Moments of the Dedekind zeta function

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Abstract

We study analytic aspects of the Dedekind zeta function of a Galois extension. Specifically, we are interested in its mean values. In the first part of this thesis we give a formula for the second moment of the Dedekind zeta function of a quadratic field times an arbitrary Dirichlet polynomial of length $T^{1/11-\epsilon}$. In the second part, we derive a hybrid Euler-Hadamard product for the Dedekind zeta function of an arbitrary number field. We rigorously calculate the $2k$th moment of the Euler product part as well as conjecture the $2k$th moment of the Hadamard product using random matrix theory. In both instances we are restricted to Galois extensions. We then conjecture that the $2k$th moment of the Dedekind zeta function of a Galois extension is given by the product of the two. By using our results from the first part of this thesis we are able to prove both conjectures in the case $k = 1$ for quadratic fields. We also re-derive our conjecture for the $2k$th moment of quadratic Dedekind zeta functions by using a modification of the moments recipe. Finally, we apply our methods to general non-primitive L-functions and gain a conjecture regarding their moments. Our main idea is that, to leading order, the moment of a product of distinct L-functions should be the product of the individual moments of the constituent L-functions.

Contents

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Author's Declaration

I hereby declare that the work included in this thesis is original, except where indicated by special reference, and has solely been submitted for the degree of Doctor of Philosophy in Mathematics at the University of York only. All views expressed are those of the author.

Signed ....

Date ....

CHAPTER 1

Introduction

1.1. L-functions and the Selberg Class

L-functions play a central role in analytic number theory. They can be associated to many different objects, and in each case encode much of the object's key information. The prototypical example is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1}$$

From an algebraic point of view, the object to which the zeta function is associated is the field of rationals $\mathbb{Q}$. The unique factorisation of the integers is realised in the equality between the series and product representations, and so one can clearly see that the zeta function embodies something fundamental. Other examples of its arithmetic nature are given by the following identities involving well known arithmetic functions:

$$\zeta(s)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \tag{2}$$

where $d(n)$ is the number of divisors of n;

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}, \tag{3}$$

where $\phi(n)$ is Euler's totient function, and

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \tag{4}$$

where $\mu(n)$ is the Möbius function. Our previous comment along with the above examples demonstrate, at least superficially, that the zeta function is of some arithmetic significance. The key point is that $\zeta(s)$ can be analytically continued; the new regions of the continuation allow for a different perspective, no longer superficial.

The analytic continuation of $\zeta(s)$ was first demonstrated by Riemann [48]. He showed $\zeta(s)$ continued to a function analytic on all of $\mathbb{C}$ except for a simple pole at $s = 1$ with residue 1 and that satisfied the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{5}$$

where $\Gamma(s)$ is the gamma function. The Euler product representation in (1) implies that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$. By the functional equation and the fact that $\Gamma(s)$ has poles at the negative integers, we see that $\zeta(s)$ must have zeros at the negative even integers – the so-called trivial zeros. If any other zeros are to occur, they must be in the *critical strip* $0 \le \sigma \le 1$. Riemann conjectured that all zeros in the critical strip lie on line of symmetry of the functional equation: $\sigma = 1/2$. This is the Riemann hypothesis. He also stated that the number of zeros in the critical strip was given by

$$N(T) = \{t \in [0,T] : \zeta(\sigma + it) = 0,\ 0 \le \sigma \le 1\} = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T). \tag{6}$$

A published proof was later given by von-Mangoldt and consequently the formula is referred to as the Riemann-von-Mangoldt formula.

One can use these facts on the analytic character of $\zeta(s)$ to demonstrate a connection with the prime number theorem. Let us outline an argument showing

$$\psi(x) = \sum_{n \le x} \Lambda(n) \sim x \tag{7}$$

where $\Lambda(n)$ is the von-Mangoldt function. This is easily seen [1] to be equivalent to the prime number theorem in the form $\sum_{p \le x} 1 \sim x/\log x$. The function $\Lambda(n)$ essentially gives a weight of $\log p$ which allows cleaner expressions. We first note for $\sigma > 1$

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds}\log\left[\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\right] = -\sum_p\frac{\log p}{p^s-1} \\ &= -\sum_p \log p\sum_{m=1}^{\infty} p^{-ms} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.\end{aligned} \tag{8}$$

Given that

$$\frac{1}{2\pi i}\int_{(c)} y^s\frac{ds}{s} = \begin{cases} 1 & \text{if } y > 1 \\ 1/2 & \text{if } y = 1 \\ 0 & \text{if } y < 1 \end{cases} \tag{9}$$

where the integral is over the vertical line $\Re(s) = c > 0$, we see that

$$\psi_0(x) = \frac{1}{2\pi i}\int_{(c)} -\frac{\zeta'(s)}{\zeta(s)}x^s\frac{ds}{s} \tag{10}$$

where $\psi_0(x)$ is the 'dashed' sum version of $\psi(x)$. This means the last term of the sum is multiplied by $1/2$ if x is an integer. Here, we have chosen $c > 1$ so that the Dirichlet series converges absolutely. We can view this integral as the limit of some finite integral, which we then view as part of a rectangular contour whose left edge is left of the line $\Re(s) = 1$. By Cauchy's residue formula, the value of the integral over the rectangle is approximately $x - \sum_\rho x^\rho/\rho$ where the sum is over the non-trivial zeros ρ inside the rectangle. After some estimates we can take limits and we're left with

$$\psi_0(x) = x + O\left(\sum_\rho \frac{x^\rho}{\rho}\right). \tag{11}$$

The fact that the error term is indeed an error term is due to the non-trivial fact that $\zeta(s)$ has no zeros on the line $\sigma = 1$, and hence $x^\rho \ll x$. The full details of this kind of argument can be found in Proposition 3.1.3. One can note that the Riemann Hypothesis implies the primes are distributed as regularly as possible, in the sense that error term in the prime number theorem is minimal. For something as equally deep, we also have the following lesser known example which better exemplifies the association with the rationals. Let $x_1, \ldots, x_N \in (0, 1]$ be the set of rationals of height at most Q. Then for each integer h we have

$$\sum_{n=1}^{N} e(hx_n) = \sum_{d|h} dM(\tfrac{Q}{d}) \tag{12}$$

where $M(x) = \sum_{n \leq x} \mu(n)$ is the sum function of the Möbius function (see for example [24]). Now, $M(x) \ll x^{\theta+\epsilon}$ if and only if the Dirichlet series for $1/\zeta(s)$ converges for $\sigma > \theta$. The Riemann hypothesis implies that we may take $\theta = 1/2$, and as a consequence one gets

$$\sum_{n=1}^{N} e(hx_n) \ll (Q|h|)^{1/2+\epsilon} \tag{13}$$

when (x_n) are the complete set of rationals of height at most Q. We can therefore we view the Riemann hypothesis as the implication that the rationals are distributed as uniformly as possible. We are now satisfied that the zeta function is an object worthy of study, and so we move on to its generalisations.

To define a general L-function, we take the axiomatic approach via the Selberg Class. This was originally defined by Selberg [53]. Along with his article our main references are [28, 56]. Although most of this thesis is concerned with the Dedekind zeta function, the approach via the Selberg class will help us describe our problems in a fuller generality whilst also setting notation and defining terms. Let

$$L(s) = \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s} \tag{14}$$

for some coefficients $a_L(n)$. The axioms are as follows:

- Ramanujan Hypothesis: $a_L(n) \ll n^\epsilon$ for any $\epsilon > 0$.
- Analytic continuation: There exists a non-negative integer k such that $(s-1)^k L(s)$ is entire and of finite order.
- Functional equation: There exists a positive integer d and for $1 \leq j \leq d$ there exist Q, $\lambda_j \in \mathbb{R}_{>0}$, and ϵ_L, $\mu_j \in \mathbb{C}$ with $|\epsilon_L| = 1$ and $\Re(\mu_j) \geq 0$ such

that

$$\Lambda_L(s) := \gamma_L(s)L(s) = \epsilon_L\overline{\Lambda}_L(1-s) \tag{15}$$

where

$$\gamma_L(s) = Q^{s/2}\prod_{j=1}^{d}\Gamma(\lambda_j s + \mu_j)$$

and $\overline{\Lambda}_L(s) = \overline{\Lambda_L(\overline{s})}$.

- Euler product: $L(s)$ satisfies

$$L(s) = \prod_p L_p(s)$$

where the product is over primes and

$$L_p(s) = \exp\left(\sum_{m=1}^{\infty}\frac{b(p^m)}{p^{ms}}\right)$$

for some coefficients satisfying $b(p^m) \ll p^{m\theta}$ for some $\theta < 1/2$.

We define the *degree* of an L-function as the quantity

$$d_L := 2\sum_{j=1}^{d}\lambda_j. \tag{16}$$

It is conjectured that $\lambda_j = 1/2$, $j = 1, \ldots, d$ for all L-functions in the Selberg class, which amounts to saying that the degree is just the number of gamma factors in the functional equation. It often occurs that the coefficients $b(p^m)$ allow for an expression of the form $L_p(s) = f(p^{-s})^{-1}$ where f is some polynomial. Consequently, some authors choose to define the degree of an L-function as the degree of f. Also, authors following [**43**] may refer to the *dimension* of an L-function as what we have defined as the degree.

Let us complement these axioms with some examples. First we have the Riemann zeta function. This is a degree 1 L-function with $\gamma_\zeta(s) = \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\epsilon_\zeta = 1$. We also have $b(n) \equiv 1$ so that

$$\zeta_p(s) = \exp\left(\sum_{m=1}^{\infty}\frac{1}{p^{ks}}\right) = \exp\left(-\log\left(1-p^{-s}\right)\right) = \left(1-\frac{1}{p^s}\right)^{-1} \tag{17}$$

as seen previously. Another example is given by Dirichlet L-functions which are also of degree 1. Let χ be a primitive Dirichlet character mod q and let

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}. \tag{18}$$

The functional equation for Dirichlet L-functions reads

$$\Lambda(s,\chi):=\left(\frac{\pi}{q}\right)^{-\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi)=\frac{G(\chi)}{i^{\mathfrak{a}}\sqrt{q}}\Lambda(1-s,\overline{\chi}) \tag{19}$$

where $G(\chi)=\sum_{k=1}^{q}\chi(k)e_q(k)$ is the Gauss sum and $\mathfrak{a}$ is defined by $\chi(-1)=(-1)^{\mathfrak{a}}$. Since $|G(\chi)|=\sqrt{q}$ (see [1] for example) we have $|\epsilon_L|=1$. We will see later that Dedekind zeta functions and Artin L-functions provide examples of L-functions with degree greater than 1.

For a given L-function in the Selberg class one can deduce equivalents of the famous results concerning the Riemann zeta function. These are derived in a similar fashion, the main tool being contour integration along with basic upper bounds on the L-function. In deriving the upper bounds, it will frequently occur that we require an estimate for some ratio of the γ_L factors. For this, we require an asymptotic expansion of the Gamma function.

Lemma 1.1.1 (Stirling's formula)**.** [58] *For s in the range $|\arg s|\le\pi-\epsilon$ we have the asymptotic formula*

$$\log\Gamma(s)=\left(s-\frac{1}{2}\right)\log s-s+\frac{1}{2}\log 2\pi+O\left(\frac{1}{s}\right), \tag{20}$$

or equivalently,

$$\Gamma(s)=\left(\frac{2\pi}{s}\right)^{\frac{1}{2}}\left(\frac{s}{e}\right)^{s}\left(1+O\left(\frac{1}{s}\right)\right). \tag{21}$$

Let $s=\sigma+it$ with σ fixed. Writing $i=e^{\pi i/2}$ and expanding the above with a consideration for $|t|$ large, we get

$$\Gamma(\sigma+it)=\sqrt{2\pi}\,t^{\sigma-1/2}e^{-\frac{\pi}{2}|t|}\left(\frac{|t|}{e}\right)^{it}e^{\frac{\pi}{2}i(\sigma-1/2)}\left(1+O\left(\frac{1}{t}\right)\right). \tag{22}$$

The details of this kind of argument can be found in Lemma 1.1.3 below. Taking moduli gives the following.

Lemma 1.1.2 (Rapid decay in vertical strips)**.** *For σ fixed and $|t| \to \infty$ we have*

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\frac{\pi}{2}|t|}\left(1 + O\left(\frac{1}{t}\right)\right). \tag{23}$$

Given that $L(s)$ is absolutely convergent for $\sigma > 1$ it is clearly bounded in this region. Therefore, by the functional equation in its *asymmetric form*:

$$L(s) = \epsilon_L \varkappa_L(s) L(1-s), \quad \varkappa_L(s) = \frac{\overline{\gamma}_L(1-s)}{\gamma_L(s)}, \tag{24}$$

we can deduce an estimate for $\sigma < 0$ provided we have an upper bound on $\varkappa_L(s)$.

Lemma 1.1.3. *Let L be a member of the Selberg class. Then for $t \geq 1$ we have uniformly in σ*

$$\varkappa_L(s) = (\lambda Q t^{d_L})^{\frac{1}{2}-\sigma-it}\lambda' t^{-2i\sum \Im(\mu_j)} e^{itd_L + \frac{\pi i}{4}(\mu - d_L)}\left(1 + O\left(\frac{1}{t}\right)\right) \tag{25}$$

where $\lambda = \prod_{j=1}^d \lambda_j^{2\lambda_j}$, $\lambda' = \prod_{j=1}^d \lambda_j^{-2i\Im(\mu_j)}$ and $\mu = 2\sum_{j=1}^d (1 - 2\Re(\mu_j))$. In particular, $|\varkappa_L(\frac{1}{2} + it)| = 1$. Also, for $z \in \mathbb{C}$ we have

$$\frac{\gamma_L(s+z)}{\gamma_L(s)} = \lambda^{1/2}\big(Q(it)^{d_L}\big)^{z/2}\left(1 + O_z\left(\frac{1}{t}\right)\right) \tag{26}$$

as $t \to \infty$. The dependence on z in the error term is at most polynomial.

