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Constructions of Uncountable Rigid Structures

by

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ABSTRACT

We construct some uncountable set theoretical structures with trivial automorphism group but admitting non-trivial epimorphism and/or embedding monoids.

The structures we consider are Suslin trees, dense subchains of the real line and graphs with vertices in ω_1.

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Mayra Montalvo Ballesteros

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Chapter 1

Introduction

"Begin at the beginning," the King said, very gravely, "and go on till you come to an end; then stop."

– Lewis Carroll, *Alice in Wonderland*

From the beginning of time, the automorphism group of a structure has been a focus of study. It is often used as an important invariant of the structure and it enshrines a great deal of information about it. Thus, one method for classifying structures is via their automorphism groups, and indeed much work has been done concerning the reconstruction of various structures from their automorphism groups. See [Rub93] and [Rub94] for an account of some of these results.

Nevertheless, apart from $\mathrm{Aut}(P)$, there are five monoids naturally associated in [LT12] with every partial order $(P, \leqslant)$, these are: the monoids of embeddings, bimorphisms, monomorphisms, epimorphisms and endomorphisms, denoted by $\mathrm{Emb}(P, \leqslant)$, $\mathrm{Bi}(P, \leqslant)$, $\mathrm{Mon}(P, \leqslant)$, $\mathrm{Epi}(P, \leqslant)$ and $\mathrm{End}(P, \leqslant)$ respectively. Here an **endomorphism** of a relational structure is a map from the structure to itself which preserves all relations (but not necessarily their negations); it is a **monomorphism** if it is also injective, an **epimorphism** if it is surjective, a **bimorphism** if it is both, and an **embedding** if it is an isomorphism to a substructure (in which case it must also preserve the negations of relations). In fact, these monoids form a lattice -see **Figure 1.1**.

A structure that has a *rich* automorphism group is called **homogeneous** and this topic has been widely developed. The diametrically opposed notion of *rigid* struc-

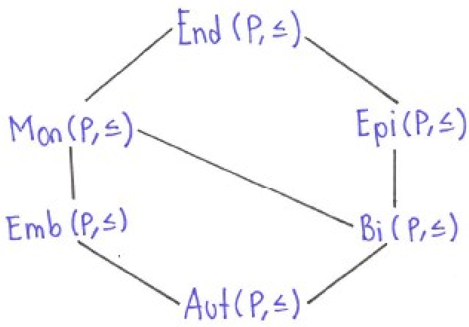

FIGURE 1.1: Lattice for monoids of order preserving maps on posets.

tures then becomes a natural subject of study. A structure is said to be **rigid** if it admits only the identity as an automorphism. Some rigid structures can be trivially seen to be so, such as any finite linear order. Moreover, any well ordered chain is rigid, so to get more interesting examples we require $(X, \leqslant)$ to be dense without endpoints. It is now clear that $(X, \leqslant)$ cannot be countable, as if so it would have to be isomorphic to $(\mathbb{Q}, \leqslant)$, which has $2^{\aleph_0}$ automorphisms. However, for some structures it may be highly non-trivial to decide whether or not it is rigid, or indeed the construction of rigid structures of certain kinds may be quite involved. A wide variety of rigid structures has been studied; Gaifman and Specker constructed $2^{\aleph_1}$ rigid *Aronszajn* trees in [GS64], and there are Shelah's absolute rigid trees in [She82], Nešetřil's rigid graphs in [Nes02] and many more.

In this work, we focus on particular cases of the following structures: trees, linear orders and graphs. First concentrating on the study of trees, we chiefly consider Suslin trees at cardinality ω_1 and above. The classical construction of Jensen of a Suslin tree in the constructible universe $\mathbf{L}$ [Jen68], had the additional property of being rigid, though he provided some modifications of the method to get Suslin trees with a specified number of automorphisms [DJ74] and Jech gave a classification of possible cardinalities of the automorphism group of any ω_1 tree [Jec72]. Also, Avraham [Avr79] and Todorcevic [Tod78] obtained results on rigid Aronzajn trees which I shall recall. In addition, some of the methods used to construct a Suslin tree in forcing extensions result in rigid structures.

In the chapters of the thesis treating Suslin trees, **Chapter 2** and **Chapter 3**, we recall some of the classical results in which rigid or homogeneous trees are constructed using the combinatorial principle $\Diamond$ or forcing and we extend these by considering some kinds of endomorphisms in place of automorphisms. In some cases, the existing models already provide examples of what is desired, for instance of a Suslin tree which admits no level preserving endomorphism, in other cases the existing constructions are adapted. The two main topics, constructions using $\Diamond$ or its relatives at higher cardinalities, and using the method of forcing, form the subjects of **Chapter 2** and **Chapter 3** respectively. Linking this material with the work on chains in **Chapter 4**, we also consider the existence of Suslin lines which are rigid or partially rigid as well as the way in which some of the results on Suslin trees transfer to ones for Suslin lines .

For linear orders a classical result of Dushnik and Miller provides us with a dense rigid subset of the real line. At higher cardinalities, methods of Shelah give constructions of several rigid chains using stationary sets to 'encode' gaps (Dedekind cuts) and stop them from being moved. When working with chains we have that Mon = Emb and Bi = Aut, so that the diagram in **Figure 1.1** reduces to that in **Figure 1.2**.

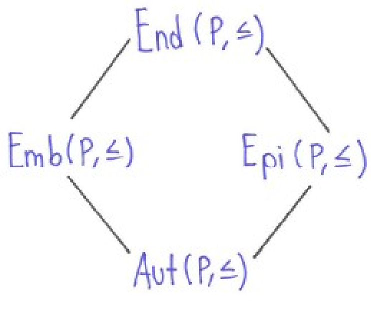

FIGURE 1.2: Lattice for moniods of order preserving maps on chains

This is because if f is a monomorphism of a chain then it must actually preserve $<$ (as well as $\leqslant$) so that Mon = Emb (and similarly Bi = Aut). Also in the cases we consider in **Chapter 4**, if a function has dense image then it belongs to the monoid Epi, which contains only continuous functions (for if we have an order preserving function with dense image, it must be a continuous surjection -see **Lemma 4.4**).

Dushnik and Miller in [DM40] showed that there is a dense subset $X \subseteq \mathbb{R}$ of size continuum which is rigid but said nothing about the other monoids. Droste and Truss in [DT01], using a similar method, found a dense subset of $\mathbb{R}$ of size continuum which is rigid, but it admits *many* embeddings, i.e. meaning that the whole chain can be embedded between any two elements of $(X, \leqslant)$. In **Chapter 4** we investigate the structure of these monoids when insisting that $\mathrm{Aut}(X, \leqslant)$ is trivial. Moreover, we can even retain a trivial $\mathrm{Epi}(X, \leqslant)$ while having many non-trivial embeddings, though if $\mathrm{Emb}(X, \leqslant)$ is trivial, so is $\mathrm{Epi}(X, \leqslant)$ (a fact that is true for any chain X as we remark below, assuming **AC**).

There is no way of getting rid of all endomorphisms using this method, for instance any constant map will always lie in $\mathrm{End}(X, \leqslant)$, but we can ask if we still can preserve some "significant" ones without having any epimorphism or embedding other than the identity, that is keeping both $\mathrm{Epi}(X, \leqslant)$ and $\mathrm{Emb}(X, \leqslant)$ trivial. It turns out that the answer to this question is yes.

Finally in **Chapter 5** we turn our attention to graphs. Here again it is not hard to construct rigid graphs in which points are distinguished by their distinct degrees, so the real challenge comes about in constructing graphs with various degrees of rigidity which are elementarily equivalent to the random graph -the analogue for chains in this context of dense linear orders without endpoints. Two main methods are considered, forcing with finite or countable conditions, giving rise to uncountable graphs Γ_{ω_1} and Δ_{ω_1}, respectively in generic extensions. These graphs share some properties, for instance as we said before, they are elementarily equivalent to the Random graph, but they also differ in certain respects, for example Δ_{ω_1} is saturated but Γ_{ω_1} is not; in $M[\Delta_{\omega_1}]$, **CH** necessarily holds (even if it didn't hold in the ground model M) but as the extension from M to $M[\Gamma_{\omega_1}]$ is *c.c.c.*, any failure of **CH** in M is preserved in $M[\Gamma_{\omega_1}]$.

Chapter 2

Suslin Trees and Lines constructed using Diamond

"Once there was a tree, and she loved a little boy."

– Shel Silverstein, *The Giving Tree*

In the first construction that Jensen gave of a Suslin tree, $(T, \leqslant)$ turned out to be automorphism rigid [Jen68]. Our aim in this chapter is to investigate the rigidity properties of Suslin trees constructed assuming the combinatorial principle $\Diamond$.

We show that Jensen's tree not only has a trivial automorphism group, but it does not admit any level preserving endomorphism; we modify this construction to obtain a Suslin tree with no non-identity-embeddings and one that is totally rigid. Also, we discuss some of Jensen's methods for transferring results about order preserving functions from Suslin trees to Suslin lines, in particular we modify one of his arguments to get **Lemma 2.14** which connects level preserving epimorphisms in a Suslin tree to epimorphisms in a Suslin line. Our main constructions are of a rigid Suslin tree admitting a non-identity embedding (**Section 2.3**) and a rigid Suslin tree admitting a non-identity epimorphism (**Section 2.4**). Later on in **Section 2.5**, we highlight some remarks linking these results with constructions in **Chapter 4**, in particular that the last one also admits non-identity embeddings and finish this chapter with some of the earlier results in **Section 2.1**, generalized to κ^+-Suslin trees for *any* uncountable κ.

2.1 Background and Preliminaries

A **tree** $(T, \leqslant)$ is a partially ordered set with the requirement that for any point x in T, the set $x_\downarrow = \{y \in T \mid y \leqslant x\}$ of predecessors of x is well ordered by the relation $\leqslant$. We usually abuse notation and let T stand for $(T, \leqslant)$ and $x \leqslant y$ if either $x = y$ or $x < y$, in which case we say that y **extends** x. For an ordinal α, the **α-th level of** T, denoted by T_α, is the set of all points (or *nodes*) of T such that the corresponding set $x_\downarrow$ has order type α, i.e. $T_\alpha = \{x \in T \mid ot(x_\downarrow) = \alpha\}$.

We let $T \restriction C = \{x \in T \mid x \in T_\alpha, \alpha \in C\}$ be the **restriction of** T **to** C, where C is a set of ordinals, and if $x \in T$ we let T^x be the set of all extensions of x. The set $ht(T) = \sup\{ot(x) + 1 \mid x \in T\}$ is the **height** of T and if $ht(T) = \alpha$ we say T is an α-tree. A **branch** of T is a maximal linearly ordered set of T and if the branch has order type α we say that it is an α-branch. We say that an **α-branch** b **has been extended**, if there is $x \in T_\alpha$ such that $x > t$, for all $t \in b$. We denote by $[T]$ the set of all branches of T and similarly $[T \restriction \alpha]$ is the set of α-branches in T, for $\alpha < ht(T)$. A set of pairwise incomparable elements under $\leqslant$ is an **antichain** of T. An antichain A has been **sealed at level** α if for every $x \in A$ there is $t_x \in T_\alpha$ which is compatible with x.

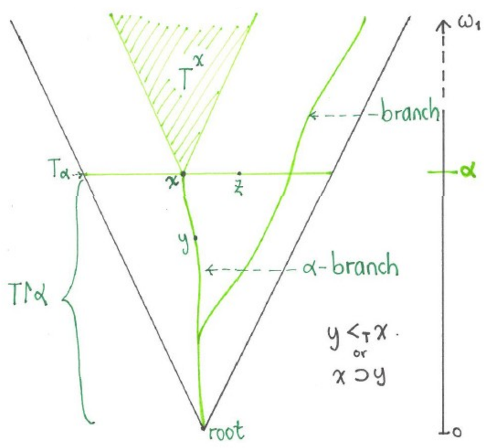

FIGURE 2.1: Notation on a Suslin Tree

A simple example of a tree is $2^{<\omega}$ (the set of finite binary sequences, whose branches form the Cantor set) ordered by extension. It is an ω-tree, and for each $n \in \omega$ the sets $T \upharpoonright n$ and T_n are countable. Also, it has a unique minimal element, which we call the *root* and 2^ω-many (ω)-branches.

A useful property to have in a tree is normality. A **normal** α-tree T is a tree satisfying the following properties:

1. It has a unique minimal element called the *root*.
2. $ht(T) = \alpha$.
3. Each level of T has cardinality $< \alpha$.
4. If x is in level T_β, for $\beta < \alpha$, then x has extensions at each higher level less than α.
5. If x is not maximal in T, then it has (at least) two extensions in the next level. We say that the tree T **splits**.
6. If x, y are in the same level, $\beta \in \mathrm{Lim}(\alpha)$ (β is a limit ordinal less than α), and $x_\downarrow = y_\downarrow$, then $x = y$, that is, an element in a limit level is identified with the set of its predecessors.

If in 5. above we ask for only two immediate successors then the resulting tree will be a **normal binary tree**. If we ask for γ-many elements immediately above each point in the tree then we say that T is a γ-**splitting normal tree**.

Continuing to fix notation, if κ is any ordinal, we write $\alpha \in \mathbf{Lim}(\kappa)$ instead of "α is a limit ordinal less than κ".

A **Suslin tree** (ST) is a normal ω_1-tree where every antichain is at most countable. This implies that a ST has no ω_1-branches. If we weaken the condition of having only countable antichains to having every level of the tree countable but still having no ω_1-branches, then we get an **Aronszajn tree** (AT). The existence of normal ω_1 Aronszajn trees is provable within **ZFC**. Moreover, it was proved (independently) by

Stevo Todorcevic [Tod78] and Uri Avraham [Avr79] that it is possible to construct a rigid AT in **ZFC**. In fact, Todorcevic's AT is **totally rigid**, meaning that whenever x and y are two distinct nodes in T then $T^x \ncong T^y$. Both proofs make use of the $2^{\aleph_1}$ non-isomorphic ATs given by Gaifman and Specker in [GS64] but differ slightly in details.

The intuitive idea is that we want to use the trees in [GS64] to code the elements of the final tree and then use the fact that they are non-isomorphic to stop any automorphism from sending a node to another.

Avraham's construction defines a final tree R, as the union of countably many trees R^n, with $n \in \omega$. We start by letting $\{X\} \cup \{X_{\gamma,n} \mid \gamma \in \omega_1,\ n \in \omega\}$ be a collection of pairwise disjoint uncountable subsets of ω_1, so that $T(X)$ and $T(X_{\gamma,n})$ are copies of the corresponding non-isomorphic ATs as in [GS64] on X and $X_{\gamma,n}$. The intuitive idea is that we want to use the trees in [GS64] to code the elements of the final tree and then use the fact that they are non-isomorphic to stop any automorphism from sending a node to another. But just planting the trees above each node is not enough because at the end the tree above a node can still be isomorphic to the tree above another node, so we need to look for different properties that keep the tree above each node in some sense "unique" and at the same time preserved under automorphisms. The trees in [GS64] have precisely what we need in the following additional properties.

P1. For every $x \in T(X_{\gamma,n})$ there is an uncountable $A_x \subseteq T(X_{\gamma,n})^x$ every two elements of which meet at a level in $X_{\gamma,n}$, and

P2. There is no uncountable subset of $T(X_{\gamma,n})^x$ every two elements of which meet at a level in $\omega_1 \setminus X_{\gamma,n}$.

We let $R^0 = T(X)$ and assume we have defined R^n. Then we enumerate the elements of R^n as $\{a^n_\gamma : \gamma \in \omega_1, n \in \omega\}$. Notice that each a^n_γ is in R^n_α for some $\alpha \in \omega_1$. Now, for every $a^n_\gamma \in R^n_\alpha$ we look at an element (any element) $b^n_\gamma \in T(X_{\gamma,n})$ at level

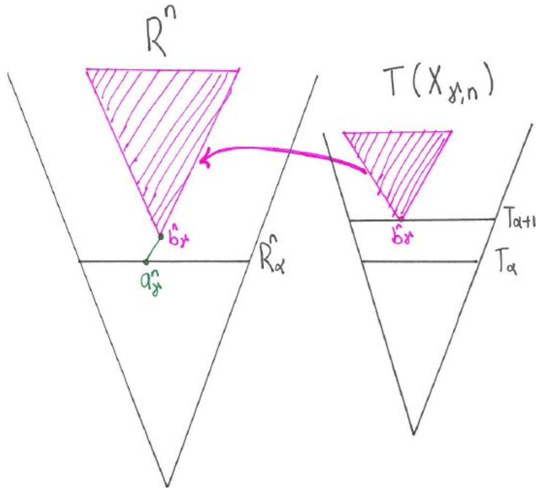

FIGURE 2.2: Avraham's construction

$\alpha + 1$ and place $T(X_{\gamma,n})^{b^n_\gamma}$ (the tree above b^n_γ together with b^n_γ) above a^n_γ, so that $R^{n+1} = R^n \cup \{T(X_{\gamma,n})^{b^n_\gamma} \mid \gamma \in \omega_1\}$ (see Figure 2.2), where

$$x < y \text{ in } R^{n+1} \longleftrightarrow x < y \text{ in } R^n,$$
$$x \in (a^n_\gamma)_\downarrow \text{ and } y = b^n_\gamma \text{ or}$$
$$x < y \text{ in } T(X_{\gamma,n})^{b^n_\gamma}$$

To see that R is rigid suppose f is a non-trivial automorphism of R sending $f(x)$ to $y \neq x$. Then we can find a subset A_x as in P1 and this translates into a subset $A_y = f[A_x]$ above y, which will contradict P2.

Todorcevic's construction takes T as the union $\bigcup\{T^\alpha \mid \alpha \in \omega_1\}$ of Aronszajn trees and also makes use of a family $\mathcal{F} = \{T(X_\delta) \mid \delta \in \omega_1\}$ of non-isomorphic AT's. $T^0 = T(X_0)$ and having defined the trees T^β for every $\beta < \alpha$ such that $\beta < \gamma$ implies $T^\beta < T^\gamma$, if $\alpha \in \mathrm{Lim}(\omega_1)$, we define T^α as $T^\alpha = \bigcup_{\beta<\alpha} T^\beta$. If $\alpha = \beta + 1$, then, for every $x \in T^\beta_\beta$ (the β-th level of T^β), we choose $T(X_{\delta_x}) \in \mathcal{F}$ that hasn't yet been used in the construction and T^α is the tree $T^\beta \cup \{T(X_{\delta_x}) \mid x \in T^\beta_\beta\}$ and the ordering: $\leqslant_{T^\alpha} \restriction T^\beta = \leqslant_{T^\beta}$, $\leqslant_{T^\alpha} \restriction T(X_{\delta_x})^\beta = \leqslant_{T(X_{\delta_x})^\beta}$ and $x \leqslant_{T^\alpha} T(X_{\delta_x})$ and $T(X_{\delta_x})$ is incompatible in T^α with everything not in $x_\downarrow$.

However, the existence of a Suslin tree is independent of the axioms of **ZFC**. The first step towards showing this consistency was made by Tennenbaum in 1963 * [Ten68] when he proved the consistency result **Con(ZFC)** implies **Con(ZFC** + ¬**SH)** by forcing using finite trees as conditions to generate a generic Suslin tree. On the other hand, Solovay and Tennenbaum in [ST71] constructed a generic model using (iterated) forcing where there are no Suslin trees using (ironically) Suslin trees as conditions for the partial order: if we force using a ST, then in the generic extension it acquires an ω_1-branch, so it is no longer Suslin.

A little after Tennenbaum, Jech [Jec67] and Jensen [Jen68] also constructed Suslin trees. The former also used the method of forcing but now with countable trees as conditions and the latter showed that in Gödel's constructible universe **L**, there is a Suslin tree. In order to construct his tree, Jensen works inside **L** using ideas from *Gödel's Condensation Lemma*†. He later formulated a general combinatorial principle denoted by $\Diamond$ (**diamond**), which captures the essence of the argument used for his construction.

Recall that if κ is an ordinal, a **club in** κ is a closed (it contains all its accumulation points) and unbounded subset of κ and a set is **stationary** in κ if it intersects every club in κ. Then $\Diamond$ stands for the following statement,

$\Diamond$: There is a sequence $(S_\alpha \mid \alpha < \omega_1)$ such that $S_\alpha \subseteq \alpha$, with the property that whenever $X \subseteq \omega_1$, the set $S = \{\alpha \in \omega_1 \mid X \cap \alpha = S_\alpha\}$ is stationary in ω_1.

We call the sequence $(S_\alpha \mid \alpha < \omega_1)$ a $\Diamond$-sequence. Intuitively, this principle gives us an approximation of any subset X of ω_1 by its intersection with a large enough subset of ω_1. This principle implies the existence of a Suslin tree, it is independent of **ZFC** and it was proved by Jensen [Jen68] to be true under the assumption $\mathbf{V} = \mathbf{L}$.

*This information was taken from [Kan06]

†Gödel's Condensation Lemma states that for every limit ordinal δ, if $M \prec (\mathrm{L}_\delta, \in)$ then the transitive collapse of M is L_γ for some $\gamma \leqslant \delta$.

Although Jensen carried out the construction of a Suslin tree inside **L**, the result can be obtained solely from the assumption of the existence of a $\Diamond$-sequence. The proof uses the fact that $\Diamond$ implies **CH**: Let $(S_\alpha \mid \alpha < \omega_1)$ be a $\Diamond$- sequence. Then for every $X \subseteq \omega$ (which is also a subset of ω_1), there is $\alpha \in \omega_1$ such that $X \cap \alpha = S_\alpha$ using the principle, but in fact since X is countable, there is some α satisfying $X \cap \alpha = X = S_\alpha$. If we define $f : \mathcal{P}(\omega) \longrightarrow \omega_1$ by $f(X) = \min\{\alpha \mid X = S_\alpha\}$, then it follows that f is an injective function.

Using the same basic construction, Jensen produced a rigid Suslin tree. The following is our generalization of his argument to level preserving endomorphisms and will be shown using **Lemma 2.25** (here we use the particular case when $\kappa = \omega_1$). That is, there is a $\Diamond$-sequence if and only if there is a $\Diamond_g$-sequence, where $\Diamond_g$ is defined analogously to $\Diamond$,

$\Diamond_g$: There is a sequence $(g_\alpha \mid \alpha \in \omega_1)$ such that $g_\alpha : \alpha \longrightarrow \alpha$ and if $g : \omega_1 \longrightarrow \omega_1$ is any function, then the set $G = \{\alpha \in \omega_1 \mid g \restriction \alpha = g_\alpha\}$ is stationary in ω_1.

Proposition 2.1. *If $\Diamond$ holds then there is a Suslin tree that admits no non-trivial level preserving endomorphisms.*

Proof. The resulting tree T, will be a normal ω_1-Suslin tree which is ω-splitting and will be constructed by recursion on $\alpha < \omega_1$ (the levels of T). At stage α we choose which elements to add to T_α from the set W_α (defined below), so that $T \restriction \beta$ is an end-extension of $T \restriction \alpha$ for all $\beta > \alpha$.

We define the sequence $(W_\alpha)_{\alpha \in \omega_1}$ as follows,

$$W_0 = root = zero$$

$$W_{n+1} = [\omega^n, \omega^{n+1}), \text{ for } n \in \omega,$$

$$W_\alpha = [\omega^\alpha, \omega^{\alpha+1}) \text{ for } \alpha \geqslant \omega$$

T_0 consists the *root* element. Assume we have defined T_α. To define $T_{\alpha+1}$, for each element $x \in T_\alpha$ we place ω-many elements of $W_{\alpha+1}$ as immediate successors of x. If $\alpha \in \mathrm{Lim}(\omega_1)$ and we want to define T_α, we follow a different approach.

We turn to look at both, our $\Diamond$-sequence $(S_\alpha)_{\alpha\in\omega_1}$ and our $\Diamond_g$-sequence $(g_\alpha)_{\alpha\in\omega_1}$. We will use the $\Diamond$-sequence to seal maximal antichains of the form S_α (and hence produce a Suslin tree) and the $\Diamond_g$-sequence to stop g_α from being a level preserving endomorphism of the resulting Suslin tree T. We have the following possible outcomes.

$\star_1$. ***S_α is not a maximal antichain of $T \restriction \alpha$ and g_α is not a non-trivial order preserving endomorphism of $T \restriction \alpha$.***

In this case, for each $t \in T \restriction \alpha$ we choose a branch b_t containing t and we extend it using elements from W_α. Since $T \restriction \alpha$ is countable so is T_α.

$\star_2$. ***S_α is a maximal antichain of $T \restriction \alpha$ and g_α is not a non-trivial order preserving endomorphism of $T \restriction \alpha$.***

Then, for each $t \in T \restriction \alpha$, there is $a \in S_\alpha$ that is compatible with t. Hence, if we let $b_t^* \in [T \restriction \alpha]$ contain t and a, we extend every branch in the set $\{b_t^* \mid t \in T \restriction \alpha\}$ which is countable, taking elements from W_α. So we have sealed the antichain S_α ensuring that it stays maximal in $T \restriction (\alpha + 1)$.

$\star_3$. ***S_α is not a maximal antichain of $T \restriction \alpha$ and g_α is a non-trivial order preserving endomorphism of $T \restriction \alpha$.***

Then there is a node $x_\alpha \in T \restriction \alpha$ which is moved by g_α, so we let $b \in [T \restriction \alpha]$ be such that $x_\alpha \in b$ and $g_\alpha[b] \neq b$. Notice that since g_α is a level preserving homomorphism, $g_\alpha[b]$ is also an element of $[T \restriction \alpha]$ and hence it must be the case that $g_\alpha[\bigcup b] = \bigcup g_\alpha[b]$. For each $t \in T \restriction \alpha$, we choose an α-branch b_t that contains t and such that $b_t \neq g_\alpha[b]$; this can be done since our tree $T \restriction \alpha$ is ω-splitting and normal, so for each $t \in T \restriction \alpha$ there are ω-many choices for b_t. In this way, we extend every α-branch in the set $\{b_t \mid t \in T \restriction \alpha\} \cup \{b\}$ using the elements in W_α.

$\star$**4.** S_α ***is a maximal antichain of*** $T \restriction \alpha$ ***and*** g_α ***is a non-trivial order preserving endomorphism of*** $T \restriction \alpha$**.**

Again, we let b be an element of $[T \restriction \alpha]$ such that $b \neq g_\alpha[b]$, and choose an α-branch b_t^* for each $t \in T \restriction \alpha$ containing t and an element of S_α which is comparable with t and such that $b_t^* \neq g_\alpha[b]$. This choice is possible, for the only problem that could arise is when $t \in g_\alpha[b]$; then there is $a \in S_\alpha$ which is compatible with t. If a is also in $g_\alpha[b]$, we let $y > t$ in $T \restriction \alpha$ be such that $y \notin g_\alpha[b]$. Then there is $a_y \in S_\alpha$ which is compatible with y, but $a_y \notin g_\alpha[b]$ since otherwise a and a_y would be compatible. So we let $b_t^* \in [T \restriction \alpha]$ be the α-branch containing, t, y and a_y and we extend all the α-branches in the set $\{b_t^* \mid t \in T \restriction \alpha\} \cup \{b\}$.

Having taken care of all the possible cases, we let $T = \bigcup_{\alpha \in \omega_1} T \restriction \alpha$. Notice that we have ensured that T is a normal ω-splitting tree, so it only remains to show that it has no uncountable antichains.

T has no uncountable antichains.

This will follow from the next couple of lemmas.

Lemma 2.2. *Let A be a maximal antichain of T. Then the following set is a club subset of ω_1.*

$$C = \{\alpha \in \omega_1 \mid A \cap (T \restriction \alpha) \textit{ is a maximal antichain of } T \restriction \alpha\}$$

Proof. Closure. Let $\lambda \in \mathrm{Lim}(\omega_1)$ and $(\alpha_\eta)_{\eta \in \lambda}$ be a sequence of elements in C. Let $\alpha = \sup_{\eta \in \lambda} \alpha_\eta$ and $x \in (T \restriction \alpha) \smallsetminus A$. Then $x \in T \restriction \eta$ for some $\eta \in \lambda$ and $A \cap (T \restriction \eta)$ is a maximal antichain of $T \restriction \alpha_\eta$. So x is compatible with an element of A, hence $A \cap (T \restriction \alpha)$ is a maximal antichain of $T \restriction \alpha$.

Unboundedness. Let $\gamma \in \omega_1$. Since $T \restriction \gamma$ is countable, we can find $\alpha_1 \in \omega_1$ such that

$\alpha_1 > \gamma$ and every element of $T \upharpoonright \gamma$ is compatible with some element in $A \cap (T \upharpoonright \alpha_1)$. Therefore we can construct an increasing sequence $(\alpha_n)_{n\in\omega}$ such that $\alpha_0 = \gamma$ and

$$(\forall x \in T \upharpoonright \alpha_n)(\exists a \in A \cap (T \upharpoonright \alpha_{n+1})[x, a \text{ are compatible }].$$

Let $\alpha = \sup_{n\in\omega} \alpha_n$. Then, by the same argument as in the proof of closure of C, $\alpha \in C$. □

The next claim allows us to assume that $T \upharpoonright \alpha = \alpha$.

Lemma 2.3. *The set $C' = \{\alpha \in \omega_1 \mid \omega^\alpha = \alpha\}$ is a club in ω_1.*

Proof. This follows because the function $f : \omega_1 \longrightarrow \omega_1$ defined by $f(\beta) = \omega^\beta$ is clearly a normal function and hence the set of its fixed points forms a club. □

So, let A be a maximal antichain in T. Recall that $S = \{\alpha \in \omega_1 \mid A \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1. Then there is $\alpha \in C \cap C' \cap S$ such that $A \cap (T \upharpoonright \alpha) = S_\alpha$ is a maximal antichain of $T \upharpoonright \alpha$, but in $\star_1$ and $\star_2$ we have sealed this antichain to stay maximal in $T \upharpoonright (\alpha + 1)$ and hence in T. Hence $A = S_\alpha$ and A is countable.

The last claim of this construction tells us that our tree T does not admit any non-trivial level preserving endomorphism.

Claim 2.4. *If f is a level preserving endomorphism of T, then f is the identity.*

Proof. Let f be a non-trivial level preserving endomorphism of T. Then $f \subseteq \omega_1 \times \omega_1$ and hence the set $G = \{\alpha \in \omega_1 \mid f \upharpoonright \alpha = g_\alpha\}$ is stationary in ω_1. Using **Lemma 2.3** there is $\alpha \in \omega_1$ such that

$$f \upharpoonright (T \upharpoonright \alpha) = g_\alpha.$$

Now, since f is a non-trivial level preserving endomorphism, there is $\gamma \in \omega_1$ such that $f(x) \neq x$ for some $x \in T_\gamma$. Hence $f \restriction (T \restriction \beta)$ is non-trivial for all $\beta > \gamma$, so

$$C_2 = \{\beta > \gamma \mid f \restriction (T \restriction \beta) \text{ is a non-trivial endomorphism of } T \restriction \beta\}$$

is a club. Therefore there is $\alpha \in C' \cap C_2 \cap G$ such that $f \restriction (T \restriction \alpha) = g_\alpha$ is a non-trivial endomorphism of $T \restriction \alpha$.

But during our construction (*case* $\star_\mathbf{3}$ and *case* $\star_\mathbf{4}$) of T, we made sure of stopping g_α (and hence$f$) from being an endomorphism of $T \restriction \alpha$. This gives a contradiction. ☺ □

Using a slight modification of the above argument we can also get a Suslin tree that admits no non-trivial embedding.

Proposition 2.5. *If $\Diamond$ holds, then there is an embedding-rigid Suslin tree.*

Proof. To prove this we modify the construction in **Proposition 2.1** only in *case* $\star_\mathbf{3}$ and *case* $\star_\mathbf{4}$ for $\alpha \in \text{Lim}(\omega_1)$ in the following manner.

If g_α is a non-trivial embedding of $T \restriction \alpha$, we choose an α-branch $b \in [T \restriction \alpha]$ such that b contains a point which is moved by g_α and with the property that $g_\alpha[b] \nsubseteq b$ using the next claim.

Claim 2.6. *There is $y \in T \restriction \alpha$ such that y and $g_\alpha(y)$ are incompatible.*

Proof. First notice that g_α is injective so it has to preserve order-types. Hence g_α can't move points to a lower level, so we can assume there is $x \in T \restriction \alpha$ with $ht(x) < ht(g_\alpha(x))$. Then $x \in T_\xi$ for some $\xi \in \alpha$. Let ξ be minimal with this property, so that g_α is the identity on $T \restriction \alpha$. Let $y > x$ be in the same level as $g_\alpha(x)$. Then $g_\alpha(y) > g_\alpha(x)$ and hence $g_\alpha(y) \in T^{g_\alpha(x)}$ but y is incompatible with $g_\alpha(x)$. Thus y is incompatible with $g_\alpha(y)$. See **Figure 2.3**.

☺

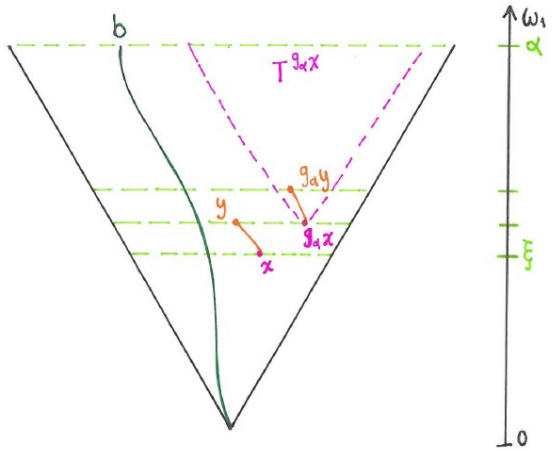

FIGURE 2.3: Picture to illustrate **Claim 2.6**

So, we let b be an α-branch containing y as in **Claim 2.6** and $g_\alpha[b]$ satisfies $g_\alpha[b] \nsubseteq b$. Observe that $g_\alpha[b]$ may not be an element of $[T \upharpoonright \alpha]$, but we can choose an α-branch $b_{g_\alpha(y)}$ containing $g_\alpha[b]$. Then we extend all branches in the set $\{b_t \mid t \in T \upharpoonright \alpha\} \cup \{b\}$ (where b_t contains t) if we are in *case* $\star_\mathbf{3}$ and if we are in *case* $\star_\mathbf{4}$ we extend all branches in the set $\{b_t^* \mid t \in T \upharpoonright \alpha\} \cup \{b\}$ (where b_t^* contains both, t and an element $a \in S_\alpha$), taking elements from W_α as in **Proposition 2.1**.

