Geometric Flows on Soliton Moduli Spaces

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Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

The University of Leeds
Department of Pure Mathematics

Submitted June 2013

Acknowledgements

Above all, I would like to thank my supervisor, Martin Speight, for his patience, his valuable guidance and all the support he has offered me generously over the last four years. He has been an indispensable source of knowledge and experience for me, for which I am the most grateful.

I am greatly indebted to King Abdulaziz University (Jeddah) for granting me a scholarship, giving me the opportunity to be entirely devoted to my PhD research. Moreover, I would like to thank the University of Leeds and in particular the School of Mathematics, for the friendly-working environment, seminars and workshops which were very important in my training procedure.

Finally, my deepest thanks to my parents, Saeedah and Saeed, who encouraged me to study abroad, and for their indescribable support in everything I do in my life and for everything they have done for me.

Abstract

It is well known that the low energy dynamics of many types of soliton can be approximated by geodesic motion on M_n, the moduli space of static n-solitons, which is usually a Kähler manifold. This thesis presents a detailed study of magnetic geodesic motion on a Kähler manifold in the case where the magnetic field 2-form is the Ricci form. This flow, which we call Ricci Magnetic Geodesic (RMG) flow, is first studied in general. A symmetry reduction result is proved which allows one to localize the flow onto the fixed point set of any group of holomorphic isometries of a Kähler manifold M. A subtlety of this reduction, which was overlooked by Krusch and Speight, is pointed out. Since RMG flow occurs at constant speed, it follows immediately that the flow is complete if M is geodesically complete. We show, by means of an explicit counterexample that, contrary to a conjecture of Krusch and Speight, the converse is false: it is possible for a geodesically incomplete manifold to be RMG complete. RMG completeness of metrically incomplete manifolds is therefore a nontrivial issue, and one which will be addressed repeatedly in later chapters.

We then specialize to the case where M_n is the moduli space of abelian Higgs n-vortices, which is the context in which RMG flow was first proposed, by Collie and Tong, as a low energy model of the dynamics of a certain type of Chern-Simons n-vortices on $\mathbb{R}^2$. The unit vortex is constructed numerically, and its asymptotics is studied. It is shown that, contrary to an assertion of Collie and Tong, RMG flow does not coincide with an earlier proposed magnetic geodesic model of vortex motion due to Kim and Lee. It is further shown that Kim and Lee's model is ill-defined on the vortex coincidence set. An asymptotic formula for the scattering angle of well-separated vortices executing RMG flow is computed. We then change the spatial geometry, placing the vortices on the hyperbolic plane of critical curvature. An explicit formula for the two-vortex metric is derived, extending the results of Strachan, who computed the metric on a submanifold of centred 2-vortices. The RMG flow localized on this submanifold is compared with its intrinsic RMG flow, revealing strong qualitative differences.

We then study the moduli space $\mathcal{H}_{n,k}(\Sigma)$ of degree n $\mathbb{C}P^k$ lumps on a compact Riemann surface Σ. It is shown that $\text{Rat}_1 = \mathcal{H}_{1,1}(S^2)$ is RMG complete (despite being metrically incomplete). The Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ is computed, supporting (for $k > 1$) a conjecture of Baptista. A natural class of topologically cylindrical submanifolds of $\mathcal{H}_{n,1}(\Sigma)$, called dilation cylinders, is studied: their volumes are computed, and it is shown that they are all isometrically embeddable as surfaces of revolution in $\mathbb{R}^3$. Conditions under which they are totally geodesic, for $\Sigma = S^2$ and T^2, are found, and RMG flow on some examples is studied.

Finally, a new metric on $\mathcal{H}_{n,1}(\Sigma)$, derived from the Baby-Skyrme model, is introduced. On Rat_1, this metric is determined explicitly and some geometric aspects such as the volume, geodesic flow and the spectral problem with respect to this metric are studied.

Contents

Acknowledgements ii
Abstract iii
Contents v
List of figures viii
List of tables x

1 Ricci Magnetic Geodesics **1**
1.1 Preliminaries 1
1.2 Ricci Magnetic Geodesics 3
1.3 Ricci Magnetic Geodesics and Symmetry Reduction 8
1.4 Ricci Magnetic Geodesics on Surfaces of Revolution 12

2 Chern-Simons Abelian Higgs Vortices **25**
2.1 Field Theories 25
2.2 Abelian Higgs Model on $\mathbb{R}^2$ 28
2.3 Chern-Simons Abelian Higgs Model on $\mathbb{R}^2$ 33
2.3.1 Vortex Asymptotics 36
2.3.2 Numerical Calculation of the 1-Vortex 41
2.3.3 The Point Vortex 43
2.3.4 n-Vortex Low Energy Dynamics 47
2.3.5 2-Vortex Scattering 50
2.4 Abelian Higgs Model on $\mathbb{H}^2$ 56
2.4.1 The L^2 Metric on the Hyperbolic 2-Vortex Moduli Space 59

2.4.2 RMG Flow on the Hyperbolic 2-Vortex Relative Moduli Space . . 69

3 Spaces of Holomorphic Maps on Compact Riemann Surfaces 76

3.1 $\mathbb{C}P^k$ model and L^2 Metric . . . 76

3.2 Degree 1 holomorphic maps $S^2 \to \mathbb{C}P^k$ . . . 79

3.2.1 $\mathcal{H}_{1,k}(S^2)$ Review . . . 79

3.2.2 Ricci Curvature Tensor . . . 83

3.2.3 Scalar Curvature . . . 88

3.2.4 Einstein-Hilbert Action of $\mathcal{H}_{1,k}(S^2)$ . . . 89

3.3 Degree 1 holomorphic maps $S^2 \to S^2$ . . . 93

3.3.1 Rat_1 Review . . . 93

3.3.2 Einstein-Hilbert Action of Rat_1 . . . 98

3.3.3 RMG Flow on Rat_1 . . . 100

3.4 Dilation Cylinders . . . 113

3.4.1 Rat_n^{eq} Submanifolds Review . . . 124

3.4.2 RMG Flow on Rat_n^{eq} Submanifolds . . . 126

3.5 Degree n Holomorphic maps $T^2 \to S^2$ . . . 134

3.5.1 $\mathcal{H}_{n,1}(T^2)$ Review . . . 134

3.5.2 2-Dimensional Totally Geodesic Submanifolds of $\mathcal{H}_{n,1}(T^2)$ . . . 137

4 Baby-Skyrme Metric 143

4.1 Baby-Skyrme Metric . . . 143

4.2 Baby-Skyrme Metric on Rat_1 . . . 147

4.2.1 Geodesic Completeness of Rat_1 with Baby-Skyrme Metric . . . 149

4.2.2 Weighted L^2 Metric on $\mathrm{Fix}_{\mathrm{P}}$ . . . 152

4.3 Rat_n^{eq} Submanifolds Revisited . . . 160

5 Baby-Skyrme Laplacian 170

5.1 Laplacian on Rat_1 . . . 170

5.2 Spectrum of Baby-Skyrme Quantum Hamiltonian on Rat_1 . . . 176

5.2.1 Reduction to Sturm-Liouville Problem . . . 177

5.2.2 Spectrum Classification 178
5.2.3 L^2 Finiteness and Boundary Conditions 181
5.2.4 Numerical Results 183
5.2.5 Spectrum with Skyrme Potential 186

Bibliography **190**

List of figures

2.1 The profile functions $f(r)$ (solid line), $a(r)$ (dashed line) and $g(r)$ (dots line) of the 1-vortex. 43

2.2 Approximated 2-vortices trajectories on $\mathcal{M}_2^0$ with large impact parameter a and small Chern-Simons parameter κ. 53

2.3 2-vortex scattering angle for large $a \geq 2$. 56

2.4 Plot of the scalar curvature $\tilde{\kappa}(s)$ of $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ 70

2.5 The angular velocities of the extrinsic RMG parallel on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ (left) and the intrinsic RMG parallel (right). 73

2.6 Plots of vortex trajectories under the extrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ (top) and the intrinsic RMG flow (bottom) with $\lambda_{\text{intrinsic}} = \lambda_{\text{extrinsic}}$ and various initial values 74

2.7 Plots of vortex trajectories under the intrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ (blue) and the extrinsic RMG flow (red) with $\lambda_{\text{extrinsic}} = \lambda_{\text{intrinsic}}/2$ and various initial values 74

4.1 Plots of the functions $V_\theta(\lambda)$ and $V_\alpha(\lambda)$. 155

4.2 The (P_θ, P_α)-plane's division according to the effective potential function's minimum as a function of the conserved angular momenta P_θ and P_α. . . . 157

4.3 The (P_θ, P_α) plane's division into: blue region where the unit energy geodesics on Fix_P are unbounded and gray region where the unit energy geodesics on Fix_P are oscillatory. 159

4.4 Plots of the effective potential function $V(\lambda; P_\theta, P_\alpha)$ for certain values of (P_θ, P_α). 160
4.5 Rat_n^{eq} submanifolds with the Baby-Skyrme metric g_ε for $\varepsilon = 0$ (the L^2 metric). 166
4.6 Rat_n^{eq} submanifolds with the Baby-Skyrme metric g_ε for $\varepsilon = 0.1$. 167
4.7 Rat_n^{eq} submanifolds with the Baby-Skyrme metric g_ε for $\varepsilon = 0.5$. 168

5.1 Plot of the lowest eigenvalue of $\hat{H}$ as a function of the Baby-Skyrme parameter $\varepsilon \in [0, 1]$ with $l = 0$. 184
5.2 Plots of the eigenvalues (red points) of $\hat{H}$ bounded above by E_ε (dots curve) for some values of $\varepsilon \in (0, 1]$. 185
5.3 Plots of the first eigenfunctions of $\hat{H}_* = \hat{H} + \varepsilon\, V_*(\lambda)$ for $l = 0$ and various value of $\varepsilon \in [0, 1]$ varying from $\varepsilon = 0$ (top horizontal line) to $\varepsilon = 1$ (bottom curve). 188

List of tables

2.1 Numerical values of the asymptotic 1-vortex parameters for various values of κ. 42

3.1 Division of $\mathsf{Fix}_R \subset \mathcal{H}_{n,1}(T^2_\triangle)$ into subsets according to the poles and zeros of their elements for $n = 2, 3, 4, 5$. 139

3.2 Division of $\mathsf{Fix}_R \subset \mathcal{H}_{n,1}(T^2_\square)$ into subsets according to the poles and zeros of their elements for $n = 2, 3$. 142

5.1 Numerical values of the lowest eigenvalues of the quantum Hamiltonian $\hat{H}$ for various values of l and ε. 184

5.2 Numerical values of the lowest eigenvalues of the modified quantum Hamiltonian $\hat{H}_* = \hat{H} + \varepsilon\, V_*(\lambda)$ for various values of l and ε. 188

Chapter 1

Ricci Magnetic Geodesics

1.1 Preliminaries

Let M be an n-dimensional Riemannian manifold with a Riemannian metric g and let ∇ be the Levi-Civita connexion with respect to g. Let $\gamma : I \to M$ be a smooth curve on M. The tangent bundle TM over M can be pulled back by γ to a vector bundle over I whose fibre at $t \in I$ is the tangent space $T_{\gamma(t)}M$. This bundle is called the pullback bundle of TM to I by γ and is denoted by γ^*TM. There is a unique connexion ∇^γ on γ^*TM defined by ∇ as follows

$$\nabla^\gamma_{d/dt}(\xi \circ \gamma) = (\nabla_{\mathrm{d}\gamma\ (d/dt)}\xi) \circ \gamma, \tag{1.1.1}$$

for any $\xi \in \Gamma(TM)$ where $\Gamma(TM)$ denotes the space of sections, called vector fields on M, of TM. Since ∇ is metric compatible on TM with g, then ∇^γ is metric connexion on γ^*TM compatible with γ^*g.

A geodesic on M is a smooth curve $\gamma(t)$ on M which satisfies

$$\nabla^\gamma_{d/dt}\dot{\gamma} = 0, \tag{1.1.2}$$

where $\dot{\gamma}(t)$ denotes the velocity of $\gamma(t)$. Since ∇^γ is metric compatible, then the speed

$\|\dot{\gamma}(t)\|$ satisfies

$$\frac{d}{dt}\|\dot{\gamma}(t)\|^2 = \frac{d}{dt}g(\dot{\gamma}(t), \dot{\gamma}(t)) = 2g(\dot{\gamma}(t), (\nabla^{\gamma}_{d/dt}\dot{\gamma})(t)) = 0. \tag{1.1.3}$$

This implies that geodesics on M are traversed at constant speed. In terms of local coordinates $(x^1, \ldots, x^n)$ on M, the geodesic equation (1.1.2) is given by

$$\ddot{x}^k + \Gamma^k_{ij}\,\dot{x}^i\dot{x}^j = 0, \tag{1.1.4}$$

where Γ^k_{ij} are smooth real valued functions called Christoffel symbols with respect to g. These functions are defined locally as

$$\Gamma^k_{ij} = \frac{1}{2}g^{mk}\left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right), \tag{1.1.5}$$

where g_{ij} are the metric components of g and g^{ij} are the components of the inverse metric g^{-1}. Clearly, the smoothness of Γ^k_{ij} guarantees the existence of a unique local solution of (1.1.4). If the geodesic equation on M has a global solution for all choices of initial data $\dot{\gamma}(0) \in T_{\gamma(0)}M$, that is, the solution exists for all time, then M is called geodesically complete. Otherwise, M is geodesically incomplete.

Any Riemannian manifold (M, g) can be thought as a metric space with a metric distance function d_g defined as

$$d_g(p, q) = \inf\{L_g[\gamma] : \gamma \text{ is a piecewise } \mathcal{C}^1 \text{ curve from } p \text{ to } q\}, \tag{1.1.6}$$

where $L_g[\gamma]$ is the length of $\gamma : [t_1, t_2] \to M$ given by

$$L_g[\gamma] = \int_{t_1}^{t_2} \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt = \int_{t_1}^{t_2} \|\dot{\gamma}(t)\|\, dt. \tag{1.1.7}$$

An important theorem in Riemannian geometry is the Hopf-Rinow Theorem (see, for example, [22, p.98]). This states that a Riemannian manifold (M, g) is geodesically complete if and only if (M, d_g) is a complete metric space.

1.2 Ricci Magnetic Geodesics

The motion of a particle on a Riemannian manifold (M, g) under the effect of a magnetic field given by a 2-form B on M is described by a curve $\gamma(t)$ on M which satisfies the following equation

$$\nabla^{\gamma}_{d/dt}\dot{\gamma} = \lambda \, \sharp\iota_{\dot{\gamma}}B, \tag{1.2.1}$$

where λ is a nonzero real constant. Here, the symbol $\sharp$ denotes the musical isomorphism with respect to g which assigns to any 1-form μ on M a vector field $\sharp\mu$ defined as

$$g(\sharp\mu, X) = \mu(X), \qquad \forall \, X \in \Gamma(TM). \tag{1.2.2}$$

The map $\iota_{\dot{\gamma}} : \Omega^p(M) \to \Omega^{p-1}(M)$ is the interior product of $\Omega^p(M)$, the space of p-forms on M, with respect to $\dot{\gamma}$ defined as

$$(\iota_{\dot{\gamma}}\eta)(X_1, \ldots, X_{p-1}) = \eta(\dot{\gamma}, X_1, \ldots, X_{p-1}), \tag{1.2.3}$$

for any $(X_1, \ldots, X_{p-1})$ where $X_i \in \Gamma(TM)$. A curve $\gamma(t)$ which solves (1.2.1) is called a magnetic geodesic on M. We referred to [18, 44] for the definition of magnetic geodesics.

Let (M, g) be a Kähler manifold with respect to an almost complex structure J. That is, M is a complex manifold whose Riemannian metric g is Hermitian, $g(X, Y) = g(JX, JY)$ and its associated 2-form

$$\omega(X, Y) = g(JX, Y), \qquad \forall \, X, Y \in \Gamma(TM), \tag{1.2.4}$$

is closed. The 2-form ω is called the Kähler form on M. On the Kähler manifold M, the Ricci curvature tensor, denoted Ric, with respect to g satisfies

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}(JX, JY), \qquad \forall \, X, Y \in \Gamma(TM). \tag{1.2.5}$$

There is a closed 2-form ρ, associated with the Ricci curvature tensor on M, defined as

$$\rho(X,Y) = \text{Ric}(JX,Y), \qquad \forall\, X,Y \in \Gamma(TM). \tag{1.2.6}$$

This is the so-called Ricci form on M. Choosing the 2-form B in (1.2.1) to be the Ricci form ρ,

$$\nabla^{\gamma}_{d/dt}\dot{\gamma} = \lambda\, \sharp\iota_{\dot{\gamma}}\rho, \tag{1.2.7}$$

a smooth curve $\gamma(t)$ on a Kähler manifold M is called Ricci magnetic geodesic if it satisfies the above equation (1.2.7). This is called Ricci magnetic geodesic equation and it is given locally by

$$g_{kl}(\ddot{x}^k + \Gamma^k_{ij}\,\dot{x}^i\dot{x}^j) = \lambda\,\rho_{kl}\,\dot{x}^k, \qquad l = 1,\ldots,n, \tag{1.2.8}$$

where ρ_{kl} are the components of the Ricci form ρ. This flow has been derived by Collie and Tong in [17] through a study of the dynamics of certain topological solitons, as we shall see in Chapter 2. The geometric formulation given in (1.2.7) analogous to the Ricci magnetic geodesic motion is due to Krusch and Speight [27] who determined some features of Ricci magnetic geodesic motion.

The metric compatibility of ∇^{γ} and the skew-symmetry of the Ricci form ρ imply

$$\begin{aligned}\frac{d}{dt}\|\dot{\gamma}(t)\|^2 = \frac{d}{dt}g(\dot{\gamma}(t),\dot{\gamma}(t)) = 2g(\dot{\gamma}(t),\nabla^{\gamma}_{d/dt}\dot{\gamma}(t)) &= 2g(\dot{\gamma}(t),\lambda\sharp\,\iota_{\dot{\gamma}(t)}\rho),\\ &= 2\lambda(\iota_{\dot{\gamma}(t)}\rho)(\dot{\gamma}(t)),\\ &= 2\lambda\rho(\dot{\gamma}(t),\dot{\gamma}(t)) = 0, \end{aligned}\tag{1.2.9}$$

from which it follows that Ricci magnetic geodesics are traversed at constant speed. Clearly, by (1.2.8), Ricci magnetic geodesic $\gamma(t)$ does not only depend on the direction of the initial velocity $\dot{\gamma}(0)$ but it also depends on the initial speed $\|\dot{\gamma}(0)\|$. Furthermore, the parameter λ in (1.2.7) can be rescaled to any convenient value by rescaling the time. That is, if $\gamma(t)$ is a solution of (1.2.7), then $\tilde{\gamma}(t) = \gamma(\lambda_* t)$ is also a solution of (1.2.7) but with a new parameter $\tilde{\lambda} = \lambda_* \lambda$. Hence, one expects that the Ricci magnetic geodesic $\gamma(t)$ to approach geodesic behaviour as $\|\dot{\gamma}(0)\| \to \infty$ [27].

Let M be a Kähler manifold whose second de Rham cohomology group $H^2_{dR}(M, \mathbb{R})$ is trivial. Then, by definition, the closed 2-form ρ on M is exact, namely, there exists a 1-form a on M such that the exterior derivative $da = \rho$.

Now, consider the dynamical Lagrangian

$$L = \frac{1}{2}\|\dot{\gamma}(t)\|^2 - \lambda a(\dot{\gamma}(t)) = \frac{1}{2} g_{ij}\, \dot{x}^i \dot{x}^j - \lambda\, a_i\, \dot{x}^i. \tag{1.2.10}$$

The action $S[\gamma]$ with respect to the Lagrangian L for any trajectory $\gamma(t)$ on M beginning at $\gamma_1 = \gamma(t_1)$ and ending at $\gamma_2 = \gamma(t_2)$ is defined as

$$S[\gamma] = \int_{t_1}^{t_2} L\, dt. \tag{1.2.11}$$

The actual motion determined by L is a critical point of the action $S[\gamma]$ [35, p.16]. This means that

$$\frac{d}{ds} S[\gamma_s]\bigg|_{s=0} = 0, \tag{1.2.12}$$

where γ_s is the curve $t \mapsto K(s,t)$ for a given variation $K : (-\epsilon, \epsilon) \times [t_1, t_2] \to M$ of γ, that is, $K(0,t) = \gamma(t)$, with fixed points $K(s, t_1) = \gamma_1$ and $K(s, t_2) = \gamma_2$ for all $s \in (-\epsilon, \epsilon)$. Now, let $X(s,t) = dK(\partial/\partial t)$ and $Y(s,t) = dK(\partial/\partial s)$ be sections of the pullback bundle K^*TM. For fixed s, $X(s,t)$ can be thought as the tangent vector of the curve $\gamma_s : t \mapsto K(s,t)$ and for fixed t, $Y(s,t)$ can be seen as the tangent vector of the curve $s \mapsto K(s,t)$.

Now, to find the critical point of the action $S[\gamma]$, we will compute the first variation formula of $S[\gamma]$ as follows

$$\begin{aligned}\frac{d}{ds}S[\gamma_s] &= \int_{t_1}^{t_2}\left(\frac{1}{2}\frac{\partial}{\partial s}g(X(s,t),X(s,t)) - \lambda\frac{\partial}{\partial s}a(X(s,t))\right)\,dt,\\ &= \int_{t_1}^{t_2}\left(g(X(s,t),\nabla^K_{\partial/\partial s}X(s,t)) - \lambda\frac{\partial}{\partial s}a(X(s,t))\right)\,dt. \end{aligned} \tag{1.2.13}$$

Here, we have used the metric compatibility of ∇^K. Since ∇ is torsion free, one can see that

$$\nabla^K_{\partial/\partial s}X - \nabla^K_{\partial/\partial t}Y = 0. \tag{1.2.14}$$

It follows that

$$\frac{d}{ds}S[\gamma_s] = \int_{t_1}^{t_2}\left(g(X(s,t),\nabla^K_{\partial/\partial t}Y(s,t)) - \lambda\frac{\partial}{\partial s}a(X(s,t))\right)\,dt. \tag{1.2.15}$$

Since ∇^K is metric compatible, then

$$\frac{\partial}{\partial t}g(X(s,t),Y(s,t)) = g(\nabla^K_{\partial/\partial t}X(s,t),Y(s,t)) + g(X(s,t),\nabla^K_{\partial/\partial t}Y(s,t)). \tag{1.2.16}$$

Moreover, it follows from the definition of the exterior derivative of a that

$$da(Y(s,t),X(s,t)) = \frac{\partial}{\partial s}a(X(s,t)) - \frac{\partial}{\partial t}a(Y(s,t)). \tag{1.2.17}$$

Then, using (1.2.16) and (1.2.17) in (1.2.15), we have

$$\begin{aligned}\frac{d}{ds}S[\gamma_s] &= \int_{t_1}^{t_2} \frac{\partial}{\partial t}\Big[g(X(s,t),Y(s,t)) - \lambda a(Y(s,t))\Big]\,dt \\ &\quad + \int_{t_1}^{t_2}\Big(\lambda da(X(s,t),Y(s,t)) - g(\nabla^K_{\partial/\partial t}X(s,t),Y(s,t))\Big)\,dt, \\ &= \Big[g(X(s,t),Y(s,t)) - \lambda a(Y(s,t))\Big]\Big|_{t_1}^{t_2} \\ &\quad + \int_{t_1}^{t_2}\Big(\lambda\rho(X(s,t),Y(s,t)) - g(\nabla^K_{\partial/\partial t}X(s,t),Y(s,t))\Big)\,dt. \end{aligned} \tag{1.2.18}$$

Since the variation has fixed endpoints, $Y(s,t_1) = Y(s,t_2) = 0$. Thus,

$$\frac{d}{ds}S[\gamma_s] = \int_{t_0}^{t_1}\Big(\lambda\rho(X(s,t),Y(s,t)) - g(\nabla^K_{\partial/\partial t}X(s,t),Y(s,t))\Big)\,dt. \tag{1.2.19}$$

For $s = 0$, $X(0,t) = \dot\gamma_0(t) = \dot\gamma(t)$, and so

$$\begin{aligned}\frac{d}{ds}S[\gamma_s]\Big|_{s=0} &= \int_{t_1}^{t_2}\Big(\lambda\rho(\dot\gamma(t),Y(0,t)) - g(\nabla^\gamma_{d/dt}\dot\gamma(t),Y(0,t))\Big)\,dt, \\ &= \int_{t_1}^{t_2}\Big(\lambda\big(\iota_{\dot\gamma(t)}\rho\big)(Y(0,t)) - g(\nabla^\gamma_{d/dt}\dot\gamma(t),Y(0,t))\Big)\,dt, \\ &= \int_{t_1}^{t_2}\Big(g(\lambda\sharp\iota_{\dot\gamma(t)}\rho,Y(0,t)) - g(\nabla^\gamma_{d/dt}\dot\gamma(t),Y(0,t))\Big)\,dt, \\ &= \int_{t_1}^{t_2} g(\lambda\sharp\iota_{\dot\gamma(t)}\rho - \nabla^\gamma_{d/dt}\dot\gamma(t),Y(0,t))\,dt. \end{aligned} \tag{1.2.20}$$

It follows from (1.2.20) that if $\gamma(t)$ is a Ricci magnetic geodesic, then the right hand side vanishes for all variations, and hence the Ricci magnetic geodesic $\gamma(t)$ is a critical point of $S[\gamma]$. Conversely, if $\gamma(t)$ is a critical point of $S[\gamma]$, then the left hand side vanishes. Then, the Fundamental Lemma of Calculus of Variations implies that $g(\lambda\sharp\iota_{\dot\gamma(t)}\rho - \nabla^\gamma_{d/dt}\dot\gamma(t),Y(0,t)) = 0$ for all $0 \neq Y(0,t) \in \Gamma(\gamma^*TM)$. Since g is non-degenerate, then $\lambda\sharp\iota_{\dot\gamma(t)}\rho - \nabla^\gamma_{d/dt}\dot\gamma(t) = 0$, and therefore $\gamma(t)$ is Ricci magnetic geodesic.

We conclude that on a Kähler manifold M with $H^2_{dR}(M,\mathbb{R}) = 0$, the curve $\gamma(t)$ on M is a critical point of the action $S[\gamma]$ if and only if it is Ricci magnetic geodesic on M.

It is a standard result in the Calculus of Variations that the Euler-Lagrange equations with respect to L,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}, \qquad i = 1, \dots, n, \tag{1.2.21}$$

are the motion equations. Since Ricci magnetic geodesic is the actual motion with respect to L, then the Euler-Lagrange equations are the Ricci magnetic geodesic equations given in (1.2.8).

In the Ricci magnetic geodesic dynamical system, the total energy

$$E = \frac{1}{2}\|\dot{\gamma}(t)\|^2 = \frac{1}{2}g(\dot{\gamma}(t), \dot{\gamma}(t)) > 0, \tag{1.2.22}$$

is conserved since Ricci magnetic geodesics are traversed at constant speed. Henceforth, we will use the letters "RMG" to indicate briefly "Ricci magnetic geodesic".

1.3 Ricci Magnetic Geodesics and Symmetry Reduction

Let M and N be n-dimensional Riemannian manifolds equipped with Riemannian metrics g_M and g_N, respectively. An isometry $\varphi : M \to N$ is a diffeomorphism such that the pullback of g_N by φ is g_M, namely, $\varphi^* \, g_N = g_M$.

Proposition 1.3.1. *Let $\varphi : M \to N$ be a holomorphic isometry between two Kähler manifolds. Then, φ maps an RMG curve on M to an RMG curve on N.*

Proof: Let ∇ and $\bar{\nabla}$ be the Levi-Civita connexions with respect to the Kähler metrics g_M and g_N on M and N, respectively. Let also $\gamma(t)$ be an RMG curve on M. Since

$\varphi : M \to N$ is an isometry, then [39, p.90]

$$d\varphi\big(\nabla^{\gamma}_{d/dt}\dot{\gamma}\big) = \bar{\nabla}^{\varphi\circ\gamma}_{d/dt}\, d\varphi\dot{\gamma}, \qquad \text{where} \qquad \big(d\varphi\,\dot{\gamma}\big)(t) = d\varphi_{\gamma(t)}\,(\dot{\gamma}(t)). \tag{1.3.1}$$

Now, we show that if $\gamma(t)$ is an RMG curve on M, then $(\varphi \circ \gamma)(t)$ is an RMG curve on N. For any $X \in \Gamma(TM)$,

$$g_N(\bar{\nabla}^{\varphi\circ\gamma}_{d/dt}\, d\varphi\dot{\gamma}, d\varphi X) = g_N(d\varphi\big(\nabla^{\gamma}_{d/dt}\dot{\gamma}\big), d\varphi X) = g_M(\nabla^{\gamma}_{d/dt}\dot{\gamma}, X). \tag{1.3.2}$$

Let ρ_M and ρ_N be the Ricci forms on M and N, respectively. Since $\gamma(t)$ is an RMG curve on M, then

$$g_M(\nabla^{\gamma}_{d/dt}\dot{\gamma}, X) = g_M(\lambda\sharp_M\iota_{\dot{\gamma}}\rho_M, X) = \lambda\iota_{\dot{\gamma}}\rho_M(X) = \lambda\rho_M(\dot{\gamma}, X) = \lambda\mathrm{Ric}_M(J_M\dot{\gamma}, X). \tag{1.3.3}$$

But, by the definition of Ricci curvature tensor, for all $X, Y \in \Gamma(TM)$,

$$\begin{aligned}\mathrm{Ric}_M(X,Y) = \sum_{i=1}^{n} g_M(R_M(E_i,X)Y, E_i) &= \sum_{i=1}^{n} g_N(d\varphi\big(R_M(E_i,X)Y\big), d\varphi\, E_i),\\ &= \sum_{i=1}^{n} g_N(R_N(d\varphi\, E_i, d\varphi\, X)d\varphi\, Y, d\varphi\, E_i),\\ &= \mathrm{Ric}_N(d\varphi X, d\varphi Y),\end{aligned} \tag{1.3.4}$$

where $\{E_i, i = 1, \ldots, n\}$ is an orthonormal basis for vector fields on M, and R_M, R_N are the Riemannian curvature tensors on M and N, respectively. Then, it follows from (1.3.2), (1.3.3) and (1.3.4) that

$$g_N(\bar{\nabla}^{\varphi\circ\gamma}_{d/dt}\, d\varphi\dot{\gamma}, d\phi X) = \lambda\,\mathrm{Ric}_M(J_M\dot{\gamma}, X) = \lambda\,\mathrm{Ric}_N(d\varphi(J_M\dot{\gamma}), d\varphi X). \tag{1.3.5}$$

Since φ is holomorphic, then $d\varphi \circ\ J_M = J_N\ \circ d\varphi$. Thus,

$$g_N(\bar{\nabla}^{\varphi\circ\gamma}_{\frac{d}{dt}} d\varphi\dot{\gamma}, d\phi X) = \lambda\,\mathrm{Ric}_N(J_N(d\varphi\dot{\gamma}), d\varphi X) = \lambda\rho_N(d\phi\dot{\gamma}, d\phi X),$$
$$= \lambda\iota_{d\varphi\dot{\gamma}}\rho_N(d\varphi X),$$
$$= g_N(\lambda\sharp_N\iota_{d\varphi\dot{\gamma}}\rho_N, d\varphi X). \quad (1.3.6)$$

Since g_N is non-degenerate and $d\varphi$ is surjective, then

$$\bar{\nabla}^{\varphi\circ\gamma}_{d/dt}\, d\varphi\dot{\gamma} = \lambda\sharp_N\iota_{d\varphi\dot{\gamma}}\rho_N. \quad (1.3.7)$$

Hence, $(\varphi\circ\gamma)(t)$ is an RMG curve on N.

□

Let M be a submanifold of a Riemannian manifold $\overline{M}$. Then, M is totally geodesic if every geodesic on M is also geodesic on $\overline{M}$. From this definition, we define a totally RMG submanifold as follows

Definition 1.3.2. *A complex submanifold M of a Kähler manifold $\overline{M}$ is totally RMG if every RMG curve on M is also an RMG curve on $\overline{M}$.*

Corollary 1.3.3. *Let $\overline{M}$ be a Kähler manifold and let M be a connected component of the fixed point set of a group of holomorphic isometries $\overline{M}\to\overline{M}$. Then, any RMG curve $\gamma : I \to \overline{M}$ with initial data $\dot{\gamma}(0)\in T_{\gamma(0)}M$ remains on M for all time.*

Proof: Let G be a group of holomorphic isometries $\overline{M}\to\overline{M}$ and let M be a connected component of the fixed point set of G. Let also

$$V_p = \{u \in T_p\overline{M} : d\varphi_p u = u,\ \forall\varphi\in G\}, \qquad \forall\, p\in M. \quad (1.3.8)$$

We know that M is a totally geodesic submanifold of $\overline{M}$ and $T_pM = V_p$ for all $p\in M$ [8, p.235]. Now, let $\gamma : I\to M$ be an RMG curve on $\overline{M}$ with initial data

$$\gamma(0) = p \in M, \qquad \dot{\gamma}(0) = v \in T_pM. \tag{1.3.9}$$

By Proposition 1.3.1, for all $\varphi \in G$, the curve $(\varphi \circ \gamma)(t)$ is RMG on $\overline{M}$. Its initial data are

$$(\varphi \circ \gamma)(0) = \varphi(p) = p \in M, \qquad (d\varphi\dot{\gamma})(0) = d\varphi_p(v) = v \in T_pM. \tag{1.3.10}$$

Thus, both $\gamma(t)$ and $(\varphi \circ \gamma)(t)$ satisfy the RMG equation on $\overline{M}$ with the same initial data, and so the uniqueness of RMG solution, as an ODE initial value problem, implies

$$(\varphi \circ \gamma)(t) = \gamma(t), \qquad \forall\, \varphi \in G. \tag{1.3.11}$$

Hence, $\gamma(t) \in M$ for all time.

□

Remark 1.3.4. *One can see that the connected component M of the fixed point set of a group G of holomorphic isometries on a Kähler manifold $\overline{M}$ is a complex submanifold of $\overline{M}$, and so is Kähler. This follows from that for all $u \in T_pM = V_p$ and all $\varphi \in G$, the almost complex structure J on $\overline{M}$ satisfies*

$$J_pu = J_p\,(d\varphi_p u) = d\varphi_p\,(J_pu), \tag{1.3.12}$$

and so, $J_pu \in V_p = T_pM$ for all $u \in T_pM$. However, Corollary 1.3.3 does not imply that M is a totally RMG submanifold of $\overline{M}$, since in general

$$\rho(X, Y) \neq \rho_M(X, Y), \qquad \forall\, X, Y \in \Gamma(TM), \tag{1.3.13}$$

where ρ is the Ricci form on $\overline{M}$ and ρ_M is the Ricci form on M defined by the induced metric on M.

1.4 Ricci Magnetic Geodesics on Surfaces of Revolution

On a Riemannian manifold (M, g) of real dimension 2, the Ricci curvature tensor is given by [7, p.4]

$$\text{Ric}(X, Y) = \frac{\kappa}{2}\, g(X, Y), \qquad \forall\, X, Y \in \Gamma(TM), \tag{1.4.1}$$

where κ is the scalar curvature with respect to g. If (M, g) is Kähler, then for any RMG curve $\gamma(t)$ on M, it follows from (1.4.1) that

$$\begin{aligned} g(\nabla^{\gamma}_{d/dt}\, \dot{\gamma}, X) = g(\lambda \sharp \iota_{\dot{\gamma}} \rho, X) = \lambda \iota_{\dot{\gamma}} \rho(X) = \lambda \rho(\dot{\gamma}, X) &= \lambda\, \text{Ric}(J\dot{\gamma}, X), \\ &= \lambda \frac{\kappa}{2}\, g(J\dot{\gamma}, X), \end{aligned} \tag{1.4.2}$$

for all $X \in \Gamma(M)$. Thus, on Kähler manifolds of real dimension 2, the RMG equation (1.2.7) can be written as [27]

$$\nabla^{\gamma}_{d/dt}\, \dot{\gamma} = \lambda\, \frac{\kappa}{2}\, J\dot{\gamma}. \tag{1.4.3}$$

In this section, we are interested in RMG flow on a class of 2-dimensional Riemannian manifolds which have an isometric circle action. They are the so-called surfaces of revolution and denoted M_{sor}. Here, the Riemannian metric g on M_{sor} is given, in terms of the polar local coordinates (r, θ), by

$$g = f(r)\ (dr^2 + r^2\, d\theta^2), \tag{1.4.4}$$

for some smooth positive real valued function $f(r)$. We find that the scalar curvature with respect to g is

$$\kappa(r) = -\frac{1}{r f(r)}\, \frac{d}{dr}\left(\frac{r f'(r)}{f(r)}\right). \tag{1.4.5}$$

Using (1.4.5) in (1.4.1), the Ricci form with respect to g is

$$\rho = -\frac{1}{2}\,\frac{d}{dr}\left(\frac{rf'(r)}{f(r)}\right)\, dr \wedge d\theta. \tag{1.4.6}$$

Thus, the RMG equations on (M_{sor}, g) are

$$f(r)\ddot{r} + \frac{1}{2}f'(r)\dot{r}^2 - \frac{1}{2}r\big(2f(r) + rf'(r)\big)\dot{\theta}^2 = \frac{\lambda}{2}\,\frac{d}{dr}\left(\frac{rf(r)}{f(r)}\right)\dot{\theta}, \tag{1.4.7}$$

$$r^2 f(r)\,\ddot{\theta} + r\big(2f(r) + rf'(r)\big)\,\dot{r}\,\dot{\theta} = -\frac{\lambda}{2}\,\frac{d}{dr}\left(\frac{rf(r)}{f(r)}\right)\dot{r}. \tag{1.4.8}$$

Let $H^2_{dR}(M_{\text{sor}}, \mathbb{R})$ be trivial, then there exists a 1-form a such that $da = \rho$. It follows from (1.4.6) that

$$a = -\frac{1}{2}\left(\frac{rf'(r)}{f(r)}\right) d\theta. \tag{1.4.9}$$

Hence, the RMG motion on M_{sor} is determined by the following Lagrangian

$$L = \frac{1}{2}f(r)(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{\lambda}{2}\left(\frac{rf'(r)}{f(r)}\right)\dot{\theta}. \tag{1.4.10}$$

The conserved total energy E with respect to L is

$$E = \frac{1}{2}f(r)(\dot{r}^2 + r^2\dot{\theta}^2). \tag{1.4.11}$$

Furthermore, since θ is cyclic, i.e. L does not depend on θ, there is a conserved angular momentum $P \in \mathbb{R}$ given by

$$P = \frac{\partial L}{\partial \dot{\theta}} = r^2 f(r)\,\dot{\theta} + \frac{\lambda}{2}\left(\frac{rf'(r)}{f(r)}\right). \tag{1.4.12}$$

One can use (1.4.12) to eliminate $\dot{\theta}$ from (1.4.11), that is,

$$E = \frac{1}{2}f(r)\dot{r}^2 + \frac{1}{2r^2 f(r)}\left(P - \lambda\,\frac{rf'(r)}{2f(r)}\right)^2. \tag{1.4.13}$$

Note that using conservation of angular momentum and energy, we can reduce the equations of motions to a single first order ODE of the form

$$E = \frac{1}{2} f(r)\dot{r}^2 + V_P(r), \tag{1.4.14}$$

where $V_P(r)$ is a non-negative real valued function given by

$$V_P(r) = \frac{1}{2r^2 f(r)} \left(P - \lambda \frac{r f'(r)}{2f(r)} \right)^2. \tag{1.4.15}$$

Note that this is identical to the equation satisfied by a particle moving on $[0, \infty)$ with metric $f(r)\ dr^2$ subject to a potential $V_P(r)$. Hence, we call $V_P(r)$ the **effective potential function**.

Now, for certain energy $E > 0$ and angular momentum $P \in \mathbb{R}$, let $D_{E,P}$ denote the image of the level set of RMG flow on TM_{sor}, with E and P, under the projection $TM_{\text{sor}} \backslash \{0\} \to T(0, \infty)$. That is,

$$D_{E,P} = \{(r, \dot{r}) : E = \frac{1}{2} f(r)\dot{r}^2 + V_P(r)\}. \tag{1.4.16}$$

Definition 1.4.1. *A Kähler manifold M is called RMG complete if every RMG curve on M exists for all time. Otherwise, M is RMG incomplete.*

It has been conjectured in [27] that the geodesically incompleteness of a Kähler manifold guarantees its RMG incompleteness. In fact, we can show that this is not always true by discussing the following counterexample:

Consider a surface of revolution with the Riemannian metric

$$g = \text{sech}\ r\ (dr^2 + r^2 d\theta^2). \tag{1.4.17}$$

This surface of revolution will be denoted Σ_{sech}. The length of a curve $\gamma(t)$ on Σ_{sech} is given by

$$L_g[\gamma] = \int_{t_1}^{t_2} \sqrt{\text{sech } r(\dot{r}^2 + r^2\dot{\theta}^2)}\, dt. \tag{1.4.18}$$

One can see that the metric space $(\Sigma_{\text{sech}}, d_g)$ is incomplete by considering the sequence $\{p_n\}$ with $p_n = (n, 0)$ on Σ_{sech}. Then,

$$d_g(p_n, p_m) \leq \left| \int_n^m \sqrt{\text{sech } r}\ dr \right| = \left| \int_n^m \sqrt{\frac{2}{e^r + e^{-r}}}\ dr \right| < \left| \int_n^m \sqrt{\frac{2}{e^r}}\ dr \right|. \tag{1.4.19}$$

Let N be a positive integer such that $m, n \geq N$. By choosing N sufficiently large, it follows from (1.4.19) that $\{p_n\}$ is a Cauchy sequence on Σ_{sech}. Clearly, $\{p_n\}$ is divergent, and so $(\Sigma_{\text{sech}}, d_g)$ is an incomplete metric space. Consequently, by Hopf-Rinow Theorem [22, p.98], Σ_{sech} is geodesically incomplete.

Theorem 1.4.2. *The surface of revolution Σ_{sech} is RMG complete.*

The proof of this Theorem proceeds in several steps:

Proposition 1.4.3. *For any energy E and angular momentum P, the set $D_{E,P}$, given in (1.4.16), of an RMG flow on Σ_{sech} is bounded.*

Proof: For an RMG flow on Σ_{sech} with E and P, the set $D_{E,P}$, defined in (1.4.16), is

$$D_{E,P} = \{(r, \dot{r}) : E = \frac{1}{2}\, \text{sech } r\ \dot{r}^2 + V_P(r)\}, \tag{1.4.20}$$

where

$$V_P(r) = \frac{1}{2\, r^2\ \text{sech } r}\left(P + \frac{1}{2} r \tanh r\right)^2. \tag{1.4.21}$$

Without loss of generality, we chose $\lambda = 1$. Let $\mathbf{p_1}$ and $\mathbf{p_2}$ be the projections to the first and second components on $D_{E,P}$, respectively. First, we show that for all $r \in \mathbf{p_1}(D_{E,P})$, $r \leq M$ where

$$M = \max\{24E, 1/E, 8\,|P|\}. \tag{1.4.22}$$

We will assume to the contrary that there exists $r \in \mathbf{p_1}(D_{E,P})$ such that $r > M$. Then, it follows from (1.4.22) that

$$r > 24E, \qquad \frac{1}{r} < E, \qquad \frac{|P|}{r} < \frac{1}{8}. \tag{1.4.23}$$

From (1.4.20) and (1.4.21), we have

$$\begin{aligned}\text{sech } r\ \dot{r}^2 &= 2E - \frac{P^2}{r^2}\cosh r - \frac{1}{4}\frac{\sinh^2 r}{\cosh r} - \frac{P}{r}\sinh r,\\ &\leq 2E - \frac{P^2}{r^2}\cosh r - \frac{1}{4}\frac{\sinh^2 r}{\cosh r} + \frac{|P|}{r}\sinh r,\\ &< 2E - \frac{1}{4}\frac{\sinh^2 r}{\cosh r} + \frac{|P|}{r}\sinh r,\end{aligned} \tag{1.4.24}$$

for all $(r, \dot{r}) \in D_{E,P}$. Since ($\sinh^2 r = \cosh^2 r - 1$) and ($\sinh r < \cosh r$) for all $r \geq 0$, then for all $(r, \dot{r}) \in D_{E,P}$,

$$\begin{aligned}\text{sech } r\ \dot{r}^2 &< 2E - \frac{1}{4}\cosh r + \frac{1}{4}\frac{1}{\cosh r} + \frac{|P|}{r}\cosh r,\\ &= 2E + \left(\frac{|P|}{r} - \frac{1}{4}\right)\cosh r + \frac{1}{4}\frac{1}{\cosh r}.\end{aligned} \tag{1.4.25}$$

Since ($\cosh r \geq r$) for all $r \geq 0$, then, by using (1.4.23), we have

$$\begin{aligned}\text{sech } r\ \dot{r}^2 &< 2E - \frac{1}{8}\cosh r + \frac{1}{4}\frac{1}{\cosh r},\\ &\leq 2E - \frac{1}{8}r + \frac{1}{4r},\\ &< 2E - \frac{1}{8}(24E) + \frac{1}{4}E = -\frac{3}{4}E < 0.\end{aligned} \tag{1.4.26}$$

This is clearly false, and hence, for all $r \in \mathbf{p_1}(D_{E,P})$, $r \leq M$.

Consequently, we have

$$\dot{r}^2 = 2\cosh r\big(E - V_P(r)\big) \leq 2E\cosh r \leq 2E\cosh M. \tag{1.4.27}$$

Thus, $\mathbf{p_2}(D_{E,P})$ is also bounded. Therefore, for any E and P, the set $D_{E,P}$, given in (1.4.20), is bounded.

$\square$

Now, let us re-express the RMG problem on M_{sor} in terms of Cartesian coordinates $x = r\cos\theta$ and $y = r\sin\theta$, for convenience. The metric g on M_{sor} becomes

$$g = F(x,y)\ (dx^2 + dy^2), \quad \text{where} \quad F(x,y) = f(r). \tag{1.4.28}$$

Hence, RMG equations on M_{sor} in Cartesian coordinates (x, y) are

$$\ddot{x} = -\frac{1}{2}\frac{F_x}{F}\dot{x}^2 - \frac{F_y}{F}\dot{x}\dot{y} + \frac{1}{2}\frac{F_x}{F}\dot{y}^2 + \frac{1}{2F^3}\Big[F(F_{xx} + F_{yy}) - (F_x^2 + F_y^2)\Big]\dot{y}, \tag{1.4.29}$$

$$=: h_1(x, y, \dot{x}, \dot{y}),$$

$$\ddot{y} = \frac{1}{2}\frac{F_y}{F}\dot{x}^2 - \frac{F_x}{F}\dot{x}\dot{y} - \frac{1}{2}\frac{F_y}{F}\dot{y}^2 - \frac{1}{2F^3}\Big[F(F_{xx} + F_{yy}) - (F_x^2 + F_y^2)\Big]\dot{x}, \tag{1.4.30}$$

$$=: h_2(x, y, \dot{x}, \dot{y}),$$

where F_x and F_y denote the partial derivatives of $F(x, y)$ with respect to x and y, respectively. Let $H : \mathbb{R}^4 \to \mathbb{R}^4$ be a function given by

$$H\big(z(t)\big) = \big(z_3, z_4, h_1\big(z(t)\big), h_2\big(z(t)\big)\big), \tag{1.4.31}$$

where

$$z(t) = (z_1, z_2, z_3, z_4) = (x(t), y(t), \dot{x}(t), \dot{y}(t)). \tag{1.4.32}$$

Then, $z(t)$ solves the RMG equations (1.4.29) and (1.4.30) if and only if

$$\dot{z}(t) = H\big(z(t)\big). \tag{1.4.33}$$

The function $H(z(t))$ is called the RMG flow function of M_{sor}.

Let d_0 be the Euclidean distance function on $\mathbb{R}^n$ which measures the distance between $z, y \in \mathbb{R}^n$ as

$$d_0(z, y) = \left(\sum_{i=1}^{n} (z_i - y_i)^2\right)^{1/2}. \tag{1.4.34}$$

Then, the closed ball of $\mathbb{R}^n$ with radius $R > 0$ centred at c is defined as

$$\bar{B}_R(c) = \{z \in \mathbb{R}^n : d_0(z, c) \leq R\} \subset \mathbb{R}^n. \tag{1.4.35}$$

Proposition 1.4.4. *For any energy E and angular momentum P, there exists a real number $R = R(E, P) > 0$ depending on E and P such that for any initial value z_0 with E and P, the solution (if it exists) of the RMG problem on Σ_{sech} is in $\bar{B}_R(0) \subset \mathbb{R}^4$.*

Proof: Let $z(t)$ be a solution of the RMG initial value problem

$$\dot{z}(t) = H(z(t)), \qquad z(0) = z_0, \tag{1.4.36}$$

on Σ_{sech}. We claim that $z(t)$ is bounded. It follows from Proposition 1.4.3 that for any E and P, there exists $M(E, P) > 0$ such that

$$z_1^2 + z_2^2 = x^2(t) + y^2(t) = r^2 \leq M^2. \tag{1.4.37}$$

Furthermore, using (1.4.11), we have

$$z_3^2 + z_4^2 = \dot{x}^2(t) + \dot{y}^2(t) = \dot{r}^2 + r^2\dot{\theta}^2 = 2E \cosh r \leq 2E \cosh M. \tag{1.4.38}$$

It follows from (1.4.37) and (1.4.38) that

$$\Big(d_0(z(t),0)\Big)^2 \leq M^2 + 2E\cosh M =: R^2, \tag{1.4.39}$$

which establishes the claim.

□

We will combine Proposition 1.4.3 and Proposition 1.4.4 with a short time existence result for RMG flow, proved using a standard Picard's method argument [10, p.78-88].

Lemma 1.4.5. *Let $H : \mathbb{R}^n \to \mathbb{R}^n$ be a function of class $\mathcal{C}^1$. Then, given a real number $R > 0$, there exists $T_*(R) > 0$ such that for all $z_0 \in \bar{B}_R(0) \subset \mathbb{R}^n$, there exists a unique local solution of the intial value problem*

$$\dot{z}(t) = H(z(t)), \qquad z(0) = z_0, \tag{1.4.40}$$

on $[0, T_*]$ such that $z(t) \in \bar{B}_{R+1}(0)$ for all $t \in [0, T_*]$.

Proof: Let δ be the max-metric distance function on $\mathbb{R}^n$ given by

$$\delta(z,y) = \max\{|z_i - y_i|, i = 1,\ldots,n\}, \qquad \forall\, z,y \in \mathbb{R}^n. \tag{1.4.41}$$

Then, for $R > 0$, the closed ball of radius $R+1$ centred at the origin is defined with respect to δ as

$$\bar{B}^{\delta}_{R+1}(0) = \{z \in \mathbb{R}^n : \delta(z,0) \leq R+1\} \subset \mathbb{R}^n. \tag{1.4.42}$$

Note that $\delta(z,0) \leq d_0(z,0)$ for all $z \in \mathbb{R}^n$, and so $\bar{B}^{\delta}_{R+1}(0) \subseteq \bar{B}_{R+1}(0)$. Since H is of class $\mathcal{C}^1$, each component $H_i : \mathbb{R}^n \to \mathbb{R}$ of H and its partial derivatives $G_{ij} := \partial H_i/\partial z_j : \mathbb{R}^n \to \mathbb{R}$, $i,j = 1,\ldots,n$, are continuous functions. Restricting H to $\bar{B}^{\delta}_{R+1}(0)$ implies that both H_i and G_{ij} are bounded. This means that there exist $H_*, G_* > 0$ such that for all $z \in \bar{B}^{\delta}_{R+1}(0)$,

$$|H_i(z)| \leq H_*, \qquad |G_{ij}(z)| \leq G_*, \qquad \forall\, i,j = 1,\ldots,n. \tag{1.4.43}$$

We show first that $H : \bar{B}^{\delta}_{R+1}(0) \to \mathbb{R}^n$ is globally Lipschitz with respect to the max-metric δ. For all $z, y \in \bar{B}^{\delta}_{R+1}(0)$, we have

$$\begin{aligned} |H_i(z) - H_i(y)| &= |H_i(z_1, z_2, \dots, z_n) - H_i(y_1, y_2, \dots, y_n)|, \\ &\leq |H_i(z_1, z_2, \dots, z_n) - H_i(y_1, z_2, \dots, z_n)|, \\ &\quad + |H_i(y_1, z_2, \dots, z_n) - H_i(y_1, y_2, \dots, z_n)| \\ &\quad + \cdots + |H_i(y_1, y_2, \dots, y_{n-1}, z_n) - H_i(y_1, y_2, \dots, y_{n-1}, y_n)|. \end{aligned} \tag{1.4.44}$$

By the Mean Value Theorem, there exists $q \in \bar{B}^{\delta}_{R+1}(0)$ such that

$$\begin{aligned} G_{ij}(z_1, \dots, z_{j-1}, q_j, z_{j+1}, \dots, z_n)\ (z_j - y_j) =& H_i(z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n) \\ &- H_i(z_1, \dots, z_{j-1}, y_j, z_{j+1}, \dots, z_n). \end{aligned} \tag{1.4.45}$$

Using (1.4.45) in (1.4.44), we have

$$\begin{aligned} |H_i(z) - H_i(y)| &\leq |G_{i1}(q_1, z_2, \dots, z_n)||z_1 - y_1| + |G_{i2}(y_1, q_2, z_3, \dots, z_n)||z_2 - y_2| \\ &\quad + \cdots + |G_{in}(y_1, y_2, \dots, q_n)||z_n - y_n|, \\ &\leq G_* \left[|z_1 - y_1| + |z_2 - y_2| + \cdots + |z_n - y_n|\right], \\ &\leq n\, G_*\ \max\{|z_i - y_i|, i = 1, \dots, n\}, \\ &= n\, G_*\, \delta(z, y). \end{aligned} \tag{1.4.46}$$

It follows from (1.4.46) that for all $z, y \in \bar{B}^{\delta}_{R+1}(0)$,

$$\delta(H(z), H(y)) = \max\{|H_i(z) - H_i(y)|, i = 1, \dots, n\} \leq n\, G_*\, \delta(z, y). \tag{1.4.47}$$

Hence, $H : \bar{B}^{\delta}_{R+1}(0) \to \mathbb{R}^n$ is globally Lipschitz with respect to the max-metric δ.

Now, let

$$T_* := \min\{\frac{1}{H_*}, \frac{1}{2nG_*}\} > 0. \tag{1.4.48}$$

Let also $\mathcal{C}^0_{R+1} := \mathcal{C}^0([0, T_*], \bar{B}^{\delta}_{R+1}(0))$ be the set of all continuous functions $[0, T_*] \to \bar{B}^{\delta}_{R+1}(0)$. Note that $[0, T_*]$ is a closed interval and $\bar{B}^{\delta}_{R+1}(0)$ is a complete metric space with respect to δ. Hence, $\mathcal{C}^0_{R+1}$ is a complete metric space with respect to the sup-metric distance function

$$d_{\sup}(h, g) := \sup\{\delta(h(t), g(t)), t \in [0, T_*]\}, \qquad \forall\, h, g \in \mathcal{C}^0_{R+1}. \tag{1.4.49}$$

Now, consider the map

$$\psi : \mathcal{C}^0_{R+1} \to \mathcal{C}^0([0, T_*], \mathbb{R}^n), \qquad \psi(h) := t \mapsto z_0 + \int_0^t H(h(s))ds. \tag{1.4.50}$$

We will show that $\psi(\mathcal{C}^0_{R+1}) \subset \mathcal{C}^0_{R+1}$. It follows from (1.4.50) that $\psi(h)$ is continuous. So, we need to show that $\psi(h)(t) \in \bar{B}^{\delta}_{R+1}(0)$ for all $t \in [0, T_*]$ as follows

$$\begin{aligned} |\psi(h)(t)_i| &= \left| z_{0_i} + \int_0^t H_i(h(s))ds \right|, \\ &\le \left| z_{0_i} \right| + \left| \int_0^t H_i(h(s))ds \right|, \\ &\le \left| z_{0_i} \right| + \int_0^t |H_i(h(s))|ds, \\ &\le \left| z_{0_i} \right| + \int_0^t H_* ds = \left| z_{0_i} \right| + H_* t. \end{aligned} \tag{1.4.51}$$

Since $z_0 \in \bar{B}_R(0)$ and $T_* \le 1/H_*$,

$$|\psi(h)(t)_i| \le R + H_* t \le R + H_* T_* \le R + 1. \tag{1.4.52}$$

Hence, for all $t \in [0, T_*]$,

$$\delta(\psi(h)(t), 0) = \max\{|\psi(h)(t)_i|, i = 1, \dots, n\} \le R + 1, \tag{1.4.53}$$

so $\psi(h)(t) \in \bar{B}^{\delta}_{R+1}(0)$, and the claim is proved.

Furthermore, we claim that $\psi : \mathcal{C}^0_{R+1} \to \mathcal{C}^0_{R+1}$ is a contraction map, which means that there exists a positive real number $k < 1$ such that for all $g, h \in \mathcal{C}^0_{R+1}$, $d_{\text{sup}}(\psi(h), \psi(g)) \le k d_{\text{sup}}(h, g)$ [13, p.139]. For all $h, g \in \mathcal{C}^0_{R+1}$ and $t \in [0, T_*]$, we have

$$\begin{aligned} |\psi(h)(t)_i - \psi(g)(t)_i| &= \left| \int_0^t \big(H_i(h(s)) - H_i(g(s))\big)\, ds \right|, \\ &\le \int_0^t \big|H_i(h(s)) - H_i(g(s))\big|\, ds. \end{aligned} \tag{1.4.54}$$

Using (1.4.46) in (1.4.54) and since $T_* \le 1/(2\, n\, G_*)$, then

$$\begin{aligned} |\psi(h)(t)_i - \psi(g)(t)_i| &\le \int_0^t n\, G_*\, \delta(h(s), g(s))\, ds, \\ &\le \int_0^t n\, G_*\, d_{\text{sup}}(h, g)\, ds, \\ &= n\, G_*\, d_{\text{sup}}(h, g)\, t, \\ &\le n\, G_*\, d_{\text{sup}}(h, g)\, T_*, \\ &\le \frac{1}{2}\, d_{\text{sup}}(h, g). \end{aligned} \tag{1.4.55}$$

Thus, for all $h, g \in \mathcal{C}^0_{R+1}$ and $t \in [0, T_*]$,

$$\delta(\psi(h)(t), \psi(g)(t)) = \max\{|\psi(h)(t)_i - \psi(g)(t)_i|, i = 1, \dots, n\} \le \frac{1}{2} d_{\text{sup}}(h, g), \tag{1.4.56}$$

and so,

$$d_{\text{sup}}(\varphi(h), \varphi(g)) = \sup\{\delta(\varphi(h)(t), \varphi(g)(t)), t \in [0, T_*]\} \leq \frac{1}{2} d_{\text{sup}}(h, g). \qquad (1.4.57)$$

Hence, ψ is a contraction map from the complete metric space $\mathcal{C}^0_{R+1}$ to itself. By the Contraction Mapping Theorem [13, p.140], ψ has a unique fixed point h. Clearly, $h(t)$ is a solution of the initial value problem on $[0, T_*]$ satisfying $h(0) = z_0$, so the solution exists. Conversely, any solution $z(t)$ of the initial value problem on $[0, T_*]$ with $z(t) \in \bar{B}^\delta_{R+1}(0)$ for all $t \in [0, T_*]$ is a fixed point of ψ, so this solution is unique.