Proof. Since $\overline{\Gamma(s)} = \Gamma(\overline{s})$ we have

$$\varkappa_L(s) = \frac{\overline{\gamma}_L(1-s)}{\gamma_L(s)} = Q^{\frac{1}{2}-s}\prod_{j=1}^d \frac{\Gamma\big(\lambda_j(1-\sigma) + \Re(\mu_j) - i(\lambda_j t - \Im(\overline{\mu_j}))\big)}{\Gamma\big(\lambda_j\sigma + \Re(\mu_j) + i(\lambda_j t + \Im(\mu_j))\big)}. \tag{27}$$

Let $u = \lambda_j(1-\sigma) + \Re(\mu_j)$, $u' = \lambda_j\sigma + \Re(\mu_j)$ and $v = \lambda_j t + \Im(\mu_j)$ so that a single term of the product is given by $\Gamma(u - iv)/\Gamma(u' + iv)$. Since we are considering t to be large we are also considering v to be large. With this in mind, Stirling's formula gives

$$\begin{aligned}\log\frac{\Gamma(u-iv)}{\Gamma(u'+iv)} =&(u-iv-\tfrac{1}{2})\log(u-iv)-(u-iv)\\&-(u'+iv-\tfrac{1}{2})\log(u'+iv)+u'+iv+O\left(\frac{1}{v}\right)\\=&(u-iv-\tfrac{1}{2})\big(\log(-iv)+\log(1-\tfrac{u}{iv})\big)-(u'+iv-\tfrac{1}{2})\\&\times\big(\log(iv)+\log(1+\tfrac{u'}{iv})\big)-u+u'+2iv+O\left(\frac{1}{v}\right)\\=&(u-iv-\tfrac{1}{2})\log(-iv)-(u'+iv-\tfrac{1}{2})\log(iv)+2iv+O\left(\frac{1}{v}\right).\end{aligned}$$

Now,

$$\begin{aligned}(u-iv-\tfrac{1}{2})\log(-iv)=&(u-iv-\tfrac{1}{2})\big(\log(-i\lambda_j t)+\log(1+\tfrac{\Im(\mu_j)}{\lambda_j t})\big)\\=&(u-iv-\tfrac{1}{2})\log(-i\lambda_j t)-i\Im(\mu_j)+O\left(\frac{1}{t}\right).\end{aligned}\tag{28}$$

The expression now becomes

$$\begin{aligned}&\log\left((-i\lambda_j t)^{u-iv-\frac{1}{2}}(i\lambda_j t)^{-(u'+iv-\frac{1}{2})}e^{2iv-2i\Im(\mu_j)}\right)+O\left(\frac{1}{t}\right)\\=&\log\left((\lambda_j t)^{u-u'-2iv}e^{2iv-2i\Im(\mu_j)-\frac{\pi i}{2}(u+u'-1)}\right)+O\left(\frac{1}{t}\right)\end{aligned}\tag{29}$$

and the first result now follows on exponentiating, inputting the values of u, u', v and then taking the product over j. For the second result we proceed similarly. Let $u = \lambda_j s + \mu_j$ and $v = \lambda_j z$ and take consideration for u large. Via the same procedure as before we find

$$\begin{aligned}\log\frac{\Gamma(u+v)}{\Gamma(u)} =&v\log u+(u+v-\tfrac{1}{2})\log(1+\tfrac{v}{u})-v+O\left(\frac{1}{u+v}\right)+O\left(\frac{1}{u}\right)\\=&v\log u+O_v\left(\frac{1}{u}\right)=\log\left((\lambda_j s+\mu_j)^{\lambda_j z}\right)+O_z\left(\frac{1}{s}\right)\\=&\log\left((\lambda_j it)^{\lambda_j z}\right)+O_z\left(\frac{1}{t}\right).\end{aligned}\tag{30}$$

The result now follows upon exponentiating and taking the product over j. □

By formula (25) and the functional equation in the form (24) we see that $L(s)$ is of polynomial growth in the half-plane $\sigma < 0$. It remains to find the order of growth inside the critical strip. Now, if we were dealing with analytic functions bounded on the boundary of some closed region, we could apply the maximum-modulus principle to show the function is bounded on the interior of that region. The so-called Phragmén-Lindelöf principle can be considered an extension of the maximum modulus principle to vertical strips, and so is useful for Dirichlet series.

Lemma 1.1.4 (Phragmén-Lindelöf for strips)**.** [**28**] *Suppose $f(s)$ is analytic and of finite order in the strip $\sigma_1 \leq \sigma \leq \sigma_2$. Assume that*

$$|f(\sigma_1 + it)| \leq M_{\sigma_1}(1 + |t|)^{\alpha},$$

$$|f(\sigma_2 + it)| \leq M_{\sigma_2}(1 + |t|)^{\beta}$$

for $t \in \mathbb{R}$. Then

$$|f(\sigma + it)| \leq M_{\sigma_1}^{l(\sigma)} M_{\sigma_2}^{1-l(\sigma)} (1 + |t|)^{\alpha l(\sigma) + \beta(1 - l(\sigma))} \tag{31}$$

for all s in the strip, where l is the linear function such that $l(\sigma_1) = 1$, $l(\sigma_2) = 0$.

Proposition 1.1.5. *Let $L(s)$ be an element of the Selberg class. Then for $|t| \geq 1$, $L(s)$ is at most of polynomial growth in vertical strips.*

The Phragmén-Lindelöf principle actually allows for a more precise statement on the growth of $L(s)$ in vertical strips. Note that for a polynomial $f(t)$ of degree d, the quantity $\log f(t)/\log t \to d$ as $t \to \infty$. Accordingly, for a given L-function we define

$$\mu_L(\sigma) = \limsup_{t \to \pm\infty} \frac{\log |L(\sigma + it)|}{\log |t|} = \liminf_{t \to \pm\infty} \{a : L(\sigma + it) \ll |t|^a\}. \tag{32}$$

By the absolute convergence of the Dirichlet series we have $\mu_L(\sigma) = 0$ for $\sigma > 1$, and by (25) we have $\mu_L(\sigma) = d_L(1/2 - \sigma)$ for $\sigma < 0$. The Phragmén-Lindelöf principle therefore gives $\mu_L(\sigma) = d_L(1 - \sigma)/2$ for $-\epsilon \leq \sigma \leq 1 + \epsilon$. By convexity we have continuity and therefore this last region can be extended to $0 \leq \sigma \leq 1$. In particular, on the half-line

$$L(\tfrac{1}{2} + it) \ll |t|^{\frac{d_L}{4}+\epsilon}. \tag{33}$$

Any improvement to this bound is known as a subconvexity bound, the best possible is the Lindelöf hypothesis which states $L(1/2 + it) \ll |t|^{\epsilon}$.

For every L-function in the Selberg class there exists an equivalent version of the Riemann-von-Mangoldt formula (6). This is derived in the same way as for the Riemann zeta function by considering contour integrals of the logarithmic derivative. The proof can be found in most books on analytic number theory, we reference [**28**] for the general case.

Proposition 1.1.6. *Let $N_L(T)$ denote the number of zeros of $L(s)$ in the region $0 \leq \sigma \leq 1$, $-T \leq t \leq T$. Then*

$$N_L(T) = \frac{T}{\pi} \log\left(\lambda Q \left(\frac{T}{e}\right)^{d_L}\right) + O(\log T) \tag{34}$$

where $d_L = 2\sum_{j=1}^{d} \lambda_j$ and $\lambda = \prod_{j=1}^{d} \lambda_j^{2\lambda_j}$.

There are several conjectures surrounding the Selberg class besides the aforementioned degree conjecture. For instance, it is expected that the Riemann hypothesis holds for all members i.e. all non-trivial zeros of $L(s)$ lie on the line $\sigma = 1/2$. There are also some conjectures due to Selberg [**53**] which concern the value distribution of these functions. To state these we must define a notion of irreducibility. We say an L-function is *primitive* if it cannot be written as the product of two non-trivial (i.e. $\equiv 1$) L-functions. The first of Selberg's conjectures states that for two primitive L-functions L_1, L_2 we have

$$\sum_{p \leq x} \frac{a_{L_1}(p)\overline{a_{L_2}(p)}}{p} = \begin{cases} \log\log x + O(1) & \text{if } L_1 = L_2 \\ O(1) & \text{otherwise.} \end{cases} \tag{35}$$

The second states that for a given L-function, not necessarily primitive, there exists an integer n_L such that

$$\sum_{p \leq x} \frac{|a_L(p)|^2}{p} = n_L \log\log x + O(1). \tag{36}$$

These conjectures are quite profound and have many consequences (see[14] for example). In the final chapter we show that these conjectures have implications on the moments of non-primitive L-functions. We now move on to our principal L-function of study: the Dedekind zeta function.

1.2. Algebraic number theory

We first review the necessary algebraic number theory. For the early introductory material we refer to Stewart and Tall's book [57], our main source is Neukirch's book [44], and we also reference [6, 19, 33]. We only include information pertinent to the definition and ensuing properties of the Dedekind zeta function, extraneous information is omitted.

A *number field* is a finite extension $\mathbb{K}$ of the rationals. The index $n = [\mathbb{K} : \mathbb{Q}]$ is the *degree* of the extension, or as we sometimes refer to it, the degree of the number field. By the Primitive Element Theorem there exists an algebraic number α such that $\mathbb{K} = \mathbb{Q}(\alpha)$. Since $\mathbb{Q}$ is of characteristic zero, $\mathbb{K}$ is a separable extension and therefore the minimal polynomial of α has n distinct roots in $\mathbb{C}$. Consequently, there exist n distinct embeddings

$$\tau_i : \mathbb{K} \to \mathbb{C}, \quad i = 1, \ldots, n \tag{37}$$

which fix $\mathbb{Q}$. Since the n embeddings correspond to the conjugates of α we see that for each $\tau : \mathbb{K} \to \mathbb{C}$ there corresponds a complex conjugate $\overline{\tau} : \mathbb{K} \to \mathbb{C}$ (since complex roots of the minimal polynomial occur in complex conjugate pairs) although we may have $\tau = \overline{\tau}$ (in the case of the associated root being real). We denote the number of real embeddings by r_1 and the number of complex conjugate pairs by r_2 so that $n = r_1 + 2r_2$. For an element $a \in \mathbb{K}$ we define its *norm* as

$$N_{\mathbb{K}/\mathbb{Q}}(a) = N(a) = \prod_{i=1}^{n} \tau_i(a)$$

and note that for $a \in \mathbb{Q}$ we have $N(a) = a^n$.

If $\mathbb{K}$ is viewed in analogy with the rationals then the object providing the analogy with the integers is known as the *ring of integers* of $\mathbb{K}$, denoted $\mathcal{O}_{\mathbb{K}}$. This is comprised of all the algebraic integers that lie in $\mathbb{K}$, that is, the set of all elements in $\mathbb{K}$ that are the root of some monic polynomial with coefficients in $\mathbb{Z}$. This does indeed form a ring, more precisely, it is a free $\mathbb{Z}$-module of rank n. Also, the field of fractions of $\mathcal{O}_{\mathbb{K}}$ is $\mathbb{K}$ and so the analogy with the rationals and integers is fairly complete. An important quantity associated to $\mathcal{O}_{\mathbb{K}}$ is the *discriminant*. Informally, this measures the size of the ring of integers and also has ramifications on the behaviour of its primes. If we choose a basis $a_1, \ldots, a_n$ for $\mathcal{O}_{\mathbb{K}}$ as a $\mathbb{Z}$-module, then the discriminant is defined as

$$d_{\mathbb{K}} = \left[\det(\tau_i(a_j))\right]^2 \tag{38}$$

where the τ_i are the n distinct embeddings $\mathbb{K} \to \mathbb{C}$.

It may be that $\mathcal{O}_{\mathbb{K}}$ does not have unique factorisation, however this can be recovered in some sense by considering the ideals of $\mathcal{O}_{\mathbb{K}}$ as opposed to the elements. Given an integral domain R with ideals $\mathfrak{a}$, $\mathfrak{b}$ we can form the product

$$\mathfrak{a}\mathfrak{b} := \left\{ \sum_{i=1}^{k} a_i b_i : a_i \in \mathfrak{a}, b_i \in \mathfrak{b}, k \geq 1 \right\}.$$

Similarly to the integers, we can then define the usual notions of divisibility, greatest common divisor, coprime etc. A fundamental property of $\mathcal{O}_{\mathbb{K}}$ is that any ideal has a unique factorisation in terms of prime ideals. So, for a given ideal $\mathfrak{a} \subset \mathcal{O}_{\mathbb{K}}$ we may write

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}(\mathfrak{a})} \tag{39}$$

where the product is over the prime ideals of $\mathcal{O}_{\mathbb{K}}$ and $v_{\mathfrak{p}}(\mathfrak{a})$ is some non-negative integer, non-zero for only finitely many prime ideals.

We would like to define an L-function related to the ideals of $\mathcal{O}_{\mathbb{K}}$ so that we can encapsulate the property of unique factorisation in a similar fashion to that of the Riemann zeta function. For this we first need a map from the ideals to the natural numbers. Given an ideal $\mathfrak{a}$, it turns out that the quotient ring $\mathcal{O}_{\mathbb{K}}/\mathfrak{a}$ is finite. We therefore define the *ideal norm* of an ideal $\mathfrak{a}$ to be $|\mathcal{O}_{\mathbb{K}}/\mathfrak{a}|$ which we denote $\mathfrak{N}(\mathfrak{a})$. We note for a principal ideal with generator β, we have $\mathfrak{N}((\beta)) = N_{\mathbb{K}/\mathbb{Q}}(\beta)$. The ideal norm is completely multiplicative and so given the above product representation (39), we have

$$\mathfrak{N}(\mathfrak{a}) = \prod_{\mathfrak{p}} \mathfrak{N}(\mathfrak{p})^{v_{\mathfrak{p}}(\mathfrak{a})}. \tag{40}$$

At this point it should be stated that all prime ideals of $\mathcal{O}_{\mathbb{K}}$ are in fact maximal. This implies that $\mathcal{O}_{\mathbb{K}}/\mathfrak{p}$ is a finite field and so $\mathfrak{N}(\mathfrak{p}) = p^f$ for some prime p and some natural number f. Let us examine this in more detail.

