypothesis; a contradiction. Thus $\beta_t < \beta$ does not happen.

*Remark.* We can now assume that $\beta \leq \beta_t$ and, by Lemma 6.2.6, that $x(\beta, t) = x(\beta, t-1)$.

*Subcase* $\beta_t < \gamma$. As above we can suppose that $\beta_t <_L \gamma$. As $y = x(\gamma, t-1)$, for $y$ to survive in $S(\alpha, t) \subseteq F_A[t]$ it must be the case that $y = x(\beta_t, t)$ (since otherwise $y \in D(t)$) via Case 2.4 applied to $z(t, |\beta_t|) = y$ at substage $|\beta_t|+1$. Thus Case 2.2 does not occur at any substage[8] of stage $t$. In particular (under the inductive assumption that $x \in I_A[t-1]$) this means that $x \in I_A[t]$. Also $\beta_t \neq \beta$ (as $x(\beta_t, t-1) =$ void by definition of Case 2.4). Hence $\beta < \beta_t$.

[7] $x(\mu, t-1) \notin D(t)$ if and only if (i) Case 1.1 or 1.2 applies at some stage $r \leq |\beta_t|$ and (ii) $z(t,p) = x(\mu, t-1)$ for all $r \leq p \leq |\beta_t|$ and (iii) $x(\beta_t, t) = x(\mu, t-1)$ via Case 2.4 at substage $|\beta_t|+1$.

[8] Case 2.2 can only happen at substage $|\beta_t|+1$ since it induces $z(t, |\beta_t|+1) =$ break.

*Subcase* $\beta_t \geq \gamma$. In this subcase, Case 2.2 of the construction does not apply to node $\beta$ during stage $t$ since this would force $\beta_t = \beta < \gamma$. Moreover, $x(\beta, t) = x(\beta, t-1) = x$ and $x(\gamma, t) = x(\gamma, t-1) = y$ (by Lemma 6.2.6(1)(c)). Combining these two observations we see that $x \in I_A[t]$ and that the extended induction hypothesis is again satisfied. □

*Notation, Assumptions and Definitions.* For $n \geq 0$ we define

$$\mathrm{True}_{\infty,n} \quad := \quad \{\,\alpha \mid |\alpha| = n \ \ \& \ \ \forall t(\exists s \geq t)[\,\alpha \subseteq \beta_s\,]\,\}\,.$$

If $\mathrm{True}_{\infty,n} \neq \emptyset$, letting $\beta = \min_{<_L} \mathrm{True}_{\infty,n}$ (i.e. the least strategy of length $n$ under $<_L$), we define $\delta_n = \beta$ if there exists $s_\beta$ such that, for all $s \geq s_\beta$, $\beta$ is not initialised at stage[9] $s$. Otherwise $\delta_n$ is undefined.

For any $\gamma \in \mathcal{T}$ and parameter $p(\gamma, s)$, if $\lim_{s \to \infty} p(\gamma, s)$ exists we define $p(\gamma)$ to be this value (otherwise we say that $p(\gamma)$ is undefined). We define

$$D_A \quad = \quad \bigcup_{s \in \omega} D_A[s]$$

$$F_A \quad = \quad \{\,n \mid \exists s(\forall t \geq s)[\,n \in F_A[t]\,]\,\}$$

and define $I_A$ and $O_A$ likewise (so that $F_A = I_A \cup O_A$). Define

$$A \quad = \quad \{\,n \mid \exists s(\forall t \geq s)[\,n \in A[t]\,]\,\}\,.$$

Also for all $\alpha \in \mathcal{T}$ define

$$S(\alpha) \quad = \quad \{\,n \mid \exists s(\forall t \geq s)[\,n \in S(\alpha, t)\,]\,\}\,.$$

**Lemma 6.2.13.** *For all $n \geq 0$, $\delta_n$ is defined.*

[9] I.e. such that for all $s \geq s_\beta$, $\beta_s \not<_L \beta$ and, if $|\beta_s| < |\beta|$, then $\beta <_L \beta_s$.

*Proof.* By induction on $n$. The case $n = 0$ is obvious. So suppose that $\alpha = \delta_n$ is defined and let $s_n$ be a stage such that $\beta_s \geq \alpha$ for all $s \geq s_n$. There are three cases to consider.