□

Since RMG equations involve second partial derivatives of the metric components, the local existence and uniqueness of the solution of RMG initial value problem requires that the metric is, at least, of class $\mathcal{C}^3$ to ensure that the RMG flow function is of class $\mathcal{C}^1$.

Recall that we wish to prove:

Theorem 1.4.2. *Given any initial value $z_0 \in \mathbb{R}^4$, the RMG problem on Σ_{sech} has a unique solution $z(t)$, with $z(0) = z_0$, which exists for all time. Hence, Σ_{sech} is RMG complete.*

Proof: On Σ_{sech}, the metric component $F(x, y) = \text{sech}(\sqrt{x^2 + y^2})$ is smooth for all $(x, y) \in \mathbb{R}^2$. This guarantees that the flow function H, given in (1.4.31), is of class $\mathcal{C}^1$.

For any initial value z_0 with energy E and angular momentum P, by Proposition 1.4.4, there exists $R(E, P) > 0$ such that the solution of the RMG problem on Σ_{sech} is in $\bar{B}_R(0) \subset \mathbb{R}^4$.

We claim that the RMG problem on Σ_{sech} has a unique solution $z(t) \in \bar{B}_R(0)$ for all $t \in [0, nT_*]$ and all $n \in \mathbb{Z}^+$, where $T_* > 0$ depends only on R.

We prove the claim by mathematical induction. First, we show that the statement is true when $n = 1$. Since the flow function H is of class $\mathcal{C}^1$, then the RMG problem on Σ_{sech} is of type covered by Lemma 1.4.5. Therefore, there exists $T_*(R) > 0$ such that the RMG problem on Σ_{sech} with $z(0) = z_0 \in \bar{B}_R(0)$ has a unique solution $z(t) \in \bar{B}_{R+1}(0)$ for all $t \in [0, T_*]$. By Proposition 1.4.4, in fact $z(t) \in \bar{B}_R(0)$ for all $t \in [0, T_*]$.

Now, we assume that the statement is true for some n and we show that it is true for $n+1$. By assumption, $z(nT_*) \in \bar{B}_R(0)$, and then the RMG problem with initial value $z_* = z(nT_*)$, by Lemma 1.4.5, has a unique solution $z(t)$ for all $t \in [nT_*, (n+1)T_*]$. Again, by Proposition 1.4.4, since z_* has the same energy E and angular momentum P as z_0, then $z(t) \in \bar{B}_R(0)$ for all $t \in [nT_*, (n+1)T_*]$.

Since the solution exists on $[0, nT_*]$ and $[nT_*, (n+1)T_*]$, then it is valid for all $t \in [0, (n+1)T_*]$. Thus, the statement is true for $n+1$. Hence, the statement is true for all $n \in \mathbb{Z}^+$. Since $z(t)$ exists and is unique on $[0, nT_*]$ for all $n \in \mathbb{Z}^+$, it exists and is unique for all t.

□

Remark 1.4.6. *The geometric reason behind this counterexample is that in this particular surface of revolution, the scalar curvature is unbounded above as $r \to \infty$, and so the force (acceleration) becomes more effective near the boundary at infinity. This prevents RMG trajectories on Σ_{sech} from hitting the boundary at infinity in finite time, unlike geodesics.*

Chapter 2

Chern-Simons Abelian Higgs Vortices

2.1 Field Theories

Let M and N be Riemannian manifolds of dimension m and n equipped with Riemannian metrics g and h, respectively. Let $\mathbf{x} = (x^1, \ldots, x^m)$ denote local coordinates on M and ∂_i be the partial differentiation with respect to x^i. Consider the spacetime $\mathbb{R} \times M$ with Lorentzian metric $\eta = dt^2 - g$. A scalar field theory on $\mathbb{R} \times M$ consists of a scalar field $\phi : \mathbb{R} \times M \to N$ and a Lagrangian density $\mathcal{L}$ concerned with the dynamics of the field ϕ. Generally, $\mathcal{L}$ is a function of ϕ and its partial derivatives $\partial_\mu \phi$, $\mu = 0, \ldots, m$, where $\partial_0 \phi$ denotes the time derivative of ϕ. The action functional $S[\phi]$ of a scalar field theory on $\mathbb{R} \times M$ is defined as an integral of the Lagrangian density $\mathcal{L}$ over $\mathbb{R} \times M$ whose extremals are those fields which solve the motion equations of the field system. Explicitly,

$$S[\phi] = \int_{t_1}^{t_2} \int_M \mathcal{L}(\partial_\mu \phi, \phi) \, d^m\mathbf{x} \, dt, \tag{2.1.1}$$

which is stationary for the scalar fields, beginning at $\phi(t_1, \mathbf{x})$ and ending at $\phi(t_2, \mathbf{x})$, which satisfy the Euler-Lagrange equations derived from $\mathcal{L}$. The scalar field ϕ takes values in the target manifold N, $\phi(t, \mathbf{x}) = (\phi^1(t, \mathbf{x}), \ldots, \phi^n(t, \mathbf{x}))$, and so the Euler-Lagrange equations of $\mathcal{L}$ are

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^l)} \right) = \frac{\partial \mathcal{L}}{\partial \phi^l}, \quad l = 1, \ldots, n, \tag{2.1.2}$$

where $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, $\mu, \nu = 0, \ldots, m$. The integral of $\mathcal{L}$ over M is called the field Lagrangian, denoted $L[\phi]$, and the Euler-Lagrange equations with respect to $\mathcal{L}$ are called the field equations of the theory.

A scalar field theory turns out to be gauged if there is a Lie group G acting on N such that the action $S[\phi]$, defined in (2.1.1), is invariant under a set of local G-internal transformations, that is, maps from $\mathbb{R} \times M$ to G. These are called gauge transformations and they form a group $\mathcal{G}$ known as the gauge group of the theory. Generically, it is essential to couple the scalar field ϕ with a gauge potential field $A = (A_\mu)$, $\mu = 0, \ldots, m$, which is interpreted locally as a 1-form on $\mathbb{R} \times M$ and then replace the partial derivatives of ϕ in $\mathcal{L}$ by its gauge covariant derivative $D_\mu \phi = \partial_\mu \phi - iA_\mu \phi$ to ensure the invariance of $S[\phi]$ (sometimes $\mathcal{L}$) under the gauge group action. Here, the field equations are

$$D^\mu \left(\frac{\partial \mathcal{L}}{\partial(D^\mu \phi^l)} \right) = \frac{\partial \mathcal{L}}{\partial \phi^l}, \quad l = 1, \ldots, n, \tag{2.1.3}$$

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu}, \quad \nu = 0, 1, \ldots, m, \tag{2.1.4}$$

where $D^\mu = \eta^{\mu\nu} D_\nu$. If the group G is Abelian (non-Abelian), then the field theory is called Abelian (non-Abelian) gauged field theory.

Whenever the Lagrangian field theory is symmetric under a local transformation on $\mathbb{R} \times M$, then there is a conserved current $\mathrm{J} = (\mathrm{J}^\mu)$, that is $\partial_\mu \mathrm{J}^\mu = 0$, associated to this symmetry. This is known as Noether's Theorem (see, for example, [35, p.28-30]). The integral of the time component J^0 over M is called the Noether's charge of J. In the scalar field theory, the Noether's charge associated with the time-translation symmetry is

$$E[\phi] = \int_M \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \, \partial_0 \phi - \mathcal{L} \right) d^m \mathbf{x}. \tag{2.1.5}$$

If there are several fields, then the first term in (2.1.5) becomes a sum over terms, one for each field. The Noether's charge $E[\phi]$ gives the total energy of the field system.

The configuration space of the field theory, denoted $\mathscr{C}$, is the space of all finite energy field configurations. In the scalar field theory, the field configuration $\phi(t, \mathbf{x})$ can be thought as a curve on the space $\mathcal{C}^\infty(M, N)$, and so the configuration space $\mathscr{C}$ of the scalar field theory is the space of finite energy smooth maps from M to N. In the gauged field theory, two field configurations are called gauge equivalent if they differ only by a gauge transformation in $\mathcal{G}$. Hence, the configuration space $\mathscr{C}$ is reduced to the orbit space $\mathscr{C}/\mathcal{G}$.

One of the most interesting field configurations are topological solitons. They are defined to be the finite energy, stable (minimal energy), static solutions of the field equations. The most well-known examples of topological solitons are kinks in dimension $n = 1$, gauged vortices, $\mathbb{C}P^1$-lumps, Baby-Skyrmions in dimension $n = 2$, and monopoles, Skyrmions in dimension $n = 3$. The set of topological solitons is called the moduli space of the field theory which is often a finite-dimensional smooth manifold. The vacuum of the field theory is the field configuration whose total energy $E[\phi]$ is zero. The non-existence of topological solitons in a field theory, except the vacuum solution, can be checked by Derrick test [19] which gives a necessary condition for a field theory defined on a flat space to have non-vacuum minimal energy static solutions. This condition is that for a given variation $\phi_\lambda(\mathbf{x}) = \phi(\lambda \mathbf{x})$, with $\lambda > 0$, of a field configuration $\phi(\mathbf{x})$, the total energy $E[\phi_\lambda]$ must have a minimal point at $\lambda = 1$, for further details see [35, p.82-87].

Let U be a smooth function of ϕ which has minimum value zero and let $\mathcal{V}$ be a subset of the target N which is where the function U takes its zero minimum value. In some field theories, specially in dimension $n = 2$, the presence of the function U in the field Lagrangian plays a significant role in the existence of non-vacuum topological solitons.

In such field theories, U is called the potential function and $\mathcal{V}$ is the vacuum manifold of the theory.

An argument due to Bogomolny [12] has helped studying the topological solitons in many field theories whose total energy $E[\phi]$, for static fields, has a non-trivial lower bound called the Bogomolny bound. In such field theories, Bogomolny's argument showed that the total energy $E[\phi]$ attains its Bogomolny bound if and only if a set of first order equations hold and their solutions satisfy the static version of the full field equations. Hence, the field equations can be reduced to these first order equations which are called the Bogomolny equations of the theory. In these field theories, which are known as of Bogomolny type, the solutions of the Bogomolny equations are minimal energy static solutions, and so they form the moduli space of the field theory.

In [33], Manton has conjectured, through studying the BPS monopole dynamics, that the low energy dynamics of the field system can be approximated by restricting the dynamics of the field system to the theory's moduli space. This conjectural approximation has been applied in some field theories such as [45, 56] and shown in other field theories as in [50, 52, 53]. In many field theories, the low energy dynamics of the field system is approximated by geodesic motion on the moduli space with respect to a natural Riemannian metric defined by the restricting of the kinetic energy of the field system to the moduli space, namely, it is the L^2 metric. We shall see in this chapter a novel example of a field theory whose low energy dynamics is no longer governed by geodesic motion on the moduli space.

2.2 Abelian Higgs Model on $\mathbb{R}^2$

The Abelian Higgs model on the spacetime $\mathbb{R}^{1+2}$ is a gauged field theory defined by the field Lagrangian [23],

$$L[\phi] = \frac{1}{2}\int_{\mathbb{R}^2}\left(\sum_{\mu=0}^{2} D_\mu\phi\overline{D^\mu\phi} - \frac{1}{2}\sum_{\mu,\nu=0}^{2} F_{\mu\nu}F^{\mu\nu} - 2U(|\phi|)\right) d^2\mathbf{x}, \quad (2.2.1)$$

where ϕ is a complex valued scalar field on $\mathbb{R}^{1+2}$, $\phi(t,\mathbf{x}) = \phi_1(t,\mathbf{x}) + i\phi_2(t,\mathbf{x})$, coupled to a gauge potential field $A = (A_\mu) = (A_0, \mathbf{A})$, and U is a potential function given by

$$U = \frac{1}{8}(|\phi|^2 - 1)^2. \quad (2.2.2)$$

The expressions $D_\mu\phi = (\partial_\mu - iA_\mu)\phi$ form the gauge covariant derivative of ϕ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are the components of the field strength tensor F. This gauged field theory is based on the Abelian group $U(1)$ as the Lagrangian density in (2.2.1) is invariant under the gauge transformation

$$\begin{aligned} \phi(t,\mathbf{x}) &\to e^{i\alpha(t,\mathbf{x})}\phi(t,\mathbf{x}), \\ A_\mu(t,\mathbf{x}) &\to A_\mu(t,\mathbf{x}) + \partial_\mu\alpha(t,\mathbf{x}), \end{aligned} \quad (2.2.3)$$

where $\alpha(t,\mathbf{x})$ is an arbitrary function on $\mathbb{R}^{1+2}$. The field equations of the theory are

$$\begin{aligned} D^\mu D_\mu\phi &= -\frac{1}{2}(|\phi|^2 - 1)\,\phi, \\ \partial_\nu F^{\nu\mu} &= -\frac{i}{2}(\bar{\phi}D^\mu\phi - \phi\overline{D^\mu\phi}). \end{aligned} \quad (2.2.4)$$

The spacetime $\mathbb{R}^{1+2}$ is equipped with the Minkowski metric whose signature is $(+,-,-)$, and so

$$D_\mu\phi\overline{D^\mu\phi} = D_0\phi\overline{D_0\phi} - D_1\phi\overline{D_1\phi} - D_2\phi\overline{D_2\phi}, \quad (2.2.5)$$

$$F_{\mu\nu}F^{\mu\nu} = -2(E_1^2 + E_2^2 - B^2), \quad (2.2.6)$$

where $E_1 = F_{01}$, $E_2 = F_{02}$ are the electric components and $B = F_{12}$ is the magnetic component of the field F. Hence, one can extract the kinetic energy $T[\phi]$ and the potential energy $V[\phi]$ of this model from $L[\phi]$ as

$$T[\phi] = \frac{1}{2}\int_{\mathbb{R}^2}\left(D_0\phi\overline{D_0\phi} + E_1^2 + E_2^2\right) d^2\mathbf{x}, \quad (2.2.7)$$

$$V[\phi] = \frac{1}{2}\int_{\mathbb{R}^2}\left(D_1\phi\overline{D_1\phi} + D_2\phi\overline{D_2\phi} + B^2 + \frac{1}{4}(|\phi|^2 - 1)^2\right) d^2\mathbf{x}. \quad (2.2.8)$$

The total energy of this model is $E[\phi] = T[\phi] + V[\phi]$, and so for static fields, $E[\phi]$ is just the potential energy $V[\phi]$. The boundary of the domain $\mathbb{R}^2$ is a circle S^1_∞ at infinity. Since the finite energy field configuration is of our interest, then there is a constraint on the field configuration at S^1_∞ to ensure the energy finiteness there. This condition is given by imposing ϕ to take its value in the vacuum manifold $\mathcal{V}$ of the theory and its gauge covariant derivative to vanish. Explicitly,

$$|\phi| \to 1, \qquad D_\mu\phi \to 0, \qquad \text{as} \quad |\mathbf{x}| \to \infty. \quad (2.2.9)$$

Thus, the vacuum manifold $\mathcal{V}$ is the unit circle S^1. Here, two field configurations on the target $\mathbb{R}^2$ are homotopic if their asymptotic maps from S^1_∞ to $\mathcal{V}$ are homotopic. Hence, the finite energy field configurations of this model fall into the homotopy classes of the asymptotic maps $S^1_\infty \to \mathcal{V}$ and each class is labelled by an integer $n \in \mathbb{Z} = \pi_1(\mathcal{V})$, which represents the winding number of the field ϕ around the circle S^1_∞. Using polar coordinates (r, θ) in the domain $\mathbb{R}^2$, the finite energy conditions in (2.2.9) imply that $\phi \sim e^{i\chi(r,\theta)}$ and $A_\mu \sim \partial_\mu\chi(r,\theta)$, for some real valued function $\chi(r,\theta)$, as $r \to \infty$. Recall that the time-independent potential gauge field A can be re-expressed locally as a 1-form on $\mathbb{R}^2$, and so the field strength tensor $F = dA$. Thus, one can obtain, by Stokes' Theorem [15, p.95], that

$$\int_{\mathbb{R}^2} F = \int_{\mathbb{R}^2} dA = \int_{S^1_\infty} A = \int_0^{2\pi} A_\theta \, d\theta\Big|_{r=\infty} = \int_0^{2\pi} \partial_\theta\chi(r,\theta)\, d\theta\Big|_{r=\infty} = 2\pi n, \quad (2.2.10)$$

where the winding number of ϕ around S^1_∞ is $n = [\chi(\infty, 2\pi) - \chi(\infty, 0)]/(2\pi)$.

For static fields, the total energy $E[\phi] = V[\phi]$ can be written as [35, p.197]

$$E[\phi] = \frac{1}{2}\int_{\mathbb{R}^2}\left((D_1 \pm iD_2)\phi\,\overline{(D_1 \pm iD_2)\phi} + \left(B \pm \frac{1}{2}(|\phi|^2 - 1)\right)^2 \pm B\right) d^2\mathbf{x}, \quad (2.2.11)$$

with all positive or all negative signs. For all finite energy static fields in the n^{th} homotopy class, the total energy $E[\phi]$ is bounded below by

$$E[\phi] \;\; \geq \frac{1}{2}\left|\int_{\mathbb{R}^2} B\, d^2\mathbf{x}\right| = \frac{1}{2}\left|\int_{\mathbb{R}^2} F\right| = \pi\,|n|\,. \quad (2.2.12)$$

The equality in (2.2.12) holds if and only if

$$\begin{aligned} (D_1 \pm iD_2)\phi &= 0, \\ B \pm \frac{1}{2}(|\phi|^2 - 1) &= 0, \end{aligned} \quad (2.2.13)$$

where the upper (lower) sign corresponds to $n > 0$ ($n < 0$). One can check that the solutions of the equations (2.2.13) satisfy the static version of the field equations (2.2.4). Hence, the equations (2.2.13) are the Bogomolny equations of the model and the solutions of these equations are topological solitons called n-vortices for $n > 0$ (n-antivortices for $n < 0$).

The n-vortex moduli space of this model, denoted $\mathcal{M}_n$, is formed by the solutions set of the Bogomolny equations with winding number n. Although, no explicit formula for such solutions is known, Taubes [54] has shown that each solution is uniquely specified by n unordered points $\{z_r = x_r + iy_r : r = 1, \ldots, n\}$ on $\mathbb{R}^2$ where the field ϕ vanishes. These points are interpreted as vortex positions on $\mathbb{R}^2$. Hence, the n-vortex moduli space $\mathcal{M}_n$ is $\mathbb{C}^n/S_n$ where S_n is the permutation group of n objects. One can use these points as local complex coordinates on $\mathcal{M}_n$ except on $\triangle_n$ which is where zeros of ϕ coincide.

However, one can define global complex coordinates on $\mathcal{M}_n$ by the coefficients $\{w_k \in \mathbb{C} : k = 0, \dots, n-1\}$ of a degree n complex polynomial $P_n(z)$ whose roots are $\{z_r : r = 1, \dots, n\}$, that is,

$$P_n(z) = z^n + w_{n-1}z^{n-1} + \cdots + w_1 z + w_0 = \prod_{r=1}^{n}(z - z_r), \tag{2.2.14}$$

where $z = x + iy$ is the standard complex coordinate on $\mathbb{C} \cong \mathbb{R}^2$. Hence, the n-vortex moduli space $\mathcal{M}_n$ is an n-dimensional complex manifold identified with the complex space $\mathbb{C}^n$.

The interesting metric on the moduli space $\mathcal{M}_n$ is the L^2 metric which is derived from the restriction of the kinetic energy $T[\phi]$ to $\mathcal{M}_n$. Following Strachan [51], Samols gave a general formula for the L^2 metric, in terms of the local complex coordinates $\{z_r : r = 1, \dots, n\}$, on $\mathcal{M}_n$, assuming all zeros of ϕ are distinct. This is [43],

$$\gamma_{L^2} = \pi \sum_{r,s=1}^{n} \left(\delta_{rs} + 2\frac{\partial b_s}{\partial z_r} \right) dz_r d\bar{z}_s, \tag{2.2.15}$$

where δ_{rs} is the Kronecker delta, $\delta_{ss} = 1$ and zero otherwise, and b_s is the coefficient of $(\bar{z} - \bar{z}_s)/2$ in the Taylor expansion of $\log|\phi|^2$ about z_s. The coefficient b_s is a complex valued function and it satisfies

$$\frac{\partial b_s}{\partial z_r} = \frac{\partial \bar{b}_r}{\partial \bar{z}_s}. \tag{2.2.16}$$

The function b_s is not known explicitly. However one can see from (2.2.16) that γ_{L^2} is Hermitian ($\gamma_{rs} \equiv \bar{\gamma}_{sr}$), and in fact is Kähler, that is,

$$\frac{\partial \gamma_{rs}}{\partial z_\delta} \equiv \frac{\partial \gamma_{\delta s}}{\partial z_r}, \qquad \frac{\partial \gamma_{rs}}{\partial \bar{z}_\delta} \equiv \frac{\partial \gamma_{r\delta}}{\partial \bar{z}_s}. \tag{2.2.17}$$

In the case $n = 1$, this metric is just a flat metric on $\mathbb{C}$. For two vortices located at

$$z_1 = Z + \sigma e^{i\vartheta}, \qquad z_2 = Z - \sigma e^{i\vartheta}, \tag{2.2.18}$$

where $Z = (z_1 + z_2)/2$ is the centre of mass of the vortex system and $\xi := \sigma e^{i\vartheta}$ is the relative coordinate given by the polar coordinates (σ, ϑ) in the plane. Here, the functions b_1 and b_2 in the L^2 metric are related by $b_1 = -b_2 = b(\sigma)e^{i\vartheta}$ where $b(\sigma)$ is a real valued function. Thus, the L^2 metric in the case $n = 2$ can be written as

$$\gamma_{L^2} = 2\pi dZd\bar{Z} + \eta(\sigma)(d\sigma^2 + \sigma^2 d\vartheta^2), \tag{2.2.19}$$

where

$$\eta(\sigma) = 2\pi\left(1 + \frac{1}{\sigma}\frac{d}{d\sigma}\big(\sigma b(\sigma)\big)\right). \tag{2.2.20}$$

Hence, the 2-vortex moduli space $\mathcal{M}_2$ is the isometric product $\mathbb{C} \times_{\text{iso}} \mathcal{M}_2^0$, where $\mathcal{M}_2^0$ is the 2-vortex relative moduli space, the space of two vortices with fixed centre. Numerical results in [43] suggest that $\mathcal{M}_2^0$ can be isometrically embedded as a surface of revolution in $\mathbb{R}^3$. In fact, it is a smooth cone asymptotic to the singular cone of deficit angle π.

2.3 Chern-Simons Abelian Higgs Model on $\mathbb{R}^2$

The Chern-Simons Abelian Higgs model on the spacetime $\mathbb{R}^{1+2}$ is defined by the field Lagrangian [30],

$$L[\phi] = \frac{1}{2}\int_{\mathbb{R}^2}\left(D_\mu\phi\overline{D^\mu\phi} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \kappa\, \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \partial_\mu N\partial^\mu N - 2U(|\phi|\,, N)\right)d^2\mathbf{x}, \tag{2.3.1}$$

where ϕ and A_μ are defined as in the Abelian Higgs model, but here they are coupled to a real valued scalar field N defined on $\mathbb{R}^{1+2}$. The symbol κ is a real constant called Chern-Simons parameter and $\epsilon^{\mu\nu\rho}$ is the Levi-Civita tensor, i.e.

$$\epsilon^{\mu\nu\rho} = \begin{cases} 1 & \text{if } \mu\nu\rho \text{ is an even permutation of } 012 \\ -1 & \text{if } \mu\nu\rho \text{ is an odd permutation of } 012 \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.2}$$

The potential function of this model is

$$U = \frac{1}{8}(|\phi|^2 - 1 - 2\kappa N)^2 + \frac{1}{2}|\phi|^2 N^2. \tag{2.3.3}$$

The term in (2.3.1) with the parameter κ is the so-called Chern-Simons 3-form. This term itself is not gauge-invariant, however its integral over $\mathbb{R}^{1+2}$ is. One notes that the absence of both the Chern-Simons term and the field N reduce this model to the Abelian Higgs model on $\mathbb{R}^2$, which has been introduced in section 2.2.

The field equations of this model are

$$\begin{aligned} D^\mu D_\mu \phi &= -\frac{1}{2}(|\phi|^2 - 1 - 2\kappa N)\,\phi - \phi N^2, \\ \partial_\nu F^{\nu\mu} &= \kappa\, \epsilon^{\mu\nu\rho}\partial_\nu A_\rho - \frac{i}{2}(\bar{\phi} D^\mu \phi - \phi \overline{D^\mu \phi}), \\ \partial^\mu \partial_\mu N &= \frac{\kappa}{2}(|\phi|^2 - 1 - 2\kappa N) - |\phi|^2 N, \end{aligned} \tag{2.3.4}$$

which form a set of second order nonlinear partial differential equations. For static fields, the total energy is

$$\begin{aligned} E[\phi] = \frac{1}{2}\int_{\mathbb{R}^2} \Bigg(&(D_1 \pm iD_2)\phi\,\overline{(D_1 \pm iD_2)\phi} + \big(B \pm \frac{1}{2}(|\phi|^2 - 1 - 2\kappa N)\big)^2 \\ &+ |\phi|^2 (A_0 \mp N)^2 \mp 2N(\nabla^2 A_0 - |\phi|^2 A_0 - \kappa B) \\ &+ (\nabla A_0 \mp \nabla N)^2 \pm B \Bigg)\, d^2\mathbf{x}. \end{aligned} \tag{2.3.5}$$

The static version of the second equation in (2.3.4) with $\mu = 0$ is known as the Gauss law of the model, that is,

$$\nabla^2 A_0 - |\phi|^2 A_0 - \kappa B = 0. \tag{2.3.6}$$

Using (2.3.6) in (2.3.5), the total energy $E[\phi]$ becomes

$$E[\phi] = \frac{1}{2}\int_{\mathbb{R}^2}\Big((D_1 \pm iD_2)\phi\,\overline{(D_1 \pm iD_2)\phi} + \big(B \pm \frac{1}{2}(|\phi|^2 - 1 - 2\kappa N)\big)^2 + |\phi|^2 (A_0 \mp N)^2 + (\nabla A_0 \mp \nabla N)^2 \pm B\Big)\, d^2\mathbf{x}. \tag{2.3.7}$$

The finite energy boundary conditions on the field configuration at spatial infinity are

$$|\phi| \to 1, \qquad D_\mu\phi \to 0, \qquad N \to 0, \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{2.3.8}$$

This implies that the vacuum manifold $\mathcal{V}$ of the theory is the unit circle S^1. Hence, the configuration space is decomposed into the homotopy classes of $S^1_\infty \to \mathcal{V}$ labelled by the winding number $n \in \pi_1(\mathcal{V})$ of ϕ around S^1_∞, as in the Abelian Higgs model. It follows from (2.3.7) that the total energy $E[\phi]$ in the n^{th} sector satisfies

$$E[\phi] \geq \frac{1}{2}\Big|\int_{\mathbb{R}^2} B\, d^2\mathbf{x}\Big| = \pi\,|n|\,. \tag{2.3.9}$$

The equality in (2.3.9) occurs if and only if

$$\begin{aligned} (D_1 \pm iD_2)\phi &= 0, \\ B \pm \frac{1}{2}(|\phi|^2 - 1 - 2\kappa N) &= 0, \\ A_0 \mp N &= 0, \end{aligned} \tag{2.3.10}$$

which, in addition to the Gauss law (2.3.6), are the Bogomolny equations of the model. The solutions of the Bogomolny equations are also called n-vortices for $n > 0$ (n-antivortices for $n < 0$). By substituting the equations (2.3.6) and (2.3.10) in the field equations (2.3.4), it can be checked that the solutions of the Bogomolny equations satisfy the whole static field equations. As in the Abelian Higgs model, the solutions of the Bogomolny equations are not known explicitly. However, in [24], it is conjectured that each of these solutions with winding number n corresponds uniquely to n unordered points on $\mathbb{R}^2$ given by the zeros of ϕ, vortex positions. Moreover, for small κ, there is a conjectured one-to-one correspondence between the solutions of (2.3.6) and (2.3.10) and the Abelian Higgs n-vortices . Hence, one expects that the n-vortex moduli space of the Chern-Simons Abelian Higgs model is identified with the n-vortex moduli space of the Abelian Higgs model $\mathcal{M}_n \cong \mathbb{C}^n$.

2.3.1 Vortex Asymptotics

A natural question to ask about an n-vortex in the Chern-Simons Abelian Higgs model is how it behaves away from and close to the positions of the vortices. Let (ϕ, A_μ, N) be an n-vortex all of whose zeros are located at the origin. Let (r, θ) be the polar coordinates in the plane and define the following n-vortex ansatz

$$\begin{aligned} \phi(r,\theta) &= f(r)e^{in\theta}, \\ (A_0, A_r, A_\theta) &= (h(r), 0, a(r)), \\ N(r,\theta) &= g(r), \end{aligned} \tag{2.3.11}$$

where the potential gauge field $A = A_0 dt + A_r dr + A_\theta d\theta$. Note that the n-vortex of the form $(2.3.11)$ is invariant under the combined global $U(1)$-action $\phi(r,\theta) \mapsto e^{-in\alpha}\phi(r,\theta+\alpha)$. Substituting $(2.3.11)$ in (2.3.6) and (2.3.10), the Bogomolny equations reduce to the non-linear ordinary differential equations

$$\begin{aligned}
\frac{df}{dr} &= \frac{1}{r}f(n-a), \\
\frac{da}{dr} &= -\frac{r}{2}(f^2-1-2\kappa g), \\
\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr} &= f^2g-\frac{\kappa}{2}(f^2-1-2\kappa g),
\end{aligned} \tag{2.3.12}$$

with $h(r) = g(r)$. Note that the above equations are invariant under $(f, a, g, k) \rightarrow (f, a, -g, -\kappa)$, and so, without loss of generality we take $\kappa \geq 0$. For static fields of the form (2.3.11), the finite energy boundary conditions, given in (2.3.8), at spatial infinity become

$$f \rightarrow 1, \qquad a \rightarrow n, \qquad g, h \rightarrow 0, \qquad \text{as} \qquad r \rightarrow \infty. \tag{2.3.13}$$

To understand the behaviour of $(f(r), a(r), g(r))$ for large r, let

$$f(r) = 1+\tilde{f}(r), \qquad a(r) = n+\tilde{a}(r), \qquad g(r) = \tilde{g}(r). \tag{2.3.14}$$

Then, at large r, $\tilde{f}(r)$, $\tilde{a}(r)$ and $\tilde{g}(r)$ are very small. Substituting (2.3.14) in (2.3.12), the linearized forms of these equations are

$$\frac{d\tilde{f}}{dr} = -\frac{\tilde{a}}{r}, \tag{2.3.15}$$

$$\frac{d\tilde{a}}{dr} = r(\kappa\tilde{g}-\tilde{f}), \tag{2.3.16}$$

$$\frac{d^2\tilde{g}}{dr^2}+\frac{1}{r}\frac{d\tilde{g}}{dr} = (\kappa^2+1)\tilde{g}-\kappa\tilde{f}. \tag{2.3.17}$$

We can use (2.3.15) to eliminate $\tilde{a}(r)$, yielding a set of second order coupled linear ordinary differential equations

$$\begin{aligned}\frac{d^2\tilde{g}}{dr^2}+\frac{1}{r}\frac{d\tilde{g}}{dr} &= (\kappa^2+1)\tilde{g}-\kappa\tilde{f},\\ \frac{d^2\tilde{f}}{dr^2}+\frac{1}{r}\frac{d\tilde{f}}{dr} &= -\kappa\tilde{g}+\tilde{f}.\end{aligned} \tag{2.3.18}$$

The matrix version of (2.3.18) is

$$\frac{d^2W}{dr^2}+\frac{1}{r}\frac{dW}{dr}=MW, \tag{2.3.19}$$

where

$$W=\begin{pmatrix}\tilde{g}\\ \tilde{f}\end{pmatrix}, \qquad M=\begin{pmatrix}\kappa^2+1 & -\kappa\\ -\kappa & 1\end{pmatrix}. \tag{2.3.20}$$

The eigenvalue problem $MW=\lambda W$ has the following eigenvectors

$$E_1=\frac{1}{2}\begin{pmatrix}-\kappa+\sqrt{\kappa^2+4}\\ 2\end{pmatrix}, \qquad E_2=\frac{1}{2}\begin{pmatrix}-\kappa-\sqrt{\kappa^2+4}\\ 2\end{pmatrix}, \tag{2.3.21}$$

associated with the eigenvalues

$$\lambda_1=\frac{1}{4}(\kappa-\sqrt{\kappa^2+4})^2, \qquad \lambda_2=\frac{1}{4}(\kappa+\sqrt{\kappa^2+4})^2, \tag{2.3.22}$$

respectively. Hence, W can be written as a linear combination of E_1 and E_2, namely,

$$W=W_1(r)E_1+W_2(r)E_2. \tag{2.3.23}$$

Substituting (2.3.23) in (2.3.19), one gets

$$\frac{d^2W_1}{dr^2}+\frac{1}{r}\frac{dW_1}{dr} = \lambda_1 W_1, \tag{2.3.24}$$

$$\frac{d^2W_2}{dr^2}+\frac{1}{r}\frac{dW_2}{dr} = \lambda_2 W_2, \tag{2.3.25}$$

which are the modified Bessel equations of zero order. The general solution of this type of equations is a linear combination of exponential decaying and growing functions K_0 and I_0, respectively [1]. Since we are interested in the finite energy solutions, we consider only the decaying term, that is,

$$W_1 \sim q_1 K_0(\sqrt{\lambda_1}\, r), \qquad W_2 \sim q_2 K_0(\sqrt{\lambda_2}\, r), \tag{2.3.26}$$

where q_1, q_2 are real constants and K_0 is called the modified Bessel's function of the second kind which satisfies [1]

$$\begin{aligned} K_0'(s) &= -K_1(s), && (2.3.27)\\ K_0(s) &= -[\frac{d}{ds}K_1(s) + K_1(s)/s]. && (2.3.28) \end{aligned}$$

Substituting (2.3.26) in (2.3.23), we get

$$\begin{aligned} \tilde{f}(r) &\sim q_1 K_0(\sqrt{\lambda_1}\, r) + q_2 K_0(\sqrt{\lambda_2}\, r), && (2.3.29)\\ \tilde{g}(r) &\sim q_1\left(\frac{1-\lambda_1}{\kappa}\right)K_0(\sqrt{\lambda_1}\, r) + q_2\left(\frac{1-\lambda_2}{\kappa}\right)K_0(\sqrt{\lambda_2}\, r). && (2.3.30) \end{aligned}$$

There is a constraint on the range of κ given by $\kappa \leq 1/\sqrt{2}$ which is where the linearization is valid. It follows from (2.3.16) and (2.3.29), and by using (2.3.28) that

$$\tilde{a}(r) \sim q_1\sqrt{\lambda_1}\, rK_1(\sqrt{\lambda_1}\, r) + q_2\sqrt{\lambda_2}\, rK_1(\sqrt{\lambda_2}\, r). \tag{2.3.31}$$

Hence, for large r, the asymptotic formulae for the n-vortex (ϕ, A_μ, N), given in (2.3.11), are determined by

$$
\begin{aligned}
f(r) &\sim 1 + q_1 K_0(\sqrt{\lambda_1}\, r) + q_2 K_0(\sqrt{\lambda_2}\, r), \\
a(r) &\sim n + q_1 \sqrt{\lambda_1}\, r K_1(\sqrt{\lambda_1}\, r) + q_2 \sqrt{\lambda_2}\, r K_1(\sqrt{\lambda_2}\, r), \\
h(r), g(r) &\sim q_1 \left(\frac{1-\lambda_1}{\kappa}\right) K_0(\sqrt{\lambda_1}\, r) + q_2 \left(\frac{1-\lambda_2}{\kappa}\right) K_0(\sqrt{\lambda_2}\, r).
\end{aligned}
\tag{2.3.32}
$$

The constants q_1 and q_2 can be determined by solving the Bogomolny equations (2.3.12) numerically with appropriate boundary conditions for large and small r. Thus, it is useful to find the asymptotic formulae for the n-vortex (ϕ, A_μ, N), given in (2.3.11), for small r. The boundary conditions on $(f(r), a(r), g(r))$ at $r = 0$ are

$$
f(0) = 0, \qquad a(0) = 0, \qquad g(0) = g_0. \tag{2.3.33}
$$

where g_0 is a real constant. It follows from (2.3.33) and (2.3.12) that the functions $f(r)$, $a(r)$ and $g(r)$ must have the form

$$
f(r) = r^n\, \hat{f}(r), \quad a(r) = r^2\, \hat{a}(r), \qquad g(r) = g_0 + r^2\, \hat{g}(r), \tag{2.3.34}
$$

where $\hat{f}(r)$, $\hat{a}(r)$ and $\hat{g}(r)$ are series in r whose leading terms are nonzero constants. Substituting (2.3.34) in the Bogomolny equations (2.3.12), we obtain that

$$
\begin{aligned}
f(r) &= f_0 r^n - \frac{1}{8} f_0 (1 + 2\kappa g_0) r^{n+2} + O(r^{n+4}), \\
a(r) &= \frac{1}{4}(1 + 2\kappa g_0) r^2 + O(r^4), \\
g(r) &= g_0 + \frac{1}{8}\kappa (1 + 2\kappa g_0) r^2 + O(r^4),
\end{aligned}
\tag{2.3.35}
$$

which determine the asymptotic formulae for the n-vortex (ϕ, A_μ, N), given in (2.3.11), for small r.

2.3.2 Numerical Calculation of the 1-Vortex

Having found the boundary conditions of the Bogomolny equations for large and small r, it is straightforward to solve numerically the boundary value problem

$$\frac{d}{dr}Z(r) = H(r, Z(r)), \tag{2.3.36}$$

where

$$Z(r) = \begin{pmatrix} f(r) \\ a(r) \\ g(r) \\ g'(r) \end{pmatrix}, \qquad H(r, Z(r)) = \begin{pmatrix} f(n-a)/r \\ -r(f^2 - 1 - 2\kappa g)/2 \\ g'(r) \\ f^2 g - \big(2g'(r) + \kappa r(f^2 - 1 - 2\kappa g)\big)/(2r) \end{pmatrix}, \tag{2.3.37}$$

with the boundary conditions given in (2.3.32) and (2.3.35).

To solve this boundary value problem on $[r_0, r_\infty]$, one can use the shooting method [14, p.545-548] with (f_0, g_0) as shooting parameters to shoot forward from r_0 to $r_1 \in (r_0, r_\infty)$ and (q_1, q_2) as shooting parameters to shoot backward from r_∞ to r_1. We have chosen $r_0 = 10^{-5}$, $r_1 = 2$ and $r_\infty = 10$. The choice of r_0 is due to the singularity of the Bogomolny equations at the origin. The reason behind the small choice of r_∞ is that the solution for large r contains a growing term, which is impossible to be ignored during the numerical procedure, prevents r_∞ from taking large value.

Let Z_L be the solution of the left initial value problem on $[r_0, r_1]$ with initial condition (2.3.35), and Z_R be the solution of the right initial value problem on $[r_1, r_\infty]$ with initial condition (2.3.32). Then, one can define a new function F as

$$\mathrm{F} := Z_L(r_1) - Z_R(r_1), \tag{2.3.38}$$

which maps from $\mathbb{R}^4$ to $\mathbb{R}^4$. This function is called the mismatch function of the boundary value problem. The zero of this function is $\mathbf{p}_0 := (f_0, g_0, q_1, q_2)$. To find $\mathbf{p}_0$, one can use Newton-Raphson method [14, p.144-149] which needs a sensible first guess for $\mathbf{p}_0$. We have chosen $q_1 = q_2 = -0.842$, which are the coefficients of the asymptotic Abelian Higgs 1-vortex for large r [46], and $f_0 = g_0 = 0$ as a first guess of $\mathbf{p}_0$.

Following the above procedure with using Runge-Kutta $(4, 5)$ method [14, p.459-498] to solve the left and right initial value problems, we get the following results for $n = 1$ and various values of κ, Table 2.1.

κ	f_0	g_0	q_1	q_2
0	0.6033	0.0000	-0.8538	-0.8538
0.1	0.5998	-0.0498	-0.9013	-0.8158
0.2	0.5895	-0.0985	-0.9587	-0.7876
0.3	0.5728	-0.1448	-1.0267	-0.7705
0.4	0.5505	-0.1877	-1.1062	-0.7678
0.5	0.5233	-0.2261	-1.1981	-0.7884
0.6	0.4927	-0.2589	-1.3031	-0.8585
0.7	0.4597	-0.2855	-1.4214	-1.0652

Table 2.1: Numerical values of the asymptotic 1-vortex parameters for various values of κ.

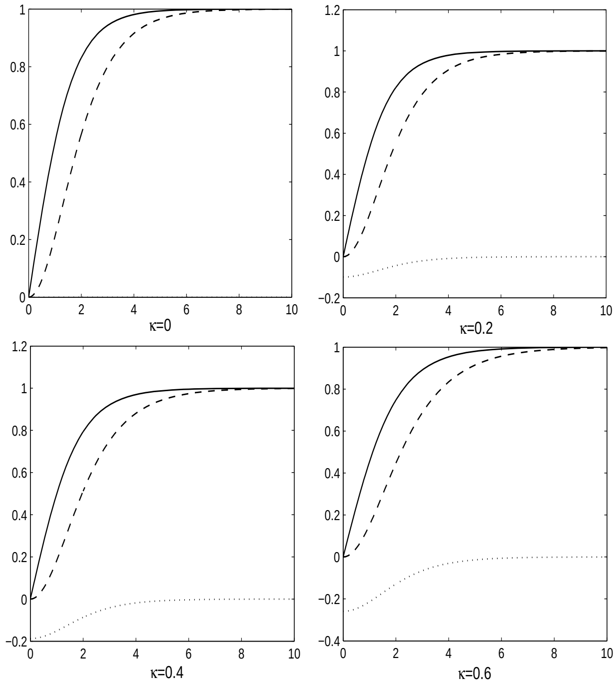


Figure 2.1: The profile functions $f(r)$ (solid line), $a(r)$ (dashed line) and $g(r)$ (dots line) of the 1-vortex.

2.3.3 The Point Vortex

Far away from the vortex position, the n-vortex is viewed as a static solution of the linear version of the original gauged field theory with a singularity at the vortex position [46]. By this view, we shall determine the appropriate point source corresponding to the Chern-Simons Abelian Higgs n-vortex located at the origin.

The linearized model of the Chern-Simons Abelian Higgs field theory is defined by a field Lagrangian whose density is the expansion of the Lagrangian density in (2.3.1) for a scalar field $\phi = 1 - \psi$, where ψ is real valued, up to the quadratic order terms. That is,

$$\begin{aligned}\mathcal{L}_{\text{linear}} &= \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}\psi^2 + \frac{1}{2}\partial_\mu N\partial^\mu N - \frac{1}{2}(\kappa^2+1)N^2 - \kappa\psi N \\ &\quad -\frac{1}{4}F_{\nu\mu}F^{\nu\mu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2}A_\mu A^\mu. \qquad (2.3.39)\end{aligned}$$

It is convenient to define two real valued scalar fields X and Y as follows

$$\psi = \frac{1}{\sqrt{\lambda_1+1}}\,X + \frac{1}{\sqrt{\lambda_2+1}}\,Y, \tag{2.3.40}$$

$$N = \frac{\lambda_1-1}{\sqrt{\lambda_1+1}}\,X + \frac{\lambda_2-1}{\sqrt{\lambda_2+1}}\,Y, \tag{2.3.41}$$

where λ_1 and λ_2 are real constants defined as in (2.3.22). Substituting (2.3.40) and (2.3.41) in (2.3.39), $\mathcal{L}_{\text{linear}}$ becomes

$$\begin{aligned}\mathcal{L}_{\text{linear}} &= \frac{1}{2}\partial_\mu X\partial^\mu X - \frac{\lambda_1}{2}X^2 + \frac{1}{2}\partial_\mu Y\partial^\mu Y - \frac{\lambda_2}{2}Y^2 \\ &\quad -\frac{1}{4}F_{\nu\mu}F^{\nu\mu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2}A_\mu A^\mu.\end{aligned} \tag{2.3.42}$$

Let $(\alpha, \beta, \mathrm{j}^\mu)$ be the point sources coupled to (X, Y, A_μ), respectively. Then, the source Lagrangian is given by

$$\mathcal{L}_{\text{source}} = \alpha\ X + \beta\ Y - \mathrm{j}^\mu\ A_\mu. \tag{2.3.43}$$

The Euler-Lagrange equations derived from $\mathcal{L}_{\text{Linear}} + \mathcal{L}_{\text{source}}$ are

$$\begin{aligned}(\Box + \lambda_1)\,X &= \alpha, \\ (\Box + \lambda_2)\,Y &= \beta, \\ (\Box + 1)\ A^\mu + \kappa\epsilon^{\mu\nu\rho}\partial_\nu A_\rho &= \mathrm{j}^\mu + \partial^\mu\partial_\nu \mathrm{j}^\nu,\end{aligned} \tag{2.3.44}$$

where $\Box = \partial^2/\partial t^2 - \nabla^2$ is the d'Alembert operator on the spacetime $\mathbb{R}^{1+2}$.

With respect to the polar coordinates (r, θ) in the plane, the asymptotic formulae for X, Y and A_μ as $r \to \infty$ can be extracted from the n-vortex asymptotics given by (2.3.32) as follows:

Recall that ϕ defined in (2.3.11) is a complex valued field of nonzero winding number n whereas ψ is a real valued field. Then, by (2.3.11), we can convert ϕ to a real field by the gauge transformation

$$\phi(r,\theta) \to e^{-in\theta}\,\phi(r,\theta), \tag{2.3.45}$$

$$A_\mu(r,\theta) \to A_\mu(r,\theta) - n\partial_\mu\theta. \tag{2.3.46}$$

Here, A_0 and N remain unchanged. Hence, it follows from (2.3.11) that

$$\psi = 1 - f(r), \qquad (A_0, A_r, A_\theta) = (h(r), 0, a(r) - n), \qquad N = g(r). \tag{2.3.47}$$

Using (2.3.32) in (2.3.47), we get

$$\psi \;\sim\; -q_1 K_0(\sqrt{\lambda_1}\, r) - q_2 K_0(\sqrt{\lambda_2}\, r), \tag{2.3.48}$$

$$(A_r, A_\theta) \;\sim\; (0, q_1\sqrt{\lambda_1}\, r K_1(\sqrt{\lambda_1}\, r) + q_2\sqrt{\lambda_2}\, r K_1(\sqrt{\lambda_2}\, r)), \tag{2.3.49}$$

$$A_0, N \;\sim\; q_1\left(\frac{1-\lambda_1}{\kappa}\right)K_0(\sqrt{\lambda_1}\, r) + q_2\left(\frac{1-\lambda_2}{\kappa}\right)K_0(\sqrt{\lambda_2}\, r). \tag{2.3.50}$$

Using (2.3.48) and (2.3.50) in (2.3.40) and (2.3.41), respectively, we obtain that

$$X \;\sim\; -q_1\sqrt{\lambda_1 + 1}\; K_0(\sqrt{\lambda_1}\, r), \tag{2.3.51}$$

$$Y \;\sim\; -q_2\sqrt{\lambda_2 + 1}\; K_0(\sqrt{\lambda_2}\, r). \tag{2.3.52}$$

Hence, the asymptotic n-vortex (X, Y, A_μ) for large r are given by (2.3.49-2.3.52).

Now, since the physical space $\mathbb{R}^2$ can be considered as a subspace of $\mathbb{R}^3$ with $\hat{\mathbf{k}}$ denoting the unit vector in the (fictitious) third spatial direction, the vector field $\mathbf{A} = (A^r, A^\theta)$ can be written as a cross product on $\mathbb{R}^3$ as follows:

$$\begin{aligned}
\mathbf{A} &= A^r \frac{\partial}{\partial r} + A^\theta \frac{\partial}{\partial \theta}, \\
&= -A_r \frac{\partial}{\partial r} - \frac{A_\theta}{r^2}\frac{\partial}{\partial \theta}, \\
&= -\Big(q_1 \sqrt{\lambda_1} K_1(\sqrt{\lambda_1}\, r) + q_2 \sqrt{\lambda_2} K_1(\sqrt{\lambda_2}\, r) \Big)\, \hat{\boldsymbol{\theta}}, \\
&= \Big(q_1 \sqrt{\lambda_1} K_0'(\sqrt{\lambda_1}\, r) + q_2 \sqrt{\lambda_2} K_0'(\sqrt{\lambda_2}\, r) \Big)\, \hat{\boldsymbol{\theta}}, \\
&= \hat{\mathbf{k}} \times \Big(q_1 \nabla K_0(\sqrt{\lambda_1}\, r) + q_2 \nabla K_0(\sqrt{\lambda_2}\, r) \Big), \qquad (2.3.53)
\end{aligned}$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the θ-direction on $\mathbb{R}^2$ and ∇K_0 is the gradient of the modified Bessel's function K_0.

The static version of the field equations (2.3.44) are

$$\begin{aligned}
(-\nabla^2 + \lambda_1)\, X &= \alpha, \qquad (2.3.54) \\
(-\nabla^2 + \lambda_2)\, Y &= \beta, \qquad (2.3.55) \\
(-\nabla^2 + 1)\, A^0 - \kappa\, \frac{1}{r}\frac{d}{dr}(r^2 A^\theta) &= \mathrm{j}^0, \qquad (2.3.56) \\
(-\nabla^2 + 1)\, \mathbf{A} - \kappa\, \hat{\mathbf{k}} \times \nabla A^0 &= \mathbf{j} - \nabla(\nabla \cdot \mathbf{j}), \qquad (2.3.57)
\end{aligned}$$

where $\mathbf{j} = (\mathbf{j^1}, \mathbf{j^2})$. The time-independent Klein-Gordon equation in dimension 2 has Green's function K_0 [46],

$$(-\nabla^2 + \lambda)\, K_0(\sqrt{\lambda}\, r) = 2\pi\delta(\mathbf{x}), \qquad (2.3.58)$$

where $\delta(\mathbf{x})$ is the Dirac delta on $\mathbb{R}^2$, $\delta(\mathbf{x}) = 0$ everywhere except at $\mathbf{x} = 0$. Substituting (2.3.51) and (2.3.52) in (2.3.54) and (2.3.55), respectively, and using (2.3.58), we obtain that

$$\alpha = -2\pi q_1\sqrt{\lambda_1+1}\,\delta(\mathbf{x}), \tag{2.3.59}$$

$$\beta = -2\pi q_2\sqrt{\lambda_2+1}\,\delta(\mathbf{x}). \tag{2.3.60}$$

Substituting (2.3.50) in (2.3.56) and using (2.3.58), we get

$$\mathrm{j}^0 = 2\pi\kappa\left(\frac{q_2}{\lambda_1-1}+\frac{q_1}{\lambda_2-1}\right)\delta(\mathbf{x}). \tag{2.3.61}$$

Similarly, we have

$$\mathbf{j} - \nabla(\nabla\cdot\mathbf{j}) = 2\pi(q_1+q_2)\,\hat{\mathbf{k}}\times\ \nabla\delta(\mathbf{x}). \tag{2.3.62}$$

Taking the divergence of (2.3.62), we get

$$(-\nabla^2+1)\,(\nabla\cdot\mathbf{j}) = 0, \tag{2.3.63}$$

which is the homogeneous time-independent Klein-Gordon equation. Therefore, if **j** is a point source, then it is divergenceless, $\nabla\cdot\mathbf{j}=0$, and so

$$\mathbf{j} = 2\pi(q_1+q_2)\,\hat{\mathbf{k}}\times\ \nabla\delta(\mathbf{x}). \tag{2.3.64}$$

Hence, $(\alpha,\beta,\mathrm{j}^0,\mathbf{j})$, given in (2.3.59-2.3.64), interpret the n-vortex $(X,Y,A^0,\mathbf{A})$ as a point source called the point vortex. In the case $\kappa=0$, our point vortex coincides with the one obtained in [46].

2.3.4 n-Vortex Low Energy Dynamics

The low energy dynamics of n-vortices in the Abelian Higgs model is approximated by geodesic motion on the n-vortex moduli space $\mathcal{M}_n$ with respect to the Samols's metric, given in (2.2.15). Namely, the motion equations are derived from the purely kinetic

Lagrangian [52],

$$L = \frac{1}{2}\gamma_{L^2_{rs}}\dot{z}_r\dot{z}_s. \tag{2.3.65}$$

In the presence of the Chern-Simons term in the field Lagrangian as in the Chern-Simons Abelian Higgs model, the low energy dynamics of n-vortices is no longer governed by geodesic motion. This is because that the Chern-Simons term implies the appearance of a magnetic field $\mathcal{F} \in \Omega^2(\mathcal{M}_n)$ which affects significantly the n-vortex dynamics. For small κ, Kim and Lee expect that the Lagrangian which describes the motion of n-vortices on the moduli space $\mathcal{M}_n$ takes the form [24],

$$L = \frac{1}{2}\gamma_{L^2_{rs}}\dot{z}_r\dot{z}_s + \lambda(\mathcal{A}_{z_r}\dot{z}_r + \mathcal{A}_{\bar{z}_r}\dot{\bar{z}}_r), \qquad \lambda = 2\pi\kappa, \tag{2.3.66}$$

where $\mathcal{A} = \mathcal{A}_{z_r}dz_r + \mathcal{A}_{\bar{z}_r}d\bar{z}_r$ is a 1-form on the moduli space $\mathcal{M}_n$ whose exterior derivative $d\mathcal{A}$ is locally identified with $\mathcal{F}$. This 1-form $\mathcal{A}$ has been described as follows:

Kim-Lee approximation

Kim and Lee [24] introduced a general formula for the 1-form $\mathcal{A}$ in (2.3.66). This is

$$\begin{aligned}\mathcal{A} =& \frac{i}{16}\left[3\bar{b}_r - \sum_{s\neq r}\left(\frac{8}{(z_r - z_s)} + (z_r - z_s)\frac{\partial\bar{b}_r}{\partial z_s} + (\bar{z}_r - \bar{z}_s)\frac{\partial\bar{b}_r}{\partial\bar{z}_s}\right)\right] dz_r \\ & - \frac{i}{4}\left[b_r - \sum_{s\neq r}\frac{2}{(\bar{z}_r - \bar{z}_s)}\right] d\bar{z}_r.\end{aligned} \tag{2.3.67}$$

For 2-vortices, the Lagrangian (2.3.66) can be written, in terms of the centre of mass Z on $\mathbb{C}$ and the relative coordinate $\xi = \sigma e^{i\vartheta}$ on $\mathcal{M}^0_2$, as [24],

$$L = \pi\dot{Z}\dot{\bar{Z}} + \frac{1}{2}\eta(\sigma)\ (\dot{\sigma}^2 + \sigma^2\dot{\vartheta}^2) + \lambda\ \mathcal{A}_\vartheta(\sigma)\dot{\vartheta}, \tag{2.3.68}$$

where $\mathcal{A} = \mathcal{A}_\vartheta(\sigma)\ d\vartheta$ is a 1-form on $\mathcal{M}^0_2$. The component $\mathcal{A}_\vartheta(\sigma)$ is a real valued function given by

$$\mathcal{A}_\vartheta(\sigma) = -[\sigma b(\sigma) - 1] + \frac{\sigma}{4}\frac{d}{d\sigma}[\sigma b(\sigma)]. \tag{2.3.69}$$

For small σ, the function $b(\sigma)$ is asymptotically given by [43],

$$b(\sigma) = \frac{1}{\sigma} - \frac{\sigma}{2} + O(\sigma^3). \tag{2.3.70}$$

Then, the exterior derivative of $\mathcal{A}$ is locally given by

$$d\mathcal{A} = \mathcal{A}'_\vartheta(\sigma)\, d\sigma \wedge d\vartheta, \quad \text{with} \quad \mathcal{A}'_\vartheta(\sigma) \sim \frac{\sigma}{2}, \quad \text{as} \quad \sigma \to 0. \tag{2.3.71}$$

One can re-express $d\mathcal{A}$ in terms of the global coordinate $w = z_1 z_2 = \xi^2 = (\sigma e^{i\vartheta})^2$ on $\mathcal{M}_2^0$, that is,

$$d\mathcal{A} = \frac{i}{8}\,\frac{\mathcal{A}'_\vartheta(|w|^{1/2})}{|w|^{3/2}}\, dw \wedge d\bar{w}, \quad \text{with} \quad \mathcal{A}'_\vartheta(|w|^{1/2}) \sim \frac{|w|^{1/2}}{2}, \quad \text{as} \quad |w| \to 0. \tag{2.3.72}$$

It follows from (2.3.72) that the component of the magnetic field $\mathcal{F} = d\mathcal{A}$ diverges like $1/(16\sigma^2)$ as $\sigma \to 0$. Hence, the Kim-Lee dynamical system is ill-defined.