Therefore, if f is a non-identity embedding of the resulting tree T, then there is a point that is moved by f say at level ξ. But then, there is an $\alpha \in G$, given by our $\Diamond_g$-sequence, where $\alpha > \xi$ and $f \upharpoonright \alpha = g_\alpha$. Using **Lemma 2.3**, we can choose α such that $f \upharpoonright (T \upharpoonright \alpha) = g_\alpha$ is a non-trivial embedding of $T \upharpoonright \alpha$ and we have chosen g_α such that $g_\alpha[b] \subseteq b_{g_\alpha(y)} \nsubseteq b$ for some $b \in [T \upharpoonright \alpha]$. Since g_α preserves order-types, $g_\alpha(\bigcup b) \in T_\beta$ for some $\beta \geqslant \alpha$. But g_α is an embedding of $T \upharpoonright \alpha$, hence $g_\alpha[\bigcup b] = \bigcup b_{g_\alpha(y)}$ and we have defined T_α so that b is extended but not $b_{g_\alpha(y)}$, hence $g_\alpha = f \upharpoonright (T \upharpoonright \alpha)$ cannot be an embedding, giving us a contradiction. □

The next result shows that there is a Suslin tree that does not admit any isomorphism between cones.

Proposition 2.7. *Assume $\Diamond$. Then there is a totally rigid Suslin tree $\mathcal{T}$.*

Proof. For this proof, we will use the following principle,

$\Diamond_k$: There is a sequence $(K_\alpha \mid \alpha \in \omega_1)$ such that $K_\alpha \subseteq \alpha \times \alpha$ and for any subset $X \subseteq \omega_1 \times \omega_1$ the set $K = \{\alpha \in \omega_1 \mid X \cap (\alpha \times \alpha) = K_\alpha\}$ is stationary in ω_1.

The $\Diamond_k$ principle is equivalent to $\Diamond$. This equivalence can be seen in the proof of **Lemma 2.25**. Once more, we will modify the proof of **Proposition 2.1** altering only *case* $\star_\mathbf{3}$ and *case* $\star_\mathbf{4}$, which is where $\Diamond_k$ will take the role of $\Diamond_g$.

So, assume $\alpha \in \mathrm{Lim}(\omega_1)$ and we are trying to define which branches to extend in T_α. We look at our $\Diamond_k$-sequence, and if for two different $x, y \in T \restriction \alpha$, K_α is a well-defined isomorphism between $(T \restriction \alpha)^x$ and $(T \restriction \alpha)^y$, that is,

$$K_\alpha = \{\langle z, f_\alpha(z)\rangle \in (T \restriction \alpha)^2 \mid z \geqslant x \wedge f_\alpha(z) \geqslant y\}$$

for some isomorphism f_α, then we let $b \in [T \restriction \alpha]$ be a branch containing x and $b_{f_\alpha} \in T \restriction \alpha$ a branch containing $f_\alpha[b \cap (T \restriction \alpha)^x]$. Using **Claim 2.6** we can choose b such that $b \neq b_{f_\alpha}$.

If we are in *case* $\star_\mathbf{3}$, then S_α is not a maximal antichain of $T \restriction \alpha$ and we use the element of W_α to extend all branches in the set $\{b_t \mid t \in T \restriction \alpha\} \cup \{b\}$, where b_t is any α-branch containing t. Since $T \restriction \alpha$ is a normal ω-splitting tree, we can choose these b_t so that $b_t \neq b_{f_\alpha}$.

If S_α is a maximal antichain of $T \restriction \alpha$, then we are in *case* $\star_\mathbf{4}$. Here we extend all branches in the set $\{b_t^* \mid t \in T \restriction \alpha\} \cup \{b\}$, where $t \in b_t^* \in [T \restriction \alpha]$, taking elements from W_α. Using the same argument as in *case* $\star_\mathbf{4}$ in **Proposition 2.1**, we can choose $b_t^* \neq b_{f_\alpha}$ so that it contains an element of S_α. Then $T = \bigcup_{\alpha \in \omega_1} T \restriction \alpha$, as usual.

To see that our construction is enough, assume $f : T^x \longrightarrow T^y$ is an isomorphism, for distinct $x, y \in T$. Then $x \in T_\xi$, $y \in T_\eta$, $f(x) = y$ and without loss of generality we can assume $\xi < \eta$.

Now, notice that f will eventually fix a higher level: let $\beta \in \omega_1$ be the unique ordinal satisfying $\xi + \beta = \eta$ and set $\gamma = \xi + \omega^{\beta+1}$. Then T_γ is the desired level: let $z > x$ and $z \in T_\gamma$. Then $ot(x) < ot(z) = \xi + \omega^{\beta+1}$ and since f is an isomorphism, $ot(f(z)) = \eta + \omega^{\beta+1}$. Hence

$$\gamma = \xi + \omega^{\beta+1} = \xi + (\beta + \omega^{\beta+1}) = (\xi + \beta) + \omega^{\beta+1} = \eta + \omega^{\beta+1},$$

since $\beta \in \omega^{\beta+1}$ implies $\beta + \omega^{\beta+1} = \omega^{\beta+1}$.

Since f is an isomorphism it will preserve levels at every level T_α for $\alpha > \gamma$. Thus, $f[T^x \restriction \alpha] = T^y \restriction \alpha$ so $f \restriction \alpha$ is an isomorphism between the cones $T^x \restriction \alpha$ and $T^y \restriction \alpha$, for every $\alpha > \gamma$. Now we use $\Diamond_k$ together with **Lemma 2.3** to find $\delta \in \omega_1$, $\delta > \gamma$ such that $f \cap \delta = K_\delta$ and such that $f \cap \delta = f \restriction \delta$. Thus $f \restriction \delta = f_\delta$ is an isomorphism between $T^x \cap \delta$ and $T^y \cap \delta$ and we are now in *case* $\star_4$, where we constructed our tree such that f_δ cannot be extended to an isomorphism of T^x and T^y, contradicting our original assumption. □

After looking at the above constructions, we could try modifying them so that we have a Suslin tree with a non-identity embedding while preserving its rigidity with respect to automorphisms. There are simple ways of giving embeddings to the tree; we start with a node, add ω-many immediate successors and put Jensen's rigid tree T above each of them, then we get a Suslin tree with an embedding sending each copy of T to the right. However, this also has many automorphisms (See **Figure 2.4**).

Moreover, Jensen already constructed a Suslin tree with exactly two automorphisms, in a slightly less trivial manner which he used to give the associated Suslin line a reversible ordering. The idea is to get rid off all the unwanted automorphisms in the same way that we got rid of the non-identity ones, by not extending the images of some branch under the unwanted automorphisms and making sure we close under the automorphism we wish to preserve. This clearly works if we want to preserve any finite number of automorphisms but we may run into trouble if we try to use the same technique to preserve countably many of them, a problem that is consistent with the result of [Jec72] stating that a normal ω_1-tree can only have either finitely many automorphisms or between $2^{\aleph_0}$ and $2^{\aleph_1}$, inclusive.

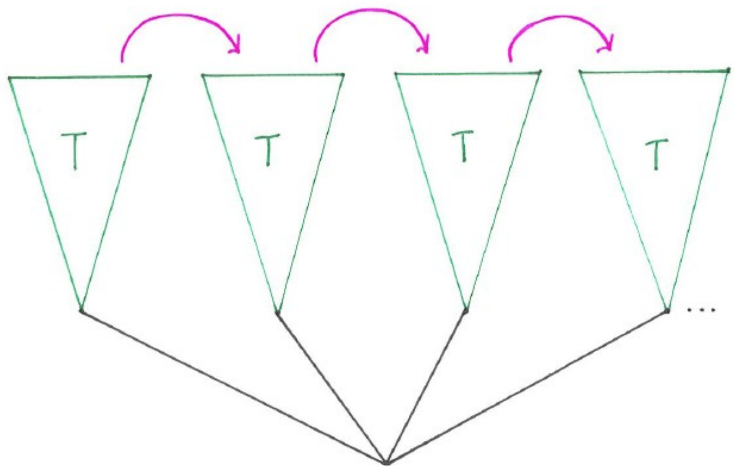

FIGURE 2.4: Suslin tree with embedding and automorphism monoids non-trivial.

As for an example of a Suslin tree that admits exactly $\aleph_1$ automorphisms, Jensen constructed a homogeneous Suslin tree (where **homogeneous** means that for any two points $x, y \in T_\alpha$ and any $\alpha \in \omega_1$, there is an automorphism of $(T, \leqslant_T)$ that sends x to y). We will sketch the construction to illustrate the method.

The construction is also done by transfinite induction on the levels of the tree and we regard a point $x \in T$ at level $\alpha < \omega_1$ as an α-binary sequence (that is, an element of 2^α). For T_0 and successor levels the construction is exactly the same as we have seen, and at limit stages $\alpha \in \mathrm{Lim}(\omega_1)$ we use $\diamondsuit$ to make the resulting tree Suslin (as in **Proposition 2.1** for *case* $\star_1$ and *case* $\star_2$) and in addition to the usual construction we choose a branch, $\mathbf{b}$ in $[T \upharpoonright \alpha]$ and we extend all branches in the set

$$B_\alpha = \{d \in [T \upharpoonright \alpha] \mid d \text{ and } \mathbf{b} \text{ differ only in an initial segment } \}$$

So we choose to adjoin a point above *all* those branches that are eventually the same as $\mathbf{b}$. The final tree is $T = \bigcup_{\alpha<\omega_1} T_\alpha$.

To see that this tree is indeed homogeneous, we notice that if two α-branches differ only in an initial segment then they differ in finitely many entries (as sequences), and then the tree T is closed under the automorphisms f_N, for finite $N \subseteq \omega_1$, defined by,

$$f_N(x(\nu)) = \begin{cases} x(\nu) & \text{if } \nu \notin N \\ 1 - x(\nu) & \text{if } \nu \in N \end{cases}$$

Recalling that $\Diamond$ implies **CH**, we get $\aleph_1$ automorphisms.

2.2 Suslin Lines

In 1920 there appeared for the first time a problem that is now known as the Suslin Problem, due to a Russian mathematician called Mikhail Suslin [Mik20]. The problem asks whether the following statement is true:

SH - *Every complete dense linear order without endpoints and with the countable chain condition*[‡] *is isomorphic to the real line*

This assertion is known as **Suslin's Hypothesis** (**SH**). This was a natural question since Cantor proved that we can characterize the real line as the unique complete separable dense linear order without endpoints. Then the hypothesis asks whether we can weaken the requirement of separability to that of the *c.c.c.*

A counterexample to **SH** is called a Suslin line (SL) and the existence of such a structure is equivalent to the existence of a Suslin tree (ST) and hence independent of **ZFC**. We will prove this equivalence for completeness.

Lemma 2.8 (Kurepa)**.** *There is a Suslin tree if and only if there is a Suslin line.*

Proof. First we'll show how to get from a SL to a ST following a very standard construction.

[‡]Here the *countable chain condition (c.c.c)* states that every family of pairwise disjoint open intervals is at most countable.

Let $\mathbb{S}$ be a SL. We will construct a binary ω_1-normal ST by recursively defining the levels of the tree. The tree will consist of closed intervals of $\mathbb{S}$ ordered by reverse inclusion.

- $T_0 = \{\mathbb{S}\}$,
- If $\alpha = \beta + 1$, then for each interval $I \in T_\beta$ we choose I_0, I_1 such that $I = I_0 \cup I_1$ and $I_0 \cap I_1 = \varnothing$, and let $T_\alpha = \bigcup\{\{I_0, I_1\} \mid I \in T_\beta \text{ and } |I| > 1\}$,
- If $\alpha \in \mathrm{Lim}(\omega_1)$, then

  $T_\alpha = \{\bigcap b \mid b \subset \bigcup\{T_\beta \mid \beta < \alpha\},\ \text{for all } \beta < \alpha, b \cap T_\beta \neq \varnothing,\ \text{and } \bigcap b \neq \varnothing\}$

Then $T = \bigcup\limits_{\alpha<\omega_1} T_\alpha$. Now we assume that there is an uncountable branch $\mathbf{b} = \{I_\alpha \mid \alpha < \omega_1\}$ in T and let $B = \{a_\alpha \mid \alpha < \omega_1\}$ be the set of left points of the elements of $\mathbf{b}$. Then, since two intervals are comparable if one of them contains the other, then B is a strictly increasing sequence, giving rise to uncountably many disjoint open intervals in $\mathbb{S}$.

Moreover, if we have an uncountable antichain $A = \{I_\alpha \mid \alpha < \omega_1\}$ in T, then each I_α contains an open interval (a_α, b_α) so that $\{(a_\alpha, b_\alpha) \mid \alpha \in \omega_1\}$ is an uncountable set of pairwise disjoint open intervals in $\mathbb{S}$. The fact that the height of T is indeed ω_1 comes from the remark that each level of T forms an antichain and that each level of T is countable. Therefore T is a ST.

Next, let T be a normal ST. The resulting SL, $\mathbb{S}$, will consist of branches of T ordered lexicographically,

Definition 2.9. *Let $(T, \leqslant_T)$ be a tree.*

*a) The **lexicographical ordering**, $\preceq_{lex}$ of $[T]$ is defined as follows for $b, d \in [T]$; given an increasing ordering to the right on each level of T, let s be the least point where b and d differ and s_0, s_1 the two immediate successors of s. Then $b \preceq_{lex} d$ iff $s_0 \in b$.*

b) The ***lexicographical ordering****, $\preceq_{lex}$ of T is the ordering: for $t, s \in T$, $s \preceq_{lex} t$ iff either $s \leqslant_T t$ or $s_\downarrow \preceq_{lex} t_\downarrow$.*

The lexicographical ordering on the levels of T that we use is the one arising when we order the successors of every node as elements of $\mathbb{N}$. Then, $\mathbb{S}$ is a complete linearly ordered dense set. It is clearly a linear ordering and if we insist on eliminating the branch of T containing only zeros (as elements of $\mathbb{N}$) then it has no end points. For completeness, let $A \subseteq \mathbb{S}$ be a subset bounded by B. We shall construct a least upper bound $b = \{b_\nu : \nu < \gamma\}$ as a branch in T by recursion on $\gamma < \omega_1$.

For $b_0 = root$. If ν is a successor, then $b_\nu = \max\{x \in T_\nu \cap A\}$, which exists as A is bounded, then $T_\nu \cap A$ is bounded by $T_\nu \cap B$. Notice that $b_\beta < b_{\beta+1}$, since otherwise $b_{\nu+1} > x$ for all $x \in T_{\nu+1} \cap A$ and $x > b_\nu$ (this is possible since we are in a successor level and b_ν belongs to a branch in A), but then the immediate predecessor of $b_{\nu+1}$,y will be greater than b_n but $b_{\nu+1} \in A$ and so must y, contradicting maximality of b_ν.

If $\nu \leqslant \mathrm{Lim}(\gamma)$, we look at our already constructed $b \restriction \nu$. If $b \restriction \nu$ has no extension on T_ν, then $b \restriction \nu$ is a maximal chain in T and hence it is an element of $\mathbb{S}$, so we let b be this branch.

Claim 2.10. *For every $s \in A$, this branch b satisfies $b \geqslant s$.*

Proof. Let $s \in A$, then by construction $s_\beta \leqslant_{lex} b_\beta$ for $s_\beta, b_\beta \in \omega$ and $\beta < \nu$ and $s_\nu \leqslant_{lex} b_\beta$ for every $\beta < \nu$, otherwise $s_\nu >_{lex} b_\beta$ and hence $s > b$ in $\mathbb{S}$ by normality of T, so there is $\eta < \nu$ such that $s_\eta > b_\eta$ contradicting maximality of b_η. ☺

Otherwise, $b \restriction \nu$ has an extension on T_ν, and we want to let this extension be b_ν, and by the last claim $b_\nu \geqslant_{lex} x_\beta$, for every $x_\beta \in x \in A$.

Claim 2.11. $b_\nu \in A \cap T_\nu$.

Proof. Let $s^*_\nu = \max\{s_\nu \mid s \in A\}$, which exists since $B \cap T_\nu$ is a upper bound of $\{s_\nu \mid s \in A\}$. Then by the claim above we have that $s^*_\nu \leqslant_{lex} b_\nu$, so let's assume $s^*_\nu <_{lex} b_\nu$.

Then, by normality of T, there is $t \in T_\eta$, for $\eta < \nu$ such that $t = \max\{s^*_\beta \mid s^*_\beta = b_\beta\}$ so that $s^*_{\eta+1} <_{lex} b_{\eta+1} = \max\{x \in A \cap T_{\eta+1} \in A\}$. So there is a branch containing $b_{\eta+1}$, say s, and hence $s >_{lex} s^*$ in $\mathbb{S}$. Therefore $s_\nu > s^*_\nu$ since $s \neq s^*$ below T_ν, giving a contradiction - See **Figure 2.5**. So $s^*_\nu = b_\nu$. ☺

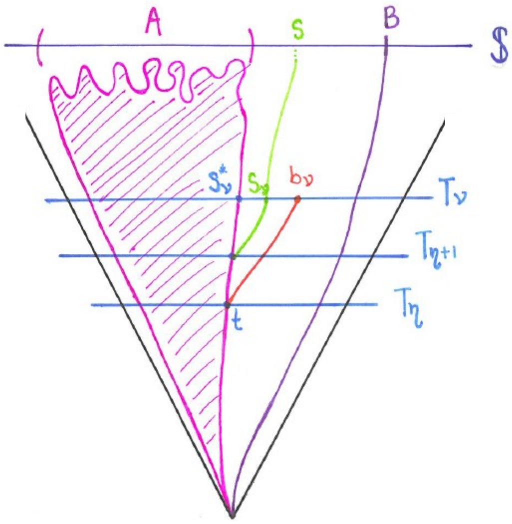

Figure 2.5: Picture to illustrate **Claim 2.11**.

This concludes our construction of $b = \sup A$. Notice that it must terminate at some level $\gamma < \omega_1$ since T is Suslin and $b \cap T_\nu$ is contained always in some element of A which means b belongs to A.

To show $\mathbb{S}$ is dense, let $s < t$ for $s \neq t$ in $\mathbb{S}$. Then $s <_{leq} t$ are two elements of $[T]$ and hence there is $\rho \in T_\eta$ such that $\rho = \max\{s_\beta \mid s_\beta = t_\beta\}$ for some $\eta < \min\{ht(s), ht(t)\}$. Then $\rho^\frown s_{\eta+1} \in s$ and $\rho^\frown t_{\eta+1} \in t$, for $s_{\eta+1}, t_{\eta+1} \in \mathbb{N}$, and $s_{\eta+1} <_{lex} t_{\eta+1}$. If $t_{\nu+1} >_{lex} s_{\eta+1} + 1$, we can find $a \in (s_{\eta+1}, t_{\eta+1})$ and we let s^* be any branch containing a. Otherwise $t_{\nu+1} = s_{\eta+1} + 1$ and we let s^* be a branch containing $\rho^\frown s_{\eta+1}{}^\frown b$ for $b >_{lex} s_{\eta+2}$. Then $s < s^* < t$ in $\mathbb{S}$.

It has also the *c.c.c.*, for if (s,t) is an open interval in $\mathbb{S}$, there is $u \in T$ such that $I_u = \{x \in \mathbb{S} \mid u \in x\}$ is an open interval contained in (s,t), and if $I_u \cap I_v = \varnothing$ then u, v are incomparable. So every uncountable family of disjoint open intervals in $\mathbb{S}$ gives rise to an uncountable antichain in T.

Furthermore, $\mathbb{S}$ is not separable. If A is a set of countable subset of $[T]$ let α be an ordinal which is above all branches in A, then if $s \in T_\alpha$, I_x is an open interval on $\mathbb{S}$ that does not contain any element on A, hence A cannot be dense. □

With this result we transform **SH** into a purely combinatorial problem since a Suslin line has the *c.c.c.* if and only if the corresponding Suslin tree satisfies the *c.c.c.*. Seeing this close relation between Suslin trees and lines, we can ask what other properties are preserved from the line to the tree and vice versa or in what way these properties are manifested in both structures.

In this sense, much has been done in terms of the automorphism group of both of them [DJ74], [FH09], [Jec72] and in this section we will investigate other monoids associated with the line (e.g. embeddings and epimorphisms).

Let's remark that if in the above construction of a Suslin line from the tree we also insist in not adding to the line the branches which contain a subchain of the form $s^\frown(0,0,0,...)$, for $s \in [T \upharpoonright \alpha]$ for some $\alpha < \omega_1$ and $(0,0,0,...)$, a chain of limit height containing only zeros, then we can also make the set $S_\alpha = \{s \upharpoonright \alpha \in [T \upharpoonright \alpha] \mid s \in \mathbb{S}\}$ an ordered set without endpoints. Notice that this won't interfere with $\mathbb{S}$ being complete or even dense. If instead of an $\aleph_0$-splitting tree we want a n-splitting one for $n \in \omega$, then in addition and in order to make the resulting tree dense, we would have to identify 'adjacent' branches, that is if s is a node in T, then only one of $s^\frown(m, n-1, n-1, ...)$ or $s^\frown(m+1, 0, 0, ..)$ will be element of $\mathbb{S}$.

Notice that when defining the lexicographical ordering on T to construct the line in **Lemma 2.8**, we could have asked for the ordering of the immediate successors of a node in T to be that of $\mathbb{Z}$ instead, and it will still work, but if we try to use $\mathbb{Q}$ we would have problems with completeness. In fact, the resulting Suslin line $\mathbb{S}$ will not be complete since we can't define a least upper bound at successor levels of T.

The following two lemmas (taken from [Tod84], but also appearing in [DJ74]) show how can we relate automorphisms of Suslin tree and of lines, but actually **Lemma**

2.12 works for any normal Aronszajn tree.

Lemma 2.12. *Let $C \subseteq \omega_1$ be a club. Then every lexicographical automorphism σ of $T \upharpoonright C$ determines a unique automorphism $\overline{\sigma}$ of the corresponding Suslin line.*

Lemma 2.13. *If T is a Suslin tree and f is an automorphism of the corresponding Suslin line, then there is a club $C \subseteq \omega_1$ and a lexicographical automorphism σ of $T \upharpoonright C$ such that $f = \overline{\sigma}$.*

The first result is actually a particular case of the following result that we extend and prove here.

Lemma 2.14. *Let $C \subseteq \omega_1$ be a club. If σ is a lexicographical level preserving epimorphism on $T \upharpoonright C$, then there is a unique epimorphism $\overline{\sigma}$ on the corresponding Suslin line $\mathbb{S}$ constructed using T.*

Proof. Without loss of generality we can assume C contains only limit ordinals. Let $\alpha \in C$ and let $S_\alpha = \{b \upharpoonright \alpha \mid b \in \mathbb{S}\}$ be the set of all α-initial segments of branches that we decided to add to $\mathbb{S}$ to make it a linear continuum. Then S_α is itself an ordered continuum. We have remarked before that it is an ordered set without endpoints. It is complete by construction and the same argument in **Lemma 2.12** used to prove that $\mathbb{S}$ is dense, applies to show that S_α is dense.

Since in addition T_α is dense in S_α, S_α is isomorphic to $\mathbb{R}$. To show that the α-th level of T is indeed dense in S_α, let $s <_{lex} t$ be in S_α and assume towards a contradiction that there is no $b \in T \upharpoonright \alpha$ such that $b \in (s, t)$ and $b \cap T_\alpha \neq \emptyset$. Let s_0, t_0 be the first two points where they differ, and let $x >_{lex} s_1 = s_0^\frown a$, for some $a \in \mathbb{N}$ and $s_1 \in s$. Then, since no branch between s and t is extended to level T_α, this violates the normality of T.

Therefore S_α is isomorphic to $\mathbb{R}$ and thus, by **Lemma 4.4** there is a unique epimorphim $\overline{\sigma}_\alpha$ on S_α extending $\sigma \upharpoonright T_\alpha$.

Claim 2.15. *If $\alpha < \beta$, then $\overline{\sigma}_\alpha \upharpoonright (S_\alpha \setminus T_\alpha) \subseteq \overline{\sigma}_\beta(S_\beta \setminus T_\beta)$.*

Proof. Let $s \in S_\alpha \smallsetminus T_\alpha$ and such that $s = \sup\{s_n \in T_\alpha \mid n \in \omega\}$, then s is also the supremum of the set $\{s'_n \in T_\beta \mid n \in \omega\}$, where $s_n < s'_n$ (possible since T_β is dense in S_β and the branches extending each s_n form an open interval on S_β) and these supremums are taken from the lexicographical ordering that we have on the T. This is because $\overline{\sigma}_\alpha$ is continuous. Hence $\overline{\sigma}_\alpha(s) = \sup\{(\sigma \upharpoonright T_\alpha)(s_n) \mid n \in \omega\}$ $= \sup\{(\sigma \upharpoonright T_\alpha)(s'_n) \mid n \in \omega\} = \overline{\sigma}_\beta(s)$. ☺

Therefore $\overline{\sigma} = \bigcup_{\alpha \in C} \overline{\sigma}_\alpha(S_\alpha \smallsetminus T_\alpha)$ is an epimorphism of $\mathbb{S}$. □

In general it is not the case that every map on T whose restriction to a club set is an automorphism is also an automorphism of T.

Proposition 2.16. *Let $C \subseteq \omega_1$ be a club. There is a normal ω_1-tree $(T, \leqslant)$ with a map $\sigma_C : T \longrightarrow T$ such that $\sigma_C \upharpoonright (T \upharpoonright C)$ is a non-trivial automorphism of $T \upharpoonright C$, but σ_C is not an automorphism of T.*

Proof. Let σ be any non-trivial automorphism of T, $C = \{c_\alpha \mid \alpha \in \omega_1\}$ a club on ω_1 and assume without loss of generality that it consists only of limit ordinals.

Then we can define $\sigma_C : T \longrightarrow T$ to agree with σ in $T \upharpoonright C$ and if s is not in $T \upharpoonright C$ then $s \in T^{s_0} \cap (T \upharpoonright c_{\alpha+1})$ for some s_0, in $T \upharpoonright c_\alpha$ and we let $\sigma_C(s) = \sigma(s_0)$.

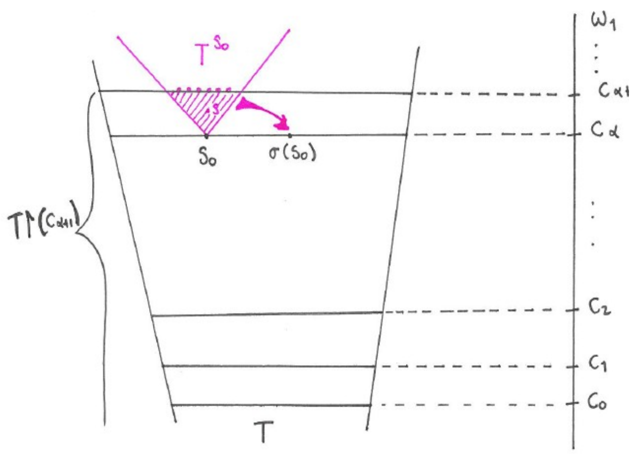

FIGURE 2.6: σ acting on T.

Then σ_C is clearly not an automorphism of T but it restricts to the automorphism of $T \upharpoonright C$. $\square$

It is clear that an automorphism of $(T, \leqslant_T)$ cannot be an automorphism of $(T, \preceq_{lex})$, since the latter is a linear ordering and we always have incomparable elements with respect to $\leqslant_T$.

So any automorphism of a Suslin tree T that cannot be extended to a lexicographical automorphism of $T \upharpoonright C$ cannot be an automorphism of a Suslin line $\mathbb{S}$. Therefore not every automorphism of T will give rise to an automorphism of the corresponding $\mathbb{S}$.

Also, if σ is a lexicographical automorphism of $T \upharpoonright C$, (where T is a normal tree), then it *can* be extended to a lexicographical automorphism of T and if we have a lexicographical automorphism of T then the restriction to $T \upharpoonright C$ is clearly a lexicographical automorphism. But if σ is an automorphism of $(T \upharpoonright C, \preceq_{lex})$ not every extension of σ will be an automorphism of $(T, \preceq_{lex})$, so we cannot in general compare the automorphism group of T with the one of the corresponding line, and looking at automorphisms on a club subset of the tree is enough to determine an automorphism of the line.

2.3 An automorphism rigid Suslin tree that admits a non-identity embedding

Now, we will use the construction of a rigid Suslin tree $\mathcal{T}$ to get a tree $\mathbb{T}$ that admits an embedding to a proper cone, that is $\mathbb{T} \cong \mathbb{T}^{x_0}$ for some $x_0 \in \mathbb{T}_\xi$ with $\xi \in \mathrm{Lim}(\omega_1)$.

Proposition 2.17. *If $\Diamond$ holds, then there is a Suslin tree $\mathbb{T}$, with trivial automorphism group but with a non identity embedding.*

Proof. Let $\mathcal{T}$ be the totally rigid Suslin tree of **Proposition 2.7** and fix $x_0 \in \mathcal{T}_\xi$

for $\xi \in \mathrm{Lim}(\omega_1)$. Let $U_0 = \mathcal{T} \setminus \mathcal{T}^{x_0}$. The desired Suslin tree will be the union of countably many sets U_n defined as follows (see **Figure 2.7**),

$U_n = \{\bigcup (x_{0\downarrow})^{n\frown} s_\downarrow \mid s \in U_0\}$, where $(x_{0\downarrow})^n$ is the concatenation of $x_{0\downarrow}$, n times.

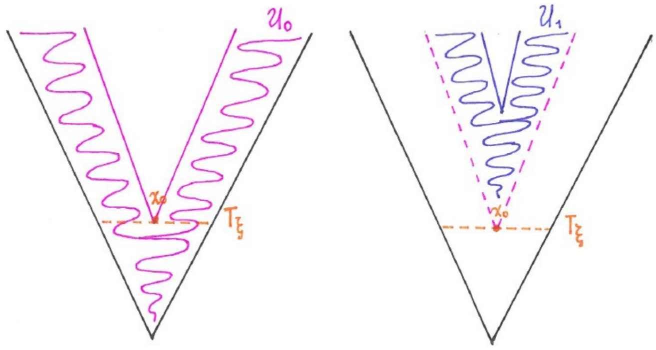

FIGURE 2.7: U_0 and U_1

Notice that we still need to take care that the elements we put above the branches $(x_{0\downarrow})^{n\frown} s_\downarrow$ are ordinals chosen in such a way that $\mathbb{T} \upharpoonright \alpha$ is an initial segment of ω_1, for every $\alpha \in \omega_1$. Then $T = \bigcup_{n \in \omega} U_n$.

Notice that since $(x_{0\downarrow})^\omega \notin T$, T is clearly a normal ω_1-tree, being the countable union of ω_1-subtrees taken from $\mathcal{T}$, which is normal.

To see that it is indeed a Suslin tree, we need only to verify that any antichain in $\mathbb{T}$ is at most countable. Since U_0 is a subset of $\mathcal{T}$ it has no uncountable antichains, and therefore, every antichain of U_n is at most countable, for all $n \in \omega$. Hence every antichain of $\mathbb{T}$ must be at most countable, being the countable union of antichains that are at most countable. Now that we have a Suslin tree we will equip it with the natural embedding, the one sending $\mathbb{T}$ to the cone above x_0.

Claim 2.18. *The tree $\mathbb{T}$ admits a (continuous) non-identity embedding.*

Proof. We will define a function σ from $\mathbb{T}$ into $\mathbb{T}^{x_0}$ that lifts the set U_n to U_{n+1} in the obvious way:

$$\text{If } x \in U_n \text{ then } \sigma(x) = \bigcup (x_{0\downarrow})^\frown x_\downarrow,$$

This is evidently an injective function and it is additionally a continuous function. For let $(x_\nu : \nu \in \gamma)$ be a sequence of length $\gamma \in \mathrm{Lim}(\omega_1)$ with $\bigcup_{\nu\in\gamma} x_\nu = x$. Then,

$$\begin{aligned}\bigcup_{\nu\in\gamma} \sigma(x_\nu) &= \bigcup_{\nu\in\gamma} (x_{0\downarrow})^\frown x_{\nu\downarrow} \\ &= (x_{0\downarrow})^\frown (\bigcup_{\nu\in\gamma} x_{\nu\downarrow}) \\ &= (x_{0\downarrow})^\frown x_\downarrow \\ &= \sigma(x)\end{aligned}$$

To see that is order preserving let $x \in U_n$ and $y > x$. Then either $y \in U_n$ or $x = \bigcup (x_{0\downarrow})^{n+1}$. Hence

$$\sigma(x) = \bigcup (x_{0\downarrow})^\frown x_\downarrow \wedge \sigma(y) = \bigcup (x_{0\downarrow})^\frown y_\downarrow$$

and thus $\sigma(x) < \sigma(y)$, or $x = \bigcup (x_{0\downarrow})^{n+1}$ and $\sigma(x) = \bigcup (x_{0\downarrow})^{n+2}$ and $y \in U_m$ for some $m \geqslant n+1$, so $\sigma(x) < \sigma(y)$. ☺

In view of the above claim, we remark that in **Proposition 2.5** we showed that there is a level that is fixed setwise. In this case, this level is the $\xi.\omega$-the level of T: say $x = (x_{0\downarrow})^{n\frown} s_\downarrow \in U_n$ for $n \in \omega$ and $s \in U_0$, then x has order type $ot(x) = \xi.n + ot(s) = \xi.\omega$ and hence $ot(s) = \xi.\omega$. But σ is an injective function, so $ot(\sigma(x)) = \xi.(n+1) + ot(s) = \xi.(n+1) + \xi.\omega = \xi.\omega$, since $\sigma(x)$ is in U_{n+1}. Therefore $\sigma(x)$ is also in level $\xi.\omega$. Then, by continuity σ will fix all levels above $\xi.\omega$ and no level will be fixed below it.