Note $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal in $\mathbb{Z}$ and hence $\mathfrak{p} \cap \mathbb{Z} = (p) = p\mathbb{Z}$ for some prime p. We therefore have $\mathfrak{p} \supset p\mathcal{O}_{\mathbb{K}}$. We say that $\mathfrak{p}$ *lies above* the associated prime p and we write $\mathfrak{p}|p$. Suppose that

$$p\mathcal{O}_{\mathbb{K}} = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_g^{e_g}$$

for some positive integers e_i, g where the $\mathfrak{p}_i$ are the distinct prime ideals in $\mathcal{O}_{\mathbb{K}}$ lying above p. Taking norms gives

$$p^n = \mathfrak{N}(\mathfrak{p}_1)^{e_1} \dots \mathfrak{N}(\mathfrak{p}_g)^{e_g}.$$

Hence each $\mathfrak{N}(\mathfrak{p}_i) = p^{f_i}$ for some positive integer f_i and $\sum_{i=1}^{g} e_i f_i = n$. In particular note that

$$\sum_{\mathfrak{p}|p} f_{\mathfrak{p}} \leq n. \tag{41}$$

The integer e_i is called the *ramification index* of $\mathfrak{p}_i$. If $e_i > 1$ we say that p *ramifies* in $\mathbb{K}$ and an important point is that a prime is ramified if and only if it divides the discriminant $d_{\mathbb{K}}$. The integer f_i is called the *degree* of $\mathfrak{p}_i$. If $e_i = f_i = 1$ for all $\mathfrak{p}_i|p$ then we say p *splits completely* in $\mathbb{K}$. If $g = 1$ and $e_1 = 1$ then we say p is *inert* in $\mathbb{K}$. We note that for quadratic extensions $\mathbb{K} = \mathbb{Q}(\sqrt{m})$ with m squarefree, the splitting of ideals can be described in terms of the Kronecker symbol:

$$\begin{cases} p \text{ ramifies} & \iff p|d_{\mathbb{K}} \\ p \text{ splits} & \iff \left(\frac{d_{\mathbb{K}}}{p}\right) = 1 \\ p \text{ inert} & \iff \left(\frac{d_{\mathbb{K}}}{p}\right) = -1 \end{cases} \tag{42}$$

1.2.1. The Dedekind zeta function. The Dedekind zeta function of $\mathbb{K}$ is defined as the series

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} \tag{43}$$

where the sum is over all ideals $\mathfrak{a} \subset \mathcal{O}_{\mathbb{K}}$. Upon formally expanding the product

$$\prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} \tag{44}$$

we see that the worst case of divergence is presented by the sums $\sum_{\mathfrak{p}} \mathfrak{N}(\mathfrak{p})^{-s}$. Since there are at most n prime ideals above a given rational prime p and since $|\mathfrak{N}(\mathfrak{p})^s| = p^{f\sigma} \geq p^{\sigma}$, we see that the product converges absolutely and uniformly for $\sigma \geq 1 + \delta$. By the unique factorisation of prime ideals and the usual argument we therefore have

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{\mathfrak{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1}. \tag{45}$$

Since this converges absolutely and locally uniformly for $\sigma > 1$ it defines an analytic function there.

To demonstrate that $\zeta_{\mathbb{K}}(s)$ is a member of the Selberg class we first write $\zeta_{\mathbb{K}}(s)$ as a normal Dirichlet series,

$$\zeta_{\mathbb{K}}(s) = \sum_{m=1}^{\infty} \frac{r_{\mathbb{K}}(m)}{m^s}. \tag{46}$$

The coefficients $r_{\mathbb{K}}(n)$ now represent the number of ideals of norm n and note $r_{\mathbb{K}}(1) = 1$ (corresponding to the ideal $\mathcal{O}_{\mathbb{K}}$). To get a grasp of these coefficients we write the product over prime ideals as a standard Euler product:

$$\prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} = \prod_{p} \prod_{\mathfrak{p}|p} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})^s}\right)^{-1} = \prod_{p} \prod_{i=1}^{g} \left(1 - \frac{1}{p^{f_i s}}\right)^{-1}. \tag{47}$$

Expanding this into the Dirichlet series (46) we see that the coefficient $r_{\mathbb{K}}(m)$ is given as a product of divisor functions and therefore $r_{\mathbb{K}}(m) \ll m^{\epsilon}$. The Ramanujan hypothesis is therefore satisfied and note we have incidentally satisfied the Euler product hypothesis.

Similarly to the Riemann Zeta function, we can continue $\zeta_{\mathbb{K}}(s)$ to a meromorphic function on $\mathbb{C}$ and this function then satisfies a functional equation. If we define the completed Dedekind Zeta function as

$$\Lambda_{\mathbb{K}}(s) = |d_{\mathbb{K}}|^{s/2} \pi^{-ns/2} \Gamma(\tfrac{s}{2})^{r_1+r_2} \Gamma(\tfrac{s+1}{2})^{r_2} \zeta_{\mathbb{K}}(s) \tag{48}$$

then this functional equation reads

$$\Lambda_{\mathbb{K}}(s) = \Lambda_{\mathbb{K}}(1-s). \tag{49}$$

So, in the notation of section 1.1 we have $\epsilon = 1$, $Q = |d_{\mathbb{K}}|\pi^{-n}$ and $\gamma(s) = \prod_{j=1}^{n} \Gamma((s+\mu_j)/2)$ where μ_j is zero for $r_1 + r_2$ of the j's and 1 for the remaining r_2. The only pole of $\zeta_{\mathbb{K}}(s)$ is at $s = 1$. It is simple and has residue given by the analytic class number formula

$$\psi_{\mathbb{K}} = \mathrm{Res}_{s=1}(\zeta_{\mathbb{K}}(s)) = \frac{2^{r_1}(2\pi)^{r_2} h_{\mathbb{K}} R_{\mathbb{K}}}{w\sqrt{|d_{\mathbb{K}}|}} \tag{50}$$

where $R_{\mathbb{K}}$ is the regulator, $h_{\mathbb{K}}$ is the class number and w is the number of roots of unity in $\mathbb{K}$ (see [44]). We therefore see that the Dedekind zeta function is a member of the Selberg class of degree n.

In Chapter 3 we will frequently make use of the following two results. The first is the equivalent of the prime number theorem for ideals. This is known as the prime ideal theorem and was first derived by Landau [31]. It states

$$\pi_{\mathbb{K}}(x) := |\{\mathfrak{p} \subset \mathcal{O}_{\mathbb{K}} : \mathfrak{N}(\mathfrak{p}) \leq x\}| = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{51}$$

The second result is an equivalent of Merten's theorem for number fields [46]. This is given by

$$\prod_{\mathfrak{N}(\mathfrak{p}) \leq x} \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{-1} = \psi_{\mathbb{K}} e^{\gamma} \log x \left(1 + O\left(\frac{1}{\log x}\right)\right) \tag{52}$$

where γ is the Euler-Mascheroni constant.

Another important tool is the Hadamard product for $\zeta_{\mathbb{K}}(s)$. First note that the function

$$\Lambda_{\mathbb{K}}^*(s) := \left(\tfrac{s}{2}\right)^{r_1+r_2} \left(\tfrac{s+1}{2}\right)^{r_2} (s-1)\Lambda_{\mathbb{K}}(s) \tag{53}$$

is entire and of order 1. Therefore, by the theory of entire functions ([58], chapter 8) we may write

$$\Lambda_{\mathbb{K}}^*(s) = e^{a+bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{54}$$

for some constants a, b. Here, the product is over the zeros of $\Lambda_{\mathbb{K}}^*(s)$. Accordingly,

$$\zeta_{\mathbb{K}}(s) = \frac{e^{A+Bs}}{(s-1)\Gamma(\frac{s+2}{2})^{r_1+r_2}\Gamma(\frac{s+3}{2})^{r_2}} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \tag{55}$$

for some constants A, B where the product is now over the non-trivial zeros of $\zeta_{\mathbb{K}}(s)$.

An interesting question arises when considering the holomorphy of $\zeta_{\mathbb{K}}(s)/\zeta(s)$. From the Hadamard product we can clearly see that the trivial zeros of $\zeta_{\mathbb{K}}(s)$ have multiplicity no less than those of $\zeta(s)$. We also see that the poles at $s = 1$ cancel.

The question then reduces to whether $\zeta(s)$ has any non-trivial zeros that $\zeta_{\mathbb{K}}(s)$ does not (accounting for multiplicities too). The answer to this lies in Artin L-functions. These L-functions provide a factorisation of $\zeta_{\mathbb{K}}$ where $\zeta(s)$ is always a factor, however, they are not fully understood and so there is not yet a complete answer to our question. Nevertheless, this factorisation will provide a useful tool for us and so we devote the next section to its derivation.

1.2.2. Artin L-functions. Artin L-functions are a generalisation of Dirichlet L-functions to arbitrary complex-valued characters and arbitrary Galois extensions of number fields. The following example demonstrates the possibility of such functions. Let $\mathbb{K} = \mathbb{Q}(\zeta_m)$ where $\zeta_m = e^{2\pi i/m}$. Then for an element $\tau \in G = \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ we have $\tau(\zeta_m) = \zeta_m^{j_\tau}$ for some j_τ. It is hard not to show that $\tau(\zeta_m)$ is also a primitive mth root of unity and so $(j_\tau, m) = 1$. It is also hard not to show that the identification $v : G \to (\mathbb{Z}/m\mathbb{Z})^\times$, $\tau \mapsto j_\tau$ provides an injective homomorphism. Since $|G| = [\mathbb{K} : \mathbb{Q}] = \phi(m) = |(\mathbb{Z}/m\mathbb{Z})^\times|$ (see [60] for example) we have an isomorphism

$$\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \cong (\mathbb{Z}/m\mathbb{Z})^\times.$$

By the above identification, given a prime $(p, m) = 1$, we have a corresponding element in G which we denote φ_p. Therefore, for a primitive Dirichlet character $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ we can attach a character $\chi_{\mathrm{Gal}} : G \to \mathbb{C}^\times$ via

$$\chi_{\mathrm{Gal}}(\varphi_p) = \chi(p).$$

The Dirichlet L-function can now be written in terms of purely Galois theoretic information:

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi_{\mathrm{Gal}}(\varphi_p)}{p^s}\right)^{-1},$$

the product over primes being considered as a product over the prime ideals of the ring of integers of the base field.

In defining Artin L-functions our first goal is to describe a generalisation of φ_p. This can in fact be defined for a general finite Galois extension $\mathbb{L}/\mathbb{K}$ to which the L-function is consequently related. We then let the character be an arbitrary character of the Galois group and we take the product over prime ideals.

So let $\mathbb{L}/\mathbb{K}$ be a finite Galois extension of number fields and let $\mathcal{O}_\mathbb{L}$, $\mathcal{O}_\mathbb{K}$ be their respective rings of integers. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_\mathbb{K}$. Then $\mathfrak{p}\mathcal{O}_\mathbb{L}$ is an ideal of $\mathcal{O}_\mathbb{L}$ and so we have the unique decomposition into prime ideals

$$\mathfrak{p}\mathcal{O}_\mathbb{L} = \prod_{j=1}^{g} \mathfrak{P}_j^{e_j}.$$

Once again; we say that the $\mathfrak{P}_j$ lie above $\mathfrak{p}$, which we denote by $\mathfrak{P}|\mathfrak{p}$, and we refer to the e_j as the ramification indices. Recall that $\mathcal{O}_\mathbb{L}/\mathfrak{P}_j$ and $\mathcal{O}_\mathbb{K}/\mathfrak{p}$ are both finite fields and therefore $\mathcal{O}_\mathbb{L}/\mathfrak{P}_j$ is a finite extension of $\mathcal{O}_\mathbb{K}/\mathfrak{p}$. The degree

$$f_j = [\mathcal{O}_\mathbb{L}/\mathfrak{P}_j : \mathcal{O}_\mathbb{K}/\mathfrak{p}]$$

is called the *inertia degree* and similarly to before, we have the relation $\sum_j e_j f_j = n = [\mathbb{L} : \mathbb{K}]$.

Since $\mathbb{L}/\mathbb{K}$ is Galois, the ideals are subject to the action of the Galois group. Indeed, for an element $a \in \mathcal{O}_\mathbb{L}$ and for $\tau \in G = \mathrm{Gal}(\mathbb{L}/\mathbb{K})$, the conjugate $\tau(a)$ is again in $\mathcal{O}_\mathbb{L}$ and hence G acts on $\mathcal{O}_\mathbb{L}$. For a prime $\mathfrak{P}$ lying above $\mathfrak{p}$, the *conjugate* $\tau(\mathfrak{P})$ also lies above $\mathfrak{p}$ since

$$\tau(\mathfrak{P}) \cap \mathcal{O}_\mathbb{K} = \tau(\mathfrak{P} \cap \mathcal{O}_\mathbb{K}) = \tau(\mathfrak{p}) = \mathfrak{p}.$$

What's more is that G acts transitively on the primes $\mathfrak{P}$ of $\mathcal{O}_\mathbb{K}$ lying above $\mathfrak{p}$ (Proposition 9.1, [**44**]). Consequently, the ramification indices e_j are all equal since $\tau(\mathfrak{P}_j^{e_j}) = (\tau(\mathfrak{P}_j))^{e_j}$. Also, for $\mathfrak{P}_i$, $\mathfrak{P}_j$ lying above $\mathfrak{p}$ there exists a τ_{ij} in G such that $\tau_{ij}(\mathfrak{P}_i) = \mathfrak{P}_j$ which induces an isomorphism

$$\mathcal{O}_\mathbb{L}/\mathfrak{P}_i \to \mathcal{O}_\mathbb{L}/\tau_{ij}(\mathfrak{P}_i), \quad a \mod \mathfrak{P}_i \mapsto \tau_{ij}(a) \mod \tau_{ij}(\mathfrak{P}_i)$$

so that

$$f_i = [\mathcal{O}_\mathbb{L}/\mathfrak{P}_i : \mathcal{O}_\mathbb{K}/\mathfrak{p}] = [\mathcal{O}_\mathbb{L}/\tau_{ij}(\mathfrak{P}_i) : \mathcal{O}_\mathbb{K}/\mathfrak{p}] = f_j.$$

We may therefore write $e_j = e$ and $f_j = f$ and we have the identity

$$efg = n = [\mathbb{L} : \mathbb{K}]. \tag{56}$$

We now define an object which, amongst other things, helps describe the number g of ideals $\mathfrak{P}$ lying above $\mathfrak{p}$. For a given prime ideal $\mathfrak{P}$ of $\mathcal{O}_{\mathbb{L}}$ we define the *decomposition group* of $\mathfrak{P}$ over $\mathbb{K}$ as

$$G_{\mathfrak{P}} = \{\tau \in G : \tau(\mathfrak{P}) = \mathfrak{P}\}. \tag{57}$$

Suppose $\mathfrak{P}$ lies above $\mathfrak{p} \subset \mathcal{O}_{\mathbb{K}}$ and let τ vary over the representatives of $G/G_{\mathfrak{P}}$. Since G acts transitively and τ varies over elements of G, $\tau(\mathfrak{P})$ varies over the prime ideals lying above $\mathfrak{p}$. If $\tau(\mathfrak{P}) = \sigma(\mathfrak{P})$ for $\sigma, \tau \in G$, then $\sigma^{-1}\tau = \mathrm{id}$ in $G/G_{\mathfrak{P}}$ i.e. σ and τ belong to the same equivalence class. Therefore, as τ varies over the representatives of $G/G_{\mathfrak{P}}$, each of the remaining prime ideals lying above $\mathfrak{p}$ is hit exactly once. Hence $g = (G : G_{\mathfrak{P}})$ and by the orbit-stabiliser theorem we also have $|G_{\mathfrak{P}}| = n/g = ef$.