Collie-Tong approximation

It has been argued by Collie and Tong [17] that the magnetic field $\mathcal{F}$ is in fact the Ricci form ρ on the moduli space $\mathcal{M}_n$ with respect to the Samols's metric. That is,

$$\mathcal{F} = \rho = i\partial\bar{\partial}\log(\sqrt{|\gamma_{L^2}|}), \tag{2.3.73}$$

where $|\gamma_{L^2}|$ denotes the determinant of the matrix associated to γ_{L^2}. Hence, the 1-form $\mathcal{A}$ is given, in terms of the complex local coordinates $\{z_r : r = 1, \ldots, n\}$, by

$$\mathcal{A}_{z_r} = \bar{\mathcal{A}}_{\bar{z}_r} = \frac{i}{2}\frac{\partial}{\partial z_r}\log(\sqrt{|\gamma_{L^2}|}). \tag{2.3.74}$$

From this, we deduce that the low energy dynamics of n-vortices in the Chern-Simons Abelian Higgs model is governed by the RMG motion on the moduli space $\mathcal{M}_n$. Here, the n-vortex trajectory $\gamma(t)$, say, solves

$$\nabla^{\gamma}_{d/dt}\dot{\gamma} = -2\pi\kappa \; \sharp\iota_{\dot{\gamma}}\rho. \tag{2.3.75}$$

Collie and Tong claimed that the Kim and Lee magnetic field $\mathcal{F}$ coincides with the Ricci form ρ. One can check this is not true as, for example, in the case $n = 2$, the RMG motion on $\mathcal{M}_2$ is described by a Lagrangian defined as in (2.3.68) but with

$$\mathcal{A}_{\vartheta}(\sigma) = -\frac{\sigma}{2}\frac{d}{d\sigma}\log(\eta(\sigma)) = -\frac{\sigma\eta'(\sigma)}{2\eta(\sigma)} = -\frac{1}{2}\frac{\sigma(\sigma b(\sigma))'' - (\sigma b)'}{\sigma + (\sigma b)'}. \tag{2.3.76}$$

From (2.3.76) and (2.3.69), one obtained that the magnetic field given by Kim and Lee is completely different from the one given by Collie and Tong. In particular, the Ricci form ρ, unlike the Kim-Lee field $\mathcal{F}$, is smooth on the whole of $\mathcal{M}_2$. Henceforth, we shall consider Collie-Tong approximation for the low energy dynamics of Chern-Simons Abelian Higgs vortices.

2.3.5 2-Vortex Scattering

In this section, we shall give an estimate of the scattering angle of two vortices at large separation on $\mathcal{M}_2$. It can be seen that the 2-vortex relative moduli space $\mathcal{M}_2^0$ is invariant under the vortex position's rotation by angle π about the origin. This is a holomorphic isometry on $\mathcal{M}_2$. Since $\mathcal{M}_2 \cong \mathbb{C} \times_{\text{iso}} \mathcal{M}_2^0$, then $\mathcal{M}_2^0$ is a totally RMG submanifold of $\mathcal{M}_2$. Hence, it is sufficient to study the scattering angle of 2-vortices on $\mathcal{M}_2^0$ where the motion is described by the RMG Lagrangian [17],

$$L_{\text{relv}} = \frac{1}{2}\eta(\sigma)\,(\dot{\sigma}^2 + \sigma^2\dot{\vartheta}^2) + \lambda\,\mathcal{A}_{\vartheta}(\sigma)\dot{\vartheta}, \qquad \lambda = 2\pi\kappa, \tag{2.3.77}$$

where $\mathcal{A}_{\vartheta}(\sigma)$ is given in (2.3.76). The functions $\eta(\sigma)$ and $\mathcal{A}_{\vartheta}(\sigma)$ depend on the single function $b(\sigma)$. For large σ, an explicit formula of $b(\sigma)$ has been given in [34],

$$b(\sigma) = \frac{q^2}{2\pi^2} K_1(2\sigma), \tag{2.3.78}$$

where q is a real constant approximately -10.6 determined numerically in [46]. Substituting (2.3.78) in both (2.2.20) and (2.3.76), the functions $\eta(\sigma)$ and $\mathcal{A}_\vartheta(\sigma)$ for large σ are

$$\begin{aligned} \eta(\sigma) &= 2\pi\left(1 - \frac{q^2}{\pi^2} K_0(2\sigma)\right), & (2.3.79) \\ \mathcal{A}_\vartheta(\sigma) &= -\frac{q^2}{\pi^2}\sigma K_1(2\sigma). & (2.3.80) \end{aligned}$$

As stated in Chapter 1, there are two conserved quantities associated with the RMG motion on the surface of revolution $\mathcal{M}_2^0$: the energy E and the angular momentum P which are given by

$$E = \frac{1}{2}\eta(\sigma)(\dot{\sigma}^2 + \sigma^2\dot{\vartheta}^2), \qquad P = \eta(\sigma)\sigma^2\dot{\vartheta} + \lambda\mathcal{A}_\vartheta(\sigma). \tag{2.3.81}$$

It follows from (2.3.79) and (2.3.80) that when the vortices approach from infinity, then

$$E = \pi v^2, \qquad P = 2\pi v a, \tag{2.3.82}$$

where v is the initial relative speed and a is the impact parameter which is defined here to be the half of the perpendicular distance between two vortices at large separation. From (2.3.81), we have

$$\begin{aligned} \dot{\vartheta}^2 &= \frac{1}{\sigma^4\eta(\sigma)^2}\big(P - \lambda\mathcal{A}_\vartheta(\sigma)\big)^2, & (2.3.83) \\ \dot{\sigma}^2 &= \frac{1}{\eta(\sigma)}\left(2E - \frac{1}{\sigma^2\eta(\sigma)}\big(P - \lambda\mathcal{A}_\vartheta(\sigma)\big)^2\right). & (2.3.84) \end{aligned}$$

It follows from (2.3.83) and (2.3.84) that

$$\frac{d\vartheta}{d\sigma} = \frac{2\pi}{\sigma^4\eta(\sigma)}\left(a - \frac{\lambda}{2\pi\nu}\mathcal{A}_\vartheta\right)^2\left[1 - \frac{2\pi}{\sigma^2\eta(\sigma)}\left(a - \frac{\lambda}{2\pi\nu}\mathcal{A}_\vartheta\right)^2\right]^{-1}. \tag{2.3.85}$$

The 2-vortex scattering angle is the angle between the incoming velocity and the outgoing velocity along the vortex trajectory on $\mathcal{M}_2^0$. This angle is denoted by Θ and it may be determined numerically by the integration of $d\vartheta/d\sigma$ which is practically intractable.

For large σ, it follows from (2.3.79) that the relative moduli space $\mathcal{M}_2^0$ is approximately flat. Then, for small Chern-Simons parameter κ, the RMG vortex trajectories on $\mathcal{M}_2^0$ are almost geodesics. In [34], Manton and Speight have estimated the scattering angle of 2-vortex in the Abelian Higgs model, whose motion is approximated by geodesic on $\mathcal{M}_2^0$, for large impact parameter a. Their result agreed with Samols's numerical computation of the same scattering angle [43]. Hence, for small κ, by following the same strategy in [34], we can estimate the 2-vortex scattering angle Θ of the RMG motion on $\mathcal{M}_2^0$, which is expected to be very small, for large impact parameter a, rather than calculating the integral in (2.3.85) numerically. This is as follows: first, it is convenient to re-express the Lagrangian in (2.3.77) in terms of Cartesian coordinates $x = \sigma\cos\vartheta$ and $y = \sigma\sin\vartheta$. That is,

$$L_{\text{relv}} = \frac{1}{2}\eta(\sigma)(\dot{x}^2 + \dot{y}^2) + \lambda\,\frac{\mathcal{A}_\vartheta(\sigma)}{\sigma^2}(x\dot{y} - y\dot{x}), \qquad \lambda = 2\pi\kappa, \tag{2.3.86}$$

where $\sigma = \sqrt{x^2 + y^2}$. The Euler-Lagrange equations derived from L_{relv} are

$$\eta(\sigma)\ddot{x} + \frac{\eta'(\sigma)}{2\sigma}(x\dot{x}^2 + 2y\dot{x}\dot{y} - x\dot{y}^2) \;=\; \lambda\,\frac{\mathcal{A}'_\vartheta(\sigma)}{\sigma}\dot{y}, \tag{2.3.87}$$

$$\eta(\sigma)\ddot{y} + \frac{\eta'(\sigma)}{2\sigma}(y\dot{y}^2 + 2x\dot{x}\dot{y} - y\dot{x}^2) \;=\; -\lambda\,\frac{\mathcal{A}'_\vartheta(\sigma)}{\sigma}\dot{x}. \tag{2.3.88}$$

Since for large a and small κ, vortex trajectories on $\mathcal{M}_2^0$ are almost geodesics, which are approximately straight lines on $\mathcal{M}_2^0$, we assume that the two vortices move approximately along the lines $y = a$ and $y = -a$ in opposite directions. Let vortex 1 be the one which moves along $y = a$ with x decreasing from ∞ to $-\infty$ at approximately constant speed v. That is, the initial data of vortex 1 is taken to be

$$x(-\infty) = \infty, \qquad y(-\infty) = a,$$
$$\dot{x}(-\infty) = -v, \qquad \dot{y}(-\infty) = 0. \tag{2.3.89}$$

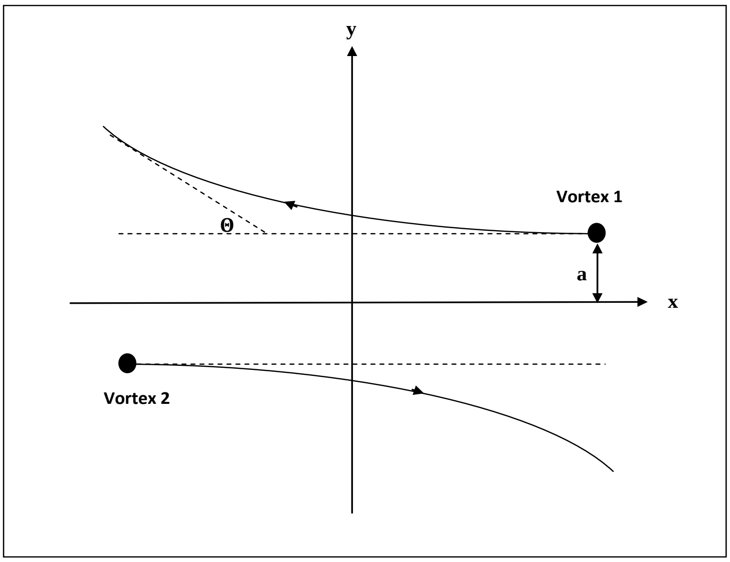


Figure 2.2: Approximated 2-vortices trajectories on $\mathcal{M}_2^0$ with large impact parameter a and small Chern-Simons parameter κ.

Hence, by calculating the total change in $\dot{y}$, $\triangle\dot{y} = \dot{y}(\infty) - \dot{y}(-\infty)$, the scattering angle Θ, which is expected to be small, is given by

$$\Theta \sim \sin\Theta = \frac{\triangle\dot{y}}{v}. \tag{2.3.90}$$

We will write the motion equations (2.3.87) and (2.3.88) up to the first order of $e^{-2\sigma}$. From (2.3.79) and (2.3.80), we have

$$\eta'(\sigma) = \frac{4q^2}{\pi} K_1(2\sigma), \tag{2.3.91}$$

$$\mathcal{A}'_\vartheta(\sigma) = \frac{2q^2}{\pi^2} \sigma K_0(2\sigma). \tag{2.3.92}$$

Both $\eta'(\sigma)$ and $\mathcal{A}'_\vartheta(\sigma)$ are of the first order of $e^{-2\sigma}$. Hence, the equations (2.3.87) and (2.3.88), up to the first order of $e^{-2\sigma}$, are

$$2\pi\ddot{x} + \frac{\eta'(\sigma)}{2\sigma} x\dot{x}^2 = 0. \tag{2.3.93}$$

$$2\pi\ddot{y} - \frac{\eta'(\sigma)}{2\sigma} y\dot{x}^2 = -\lambda \frac{\mathcal{A}'_\vartheta(\sigma)}{\sigma} \dot{x}, \tag{2.3.94}$$

where any terms proportional to $\eta'(\sigma)\dot{y}$ or $\mathcal{A}'_\vartheta(\sigma)\dot{y}$ are negligible. By differentiating the both sides of E in (2.3.81) with respect to x, one gets

$$\frac{\eta'(\sigma)}{\sigma} x(\dot{x}^2 + \dot{y}^2) = 0. \tag{2.3.95}$$

Since any terms proportional to $\eta'(\sigma)\dot{y}$ are negligible, then

$$\frac{\eta'(\sigma)}{\sigma} x\dot{x}^2 \simeq 0. \tag{2.3.96}$$

Using (2.3.96) in (2.3.93), one gets that it is sufficient to take $x = -vt$ as a solution of (2.3.93). Hence, the equation (2.3.94) becomes

$$\ddot{y} = \frac{av^2}{4\pi} \frac{\eta'(\sigma)}{\sigma} + \frac{\lambda v}{2\pi} \frac{\mathcal{A}'_\vartheta(\sigma)}{\sigma} = \frac{av^2}{4\pi} \frac{\eta'(\sigma)}{\sigma} + \kappa v \frac{\mathcal{A}'_\vartheta(\sigma)}{\sigma}. \tag{2.3.97}$$

Therefore, the total change in $\dot{y}$ is

$$\triangle\dot{y} = \int_{-\infty}^{\infty} \left(\frac{av^2}{4\pi} \frac{\eta'(\sigma)}{\sigma} + \kappa v \frac{\mathcal{A}'_\vartheta(\sigma)}{\sigma} \right) dt. \tag{2.3.98}$$

Substituting (2.3.91) and (2.3.92) in (2.3.98), one gets

$$\triangle\dot{y} = \frac{q^2 a\ v^2}{\pi^2}\int_{-\infty}^{\infty}\frac{K_1(2\sigma)}{\sigma}\ dt + \frac{2\ q^2\ \kappa\ v}{\pi^2}\int_{-\infty}^{\infty} K_0(2\sigma)\ dt. \tag{2.3.99}$$

It follows from (2.3.90) and (2.3.99) that the scattering angle is

$$\Theta = \frac{q^2 a\ v}{\pi^2}\int_{-\infty}^{\infty}\frac{K_1(2\sigma)}{\sigma}\ dt + \frac{2\ q^2\ \kappa}{\pi^2}\int_{-\infty}^{\infty} K_0(2\sigma)\ dt. \tag{2.3.100}$$

Since $dx = -v\ dt$ and $\sigma = \sqrt{x^2+a^2}$, then

$$\begin{aligned}\Theta &= -\frac{q^2 a}{\pi^2}\int_{\infty}^{-\infty}\frac{K_1(2\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}}\ dx - \frac{2\ q^2\ \kappa}{\pi^2 v}\int_{\infty}^{-\infty} K_0(2\sqrt{x^2+a^2})\ dx,\\ &= \frac{q^2 a}{\pi^2}\int_{-\infty}^{\infty}\frac{K_1(2\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}}\ dx + \frac{2\ q^2\ \kappa}{\pi^2 v}\int_{-\infty}^{\infty} K_0(2\sqrt{x^2+a^2})\ dx. \end{aligned}\tag{2.3.101}$$

From (2.3.27), we have

$$\frac{d}{da}K_0(2\sqrt{x^2+a^2}) = -2a\ \frac{K_1(2\sqrt{x^2+a^2})}{\sqrt{x^2+a^2}}. \tag{2.3.102}$$

Using (2.3.102) in (2.3.101), we get

$$\Theta = -\frac{q^2}{2\pi^2}\frac{d}{da}\left(\int_{-\infty}^{\infty} K_0(2\sqrt{x^2+a^2})\ dx\right) + \frac{2\ q^2\ \kappa}{\pi^2 v}\int_{-\infty}^{\infty} K_0(2\sqrt{x^2+a^2})\ dx. \tag{2.3.103}$$

The integral in (2.3.103) was calculated in [34],

$$\int_{-\infty}^{\infty} K_0(2\sqrt{x^2+a^2})\ dx = \frac{\pi}{2}e^{-2a}. \tag{2.3.104}$$

Hence, the 2-vortex scattering angle is

$$\Theta_\kappa(a,v) := \Theta = \frac{q^2}{2\pi}\left(1+\frac{2\kappa}{v}\right)\ e^{-2a}. \tag{2.3.105}$$

Clearly the scattering angle $\Theta_\kappa(a,v)$ is increasing as a function of κ. Our scattering angle with $\kappa = 0$ agrees with that computed in [34]. The scattering angle $\Theta_\kappa(a,v)$ must be invariant under time rescaling. This can be checked by exploiting the RMG parameter rescaling property. That is, if $\gamma(t)$ is the RMG vortex trajectory with parameter λ and initial speed v, then for real number λ_*, the trajectory $\gamma(\lambda_* t)$ is an RMG with parameter $\tilde{\lambda} = \lambda_* \lambda$ and initial speed $\tilde{v} = \lambda_* v$. Hence, it follows that $\Theta_{\lambda_* \lambda}(a, \lambda_* v) = \Theta_\lambda(a, v)$. We see from (2.3.105) that our asymptotic formula has this symmetry.

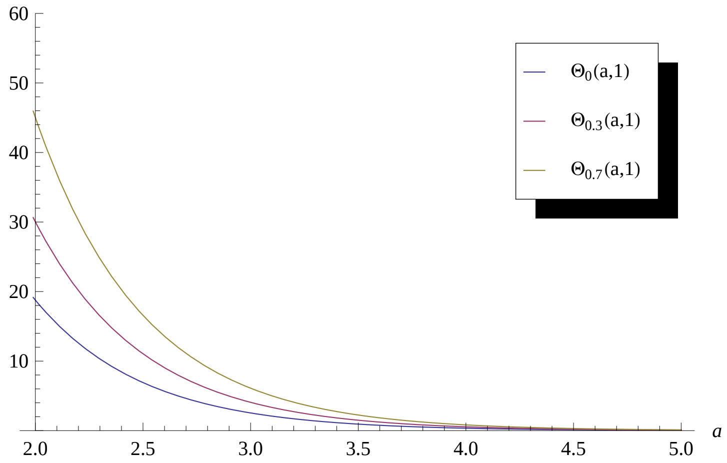


Figure 2.3: 2-vortex scattering angle for large $a \geq 2$.

2.4 Abelian Higgs Model on $\mathbb{H}^2$

Let $\mathbb{H}^2$ be the hyperbolic plane of scalar curvature -1 given by the upper half plane model, namely,

$$\mathbb{H}^2 = \{z = x + iy \in \mathbb{C} : y > 0\}, \tag{2.4.1}$$

with the Riemannian metric

$$g = \Omega^2 \, (dx^2 + dy^2), \qquad \Omega^2 = \frac{2}{y^2}. \tag{2.4.2}$$

Consider the Abelian Higgs model with a complex valued scalar field ϕ defined on $\mathbb{H}^2$ and a potential 1-form A on $\mathbb{H}^2$ whose exterior derivative is given by $dA = B \; dx \wedge dy$. Then, the total energy of the field system is

$$E[\phi] = \frac{1}{2}\int_{\mathbb{H}^2}\left(D_1\phi\overline{D_1\phi} + D_2\phi\overline{D_2\phi} + \Omega^{-2}B^2 + \frac{\Omega^2}{4}(|\phi|^2 - 1)^2\right) d^2\mathbf{x}. \tag{2.4.3}$$

As in the planar case, the finite energy requirement imposes boundary conditions on the field (ϕ, A) at spatial infinity. These boundary conditions are given as in (2.2.9). Let ϕ have winding number n, then the total energy $E[\phi]$ is [35, p.197]

$$\begin{aligned} E[\phi] &= \frac{1}{2}\int_{\mathbb{H}^2}\left((D_1 \pm iD_2)\phi\,\overline{(D_1 \pm iD_2)\phi} + \Omega^{-2}(B \pm \frac{\Omega^2}{2}(|\phi|^2 - 1))^2 \pm B\right) d^2\mathbf{x}, \\ &\geq \frac{1}{2}\left|\int_{\mathbb{H}^2} B\; d^2\mathbf{x}\right| = \pi\,|n|\,. \end{aligned} \tag{2.4.4}$$

Here, we have used the finite energy boundary conditions to compute the right hand side as in (2.2.10). The equality in (2.4.4) occurs when

$$\begin{aligned} (D_1 \pm iD_2)\phi &= 0, \\ B \pm \frac{\Omega^2}{2}(|\phi|^2 - 1) &= 0, \end{aligned} \tag{2.4.5}$$

hold. These are the Bogomolny equations of the model and their solutions are called, here, hyperbolic n-vortices for $n > 0$ (hyperbolic n-antivortices for $n < 0$). Setting $\phi = e^{h/2+i\chi}$, then the Bogomolny equations (2.4.5) imply [35, p.199]

$$\nabla^2 h + \Omega^2 - \Omega^2 e^h = 0, \tag{2.4.6}$$

which is valid away from the zeros of ϕ. The above reduced Bogomolny equation has been studied by Strachan [51], following Witten [57] who found that this equation can be written as

$$\nabla^2 \mathrm{g} - e^{2\mathrm{g}} = 0, \tag{2.4.7}$$

where $h = 2\mathrm{g} + \log(1/\Omega^2)$. This is Liouville's equation whose general solution is

$$\mathrm{g} = \log\left(2\,\frac{|df/dz|}{1-|f(z)|^2}\right), \tag{2.4.8}$$

where $f(z)$ is an arbitrary complex valued meromorphic function on $\mathbb{H}^2$. Hence, with a simple choice of the phase χ, the field ϕ which satisfies the Bogomolony equations is

$$\phi = \frac{2}{\Omega}\left(\frac{df/dz}{1-|f|^2}\right). \tag{2.4.9}$$

The function $f(z)$ must be chosen such that ϕ is not singular for all $z \in \mathbb{H}^2$ and it satisfies the finite energy boundary conditions (2.2.9) with winding number n. In fact, it has been deduced in [51] that $f(z)$ has to be of the form

$$f(z) = \left(\frac{z-i}{z+i}\right)\prod_{r=1}^{n}\left[\left(\frac{z-b_r}{z-\bar{b}_r}\right)\left(\frac{\bar{b}_r-i}{b_r+i}\right)\right], \qquad b_r \in \mathbb{H}^2. \tag{2.4.10}$$

Let $\{z_r : r = 1,\ldots,n\}$ be the zeros of the field ϕ on $\mathbb{H}^2$, then it follows from (2.4.9) that these zeros are uniquely related to the parameters $\{b_r : r = 1,\ldots,n\}$ by

$$\left.\frac{df}{dz}\right|_{z=z_r} = 0, \qquad r = 1,\ldots,n. \tag{2.4.11}$$

It can be seen from (2.4.10) that the solution of the Bogomolny equations are in one-to-one correspondence with the parameters $\{b_r : r = 1,\ldots,n\}$ which are specified uniquely by ϕ's zeros $\{z_r : r = 1,\ldots,n\}$, interpreted as hyperbolic vortex positions on $\mathbb{H}^2$. This confirms Taubes's argument in [54], and so the hyperbolic n-vortex moduli space is the n-dimensional complex manifold $\mathcal{M}_n(\mathbb{H}^2) = (\mathbb{H}^2)^n/S_n$, where S_n is the permutation group of n objects. In terms of $\{z_r : r = 1,\ldots,n\}$, as complex local coordinates on $\mathcal{M}_n(\mathbb{H}^2)$, assuming all vortex positions are distinct, a general formula for the L^2 metric on $\mathcal{M}_n(\mathbb{H}^2)$ has been given, first, by Strachan [51], and then it has been modified to be in the structure of Samols's formula, that is,

$$\gamma_{L^2} = \pi\sum_{r,s=1}^{n}\left(\Omega^2\delta_{rs} + 2\frac{\partial b_s}{\partial z_r}\right)\, dz_r\, d\bar{z}_s, \tag{2.4.12}$$

which is again Kähler. Here, since the solutions of the Bogomolny equations are known, one wishes to determine the functions b_s by solving (2.4.11) and expanding $h = \log|\phi|^2$ around z_s. For $n = 1$, this can be easily calculated and the L^2 metric on $\mathcal{M}_1(\mathbb{H}^2)$ is

$$\gamma_{L^2} = \frac{3\pi}{y^2}\,(dx^2 + dy^2). \tag{2.4.13}$$

But, for $n \geq 2$, this is practically intractable. Therefore, our aim in the next section is to determine an explicit formula for the L^2 metric on the hyperbolic 2-vortex moduli space $\mathcal{M}_2(\mathbb{H}^2)$, assuming all $\{z_r : r = 1, \ldots, n\}$ are distinct, by exploiting the isometric properties of the physical space $(\mathbb{H}^2, g)$.

2.4.1 The L^2 Metric on the Hyperbolic 2-Vortex Moduli Space

Let $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ denote double cover of the space of all 2-vortices with distinct positions on $\mathbb{H}^2$. Namely, $\tilde{\mathcal{M}}_2(\mathbb{H}^2) = (\mathbb{H}^2 \times \mathbb{H}^2)\backslash\triangle_2$ where $\triangle_2$ is the vortex coincidence set. Let $\tilde{\gamma}_{L^2}$ be the pullback of the L^2 metric γ_{L^2} on $\tilde{M}_2(\mathbb{H}^2)\backslash\triangle_2$ by the covering map. We call $\tilde{\gamma}_{L^2}$, the L^2 metric on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$.

The hyperbolic plane $(\mathbb{H}^2, g)$ has an isometric action of the projective linear group $PL(2, \mathbb{R})$ given by

$$z \to \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot z =: M \odot z, \tag{2.4.14}$$

where $[M] \in PL(2, \mathbb{R})$. This builds an isometric action on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, with respect to the L^2 metric $\tilde{\gamma}_{L^2}$, defined as

$$(z_1, z_2) \to (M \odot z_1, M \odot z_2). \tag{2.4.15}$$

For a generic element on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, the isotropy group of $PL(2, \mathbb{R})$ action, defined in (2.4.15), is trivial. By the Orbit-Stabilizer Theorem [2, p.94], it follows that each generic orbit is diffeomorphic to $PL(2, \mathbb{R})$ itself. Hence, the isometric action of

$PL(2,\mathbb{R})$ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ has cohomogeneity 1, that is, all generic orbits of this action are submanifolds of $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ with real codimension 1. Let s denote the distance between two vortices in $\mathbb{H}^2$. Then one can see that each orbit has a unique element $w_s = (ie^{s/2}, ie^{-s/2}) \in \tilde{\mathcal{M}}_2(\mathbb{H}^2)$. Thus, this action decomposes $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ into a one parameter family of orbits parameterized by $s \geq 0$. Note that there is an exceptional orbit of codimension 2, when $s = 0$.

Consider the coframe $\{ds, \sigma_k : k = 1,2,3\}$ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ where σ_k are the left-invariant 1-forms dual to the basis $\{e_k : k = 1,2,3\}$ of $T_{[\mathbb{I}_2]}PL(2,\mathbb{R})$ given by

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4.16}$$

Any $PL(2,\mathbb{R})$-invariant metric $\tilde{\gamma}$ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ is determined by a one-parameter family of symmetric bilinear forms $\tilde{\gamma}_s : V_s \times V_s \to \mathbb{R}$ where $V_s := \partial/\partial s \oplus T_{[\mathbb{I}_2]}PL(2,\mathbb{R})$ is the tangent space to $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ at the element w_s.

In terms of the complex coordinate system (z_1, z_2) on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, one finds that

$$\begin{aligned} e_1 &= (1+e^s)\frac{\partial}{\partial x_1} + (1+e^{-s})\frac{\partial}{\partial x_2}, & e_2 &= (1-e^s)\frac{\partial}{\partial x_1} + (1-e^{-s})\frac{\partial}{\partial x_2}, \\ e_3 &= 2\,e^{s/2}\frac{\partial}{\partial y_1} + 2\,e^{-s/2}\frac{\partial}{\partial y_2}, & \frac{\partial}{\partial s} &= \frac{1}{2}\,e^{s/2}\frac{\partial}{\partial y_1} - \frac{1}{2}\,e^{-s/2}\frac{\partial}{\partial y_2}. \end{aligned} \tag{2.4.17}$$

The action of the almost complex structure J on the basis $\{\partial/\partial s, e_k : k = 1,2,3\}$ of V_s can be determined as follows:

$$\begin{aligned} Je_1 &= (1+e^s)J\frac{\partial}{\partial x_1} + (1+e^{-s})J\frac{\partial}{\partial x_2} = (1+e^s)\frac{\partial}{\partial y_1} + (1+e^{-s})\frac{\partial}{\partial y_2}, \\ Je_2 &= (1-e^s)J\frac{\partial}{\partial x_1} + (1-e^{-s})J\frac{\partial}{\partial x_2} = (1-e^s)\frac{\partial}{\partial y_1} + (1-e^{-s})\frac{\partial}{\partial y_2}. \end{aligned} \tag{2.4.18}$$

It follows from (2.4.17) that

$$\frac{\partial}{\partial y_1} = e^{-s/2}\left(\frac{e_3}{4} + \frac{\partial}{\partial s}\right), \qquad \frac{\partial}{\partial y_2} = e^{s/2}\left(\frac{e_3}{4} - \frac{\partial}{\partial s}\right). \tag{2.4.19}$$

Substituting (2.4.19) in (2.4.18), one gets

$$Je_1 = \cosh(s/2)\ e_3, \qquad Je_2 = -4\sinh(s/2)\ \frac{\partial}{\partial s}. \tag{2.4.20}$$

Since $J^2 = -1$, then we have

$$Je_3 = -\frac{1}{\cosh(s/2)}\ e_1, \qquad J\frac{\partial}{\partial s} = \frac{1}{4\sinh(s/2)}\ e_2. \tag{2.4.21}$$

In addition to the $PL(2,\mathbb{R})$ isometric action on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, there is a discrete isometry on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ defined as P : $(z_1, z_2) \to (z_2, z_1)$. Hence, the group $G := PL(2,\mathbb{R}) \times \{\text{Id},\ \text{P}\}$, where Id is the identity map, acts isometrically on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$.

Lemma 2.4.1. *Let $\tilde{\gamma}$ be a G-invariant Kähler metric on the hyperbolic 2-vortex moduli space $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$. Then, there exists a smooth function $A : (0,\infty) \to \mathbb{R}$ such that*

$$\gamma = A_1(s)\ ds^2 + A_2(s)\ \sigma_1^2 + A_3(s)\ \sigma_2^2 + A_4(s)\ \sigma_3^2, \tag{2.4.22}$$

where $A_1(s), \ldots, A_4(s)$ are related to $A(s)$ by

$$
\begin{aligned}
A_1 &= \frac{1}{8\sinh(s/2)}\frac{d}{ds}\left(\frac{A(s)}{\cosh(s/2)}\right), & A_2 &= A(s), \\
A_3 &= 2\sinh(s/2)\frac{d}{ds}\left(\frac{A(s)}{\cosh(s/2)}\right), & A_4 &= \frac{A(s)}{\cosh^2(s/2)}.
\end{aligned} \tag{2.4.23}
$$

Proof: With respect to the coframe $\{ds, \sigma_k : k = 1,2,3\}$ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, any $PL(2,\mathbb{R})$ invariant symmetric $(0,2)$ tensor has the form

$$
\begin{aligned}
\tilde{\gamma} &= A_1\, ds^2 + A_2\, \sigma_1^2 + A_3\, \sigma_2^2 + A_4\, \sigma_3^2 + 2A_5\, ds\, \sigma_1 + 2A_6\, ds\, \sigma_2 + 2A_7\, ds\, \sigma_3 \\
&\quad + 2A_8\, \sigma_1\sigma_2 + 2A_9\, \sigma_1\sigma_3 + 2A_{10}\, \sigma_2\sigma_3,
\end{aligned} \tag{2.4.24}
$$

where $A_1, \ldots, A_{10}$ are smooth functions of s only.

Now, since $\tilde{\gamma}$ is Hermitian, then $\tilde{\gamma}_s(u,v) = \tilde{\gamma}_s(Ju, Jv)$ for all $u, v \in V_s$. This implies a set of relations between the coefficients $A_1, \ldots, A_{10}$. In fact, we obtain that

$$
\begin{aligned}
&A_3 \equiv 16\sinh^2(s/2)A_1, && A_4 \equiv \frac{A_2}{\cosh^2(s/2)}, && A_9 \equiv A_6 \equiv 0, \\
&A_8 \equiv -4\sinh(s/2)\cosh(s/2)A_7, && A_{10} \equiv 4\tanh(s/2)A_5. &&
\end{aligned} \tag{2.4.25}
$$

Hence, any $PL(2,\mathbb{R})$ invariant Hermitian metric on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ of the form (2.4.24) is determined by four functions A_1, A_2, A_5 and A_7.

The associated 2-form $\omega(.,.) = \tilde{\gamma}(J.,.)$ of a Hermitian metric $\tilde{\gamma}$ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, given as in (2.4.24), is

$$\omega = -\cosh(s/2)A_7\ ds \wedge \sigma_1 + 4\sinh(s/2)A_1\ ds \wedge \sigma_2 + \frac{A_5}{\cosh(s/2)}\ ds \wedge \sigma_3$$
$$+ 4\sinh(s/2)A_5\ \sigma_1 \wedge \sigma_2 + \frac{A_2}{\cosh(s/2)}\ \sigma_1 \wedge \sigma_3 - 4\sinh(s/2)A_7\ \sigma_2 \wedge \sigma_3. \quad (2.4.26)$$

The exterior derivative of ω is a 3-form on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ given by

$$d\omega = \cosh(s/2)A_7\ ds \wedge d\sigma_1 - 4\sinh(s/2)A_1\ ds \wedge d\sigma_2 - \frac{A_5}{\cosh(s/2)}\ ds \wedge d\sigma_3$$
$$+ \frac{d}{ds}\left(4\sinh(s/2)A_5\right)\ ds \wedge \sigma_1 \wedge \sigma_2 + \frac{d}{ds}\left(\frac{A_2}{\cosh(s/2)}\right)\ ds \wedge \sigma_1 \wedge \sigma_3$$
$$- \frac{d}{ds}\left(4\sinh(s/2)A_7\right)\ ds \wedge \sigma_2 \wedge \sigma_3. \quad (2.4.27)$$

The exterior derivatives of the 1-forms $\{\sigma_k, k = 1, 2, 3\}$ can be determined as follows: for any vector fields X, Y on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, we have

$$d\sigma_k(X, Y) = X[\sigma_k(Y)] - Y[\sigma_k(X)] - \sigma_k([X, Y]). \quad (2.4.28)$$

Let $\{X_k : k = 1, 2, 3\}$ be the left-invariant vector fields on $PL(2, \mathbb{R})$ such that $X_k([\mathbb{I}_2]) = e_k$. Then,

$$[X_i, X_j]\Big|_{[\mathbb{I}_2]} = [e_i, e_j]_{\mathfrak{p}}, \quad (2.4.29)$$

where $[.,.]_{\mathfrak{p}}$ is the Lie algebra bracket on $\mathfrak{p} := T_{[\mathbb{I}_2]}PL(2, \mathbb{R})$. From the definition of the Lie algebra bracket on matrix group, we get

$$[e_1, e_2]_{\mathfrak{p}} = -2\ e_3, \qquad [e_2, e_3]_{\mathfrak{p}} = -2\ e_1, \qquad [e_1, e_3]_{\mathfrak{p}} = -2\ e_2, \quad (2.4.30)$$

and $[e_i, e_i]_{\mathfrak{p}} = 0$. This implies that for all $i < j$,

$$\sigma_k([X_i, X_j]\Big|_{w_s}) = \sigma_k([e_i, e_j]_{\mathfrak{p}}) = \begin{cases} -2 & i \neq j \neq k \\ 0 & i = k \text{ or } j = k. \end{cases} \quad (2.4.31)$$

Hence, it follows from (2.4.28) and (2.4.31) that

$$d\sigma_1 = 2\,\sigma_2 \wedge \sigma_3, \qquad d\sigma_2 = 2\,\sigma_1 \wedge \sigma_3, \qquad d\sigma_3 = 2\,\sigma_1 \wedge \sigma_2. \tag{2.4.32}$$

Substituting (2.4.32) in (2.4.27), we obtain that

$$\begin{aligned} d\omega = {} & 2\left[\cosh(s/2)A_7 - 2\frac{d}{ds}\Big(\sinh(s/2)A_7\Big)\right] ds \wedge \sigma_2 \wedge \sigma_3 \\ & - 2\left[\frac{A_5}{\cosh(s/2)} - 2\frac{d}{ds}\Big(\sinh(s/2)A_5\Big)\right] ds \wedge \sigma_1 \wedge \sigma_2 \\ & - \left[8\sinh(s/2)A_1 - \frac{d}{ds}\left(\frac{A_2}{\cosh(s/2)}\right)\right] ds \wedge \sigma_2 \wedge \sigma_3. \end{aligned} \tag{2.4.33}$$

Since $\tilde{\gamma}$ is Kähler, then the associated 2-form ω must be closed, $d\omega = 0$. Hence, it follows from (2.4.33) that ω is closed if and only if the following three equations

$$\frac{d}{ds}\Big(\sinh(s/2)A_7\Big) - \frac{1}{2}\cosh(s/2)A_7 = 0, \tag{2.4.34}$$

$$\frac{d}{ds}\Big(\sinh(s/2)A_5\Big) - \frac{1}{2}\frac{A_5}{\cosh(s/2)} = 0, \tag{2.4.35}$$

$$\frac{d}{ds}\left(\frac{A_2}{\cosh(s/2)}\right) - 8\sinh(s/2)A_1 = 0, \tag{2.4.36}$$

hold. The general solutions of equations (2.4.34) and (2.4.35) are

$$A_7 = c_1, \qquad A_5 = \frac{c_2}{\cosh(s/2)}, \tag{2.4.37}$$

where c_1 and c_2 are constants. Since $\tilde{\gamma}$ is also invariant under the discrete isometry P : $s \to -s$, then this requires that both A_5 and A_7 are odd functions. It follows from (2.4.37) that c_1 and c_2 must be zero, and so $A_5 = A_7 = 0$. Thus, the associated 2-form ω, the Kähler form, on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ is of the form

$$\omega = 4\sinh(s/2)A_1\, ds \wedge \sigma_2 + \frac{A_2}{\cosh(s/2)}\,\sigma_1 \wedge \sigma_3. \tag{2.4.38}$$

Hence, any G-invariant Kähler metric $\tilde{\gamma}$ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ is of the form (2.4.22) and its coefficients are related by

$$A_1 = \frac{1}{8\sinh(s/2)}\frac{d}{ds}\left(\frac{A_2}{\cosh(s/2)}\right), \tag{2.4.39}$$

$$A_3 = 16\sinh^2(s/2)A_1, \tag{2.4.40}$$

$$A_4 = \frac{A_2}{\cosh^2(s/2)}. \tag{2.4.41}$$

Clearly, these coefficients are defined by a single function $A(s) := A_2$, and hence the claim is established.

□

Now, consider the isometry of $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ given by

$$\tilde{K} : (z_1, z_2) \to \mathrm{P}(M \odot z_1, M \odot z_2) = (-\frac{1}{z_2}, -\frac{1}{z_1}), \quad M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4.42}$$

The fixed point set in $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ under this isometry is

$$\tilde{\mathcal{M}}_2^0(\mathbb{H}^2) = \{(\xi, -\frac{1}{\xi}) : i \neq \xi \in \mathbb{H}^2\} \subset \tilde{\mathcal{M}}_2(\mathbb{H}^2), \tag{2.4.43}$$

which is the so-called hyperbolic 2-vortex relative moduli space. Clearly, $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ is a non-compact 1-dimensional complex submanifold of $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$. The induced metric of any G-invariant Kähler metric $\tilde{\gamma}$, determined as in (2.4.22), to the hyperbolic 2-vortex relative moduli space $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ is

$$\tilde{g} = A_1(s)\, ds^2 + A_3(s)\, \sigma_2^2 = A_1(s)\, (ds^2 + 16\sinh^2(s/2)\sigma_2^2). \tag{2.4.44}$$

For the L^2 metric, an explicit formula for the induced metric $\tilde{g}$ on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ has been determined by Strachan in [51], where $\mathbb{H}^2$ is given in terms of the Poincaré disk model.

With respect to the coordinate transformation, one can extract the function $A_1(s)$ for the L^2 metric by comparing the formula of Strachan's metric on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ in [51] and (2.4.44). In fact,

$$\begin{aligned} A_1(s) &= 2\pi \frac{\tanh^2(s/4)}{(1+\tanh^2(s/4))^2}\left[1+\frac{4(1+\tanh^4(s/4))}{\sqrt{1+\tanh^8(s/4)+14\tanh^4(s/4)}}\right], \\ &= 2\pi \frac{\tanh^2(s/4)}{(1+\tanh^2(s/4))^2}\left[1+\frac{4(\coth^2(s/4)+\tanh^2(s/4))}{\sqrt{[\coth^2(s/4)-\tanh^2(s/4)]^2+16}}\right]. \end{aligned} \tag{2.4.45}$$

From the half angle identities of the hyperbolic functions, we have

$$\begin{aligned} \frac{\tanh^2(s/4)}{(1+\tanh^2(s/4))^2} &= \frac{1}{4}\tanh^2(s/2), \\ \coth^2(s/4)-\tanh^2(s/4) &= 4\,\frac{\cosh(s/2)}{\sinh^2(s/2)}, \\ \coth^2(s/4)+\tanh^2(s/4) &= 2\,\frac{(\cosh^2(s/2)+1)}{\sinh^2(s/2)}. \end{aligned} \tag{2.4.46}$$

Using (2.4.46) in (2.4.45), the function $A_1(s)$ becomes

$$A_1(s) = \frac{\pi}{2}\tanh^2(s/2)\left[1+\frac{2(\cosh^2(s/2)+1)/\sinh^2(s/2)}{\sqrt{[\cosh(s/2)/\sinh^2(s/2)]^2+1}}\right]. \tag{2.4.47}$$

Since the functions $A_1(s)$ and $A(s)$ are related by

$$\frac{d}{ds}\left(\frac{A(s)}{\cosh(s/2)}\right) = 8\sinh(s/2)\;A_1(s), \tag{2.4.48}$$

it follows from (2.4.47) that

$$\frac{d}{ds}\left(\frac{A(s)}{\cosh(s/2)}\right) = 4\pi\sinh(s/2)\tanh^2(s/2)\left[1+\frac{2(\cosh^2(s/2)+1)/\sinh^2(s/2)}{\sqrt{[\cosh(s/2)/\sinh^2(s/2)]^2+1}}\right]. \tag{2.4.49}$$

Let $\beta(s) = \cosh(s/2)/\sinh^2(s/2)$, then one can write (2.4.49) as

$$\frac{d}{ds}\left(\frac{A(s)}{\cosh(s/2)}\right) = 2\pi\sinh(s/2)\tanh^2(s/2) - 8\pi\,\frac{\beta'(s)}{\beta^2\sqrt{\beta^2+1}}. \tag{2.4.50}$$

By integrating (2.4.50), we obtain that

$$A(s) = 8\pi\Big[(\cosh^2(s/2)+1)+2\sinh^2(s/2)\sqrt{[\cosh(s/2)/\sinh^2(s/2)]^2+1}\,\Big]+c\cosh(s/2), \tag{2.4.51}$$

where c is an integration constant. When two vortices approach each other, that is, $z_1, z_2 \to z = x + iy$ on $\mathbb{H}^2$, then, the L^2 metric on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ is asymptotically given by

$$\gamma_{L^2} \approx (8\pi + \frac{c}{2})\,\frac{1}{y^2}(dx^2+dy^2), \qquad \text{as } s \to 0. \tag{2.4.52}$$

In fact, Strachan [51] has given an explicit formula for the L^2 metric on $\mathcal{M}_n(\mathbb{H}^2)$ when vortices exactly coincide at one position,

$$\gamma_{L^2} = \pi n(n+2)\,\frac{1}{y^2}(dx^2+dy^2), \qquad s = 0. \tag{2.4.53}$$

Inspecting the formulae for the coefficients A_i, we see that the L^2 metric is continuous at $s = 0$, hence by comparing (2.4.52) with (2.4.53) in the case $n = 2$, it follows that the constant c must be zero. Hence, the L^2 metric on the moduli space $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ has the form (2.4.22) and its coefficients are determined as in (2.4.23) by a single function, call it $A_{L^2}(s)$, given by

$$A_{L^2}(s) = 8\pi\Big[(\cosh^2(s/2)+1)+2\sinh^2(s/2)\sqrt{[\cosh(s/2)/\sinh^2(s/2)]^2+1}\,\Big]. \tag{2.4.54}$$

Proposition 2.4.2. *Let $\tilde{\gamma}$ be a G-invariant Kähler metric on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, determined as in Lemma 2.4.1. Then, the Ricci curvature tensor with respect to $\tilde{\gamma}$ is given by,*

$$\mathit{Ric} = C_1(s)\, ds^2 + C_2(s)\, \sigma_1^2 + C_3(s)\, \sigma_2^2 + C_4(s)\, \sigma_3^2, \tag{2.4.55}$$

where $C_1, \ldots, C_4$ are smooth functions of $s \in (0, \infty)$, defined as in (2.4.23), by a single function $C(s)$ given by

$$C(s) = -4\sinh(s/2)\cosh(s/2)\frac{d}{ds}\log\left(\frac{A_1 A_2}{\cosh^2(s/2)}\right) - 8\cosh^2(s/2). \tag{2.4.56}$$

Proof: The Ricci curvature tensor with respect to $\tilde{\gamma}$ is a G-invariant symmetric $(0,2)$ tensor on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ which is Hermitian and whose associated 2-form $\rho(.,.) = \mathrm{Ric}(J.,.)$, Ricci form, is closed. Thus, it is covered by Lemma 2.4.1, that is, Ric has the same structure as $\tilde{\gamma}$ and its coefficients $C_1, \ldots, C_4$ are related, as in (2.4.23), to the function $C(s) := C_2(s) = \mathrm{Ric}_s(e_1, e_1)$. Introducing an orthonormal basis $\{E_k : k = 0, 1, 2, 3\}$ of (V_s, γ_s) as

$$E_0 = \frac{1}{\sqrt{A_1}}\frac{\partial}{\partial s}, \quad E_1 = \frac{1}{\sqrt{A_2}}\, e_1, \quad E_2 = \frac{1}{\sqrt{A_3}}\, e_2, \quad E_3 = \frac{1}{\sqrt{A_4}}\, e_3, \tag{2.4.57}$$

then, by the definition of the Ricci curvature tensor, we obtain that

$$\begin{aligned} \mathrm{Ric}_s(e_1, e_1) &= \sum_{i=0}^{3} \gamma(R(E_i, e_1)e_1, E_i), \\ &= -4\sinh(s/2)\cosh(s/2)\frac{d}{ds}\log\left(\frac{A_1 A_2}{\cosh^2(s/2)}\right) - 8\cosh^2(s/2), \end{aligned} \tag{2.4.58}$$

where R is the Riemannian curvature tensor associated with $\tilde{\gamma}$. Hence, the claim is proved. □

The Ricci form ρ associated with the Ricci curvature tensor, given in (2.4.55), has the same structure as the Kähler form ω associated with $\tilde{\gamma}$ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$. That is,

$$\rho = 4\sinh(s/2)C_1\, ds \wedge \sigma_2 + \frac{C_2}{\cosh(s/2)}\, \sigma_1 \wedge \sigma_3. \tag{2.4.59}$$

The function $C_1(s)$ is related to $C(s)$ by

$$C_1(s) = \frac{1}{8\sinh(s/2)}\frac{d}{ds}\left(\frac{C(s)}{\cosh(s/2)}\right). \tag{2.4.60}$$

Substituting (2.4.56) in (2.4.60), we obtain that

$$C_1(s) = -\frac{1}{2}\frac{d^2}{ds^2}\log\left(\frac{A_1A_2}{\cosh^2(s/2)}\right) - \frac{1}{4}\coth(s/2)\frac{d}{ds}\log\left(\frac{A_1A_2}{\cosh^2(s/2)}\right) - \frac{1}{2}. \tag{2.4.61}$$

Clearly, the Ricci form ρ is exact on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ and its potential 1-form is

$$a = \frac{1}{2}\,\frac{C(s)}{\cosh(s/2)}\,\sigma_2. \tag{2.4.62}$$

2.4.2 RMG Flow on the Hyperbolic 2-Vortex Relative Moduli Space

The relative moduli space $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ is the fixed point set in $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ under the holomorphic isometry $\tilde{K}$, defined in (2.4.42). Hence, by Corollary 1.3.3, RMG curves with initial data in $T\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ remain on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ for all time. So RMG flow localizes to $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$. However, the restriction of the Ricci form ρ on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ to $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ does not coincide with $\tilde{\rho}$, the Ricci form on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ defined by its induced metric $\tilde{g}_{L^2}$. Hence, the RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$, thought of as a submanifold of $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$, does not coincide with the RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$, thought of as a Kähler manifold in its own right. In this section, we will compare these flows on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$, which we call the extrinsic and intrinsic RMG flow, respectively.

It follows from (2.4.59) that the Ricci form on $\tilde{\mathcal{M}}_2(\mathbb{H}^2)$ restricted to $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ is

$$
\begin{aligned}
\rho|_{\mathcal{M}_2^0} &= 4\sinh(s/2)C_1(s)\ ds \wedge \sigma_2, \\
&= \tilde{\rho}\ + 4\sinh(s/2)\left[-\frac{1}{2}\frac{d^2}{ds^2}\log\left(\frac{A_2}{\cosh^2(s/2)}\right)\right. \\
&\quad \left. -\frac{1}{4}\coth(s/2)\frac{d}{ds}\log\left(\frac{A_2}{\cosh^2(s/2)}\right)-\frac{1}{4}\right] ds \wedge \sigma_2,
\end{aligned} \tag{2.4.63}
$$

where $\tilde{\rho}$ is the Ricci form on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$, defined by its induced metric $\tilde{g}_{L^2}$, given by

$$
\tilde{\rho} = \frac{\tilde{\kappa}(s)}{2}\ \tilde{\omega} = 2\ \tilde{\kappa}(s)\sinh(s/2)A_1(s)\ ds \wedge \sigma_2. \tag{2.4.64}
$$

Here, $\tilde{\kappa}(s)$ is the scalar curvature of $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ given by

$$
\tilde{\kappa}(s) = \frac{2}{A_1}\left[-\frac{1}{2}\frac{d^2}{ds^2}\log A_1 - \frac{1}{4}\coth(s/2)\frac{d}{ds}\log A_1 - \frac{1}{4}\right] =: \frac{2}{A_1}\tilde{C}_1(s). \tag{2.4.65}
$$

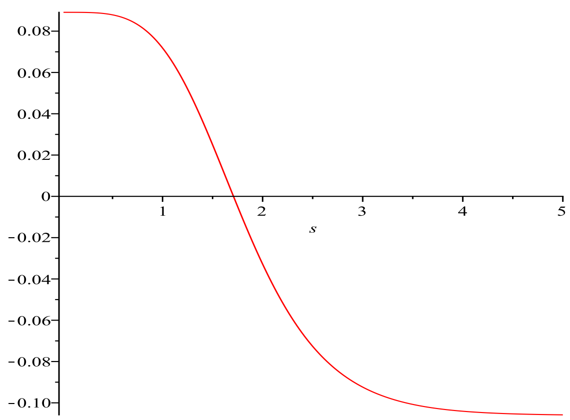


Figure 2.4: Plot of the scalar curvature $\tilde{\kappa}(s)$ of $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$

.

The intrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ was studied numerically in [27] where it was supposed to be coincided with the extrinsic one. Clearly, this is not true as from (2.4.64) and (2.4.65), one can see that the forms $\rho|_{\tilde{\mathcal{M}}_2^0}$ and $\tilde{\rho}$ are different from each other. Asymptotically, they behave like

$$\rho|_{\tilde{\mathcal{M}}_2^0} \sim -\frac{1}{5}\, s^3\, ds \wedge \sigma_2, \qquad \tilde{\rho} \sim \frac{7}{40}\, s^3\, ds \wedge \sigma_2, \qquad \text{as } s \to 0. \tag{2.4.66}$$

$$\rho|_{\tilde{\mathcal{M}}_2^0} \sim -\, e^{s/2}\, ds \wedge \sigma_2, \qquad \tilde{\rho} \sim -\frac{1}{2}\, e^{s/2}\, ds \wedge \sigma_2, \quad \text{as } s \to \infty. \tag{2.4.67}$$

(i) From (2.4.67), one can see that even as $s \to \infty$, the extrinsic and intrinsic RMG flows do not coincide, see Figure 2.6 (d). Comparing $\rho|_{\tilde{\mathcal{M}}_2^0}$ with $\tilde{\rho}$ in (2.4.67), one expects that the extrinsic and intrinsic RMG flows coincide for large s if the RMG parameters in both are related by

$$\lambda_{\text{extrinsic}} = \frac{1}{2}\lambda_{\text{intrinsic}}. \tag{2.4.68}$$

(ii) In the core region, the extrinsic and intrinsic RMG flows have qualitative differences in curves, see Figure 2.6 (a).

The RMG equations on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ are given, in terms of the coordinate system (s, ψ) where $d\psi = \sigma_2$, by

$$\begin{aligned} \ddot{s} &= -\frac{1}{2A_1(s)}\big[A_1'(s)\dot{s}^2 - A_3'(s)\dot{\psi}^2 + 8\lambda \sinh(s/2) C_1(s)\dot{\psi}\big], \\ \ddot{\psi} &= -\frac{1}{A_3(s)}\big[A_3'(s)\dot{s}\;\dot{\psi} - 4\lambda \sinh(s/2) C_1(s)\dot{s}\big]. \end{aligned} \tag{2.4.69}$$

These are the extrinsic RMG equations. Replacing $C_1(s)$ in (2.4.69) by $\tilde{C}_1(s)$, we get the equations of the intrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$.

(iii) Clearly, the scalar curvature $\tilde{\kappa}(s)$ changes its sign from positive to negative as s moves from 0 to ∞, Figure 2.4. Unlike extrinsic RMG flow, this makes a significant effect in the behaviour of the intrinsic RMG curve in which it has an inflexion point where the curve changes from having positive geodesic curvature to negative geodesic curvature. Recall that the RMG equation of $\gamma(t)$ on a 2-dimensional Kähler manifold such as $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ is $\nabla^{\gamma}_{d/dt}\dot{\gamma} = \lambda\tilde{\kappa}(s)J\dot{\gamma}/2$. Particularly, this can be seen in a certain curve on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ whose radial velocity is zero. This is the so-called parallel on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$. It follows from the RMG equations (2.4.69) that a parallel on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ is extrinsic RMG if and only if its angular velocity satisfies

$$\dot{\psi} = \frac{\lambda}{2}\Big(\sinh(s/2)C_1(s)\Big)\Big/\frac{d}{ds}\Big(\sinh^2(s/2)A_1(s)\Big), \tag{2.4.70}$$

whereas the angular velocity, call it $\tilde{\dot{\psi}}$, of the intrinsic RMG parallel is

$$\begin{aligned}\tilde{\dot{\psi}} &= \frac{\lambda}{2}\Big(\sinh(s/2)\tilde{C}_1(s)\Big)\Big/\frac{d}{ds}\Big(\sinh^2(s/2)A_1(s)\Big),\\ &= \frac{\lambda}{4}\,\tilde{\kappa}(s)\sinh(s/2)A_1\Big/\frac{d}{ds}\Big(\sinh^2(s/2)A_1(s)\Big).\end{aligned} \tag{2.4.71}$$

The graphs of $\dot{\psi}$ and $\tilde{\dot{\psi}}$ as functions of s are drawn in Figure 2.5. Clearly, there is s_0 in $(0,\infty)$ where the angular velocity $\tilde{\dot{\psi}}$ changes its sign from positive, as it started, to negative. This behaviour is expected from appearing the scalar curvature $\tilde{\kappa}(s)$ in (2.4.71). Unlike $\tilde{\dot{\psi}}$, the angular velocity $\dot{\psi}$ keeps being negative for all $s > 0$. Hence, the extrinsic RMG parallels on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ keep spinning in the same direction they started with whereas the spin direction of the intrinsic RMG parallels is reversed at $s = s_0$.

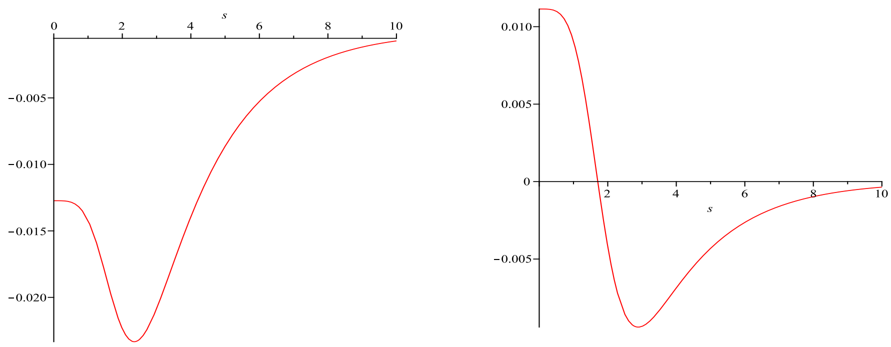


Figure 2.5: The angular velocities of the extrinsic RMG parallel on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ (left) and the intrinsic RMG parallel (right).

Numerically, we have solved the equations (2.4.69) for the extrinsic and intrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ with various initial values. The corresponding RMG trajectories of one of the vortices on the Poincaré disk are depicted in Figure 2.6. Note that inflexion points in $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ do not necessarily coincide with inflexion points of vortex trajectories in $\mathbb{H}^2$.