*Case* $\alpha \in \mathcal{R}$. By construction, at each $\alpha$-true stage $s$, $|\beta_s| > \alpha$. Hence, by the induction hypothesis $\beta^\frown\langle i\rangle \in \mathrm{True}_{\infty,n+1}$ for some $i \in \{0, 1, 2\}$. Thus $\delta_{n+1}$ is defined.

*Case* $\alpha \in \mathcal{N}$. Inspection of the construction shows that, for any $\alpha$-true stage $s > 0$, if $m = z(s, |\alpha|)$ then either $m = s-1$, or $m = x(\gamma, s-1)$ for some $\alpha <_L \gamma$ whereas, for all $t \geq s$, either $m \in D_A[t]$ or $m = x(\beta, t)$ for some[10] $\alpha \subseteq \beta$. It follows that for all $\alpha$-true stages $r > p > s_n$, $z(p, |\alpha|) \neq z(r, |\alpha|)$ (and in fact $z(p, |\alpha|) < z(r, |\alpha|)$). Hence at one such $\alpha$-true stage $s$ (if $N(\alpha, s-1) = \text{void}$), Case 2.4. of the construction will apply, so that $x(\alpha, s) = z(s, |\alpha|)$. Moreover, clearly for all $t \geq s$, $x(\alpha, t) = x(\alpha, s)$. Notice also that Case 2.2 can apply at most once after stage $s$. In other words, there is a stage $s'$ such that at every $\alpha$ true stage $t \geq s'$, $|\beta_t| > |\alpha|$. Thus (as in the first case) $\delta_{n+1}$ is defined to be $\alpha^\frown\langle i\rangle$ for some $i \in \{0, 1\}$.

*Case* $\alpha \in \mathcal{P}$. Clearly Case 3.2 applies at most once after stage $s_n$. Thus, as above, $\delta_{n+1}$ is defined to be $\alpha^\frown\langle i\rangle$ for some $i \in \{0, 1\}$.

Note that to each case there corresponds a stage $s_{n+1}$ as in the induction hypothesis. Thus the latter is validated. This concludes the proof of the Lemma. □

**Corollary 6.2.14.** *For all $n \geq 0$, $S(\delta_n)$ is infinite.*

*Proof.* It follows from Lemma 6.2.13 that, for all $n$ such that $\delta_n \in \mathcal{N}$, $x(\delta_n)$ is defined (with value in $\omega$). Moreover, a straightforward argument by induction using Lemma 6.2.6(2) implies that, for all such $p \neq m$, $x(\delta_p) \neq x(\delta_m)$. It now suffices to notice that $\{\, x(\delta_m) \mid \delta_m \in \mathcal{N} \;\&\; m > n \,\} \subseteq S(\delta_n)$. □

**Lemma 6.2.15.** *The following statements are true.*

[10]Note that $\alpha \subset \beta$ implies that $x(\alpha, t-1) \geq 0$, i.e. is already defined (see Lemma 6.2.8).

1. $A = D_A \cup I_A$.

2. $F_A = I_A \cup O_A$ *and* $I_A \cap O_A = \emptyset$.

3. $D_A \cap F_A = \emptyset$.

4. $\omega = F_A \cup D_A$.

5. *For any* $\alpha \in \mathcal{T}$ *such that* $\alpha \subseteq \delta$,

$$F_A \quad = \quad S(\alpha) \cup \{\, x(\gamma) \mid x(\gamma) \geq 0 \ \& \ \gamma < \alpha \,\}.$$

*Proof.* (1) and (2) are obvious by definition, whereas (3), (4) and (5) follow by application of Lemma 6.2.7 using induction over the stages of the construction. □

*Notation.* For $G \in \{F, I, O\}$ and $\alpha \in \mathcal{T}$ we use the notation $G_A^{<\alpha}[s]$ to denote the set $G_A[s] \cap \{\, x(\gamma, s) \mid \beta < \alpha \,\}$.

**Lemma 6.2.16.** *For* $G \in \{F, I, O\}$, *any* $\alpha \subseteq \delta$ *and stage* $s_\alpha$ *such that* $\alpha \leq \beta_s$ *for all* $s \geq s_\alpha$, $G_A^{<\alpha}[s] = G_A^{<\alpha}[s_\alpha]$.

*Proof.* A straightforward induction over $s \geq s_\alpha$. □

By definition of $\mathcal{T}$ and $\delta$, for any requirement $Q$ there is precisely one strategy $\alpha$ associated with $Q$ such that $\alpha \subseteq \delta$. Accordingly we consider each such $\alpha$ by cases.