Now, the tree T not only satisfies that the embedding monoid is non-trivial, but the automorphism group stays trivial.

Claim 2.19. *The tree $\mathbb{T}$ is automorphism rigid.*

Proof. Let θ be a non-identity automorphism of $\mathbb{T}$. Then we can find $x \in U_n$ with $\theta(x) \neq x$ and by the above remark also both in the same limit level.

Now, if $\theta(x)$ and x are always in the same U_n, say $x = \bigcup (x_{0\downarrow})^{n\frown} s_1$ and $\sigma(x) = \bigcup (x_{0\downarrow})^{n\frown} s_2$ for two distinct $s_1, s_2 \in U_0$, then s_1, s_2 are also in the same level. Hence

θ takes s_1 to s_2, and the cones above s_1 and s_2 are isomorphic. Therefore we can find an $\alpha \in K$ such that $K_\alpha = \sigma \upharpoonright \alpha$ where we made sure in **Proposition 2.7**, that σ cannot be extended to be an isomorphism in the α-th level.

If on the other hand there is $x \in U_n$ with $\theta(x)$ in U_m for some $m \neq n$, let n be the least with this property. Assume without loss of generality that $n < m$, and let $l > 0$ be the difference between them. Then for s_1 and s_2 in U_0,

$$x = \bigcup (x_{0\downarrow})^{n\frown} s_{1\downarrow} \text{ and}$$
$$\theta(x) = \bigcup (x_{0\downarrow})^{n\frown} (x_{0\downarrow})^{l\frown} s_{2\downarrow}$$

Notice that $(x_{0\downarrow})^n$ and $(x_{0\downarrow})^l$ must be fixed by the automorphism θ since x_0 is in U_0 which is rigid. Therefore

$$\theta(s_{1\downarrow}) = (x_{0\downarrow})^{l\frown} s_{2\downarrow} \text{ so for some } s_3 \in U_0$$
$$s_{1\downarrow} = (x_{0\downarrow})^{l\frown} s_{3\downarrow}$$

But s_1 is in U_0 and hence it cannot extend x_0, contradiction. Therefore θ cannot be a non-identity automorphism of $\mathbb{T}$. ☺ □

2.4 An automorphism rigid Suslin tree admitting a non-trivial level preserving epimorphism

We start the construction of a Suslin tree by transfinite induction on the levels of the tree and at the same time we will be defining a function σ that will be our desired non-trivial epimorphism. The epimorphism will violate injectivity 'at the beginning' and the effect will 'propagate' throughout the entire tree. However, by the usual method we can ensure (automorphism-)rigidity of our resulting tree. (See **Figure 2.8**). We will construct a normal $\aleph_0$-splitting tree whose elements we will regard as countable sequences with entries in ω (to facilitate the definition of σ), but also as elements of ω_1 (to make easier the construction of a normal ST) as we have been using during this section and the distinction should be clear from context.

Let $T_0 = \{root\}$, or the empty sequence and $\sigma(root) = root$.

Having defined T_β and σ on T_β, for every $\beta < \alpha$ and some $\alpha \in \omega_1$, let $T_{\alpha+1} = \{x^\frown k \mid x \in T_\alpha, k \in \omega\}$ and if $x^\frown k = t \in T_{\alpha+1}$ then we define $\sigma(t) = \sigma(x)^\frown(k \dot{-} 1)$ where,

$$k \dot{-} 1 = \begin{cases} 0 & \text{if } k = 0 \\ k - 1 & \text{if } k > 0 \end{cases}$$

From this it follows that $\sigma^m(t) = \sigma(x)^{m\frown}(k \dot{-} m)$ and $\sigma^{-m}(t) = \sigma(x)^{-m\frown}(k \dot{+} m)$ where,

$$k \dot{-} m = \begin{cases} 0 & \text{if } k \in \{0, 1, ...m\} \\ k - m & \text{if } k > m \end{cases} \qquad k \dot{+} m = \begin{cases} \{0, 1, ...m\} & \text{if } k \in \{0, 1, ...m\} \\ k + m & \text{if } k > m \end{cases}$$

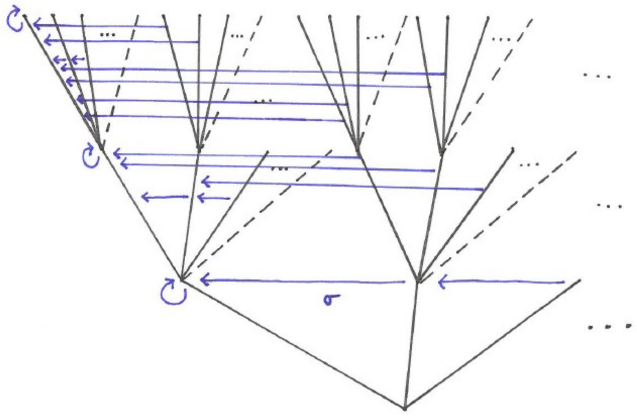

FIGURE 2.8: σ acting on T

Now, if $\alpha \in \mathrm{Lim}(\omega_1)$, we will choose which α-branches we want to extend in order to maintain $T \restriction (\alpha + 1)$ closed under σ and σ surjective, so it will have to satisfy these two conditions:

1. For every $t \in T \restriction (\alpha + 1)$, $\sigma(t) \in T \restriction (\alpha + 1)$ for every positive n.

2. For every $t \in T \restriction (\alpha + 1)$ there is $s \in T \restriction (\alpha + 1)$ such that $\sigma(s) = t$.

In addition, we want to get rid of every non-identity automorphism on $T \restriction (\alpha + 1)$, so we look at the α-th element of the $\Diamond$-sequence $(g_\nu : \nu \in \omega_1)$ in order to ensure rigidity and then we take care of sealing maximal antichains with the help of the α-th element in the $\diamondsuit$-sequence $(S_\nu : \nu \in \omega_1)$. We shall do this in cases, but first let's define what we mean by the set $\sigma^{-m}[b]$ for every $m \geqslant 1$, where b is an α-branch, which we define by induction on the elements of b; first we will define what $\sigma^{-1}[b]$ is for an α-branch and then we extend to every $m > 1$. The idea is that this set consists of countably many *branches* whose restriction to a level $\beta < \alpha$ contains only elements of $\sigma^{-m}(x)$ for $x \in b$ in level β.

For $x = root$, $\sigma^{-1}(root) = root$.

Having defined $\sigma^{-1}[b \cap T_\beta]$ for every $\beta < \alpha$ (that is, $\forall w \in b \cap (T \restriction \alpha)$) we want now to define $\sigma^{-1}[b]$:

a. If $\alpha = \beta + 1$. If x is in the α-th level of b, that is if $x \in T_\alpha \cap b$, then $\sigma^{-1}(x) \subset T_\alpha$ and choose elements in this set such that for each $y \in \sigma^{-1}(w)$ already chosen for $w \in T_\beta \cap b$, pick two different elements y_1, y_2 in $\sigma^{-1}(x)$ (or one if there is only one element) such that they both extend y, and let y_1, y_2 be the elements of $\sigma^{-1}[b \cap T_\alpha]$. Notice that y_1 and y_2 are quite arbitrary so we have some freedom to choose them carefully, freedom that we may use further in the construction.

b. If $\alpha \in \mathrm{Lim}(\omega_1)$, let $\sigma^{-1}[b]$ consist of countably many branches b_x in $\sigma^{-1}[b \cap T \restriction \alpha]$ for every $x \in T \restriction \alpha$, such that $x \in b_x$.

Now, once we have defined what $\sigma^{-1}[b]$ is for $\alpha \in \omega_1$, we assume we have the definition of $\sigma^{-p}[b]$ for $p = m - 1$. We look at every branch $d \in \sigma^{-p}[b]$ and define $\sigma^{-1}[d]$ as above. Thus, the following is clearly countable

$$\sigma^{-m}[b] = \bigcup_{d \in \sigma^{-p}[b]} \sigma^{-1}[d].$$

Hence, if we include a branch d in $\sigma^{-(m+1)}[b]$ for some α-branch b, we have ensured that $\sigma[d]$ is in $\sigma^{-m}[b]$

$\star_1$. ***S_α is not a maximal antichain and g_α is the identity automorphism on $T \upharpoonright \alpha$.***

In this case we do not need to take care of either of S_α or g_α so we concentrate on trying to preserve σ as an epimorphism of $T \upharpoonright \alpha$.

For each $t \in T \upharpoonright \alpha$ we extend an α-branch b_t containing t and let $\sigma(\bigcup b_t) = \bigcup_{x \in b_t} \sigma(x)$. Also, we extend the branches in the set $\sigma^m[b_t]$ for every $m \in \omega$ (which is exactly one for each m and t) and for every $d_{mt} \in \sigma^m[b_t]$, we extend the set $\sigma^{-1}[d_{mt}]$ as explained before and then we iterate to get $\sigma^{-p}[d_{mt}]$ for every $p \geqslant 1$.

$\star_2$. ***S_α is a maximal antichain but g_α is the identity automorphism on $T \upharpoonright \alpha$.***

Here we need only seal the maximal antichain so that it stays maximal from here on. In order to do this, we notice that because S_α is maximal, for every $t \in T \upharpoonright \alpha$, there is $a \in S_\alpha$ compatible with t. Let b_t^* be a chosen α-branch containing t and a. Then we extend every branch in the set $\{b_t^* \mid t \in T \upharpoonright \alpha\}$ and proceed as in case $\star_1$ with b_t^* instead of b_t.

From now on b_t^* will denote a branch going through t and an element of the antichain S_α.

$\star_2$. ***S_α is a maximal antichain of $T \upharpoonright \alpha$ and g_α is a non-trivial automorphism of $T \upharpoonright \alpha$.***

In this case there is an element $t_0 \in T \upharpoonright \alpha$ which is moved by g_α.

The idea is to destroy the automorphism g_α by finding a branch b that contains t_0 whose image under g_α is contained in a branch going through $g_\alpha(t_0)$ (and therefore different from b) and then extend the former but not the latter, stopping g_α from becoming an automorphism of $T \upharpoonright (\alpha + 1)$ and hence of T.

Notice that since g_α is an automorphism of $T \upharpoonright \alpha$, g_α preserves levels.

So, we want to choose $b_{t_0} \in [T \restriction \alpha]$ so that $g_\alpha[b_{t_0}] \neq b_{t_0}$ as branches, and

A. $g_\alpha[b_{t_0}] \nsubseteq \sigma^m[b_{t_0}]$ and $g_\alpha[b_{t_0}] \notin \sigma^{-m}[b_{t_0}]$ for any positive m. This is because once we extend b_{t_0} we need to extend its image under σ to ensure σ stays surjective, but at the same time we need to extend the image of this image and so on.

If b_t^* is an α-branch containing t and an element of S_α, then for every branch $b_t^* \in [T \restriction \alpha]$ we have to extend $\sigma^m[b_t^*]$ for every $m > 0$, so we need to choose b_t^* for every $t \in T \restriction \alpha$ such that:

B. $g_\alpha[b_{t_0}^*] \neq \sigma^m[b_t^*]$ as branches, for any positive m,

C. There exists $b_x^* \in [T \restriction \alpha]$ such that $\sigma^m[b_x^*] = b_t^*$,

D. $g_\alpha[b_{t_0}]$ is not a branch of $\sigma^{-m}[b_t^*]$ for any positive m.

In order to do this, we will use the following lemma.

Lemma 2.20. *It is always possible to find two different branches b_t and b_t' so that both contain t and an element of S_α.*

Proof. Let b_t be chosen so that $t, a \in b_t$ and $a \in S_\alpha$. Let $x = max\{t, a\}$ and $y \notin b_t$ be an extension of x (which is possible since x is ω-splitting). Because S_α is maximal, there is $a_y \in S_\alpha$ which is compatible with y so that $b_y^* \neq b_t$. □

First, we will concentrate on finding b_{t_0} satisfying condition **A** and then we will use a similar method to choose carefully the remaining branches to extend so that they satisfy the rest of the conditions.

- *Assume that for some $t_0 \in T \restriction \alpha$, $g_\alpha(t_0) \neq \sigma^m(t_0)$ and $g_\alpha(t_0) \notin \sigma^{-m}(t_0)$ for all $m > 0$.*

Here we need only to worry about sealing the maximal antichain S_α since our assumption on t_0 is precisely what we need for the requirement in **A** to be satisfied by any branch containing t_0. For this, we let our b_{t_0} be a branch $b^*_{t_0}$ but using **Lemma 2.20** to ensure $b^*_{g_\alpha(t_0)} \neq g_\alpha[b_{t_0}]$, and since $g_\alpha(t_0) \notin \sigma^{-m}(t_0)$ for any $m > 0$, we can also choose $b^*_{\sigma^{-m}(t_0)} \neq g_\alpha[b_{t_0}]$.

If our assumption is not satisfied, we have either $g_\alpha(t_0) = \sigma^m(t_0)$ or $g_\alpha(t_0) \in \sigma^{-m}(t_0)$ for some $m > 0$. Notice that in either case, we need only to take care of one part of condition **A** : if $g_\alpha(t_0) = \sigma^m(t_0)$, then $\sigma^{-m}(t_0) \neq g_\alpha(t_0)$ since our epimorphism σ sends nodes in the tree "to the left" whereas the inverse moves "to the right" (and they are both level preserving maps) and thus, the inverse image of a point above t_0 will always be "to the right" but g_α has to preserve order and therefore will stay "to the left" above $\sigma(t_0)$. Hence, $\sigma^{-m}[b_{t_0}]$ cannot contain $g_\alpha[b_{t_0}]$, as desired. The same argument applies when $g_\alpha(t_0) \in \sigma^{-m}(t_0)$.

- *If* $g_\alpha(t_0) = \sigma^m(t_0)$ *for some* $m > 0$.

Then we let m be the minimum such that this happens. In this case we will also take care to choose a branch in which the epimorphisms σ^m and g_α differ at some point, so we can close $T \restriction \alpha$ under σ. This will be possible because we have chosen our epimorphism to be not injective, unlike the automorphism g_α, so we will take advantage of this feature to define b_{t_0}.

So, let's first take care of S_α. Let $t^* = \max\{t_0, a\}$ for some $a \in S_\alpha$ that is comparable with t_0. Notice that $g_\alpha(t^*) \neq t^*$.

Next, take $y_0 = t^{*\frown}0$ and $y_1 = t^{*\frown}1$ be the first two immediate successors of t^*, then their image under σ is the same $\sigma^m(y_0) = \sigma^m(y_1) = \sigma^m(t^*)^\frown 0$ (in fact, we can take any of the first m successors of t^* for this to hold). Since g_α is an automorphism, $t_0 \leqslant t^* < y_0$ and $t_0 \leqslant t^* < y_1$ imply

$$g_\alpha(t_0) < g_\alpha(y_0) \neq y_0,$$

$g_\alpha(t_0) < g_\alpha(y_1) \neq y_1$,

$g_\alpha(y_0) \neq g_\alpha(y_1)$

Therefore we can't have both $\sigma^m(y_0) = g_\alpha(y_0)$ and $\sigma^m(y_1) = g_\alpha(y_1)$, so we let y_m be one among y_0 and y_1 satisfying $\sigma^m(y_m) \neq g_\alpha(y_m)$.

Choosing a branch containing y_m will take care of condition **A** for m, but we need to make sure we can do this for all $n > 0$. Notice that since m is the minimum satisfying $g_\alpha(t_0) = \sigma^m(t_0)$, we have that $g_\alpha(t_0) \neq \sigma^n(t_0)$ for all $n < m$, so we need only take care of $n \geqslant m$.

Claim 2.21. *For all $n \geqslant m$ there is a sequence $t^* < y_m < t_1 < ... < t_l$ such that $\sigma^n(t_i) \neq g_\alpha(t_i)$ for all $i \in \{1, 2, ...l\}$ and $l = m + n$.*

Proof. We will prove this by induction. We have proved the base case $n = m$ above so we will prove the case for $n + 1$ assuming $\sigma^n(t_l) \neq g_\alpha(t_l)$. Let $t_j = t_l^\frown j$ be the j-th immediate successor of t_l, with $j \in \{0, 1, ...n + 1\}$, then $\sigma^{n+1}(t_j) = 0$ for every j but this is not the case for g_α as $g_\alpha(t_j) \neq g_\alpha(t_h)$ for all $j \neq h$, therefore not all the t_j's can have $g_\alpha(t_j) = \sigma^{n+1}(t_j)$ and we let s be any element of $\{t_j \mid j \in \{0, 1, ...n + 1\}\}$ satisfying $\sigma^{n+1}(s) \neq g_\alpha(s)$. ☺

Hence we have shown that for every $m > 0$, there is a chain b of order type $ot(b) = ot(t^*{\downarrow}) + \omega$ that contains at least one element $y_m > t^*$ for which the m-th iteration of the epimorphism σ is not equal to the image of y_m under g_α, so taking an α-branch containing this chain will give us our desired b_{t_0} with the property $g_\alpha[b_{t_0}] \neq \sigma^m[b_{t_0}]$.

- *If $g_\alpha(t_0) = \sigma^{-m}(t_0)$ for some $m > 0$.*

In this case we argue in a similar way as in the above case, this time concentrating on σ^{-m} instead. Let m be the minimum value for which the above assumption holds and let t^* be as above. Then there is an immediate successor of t^*, $y_m = t^{*\frown} i$ for $i \in \omega$, such that $g_\alpha(y_m) \notin \sigma^{-m}(y_m)$.

Otherwise, if we let $\{y_i \mid i \in \omega\}$ be an enumeration of the immediate successors of t^* and assume $g_\alpha(y_i) \in \sigma^{-m}(y_i)$ for all $i \in \omega$, then for $i \neq j$ we have

$$y_i \neq y_j,$$

$$\sigma^{-m}(y_i) \cap \sigma^{-m}(y_j) = \varnothing,$$

$$g_\alpha(y_i) \in \sigma^{-m}(y_i),$$

$$g_\alpha(y_j) \in \sigma^{-m}(y_j)$$

So if we take two different elements $w_1 \neq w_2$ in $\sigma^{-m}(y_0)$ (we take y_0 because it is an element that we can be sure will have more than one pre-image according to our definition of σ, but we could've taken any $i \in \{0, 1, ...m\}$), then $g_\alpha^{-1}(w_1) \neq g_\alpha^{-1}(w_2)$ but $g_\alpha^{-1}(w_1)$ corresponds to some y_i and $g_\alpha^{-1}(w_2)$ to a different y_j, thus $g_\alpha(y_i) = w_1$ and $g_\alpha(y_j) = w_2$, so $w_1 \in \sigma^{-m}(y_i)$ and $w_2 \in \sigma^{-m}(y_j)$, which is not possible.

Therefore there is some $y_m > t^*$ such that $g_\alpha(y_m) \notin \sigma{-}m(y_m)$ for some $m > 0$. Again, this property holds for every $n \leqslant m$ so we just need to take care of everything above m.

Claim 2.22. *For all $n \geqslant m$ there is a sequence $t^* < y_m < t_1 < t_2 ... < t_{n-m}$ such that $g_\alpha(t_i) \notin \sigma^{-n}(t_i)$.*

Proof. We will prove this by induction and the base case $n = m$ has been done above. Therefore, assuming the premise above we will prove it for $n+1$. We know that $\sigma^{-m}(t_{n-m})$ does not contain $g_\alpha(t_{n-m})$. Assume towards a contradiction that we can't find such t. Let $\{y_i \mid i \in \omega\}$ be an enumeration of the immediate successors of t_{n-m}, and proceed as in the base case above. ☺

Hence, we take b_{t_0} to be a branch containing t^* and y_m together with the sequence $\{t_i \mid i \in \omega\}$ which satisfies $g_\alpha[b_{t_0}] \notin \sigma^{-n}[b_{t_0}]$ for all $n > 0$.

This concludes the search for our branch b_{t_0} satisfying the requirements on **A**. We will extend b_{t_0}, as well as $b^*_{g_\alpha(t^*)} \neq g_\alpha[b_{t_0}]$. Next we will take care of **B**,**C**, and **D**.

Let $t \in (T \upharpoonright \alpha) \setminus (b_{t_0} \cup g_\alpha[b_{t_0}])$. To satisfy condition **B**, we want to extend b^*_t and $\sigma^m[b^*_t]$ for all $m > 0$.

I shall make use of the following terminology: Let $t, y \in T \upharpoonright \alpha$ be two α-sequences, then we say that 't is *to the right of* y' if and only if on the least entry where they differ, say $\nu < \alpha$, $y(\nu) < t(\nu)$. In this case we also say that 'y is *to the left of* t'.

So, we want $g_\alpha[b_{t_0}] \neq \sigma^m[b^*_t]$ for all $m > 0$, and this is given by the next claim.

Claim 2.23. *For all $m > 0$, there is $y_{B,m} \in T \upharpoonright \alpha$ extending t^* such that $\sigma^m(y_m) \notin g_\alpha[b_{t_0}]$.*

Proof. If there is some element of $g_\alpha[b_{t_0}]$ to the right of t^*, then any extension of t^* will satisfy the claim since σ^m moves points to the left, so assume t^* is to the left of every element of $g_\alpha[b_{t_0}]$. Let $x_0 \in g_\alpha[b_{t_0}]$ be the unique element on the branch $g_\alpha[b_{t_0}]$ in the same level as the immediate successors of t^* (the next level above t^*) and look at $\sigma^{-m}(x_0)$. Then not all the immediate successors of t^* are part of this set (otherwise, for all $y = t^{*i}$ for all $i \in \omega$, $x_0 = \sigma^m(y) = \sigma^m(t^*)^\frown \sigma^m(i) = \sigma^m(t^*)^\frown (i \dot{-} m)$, but $i \dot{-} m$ is equal to zero for $i \in \{0, ...m\}$ and equal to $i - m$ for $i > m$). So let $y_{B,m} \in \{t^{*\frown} i \mid i \in \omega\} \setminus \sigma^{-m}(x_0)$. Then $\sigma^m(y_{B,m}) \neq x_0$ and hence not in $g_\alpha[b_{t_0}]$, since σ preserves levels. ☺

Therefore, if we choose $y_{B,m} > t^*$ as above for each $t \in T \upharpoonright \alpha$ so that t^* is to the left of $g_\alpha[b_{t_0}]$, and any extension of t^* if t^* is to the right of $g_\alpha[b_{t_0}]$, then we satisfy **B** for some $m > 0$.

Next, in addition to this, we have to find an extension $y_{D,m}$ of $y_{B,m}$, such that $g_\alpha[b_{t_0}] \nsubseteq \sigma^{-m}[b^*_t]$, that is, $\sigma^{-m}(y_{D,m}) \cap g_\alpha[b_{t_0}] = \varnothing$.

Once more, if $y_{B,m}$ is to the left of $g_\alpha[b_{t_0}]$, then any extension will work, so we assume it is to the right. Let $x \in g_\alpha[b_{t_0}]$ so that x is in the same level as the immediate successors of t^* and look at $\sigma^m(x)$. Then any element $y_{D,m}$ in $\{t^{*\frown} i \mid i \in \omega\} \setminus \{\sigma^m(x)\}$ will satisfy our requirement in **D** for $m > 0$.

Hence, the sequence $y_{D,m} > y_{B,m} > t^* \geqslant t$ satisfies **B** and **D** for a given $m > 0$. Using the above technique we can construct by induction on m a sequence of elements that will define our branch b_t^*, so that at each step m we add two new elements $y_{D,m}, y_{B,m}$ that will take care of conditions **B** and **D** for that given m, and such that $y_{D,m} > y_{B,m} > y_{D,m-1} > y_{B,m-1} > ...y_{D,0} > y_{B,0} > t^*$.

Then we choose b_t^* to be a branch containing this sequence.

To get **C** we need: given $t \in T \restriction \alpha$ and $m > 0$, there is $x \in T \restriction \alpha$ such that $\sigma^m[b_x^*] = b_t^*$. Here we will reconstruct the set of branches $\sigma^{-m}[b_t^*]$ by induction on the elements of b_t^*.

As usual, for $x = root$, $\sigma^{-m}(root) = root$. Assume we have defined $\sigma^{-m}[x_\downarrow \setminus \{x\}]$ for $x \in b_t^*$ in T_γ. Assume further that x is in a level corresponding to a successor ordinal, that is, $\gamma = \beta + 1$. If $|\sigma^{-m}(x)| > 1$ and we choose elements in $\sigma^{-m}(x)$ such that for each $z \in T_\beta$ already chosen, there are two elements in $\sigma^{-m}(x)$ extending z, if there is only one point in $\sigma^{-m}(x)$, we add it . The worst case scenario would be if $\sigma^{-m}(x)$ contains only one element and it happens to be in $g_\alpha[b_{t_0}] \cup b_{t_0}$, but even if this is the case, once we run into $y_{D,m}$ we will make sure there is some $z \in b_t^*$ satisfying $g_\alpha[b_{t_0}] \notin \sigma^{-m}[b_t^*]$. If γ is a limit level, then we choose to extend $\aleph_0$ branches from the ones we have collected previously so that their extensions are elements of $\sigma^{-m}(x)$.

Thus, $\sigma^{-m}[b_t^*]$ will be the union of countably-many of these branches, and in order to satisfy **C**. we extend every branch in this set, and this concludes *case* $\star_\mathbf{3}$.

$\star_\mathbf{4}$. ***S_α is not a maximal antichain of $T \restriction \alpha$ and g_α is a non-trivial automorphism of $T \restriction \alpha$.***

This case is completely analogous to *case* $\star_\mathbf{4}$, the only difference being that we need not to stop S_α from being a maximal antichain and hence t^* becomes just t.

Hence, to define T_α we extend the following branches:

$$b_{t_0} \cup b^*_{g_\alpha[t_0]} \cup \{b_t^* \mid t \in T \restriction \alpha\} \cup \bigcup_{m>0} \{\sigma^m[b_t^*] \mid t \in T \restriction \alpha\} \cup \bigcup_{m>0} \{\sigma^{-m}[b_t^*] \mid t \in T \restriction \alpha\}$$

which is a countable set. The resulting tree is clearly Suslin since we have been taking care of maximal antichains in the usual way using our $\Diamond$-sequence.

2.5 Remarks

Notice that by contrast with the result of **Lemma 4.8** obtained for linear orderings presented in **Chapter 4**, in general it is not the case that an epimorphism of a tree $(T, \leqslant)$ gives rise to an embedding of $(T, \leqslant)$, as illustrated by the following example.

Let $(T, \leqslant)$ be defined as: $T_0 = root$ consisting of a single element following by T_1, a set of nodes of order-type $(\omega + 1).\omega$. Then we add an n-branch above every node in $n \times m$ for every $n, m \in \omega$ and ω-branches above each (ω, m)-node in T_1, as seen in **Figure 2.9**.

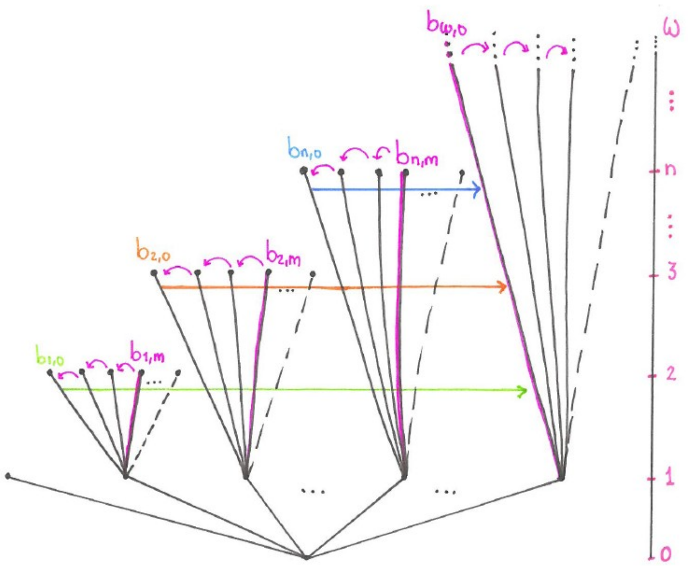

FIGURE 2.9: A tree with an epimorphism which does not give rise to any embedding.

Now, we will define an epimorphism f on $(T, \leqslant)$ by its actions on the branches of the tree. Let $b_{n,m}$ be the m-th branch of height n, for $m \in (\omega + 1)$ and $n \geqslant 1$.

$$f(b_{n,m}) = \begin{cases} b_{n,m-1} & \text{if } n < \omega, m > 0 \\ b_{n,m+1} & \text{if } n = \omega \\ \text{first } n+1 \text{ elements of } b_{\omega,0} & \text{if } m = 0 \end{cases}$$

Then, let's try defining an embedding g from f by choosing an element of each inverse image under f^{-1}. Since f is injective on most of the elements on T we just need to check how the action of g on the branch $b_{\omega,0}$ could be defined. Since g must preserve levels, it maps $b_{\omega,0} \cap T_1$ to some $b_{n,m} \cap T_1$ for finite n. But this implies that it maps the whole of $b_{\omega,0}$ to $b_{n,m}$ which is impossible while preserving levels, since $b_{\omega,0}$ is infinite but $b_{n,m}$ is finite.

However, in the case of the tree we constructed with the non-trivial epimorphism it is possible to define this embedding with a couple of modifications to the construction. The trouble could be that once we take the set of branches $\sigma^{-1}[b]$ for an α-branch on a limit level, we can't simply let $g[b]$ be any branch on this set since for two different α-branches $b_1 \neq b_2$, sharing a point x, there are two branches in $\sigma^{-1}[b]$ containing two different point of $\sigma^{-1}(x)$.

To solve this problem, let us redefine $\sigma^{-1}[b_t^*]$ for $b_t^* \in [T \upharpoonright \alpha]$ and $\alpha \in \mathrm{Lim}(\omega_1)$, for any branches b_t^* that have extensions t^* on T_α. Looking at t^* as α-sequences, order them lexicographically, and let $\{t_\beta^* : \beta \in \omega\}$ be an increasing sequence formed by all these extensions.

Now, choose as usual $\sigma^{-1}(root) = root$ and $\sigma^{-1}(x)$ for all $x \in b_{t_0}^*$. If $\sigma^{-1}(x)$ has been chosen for every $x \in b_{t_\beta}^*$, to define the elements of $\sigma^{-1}(x)$ for $x \in b_{t_{\beta+1}}^*$, if x belongs to some of the branches $b_{t_\gamma}^*$ for $\gamma \leqslant \beta$, then $\sigma^{-1}(x)$ has been chosen and if x is not a member of these branches we choose some different elements in $\sigma^{-1}(x)$ following the same pattern as in our construction. Finally, to define the set of branches in $\sigma^{-1}[b_{t_\beta}^*]$ for $\beta \in \omega$, choose countably many branches from $\bigcup_{x \in b_{t_\beta}^*} \sigma^{-1}(x)$.

The idea is that the branches that we extend for $\sigma^{-1}[b_t^*]$, will now have the property that they agree, i.e. if $x \in b_t^* \cap b_u^*$ then $\sigma^{-1}(x) = \sigma^{-1}[b_t^*] \cap \sigma^{-1}[b_u^*] \cap T_{ht(x)}$.

To define g we look at $t^* \in T_\alpha$, for $\alpha \in \mathrm{Lim}(\omega_1)$ and let $g[b_t^*]$ be any element of $\sigma^{-1}[b_t^*]$ and $g(t^*) = \bigcup g[b_t^*]$. This will take care thus of all the elements in $T \restriction \alpha$ since we have extended an α-branch b_t^* for each $t \in T \restriction \alpha$. It is clear from the construction that g is indeed an embedding.

The reason why it was possible for us to define g from our σ, is that the following property holds. For every $z \supset y$ and every $x \in \sigma^{-1}(y)$, there is an extension $w \supset x$ such that $\sigma(w) = z$, for let $z = y^\frown k$ for $k \in \omega$, and let $x \in \sigma^{-1}(y)$, then $w = x^\frown(k \dot{+} 1)$. If $k > 0$, $\sigma(w) = \sigma(x)^\frown k = y^\frown k = z$ and if $k = 0$, $w = \sigma(x)^\frown 0 = y^\frown 0 = z$.

Therefore, if we try to construct a tree that will admit a non-trivial epimorphism but that does not admit a non-trivial embedding the best place to start is to try and construct something that violates this property.

2.6 κ^+-Suslin trees

In this section we generalize the results of **Section 2.1** to Suslin trees with height κ^+ for an infinite cardinal κ. For this, we define a κ-Suslin tree T as a κ-tree such that:

1. T is normal
2. $ht(T) = \kappa$
3. $|T_\alpha| < \kappa$ for every $\alpha < \kappa$.

But when studying infinite cardinals we are faced with the following splitting categories:

1. *singular cardinals*

2. *regular cardinals* $\begin{cases} \text{successors of other regulars} \\ \text{limit cardinals} \\ \text{successors of singular cardinals} \end{cases}$

As a matter of fact, the mere existence of a regular limit cardinal (also called **weakly inaccessible cardinal**) is independent of **ZFC**. On the other hand, if κ is a singular cardinal, the situation is quite strange: we can find a κ-tree which is not normal but it has no antichains nor chains of cardinality κ, but if we require the tree to be normal then it necessarily has an antichain of cardinality κ, preventing it from being Suslin.