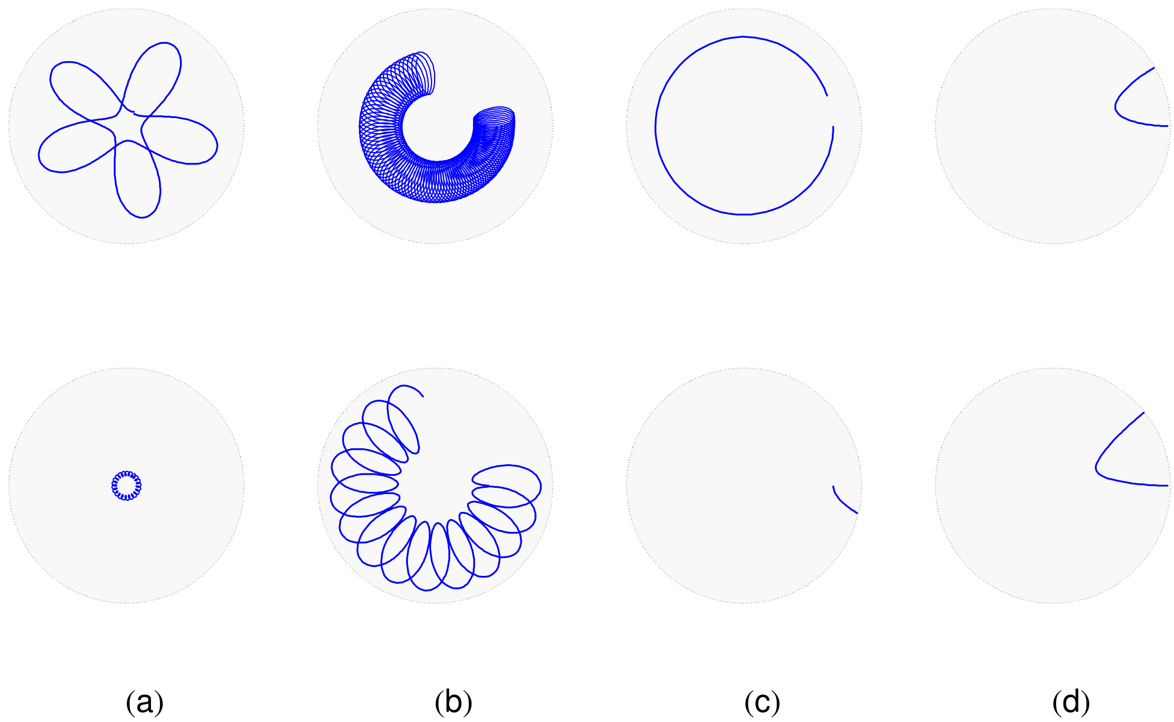


Figure 2.6: Plots of vortex trajectories under the extrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ (top) and the intrinsic RMG flow (bottom) with $\lambda_{\text{intrinsic}} = \lambda_{\text{extrinsic}}$ and various initial values .

(iv) Furthermore, we compare between the intrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ and the extrinsic RMG flow with RMG parameter $\lambda_{\text{extrinsic}} = \lambda_{\text{intrinsic}}/2$. The corresponding trajectories are given in Figure 2.7.

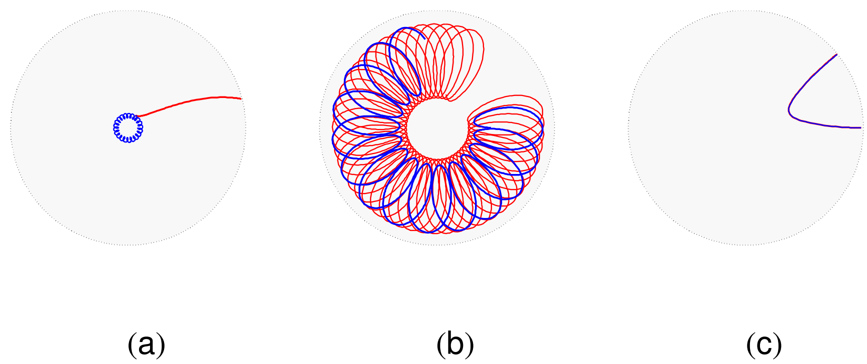


Figure 2.7: Plots of vortex trajectories under the intrinsic RMG flow on $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$ (blue) and the extrinsic RMG flow (red) with $\lambda_{\text{extrinsic}} = \lambda_{\text{intrinsic}}/2$ and various initial values .

From Figure 2.7 (c), one can see that by choosing s to be initially large, the extrinsic and intrinsic RMG trajectories coincide which confirms the suggestion given in (2.4.68). In contrast, those which begin with small s have a significant difference in their qualitative behaviour as in Figure 2.7 (a).

(v) Unlike geodesics, the features of RMG flow on a fixed point set, of a holomorphic isometry, such as $\tilde{\mathcal{M}}_2^0(\mathbb{H}^2)$, can not be deduced by knowing only the metric on such a set. This difference makes RMG flow much harder to study.

Chapter 3

Spaces of Holomorphic Maps on Compact Riemann Surfaces

3.1 $\mathbb{C}P^k$ model and L^2 Metric

Let Σ be a compact Riemann surface equipped with a Riemannian metric g given, in terms of isothermal local coordinates $\mathbf{x} = (x_1, x_2)$, by

$$g = \Omega^2(x_1, x_2)\ (dx_1^2 + dx_2^2), \tag{3.1.1}$$

for some smooth function Ω. The $\mathbb{C}P^k$ model on the spacetime $\mathbb{R}\times\Sigma$, with the Lorentzian metric $\eta = dt^2 - g$, is a scalar field theory defined by the field Lagrangian [35],

$$L[\phi] = \frac{1}{2}\int_\Sigma h_{ab}\,\partial_\mu\phi^a\partial_\nu\phi^b\,\eta^{\mu\nu}\ \mathrm{vol}_g, \tag{3.1.2}$$

for a scalar field ϕ on $\mathbb{R}\times\Sigma$ taking values in the target space $\mathbb{C}P^k$ given the Fubini-Study metric h. In this model, the kinetic and potential energy of the field system are given, respectively, by

$$T[\phi] = \frac{1}{2}\int_{\Sigma} h_{ab}\,\partial_0\phi^a\partial_0\phi^b\;\mathrm{vol}_g, \tag{3.1.3}$$

$$V[\phi] = \frac{1}{2}\int_{\Sigma}\sum_{i=1,2} h_{ab}\,\partial_i\phi^a\partial_i\phi^b\,\Omega^{-2}\;\mathrm{vol}_g. \tag{3.1.4}$$

Here, the static field equations are the Euler Lagrange equations derived from $V[\phi]$, that is,

$$h_{cb}\partial_i^2\phi^b + \frac{\partial h_{cb}}{\partial\phi^d}\partial_i\phi^d\partial_i\phi^b = \frac{1}{2}\frac{\partial h_{ab}}{\partial\phi^c}\partial_i\phi^a\partial_i\phi^b, \qquad c = 1,\ldots 2k. \tag{3.1.5}$$

The configuration space $\mathscr{C}$ of the $\mathbb{C}P^k$ model consists of all finite energy smooth maps from Σ to $\mathbb{C}P^k$. These are classified into equivalence classes $\mathscr{C}_n$ labelled by an integer n interpreted as the topological degree of the map $\phi \in \mathscr{C}_n$, namely,

$$n = \int_{\Sigma}\phi^*\omega_0, \tag{3.1.6}$$

where ω_0 is the normalized Kähler form with respect to h. An argument due to Lichnerowicz [31] shows that in each $\mathscr{C}_n$, the total energy $E[\phi] = V[\phi]$ has a non-trivial lower bound given by

$$E[\phi] \geq 2\left(\frac{4\pi}{c}\right)|n|\,. \tag{3.1.7}$$

The total energy $E[\phi]$ attains its lower bound in (3.1.7) if $\phi \in \mathscr{C}_n$ is holomorphic (antiholomorphic) map of degree $n > 0$ ($n < 0$). Hence, the degree n moduli space M_n consists of all degree $n > 0$ holomorphic (degree $n < 0$ antiholomorphic) maps from Σ to $\mathbb{C}P^k$.

Let $z = x_1 + ix_2$ denote a complex local coordinate on Σ and let $W = (W_1,\ldots,W_k)$ be inhomogeneous coordinates on $\mathbb{C}P^k$. Then, the kinetic energy $T[\phi]$ can be written, in terms of the inhomogeneous coordinates $W = (W_1,\ldots,W_k)$, as

$$T[W] = \frac{1}{2}\int_{\Sigma} \frac{4}{c}\left(\frac{(1+|W|^2)\delta_{\alpha\beta} - W_\alpha \overline{W}_\beta}{(1+|W|^2)^2}\right) \dot{W}_\alpha \dot{\overline{W}}_\beta \text{ vol}_g, \tag{3.1.8}$$

where $c > 0$ is the constant holomorphic sectional curvature of the Fubini-Study metric h on $\mathbb{C}P^k$. Introducing complex local coordinates $\{b^i : i = 1, 2, \ldots\}$ on the degree n moduli space M_n, then the restriction of the kinetic energy $T[W]$ to the moduli space M_n is

$$T[W] = \frac{1}{2}\int_{\Sigma} \frac{4}{c}\left(\frac{(1+|W|^2)\delta_{\alpha\beta} - W_\alpha \overline{W}_\beta}{(1+|W|^2)^2}\right) \frac{\partial W_\alpha}{\partial b^i}\frac{\partial \overline{W}_\beta}{\partial \bar{b}^j} \dot{b}^i \dot{\bar{b}}^j \text{ vol}_g. \tag{3.1.9}$$

In the $\mathbb{C}P^k$ model, the low energy dynamics of degree n $\mathbb{C}P^k$ lumps, the degree n minimal energy static solutions of the field equations in the $\mathbb{C}P^k$ model, is conjecturally approximated by geodesic motion, as in [45, 56], on the moduli space M_n with respect to the metric

$$\gamma_{L^2} = \gamma_{L^2 ij}\, db^i d\bar{b}^j, \quad \gamma_{L^2 ij} = \frac{4}{c}\int_{\Sigma}\left(\frac{(1+|W|^2)\delta_{\alpha\beta} - W_\alpha \overline{W}_\beta}{(1+|W|^2)^2}\right) \frac{\partial W_\alpha}{\partial b^i}\frac{\partial \overline{W}_\beta}{\partial \bar{b}^j} \text{ vol}_g. \tag{3.1.10}$$

A precise version of this conjecture is proved for $\Sigma = T^2$, $k = 1$ and $n \geq 2$ by Speight in [50]. The metric given in (3.1.10) is the L^2 metric on M_n. Formally, it assigns to any tangent vectors $X, Y \in T_\phi \mathsf{M}_n \subset \Gamma(\phi^* T\mathbb{C}P^k)$, the inner product

$$\gamma_{L^2}(X, Y) = \int_{\Sigma} h(X, Y) \text{ vol}_g. \tag{3.1.11}$$

This metric has many aspects of interest on the spaces of degree n holomorphic maps from Σ to $\mathbb{C}P^k$, denoted $\mathcal{H}_{n,k}(\Sigma)$, as we shall see throughout this chapter.

3.2 Degree 1 holomorphic maps $S^2 \to \mathbb{C}P^k$

3.2.1 $\mathcal{H}_{1,k}(S^2)$ Review

This section reviews the geometric structure of $\mathcal{H}_{1,k}(S^2)$ introduced in [49]. Let S^2 be the 2-sphere equipped with its standard round metric and let ϕ be a degree 1 holomorphic map from S^2 to $\mathbb{C}P^k$. Then, by introducing homogeneous coordinates (z_0, z_1) on $\mathbb{C}P^1 \cong S^2$, such degree 1 map has the form

$$\phi([z_0, z_1]) = [a_0 z_0 + b_0 z_1, \dots, a_k z_0 + b_k z_1], \tag{3.2.1}$$

where $(a_0, \dots, a_k)$ and $(b_0, \dots, b_k)$ are linearly independent in $\mathbb{C}^{k+1}$. For $\xi \in \mathbb{C}^\times$, the element $(\xi a_0, \xi b_0, \dots, \xi a_k, \xi b_k) \in \mathbb{C}^{2k+2} \backslash \{0\}$ determines the same holomorphic map ϕ, and so this induces an open inclusion $\mathcal{H}_{1,k}(S^2) \hookrightarrow \mathbb{C}P^{2k+1}$ whose complement in $\mathbb{C}P^{2k+1}$ is a complex codimension k algebraic variety, which is where the linear independence fails. This inclusion is used to equip $\mathcal{H}_{1,k}(S^2)$ with a topology, differentiable and complex structures.

Let g and h be the Fubini-Study metrics on the domain $S^2 \cong \mathbb{C}P^1$ and the codomain $\mathbb{C}P^k$ with constant holomorphic sectional curvatures c_1 and c_2, respectively. Consider the space $\mathcal{H}_{1,k}(S^2)$ equipped with the L^2 metric γ_{L^2} defined as in (3.1.11). The isometry groups $U(2)$ and $U(k+1)$ of $(\mathbb{C}P^1, g)$ and $(\mathbb{C}P^k, h)$, respectively, build an isometric action of $G = U(k+1) \times U(2)$ on $\mathcal{H}_{1,k}(S^2)$ given by $\phi \to \delta_2 \circ \phi \circ \delta_1^{-1}$ where δ_1 and δ_2 are isometries of $\mathbb{C}P^1$ and $\mathbb{C}P^k$. Generically, each orbit of G on $\mathcal{H}_{1,k}(S^2)$ is a real codimension 1 submanifold of $\mathcal{H}_{1,k}(S^2)$ and has a unique element ϕ_μ given by

$$\phi_\mu([z_0, z_1]) = [\mu z_0, z_1, 0, \dots, 0], \qquad \mu > 1. \tag{3.2.2}$$

An exceptional orbit of real codimension 3 occurs when $\mu = 1$. This action decomposes $\mathcal{H}_{1,k}(S^2)$ into a one parameter family of orbits parametrized by $\mu \in [1, \infty)$. The isotropy group of a generic orbit G_μ of ϕ_μ is

$$K = \left\{ \left(\begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & e^{i\beta} & 0 \\ 0 & 0 & U \end{pmatrix}, \begin{pmatrix} e^{i(\alpha+\delta)} & 0 \\ 0 & e^{i(\beta+\delta)} \end{pmatrix} \right) : \alpha, \beta, \delta \in \mathbb{R}, U \in U(k-1) \right\}. \quad (3.2.3)$$

By the Orbit-Stabilizer Theorem [2, p.94], each orbit G_μ is diffeomorphic to G/K. Now, let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Let also $\langle , \rangle$ be the $Ad(G)$ invariant inner product on $\mathfrak{g}$, that is,

$$\langle (M_1, m_1), (M_2, m_2) \rangle = -\frac{1}{2}(\text{tr} M_1 M_2 + \text{tr} m_1 m_2), \quad (3.2.4)$$

where $M_i \in \mathfrak{u}(k+1)$ and $m_i \in \mathfrak{u}(2)$. It follows from (3.2.3) that

$$\mathfrak{k} = \left\{ \left(\begin{pmatrix} i\alpha & 0 & 0 \\ 0 & i\beta & 0 \\ 0 & 0 & \mathsf{u} \end{pmatrix}, \begin{pmatrix} i(\alpha+\delta) & 0 \\ 0 & i(\beta+\delta) \end{pmatrix} \right) : \alpha, \beta, \delta \in \mathbb{R}, \mathsf{u} \in \mathfrak{u}(\mathsf{k}-1) \right\}. \quad (3.2.5)$$

The tangent space to $\mathcal{H}_{1,k}(S^2)$ at ϕ_μ is

$$V_\mu := T_{\phi_\mu} \mathcal{H}_{1,k}(S^2) \cong \langle \frac{\partial}{\partial \mu} \rangle \oplus \mathfrak{p}, \quad (3.2.6)$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $\langle , \rangle$. The space $\mathfrak{p}$ can be decomposed into $Ad(K)$ invariant subspaces

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_\mu \oplus \tilde{\mathfrak{p}}_\mu \oplus \hat{\mathfrak{p}} \oplus \check{\mathfrak{p}}, \quad (3.2.7)$$

where

$$\mathfrak{p}_0 = \left\{(\lambda(\mathrm{diag}(i,-i,0,\ldots,0),\mathrm{diag}(-i,i)) : \lambda \in \mathbb{R}\right\} \equiv \mathbb{R}, \tag{3.2.8}$$

$$\mathfrak{p}_\mu = \left\{\left(\begin{pmatrix} 0 & x & 0 & \ldots \\ -\bar{x} & 0 & 0 & \ldots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix}, \begin{pmatrix} 0 & \mu x \\ -\mu\bar{x} & 0 \end{pmatrix}\right) : x \in \mathbb{C}\right\} \equiv \mathbb{C}, \tag{3.2.9}$$

$$\tilde{\mathfrak{p}}_\mu = \left\{\left(\begin{pmatrix} 0 & -\mu\bar{y} & 0 & \ldots \\ \mu y & 0 & 0 & \ldots \\ 0 & 0 & & \\ \vdots & \vdots & & \end{pmatrix}, \begin{pmatrix} 0 & -\bar{y} \\ y & 0 \end{pmatrix}\right) : y \in \mathbb{C}\right\} \equiv \mathbb{C}, \tag{3.2.10}$$

$$\hat{\mathfrak{p}} = \left\{\left(\begin{pmatrix} 0 & 0 & -\mathbf{u}^\dagger \\ 0 & 0 & \ldots \\ \mathbf{u} & \vdots & \end{pmatrix}, \mathbf{0}\right) : \mathbf{u} \in \mathbb{C}^{k-1}\right\} \equiv \mathbb{C}^{k-1}, \tag{3.2.11}$$

$$\check{\mathfrak{p}} = \left\{\left(\begin{pmatrix} 0 & 0 & \ldots \\ 0 & 0 & -\mathbf{v}^\dagger \\ \vdots & \mathbf{v} & \end{pmatrix}, \mathbf{0}\right) : \mathbf{v} \in \mathbb{C}^{k-1}\right\} \equiv \mathbb{C}^{k-1}. \tag{3.2.12}$$

The almost complex structure J acts on $\mathfrak{p}$ as

$$J : (\lambda, x, y, \mathbf{u}, \mathbf{v}) \to 4\mu\lambda\frac{\partial}{\partial\mu} + (0, ix, iy, i\mathbf{u}, i\mathbf{v}). \tag{3.2.13}$$

It was shown in [49] that

Proposition 3.2.1. *Let γ be a G-invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$. Then, for $k \geq 2$, γ is uniquely determined by the one parameter family of symmetric bilinear forms γ_μ : $V_\mu \times V_\mu \to \mathbb{R}$ given by*

$$\gamma_\mu = A_0(\mu)d\mu^2 + 8\mu^2 A_0(\mu)\langle,\rangle_{\mathfrak{p}_0} + A_1(\mu)\ \langle,\rangle_{\mathfrak{p}_\mu} + A_2(\mu)\langle,\rangle_{\tilde{\mathfrak{p}}_\mu} + A_3(\mu)\ \langle,\rangle_{\hat{\mathfrak{p}}} + A_4(\mu)\ \langle,\rangle_{\check{\mathfrak{p}}}, \tag{3.2.14}$$

where $A_0, \ldots, A_4$ are smooth positive functions of μ defined by a single function $A(\mu)$ and a positive constant B as follows

$$\begin{aligned} A_0(\mu) &= \frac{1}{4\mu}\, A'(\mu), & A_1(\mu) &= A_2(\mu) = \frac{\mu^2-1}{\mu^2+1}\, A(\mu), \\ A_3(\mu) &= B + \frac{A(\mu)}{2}, & A_4(\mu) &= B - \frac{A(\mu)}{2}. \end{aligned} \tag{3.2.15}$$

Here, $\langle , \rangle_{\mathfrak{p}_i}$ denote the induced inner products of $\langle , \rangle$ to the $Ad(K)$ invariant subspaces, given in (3.2.7).

The L^2 metric γ_{L^2} on $\mathcal{H}_{1,k}(S^2)$ is Kähler and invariant under the action of G, so it is covered by Proposition 3.2.1. It is determined by

$$A_{L^2}(\mu) = \frac{16\pi}{c_1 c_2}\, \frac{\mu^4 - 4\mu^2 \log\mu - 1}{(\mu^2-1)^2}, \qquad B_{L^2} = \frac{8\pi}{c_1 c_2}. \tag{3.2.16}$$

Another G-invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$ is the induced metric γ_{FS} defined by the inclusion $\mathcal{H}_{1,k}(S^2) \hookrightarrow \mathbb{C}P^{2k+1}$, where $\mathbb{C}P^{2k+1}$ is given the Fubini-Study metric of constant holomorphic sectional curvature c, say. We call this the Fubini-Study metric on $\mathcal{H}_{1,k}(S^2)$ denoted γ_{FS}. It is determined by

$$A_{FS}(\mu) = \frac{4}{c}\, \frac{\mu^2-1}{\mu^2+1}, \qquad B_{FS} = \frac{2}{c}. \tag{3.2.17}$$

The volume form of a G-invariant Kähler metric γ, determined as in (3.2.14) by the function $A(\mu)$ and the constant B, on $\mathcal{H}_{1,k}(S^2)$ is

$$\mathrm{vol}_\gamma = V(\mu)\, d\mu \wedge \mathrm{vol}_{G/K}, \tag{3.2.18}$$

where

$$V(\mu) = \frac{1}{\sqrt{2}}\, A^2(\mu) \left(B^2 - \frac{A^2(\mu)}{4}\right)^{k-1} A'(\mu), \tag{3.2.19}$$

and $\mathrm{vol}_{G/K}$ is the volume form of G/K with respect to the inner product $\langle , \rangle$, defined in

(3.2.4). It was shown that for $k \geq 2$, every G-invariant Kähler metric γ on $\mathcal{H}_{1,k}(S^2)$ has finite volume [49]. In the special case that $\lim_{\mu\to\infty} A(\mu) = 2B$, this volume is

$$\text{Vol}(\mathcal{H}_{1,k}(S^2), \gamma) = \sqrt{2}\,(2B)^{2k+1}\frac{(k-1)!k!}{(2k+1)!}\,\text{Vol}(G/K) = \frac{(2B\pi)^{2k+1}}{(2k+1)!}, \tag{3.2.20}$$

where Vol(G/K) is the volume of G/K with respect to $\langle,\rangle$. This formula agreed with Baptista's conjectured formula in [5] for the volume of $\mathcal{H}_{n,k}(\Sigma)$.

3.2.2 Ricci Curvature Tensor

With respect to any G-invariant Kähler metric γ, determined as in Proposition 3.2.1, on $\mathcal{H}_{1,k}(S^2)$, the Ricci curvature tensor Ric is given explicitly by the following Proposition

Proposition 3.2.2. *Let γ be a G invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$, determined as in (3.2.14) by the function $A(\mu)$ and the constant B. Then, the Ricci curvature tensor of $(\mathcal{H}_{1,k}(S^2), \gamma)$ with $k \geq 2$ is uniquely determined by the one-parameter family of symmetric bilinear forms $Ric_\mu : V_\mu \times V_\mu \to \mathbb{R}$ given by*

$$Ric_\mu = C_0 d\mu^2 + 8\mu^2 C_0 \langle,\rangle_{\mathfrak{p}_0} + C_1(\mu)\,\langle,\rangle_{\mathfrak{p}_\mu} + C_2(\mu)\langle,\rangle_{\tilde{\mathfrak{p}}_\mu} + C_3(\mu)\,\langle,\rangle_{\hat{\mathfrak{p}}} + C_4(\mu)\,\langle,\rangle_{\check{\mathfrak{p}}}, \tag{3.2.21}$$

where $C_0, \ldots, C_4$ are smooth functions of μ, determined as in (3.2.15), by the function $C(\mu)$ and the constant D given by

$$C(\mu) = 4(k+1)\frac{\mu^2-1}{\mu^2+1} - 2\mu\frac{F'(\mu)}{F(\mu)}, \qquad D = 2(k+1), \tag{3.2.22}$$

where

$$F(\mu) = \frac{A^2(\mu)A'(\mu)}{A_{FS}^2(\mu)A'_{FS}(\mu)}\left(B^2 - \frac{A^2(\mu)}{4}\right)^{k-1}\left(B_{FS}^2 - \frac{A_{FS}^2(\mu)}{4}\right)^{-(k-1)}. \tag{3.2.23}$$

Proof: The Ricci curvature tensor on $(\mathcal{H}_{1,k}(S^2), \gamma)$ is a G-invariant symmetric $(0, 2)$ tensor which is Hermitian and whose associated 2-form $\rho(.,.) = \mathrm{Ric}(J.,.)$ is closed. Hence, Ric has the same structure as γ, namely, it is uniquely determined by the one parameter family of symmetric bilinear forms $\mathrm{Ric}_\mu : V_\mu \times V_\mu \to \mathbb{R}$ given as in (3.2.14). Since the coefficients $C_0, \ldots, C_4$ of Ric are defined as in (3.2.15) by a single function $C(\mu)$ and a constant D, then we only need to determine $C(\mu)$ and D. By Proposition 3.2.1, we have

$$C(\mu) = C_3(\mu) - C_4(\mu), \qquad D = \frac{1}{2}[C_3(\mu) + C_4(\mu)]. \tag{3.2.24}$$

To compute $C(\mu)$ and D, we need first an orthonormal basis for $\mathfrak{p}$ with respect to the inner product $\langle , \rangle$. We shall use the orthonormal basis introduced in [49] as follows

$$\begin{aligned}
&Y_0 = \frac{i}{\sqrt{2}}\,(E_{11} - E_{22}, -e_{11} + e_{22}), && \\
&Y_1 = (E_{12} - E_{21}, \mathbf{0}), && Y_2 = i(E_{12} + E_{21}, \mathbf{0}), \\
&Y_3 = (\mathbf{0}, -e_{12} + e_{21}), && Y_4 = i(\mathbf{0}, e_{12} + e_{21}), \\
&\hat{Y}_{2i-1} = (-E_{1,i+2} + E_{i+2,1}, \mathbf{0}), && \hat{Y}_{2i} = i(E_{1,i+2} + E_{i+2,1}, \mathbf{0}), \quad i = 1, \ldots, k-1, \\
&\check{Y}_{2i-1} = (-E_{2,i+2} + E_{i+2,2}, \mathbf{0}), && \check{Y}_{2i} = i(E_{2,i+2} + E_{i+2,2}, \mathbf{0}), \quad i = 1, \ldots, k-1,
\end{aligned} \tag{3.2.25}$$

where $E_{\alpha\beta}$ and $e_{\alpha\beta}$ denote $(k+1) \times (k+1)$ and 2×2 matrices, respectively, whose element (α, β) is 1 and the others being zero.

Hence, the functions C_3 and C_4 can be given, for example, by

$$\begin{aligned}
C_3 &= \mathrm{Ric}_\mu(\hat{Y}_1, \hat{Y}_1) = -\mathrm{Ric}_\mu(J\hat{Y}_2, \hat{Y}_1) = \rho_\mu(\hat{Y}_1, \hat{Y}_2), \\
C_4 &= \mathrm{Ric}_\mu(\check{Y}_1, \check{Y}_1) = -\mathrm{Ric}_\mu(J\check{Y}_2, \check{Y}_1) = \rho_\mu(\check{Y}_1, \check{Y}_2).
\end{aligned} \tag{3.2.26}$$

Now, the volume form, given in (3.2.18), of any G-invariant Kähler metric γ on $\mathcal{H}_{1,k}(S^2)$ can be written as

$$\mathrm{vol}_\gamma = F(\mu)\,\mathrm{vol}_{\gamma_{FS}}, \tag{3.2.27}$$

where

$$F(\mu) = \frac{A^2(\mu)A'(\mu)}{A_{FS}^2(\mu)A'_{FS}(\mu)}\left(B^2 - \frac{A^2(\mu)}{4}\right)^{k-1}\left(B_{FS}^2 - \frac{A_{FS}^2(\mu)}{4}\right)^{-(k-1)}. \tag{3.2.28}$$

Hence, the Ricci form ρ with respect to γ is [9, p.83],

$$\rho = \rho_{FS} - i\partial\bar{\partial}f, \qquad f(\mu) := \log F(\mu), \tag{3.2.29}$$

where ρ_{FS} is the Ricci form with respect to γ_{FS}, $\partial : \Omega^{(p,q)} \to \Omega^{(p+1,q)}$ and $\bar{\partial} : \Omega^{(p,q)} \to \Omega^{(p,q+1)}$ are the partial exterior derivatives on the space of (p,q)-forms $\Omega^{(p,q)}$ on $\mathcal{H}_{1,k}(S^2)$. Using (3.2.29) in (3.2.26), we have

$$\begin{aligned} C(\mu) &= \rho_{\mu_{FS}}(\hat{Y}_1,\hat{Y}_2) - \rho_{\mu_{FS}}(\check{Y}_1,\check{Y}_2) - i\big[(\partial\bar{\partial}f)_\mu(\hat{Y}_1,\hat{Y}_2) - (\partial\bar{\partial}f)_\mu(\check{Y}_1,\check{Y}_2)\big],\\ &= C_{FS}(\mu) - i\big[(\partial\bar{\partial}f)_\mu(\hat{Y}_1,\hat{Y}_2) - (\partial\bar{\partial}f)_\mu(\check{Y}_1,\check{Y}_2)\big], \end{aligned} \tag{3.2.30}$$

and

$$\begin{aligned} 2D &= \rho_{\mu_{FS}}(\hat{Y}_1,\hat{Y}_2) + \rho_{\mu_{FS}}(\check{Y}_1,\check{Y}_2) - i\big[(\partial\bar{\partial}f)(\mu)(\hat{Y}_1,\hat{Y}_2) + (\partial\bar{\partial}f)(\mu)(\check{Y}_1,\check{Y}_2)\big],\\ &= 2D_{FS} - i\big[(\partial\bar{\partial}f)_\mu(\hat{Y}_1,\hat{Y}_2) + (\partial\bar{\partial}f)_\mu(\check{Y}_1,\check{Y}_2)\big]. \end{aligned} \tag{3.2.31}$$

Since $(\mathcal{H}_{1,k}(S^2),\gamma_{FS})$ is a complex $(2k+1)$-dimensional Kähler-Einstein manifold, then [26, p.168]

$$\rho_{FS} = c\,(k+1)\,\omega_{FS}, \tag{3.2.32}$$

where ω_{FS} is the Kähler form of γ_{FS}. Hence, the function $C_{FS}(\mu)$ and the constant D_{FS} are

$$C_{FS}(\mu) = c(k+1)A_{FS} = 4(k+1)\frac{\mu^2-1}{\mu^2+1}, \quad D_{FS} = c(k+1)B_{FS} = 2(k+1). \tag{3.2.33}$$

It remains to compute $(\partial\bar{\partial}f)_\mu(\hat{Y}_1, \hat{Y}_2)$ and $(\partial\bar{\partial}f)_\mu(\check{Y}_1, \check{Y}_2)$. Let $\xi_0 = -Y_0/(2\sqrt{2}\ \mu)$, then it follows from (3.3.13) that $J\xi_0 = -\partial/\partial\mu$, and so,

$$J^* d\mu(\xi_0) = d\mu(J\xi_0) = d\mu(-\frac{\partial}{\partial\mu}) = -1, \tag{3.2.34}$$

where J^* is the induced almost complex structure on V_μ^*. This means that $\eta_0 = -J^* d\mu$ is a covector with $\eta_0(\xi_0) = 1$. The exterior derivative of f is

$$df = \frac{1}{2}\ f'(\mu)\ [(d\mu+i\eta_0)+(d\mu-i\eta_0)] = \frac{1}{2}\ f'(\mu)\ [(d\mu-iJ^*d\mu)+(d\mu+iJ^*d\mu)]. \tag{3.2.35}$$

By definition, the $(1,0)$-part ∂f and the $(0,1)$-part $\bar{\partial}f$ of the 1-form df are

$$\partial f = \frac{1}{2}\ f'(\mu)\ (d\mu + i\eta_0), \qquad \bar{\partial} f = \frac{1}{2}\ f'(\mu)\ (d\mu - i\eta_0). \tag{3.2.36}$$

Since $d = \partial + \bar{\partial}$ and $\bar{\partial}^2 = 0$, then

$$\partial\bar{\partial} f = d\bar{\partial} f = -\frac{i}{2} f''(\mu)\ d\mu \wedge \eta_0 - \frac{i}{2} f'(\mu) d\eta_0, \tag{3.2.37}$$

where $d\eta_0$ is a 2-form on $\mathcal{H}_{1,k}(S^2)$ given for any vector fields X, Y on $\mathcal{H}_{1,k}(S^2)$ by

$$d\eta_0(X,Y) = X[\eta_0(Y)] - Y[\eta_0(X)] - \eta_0([X,Y]). \tag{3.2.38}$$

Let ξ_1, ξ_2 be the extension of $\hat{Y}_1$ and $\hat{Y}_2$ as Killing vector fields on $\mathcal{H}_{1,k}(S^2)$. Then, from (3.2.37) and (3.2.38), we have

$$(\partial\bar{\partial} f)_\mu(\hat{Y}_1, \hat{Y}_2) = \frac{i}{2} f'(\mu)\ \eta_0([\xi_1, \xi_2]\Big|_{\phi=\phi_\mu}). \tag{3.2.39}$$

The Lie bracket of Killing vector fields on $\mathcal{H}_{1,k}(S^2)$ can be calculated from the Lie algebra bracket $[\ ,\ ]_{\mathfrak{g}}$ of $\mathfrak{g}$ as follows [9, p.182]

$$[\xi_1, \xi_2]\Big|_{\phi=\phi_\mu} = -P_{\mathfrak{p}}([\hat{Y}_1, \hat{Y}_2]_{\mathfrak{g}}), \tag{3.2.40}$$

where $P_{\mathfrak{p}}$ is the projection of $\mathfrak{g}$ to $\mathfrak{p}$. From (3.2.25), we have

$$\hat{Y}_1 = (-E_{13} + E_{31}, \mathbf{0}), \qquad \hat{Y}_2 = i(E_{13} + E_{31}, \mathbf{0}). \tag{3.2.41}$$

Then, it follows that

$$\begin{aligned} [\hat{Y}_1, \hat{Y}_2]_{\mathfrak{g}} &= -2i(E_{13}E_{31} - E_{31}E_{13}, \mathbf{0}), \\ &= -i(2E_{11} - 2E_{33}, \mathbf{0}), \\ &= -\frac{i}{2}(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22}) + \frac{i}{2}(E_{11} - E_{22}, -e_{11} + e_{22}), \\ &= -\frac{i}{2}(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22}) + \frac{1}{\sqrt{2}} Y_0. \end{aligned} \tag{3.2.42}$$

Since the element $i(3E_{11} + E_{22} - 2E_{33}, e_{11} - e_{22})/2 \in \mathfrak{k}$, then it vanishes under $P_{\mathfrak{p}}$, and so

$$[\xi_1, \xi_2]\Big|_{\phi=\phi_\mu} = -\frac{1}{\sqrt{2}} Y_0. \tag{3.2.43}$$

Substituting (3.2.43) in (3.2.39), we get

$$(\partial\bar{\partial} f)_\mu(\hat{Y}_1, \hat{Y}_2) = i\mu f'(\mu). \tag{3.2.44}$$

Similarly, one can find that

$$(\partial\bar{\partial} f)_\mu(\check{Y}_1, \check{Y}_2) = -i\mu f'(\mu). \tag{3.2.45}$$

Substituting (3.2.33), (3.2.44) and (3.2.45) in (3.2.30) and (3.2.31), we obtain the function $C(\mu)$ and the constant D as in (3.2.22).

□

3.2.3 Scalar Curvature

An orthonormal basis for (V_μ, γ_μ) can be defined, using (3.2.25), as follows [49],

$$\begin{aligned}
X &= \frac{1}{\sqrt{A_0}}\frac{\partial}{\partial\mu}, & X_0 &= \frac{1}{\sqrt{8\mu^2 A_0}}\,Y_0, \\
X_1 &= \frac{Y_1-\mu Y_3}{\sqrt{(1+\mu^2)A_1}}, & X_2 &= \frac{Y_2+\mu Y_4}{\sqrt{(1+\mu^2)A_1}}, \\
X_3 &= \frac{-\mu Y_1+Y_3}{\sqrt{(1+\mu^2)A_1}}, & X_4 &= \frac{\mu Y_2+Y_4}{\sqrt{(1+\mu^2)A_1}}, \\
\hat{X}_j &= \frac{1}{\sqrt{A_3}}\,\hat{Y}_j, & \check{X}_j &= \frac{1}{\sqrt{A_4}}\,\check{Y}_j, \quad j=1,\dots,2k-2.
\end{aligned} \tag{3.2.46}$$

Proposition 3.2.3. *Let γ be a G invariant Kähler metric on $\mathcal{H}_{1,k}(S^2)$, determined as in (3.2.14) by the function $A(\mu)$ and the constant B. Then, the scalar curvature of $(\mathcal{H}_{1,k}(S^2),\gamma)$ for $k\geq 2$ is*

$$\kappa(\mu) = 2\left[2\frac{C(\mu)}{A(\mu)}+\frac{C'(\mu)}{A'(\mu)}\right]+2(k-1)\left[\frac{4(k+1)+C(\mu)}{2B+A(\mu)}+\frac{4(k+1)-C(\mu)}{2B-A(\mu)}\right]. \tag{3.2.47}$$

Proof: The scalar curvature of a G-invariant Kähler metric γ, determined as in (3.2.14), with respect to the orthonormal basis (3.2.46) is

$$\begin{aligned}\kappa(\mu) =&\mathrm{Ric}_\mu(X,X) + \sum_{i=0}^{4}\mathrm{Ric}_\mu(X_i,X_i) + \sum_{j=1}^{2k-2}\big[\mathrm{Ric}_\mu(\hat{X}_j,\hat{X}_j) + \mathrm{Ric}_\mu(\check{X}_j,\check{X}_j)\big],\\ =&\frac{1}{A_0}\mathrm{Ric}_\mu(\frac{\partial}{\partial\mu},\frac{\partial}{\partial\mu}) + \frac{1}{8\mu^2A_0}\mathrm{Ric}_\mu(Y_0,Y_0) + \frac{1}{A_1}\sum_{i=1}^{4}\mathrm{Ric}_\mu(Y_i,Y_i)\\ &+\frac{1}{A_3}\sum_{j=1}^{2k-2}\mathrm{Ric}_\mu(\hat{Y}_j,\hat{Y}_j) + \frac{1}{A_4}\sum_{j=1}^{2k-2}\mathrm{Ric}_\mu(\check{Y}_j,\check{Y}_j). \qquad (3.2.48)\end{aligned}$$

Using (3.2.21) in (3.2.48), we get

$$\kappa(\mu) = 2\,\frac{C_0}{A_0} + 4\,\frac{C_1}{A_1} + 2(k-1)\left[\frac{C_3}{A_3} + \frac{C_4}{A_4}\right]. \qquad (3.2.49)$$

Using the relations between the functions $A_i(\mu)$ and $C_i(\mu)$ with $A(\mu)$ and $C(\mu)$, respectively, as in (3.2.15), we obtain that the scalar curvature of a G invariant Kähler metric γ on $\mathcal{H}_{1,k}(S^2)$ has the formula (3.2.47).

□

3.2.4 Einstein-Hilbert Action of $\mathcal{H}_{1,k}(S^2)$

The Einstein-Hilbert action of a Riemannian manifold (M,g) is defined to be the integral

$$H(M,g) = \int_M \kappa\,\mathrm{vol}_g, \qquad (3.2.50)$$

where κ and vol_g are the scalar curvature and the volume form, respectively, with respect to the Riemannian metric g on M.

Theorem 3.2.4. *The Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to the L^2 metric γ_{L^2} is*

$$H(\mathcal{H}_{1,k}(S^2),\gamma_{L^2}) = \frac{2^{2k+2}\,\pi^{2k+1}(k+1)B_{L^2}^{2k}}{(2k)!}, \qquad \forall\, k\geq 2. \qquad (3.2.51)$$

Proof: The Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to any G invariant Kähler metric γ is

$$H(\mathcal{H}_{1,k}(S^2),\gamma) = \int_{\mathcal{H}_{1,k}(S^2)} \kappa(\mu)\, V(\mu)\, d\mu \wedge \mathrm{vol}_{G/K}, \\ = \mathrm{Vol}(G/K) \int_1^\infty \kappa(\mu)\, V(\mu)\, d\mu. \tag{3.2.52}$$

The scalar curvature of $(\mathcal{H}_{1,k}(S^2),\gamma)$, given in (3.2.47), can be written as

$$\kappa(\mu) = \frac{2}{AA'(\mu)}\big[2CA'(\mu)+AC'(\mu)\big]+(k-1)\left(B^2-\frac{A^2}{4}\right)^{-1}\big[4(k+1)B-AC\big], \tag{3.2.53}$$

and then, by (3.2.19), we have

$$\kappa(\mu)\, V(\mu) = \frac{2}{\sqrt{2}}\big[2ACA'(\mu) + A^2C'(\mu)\big]\left(B^2-\frac{A^2}{4}\right)^{k-1} \\ + \frac{(k-1)}{\sqrt{2}}A^2A'(\mu)\,\big[4(k+1)B-AC\big]\left(B^2-\frac{A^2}{4}\right)^{k-2}, \\ = \frac{2}{\sqrt{2}}\left(B^2-\frac{A^2}{4}\right)^{k-1}\frac{d}{d\mu}(A^2\,C) - \frac{(k-1)}{\sqrt{2}}\,CA^3A'(\mu)\left(B^2-\frac{A^2}{4}\right)^{k-2} \\ + \frac{4(k^2-1)B}{\sqrt{2}}A^2A'(\mu)\left(B^2-\frac{A^2}{4}\right)^{k-2}. \tag{3.2.54}$$

Since

$$\frac{d}{d\mu}\left[\left(B^2-\frac{A^2}{4}\right)^{k-1}\right] = -\frac{(k-1)}{2}A\,A'(\mu)\left(B^2-\frac{A^2}{4}\right)^{k-2}, \tag{3.2.55}$$

then,

$$\kappa(\mu)\, V(\mu) = \frac{2}{\sqrt{2}}\,\frac{d}{d\mu}\left[A^2\,C\left(B^2-\frac{A^2}{4}\right)^{k-1}\right]+2\sqrt{2}(k^2-1)BA^2\,A'(\mu)\left(B^2-\frac{A^2}{4}\right)^{k-2}. \tag{3.2.56}$$

Hence, the Einstein-Hilbert Action $H(\mathcal{H}_{1,k}(S^2),\gamma)$ is

$$\begin{aligned} H(\mathcal{H}_{1,k}(S^2),\gamma) =& \frac{2}{\sqrt{2}}\text{Vol}(G/K)\left[A^2\, C\left(B^2-\frac{A^2}{4}\right)^{k-1}\right]_1^\infty \\ &+ 2\sqrt{2}\,(k^2-1)B^{2k-3}\text{Vol}(G/K)\int_{A(1)}^{A(\infty)} A^2\left(1-\frac{A^2}{4B^2}\right)^{k-2} dA. \end{aligned} \tag{3.2.57}$$

For the L^2 metric on $\mathcal{H}_{1,k}(S^2)$, the following limits follow from (3.2.16),

$$\begin{aligned} &\lim_{\mu\to 1} A_{L^2}(\mu) = 0, && \lim_{\mu\to\infty} A_{L^2}(\mu) = 2B_{L^2}, \\ &\lim_{\mu\to 1} C_{L^2}(\mu) = 0, && \lim_{\mu\to\infty} C_{L^2}(\mu) = 4(k+1), \end{aligned} \tag{3.2.58}$$

and so,

$$\lim_{\mu\to 1}\left[A_{L^2}^2\, C_{L^2}\left(B_{L^2}^2-\frac{A_{L^2}^2}{4}\right)^{k-1}\right] = \lim_{\mu\to\infty}\left[A_{L^2}^2\, C_{L^2}\left(B_{L^2}^2-\frac{A_{L^2}^2}{4}\right)^{k-1}\right] = 0. \tag{3.2.59}$$

Thus, the Einstein-Hilbert Action with respect to the L^2 metric γ_{L^2} on $\mathcal{H}_{1,k}(S^2)$ is

$$H(\mathcal{H}_{1,k}(S^2),\gamma_{L^2}) = 2\sqrt{2}\,(k^2-1)B_{L^2}^{2k-3}\,\text{Vol}(G/K)\int_{A_{L^2}(1)}^{A_{L^2}(\infty)} A_{L^2}^2\left(1-\frac{A_{L^2}^2}{4B_{L^2}^2}\right)^{k-2} dA_{L^2}. \tag{3.2.60}$$

To compute the integral above, let $\tau = A_{L^2}/2B_{L^2}$, then

$$H(\mathcal{H}_{1,k}(S^2),\gamma_{L^2}) = 2^4\sqrt{2}\,(k^2-1)B_{L^2}^{2k}\,\text{Vol}(G/K)\int_0^1 \tau^2\left(1-\tau^2\right)^{k-2} d\tau. \tag{3.2.61}$$

The integral in (3.2.61) is finite for all $k \geq 2$. In fact

$$\int_0^1 \tau^2 \left[1-\tau^2\right]^{k-2} d\tau = \frac{2^{2k-2}(k-2)!\; k!}{(2k)!}, \qquad \forall\, k \geq 2. \tag{3.2.62}$$

The volume of G/K can be extracted from the formula of Vol$(\mathcal{H}_{1,k}(S^2), \gamma)$ in (3.2.20),

$$\text{Vol}(G/K) = \frac{1}{\sqrt{2}} \frac{\pi^{2k+1}}{(k-1)!\; k!}. \tag{3.2.63}$$

Substituting (3.2.62) and (3.2.63) in (3.2.61), we get

$$H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = \frac{2^{2k+2}\; \pi^{2k+1}(k+1) B_{L^2}^{2k}}{(2k)!}. \tag{3.2.64}$$

□

By taking the holomorphic sectional curvatures $c_1 = c_2 = 4$, then the constant $B_{L^2} = \pi/2$, and so the Einstein-Hilbert action of $\mathcal{H}_{1,k}(S^2)$ with respect to the L^2 metric becomes

$$H(\mathcal{H}_{1,k}(S^2), \gamma_{L^2}) = \frac{2^2\; \pi^{4k+1}(k+1)}{(2k)!}. \tag{3.2.65}$$

This confirms Baptista's conjectured formula in [5] for the the Einstein-Hilbert action of $\mathcal{H}_{n,k}(\Sigma)$, provided Σ has genus $\mathrm{g} \leq n/2$,

$$H(\mathcal{H}_{n,k}(\Sigma), \gamma_{L^2}) = \frac{2\pi(k+1)^{\mathrm{g}}[m-2\mathrm{g}+1]}{(m-1)!} \Big(\pi \text{Vol}(\Sigma, g)\Big)^{m-1}, \tag{3.2.66}$$

where $m = (k+1)(n+1-\mathrm{g}) + \mathrm{g} - 1$ and Vol(Σ, g) is the volume of Σ. This conjecture is based on a singular limit relating the $\mathbb{C}P^k$ model on Σ with a gauged sigma model whose fields take values in $\mathbb{C}^{k+1}$ [5]. More precisely, a one parameter family of metrics on the n-vortex moduli space, which is a compact Kähler manifold, are conjectured to converge, in a certain limit, to the L^2 metric on $\mathcal{H}_{n,k}(\Sigma)$. Such convergence has recently been established rigorously by Lui [32] in the sense of Cheeger-Gromov, that is, on each open set in some locally finite open cover of $\mathcal{H}_{n,k}(\Sigma)$. This convergence does not directly imply Baptista's conjectured formula (3.2.66) for the Einstein-Hilbert action of $\mathcal{H}_{n,k}(\Sigma)$, however.

3.3 Degree 1 holomorphic maps $S^2 \to S^2$

3.3.1 Rat_1 Review

This section reviews the work done by Speight in [48]. Consider the space of degree $n > 0$ holomorphic maps $(S^2_{do}, g) \to (S^2_{co}, h)$ where g and h are the round metrics on the 2-spheres S^2_{do} and S^2_{co} of radius $1/2$. Introducing stereographic coordinates z and W on the domain and codomain, respectively, then such degree n holomorphic maps are of the form

$$z \mapsto W(z) = \frac{a_1 + a_2 z + \cdots + a_{n+1} z^n}{a_{n+2} + a_{n+3} z + \cdots + a_{2n+2} z^n}, \tag{3.3.1}$$

where a_i are complex constants, $a_{n+1}, a_{2n+2} \neq 0$ simultaneously, and the numerator and denominator have no common roots. Hence, this space is just the space of degree n rational maps and is denoted Rat_n. It is clear that $(\xi a_1, \ldots, \xi a_{2n+2}) \in \mathbb{C}^{2n+2} \backslash \{0\}$ for $\xi \in \mathbb{C}^\times$ determines the same rational map $W(z)$. This leads to a natural open inclusion $\text{Rat}_n \hookrightarrow \mathbb{C}P^{2n+1}$ whose complement is a complex codimension 1 algebraic variety, which is where the numerator and denominator in $W(z)$ are sharing roots. Thus, Rat_n inherits a natural topology, differentiable structure and complex structure from $\mathbb{C}P^{2n+1}$.

Since S^2_{co} is identified with $\mathbb{C}P^1$, then the space Rat_n is regarded as the moduli space M_n of the $\mathbb{C}P^1$ model on S^2_{do} whose field Lagrangian is given, in terms of W, by

$$\begin{aligned} L[W] &= \frac{1}{2}\int_\Sigma \frac{\partial_\mu W \partial^\mu \overline{W}}{(1+|W|^2)^2}\,\text{vol}_g, \\ &= \frac{i}{4}\int_\Sigma \frac{|\dot{W}|^2}{(1+|W|^2)^2}\,\frac{dz d\bar{z}}{(1+|z|^2)^2} - \frac{i}{2}\int_\Sigma \frac{|\partial_z W|^2 + |\partial_{\bar{z}} W|^2}{(1+|W|^2)^2}\,dz d\bar{z}, \\ &=: T[W] - V[W]. \end{aligned} \tag{3.3.2}$$

With respect to the identification $\text{Rat}_n \cong \mathsf{M}_n$, the L^2 metric on Rat_n is the one given in (3.1.10). That is, by introducing complex local coordinates $\{b^i = a_i/a_{2n+2} : i =$

$1, \ldots, 2n+1\}$ on Rat_n, then the L^2 metric on Rat_n is

$$\gamma_{L^2} = \gamma_{L^2_{ij}}\, db^i d\bar{b}^j, \qquad \gamma_{L^2_{ij}} = \frac{i}{2}\int_{S^2_{do}} \frac{1}{(1+|W|^2)^2}\frac{\partial W}{\partial b^i}\frac{\partial \overline{W}}{\partial \bar{b}^i}\frac{dz d\bar{z}}{(1+|z|^2)^2}. \tag{3.3.3}$$

It follows from (3.3.3) that γ_{L^2} on Rat_n is Hermitian and in fact is Kähler [48].

Some significant features of $(\mathrm{Rat}_n, \gamma_{L^2})$ are already known: it is geodesically incomplete [42], and it has an isometric action of $G_0 = SO(3)\times SO(3)$ descending from the isometric $SO(3)$ action on the domain (S^2_{do}, g) and the codomain (S^2_{co}, h). The G_0 action on $\phi \in \mathrm{Rat}_n$ is defined as

$$\phi \mapsto \delta_2 \circ \phi \circ \delta_1^{-1}, \tag{3.3.4}$$

where δ_1 and δ_2 are isometries of S^2_{do} and S^2_{co}, respectively. Let P be the map on Rat_n which maps $W(z)$ to $\overline{W}(\bar{z})$. It follows from (3.3.3) that P acts isometrically on Rat_n. This is called the discrete isometry of Rat_n. Hence, $G = G_0 \times \{\mathrm{Id}, \mathrm{P}\}$, where Id is the identity map, acts isometrically on Rat_n.

Now, we consider the case $n = 1$ in more detail. Identifying $S^2 \cong \mathbb{C}\cup\{\infty\}$, then any degree 1 rational map on S^2 can be thought as a Möbius transformation on $\mathbb{C}\cup\{\infty\}$. Namely,

$$W(z) = \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot z, \qquad ad - bc \neq 0, \tag{3.3.5}$$

which is identified with an equivalence class of $GL(2,\mathbb{C})$ matrices

$$[M] = \{\xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \xi \in \mathbb{C}^\times\}. \tag{3.3.6}$$

Thus, Rat_1 is identified with $PL(2,\mathbb{C})$. Also, it is known that any $[M] \in PL(2,\mathbb{C})$ can be uniquely decomposed as

$$[M] = [U]\big(\Lambda \mathbb{I}_2 + \boldsymbol{\lambda} \cdot \boldsymbol{\tau}\big), \tag{3.3.7}$$

where $U \in SU(2)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, $\lambda = \|\boldsymbol{\lambda}\|$, $\Lambda = \sqrt{1+\lambda^2}$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the standard Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3.8}$$

Hence, $\mathrm{Rat}_1 \cong PU(2) \times \mathbb{R}^3 \cong SU(2)/\mathbb{Z}_2 \times \mathbb{R}^3$. There is a well-known identification between $SU(2)/\mathbb{Z}_2$ and $SO(3)$ given by the map

$$\Phi : SU(2) \to SO(3), \qquad \Phi(U) = \Psi^{-1} \circ Ad_U \circ \Psi, \tag{3.3.9}$$

where Ad_U is the adjoint representation of U in $SU(2)$ and Ψ is a linear isomorphism given by

$$\Psi : \mathbb{R}^3 \to \mathfrak{su}(2), \qquad \Psi(\mathbf{x}) = \frac{i}{2}(\mathbf{x} \cdot \boldsymbol{\tau}). \tag{3.3.10}$$

The map Φ is a surjective homomorphism with $\mathrm{Ker}(\Phi) = \mathbb{Z}_2$, and so $SU(2)/\mathbb{Z}_2 \cong SO(3)$ [20, p.303]. Hence, Rat_1 is identified with $SO(3) \times \mathbb{R}^3$. In the coordinate system $([U], \boldsymbol{\lambda})$ on Rat_1, the G_0 action on Rat_1 is given by

$$(\mathcal{L}, \mathcal{R}) : ([U], \boldsymbol{\lambda}) \to ([LUR^{-1}], \mathcal{R}\boldsymbol{\lambda}), \tag{3.3.11}$$

where $\mathcal{L} = \Phi(L)$ and $\mathcal{R} = \Phi(R)$ for $L, R \in SU(2)$. This action has cohomogeneity 1, that is, generic G_0 orbits are submanifolds of Rat_1 with real codimension 1. One finds that the orbit space Rat_1/G_0 is identified with $\Gamma = \{([\mathbb{I}_2], (0, 0, \lambda)) : \lambda \geq 0\}$.

Consider the coframe on Rat_1 defined by $\{d\lambda_k, \sigma_k : k = 1, 2, 3\}$, where σ_k are the left-invariant 1-forms dual to the basis $\{\theta_k = i\tau_k/2 : k = 1, 2, 3\}$ for $\mathfrak{su}(2)$. From (3.3.11), one can obtain that the induced action of G_0 on this coframe is

$$(\mathcal{L}, \mathcal{R}) : (\boldsymbol{\sigma}, \boldsymbol{d\lambda}) \to (\mathcal{R}\boldsymbol{\sigma}, \mathcal{R}\boldsymbol{d\lambda}). \tag{3.3.12}$$

The action of the almost complex structure J on the basis $\{\partial/\partial\lambda_k, \theta_k : k = 1, 2, 3\}$ at $W_\lambda = ([\mathbb{I}_2], (0, 0, \lambda))$ is,

$$J\frac{\partial}{\partial\lambda_1} = \frac{2}{\Lambda}\left(\theta_1 - \frac{\lambda}{2}\frac{\partial}{\partial\lambda_2}\right), \qquad J\frac{\partial}{\partial\lambda_2} = \frac{2}{\Lambda}\left(\theta_2 + \frac{\lambda}{2}\frac{\partial}{\partial\lambda_1}\right), \qquad J\frac{\partial}{\partial\lambda_3} = \frac{2}{\Lambda}\theta_3. \tag{3.3.13}$$

Using $J^2 = -1$, one can determine $\{J\theta_k : k = 1, 2, 3\}$ from (3.3.13).

Speight has shown in [48] that any G-invariant Hermitian metric γ on Rat_1 has the form,

$$\gamma = A_1\boldsymbol{d\lambda}\cdot\boldsymbol{d\lambda} + A_2(\boldsymbol{\lambda}\cdot\boldsymbol{d\lambda})^2 + A_3\boldsymbol{\sigma}\cdot\boldsymbol{\sigma} + A_4(\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})^2 + A_5\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\boldsymbol{d\lambda}), \tag{3.3.14}$$

where $A_1, \ldots, A_5$ are smooth functions of λ which satisfy

$$A_3 \equiv \frac{A_4}{4} + \frac{\lambda^2}{2}A_5, \qquad A_1 + \lambda^2 A_2 \equiv \frac{4}{\Lambda^2}(A_3 + \lambda^2 A_4), \tag{3.3.15}$$

which are the Hermiticity constraints on the coefficients of any G-invariant metric, determined as in (3.3.14), on Rat_1. If γ is Kähler, then there exists a smooth function $A(\lambda)$ such that the coefficients $A_1, \ldots, A_5$ are defined by $A(\lambda)$ as follows

$$\begin{aligned} &A_1 = A(\lambda), \qquad A_2 = \frac{A(\lambda)}{\Lambda^2} + \frac{A'(\lambda)}{\lambda}, \qquad A_3 = (\frac{1+2\lambda^2}{4})A(\lambda), \\ &A_4 = \left(\frac{\Lambda^2}{4\lambda}\right)A'(\lambda), \qquad A_5 = A(\lambda). \end{aligned} \tag{3.3.16}$$

The associated G-invariant closed 2-form of γ, the Kähler form, is of the form

$$\omega = \hat{A}_1\, \boldsymbol{d\lambda} \dot{\wedge} \boldsymbol{\sigma} + \hat{A}_2\, (\boldsymbol{\lambda} \cdot \boldsymbol{d\lambda}) \wedge (\boldsymbol{\lambda} \cdot \boldsymbol{\sigma}) + \hat{A}_3\, \boldsymbol{\lambda} \cdot (\boldsymbol{\sigma} \times \boldsymbol{\sigma}), \tag{3.3.17}$$

where $\boldsymbol{d\lambda} \dot{\wedge} \boldsymbol{\sigma} = \sum_i d\lambda_i \ \wedge \ \sigma_i$, and

$$\hat{A}_1(\lambda) = \frac{\Lambda}{2} A(\lambda), \qquad \hat{A}_2(\lambda) = \frac{\hat{A}_1'(\lambda)}{\lambda}, \qquad \hat{A}_3(\lambda) = \hat{A}_1(\lambda). \tag{3.3.18}$$

For the L^2 metric γ_{L^2} on Rat_1, the function $A(\lambda)$, call it $A_{L^2}(\lambda)$, is determined explicitly by

$$A_{L^2}(\lambda) = 2\pi\mu\, \frac{[\mu^4 - 4\mu^2 \log\mu - 1]}{(\mu^2 - 1)^3}, \tag{3.3.19}$$

where $\mu = (\Lambda + \lambda)^2$. The above expression appears in [45] with a factor $4\pi\mu$ instead of $2\pi\mu$. This is due to a rescaling of the kinetic energy $T[W]$ in [45] by factor $1/2$.