An alternative interpretation of the decomposition group exists in some cases and goes as follows. First note for $\tau \in G$ we have $\tau(\mathcal{O}_{\mathbb{L}}) = \mathcal{O}_{\mathbb{L}}$ and therefore given $\tau \in G_{\mathfrak{P}}$ (for which $\tau(\mathfrak{P}) = \mathfrak{P}$) we have an induced automorphism

$$\tau^* : \mathcal{O}_{\mathbb{L}}/\mathfrak{P} \to \mathcal{O}_{\mathbb{L}}/\mathfrak{P}, \quad a \mod \mathfrak{P} \mapsto \tau(a) \mod \mathfrak{P}.$$

Write $\kappa(\mathfrak{P}) = \mathcal{O}_{\mathbb{L}}/\mathfrak{P}$ and $\kappa(\mathfrak{p}) = \mathcal{O}_{\mathbb{K}}/\mathfrak{p}$ so that $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ is an extension of finite fields. This turns out (Proposition 9.4, [44]) to be a Galois extension and note that τ^* fixes the elements of $\kappa(\mathfrak{p})$ so it can therefore be viewed as an element of $\mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$. Also, the map $\tau \mapsto \tau^*$ provides a surjective homomorphism

$$G_{\mathfrak{P}} \to \mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p})).$$

The kernel of this homomorphism is defined as the *inertia group* of $\mathfrak{P}$ over $\mathbb{K}$ and is denoted $I_{\mathfrak{P}}$. Explicitly, we have

$$I_{\mathfrak{P}} = \{\tau \in G_{\mathfrak{P}} : \tau(a) \equiv a \mod \mathfrak{P} \quad \forall a \in \mathcal{O}_{\mathbb{L}}\}.$$

So we have the isomorphism $G_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ and therefore

$$|I_{\mathfrak{P}}| = |G_{\mathfrak{P}}|/|\mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))| = ef/f = e, \qquad (G_{\mathfrak{P}} : I_{\mathfrak{P}}) = f.$$

We now see that if $e = 1$ i.e. $\mathfrak{p}$ is unramified in $\mathcal{O}_{\mathbb{L}}$, then $G_{\mathfrak{P}} \cong \mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$. Since $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ is an extension of finite fields, its Galois group is cyclic and generated by the Frobenius

$$a \mod \mathfrak{P} \mapsto a^q \mod \mathfrak{P}.$$

where $q = |\mathcal{O}_{\mathbb{K}}/\mathfrak{p}| = \mathfrak{N}(\mathfrak{p})$ (see [32] for example). The corresponding element of $G_{\mathfrak{P}}$ is called the *Frobenius automorphism* of $\mathfrak{P}$ relative to the extension $\mathbb{L}/\mathbb{K}$ and is denoted $\varphi_{\mathfrak{P}}$. So, for every $a \in \mathcal{O}_{\mathbb{L}}$, we have

$$\varphi_{\mathfrak{P}}(a) \equiv a^q \mod \mathfrak{P}. \tag{58}$$

By computing $\varphi_{\mathfrak{P}}\tau^{-1}$ for $\tau \in G$ and then applying τ we see that $\tau\varphi_{\mathfrak{P}}\tau^{-1} = \varphi_{\tau(\mathfrak{P})}$. Since G acts transitively on the primes lying above $\mathfrak{p}$, we see that for two different primes $\mathfrak{P}, \mathfrak{P}'$, the automorphisms $\varphi_{\mathfrak{P}}$ and $\varphi_{\mathfrak{P}'}$ are conjugate. In particular, if G is abelian then $\varphi_{\mathfrak{P}}$ does not depend on the choice of $\mathfrak{P}$ lying above $\mathfrak{p}$ and we may write $\varphi_{\mathfrak{p}}$ instead of $\varphi_{\mathfrak{P}}$. In this case $\varphi_{\mathfrak{p}}$ is sometimes referred to as the *Artin symbol* which we denote $\left(\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right)$.

We are more interested in $\varphi_{\mathfrak{P}}$ as a map from the unramified prime ideals to G, as opposed to its effects on the elements of $\mathcal{O}_{\mathbb{L}}$. Let $\mathbb{L}/\mathbb{K}$ be an abelian extension and let $I_{\mathbb{K}}$ denote the set of ideals in $\mathcal{O}_{\mathbb{K}}$ whose prime decomposition contains only primes that are unramified in $\mathcal{O}_{\mathbb{L}}$. Then we may extend $\left(\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right)$ by multiplicativity to a map

$$\left(\frac{\mathbb{L}/\mathbb{K}}{\cdot}\right) : I_{\mathbb{K}} \to G, \qquad \prod_{j=1}^{m} \mathfrak{p}_j^{n_j} \mapsto \prod_{j=1}^{m} \left(\frac{\mathbb{L}/\mathbb{K}}{\mathfrak{p}}\right)^{n_j}.$$

Let us furnish these definitions with some examples. Let $\mathbb{L} = \mathbb{Q}(\sqrt{m})$ for some squarefree m and let $\mathbb{K} = \mathbb{Q}$. Then $G = \{\pm 1\}$ up to isomorphism and so there are two possibilities for the subgroup $G_{\mathfrak{p}}$; namely G itself and $H = \{1\}$. If $G_{\mathfrak{p}} = G$ then $g = |G|/|G_{\mathfrak{p}}| = 1$ and so the prime p lying under $\mathfrak{p}$ is inert in $\mathcal{O}_{\mathbb{L}}$. In this case $G_\mathfrak{p}$ is generated by -1, and so inert primes are sent to -1 by the Frobenius. If $G_\mathfrak{p} = H$ then $g = 2$ and so p splits. In this case, H is generated by 1 and so split primes map to 1. If p ramifies we set $\varphi_p = 0$. By (42), we see that the Artin symbol equals the Kronecker symbol in this case.

For the extension $\mathbb{Q}(\zeta_m)/\mathbb{Q}$ where $\zeta_m = e^{2\pi i/m}$ we have already noted that $G \cong (\mathbb{Z}/m\mathbb{Z})^\times$ via the identification $\tau \mapsto j_\tau$ where j_τ is the integer such that $\tau(\zeta_m) = \zeta_m^{j_\tau}$. We also have that a prime ramifies in $\mathbb{K}$ if and only if it divides m (Proposition 2.3, [**60**]). So assume $p \nmid m$ and let $\mathfrak{p}$ lie above p. Then by (58) we have

$$\varphi_p(a) \equiv a^p \bmod \mathfrak{p} \quad \forall a \in \mathcal{O}_{\mathbb{Q}(\zeta_m)} = \mathbb{Z}[\zeta_m].$$

Since $\varphi_p(\zeta_m)$ is an mth root of unity and since the mth roots of unity are distinct mod $\mathfrak{p}$ (since $m = \prod_{j=1}^{m-1}(1 - \zeta_m^j)$) we see that

$$\varphi(\zeta_m) = \zeta_m^p,$$

i.e. the Frobenius of p maps p to the equivalence class $[p]$ in $(\mathbb{Z}/m\mathbb{Z})^\times$. This is of course the element φ_p described in the introduction to this section.

We are now ready to define Artin L-functions. For a given Galois extension $\mathbb{L}/\mathbb{K}$ with Galois group G, let

$$\rho : G \to GL(V)$$

be a representation of G on some finite dimensional $\mathbb{C}$-vector space V. For unramified primes $\mathfrak{p}$ of $\mathcal{O}_\mathbb{K}$ with divisor $\mathfrak{P}$ in $\mathcal{O}_\mathbb{L}$ let

(59) $$L_\mathfrak{p}(s, \rho, \mathbb{L}/\mathbb{K}) = \det(I - \rho(\varphi_\mathfrak{P})\mathfrak{N}(\mathfrak{p})^{-s})^{-1}$$

and form the product over such primes, which we denote

(60) $$L_{\mathrm{un}}(s, \rho, \mathbb{L}/\mathbb{K}) = \prod_{\mathfrak{p} \text{ unramified}} L_\mathfrak{p}(s, \rho, \mathbb{L}/\mathbb{K}).$$

Since the choice of $\mathfrak{P}|\mathfrak{p}$ only affects $\varphi_\mathfrak{P}$ up to conjugation, under which the determinant is invariant, the above quantity is well defined. Note also that we have invariance under equivalent representations and that we may also take $\rho(\varphi_{\mathfrak{P}})$ in its diagonalised form. For the ramified primes we still have an isomorphism $G_{\mathfrak{P}}/I_{\mathfrak{P}} \cong \mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ and so we can lift $\varphi_{\mathfrak{P}}$ from $\mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$ except that it must be defined modulo $I_{\mathfrak{P}}$. Consequently, we restrict ρ to the subspace $V^{I_{\mathfrak{P}}}$ of V on which $I_{\mathfrak{P}}$ acts trivially i.e. the subspace

$$V^{I_{\mathfrak{P}}} = \{v \in V : \rho(\tau)(v) = v \ \ \forall \tau \in I_{\mathfrak{P}}\}. \tag{61}$$

We denote the new local factor by

$$L_{\mathfrak{p}}(s, \rho, \mathbb{L}/\mathbb{K}) = \det(I - \rho(\varphi_{\mathfrak{P}})|_{V^{I_{\mathfrak{P}}}} \mathfrak{N}(\mathfrak{p})^{-s})^{-1} \tag{62}$$

and let

$$L(s, \rho, \mathbb{L}/\mathbb{K}) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s, \rho, \mathbb{L}/\mathbb{K}). \tag{63}$$

We will sometimes write $L(s, \rho)$ for $L(s, \rho, \mathbb{L}/\mathbb{K})$ if the context is clear. Now, for each local factor we diagonalise $\rho(\varphi_{\mathfrak{P}})$ so that

$$\det(I - \rho(\varphi_{\mathfrak{P}})|_{V^{I_{\mathfrak{P}}}} \mathfrak{N}(\mathfrak{p})^{-s})^{-1} = \prod_{j=1}^{\dim V} (1 - \lambda_j \mathfrak{N}(\mathfrak{p})^{-s})^{-1} \tag{64}$$

where the λ_j are the eigenvalues of $\rho|_{V^{I_{\mathfrak{P}}}}(\varphi_{\mathfrak{P}})$. Since G is finite, the linear automorphism $\rho(\tau)$ is of finite order for all $\tau \in G$, and in particular, for $\varphi_{\mathfrak{P}}$. Therefore, $|\lambda_j| = 1$ and so $L(s, \rho)$ is bounded by $\zeta_{\mathbb{K}}(s)^{\dim V}$. Hence $L(s, \rho)$ converges absolutely and uniformly for $\sigma \geq 1 + \delta$ and consequently defines an analytic function there. As is also made clear by (64): if V is the trivial representation (i.e. $\rho(\tau) \equiv I$) then $L(s, \rho) = \zeta_{\mathbb{K}}(s)$.

Recall that the *character* of a representation ρ is defined as the trace $\chi = \mathrm{tr}(\rho)$ and note that $\mathrm{degree}(\rho) := \dim V = \chi(1)$. Two representations are equivalent if and only if their characters are equal. Since the local factor $L_{\mathfrak{p}}(s, \rho)$ is invariant under equivalent representations, we see that $L(s, \rho)$ is only dependent on χ. To gain an explicit expression in terms of characters let λ_j be the eigenvalues of $\rho|_{V^{I_\mathfrak{P}}}(\varphi_\mathfrak{P})$ and denote the character of this representation by $\chi_{V^{I_\mathfrak{P}}}$. Then

$$\begin{aligned}\log L(s,\rho) &= -\sum_{\mathfrak{p}}\sum_{j=1}^{\dim V}\log(1-\lambda_j\mathfrak{N}(\mathfrak{p})^{-s}) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}\sum_{j=1}^{\dim V}\frac{\lambda_j^m}{m\mathfrak{N}(\mathfrak{p})^{ms}}\\ &= \sum_{\mathfrak{p},m}\frac{\chi_{V^{I_\mathfrak{P}}}(\varphi_\mathfrak{P}^m)}{m\mathfrak{N}(\mathfrak{p})^{ms}}.\end{aligned} \tag{65}$$

It is known that $L(s,\chi)$ can be continued to a meromorphic function on $\mathbb{C}$ with functional equation of the usual type. This is achieved by expressing it as a quotient of Hecke L-functions via theorems of class field theory. From this expression, we see that it is possible for $L(s,\chi)$ to contain an infinite amount of poles in the region $0<\sigma<1$. Artin conjectured that if χ is non-trivial, then $L(s,\chi)$ is in fact entire. Assuming this, it folows that $L(s,\chi)$ is a member of the Selberg class. The functional equation for $L(s,\chi)$ is given by

$$\Lambda(s,\chi) := \gamma(s,\chi)L(s,\chi) = W(\chi)\Lambda(1-s,\overline{\chi}) \tag{66}$$

where $W(\chi)$ is some complex number of modulus one. The gamma factor is

$$\gamma(s,\chi) = \left(\frac{q(\chi)}{\pi^{n\chi(1)}}\right)^{s/2}\prod_{j=1}^{n\chi(1)}\Gamma\left(\frac{s+\mu_j}{2}\right) \tag{67}$$

where $n=[\mathbb{K}:\mathbb{Q}]$, μ_j is equal to 0 or 1, and $q(\chi)$ is the conductor, for which we will not require an explicit expression.