Lemma 2.24. *If κ is a singular cardinal and T is a normal κ-tree then there is an antichain of cardinality κ.*

Proof. Let κ and T be as in the statement above and let $cf(\kappa) = \lambda < \kappa$. Then we can find a sequence $(c_\xi)_{\xi<\lambda}$ of cardinals less than κ such that

$$\kappa = \bigcup_{\xi<\lambda} c_\xi$$

Without loss of generality we can assume all of the elements in this sequence to be cardinals greater than λ. Now, we will construct an antichain A of cardinality κ in stages and we'll let A be the union,

$$A = \bigcup_{\alpha<\lambda} A_\alpha$$

We look at T_{c_0}, the c_0-th level of T, and find a partition of it into the disjoint union of λ-many subsets U_{c_ξ}, for each $\xi < \lambda$

$$T_{c_0} = \dot{\bigcup}_{\xi<\lambda} U_{c_\xi} \text{ such that } |U_{c_\xi}| = c_\xi$$

We can definitely do this because c_0 is greater than λ and since T is normal each level has cardinality at least $c_\nu < \kappa$. Now we define A_α for $\xi < \lambda$ as follows,

$$A_0 = U_0 \text{ and } A_\xi = \{s \geqslant_T t \mid t \in U_{c_\xi} \wedge s \in T_{c_\xi}\}$$

Intuitively what we are doing here is take advantage of the fact that each level is itself an antichain of T and the fact that we have a sequence of $cf(\kappa)$-many cardinals whose limit is κ, and since we are requiring these cardinals to be above $cf(\kappa)$ and we know that the cardinality of the c_ν-th level is at least c_ν, so each A_ξ is at least of cardinality c_ξ. See **Figure 2.10**.

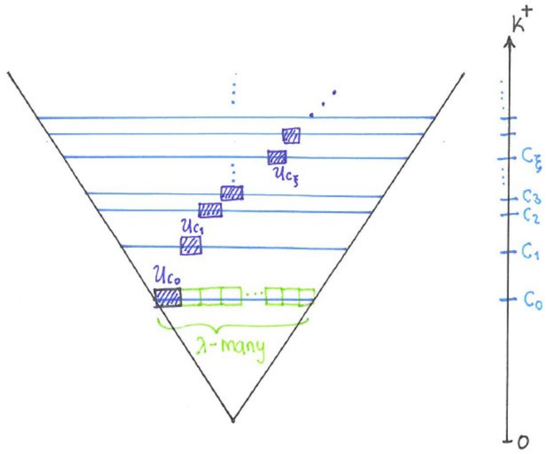

FIGURE 2.10: The elements of the sequence $(A_\xi : \xi \in \lambda)$

Then clearly A_α being a subset of an antichain its itself an antichain. In addition, if $\alpha < \beta$ then $A_\alpha \cap A_\beta = \varnothing$ and moreover they are incomparable since A_β was chosen to be in a higher level and its elements are extensions of elements that are incomparable with A_α (namely extensions of nodes in the c_0-th level). Therefore A is an antichain and

$$\kappa \geqslant |A| = \bigcup_{\alpha<\lambda} |A_\alpha| \geqslant \bigcup_{\alpha<\lambda} c_\alpha = \kappa$$

The first inequality holds because T has only κ-many elements, and so $|A| = \kappa$ as required. □

Therefore, the existence of a κ^+-Suslin tree seems to be more involved than the existence of an ω_1-Suslin tree (which follows from $\diamondsuit$). For the above reasons, we concentrate on constructing Suslin trees only for regular cardinals of the form κ^+.

Moreover, if κ is a regular cardinal the existence of a κ^+-Suslin tree follows from '**GCH** + $\Diamond_{\kappa^+}(E)$' ([Jen68]) for a suitable stationary $E \subseteq \kappa^+$. The principle $\Diamond_{\kappa^+}(E)$ is a generalization of $\Diamond$ and is defined as follows.

Let $\kappa > \omega$ be a regular cardinal and E a stationary subset of κ,

$\Diamond_\kappa(E)$: There is a sequence $(S_\alpha \mid \alpha \in E)$ such that $S_\alpha \subseteq \alpha$, with the property that whenever $X \subseteq \kappa$, the set $\{\alpha \in E \mid X \cap \alpha = S_\alpha\}$ is stationary in κ.

$\Diamond_\kappa(E)_g$: There is a sequence $(g_\alpha \mid \alpha \in E)$ such that $g_\alpha \in \alpha^\alpha$, with the property that whenever $g \subseteq \kappa^\kappa$, the set $\{\alpha \in E \mid g \upharpoonright \alpha = g_\alpha\}$ is stationary in κ.

Our principle $\Diamond$ is precisely $\Diamond_{\omega_1}(\omega_1)$. Regarding these results, Jensen [DJ74] showed **Con(GCH** + $\neg\Diamond_{\omega_2}(E)$**)** and Shelah [50] established **Con(**$\Diamond_{\omega_1}$ + $\neg\Diamond_{\omega_1}(E)$**)**.

The next lemma shows that $\Diamond_\kappa(E)$ and $\Diamond_\kappa(E)_g$ are actually equivalent.

Lemma 2.25. *$\Diamond_\kappa(E)$ holds if and only if $\Diamond_\kappa(E)_g$ does.*

Proof. First we'll show $\Diamond_\kappa(E)_g \to \Diamond_\kappa(E)$. Let $(g_\alpha)_{\alpha\in E}$ be a $\Diamond_\kappa(E)_g$-sequence and define

$$S_\alpha = \{\beta < \alpha \mid g_\alpha(\beta) = 1\}$$

for each $\alpha \in E$. Let $X \subseteq \kappa$ be arbitrary and $f = \chi_X$ be the characteristic function of X. Then $G = \{\alpha \in E \mid f \upharpoonright \alpha = g_\alpha\}$ is stationary. We want to show for each $\alpha \in E$:

$$\chi_X \upharpoonright \alpha = g_\alpha \to X \cap \alpha = S_\alpha.$$

Let $\alpha \in E$ and assume $\chi_X \upharpoonright \alpha = g_\alpha$. If $\beta \in X \cap \alpha$, then $\chi_X(\beta) = 1 = g_\alpha(\beta)$ and hence $\beta \in S_\alpha$. On the other hand, if $\beta \in S_\alpha$, then $\chi_X(\beta) = g_\alpha(\beta) = 1$, so $\beta \in X \cap \alpha$. Therefore,

$$G \subseteq \{\alpha \in E \mid X \cap \alpha = S_\alpha\} = S$$

and since G is stationary in κ, so is S.

Now, we shall show that $\Diamond_\kappa(E) \to \Diamond_\kappa(E)_g$. Let $(S_\alpha)_{\alpha\in E}$ be a $\Diamond_\kappa(E)$-sequence. Let $\pi : \kappa \longrightarrow \kappa \times \kappa$ be an order isomorphism from $(\kappa, \leqslant)$ (the ordinal order) to $(\kappa \times \kappa, \preceq)$, where we define $\prec$ on $\kappa \times \kappa$ by,

$$\begin{aligned}(\alpha_1,\beta_1) \preceq (\alpha_2,\beta_2) \longleftrightarrow &\max\{\alpha_1,\beta_1\} < \max\{\alpha_2,\beta_2\} \text{ or}\\ &\max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} \wedge (\beta_1 < \beta_2) \text{ or}\\ &\max\{\alpha_1,\beta_1\} = \max\{\alpha_2,\beta_2\} \wedge (\beta_1 = \beta_2) \wedge (\alpha_1 < \alpha_2).\end{aligned}$$

Define, for each $\alpha \in E$,

$$g_\alpha = \begin{cases} \pi[S_\alpha] & \text{if it is a well-defined function on } \alpha \\ id & \text{otherwise} \end{cases}$$

This means, $g_\alpha(x) = \beta$ if $(x, \beta) \in \pi[S_\alpha]$ when $\pi[S_\alpha] \subseteq \kappa \times \kappa$ is a well-defined function. To show that this sequence of functions satisfies the requirements of $\Diamond_\kappa(E)_g$, we shall need a couple of claims.

Claim 2.26. *The set $C = \{\alpha < \kappa \mid \pi[\alpha] = \alpha \times \alpha$ is a bijection $\}$ is a club.*

Proof. To show this, we'll show that the following function $g : \kappa \longrightarrow \kappa$, is a **normal function**, that is, that it is increasing and continuous.

$$g(\alpha) = ot(\alpha \times \alpha).$$

To see that g is continuous, let $\lambda \in \mathrm{Lim}(\kappa)$ and $(\alpha_\eta)_{\eta\in\lambda}$ be an increasing sequence with $\alpha = \sup_{\eta\in\lambda} \alpha_\eta$. Then

$$\bigcup_{\eta\in\lambda} g(\alpha_\eta) = \bigcup_{\eta\in\lambda} ot(\alpha_\eta \times \alpha_\eta) = ot(\alpha \times \alpha) = g(\alpha).$$

To show that $\alpha \leqslant g(\alpha)$, notice that $(\beta, 0) \preceq (\alpha, \alpha)$ for every $\beta < \alpha$ and the set

$\{(\beta,0) \mid \beta \in \alpha\} \subseteq \alpha \times \alpha$ has clearly order-type α. Hence $\alpha \leqslant ot(\alpha \times \alpha) = g(\alpha)$.

Since g is a normal function the set

$$C = \{\alpha \in \kappa \mid g(\alpha) = \alpha\}$$

is a club subset of κ [§], proving the claim. ☺

Let $f \subseteq \kappa \times \kappa$ be an arbitrary function and set $X \subseteq \kappa$ to be $X = \pi^{-1}[f]$. Then, the set $S = \{\alpha \in E \mid \pi^{-1}[f] \cap \alpha = S_\alpha\}$ is a stationary subset of κ.

Claim 2.27. *The set $A = \{\alpha \in \kappa \mid f[\alpha] \subseteq \alpha\}$ is a club subset of κ.*

Proof. Closure clear since f preserves unions. To see that A is unbounded, let $\gamma \in \kappa$. Construct a sequence $(\alpha_n)_{n\in\omega}$ such that $\alpha_0 = \gamma$ and $\alpha_{n+1} = \sup_{n\in\omega} f[\alpha_n]$, so that $f[\alpha_n] \subseteq \alpha_{n+1}$. Hence, if $\alpha = \sup_{n\in\omega} \alpha_n$,

$$f[\alpha] = f[\bigcup_{n\in\omega} \alpha_n] = \bigcup_{n\in\omega} f[\alpha_n] \subseteq \bigcup_{n\in\omega} \alpha_n = \alpha.$$

☺

Moreover, the set $I = A \cap C \cap S \subseteq E$ is a stationary subset of κ (it is the intersection of a club and a stationary set), so it is enough to show that if $\alpha \in I$, then $f \restriction \alpha = \alpha$, for then $I \subseteq \{\alpha \in E \mid f \restriction \alpha = \alpha\}$.

Let $\alpha \in I$. Then $\pi^{-1}[f] \cap \alpha = S_\alpha$ and since π is a bijection $\pi[S_\alpha] = \pi[\pi^{-1}[f] \cap \alpha] = f \cap \pi[\alpha] = f \cap \alpha \times \alpha = f \restriction \alpha$, and since $S_\alpha \subseteq \alpha$, $g_\alpha = \pi[S_\alpha] \subseteq \pi[\alpha] = \alpha \times \alpha$, as required. □

But Jensen's construction using $\diamondsuit_{\kappa^+}(E)$ works only for regular κ. In order to expand this result to allow also singular cardinals, he made use of the principle $\square_\kappa(E)$, which we now describe.

[§] This is a standard result, see [Jec02]

Let κ be an infinite cardinal and E a subset of κ^+. By $\square_\kappa(E)$ we mean the following statement,

$\square_\kappa(E)$: There is a sequence $(C_\alpha \mid \alpha \in \text{ Lim}(\kappa^+))$ such that

a) C_α is a club in α,

b) if $cf(\alpha)$ is less than κ, then $ot(C_\alpha)$ is less than κ,

c) whenever $\beta < \alpha$ is a limit point of C_α, then $\beta \notin E$ and $C_\beta = C_\alpha \cap \beta$.

Although this statement is provable in **ZFC** for $\kappa = \omega$, the same is not possible for $\kappa = \omega_1$. However, if $\mathbf{V} = \mathbf{L}$ then $\square_\kappa(\varnothing)$ (which we denote by $\square_\kappa$) holds for any infinite κ. Furthermore $\square_\kappa$ gives a stationary E for which $\square_\kappa(E)$ holds.

According to T. Eisworth in [Eis12], $\square_\kappa$ 'is quite persistent'. This is because once $\square_\kappa$ holds in a model, as long as we preserve κ and κ^+ we will have it in any extension.

Continuing with our survey of results, Avraham, Shelah and Solovay showed in [AS87] that if λ is a strong limit singular, then 'CH_λ + $\exists\lambda^+$-**Suslin tree**', even though it is consistent with **GCH** that there are no λ^+- Suslin trees [She84a]. **GCH** also implies some of the consequences of $\Diamond$ even when $\Diamond$ fails in the model [She03].

In **Proposition 2.28** further on in this section, we construct a κ^+-Suslin tree for any infinite cardinal κ (satisfying some rigidity properties), if $\square_\kappa(E)$ and $\Diamond_{\kappa^+}(E)$ hold together in our model for some stationary $E \subseteq \kappa^+$. Our result will be an extension of Jensen's work in [Jen68], who showed that both principles, $\square_\kappa(E)$ and $\Diamond_{\kappa^+}(E)$ hold in **L**.

Nevertheless, in Shelah [She84b], the consistency of '$\square^*_\lambda$ + **GCH**' with $\neg\Diamond_{\lambda^+}(E)$ is shown, for a strong limit singular λ and $\square^*_\lambda$ a weakening of $\square_\lambda$. Also, Gitik and Rinot in [GR12] showed that **'GCH holds above** ω + $\neg\Diamond_\omega(E)$**'** is consistent relative to the existence of a super compact cardinal for a suitable E. For an extensive review of the work done around these principles see [Rin10], and for a clear exposition of some of the results concerning $\square$ and $\Diamond$ in L, see [Dev84].

Proposition 2.28. *Let κ be an infinite cardinal. If there is a stationary subset E of κ^+ such that both $\square_\kappa(E)$ and $\Diamond_{\kappa^+}(E)$ hold, then,*

C1. *there is a κ^+-Suslin tree which admits no level preserving endomorphism other than the identity,*

C2. *there is a κ^+-Suslin tree which admits no embedding other than the identity,*

C3. *there is a κ^+-Suslin tree which is totally rigid.*

Proof. We will show **C1** first and state how to modify the construction to obtain **C2** and **C3**. This proof is a generalization of that of **Proposition 2.1** and the basic idea is the same. We construct a normal ω-splitting κ^+-Suslin tree whose elements are those of κ^+.

The construction is as usual by transfinite induction on κ^+. At stage $\alpha < \kappa^+$we decide which α-branches to extend in level α so that the tree $T \restriction (\alpha+1)$ is a normal ω-splitting tree, and if $\alpha < \beta$ as ordinals, we define $T \restriction \beta$ as an end extension of $T \restriction \alpha$ and the ordinals appearing in $T \restriction \alpha$ form an initial segment of the ordinals in $T \restriction \beta$. The resulting tree T is the union

$$T = \bigcup_{\alpha\in\kappa^+} T \restriction \alpha = \bigcup_{\alpha\in\kappa^+} T_\alpha.$$

We start by defining $T_0 = root$. T_1 contains the elements of ω as immediate successors of the root. Now, we recall our definition of W, a family of subsets of ω_1 as in **Proposition 2.1**:

$$W_0 = root = zero$$
$$W_{n+1} = [\omega^n, \omega^{n+1}),\ \text{for } n \in \omega,$$
$$W_\alpha = [\omega^\alpha, \omega^{\alpha+1}) \text{ for } \alpha \geqslant \omega$$

So this definition still works for $\alpha \in \kappa^+$. Also notice that **Lemma 2.3** still holds for κ^+ in place of ω_1, since $\omega^\alpha = \alpha$ for every $\alpha \geqslant \omega_1$.

Hence, assume we have defined $T \restriction \alpha$. If $\alpha = \beta + 1$, we take elements from W_α and place ω-many immediate successors on top of each $x \in T_\beta$. If $\alpha \in \mathrm{Lim}(\kappa^+)$, we use our $\Diamond_{\kappa^+}(E)$-sequence, $(S_\alpha : \alpha \in E)$, to ensure that $T \restriction (\alpha + 1)$ will be a normal ω-splitting $(\alpha+1)$-tree; our $\Diamond_{\kappa^+}(E)_g$-sequence, $(g_\alpha : \alpha \in E)$, to stop a potential level preserving endomorphism ($\doteq$ l.p.-end) of $T \restriction \alpha$ extending to a l.p.-end of $T \restriction (\alpha+1)$; finally, our $\square_\kappa(E)$-sequence $(C_\alpha : \alpha \in \kappa^+)$ will be used to make sure we can always extend the desired α-branches.

We will extend α-branches in the set

$$B_\alpha = \{b_x(\alpha) \mid x \in T \restriction \alpha\}.$$

We shall define precisely how these branches are constructed later on. For now, assume we have constructed the set B_α. To decide which branches to extend in T_α, we distinguish various cases.

If $\alpha \notin E$, we need only to extend every branch in the set B_α, taking elements from W_α. Otherwise $\alpha \in E$ and the cases are analogous to the ones in **Proposition 2.1**.

$\star_1$. $\alpha \in E \wedge S_\alpha$ ***is not a maximal antichain of*** $T \restriction \alpha \wedge g_\alpha$ ***is not a non-trivial l.p.-end of*** $T \restriction \alpha$.

Then we proceed as if α was not in E, that is, extending every branch in B_α.

$\star_2$. $\alpha \in E \wedge S_\alpha$ ***is a maximal antichain of*** $T \restriction \alpha \wedge g_\alpha$ ***is not a non-trivial l.p.-end of*** $T \restriction \alpha$.

Then we extend the branches in the subset $B^*_\alpha \subseteq B_\alpha$ consisting of all the branches $b_x(\alpha) \in B_\alpha$ for those $x \in T \restriction \alpha$ extending some element of S_α. That is,

$$B^*_\alpha = \{b_x(\alpha) \in B_\alpha \mid (\exists a \in S_\alpha)[x >_T a]\}.$$

The next claim shows that it is enough to extend the branches in B^*_α to have for every $y \in T \upharpoonright \alpha$, a branch $b_x(\alpha) \in B^*_\alpha$ containing x and which is extended in T_α (which we require for normality).

Claim 2.29. $(\forall y \in T \upharpoonright \alpha)(\exists x >_T y)(\exists a \in S_\alpha)[a <_T x]$.

Proof. Let $y \in T \upharpoonright \alpha$. Since S_α is a maximal antichain of $T \upharpoonright \alpha$, there is $a \in S_\alpha$ which is compatible with y. If $a <_T y$ we are done, so assume $a >_T y$ and let x be any immediate successor of a. ☺

$\star_3$. ***$\alpha \in E \wedge S_\alpha$ is not a maximal antichain of $T \upharpoonright \alpha \wedge g_\alpha$ is a non-trivial l.p.-end of $T \upharpoonright \alpha$.***

Here, we will stop g_α from becoming a l.p.-end of $T \upharpoonright \alpha$. Since g_α is non-trivial, there are two distinct points $x, y \in T \upharpoonright \alpha$ such that $g_\alpha(x) = y$ and hence $g_\alpha[(T \upharpoonright \alpha)^x] \subseteq (T \upharpoonright \alpha)^y$. So, there is $z \in (T \upharpoonright \alpha)^x$ such that $b_z(\alpha) \in B_\alpha \wedge g_\alpha[b_z(\alpha) \neq b_z(\alpha)]$ and we extend all branches in $B_\alpha \setminus \{g_\alpha[b_z(\alpha)]\}$.

We need only to check that for all $v \in g_\alpha[b_z(\alpha)]$, there is $w \in T \upharpoonright \alpha$ such that $v \in b_w(\alpha)$. So, let $v \in g_\alpha[b_z(\alpha)]$, then let $w >_T v$ be an immediate successor of v which is not in $g_\alpha[b_z(\alpha)]$. This can be done because $T \upharpoonright \alpha$ is ω-splitting. Hence $v \in b_w(\alpha)$ and $b_w(\alpha) \neq g_\alpha[b_z(\alpha)]$.

$\star_4$. ***$\alpha \in E \wedge S_\alpha$ is a maximal antichain of $T \upharpoonright \alpha \wedge g_\alpha$ is a non-trivial l.p.-end of $T \upharpoonright \alpha$.***

For this, we will combine *cases* $\star_2$ and $\star_3$ to choose a branch $b \in B^*_\alpha$ such that $g_\alpha[b] \neq b$ and then extend all branches in $B^*_\alpha \setminus \{g_\alpha[b]\}$. So, let $x \neq y$ witness the non-triviality of g_α. Then there is $z \in (T \upharpoonright \alpha)^x$ such that $b_z(\alpha) \in B^*_\alpha$ and $g_\alpha[b_z(\alpha)]$ using **Claim 2.29**. Similarly, if $v \in g_\alpha[b_z(\alpha)]$, we let $w >_T v$ be an immediate successor of v which is not in $g_\alpha[b_z(\alpha)]$. Hence there is $b^* \in B^*_\alpha$ containing w and satisfying $b^* \neq g_\alpha[b_z(\alpha)]$.

This finishes all the cases so now we show how to choose the elements of B_α for $\alpha \in$

$\mathrm{Lim}(\kappa^+)$.

Let $x \in T \restriction \alpha$. Then we look at our $\square_\kappa(E)$-sequence and let $\{\gamma_\nu \mid \nu \in \lambda\}$ be an enumeration of C_α, for $ot(C_\alpha) = \lambda$. Set $\nu_x = \min\{\nu \mid x \in T \restriction \gamma_\nu\}$, and define a sequence $(p_x(\nu) : \nu_x \leqslant \nu < \lambda)$ of compatible elements of $T \restriction \alpha$ as follows,

$$p_x(\nu_x) = \min\{y \in T_{\gamma_{\nu_x}} \mid y >_T x\} \text{ as ordinals}$$

$$p_x(\nu+1) = \min\{y \in T_{\gamma_{\nu+1}} \mid y >_T p_x(\nu)\} \text{ as ordinals}$$

$$\text{and if } \eta \in \mathrm{Lim}(\lambda)$$

$$p_x(\eta) = y \in T_{\gamma_\eta} \text{ such that } (\forall \nu < \eta)[\nu \geqslant \nu_x \longrightarrow p_x(\nu) <_T y]$$

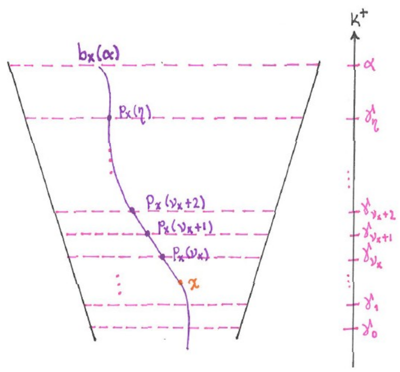

FIGURE 2.11: Defining $b_x(\alpha)$.

Then,

$$b_x(\alpha) = \{y \in T \restriction \alpha \mid (\exists \nu < \eta)[y \leqslant_T p_x(\nu)]\}$$

is an α-branch extending every element of the sequence $(p_x(\nu) : \nu_x \leqslant \nu < \lambda)$.

Now, this construction is consistent, in the sense that for every $\eta \in \mathrm{Lim}(\lambda)$, we can find $p_x(\eta) \in T_{\gamma_\eta}$ as above.

Claim 2.30. $(\exists! y \in T_{\gamma_\eta})[\nu_x \leqslant \nu < \eta \wedge p_x(\nu) <_T y]$.

Proof. By part $c)$ in $\square_\kappa(E)$, since γ_η is a limit point of C_α, we have $\gamma_\eta \notin E$ and $C_{\gamma_\eta} = \gamma_\eta \cap C_\alpha$.

Now, we have defined $p_x(\nu)$ so that $p_x(\nu) \in T_{\gamma_\nu}$ and hence $\{p_x(\nu) \mid \nu_x \leqslant \nu < \eta\} \subseteq b_x(\gamma_\eta)$. Since γ_η is not in E and we have asked to extend every branch $b_x(\gamma_\eta)$ on level γ_η, there is $y \in T_{\gamma_\eta}$ extending $b_x(\gamma_\eta)$. The uniqueness follows from the fact that $T \restriction \alpha$ was constructed to be normal. ☺

Next, to show that T is a Suslin tree we notice that **Lemma 2.2** and **Lemma 2.3** in **Proposition 2.1** are independent of κ^+ and hence they hold in this construction.

The same arguments used in **Claim 2.4** to show that our ω_1-Suslin tree does not admit a non-trivial l.p-end apply here for our κ^+-Suslin tree, since E is stationary.

For the proof of point **C2**, **Claim 2.6** still holds for $\alpha \in \kappa^+$. So if we are in *case* $\star_\mathbf{3}$ as above, g_α is non-trivial and we can choose a branch $b \in B_\alpha$ that contains y (a point moved by g_α) and such that $g_\alpha[b] \nsubseteq b$. Now we choose an α-branch e that contains $g_\alpha[b]$ and extend every branch in $b_\alpha \smallsetminus \{e\}$ and there is an extended branch in B_α that contains every element of e, so we have normality.

If, on the other hand, we are in *case* $\star_\mathbf{4}$, we still can find $b^* \in B^*_\alpha$ containing y as we did for **C1** above, using **Claim 2.29**. So $g_\alpha[b^*] \subseteq e^*$ for some $e^* \in [T \restriction \alpha]$ and we extend every branch in $B^*_\alpha \smallsetminus \{e^*\}$ as before.

To show that the resulting tree T admits no non-trivial embedding, assume f is a non-trivial embedding of T. Then there is a level, say $\xi < \kappa^+$ that contains a point moved by f. Then by $\lozenge_{\kappa^+}(E)$, there is $\alpha > \xi$ such that $f \restriction \alpha = g_\alpha$. Using **Lemma 2.3** we can choose α so that $f \restriction (T \restriction \alpha) = g_\alpha$, since in the lemma C' is a club and E is stationary. The rest of the argument is the same as in **Proposition 2.5**.

To show that **C3** holds, we need the following principle which is the analogue of $\lozenge_{k^+}$. For $\kappa^+ > \omega$ a regular cardinal and E a stationary subset of κ^+,

$\lozenge_\kappa(E)_k$: There is a sequence $(K_\alpha \mid \alpha \in E)$ such that $K_\alpha \subseteq \alpha \times \alpha$, with the property that whenever $X \subseteq \kappa^+ \times \kappa^+$, the set $\{\alpha \in E \mid X \cap \alpha \times \alpha = K_\alpha\}$ is stationary in κ^+.

which can be seen by the proof of **Lemma 2.25** to be equivalent to $\Diamond_\kappa(E)$. We want to construct a tree where any isomorphism between different cones must be the identity. To define this tree we just change the construction on *case* $\star_\mathbf{3}$ and *case* $\star_\mathbf{4}$ above, but we use $\Diamond_\kappa(E)_k$ instead of $\Diamond_\kappa(E)_g$.

If we find ourselves in *case* $\star_\mathbf{3}$, then S_α is not a maximal antichain of $T \upharpoonright \alpha$ and there are distinct $x, y \in T \upharpoonright \alpha$ such that K_α is a well-defined isomorphism f between the cones $(T \upharpoonright \alpha)^x$ and $(T \upharpoonright \alpha)^y$, that is

$$K_\alpha = \{\langle z, f(z)\rangle \in (T \upharpoonright \alpha) \times (T \upharpoonright \alpha) \mid z \geqslant_T y \wedge f(z) \geqslant_T y\}.$$

So in this case we choose a branch $b \in B_\alpha$ and an α-branch $e \in T \upharpoonright \alpha$ such that $f[b \cap (T \upharpoonright \alpha)^x] \subseteq e$ and $b \neq e$ (using **Claim 2.6**) and we extend all branches in $B_\alpha \smallsetminus \{e\}$. By the same argument as in *case* $\star_\mathbf{3}$ for **C1**, we assure normality.

If we are in *case* $\star_\mathbf{4}$, now S_α is in addition a maximal antichain of $T \upharpoonright \alpha$ and we choose a branch $b^* \in B^*_\alpha$ that contains x. By again using **Claim 2.29** we extend all branches in $B^*_\alpha \smallsetminus \{e^*\}$, where e^* is an α-branch containing $f[b^* \cap (T \upharpoonright \alpha)^x] \subseteq e^*$.

To show that the resulting tree is in fact totally rigid, assume f is an isomorphism between two different cones and the argument is completely analogous to that of **Proposition 2.7**. $\square$

Chapter 3

Suslin Trees constructed by Forcing

"Trees are poems the earth writes upon the sky, we fell them down and turn them into paper, that we may record our emptiness."

– Kahlil Gibran

Within the present chapter, we will be using the method of **forcing** to construct Suslin trees with certain rigidity properties. This method has been widely studied since its beginning in the earlies '60's [Coh66] and there are many books in the literature explaining this method, [Jec02] and [Jec86] for example. Nevertheless, we still write in **Section 3.1** a bit of the basics if only to fix some notation.

The two main constructions are due to Jech [Jec67] and Tennenbaum [Ten68], and use countable and finite conditions respectively. Since Jech's generic extension adds a Suslin tree and a $\diamondsuit$-sequence (see **Lemma 3.2**) we concentrate only on Tennenbaum's generic extension using finite conditions. The resulting Suslin tree can be proved to be rigid and level preserving rigid using suitable density arguments, so in **Section 3.2** we give a modification of this forcing notion to obtain a rigid Suslin tree that admits a non-trivial level preserving endomorphism.

3.1 Preliminaries

Let $\mathbb{P}$ be a partial order over a countable transitive model M. Then we call M our **ground model**, $\mathbb{P}$ is going to be our **forcing notion** and the elements of $\mathbb{P}$ are the **forcing conditions**. If two conditions $p, t \in \mathbb{P}$ satisfy $p \leq t$ in $\mathbb{P}$, then we say that p **extends (or is stronger than)** t. If

$$(\exists r \in \mathbb{P})[r \leq p \wedge r \leq t],$$

then p, t are **compatible** conditions. If two conditions are not compatible, then they are incompatible and a set $A \subseteq \mathbb{P}$ of incompatible conditions is an **antichain**. The set $\mathbb{P}$ has the ***c.c.c.*** if every antichain is at most countable.

Notice that if our notion of forcing $\mathbb{P}$ is a tree, two conditions are compatible precisely when they are comparable.

Now, the idea of forcing is to extend our ground model M to a model $M[G]$ containing M, by adding a *generic* set G. To define what G is, first let's say $D \subseteq \mathbb{P}$ is **dense** in $\mathbb{P}$ if,

$$(\forall p \in \mathbb{P})(\exists t \in D)[t \leq p].$$

If $p \in \mathbb{P}$, the set D' is said to be **dense below** p if

$$(\forall q \leq p)(\exists t \in D')[t \leq p].$$

A set $G \subseteq \mathbb{P}$ is **generic over** M if

1. G is a filter,
2. $G \cap D \neq \varnothing$ for every dense $D \subseteq \mathbb{P}$ lying in M.

The generic set G will in general not be in the ground model M and hence $M[G]$ is a proper extension of M. A nice feature of this method is that we can still get some information about $M[G]$ from elements in M. For instance, if $a \in M[G]$, then there is $\dot{a} \in M$ called the **name** for a, with an *interpretation in* $M[G]$, $\dot{a}^G$ for all generic $G \subseteq \mathbb{P}$. Naturally, elements in M also have names and a **canonical name** for an element $a \in M$ is such that $\dot{a}^G = a$, so we won't distinguish between the element and its canonical name. We also have a **forcing relation** $\Vdash$ and a **forcing language**, but we are not discussing these here.

Finally, the main theorem about forcing gives us a relation between formulas in $M[G]$ and in M,

$$M[G] \vDash \varphi(a) \text{ if and only if } (\exists p \in G)[p \Vdash \ulcorner \varphi(\dot{a}) \urcorner],$$

where $p \Vdash \ulcorner \varphi(\dot{a}) \urcorner$ is meant to be read as 'p *forces* $\varphi(\dot{a})$'.

Now we are ready to start our constructions. The next theorem shows how Jech got a Suslin tree using a partial order with countable conditions. This is a very well known result and can be found in [Jec02], so we only sketch the construction.

Theorem 3.1 (Jech, countable conditions)**.** *There is a generic model in which there is a Suslin tree.*

Proof. The forcing notion $\mathbb{P}$ consists of normal α-trees T for $\alpha < \omega_1$, such that

- the elements of T are functions from $\beta < \alpha$ to ω,
- T is closed under initial segments,
- T is ω-splitting,
- $T \leq S$ iff there is $\alpha < ht(T)$ such that $S = T \upharpoonright \alpha$.

This forcing notion is clearly $\aleph_0$-closed since $T = \bigcup_{n \in \omega} T_n$ is an extension of a sequence $T_0 \geq T_1 \geq ...T_n...$ of countably many of these normal trees. Therefore ω_1 is preserved.

If we let G be any generic set of conditions and we let $\mathbf{T} = \bigcup\{T \in G\}$.

The proof that **T** is a normal ω_1-tree follows similar patterns as in **Proposition 2.1**, and every antichain A of **T** is at most countable because the next set is dense in $\mathbb{P}$,

$$D = \{T' \leq T \mid \text{there is a maximal antichain } A' \text{ in } T' \text{ with all elements in } A' \text{ below some fixed } \alpha < ht(T') \text{ and } T' \Vdash \ulcorner \dot{A}' \subset \dot{A} \urcorner\}$$

□

It turns out that this notion of forcing also adds a $\diamondsuit$-sequence (this follows from **Lemma 3.2** below, since the notions of forcing are equivalent) and hence our work in the previous chapter shows that we can get Suslin trees with the same rigidity properties that we got in **Section 2.1** in this generic model.