Since the Ricci curvature tensor Ric is a G-invariant symmetric $(0, 2)$ tensor which is Hermitian and whose associated 2-form ρ, the Ricci form, is closed, then it has the same formula (3.3.14) with coefficients $\bar{A}_1, \ldots, \bar{A}_5$ defined, as in (3.3.16), by

$$\bar{A}(\lambda) = -\frac{1}{2\lambda}\frac{d}{d\lambda} \log(A^2(\lambda)\ B(\lambda)), \tag{3.3.20}$$

where

$$B(\lambda) := A_3(\lambda) + \lambda^2 A_4(\lambda) = \frac{1 + 2\lambda^2}{4} A(\lambda) + \frac{\lambda\Lambda^2}{4} A'(\lambda) = \frac{\Lambda}{4}\frac{d}{d\lambda}(\lambda\Lambda A(\lambda)). \tag{3.3.21}$$

With respect to any G-invariant Kähler metric γ, determined as in (3.3.14), on Rat_1, the volume form and the scalar curvature are of the form

$$\mathrm{vol}_\gamma = \frac{\Lambda}{2} B(\lambda)\ A^2(\lambda) \quad d\lambda_1 \wedge d\lambda_2 \wedge \lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3, \tag{3.3.22}$$

$$\kappa(\lambda) = 4\frac{\bar{A}(\lambda)}{A(\lambda)} + 2\frac{\bar{B}(\lambda)}{B(\lambda)}, \tag{3.3.23}$$

respectively, where

$$\bar{B}(\lambda) := \bar{A}_3(\lambda) + \lambda^2 \bar{A}_4(\lambda) = \frac{1+2\lambda^2}{4}\bar{A}(\lambda) + \frac{\lambda\Lambda^2}{4}\bar{A}'(\lambda) = \frac{\Lambda}{4}\frac{d}{d\lambda}(\lambda\Lambda\bar{A}(\lambda)). \tag{3.3.24}$$

From the formula for $A_{L^2}(\lambda)$, it has been proved that $(\mathrm{Rat}_1, \gamma_{L^2})$ has finite volume and its scalar curvature is unbounded above [48].

3.3.2 Einstein-Hilbert Action of Rat_1

The Einstein-Hilbert action of Rat_1 with respect to a G-invariant Kähler metric γ, determined as in (3.3.14), is

$$H(\mathrm{Rat}_1, \gamma) = \int_{\mathrm{Rat}_1} \kappa(\lambda)\ \mathrm{vol}_\gamma. \tag{3.3.25}$$

Substituting (3.3.22) and (3.3.23) in (3.3.25), we have

$$\begin{aligned} H(\mathrm{Rat}_1, \gamma) &= \int_{SO(3)\times\mathbb{R}^3} \Lambda A[A\bar{B} + 2\bar{A}B]\ d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3, \\ &= \mathrm{Vol}(SO(3)) \int_{\mathbb{R}^3} \Lambda A[A\bar{B} + 2\bar{A}B]\ d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3, \\ &= 8\,\pi^2 \int_{\mathbb{R}^3} \Lambda A[A\bar{B} + 2\bar{A}B]\ d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3, \end{aligned} \tag{3.3.26}$$

where

$$\mathrm{Vol}(SO(3)) = \int_{SO(3)} \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 8\,\pi^2, \tag{3.3.27}$$

is the volume of $SO(3) \cong S^3/\mathbb{Z}_2$ with respect to the round metric of radius 2. To simplify the integral (3.3.26), we use the spherical coordinates $(\lambda, \vartheta, \varphi)$ on $\mathbb{R}^3$ which implies

$$d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3 = \lambda^2 \ \sin\vartheta \ \ d\vartheta \wedge d\varphi \wedge d\lambda. \tag{3.3.28}$$

Hence, the Einstein-Hilbert action on (Rat_1, γ) becomes

$$H(\text{Rat}_1, \gamma) = \ 32\ \pi^3 \ \int_0^\infty \ \lambda^2 \Lambda A[A\bar{B} + 2\bar{A}B] \ \ d\lambda. \tag{3.3.29}$$

we note that

$$\begin{aligned}\frac{d}{d\lambda}[(\lambda\Lambda)^3 A^2 \bar{A}] &= \frac{d}{d\lambda}[(\lambda\Lambda A)^2 (\lambda\Lambda\bar{A})],\\ &= 2(\lambda\Lambda A)\ (\lambda\Lambda\bar{A})\ \frac{d}{d\lambda}\Big(\lambda\Lambda A\Big) + (\lambda\Lambda A)^2\ \frac{d}{d\lambda}\Big(\lambda\Lambda\bar{A}\Big),\\ &= (\lambda\Lambda A)\ \Big[2(\lambda\Lambda\bar{A})\ \frac{d}{d\lambda}\Big(\lambda\Lambda A\Big) + (\lambda\Lambda A)\ \frac{d}{d\lambda}\Big(\lambda\Lambda\bar{A}\Big)\Big].\end{aligned} \tag{3.3.30}$$

Using (3.3.21) and (3.3.24) in (3.3.30), we obtain that

$$\frac{d}{d\lambda}[(\lambda\Lambda)^3 A^2 \bar{A}] = 4\lambda^2 \Lambda A[A\bar{B} + 2\bar{A}B]. \tag{3.3.31}$$

Thus, the Einstein-Hilbert action of any G-invariant Kähler metric γ, determined as in (3.3.14), of Rat_1 is

$$H(\text{Rat}_1, \gamma) = \ 8\ \pi^3 \ [(\lambda\Lambda)^3 A^2 \bar{A}]\Big|_0^\infty. \tag{3.3.32}$$

From the formula for $A_{L^2}(\lambda)$, we have the following limits

$$\lim_{\lambda\to\infty}[(\lambda\Lambda)^3 A_{L^2}^2 \bar{A}] = \pi^2 \quad \text{and} \quad \lim_{\lambda\to 0}[(\lambda\Lambda)^3 A_{L^2}^2 \bar{A}] = 0. \tag{3.3.33}$$

Hence, the Einstein-Hilbert action of $(\text{Rat}_1, \gamma_{L^2})$ is

$$H(\text{Rat}_1, \gamma_{L^2}) = 8\ \pi^5, \tag{3.3.34}$$

which is twice the value conjectured by Baptista in [5]. This may be due to the difference in the scalar curvature definition in [5] from the one in [48] by a factor $1/2$.

The volume of Rat_1 with respect to the L^2 metric has been computed by Baptista in [4]. Here, it is convenient to remark that the volume of Rat_1 with respect to any G-invariant Kähler metric γ, determined as in (3.3.14), can be easily computed by noting that

$$\frac{d}{d\lambda}[\lambda\Lambda A]^3 = \lambda^2\Lambda A^2 B. \tag{3.3.35}$$

Hence, the volume of (Rat_1, γ) is

$$\begin{aligned} \mathrm{Vol}(\mathrm{Rat}_1, \gamma) &= \int_{\mathrm{Rat}_1} \frac{\Lambda}{2} BA^2\, d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3, \\ &= \mathrm{Vol}(SO(3)) \int_{\mathbb{R}^3} \frac{\Lambda}{2} BA^2\;\; d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3, \\ &= 16\pi^3 \int_0^\infty \lambda^2\Lambda A^2 B\, d\lambda = \frac{4\pi^3}{3}[\lambda\Lambda A]^3\Big|_0^\infty . \end{aligned} \tag{3.3.36}$$

For the L^2 metric, we have

$$\lim_{\lambda\to\infty}[\lambda\Lambda A_{L^2}]^3 = \frac{\pi^3}{8} \quad \text{and} \quad \lim_{\lambda\to 0}[\lambda\Lambda A_{L^2}]^3 = 0, \tag{3.3.37}$$

and hence $\mathrm{Vol}(\mathrm{Rat}_1, \gamma_{L^2}) = \pi^6/6$ which coincides with the value found by Baptista in [4].

3.3.3 RMG Flow on Rat_1

Whenever the formulae for the metric g and the Ricci potential 1-form a on a Kähler manifold M are known explicitly, then one can discuss the RMG motion on M, given by an RMG curve $\chi(t)$, by the following Lagrangian

$$L = \frac{1}{2}g(\dot\chi(t), \dot\chi(t)) - a(\dot\chi(t)). \tag{3.3.38}$$

Clearly from the identification $\mathrm{Rat}_1 \cong SO(3)\times\mathbb{R}^3$, one can see that the second de Rham cohomology of Rat_1 is trivial, and so the Ricci form ρ on Rat_1 is exact. Moreover, there exists a G_0-invariant 1-form a such that $\rho = da$. In fact, we obtain that

$$a = \frac{\Lambda}{2}\bar{A}(\lambda)\ \ (\boldsymbol{\lambda}\cdot\boldsymbol{\sigma}). \tag{3.3.39}$$

This follows from the following calculation:

$$\begin{aligned}\rho &= \hat{\hat{A}}_1\ \boldsymbol{d\lambda}\ \dot{\wedge}\ \boldsymbol{\sigma} + \hat{\hat{A}}_2(\boldsymbol{\lambda}\cdot\boldsymbol{d\lambda})\wedge(\boldsymbol{\lambda}\cdot\boldsymbol{\sigma}) + \hat{\hat{A}}_3\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\boldsymbol{\sigma}),\\ &= \hat{\hat{A}}_1\ \boldsymbol{d\lambda}\ \dot{\wedge}\ \boldsymbol{\sigma} + \frac{1}{\lambda}\frac{d\hat{\hat{A}}_1}{d\lambda}(\boldsymbol{\lambda}\cdot\boldsymbol{d\lambda})\wedge(\boldsymbol{\lambda}\cdot\boldsymbol{\sigma}) + \hat{\hat{A}}_1\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\boldsymbol{\sigma}).\end{aligned} \tag{3.3.40}$$

Since

$$d\hat{\hat{A}}_1 = \frac{\partial\hat{\hat{A}}_1}{\partial\lambda_i}\ d\lambda_i = \frac{\lambda_i}{\lambda}\ \frac{d\hat{\hat{A}}_1}{d\lambda}\ d\lambda_i = \frac{1}{\lambda}\frac{d\hat{\hat{A}}_1}{d\lambda}\ (\boldsymbol{\lambda}\cdot\boldsymbol{d\lambda}), \tag{3.3.41}$$

then,

$$\begin{aligned}\rho &= \hat{\hat{A}}_1\ \boldsymbol{d\lambda}\ \dot{\wedge}\ \boldsymbol{\sigma} + d\hat{\hat{A}}_1\wedge(\boldsymbol{\lambda}\cdot\boldsymbol{\sigma}) + \hat{\hat{A}}_1\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\boldsymbol{\sigma}),\\ &= \hat{\hat{A}}_1\ \boldsymbol{d\lambda}\ \dot{\wedge}\ \boldsymbol{\sigma} + d\hat{\hat{A}}_1\ \boldsymbol{\lambda}\ \dot{\wedge}\ \boldsymbol{\sigma} + \hat{\hat{A}}_1\boldsymbol{\lambda}\cdot(\boldsymbol{\sigma}\times\boldsymbol{\sigma}),\\ &= d(\hat{\hat{A}}_1\ \boldsymbol{\lambda})\ \dot{\wedge}\ \boldsymbol{\sigma} + \hat{\hat{A}}_1\boldsymbol{\lambda}\cdot\boldsymbol{d\sigma},\\ &= d(\hat{\hat{A}}_1\ \boldsymbol{\lambda}\cdot\boldsymbol{\sigma}) = d(\frac{\Lambda}{2}\bar{A}\ \boldsymbol{\lambda}\cdot\boldsymbol{\sigma}).\end{aligned} \tag{3.3.42}$$

Here, $\boldsymbol{d\sigma} = \boldsymbol{\sigma}\times\boldsymbol{\sigma}$ gives the exterior derivatives of the left-invariant 1-forms $\{\sigma_k : k = 1,2,3\}$ on $SO(3)$, that is, [48]

$$d\sigma_1 = \sigma_2\wedge\sigma_3, \qquad d\sigma_2 = \sigma_3\wedge\sigma_1, \qquad d\sigma_3 = \sigma_1\wedge\sigma_2. \tag{3.3.43}$$

Hence, the RMG motion on Rat_1 with respect to a G_0-invariant Kähler metric γ, given as in (3.3.14), is determined by the following Lagrangian

$$L = \frac{1}{2}[A_1(\dot{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}}) + A_2(\boldsymbol{\lambda} \cdot \dot{\boldsymbol{\lambda}})^2 + A_3(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) + A_4(\boldsymbol{\lambda} \cdot \boldsymbol{\Omega})^2 + A_5\boldsymbol{\lambda} \cdot (\boldsymbol{\Omega} \times \dot{\boldsymbol{\lambda}})]$$
$$- \frac{\Lambda}{2}\bar{A}(\boldsymbol{\lambda} \cdot \boldsymbol{\Omega}). \tag{3.3.44}$$

for an RMG curve $\chi(t) = ([U(t)], \boldsymbol{\lambda}(t))$ on Rat_1 whose velocity is $\dot{\chi}(t) = ([U(t)\boldsymbol{\Omega}], \dot{\boldsymbol{\lambda}}(t))$, where

$$\dot{\boldsymbol{\lambda}}(t) = (\dot{\lambda}_1(t), \dot{\lambda}_2(t), \dot{\lambda}_3(t)) \in T_{\lambda(t)}\mathbb{R}^3, \tag{3.3.45}$$
$$\boldsymbol{\Omega} = U^{-1}(t)(\dot{U}(t)) \in \mathfrak{su}(2). \tag{3.3.46}$$

Since RMG motion preserves speed, then there is a conserved energy

$$E = \frac{1}{2}[A_1(\dot{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}}) + A_2(\boldsymbol{\lambda} \cdot \dot{\boldsymbol{\lambda}})^2 + A_3(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) + A_4(\boldsymbol{\lambda} \cdot \boldsymbol{\Omega})^2 + A_5\boldsymbol{\lambda} \cdot (\boldsymbol{\Omega} \times \dot{\boldsymbol{\lambda}})]. \tag{3.3.47}$$

Furthermore, since the Lagrangian, given in (3.3.44), has G_0 symmetry, there is a set of six conserved angular momenta, one for each generator of G_0 action. A conserved angular momentum for an arbitrary generator Y of G_0 action, Killing vector field on Rat_1, is given by [35, p.21-23]

$$\mathrm{J}_Y = \gamma(Y, \dot{\chi}) - a(Y) + \alpha_Y, \tag{3.3.48}$$

where α_Y is a scalar function on Rat_1 whose exterior derivative is the Lie derivative of the 1-form a with respect to the Killing vector field Y, that is,

$$d\alpha_Y = \mathcal{L}_Y a. \tag{3.3.49}$$

Since a is G_0-invariant, then $\mathcal{L}_Y a = 0$ for all Y, so we may take $\alpha_Y = 0$ for all Y.

In order to determine the generators of G_0 action, one can think of the left and right $SO(3)$ action on Rat$_1$ separately. It follows from (3.3.11) that the left $SO(3)$ action $\mathcal{L}$ acts on $([U], \boldsymbol{\lambda}) \in$ Rat$_1$ by taking U to its left multiplication by L and leaving $\boldsymbol{\lambda}$ unchanged. Thus, the left $SO(3)$ action on Rat$_1$ acts only on $SO(3)$, and so it is generated by the right-invariant vector fields $\{\xi_k : k = 1, 2, 3\}$ on $SO(3)$ which are related to the left-invariant vector fields $\{\theta_k : k = 1, 2, 3\}$ by

$$\xi_j = \mathcal{U}_{ij}\, \theta_i, \qquad \text{and} \qquad [\xi_i, \theta_j] = 0, \tag{3.3.50}$$

where $\mathcal{U}_{ij} = \text{tr}(\tau_i U \tau_j U^{-1})/2$ are the elements of the matrix $\mathcal{U} = \Phi(U) \in SO(3)$. The relation between such vector fields and $\{\partial/\partial\lambda_k : k = 1, 2, 3\}$ is

$$[\xi_i, \partial/\partial\lambda_j] = [\theta_i, \partial/\partial\lambda_j] = 0, \quad \forall\, i, j = 1, 2, 3. \tag{3.3.51}$$

Hence, the first three conserved angular momenta descending from the left $SO(3)$ action on Rat$_1$ are

$$P_k := \gamma(\xi_k, \chi(t)) - a(\xi_k), \qquad k = 1, 2, 3. \tag{3.3.52}$$

Using (3.3.14), (3.3.39), (3.3.50) and (3.3.51), we obtain that

$$\begin{aligned} P_1 = \mathcal{U}_{11}\, &[A_3\Omega_1 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_1 + \frac{1}{2}A_1(\dot{\lambda}_2\lambda_3 - \dot{\lambda}_3\lambda_2) - \frac{1}{2}\Lambda\bar{A}\lambda_1] \\ + \mathcal{U}_{21}\, &[A_3\Omega_2 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_2 + \frac{1}{2}A_1(\dot{\lambda}_3\lambda_1 - \dot{\lambda}_1\lambda_3) - \frac{1}{2}\Lambda\bar{A}\lambda_2] \\ + \mathcal{U}_{31}\, &[A_3\Omega_3 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_3 + \frac{1}{2}A_1(\dot{\lambda}_1\lambda_2 - \dot{\lambda}_2\lambda_1) - \frac{1}{2}\Lambda\bar{A}\lambda_3], \end{aligned}$$

$$\begin{aligned}P_2 =& \,\mathcal{U}_{12}\,[A_3\Omega_1 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_1 + \frac{1}{2}A_1(\dot\lambda_2\lambda_3 - \dot\lambda_3\lambda_2) - \frac{1}{2}\Lambda\bar{A}\lambda_1]\\ &+\mathcal{U}_{22}\,[A_3\Omega_2 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_2 + \frac{1}{2}A_1(\dot\lambda_3\lambda_1 - \dot\lambda_1\lambda_3) - \frac{1}{2}\Lambda\bar{A}\lambda_2]\\ &+\mathcal{U}_{32}\,[A_3\Omega_3 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_3 + \frac{1}{2}A_1(\dot\lambda_1\lambda_2 - \dot\lambda_2\lambda_1) - \frac{1}{2}\Lambda\bar{A}\lambda_3],\end{aligned}$$

$$\begin{aligned}P_3 =& \,\mathcal{U}_{13}\,[A_3\Omega_1 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_1 + \frac{1}{2}A_1(\dot\lambda_2\lambda_3 - \dot\lambda_3\lambda_2) - \frac{1}{2}\Lambda\bar{A}\lambda_1]\\ &+\mathcal{U}_{23}\,[A_3\Omega_2 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_2 + \frac{1}{2}A_1(\dot\lambda_3\lambda_1 - \dot\lambda_1\lambda_3) - \frac{1}{2}\Lambda\bar{A}\lambda_2]\\ &+\mathcal{U}_{33}\,[A_3\Omega_3 + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_3 + \frac{1}{2}A_1(\dot\lambda_1\lambda_2 - \dot\lambda_2\lambda_1) - \frac{1}{2}\Lambda\bar{A}\lambda_3].\end{aligned}$$

One can write the above angular momenta as a vector

$$\boldsymbol{P} = \mathcal{U}^T[A_3\boldsymbol{\Omega} + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\boldsymbol{\lambda} + \frac{1}{2}A_1(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda})) - \frac{1}{2}\Lambda\bar{A}\boldsymbol{\lambda}], \tag{3.3.53}$$

where $\mathcal{U}^T$ is the transpose of the matrix $\mathcal{U} \in SO(3)$, defined in (3.3.50).

In contrast, the right $SO(3)$ action $\mathcal{R}$ on Rat_1 does not only map U to its right multiplication by R^{-1} but also acts on $\mathbb{R}^3$ by taking $\boldsymbol{\lambda}$ to $\mathcal{R}\boldsymbol{\lambda}$. This implies that the right $SO(3)$ action on Rat_1 is generated by a set of vector fields X_k constructed as

$$\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\lambda}\times\boldsymbol{\partial}, \tag{3.3.54}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ are the left-invariant vector fields on $SO(3)$ and $\boldsymbol{\partial} = (\partial/\partial\lambda_1, \partial/\partial\lambda_2, \partial/\partial\lambda_3)$. Hence, the other three conserved angular momenta are

$$Q_k := \gamma(X_k, \dot\chi) - a(X_k), \qquad k = 1, 2, 3. \tag{3.3.55}$$

Using (3.3.14), (3.3.39), (3.3.50) and (3.3.51), we get

$$Q_1 = (A_3 - \frac{1}{2}\lambda^2 A_1)\Omega_1 + (A_4 + \frac{1}{2}A_1)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_1 - \frac{1}{2}A_1(\dot{\lambda}_2\lambda_3 - \dot{\lambda}_3\lambda_2) - \frac{1}{2}\Lambda\bar{A}\lambda_1,$$
$$Q_2 = (A_3 - \frac{1}{2}\lambda^2 A_1)\Omega_2 + (A_4 + \frac{1}{2}A_1)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_2 - \frac{1}{2}A_1(\dot{\lambda}_3\lambda_1 - \dot{\lambda}_1\lambda_3) - \frac{1}{2}\Lambda\bar{A}\lambda_2,$$
$$Q_3 = (A_3 - \frac{1}{2}\lambda^2 A_1)\Omega_3 + (A_4 + \frac{1}{2}A_1)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\lambda_3 - \frac{1}{2}A_1(\dot{\lambda}_1\lambda_2 - \dot{\lambda}_2\lambda_1) - \frac{1}{2}\Lambda\bar{A}\lambda_3.$$

These also can be written as a vector

$$\boldsymbol{Q} = (A_3 - \frac{1}{2}\lambda^2 A_1)\boldsymbol{\Omega} + (A_4 + \frac{1}{2}A_1)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\boldsymbol{\lambda} - \frac{1}{2}A_1(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}) - \frac{1}{2}\Lambda\bar{A}\boldsymbol{\lambda}. \quad (3.3.56)$$

Hence, we have determined the six conserved angular momenta associated with the RMG motion on Rat_1 for the six Killing vector fields $\{\xi_k, X_k : k = 1, 2, 3\}$ on Rat_1.

Having determined the conserved quantities E, $\boldsymbol{P}$ and $\boldsymbol{Q}$ associated with the RMG dynamical system on (Rat_1, γ), one can eliminate $\boldsymbol{\Omega}$ from (3.3.47) as follows: first, we take the dot product of (3.3.56) by $\boldsymbol{\lambda}$,

$$\begin{aligned}\boldsymbol{Q}\cdot\boldsymbol{\lambda} &= (A_3 - \frac{1}{2}\lambda^2 A_1)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}) + \lambda^2(A_4 + \frac{1}{2}A_1)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}) - \frac{1}{2}\lambda^2\Lambda\bar{A},\\ &= (A_3 + \lambda^2 A_4)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}) - \frac{1}{2}\lambda^2\Lambda\bar{A},\\ &= B(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}) - \frac{1}{2}\lambda^2\Lambda\bar{A}. \end{aligned} \quad (3.3.57)$$

This follows that,

$$\boldsymbol{\Omega}\cdot\boldsymbol{\lambda} = \frac{1}{B}[(\boldsymbol{Q}\cdot\boldsymbol{\lambda}) + \frac{1}{2}\lambda^2\Lambda\bar{A}]. \quad (3.3.58)$$

Second, we compute the squared length of $\boldsymbol{P}$,

$$\begin{aligned}
\|\boldsymbol{P}\|^2 &= \|A_3\boldsymbol{\Omega} + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})\boldsymbol{\lambda} + \frac{1}{2}A_1(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}) - \frac{1}{2}\Lambda\bar{A}\boldsymbol{\lambda}\|^2,\\
&= A_3^2(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}) + (2A_3A_4 + \lambda^2A_4^2)(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^2 + A_3A_1(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda})\cdot\boldsymbol{\Omega} + \frac{1}{4}A_1^2\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2\\
&\quad - \Lambda\bar{A}B(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}) + \frac{1}{4}\lambda^2\Lambda^2\bar{A}^2,\\
&= A_3[A_3(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}) + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^2 + A_1(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda})\cdot\boldsymbol{\Omega}] + \frac{1}{4}A_1^2\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2\\
&\quad + A_4B(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^2 - \Lambda\bar{A}B(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}) + \frac{1}{4}\lambda^2\Lambda^2\bar{A}^2. \qquad (3.3.59)
\end{aligned}$$

It follows from (3.3.59) that

$$\begin{aligned}
&[A_3(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}) + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^2 + A_1(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda})\cdot\boldsymbol{\Omega}] =\\
&\qquad \frac{1}{A_3}[\|\boldsymbol{P}\|^2 - \frac{1}{4}A_1^2\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2 - A_4B(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^2 + \Lambda\bar{A}B(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda}) - \frac{1}{4}\lambda^2\Lambda^2\bar{A}^2]. \qquad (3.3.60)
\end{aligned}$$

Substituting (3.3.58) in (3.3.60), we get

$$\begin{aligned}
&[A_3(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}) + A_4(\boldsymbol{\Omega}\cdot\boldsymbol{\lambda})^2 + A_1(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda})\cdot\boldsymbol{\Omega}] =\\
&\qquad \frac{1}{A_3}[\|\boldsymbol{P}\|^2 - \frac{1}{4}A_1^2\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2 - \frac{A_4}{B}\,[(\boldsymbol{Q}\cdot\boldsymbol{\lambda}) + \lambda^2\hat{\bar{A}}_1]^2 + \Lambda\bar{A}(\boldsymbol{Q}\cdot\boldsymbol{\lambda}) + \frac{1}{4}\lambda^2\Lambda^2\bar{A}^2]. \qquad (3.3.61)
\end{aligned}$$

Then, the conserved energy E in (3.3.47) becomes

$$\begin{aligned}
2E =& A_1\|\dot{\boldsymbol{\lambda}}\|^2 + A_2(\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}})^2 + \frac{1}{A_3}\|\boldsymbol{P}\|^2 - \frac{A_1^2}{4A_3}\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2 + \frac{\Lambda\bar{A}}{A_3}(\boldsymbol{Q}\cdot\boldsymbol{\lambda})\\
&- \frac{A_4}{A_3B}\,[(\boldsymbol{Q}\cdot\boldsymbol{\lambda}) + \frac{1}{2}\lambda^2\Lambda\bar{A}]^2 + \frac{\lambda^2\Lambda^2\bar{A}^2}{4A_3}. \qquad (3.3.62)
\end{aligned}$$

Now, for certain E, $\boldsymbol{P}$ and $\boldsymbol{Q}$, let $X_{E,\boldsymbol{P},\boldsymbol{Q}}$ denote the subset of $T\mathbb{R}^3$ which is defined as the image of the level set of RMG flow on $T\text{Rat}_1$, with E, $\boldsymbol{P}$ and $\boldsymbol{Q}$, under the projection map $T\text{Rat}_1 \to T\mathbb{R}^3$. That is,

$$X_{E,\boldsymbol{P},\boldsymbol{Q}} = \Big\{(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) : 2E = A_1\|\dot{\boldsymbol{\lambda}}\|^2 + A_2(\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}})^2 - \frac{A_1^2}{4A_3}\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2 + \frac{1}{A_3}\|\boldsymbol{P}\|^2 - \frac{A_4}{A_3B}(\boldsymbol{Q}\cdot\boldsymbol{\lambda})^2 + \frac{\Lambda\bar{A}}{A_3}(1-\frac{\lambda^2A_4}{B})(\boldsymbol{Q}\cdot\boldsymbol{\lambda}) + \frac{\lambda^2\Lambda^2\bar{A}^2}{4B}\Big\}. \tag{3.3.63}$$

Theorem 3.3.1. *Rat_1 is RMG complete with respect to the L^2 metric.*

Proof: Let $\mathbf{p_1}$ and $\mathbf{p_2}$ be the projections of the set $X_{E,\boldsymbol{P},\boldsymbol{Q}}$, given in (3.3.63), to the first and second components, respectively. We shall prove the RMG completeness of Rat_1 with respect to the L^2 metric by showing that $\mathbf{p_1}(X_{E,\boldsymbol{P},\boldsymbol{Q}})$ and $\mathbf{p_2}(X_{E,\boldsymbol{P},\boldsymbol{Q}})$ are bounded for all E, $\boldsymbol{P}$ and $\boldsymbol{Q}$.

Let $\hat{\boldsymbol{\lambda}}$ denote the unit vector of $\boldsymbol{\lambda}$ and

$$H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) := A_1\|\dot{\boldsymbol{\lambda}}\|^2 + \lambda^2A_2(\hat{\boldsymbol{\lambda}}\cdot\dot{\boldsymbol{\lambda}})^2 - \frac{\lambda^2A_1^2}{4A_3}\|\hat{\boldsymbol{\lambda}}\times\dot{\boldsymbol{\lambda}}\|^2, \tag{3.3.64}$$

$$G(\boldsymbol{\lambda}, \boldsymbol{Q}) := F_1(\lambda)\ (\boldsymbol{Q}\cdot\hat{\boldsymbol{\lambda}})^2\ + F_2(\lambda)\ (\boldsymbol{Q}\cdot\hat{\boldsymbol{\lambda}})\ + F_3(\lambda), \tag{3.3.65}$$

where

$$F_1(\lambda) = -\frac{\lambda^2A_4}{A_3B}, \qquad F_2(\lambda) = \frac{\lambda\Lambda\bar{A}}{A_3}\left(1-\frac{\lambda^2A_4}{B}\right), \qquad F_3(\lambda) = \frac{\lambda^2\Lambda^2\bar{A}^2}{4B}. \tag{3.3.66}$$

Then, the conserved energy E, given in (3.3.62), can be written as

$$2E = H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) + \frac{1}{A_3}\|\boldsymbol{P}\|^2 + G(\boldsymbol{\lambda}, \boldsymbol{Q}). \tag{3.3.67}$$

Since the cross and dot products on $\mathbb{R}^3$ are related by

$$\|\hat{\boldsymbol{\lambda}} \times \dot{\boldsymbol{\lambda}}\|^2 = \|\dot{\boldsymbol{\lambda}}\|^2 - (\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2, \tag{3.3.68}$$

then,

$$\begin{aligned} H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) &= \left(A_1 - \frac{\lambda^2 A_1^2}{4A_3}\right)\|\dot{\boldsymbol{\lambda}}\|^2 + \lambda^2\left(A_2 + \frac{A_1^2}{4A_3}\right)(\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2, \\ &= \frac{\Lambda^2 A}{1+2\lambda^2}\,\|\dot{\boldsymbol{\lambda}}\|^2 + \lambda^2\left(\frac{2+3\lambda^2}{(1+2\lambda^2)\Lambda^2}A + \frac{A'(\lambda)}{\lambda}\right)(\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2. \end{aligned} \tag{3.3.69}$$

Here, we have used the definition of A_1, A_2 and A_3, as in (3.3.16). Since $B(\lambda)$, given in (3.3.21), is positive, then

$$\frac{A'(\lambda)}{\lambda} > -\frac{1+2\lambda^2}{\lambda^2\Lambda^2}A. \tag{3.3.70}$$

Using (3.3.70) in (3.3.69), we get

$$H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) > \frac{\Lambda^2 A}{1+2\lambda^2}\,[\|\dot{\boldsymbol{\lambda}}\|^2 - (\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2] = \frac{\Lambda^2 A}{1+2\lambda^2}\,\|\hat{\boldsymbol{\lambda}} \times \dot{\boldsymbol{\lambda}}\|^2. \tag{3.3.71}$$

Since $H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) > 0$ and $A_3 > 0$, then it follows from (3.3.67) that

$$2E > G(\boldsymbol{\lambda}, \boldsymbol{Q}). \tag{3.3.72}$$

From (3.3.66) and the formula for $A_{L^2}(\lambda)$ in (3.3.19), one has the limit

$$\lim_{\lambda\to\infty} \frac{\log\lambda}{\lambda^4}\, G(\boldsymbol{\lambda}, \boldsymbol{Q}) = \frac{4}{\pi}[(\boldsymbol{Q} \cdot \hat{\boldsymbol{\lambda}}) + 2]^2. \tag{3.3.73}$$

Assume that $\|\boldsymbol{Q}\| \neq 2$. Then, there is $\delta > 0$ such that

$$[(\boldsymbol{Q} \cdot \hat{\boldsymbol{\lambda}}) + 2]^2 \geq \delta. \tag{3.3.74}$$

Then, by (3.3.73), there exists $\lambda_* > 0$ such that for all $\lambda \geq \lambda_*$

$$G(\boldsymbol{\lambda}, \boldsymbol{Q}) > \frac{\lambda^4}{\log \lambda} \left[(\boldsymbol{Q} \cdot \hat{\boldsymbol{\lambda}}) + 2 \right]^2 \geq \frac{\delta \lambda^4}{\log \lambda}. \tag{3.3.75}$$

Hence, whenever solutions of RMG equations exist, either $\lambda < \lambda_*$ or

$$\frac{2E}{\delta} > \frac{\lambda^4}{\log \lambda}, \qquad \forall\, \lambda \geq \lambda_*. \tag{3.3.76}$$

In either cases, $\mathbf{p_1}(X_{E,\boldsymbol{P},\boldsymbol{Q}})$ is bounded for all E, $\boldsymbol{P}$ and $\boldsymbol{Q}$ with $\|\boldsymbol{Q}\| \neq 2$.

Now, assume that $\|\boldsymbol{Q}\| = 2$ and let θ be the angle between $\boldsymbol{\lambda}$ and $\boldsymbol{Q}$, namely,

$$\boldsymbol{Q} \cdot \hat{\boldsymbol{\lambda}} = \|\boldsymbol{Q}\| \cos \theta = 2 \cos \theta. \tag{3.3.77}$$

Then, it follows from (3.3.65) that

$$G(\boldsymbol{\lambda}, \boldsymbol{Q}) = 4F_1(\lambda) \cos^2 \theta + 2F_2(\lambda) \cos \theta + F_3(\lambda) =: Z(\lambda, \theta). \tag{3.3.78}$$

We shall appeal to the following technical lemma whose proof we postpone until the end of the theorem.

Lemma 3.3.2. *On (Rat_1, γ_{L^2}), there exist $c_0, \lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, $Z(\lambda, \theta)$, given in (3.3.78), satisfies*

$$Z(\lambda, \theta) > \frac{c_0 \lambda^4}{(\log \lambda)^3}, \quad \forall\, \theta \in \mathbb{R}. \tag{3.3.79}$$

Using the above lemma, it follows from (3.3.72), (3.3.78) and (3.3.79) that for all $\lambda \geq \lambda_0$,

$$2E > G(\boldsymbol{\lambda}, \boldsymbol{Q}) > \frac{c_0 \lambda^4}{(\log \lambda)^3}, \tag{3.3.80}$$

for all $\boldsymbol{Q}$ with $\|\boldsymbol{Q}\| = 2$. Thus, whenever solutions of RMG equations exist, either $\lambda < \lambda_0$ or

$$\frac{2E}{c_0} > \frac{\lambda^4}{(\log \lambda)^3}, \qquad \forall\, \lambda \geq \lambda_0. \tag{3.3.81}$$

In either cases, $\mathbf{p_1}(X_{E,\boldsymbol{P},\boldsymbol{Q}})$ is bounded for all E, $\boldsymbol{P}$ and $\boldsymbol{Q}$ with $\|\boldsymbol{Q}\| = 2$. Hence, for all E, $\boldsymbol{P}$ and $\boldsymbol{Q}$, $\mathbf{p_1}(X_{E,\boldsymbol{P},\boldsymbol{Q}})$ is bounded.

Now, we show that $\mathbf{p_2}(X_{E,\boldsymbol{P},\boldsymbol{Q}})$ is also bounded for all E, $\boldsymbol{P}$ and $\boldsymbol{Q}$. By using (3.3.68), it follows from (3.3.69) that $H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}})$ can be written as

$$H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) = K_1(\lambda)\|\hat{\boldsymbol{\lambda}} \times \dot{\boldsymbol{\lambda}}\|^2 + K_2(\lambda)(\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2, \tag{3.3.82}$$

where

$$K_1(\lambda) = \frac{\Lambda^2 A}{1 + 2\lambda^2} > 0, \quad K_2(\lambda) = \lambda^2\left(\frac{A'(\lambda)}{\lambda} + \frac{1 + 2\lambda^2}{\lambda^2\Lambda^2}A\right) > 0. \tag{3.3.83}$$

Since $K_1(\lambda)$ and $K_2(\lambda)$ are continuous functions defined on a closed interval of λ, then they are bounded, and so there exist $c_1, c_2 > 0$ such that

$$K_1(\lambda) \geq c_1 \quad \text{and} \quad K_2(\lambda) \geq c_2. \tag{3.3.84}$$

Using (3.3.84) in (3.3.82), we get

$$H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) \geq c_1\|\hat{\boldsymbol{\lambda}} \times \dot{\boldsymbol{\lambda}}\|^2 + c_2(\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2 > c_3\|\dot{\boldsymbol{\lambda}}\|^2, \tag{3.3.85}$$

where $c3 = \min c_1, c_2$. Also, $H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) > 0$ is a continuous function defined on a closed interval of λ, then it is bounded, and so there exist $c_4 > 0$ such that for all $(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) \in X_{E,\boldsymbol{P},\boldsymbol{Q}}$,

$$H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) \leq c_4. \tag{3.3.86}$$

Thus, for all $(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) \in X_{E,\boldsymbol{P},\boldsymbol{Q}}$, by (3.3.85) and (3.3.86), we have

$$\|\dot{\boldsymbol{\lambda}}\|^2 < \frac{H(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}})}{c_3} \leq \frac{c_4}{c_3} := c_5, \tag{3.3.87}$$

which implies that for all $E, \boldsymbol{P}, \boldsymbol{Q}$, $\mathbf{p_2}(X_{E,\boldsymbol{P},\boldsymbol{Q}})$ is bounded. Hence, for all $E, \boldsymbol{P}, \boldsymbol{Q}$, the whole $X_{E,\boldsymbol{P},\boldsymbol{Q}}$ is bounded.

Consequently, every RMG solution remains in a compact set, whenever it exists. Hence, by applying Picard's method [10, 78-88], we extend RMG solutions by a time $T > 0$, depending only on the compact set. Hence, RMG solutions can be extended for all time.

□

Proof of Lemma 3.3.2: One can obtain using, for example, Maple the following asymptotic formulae for $F_i(\lambda)$, given in (3.3.66), with respect to the L^2 metric on Rat_1 as $\lambda \to \infty$:

$$\begin{aligned} F_1(\lambda) &= \frac{\lambda^4}{\log\lambda}\left[a_1 + \frac{a_2}{\log\lambda} + \frac{a_3}{(\log\lambda)^2} + O\left(\frac{1}{(\log\lambda)^3}\right)\right], \\ F_2(\lambda) &= \frac{\lambda^4}{\log\lambda}\left[b_1 + \frac{b_2}{\log\lambda} + \frac{b_3}{(\log\lambda)^2} + O\left(\frac{1}{(\log\lambda)^3}\right)\right], \\ F_3(\lambda) &= \frac{\lambda^4}{\log\lambda}\left[c_1 + \frac{c_2}{\log\lambda} + \frac{c_3}{(\log\lambda)^2} + O\left(\frac{1}{(\log\lambda)^3}\right)\right], \end{aligned} \tag{3.3.88}$$

where

$$\begin{aligned} a_1 &= \frac{4}{\pi}, & a_2 &= \frac{2}{\pi}[1 - 2\log 2], & a_3 &= \frac{1}{\pi}[1 - 4\log 2 + 4(\log 2)^2], \\ b_1 &= \frac{16}{\pi}, & b_2 &= \frac{2}{\pi}[3 - 8\log 2], & b_3 &= \frac{1}{\pi}[2 - 12\log 2 + 16(\log 2)^2], \\ c_1 &= \frac{16}{\pi}, & c_2 &= \frac{4}{\pi}[1 - 4\log 2], & c_3 &= \frac{1}{\pi}[1 - 32\log 2 + 64(\log 2)^2]. \end{aligned} \tag{3.3.89}$$

It follows from (3.3.78) and (3.3.88) that

$$Z(\lambda,\theta) = Z_0(\lambda,\theta) + Z_{\text{error}}(\lambda,\theta), \tag{3.3.90}$$

where

$$Z_0(\lambda,\theta) = \frac{\lambda^4}{\log\lambda}\Bigg[(4a_1\cos^2\theta + 2b_1\cos\theta + c_1) + \frac{1}{\log\lambda}(4a_2\cos^2\theta + 2b_2\cos\theta + c_2) + \frac{1}{(\log\lambda)^2}(4a_3\cos^2\theta + 2b_3\cos\theta + c_3)\Bigg], \tag{3.3.91}$$

and $Z_{\text{error}}(\lambda,\theta)$ satisfies the following estimate: there exist $c_*, \lambda_* > 0$ such that for all $\lambda \geq \lambda_*$,

$$\big|Z_{\text{error}}(\lambda,\theta)\big| < \frac{c_*\lambda^4}{(\log\lambda)^4}, \quad \forall\,\theta\in\mathbb{R}. \tag{3.3.92}$$

Hence, it suffices to prove that $Z_0(\lambda,\theta)$ satisfies an estimate of the form (3.3.79).

Defining $\tau = 1+\cos\theta$ and $x = 1/\log\lambda$. Then

$$\frac{\log\lambda}{\lambda^4}\,Z_0(\lambda,\theta) = P_x(\tau), \tag{3.3.93}$$

where

$$P_x(\tau) = \alpha_1(x)\tau^2 + \alpha_2(x)\tau + \alpha_3(x), \tag{3.3.94}$$

and the coefficients α_1, α_2 and α_3 given by

$$\begin{aligned}\alpha_1(x) &= 4(a_1 + a_2x + a_3x^2),\\ \alpha_2(x) &= 2(b_2-4a_2)x + 2(b_3-4a_3)x^2,\\ \alpha_3(x) &= (4a_3 - 2b_3 + c_3)x^2.\end{aligned} \tag{3.3.95}$$

Since $\alpha_1(0) > 0$, then there exists $x_* > 0$ such that for all $x \in (-x_*, x_*)$, $P_x(\tau)$ has a minimum, occurs at $\tau = \tau_*$, where $dP_x(\tau)/d\tau\Big|_{\tau=\tau_*} = 0$, that is,

$$\tau_*(x) = -\frac{1}{2}\frac{\alpha_2(x)}{\alpha_1(x)}. \tag{3.3.96}$$

So, for all $x \in (-x_*, x_*)$, the minimum value of $P_x(\tau)$ is

$$P_x(\tau_*(x)) = -\frac{1}{4\alpha_1(x)}[\alpha_2(x)^2 - 4\alpha_1(x)\alpha_3(x)]. \tag{3.3.97}$$

Note that $P_x(\tau_*(x))$ is a rational function of x, and hence is analytic. Using (4.1.4), one finds that

$$P_0(\tau_*(0)) = 0, \qquad \frac{d}{dx}P_x(\tau_*(x))\Big|_{x=0} = 0, \tag{3.3.98}$$

and

$$\frac{d^2}{dx^2}P_x(\tau_*(x))\Big|_{x=0} = -\frac{1}{8a_1}[(b_2 - 4a_2)^2 - 16a_1(4a_3 - 2b_3 + c_3)] > 0. \tag{3.3.99}$$

Thus, there exist $\varepsilon > 0$ and $0 < x_0 < x_*$ such that for all $x \in (-x_0, x_0)$,

$$P_x(\tau_*(x)) \geq \varepsilon\, x^2. \tag{3.3.100}$$

Hence, for all $x \in (0, x_0)$,

$$P_x(\tau) \geq \varepsilon\, x^2, \qquad \forall\, \tau \in \mathbb{R}. \tag{3.3.101}$$

Hence, it follows from (3.3.93) that for all $\lambda > e^{1/x_0}$,

$$\frac{\log\lambda}{\lambda^4}\, Z_0(\lambda,\theta) = P_x(\tau) \geq \varepsilon\, x^2 = \frac{\varepsilon}{(\log\lambda)^2}, \quad \forall\, \theta \in \mathbb{R}, \tag{3.3.102}$$

which implies that $Z_0(\lambda, \theta)$ satisfies the estimate (3.3.79).

□

3.4 Dilation Cylinders

In this section, we are interested in studying some geometric properties of a certain submanifold of $\mathcal{H}_{n,1}(\Sigma)$, the space of degree n holomorphic maps from a compact Riemann surface Σ to $\mathbb{C}P^1$.

Let Q be a degree n meromorphic function on Σ, then the subset

$$C_Q = \{a\, Q : a \in \mathbb{C}^{\times}\} \subset \mathcal{H}_{n,1}(\Sigma), \tag{3.4.1}$$

is a non-compact 1-dimensional complex submanifold of $\mathcal{H}_{n,1}(\Sigma)$. The induced L^2 metric on C_Q can be determined by computing the kinetic energy $T[W]$, given in (3.1.8), for $W(t) = a(t)\, Q$. That is,

$$T[W] = \frac{1}{2}\int_{\Sigma} \frac{|\dot{W}|^2}{(1+|W|^2)^2}\,\mathrm{vol}_g = \frac{1}{2}\int_{\Sigma} \frac{|Q|^2\,|\dot{a}|^2}{(1+|a|^2\,|Q|^2)^2}\,\mathrm{vol}_g, \tag{3.4.2}$$

where the codomain $\mathbb{C}P^1$, here, is equipped with the Fubini-Study metric of constant holomorphic sectional curvature $c = 4$. Hence, the induced L^2 metric on C_Q is

$$g_{L^2} = \left(\int_{\Sigma} \frac{|Q|^2}{(1+|a|^2\,|Q|^2)^2}\,\mathrm{vol}_g\right) da d\bar{a}. \tag{3.4.3}$$

Let $a = \chi\, e^{i\psi}$, where (χ, ψ) are the polar coordinates on $\mathbb{R}^2 \backslash \{0\}$. Then, the L^2 metric g_{L^2} can be written as

$$g_{L^2} = F(\chi)(d\chi^2 + \chi^2 d\psi^2), \tag{3.4.4}$$

where $F(\chi)$ is given by

$$F(\chi) = \int_{\Sigma} \frac{|Q|^2}{(1+\chi^2\,|Q|^2)^2}\,\mathrm{vol}_g. \tag{3.4.5}$$

We call this submanifold, with the induced L^2 metric g_{L^2}, a dilation cylinder of $\mathcal{H}_{n,1}(\Sigma)$. Note that special cases of these cylinders has been studied before in [38] where $\Sigma = S^2$ and $Q(z) = z^n$, and in [47] with $\Sigma = T^2$ and $Q(z)$ is the Weierstrass elliptic function of degree 2.

Proposition 3.4.1. *Dilation cylinders of $\mathcal{H}_{n,1}(\Sigma)$ have Q-independent finite volume,*

$$\mathit{Vol}(C_Q, g_{L^2}) = \pi\ \mathit{Vol}(\Sigma, g). \tag{3.4.6}$$

Proof: The volume of C_Q with respect to g_{L^2} is

$$\begin{aligned}\mathrm{Vol}(C_Q, g_{L^2}) = 2\pi \int_0^\infty \chi\, F(\chi)\, d\chi &= 2\pi \int_0^\infty \left(\int_\Sigma \frac{\chi \,|Q|^2}{(1+\chi^2\,|Q|^2)^2} \mathrm{vol}_g \right) d\chi, \\ &= 2\pi \int_0^\infty \left(\int_\Sigma \frac{s}{(1+s^2)^2} \mathrm{vol}_g \right) ds, \end{aligned} \tag{3.4.7}$$

where $s = \chi\,|Q|$. Since Σ is compact, then $\mathrm{Vol}(\Sigma, g)$ is finite, and so

$$\int_0^\infty \left(\int_\Sigma \frac{s}{(1+s^2)^2} \mathrm{vol}_g \right) ds < \infty, \qquad \int_\Sigma \left(\int_0^\infty \frac{s}{(1+s^2)^2} ds \right) \mathrm{vol}_g < \infty. \tag{3.4.8}$$

Since both Σ and $(0, \infty)$ are σ-finite measure spaces, then we can apply the Fubini Theorem [11, p.185] in (3.4.7),

$$\mathrm{Vol}(C_Q, g_{L^2}) = 2\pi \int_\Sigma \mathrm{vol}_g \int_0^\infty \frac{s}{(1+s^2)^2}\, ds = \pi\, \mathrm{Vol}(\Sigma, g). \tag{3.4.9}$$

□

Note that Proposition 3.4.1 generalizes a result for $\Sigma = S^2$ in [38].

Proposition 3.4.2. *Dilation cylinders of $\mathcal{H}_{n,1}(\Sigma)$ can be isometrically embedded as surfaces of revolution in $\mathbb{R}^3$.*

Proof: Assume that there is an embedding $\mathbf{X} : C_Q \to \mathbb{R}^3$ given by

$$\mathbf{X}(\chi, \psi) = (\alpha(\chi), \beta(\chi)\cos\psi, \beta(\chi)\sin\psi). \tag{3.4.10}$$

Then, the induced metric on $\mathbf{X}(C_Q) \subset \mathbb{R}^3$ is

$$g_{\mathrm{sor}} = \left(\alpha'(\chi)^2 + \beta'(\chi)^2\right) d\chi^2 + \beta(\chi)^2\, d\psi^2. \tag{3.4.11}$$

Hence, $\mathbf{X}$ is an isometric embedding if and only if

$$\beta(\chi) = \chi\sqrt{F(\chi)}, \tag{3.4.12}$$

$$\frac{d\alpha}{d\chi} = \sqrt{F(\chi)}\sqrt{1 - \left(1 + \frac{\chi F'(\chi)}{2F(\chi)}\right)^2}. \tag{3.4.13}$$

Clearly, the solution of (3.4.13) exists if

$$-1 < 1 + \frac{\chi F'(\chi)}{2F(\chi)} < 1, \qquad \forall\, \chi > 0. \tag{3.4.14}$$

One can show that the function $F(\chi)$ indeed satisfies (3.4.14) for all $\chi > 0$ as follows:

$$1 + \frac{\chi F'(\chi)}{2F(\chi)} = \frac{1}{2F(\chi)}\big[2F(\chi) + \chi F'(\chi)\big]. \tag{3.4.15}$$

It follows from (3.4.5) that

$$F'(\chi) = -4\chi \int_\Sigma \frac{|Q|^4}{(1+\chi^2|Q|^2)^3}\,\mathrm{vol}_g < 0. \tag{3.4.16}$$

Then, from (3.4.5) and (3.4.16), we have

$$2F(\chi) + \chi F'(\chi) = 2\int_\Sigma \frac{|Q|^2}{(1+\chi^2|Q|^2)^2}\left[1 - \frac{2\chi^2|Q|^2}{1+\chi^2|Q|^2}\right]\mathrm{vol}_g. \tag{3.4.17}$$

Since for all $\chi > 0$,

$$1 - \frac{2\chi^2|Q|^2}{1+\chi^2|Q|^2} > -1. \tag{3.4.18}$$

Then, using (3.4.18) in (3.4.17), we get

$$2F(\chi) + \chi F'(\chi) > -2F(\chi). \tag{3.4.19}$$

Also, it follows directly as $F'(\chi) < 0$ that

$$2F(\chi) + \chi F'(\chi) < 2F(\chi). \tag{3.4.20}$$

Hence, Using (3.4.19) and (3.4.20) in (3.4.15), the claim is established.

□

Note that Proposition 3.4.2 proves (and generalizes) a conjecture in [38] with $\Sigma = S^2$ and $Q(z) = z^n$.

Proposition 3.4.3. *Let Q be a meromorphic function on Σ with no simple poles. For each $k \in \mathbb{Z}^+$, let $F_k : (0, \infty) \to \mathbb{R}$ be*

$$F_k(\chi) = \int_\Sigma \frac{|Q|^2}{(1 + \chi^2 |Q|^2)^k} \, vol_g. \tag{3.4.21}$$

Then, there exists a real number $\mathscr{R} > 0$ such that for all $k \in \mathbb{Z}^+$, $F_k(\chi)$ is asymptotically given by

$$F_k(\chi) \sim \frac{\mathscr{R}\, C_k}{\chi^{2-2/p}}, \qquad \text{as } \chi \to 0, \tag{3.4.22}$$

where p is the maximum order of the poles of Q, and

$$C_k = \int_0^\infty \frac{r^{p(k-1)}}{(r^p + 1)^k} \, dr. \tag{3.4.23}$$

Proof: We adapt the argument used to prove Lemma 2 in [47]. Let $Q(z)$ be a meromorphic function with poles z_i of order $p_i \geq 2$, and let $p = \max\{p_i\}$. For $\epsilon_i > 0$, we define open disks of radius ϵ_i centred at z_i as

$$D_{\epsilon_i}(z_i) = \{z \in \Sigma : |z - z_i| < \epsilon_i\}. \tag{3.4.24}$$

It is convenient to split Σ into the union of $D_{\epsilon_i}(z_i)$ and its complement, namely,

$$\Sigma = \bigcup_i D_{\epsilon_i}(z_i) \cup \left(\Sigma \backslash \bigcup_i D_{\epsilon_i}(z_i)\right). \tag{3.4.25}$$

Then,

$$\sum_i \int_{D_{\epsilon_i}(z_i)} \frac{|Q(z)|^2}{\left(1+\chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g < F_k(\chi) < \sum_i \int_{D_{\epsilon_i}(z_i)} \frac{|Q(z)|^2}{\left(1+\chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g + \int_{\Sigma \backslash \bigcup_i D_{\epsilon_i}(z_i)} \frac{|Q(z)|^2}{\left(1+\chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g. \tag{3.4.26}$$

Since $|Q(z)|$ is bounded on $\Sigma \backslash \bigcup_i D_{\epsilon_i}(z_i)$, then there exists $M > 0$ such that

$$\int_{\Sigma \backslash \bigcup_i D_{\epsilon_i}(z_i)} \frac{|Q(z)|^2}{\left(1+\chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g < \int_{\Sigma \backslash \bigcup_i D_{\epsilon_i}(z_i)} |Q(z)|^2 \operatorname{vol}_g < M. \tag{3.4.27}$$

Thus,

$$\sum_i \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p} |Q(z)|^2}{\left(1+\chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g < \chi^{2-2/p} F_k(\chi) < \sum_i \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p} |Q(z)|^2}{\left(1+\chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g + \chi^{2-2/p} M. \tag{3.4.28}$$

Hence, we need to compute

$$\lim_{\chi \to 0} \sum_i \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p} |Q(z)|^2}{\left(1+\chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g. \tag{3.4.29}$$

The Laurent expansion of $Q(z)$ about the pole z_i of order p_i is

$$Q(z) = a_i (z-z_i)^{-p_i} + O((z-z_i)^{-(p_i-1)}), \tag{3.4.30}$$

The function $h(z) = (z-z_i)^{p_i} Q(z)$ is analytic and bounded on $D_{\epsilon_i}(z_i)$. One can choose ϵ_i such that $h(z)$ is bounded away from zero. Thus, there exist $c_1, c_2 > 0$ such that

$0 < c_1 < |h(z)| < c_2 < \infty$. Now, the integrand in (3.4.29) can be written in terms of $h(z)$ as

$$\frac{|Q(z)|^2}{(1+\chi^2 |Q(z)|^2)^k} = \frac{|h(z)|^2 \left|(z-z_i)^{p_i}/\chi\right|^{2k} |z-z_i|^{-2p_i}}{(\left|(z-z_i)^{p_i}/\chi\right|^2 + |h(z)|^2)^k}. \tag{3.4.31}$$

Defining $u = (z-z_i)/\chi^{1/p_i}$ and $\delta_i = \epsilon_i/\chi^{1/p_i}$, then

$$\begin{aligned}
&\int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p_i} |Q(z)|^2}{(1+\chi^2 |Q(z)|^2)^k} \operatorname{vol}_g \\
&\quad = \frac{i}{2}\int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p_i} |h(z)|^2 \left|(z-z_i)^{p_i}/\chi\right|^{2k} |z-z_i|^{-2p_i}}{(\left|(z-z_i)^{p_i}/\chi\right|^2 + |h(z)|^2)^k}\, V^2(z) dz d\bar{z}, \\
&\quad = \frac{i}{2}\int_{D_{\delta_i}(0)} \frac{\left|h(\chi^{1/p_i}u + z_i)\right|^2 |u|^{2p_i(k-1)}}{(|u|^{2p_i} + \left|h(\chi^{1/p_i}u+z_i)\right|^2)^k}\, V^2(\chi^{1/p_i}u + z_i) du\, d\bar{u},
\end{aligned} \tag{3.4.32}$$

where $\operatorname{vol}_g = iV^2(z) dz \wedge d\bar{z}/2$. Clearly, the integrand in (3.4.32) is bounded above by

$$\frac{c_2^2\, V_{\max}^2\, |u|^{2p_i(k-1)}}{(|u|^{2p_i} + c_1^2)^k}, \tag{3.4.33}$$

which is integrable on $\mathbb{C}$ for all $k \in \mathbb{Z}^+$. Then, one can apply the Lebesgue Dominated Convergence Theorem (LDCT) [16, p.43-44] in (3.4.32),

$$\lim_{\chi\to 0} \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p_i} |Q(z)|^2}{(1+\chi^2 |Q(z)|^2)^k} \operatorname{vol}_g \tag{3.4.34}$$

$$= \frac{i}{2}\int_{\mathbb{C}} \lim_{\chi\to 0}\left(\frac{\left|h(\chi^{1/p_i}u + z_i)\right|^2 |u|^{2p_i(k-1)}}{(|u|^{2p_i} + \left|h(\chi^{1/p_i}u+z_i)\right|^2)^k}\, V^2\big(\chi^{1/p_i}u + z_i\big)\right) \tau_{\delta_i}(u) du\, d\bar{u}, \tag{3.4.35}$$

where $\tau_{\delta_i}(u)$ is the characteristic function of the disk $D_{\delta_i}(0)$, that is,

$$\tau_{\delta_i}(u) = \begin{cases} 1 & |u| < \delta_i \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.36}$$

Recall that $h(z) = (z - z_i)^{p_i} Q(z)$, and so $\lim_{z \to z_i} h(z) = a_i$. Hence,

$$\lim_{\chi \to 0} \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p_i} |Q(z)|^2}{(1 + \chi^2 |Q(z)|^2)^k} \operatorname{vol}_g = \frac{i}{2} a_i^2 \, V^2(z_i) \int_{\mathbb{C}} \frac{|u|^{2p_i(k-1)}}{(|u|^{2p_i} + a_i^2)^k} \, du \, d\bar{u}, \tag{3.4.37}$$

To simplify the integral above, let $u = a_i^{1/p_i} r^{1/2} \, e^{i\theta}$, where $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$, then

$$\lim_{\chi \to 0} \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p_i} |Q(z)|^2}{(1 + \chi^2 |Q(z)|^2)^k} \operatorname{vol}_g = \pi \, a_i^{2/p_i} \, V^2(z_i) \int_0^\infty \frac{r^{p_i(k-1)}}{(r^{p_i} + 1)^k} \, dr. \tag{3.4.38}$$

Since $p \geq p_i$, then $\chi^{2-2/p} \geq \chi^{2-2/p_i}$. Thus, it follows from (3.4.38) that

$$\lim_{\chi \to 0} \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p} |Q(z)|^2}{(1 + \chi^2 |Q(z)|^2)^k} \operatorname{vol}_g = \begin{cases} \pi \, a_i^{2/p} \, V^2(z_i) \displaystyle\int_0^\infty \frac{r^{p(k-1)}}{(r^p + 1)^k} \, dr & p_i = p \\ 0 & p_i < p. \end{cases} \tag{3.4.39}$$

Hence,

$$\begin{aligned} \lim_{\chi \to 0} \sum_i \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p} |Q(z)|^2}{\left(1 + \chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g &= \sum_i \left(\lim_{\chi \to 0} \int_{D_{\epsilon_i}(z_i)} \frac{\chi^{2-2/p} |Q(z)|^2}{\left(1 + \chi^2 |Q(z)|^2\right)^k} \operatorname{vol}_g \right), \\ &= \left(\pi \sum_{z_i,\, p_i = p} a_i^{2/p} \, V^2(z_i) \right) \int_0^\infty \frac{r^{p(k-1)}}{(r^p + 1)^k} \, dr, \end{aligned} \tag{3.4.40}$$

$$=: \mathscr{R} \int_0^\infty \frac{r^{p(k-1)}}{(r^p + 1)^k} \, dr. \tag{3.4.41}$$

Hence, the claim is proved.