Our main reason for describing Artin L-functions is the fact that they provide factorisations of the Dedekind zeta function. In order to explain this we briefly review some representation theory, the details can be found in Serre's book [**55**]. Let G be a finite group and let V be a vector space of dimension equal to the order of G with basis $(e_\tau)_{\tau\in G}$ indexed by the elements of G. We let G act on V by permuting the indices of the basis via left multiplication, that is, for σ in G we define a linear map $\rho(\sigma)$ which sends e_τ to $e_{\sigma\tau}$. This is a representation of G

called the *regular representation*. The character r_G of the regular representation is given by

$$r_G = \sum_{\chi} \chi(1)\chi \tag{68}$$

where the sum is over all distinct irreducible characters of G. For a subgroup H with character ψ we can define the *induced character* of G by the formula

$$\mathrm{Ind}_H^G(\psi)(\tau) = \frac{1}{|H|} \sum_{\substack{\sigma\in G \\ \sigma^{-1}\tau\sigma\in H}} \psi(\sigma^{-1}\tau\sigma). \tag{69}$$

In terms of induced characters we have $r_G = \mathrm{Ind}_{\{1\}}^G(\mathbf{1}_{\{1\}})$ where $\mathbf{1}_G$ is the trivial character. Now, given two characters χ_1, χ_2 we have

$$L(s, \chi_1 + \chi_2) = L(s, \chi_1)L(s, \chi_2). \tag{70}$$

To see this take two representations (ρ_1, V_1), (ρ_2, V_2) with characters χ_1, χ_2. Then $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$ is a representation with character $\chi_1 + \chi_2$ and we have

$$\begin{aligned} &\det(I - (\rho_1 \oplus \rho_2)(\varphi_{\mathfrak{P}})|(V_1 \oplus V_2)^{I_{\mathfrak{P}}} z) \\ &\qquad = \det(I - \rho_1(\varphi_{\mathfrak{P}})|V_1^{I_{\mathfrak{P}}} z) \det(I - \rho_2(\varphi_{\mathfrak{P}})|V_2^{I_{\mathfrak{P}}} z). \end{aligned} \tag{71}$$

Slightly more involved (and hence omitted) is the proof of the fact that for a given subgroup H of G we have

$$L(s, \chi, \mathbb{L}/\mathbb{L}^H) = L(s, \mathrm{Ind}_H^G(\chi), \mathbb{L}/\mathbb{K}) \tag{72}$$

where $\mathbb{L}^H$ is the fixed field of H. Writing r_G as an induced character we see

$$L(s, r_G, \mathbb{L}/\mathbb{K}) = L(s, \mathbf{1}_{\{1\}}, \mathbb{L}/\mathbb{L}^{\{1\}}) = L(s, \mathbf{1}, \mathbb{L}/\mathbb{L}) = \zeta_{\mathbb{L}}(s). \tag{73}$$

On the other hand, by formula (68) we have

$$L(s, r_G, \mathbb{L}/\mathbb{K}) = \prod_{\chi} L(s, \chi, \mathbb{L}/\mathbb{K})^{\chi(1)} = \zeta_{\mathbb{K}}(s) \prod_{\chi\neq\mathbf{1}} L(s, \chi, \mathbb{L}/\mathbb{K})^{\chi(1)}. \tag{74}$$

Equating gives the formula

$$\zeta_{\mathbb{L}}(s) = \zeta_{\mathbb{K}}(s) \prod_{\chi \neq \mathbf{1}} L(s, \chi, \mathbb{L}/\mathbb{K})^{\chi(1)}. \tag{75}$$

1.2.3. The Dedekind zeta function (bis). We now return to the Dedekind zeta function with its factorisation in terms of Artin L-functions. We renew the old setup with $\mathbb{Q}$ as the base field but we now assume the finite extension $\mathbb{K}$ to be Galois. The factorisation now reads

$$\zeta_{\mathbb{K}}(s) = \zeta(s) \prod_{\chi \neq \mathbf{1}} L(s, \chi, \mathbb{K}/\mathbb{Q})^{\chi(1)}. \tag{76}$$

Recall that for $\mathbb{K} = \mathbb{Q}(\sqrt{m})$ with m squarefree, $G = \{\pm 1\}$ and the frobenius sends split primes to 1 and inert primes to -1. The only non-trivial representation of G is the one dimensional signature representation sending 1 to 1 and -1 to -1. If $I_{\mathfrak{p}}$ is non-trivial then $V^{I_{\mathfrak{p}}} = V^G$ which is trivial and therefore there is no local factor for ramified primes. We therefore have

$$\begin{aligned} \zeta_{\mathbb{Q}(\sqrt{m})}(s) =& \zeta(s) \prod_{p \text{ split}} \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p \text{ inert}} \left(1 + \frac{1}{p^s}\right)^{-1} \\ =& \zeta(s) \prod_{p} \left(1 - \left(\frac{d_{\mathbb{K}}}{p}\right) p^{-s}\right)^{-1} \\ =& \zeta(s) L(s, \chi) \end{aligned} \tag{77}$$

where $\chi = (d_{\mathbb{K}}|\cdot)$ is the Kronecker symbol (n.b. (42)) and $L(s, \chi)$ is the usual Dirichlet L-function. The modulus q of χ is given by the formula

$$q = \begin{cases} 4|d_{\mathbb{K}}| & \text{if } d_{\mathbb{K}} \equiv 2 (\text{mod } 4), \\ |d_{\mathbb{K}}| & \text{otherwise.} \end{cases} \tag{78}$$

The main aspects of the Dedekind zeta function that will be of interest to us are its mean values on the half line, or as we shall refer to them, its moments. We briefly review the results in this area before describing the moments problem for general L-functions. The first major result concerning the Dedekind zeta function was given by Motohashi [39]. He showed that for a quadratic extension $\mathbb{K}$,

$$\frac{1}{T}\int_T^{2T}|\zeta_{\mathbb{K}}(\tfrac{1}{2}+it)|^2dt \sim \frac{6}{\pi^2}L(1,\chi)^2\prod_{p|d_{\mathbb{K}}}\left(1+\frac{1}{p}\right)^{-1}\log^2 T \tag{79}$$

where $L(s,\chi)$ is the Dirichlet L-function appearing in (77). This was subsequently improved by Müller in [42] where, by employing the methods of Heath-Brown [22], he found the lower order terms. For higher power moments or higher degree extensions little is known. In the papers [2, 3], Bruggeman and Motohashi give an explicit formula for the fourth moment of the Dedekind zeta function of particular quadratic fields, however this does not immediately yield an asymptotic. Their methods rely on a spectral analysis of Kloosterman sums which echos the methods of Motohashi used for the Riemann zeta function [40].

In terms of higher degree extensions, the most that is known is either an upper bound on the moments, or the asymptotic value of the sum $\sum_{m\le x} r_{\mathbb{K}}(m)^2$ where the coefficients are those of the Dedekind zeta function when written as a Dirichlet series. As we shall see later, the value of this sum has a bearing on the moments. For a general Galois extension of degree n, Chandrasekharan and Narasimhan showed [7]

$$\sum_{m\le x} r_{\mathbb{K}}(m)^2 \sim cx\log^{n-1}x \tag{80}$$

for some constant c. The strength of this result is essentially allowed by the simple description of how primes split in Galois extensions. For non-Galois extensions they must replace $\sim$ with $\le$. In the same paper they also give upper bounds on the moments as well as conjecture the asymptotic (79) with an unspecified constant, which Motohashi then deduced.

An asymptotic relation of the form (80) for non-Galois extensions was recently given by Fomenko [17]. For $\mathbb{K}$ a (non-Galois) cubic field, he obtains the formula

$$\sum_{m\le x} r_{\mathbb{K}}(m)^2 = c_1 x\log x + c_2 x + O(x^{9/11+\epsilon}) \tag{81}$$

for some constants c_1, c_2. Here, the method relies on expressing $\zeta_{\mathbb{K}}(s)$ as a product of the Riemann zeta function and the L-function of a modular form of weight 1. This can be deduced from the factorisation (75) by considering a higher degree extension in which the cubic field $\mathbb{K}$ is normal. One then obtains $\zeta_{\mathbb{K}}(s)$ as a product of $\zeta(s)$ and the Artin L-function of a 2-dimensional representation (this particular factorisation is described in Heilbronn's section of [6]). By the work of Weil-Langlands and Deligne-Serre [54] we can write this Artin L-function as the L-function of a modular form of weight 1 and obtain the aforementioned factorisation.

1.3. Moments of L-functions

Let L be a member of the Selberg class. The quantity of interest is

$$\int_0^T |L(\tfrac{1}{2}+it)|^{2k} dt, \tag{82}$$

which we refer to as the $2k$th moment of L. We first present the results of Hardy-Littlewood [21] and Ingham [25] concerning the 2nd and 4th moment of the zeta function. We then describe the general moments problem in a modern setting (smooth functions etc.) which helps clarify the main obstacles to achieving higher moments.

The classical approach to evaluating moments first involves expressing $\zeta(s)$ as a combination of sums that converge within the critical strip. This representation is derived from the functional equation and consequently embodies some its character, for example, it possesses a symmetry about the critical point. We refer to these expressions as approximate functional equations. The first of these takes the form

$$\zeta(s) = \sum_{m \le x} \frac{1}{m^s} + \varkappa(s) \sum_{n \le y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(t^{1/2-\sigma} y^{\sigma-1}) \tag{83}$$

which is valid for $0 \le \sigma \le 1$ (uniformly) and for $x, y, t > 0$ with $2\pi xy = t$. Here,

$$\varkappa(s) = \frac{\gamma(1-s)}{\gamma(s)} = \pi^{1/2-s} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \tag{84}$$

which is the factor appearing in the functional equation $\zeta(s) = \varkappa(s)\zeta(1-s)$. Specialising to the 1/2-line we have

$$\zeta(s) = \sum_{m \le N} \frac{1}{m^{1/2+it}} + \varkappa(\tfrac{1}{2}+it) \sum_{n \le N} \frac{1}{n^{1/2-it}} + O(N^{-1/2}) \tag{85}$$

where $N = N(t) = \sqrt{t/2\pi}$. Writing $\zeta(s) = S + \varkappa \overline{S} + E$ we see

$$\begin{aligned} \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt = \int_0^T \Big[& 2|S|^2 + 2\Re(\varkappa(\tfrac{1}{2}-it)S^2) + (S + \varkappa(\tfrac{1}{2}+it)\overline{S})\overline{E} \\ & + E(\overline{S} + \varkappa(\tfrac{1}{2}-it)S + \overline{E}) \Big] dt \end{aligned} \tag{86}$$

where we have used $\overline{\varkappa(\frac{1}{2}+it)} = \varkappa(\frac{1}{2}-it)$ and $|\varkappa(\frac{1}{2}+it)| = 1$ (the first of these follows from $\overline{\Gamma(s)} = \Gamma(\overline{s})$ whilst the latter follows from (25)). In order to evaluate the first term, which turns out to be the main term, we appeal to the following Theorem of Montgomery and Vaughan.

Proposition 1.3.1. (Montgomery–Vaughan mean value Theorem) [**37**] *We have*

$$\int_0^T \Big| \sum_{n \le M} \frac{a(n)}{n^{it}} \Big|^2 dt = \sum_{n \le M} |a(n)|^2 (T + O(n)). \tag{87}$$

Note that if $M = o(T)$ then the error term is smaller than the main term. Applying the Theorem to our above situation gives

$$2 \int_0^T |S|^2 dt \sim T \log T \tag{88}$$

and it remains to evaluate the remaining terms. For the term involving $\varkappa(\frac{1}{2}-it)S^2$ one can use Stirling's formula to give an asymptotic expansion for $\varkappa(\frac{1}{2}-it)$. The resultant integral can then be evaluated by the theory of exponential integrals (see [**59**]). The term $|E|^2$ is easily evaluated whilst for the cross terms one can use the Cauchy-Schwarz inequality. As a result, we acquire

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt \sim T \log T. \tag{89}$$

For the fourth moment one can apply a similar process starting from the equation

$$\zeta(s)^2 = \sum_{m \le x} \frac{d(m)}{m^s} + \varkappa(s)^2 \sum_{n \le y} \frac{d(n)}{n^{1-s}} + O(x^{1/2-\sigma} \log t) \tag{90}$$

which can be found in section 4.2 of [**26**]. Again, this is valid uniformly for $0 \le \sigma \le 1$ and for $x, y, t > 0$ such that $xy = (t/2\pi)^2$. After applying the Montgomery-Vaughan mean value Theorem and using

$$\sum_{n \le M} \frac{d(n)^2}{n} \sim \frac{1}{4\pi^2} \log^4 M \tag{91}$$

we get

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt \sim \frac{1}{2\pi^2} T \log^4 T. \tag{92}$$

There are several issues that arise when considering higher moments of the zeta function. As one might expect, there exist approximate functional equations for $\zeta(s)^k$ involving sums with the coefficients $d_k(n)$. The first issue is that for some approximate functional equations the error term increases with k making the integral of $|E|^2$ large. Another is that the sums are of length $\le (t/2\pi)^{k/2}$ and so when applying the Montgomery-Vaughan mean value Theorem, the error term becomes too large. For moments of general L-functions these problems often occur in one form or another, indeed, it seems they are systemic to the approximate functional equation approach. Nevertheless, we continue with this approach and give our treatment of general L-functions in all its modern cleanliness.

Let $L(s)$ be an L-function in the Selberg class with at most one pole at $s = 1$. Let $G(s)$ be an even entire function of rapid decay in vertical strips with $G(0) = 1$.

We keep in mind something like $G(z) = e^{z^2}$. Let s lie in the strip $0 \le \sigma \le 1$ and consider the integral

$$I(s, X, \Lambda) = \frac{1}{2\pi i} \int_{(c)} \Lambda_L(s+z) G(z) X^z \frac{dz}{z} \tag{93}$$

where X is some parameter, $\Lambda_L(s)$ is given by (15) and c is some number greater than 1 but not too large. Since $L(s)$ is of at most polynomial growth in vertical strips, this integral exists due to the rapid decay of $G(s)$ and $\Gamma(s)$. Shifting the contour to $\Re(z) = -c$ gives

$$I(s, X, \Lambda) = \Lambda(s) + R(s, X) + \int_{(-c)} \Lambda_L(s+z) G(z) X^z \frac{dz}{z} \tag{94}$$

where

$$R(s, X) = (\mathrm{res}_{z=1-s} + \mathrm{res}_{z=-s}) \Lambda_L(s+z) G(z) \frac{X^z}{z} \tag{95}$$

(recall $\Lambda(s)$ may have a pole at $s = 0$ arising from the Gamma function, hence the residue term at $z = -s$). Note that we have essentially considered the integral as the limit of some finite integral, which we then considered as part of a rectangular contour. Since $L(s)$ is of at most polynomial growth in vertical strips, the horizontal sections vanish in the limit due to the rapid decay of G. We now apply the functional equation in this last integral to give

$$\Lambda(s) = I(s, X, \Lambda) + \epsilon_L I(1-s, X^{-1}, \overline{\Lambda}) - R(s, X). \tag{96}$$

On expanding the absolutely convergent Dirichlet series we get the following.