Lemma 3.2. *There is a notion of forcing that adds a $\diamondsuit$-sequence.*

Proof. The notion of forcing $\mathbb{P}$ consists of countable sequences $p = (S_\xi : \xi < \alpha)$ for $\alpha < \omega_1$ satisfying,

- $S_\xi \subseteq \xi$,
- $p \leq q$ iff p extends q as a sequence.

This notion is countably closed and hence it preserves ω_1. If we let $G \subseteq \mathbb{P}$ be generic over M, then we claim that our $\diamondsuit$-sequence is defined as $S = \bigcup_{p \in G} p$.

Let $M[G] \Vdash \ulcorner X \subseteq \omega_1 \urcorner$. We want to show that

$$M[G] \Vdash \ulcorner \{\alpha \in \omega_1 \mid X \cap \alpha = S_\alpha\} \text{ is stationary in } \omega_1 \urcorner.$$

So let C be a club in ω_1. Then there is a condition $p \in G$ with

$$p \Vdash \ulcorner \dot{C} \text{ is a club } \wedge \dot{X} \subseteq \omega_1 \urcorner.$$

We will show that the following set is dense below p,

$$D = \{q \leq p \mid (\exists \alpha \in \omega_1)[q \Vdash \ulcorner \alpha \in \dot{C} \wedge \dot{X} \cap \alpha = S_\alpha \urcorner]\}.$$

Let $p = (S_\xi : \xi < \beta)$. Since each of the elements of p are countable subsets of ω_1 and $\mathbb{P}$ is countably closed, $X \cap \beta$ is a countable subset that lies in the ground model. So, if we define $S_\beta = X \cap \beta$ then $S_\beta \in M$.

Hence, if we let $p' = p \cup \{S_\beta\}$ then $p' \leq p$ and $p' \Vdash \ulcorner \dot{X} \cap \beta = S_\beta \urcorner$.

Since p' forces C to be unbounded, there is $\beta_0 > \beta$ and $q_0 \leq p'$ such that

$$q_0 \Vdash \ulcorner \beta_0 \in \dot{C} \wedge \dot{X} \cap \beta = S_\beta \urcorner.$$

Now let β_0' be such that $q_0 = (S_\xi : \xi < \beta_0')$ and define $S_{\beta_0'} = X \cap \beta_0'$. Hence, if $q_0' = q_0 \cup \{S_{\beta_0'}\}$ then there is $\beta_1 > \beta_0'$ and $q_1 \leq q_0'$ such that

$$q_1 \Vdash \ulcorner \beta_1 \in \dot{C} \wedge \dot{X} \cap \beta_0' = S_{\beta_0'} \urcorner.$$

Following this procedure, we can find decreasing countable sequences of conditions $(q_n : n \in \omega)$, $(q_n' : n \in \omega)$ and increasing countable sequences of ordinals $(\beta_n : n \in \omega)$, $(\beta_n' : n \in \omega)$ such that,

1. $\beta < \beta_0 < \beta_0' < \beta_1 < \beta_1' \ldots$
2. $p > p' \geq q_0 > q_0' \geq q_1 \ldots$
3. $S_{\beta_n'} \in q_n'$
4. $q_{n+1} \Vdash \ulcorner \beta_{n+1} \in \dot{C} \wedge \dot{X} \cap \beta_n' = S_{\beta_n'} \urcorner$
5. $\sup\limits_{n \in \omega} q_n = \sup\limits_{n \in \omega} q_n' = q_\omega$

Hence $\bigcup\limits_{n \in \omega} q_n = \bigcup\limits_{n \in \omega} q_n' = q_\omega$ and $\sup\limits_{n \in \omega} \beta_n = \sup\limits_{n \in \omega} \beta_n' = \beta_\omega$. So, if we define $S_{\beta_\omega} = \bigcup S_{\beta_n}$

$(= \bigcup S_{\beta'_n})$, then $S_{\beta_\omega} \subseteq \beta_\omega$ and it satisfies

$$q_\omega \Vdash \ulcorner \beta_\omega \in \dot{C} \wedge \dot{X} \cap \beta_\omega = S_{\beta_\omega} \urcorner.$$

Therefore D is dense below p and hence there is $q \in D \cap G$ witnessing

$$M[G] \Vdash \ulcorner C \cap \{\alpha \in \omega_1 \mid X \cap \alpha = S_\alpha\} \neq \varnothing \urcorner.$$

□

Another way to get a Suslin tree by forcing is by using finite conditions, and when working with uncountable many finite sets the following lemma is often useful.

Lemma 3.3 (**Δ- System Lemma**). *Let W be an uncountable collection of finite sets. Then there is an uncountable $Z \subset W$ and a finite set S such that $X \cap Y = S$ for any two distinct $X, Y \in Z$.*

Theorem 3.4 (Tennenbaum, finite conditions). *There is a generic model in which there is a Suslin tree.*

Proof. We provide only the notion of forcing used to prove this result and skip the details since they are similar to the arguments used to construct the tree in **Section 3.2**.

The notion of forcing is the set $\mathbb{P}$ of finite trees $(t, <_t)$ satisfying

- $t \subseteq \omega_1$
- $\alpha <_t \beta \to \alpha < \beta$
- $(t_1, <_{t_1})$ extends $(t_2, <_{t_2})$ iff t_1 extends t_2 as trees.

We let G be a generic set of conditions and define $\mathbf{T}$ as the union $\bigcup\{t : t \in G\}$. Then $\mathbf{T}$ is a Suslin tree. We shall identify the tree $(t_1, <_{t_1})$ with its underlying set t_1.

□

3.2 There is a forcing extension M[G], where there is a Suslin tree which is automorphism rigid but admits a non-trivial level preserving epimorphism (using finite conditions)

Let $\mathbb{P}$ be the notion of forcing consisting of ordered pairs (t, f), where t is a finite tree and f is a homomorphism of $(t, <)$ that preserves the first coordinate, where the finite tree t is required to satisfy $t \subseteq \omega_1 \times \omega$ and $(0, n) \in t \longrightarrow n = 0$ (then $(0,0) \leqslant (\alpha, n)$ for every $(\alpha, n) \in \omega_1 \times \omega$).

Notation.

- Given an enumeration of elements of $\mathbb{P}$, we write $(t, f)_\eta$ in place of (t_η, f_η).
- If t is a finite tree defined as above, then t^0 is the projection to the first coordinate and similarly for t^1. That is

$$t^0 = \{\alpha \mid (\exists n)(\alpha, n) \in t\} \text{ and } t^1 = \{n \mid (\exists \alpha)(\alpha, n) \in t\}$$

The ordering on t must satisfy

$$(\alpha, n) <_t (\beta, m) \longrightarrow \alpha < \beta \text{ as ordinals}$$

so that (α, n) is incompatible with (α, m) in t if $n \neq m$. The partial ordering of $\mathbb{P}$ is given by

$$(t, f)_2 \leq_{\mathbb{P}} (t, f)_1 \longleftrightarrow t_2 \text{ is an extension of } t_1, \text{ and}$$
$$f_1 = f_2 \upharpoonright t_1.$$

The idea is that the first coordinate of an element (α, n) of a finite tree t corresponding to a condition (t, f) will denote the *level* in which the element lies in the tree T

in the generic extension, whereas the second is just part of an enumeration of the level. If (t, f) is in a generic subset of $\mathbb{P}$, then t is a finite subtree of our desired Suslin tree T and f is a partial homomorphism of T which will be extended to an epimorphism of the entire T.

Claim 3.5. *$\mathbb{P}$ satisfies the* c.c.c.

Proof. Let $\mathcal{A}$ be an uncountable antichain in $\mathbb{P}$, $\mathcal{A} = \{(t, f)_\eta \mid \eta \in \omega_1\}$. Let $\mathcal{A}_0 = \{t_\eta \mid (t, f)_\eta \in \mathcal{A}\}$. Then $(t_\eta, \leqslant_{t_\eta})$ is a finite tree for each $\eta \in \omega_1$ and $\mathcal{A}_0$ is also uncountable.

Using the **Δ-System Lemma** we can find $W \subseteq \mathcal{A}_0$ which is also uncountable and a finite set S such that $t_\eta \cap t_\nu = S$ for all $\eta \neq \nu$ in ω_1.

We want S to be an 'initial' segment of uncountably many elements of W. For this, we need the conditions to agree in the ordering and also push down S.

So, let $t_\eta \cong t_\nu$ whenever $(\leqslant_{t_\eta} \upharpoonright S) = (\leqslant_{t_\nu} \upharpoonright S)$. Since S is finite there are only finitely many possibilities for the order on S and hence there is an uncountable equivalence class $W_1 \subseteq W$.

The next lemma will be used very much during the rest of the construction.

Lemma 3.6 (($\star$): **Pushing down** X)**.** *For each $\alpha \in \omega_1$ and a finite X with $S^0 \subseteq X \subseteq \alpha$ and $\max(X) < \alpha$, there is an uncountable subset $Z \subseteq W_1$ such that $\min(t^0 \smallsetminus X) \geqslant \alpha$, for every $t \in Z$.*

Proof. Let α and X as stated. Then $\alpha \smallsetminus X$ has only countably many finite subsets and thus the set $B = \{t^0 \cap \alpha \neq X \mid t \in W_1\}$ is a family of finite subsets of α that is disjoint outside X (since X contains S^0 and $t_\eta \cap t_\nu = S$ for every $t_\eta, t_\nu \in W_1$) and hence it is countable. See **Figure 3.1**.

Therefore, the set $Z = \{t \in W_1 \mid t \cap \alpha = X\}$ is uncountable, and since all of them agree only on S, below α, we get $\min(t^0 \smallsetminus S^0) \geqslant \alpha$ for every $t \in Z$. □

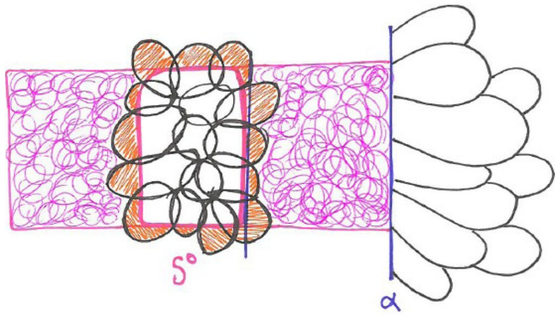

FIGURE 3.1: Pushing down X

Hence, using **Lemma** $(\star)$, we can find an uncountable $W_2 \subseteq W_1$ such that, if $\beta_{\max} = \max\{\beta : (\beta, k) \in S\}$ and $\alpha_\eta = \min\{\alpha \in t_\eta : \alpha \notin S^0 \wedge t_\eta \in W_1\}$, then

$$W_2 = \{t_\eta \mid \beta_{\max} < \alpha_\eta\}$$

is uncountable, and so is

$$W_2' = \{(t, f)_\eta \in \mathcal{A} \mid t_\eta \in W_2\}$$

Now, let $(t, f)_\eta, (t, f)_\nu$ be elements of W_2'. Then $(t, f)_\eta \cong_1 (t, f)_\nu$ if and only if $(f_\eta \upharpoonright S) = (f_\nu \upharpoonright S)$, that is, if f_η and f_ν agree on how they act on S. Since S is finite, there are finitely many ways in which a homomorphism can act on S and hence there is an uncountable $\cong_1$-equivalence class. Let $W_3 \subseteq W_2'$ be such an equivalence class.

Thus, for every $(t, f)_\eta \neq (t, f)_\nu$ in W_3, S is an initial segment of t_ν, t_η and $(f_\nu \upharpoonright S) = (f_\eta \upharpoonright S)$.

Let $(q, g)_1$ be any element of W_3. Then there is $(q, g)_2 \in W_3$ such that $(q, g)_1$ is an initial segment of some $(q, g)_2$ using **Lemma** $(\star)$ and $X = q_1$, with the additional property that $(g_1 \upharpoonright q_1) = (g_2 \upharpoonright q_1)$. Considering $q = q_1 \cup q_2$ with $g = g_1$ on q_1 and $g = g_2$ on q_2 shows that q_1, q_2 are compatible. However, q_1 and q_2 are both in $\mathcal{A}$, so $\mathcal{A}$ cannot be an antichain of $\mathbb{P}$. □

Hence $\mathbb{P}$ preserves ω_1 and we can define the following structure

$$T = \bigcup_{(t,f)\in G} t$$

for some generic subset G over $\mathbb{P}$. Then T is clearly a tree and we can show that it is a splitting tree. Let

$$M[G] \vDash \ulcorner(\gamma, k) \in T\urcorner$$

for some $\gamma \in \omega_1$ and $k \in \omega$. Then there is $(t, f) \in G$ satisfying,

$$(t, f) \Vdash \ulcorner(\gamma, k) \in \dot{T}\urcorner$$

Notice that we can choose $(t, f)'$ extending (t, f) so that $(\gamma, k) \in t'$. To do this, let

$$\alpha_{max} = \max\{\alpha : (\alpha, n) \in t\} \text{ and } n_{max} = \max\{n : (\alpha_{max}, n) \in t\}$$

Assume that (γ, k) is not in t. Then we can let $t' = t \cup \{(\gamma, k)\}$ and the order is given by,

$$\leqslant_{t'} \restriction t = \leqslant_t \text{ and } (\gamma, k) >_{t'} (\alpha, n)_{max}$$

We let f' act on t' by defining,

$$f' \restriction t = f \text{ and } f'(\gamma, k) = f(\alpha, n)_{max}$$

Therefore $(t, f)'$ extends (t, f) and t' contains the element (γ, k).

Next, we shall show that the following set is dense,

$$D = \{(q, g) \leq (t, f)' \mid (\exists \alpha, \beta \in \omega_1)[\alpha, \beta > \gamma \wedge n_\alpha, n_\beta > k,$$
$$(\alpha, n_\alpha), (\beta, n_\beta) \in q \wedge n_\alpha \neq n_\beta]\}$$

Let $\alpha = \beta$ be greater than every ordinal appearing in t' and choose any $n_\alpha \in \omega$, say $n_\alpha = 0$. Then define $q = t' \cup \{(\alpha, 0), (\alpha, 1)\}$ with the ordering $\leqslant_{t'} = \leqslant_q \restriction t'$ and

$(\gamma, k) \leqslant_q (\alpha, 0), (\alpha, 1)$. Additionally, we let g be equal to f' on t' and $g(\alpha, 0) = g(\alpha, 1) = f'(\gamma, k)$.

So $(q, g) \in D$, D is a dense subset of $\mathbb{P}$ and hence $G \cap D$ is non-empty and thus there is an extension (q, g) of (t, f) in G such that $(q, g) \Vdash \ulcorner \dot{T}$ is a splitting tree $\urcorner$.

Now we want to show that T is Suslin.

Claim 3.7. *Every antichain in T is at most countable.*

Proof. Assume towards a contradiction that there is $A \subseteq T$ in the generic extension such that,

$$M[G] \models \ulcorner \dot{A} \text{ is an uncountable antichain of } \dot{T} \urcorner$$

Then there is a condition $(f, t)_0 \in G$ such that

$$(t, f)_0 \Vdash \ulcorner \dot{A} \text{ is an uncountable antichain of } \dot{T} \urcorner$$

Then t_0 is a finite approximation to T. Let $(t, f)_1$ be an extension of $(t, f)_0$ and $(\alpha, n)_1 \in \omega_1 \times \omega$ with the properties,

- $(\alpha, n)_1 \notin t_0$
- $(\alpha, n)_1 \in t_1$
- $(t, f)_1 \Vdash \ulcorner \dot{A}$ is uncountable and $(\alpha, n)_1 \in \dot{A} \urcorner$

Similarly, we can get an uncountable set $W = \{(t, f)_\eta \in \mathbb{P} \mid \eta \in \omega_1\}$ of conditions and a corresponding set $Z = \{(\alpha, n)_\eta \in \omega_1 \times \omega \mid \eta \in \omega_1\}$ (all of whose elements are different) such that,

- $(f,t)_\eta$ extends $(f,t)_0$
- $(\alpha,n)_\eta \notin t_0$
- $(\alpha,n)_\eta \in t_\eta$
- $(t,f)_\eta \Vdash \ulcorner \dot{A}$ is uncountable and $(\alpha,n)_\eta \in \dot{A}\urcorner$
- And $(\alpha,n)_\eta$ is maximal in t_η with this property.

Then the set $W^0 = \{t_\eta \mid (t,f)_\eta \in W\}$ is also uncountable and the **Δ-System Lemma** gives us an uncountable subset $W^0_1 \subseteq W^0$ and a finite S such that $t_\eta \cap t_\nu = S$ for every $t_\eta, t_\nu \in W^0_1$.

Since there are only finitely many ways in which we can order S, there is an uncountable subset $W^0_2 \subseteq W^0_1$ such that

$$(\leqslant_{t_\eta} \upharpoonright S) = (\leqslant_{t_\nu} \upharpoonright S) \text{ for every two different elements } t_\eta, t_\nu \in W^0_2$$

Now we 'push down' the subset S so that it is an initial subset of uncountably many elements in W^0_2 using **Lemma (⋆)**. So, if we define $\min(t_\eta \smallsetminus S)$ and $\max(S)$ as follows,

$$\min(t_\eta \smallsetminus S) = (\alpha,n)_{min} \text{ and } \max(t_\eta \smallsetminus S) = (\alpha,n)_{max} \text{ with}$$
$$\alpha_{min} = \min(t^0_\eta \smallsetminus S^0), n_{min} = \min\{k : (\alpha_{min},k) \in t_\eta \smallsetminus S\} \text{ and}$$
$$\alpha_{max} = \max(S^0), n_{max} = \max\{k : (\alpha_{max},k) \in S\}$$

we can find an uncountable subset $W^0_3 \subseteq W^0_2$ with

$$W^0_3 = \{t_\eta \in W^0_2 \mid \min(t_\eta \smallsetminus S) > \max(S)\}.$$

Since S is finite there are finitely many ways in which an homomorphism can act on

S, so we let $(t,f)_\eta \cong_1 (t,f)_\nu$ whenever $t_\eta, t_\nu \in W_3^0$ and $(f_\eta \upharpoonright S) = (f_\nu \upharpoonright S)$. Thus, there is an uncountable $\cong_1$-equivalence class W_4.

Therefore for every two elements $(t,f)_\eta \neq (t,f)_\nu$ of W_4 the following properties, which we denote by $[*]$ hold,

1. S is an initial segment of t_η,
2. $(f_\eta \upharpoonright S) = (f_\nu \upharpoonright S)$,
3. $t_\eta \cap t_\nu = S$
4. $(\alpha,n)_\eta \in t_\eta$,
5. $(\alpha,n)_\eta \notin t_0$
6. $(t,f)_\eta \Vdash \ulcorner \dot{A}$ is uncountable and $(\alpha,n)_\eta \in \dot{A}\urcorner$

Moreover, we can find an uncountable subset $W_5 \subseteq W_4$ with the additional property that $(\alpha,n)_\eta$ is not in S. To show this, we assume to the contrary that for every uncountable $Z \subseteq W_4$, there is a condition $(t,f)_eta \in Z$ such that $(\alpha,n)_\eta \in S$. Let $W_4 = \bigcup_{n\in\omega} Z_n$ be a partition into countably many subsets $Z_n \subset W_4$ such that $|Z_n| = \aleph_1$. Then for each Z_m, there us a corresponding $(t,f)_{\eta_m} \in Z_m$ with $(\alpha,n)_{\eta_m} \in S$. Since S is finite and the Z_m's disjoint, there are two distinct $(\alpha,n)_{\eta_m}$ and $(\alpha,n)_{\eta_k}$ which are comparable in S and such that

$$(t,f)_{\eta_m} \Vdash \ulcorner \dot{A} \text{ is an antichain} \wedge (\alpha,n)_{\eta_m} \in S\urcorner$$

and

$$(t,f)_{\eta_k} \Vdash \ulcorner \dot{A} \text{ is an antichain} \wedge (\alpha,n)_{\eta_k} \in S\urcorner.$$

Now we can construct a condition (r,h) that extends both $(t,f)_{\eta_m}$ and $(t,f)_{\eta_k}$ which will give us a contradiction. Since $t_{eta_m} \cap t_{\eta_k} = S$ we define (r,h) by

$$r = t_{eta_m} \cup t_{\eta_k} \text{ with the ordering}$$

$$\leqslant_r \restriction t_{eta_m} = \leqslant_{t_{eta_m}}$$

$$\leqslant_r \restriction t_{eta_k} = \leqslant_{t_{eta_k}}$$

And the homomorphism acting by

$$h \restriction t_{\eta_m} = t_{\eta_m} \text{ and } h \restriction t_{\eta_k} = t_{\eta_k}$$

And finally (r,h) satisfies,

$$(r,h) \Vdash \ulcorner \dot{A} \text{ is an antichain} \wedge (\alpha,n)_{\eta_m}, (\alpha,n)_{\eta_m} \in \dot{A} \in \dot{A} \text{ and they are compatible } \urcorner.$$

This gives a contradiction.

Thus W_5 satisfies the properties in [$\circledast$] with the following modification,

$$4'. \, (\alpha,n)_\eta \in t_\eta \smallsetminus S,$$

Let $(q,g)_0 \in W_5$ be arbitrary. Then, it follows from **Lemma** ($\star$) that there there is an uncountable $W_6 \subseteq W_5$ with the additional property,

$$7. \, \min(t_\eta \smallsetminus q_0) = \min(t_\eta \smallsetminus S) > \max(q_0)$$

Also, it is possible to have $(\alpha,n)_\eta >_{t_\eta} (\beta,k)$ for uncountably many ν's and a fixed maximal $(\beta,k) \in S$, since S is finite. So let $W_7 \subseteq W_6$ be the set satisfying also the following condition for b_η, b_ν maximal branches of t_η, t_ν respectively and $(t,f)_\eta, (t,f)_\nu \in W_7$,

$$8. \, b_\eta \cap S = b_\nu \cap S$$

Let b_{q_0} be a branch on q_0 containing $(\alpha, n)_{q_0}$ (where $(q,g)_0 \Vdash \ulcorner(\alpha,n)_{q_0} \in \dot{A}\urcorner$). Then, condition 8. assures us that every branch on an element $t_\eta \in W_7^0$ containing $(\alpha, n)_\eta$, agrees on S. So, let $(t, f)_\tau$ be an arbitrary element of W_7. We will construct a condition (q, g) extending $(q, g)_0$ and $(t, f)_\tau$ (see **Figure 3.2**),

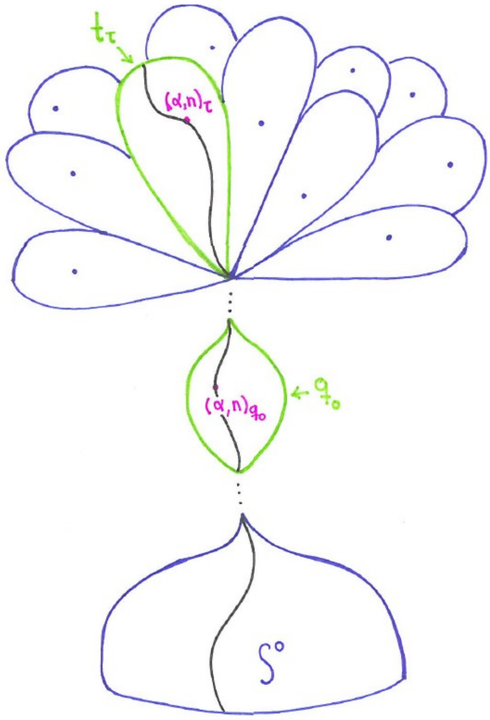

FIGURE 3.2: Choosing q.

$q = q_0 \cup t_\tau$ with the ordering

$$\leqslant_q \upharpoonright q_0 = \leqslant_{q_0}$$

$$\leqslant_q \upharpoonright t_\tau = \leqslant_{t_\tau}$$

$$\max(b_{q_0}) <_q \min(b_\tau \setminus S)$$

Then $(q, \leqslant_q)$ is a finite tree and we let g be defined by

$$g \upharpoonright q_0 = g_0 \text{ and } g \upharpoonright t_\tau = f_\tau$$

Thus,

$$(q,g) \Vdash \ulcorner \dot{A} \text{ is an uncountable antichain of } \dot{T}, (\alpha,n)_\tau, (\alpha,n)_{q_0} \in \dot{A}$$
$$\text{and } (\alpha,n)_\tau, (\alpha,n)_{q_0} \text{ are compatible } \urcorner$$

Giving clearly a contradiction. □

Hence, there are no uncountable antichains in T. With this and the fact that T is splitting we get that T is actually an ω_1- Suslin tree.

Now, remembering that $T = \bigcup_{(t,f) \in G} t$ for some generic $G \subseteq \mathbb{P}$, if we let F be the function on T defined by

$$F(\alpha,n) = f(\alpha,n) \text{ for } (\alpha,n) \in t \text{ and } (t,f) \in G,$$

then, we can prove that this is a non-identity level preserving epimorphism of T.

Claim 3.8. *The α-th level of T is the set $\{(\alpha,n) \in T \mid n \in \omega\}$.*

Proof. Let $M[G] \vDash \ulcorner(\alpha,n) \in T\urcorner$. Then there is a condition $(t,f) \in G$ with $(t,f) \Vdash \ulcorner(\alpha,n) \in \dot{T}\urcorner$. We have to show that the set $C = \{s \in T \mid s < (\alpha,n)\}$ is a chain of order-type α, so we can assume $(\alpha,n) \in t$. Since we have asked for $(0,n) \in t \longrightarrow n = 0$, we can also assume $\alpha > 0$.

We shall prove that C is in fact the set $\{(\beta,m_\beta) \mid \beta < \alpha\}$ for some $m_\beta \in \omega$,* that is, for each $\beta < \alpha$, the following set is dense below (t,f),

$$D_\beta = \{(q,g) \leq (t,f) \mid (\exists m)(\beta,m) \in C \cap q\}.$$

Let $(q,g) \leq (t,f)$. We look at the first projection of the *root* of q, α_{min}. Take $\gamma^* = \max\{\gamma \leqslant \beta : (\gamma,k) <_q (\alpha,n)\}$ and $n^* = \max\{n : (\gamma,k) \in q\}$. If $\gamma^* = \beta$ we are done. If $\alpha_{min} < \beta < \alpha$ as ordinals, define $q' = q \cup \{(\beta,n_\beta)\}$ with $n_\beta = n^*+1$, ordered by

*This works because we have defined our finite trees so that C cannot contain (β,m_1) and (β,m_2) for $m_1 \neq m_2$, so $(\beta,m_\beta) < (\gamma,n_\gamma) \longleftrightarrow \beta < \gamma$

$(\gamma, n)^* <_q (\beta, n_\beta) <_q (\alpha, n)$ and g' extending g as $g'(\beta, n_\beta) = g(\gamma, n)^*$. If $\beta \leqslant \alpha_{min}$, we just let (β, n_β) be the *new* root of $q' = q \cup \{(\beta, n_\beta)\}$ with $g'(\beta, n_\beta) = (\beta, n_\beta)$. □

Hence, the resulting map F is actually a level preserving map on T, since we have required from a condition $(t, f) \in \mathbb{P}$ that the homomorphism f preserves the first coordinate, so the next proposition shows that F is an epimorphism.

Proposition 3.9. *F is a non-trivial level preserving epimorphism of T.*

Proof. Let $M[G] \vDash \ulcorner(\alpha, n)_0 \in \dot{T}\urcorner$. Then there is a condition $(t, f)_0 \in G$ such that

$$(t, f)_0 \Vdash \ulcorner(\alpha, n)_0 \in \dot{T}\urcorner$$

and by extending $(t, f)_0$ if necessary we can assume $(\alpha, n)_0 \in t_0$. We want to show that there is a condition $(q, g) \in G$ and $(\alpha_0, m) \in \omega_1 \times \omega$ such that

$$(q, g) \Vdash \ulcorner\dot{F}(\alpha_0, m) = (\alpha, n)_0\urcorner$$

For this, we shall show that the following set is dense in $\mathbb{P}$.

$$D = \{(t, f) \leq (t, f)_0 \mid (\exists m \in \omega)[(\alpha_0, m) \in t \wedge f(\alpha_0, m) = (\alpha, n)_0 \wedge m \neq n_0]\}$$

Take (t, f) extending $(t, f)_0$. If there is $m \neq n_0$ such that $(\alpha_0, m) \in f^{-1}(\alpha, n)_0 \subseteq t$, then $(t, f) \in D$. So assume there is no such $(\alpha_0, m) \in t$, that is $f^{-1}(\alpha, n)_0 = \varnothing$ or $f(\alpha, n)_0 = (\alpha, n)_0$.

Since we have asked from our conditions that $(0, n) \longrightarrow n = 0$, and f preserves the first coordinate, $f(0, 0) = (0, 0)$ and hence we are also assuming $(0, 0) \notin t$. Look at the predecessors of $(\alpha, n)_0$. If there is some (maximal) $(\alpha, n)_1 <_t (\alpha, n)_0$ such that $f(\alpha_1, k) = (\alpha, n)_1$ for some $k \neq n_1$ in ω, let (α_0, m) be such that $m > \max\{n : (\alpha_0, n) \in t\}$ and define (q, g) as follows (see **Figure 3.3**);

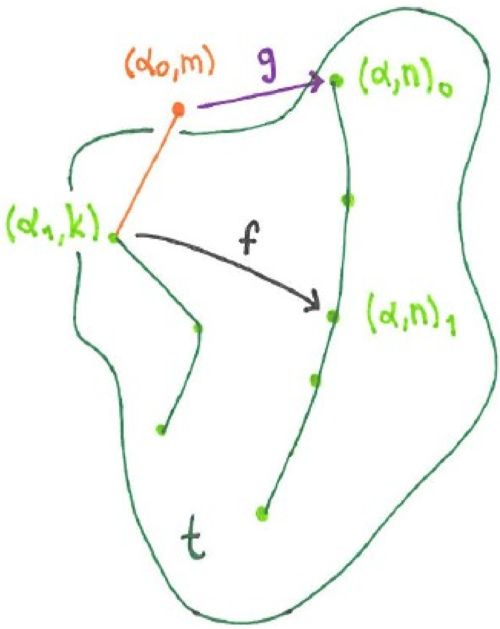

FIGURE 3.3: Extending t when $(\exists(\alpha,n)_1 <_t (\alpha,n)_0)[f(\alpha_1,k)=(\alpha,n)_1]$

$$q = t \cup \{(\alpha_0, m)\} \text{ with the ordering}$$

$$\leqslant_q \restriction t = \leqslant_t$$

$$(\alpha_0, m) >_q (\alpha_0, k)$$

Then $(q, \leqslant_q)$ is a finite tree and define g by

$$g \restriction t = f \text{ and } g(\alpha_0, m) = (\alpha, n)_0$$

Then g is an homomorphism since $(\alpha_1, k) <_q (\alpha_0, m)$ implies $g(\alpha_1, k) = (\alpha, n)_1 <_q (\alpha, n)_0 = g(\alpha_0, m)$.

Otherwise, f^{-1} is either empty or f is the identity on every element $(\alpha, n)_i <_t (\alpha, n)_0$, for $i \in I$ and some finite index set I. Then, for each $(\alpha, n)_i \leqslant_t (\alpha, n)_0$, we choose (α_i, m), with $m > \max\{n_i : i \in I\}$, and let (q, g) be defined as follows (See **Figure**

3.4),

$$q = t \cup \{(0,0\} \cup \{(\alpha_i, m) \mid (\alpha, n)_i \leqslant_t (\alpha, n)_0\} \text{ with the ordering}$$

$$\leqslant_q \restriction t = \leqslant_t$$

$$(\alpha_i, m) <_q (\alpha_{i+1}, m)$$

$$(0,0) <_q (\alpha, n)_{\max(I)}, (\alpha_{\max(I)}, m)$$

So $(q, \leqslant_q)$ is a finite tree and we can define g as,

$$g \restriction t = f \text{ and } g(\alpha_i, m) = (\alpha, n)_i$$

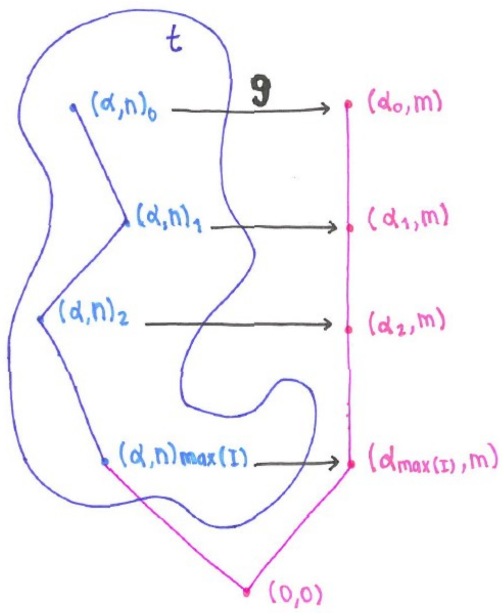

FIGURE 3.4: Extending t when f^{-1} is either empty or f is the identity on $(\alpha, n)_{0\downarrow}$

Hence we have found that (q, g) is in D, and since D is dense in $\mathbb{P}$, there is (q, g) in the intersection $D \cap G$ for some generic $G \subseteq \mathbb{P}$ and thus $(q, g) \Vdash \ulcorner \dot{F}(\alpha_0, m) = (\alpha, n)_0 \urcorner$. $\square$

Hence, T is a Suslin tree that admits a non-trivial level preserving epimorphism F, and now we show that in addition it stays rigid.