□

We can use Proposition 3.4.3 to compute the leading asymptotic of $F_k(\chi)$'s derivatives as follows: the first derivative of $F_k(\chi)$ with respect to χ is

$$F_k'(\chi) = -2k \int_\Sigma \frac{\chi\,|Q(z)|^4}{(1+\chi^2\,|Q(z)|^2)^{k+1}}\ \mathrm{vol}_g = -\frac{2k}{\chi}\Big[F_k(\chi) - F_{k+1}(\chi)\Big]. \tag{3.4.42}$$

Hence, it follows from Proposition 3.4.3, the asymptotic formula of $F_k'(\chi)$ as $\chi \to 0$ is

$$F_k'(\chi) \sim -2k\,\mathscr{R}\,[C_k - C_{k+1}]\,\chi^{-3+2/p}. \tag{3.4.43}$$

The second derivative of $F_k(\chi)$ is

$$F_k''(\chi) = \frac{2k}{\chi^2}\Big[F_k(\chi) - F_{k+1}(\chi)\Big] - \frac{2k}{\chi}\Big[F_k'(\chi) - F_{k+1}'(\chi)\Big]. \tag{3.4.44}$$

It follows from (3.4.42) that

$$F_k''(\chi) = \frac{2k(2k+1)}{\chi^2}\Big[F_k(\chi) - F_{k+1}(\chi)\Big] - \frac{4k(k+1)}{\chi^2}\Big[F_{k+1}(\chi) - F_{k+2}(\chi)\Big]. \tag{3.4.45}$$

Hence, as $\chi \to 0$,

$$F_k''(\chi) \sim \mathscr{R}\Big[2k(2k+1)[C_k - C_{k+1}] - 4k(k+1)[C_{k+1} - C_{k+2}]\Big]\ \chi^{-4+2/p}. \tag{3.4.46}$$

Following this procedure, one can compute the leading asymptotic of the derivative $F_k^{(n)}(\chi)$ for all $n \in \mathbb{N}$.

Proposition 3.4.4. *A dilation cylinder C_Q of a meromorphic function Q with no simple poles is RMG incomplete if its scalar curvature is bounded above in the neighbourhood of zero.*

Proof: Comparing the metric component $F(\chi)$, given in (3.4.5), of the induced L^2 metric on C_Q with the function $F_k(\chi)$, defined in (3.4.22), we note that

$$F(\chi) \equiv F_2(\chi). \tag{3.4.47}$$

Let Q have no simple poles. Then, by Proposition 3.4.3, we have the following limits

$$\lim_{\chi\to 0} \chi^{2-2/p} F(\chi) = \mathscr{R}\, C_2, \qquad \lim_{\chi\to 0} \frac{\chi F'(\chi)}{2F(\chi)} = -2\left[1 - \frac{C_3}{C_2}\right] =: P_0. \tag{3.4.48}$$

Now, let C_Q have a scalar curvature which is bounded above in the neighbourhood of 0. We show that C_Q is RMG incomplete. It is sufficient to check that there exists at least one RMG curve on C_Q which escapes to the boundary at 0 with finite length.

Associated with the RMG dynamical system on C_Q, there are two conserved quantities: energy $E > 0$ and angular momentum $P \in \mathbb{R}$ given by

$$E = \frac{1}{2} F(\chi)\dot{\chi}^2 + V_P(\chi), \qquad P = \chi^2 F(\chi)\dot{\psi}^2 + \frac{\chi F'(\chi)}{2F(\chi)}, \tag{3.4.49}$$

where $V_P(\chi)$ is the effective potential defined as in (1.4.15). Consider $V_P(\chi)$ with the angular momentum $P = P_0$, that is,

$$\begin{aligned} V_{P_0}(\chi) &= \frac{1}{2\chi^2 F(\chi)} \left(P_0 - \frac{\chi F'(\chi)}{2F(\chi)} \right)^2, \\ &= \frac{1}{2\chi^{2-2/p} F(\chi)} \left[\frac{1}{\chi^{1/p}} \left(P_0 - \frac{\chi F'(\chi)}{2F(\chi)} \right) \right]^2. \end{aligned} \tag{3.4.50}$$

It follows from (3.4.48) that

$$\lim_{\chi\to 0} \left(P_0 - \frac{\chi F'(\chi)}{2F(\chi)} \right) = 0. \tag{3.4.51}$$

Then, by L'Hôpital's rule, we have

$$\lim_{\chi\to 0}\frac{1}{\chi^{1/p}}\left(P_0-\frac{\chi F'(\chi)}{2F(\chi)}\right)=-p\lim_{\chi\to 0}\chi^{1-1/p}\frac{d}{d\chi}\left(\frac{\chi F'(\chi)}{2F(\chi)}\right). \tag{3.4.52}$$

Since the scalar curvature on C_Q is

$$\kappa(\chi)=-\frac{1}{\chi F(\chi)}\frac{d}{d\chi}\left(\frac{\chi F'(\chi)}{2F(\chi)}\right), \tag{3.4.53}$$

then, we have

$$\begin{aligned}\lim_{\chi\to 0}\frac{1}{\chi^{1/p}}\left(P_0-\frac{\chi F'(\chi)}{2F(\chi)}\right)&=p\lim_{\chi\to 0}\chi^{2-1/p}F(\chi)\ \kappa(\chi),\\ &=p\lim_{\chi\to 0}\big(\chi^{2-2/p}F(\chi)\big)\ \lim_{\chi\to 0}\big(\chi^{1/p}\kappa(\chi)\big),\\ &=p\,\mathscr{R}\,C_2\lim_{\chi\to 0}\big(\chi^{1/p}\kappa(\chi)\big). \end{aligned} \tag{3.4.54}$$

Since the scalar curvature $\kappa(\chi)$ is bounded above in the neighbourhood of 0, then

$$\lim_{\chi\to 0}\chi^{1/p}\kappa(\chi)=0. \tag{3.4.55}$$

Hence, for $P=P_0$, it follows from (3.4.48), (3.4.54) and (3.4.55) that if $\kappa(\chi)$ is bounded above in the neighbourhood of 0, then

$$\lim_{\chi\to 0}V_{P_0}(\chi)=0. \tag{3.4.56}$$

This means that for all $E_*>0$, there exists $\chi_*>0$ such that $V_{P_0}(\chi)\leq E_*$ for all $\chi\leq\chi_*$. Now, by choosing $E>2E_*$, then

$$E>2E_*\geq 2V_{P_0}(\chi),\qquad \forall\,\chi\leq\chi_*, \tag{3.4.57}$$

and so, $(0,\chi_*]\subset\mathbf{p_1}(D_{E,P_0})$, where $\mathbf{p_1}$ is the projection on D_{E,P_0}, defined as in (1.4.16), to the first component. Also, for all $\chi\leq\chi_*$, we have

$$\dot{\chi}^2F(\chi)=2[E-V_{P_0}(\chi)]>2E_*>0, \tag{3.4.58}$$

from which it follows that $\dot{\chi}$ will never be zero.

Thus, for an RMG curve $c(t) = (\chi(t), \psi(t))$ on C_Q, starting at $t = 0$ away from 0, with $P = P_0$ and $E > 2E_*$, there exists $t_* \in (0, \infty)$ such that $\lim_{t \to t_*} \chi(t) = 0$. Hence, the RMG curve $c(t)$ in $[0, t_*]$ hits the boundary at 0 with finite length. Hence, C_Q is RMG incomplete, providing its scalar curvature is bounded above about 0.

□

3.4.1 Rat_n^{eq} Submanifolds Review

This section deals with an interesting example of dilation cylinders in the space of degree n holomorphic maps from S^2 to $\mathbb{C}P^1$, $\mathcal{H}_{n,1}(S^2) \cong \text{Rat}_n$.

Let K_0 be the subgroup of $G_0 = SO(3) \times SO(3)$ defined as

$$K_0 = \left\{ (R_{n\alpha}, R_\alpha) : R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \alpha \in \mathbb{R} \right\} \cong SO(2). \tag{3.4.59}$$

The action of the subgroup K_0 on Rat_n, defined by (3.3.4), is given in terms of stereographic coordinates by

$$W(z) \mapsto e^{in\alpha} W(e^{-i\alpha} z). \tag{3.4.60}$$

By (3.3.1), this means that

$$\frac{a_1 + a_2 z + \cdots + a_{n+1} z^n}{a_{n+2} + a_{n+3} z + \cdots + a_{2n+2} z^n} \mapsto \frac{a_1 e^{in\alpha} + a_2 e^{i(n-1)\alpha} z + \cdots + a_{n+1} z^n}{a_{n+2} + a_{n+3} e^{-i\alpha} z + \cdots + a_{2n+2} e^{-in\alpha} z^n}. \tag{3.4.61}$$

The fixed point set of the K_0 action on Rat_n, denoted Rat_n^{eq}, has been shown in [38] to be

$$\text{Rat}_n^{eq} = \{ a z^n : a \in \mathbb{C}^\times \} \cong \mathbb{C}^\times. \tag{3.4.62}$$

This follows by showing that the fixed point set of K_0 is a subset of $\{W(z) \in \mathrm{Rat}_n : a_{n+2} \neq 0\}$, and so, by (3.4.61), the elements of Rat_n with $a_{n+2} \neq 0$ are fixed under the K_0 action if and only if

$$W(z) = \frac{a_{n+1}}{a_{n+2}}\, z^n =: a z^n. \tag{3.4.63}$$

It is clear that Rat_n^{eq} is a non-compact 1-dimensional complex submanifold of Rat_n. The induced L^2 metric, in terms of the polar coordinates (χ, ψ) in $\mathbb{R}^2\backslash\{0\}$, is

$$g_{L^2} = F(\chi)\, (d\chi^2 + \chi^2\, d\psi^2), \tag{3.4.64}$$

where

$$F(\chi) = \frac{i}{2} \int_{\mathbb{C}} \frac{|z|^{2n}}{(1+\chi^2\,|z|^{2n})^2}\, \frac{dz d\bar{z}}{(1+|z|^2)^2} = 2\pi \int_0^\infty \frac{r^{2n+1}}{(1+\chi^2 r^{2n})^2}\, \frac{dr}{(1+r^2)^2}. \tag{3.4.65}$$

The geometry and geodesic flow on Rat_n^{eq}, equipped with the L^2 metric g_{L^2}, were studied in detail in [38]. The isometry $\delta : z \mapsto z^{-1}$ of S^2 induces an isometry $\hat{\delta} : \phi \mapsto \delta \circ \phi \circ \delta^{-1}$ of Rat_n. The restriction of $\hat{\delta}$ to Rat_n^{eq}, given in the coordinates (χ, ψ) by $\hat{\delta} : (\chi, \psi) \mapsto (\chi^{-1}, -\psi)$, is an isometry of Rat_n^{eq}. Hence,

$$\hat{\delta}^* g_{L^2} = \chi^{-4} F(\chi^{-1})\, (d\chi^2 + \chi^2\, d\psi^2) = g_{L^2}. \tag{3.4.66}$$

This means that,

$$F(\chi^{-1}) = \chi^4 F(\chi), \qquad \forall\, \chi > 0. \tag{3.4.67}$$

Clearly, Rat_n^{eq} is an example of a dilation cylinder of $\mathcal{H}_{n,1}(S^2)$ with a meromorphic function $Q(z) = z^n$ of one pole of order n. Since Rat_n^{eq} is the fixed point set of the group K_0, which is a group of holomorphic isometries of Rat_n, then, by Corollary 1.3.3, RMG flow is localized to Rat_n^{eq}. But Rat_n^{eq} is not totally RMG (for example, one can see this explicitly for $n = 1$).

3.4.2 RMG Flow on Rat_n^{eq} Submanifolds

A natural question to ask about Rat_n^{eq} is whether it is RMG complete with respect to the induced L^2 metric, and how this depends on the degree n.

First, it is convenient to introduce the natural n-fold cover $\pi : \mathbb{C}^\times \to \mathrm{Rat}_n^{eq}$ defined, in terms of the polar coordinates (ρ, ϑ), by

$$\pi(\rho, \vartheta) = (\rho^n, n\psi), \tag{3.4.68}$$

namely, $W(z) = az^n = (\rho\, e^{i\vartheta} z)^n$. The pullback of the metric g_{L^2} on Rat_n^{eq} by π is

$$\tilde{g}_{L^2} := \pi^* g_{L^2} = \tilde{F}(\rho)\; (d\rho^2 + \rho^2\; d\vartheta^2), \tag{3.4.69}$$

where

$$\tilde{F}(\rho) = n^2 \rho^{2n-2} F(\rho^n) = \pi n^2 \int_0^\infty \frac{s^n}{(1+s^n)^2} \frac{ds}{(\rho^2+s)^2}, \tag{3.4.70}$$

and $s = (\rho\, r)^2$. It follows from (3.4.67) and (3.4.70) that

$$\tilde{F}(\rho^{-1}) = \rho^4 \tilde{F}(\rho). \tag{3.4.71}$$

For all $\rho > 0$, the integrand in (3.4.70) is bounded above by $s^{n-2}(1+s^n)^{-2}$ which is integrable for all $n \geq 2$. Thus, the limit of $\tilde{F}(p)$ as $\rho \to 0$ can be obtained by the Lebesgue Dominated Convergence Theorem (LDCT) [16],

$$\lim_{\rho \to 0} \tilde{F}(p) = \pi n^2 \int_0^\infty \lim_{\rho \to 0} \left(\frac{s^n}{(1+s^n)^2} \frac{1}{(\rho^2+s)^2} \right) ds = \pi n^2 \int_0^\infty \frac{s^{n-2}}{(1+s^n)^2}\, ds = \tilde{F}(0). \tag{3.4.72}$$

Clearly, for all $n \geq 2$, $\tilde{F}(0)$ is finite. Hence, for all $n \geq 2$, the lifted metric $\tilde{g}_{L^2}$ on $\mathbb{C}^\times$ extends to a $\mathcal{C}^0$ metric, denoted $\bar{g}_{L^2}$, on $S^2 = \mathbb{C}^\times \cup \{0, \infty\}$.

In [38], it was proved that $\bar{g}_{L^2}$ is a $\mathcal{C}^2$ metric on S^2 for all $n \geq 4$. This is enough regularity to show that lifted geodesic flow extends to S^2. For RMG flow, we need the metric to be $\mathcal{C}^3$, which motivates us to prove the following Lemma, improving on [38].

Lemma 3.4.5. *The $\mathcal{C}^0$ metric $\bar{g}_{L^2} = \tilde{F}(\rho)(d\rho^2 + \rho^2 d\vartheta^2)$ on S^2 is $\mathcal{C}^3$ if $n \geq 5$.*

Proof: It is convenient to re-express the metric $\bar{g}_{L^2}$ in terms of the Cartesian coordinates $x = \rho\cos\vartheta$ and $y = \rho\sin\vartheta$. That is,

$$\bar{g}_{L^2} = f(x,y)(dx^2 + dy^2), \quad \text{where} \quad f(x,y) = \tilde{F}(\sqrt{x^2+y^2}) = \tilde{F}(\rho). \tag{3.4.73}$$

The metric $\bar{g}_{L^2}$ is $\mathcal{C}^3$ if and only if all partial derivatives of $f(x,y)$ up to 3 exist and are continuous. Since $\tilde{F}(\rho)$ is smooth away from $\{0,\infty\}$, then, by (3.4.71), it is sufficient to check that f is $\mathcal{C}^3$ at 0.

In [38], it has been proved that $\bar{g}_{L^2}$ is $\mathcal{C}^2$ if $n \geq 4$. Thus, consider only the third partial derivatives of f which are

$$\begin{aligned}
f_{xxx} &= \cos\vartheta\,\frac{3}{\rho}\left[\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right] + \cos^3\vartheta\left[\tilde{F}'''(\rho) - \frac{3}{\rho}\left(\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right)\right],\\
f_{yyy} &= \sin\vartheta\,\frac{3}{\rho}\left[\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right] + \sin^3\vartheta\left[\tilde{F}'''(\rho) - \frac{3}{\rho}\left(\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right)\right],\\
f_{xxy} &= \sin\vartheta\,\frac{3}{\rho}\left[\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right] + \sin\vartheta\cos^2\vartheta\left[\tilde{F}'''(\rho) - \frac{3}{\rho}\left(\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right)\right],\\
f_{yyx} &= \cos\vartheta\,\frac{3}{\rho}\left[\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right] + \cos\vartheta\sin^2\vartheta\left[\tilde{F}'''(\rho) - \frac{3}{\rho}\left(\tilde{F}''(\rho) - \frac{\tilde{F}'(\rho)}{\rho}\right)\right].
\end{aligned} \tag{3.4.74}$$

Clearly, $\lim\limits_{(x,y)\to(0,0)} f_{xxx}, f_{yyy}, f_{xxy}, f_{yyx}$ exist if and only if

$$\lim_{\rho\to 0}\frac{1}{\rho}\left(\tilde{F}''(\rho)-\frac{\tilde{F}'(\rho)}{\rho}\right)=0, \tag{3.4.75}$$

$$\lim_{\rho\to 0}\left[\tilde{F}'''(\rho)-\frac{3}{\rho}\left(\tilde{F}''(\rho)-\frac{\tilde{F}'(\rho)}{\rho}\right)\right]=0. \tag{3.4.76}$$

Hence, $\bar{g}_{L^2}$ is $\mathcal{C}^3$ if and only if (3.4.75) and (3.4.76) hold.

Now, from (3.4.70), we have

$$\tilde{F}'(\rho)=-4\pi n^2\rho\,\eta(\rho), \qquad \text{where} \qquad \eta(\rho)=\int_0^\infty \frac{s^n}{(1+s^n)^2}\frac{ds}{(\rho^2+s)^3}. \tag{3.4.77}$$

It follows from (3.4.77) that

$$\frac{1}{\rho}\left(\tilde{F}''(\rho)-\frac{\tilde{F}'(\rho)}{\rho}\right)=-4\pi n^2\eta'(\rho)=24\pi n^2\rho\int_0^\infty \frac{s^n}{(1+s^n)^2}\frac{ds}{(\rho^2+s)^4}, \tag{3.4.78}$$

$$\begin{aligned}\tilde{F}'''(\rho)-\frac{3}{\rho}\left(\tilde{F}''(\rho)-\frac{\tilde{F}'(\rho)}{\rho}\right)&=-4\pi n^2[\rho\,\eta''(\rho)-\eta'(\rho)],\\ &=-129\;\pi n^2\rho^3\int_0^\infty \frac{s^n}{(1+s^n)^2}\frac{ds}{(\rho^2+s)^5}.\end{aligned} \tag{3.4.79}$$

From the LDCT [16], we have

$$\lim_{\rho\to 0}\int_0^\infty \frac{s^n}{(1+s^n)^2}\frac{ds}{(\rho^2+s)^k}=\int_0^\infty \frac{s^{n-k}}{(1+s^n)^2}\,ds<\infty \;\text{ if }\; 0<k\le n. \tag{3.4.80}$$

Thus, it follows from (3.4.78), (3.4.79) and (3.4.80) that the conditions (3.4.75) and (3.4.76) hold if $n\ge 5$. Hence, the claim is proved.

□

Since the n-fold cover $\pi:\mathbb{C}^\times\to\mathrm{Rat}_n^{eq}$ is a holomorphic isometry, then, by Proposition

1.3.1, it maps an RMG curve $c(t)$ on $\mathbb{C}^\times$ to an RMG curve $(\pi \circ c)(t) = c(t)^n$ on Rat_n^{eq}. Hence, in some cases, it may be convenient to study the RMG problem on $(\mathbb{C}^\times, \tilde{g}_{L^2})$ instead of $(\text{Rat}_n^{eq}, g_{L^2})$.

Proposition 3.4.6. *For all $n \geq 5$, the submanifold (Rat_n^{eq}, g_{L^2}) is RMG incomplete.*

Proof: Since RMG equations involve second partial derivatives of the metric components, then the local existence and uniqueness of the RMG solution requires that the metric is, at least, $\mathcal{C}^3$ to ensure that the RMG flow function is $\mathcal{C}^1$. Therefore, for all $n \geq 5$, by Lemma 3.4.5, there is a unique local solution of RMG equations on $(S^2, \bar{g}_{L^2})$.

Now, consider an RMG curve $c(t)$ on $(S^2, \bar{g}_{L^2})$ with an initial data $c(0)$ is the north pole and $\dot{c}(0) \neq 0$. This curve exists on the interval $(-\epsilon, \epsilon)$ for $\epsilon > 0$. Consider the projection $a := \pi \circ c : (-\epsilon, 0) \to \text{Rat}_n^{eq}$. Since π is a holomorphic isometry, then, by Proposition 1.3.1, $a(t)$ is an RMG curve on $(\text{Rat}_n^{eq}, g_{L^2})$. Using the initial values of $c(t)$, then $a(t)$ escapes to the boundary at infinity with finite length. Hence, for all $n \geq 5$, $(\text{Rat}_n^{eq}, g_{L^2})$ is RMG incomplete.

□

Proposition 3.4.7. *For all $n \geq 3$, the scalar curvature of (Rat_n^{eq}, g_{L^2}) is bounded above.*

Proof: Since $\tilde{F}(\rho) = n^2 \rho^{2n-2} F(\chi)$, one can obtain that

$$\frac{d}{d\chi}\left(\frac{\chi F'(\chi)}{2F(\chi)}\right) = \frac{1}{n^2 \rho^{n-1}} \frac{d}{d\rho}\left(\frac{\rho \tilde{F}'(\rho)}{2\tilde{F}(\rho)}\right). \tag{3.4.81}$$

One gets by using, for example, Maple that

$$\lim_{\rho \to 0} \frac{1}{\rho} \frac{d}{d\rho}\left(\frac{\rho \tilde{F}'(\rho)}{2\tilde{F}(\rho)}\right) = -8\, \frac{\sin(\pi(1 - 1/n))}{\sin(\pi(1 - 2/n))} =: B_n, \tag{3.4.82}$$

which exists for all $n \geq 3$. It follows from (3.4.81) and (3.4.82) that

$$\lim_{\chi\to 0}\chi^{1-2/n}\frac{d}{d\chi}\left(\frac{\chi F'(\chi)}{2F(\chi)}\right)=\frac{B_n}{n^2}<0. \tag{3.4.83}$$

For all $n\geq 2$, Rat_n^{eq} is a dilation cylinder with a meromorphic function of one pole of order $n\geq 2$. Hence, by Proposition 3.4.3, there exists $\mathscr{R}>0$ such that

$$\lim_{\chi\to 0}\chi^{2-2/n}F(\chi)=\mathscr{R}\,C_2,\qquad C_2=\int_0^\infty\frac{r^n}{(r^n+1)^2}\,dr. \tag{3.4.84}$$

It follows from (3.4.83) and (3.4.84) that the scalar curvature $\kappa(\chi)$ on $(\mathrm{Rat}_n^{eq},g_{L^2})$ has the following limit

$$\begin{aligned}\lim_{\chi\to 0}\kappa(\chi)&=-\lim_{\chi\to 0}\frac{1}{\chi F(\chi)}\frac{d}{d\chi}\left(\frac{\chi F'(\chi)}{2F(\chi)}\right),\\&=-\lim_{\chi\to 0}\frac{1}{\chi^{2-2/n}F(\chi)}\lim_{\chi\to 0}\chi^{1-2/n}\frac{d}{d\chi}\left(\frac{\chi F'(\chi)}{2F(\chi)}\right)=-\frac{B_n}{n^2\mathscr{R}\,C_2}.\end{aligned} \tag{3.4.85}$$

The above limit exists for all $n\geq 3$. Hence, $\kappa(\chi)$ is bounded above in the neighbourhood of 0. Exploiting the property in (3.4.67), we get that the scalar curvature $\kappa(\chi)$ of Rat_n^{eq} with $n\geq 3$ is bounded above for all $\chi>0$.

□

Corollary 3.4.8. *For all $n\geq 3$, the submanifold (Rat_n^{eq},g_{L^2}) is RMG incomplete.*

Proof: For all $n\geq 3$, the submanifold Rat_n^{eq} is a dilation cylinder of a meromorphic function with no simple poles. Furthermore, its scalar curvature, by Proposition 3.4.9, is bounded above. Hence, it follows immediately from Proposition 3.4.4 that $(\mathrm{Rat}_n^{eq},g_{L^2})$ is RMG incomplete for all $n\geq 3$.

□

Unlike Rat_n^{eq} with $n\geq 3$, the case $n=2$ shows that the converse of Proposition 3.4.4 is not true. That is, the RMG incompleteness of a dilation cylinder, of a meromorphic function with no simple poles, does not imply that its scalar curvature is bounded above in the neighborhood of 0. Note that on Rat_2^{eq}, we have the following limit

$$\lim_{\chi\to 0}\frac{1}{\log\chi}\frac{d}{d\chi}\left(\frac{\chi F'(\chi)}{2F(\chi)}\right)=\frac{4}{\pi}, \tag{3.4.86}$$

and so the scalar curvature $\kappa(\chi)$ of Rat_2^{eq} is unbounded above in the neighborhood of 0. However,

Proposition 3.4.9. *The submanifold* (Rat_2^{eq}, g_{L^2}) *is RMG incomplete.*

Proof: To show that $(\mathrm{Rat}_2^{eq}, g_{L^2})$ is RMG incomplete, it is sufficient to check the existence of, at least, an RMG curve on $(\mathrm{Rat}_2^{eq}, g_{L^2})$ which hits the boundary at 0 with finite length. It follows from (3.4.65) that

$$\lim_{\chi\to 0}\chi F(\chi)=\frac{\pi^2}{4}, \qquad \lim_{\chi\to 0}\frac{\chi F'(\chi)}{2F(\chi)}=-\frac{1}{2}, \tag{3.4.87}$$

and so,

$$\lim_{\chi\to 0}\left(P-\frac{\chi F(\chi)}{2\,F(\chi)}\right)=P+\frac{1}{2}. \tag{3.4.88}$$

Consider the RMG effective potential $V_P(\chi)$, defined in (1.4.15), on Rat_2^{eq} with the conserved angular momentum $P=-1/2$,

$$V_{-\frac{1}{2}}(\chi)=\frac{1}{2\chi^2F(\chi)}\left(\frac{1}{2}+\frac{\chi F(\chi)}{2\,F(\chi)}\right)^2. \tag{3.4.89}$$

Using L'Hôpital's rule, one has the following limit

$$\begin{aligned}\lim_{\chi\to 0}\frac{1}{\sqrt{\chi}}\left(\frac{1}{2}+\frac{\chi F(\chi)}{2\,F(\chi)}\right)&=2\lim_{\chi\to 0}\sqrt{\chi}\,\frac{d}{d\chi}\left(\frac{\chi F(\chi)}{2\,F(\chi)}\right),\\ &=2\lim_{\chi\to 0}\left(\sqrt{\chi}\log\chi\right)\lim_{\chi\to 0}\frac{1}{\log\chi}\frac{d}{d\chi}\left(\frac{\chi F(\chi)}{2\,F(\chi)}\right)=0.\end{aligned} \tag{3.4.90}$$

Hence, it follows from (3.4.87) and (3.4.90) that

$$\lim_{\chi\to 0} V_{-\frac{1}{2}}(\chi) = \lim_{\chi\to 0} \frac{1}{2\chi F(\chi)} \left[\lim_{\chi\to 0} \frac{1}{\sqrt{\chi}} \left(\frac{1}{2} + \frac{\chi F(\chi)}{2\,F(\chi)} \right) \right]^2 = 0. \tag{3.4.91}$$

This means that for all $E_* > 0$, there exists $\chi_* > 0$ such that $V_{-\frac{1}{2}}(\chi)$ is bounded above by E_* for all $\chi \leq \chi_*$. The conserved energy associated with the RMG dynamical system on Rat_2^{eq} is

$$E = \frac{1}{2} F(\chi)\dot{\chi}^2 + V_P(\chi). \tag{3.4.92}$$

By choosing $E > 2E_*$, then

$$E > 2E_* \geq V_{-\frac{1}{2}}(\chi), \qquad \forall\, \chi \geq \chi_*, \tag{3.4.93}$$

and so, $(0, \chi_*] \subset \mathbf{p_1}(D_{E,-\frac{1}{2}})$, where $D_{E,-\frac{1}{2}}$ is defined as in (1.4.16) for angular momentum $P = -1/2$ and energy $E > 2E_*$. Also, for all $\chi \leq \chi_*$, we have

$$\dot{\chi}^2 F(\chi) = 2[E - V_{-\frac{1}{2}}(\chi)] > 2E_* > 0. \tag{3.4.94}$$

Thus, for an RMG curve $c(t) = (\chi(t), \psi(t))$ on Rat_2^{eq}, starting at $t = 0$ away from 0, with $P = -1/2$ and $E > 2E_*$, there exists $t_* \in (0, \infty)$ such that $\lim_{t\to t_*} \chi(t) = 0$. Hence, the RMG curve $c(t)$ in $[0, t_*]$ hits the boundary at 0 with finite length. Hence, Rat_2^{eq} is RMG incomplete, despite its unbounded scalar curvature about 0.

□

Here, Rat_1^{eq} is different from others by

Proposition 3.4.10. *The submanifold* (Rat_1^{eq}, g_{L^2}) *is RMG complete.*

Proof: Note that this does not follow immediately from the RMG completeness of Rat_1 with respect to the L^2 metric, given in Theorem 3.3.1.

The induced L^2 metric on Rat_1^{eq} can be written explicitly as

$$g_{L^2} = F(\chi)(d\chi^2 + \chi^2 d\psi^2) = 2\pi\, \frac{[1 - \chi^2 + (\chi^2 + 1)\log \chi]}{(\chi^2 - 1)^3}\, (d\chi^2 + \chi^2 d\psi^2). \quad (3.4.95)$$

Let P be the conserved angular momentum associated with the RMG motion on $(\mathrm{Rat}_1^{eq}, g_{L^2})$. Then, the RMG effective potential function, defined as in (1.4.15), has the following limit

$$\lim_{\chi \to \infty} \frac{\log \chi}{\chi^2} V_P(\chi) = \frac{1}{4\pi}\,(P+2)^2, \qquad \forall\, P \in \mathbb{R}. \quad (3.4.96)$$

This means that there exists $\chi_0 > 0$ such that for all $\chi \geq \chi_0$, there is $\varepsilon > 0$ such that

$$V_P(\chi) \geq \frac{\varepsilon \chi^2}{\log \chi}, \qquad \forall\, P \in \mathbb{R}. \quad (3.4.97)$$

Thus, whenever solutions of RMG equations exist, either $\chi < \chi_0$ or the conserved energy E, associated with the RMG motion on $(\mathrm{Rat}_1^{eq}, g_{L^2})$, satisfies

$$E = \frac{1}{2} F(\chi)\dot{\chi}^2 + V_P(\chi) \geq V_P(\chi) \geq \frac{\varepsilon \chi^2}{\log \chi}, \qquad \forall\, \chi \geq \chi_0. \quad (3.4.98)$$

Hence, in either cases, the radial component $\chi(t)$, in the RMG solution, is bounded above for all $t \in \mathbb{R}$. Similarly, or by exploiting the property (3.4.67), one can obtain that $\chi(t)$ is also bounded below away from 0 for all $t \in \mathbb{R}$.

Consequently, it is easy to check that $\dot{\chi}(t)$ is also bounded for all $t \in \mathbb{R}$. Thus, every RMG solution remains in a compact set. Hence, it can be extended to be defined for all time by applying Picard's method as usual.

□

Conjecture 3.4.11. *For all $n \geq 2$, Rat$_n$ is RMG incomplete with respect to the L^2 metric.*

3.5 Degree n Holomorphic maps $T^2 \to S^2$

3.5.1 $\mathcal{H}_{n,1}(T^2)$ Review

Let T^2 be the flat torus, the familiar torus equipped with Euclidean metric. Consider the space of degree n holomorphic maps from T^2 to S^2, namely, $\mathcal{H}_{n,1}(T^2)$. The torus T^2 is identified with $\mathbb{C}/\Omega$ where Ω is the period lattice given by

$$\Omega = \{m_1 + \tau m_2 : m_1, m_2 \in \mathbb{Z}\}, \tag{3.5.1}$$

of period τ in the upper half plane. Then, one can see that a stereographic coordinate W on the codomain S^2 is doubly periodic, that is, $W(z) \equiv W(z + m_1 + \tau m_2)$, and so is elliptic. Hence, $\mathcal{H}_{n,1}(T^2)$ is just the space of degree n elliptic functions.

Note that $\mathcal{H}_{1,1}(T^2)$ is empty since there is no elliptic function of degree 1 [25, p.77]. Thus, we shall start by discussing the first non-trivial case $n = 2$ in more detail. Let $\wp(z)$ be the Weierstrass elliptic function of degree 2, explicitly

$$\wp(z) = \frac{1}{z^2} + \sum_{\nu \in \Omega \setminus \{0\}} \left[\frac{1}{(z-\nu)^2} - \frac{1}{\nu^2} \right]. \tag{3.5.2}$$

It follows from (3.5.2) that

$$\wp(-z) = \wp(z), \qquad \wp(\bar{z}) = \bar{\wp}(z). \tag{3.5.3}$$

Also, this function satisfies the following property [29, p.162]:

$$\wp(z_1 + z_2) = \frac{1}{4} \left[\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} \right]^2 - [\wp(z_1) + \wp(z_2)]. \tag{3.5.4}$$

Now, let $T^2_\square$ denote the torus $T^2 \cong \mathbb{C}/\Omega$ with period $\tau_1 = e^{i\pi/2} = i$ and $T^2_\triangle$ be the one with period $\tau_2 = e^{i\pi/3} = (1 + i\sqrt{3})/2$. These are known by the lemniscatic and equianharmonic tori, respectively. On $T^2_\square$ and $T^2_\triangle$, the function $\wp(z)$ has the following properties [29, p.159],

$$\wp(\tau z) = \frac{1}{\tau^2}\,\wp(z), \qquad \forall z \in T^2_\Box, T^2_\triangle, \tag{3.5.5}$$

$$\wp'(z)^2 = 4\,\wp(z)\,(\wp(z)^2 - e_1^2), \qquad \forall z \in T^2_\Box, \tag{3.5.6}$$

$$\wp'(z)^2 = 4\,(\wp(z)^3 - \varepsilon_1^3), \qquad \forall z \in T^2_\triangle, \tag{3.5.7}$$

where $e_1 = \wp(1/2;\tau_1) \approx 6.875$ and $\varepsilon_1 = \wp(1/2;\tau_2) \approx 5.898$. Let G denote the 8-dimensional noncompact Lie group $PL(2,\mathbb{C}) \times T^2$ whose action on $\mathcal{H}_{2,1}(T^2)$ is defined as

$$(M, s) : W(z) \mapsto M \odot W(z - s), \tag{3.5.8}$$

where $[M] \in PL(2,\mathbb{C})$ and $s \in T^2$. It was shown in [47] that for each $W \in \mathcal{H}_{2,1}(T^2)$, there exists $([M], s) \in G = PL(2,\mathbb{C}) \times T^2$ such that $W(z)$ can be written as

$$W(z) = M \ \odot \wp(z - s). \tag{3.5.9}$$

The element $([M], s)$ which satisfies (3.5.9) is not unique. Also, it is known that each degree 2 holomorphic map in $\mathcal{H}_{2,1}(T^2)$ has exactly four distinct double valency points [6, p.81]. For the Weierstrass elliptic function $\wp(z)$, the double valency points are

$$s_0 = 0, \qquad s_1 = \frac{1}{2}, \qquad s_2 = \frac{\tau}{2}, \qquad s_3 = \frac{1+\tau}{2}. \tag{3.5.10}$$

One can use the double valency points of $\wp(z)$ to construct elements in G which determine the same elliptic function $\wp(z)$. For lemniscatic case, this has been done by Speight in [47] where he determined first the isotropy group G_0 of the element $\wp(z)/e_1$, and then showed that $\mathcal{H}_{2,1}(T^2_\Box)$ is homeomorphic to G/G_0. The argument which was used in [47] to show the former will cover the equianharmonic case as we shall see.

Henceforth, we consider only the equianharmonic case. Using the double valency points of $\wp(z)$ with $\tau = \tau_2$ and the properties of $\wp(z)$, given in (3.5.3), (3.5.5) and (3.5.7), we obtain the following identities:

$$\begin{aligned}\wp(z) &\equiv \varepsilon_1\left[\frac{\wp(z-s_1)+2\varepsilon_1}{\wp(z-s_1)-\varepsilon_1}\right],\\ &\equiv -\tau_2\varepsilon_1\left[\frac{\wp(z-s_2)-2\tau_2\varepsilon_1}{\wp(z-s_2)+\tau_2\varepsilon_1}\right],\\ &\equiv -\bar{\tau}_2\varepsilon_1\left[\frac{\wp(z-s_3)-2\bar{\tau}_2\varepsilon_1}{\wp(z-s_3)+\bar{\tau}_2\varepsilon_1}\right].\end{aligned} \tag{3.5.11}$$

These three identities can be written as

$$\frac{\wp(z)}{\varepsilon_1\sqrt{2}} = U_i \odot \frac{\wp(z-s_i)}{\varepsilon_1\sqrt{2}}, \qquad i = 1,2,3, \tag{3.5.12}$$

where U_i are the following $SU(2)$ matrices

$$U_1 = \frac{i}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \qquad U_2 = \frac{i}{\tau_2\sqrt{3}}\begin{pmatrix} -\tau_2 & \sqrt{2}\,\tau_2^2 \\ \sqrt{2} & \tau_2 \end{pmatrix},$$
$$U_3 = \frac{i}{\bar{\tau}_2\sqrt{3}}\begin{pmatrix} -\bar{\tau}_2 & \sqrt{2}\bar{\tau}_2^2 \\ \sqrt{2} & \bar{\tau}_2 \end{pmatrix}. \tag{3.5.13}$$

This implies that the isotropy group of the element $\wp(z)/(\varepsilon_1\sqrt{2})$ under G action on $\mathcal{H}_{2,1}(T^2_{\triangle})$ is

$$G_* = \{([\mathbb{I}_2],0),([U_1],s_1),([U_2],s_2),([U_3],s_3)\}. \tag{3.5.14}$$

Clearly, G_* is a finite discrete subgroup of G, and so it acts freely and properly discontinuously on G. It follows that G/G_* is sooth manifold and G is a covering space of G/G_* by the projection map $p(g) = [g]$ [55, p.153-157]. Comparing G_* with G_0, determined in [47], one can see that G_* has the main features of G_0 which was used in the argument proving $\mathcal{H}_{2,1}(T^2_{\square}) \cong G/G_0$. Therefore, an identical argument in the equianharmonic case implies that $\mathcal{H}_{2,1}(T^2_{\triangle})$ is homeomorphic to G/G_* by the homeomorphism

$$[(M,s)] \mapsto M \odot \frac{\wp(z-s)}{\varepsilon_1\sqrt{2}}. \tag{3.5.15}$$

This can be used to equip $\mathcal{H}_{2,1}(T^2_\triangle)$ with a differentiable structure from G/G_*.

3.5.2 2-Dimensional Totally Geodesic Submanifolds of $\mathcal{H}_{n,1}(T^2)$

Let $\wp^{(k)}(z)$ denote the k^{th} derivative of the Weierstrass elliptic function $\wp(z)$ on T^2. Clearly, if $k = n-2$, then for all $n \geq 2$, $\wp^{(n-2)}(z)$ is an elliptic function of degree n. For $n \geq 2$, consider the subset

$$\Gamma_n = \{a\, \wp^{(n-2)}(z) : a \in \mathbb{C}^\times\} \subset \mathcal{H}_{n,1}(T^2). \tag{3.5.16}$$

This is a non-compact 1-dimensional complex submanifold of $\mathcal{H}_{n,1}(T^2)$. The induced L^2 metric on Γ_n is

$$g_{L^2} = \left(\int_{T^2} \frac{\left|\wp^{(n-2)}\right|^2}{(1+|a|^2\left|\wp^{n-2)}(z)\right|^2)^2}\, \mathrm{vol}_g\right) da d\bar{a}. \tag{3.5.17}$$

This is an example of dilation cylinders in the space of degree n holomorphic maps from T^2 to $\mathbb{C}P^1$.

Lemma 3.5.1. *For $k \in \mathbb{Z}^+ \cup \{0\}$, the Weierstrass elliptic function $\wp(z)$ satisfies*

$$\wp^{(k)}(\tau z) = \frac{1}{\tau^{k+2}}\, \wp^{(k)}(z), \qquad \forall z \in T^2_\square, T^2_\triangle. \tag{3.5.18}$$

Proof: Immediately from (3.5.3), it follows that (3.5.18) is true for $k = 0$. Assuming that the statement (3.5.18) is true for some k, then

$$\wp^{(k+1)}(z) = \frac{d}{dz}\left(\wp^{(k)}(z)\right) = \frac{d}{dz}\left(\tau^{k+2}\wp^{(k)}(\tau z)\right) = \tau^{k+3}\wp^{(k+1)}(\tau z), \tag{3.5.19}$$

which, by mathematical induction, proves the claim for all $k \in \mathbb{Z} \cup \{0\}$.

□

Now, let R be the map on $\mathcal{H}_{n,1}(T^2)$ defined as

$$R : W(z) \mapsto \tau^n \, W(\tau z). \tag{3.5.20}$$

This is holomorphic and it acts, with $\tau = \tau_2$ ($\tau = \tau_1$), isometrically on $\mathcal{H}_{n,1}(T^2_\triangle)$ (on $\mathcal{H}_{n,1}(T^2_\square)$) with respect to the L^2 metric, defined in (3.1.10). Let Fix_R be the fixed point set of R in $\mathcal{H}_{n,1}(T^2)$, namely

$$\mathrm{Fix}_R = \{W \in \mathcal{H}_{n,1}(T^2) : W(z) = \tau^n \, W(\tau z), \; \forall \; z \in T^2\}. \tag{3.5.21}$$

Any connected component of Fix_R with $\tau = \tau_2$ ($\tau = \tau_1$) is totally geodesic submanifold of $\mathcal{H}_{n,1}(T^2_\triangle)$ ($\mathcal{H}_{n,1}(T^2_\square)$) [8, p.235]. We shall use this to prove:

Proposition 3.5.2. *Γ_n is a totally geodesic submanifold of $\mathcal{H}_{n,1}(T^2_\triangle)$ for $n = 2, 3, 4, 5$.*

Proof: One can consider $\tau_2 = e^{i\pi/3}$ as a group action of $T^2_\triangle$, which rotates elements by angle $\pi/3$ about the origin. Hence, the τ_2 orbit of a generic point in $T^2_\triangle$ is of order 6. However, there are 3 exceptional orbits of order 1, 2 and 3

$$\{0\}, \qquad \{z_*, \tau_2 z_*\}, \qquad \{z_0, \tau_2 z_0, \tau_2^2 z_0\}, \tag{3.5.22}$$

for $z = 0$, $z_* = -i\tau_2^2/\sqrt{3}$ and $z_0 = (1 + \tau_2)/2$, respectively.

Let $W \in \mathrm{Fix}_R$, then for all $z \in T^2_\triangle$, we have

$$W(z) = \tau_2^n \, W(\tau_2 z). \tag{3.5.23}$$

This implies that if $z \in W^{-1}(0)$, then $\tau_2 z \in W^{-1}(0)$, and so $W^{-1}(0)$ consists of complete orbits under τ_2 action of $T^2_\triangle$. Furthermore, it follows from (3.5.23) that

$$W^{(k)}(z) = \tau_2^{n+k} \, W^{(k)}(\tau_2 z). \tag{3.5.24}$$

Hence, each zero in a given τ_2 orbit has equal multiplicity. Similarly, (3.5.23) implies that if z is a pole of W so is $\tau_2 z$, and so $W^{-1}(\infty)$ consists of complete τ_2 orbits. Again, by (3.5.24), each pole in a given τ_2 orbit has also equal multiplicity.

For degree $2 \leq n \leq 5$, zeros and poles can only lie in the exceptional orbits, since we have n of each. We have determined all possibilities of $W^{-1}(\infty)$ and $W^{-1}(0)$ for the cases $n = 2, 3, 4, 5$ in Table 3.1. The lower index ($\{\}_k$) in Table 3.1 indicates the elements' multiplicity in each set.

Degree	$W^{-1}(\infty)$	$W^{-1}(0)$
2	$\{0\}_2$	$\{z_*, \tau_2 z_*\}_1$
	$\{z_*, \tau_2 z_*\}_1$	$\{0\}_2$
3	$\{0\}_3$	$\{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$
	$\{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$	$\{0\}_3$
	$\{0\}_1 \cup \{z_*, \tau_2 z_*\}_1$	$\{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$
	$\{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$	$\{0\}_1 \cup \{z_*, \tau_2 z_*\}_1$
4	$\{0\}_4$	$\{z_*, \tau_2 z_*\}_2$
	$\{z_*, \tau_2 z_*\}_2$	$\{0\}_4$
	$\{0\}_1 \cup \{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$	$\{z_*, \tau_2 z_*\}_2$
	$\{z_*, \tau_2 z_*\}_2$	$\{0\}_1 \cup \{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$
5	$\{0\}_5$	$\{z_*, \tau_2 z_*\}_1 \cup \{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$
	$\{z_*, \tau_2 z_*\}_1 \cup \{z_0, \tau_2 z_0, \tau_2^2 z_0\}_1$	$\{0\}_5$

Table 3.1: Division of $\text{Fix}_R \subset \mathcal{H}_{n,1}(T^2_\triangle)$ into subsets according to the poles and zeros of their elements for $n = 2, 3, 4, 5$.

Now, define the subset

$$\Sigma_n = \{W \in \mathcal{H}_{n,1}(T^2_\triangle) : W(z) = (RW)(z) \ \text{ and } \ W^{-1}(\infty) = \{0\}_n\} \subset \text{Fix}_R. \quad (3.5.25)$$

From Table 3.1, we note that Σ_n is a connected component of Fix_R for $n = 2, 3, 4, 5$. Therefore, Σ_n is totally geodesic for all $n = 2, 3, 4, 5$.

It remains to show that $\Sigma_n = \Gamma_n$ for all $n = 2, 3, 4, 5$. Suppose that $W \in \Gamma_n$, that is,

$$W(z) = a\wp^{(n-2)}(z), \quad a \in \mathbb{C}^{\times}. \tag{3.5.26}$$

Clearly, $W(z)$ has n poles at $z = 0$, and

$$(RW)(z) = \tau_2^n \, W(\tau z) = a \, \tau_2^n \wp^{(n-2)}(\tau_2 z). \tag{3.5.27}$$

It follows from (3.5.27) and Lemma 3.5.1 that $(RW)(z) = W(z)$ for all $z \in T^2_{\triangle}$, and so $\Gamma_n \subset \Sigma_n$ for all n. Conversely, suppose that $W \in \Sigma_n$, then

$$W(z) - \tau_2^n W(\tau z) = 0, \qquad \forall \, z \in T^2_{\triangle}. \tag{3.5.28}$$

The Laurent expansion of $W(z)$ about 0 is of the form

$$W(z) = \frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \frac{a_{n-2}}{z^{n-2}} + \dots, \tag{3.5.29}$$

where a_i are real constants with $a_n \neq 0$. From (3.5.2), we find that

$$\wp^{(n-2)}(z) = \frac{(-1)^{n-2}(n-1)!}{z^n} + \sum_{\nu \in \Omega \setminus \{0\}} \frac{d^{(n-2)}}{dz^{(n-2)}} \left[\frac{1}{(z-\nu)^2} - \frac{1}{\nu^2} \right]. \tag{3.5.30}$$

Defining

$$W_*(z) = W(z) - \sum_{k=0}^{n-2} \frac{a_{k+2} \, \wp^{(k)}(z)}{(-1)^k (k+1)!}, \tag{3.5.31}$$

then it can be seen from (3.5.31) that $W_*(z)$ is an elliptic function of degree less than 2. Since there is no degree 1 elliptic function, then $W_*(z)$ is of degree 0, namely, $W_*(z)$ is a constant c, say. Thus,

$$W(z) = \sum_{k=0}^{n-2} \frac{a_{k+2} \, \wp^{(k)}(z)}{(-1)^k (k+1)!} + c. \tag{3.5.32}$$

Substituting (3.5.32) in (3.5.28) and using Lemma 3.5.1, one obtains that

$$W(z) - \tau_2^n W(\tau_2 z) = \sum_{k=0}^{n-3} (1 - \tau_2^{(n-2)-k}) \frac{a_{k+2}\wp^{(k)}(z)}{(-1)^k (k+1)!} + (1 - \tau_2^n)c = 0. \qquad (3.5.33)$$

Since $\tau_2^n = e^{in\pi/3} = 1$ only if $6|n$, then for $n = 2, 3, 4, 5$, the equation (3.5.33) implies that

$$a_{n-1} = a_{n-2} = \cdots = a_2 = c = 0. \qquad (3.5.34)$$

Hence,

$$W(z) = \frac{a_n \, \wp^{(n-2)}(z)}{(-1)^{n-2}(n-1)!} =: a \, \wp^{(n-2)}(z) \in \Gamma_n, \qquad (3.5.35)$$

and so, $\Sigma_n \subset \Gamma_n$ for $n = 2, 3, 4, 5$. Therefore, $\Gamma_n = \Sigma_n$ for $n = 2, 3, 4, 5$.

□

Proposition 3.5.3. Γ_n *is a totally geodesic submanifold of* $\mathcal{H}_{n,1}(T^2_\square)$ *for* $n = 2, 3$.

Proof: As a group action, τ_1 rotates $T^2_\square$ by angle $\pi/2$ about the origin. Hence, the τ_1 orbit of a generic element in $T^2_\square$ is of order 4. But, there are 3 exceptional τ_1 orbits for $z = 0$, $z_0 = (1+i)/2$ and $z_* = 1/2$ are given, respectively, by

$$\{0\}, \qquad \{z_0\}, \qquad \{z_*, iz_*\}. \qquad (3.5.36)$$

Following the same argument in the proof of Proposition 3.5.2, one can show that for $n = 2, 3$, the subset

$$\Sigma_n = \{W \in \mathcal{H}_{n,1}(T^2_\square) : W(z) = (RW)(z) \;\text{ and }\; W^{-1}(\infty) = \{0\}_n\} = \Gamma_n. \qquad (3.5.37)$$

Also, zeros and poles of $W \in \text{Fix}_R$ lie in the exponential τ_1 orbits given in (3.5.36). All possibilities of $W^{-1}(\infty)$ and $W^{-1}(0)$ for $n = 2, 3$ are given in Table 3.2 .

Degree	$W^{-1}(\infty)$	$W^{-1}(0)$
2	$\{0\}_2$	$\{z_0\}_2$
	$\{z_0\}_2$	$\{0\}_2$
3	$\{0\}_3$	$\{z_0\}_3$
	$\{z_0\}_3$	$\{0\}_3$
	$\{0\}_1 \cup \{z_*, iz_*\}_1$	$\{z_0\}_3$
	$\{z_0\}_3$	$\{0\}_1 \cup \{z_*, iz_*\}_1$

Table 3.2: Division of $\text{Fix}_R \subset \mathcal{H}_{n,1}(T^2_\square)$ into subsets according to the poles and zeros of their elements for $n = 2, 3$.

It follows from Table 3.2 that Σ_n is a connected component of Fix_R for $n = 2, 3$. Hence, Σ_n is a totally geodesic submanifold of $\mathcal{H}_{n,1}(T^2_\square)$ for $n = 2, 3$.

□

Chapter 4

Baby-Skyrme Metric

4.1 Baby-Skyrme Metric

Let Σ be a compact Riemann surface equipped with a Riemannian metric g given, in terms of isothermal local coordinates $\mathbf{x} = (x_1, x_2)$, by

$$g = \Omega^2(x_1, x_2)\ (dx_1^2 + dx_2^2), \tag{4.1.1}$$

for some smooth function Ω. The Baby-Skyrme model on the spacetime $\mathbb{R} \times \Sigma$ is defined by the following field Lagrangian [40],

$$L[\phi] = \frac{1}{2}\int_\Sigma \left(\sum_{\mu=0}^{2} \partial_\mu\phi \cdot \partial^\mu\phi - \frac{\varepsilon}{2} \sum_{\mu,\nu=0}^{2} (\partial_\mu\phi \times \partial_\nu\phi) \cdot (\partial^\mu\phi \times \partial^\nu\phi) \right) \text{vol}_g, \tag{4.1.2}$$

where ϕ is a scalar field on $\mathbb{R} \times \Sigma$ taking values in $(\mathbb{C}P^1, h)$, where h is the Fubini-Study metric of constant holomorphic sectional curvature $c = 4$. Note that we do not need to include a potential term in (4.1.2) because Σ is compact. The symbol ε is a non-negative real constant called the Baby-Skyrme parameter. This model is just the $\mathbb{C}P^1$ model on $\mathbb{R} \times \Sigma$ with additional term

$$L_{BS}[\phi] = -\frac{1}{4}\int_{\Sigma} (\partial_\mu \phi \times \partial_\nu \phi) \cdot (\partial^\mu \phi \times \partial^\nu \phi) \ \mathrm{vol}_g, \tag{4.1.3}$$

called the Skyrme term. The kinetic and potential energy of the field system in the Baby-Skyrme model are

$$T[\phi] = \frac{1}{2}\int_{\Sigma} \left(\|\partial_0 \phi\|^2 + \varepsilon \Omega^{-2} \sum_i \|\partial_0 \phi \times \partial_i \phi\|^2 \right) \mathrm{vol}_g, \tag{4.1.4}$$

$$V[\phi] = \frac{1}{2}\int_{\Sigma} \sum_{i=1,2} \left(\Omega^{-2} \|\partial_i \phi\|^2 + \frac{\varepsilon}{2}\Omega^{-4} \sum_{j=1,2} \|\partial_i \phi \times \partial_j \phi\|^2 \right) \mathrm{vol}_g, \tag{4.1.5}$$

respectively. Since the cross and dot products are related by

$$\|\partial_0 \phi \times \partial_i \phi\|^2 + (\partial_0 \phi \cdot \partial_i \phi)^2 = \|\partial_0 \phi\|^2 \|\partial_i \phi\|^2, \tag{4.1.6}$$

then the kinetic energy $T[\phi]$ can be written as

$$\begin{aligned} T[\phi] = {} & \frac{1}{2}\int_{\Sigma} \|\partial_0 \phi\|^2 \ \mathrm{vol}_g + \frac{\varepsilon}{2}\int_{\Sigma} \Omega^{-2} \|\partial_0 \phi\|^2 \big(\|\partial_1 \phi\|^2 + \|\partial_2 \phi\|^2 \big) \ \mathrm{vol}_g \\ & - \frac{\varepsilon}{2}\int_{\Sigma} \Omega^{-2} \left((\partial_0 \phi \cdot \partial_1 \phi)^2 + (\partial_0 \phi \cdot \partial_2 \phi)^2 \right) \mathrm{vol}_g. \end{aligned} \tag{4.1.7}$$

Now, let $\phi \in \mathcal{H}_{n,1}(\Sigma)$. Then, ϕ is holomorphic, and so $\partial_1 \phi$ and $\partial_2 \phi$ are orthogonal and both have the same length with respect to the Riemannian metric on $\mathbb{C}P^1$. Let

$$\partial_0 \phi = a_1 \frac{\partial_1 \phi}{\|\partial_1 \phi\|} + a_2 \frac{\partial_2 \phi}{\|\partial_2 \phi\|}, \qquad a_1, a_2 \in \mathbb{R}. \tag{4.1.8}$$

Then, we have

$$\begin{aligned}(\partial_0\phi\cdot\partial_1\phi)^2+(\partial_0\phi\cdot\partial_2\phi)^2 &= a_1^2\|\partial_1\phi\|^2+a_2^2\|\partial_2\phi\|^2,\\ &= \frac{1}{2}(a_1^2+a_2^2)\big(\|\partial_1\phi\|^2+\|\partial_2\phi\|^2\big),\\ &= \frac{1}{2}\|\partial_0\phi\|^2\big(\|\partial_1\phi\|^2+\|\partial_2\phi\|^2\big).\end{aligned} \tag{4.1.9}$$

Hence, for all $\phi\in\mathcal{H}_{n,1}(\Sigma)$, the kinetic energy $T[\phi]$ is

$$T[\phi]=\frac{1}{2}\int_\Sigma\|\partial_0\phi\|^2\ \mathrm{vol}_g+\frac{\varepsilon}{4}\int_\Sigma\Omega^{-2}\|\partial_0\phi\|^2\big(\|\partial_1\phi\|^2+\|\partial_2\phi\|^2\big)\ \mathrm{vol}_g. \tag{4.1.10}$$

Introducing a complex coordinate $z=x_1+ix_2$ on Σ and a stereographic coordinate W on $S^2\cong\mathbb{C}P^1$, then (4.1.10) becomes

$$T[W]=\frac{1}{2}\int_\Sigma\frac{|\dot{W}|^2}{(1+|W|^2)^2}\ \mathrm{vol}_g+\frac{\varepsilon}{2}\int_\Sigma\Omega^{-2}\frac{|\dot{W}|^2}{(1+|W|^2)^4}\ |\partial_z W|^2\ \mathrm{vol}_g. \tag{4.1.11}$$

Let $\{b^i : i=1,\dots,2n+1-\mathrm{g}\}$ be complex local coordinates on $\mathcal{H}_{n,1}(\Sigma)$, then

$$T[W]=\frac{1}{2}\int_\Sigma\bigg(\frac{1}{(1+|W|^2)^2}\ \frac{\partial W}{\partial b^i}\frac{\partial\overline{W}}{\partial\bar{b}^j}+\varepsilon\ \Omega^{-2}\frac{|\partial_z W|^2}{(1+|W|^2)^4}\ \frac{\partial W}{\partial b^i}\frac{\partial\overline{W}}{\partial\bar{b}^j}\bigg)\ \mathrm{vol}_g\ \dot{b}^i\ \dot{\bar{b}}^j. \tag{4.1.12}$$

Hence, the restriction of the kinetic energy $T[\phi]$ in the Baby-Skyrme model to $\mathcal{H}_{n,1}(\Sigma)$ introduces a new metric on $\mathcal{H}_{n,1}(\Sigma)$ given by

$$\gamma_\varepsilon=\sum_{i,j}\ \gamma_{\varepsilon_{ij}}db^i d\bar{b}^j, \tag{4.1.13}$$

where

$$\gamma_{\varepsilon_{ij}}=\int_\Sigma\frac{1}{(1+|W|^2)^2}\ \frac{\partial W}{\partial b^i}\frac{\partial\overline{W}}{\partial\bar{b}^j}\ \mathrm{vol}_g+\varepsilon\int_\Sigma\Omega^{-2}\frac{|\partial_z W|^2}{(1+|W|^2)^4}\ \frac{\partial W}{\partial b^i}\frac{\partial\overline{W}}{\partial\bar{b}^j}\ \mathrm{vol}_g. \tag{4.1.14}$$

We call γ_ε the Baby-Skyrme metric on $\mathcal{H}_{n,1}(\Sigma)$. For the limit $\varepsilon \to 0$, Baby-Skyrme metric γ_ε is just the L^2 metric γ_{L^2}, defined in (3.1.10), on $\mathcal{H}_{n,1}(\Sigma)$. The term with the parameter ε in (4.1.14) is a metric on $\mathcal{H}_{n,1}(\Sigma)$, denoted γ_*, descending from the kinetic energy of the Skyrme term $L_{BS}[W]$. That is,

$$\gamma_* = \sum_{i,j} \gamma_{*_{ij}} db^i d\bar{b}^j, \tag{4.1.15}$$

where

$$\gamma_{*_{ij}} = \int_\Sigma \Omega^{-2} \frac{|\partial_z W|^2}{(1+|W|^2)^4} \frac{\partial W}{\partial b^i} \frac{\partial \overline{W}}{\partial \bar{b}^j} \,\mathrm{vol}_g. \tag{4.1.16}$$

From (4.1.16), one notes that γ_* is Hermitian. Again by using (4.1.16), one finds that $\gamma_{*_{ij}}$ does not satisfy

$$\frac{\partial \gamma_{*_{ij}}}{\partial b^k} \equiv \frac{\partial \gamma_{*_{kj}}}{\partial b^i}, \qquad \frac{\partial \gamma_{*_{ij}}}{\partial \bar{b}^k} \equiv \frac{\partial \gamma_{*_{ik}}}{\partial \bar{b}^j}, \tag{4.1.17}$$

and so it is not Kähler.