Proposition 1.3.2 (Approximate functional equation)**.** *Let $L(s)$ be an element of the Selberg class and suppose it has at most one pole, situated at $s = 1$. Then for $0 \le \sigma \le 1$ and $X > 0$ we have*

$$L(s) = \sum_{m=1}^{\infty} \frac{a_L(m)}{m^s} V_s\left(\frac{m}{X}\right) + \varkappa_L(s) \sum_{n=1}^{\infty} \frac{\overline{a_L}(n)}{n^{1-s}} V_{1-s}(nX) - \frac{R(s, X)}{\gamma_L(s)} \tag{97}$$

where $R(s, X)$ is given by (95),

$$V_s(Y) = \frac{1}{2\pi i}\int_{(c)} \frac{\gamma_L(s+z)}{\gamma_L(s)} G(z) Y^{-z}\frac{dz}{z}, \tag{98}$$

with G an even, entire function of rapid decay in vertical strips with $G(0) = 1$, and

$$\varkappa_L(s) = \epsilon_L \frac{\overline{\gamma}_L(1-s)}{\gamma_L(s)}. \tag{99}$$

We can show that both sums in (97) converge in the critical strip, in fact, they are essentially finite. We first estimate $V_s(Y)$ via formula (26). This gives

$$\frac{\gamma_L(s+z)}{\gamma_L(s)} \ll (Q|t|^{d_L})^{\Re(z)/2} \tag{100}$$

and hence

$$\begin{aligned} V_s(Y) =& \frac{1}{2\pi i}\int_{(c)} \frac{\gamma_L(s+z)}{\gamma_L(s)} G(z) Y^{-z}\frac{dz}{z} \\ =& 1 + \frac{1}{2\pi i}\int_{(-c)} \frac{\gamma_L(s+z)}{\gamma_L(s)} G(z) Y^{-z}\frac{dz}{z} \\ =& 1 + O\left(\left(\frac{Y}{\sqrt{Q|t|^{d_L}}}\right)^c\right). \end{aligned} \tag{101}$$

On the other hand, taking $c = A$ with A large we derive the bound $O\big((\sqrt{Q|t|^{d_L}}/Y)^A\big)$. For convenience we write $q(t) = \sqrt{Q|t|^{d_L}}$. Then the above upper bound gives

$$\sum_{m=1}^{\infty} \frac{a_L(m)}{m^s} V_s\,(m) = \sum_{m \le q(t)^{1+\epsilon}} \frac{a_L(m)}{m^s} V_s(m) + O\Bigg(\sum_{m > q(t)^{1+\epsilon}} \frac{a_L(m)}{m^\sigma}\left(\frac{q(t)}{m}\right)^A\Bigg) \tag{102}$$

with A large. Using the Ramanujan hypothesis $a_L(n) \ll n^\epsilon$ and taking A large enough we can bound the tail of this series by $q(t)^{-B}$ with B large. Applying a similar argument to $V_{1-s}(n)$ we gain an expression for $L(s)$ as a combination of two convergent sums of length approximately $q(t)$.

Let us now return to the moment problem and attempt to evaluate

$$\int_0^T |L(\tfrac{1}{2} + it)|^{2k} dt. \tag{103}$$

The cleanest way to do this is to find an approximate functional equation for $|L(1/2+it)|^{2k}$. Clearly, $|L(s)|^{2k}$ is an L-function satisfying

$$|L(s)|^{2k}|\gamma_L(s)|^{2k} = |\gamma_L(1-s)|^{2k}|L(1-s)|^{2k} \tag{104}$$

and has the double Dirichlet series

$$\sum_{m_1,m_2=1}^{\infty} \frac{a_{L,k}(m_1)\overline{a_{L,k}(m_2)}}{m_1^s m_2^{\overline{s}}} \tag{105}$$

for some coefficients $a_{L,k}(n)$. To simplify its approximate functional equation, we set the parameter X equal to 1 and we take $G(z) = e^{z^2}$ so that the R term divided by the gamma factor is $\ll (1+|t|)^{-A}$ with A large. Then, by Proposition 1.3.2 we have

(106)

$$|L(\tfrac{1}{2}+it)|^{2k} = \sum_{m_1,m_2=1}^{\infty} \frac{a_{L,k}(m_1)\overline{a_{L,k}(m_2)}}{m_1^{1/2+it} m_2^{1/2-it}} V_t\left(m_1 m_2\right) + \varkappa_k(\tfrac{1}{2}+it) \sum_{n_1,n_2=1}^{\infty} \frac{a_{L,k}(n_1)\overline{a_{L,k}(n_2)}}{n_1^{1/2+it} n_2^{1/2-it}} V_t\left(n_1 n_2\right) + O((1+|t|)^{-A})$$

where

$$V_t(Y) = \frac{1}{2\pi i}\int_{(c)} \left(\frac{\gamma_L(\frac{1}{2}+it+z)\gamma_L(\frac{1}{2}-it+z)}{\gamma_L(\frac{1}{2}+it)\gamma_L(\frac{1}{2}-it)}\right)^k G(z) Y^{-z}\frac{dz}{z}, \tag{107}$$

and

$$\varkappa_k(\tfrac{1}{2}+it) = \left|\frac{\gamma_L(\frac{1}{2}-it)}{\gamma_L(\frac{1}{2}+it)}\right|^{2k}. \tag{108}$$

Note that by (25), $\varkappa_k(1/2+it) = 1$ and therefore the above expression simplifies to $2\sum + O(t^{-A})$. Integrating the simplified approximate functional equation gives

(109)
$$\begin{aligned}\int_0^T |L(\tfrac{1}{2}+it)|^{2k} dt =& 2\int_0^T \sum_{m_1,m_2=1}^{\infty} \frac{a_{L,k}(m_1)\overline{a_{L,k}(m_2)}}{m_1^{1/2+it} m_2^{1/2-it}} V_t\left(m_1 m_2\right) dt + O((1+|t|)^{-A}) \\ =& 2\int_0^T \Bigg[\sum_{m=1}^{\infty} \frac{|a_{L,k}(m)|^2}{m} V_t\left(m^2\right) \\ &+ \sum_{\substack{m_1,m_2=1\\ m_1\neq m_2}}^{\infty} \frac{a_{L,k}(m_1)\overline{a_{L,k}(m_2)}}{m_1^{1/2+it} m_2^{1/2-it}} V_t\left(m_1 m_2\right)\Bigg] dt + O((1+|t|)^{-A})\end{aligned}$$

As is clear, we've split the sum over terms for which $m_1 = m_2$; the diagonals, and those for which $m_1 \neq m_2$; the off-diagonals. Thus far, we have managed to avoid any large error terms and we have not needed to calculate any integrals involving $\Re(\varkappa_L(1/2+it))$, such as those that appear in (86). The remaining difficulty lies in calculating the off-diagonals. It is here that we run into a problem similar to that which prevents us from using the Montgomery-Vaughan Mean Value Theorem. For any L-function with more than 2 gamma factors it seems incredibly difficult to evaluate the off-diagonals. We shall say more on this later, but for the meanwhile let us assume these terms are no larger than the diagonals themselves. We can therefore expect

$$\int_0^T |L(\tfrac{1}{2}+it)|^{2k} dt \sim f_L(k) \int_0^T \sum_{m=1}^{\infty} \frac{|a_{L,k}(m)|^2}{m} V_t\left(m^2\right) dt \tag{110}$$

for some coefficient $f_L(k)$.

Let us evaluate this integral for $L(s) = \zeta(s)^k$. In this case we have $a_{L,k}(m) = d_k(m)$. We first push the sum through the integral in $V_t(m^2)$. This is legal since the resultant series converges absolutely for $\Re(z) > 0$ (since $d_k(n) \ll n^{\epsilon}$). Therefore,

$$\sum_{m=1}^{\infty} \frac{d_k(m)^2}{m} V_t\left(m^2\right) = \frac{1}{2\pi i}\int_{(c)} f_k(t,z) G(z) \left[\sum_{m=1}^{\infty} \frac{d_k(m)^2}{m^{1+2z}}\right] \frac{dz}{z} \tag{111}$$

where

$$f_k(t,z) = \left(\frac{\gamma(\frac{1}{2}+it+z)\gamma(\frac{1}{2}-it+z)}{\gamma(\frac{1}{2}+it)\gamma(\frac{1}{2}-it)}\right)^k \tag{112}$$

and $\gamma(s) = \pi^{-s/2}\Gamma(s/2)$. By formula (26) we have

$$f_k(t,z) = \left(\frac{t}{2\pi}\right)^{kz}\left(1 + O_z\left(\frac{1}{t}\right)\right) \tag{113}$$

where the growth of z in the error term is at most polynomial. To gain an expansion for the series we first note

$$\sum_{m=1}^{\infty}\frac{d_k(m)^2}{m^{1+2z}} = \prod_p \sum_{j=0}^{\infty}\frac{d_k(p^j)^2}{p^{j(1+2z)}} = \prod_p\left(1 + \frac{k^2}{p^{1+2z}} + O\left(\frac{1}{p^{2(1+2z)}}\right)\right). \tag{114}$$

We factor out the divergent part of this product and write it in terms of an L-function we know i.e. we write

$$\sum_{m=1}^{\infty}\frac{d_k(m)^2}{m^{1+2z}} = \zeta(1+2z)^{k^2}A_k(z) \tag{115}$$

where

$$A_k(z) = \prod_p\left(1 - \frac{1}{p^{1+2z}}\right)^{k^2}\sum_{j=0}^{\infty}\frac{d_k(p^j)^2}{p^{j(1+2z)}} = \prod_p\left(1 + O\left(\frac{1}{p^{2(1+2z)}}\right)\right). \tag{116}$$

Note this last product is absolutely convergent for $\Re(z) > -1/4$. We therefore shift the contour in (111) to $(-1/4+\epsilon)$ picking up a pole at $z=0$. Writing

$$\zeta(s) = \frac{1}{s-1} + \gamma + \cdots, \qquad \left(\frac{t}{2\pi}\right)^{kz} = \sum_{m=0}^{\infty}\frac{1}{m!}\left[kz\log\left(\frac{t}{2\pi}\right)\right]^m \tag{117}$$

we see that the residue of the integrand at $z=0$ is given by

$$\frac{a(k)}{k^2!}\left(\frac{k}{2}\right)^{k^2}\log^{k^2}\left(\frac{t}{2\pi}\right) \tag{118}$$

where $a(k) = A_k(0)$. By virtue of (113) and (115) when combined with the bound $\zeta(1/2+it) \ll t^{1/6+\epsilon}$ ([59], Theorem 5.5), the integral on the new line is $\ll t^{-k/4+1/6+\epsilon} \ll t^{-1/12+\epsilon}$. Therefore, upon integrating we acquire

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}dt \sim \frac{a(k)g(k)}{k^2!}\log^{k^2}T \tag{119}$$

for some coefficient $g(k)$. We refer to $a(k)$ as the *arithmetic factor* and $g(k)$ as the *geometric factor*.

Determining $g(k)$ is a difficult problem. By the second and fourth moments we know that $g(1) = 1$ and $g(2) = 2$, and these remain the only rigorous results. It is only recently that headway has been made in finding believable conjectures for $g(k)$ with $k > 2$. In the paper [**12**], Conrey and Ghosh conjecture, based on their work in [**13**], that g(3)=42. Their methods involve considering the moments of the zeta function times some Dirichlet polynomial, which is a sum of the form $\sum_{n \leq T^\theta} a(n)n^{-s}$ where $\theta < 1/2$. Taking $a(n) = 1$, the Dirichlet polynomial then approximates $\zeta(s)$ and accordingly, the moment then approximates the higher moments of the zeta function. Later, Conrey and Gonek [**15**] described a method that could also give a conjecture for the eighth. This was based on mean values of long Dirichlet polynomials, and it seems these methods reach their limit with the eighth moment.

An entirely different approach was recently given by Keating and Snaith [**30**]. This used random matrix theory which requires at least a brief explanation; more thorough accounts can be found in the survey article [**8**] and the conference proceedings [**35**]. The idea that the zeros of the Riemann zeta function could be the eigenvalues of some Hermitian operator was first attributed [**36**] to Hilbert-Polya. The point here is that if we write a zero as $1/2 + i\gamma$, then the fact that the γ are the eigenvalues of some Hermitian operator implies they are real, and the Riemann hypothesis follows. Little evidence suggested this was the case until Montgomery calculated the pair-correlation of the zeros [**36**]. Freeman Dyson then noticed this matched with the pair-correlation of the eigenvalues of a random hermitian matrix. This was later corroborated with the numerical evidence of Odlyzko [**45**]. Keating and Snaith then argued that if the zeros could be modeled by eigenvalues, then the zeta function on the 1/2-line should be modeled by a characteristic polynomial. By calculating the moments of this polynomial they were then led to the conjecture

$$\frac{g(k)}{k^2!} = \frac{G(k+1)^2}{G(2k+1)} \tag{120}$$

where G is the Barnes G-function which satisfies $G(1) = 1$ and $G(z+1) = \Gamma(z)G(z)$. Their formula actually allows for k to range continuously through values $> -1/2$ if we replace $k^2!$ by $\Gamma(k^2+1)$ and $d_k(p^m)$ by $\Gamma(k+m)/m!\Gamma(k)$.