Proposition 3.10. *T admits no automorphism other than the identity.*

Proof. Let $M[G] \vDash \ulcorner \sigma$ is a non-trivial automorphism of $T\urcorner$ for some generic $G \subseteq \mathbb{P}$. Then there is a condition $(q,g)_0 \in G$ with

$$(q,g)_0 \Vdash \ulcorner \dot{\sigma} \text{ is a non-trivial automorphism of } \dot{T}\urcorner.$$

Since T is an ω_1-tree which splits, if there is a point in T that is moved by σ, the whole cone above that point is moved by σ and thus, there is a condition $(g,q)_1 \in G$ extending $(q,g)_0$ with,

$$(q,g)_1 \Vdash \ulcorner \text{ supp}(\dot{\sigma}) \text{ is uncountable}\urcorner.$$

Now, because ω_1 is preserved in $M[G]$, for each different $(\tau, n_\tau) \in \text{supp}(\sigma)$ there is a condition $(t,f)_\tau \in G$ such that

$$(t,f)_\tau \Vdash \ulcorner \dot{\sigma}(\tau, n_\tau) = (\tau, n_\tau)', \tau = \tau' \wedge n_\tau \neq n'_\tau\urcorner.$$

By adjoining them if necessary we can assume that both (τ, n_τ) and (τ, n'_τ) are in t_τ. Then, the set

$$W = \{(t,f)_\tau \in G \mid (t,f)_\tau \Vdash \ulcorner \dot{\sigma}(\tau, n_\tau) = (\tau, n'_\tau) \wedge n_\tau \neq n'_\tau\urcorner\}$$

is uncountable, and so is the set $W^0 = \{t_\tau \mid (t,f)_\tau \in W\}$. Using the **Δ-System Lemma** several times just as we have been doing, we get a set W_4 satisfying conditions similar to [∗] for two different elements of W_4, $(t,f)_\tau$ and $(t,f)_\eta$,

i) S is an initial segment of t_τ,

ii) $(f_\eta \restriction S) = (f_\tau \restriction S)$,

iii) $t_\eta \cap t_\tau = S$

iv) $(\tau, n_\tau), (\tau, n'_\tau) \in t_\tau$,

v) $(\tau, n_\tau), (\tau, n'_\tau) \notin q_1$

vi) $(t,f)_\tau \Vdash \ulcorner \dot{\sigma}(\tau, n_\tau) = (\tau, n'_\tau) \wedge n_\tau \neq n'_\tau\urcorner$

Next, we find an uncountable $W_5 \subseteq W_4$ whose elements satisfy in addition,

$$vii)\ (\tau, n_\tau), (\tau, n'_\tau) \in t_\tau \setminus S.$$

For this, we assume on the contrary that there are uncountably many elements $(t, f)_\tau \in W_4$ such that $(\tau, n_\tau) \in S$. But then there is $(\beta, m) \in S$ such that $(\tau, n_\tau) = (\beta, m)$ for uncountably many elements in W_4, but this is impossible since we have asked for (τ, n_τ) to be different from (η, n_η) whenever $\tau \neq \eta$. The same argument applies for (τ, n'_τ) because σ is an automorphism. So we have an uncountable subset of W_4 as claimed.

Moreover, since S is finite, there is an uncountable $W_6 \subseteq W_5$ satisfying the following for branches $b_\tau \in [t_\tau]$ and $b_\eta \in [t_\eta]$ containing $(\tau, n_\tau), (\eta, n_\eta)$ respectively,

$$viii)\ b_\tau \cap S = b_\eta \cap S \text{ for all } \tau \neq \eta$$

So we have uncountably many elements $(t, f)_\tau$ whose corresponding branches b_τ agree on S (which is an initial segment of t_τ).

Now let $(q, g)_2 \in W_6$ with $(q, g)_2 \Vdash \ulcorner \dot{\sigma}(\alpha, k)_2 = (\alpha, k')_2 \urcorner$. Then we can get an uncountable $W_7 \subseteq W_6$ whose elements agree only on S, by **Lemma** ($\star$).

$$ix)\ \min(t_\tau \setminus q_2) = \min(t_\tau \setminus S) > \max(q_2) \text{ as ordinals.}$$

So, what we want is to find a condition $(q, g)_3$ extending $(q, g)_2 \in G$ such that

$$(q, g)_3 \Vdash \ulcorner \dot{\sigma}(\tau, n_\tau) = (\tau, n'_\tau), \dot{\sigma}(\eta, n_\eta) = (\eta, n'_\eta) \wedge n_\tau \neq n'_\tau, n_\eta \neq n'_\eta$$

$$\text{and } (\tau, n_\tau) <_{q_3} (\eta, n_\eta) \text{ but } (\tau, n'_\tau) \nleq_{q_3} (\eta, n'_\eta) \urcorner$$

which would give us that $(q, g)_3 \Vdash \ulcorner \dot{\sigma}$ is not an automorphism of $\dot{T} \urcorner$, contradicting or original assumption. So, take any element $(t, f)_\tau \in W_7$ and let$(q, g)_3$ be as follows (see **Figure 3.5**),

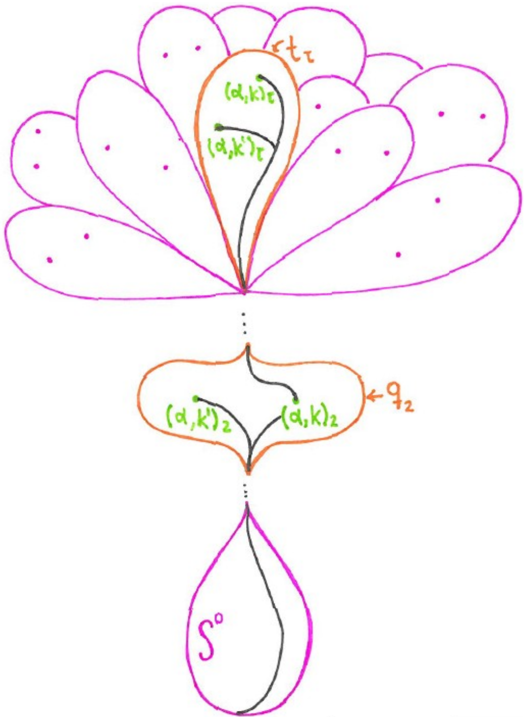

FIGURE 3.5: Choosing q_3

$q_3 = q_2 \cup t_\tau$ with the ordering

$$\leqslant_{q_3} \restriction q_2 = \leqslant_{q_2}$$

$$\leqslant_{q_3} \restriction t_\tau = \leqslant_{t_\tau}$$

$$\min(b_\tau \smallsetminus S) >_{q_3} \max(b_{q_2} \smallsetminus S) \text{ with}$$

$$(g_3 \restriction q_2) = g_2 \text{ and } (g_3 \restriction t_\tau) = f_\tau$$

where b_{q_2} is a branch containing $(\alpha, k)_2$. Then clearly $(\tau, n_\tau) >_{q_3} (\alpha, k)_2$ but (τ, n'_τ) cannot be compatible with $(\alpha, k')_2$ because this point is incompatible with $(\alpha, k)_2$, giving us a contradiction. □

Chapter 4

Dense Subchains of $\mathbb{R}$

"Each move is dictated by the previous one – that is the meaning of order."

– Tom Stoppard, *Rosencrantz and Guildenstern Are Dead*

In this chapter we are concerned with rigidity properties of subchains of the real line. **Section 4.1** introduces some notation and explains the method used by Dushnik and Miller in [DM40] to give the background that will be needed to carry out all the constructions in this chapter. The following section (**Section 4.2**) presents our construction of a dense subset X of $\mathbb{R}$ that is rigid and whose Epi monoid contains $(\mathbb{N}^2, +)$, which shows in particular that it is possible to use this method to ensure that $(X, \leqslant)$ can be rigid but $\mathrm{Epi}(X, \leqslant)$ is non-trivial. In **Section 4.3** we give an example of a densely ordered chain $(X, \leqslant)$ with trivial epimorphism monoid and non-trivial embedding monoid. **Section 4.4** treats the case of endomorphisms, showing that we can keep the epimorphism and embedding monoids trivial (and hence the structure is still rigid) while having a non-trivial endomorphism monoid. Lastly, **Section 4.5** discusses the case when our chains have cardinality $\kappa > \aleph_0$.

4.1 Background

Following the method used by Dushnik and Miller to produce a rigid dense subset of $\mathbb{R}$ we shall construct two disjoint sets X and Y. Here X will be our desired set while Y contains elements that prevent each non-identity automorphism (embedding or epimorphism depending on the case). The construction is done by a transfinite induction along an enumeration of the monoid over $\mathbb{R}$ which we want to trivialise over X, adding elements to X and Y at each successor step and taking unions at limit steps. For this to work, by a diagonal argument, we need to consider only $2^{\aleph_0} = \mathfrak{c}$ many functions. For if $\mathbb{Q} \subseteq X \subseteq \mathbb{R}$, then it is easy to see that $|\mathrm{Aut}(X, \leqslant)| = \mathfrak{c}$, since any automorphism is determined by its action on $\mathbb{Q}$, and $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$; this (continuity) argument extends easily to $\mathrm{Epi}(X, \leqslant)$. For $\mathrm{Emb}(X, \leqslant)$ a sightly more involved argument is required and it follows from the next result.

Lemma 4.1. *If $\mathbb{Q} \subseteq X \subseteq \mathbb{R}$, where Xhas cardinality $\mathfrak{c}$, then* $|\mathrm{End}(\mathrm{X}, \leqslant)| = \mathfrak{c}$.

Proof. Let $f \in \mathrm{End}(X, \leqslant)$ be an order preserving function, and let D_f be the set of its discontinuities in X. Then D_f is countable: since f is order preserving, for every $x \in X$ it has left and right limits, f_x^- and f_x^+ respectively, and by monotonicity a point x is a discontinuity if and only if $f_x^- < f_x^+$. Let U_x be the open interval (f_x^-, f_x^+), for each x in D_f. Then these intervals are disjoint for any two distinct points in D_f, for if $x < y$ are points in X and we choose $z \in (x, y)$, it must be the case that $f_x^+ \leqslant fz \leqslant f_y^-$ and hence $(f_x^-, f_x^+) \cap (f_y^-, f_y^+) = \varnothing$. Since each of these intervals contains a distinct rational there are countably many intervals U_x, therefore countably many points in D_f.

Now let C be a countable subset of X and let F_C be the set of order preserving functions $f \in \mathrm{End}(X \leqslant)$ whose set of discontinuities D_f is contained in C. Then if $f, g \in F_C$ agree on the set $C \cup \mathbb{Q}$, then they must agree on the whole of X. To prove the last statement let $x \in X \setminus (C \cup \mathbb{Q})$, and take a sequence $(x_n)_{n \in \mathbb{N}}$ of elements in $\mathbb{Q}$ converging to x. Since x is not in D_f or D_g, both fx_n and gx_n converge, but $fx_n = gx_n$ by hypothesis and the functions are continuous at x, therefore $gx = fx$.

Finally, there are $\mathfrak{c}$ many functions $f : \mathbb{Q} \cup C \to X$ and hence $|F_C|$ is at most $\mathfrak{c}$. Also, there are $\mathfrak{c}$ many possible choices of C, therefore the cardinality of the union $\bigcup_{C \subset X} F_C$, which is the set of all order-preserving functions, is at most $\mathfrak{c}$. Since there are at least as many order-preserving functions as elements in X, the cardinality of the union above is precisely $\mathfrak{c}$. □

We now turn to the construction based on Dushnk and Miller. We construct, by transfinite induction, sequences $(X_\alpha)_{\alpha<\mathfrak{c}}$,$(Y_\alpha)_{\alpha<\mathfrak{c}}$ such that for each $\alpha < \mathfrak{c}$,

1*. $X_\alpha \cap Y_\alpha = \varnothing$

2*. $|X_\alpha|, |Y_\alpha| < \mathfrak{c}$

and we set $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$ and $Y = \bigcup_{\alpha<\mathfrak{c}} Y_\alpha$. We start the constructions by defining $X_0 = \mathbb{Q}$, so that the resulting X is necessarily dense, and $Y_0 = \varnothing$. Since X is dense, every element of the monoid on X extends to an element of the same monoid on $\mathbb{R}$, as follows from the proof of **Lemma 4.1** (if $x \in \mathbb{R} \smallsetminus X$, then x is the supremum of countably many elements of X, $(x_n : n \in \omega)$, and we let $f(x) = \sup_{n\in\omega} f(x_n)$). To see that it is uniquely extended in the case of Aut and Epi see **Lemma 4.3**.

Our definition of $X_{\alpha+1}$ will depend on what are we trying to preserve on our final X, but will in all cases contain a to-be-chosen element x_α satisfying certain requirements. On the other hand $Y_{\alpha+1} = Y_\alpha \cup \{f_\alpha x_\alpha\}$ will be the same in each case (where f_α is an element of the monoid we are trying to maintain trivial). The whole construction will be based on finding a point x_α not previously in our already constructed X_α such that conditions **1*** and **2*** are also satisfied by $X_{\alpha+1}$ and $Y_{\alpha+1}$. Usually this point will be found among the elements of a set I such that $f_\alpha I \cap I = \varnothing$; so the construction is reduced to finding an appropriate I and choosing a point here such that $X_{\alpha+1}, Y_{\alpha+1}$ also satisfy the requirements imposed by conditions **1*** and **2***.

Since all members of the monoids that we will try to 'kill' are non-trivial there is always a point x^* which is moved by f_α. As a remark we prove the following useful lemma.

Lemma 4.2. *If f is an order preserving function on $\mathbb{R}$ and $fx \neq x$ for some $x \in \mathbb{R}$, then there is a non-empty open interval U such that $fU \cap U = \varnothing$.*

Proof. Assume without loss of generality that $fx < x$. Let $\epsilon = \frac{1}{2}|fx - x|$ and take $U = (x-\epsilon, x)$. Then since f is order preserving, for every $y \in U$, $y < x$ implies $fy \leqslant fx$ and hence $fU \cap U = \varnothing$. □

This lemma allows us to simplify some arguments in the constructions for if an order preserving function moves an integer, then it must move an interval containing it.

As remarked above, any automorphism of a dense subset of $\mathbb{R}$ extends uniquely to an automorphism of the whole real line; the same result is required for epimorphisms and we make use of the following lemma.

Lemma 4.3. *If* $f \in \mathrm{Epi}(\mathrm{X}, \leqslant)$ *for a dense* $X \subseteq \mathbb{R}$*, then* $f = F \restriction X$ *for a unique* $F \in \mathrm{Epi}(\mathbb{R}, \leqslant)$.

Proof. Define $F : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$Fx = \sup\{fy \mid y \in X, y \leqslant x\}.$$

Then by monotonicity of f on X it is clear that $Fx = fx$ for every $x \in X$.

1. *F is well-defined.* By uniqueness of the supremum and since f is itself well defined, F is well defined as well.

2. *F is order preserving.* Let $x < y$ be elements in $\mathbb{R}$. Then

$$Fx = \sup\{fz \mid z \in X, z \leqslant x\} \text{ and } Fy = \sup\{fw \mid w \in X, w \leqslant y\}$$

but $z \leqslant x$ implies $z \leqslant y$ by assumption, therefore

$$\{fz \mid z \in X, z \leqslant x\} \subseteq \{fw \mid w \in X, w \leqslant y\}$$

and hence $Fx \leqslant Fy$.

3. *F is surjective.* Let $y \in \mathbb{R}$. If $y \in X$ this follows from f being surjective on X, therefore we will only consider $y \in \mathbb{R} \smallsetminus X$. We will show that if $A = \{z \in$

$X \mid fz \leqslant y\}$ and $x = \sup A$ then $y = Fx$. If $z \in A$ then $fz \leqslant y$ and so $Fx \leqslant y$. Suppose for a contradiction that $Fx < y$. Since X is dense, there is $w \in X$ with $Fx < w < y$. Since f is surjective on X, there is $u \in X$ with $fu = w$. As $f(u) < y$, $u \in A$ so $u \leqslant x$ and $w = fu \leqslant Fx$, which gives a contradiction.

4. *F is continuous.* By Lemma 4.4, see below, since we now know that F is order preserving and surjective.

5. *F is unique.* It is well known that if two functions F, G are continuous on a set Y and they agree on a dense subset of Y then they actually agree on Y itself.

□

We now give the proof of what we claimed in the introduction regarding the continuity of epimorphisms.

Lemma 4.4. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an order preserving map with dense image, then f is a continuous epimorphism.*

Proof. First we show that f must be surjective. Suppose not for contradiction. Then there is an element $x \in \mathbb{R}$ such that $x \notin \mathrm{Im}(f)$. Consider

$$A = f^{-1}(-\infty, x) \text{ and } B = f^{-1}(x, \infty).$$

Then $A \cap B = \varnothing$, for otherwise there would be an element $y \in A \cap B$ so that $f(y) < x$ and $f(y) > x$. Also $A < B$, for suppose there is $a \in A$ and $b \in B$ such that $b \leqslant a$. Then

$$x < f(b) \leqslant f(a) < x$$

which is clearly a contradiction. Note also that $A \cup B = \mathbb{R}$ since $x \notin \mathrm{Im}(f)$.

Since $\mathrm{Im}(f)$ is dense, both A and B are non-empty. That B is non-empty implies A is bounded above and so it has a supremum, and similarly B has an infimum, let:

$$a^* = \sup(A) \text{ and } b^* = \inf(B).$$

Since $A \cup B = \mathbb{R}$, $A \cap B = \varnothing$ and $A < B$ we get $a^* = b^*$, for $a^* \leqslant b^*$ is clear, and if $a^* < b^*$ this would be contrary to $A \cup B = \mathbb{R}$. Again using $A \cup B = \mathbb{R}$, a^* must be either in A or B. If $a^* \in B$ then $f(a^*) > x$. As $\mathrm{Im}(f)$ is dense there exists y such that

$$x < f(y) < f(a^*).$$

This implies $y < a^*$, so $y \in A$ giving $f(y) < x$, a contradiction. In the same way, if we assume $a^* \in A$ we get a similar contradiction. Therefore our hypothesis on the existence of such an x not in the image of f is impossible, hence f must be surjective.

Now let x be in $\mathbb{R}$ and V an open interval around $f(x)$. Let $(y_1, y_2) \subset V$ be such that $y_1 < f(x) < y_2$. Since f is surjective, both y_1 and y_2 are in the image of f. Let $x_1 \in f^{-1}\{y_1\}$ and $x_2 \in f^{-1}\{y_2\}$, then

$$y_1 < f(x) < y_2 \text{ implies } x_1 < x < x_2$$

because f is order preserving. Now for each $z \in (x_1, x_2)$ we have

$$f(x_1) = y_1 \leqslant f(z) \leqslant y_2 = f(x_2),$$

so that $f(x_1, x_2)$ is completely contained in V. □

We will finish this section with a couple of results before starting the constructions.

Lemma 4.5. *There are $2^{\aleph_0}$ epimorphisms of the structure $(\mathbb{N}, \leqslant)$.*

Proof. For each $A \subseteq \mathbb{N}$ we can define a function $f_A : \mathbb{N} \to \mathbb{N}$ by

$$f_A(2n) = n \text{ and } f_A(2n+1) = \begin{cases} n+1 & \text{if } n \in A \\ n & \text{if } n \notin A\,. \end{cases}$$

Then the set $\{f_A \mid A \subseteq \mathbb{N}\}$ is a family of epimorphisms of $(\mathbb{N}, \leqslant)$ and if A, B are two distinct subsets of $\mathbb{N}$, then $f_A \neq f_B$. □

The next proposition tells us that if we have a non-trivial epimorphism on a chain, then we are bound to have many of them.

Proposition 4.6. *If $(X, \leqslant)$ is a chain having a non-identity epimorphism, then it has at least $2^{\aleph_0}$ epimorphisms. In particular, if X is a dense subset of $\mathbb{R}$, then $(X, \leqslant)$ has exactly $2^{\aleph_0}$ epimorphisms.*

Proof. Let $f \in \mathrm{Epi}(X, \leqslant)$ be non-trivial. Let a_0, a_1 in X be such that $a_0 = fa_1$ and $a_1 \neq a_0$; without loss of generality assume $a_0 < a_1$. Since f is surjective we may find a sequence $(a_n)_{n\in\mathbb{N}}$ in X such that for each $n \in \mathbb{N}$, $fa_{n+1} = a_n$. It follows that $a_0 < a_1 < ....$, for if $a_{n+1} \leqslant a_n$, then applying f n-times we get that $a_1 \leqslant a_0$.

Now, for each epimorphism θ of $(\mathbb{N}, \leqslant)$, we will define a corresponding f_θ of $(X, \leqslant)$ mapping a_n to $a_{\theta(n)}$. Then different epimorphisms of $(\mathbb{N}, \leqslant)$ give rise to distinct epimorphisms of $(X, \leqslant)$ and previous lemma will ensure there are at least $2^{\aleph_0}$ of them.

Observe that $\theta(n) \leqslant n$ and for each n, $\theta(n+1) = \theta(n)$ or $\theta(n) + 1$. Let f_θ fix all points not in $\bigcup_{n\in\mathbb{N}} [a_n, a_{n+1}]$ and set $f_\theta a_n = a_{\theta(n)}$. Finally, we have to define f_θ inside the interval $[a_n, a_{n+1}]$. This is done by mapping the entire closed interval to $\{a_{\theta(n)}\}$ if $\theta(n) = \theta(n+1)$ and mapping $[a_n, a_{n+1}]$ onto $[a_{\theta(n)}, a_{\theta(n+1)}]$ using $f^{n-\theta(n)}$ if $\theta(n) = \theta(n+1)$. Therefore we have defined f_θ as follows

$$f_\theta(x) = \begin{cases} x & \text{if } x \notin \bigcup_{n\in\omega} [a_n, a_{n+1}] \\ a_{\theta(n)} & \text{if } x \in [a_n, a_{n+1}] \text{ and } \theta(n+1) = \theta(n) \\ f^{n-\theta(n)}x & \text{if } x \in [a_n, a_{n+1}] \text{ and } \theta(n+1) = \theta(n) + 1 \end{cases}$$

□

The next corollary follows from the proof of the last proposition.

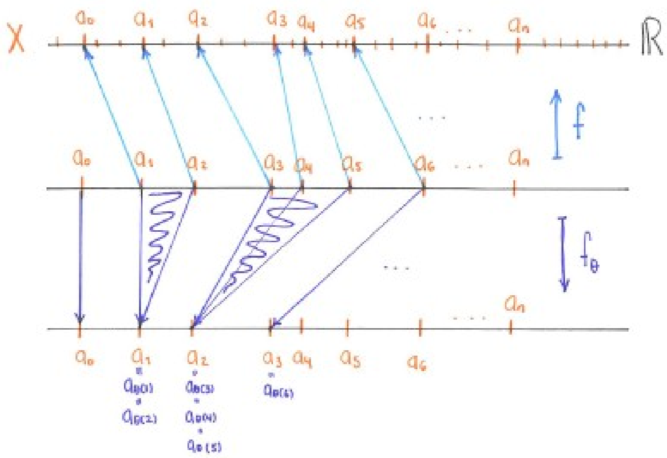

FIGURE 4.1: f_θ

Corollary 4.7. *Epi*$(\mathbb{N}, \leqslant)$ *can be embedded in Epi*$(X \leqslant)$*, and so Epi*$(X, \leqslant)$ *is not commutative.*

Proof. This follows from the fact that f_θ was defined to act on $\{a_n : n \in \mathbb{N}\}$ in precisely the same way that θ acts on $\mathbb{N}$. □

Proposition 4.8. *If Epi*$(X, \leqslant)$ *is non-trivial, then so is Emb*$(X, \leqslant)$*.*

Proof. Let $f \in \text{Epi}(X, \leqslant)$ be non-trivial. Define g by letting $g(x) \in f^{-1}x$, in such a way that if $|f^{-1}(x)| > 1$, then $g(x) \neq x$. Since f is not the identity and fg is, there is some $a \in X$ such that $f(a) \neq a$ and hence $a \notin f^{-1}a$ so $g(a) \neq a$.

Also, g is an embedding: for suppose $a < b$ but $g(b) \leqslant g(a)$, then $f(g(b)) \leqslant f(g(a))$ which implies $b \leqslant a$, giving a contradiction. □

We remark that in the proof of **Proposition 4.6** there are epimorphisms of $(X, \leqslant)$ provided by the map f and the sequence $(a_n : n \in \omega)$ other than those we have described, thanks to the density of X. Let $b_0 < b_1 < \ldots < b_{2N+1}$ or $b_0 < b_1 < \ldots\ldots$ be an infinite sequence such that $a_0 \leqslant b_n < \sup_{m \in \omega} a_m$ for each n and such that for each $n < N$ (or each n in case of an infinite sequence), $f^{k_n}(b_{2n+1}) = b_{2n}$ for some integer $k_n \geqslant 1$.

Then we can define an epimorphism φ on X corresponding to this sequence: φ fixes all points of X not in $\bigcup_{n\in\omega}[a_n, a_{n+1}]$ and otherwise,

$$\varphi_f(x) = \begin{cases} x & \text{if } a_0 \leqslant x \leqslant b_0 \\ b_0 & \text{if } b_0 \leqslant x \leqslant b_1 \\ f^{k_0}(x) & \text{if } b_1 \leqslant x \leqslant b_2 \\ f^{k_0}(b_2) & \text{if } b_2 \leqslant x \leqslant b_3 \\ f^{k_0+k_1}(x) & \text{if } b_3 \leqslant x \leqslant b_4 \\ f^{k_0+k_1}(b_4) & \text{if } b_4 \leqslant x \leqslant b_5 \\ \vdots & \end{cases} .$$

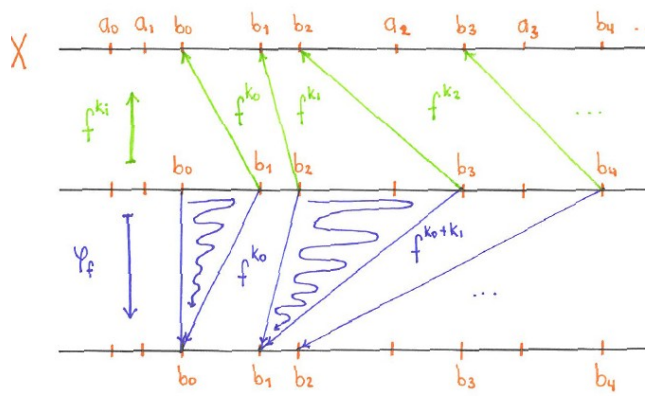

FIGURE 4.2: φ_f

Intuitively, each interval $[b_{2n}, b_{2n+1}]$ is mapped to a singleton, and other points are mapped by a suitable power of f to give continuity. In the proof of **Proposition 4.6** we glued together maps on intervals with endpoints in $\{a_n : n \in \omega\}$; here we allow ourselves to glue intermediate intervals which are still 'compatible' with f.

Finally let's say that maps φ_f of this form are generated from f in a **wide sense**, from the action of f on $\bigcup_{n\in\omega}[b_n, b_{n+1}]$, for a corresponding sequence B_φ as above.

4.2 Construction of a dense rigid subchain of $\mathbb{R}$ with specified epimorphism monoid

In this section, we are trying to define a dense rigid subchain X of $\mathbb{R}$ which admits epimorphisms generated by the following two commuting epimorphisms,

$$gx = \begin{cases} x & \text{for} \quad x \leqslant 0 \\ 0 & \text{for} \quad 0 \leqslant x \leqslant 1 \\ x-1 & \text{for} \quad x > 1 \end{cases} \quad hx = \begin{cases} x+1 & \text{for} \quad x \leqslant -1 \\ 0 & \text{for} \quad -1 \leqslant x \leqslant 0 \\ x & \text{for} \quad x > 0 \end{cases}$$

That is, we want to construct X so that any epimorphism of $(X, \leqslant)$ has the form $g^n h^m$ for some $n, m \geqslant 0$.

However, the above discussion shows that we cannot avoid also admitting epimorphisms generated from gh in a wide sense, for which the corresponding b_0 is equal to zero.

$$\varphi_f x = \begin{cases} 0 & \text{for} \quad x = 0 \\ g^{\alpha_i} x & \text{for} \quad x > 0 \wedge x \in [b_{2i-1}, b_{2i}] \\ h^{\alpha_i} x & \text{for} \quad x < 0 \wedge x \in [b_{2i-1}, b_{2i}] \\ b_i & \text{for} \quad x \in [b_{2i}, b_{2i+1}] \end{cases}$$

where $\alpha_i = \sum_{j=0}^{i-1} kj$ and $[b_{2i-1}, b_{2i}]$ is mapped to $[b_{i-1}, b_i]$ via g^{α_i} or h^{α_i} respectively.

This is still however quite a restricted class of maps and we want to show that X can be constructed so that there are no epimorphisms apart from these.

The following tables will be useful as they show the values of $g^n h^m$ for positive $m, n \in \mathbb{N}$:

Proposition 4.9. *There is a dense rigid chain $X \subset \mathbb{R}$, with $Epi(X, \leqslant) \neq \{id\}$ and which contains every epimorphism generated from g and h.*

$x \leqslant -m$	$x \in [-m, 0]$	$x \in [0, n]$	$x \geqslant n$
$h^m x = x + m$	$h^m x = 0$	$h^m x = x$	$h^m x = x$
$g^n x = x$	$g^n x = x$	$g^n x = 0$	$g^n x = x - n$

TABLE 4.1: .

$x < 0$	$x > 0$
$h^{-m} x = x - m$	$h^{-m} x = x$
$g^{-n} x = x$	$g^{-n} x = x + n$

TABLE 4.2: .

Proof. Let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all non-identity epimorphisms of $\mathbb{R}$ not generated from g and h in a wide sense or generated from g and h but which corresponding b_0 is not equal to 0. As seen in **Section 4.1**, we start our construction with $X_0 = \mathbb{Q}$ and $Y_0 = \varnothing$. Consider the inductive step, in which we are given disjoint X_α, Y_α, each of cardinality less than $\mathfrak{c}$. Then, for stage $\alpha + 1$, we have to choose $x_\alpha \notin X_\alpha \cup Y_\alpha$ such that

- $X_{\alpha+1} = X_\alpha \cup \{g^n h^m x_\alpha \mid m \in \mathbb{Z}\}$
- $Y_{\alpha+1} = Y_\alpha \cup \{f_\alpha x_\alpha\}$

So, the following conditions are required,

1. $g^n h^m x_\alpha \notin Y_\alpha$,

   So that the image (under the epimorphism gh) of the element we are trying to add to X is not already in the set of undesired elements.

2. $f_\alpha x_\alpha \notin X_\alpha$,

   So that the image (under the epimorphism) that we want to avoid having in X is not already in X_α

3. $f_\alpha x_\alpha \neq g^n h^m x_\alpha$,

   Because by adding x_α we need to add also all its images under $g^n h^m$ to preserve them as epimorphisms of X.

Let $H = \bigcup_{m,n\in\mathbb{Z}} g^n h^m Y_\alpha$.

First assume there is a non-empty open interval I such that its image J under f_α is also non-empty and open such that for all $x \in I$, $f_\alpha x - x \notin \mathbb{Z}$. Then we can pick $y_\alpha \in J \setminus (X_\alpha \cup f_\alpha(X_\alpha \cup H))$ (since $|J| = \mathfrak{c}$) and x_α such that $f_\alpha x_\alpha = y_\alpha$. Since $x_\alpha \notin X_\alpha$, $x_\alpha \neq 0$ so we can assume without loss of generality that $x_\alpha > 0$. Then $g^n h^m x_\alpha = g^n x_\alpha$ and

$$g^n x_\alpha = \begin{cases} 0 & \text{if} \quad x \in [0, n] \\ x_\alpha - n & \text{if} \quad x > n \end{cases}$$

So, if $x_\alpha > n$, $g^n h^m x_\alpha - x_\alpha \in \mathbb{Z}$ and it follows that $g^n h^m x_\alpha \neq f_\alpha x_\alpha$. If $x_\alpha \in [0, n]$, $g^n h^m x_\alpha = 0$ and hence $g^n h^m x_\alpha \neq f_\alpha x_\alpha$ since $y_\alpha \notin X_\alpha$.

Otherwise no such intervals I and J exist. Therefore we either have to find another method of choosing x_α, or show that f_α is generated from g and h in a wide sense with $b_0 = 0$. To analyse this we use the auxiliary function $j_\alpha = f_\alpha x - x$, noticing that both j_α and f_α are continuous.

Claim 4.10. *The auxiliary function j_α is decreasing, that is, if $a \leqslant b$ then $j_\alpha a \geqslant j_\alpha b$.*

Proof. Suppose on the contrary that for some $a < b$, $j_\alpha a < j_\alpha b$. Using the Intermediate Value Theorem, $j_\alpha[a, b]$ contains all values in $[j_\alpha a, j_\alpha b]$, so by cutting down $[a, b]$ we may assume that the interval $[j_\alpha a, j_\alpha b]$ does not contain any integer, i.e. $[j_\alpha a, j_\alpha b] \cap \mathbb{Z} = \varnothing$. Let

$$a^* = \sup\{x \in [a, b] : j_\alpha x = j_\alpha a\} \text{ and } b^* = \inf\{x \in [a^*, b] : j_\alpha x = j_\alpha b\}$$

Then $j_\alpha a^* < j_\alpha b^*$ and $j_\alpha[a^*, b^*] = [j_\alpha a^*, j_\alpha b^*]$. Hence for some x such that $a \leqslant x \leqslant b$, $j_\alpha x = f_\alpha x - x \notin \mathbb{Z}$. Our assumption implies that f_α is constant on $[a^*, b^*]$ for otherwise we could find intervals I and J as before. Let $f_\alpha x = c$ on $[a^*, b^*]$. Then $j_\alpha a^* = c - a^* > c - b^* = j_\alpha b^*$, giving a contradiction. ☺

Now, j_α may be constant on an interval $[a, b]$, but then the value of this constant must be an integer (as otherwise $f_\alpha x = j_\alpha x + x$ and we could find suitable I and J).

So we can consider maximal intervals on which j_α is constant, and in any interval in which j_α is nowhere constant, f_α must be constant itself as otherwise we will get our I and J again. Therefore we can think of $\mathbb{R}$ as the union of intervals on which one of the following holds,

- j_α is constant with integer value n and then $f_\alpha x = x + n$,
- f_α is constant with some value c and so $j_\alpha x = c - x$.