The geometric interpretation of the Baby-Skyrme metric γ_ε is as follows: let $X, Y \in T_\phi \mathcal{H}_{n,1}(\Sigma) \subset \Gamma(\phi^* T\mathbb{C}P^1)$, then

$$\gamma_\varepsilon(X,Y) = \gamma_{L^2}(X,Y) + \varepsilon \gamma_*(X,Y), \tag{4.1.18}$$

where

$$\begin{aligned} \gamma_{L^2}(X,Y) &= \int_\Sigma h(X,Y) \,\mathrm{vol}_g, \\ \gamma_*(X,Y) &= \int_\Sigma \mathcal{E}\, h(X,Y) \,\mathrm{vol}_g, \end{aligned} \tag{4.1.19}$$

where h is the Fubini-Study metric on $\mathbb{C}P^1$ with constant holomorphic sectional curvature $c = 4$. Here, $\mathcal{E}$ denotes the potential energy density in the $\mathbb{C}P^1$ model which is, in the language of harmonic map theory, the Dirichlet energy density of W. It can be seen that the integrand in γ_* is similar to the one in the L^2 metric γ_{L^2} but, here, it is weighted by $\mathcal{E}$, and hence, we call γ_* the weighted L^2 metric on $\mathcal{H}_{n,1}(\Sigma)$.

4.2 Baby-Skyrme Metric on Rat_1

It is interesting to consider the Baby-Skyrme metric on the spac $\mathcal{H}_{1,1}(S^2) \cong \mathrm{Rat}_1$, where S^2 is the 2-sphere given its round metric of radius $1/2$, namely, the function Ω^2 of the metric g, given in (4.1.1), is

$$\Omega^2(x_1, x_2) = \frac{1}{\left(1 + |\mathbf{x}|^2\right)^2}. \tag{4.2.1}$$

Since the Baby-Skyrme metric is the L^2 metric added to the weighted L^2 metric on Rat_1, and since the formula for the L^2 metric on Rat_1 was already given explicitly in [48], then to determine an explicit formula for the Baby-Skyrme metric on Rat_1, it is sufficient to determine the weighted L^2 metric γ_* on Rat_1.

Since the weighted L^2 metric γ_* is Hermitian and invariant under the action of $G = SO(3) \times SO(3) \times \{\mathrm{Id}, \mathrm{P}\}$ on Rat_1, where Id is the identity map and P is the discrete isometry mapping $W(z)$ to $\overline{W}(\bar{z})$ on Rat_1, then with respect to the coframe $\{d\lambda_k, \sigma_k : k = 1, 2, 3\}$ on Rat_1, γ_* has the same structure given in (3.3.14). That is,

$$\gamma_* = A_{*_1} \boldsymbol{d\lambda} \cdot \boldsymbol{d\lambda} + A_{*_2} (\boldsymbol{\lambda} \cdot \boldsymbol{d\lambda})^2 + A_{*_3} \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} + A_{*_4} (\boldsymbol{\lambda} \cdot \boldsymbol{\sigma})^2 + A_{*_5} \boldsymbol{\lambda} \cdot (\boldsymbol{\sigma} \times \boldsymbol{d\lambda}), \tag{4.2.2}$$

where $A_{*_1}, \ldots, A_{*_5}$ are smooth functions of $\lambda = \|\boldsymbol{\lambda}\|$ related by

$$A_{*_1} + \lambda^2 A_{*_2} = \frac{4}{\Lambda^2}(A_{*_3} + \lambda^2 A_{*_4}), \qquad A_{*_3} = \frac{A_{*_1}}{4} + \frac{\lambda^2}{2} A_{*_5}. \tag{4.2.3}$$

These functions can be evaluated by calculating the kinetic energy,

$$T_{BS}[W] = \frac{i}{4}\int_{S^2_{do}} \frac{|\dot{W}|^2}{(1+|W|^2)^4}\ |\partial_z W|^2\ dz d\bar{z} = \frac{i}{4}\int_{S^2_{co}} \frac{|\dot{W}|^2}{(1+|W|^2)^4}\, dW d\bar{W}, \quad (4.2.4)$$

for a chosen convenient curve $W(z,t) \equiv ([U(t)], \boldsymbol{\lambda}(t))$ on Rat_1 and comparing this with (4.2.2). In fact, we have obtained that

$$A_{*_1} = \frac{\pi}{3}\frac{(\mu^2+1)}{\mu}, \quad A_{*_2} = \frac{1}{\lambda^2}\left(\frac{2\pi}{3\Lambda^2} - A_{*_1}\right), \quad A_{*_3} = \frac{\pi}{6}, \quad A_{*_4} = 0, \quad A_{*_5} = -\frac{2\pi}{3}, \quad (4.2.5)$$

where $\Lambda = \sqrt{1+\lambda^2}$ and $\mu = (\Lambda+\lambda)^2$. Clearly, the functions $A_{*_1}, \ldots, A_{*_5}$ indeed satisfy the Hermiticity constraints given in (4.2.3). Note that $A_{*_1}, \ldots, A_{*_5}$ do not satisfy the Kähler constraints, given in (3.3.16), for any G-invariant metric, determined as (4.2.2), on Rat_1.

The volume form on Rat_1 with respect to γ_* is

$$\begin{aligned}\mathrm{vol}_{\gamma_*} &= \frac{A_{*_3}}{2\Lambda}\left(4A_{*_1}A_{*_3} - \lambda^2 A_{*_5}^2\right)\ d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3, \\ &= 2\left(\frac{\pi}{6}\right)^3 \frac{(\mu+1)^2}{\Lambda\mu}\ d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3. \end{aligned} \quad (4.2.6)$$

Proposition 4.2.1. *For all $\varepsilon > 0$, Rat_1 has infinite volume with respect to the Baby-Skyrme metric γ_ε.*

Proof: Let M_{L^2} and M_* be the 6×6 matrices associated with the metrics γ_{L^2} and γ_* on Rat_1, respectively. Then, the volume of $(\mathrm{Rat}_1, \gamma_\varepsilon)$ is

$$\begin{aligned}\text{Vol}(\text{Rat}_1, \gamma_\varepsilon) &= \text{Vol}(SO(3)) \int_{\mathbb{R}^3} \sqrt{\det(M_{L^2} + \varepsilon M_*)}\, d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3, \\ &= 4\pi\ \text{Vol}(SO(3)) \int_0^\infty \lambda^2 \sqrt{\det(M_{L^2} + \varepsilon M_*)}\, d\lambda, \end{aligned} \tag{4.2.7}$$

where we have used the spherical coordinates $(\lambda, \vartheta, \varphi)$ on $\mathbb{R}^3$. It follows from the Minkowski Determinant Theorem [36] that

$$\det(M_{L^2} + \varepsilon M_*) > \varepsilon^6 \det(M_*). \tag{4.2.8}$$

Thus, it follows from (4.2.7) that

$$\begin{aligned}\text{Vol}(\text{Rat}_1, \gamma_\varepsilon) &> 4\pi\ \varepsilon^3\ \text{Vol}(SO(3)) \int_0^\infty \lambda^2 \sqrt{\det(M_*)}\, d\lambda, \\ &= \varepsilon^3\ \text{Vol}(\text{Rat}_1, \gamma_*). \end{aligned} \tag{4.2.9}$$

But, we have

$$\begin{aligned}\text{Vol}(\text{Rat}_1, \gamma_*) &= 4\pi\ \text{Vol}(SO(3)) \int_0^\infty \lambda^2 \sqrt{\det(M_*)}\, d\lambda, \\ &= \frac{\pi^4}{3^3}\ \text{Vol}(SO(3)) \int_0^\infty \lambda^2\ \frac{(\mu+1)^2}{\Lambda\mu}\, d\lambda, \\ &= \frac{\pi^4}{6^3}\ \text{Vol}(SO(3)) \int_1^\infty \frac{(\mu^2-1)^2}{\mu^3}\, d\mu = \infty. \end{aligned} \tag{4.2.10}$$

Hence, $(\text{Rat}_1, \gamma_\varepsilon)$ has infinite volume for all $\varepsilon > 0$.

□

4.2.1 Geodesic Completeness of Rat_1 with Baby-Skyrme Metric

Geodesic motion on Rat_1 with respect to the weighted L^2 metric γ_* is determined by the Lagrangian

$$L = \frac{1}{2}\Big(A_{*_1}\|\dot{\boldsymbol{\lambda}}\|^2 + A_{*_2}(\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}})^2 + A_{*_3}\|\boldsymbol{\Omega}\|^2 + A_{*_5}\boldsymbol{\lambda}\cdot(\boldsymbol{\Omega}\times\dot{\boldsymbol{\lambda}})\Big), \tag{4.2.11}$$

for a geodesic $\chi(t) = ([U(t)], \boldsymbol{\lambda}(t))$ on Rat_1 whose velocity $\dot{\chi}(t) = ([U(t)\boldsymbol{\Omega}], \dot{\boldsymbol{\lambda}}(t))$ and $\boldsymbol{\Omega} \in \mathfrak{su}(2)$ is defined as in (3.3.46). Since geodesics are traversed at constant speed, then the energy E is conserved. In addition to this, there are six conserved angular momenta $\boldsymbol{Q}_* = (Q_{*_1}, Q_{*_2}, Q_{*_3})$ and $\boldsymbol{P}_* = (P_{*_1}, P_{*_2}, P_{*_3})$ descending from the Lagrangian symmetry under the $SO(3)\times SO(3)$ action on Rat_1, defined in (3.3.11). In fact,

$$\begin{aligned} \boldsymbol{Q}_* &= (A_{*_3} - \frac{1}{2}A_{*_5})\boldsymbol{\Omega} - (A_{*_1} - \frac{1}{2}A_{*_5})(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}) + \frac{1}{2}A_{*_5}(\boldsymbol{\lambda}\cdot\boldsymbol{\Omega})\boldsymbol{\lambda}, \\ \boldsymbol{P}_* &= \mathcal{U}^T[A_{*_3}\boldsymbol{\Omega} + \frac{1}{2}A_{*_5}(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda})], \end{aligned} \tag{4.2.12}$$

where $\mathcal{U} = (\mathcal{U}_{ij}) = (\mathrm{tr}(\tau_i U \tau_j U^{-1})/2) \in SO(3)$. The squared length of $\boldsymbol{P}_*$ is

$$\|\boldsymbol{P}_*\|^2 = A_{*_3}^2\|\boldsymbol{\Omega}\|^2 + \frac{1}{4}A_{*_5}^2\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2 + A_{*_3}A_{*_5}\boldsymbol{\Omega}\cdot(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}), \tag{4.2.13}$$

and so,

$$A_{*_3}\|\boldsymbol{\Omega}\|^2 + A_{*_5}\boldsymbol{\Omega}\cdot(\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}) = \frac{1}{A_{*_3}}\|\boldsymbol{P}_*\|^2 - \frac{1}{4}\frac{A_{*_5}^2}{A_{*_3}}\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2. \tag{4.2.14}$$

By using (4.2.14), we eliminate $\boldsymbol{\Omega}$ from (4.2.11),

$$E = L = \frac{1}{2}\Big(A_{*_1}\|\dot{\boldsymbol{\lambda}}\|^2 + A_{*_2}(\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}})^2 + \frac{1}{A_{*_3}}\|\boldsymbol{P}_*\|^2 - \frac{1}{4}\frac{A_{*_5}^2}{A_{*_3}}\|\dot{\boldsymbol{\lambda}}\times\boldsymbol{\lambda}\|^2\Big). \tag{4.2.15}$$

Since $2E = \gamma_*(\dot{\chi}(t), \dot{\chi}(t))$, then it follows from (4.2.15) that

$$\gamma_*(\dot{\chi}(t), \dot{\chi}(t)) = A_{*_1}\|\dot{\boldsymbol{\lambda}}\|^2 + A_{*_2}(\boldsymbol{\lambda}\cdot\dot{\boldsymbol{\lambda}})^2 - \frac{1}{4}\frac{A_{*_5}^2}{A_{*_3}}\|\boldsymbol{\lambda}\times\dot{\boldsymbol{\lambda}}\|^2 + \frac{1}{A_{*_3}}\|\boldsymbol{P}_*\|^2. \tag{4.2.16}$$

Using (4.2.5) and $\|\dot{\boldsymbol{\lambda}}\|^2 = \|\hat{\boldsymbol{\lambda}} \times \dot{\boldsymbol{\lambda}}\|^2 + (\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2$, then (4.2.16) becomes

$$\gamma_*(\dot{\chi}(t), \dot{\chi}(t)) = \left(A_{*_1} - \frac{2\pi}{3}\lambda^2\right)\|\hat{\boldsymbol{\lambda}} \times \dot{\boldsymbol{\lambda}}\|^2 + \frac{2\pi}{3\Lambda^2}(\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2 + \frac{6}{\pi}\|\boldsymbol{P}_*\|^2. \tag{4.2.17}$$

Since

$$A_{*_1} - \frac{2\pi}{3}\lambda^2 = \frac{\pi}{6}\,\frac{(\mu+1)^2}{\mu} > 0, \tag{4.2.18}$$

then,

$$\gamma_*(\dot{\chi}(t), \dot{\chi}(t)) > \frac{2\pi}{3\Lambda^2}(\hat{\boldsymbol{\lambda}} \cdot \dot{\boldsymbol{\lambda}})^2 = \frac{2\pi}{3\Lambda^2}\left(\frac{d\lambda}{dt}\right)^2. \tag{4.2.19}$$

The length of $\chi(t) : [t_0, t_*] \to (\mathrm{Rat}_1, \gamma_*)$, with $\lambda(t_0) = \lambda_0$ and $\lambda(t_*) = \lambda_*$, is

$$\begin{aligned} L_*[\chi] = \int_{t_0}^{t_*} \sqrt{\gamma_*(\dot{\chi}(t), \dot{\chi}(t))}\, dt &> \sqrt{\frac{2\pi}{3}} \int_{t_0}^{t_*} \frac{1}{\Lambda}\left(\frac{d\lambda}{dt}\right) dt, \\ &= \sqrt{\frac{2\pi}{3}} \int_{\lambda_0}^{\lambda_*} \frac{d\lambda}{\sqrt{1+\lambda^2}}. \end{aligned} \tag{4.2.20}$$

This implies that $\lim_{\lambda_* \to \infty} L_*[\chi] = \infty$. Hence, all geodesics on $(\mathrm{Rat}_1, \gamma_*)$ have infinite length. Since geodesics are traversed at constant speed and have infinite length on $(\mathrm{Rat}_1, \gamma_*)$, then they exist for all time. Hence, $(\mathrm{Rat}_1, \gamma_*)$ is geodesically complete.

Proposition 4.2.2. *For all $\varepsilon > 0$, Rat$_1$ is geodesically complete with respect to the Baby-Skyrme metric γ_ε.*

Proof: We assume to the contrary that $(\mathrm{Rat}_1, \gamma_\varepsilon)$ is geodesically incomplete. It follows from the Hopf-Rinow Theorem [22], there is a Cauchy sequence $\{p_n\}$ on the metric space $(\mathrm{Rat}_1, d_\varepsilon)$ such that it is divergent. Explicitly, for all $\delta > 0$, there exists an integer $N > 0$ such that

$$d_\varepsilon(p_m, p_n) < \delta, \qquad \forall m, n > N, \tag{4.2.21}$$

where d_ε denotes the distance function with respect to γ_ε. Since the length of an arbitrary curve $\chi : [t_0, t_*] \to (\mathrm{Rat}_1, \gamma_\varepsilon)$ connecting p_m and p_n is

$$\begin{aligned} L_\varepsilon[\chi] &= \int_{t_0}^{t_*} \sqrt{\gamma_\varepsilon(\dot{\chi}(t), \dot{\chi}(t))}\, dt, \\ &= \int_{t_0}^{t_*} \sqrt{\gamma_{L^2}(\dot{\chi}(t), \dot{\chi}(t)) + \varepsilon \gamma_*(\dot{\chi}(t), \dot{\chi}(t))}\, dt, \\ &> \sqrt{\varepsilon} \int_{t_0}^{t_*} \sqrt{\gamma_*(\dot{\chi}(t), \dot{\chi}(t))}\, dt = \sqrt{\varepsilon}\, L_*[\chi]. \end{aligned} \tag{4.2.22}$$

It follows from the definition of the distance function on Riemannian manifold, given in (1.1.6), and by using (4.2.21) and (4.2.22) that,

$$\delta > d_\varepsilon(p_m, p_n) > \sqrt{\varepsilon}\, d_*(p_m, p_n), \qquad \forall m, n > N. \tag{4.2.23}$$

This implies that the divergent sequence $\{p_n\}$ is also Cauchy on (Rat_1, d_*) where d_* is the distance function with respect to γ_*. Hence, (Rat_1, d_*) is an incomplete metric space, and so geodesically incomplete which is a contradiction. Hence, $(\mathrm{Rat}_1, \gamma_\varepsilon)$ is geodesically complete.

□

4.2.2 Weighted L^2 Metric on Fix_P

Throughout this section, we discuss the qualitative behaviour of geodesics on a certain 3-dimensional totally geodesic submanifold of Rat_1 equipped with its induced weighted L^2 metric.

In the coordinate system $([U], \boldsymbol{\lambda})$, the discrete isometry P acts on Rat_1 as

$$\mathrm{P} : [U](\Lambda \mathbb{I}_2 + \boldsymbol{\lambda} \cdot \boldsymbol{\tau}) \mapsto [\bar{U}](\Lambda \mathbb{I}_2 + \boldsymbol{\lambda} \cdot \bar{\boldsymbol{\tau}}), \tag{4.2.24}$$

where $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ are the standard Pauli matrices, given in (3.3.8). Identifying $SU(2)$

with the 3-sphere S^3 via the standard Euler's angles $(\alpha, \beta, \gamma) \in [0, \pi] \times [0, 4\pi] \times [0, 2\pi]$ as

$$U = \begin{pmatrix} \cos\frac{\alpha}{2} e^{i(\beta+\gamma)/2} & \sin\frac{\alpha}{2} e^{i(\beta-\gamma)/2} \\ -\sin\frac{\alpha}{2} e^{-i(\beta-\gamma)/2} & \cos\frac{\alpha}{2} e^{-i(\beta+\gamma)/2} \end{pmatrix}, \tag{4.2.25}$$

then, the action of P on Rat_1 in terms of $(\alpha, \beta, \gamma, \lambda_1, \lambda_2, \lambda_3)$ is

$$\mathrm{P} : (\alpha, \beta, \gamma, \lambda_1, \lambda_2, \lambda_3) \mapsto (\alpha, -\beta, -\gamma, \lambda_1, -\lambda_2, \lambda_3). \tag{4.2.26}$$

Hence, the fixed point set of P in Rat_1 is [45],

$$\mathrm{Fix}_{\mathrm{P}} = \left\{ \left(\left[\begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \right], (\lambda_1, 0, \lambda_3) \right) : \alpha, \lambda_1, \lambda_3 \in \mathbb{R} \right\} \cong S^1 \times \mathbb{R}^2, \tag{4.2.27}$$

which is a 3-dimensional submanifold of Rat_1. This space consists of $\mathbb{C}P^1$ lumps located on a great circle passing the poles of the domain S^2_{do} with sharpness $\lambda = \sqrt{\lambda_1^2 + \lambda_3^2}$ and internal phase given by α [45]. Since $\mathrm{Fix}_{\mathrm{P}}$ is a fixed point set of an isometry of Rat_1, then it is totally geodesic [8]. Introducing polar coordinates (λ, θ) on $\mathbb{R}^2$, then the induced weighted L^2 metric on $\mathrm{Fix}_{\mathrm{P}}$ is

$$g_* = (A_{*_1} + \lambda^2 A_{*_2})\, d\lambda^2 + \lambda^2 A_{*_1}\, d\theta^2 + A_{*_3}\, d\alpha^2 - \lambda^2 A_{*_5}\, d\theta\, d\alpha. \tag{4.2.28}$$

Geodesic motion on $(\mathrm{Fix}_{\mathrm{P}}, g_*)$ is determined by the following Lagrangian

$$L = \frac{1}{2}\left((A_{*_1} + \lambda^2 A_{*_2})\, \dot{\lambda}^2 + \lambda^2 A_{*_1}\, \dot{\theta}^2 + A_{*_3}\, \dot{\alpha}^2 - \lambda^2 A_{*_5}\, \dot{\theta}\, \dot{\alpha} \right). \tag{4.2.29}$$

Since geodesics preserve speed, then the energy $E = L$ is conserved. Furthermore, both θ and α are cyclic, L does not depend on θ and α, so there are two conserved angular momenta $P_\theta, P_\alpha \in \mathbb{R}$ given by

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = \lambda^2 A_{*_1}\ \dot{\theta} - \frac{1}{2}\lambda^2 A_{*_5}\ \dot{\alpha} = \lambda^2 A_{*_1}\ \dot{\theta} + \frac{\pi}{3}\lambda^2\ \dot{\alpha}, \tag{4.2.30}$$

$$P_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = -\frac{1}{2}\lambda^2 A_{*_5}\ \dot{\theta} + A_{*_3}\ \dot{\alpha} = \frac{\pi}{3}\lambda^2\ \dot{\theta} + \frac{\pi}{6}\ \dot{\alpha}. \tag{4.2.31}$$

It follows from (4.2.30) and (4.2.31) that

$$\dot{\theta} = P_\theta\left(1 - 2\frac{P_\alpha}{P_\theta}\lambda^2\right)\left(A_{*_1} - \frac{2\pi}{3}\lambda^2\right)^{-1}, \tag{4.2.32}$$

$$\dot{\alpha} = -2P_\theta\left(1 - \frac{3}{\pi}\frac{P_\alpha}{P_\theta}A_{*_1}\right)\left(A_{*_1} - \frac{2\pi}{3}\lambda^2\right)^{-1}. \tag{4.2.33}$$

Substituting (4.2.32) and (4.2.33) in (4.2.29), one gets

$$E = \frac{1}{2}\left((A_{*_1} + \lambda^2 A_{*_2})\ \dot{\lambda}^2 + P_\theta^2 V_\theta(\lambda) + P_\alpha(P_\alpha + P_\theta)V_\alpha(\lambda) - \frac{12}{\pi}P_\alpha P_\theta\right), \tag{4.2.34}$$

where $V_\theta(\lambda)$ and $V_\alpha(\lambda)$ are functions of λ given by

$$V_\theta(\lambda) = \frac{1}{\lambda^2}\left(A_{*_1} - \frac{2\pi}{3}\lambda^2\right)^{-1},$$
$$V_\alpha(\lambda) = \frac{6}{\pi}A_{*_1}\left(A_{*_1} - \frac{2\pi}{3}\lambda^2\right)^{-1}. \tag{4.2.35}$$

The energy E in (4.2.34) can be written as

$$E = \frac{1}{2}(A_{*_1} + \lambda^2 A_{*_2})\ \dot{\lambda}^2 + V(\lambda; P_\theta, P_\alpha), \tag{4.2.36}$$

where $V(\lambda; P_\theta, P_\alpha)$ is the effective potential function of geodesic motion on $\mathrm{Fix_P}$ given by

$$V(\lambda; P_\theta, P_\alpha) = \frac{1}{2}\left[P_\theta^2 V_\theta(\lambda) + P_\alpha(P_\alpha + P_\theta)V_\alpha(\lambda) - \frac{12}{\pi}P_\alpha P_\theta\right]. \tag{4.2.37}$$

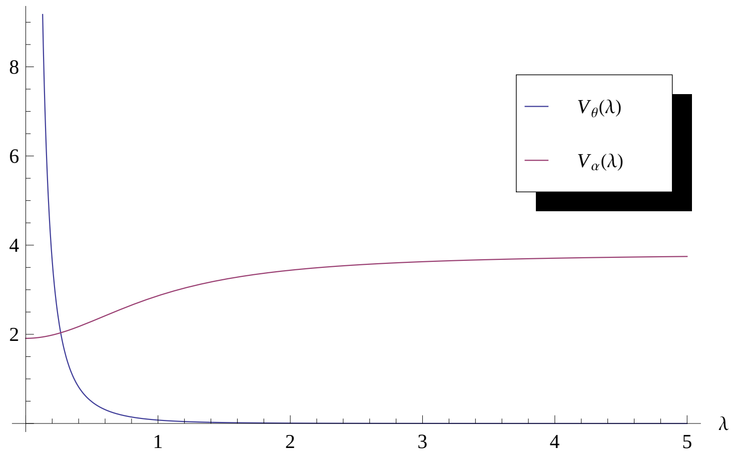


Figure 4.1: Plots of the functions $V_\theta(\lambda)$ and $V_\alpha(\lambda)$.

The minimum of the effective potential function on $\mathrm{Fix_P}$

The derivative of the effective potential function $V(\lambda; P_\theta, P_\alpha)$, given in (4.2.37), with respect to λ is

$$V'(\lambda; P_\theta, P_\alpha) = \frac{1}{2}\left[P_\theta^2 V_\theta'(\lambda) + P_\alpha(P_\alpha + P_\theta)V_\alpha'(\lambda)\right]. \tag{4.2.38}$$

Now, let

$$\mathscr{P} = \{(P_\theta, P_\alpha) \in \mathbb{R}^2 : \exists\, \lambda \geq 0 \text{ s.t. } V'(\lambda; P_\theta, P_\alpha) = 0\} \subset \mathbb{R}^2. \tag{4.2.39}$$

From (4.2.35), we note that for all $\lambda > 0$, $V_\theta'(\lambda) < 0$ and $V_\alpha'(\lambda) > 0$. Hence, the formula for $V'(\lambda; P_\theta, P_\alpha)$ in (4.2.38) implies the possibility for $V(\lambda; P_\theta, P_\alpha)$ to have minimum points. Clearly, if $P_\alpha(P_\alpha + P_\theta) \leq 0$, then $V(\lambda; P_\theta, P_\alpha)$ is monotonically decreasing, and so $\mathscr{P}$ is empty. Defining a positive real number κ depending on P_θ and P_α by

$$\kappa(P_\theta, P_\alpha) = \frac{P_\theta^2}{P_\alpha(P_\alpha + P_\theta)} = \left[\frac{P_\alpha}{P_\theta}\left(\frac{P_\alpha}{P_\theta} + 1\right)\right]^{-1}, \quad \forall\, (P_\theta, P_\alpha) \in \mathscr{P}. \tag{4.2.40}$$

Then, $V'(\lambda; P_\theta, P_\alpha)$ can be written as

$$V'(\lambda; P_\theta, P_\alpha) = P_\alpha(P_\alpha + P_\theta)\big[\kappa(P_\theta, P_\alpha)\; V'_\theta(\lambda) + V'_\alpha(\lambda)\big]. \tag{4.2.41}$$

Using (4.2.35) in (4.2.41), it can be seen that $V(\lambda; P_\theta, P_\alpha)$ has only one minimum point. It follows from (4.2.41) that for each $\lambda \geq 0$, there exists one value

$$\tilde{\kappa}(\lambda) = -\frac{V'_\alpha(\lambda)}{V'_\theta(\lambda)}, \tag{4.2.42}$$

such that for all $(P_\theta, P_\alpha) \in \mathscr{P}$, such that $\kappa(P_\theta, P_\alpha) = \tilde{\kappa},\; V(\lambda; P_\theta, P_\alpha)$ has a minimum at λ. But, from (4.2.40), each value of $\tilde{\kappa}$ corresponds to two values of P_α/P_θ denoted $\xi_\pm$ and given by

$$\xi_\pm(\lambda) = \frac{P_\alpha}{P_\theta} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{1}{\tilde{\kappa}(\lambda)}}. \tag{4.2.43}$$

Substituting (4.2.42) in ξ_+ and ξ_-, we get that for all $(P_\theta, P_\alpha) \in \mathscr{P}$, the minimum point of $V(\lambda; P_\theta, P_\alpha)$ is

$$\lambda_0 = \begin{cases} \sqrt{P_\theta/(2P_\alpha)} & \text{if} \quad P_\theta P_\alpha \geq 0 \\ \sqrt{-P_\theta/[2(P_\alpha + P_\theta)]} & \text{if} \quad P_\theta(P_\alpha + P_\theta) \leq 0, \end{cases} \tag{4.2.44}$$

where the division to these cases is because of the reality of λ_0. Recall that $P_\alpha = 0$ does not belong to $\mathscr{P}$. The minimum $V_0 := V(\lambda_0; P_\theta, P_\alpha)$ is a function of (P_θ, P_α) given by

$$V_0 = \begin{cases} \frac{3}{\pi} P_\alpha^2 & \text{if} \quad P_\theta P_\alpha \geq 0 \\ \frac{3}{\pi}(P_\alpha^2 - 2P_\alpha P_\theta - P_\theta^2) & \text{if} \quad P_\theta(P_\alpha + P_\theta) \leq 0. \end{cases} \tag{4.2.45}$$

Hence, we conclude that the (P_θ, P_α)-plane is divided as follows

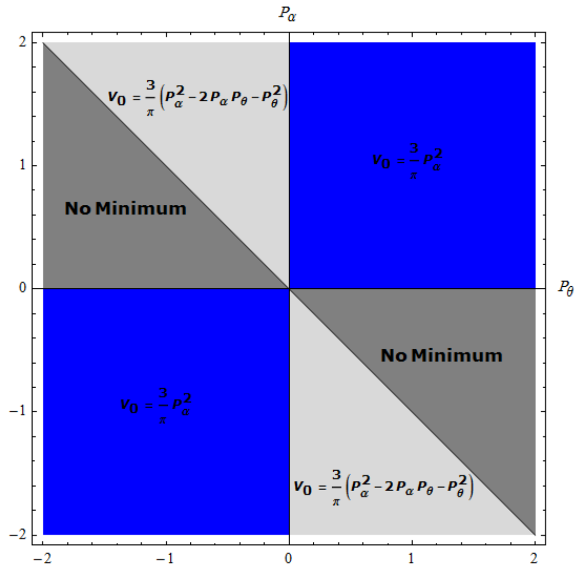


Figure 4.2: The (P_θ, P_α)-plane's division according to the effective potential function's minimum as a function of the conserved angular momenta P_θ and P_α.

Unbounded and oscillatory geodesics on Fix_P

In this section, we shall determine the region in the (P_θ, P_α)-plane where there exist geodesic trajectories with fixed energy, for example, $E = 1$, and then we divide this region into two subsets according to whether the geodesic is unbounded or oscillatory.

It follows from (4.2.37) that

$$\lim_{\lambda\to\infty} V(\lambda; P_\theta, P_\alpha) = \frac{6}{\pi}P_\alpha^2. \tag{4.2.46}$$

This implies that for all $(P_\theta, P_\alpha) \in \mathbb{R}^2$ such that $|P_\alpha| < \sqrt{\pi/6}$, there exists $\lambda_* > 0$ such that for all $\lambda \geq \lambda_*$,

$$V(\lambda; P_\theta, P_\alpha) < 1. \tag{4.2.47}$$

Hence, for all $|P_\alpha| < \sqrt{\pi/6}$, $V(\lambda; P_\theta, P_\alpha)$ will not exceed the energy $E = 1$ for all $\lambda \geq 0$, and so λ is unbounded above. Therefore, geodesics on Fix_P with energy $E = 1$ and angular momenta $(P_\theta, P_\alpha) \in \mathbb{R}^2$ such that $|P_\alpha| < \sqrt{\pi/6}$ are unbounded.

Now, to determine $(P_\theta, P_\alpha) \in \mathbb{R}^2$ in which geodesics on Fix_P with $E = 1$ are oscillating, it is sufficient to find what $(P_\theta, P_\alpha) \in \mathscr{P}$ with $|P_\alpha| \geq \sqrt{\pi/6}$ have the minimum V_0 is less than 1 .

First, if $P_\theta P_\alpha \geq 0$, it follows from (4.2.45) that $V_0 < 1$ if and only if

$$|P_\alpha| < \sqrt{\frac{\pi}{3}}. \tag{4.2.48}$$

Consequently, for all $(P_\theta, P_\alpha) \in \mathscr{P}$ such that $P_\theta P_\alpha \geq 0$, geodesics on Fix_P with energy $E = 1$ are oscillating if and only if

$$\sqrt{\frac{\pi}{6}} \leq |P_\alpha| < \sqrt{\frac{\pi}{3}}. \tag{4.2.49}$$

Second, if $P_\theta(P_\alpha + P_\theta) \leq 0$, it follows from (4.2.45) that the minimum V_0 attains $E = 1$ if and only if

$$P_\alpha = P_\theta \pm \sqrt{2P_\theta^2 + \frac{\pi}{3}}. \tag{4.2.50}$$

Furthermore, the partial derivative of V_0 with respect to P_α is

$$\frac{\partial V_0}{\partial P_\alpha} = \frac{6}{\pi}(P_\alpha - P_\theta), \tag{4.2.51}$$

and so,

$$\frac{\partial V_0}{\partial P_\alpha} \begin{cases} < 0, & \text{if} \qquad P_\theta \geq 0, (P_\alpha + P_\theta) < 0 \\ > 0, & \text{if} \qquad P_\theta \leq 0, (P_\alpha + P_\theta) > 0. \end{cases} \tag{4.2.52}$$

Thus, for all $(P_\theta, P_\alpha) \in \mathscr{P}$ such that $P_\theta(P_\alpha + P_\theta) \leq 0$, it follows from (4.2.50) and (4.2.52) that $V(\lambda_0; P_\theta, P_\alpha) < 1$ for all

$$P_\theta \geq 0 \qquad \text{and} \qquad P_\theta - \sqrt{2P_\theta^2 + \frac{\pi}{3}} < P_\alpha < -P_\theta, \tag{4.2.53}$$

or

$$P_\theta \leq 0 \qquad \text{and} \qquad -P_\theta < P_\alpha < P_\theta + \sqrt{2P_\theta^2 + \frac{\pi}{3}}. \tag{4.2.54}$$

Hence, for all $(P_\theta, P_\alpha) \in \mathscr{P}$ such that $P_\theta(P_\alpha + P_\theta) \leq 0$, geodesics on Fix_P with energy $E = 1$ are oscillating if and only if the angular momenta (P_θ, P_α) lie in the following region

$$0 \leq P_\theta < \sqrt{\frac{\pi}{6}} \qquad \text{and} \qquad P_\theta - \sqrt{2P_\theta^2 + \frac{\pi}{3}} < P_\alpha \leq -\sqrt{\frac{\pi}{6}}, \tag{4.2.55}$$

or

$$-\sqrt{\frac{\pi}{6}} < P_\theta \leq 0 \qquad \text{and} \qquad \sqrt{\frac{\pi}{6}} \leq P_\alpha < P_\theta + \sqrt{2P_\theta^2 + \frac{\pi}{3}}. \tag{4.2.56}$$

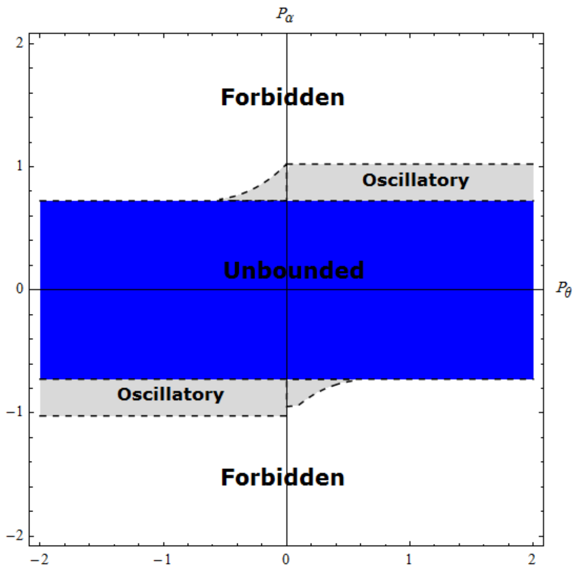


Figure 4.3: The (P_θ, P_α) plane's division into: blue region where the unit energy geodesics on Fix_P are unbounded and gray region where the unit energy geodesics on Fix_P are oscillatory.

From Figure 4.3, one can see that both subsets in the (P_θ, P_α)-plane for unbounded or oscillatory geodesics are symmetric under rotation by angle π about the origin. This is checked by noting that for all $(P_\theta, P_\alpha) \in \mathbb{R}^2$,

$$V(\lambda; -P_\theta, -P_\alpha) \equiv V(\lambda; P_\theta, P_\alpha). \tag{4.2.57}$$

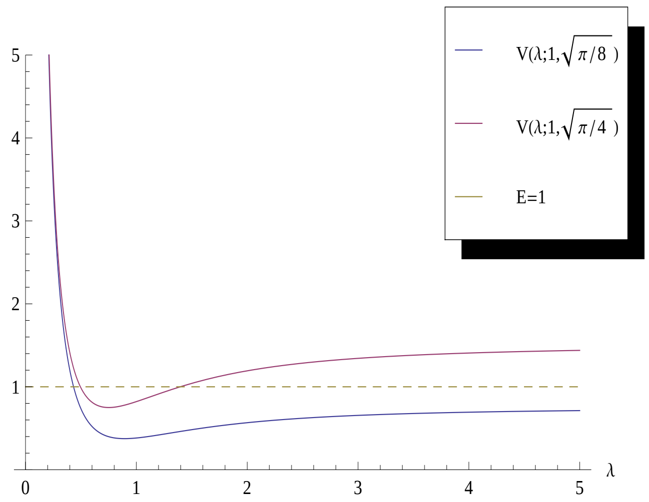


Figure 4.4: Plots of the effective potential function $V(\lambda; P_\theta, P_\alpha)$ for certain values of (P_θ, P_α).

4.3 Rat_n^{eq} Submanifolds Revisited

In this section, we consider the 2-dimensional submanifold Rat_n^{eq} of Rat_n with its induced Baby-Skyrme metric. Let $W(t,z) = \chi(t)e^{i\psi}\, z^n$ be a curve on Rat_n^{eq}. Then, the kinetic energy $T[W]$, given in (4.1.11), for $W(t,z)$ induces a metric on Rat_n^{eq} of the form

$$\begin{aligned} g_\varepsilon &= H(\chi)\,(d\chi^2 + \chi^2\, d\psi^2) = \big(F(\chi) + \varepsilon\, G(\chi)\big)\,(d\chi^2 + \chi^2\, d\psi^2), \\ &:= g_{L^2} + \varepsilon\, g_*, \end{aligned} \tag{4.3.1}$$

where

$$F(\chi) = 2\pi \int_0^\infty \frac{r^{2n+1}}{(1+\chi^2 r^{2n})^2(1+r^2)^2}\, dr, \tag{4.3.2}$$

$$G(\chi) = 2\pi n^2 \int_0^\infty \frac{\chi^2 r^{4n-1}}{(1+\chi^2 r^{2n})^4}\, dr. \tag{4.3.3}$$

Here, $z = re^{i\theta}$ with $r \in [0, \infty)$ and $\theta \in [0, 2\pi]$. In the limit $\varepsilon \to 0$, this is the induced L^2 metric g_{L^2} on Rat_n^{eq}. Since $\hat{\delta} : (\chi, \psi) \to (\chi^{-1}, -\psi)$ is an isometry of $(\text{Rat}_n^{eq}, g_\varepsilon)$, then

$$\hat{\delta}^* g_\varepsilon = \chi^{-4}\ H(\chi^{-1})\ (d\chi^2 + \chi^2 d\psi^2) = g_\varepsilon. \tag{4.3.4}$$

Namely,

$$H(\chi^{-1}) = \chi^4\ H(\chi), \qquad \forall\ \chi > 0. \tag{4.3.5}$$

Also, the pullback of g_ε under the n fold covering map π of Rat_n^{eq}, defined in (3.4.68), is

$$\tilde{g}_\varepsilon = \pi^* g_\varepsilon = \tilde{H}(\rho)\ (d\rho^2 + \rho^2 d\theta^2), \tag{4.3.6}$$

where

$$\tilde{H}(\rho) = n^2 \rho^{2n-2}\ H(\rho^n), \qquad \forall\ \rho > 0. \tag{4.3.7}$$

It follows from (4.3.2), (4.3.3) and (4.3.7) that

$$\tilde{H}(\rho) = \tilde{F}(\rho) + \varepsilon\, \tilde{G}(\rho), \tag{4.3.8}$$

where

$$\tilde{F}(\rho) = \pi n^2 \int_0^\infty \frac{s^n}{(\rho^2 + s)^2(1+s^n)^2}\, ds, \tag{4.3.9}$$

$$\tilde{G}(\rho) = \frac{\pi n^4}{\rho^2} \int_0^\infty \frac{s^{2n-1}}{(1+s^n)^4}\, ds = \frac{\pi n^3}{6\rho^2}. \tag{4.3.10}$$

Proposition 4.3.1. *For all $\varepsilon > 0$ and all $n \geq 1$, the submanifold (Rat_n^{eq}, g_ϵ) has zero total Gauss curvature.*

Proof: The scalar curvature and the volume form of $(\mathrm{Rat}_n^{eq}, g_\epsilon)$ are

$$\kappa(\chi) = -\frac{1}{\chi H(\chi)} \frac{d}{d\chi}\left(\frac{\chi H'(\chi)}{H(\chi)}\right), \qquad \mathrm{vol}_{g_\varepsilon} = \chi H(\chi)\, d\chi \wedge d\psi, \tag{4.3.11}$$

respectively. Then, the total Gauss curvature of $(\mathrm{Rat}_n^{eq}, g_\epsilon)$ is

$$\int_{\mathrm{Rat}_n^{eq}} \frac{\kappa(\chi)}{2}\, \mathrm{vol}_{g_\varepsilon} = -2\pi \left[\frac{\chi H'(\chi)}{2H(\chi)}\right]_0^\infty. \tag{4.3.12}$$

It is convenient to change the coordinate $\chi \mapsto \rho^n$ and use (4.3.7) to get

$$\frac{\chi H'(\chi)}{2H(\chi)} = \frac{\rho^n H'(\rho^n)}{2H(\rho^n)} = \frac{1}{n}\frac{\rho \tilde{H}'(\rho)}{2\tilde{H}(\rho)} + \frac{1}{n} - 1. \tag{4.3.13}$$

From (4.3.9), one can find that $\lim_{\rho\to 0} \tilde{F}(\rho)$ is a nonzero constant and $\lim_{\rho\to 0} \rho \tilde{F}'(\rho) = 0$ for all $n \geq 2$ and $\lim_{\rho\to 0} \rho \tilde{F}(\rho) = 0$ for $n = 1$. Then,

$$\begin{aligned} \lim_{\rho\to 0} \rho^2 \tilde{H}(\rho) &= \lim_{\rho\to 0} \rho^2 \tilde{F}(\rho) + \varepsilon \lim_{\rho\to 0} \rho^2 \tilde{G}(\rho) = \frac{\varepsilon\pi n^3}{6}, \\ \lim_{\rho\to 0} \rho^3 \tilde{H}'(\rho) &= \lim_{\rho\to 0} \rho^3 \tilde{F}'(\rho) + \varepsilon \lim_{\rho\to 0} \rho^3 \tilde{G}'(\rho) = -\frac{\varepsilon\pi n^3}{3}. \end{aligned} \tag{4.3.14}$$

This implies that

$$\lim_{\rho\to 0} \frac{\rho \tilde{H}'(\rho)}{2\tilde{H}(\rho)} = -1. \tag{4.3.15}$$

It follows from (4.3.13) and (4.3.15) that

$$\lim_{\chi\to 0} \frac{\chi H'(\chi)}{2H(\chi)} = \frac{1}{n} \lim_{\rho\to 0} \frac{\rho \tilde{H}'(\rho)}{2\tilde{H}(\rho)} + \frac{1}{n} - 1 = -1. \tag{4.3.16}$$

One can exploit the property (4.3.5) of $H(\chi)$ to obtain the limit of the expression $\chi H'(\chi)/2H(\chi)$ at infinity as follows:

$$H'(\chi^{-1}) = -\chi^5[4H(\chi) + \chi H'(\chi)], \tag{4.3.17}$$

and so for $\tau = \chi^{-1}$, we have

$$\lim_{\tau\to\infty} \frac{\tau H'(\tau)}{2H(\tau)} = \lim_{\chi\to 0} \frac{H'(\chi^{-1})}{2\chi H(\chi^{-1})} = -\lim_{\chi\to 0}\left[2 + \frac{\chi H'(\chi)}{2H(\chi)}\right] = -1. \tag{4.3.18}$$

Hence, for all $n \geq 1$ and all $\varepsilon > 0$, it follows from (4.3.16) and (4.3.18) that

$$\int_{\mathrm{Rat}_n^{eq}} \frac{\kappa(\chi)}{2}\, \mathrm{vol}_{g_\varepsilon} = 0. \tag{4.3.19}$$

□

In [38], it was shown that the total Gauss curvature of $(\mathrm{Rat}_n^{eq}, g_\varepsilon)$ with $\varepsilon = 0$, namely with the induced L^2 metric, depends on the degree n and in fact equals to $4\pi/n$ for all $n \geq 1$.

The following is a generalization of Proposition 3.4.2 for Rat_n^{eq} with the Baby-Skyrme metric g_ε for all $\varepsilon \geq 0$.

Proposition 4.3.2. *For all $\varepsilon \geq 0$ and all $n \geq 1$, the submanifold $(Rat_n^{eq}, g_\varepsilon)$ can be isometrically embedded as a surface of revolution in $\mathbb{R}^3$.*

Proof: Assume that there is an embedding $\mathbf{X} : \mathrm{Rat}_n^{eq} \to \mathbb{R}^3$ given by

$$\mathbf{X}(\chi, \psi) = (\alpha(\chi), \beta(\chi)\cos\psi, \beta(\chi)\sin\psi). \tag{4.3.20}$$

Then, the induced metric on $\mathbf{X}(\mathrm{Rat}_n^{eq}) \subset \mathbb{R}^3$ is

$$g_{\mathrm{sor}} = \left(\alpha'(\chi)^2 + \beta'(\chi)^2\right) d\chi^2 + \beta(\chi)^2\, d\psi^2. \tag{4.3.21}$$

Hence, $\mathbf{X}$ is an isometric embedding if and only if

$$\beta(\chi) = \chi\sqrt{H(\chi)}, \tag{4.3.22}$$

$$\frac{d\alpha(\chi)}{d\chi} = \sqrt{H(\chi)}\sqrt{1-\left(1+\frac{\chi H'(\chi)}{2H(\chi)}\right)^2}. \tag{4.3.23}$$

Clearly, the solution of (4.3.23) exists if

$$-1 < 1+\frac{\chi H'(\chi)}{2H(\chi)} < 1, \qquad \forall\, \chi > 0. \tag{4.3.24}$$

One can show that the function $H(\chi)$ indeed satisfies (4.3.24) for all $n \geq 1$ and all $\varepsilon \geq 0$ as follows:

$$\begin{aligned} 1+\frac{\chi H'(\chi)}{2H(\chi)} &= \frac{1}{2H(\chi)}\big[2H(\chi)+\chi H'(\chi)\big], \\ &= \frac{1}{2H(\chi)}\big[2F(\chi)+\chi F'(\chi)+\varepsilon\big(2G(\chi)+\chi G'(\chi)\big)\big]. \end{aligned} \tag{4.3.25}$$

As in (4.3.13), one finds that

$$\frac{\chi G'(\chi)}{2G(\chi)} = \frac{\rho^n G'(\rho^n)}{2G(\rho^n)} = \frac{1}{n}\frac{\rho\tilde{G}'(\rho)}{2\tilde{G}(\rho)} + \frac{1}{n} - 1 = -1. \tag{4.3.26}$$

Using (4.3.26) in (4.3.25), we obtain that

$$1+\frac{\chi H'(\chi)}{2H(\chi)} = \frac{1}{2H(\chi)}\big[2F(\chi)+\chi F'(\chi)\big]. \tag{4.3.27}$$

By Proposition 3.4.2, the function $F(\chi)$ on Rat_n^{eq} satisfies

$$-2F(\chi) < 2F(\chi)+\chi F'(\chi) < 2F(\chi). \tag{4.3.28}$$

Using (4.3.28) in (4.3.27), we get that

$$-\frac{F(\chi)}{H(\chi)} < 1+\frac{\chi H'(\chi)}{2H(\chi)} < \frac{F(\chi)}{H(\chi)}. \tag{4.3.29}$$

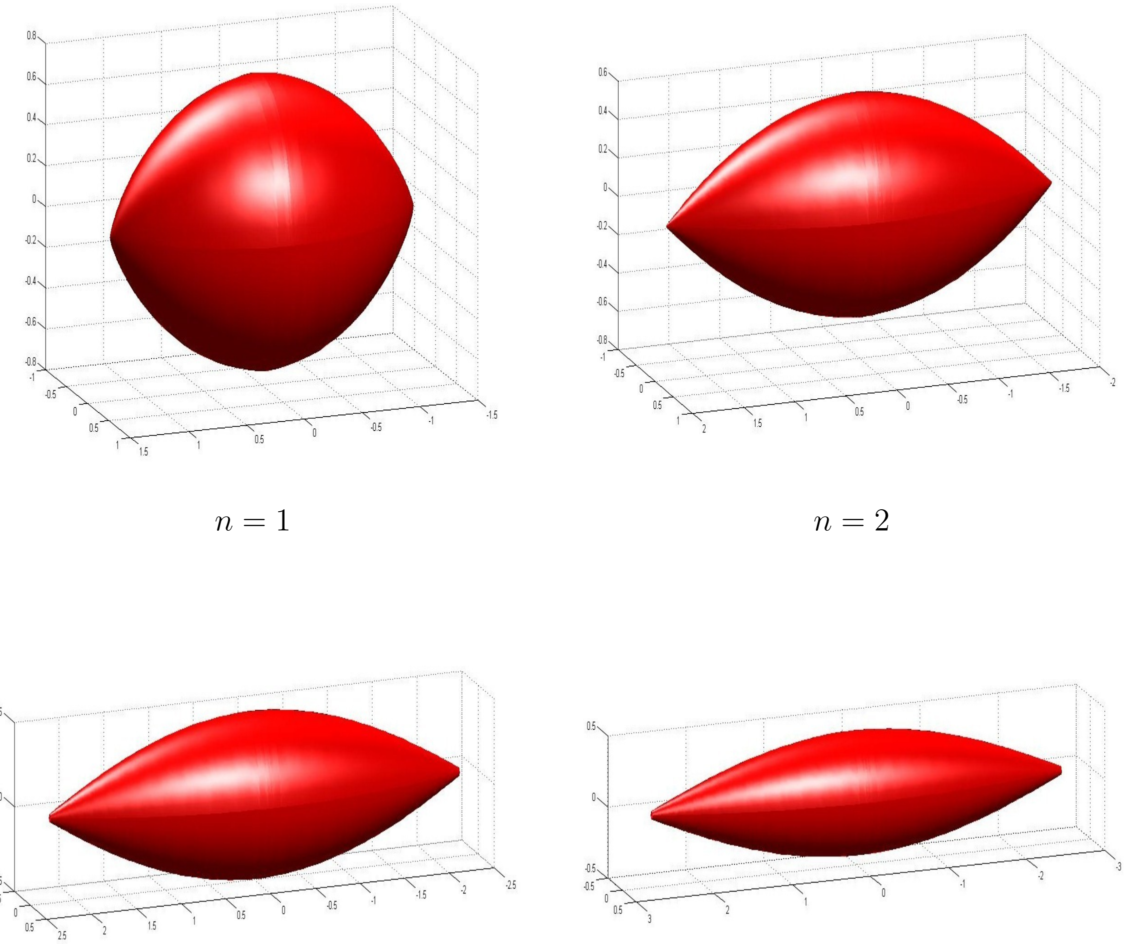
Since $F(\chi)/H(\chi) \leq 1$, then for all $n \geq 1$ and all $\varepsilon \geq 0$, we have

$$-1 < 1 + \frac{\chi H'(\chi)}{2H(\chi)} < 1, \qquad \forall\, \chi > 0. \tag{4.3.30}$$

Hence, the claim is established.

□

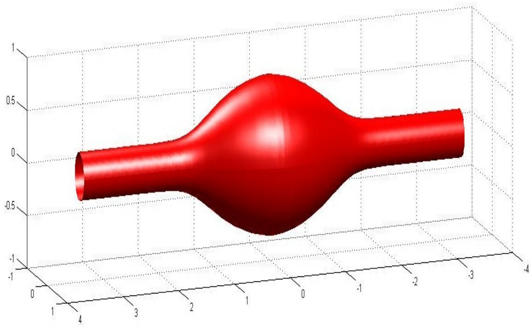
Numerically, we have solved (4.3.23) with initial data $\alpha(1) = 0$ for $n = 1, 2, 3, 4$ with several values of ε. The resulting surfaces of revolution $(\text{Rat}^{eq}_{1,2,3,4}, g_\varepsilon)$ for $\varepsilon = 0, 0.1, 0.5$ are given in the Figures 4.5, 4.6 and 4.7, respectively. Note that for all $n \geq 1$, the submanifold Rat^{eq}_n equipped with its weighted L^2 metric g_* is a flat cylinder.



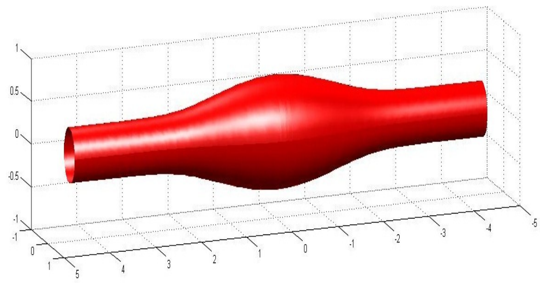
$n = 1$ $n = 2$

$n = 3$ $n = 4$

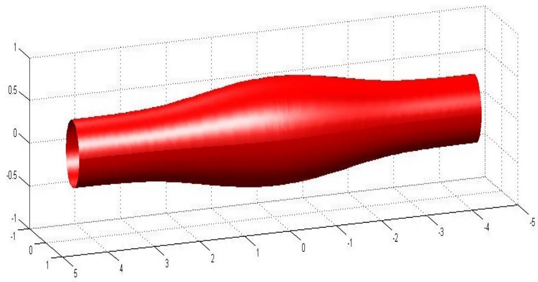
Figure 4.5: Rat_n^{eq} submanifolds with the Baby-Skyrme metric g_ε for $\varepsilon = 0$ (the L^2 metric).