The issue with the method of Keating and Snaith was that the arithmetic factor had to be incorporated in an ad hoc way. Indeed, it seems unlikely that eigenvalues would really 'know' anything about primes. In the paper [20], Gonek, Hughes and Keating were able to reproduce the conjecture whilst incorporating the primes in a more natural way. The method basically involves writing the zeta function as a product over primes times a product over zeros. The moments of the product over zeros are handled with the random matrix theory whilst the primes can handle themselves. One of the main results of this thesis is to reproduce this result for the Dedekind zeta function. In the process we raise some new questions about moments of non-primitive L-functions in general. The simplest way to describe our solution is, in fact, in terms of the characteristic polynomial method of Keating and Snaith. The basic idea is that for two distinct L-functions, the matrices associated to their zeros can be chosen independently. The expectation of the characteristic polynomial associated to the product of these two L-functions will then factorise due to this independence. Consequently, we see a factorisation in the main term of the moments. So for example, given two distinct L-functions L_1, L_2 we can expect

$$\frac{1}{T}\int_0^T |L_1(\tfrac{1}{2}+it)|^{k_1}|L_2(\tfrac{1}{2}+it)|^{k_2}dt \sim a(k_1,k_2)\left(\frac{g(k_1)}{k_1^2!}\log^{k_1^2} T\right)\cdot\left(\frac{g(k_2)}{k_2^2!}\log^{k_2^2} T\right)$$

where $g(k)$ is as above and $a(k_1,k_2)$ is some mixed arithmetic factor containing information from L_1 and L_2.

An alternative method to conjecturing moments was recently given by Conrey et al. [10] in the form of a recipe. It is capable of conjecturing several different types of moments and in particular, when applied to the Riemann zeta function, this recipe reproduces the conjecture of Keating and Snaith. The recipe is actually concerned with shifted moments, which is to say, moments of the form

$$\int_0^T L(\tfrac{1}{2}+\alpha_1+it)\cdots L(\tfrac{1}{2}+\alpha_k+it)L(\tfrac{1}{2}+\alpha_{k+1}-it)\cdots L(\tfrac{1}{2}+\alpha_{2k}-it)dt \tag{121}$$

where α_i are (usually) small (usually) complex numbers. Of course, setting the shifts to zero gives the $2k$th moment of L. The main purpose of the shifts is to give a structural viewpoint of the resultant asymptotic whilst having the added advantage of giving formulas for moments of the derivatives (which are acquired by differentiating the shifts). The other main result of this thesis gives the first example of a shifted moment of a non-primitive L-function, namely, the product of two zeta functions and two Dirichlet L-functions which is a more general form of a quadratic Dedekind zeta function. We will then use this result to suggest how the recipe should be modified to deal with non-primitive L-functions in general, and we then apply this to corroborate our conjectures based on the method of Gonek, Hughes and Keating.

Examples of moments of non-primitive L-functions do exist, although there are not many and they certainly have not been studied systematically. In our final chapter we attempt to apply both conjectural methods to general non-primitive L-functions and hopefully offer some insight into this new area.

1.4. Statement of Results

There are two main results to this thesis; the extension of the hybrid product method of Gonek, Hughes and Keating to the Dedekind zeta function, and the shifted second moment of the Dedekind zeta function times a Dirichlet polynomial. The presence of the Dirichlet polynomial takes precedence over the shifts in our nomenclature and we henceforth refer to this result as the twisted second moment. In order to prove some of the conjectures given by the hybrid product method, we require the twisted moment result as a (somewhat long) Lemma. However, this is a result of interest in its own right, and we therefore devote the first half of this thesis to its derivation.

In Chapter 2 we consider the integral

$$I(h,k) = \int_{-\infty}^{\infty} \left(\frac{h}{k}\right)^{-it} \zeta\left(\tfrac{1}{2}+\alpha+it\right) L\left(\tfrac{1}{2}+\beta+it,\chi\right) \times \zeta\left(\tfrac{1}{2}+\gamma-it\right) L\left(\tfrac{1}{2}+\delta-it,\overline{\chi}\right) w(t)dt \tag{122}$$

where h,k are coprime integers, χ is some primitive Dirichlet character mod q and $w(t)$ is some smooth function with the intention of being the characteristic function of the interval $[T/2,4T]$. Our methods will closely follow those of Hughes and Young [23] who derived a formula for $I(h,k)$ when $q=1$. Similarly to their result, our main term will be written in terms of products of shifted zeta and L-functions, as well as finite products over the primes dividing h and k. Let

$$f_{\alpha,\beta}(n,\chi) = \sum_{n_1 n_2 = n} n_1^{-\alpha} n_2^{-\beta} \chi(n_2) \tag{123}$$

and let

$$\sigma_{\alpha,\beta}(n) = \sum_{n_1 n_2 = n} n_1^{-\alpha} n_2^{-\beta}. \tag{124}$$

Then, our main term will be given in terms of

$$Z_{\alpha,\beta,\gamma,\delta,h,k}(s) = A_{\alpha,\beta,\gamma,\delta}(s) B_{\alpha,\beta,\gamma,\delta,h,k}(s) \tag{125}$$

where

$$
\begin{aligned}
A_{\alpha,\beta,\gamma,\delta}(s) =&\zeta(1+\alpha+\gamma+s)\zeta(1+\beta+\delta+s)L(1+\beta+\gamma+s,\chi) \\
&\times \frac{L(1+\alpha+\delta+s,\overline{\chi})}{\zeta(2+\alpha+\beta+\gamma+\delta+2s)}\prod_{p|q}\left(\frac{1-p^{-1-s-\beta-\delta}}{1-p^{-2-2s-\alpha-\beta-\gamma-\delta}}\right)
\end{aligned} \tag{126}
$$

and

$$
B_{\alpha,\beta,\gamma,\delta,h,k}(s) = B_{\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi})B_{\gamma,\delta,\alpha,\beta,k}(s,\chi) \tag{127}
$$

with

$$
B_{\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi}) = \prod_{p|h}\frac{\sum_{j\geq 0} f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})p^{-j(1+s)}}{\sum_{j\geq 0} f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^j,\overline{\chi})p^{-j(1+s)}}. \tag{128}
$$

Here, h_p is the highest power of p dividing h. We must also define a slight variant of the above. This is given by

$$
Z'_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi) = A'_{\alpha,\beta,\gamma,\delta}(s,\chi)B'_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi) \tag{129}
$$

where

$$
\begin{aligned}
A'_{\alpha,\beta,\gamma,\delta}(s,\chi) = L(1+\alpha+\gamma+s,\chi)L(1+\beta+\delta+s,\chi) \\
\times\frac{L(1+\alpha+\delta+s,\chi)L(1+\beta+\gamma+s,\chi)}{L(2+\alpha+\beta+\gamma+\delta+2s,\chi^2)}
\end{aligned} \tag{130}
$$

and

$$
B'_{\alpha,\beta,\gamma,\delta,h,k}(s,\chi) = B'_{\alpha,\beta,\gamma,\delta,h}(s,\chi)B'_{\gamma,\delta,\alpha,\beta,k}(s,\chi) \tag{131}
$$

with

$$
B'_{\alpha,\beta,\gamma,\delta,h}(s,\chi) = \prod_{p|h}\frac{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^{h_p+j})p^{-j(1+s)}}{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j(1+s)}}. \tag{132}
$$

Theorem 1. *Let*

$$
\begin{aligned}
I(h,k) = &\int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}\zeta\left(\frac{1}{2}+\alpha+it\right)L\left(\frac{1}{2}+\beta+it,\chi\right) \\
&\times\zeta\left(\frac{1}{2}+\gamma-it\right)L\left(\frac{1}{2}+\delta-it,\overline{\chi}\right)w(t)dt
\end{aligned} \tag{133}
$$

where $w(t)$ is a smooth, nonnegative function with support contained in $[T/2, 4T]$, satisfying $w^{(j)}(t) \ll_j T_0^{-j}$ for all $j = 0, 1, 2, \ldots,$ where $T^{1/2+\epsilon} \ll T_0 \ll T$. Suppose $(h, k) = 1$, $hk \leq T^{\frac{2}{11}-\epsilon}$, and that $\alpha, \beta, \gamma, \delta$ are complex numbers $\ll (\log T)^{-1}$. Then

(134)

$$\begin{aligned} I(h,k) =& \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \Bigg(Z_{\alpha,\beta,\gamma,\delta,h,k}(0) + \frac{1}{q^{\beta+\delta}} Z_{-\gamma,-\delta,-\alpha,-\beta,h,k}(0) \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} \\ &+ Z_{-\gamma,\beta,-\alpha,\delta,h,k}(0) \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} + \frac{1}{q^{\beta+\delta}} Z_{\alpha,-\delta,\gamma,-\beta,h,k}(0) \left(\frac{t}{2\pi}\right)^{-\beta-\delta} \\ &+ \mathbf{1}_{q|h} \frac{\chi(k) G(\overline{\chi})}{q^{\delta}} Z'_{-\delta,\beta,\gamma,-\alpha,\frac{h}{q},k}(0,\chi) \left(\frac{t}{2\pi}\right)^{-\alpha-\delta} \\ &+ \mathbf{1}_{q|k} \frac{\overline{\chi}(h) \overline{G(\overline{\chi})}}{q^{\beta}} Z'_{\alpha,-\gamma,-\beta,\delta,h,\frac{k}{q}}(0,\overline{\chi}) \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} \Bigg) dt + E(T) \end{aligned}$$

where

$$E(T) \ll T^{3/4+\epsilon} (hk)^{7/8+\epsilon} \big(q^{3/2+\epsilon} |L(1,\chi)| (T/T_0)^{7/4} + q^{1+\epsilon} (T/T_0)^{9/4} \big) \tag{135}$$

and $G(\chi)$ is the Gauss sum.

In Chapter 3 we extend the hybrid product method to the Dedekind zeta function. The starting point is the hybrid product itself of course. This takes the following form.

Theorem 2. *Let $X \geq 2$ and let l be any fixed positive integer. Let $u(x) = Xf(X \log(x/e)+1)/x$ where f is a smooth, real, nonnegative function of total mass one with support in $[0, 1]$. Thus, $u(x)$ is a real, non-negative, smooth function with mass 1 and compact support on $[e^{1-1/X}, e]$. Set*

$$U(z) = \int_0^{\infty} u(x) E_1(z \log x) dx,$$

where $E_1(z) = \int_z^{\infty} e^{-w}/w \, dw$. Then for $\sigma \geq 0$ and $|t| \geq 2$ we have

$$\zeta_{\mathbb{K}}(s) = P_{\mathbb{K}}(s, X) Z_{\mathbb{K}}(s, X) \left(1 + O\left(\frac{X^{l+2}}{(|s| \log X)^l} \right) + O(X^{-\sigma} \log X) \right) \tag{136}$$

where

$$P_{\mathbb{K}}(s,X) = \exp\left(\sum_{\substack{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}\\ \mathfrak{N}(\mathfrak{a})\le X}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log\mathfrak{N}(\mathfrak{a})}\right) \tag{137}$$

with

$$\Lambda(\mathfrak{a}) = \begin{cases} \log\mathfrak{N}(\mathfrak{p}) & \textit{if } \mathfrak{a}=\mathfrak{p}^m,\\ 0 & \textit{otherwise,}\end{cases} \tag{138}$$

and

$$Z_{\mathbb{K}}(s,X) = \exp\left(-\sum_{\rho} U((s-\rho)\log X)\right), \tag{139}$$

where the sum is over all non-trivial zeros of $\zeta_{\mathbb{K}}(s)$*.*

We will use the hybrid product to conjecture asymptotics for the $2k$th moment of $\zeta(\frac{1}{2}+it)$. This will be facilitated by the following conjecture which follows from a similar reasoning to that given in [**20**].

Conjecture 1 (Splitting Conjecture)**.** *Let* $X,T\to\infty$ *with* $X \ll (\log T)^{2-\epsilon}$*. Then for* $k>-1/2$*, we have*

$$\begin{aligned} \frac{1}{T}\int_T^{2T} |\zeta_{\mathbb{K}}(\tfrac{1}{2}+it)|^{2k}dt \\ \sim \left(\frac{1}{T}\int_T^{2T}\left|P_{\mathbb{K}}\left(\tfrac{1}{2}+it,X\right)\right|^{2k}dt\right)\left(\frac{1}{T}\int_T^{2T}\left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it,X\right)\right|^{2k}dt\right). \end{aligned} \tag{140}$$

We plan to evaluate the moments of $P_{\mathbb{K}}$ by using the Montgomery-Vaughan mean value theorem. Due to the nature of how primes split, or rather, how they are not known to split in some cases, we restrict ourselves to Galois extensions. It may be possible to remove this restriction given milder conditions on $\mathbb{K}$.

Theorem 3. *Let* $\mathbb{K}$ *be a Galois extension of degree* n *with Galois group* $G = \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ *and for a given prime* $\mathfrak{p}$ *let* $g_{\mathfrak{p}}$ *denote the index of the decomposition* *group $G_{\mathfrak{p}}$ in G. Let $1/2 \le c < 1$, $\epsilon > 0$, $k > 0$ and suppose that X and $T \to \infty$ with $X \ll (\log T)^{1/(1-c+\epsilon)}$. Then*

$$\frac{1}{T}\int_T^{2T} \left|P_{\mathbb{K}}\left(\tfrac{1}{2}+it,\, X\right)\right|^{2k} dt \sim a(k)\psi_{\mathbb{K}}^{nk^2}(e^{\gamma}\log X)^{nk^2} \tag{141}$$

where $\psi_{\mathbb{K}}$ denotes the residue of $\zeta_{\mathbb{K}}(s)$ at $s=1$ and

$$a(k) = \prod_{\mathfrak{p}\subseteq\mathcal{O}_{\mathbb{K}}} \left(\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{nk^2} \left(\sum_{m\ge 0} \frac{d_{g_{\mathfrak{p}}k}(\mathfrak{p}^m)^2}{\mathfrak{N}(\mathfrak{p})^m}\right)^{1/g_{\mathfrak{p}}} \right) \tag{142}$$

with $d_k(\mathfrak{p}^m) = d_k(p^m) = \Gamma(m+k)/(m!\Gamma(k))$.

In considering the moments of $Z_{\mathbb{K}}$ for Galois extensions we first express $\zeta_{\mathbb{K}}(s)$ as a product of Artin L-functions. For each individual L-function we then follow the heuristic argument given in section 4 of [20]. This essentially allows us to write the moments of $Z_{\mathbb{K}}$ as an expectation over the unitary group. We then assume a certain quality of independence between the Artin L-functions, namely, that the matrices associated to the zeros of $L(s,\chi,\mathbb{K}/\mathbb{Q})$ at height T, act independently for distinct χ. This allows for a factorisation of the expectation and we are led to

Conjecture 2. *Let $\mathbb{K}$ be a Galois extension of degree n. Suppose that $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then for $k > -1/2$ we have*

$$\frac{1}{T}\int_T^{2T} \left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^{2k} dt \sim (e^{\gamma}\log X)^{-nk^2} \prod_{\chi} \frac{G(\chi(1)k+1)^2}{G(2\chi(1)k+1)} \Big(\log\big(q(\chi)T^{d_\chi}\big)\Big)^{\chi(1)^2k^2} \tag{143}$$

where the product is over the irreducible characters of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, *$G$ is the Barnes G-function, $q(\chi)$ is the conductor of $L(s,\chi,\mathbb{K}/\mathbb{Q})$ and d_χ is its degree.*

By combining this with Theorem 3 and Conjecture 1 we see that the factors of $e^{\gamma}\log X$ cancel, as expected, and we acquire a full conjecture for the moments of $\zeta_{\mathbb{K}}(1/2+it)$ when $\mathbb{K}$ is Galois. Although the form of this conjecture should be fairly clear, we state it in full for the purposes of completeness.