So, suppose that $f_\alpha 0 > 0$. Since f_α is an epimorphism $f_\alpha x \longrightarrow -\infty$ when $x \longrightarrow -\infty$, therefore there is a value below zero that is mapped to a positive number by f_α. In fact, we can find an interval of $(-\infty, 0)$ with positive image. Let $I \subset (-\infty, 0)$ be a non-empty open interval such that $J = f_\alpha I$ is also a non-empty open interval and $J \subset (0, \infty)$. Then, we have to choose $x_\alpha \in I$ such that $f_\alpha x_\alpha \neq g^n h^m x_\alpha$.

Since g^n fixes the negative axis, we need just to consider the actions of h^m. But $h^m x < 0$ and $f_\alpha x > 0$ for each $x \in I$. Hence any $x \in I$ will satisfy $h^m x \neq f_\alpha x$, so we let $y \in J \smallsetminus (X_\alpha \cup f_\alpha(X_\alpha \cup H))$ and define $x_\alpha = f_\alpha y$. If $f_\alpha 0 < 0$ we argue in a similar way.

If $f_\alpha 0 = 0$, then we can write f_α as the product of two epimorphisms on $\mathbb{R}$, one that fixes the negative axis and another that fixes the positive axis. Since f_α is either constant or $f_\alpha = x + n$ for some $n \in \mathbb{Z}$, f_α is generated from gh in a wide sense. If its corresponding b_0 is zero, we are done, since we cannot avoid this function, but if $b_0 \neq 0$ (say > 0), then either the corresponding sequence $B = (b_i \mid \in \omega)$ lies in the positive axis or in the negative axis.

If $B \subseteq (0, \infty)$, then we look at g^n, since f_α and h^m agree on $[0, \infty)$. Since $0 < b_0$ and f_α fixes everything before b_0, f_α and g^n disagree on $(0, \min n, b_0)$ and here we can find suitable I and J.

For $B \subseteq (-\infty, 0)$, a similar argument applies. □

To conclude, we remark on what can be said about the value of $\mathrm{Epi}(X, \leqslant)$ in this case. From the construction, it is clear that any epimorphism of X fixes zero (from **Table 1** we can see that $g^n h^m 0 = 0$ for any $m, n \in \mathbb{Z}$ and if φ_{gh} is generated in a wide sense by g and h then φ_{gh} moves only elements that gh already moved) and any epimorphism of $\mathbb{Z}$ that fixes zero gives rise to an epimorphism of X, which already gives us $2^{\aleph_0}$ epimorphisms. However, we are still avoiding quite an extensive family of epimorphisms, and the ones that we keep are formed thanks to the fact that intervals in the real line are isomorphic to each other.

4.3 Construction of a dense rigid subset of $\mathbb{R}$ with trivial epimorphism monoid and specified embedding monoid

We remark that the method used in [DT01] can be easily adapted to give an example of a rigid chain, even with trivial epimorphism monoid, which admits many embeddings, using a method involving Baire category. However we are principally interested in controlling the behaviour of the monoids, and here $\mathrm{Emb}(\mathrm{X}, \leqslant)$ is enormous, so instead we follow the method of **Section 4.2** to find situations where $\mathrm{Emb}(\mathrm{X}, \leqslant)$ is of specified isomorphism type, e.g. $(\mathbb{N}^2, +)$.

Proposition 4.11. *There is a dense rigid subset of* $\mathbb{R}$*,* X*, with* $\mathrm{Emb}(\mathrm{X}, \leqslant) \cong (\mathbb{N}^2, +)$ *and* $\mathrm{Epi}(\mathrm{X}, \leqslant) = \{\mathrm{id}\}$.

Proof. The idea is to proceed analogously to the previous section but this time preserve all embeddings generated by the following:

$$gx = \begin{cases} x & \text{for} \quad x \leqslant 0 \\ x+1 & \text{for} \quad x > 0 \end{cases} \quad \text{and } hx = \begin{cases} x-1 & \text{for} \quad x < 0 \\ x & \text{for} \quad x \geqslant 0 \end{cases}$$

Notice that since g and h act on disjoint intervals they commute and now we want to destroy all non-identity epimorphisms and embeddings not of the form $g^n h^m$ for any integers $n, m \geqslant 0$, which we enumerate as $\{f_\alpha \mid \alpha < \mathfrak{c}\}$. To have a clearer idea of what the functions we preserve look like, see **Table 3** for $m, n \geqslant 0$.

$x < 0$	$x > 0$
$h^m x = x - m$	$h^m x = x$
$g^n x = x$	$g^n x = x + n$

TABLE 4.3: .

We start again setting $X_0 = \mathbb{Q}$ and $Y_0 = \varnothing$ and proceed by transfinite induction on $\alpha < \mathfrak{c}$. Assuming the induction hypothesis that for some $\alpha < \mathfrak{c}$ we have $X_\alpha \cap Y_\alpha = \varnothing$, where X_α and Y_α are of cardinality less than $\mathfrak{c}$, we move on to the step $\alpha + 1$, where we concentrate on choosing a point x_α not in $X_\alpha \cup Y_\alpha$ satisfying:

- $X_{\alpha+1} = X_\alpha \cup \begin{cases} G_{x_\alpha} = \{g^n x_\alpha \mid n \geqslant 0\} & \text{for} \quad x_\alpha > 0 \\ H_{x_\alpha} = \{h^n x_\alpha \mid n \geqslant 0\} & \text{for} \quad x_\alpha < 0 \end{cases}$
- $Y_{\alpha+1} = Y_\alpha \cup \{f_\alpha x_\alpha\}$

so that $X_{\alpha+1}$ and $Y_{\alpha+1}$ are disjoint and have cardinality less than $\mathfrak{c}$, which are precisely the conditions that we assume for X_α and Y_α in our induction hypothesis. This means that we will require the following conditions,

- $X_\alpha \cap Y_\alpha = \varnothing$,
- $f_\alpha x_\alpha \notin X_\alpha$, that is $x_\alpha \notin f_\alpha^{-1} X_\alpha$,
- $G_{x_\alpha} \cap Y_\alpha = \varnothing = H_{x_\alpha} \cap X_\alpha$, which means that for every $n \geqslant 0$ the following holds,

  $x_\alpha \notin g^{-n} Y_\alpha \quad \text{for} \quad x_\alpha > 0$

  $x_\alpha \notin h^{-n} Y_\alpha \quad \text{for} \quad x_\alpha < 0$
- $f_\alpha x_\alpha \notin G_{x_\alpha} \cup H_{x_\alpha}$. That is, for every $n \geqslant 0$,

  $f_\alpha x_\alpha \neq g^n x_\alpha \quad \text{for} \quad x_\alpha > 0$

  $f_\alpha x_\alpha \neq h^n x_\alpha \quad \text{for} \quad x_\alpha < 0$

Notice that 0 is already in X_0 so we do not need to take care of it as a possibility for x_α. Also, the first of the above requirements is already satisfied by induction hypothesis and since g and h are embeddings the sets $g^{-n}Y_\alpha$ and $h^{-n}Y_\alpha$ have cardinality less than $\mathfrak{c}$ for every $n \geqslant 0$. So the strategy again will be to find a set I, so that the set and its image are disjoint (which can be done since in particular none of the f_α's are the identity) and of cardinality $\mathfrak{c}$, then try to make I miss the following set M and $f_\alpha I$ miss both G_x and H_x for every $x \in I$ (although this is more than necessary, for we need only one x that satisfies these conditions),

$$M = X_\alpha \cup Y_\alpha \cup \bigcup_{n\in\omega} g^{-n}Y_\alpha \cup \bigcup_{n\in\omega} h^{-n}Y_\alpha$$

In order to do this we will consider first the case when we choose f_α as an epimorphism and then when we choose it as an embedding.

f_α is an epimorphism

We start by assuming that we have picked an epimorphism f_α and we split this case into two, and for simplicity we define a new function $j_\alpha x = f_\alpha x - x$ and we recall that it is continuous.

Case 1. *f_α moves a point below* 0.

This case tells us that our function j_α is not zero in $\mathbb{R}^-$ and we consider a further two cases,

Case (1.1). *There is a point $x < 0$ for which $j_\alpha x \neq -m$ for every integer $m \geqslant 0$.*

Then $j_\alpha x$ is in the interval $(-(n+1), -n)$ for some $n \geqslant 0$, and either j_α is constant on the interval $(-\infty, 0)$ or we can find an open interval J contained in $(-(n+1), -n)$ which is part of the image of j_α. If the latter holds, let $I_0 = j_\alpha^{-1}J$ and take $I = f_\alpha^{-1}(f_\alpha I_0 \smallsetminus f_\alpha X_\alpha)$ and let x_α be any point in $I \smallsetminus M$.

Otherwise $j_\alpha x = -k$ for some real $k \notin \mathbb{Z}$ for every $x < 0$. In this case let I be any interval of the form $(-(n+1), -n)$ for some integer $n > 0$, such that $-k \notin I$ and take x_α to be any element of $I \smallsetminus (M \cup (X_\alpha + k))$, where $X_\alpha + k = \{x + k \mid x \in X_\alpha\}$.

Case (1.2). *For all $x < 0$ there is $m \in \mathbb{N}$ such that $j_\alpha x = -m$.*

By continuity, j_α must be constant on $(-\infty, 0)$ and since we are assuming that there is at least one point for which j_α is not zero, we get $m > 0$. But f_α is not of the form h^n for any $n \in \mathbb{N}$, in particular for $n = m$, and we are assuming f_α and h^m coincide on $(-\infty, 0)$, so f_α cannot be the identity on $[0, \infty)$. Let $x \in (n, n+1)$ be moved by f_α for the least $n \geqslant 0$.

If $j_\alpha x < 0$, either one of the intervals $(j_\alpha x, -m)$ or $(-m, j_\alpha x)$ is also part of the image of j_α, or $j_\alpha x = -m$ for all $x > 0$. If the latter holds, let I be any open interval on $(0, \infty)$ and take $x_\alpha \in I \setminus (M \cup (X_\alpha + m))$. If either $(j_\alpha x, -m)$ or $(-m, j_\alpha x)$ are in the image of j_α let $I = (\min\{j_\alpha x, -(m+1)\}, -m)$ in the first case, $I = (-m, \min\{j_\alpha x, -m+1\})$ in the second and take any element of $I \setminus (M \cup (X_\alpha + m))$ as x_α.

If $j_\alpha x > 0$ then the interval $(-m, 0)$ must be part of the image of j_α. Therefore let $V = (-m, 0) \setminus f_\alpha X_\alpha$, $I = f_\alpha^{-1} V$ and $x_\alpha \in I \setminus M$.

Case 2. *f_α fixes every element below zero and moves something in $(0, \infty)$.*

In this case we proceed as in ***Case*** (1.2) with $-m = 0$, so we have strictly the case when $j_\alpha x > 0$ and we take the interval $V = (0, \min\{1, j_\alpha x\}) \setminus f_\alpha X_\alpha$.

Now we consider how to choose x_α if we pick an embedding.

f_α is an embedding

Case 3. *There is a point less than zero moved by f_α.*

A usual, we let x be moved by f_α in the interval $(-n-1, -n)$, for the least integer $n \geqslant 0$. If $f_\alpha x < x$ let $V = (-n-1, x)$, then $f_\alpha V \cap V = \varnothing$ and V is also moved by f_α. Since f_α is an embedding V is also moved by f_α^{-1}, so we let $I = f_\alpha^{-1}(V \setminus D_{f_\alpha})$. If $f_\alpha x > x$ we let $I = (x, \min\{-n, f_\alpha x\})$ and in both cases I is disjoint from its image and $I \subseteq (-n-1, -n)$. Therefore for all $x \in I$, $|f_\alpha x - x| < 1$ and hence $f_\alpha x \neq h^m x$ for any $m \geqslant 0$.

Finally, we let x_α be a point in the set $I \setminus (M \cup f_\alpha^{-1}(X_\alpha \setminus D_{f_\alpha}) \cup D_{f_\alpha})$, recalling that D_{f_α} is the set of discontinuities of f_α.

Case 4. *f_α fixes everything less than zero and moves something in* $(0, \infty)$.

Here we let $x \in (n, n+1)$ be moved by f_α for the least $n \geqslant 0$ and proceed similarly to ***Case 3***. □

The above construction for $n = 2$ is done to illustrate the method, but this can be generalized to any $n \in \mathbb{N}$, as long as the generating embeddings have disjoint support (i.e. the sets of points which are moved by the embeddings are disjoint), which will also ensure that they are distinct and are such that they are specified by their actions on $\mathbb{Q}$. For instance, define for each $n \in \mathbb{N}$,

$$g_n x = \begin{cases} x & \text{for} \quad x \notin (n, n+1) \\ n + \phi g \phi^{-1} x & \text{for} \quad x \in (n, n+1) \end{cases}$$

where g is any fixed embedding, we can take the above for example, and ϕ is an order preserving isomorphism from the real line to the interval $(0, 1)$. Then any finite set of g_n's will satisfy our requirements since their supports are clearly disjoint.

4.4 A dense set $X \subseteq \mathbb{R}$ which is embedding and epimorphism rigid with non-trivial endomorphism monoid

Since, for whichever chain $(X, \leqslant)$ we consider there are endomorphisms we cannot destroy (such as constant maps), the strongest result we can at present aim for is to find a dense chain $X \subset \mathbb{R}$ which is rigid for both epimorphisms and embeddings but which is preserved by some specific endomorphism of none of these "obvious" kinds.

Proposition 4.12. *There is a dense subchain, X, of the real line of cardinality continuum with* $\mathrm{Epi}(\mathrm{X}, \leqslant) = \{\mathrm{id}\} = \mathrm{Emb}(\mathrm{X}, \leqslant)$ *and whose endomorphism monoid contains a monoid isomorphic to* $(\mathbb{N}, +)$.

Proof. In order to prove this, we will begin the construction by "destroying" all embeddings and epimorphisms different from the identity but preserving all endomorphisms generated by the following function (which is clearly neither an embedding nor an epimorphism):

$$h(x) = \begin{cases} 0 & \text{for} \quad x \leqslant 0 \\ \frac{2}{\pi} \arctan x & \text{for} \quad x > 0 \end{cases}$$

As usual, we enumerate all epimorphisms *and* embeddings of $\mathbb{R}$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$ and hence we shall do our construction by transfinite induction on $\alpha < \mathfrak{c}$. Starting again with the rationals as X_0 and withY_0 empty. As our induction step, we assume that X_α and Y_α are disjoint and of cardinality less than $\mathfrak{c}$ and we will choose a point $x_\alpha \notin X_\alpha \cup Y_\alpha$ and define:

- $X_{\alpha+1} = X_\alpha \cup \{h^n x_\alpha \mid n \geqslant 0\}$
- $Y_{\alpha+1} = Y_\alpha \cup \{f_\alpha x_\alpha\}$

The point x_α will be chosen so that $X_{\alpha+1} \cap Y_{\alpha+1} = \varnothing$ and both $X_{\alpha+1}$ and $Y_{\alpha+1}$ are of cardinality less than $\mathfrak{c}$. Notice that the second requirement is satisfied by induction hypothesis and the fact that $\{h^n x_\alpha \mid n \geqslant 0\}$ is countable. For disjointness we will require in addition:

- $f_\alpha x_\alpha \notin X_\alpha$
- $x_\alpha \notin h^{-n} Y_\alpha$ for every $n \geqslant 0$
- $h^n x_\alpha \neq f_\alpha x_\alpha$ for every $n \geqslant 0$

If we define $H = \{h^{-n}Y_\alpha : n \geqslant 0\}$ then, since $0 \notin Y_\alpha$ by hypothesis, we have that $|H| < \mathfrak{c}$. Define the set M as:

$$M = H \cup X_\alpha \cup Y_\alpha$$

Now, the proof will be divided into cases corresponding to our choice of f_α, and the aim will be to find either an interval or a set I of cardinality $\mathfrak{c}$ which is disjoint from its image and whose elements satisfy the above requirements, so we will take x_α to be in the set I but to miss M.

f_α is an epimorphism

Case 1. *There is some point x less than zero which is moved by f_α.*

If $f_\alpha x < x$ let $V = (f_\alpha x, x) \setminus X_\alpha$. Otherwise $f_\alpha x > x$ and we take $V = (x, \min\{0, f_\alpha x\}) \setminus X_\alpha$. In either case if we let $I = f_\alpha^{-1}V$ then $I \cap f_\alpha I = \varnothing$, so any element of the set $I \setminus M$ will fulfil the requirements for x_α, in particular for any $y \in I \setminus M$ and any $n > 1$, $f_\alpha y \neq h^n y$ since either $y < 0$ and $0 \notin f_\alpha I$ or $y > 0$ and $f_\alpha y < 0$.

Case 2. *f_α fixes all elements in $(-\infty, 0]$ and moves a point greater than 1.*

Then we take x in the interval $(n, n+1)$ moved by f_α for the least $n \geqslant 1$. If $f_\alpha x < x$ let $V = (f_\alpha x, x)$ and $J = f_\alpha^{-1}V \setminus X_\alpha$. Then J and its image are disjoint and J is moved by f_α. If $f_\alpha x > x$ let $J = (x, f_\alpha x)$. Then $I = f_\alpha^{-1}J$ clearly satisfies $f_\alpha I \cap I = \varnothing$ and it is also moved by f_α, therefore $I \subseteq (n, \infty)$ and hence for every $y \in I$, $f_\alpha y \neq h^n y$ for any $n \geqslant 0$ since $\operatorname{Im} h^n \subseteq (0,1)$. Take x_α to be in $I \setminus M$.

Case 3. *f_α fixes everything not in $(0,1]$.*

Because f_α is continuous and order preserving it actually fixes everything not in $(0,1)$. Notice also that for every $x \in (0,1)$

$$h^n x < \tfrac{1}{2} \text{ so } h^n x < x \text{ for every } n \geqslant 1.$$

Now, since f_α is not the identity it must move something in the interval $(0,1)$. If there is a point x^* with $f_\alpha x^* > x^*$, let $V = (x^*, f_\alpha x^*) \setminus X_\alpha$ and $I = f_\alpha^{-1}V$. Then $f_\alpha I \cap I = \varnothing$ and for every x in I, $h^n x < x < f_\alpha x$ for every $n \geqslant 0$, hence any element in $I \setminus M$ will work as x_α.

If there is no such x^*, viz., $f_\alpha x < x$ for every $x \in (0,1)$, we split into two additional cases.

Case (3.1). *$hx < f_\alpha x < x$ for every $x \in (0,1)$.*

Then we take any element x in the interval $(0,1)$ to define $V = (f_\alpha x, x) \setminus X_\alpha$ and $I = f_\alpha^{-1}V$. Hence for every $y \in I$ and for every $n \geqslant 0$, $f_\alpha y \neq h^n y$, so take x_α in the set $I \setminus M$.

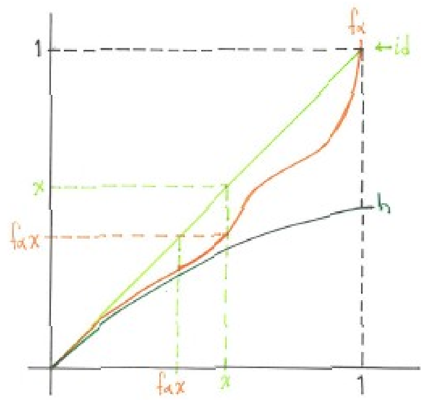

FIGURE 4.3: Diagram to illustrate ***Case*** (3.1)

Case (3.2). *$f_\alpha x^* < hx^* < x^*$ for some $x^* \in (0,1)$.*

For this case we will need the help of the following function,

$$g_\alpha x = \tfrac{1}{2}(x + hx)$$

Notice that g_α is continuous, injective and satisfies $hx < g_\alpha x < x$ for every x in $(0,1)$. Now, $f_\alpha x^* < hx^* < g_\alpha x^* < x^*$ but $f_\alpha 1 = 1 > g_\alpha 1$, therefore there is an element $z \in (x^*, 1)$ such that $f_\alpha z = g_\alpha z < 1$. Moreover, since all these functions are continuous and $h < g_\alpha < id$, there is an open interval V_1 completely contained in $(g_\alpha z, z)$ whose inverse image under f_α satisfies $hx < g_\alpha x < f_\alpha x < x$ for all $x \in f_\alpha^{-1}V_1$, which takes us back to ***Case*** (3.1) letting $V = V_1 \smallsetminus X_\alpha$.

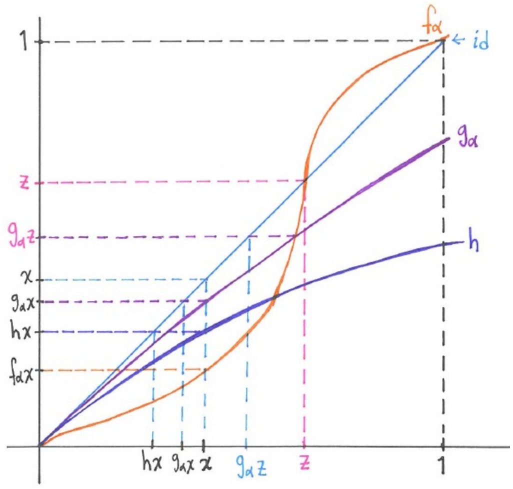

FIGURE 4.4: Diagram to illustrate ***Case*** (3.2)

f_α is an embedding

We consider now the case when f_α is an non-identity embedding and hence there is a point moved by it.

Case 1. *f_α moves a point x less than zero.*

If $f_\alpha x < x$, let $I = (f_\alpha x, x)$, otherwise $I = (x, \min\{f_\alpha x, 0\})$. Then $f_\alpha I \cap I = \varnothing$ so any element y in $I \smallsetminus (f_\alpha^{-1}(X_\alpha \smallsetminus D_{f_\alpha}) \cup M \cup D_{f_\alpha})$ satisfies $f_\alpha y \neq h^n y$ for every $n \geqslant 0$ since $f_\alpha^{-1} 0 \notin I$, so let y be x_α.

Case 2. *f_α fixes all elements less than or equal to zero and moves something in $(1, \infty)$.*

In this case we as usual let $x \in (n, n+1)$ be moved by f_α for the least natural number $n \geqslant 1$. If $f_\alpha x < x$ let $I = (\min\{f_\alpha x, n\}, x)$. Then $f_\alpha I < I$ and $f_\alpha^2 I < f_\alpha I$ since f_α is an embedding therefore $f_\alpha I$ is also moved and hence it is contained in the interval $(n, n+1)$. If $f_\alpha x > x$ let $I = (x, f_\alpha x)$ so that $f_\alpha I \subseteq (n, \infty)$. In both cases $f_\alpha y \neq h^m y$

for every $m \geqslant 0$ and every $y \in I$ since n is greater or equal to 1, so take x_α to be an element in $I \smallsetminus (f_\alpha^{-1}(X_\alpha \smallsetminus D_{f_\alpha}) \cup M \cup D_{f_\alpha})$.

Case 3. *f_α fixes everything not in* $(0,1)$.

Recall that $h^n x \leqslant hx < x$ and $0 < f_\alpha x < 1$ for all $x \in (0,1)$. Since f_α is not the identity it must move some point $x \in (0,1)$. If the right limit f_α^+ satisfies $f_\alpha^+ x > x$ we let $I = (x, f_\alpha^+ x)$ so that $f_\alpha y \neq h^n y$ for any $y \in I$ and $n \geqslant 0$ since $h^n y < y < f_\alpha y$. Otherwise, $f_\alpha^+ x < x$ and one of the following two cases holds. The strategy in these cases will be to find a set for which each point on it has image under f_α is greater than $\frac{1}{2}$, so that it cannot agree with h^n for any $n \geqslant 1$.

Case (3.1)**.** $f_\alpha^+ x \leqslant \frac{1}{2} < x$

Let $J = (\frac{1}{2}, x) \smallsetminus (D_{f_\alpha} \cup f_\alpha D_{f_\alpha})$. Then J and its image under f_α are disjoint. Moreover, $f_\alpha^{-1} J$ is also moved by f_α and hence it is contained in $(0,1)$. Let $I = f_\alpha^{-1} J$ and take x_α in $I \smallsetminus (f_\alpha^{-1}(X_\alpha \smallsetminus D_{f_\alpha}) \cup M \cup D_{f_\alpha})$.

Case (3.2)**.** $\frac{1}{2} \leqslant f_\alpha^+ x < x$.

Since x is moved by f_α, $f_\alpha x \in (0,1)$. Then either $f_\alpha x < x$ or $f_\alpha x > x$. If the former holds we let $J = (f_\alpha x, x) \smallsetminus (D_{f_\alpha} \cup f_\alpha D_{f_\alpha})$ and we are back in ***Case*** (3.1). Otherwise, let $I = (x, f_\alpha x)$ and take x_α to be any element in $I \smallsetminus (f_\alpha^{-1}(X_\alpha \smallsetminus D_{f_\alpha}) \cup M \cup D_{f_\alpha})$ □

4.5 Higher cardinalities (regular)

In [DT01], Droste and Truss constructed a densely ordered chain of cardinality κ for any uncountable cardinal which was automorphism rigid, using a different method from the diagonalization using by Dushnik and Miller in [DM40]. We will show that with a slight modification to the argument they used, we can get their chain of cardinality κ, for regular κ, epimorphism rigid. For this we shall need to use the following result.

Lemma 4.13. *Let X be the rigid chain constructed in the paper [DT01] and $\overline{X}$ its order completion. Then if $f \in Epi(\overline{X}, \leqslant)$, f is continuous. Moreover, if $f \in Epi(X, \leqslant)$, then f extends to an epimorphism of $\overline{X}$.*

Proof. This result holds because we are working in the completion of a dense chain, and it is precisely analogous to the proof of continuity in **Lemma 4.4**; and the second part is the same as in the proof of **Lemma 4.3**. □

The construction of X will be in stages, and will result in the union $\bigcup_{n\in\omega} X_n$, where each X_n is dense. Let $X_0 = \mathbb{L}_\kappa$, where $\mathbb{L}_\kappa = \kappa.\mathbb{Q}$ with the lexicographical ordering. Now, let $\mathcal{A}$ be a family of κ pairwise disjoint stationary subsets of κ, and let $\{\mathcal{A}_n \mid n \in \omega\}$ be a partition of $\mathcal{A}$, where $|\mathcal{A}_n| = \kappa$ for each $n \in \omega$.

Assuming we have constructed X_n, to define X_{n+1} we will make use of the elements of $\mathcal{A}_n$. Let $B_n = \{x \in X_n \mid x \text{ lies in a copy of } \mathbb{Q} \text{ added in the previous stage}\}$ and let's enumerate the elements of $\mathcal{A}_n$ as

$$\mathcal{A}_n = \{S_x \subseteq \kappa \mid x \in B_n\}$$

The stationary set $S_x \in \mathcal{A}_n$ is going to be used as a '*code*' for x during the construction of X_{n+1}. Thus, X_{n+1} will consist of X_n with $\mathbb{L}_{\kappa,S_x}$ added to every corresponding $x \in B_n$, where

$$\mathbb{L}_{\kappa,S_x} = \kappa.\mathbb{Q} \cup \{(\alpha, -\infty) \mid \alpha \in S_x\}$$

ordered lexicographically. That is, $\mathbb{L}_\kappa$ with a point $\{-\infty\}$ put in the gaps between the copies of $\mathbb{Q}$ corresponding to elements of S_x. Notice that the elements of B_n have cofinality ω but we shall see that after being coded in this step, each $x \in B_n$ will be the supremum of a set of order-type S_x. Hence every element of B_n will have cofinality κ and the set of points that have been coded by stage $n+1$ is dense in X_n.

So let f be a non-trivial epimorphism of X. Since the set of coded points is dense in X there is a coded point $x \in X$ such that $f(x) = y$ for $y \in X$ different from x.

Because the only cuts of X_n that are realized during the construction are those of the form $(\{z : z < x\}, \{z : z \geqslant x\})$ for $x \in X_n$, and these cuts are only realized at one stage of the construction, x is the supremum of a set of type S_x in X (this is the copy of the stationary $S_x \in \mathcal{A}$ that was put immediately to the left of x) - this is what we mean when we say that x has been **coded** by S_x.

Moreover, if we let $L_x = \{(\alpha, -\infty) \mid \alpha \in \kappa \smallsetminus \{0\}\}$ be a subset of $\overline{\mathbb{L}_{\kappa,S_x}}$, defined at stage n where x was coded, then L_x is a club of the set $(-\infty, x) \cap \overline{X}_{n+1}$, and since none of the cuts $(\{z : z < t\}, \{z : z > t\})$ for $t \in L_x$ is realized at any stage, L_x is still a club in X. In addition, $L_x \cap X$ is still the same S_x as was added at stage n. Notice that the same remarks apply to L_y.

Enumerate L_x and L_y as $L_x = \{x_\alpha \mid 0 < \alpha < \kappa\}$ and $L_y = \{y_\alpha \mid 0 < \alpha < \kappa\}$, in an increasing manner. Since $(X, \leqslant)$ is densely ordered, f extends to an epimorphism of $(\overline{X}, \leqslant)$.

Claim 4.14. *The set $C = \{\alpha \in \kappa \mid f(x_\alpha) = y_\alpha\}$ is a club in κ.*

Proof. For closure: Let $\alpha_1, \alpha_2, \ldots < \alpha_\beta < \ldots$ be an increasing sequence such that $\alpha_\beta \in C$ for all $\beta < \lambda < \kappa$. Let

$$x_{\alpha_s} = \sup_{\beta<\lambda} x_{\alpha_\beta} \text{ and } y_{\alpha_s} = \sup_{\beta<\lambda} y_{\alpha_\beta}$$

Since α_β is in C for all $\beta < \lambda$, $f_\alpha(x_{\alpha_\beta}) = y_\alpha$. Hence, because L_x and L_y are both clubs in X and f is continuous on $\overline{X}$, $x_{\alpha_s} \in L_x$, $y_{\alpha_s} \in L_y$ and

$$f(x_{\alpha_s}) = \sup_{\beta<\lambda} f(x_{\alpha_\beta}) = \sup_{\beta<\lambda} y_{\alpha_\beta} = y_{\alpha_s}$$

So $\alpha_s \in C$.

For unboundedness: Let $\lambda < \kappa$. We want to show there is $\alpha \in C$ with $\lambda < \alpha < \kappa$. Let $x_\lambda \in L_x$ and look at f_{x_λ}. We will construct two sequences $(x_{\gamma_n} : n \geqslant 1)$ and $(y_{\gamma_n} : n \geqslant 1)$ as follows.

Since L_y is unbounded in X, there is $y_{\gamma_1} \in L_y$ such that $f(x_\gamma) < y_{\gamma_1}$. Then, because L_x is unbounded, we can define $x_{\gamma_2} \in L_x$ to be the smallest element of L_x that is greater than every element of $f^{-1}\{y_{\gamma_1}\}$. Since f is order preserving, $f(x_{\gamma_2})$ must be greater than y_{γ_1} and we now let $y_{\gamma_2} \in L_y$ be the smallest element of L_y greater than y_{γ_1}, and we iterate this process to obtain the desired sequences, with the property that $x_{\gamma_{n-1}} < x_{\gamma_n}$ and $y_{\gamma_n} < f(x_{\gamma_n}) < y_{\gamma_{n+1}}$. Now, because L_x and L_y are closed and f is continuous, there are $x_\gamma \in L_x$ and $y_\gamma \in L_y$ such that

$$\text{if } x_\gamma = \sup_{n\geqslant 1} x_{\gamma_n} \text{ then } y_\gamma = \sup_{n\geqslant 1} y_{\gamma_n} = \sup_{n\geqslant 1} f(x_{\gamma_n}) = f(x_\gamma)$$

Therefore $\gamma \in C$ and since the sequence $(x_{\gamma_n} : n \geqslant 1)$ is increasing, we have $\lambda < \gamma$.

Since S_x is stationary, $S_x \cap C \neq \varnothing$. Let $\alpha \in S_x \cap C$. Then $x_\alpha \in X$ (being in L_x) and $f(x_\alpha) = y_\alpha \in X$ (since f is surjective on X). So $\alpha \in S_y$ (since $y_\alpha \in L_y$) contradicting our assumption that $S_x \cap S_y = \varnothing$, hence showing that there cannot be any non-identity epimorphism of $(X, \leqslant)$. ☺

In fact, the above proof also shows that X doesn't even have any *local* embeddings, by which we mean the following,

If $(x, y) \cong (a, b)$ are two non-empty intervals of X, then $x = a$ and $y = b$.

If we have a chain that satisfies this property for any two intervals, then we say that the chain is **strongly rigid**.

Now, it is possible to get an automorphism rigid chain which is not locally rigid: let X be as in **Lemma 4.13**, and choose two different coded elements $a, b \in X$. We let X' be the chain obtained by putting a copy $(a, b)_1$, of the interval (a, b), immediately to the right of b, so that every element in (a, b) is less than every element of $(a, b)_1$, and every element of this is less b - see **Figure 4.5**.

Then X' is clearly not strongly rigid, but it stays automorphism rigid on the whole chain, for if f is an automorphism of X' then, since f still has to preserve the

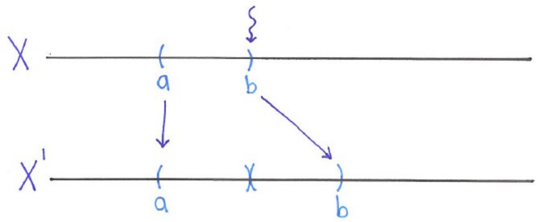

Figure 4.5: X' is an automorphism rigid chain which is not locally rigid.