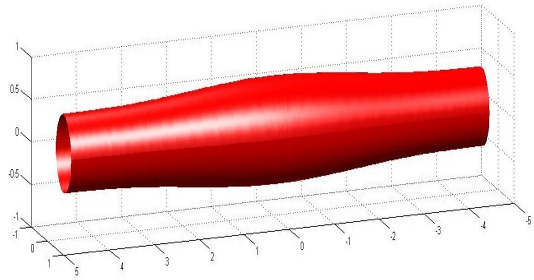
$n = 1$



$n = 2$



$n = 3$



$n = 4$

Figure 4.6: Rat_n^{eq} submanifolds with the Baby-Skyrme metric g_ε for $\varepsilon = 0.1$.

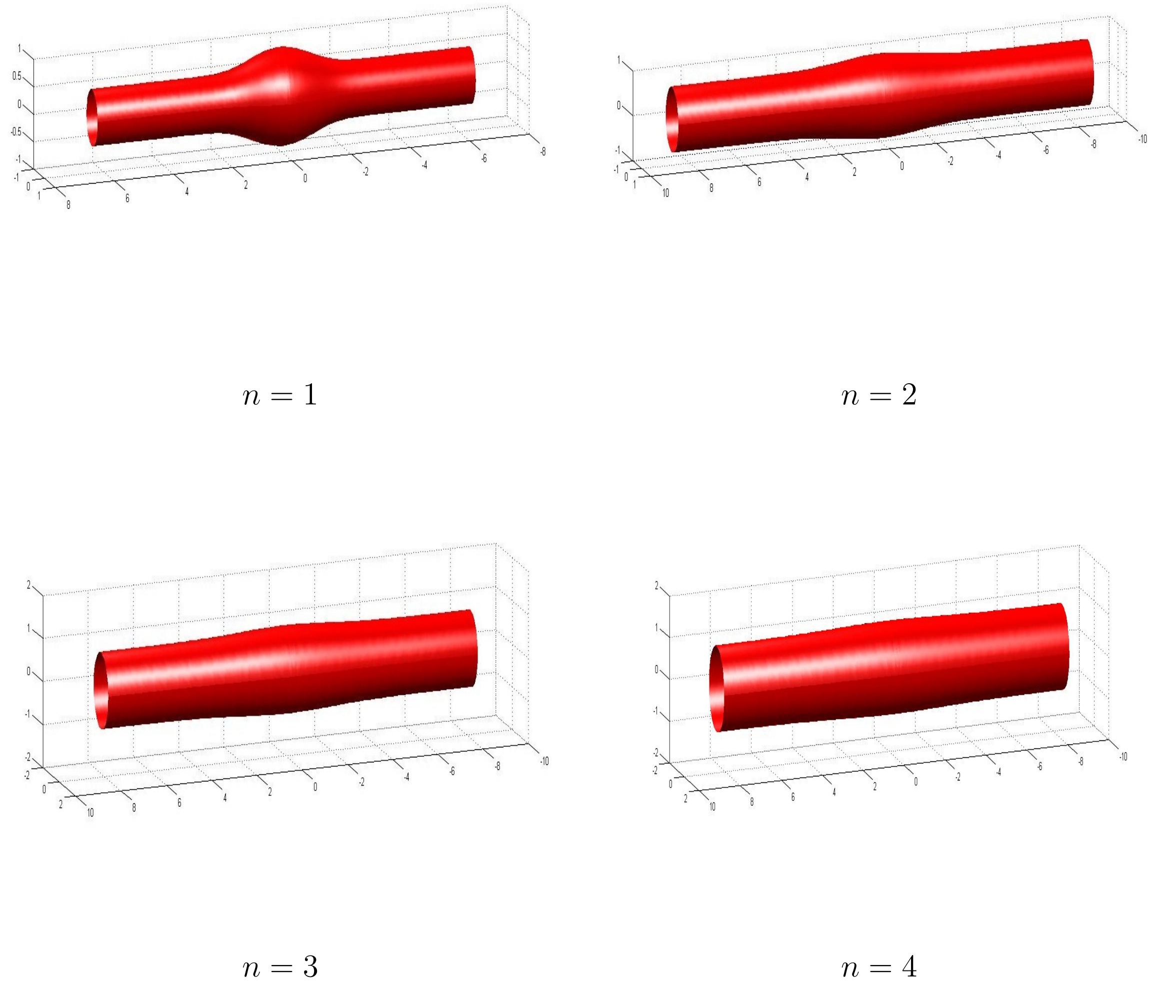


Figure 4.7: Rat_n^{eq} submanifolds with the Baby-Skyrme metric g_ε for $\varepsilon = 0.5$.

There is a geometric interpretation of the function bounded in (4.3.30) as follows: let $\xi(\chi)$ be the angle between the tangent of the generating curve $(\alpha(\chi), \beta(\chi))$ of $(\mathrm{Rat}_n^{eq}, g_\varepsilon)$ and the $\alpha(\chi)$- axis. Then,

$$\cos\xi(\chi) = \frac{(\alpha(\chi), 0)\cdot(\alpha'(\chi), \beta'(\chi))}{\alpha(\chi)\sqrt{\alpha'^2(\chi)+\beta'^2(\chi)}} = \frac{\alpha'(\chi)}{\sqrt{\alpha'^2(\chi)+\beta'^2(\chi)}}. \tag{4.3.31}$$

Using (4.3.23) in (4.3.31), one finds that

$$\cos\xi(\chi) = \sqrt{1-\left(1+\frac{\chi H'(\chi)}{2H(\chi)}\right)^2}, \tag{4.3.32}$$

and so,

$$\sin\xi(\chi) = \sqrt{1-\cos^2\xi(\chi)} = 1+\frac{\chi H'(\chi)}{2H(\chi)}. \tag{4.3.33}$$

Hence, for all $n \geq 1$ and all $\varepsilon \geq 0$,

$$-1 < \sin\xi(\chi) < 1. \tag{4.3.34}$$

A corollary of the local Gauss-Bonnet Theorem implies that

$$\int_{\mathrm{Rat}_n^{eq}} \frac{\kappa(\chi)}{2}\,\mathrm{vol}_{g_\varepsilon} = 2\pi[\sin\xi(0) - \sin\xi(\infty)]. \tag{4.3.35}$$

Since $(\mathrm{Rat}_n^{eq}, g_\varepsilon)$ is isometric under $\hat{\delta}$, then $\xi(0) = -\xi(\infty)$. For $\varepsilon > 0$, $\xi(0) = 0$, and so that the total Gauss curvature of $(\mathrm{Rat}_n^{eq}, g_\varepsilon)$ is zero. This can be considered as another proof of Proposition 4.3.2.

Chapter 5

Baby-Skyrme Laplacian

5.1 Laplacian on Rat_1

In respect to the identification $\mathrm{Rat}_1 \cong SO(3) \times \mathbb{R}^3$ with the coordinate system $([U], \boldsymbol{\lambda})$, let γ be any $SO(3) \times SO(3)$-invariant metric on Rat_1 determined as

$$\gamma = A_1 \boldsymbol{d\lambda} \cdot \boldsymbol{d\lambda} + A_2(\boldsymbol{\lambda} \cdot \boldsymbol{d\lambda})^2 + A_3 \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} + A_4(\boldsymbol{\lambda} \cdot \boldsymbol{\sigma})^2 + A_5 \boldsymbol{\lambda} \cdot (\boldsymbol{\sigma} \times \boldsymbol{d\lambda}), \quad (5.1.1)$$

where $A_1, \ldots, A_5$ are smooth functions of a single variable $\lambda = \|\boldsymbol{\lambda}\|$. Here, as usual $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the left invariant 1-forms dual to the basis $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ of $\mathfrak{so}(3)$. This metric is Hermitian if and only if [48]

$$A_1 + \lambda^2 A_2 = \frac{4}{\Lambda^2}(A_3 + \lambda^2 A_4), \qquad A_3 = \frac{A_1}{4} + \frac{\lambda^2}{2} A_5. \quad (5.1.2)$$

Using (5.1.2), the volume form of an $SO(3) \times SO(3)$-invariant Hermitian metric γ, determined as in (5.1.1), is

$$\mathrm{vol}_\gamma = \sqrt{|\gamma|}\, d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3 \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3, \quad (5.1.3)$$

where

$$|\gamma| = \frac{\Lambda^2}{64}(A_1 + \lambda^2 A_2)^2 Q(\lambda), \tag{5.1.4}$$

and

$$Q(\lambda) = (A_1^2 + \lambda^2 A_5^2)^2 + 4\lambda^2 A_1 A_5 (A_1^2 + \lambda^2 A_1 A_5 - \lambda^2 A_5^2). \tag{5.1.5}$$

Proposition 5.1.1. *The inverse of an $SO(3) \times SO(3)$-invariant Hermitian metric γ, determined as in (5.1.1), on Rat$_1$ is of the form*

$$\gamma^{-1} = C_1 \boldsymbol{\partial} \cdot \boldsymbol{\partial} + C_2(\boldsymbol{\lambda} \cdot \boldsymbol{\partial})^2 + C_3 \boldsymbol{\theta} \cdot \boldsymbol{\theta} + C_4(\boldsymbol{\lambda} \cdot \boldsymbol{\theta})^2 + C_5 \boldsymbol{\lambda} \cdot (\boldsymbol{\theta} \times \boldsymbol{\partial}), \tag{5.1.6}$$

where $C_1, \ldots, C_5$ are smooth functions of λ defined as

$$\begin{aligned} C_1 &= \frac{\Lambda^2 + \lambda^2}{\Lambda^2 A_1}, & C_1 + \lambda^2 C_2 &= \frac{1}{A_1 + \lambda^2 A_2}, & & \\ C_3 &= \frac{4}{\Lambda^2 A_1}, & C_3 + \lambda^2 C_4 &= \frac{4}{\Lambda^2} \frac{1}{A_1 + \lambda^2 A_2}, & C_5 &= -C_3. \end{aligned} \tag{5.1.7}$$

Here, $\boldsymbol{\partial} = (\partial/\partial\lambda_1, \partial/\partial\lambda_2, \partial/\partial\lambda_3)$ is the basis of vector fields on $\mathbb{R}^3$ and $\boldsymbol{\theta} \cdot \boldsymbol{\theta}$ denotes the tensor product of $\boldsymbol{\theta}$ by itself, that is, $\boldsymbol{\theta} \cdot \boldsymbol{\theta} = \sum_{i=1}^{3} \theta_i \otimes \theta_i$.

Proof: The inverse of any $SO(3) \times SO(3)$-invariant metric on Rat$_1$ is an $SO(3) \times SO(3)$-invariant symmetric $(2,0)$ tensor on Rat$_1$. Hence, by following the symmetry argument in [48], which was used to determine the general formula for an $SO(3)\times SO(3)$-invariant metric on Rat$_1$, one obtains that the inverse of an $SO(3) \times SO(3)$-invariant metric γ, determined as in (5.1.1), on Rat$_1$ has the form

$$\begin{aligned} \gamma^{-1} = & C_1 \boldsymbol{\partial} \cdot \boldsymbol{\partial} + C_2(\boldsymbol{\lambda} \cdot \boldsymbol{\partial})^2 + C_3 \boldsymbol{\theta} \cdot \boldsymbol{\theta} + C_4(\boldsymbol{\lambda} \cdot \boldsymbol{\theta})^2 + C_5 \boldsymbol{\lambda} \cdot (\boldsymbol{\theta} \times \boldsymbol{\partial}) \\ & + C_6 \boldsymbol{\theta} \cdot \boldsymbol{\partial} + C_7(\boldsymbol{\lambda} \cdot \boldsymbol{\partial})(\boldsymbol{\lambda} \cdot \boldsymbol{\theta}). \end{aligned} \tag{5.1.8}$$

The coefficients $C_1, \ldots, C_7$ can be determined as follows: let

$$e_1 = \frac{1}{\sqrt{A_1}} \frac{\partial}{\partial \lambda_1}, \quad e_2 = \frac{1}{\sqrt{A_1}} \frac{\partial}{\partial \lambda_2}, \quad e_3 = \frac{1}{\sqrt{A_1 + \lambda^2 A_2}} \frac{\partial}{\partial \lambda_3}. \tag{5.1.9}$$

Since γ is Hermitian, then one can see that $\{e_i, Je_i, i = 1, 2, 3\}$ is an orthonormal basis of $T\text{Rat}_1$ at $([\mathbb{I}_2], (0, 0, \lambda))$ with respect to γ, where J is the almost complex structure on Rat_1, defined as in (3.3.13). Now, for any 1-forms η_1 and η_2 on Rat_1, we have

$$\langle \eta_1, \eta_2 \rangle = \sum_{i=1}^{3} \Big(\eta_1(e_i)\eta_2(e_i) + \eta_1(Je_i)\eta_2(Je_i) \Big), \tag{5.1.10}$$

where $\langle ., . \rangle$ is the induced metric of γ on $\Omega^1(\text{Rat}_1)$. Then, it follows from (5.1.9) and (5.1.10) that

$$\begin{aligned}
\langle d\lambda_1, d\lambda_1 \rangle &= \frac{\Lambda^2 + \lambda^2}{\Lambda^2 A_1}, & \langle \sigma_1, \sigma_1 \rangle &= \frac{4}{\Lambda^2 A_1}, & \langle d\lambda_1, \sigma_1 \rangle &= 0, \\
\langle d\lambda_3, d\lambda_3 \rangle &= \frac{1}{A_1 + \lambda^2 A_2}, & \langle \sigma_3, \sigma_3 \rangle &= \frac{4}{\Lambda^2} \frac{1}{A_1 + \lambda^2 A_2} & \langle d\lambda_3, \sigma_3 \rangle &= 0, \\
\langle d\lambda_1, \sigma_2 \rangle &= \frac{2\lambda}{\Lambda^2 A_1}.
\end{aligned} \tag{5.1.11}$$

But, from (5.1.8), we have

$$\begin{aligned}
\langle d\lambda_1, d\lambda_1 \rangle &= C_1, & \langle \sigma_1, \sigma_1 \rangle &= C_3, & \langle d\lambda_1, \sigma_1 \rangle &= C_6, \\
\langle d\lambda_3, d\lambda_3 \rangle &= C_1 + \lambda^2 C_2, & \langle \sigma_3, \sigma_3 \rangle &= C_3 + \lambda^2 C_4 & \langle d\lambda_3, \sigma_3 \rangle &= C_6 + \lambda^2 C_7, \\
\langle d\lambda_1, \sigma_2 \rangle &= -\frac{1}{2}\lambda\, C_5.
\end{aligned} \tag{5.1.12}$$

Comparing (5.1.12) with (5.1.11), the functions $C_1, \ldots, C_7$ are determined, and hence the claim is proved.

□

Proposition 5.1.2. *The Hodge Laplacian of an $SO(3) \times SO(3)$-invariant Hermitian metric γ, determined as in (5.1.1), on Rat$_1$ is of the form*

$$\Delta f = -\frac{4}{\Lambda^2 A_1}\left\{\boldsymbol{\theta}\cdot\boldsymbol{\theta} f + \boldsymbol{\lambda}\cdot(\boldsymbol{\partial}\times\boldsymbol{\theta})f - \frac{A_2}{A_1+\lambda^2 A_2}(\boldsymbol{\lambda}\cdot\boldsymbol{\theta})^2 f\right\}$$
$$-\frac{1}{\lambda^2\sqrt{|\gamma|}}\frac{\partial}{\partial\lambda}\left[\frac{\lambda^2\sqrt{|\gamma|}}{A_1+\lambda^2 A_2}\frac{\partial f}{\partial\lambda}\right] - \frac{\Lambda^2+\lambda^2}{\lambda^2\Lambda^2 A_1}(\boldsymbol{\lambda}\times\boldsymbol{\partial})\cdot(\boldsymbol{\lambda}\times\boldsymbol{\partial})f, \qquad (5.1.13)$$

for all $f \in \mathcal{C}^\infty(Rat_1, \mathbb{C})$ where $|\gamma|$ is given by (5.1.4).

Proof: Let $\{\mu_{ij}, \eta_{ij}, i,j = 1,2,3\}$ be a local basis for $\Omega^5(\text{Rat}_1)$ defined as

$$\mu_{ij} = d\lambda_i \wedge d\lambda_j \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3,$$
$$\eta_{ij} = \sigma_i \wedge \sigma_j \wedge d\lambda_1 \wedge d\lambda_2 \wedge d\lambda_3. \qquad (5.1.14)$$

For any 1-forms $\alpha_1, \ldots, \alpha_k$ on Rat_1, the exterior derivative of the k-form $(\alpha_1 \wedge \cdots \wedge \alpha_k)$ is given by

$$d(\alpha_1 \wedge \cdots \wedge \alpha_k) = \sum_{i=1}^{k}(-1)^{i-1}\alpha_1 \wedge \cdots \wedge d\alpha_i \wedge \cdots \wedge \alpha_k. \qquad (5.1.15)$$

Now, since $d(d\lambda_i) = d^2\lambda_i = 0$ and the exterior derivatives of the left-invariant 1-forms $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, as in (3.3.43), are

$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \quad d\sigma_2 = \sigma_3 \wedge \sigma_1, \quad d\sigma_3 = \sigma_1 \wedge \sigma_2, \qquad (5.1.16)$$

then, by using (5.1.15), one gets

$$d\mu_{ij} = d\eta_{ij} = 0, \qquad \forall\, i,j = 1,2,3. \qquad (5.1.17)$$

It follows from (5.1.17) that all μ_{ij} and η_{ij} are closed. Hence, in this case, Lemma 2 in [28] states that the Hodge Laplacian $\Delta = d\,\delta + \delta\,d$ acting on functions on (Rat_1, γ) is of the form

$$\Delta f = -\frac{1}{\sqrt{|\gamma|}}\sum_{i,j}\theta_i\left[\sqrt{|\gamma|}\Big(\gamma^{-1}(\sigma_i, d\lambda_j)\partial_j[f] + \gamma^{-1}(\sigma_i,\sigma_j)\theta_j[f]\Big)\right]$$
$$-\frac{1}{\sqrt{|\gamma|}}\sum_{i,j}\partial_i\left[\sqrt{|\gamma|}\Big(\gamma^{-1}(d\lambda_i, d\lambda_j)\partial_j[f] + \gamma^{-1}(d\lambda_i,\sigma_j)\theta_j[f]\Big)\right]. \quad (5.1.18)$$

Using the formula of the inverse metric γ^{-1}, given in (5.1.6), we have

$$\Delta f = -\frac{1}{\sqrt{|\gamma|}}\,\boldsymbol{\theta}\cdot\left[\sqrt{|\gamma|}\Big(C_3\boldsymbol{\theta}f + C_4\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\boldsymbol{\theta}f) - \frac{C_5}{2}(\boldsymbol{\lambda}\times\boldsymbol{\partial}f)\Big)\right]$$
$$-\frac{1}{\sqrt{|\gamma|}}\,\boldsymbol{\partial}\cdot\left[\sqrt{|\gamma|}\Big(C_1\boldsymbol{\partial}f + C_2\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\boldsymbol{\partial}f) + \frac{C_5}{2}(\boldsymbol{\lambda}\times\boldsymbol{\theta}f)\Big)\right]. \quad (5.1.19)$$

Since $C_5 = -C_3$, then Δf can be written as

$$\Delta f = -C_3\Big\{\boldsymbol{\theta}\cdot\boldsymbol{\theta}f + \boldsymbol{\lambda}\cdot(\boldsymbol{\partial}\times\boldsymbol{\theta})f + \frac{C_4}{C_3}(\boldsymbol{\lambda}\cdot\boldsymbol{\theta})^2 f\Big\}$$
$$-\frac{1}{\sqrt{|\gamma|}}\boldsymbol{\partial}\cdot\left[\sqrt{|\gamma|}C_1\boldsymbol{\partial}f + \sqrt{|\gamma|}C_2\boldsymbol{\lambda}(\boldsymbol{\lambda}\cdot\boldsymbol{\partial}f)\right]. \quad (5.1.20)$$

Here, $\boldsymbol{\theta}\cdot\boldsymbol{\theta}$ is the trace of the $\boldsymbol{\theta}\cdot\boldsymbol{\theta}$ in (5.1.6). After straightforward manipulations, we get

$$\Delta f = -C_3\Big\{\boldsymbol{\theta}\cdot\boldsymbol{\theta}f + \boldsymbol{\lambda}\cdot(\boldsymbol{\partial}\times\boldsymbol{\theta})f + \frac{C_4}{C_3}(\boldsymbol{\lambda}\cdot\boldsymbol{\theta})^2 f\Big\} - \frac{C_1}{\lambda^2}(\boldsymbol{\lambda}\times\boldsymbol{\partial})\cdot(\boldsymbol{\lambda}\times\boldsymbol{\partial})f$$
$$-\frac{1}{\lambda^2\sqrt{|\gamma|}}\frac{\partial}{\partial\lambda}\Big(\lambda^2\sqrt{|\gamma|}(C_1 + \lambda^2 C_2)\frac{\partial f}{\partial\lambda}\Big). \quad (5.1.21)$$

Substituting (5.1.7) in (5.1.21), the claim is established.

□

The above Laplacian formula (5.1.13) for any $SO(3)\times SO(3)$-invariant Hermitian metric on Rat_1, determined as in (5.1.1), is a generalization of the the one has been determined in [28] provided the metric is Kähler.

Recall that in section 3.3.3 the generators, the Killing vector fields on Rat_1, of the $SO(3)\times SO(3)$ action on Rat_1, given in (3.3.11), were determined by studying the effect of the left and right $SO(3)$ actions on Rat_1 separately. The right $SO(3)$ action on Rat_1 is generated by the left-invariant vector fields $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ of $SO(3)$ added to $\boldsymbol{\Phi} = \boldsymbol{\lambda} \times \boldsymbol{\partial}$, that is, $\boldsymbol{X} = \boldsymbol{\theta} + \boldsymbol{\Phi}$ where $\Phi_i = \epsilon_{ijk}\ \lambda_j\ \partial/\partial\lambda_k$. These vector fields satisfy the angular momentum algebra [28],

$$[\theta_i, \theta_j] = -\epsilon_{ijk}\theta_k, \qquad [\Phi_i, \Phi_j] = -\epsilon_{ijk}\Phi_k, \qquad [X_i, X_j] = -\epsilon_{ijk}X_k. \tag{5.1.22}$$

Using the commutator identity $[X, YZ] = [X, Y]Z + Y[X, Z]$, then

$$[\theta_i, \boldsymbol{\theta}\cdot\boldsymbol{\theta}] = [\Phi_i, \boldsymbol{\Phi}\cdot\boldsymbol{\Phi}] = [X_i, \boldsymbol{X}\cdot\boldsymbol{X}] = 0. \tag{5.1.23}$$

Note that $\boldsymbol{\theta}\cdot\boldsymbol{\theta}$ now denotes a second order operator. Since $[\theta_i, \partial/\partial\lambda_j] = 0$, then $[\theta_i, \Phi_j] = 0$, and so

$$[\theta_i, \boldsymbol{\Phi}\cdot\boldsymbol{\Phi}] = [\Phi_i, \boldsymbol{\theta}\cdot\boldsymbol{\theta}] = 0. \tag{5.1.24}$$

It follows from (5.1.23) and (5.1.24) that

$$[X_i, \boldsymbol{\theta}\cdot\boldsymbol{\theta}] = [X_i, \boldsymbol{\Phi}\cdot\boldsymbol{\Phi}] = [X_i, \boldsymbol{\theta}\cdot\boldsymbol{\Phi}] = 0. \tag{5.1.25}$$

Note that $\boldsymbol{\theta}\cdot\boldsymbol{\Phi} = \boldsymbol{\theta}\cdot(\boldsymbol{\lambda}\times\boldsymbol{\partial}) = \boldsymbol{\lambda}\cdot(\boldsymbol{\partial}\times\boldsymbol{\theta})$. Also, we have

$$[X_i, \boldsymbol{\lambda}\cdot\boldsymbol{\theta}] = [\theta_i, \lambda_l\theta_l] + \epsilon_{ijk}[\lambda_j\partial_k, \lambda_l\theta_l] = -\epsilon_{ilm}\lambda_l\theta_m + \epsilon_{ijl}\lambda_j\theta_l = 0. \tag{5.1.26}$$

Unlike the right $SO(3)$ action, the left $SO(3)$ action on Rat_1 is only generated by the right-invariant vector fields $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ of $SO(3)$ which satisfy

$$[\xi_i, \theta_j] = [\xi_i, \Phi_j] = 0. \tag{5.1.27}$$

Again, by using the commutator identity, we have

$$[\xi_i, \boldsymbol{\theta} \cdot \boldsymbol{\theta}] = [\xi_i, \boldsymbol{\Phi} \cdot \boldsymbol{\theta}] = [\xi_i, \boldsymbol{\lambda} \cdot \boldsymbol{\theta}] = 0. \tag{5.1.28}$$

The vector fields X_i and ξ_i on Rat_1 satisfy

$$[X_i, \partial/\partial\lambda] = [\xi_i, \partial/\partial\lambda] = 0, \qquad X_i[f] = \xi_i[f] = 0, \tag{5.1.29}$$

for any function f of a single variable λ. It follows from (5.1.25), (5.1.26) and (5.1.29) that $[X_i, \Delta] = 0$, whereas (5.1.28) and (5.1.29) imply that $[\xi_i, \Delta] = 0$. Hence, the Laplacian Δ, given in (5.1.13), commutes with all Killing vector fields on (Rat_1, γ), and so it is $SO(3) \times SO(3)$ invariant.

5.2 Spectrum of Baby-Skyrme Quantum Hamiltonian on Rat_1

In the field theories whose static n-soliton low energy dynamics is approximated by geodesic motion on the moduli space $\mathsf{M_n}$ with respect to the metric derived from the kinetic energy of the theory, there is a well-known quantization of this dynamics given by the quantum Hamiltonian

$$\hat{H}\Psi = \frac{1}{2}\Delta\Psi, \tag{5.2.1}$$

where Ψ is a complex-valued function on $\mathsf{M_n}$ and Δ is the Hodge Laplacian on $\mathsf{M_n}$. The spectrum of $\hat{H}$, denoted $\sigma(\hat{H})$, can be derived by studying the spectral problem

$$\hat{H}\Psi = E\Psi, \qquad \forall E \geq 0. \tag{5.2.2}$$

Let $\sigma_p(\hat{H})$ denote the set of E's such that the equation (5.2.2) has non-trivial solutions on $\mathsf{M_n}$. Let also $\sigma_c(\hat{H})$ be the set of E's $\notin \sigma_p(\hat{H})$ such that the operator $(\hat{H} - E)$ is not invertible. Then, the spectrum of $\hat{H}$ is [41, p.140-141]

$$\sigma(\hat{H}) = \sigma_p(\hat{H}) \cup \sigma_c(\hat{H}). \tag{5.2.3}$$

Clearly, $\sigma_p(\hat{H})$ is just the set of eigenvalues of $\hat{H}$ which called the point spectrum of $\hat{H}$, whereas the set $\sigma_c(\hat{H})$ is called the continuous spectrum of $\hat{H}$. In the case $\sigma_c(\hat{H}) = \emptyset$ ($\sigma_p(\hat{H}) = \emptyset$), then the spectrum of $\hat{H}$ is called purely discrete (purely continuous).

On Rat_1, the spectrum of $\hat{H}$ with respect to the L^2 metric has been studied in [28] using the Kähler property of the metric. Unlike the L^2 metric, finding the spectrum of the quantum Hamiltonian $\hat{H}$ with respect to an $SO(3) \times SO(3)$ invariant Hermitian metric on Rat_1 which is not Kähler, such as the Baby-Skyrme metric, is a very difficult problem. Hence, we will restrict this problem to the space of complex-valued functions depending only on the vector $\boldsymbol{\lambda} \in \mathbb{R}^3$. Then, the spectral problem given in (5.2.2) will reduce to the Sturm-Liouville problem

$$-\frac{1}{w(\lambda)} \frac{d}{d\lambda} \left[p(\lambda) \frac{df}{d\lambda} \right] + q(\lambda) f = 2Ef, \tag{5.2.4}$$

where f is a function of a single variable $\lambda \in [0, \infty)$.

5.2.1 Reduction to Sturm-Liouville Problem

Let $(\lambda, \vartheta, \varphi)$ be the spherical coordinates on $\mathbb{R}^3$ and $F = F(\lambda, \vartheta, \varphi)$ be a function on $\mathbb{R}^3$. Then, the Laplacian on Rat_1, given in (5.1.13), for $\Psi = F$ reduces to

$$\Delta F = -\frac{1}{\lambda^2 \sqrt{|\gamma|}} \frac{\partial}{\partial \lambda} \left[\frac{\lambda^2 \sqrt{|\gamma|}}{A_1 + \lambda^2 A_2} \frac{\partial F}{\partial \lambda} \right] + \frac{\Lambda^2 + \lambda^2}{\lambda^2 \Lambda^2 A_1} \Delta_{S^2} F, \tag{5.2.5}$$

where Δ_{S^2} is the Laplacian on S^2, namely,

$$\Delta_{S^2} = -\frac{1}{\sin\vartheta} \frac{\partial}{\partial \vartheta} \left(\sin\vartheta \frac{\partial}{\partial \vartheta} \right) - \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial \varphi^2}. \tag{5.2.6}$$

The eigenfunctions of Δ_{S^2} are the so-called spherical harmonic functions $Y_{lm}(\vartheta, \varphi)$ defined such that

$$\Delta_{S^2} Y_{lm} = l(l+1) Y_{lm}, \qquad l, m \in \mathbb{N}. \tag{5.2.7}$$

Consequently, it is useful to separate the variables in (5.2.5) by decomposing $F(\lambda, \vartheta, \varphi)$ into $Y_{lm}(\vartheta, \varphi)$ multiplied by a radial function $f(\lambda)$, that is, $F(\lambda, \vartheta, \varphi) = f(\lambda)\, Y_{lm}(\vartheta, \varphi)$. Thus, ΔF in (5.2.5) can be written as

$$\Delta F = -\frac{1}{\lambda^2 \sqrt{|\gamma|}} \frac{d}{d\lambda} \left[\frac{\lambda^2 \sqrt{|\gamma|}}{A_1 + \lambda^2 A_2} \frac{df}{d\lambda} \right] Y_{lm} + \frac{\Lambda^2 + \lambda^2}{\lambda^2 \Lambda^2 A_1}\, l(l+1) f\, Y_{lm}. \tag{5.2.8}$$

Hence, solving the time-independent Schödinger equation

$$\hat{H} F = E F, \tag{5.2.9}$$

is equivalent to solving

$$-\frac{1}{\lambda^2 \sqrt{|\gamma|}} \frac{d}{d\lambda} \left[\frac{\lambda^2 \sqrt{|\gamma|}}{A_1 + \lambda^2 A_2} \frac{df}{d\lambda} \right] + \frac{\Lambda^2 + \lambda^2}{\lambda^2 \Lambda^2 A_1}\, l(l+1) f = 2 E f, \tag{5.2.10}$$

which is a second order ordinary differential equation on $[0, \infty)$ given in the form of Sturm-Liouville equation (5.2.4) with

$$p(\lambda) = \frac{\lambda^2 \sqrt{|\gamma|}}{A_1 + \lambda^2 A_2}, \quad q(\lambda) = \frac{\Lambda^2 + \lambda^2}{\lambda^2 \Lambda^2 A_1} l(l+1), \quad w(\lambda) = \lambda^2 \sqrt{|\gamma|}. \tag{5.2.11}$$

5.2.2 Spectrum Classification

A Sturm-Liouville problem (5.2.4) is said to be singular if one or both endpoints are singular, which means that $1/p(\lambda)$, $q(\lambda)$ or $w(\lambda)$ fail to be integrable at this point [41, p.137]. Now, to check whether the Sturm-Liouville problem (5.2.10) with respect to the Baby-Skyrme metric γ_ε is singular or not, we need the following limits:

$$\lim_{\lambda\to 0} A_{\varepsilon_1} = \frac{2\pi}{3}(\varepsilon+1), \qquad \lim_{\lambda\to 0} A_{\varepsilon_2} = -2\pi(\varepsilon+\frac{7}{15}), \tag{5.2.12}$$

$$\lim_{\lambda\to\infty} \frac{A_{\varepsilon_1}}{\lambda^2} = \frac{4\pi\varepsilon}{3}, \qquad \lim_{\lambda\to\infty} A_{\varepsilon_2} = -\frac{4\pi\varepsilon}{3}, \tag{5.2.13}$$

where $A_{\varepsilon_i} = A_i + \varepsilon A_{*_i}$ are the coefficients of the Baby-Skyrme metric on Rat_1. Since γ_ε is Hermitian, then its volume form is given by (5.1.4), and so we have the limits

$$\lim_{\lambda\to 0} \sqrt{|\gamma_\varepsilon|} = \frac{\pi^3}{27}\,(\varepsilon+1)^3, \qquad \lim_{\lambda\to\infty} \sqrt{|\gamma_\varepsilon|}/\lambda = \frac{\pi^3\varepsilon^2}{27}\,(\varepsilon+3). \tag{5.2.14}$$

It follows from (5.2.12), (5.2.13) and (5.2.14) that $1/p(\lambda)$ and $w(\lambda)$ are not integrable at $\lambda = 0$ and $\lambda = \infty$, respectively. Hence, our Sturm-Liouville problem (5.2.10) on $[0,\infty)$ with respect to γ_ε is singular.

In this section, we wish to classify the spectrum of the quantum Hamiltonian $\hat{H}$ with respect to the Baby-Skyrme metric γ_ε on Rat_1 depending on the behaviour of the solutions of our Sturm-Liouville problem (5.2.10) in neighbourhoods of the endpoints $\lambda = 0$ and $\lambda = \infty$ as follows:

It follows from (5.2.12) and (5.2.14) that the leading order equation of (5.2.10) as $\lambda \to 0$ is

$$\frac{d^2 f}{d\lambda^2} + \frac{2}{\lambda}\frac{df}{d\lambda} - \frac{l(l+1)}{\lambda^2} f = 0, \tag{5.2.15}$$

which has asymptotic solutions of the form

$$f(\lambda) = a_1\lambda^l + a_2\lambda^{-(l+1)} =: a_1 f_1(\lambda) + a_2 f_2(\lambda). \tag{5.2.16}$$

Note that these solutions are independent of ε and E. As $\lambda \to \infty$, the leading order equation of (5.2.10) for all $\varepsilon > 0$ is

$$\frac{d^2f}{d\lambda^2}+\frac{5}{\lambda}\frac{df}{d\lambda}+\frac{4\pi\varepsilon}{3\lambda^2}Ef=0. \tag{5.2.17}$$

The asymptotic solutions of (5.2.17) have the form

$$\begin{aligned} f(\lambda) &= b_1\lambda^{\left(-2+2\sqrt{1-\pi\varepsilon E/3}\right)}+b_2\lambda^{\left(-2-2\sqrt{1-\pi\varepsilon E/3}\right)} \\ &=: b_1g_1(\lambda)+b_2g_2(\lambda), \end{aligned} \tag{5.2.18}$$

which clearly depend on ε and E. For $\varepsilon = 0$, the leading order equation as $\lambda \to \infty$ is

$$\frac{d^2f}{d\lambda^2}+\frac{1}{\lambda}\frac{df}{d\lambda}=0, \tag{5.2.19}$$

which is the leading order equation of (5.2.10) with respect to the L^2 metric on Rat_1. The general solutions of (5.2.19) are

$$f(\lambda)=c_1+c_2\log\lambda := c_1h_1(\lambda)+c_2h_2(\lambda), \tag{5.2.20}$$

which are independent of both E and l.

At the endpoint $\lambda = 0$, the asymptotic solutions (5.2.16) are not oscillating for all E. In contrast, at the endpoint $\lambda = \infty$, the asymptotic solutions are classified to

- For $\varepsilon = 0$, the solutions (5.2.20) are not oscillating for all E.
- For $\varepsilon > 0$, the solutions (5.2.18) are not oscillating for all $E \leq 3/(\pi\varepsilon)$ whereas they oscillate for all $E > 3/(\pi\varepsilon)$.

Hence, by the classification categories of Sturm-Liouville problems given in [21], the spectrum of Baby-Skyrme quantum Hamiltonian $\hat{H}$ on Rat_1 is as following:

- For $\varepsilon = 0$, both endpoints $\lambda = 0$ and $\lambda = \infty$ are non-oscillatory. Hence, the spectrum is simple, bounded below, infinite and purely discrete.

- For $\varepsilon > 0$, the endpoint $\lambda = 0$ is non-oscillatory for all E, but at the other endpoint $\lambda = \infty$, there exists $E_\varepsilon = 3/(\pi\varepsilon)$ such that the endpoint is non-oscillatory for all $E \leq E_\varepsilon$ and it is oscillatory for all $E > E_\varepsilon$. Thus, the spectrum is simple, bounded below and it consists of finite point spectrum $\sigma_p(\hat{H})$ which is, if any, in $[0, E_\varepsilon]$ and continuous spectrum $\sigma_c(\hat{H})$ for all (E_ε, ∞).

5.2.3 L^2 Finiteness and Boundary Conditions

One of the most important requirement for a singular Sturm-Liouville problem is to have solutions f which are L^2 finite with respect to the appropriate weight function $w(\lambda)$ [41, p.138], that is,

$$\int_0^\infty |f|^2 \, w(\lambda) \, d\lambda < \infty. \tag{5.2.21}$$

If f is continuous, then it is sufficient to check that f is L^2 finite at both endpoints $\lambda = 0$ and $\lambda = \infty$.

For the Baby-Skyrme metric on Rat_1, one can see from the asymptotic solutions f_1 and f_2 in (5.2.16) that

$$|f_1|^2 \, w_\varepsilon(\lambda) \sim \tilde{a}_1 \lambda^{2l+2}, \qquad |f_2|^2 \, w_\varepsilon(\lambda) \sim \tilde{a}_2 \lambda^{-2l}, \quad \text{as } \lambda \to 0, \tag{5.2.22}$$

where $\omega_\varepsilon(\lambda)$ is the weight function with respect to γ_ε on Rat_1. Therefore, for $l = 0$, both asymptotic solutions f_1 and f_2 are L^2 finite, whereas for $l > 0$, f_1 is the only L^2 finite solution.

As $\lambda \to \infty$, the asymptotic form of the integrand in (5.2.21) for g_1 and g_2 are

$$|g_1|^2 \, w_\varepsilon(\lambda) \sim \tilde{b}_1 \lambda^{\left(-1+4\sqrt{1-E/E_\varepsilon}\right)}, \qquad |g_2|^2 \, w_\varepsilon(\lambda) \sim \tilde{b}_2 \lambda^{\left(-1-4\sqrt{1-E/E_\varepsilon}\right)}. \tag{5.2.23}$$

Clearly, for all $E < E_\varepsilon$, the asymptotic solution g_2 is the only L^2 finite solution, with no L^2 finite non-trivial solutions for all $E \geq E_\varepsilon$. When $\varepsilon = 0$, in contrast, one has from (5.2.20) that

$$|h_1|^2 \, w_0(\lambda) \sim \tilde{c}_1 \frac{\log \lambda}{\lambda^5}, \qquad |h_2|^2 \, w_0(\lambda) \sim \tilde{c}_2 \frac{(\log \lambda)^3}{\lambda^5}, \tag{5.2.24}$$

as $\lambda \to \infty$ which implies the L^2 finiteness for both h_1 and h_2.

A singular endpoint of a Sturm-Liouville problem (5.2.4) is classified to: **limit-circle** if all solutions of (5.2.4) are L^2 finite in a neighbourhood of the endpoint, whereas if there is precisely one L^2 finite non-trivial solution of (5.2.4) in an endpoint's neighbourhood, then it is called **limit-point** [41, p.147]. Hence, it follows from the above investigation of the asymptotic solutions at $\lambda = 0$ and $\lambda = \infty$ of (5.2.10) with respect to the Baby-Skyrme metric γ_ε on Rat_1 that

- The endpoint $\lambda = 0$ is limit-circle for $l = 0$ and is limit-point for all $l > 0$.
- The endpoint $\lambda = \infty$ is limit-circle for $\varepsilon = 0$, whereas it is limit-point for all $\varepsilon > 0$, with no L^2 finite non-trivial solutions for all $E \geq E_\varepsilon$.

In the presence of limit-circle case at a non-oscillatory endpoint λ_*, say, then the Sturm-Liouville problem is not self adjoint unless an boundary condition is imposed in the nieghbourhood of λ_*. The boundary condition of the Sturm-Lioville problem at such point is given by a solution $u(\lambda)$ in a neighborhood of the endpoint such that for any linearly independent solution $v(\lambda)$ [41, p.158],

$$\lim_{\lambda \to \lambda_*} \frac{u(\lambda)}{v(\lambda)} = 0. \tag{5.2.25}$$

This solution is called the **subdominant** solution at the endpoint λ_*. If no limit-circle non-oscillatory endpoints exist in the problem, then, by definition, the L^2 finite solution is the subdominant solution. For our Sturm-Liouville problem (5.2.10) with respect to γ_ε on Rat_1 , we have

- At the endpoint $\lambda = 0$, the subdominant solution is

$$f(\lambda) \sim a_1 \lambda^l, \qquad \forall\, l \geq 0. \tag{5.2.26}$$

- At the endpoint $\lambda = \infty$, the subdominant solution is

$$\begin{aligned} f(\lambda) &\sim b_2 \lambda^{\left(-2-2\sqrt{1-E/E_\varepsilon}\right)}, && \varepsilon > 0, \\ f(\lambda) &\sim c_1, && \varepsilon = 0. \end{aligned} \tag{5.2.27}$$

We conclude that the boundary conditions at 0 and ∞ for the Sturm-Liouville problem (5.2.10) with respect to the Baby-Skyrme-metric on Rat_1 are given by the above subdominant solutions.

5.2.4 Numerical Results

Solving the Sturm-Liouville equation (5.2.10) with respect to the Baby-Skyrme metric γ_ε for $\lambda \in [\lambda_0, \lambda_\infty]$ with the above boundary conditions can be reduced to solving two initial value problems, one in $[\lambda_0, \lambda_1]$ with initial condition (5.2.26) at λ_0 and the other on $[\lambda_1, \lambda_\infty]$ with intial condition (5.2.27) at λ_∞ where $\lambda_1 \in (\lambda_0, \lambda_\infty)$. Now, for fixed l and ε, let $Y_L(\lambda) = (f_L(\lambda), f_L'(\lambda))$ and $Y_R(\lambda) = (f_R(\lambda), f_R'(\lambda))$ be the solutions of the left and right initial value problems, respectively. Hence, E is an eigenvalue of (5.2.10) if

$$\text{F}(E) = f_L(\lambda_1) f_R'(\lambda_1) - f_L'(\lambda_1) f_R(\lambda_1) = 0. \tag{5.2.28}$$

For fixed ε and l, having solved the left and right initial value problems on $[\lambda_0, \lambda_1]$ and $[\lambda_1, \lambda_\infty]$, respectively, by Runge-Kutta (4,5) method [14], one can construct the function $\text{F}(E)$ numerically, and then find the roots of $\text{F}(E)$ by one of the root finding numerical methods. Here, we have used the bisection method [3, p.43-45] on the interval $[\lambda_0, \lambda_\infty] = [0.001, 30]$. The following table shows the lowest eigenvalues E_0 of the quantum Hamiltonian $\hat{H}$ for various values of l and ε.

ε	$l=0$	$l=1$	$l=2$	$l=3$	$l>3$	E_ε
0.0	0.00000	2.12055	5.52641	10.21228	$\longrightarrow\infty$	∞
0.1	2.03751	4.39516	6.86855	8.85858	$\times$	9.54929
0.2	2.05607	3.69199	4.72555	$\times$	$\times$	4.77465
0.3	1.92457	2.99012	$\times$	$\times$	$\times$	3.18309
0.4	1.75728	2.38299	$\times$	$\times$	$\times$	2.3873
0.5	1.58933	$\times$	$\times$	$\times$	$\times$	1.90986
0.6	1.43279	$\times$	$\times$	$\times$	$\times$	1.59155
0.7	1.29128	$\times$	$\times$	$\times$	$\times$	1.36419
0.8	1.16513	$\times$	$\times$	$\times$	$\times$	1.19366
0.9	1.05335	$\times$	$\times$	$\times$	$\times$	1.06103
1.0	0.95448	$\times$	$\times$	$\times$	$\times$	0.95493

Table 5.1: Numerical values of the lowest eigenvalues of the quantum Hamiltonian $\hat{H}$ for various values of l and ε.

From Table 5.1, one observes that for fixed ε, the lowest eigenvalues, if any, are increasing as l increases. For fixed l, the behaviour of the lowest eigenvalue as a function of ε is drawn in Figure 5.1. Furthermore, one may deduce, from Figure 5.2, that $\sigma_p(\hat{H}) \to \emptyset$ as $\varepsilon, l \to \infty$.

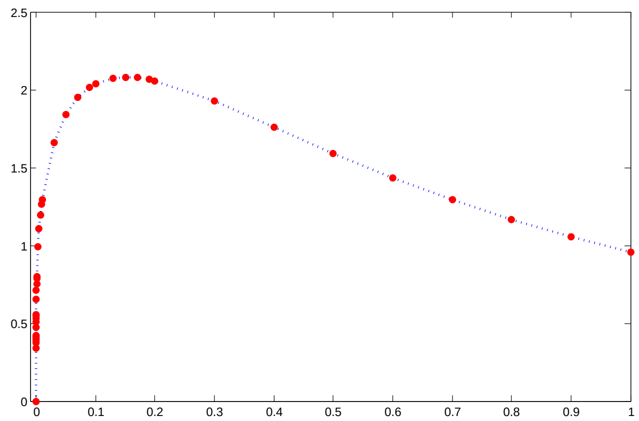


Figure 5.1: Plot of the lowest eigenvalue of $\hat{H}$ as a function of the Baby-Skyrme parameter $\varepsilon \in [0,1]$ with $l=0$.

Recall that we have determined our Baby-Skyrme metric γ_ε on $\mathrm{Rat}_1 \cong \mathcal{H}_{1,1}(S^2)$ when both the domain and codomain are 2-spheres with radius $1/2$. Changing the radii of the domain and codomain to R_1 and R_2, respectively, will scale the Baby-Skyrme metric as $\tilde{\gamma}_\varepsilon = (4R_1R_2)^2\gamma_\varepsilon$. Consequently, the Laplacian on Rat_1 with respect to $\tilde{\gamma}_\varepsilon$ is $\tilde{\Delta} = (4R_1R_2)^{-2}\Delta$. Hence, the eigenvalues of $\tilde{\hat{H}} = \tilde{\Delta}/2$ and $\hat{H} = \Delta/2$ are related by

$$\tilde{E} = (4R_1R_2)^{-2}E. \tag{5.2.29}$$

One can use (5.2.29) to compare our results for $\varepsilon = 0$ with those which have been computed with respect to the L^2 metric on Rat_1 in [28] where $R_1 = R_2 = 1$ and in [37] where $R_1 = 1/\sqrt{2}$ and $R_2 = 1/2$.

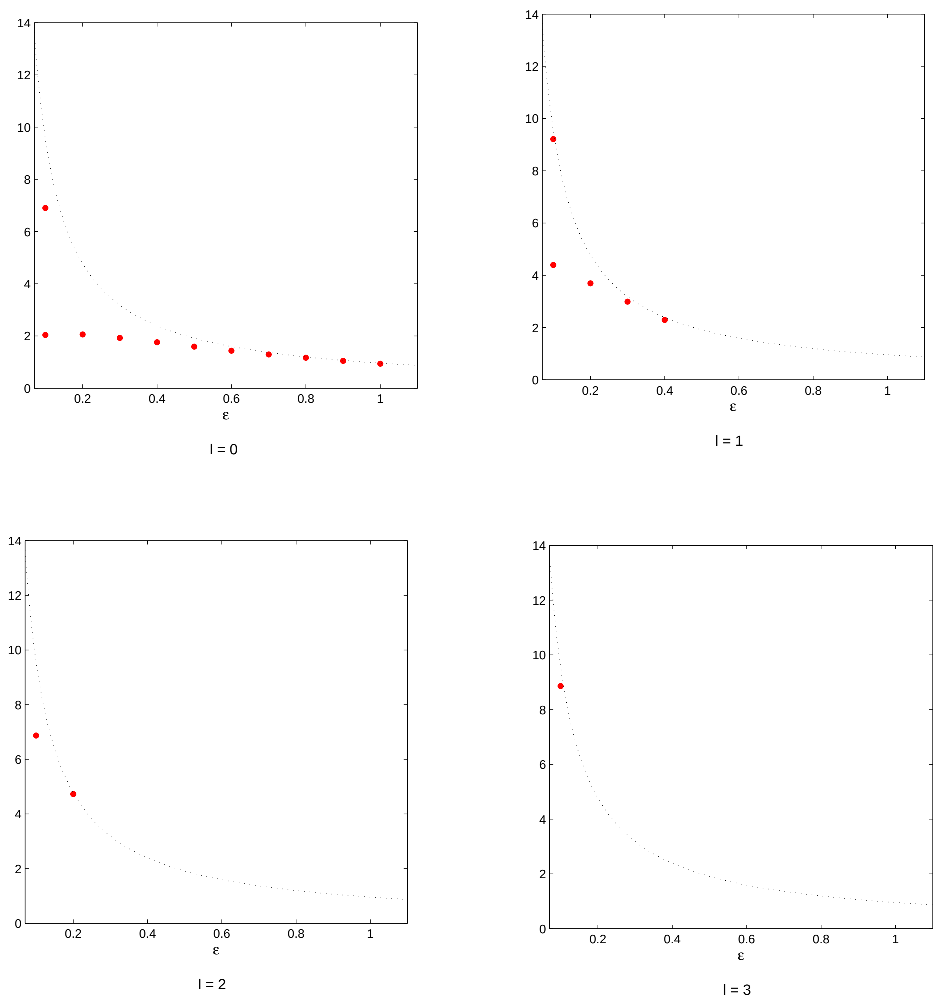


Figure 5.2: Plots of the eigenvalues (red points) of $\hat{H}$ bounded above by E_ε (dots curve) for some values of $\varepsilon \in (0, 1]$.

5.2.5 Spectrum with Skyrme Potential

Restricting the potential energy, given in (4.1.5), of the Baby-Skyrme model to Rat_1, we get

$$V[W] = 2\pi + \frac{i}{2}\,\varepsilon \int_{S^2} \frac{2\,|\partial_z W|^4}{(1+|W|^2)^4}(1+|z|^2)^2\, dzd\bar{z}, \tag{5.2.30}$$

$$=: 2\pi + \varepsilon\, V_{BS}[W]. \tag{5.2.31}$$

The term with ε is the potential energy $V_{BS}[W]$, descending from the Skyrme term. This is $SO(3)\times SO(3)$-invariant, and hence

$$V_{BS}[W] = V_{BS}[W_\lambda] = \frac{i}{2}\int_{S^2} \frac{2\mu^4}{(1+\mu^2\,|z|^2)^4}(1+|z|^2)^2\, dz\, d\bar{z}, \\ = 2\pi\left(\frac{\mu^4+\mu^2+1}{3\mu^2}\right) =: V_*(\lambda), \tag{5.2.32}$$

where $W_\lambda(z) = \mu z$, $\mu = (\Lambda+\lambda)^2$, is an element of the $SO(3)\times SO(3)$ orbit space in Rat_1. Consequently, there is a small modification in the quantum Hamiltonian $\hat{H}$ given by

$$\hat{H}_* := \hat{H} + \varepsilon\, V_*(\lambda), \qquad \forall\, \varepsilon > 0. \tag{5.2.33}$$

Clearly, as $\varepsilon \to 0$, $\hat{H}_*$ approaches the quantum Hamiltonian $\hat{H}$ with respect to the L^2 metric. The spectrum of $\hat{H}_*$ can be computed by solving the following Sturm-Liouville problem

$$-\frac{1}{\lambda^2\sqrt{|\gamma_\varepsilon|}}\frac{d}{d\lambda}\left[\frac{\lambda^2\sqrt{|\gamma_\varepsilon|}}{A_{\varepsilon_1}+\lambda^2 A_{\varepsilon_2}}\frac{df}{d\lambda}\right] + \frac{\Lambda^2+\lambda^2}{\lambda^2\Lambda^2 A_{\varepsilon_1}} l(l+1)f + \varepsilon\, V_*(\lambda) f = 2Ef. \tag{5.2.34}$$

Since $\lim_{\lambda\to 0}\lambda^2 V_*(\lambda) = 0$, then the leading order equation of (5.2.34), and so the asymptotic solutions, as $\lambda \to 0$ remain unchanged, as in (5.2.16). In contrast, the presence of the potential $V_*(\lambda)$ in (5.2.34) changes the leading order equation (5.2.17) as $\lambda \to \infty$ to

$$\frac{df^2}{d\lambda^2} + \frac{5}{\lambda}\frac{df}{d\lambda} - \frac{64\pi^2\varepsilon^2}{9}\lambda^2 f = 0, \tag{5.2.35}$$

which is clearly independent of E. The asymptotic solutions of (5.2.35) are

$$\begin{aligned} f(\lambda) &= d_1\,\frac{1}{\lambda^2} I_1\big(4\pi\varepsilon\lambda^2/3\big) + d_2\,\frac{1}{\lambda^2} K_1\big(4\pi\varepsilon\lambda^2/3\big), \\ &=: d_1\, p_1(\lambda) + d_2\, p_2(\lambda), \end{aligned} \tag{5.2.36}$$

where I_1 and K_1 are the modified Bessel functions of order 1. They have the leading asymptotic

$$I_1(s) \sim \sqrt{\frac{1}{2\pi s}}\, e^{s}, \qquad K_1(s) \sim \sqrt{\frac{\pi}{2s}}\, e^{-s}, \qquad \text{as } s \to \infty. \tag{5.2.37}$$

Therefore, the asymptotic formulae for the integrand in (5.2.21) with p_1 and p_2 are

$$|p_1|^2\, w_\varepsilon(\lambda) \sim \tilde{d}_1 \frac{1}{\lambda^3}\, e^{8\pi\varepsilon\lambda^2/3}, \qquad |p_1|^2\, w_\varepsilon(\lambda) \sim \tilde{d}_2 \frac{1}{\lambda^3}\, e^{-8\pi\varepsilon\lambda^2/3}. \tag{5.2.38}$$

Thus, for all $\varepsilon > 0$, $p_2(\lambda)$ is the only L^2 finite solution of (5.2.35), and so the endpoint $\lambda = \infty$ is limit-point. Consequently, $p_2(\lambda)$ is the subdominant solution at $\lambda = \infty$.

Clearly, the asymptotic solutions (5.2.36) are not oscillating for all $\varepsilon > 0$. Thus, adding the potential $V_*(\lambda)$ to $\hat{H}$ leads to non-oscillatory endpoints $\lambda = 0$ and $\lambda = \infty$ for all $E \geq 0$. Hence, by the classification categories of Sturm-Liouville problems given in [21], the spectrum of $\hat{H}_*$ is simple, bounded below, infinite and purely discrete.

Numerically, we have solved (5.2.34) with the boundary conditions given by the subdominant solutions at $\lambda = 0$ and $\lambda = \infty$, that is,

$$
\begin{aligned}
f(\lambda) &\sim a_1 \lambda^l, && \text{as } \lambda \to 0, \\
f(\lambda) &\sim d_2 \frac{1}{\lambda^2} K_1\big(4\pi\varepsilon\lambda^2/3\big) && \text{as } \lambda \to \infty.
\end{aligned} \tag{5.2.39}
$$

The lowest eigenvalues of the modified quantum Hamiltonian $\hat{H}_*$ with various values of ε and l are given in Table 5.2. Clearly, they are increasing as ε and l increase.

ε	$l = 0$	$l = 1$	$l = 2$	$l = 3$
0.1	3.45671	6.90506	11.04127	15.70530
0.2	4.50109	8.01558	12.02664	16.39705
0.3	5.24830	8.78819	12.72389	16.94067
0.4	5.85455	9.41107	13.29924	17.42190
0.5	6.37958	9.95039	13.80742	17.86720
0.6	6.85284	10.43710	14.27334	18.28884
0.7	7.29095	10.88825	14.71042	18.69356
0.8	7.70406	11.31403	15.12679	19.08566
0.9	8.09880	11.72103	15.52764	19.46794
1.0	8.47966	12.11369	15.91643	19.84236

Table 5.2: Numerical values of the lowest eigenvalues of the modified quantum Hamiltonian $\hat{H}_* = \hat{H} + \varepsilon\, V_*(\lambda)$ for various values of l and ε.

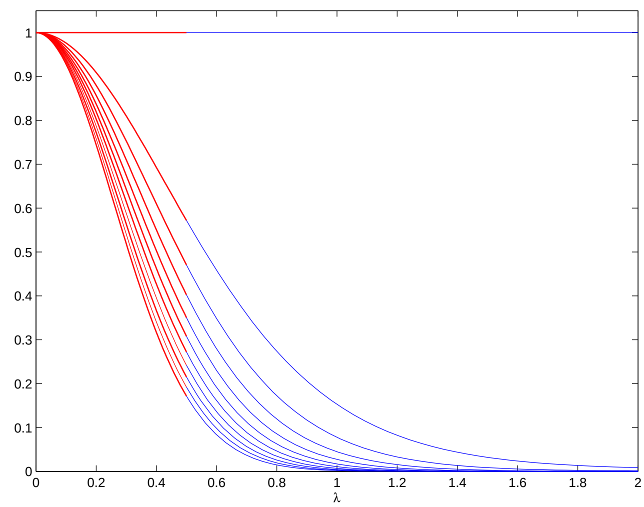


Figure 5.3: Plots of the first eigenfunctions of $\hat{H}_* = \hat{H} + \varepsilon\, V_*(\lambda)$ for $l = 0$ and various value of $\varepsilon \in [0, 1]$ varying from $\varepsilon = 0$ (top horizontal line) to $\varepsilon = 1$ (bottom curve).

We conclude that for $\varepsilon > 0$, the features of the energy spectrum of the quantum Hamiltonian $\hat{H}$ are changed by adding the potential term given by $V_*(\lambda)$ in which the point spectrum is unbounded above and the continuous spectrum is empty.

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