**Lemma 6.2.17.** $\alpha \in \mathcal{R}_{\Psi,\Omega_0,\Omega_1}$. *If* $A = \Psi^{\Omega_0^A \oplus \Omega_1^A}$ *and* $A$ *is not c.e. then* $A \leq_{\mathrm{e}} \Omega_i^A$ *for some* $i \in \{0, 1\}$.

*Proof.* Define $\Lambda = \{\, \langle z, \emptyset \rangle \mid z \in D_A \,\}$. There are three cases to consider.

*Case* $\alpha^\frown\langle 2 \rangle \subset \delta$. Consider $x \in S(\alpha^\frown\langle 2 \rangle)$. Clearly $x \notin A$. Indeed it cannot be the case that $x \in A \cap \Psi^{\Omega_0^A \oplus \Omega_1^A}$ since then $x$ would have been removed from $\alpha^\frown\langle 2 \rangle$'s stream via Case 1.2 of the construction. Moreover, if $x \in A \setminus \Psi^{\Omega_0^A \oplus \Omega_1^A}$ then $A \neq$

$\Psi^{\Omega_0^A \oplus \Omega_1^A}$. A contradiction. We see therefore that $\alpha^\frown\langle 2\rangle \subseteq \delta$ implies that $A =^* D_A$, i.e. that $A$ is c.e. Hence $\alpha^\frown\langle 2\rangle \subset \delta$ cannot apply (under the assumptions of the Lemma).

*Case* $\alpha^\frown\langle 1\rangle \subset \delta$. Consider $x \in S(\alpha^\frown\langle 1\rangle)$. By construction there exists a unique stage $s_x$ and, for $i \in \{0,1\}$, a unique axiom $\langle x, F_{i,x}\rangle$, such that $F_{i,x} =_{\text{def}} \Omega_i^A[s_x]$ was enumerated into $\Gamma_i$ at stage $s_x + 1$ via Case 1.2 of the construction. Now, it follows from Lemma 6.2.10, Corollary 6.2.11 and the dumping activity at stage $s_x + 1$ that $\{\, z \mid z > x \ \&\ z \in A[s_x]\,\} \subseteq D_A \subseteq A$. On the other hand we can also deduce from Lemma 6.2.16, Lemma 6.2.12 and the dumping activity at stage $s_x + 1$ that $\{\, z \mid z < x \ \&\ z \in A[s_x]\,\} \subseteq A$. Notice now that these observations imply that, for each $i \in \{0,1\}$,

$$F_{i,x} \ \subseteq\ \Omega_i^{A \,\cup\, \{x\}} \tag{6.2.2}$$

whereas the definition of Case 1.2 implies that

$$x \ \in\ \Psi^{F_{0,x} \oplus F_{1,x}} \tag{6.2.3}$$

- Suppose that $x \in A$. Then by (6.2.2), $F_{1,x} \subseteq \Omega_1^A$, and so $x \in \Gamma_1^{\Omega_1^A}$.
- Now suppose that $x \notin A$. Then $x \notin \Gamma_1^{\Omega_1^A}$. Indeed $x \in \Gamma_1^{\Omega_1^A}$ would imply the transfer of $x$ from $\alpha^\frown\langle 1\rangle$'s stream to $\alpha^\frown\langle 0\rangle$'s stream at some stage $s > s_x$ (via Case 1.1).

We see therefore that $\alpha^\frown\langle 1\rangle \subseteq \delta$ implies (by Lemma 6.2.15) that $A =^* \Phi_1^{\Omega_1^A}$ where $\Phi_1 =_{\text{def}} \Gamma_1 \cup \Lambda$.

*Case* $\alpha^\frown\langle 0\rangle \subset \delta$. Consider $x \in S(\alpha^\frown\langle 0\rangle)$ and (for $i \in \{0,1\}$) let $s_x$ and $F_{i,x}$ be defined as above. Also let $t_x + 1$ be the stage at which the application of Case 1.1 caused $x$ to be transferred from $\alpha^\frown\langle 1\rangle$'s stream to $\alpha^\frown\langle 0\rangle$'s stream. Note that, similarly to the argument used in the last case, it follows from Lemma 6.2.10, Corollary 6.2.11, Lemma 6.2.16, Lemma 6.2.12 and the dumping activity at stage $t_x + 1$ that

$$F_{1,x} \subseteq \Omega_1^A \tag{6.2.4}$$

(i.e. whether or not $x \in A$).