"coding" of its coded points, it will fix every element $x \leqslant a$ and every $x \geqslant b$. It cannot send $x \in (a,b)$ to a distinct point of (a,b) because of our construction, and hence the only other place that x can be sent to is the copy x_1 of x in $(a,b)_1$. But then if $x < x_1 < b$ and $f(x) = x_1$, we must have $f(x) = x_1 < f(x_1) < f(b) = b$, hence violating the fact that we can't move a point to any element of the same copy.

We can even do more than this. If we let $C = \{(a,b)_n \mid n \in \omega\}$ be a set of ω-many copies of $(a,b) = (a,b)_0$ added to X in the obvious way, then we get an automorphism rigid chain $(X', \leqslant')$ (for it must preserve each copy rigid and can't move anything upwards since it fixes everything below a) with some epimorphisms, in fact for each $\varphi \in \mathrm{Epi}(\mathbb{N}, \leqslant)$, if we write $x_{\varphi(n)}$ for the copy of $x \in (a,b)$ that lies in $(a,b)_{\varphi(n)}$, then f_φ may be defined as

$$f_\varphi(x) = \begin{cases} x_{\varphi(n)} & \text{if} \quad x \in (a,b)_n \text{ for some } n \\ x & \text{if} \quad x \notin \bigcup\limits_{n \in \omega} (a,b)_n \end{cases}$$

Then $f_\varphi[(a,b)_n] = (a,b)_{\varphi(n)}$ and f_φ is identity elsewhere,which is an epimorphism of $(X', \leqslant')$. Moreover, since any epimorphism ψ of X' must fix everything below a and below b, ψ must be generated in the wide sense from one of f_φ, for some $\varphi \in \mathrm{Epi}(\mathbb{N}, \leqslant)$ (take the sequence $(b_n : n \in \mathbb{N})$ to be such that $b_n < b_{n+1}$ in (a,b) but b_n lies in the n-th copy $(a,b)_n$ of (a,b)).

Chapter 5

Graphs

"What we call chaos is just patterns we haven't recognized. What we call random is just patterns we can't decipher. What we can't understand we call nonsense."

– Chuck Palahniuk, *Survivor*

5.1 Introduction

The countable **random graph** (also known as *Rado's graph* or the *Erdös-Rényi graph*) is the unique (up to isomorphism) countable graph with the following property,

ARP: *If U and V are two finite and disjoint sets of vertices, then there is a vertex $x \notin U \cup V$ which is joined to all elements in U and none of V.*

The initials **ARP** stand for **Alice's Restaurant Property**, and according to J. Spencer in [Spe01], this name was first used by Peter Winkler in allusion to the refrain in a song by Arlo Guthrie - you can get anything you want at Alice's Restaurant. The proof of uniqueness of Γ follows a standard *back-and-forth* argument.

The study of the random graph has been very well developed since the 1960's after Erdös and Rényi published a very influential paper [ER60] and much work has been done about it.

In this chapter, we shall denote the Random Graph by Γ. To fix some notation, we will be working with graphs $\mathcal{G}$ that have a set of vertices in some ordinal κ (for example, the vertices of Γ are in ω), and if α and β in κ are adjacent (joined) by an edge, then we write $\alpha \sim \beta$. If α is joined to every element of the set U, then we simply write $\alpha \sim U$. If $\alpha \in \kappa$ is a vertex of $\mathcal{G}$, then we denote by $E(\alpha) = \{\beta \in \kappa \mid \alpha \sim \beta\}$ the set of vertices that are joined to α and $NE(\alpha) = \{\beta \in \kappa \mid \beta \nsim \alpha\} \setminus \{\alpha\}$ the vertices that are not joined to α. If $A \subseteq \kappa$ is the set of vertices of a subgraph g of $\mathcal{G}$, then we say A is the **domain** of g and we denote it by $\mathrm{dom}(g)$.

Now, the story looks a bit different once we look at uncountable graphs. For instance, there are many uncountable graphs satisfying **ARP**, while **CH** implies that there is only one ω_1-graph up to isomorphism which is saturated (see Theorem 5.1 below). But even now, there will be many (non-saturated) **ARP** graphs of cardinality $\aleph_1$.

When we say that an uncountable graph $|G| = \aleph_1$ is **saturated**, we mean that it satisfies the following statement, which is a natural generalization of our **ARP**, in the sense that if G is saturated then it clearly satisfies **ARP**.

(*) *If U and V are two* countable *and disjoint sets of vertices, then there is a vertex $x \notin U \cup V$ which is joined to all elements in U and none of V.*

Theorem 5.1 (Literature [*]). *Assume* CH. *Then there is an ω_1-graph $\mathcal{G}$ which is saturated.*

Proof. We will construct $\mathcal{G}$ inductively. Let $\mathcal{G}_0$ be our countable random graph Γ. Let $\mathcal{A} = \{A_n \subset \omega_1 \mid n \in \omega\}$ be such that for each $n, m \in \omega$, $|A_n| = \aleph_1$, $A_n \cap A_m = \varnothing$ and $\bigcup_{n \in \omega} A_n = \omega_1 \setminus \omega$. The elements of $\mathcal{A}$ are going to comprise the family of vertices of $\mathcal{G}$. Next, we will enumerate all the pairs (U, V) of disjoint countable subsets of $\mathcal{G}_0$ and put a witness to **(*)** for each one of them from the elements in A_0. Notice that since we are assuming that **CH** holds, we have ω_1- many of them. Then we shall proceed

[*] This result is well known and appears in [CK73] page 216

in a similar way to define what to do with the rest of the elements in $\mathcal{A}$.

For each $n \in \omega$, we let $B_n = \{(U^n_\nu, V^n_\nu) \mid \nu \in \omega_1\}$ be the collection of all disjoint pairs of countable subsets of $\mathcal{G}_n$. Now for each $(U^n_\nu, V^n_\nu) \in B_n$, we look at $x^n_\nu \in A_n$ and define the relations:

$$x^n_\nu \sim U^n_\nu \text{ and } x^n_\nu \nsim V^n_\nu$$

And let $\mathcal{G}_{n+1} = \mathcal{G}_n \cup A_n$. Then the graph $\mathcal{G} = \bigcup_{n \in \omega} \mathcal{G}_n$ satisfies (*). □

In the countable case, having the property **ARP** is equivalent to being **homogeneous**[†], in the same way that being saturated is equivalent to being $\aleph_1$**-homogeneous**[‡].

Lemma 5.2. *If $\mathcal{G}$ is a saturated ω_1-graph then it is $\aleph_1$-homogeneous.*

We will see in this chapter that the notions of **ARP**, saturation and homogeneity do not coincide in general.

5.2 Generically constructed graph, Γ_{ω_1}

In this section we will construct an ω_1-graph that satisfies the **ARP** but is not homogeneous.

Proposition 5.3. *There is a model M where there is a rigid graph Γ_{ω_1} that satisfies* **ARP**.

Proof. We will make use of a forcing model to prove the above statement. Our notion of forcing is set of all finite graphs with vertices in ω_1 ordered by extension, that is

[†] Any isomorphism between finite structures, extends to an automorphism of the whole structure.

[‡] Any isomorphism between countable substructures, extends to an automorphism of the whole structure.

$$\mathbb{P}_\Gamma = \{p \mid p \text{ is a graph, } \mathrm{dom}(p) \subseteq \omega_1 \text{ and } |\mathrm{dom}(p)| < \aleph_0\} \text{ and}$$

$$q \leq p \ (q \text{ extends } p) \text{ iff } \mathrm{dom}(p) \subseteq \mathrm{dom}(q) \wedge \forall \alpha, \beta \in \mathrm{dom}(p),\ \alpha \sim \beta \text{ in } p \text{ iff } \alpha \sim \beta \text{ in } q.$$

Then $\Gamma_{\omega_1} = \bigcup G$ for some generic $G \subseteq \mathbb{P}_\Gamma$. We will prove first that any automorphism of Γ_{ω_1} will move uncountably many points, and we shall denote by **supp(f)** the set of elements moved by a function f.

Lemma 5.4. *If $\theta \in \mathrm{Epi}(\Gamma_{\omega_1})$ or $\theta \in \mathrm{Emb}(\Gamma_{\omega_1})$ and θ is not the identity map, then θ has uncountable* ***support****, that is, the set of vertices that are moved by θ is uncountable.*

Proof. Assume by way of contradiction that there is a condition $p \in G$ such that $p \Vdash \ulcorner \dot{\theta}$ has countable support $\urcorner$, i.e. there is $\xi \in M$ such that for some $\alpha \neq \beta \in supp(\theta)$

$$p \Vdash \ulcorner \dot{\theta} \in \mathrm{Epi}(\Gamma_{\omega_1}) \text{ or } \dot{\theta} \in \mathrm{Emb}(\Gamma_{\omega_1}),\ supp(\dot{\theta}) \subseteq \xi \wedge \theta(\alpha) = \beta \urcorner$$

Now, since p is finite, we can find $\delta > \xi$ and $\delta \notin \mathrm{dom}(p)$ and define a condition $q \in \mathbb{P}$ by,

$$q = p \cup \{\delta \sim \alpha\} \cup \{\delta \nsim \beta\}$$

Then, $q \Vdash \ulcorner \dot{\theta}(\delta) = \delta \wedge \alpha \sim \delta \wedge \dot{\theta}(\alpha) = \beta \nsim \dot{\theta}(\delta) \urcorner$. Since $q \leq p$ this gives a contradiction. □

So, in particular this works for $\theta \in \mathrm{Aut}(\Gamma_{\omega_1})$. Now we will see that the **ARP** holds for our generically constructed Γ_{ω_1}.

Claim 5.5. Γ_{ω_1} *satisfies* **ARP**.

Proof. For each pair of finite disjoint subsets U, V of ω_1 let

$$D_{U,V} = \{p \in \mathbb{P}_\Gamma \mid \exists \alpha \in \mathrm{dom}(p) \smallsetminus (U \cup V), U \subseteq E(\alpha) \wedge V \subseteq NE(\alpha)\}$$

Note that $D_{U,V} \in M$. We will show that this subset is dense in $\mathbb{P}_\Gamma$.

Let $q \in \mathbb{P}_\Gamma$. Let p be a finite graph extending q, with domain equal to $\mathrm{dom}(q) \cup (U \cup V)$. Now, let $\alpha \in \omega_1 \setminus \mathrm{dom}(p)$, and define p' extending p so that $\mathrm{dom}(p') = \mathrm{dom}(p) \cup \{\alpha\}$ and

$$\gamma \sim \beta \text{ in } p' \leftrightarrow (\gamma \sim \beta \text{ in } p) \text{ or } (\gamma = \alpha \wedge \beta \in U) \vee (\beta = \alpha \wedge \gamma \in U)$$

Then $p' \leq q$ and $p' \in D_{U,V}$. Hence for each finite disjoint $U, V \subseteq \omega_1$, $D_{U,V}$ is dense in $\mathbb{P}_\Gamma$ and it intersects the generic subset G, so for each U, V there is a condition in G that forces the property, therefore the generically constructed graph satisfies **ARP**. ☺

The next claim shows that Γ_{ω_1} is rigid in the generic extension.

Claim 5.6. *If θ is an automorphism of Γ_{ω_1}, then θ is the identity.*

Proof. Assume θ is a non trivial automorphism of Γ_{ω_1}, that is,

$$\mathbf{1} \Vdash \ulcorner \dot{\theta} \text{ is a non-identity automorphism of } \omega_1 \urcorner$$

For each $\alpha \in supp(\theta)$ let p_α be a condition in G such that for some $\alpha \neq \alpha_1$

$$p_\alpha \Vdash \ulcorner \dot{\theta}\alpha = \alpha_1 \urcorner.$$

By extending p_α if necessary, we can assume $\alpha \in \mathrm{dom}(p_\alpha)$. Then by **Claim 5.4** above, we have an uncountable set of finite conditions so we can use the **Δ-System Lemma** to find an uncountable $Z \subseteq \mathbb{P}$ and a finite subset S satisfying $p_\alpha \cap p_\beta = S$, for any two distinct p_α, p_β in Z.

Now choose $p_\alpha, p_\beta \in Z$ with α, β and their images under θ not contained in S (which can be done because we are working with finite sets, and both $supp(\theta)$ and its image under θ are uncountable). Let q be the graph

$$q = p_\alpha \cup p_\beta \cup \{\alpha \sim \beta\} \cup \{\alpha_1 \nsim \beta_1\}.$$

Then q extends both p_α and p_β but $q \Vdash \ulcorner\dot{\theta}$ is not an automorphism of $\dot{\Gamma}_{\omega_1}\urcorner$. ☺

This finishes the proof of **Proposition 5.3**. □

The above proof still works to show that Γ_{ω_1} does not admit embeddings or epimorphisms other than the identity and it is enough to prove that $\mathrm{Im}(supp(\theta))$ is also uncountable and follow the arguments as in **Claim 5.6**.

Claim 5.7. *If θ is either an epimorphism or an embedding of Γ_{ω_1}, then $\theta(supp(\theta))$ is uncountable.*

Proof. Consider first that we are dealing with an epimorphism. Assume by way of contradiction that $p \Vdash \ulcorner\dot{\theta} \in \mathrm{Epi}(\dot{\Gamma}_{\omega_1}) \wedge \mathrm{Im}(supp(\dot{\theta}))$ is countable$\urcorner$. Then, there is $\xi < \omega_1$ such that $p \Vdash \ulcorner\dot{\theta}(supp(\dot{\theta})) \subseteq \xi\urcorner$. Since every element of $\omega_1 \smallsetminus supp(\theta)$ is mapped to itself, we have

$$p \Vdash \ulcorner\ \mathrm{Im}(\dot{\theta}) \subseteq \xi \cup (\omega_1 \smallsetminus supp(\dot{\theta}))\urcorner$$

Now, because $\theta \in \mathrm{Epi}(\Gamma_{\omega_1})$ and we are assuming that the image of $supp(\theta)$ is countable, $p \Vdash \ulcorner supp(\dot{\theta}) \neq \omega_1\urcorner$, and since we have also shown that $supp(\theta)$ is uncountable this implies that $p \Vdash \ulcorner(supp(\dot{\theta}) \smallsetminus \xi) \neq \varnothing\urcorner$. Hence, $p \Vdash \ulcorner supp(\dot{\theta}) \smallsetminus \xi \nsubseteq \mathrm{Im}(\dot{\theta})\urcorner$, which is a contradiction with θ being surjective.

To conclude, we just remark that if θ is an embedding then it must be injective and since $supp(\theta)$ is uncountable, so is its image. ☺

The following are some properties of our generically constructed random graph, Γ_{ω_1}.

Proposition 5.8. *Γ_{ω_1} is not saturated.*

Proof. Let $A = \{n \mid n \in \omega\}$ and $B = \{\omega + n \mid n \in \omega\}$. We'll prove that for each $\alpha \geqslant \omega.2$, we can find an element x in A and another y in B so that α is joined to both x and y. For $\alpha \in \omega_1 \smallsetminus (A \cup B)$ we will show the following set is dense in $\mathbb{P}_\Gamma$

$$D^\alpha = \{p \in \mathbb{P}_\Gamma \mid (\exists x \in \mathrm{dom}(p) \cap A)(\exists y \in \mathrm{dom}(p) \cap B)[\alpha \in E(x) \wedge \alpha \in E(y)]\}$$

Let $q \in \mathbb{P}_\Gamma$. Since A, B are countable and $\mathrm{dom}(p)$ is finite, we can find $x \in A \smallsetminus (\mathrm{dom}(p) \cup \{\alpha\})$ and $y \in B \smallsetminus (\mathrm{dom}(q) \cup \{\alpha\})$. Let $p = q \cup \{\alpha \sim x\} \cup \{\alpha \sim y\}$. Then p extends q and $p \in D^\alpha$.

Hence, for each $\alpha \notin (A \cup B)$ there is a condition in G preventing α from being a witness for $\aleph_1$ saturation of Γ_{ω_1}. □

Observe that Γ_{ω_1} is specified by $E(x)$ and $NE(x)$ for all $x \in \omega_1$. We will show that these two sets are stationary for each $x \in \omega_1$. The proofs are very similar so we will only give the details for $E(x)$.

Proposition 5.9. *Let $x \in \Gamma_{\omega_1}$. Then $E(x)$ and $NE(x)$ are stationary in ω_1.*

Proof. Let $C \in M[G]$ be a club in ω_1 and $x \in \Gamma_{\omega_1}$. Then there is a condition $p \in G$ satisfying $p \Vdash \ulcorner \dot{C}$ is a club in $\omega_1 \urcorner$. We want to show that there is $q \in G$ with $q \Vdash \ulcorner \dot{E}(x) \cap \dot{C} \neq \varnothing \urcorner$, that is, there is $\alpha \in \omega_1$ such that $p \Vdash \ulcorner \alpha \in \dot{C} \wedge \alpha \sim x \urcorner$.

Since $C \in M[G]$ is a club and $\mathbb{P}_\Gamma$ is *c.c.c.*, there is $D \in M$, $D \subseteq C$ which is also a club in ω_1 and hence we need only to show $D \cap E(x)$ is non-empty. To see this, let $D = \{\alpha \mid p \Vdash \ulcorner \alpha \in \dot{C} \urcorner\} \in M$. Then clearly $D \subseteq C$, that is, $p \Vdash \ulcorner D \subseteq \dot{C} \urcorner$. We shall show D is a club.

For closure, let $(\alpha_\nu : \nu < \lambda \in \mathrm{Lim}(\omega_1))$ be an increasing sequence of elements in D with $\alpha' = \sup\{\alpha_\nu : \nu < \lambda\}$. Then, for all α_ν, $p \Vdash \ulcorner \alpha_\nu \in \dot{C} \urcorner$ and $p \Vdash \ulcorner \dot{C}$ is closed $\urcorner$, hence $p \Vdash \ulcorner \forall \alpha_\nu, \alpha_\nu \in \dot{C} \wedge \dot{C}$ is closed$\urcorner$ and thus $p \Vdash \ulcorner \alpha' \in \dot{C}$.

For unboundedness, let $\alpha_0 < \omega_1$. We want to show that there is $\alpha > \alpha_0$ such that $p \Vdash \ulcorner \alpha_0 \in \dot{C} \urcorner$. Since $p \Vdash \ulcorner \dot{C}$ is unbounded $\urcorner$, then $p \Vdash \ulcorner \alpha_0 < \dot{\gamma} \in \dot{C} \urcorner$ for some $\gamma \in M$. Now, we can get a maximal antichain A of conditions and a sequence of ordinals $(\gamma_q : q \in A)$ satisfying $q \Vdash \ulcorner \dot{\gamma} = \gamma_q \urcorner$. Since $\mathbb{P}_\Gamma$ is *c.c.c.*, A is countable so let α_1 be the supremum of the γ_q's, which is countable. Then $p \Vdash \ulcorner (\exists \gamma \in \dot{C}) \alpha_0 < \gamma \leqslant \alpha_1 \urcorner$. In the same manner, we find an increasing sequence $(\alpha_n : n \in \omega)$ such that

$$p \Vdash \ulcorner(\exists\gamma \in \dot{C})\alpha_n < \gamma \leqslant \alpha_{n+1}\urcorner$$

and let $\alpha = \sup_{n\in\omega} \alpha_n$. Then $p \Vdash \ulcorner\alpha_0 < \alpha \in \dot{C}\urcorner$, as required.

Now, we will show that the next set is dense below p,

$$B = \{q \leq p \mid q \Vdash \ulcorner\exists\alpha \in D, \alpha \sim x\urcorner\}$$

Let $q \leq p$. By extending it if necessary, we can assume $x \in \mathrm{dom}(q)$. Let $\alpha_{max} = \max\{\gamma \in q\}$ which is less than ω_1. Then, since D is unbounded in ω_1, we can find a countable $\alpha > \alpha_{max}$ in D. Define a finite graph q' by $\mathrm{dom}(q') = \mathrm{dom}(q) \cup \{\alpha\}$ and $\alpha \sim x$, then $q' \in B$.

Hence $B \in M$ is dense and $B \cap G \neq \varnothing$. Since $p \Vdash \ulcorner D \subseteq \dot{C}\urcorner$ we have $q' \Vdash \ulcorner\alpha \sim x \wedge \alpha \in \dot{C}\urcorner$, as required.

To show that $NE(x)$ is stationary, we show that the set is dense in a similar manner,

$$B' = \{q \leq p \mid q \Vdash \ulcorner\exists\alpha \in D, \alpha \nsim x\urcorner\}.$$

□

Moreover, for each pair of disjoint finite subsets U, V of ω_1 and $x \notin U \cup V$, the set $\bigcap_{x\in U} E(x) \cup \bigcap_{x\in V} NE(x) = \{\alpha \in \omega_1 \mid \alpha \sim U \wedge \alpha \nsim V\}$ is also stationary. This is shown in the same way as in the proposition above since the following set is dense in $\mathbb{P}_\Gamma$ for each U, V,

$$D_{U,V} = \{p \in \mathbb{P}_\Gamma \mid (\exists\alpha)[\alpha \notin (U \cup V) \wedge \alpha \in D \wedge \alpha \sim U \wedge \alpha \nsim V]\} \in M$$

Proposition 5.10. *For each $\alpha \in \omega_1$, let Γ_α be the subgraph of Γ_{ω_1} whose domain is given by the set $X_\alpha = \{\omega.\alpha + n \mid n \in \omega\}$. Then Γ_α is isomorphic to the countable random graph.*

Proof. Let $\alpha \in \omega_1$ and fix two finite and disjoint subsets $U, V \subseteq X_\alpha$. Let

$$D^\alpha_{U,V} = \{p \in \mathbb{P}_\Gamma \mid \exists \gamma \in (\mathrm{dom}(p) \cap X_\alpha) \smallsetminus (U \cup V)[\gamma \sim U \text{ and } \gamma \nsim V]\}.$$

Then $D^\alpha_{U,V} \in M$ and it is dense in $\mathbb{P}_\Gamma$ just as in the proof of **Claim 5.5**, so it intersects our generic set G, and this establishes the **ARP** for Γ_{ω_1}. □

5.3 Generically constructed graph, Δ_{ω_1}

In this section we turn our attention to another graph, constructed analogously to Γ_{ω_1}.

We now let $\mathbb{P}_\Delta$ be the notion of forcing that consists of all countable graphs with vertices in ω_1,

$$\mathbb{P}_\Delta = \{p \mid p \text{ is a graph, } \mathrm{dom}(p) \subset \omega_1 \text{ and } |\mathrm{dom}(p)| = \aleph_0\} \text{ and}$$

$$q \leq p \ (q \text{ extends } p) \text{ iff } \mathrm{dom}(p) \subseteq \mathrm{dom}(q) \text{ and } \forall \alpha, \beta \in \mathrm{dom}(p),$$

$$\alpha \sim \beta \text{ in } q \text{ iff } \alpha \sim \beta \text{ in } p.$$

Then $\Delta_{\omega_1} = \bigcup G$ for some generic $G \subseteq \mathbb{P}_\Delta$.

This notion of forcing is countably closed: for let $p_0 \geqslant p_1 \geqslant ..$ be a decreasing countable sequence of conditions in $\mathbb{P}_\Delta$, then $\bigcup_{n \in \omega} p_n$ is a countable graph with vertices in ω_1.

Proposition 5.11. *The graph Δ_{ω_1} is saturated.*

Proof. Let U, V be two disjoint countable subsets of ω_1 in the generic extension $M[G]$. Since $\mathbb{P}_\Delta$ is countably closed, $M[G]$ and the ground model M share the same countable subsets of ordinals, hence U, V are in M. We will show that the following set is dense in $\mathbb{P}_\Delta$ for each pair U, V of disjoint countable subsets of ω_1.

$$D_{U,V} = \{p \in \mathbb{P}_\Delta \mid (\exists \gamma \in \omega_1 \smallsetminus (U \cup V))[\gamma \sim U \text{ and } \gamma \nsim V]\}$$

Let $q \in \mathbb{P}_\Delta$. Let q' extend q such that $\mathrm{dom}(q') = \mathrm{dom}(q) \cup (U \cup V)$. Since q' is countable, $q' \in \mathbb{P}_{\omega_1}$ and we can find $\gamma \in \omega_1 \setminus \mathrm{dom}(q')$ and define a countable graph p such that,

$$\begin{aligned} \mathrm{dom}(p) &= \mathrm{dom}(q') \cup \{\gamma\} \\ \beta \sim \alpha &\leftrightarrow (\beta \sim \alpha \text{ in } q'), \text{ or} \\ &\quad (\beta = \gamma \wedge \alpha \in U) \vee (\alpha = \gamma \wedge \beta \in U) \end{aligned}$$

Then $p \leq q' \leq q$ and $p \in D_{U,V}$, therefore $D_{U,V}$ is dense in $\mathbb{P}_\Delta$ and hence for each U, V, $D_{U,V} \cap G \neq \varnothing$ and there is a condition $p \in G$ that forces (*). □

We will show that this notion of forcing collapses 2^ω to ω_1.

Proposition 5.12. $M[V] \vDash \ulcorner \mathfrak{c} = \aleph_1 \urcorner$.

Proof. We already know that $2^{\aleph_0} \geqslant \aleph_1$ so we will concentrate in showing we can find $2^{\aleph_0}$ different elements in Δ_{ω_1}, which is an ω_1 graph. Let $\{q_\eta \mid \eta \in \omega\}$ be an enumeration of the rational numbers $\mathbb{Q}$. Now, for each $\nu \in \mathbb{R} \setminus \mathbb{Q}$ (irrational), let

$$\begin{aligned} U_\nu &= \{\eta \mid q_\eta <_{\mathbb{R}} \nu\} \\ V_\nu &= \{\eta \mid q_\eta >_{\mathbb{R}} \nu\} \end{aligned}$$

which are both countable subsets of ω_1 and since Δ_{ω_1} is saturated, there is $x_\nu \in \Delta_{\omega_1}$ such that $x_\nu \sim U_\nu$ and $x_\nu \nsim V_\nu$. There are $2^{\aleph_0}$ irrationals, so this gives us $2^{\aleph_0}$ distinct members of Δ_{ω_1}, for if $\nu < \upsilon$ are both irrational numbers, then there is ρ such that $\rho \in (\nu, \upsilon)$ and ρ is joined to x_υ but not to x_ν. Therefore $x_\nu \neq x_\upsilon$ and hence $2^{\aleph_0} = \aleph_1$. □

Proposition 5.13. *Let $x \in \Delta_{\omega_1}$. Then $E(x)$ and $NE(x)$ are stationary in ω_1.*

Proof. We will only show the statement for $E(x)$ since the proof for $NE(x)$ is completely analogous. Let C be a club in ω_1. Then, there is $p \in G$ such that $p \Vdash \ulcorner \dot{C}$ is a club in $\omega_1 \urcorner$. We want to find a condition extending p that forces the intersection $E(x) \cap C$ to be non-empty, so will show that the following set is dense below p.

$$D = \{q \leq p \mid (\exists \alpha < \omega_1)[q \Vdash \ulcorner \alpha \in \dot{C} \wedge \alpha \sim x] \urcorner\}$$

Let $q \leq p$. Let $\alpha > \sup\{\gamma : \gamma \in q\}$. Then $\alpha \notin q$. Since q extends p, there is an ordinal $\beta_0 > \alpha$ (and hence not in q) and a condition $r_0 \leq q$ such that $r_0 \Vdash \ulcorner \alpha < \beta_0 \in \dot{C} \urcorner$. Similarly, there is $\beta_1 > \sup(r_0)$, $\beta_1 > \beta_0$ and $r_1 \leq r_0$ such that $r_1 \Vdash \ulcorner \beta_0 < \beta_1 \in \dot{C} \urcorner$ (and hence $\beta_1 \notin r_0$).

Hence, we can construct in this manner a sequence of conditions $r_0 \geq r_1 \geq ... r_n \geq ...$ and an increasing sequence of ordinals $\beta_0 < \beta_1 ... \beta_n < ...$ satisfying

$$r_n \Vdash \ulcorner \beta_{n-1} < \beta_n \in \dot{C} \urcorner \text{ and } \beta_n > \sup(r_{n-1}).$$

Since $\mathbb{P}_\Delta$ is countably closed, $r_\omega = \bigcup_{n \in \omega} r_n$ is a condition in $\mathbb{P}_\Delta$ extending every r_n, and if $\beta_\omega = \sup_{n \in \omega} \beta_n$, then $\beta_\omega > \sup(r_n)$ so $\beta_\omega \notin r_n$, for each $n \in \omega$ and hence $\beta_\omega \notin r_\omega$. Since all these conditions force C to be a club we have $r_\omega \Vdash \ulcorner \beta_\omega \in \dot{C} \urcorner$.

Finally we define $q' \in \mathbb{P}_\Delta$ by,

$$q' = r_\omega \cup \{x \sim \beta_\omega\}$$

$\square$

Bibliography

[AS87] Shelah S. Avraham, U. and R.M. Solovay. Squares with diamonds and Suslin trees with special squares. *Fund. Math.*, 2:133–162, 1987.

[Avr79] Uri Avraham. Construction of a rigid Aronszajn tree. *Proc. Amer. Math. Soc.*, 77:136–137, 1979.

[CK73] C.C. Chang and H.J. Keisler. *Model Theory*. North-Holland, Amsterdam, 1973.

[Coh66] J. Paul Cohen. *Set Theory and the Continuum Hypothesis*. W. A. Benjamin, Inc, 1966.

[Dev84] Keith J. Devlin. Constructibility. In *Perspectives in Mathematical Logic*, chapter 6. Berlin: Springer-Verlag, 1984.

[DJ74] Keith J. Devlin and Havard Johnsbraten. The Souslin Problem. In *Lecture Notes in Mathematics*, volume 405. Springer-Verlag Berlin-Heidelberg, 1974.

[DM40] B. Dushnik and E.W. Miller. Concerning similarity transformations of linearly ordered sets. *Bull. Amer. Math. Soc*, 46:322–326, 1940.

[DT01] M. Droste and J. Truss. Rigid chains admitting many embeddings. *Proc. Amer. Math. Soc.*, 129:1601–1608, 2001.

[Eis12] Todd Eisworth. *Handbook of Set Theory*, volume 2. Springer, 2012.

[ER60] Paul Erdös and Alfred Rényi. On the evolution of Random Graphs. *Mat. Kutató Int. Közl*, 5:17–60, 1960.

[FH09] Gunter Fush and David Hamkins. Degrees of rigidity for Suslin trees. *The Journal of Symbolic Logic*, 74:423–454, 2009.

[GR12] Moti Gitik and Assaf Rinot. The failure of diamond on a reflecting stationary set. *Trans. Amer. Math.*, 4:771–1795, 2012.

[GS64] H Gaifman and E.P. Specker. Isomorphism types of trees. *Proc. Amer. Math. Soc.*, 15:1–7, 1964.

[Jec67] Thomas Jech. Non-provability of Souslin's Hypothesis. *Commentationes mathematicae Universitatis Carolinae*, 8:291–305, 1967.

[Jec72] Thomas Jech. Automorphisms of ω_1-Trees. *Transactions of the American Mathematical Society*, 173:57–70, 1972.

[Jec86] Thomas Jech. *Multiple Forcing.* Cambridge University Press, 1986.

[Jec02] Thomas Jech. *Set Theory.* Springer, 2002.

[Jen68] Ronald Jensen. Souslin's Hypothesis is incompatible with V = L. *Notices of the American Mathematical Society*, 16:935, 1968.

[Kan06] Akihiko Kanamori. *Historical Remarks on Suslin's Problem.* Invited address given at a memorial conference commemorating the life and work of Stanley Tennenbaum held at the Graduate Center of the City University of New York, April 2006.

[LT12] D.C. Lockett and J.K. Truss. Generic endomorphisms of homogeneous Structures. In Fuchs Tent Strungmann, Droste, editor, *Groups and Model Theory*, Contemporary Mathematics, pages 217–237. American Mathematical Society, 2012.

[Mik20] Suslin Mikhail. Probléme 3. *Fundamenta Mathematicae*, 1:223, 1920.

[Nes02] Jaroslav Nesetril. A rigid graph for every set. *Journal of Graph Theory*, 39:108–110, 2002.

[Rin10] Assaf Rinot. Jensen's Diamond Principle and its relatives. *Notes for 18th annual B.E.S.T. conference*, 2010.

[Rub93] Matatyahu. Rubin. The reconstruction of trees from their automorphism group. In *Contemporary Mathematics*. American Mathematical Society, 1993.

[Rub94] Matatyahu Rubin. On the reconstruction of $\aleph_0$-categorical structures from their automorphism group. *Proc. London Math. Soc, Series 3 (69)*, 2:225–249, 1994.

[She82] Saharon Shelah. Better quasi-orders for uncountable cardinals. *Israel J. Math*, 42:177–226, 1982.

[She84a] Saharon Shelah. Can you take Solovays inaccessible away? *Israel J. Math*, 1:1–47, 1984.

[She84b] Saharon Shelah. Diamonds, uniformization. *J. Symbolic Logic*, 4:1022–1033, 1984.

[She03] Saharon Shelah. Successor of singulars: combinatorics and not collapsing cardinals $\leqslant \kappa$ in $(< \kappa)$-support iterations. *Israel J. Math*, 134:127–155, 2003.

[Spe01] Joel Spencer. The Strange Logic of Random Graphs. In *Algorithms and Combinatorics*, chapter 22. Springer-Verlag Berlin-Heidelberg, 2001.

[ST71] R. M. Solovay and S. Tennenbaum. Iterated Cohen extensions and Souslin's problem. *Annals of Mathematics*, 94:201–245, 1971.

[Ten68] S. Tennenbaum. Souslin's Problem. *Proceedings of the National Academy of Sciences of the United States of America*, 59:60–63, 1968.

[Tod78] Todorčević. Rigid Aronszajn Trees. *Publications de L'institut Mathematique*, 27 (41):259–265, 1978.

[Tod84] Todorčević. Trees and Linearly Ordered Sets. In *Handbook of Set-Theoretic Topology*, chapter 6, pages 235–293. Elsevier Science Publishers B.V., 1984.

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