• Suppose that $x \in A$. Then by (6.2.2), $F_{0,x} \subseteq \Omega_0^A$, and so $x \in \Gamma_0^{\Omega_0^A}$.

• Now suppose that $x \notin A$. Then $x \notin \Gamma_0^{\Omega_0^A}$. Indeed, if $x \in \Gamma_0^{\Omega_0^A}$, then $F_{0,x} \subseteq \Omega_0^A$. However, by (6.2.4), $F_{1,x} \subseteq \Omega_1^A$ and by (6.2.3) $x \in \Psi^{F_{0,x} \oplus F_{1,x}}$. Thus $x \in \Psi^{\Omega_0^A \oplus \Omega_1^A} \setminus A$. A contradiction.

We see therefore that $\alpha^\frown\langle 0\rangle \subseteq \delta$ implies (by Lemma 6.2.15) that $A =^* \Phi_0^{\Omega_0^A}$ where $\Phi_0 =_{\text{def}} \Gamma_0 \cup \Lambda$. □

**Lemma 6.2.18.** $\alpha \in \mathcal{N}_W$. *Then* $x(\alpha) \in A$ *if and only if* $x(\alpha) \notin W$.

*Proof.* Inspection of the construction shows that if $\alpha^\frown\langle 1\rangle \subseteq \delta$, then $x(\alpha) \in A \setminus W$ whereas if $\alpha^\frown\langle 0\rangle \subseteq \delta$ then $x(\alpha) \in W \setminus A$. □

*Notation.* For $G \in \{F, I, O\}$ and $\alpha \subseteq \delta$ we define (on the strength of Lemma 6.2.16) $G_A^{<\alpha} = \lim_{s\to\infty} G_A^{<\alpha}[s]$.

Note that, for any $\alpha \subseteq \delta$, $O_A^{<\alpha} \subseteq \overline{A}$.

**Lemma 6.2.19.** $\alpha \in \mathcal{P}_W$. *Let* $E = O_A^{<\alpha}$. *If there is no* $D \in W$ *such that* $D \subseteq A$ *then, for all* $D \in W$, $D \cap E \neq \emptyset$.

*Proof.* Let $s_\alpha$ be the least stage such that $\beta_s \geq \alpha$ for all $s \geq s_\alpha$.

*Case* $\alpha^\frown\langle 0\rangle \subseteq \delta$. Then Case 3.2 applied relative to $\alpha$ at some stage $s \geq s_\alpha$ and it follows by Lemma 6.2.16 and the dumping activity at stage $s$ that there is a finite set $D \in W$ such that $D \subseteq A$.

*Case* $\alpha^\frown\langle 1\rangle \subseteq \delta$. Then Case 3.2 applies at no stage $s \geq s_\alpha$ and we can deduce from

Lemmas 6.2.7 and 6.2.15 in conjunction with Lemma 6.2.16 that $D \cap E \neq \emptyset$ for all $D \in W$. $\square$

**Lemma 6.2.20.** *All the requirements are satisfied.*

*Proof.* For the $N$ and $P$ requirements this is immediate by Lemmas 6.2.18 and 6.2.19. Satisfaction of each $R$ requirement follows from the conjunction of Lemma 6.2.17 with the fact that all the $N$ requirements are satisfied (and hence $A$ is not c.e.). $\square$

**Lemma 6.2.21.** *$A$ is low.*

*Proof.* Consider $n \in \omega$. Notice that by construction $n$ can only be extracted from $A$ by $N$ strategies of length $\leq n$ and moreover that each such strategy extracts $n$ at most once. It follows that $n$ can be extracted from $A$ at most $2^n + 1$ times. Since this is true for all $n \in \omega$, the construction defines a $\Delta^0_2$ approximation to $A$. Since $A$ is also enumeration 1-generic, $A$ is low (by Corollary 5.3.3). $\square$

We can now conclude the proof of Theorem 6.2.5 by setting $\boldsymbol{a} = deg_{\mathrm{e}}(A)$.

**Corollary 6.2.22.** *There exists a low enumeration 1-generic degree $\boldsymbol{a}$ such that $\mathbf{0}_{\mathrm{e}} < \boldsymbol{a} < \mathbf{0}'_{\mathrm{e}}$ and is not 1-generic.*

*Proof.* Apply Theorem 6.2.5 together with the fact that every 1-generic degree is splittable (Proposition 6.2.2). $\square$

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