Conjecture 3. *Let $a(k)$ be given by (142) and let $\psi_{\mathbb{K}}$ denote the residue of $\zeta_{\mathbb{K}}(s)$ at $s = 1$. Let $\mathbb{K}$ be a Galois extension of degree n and suppose that $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then for $k > -1/2$ we have*

(144)
$$\frac{1}{T}\int_T^{2T} |\zeta_{\mathbb{K}}(\tfrac{1}{2}+it)|^{2k} dt \sim a(k)\psi_{\mathbb{K}}^{nk^2} \prod_{\chi} \frac{G(\chi(1)k+1)^2}{G(2\chi(1)k+1)} \Big(\log\big(q(\chi)T^{d_\chi}\big)\Big)^{\chi(1)^2k^2}$$

where the product is over the irreducible characters of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$, G is the Barnes G-function, $q(\chi)$ is the conductor of $L(s,\chi,\mathbb{K}/\mathbb{Q})$ and d_χ is its degree.

In section 3.4 of Chapter 3 we use Theorem 1 to prove Conjecture 2 for $k = 1$ in the case of quadratic extensions. That is, we prove

Theorem 4. *Let $\mathbb{K}$ be a quadratic extension. Suppose that $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then*

$$\frac{1}{T}\int_T^{2T} \left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^2 dt \sim \frac{\log T \cdot \log qT}{(e^\gamma \log X)^2}. \tag{145}$$

where q is the modulus of the character χ in the equation $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi)$.

By combining this with Theorem 3 and then comparing with Motohashi's result (79), we see that Conjecture 1, the splitting conjecture, is true for $k = 1$ in the case of quadratic extensions.

As previously mentioned, we can use our twisted moment theorem as a means to extend the moments recipe of Conrey et al. [10] to non-primitive L-functions. We then use our modified recipe to reproduce our main moments conjecture in the case of quadratic extensions.

Conjecture 4. *Let $\mathbb{K}$ be a quadratic extension and let $a(k)$ be given by (142). Then*

$$\frac{1}{T}\int_T^{2T} |\zeta_{\mathbb{K}}(\tfrac{1}{2}+it)|^{2k} dt \sim a(k)L(1,\chi)^{2k^2}\left(\frac{G(k+1)^2}{G(2k+1)}\right)^2 (\log T \cdot \log qT)^{k^2}. \tag{146}$$

In the final chapter we attempt to generalise the main ideas of this paper to non-primitive L-functions. The functions under consideration are of the form

$$L(s) = \sum \alpha_L(n) n^{-s} = \prod_{j=1}^{m} L_j(s)^{e_j} \tag{147}$$

where $e_j \in \mathbb{N}$ and the $L_j(s)$ are distinct, primitive members of the Selberg class. We require that the 'convolution' L-functions

$$M_j(s) = \sum_{n=1}^{\infty} \frac{|\alpha_{L_j}(n)|^2}{n^s} \tag{148}$$

behave reasonably, in particular, that they have an analytic continuation. We then claim

Conjecture 5. *With the notation as above, let $\alpha_{L,k}(n)$ be the Dirichlet coefficients of $L(s)^k$. Then for $k > -1/2$,*

$$\frac{1}{T}\int_0^T \left|L\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim a_L(k) \prod_{j=1}^{m} \frac{G^2(e_j k+1)}{G(2e_j k+1)} \left(\log\left(Q_j T^{d_j}\right)\right)^{(e_j k)^2} \tag{149}$$

where

$$a_L(k) = \prod_p \left(1-\frac{1}{p}\right)^{n_L k^2} \sum_{n=0}^{\infty} \frac{|\alpha_{L,k}(p^n)|^2}{p^n} \tag{150}$$

with $n_L = \sum_{j=1}^m e_j^2$.

We remark that if $L(s) = \zeta_{\mathbb{K}}(s)$ with $\mathbb{K}$ Galois and we have a factorisation in terms of Dirichlet series, then the residue term $\chi_{\mathbb{K}}^{nk^2}$ of (141) is a factor of $a_L(k)$. For example, see the discussion following (556) in Chapter 4

CHAPTER 2

The Twisted Second Moment

Let $M(s)$ be an arbitrary Dirichlet polynomial of length T^θ. So,

$$M(s) = \sum_{n \le T^\theta} \frac{a(n)}{n^s} \tag{151}$$

for some complex valued coefficients $a(n)$. The aim of this chapter is to find an asymptotic formula for the integral

$$\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^2 \left|M\left(\tfrac{1}{2}+it\right)\right|^2 dt \tag{152}$$

when $\mathbb{K}$ is a quadratic number field. On expanding $|M|^2$ and pushing the integral through the sum this becomes

$$\sum_{h,k \le T^\theta} \frac{a(h)\overline{a(k)}}{\sqrt{hk}} \int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^2 \left(\frac{h}{k}\right)^{-it} dt. \tag{153}$$

When evaluating this inner integral we assume h/k is in its reduced form i.e. that h and k are coprime. The formula we acquire can then be applied to the above by writing $h/k = (h/(h,k))/(k/(h,k))$.

In a bid for greater applicability, we generalise the integral in several ways. First, we replace $\zeta_{\mathbb{K}}(s)$ with $\zeta(s)L(s,\chi)$ where χ is an arbitrary Dirichlet character mod q. This makes little difference to the ensuing arguments. Second, we include small shifts in the argument's of $\zeta(s)$ and $L(s,\chi)$. There are several benefits to the shifts; one is that they allow for formulas involving the derivatives of $\zeta_{\mathbb{K}}(s)$, another is that they make residue calculations easier since we only have to deal with functions with simple poles. Finally, we approximate the range of integration by incorporating some smooth function in the integrand. The generalised integral is then given by

$$
(154) \qquad \begin{aligned} I(h,k) = \int_{-\infty}^{\infty} & w(t)\zeta\left(\tfrac{1}{2}+\alpha+it\right)L\left(\tfrac{1}{2}+\beta+it,\chi\right) \\ & \times \zeta\left(\tfrac{1}{2}+\gamma-it\right)L\left(\tfrac{1}{2}+\delta-it,\overline{\chi}\right)\left(\frac{h}{k}\right)^{-it} dt \end{aligned}
$$

where $\alpha, \beta, \gamma, \delta$ are small complex numbers and $w(t)$ is some smooth function having the intention of being the characteristic function of the interval $[T/2, 4T]$.

The formula for $I(h,k)$ given in Theorem 1 states that

$$
\begin{aligned} I(h,k) = & \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} w(t)\Bigg(Z_{\alpha,\beta,\gamma,\delta,h,k}(0) + \frac{1}{q^{\beta+\delta}}Z_{-\gamma,-\delta,-\alpha,-\beta,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} \\ & + Z_{-\gamma,\beta,-\alpha,\delta,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} + \frac{1}{q^{\beta+\delta}}Z_{\alpha,-\delta,\gamma,-\beta,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\beta-\delta} \\ & + \cdots\Bigg)dt + E(T) \end{aligned}
$$

where the dots represent the two extra Z' terms and where

$$
E(T) \ll T^{3/4+\epsilon}(hk)^{7/8+\epsilon}q^{1+\epsilon}(T/T_0)^{9/4}.
$$

As we shall see shortly, the main terms in our formula for $I(h,k)$ are of size $\approx (hk)^{-1/2}T\log^2 T$. Therefore, on taking $T_0 \gg T^{1-\epsilon}$ we see that the error term has a power saving provided $hk \ll T^{2/7-\epsilon}$. However, when applying our formula for $I(h,k)$ in (153), one must take into account the size of the coefficients when determining the length of the polynomial, if one is to obtain a power saving in the error. In particular, for coefficients $a(n) \ll n^{\epsilon}$, we must take the length of the polynomial θ to be $\leq 1/11-\epsilon$ i.e. $hk \ll T^{2/11-\epsilon}$ (see formula (498) for example).

For $q > 1$, the two Z' terms are entire since they are comprised solely of Dirichlet L-functions times finite products over primes dividing hk. The main terms are given by

$$
Z_{\alpha,\beta,\gamma,\delta,h,k}(0) = A_{\alpha,\beta,\gamma,\delta}(0)B_{\alpha,\beta,\gamma,\delta,h,k}(0)
$$

where

$$A_{\alpha,\beta,\gamma,\delta}(0) = \frac{\zeta(1+\alpha+\gamma)\zeta(1+\beta+\delta)L(1+\beta+\gamma,\chi)L(1+\alpha+\delta,\overline{\chi})}{\zeta(2+\alpha+\beta+\gamma+\delta)} \times \prod_{p|q}\left(\frac{1-p^{-1-\beta-\delta}}{1-p^{-2-\alpha-\beta-\gamma-\delta}}\right)$$

and $B_{\alpha,\beta,\gamma,\delta,h,k}(0)$ is some product over the primes dividing hk.

In order to make sense of the formula, let us sketch the evaluation of $I(1,1)$. As can be seen from the formula for $A_{\alpha,\beta,\gamma,\delta}(0)$, the only cause of non-holomorphy is due to the zeta functions. In particular, the integrand is undefined as $\alpha+\gamma \to 0$ and $\beta+\delta \to 0$. We first simplify by letting $\gamma = \delta = 0$. Writing $\zeta(1+s) = 1/s + \cdots$ and

$$\frac{L(1+\beta,\chi)L(1+\alpha,\overline{\chi})}{\zeta(2+\alpha+\beta)} \prod_{p|q}\left(\frac{1-p^{-1-\beta}}{1-p^{-2-\alpha-\beta}}\right) = c_{00} + \alpha c_{10} + \beta c_{01} + \cdots \tag{155}$$

with

$$c_{00} = \frac{|L(1,\chi)|^2}{\zeta(2)} \prod_{p|q}\left(1+\frac{1}{p}\right)^{-1}, \tag{156}$$

we see that $Z_{\alpha,\beta,0,0,1,1}(0) = \frac{c_{00}}{\alpha\beta} + \cdots$. Performing a similar procedure on the remaining Z terms we see that the integrand is given by

$$\frac{c_{00}}{\alpha\beta}\left[1 + \left(\frac{t}{2\pi}\right)^{-\alpha}\left(\frac{qt}{2\pi}\right)^{-\beta} - \left(\frac{t}{2\pi}\right)^{-\alpha} - \left(\frac{qt}{2\pi}\right)^{-\beta} + \cdots\right] = c_{00}\log\left(\frac{t}{2\pi}\right)\log\left(\frac{qt}{2\pi}\right) + \cdots$$

where the remaining terms are of a lower order (in t) and holomorphic in $\alpha\beta$. On taking $w(t)$ as a smooth approximation to the characteristic function of interval $[T, 2T]$ with $T_0 = T^{1-\epsilon}$ we recover Motohashi's formula (79). Being more diligent with the lower order terms we can show that

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)L(\tfrac{1}{2}+it,\chi)|^2 dt = \frac{1}{T}\int_T^{2T} P\left(\log\left(\tfrac{t}{2\pi}\right), \log\left(\tfrac{qt}{2\pi}\right)\right) dt + E'(T) \tag{157}$$

where $P(x, y)$ is some quadratic polynomial with leading coefficient $c_{00}xy$, and $E'(T) \ll T^{-1/4+\epsilon}$. Note that the main term has the same order of magnitude as the product

$$\frac{1}{T}\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^2 dt \times \frac{1}{T}\int_T^{2T} |L(\tfrac{1}{2}+it,\chi)|^2 dt.$$

We expect this sort of factorisation to hold for an arbitrary product of L-functions, the above example provides a building block for this philosophy which we detail in the final chapter.

The proof of Theorem 1 follows the same line of reasoning to that taken in section 1.3 of Chapter 1. We first find an approximate functional equation for

$$\zeta\left(\tfrac{1}{2}+\alpha+it\right) L\left(\tfrac{1}{2}+\beta+it,\chi\right) \zeta\left(\tfrac{1}{2}+\gamma-it\right) L\left(\tfrac{1}{2}+\delta-it,\overline{\chi}\right). \tag{158}$$

As usual, this takes the form

$$\sum{}^{(1)} + \varkappa \sum{}^{(2)}.$$

where $\varkappa$ is the factor appearing in the asymmetric functional equation of (158). Due to the presence of the shifts $\varkappa \neq 1$ as it would do otherwise (cf. (108)). However, the two sums are sufficiently similar that we only need treat one of them. We split each sum into its diagonal and off-diagonal components. The diagonals are evaluated similarly to before by shifting contours and computing residues. The diagonals are then seen to contribute the first two terms:

$$Z_{\alpha,\beta,\gamma,\delta,h,k}(0) + \frac{1}{q^{\beta+\delta}} Z_{-\gamma,-\delta,-\alpha,-\beta,h,k}(0) \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta}. \tag{159}$$

They also contribute four extra terms which are dependent on the shifts. These need to be shown to cancel with terms arising from the off-diagonal contribution.

To evaluate the off-diagonals we apply the 'delta-method' of Duke, Friedlander and Iwaniec [**16**]. This first requires a little preparation however most of the work lies in finding, and then applying, a Voronoi-type sum formula. Specifically, we need a formula for the sum

$$\sum_{n=1}^{\infty} f_{\alpha,\beta}(n,\chi)e_d(chn)g(n) \tag{160}$$

where $f_{\alpha,\beta}(n,\chi)$ is given by (123), c and d are coprime integers and g is some smooth function of rapid decay. After an application of the delta-method we see that the total contribution of the off-diagonal terms is given by a sum of eight main terms plus the error term $E(T)$. Each of these main terms then combines with another to give either a Z or Z' term minus one of four the extra terms acquired from the diagonals.

2.1. Setup

2.1.1. The Approximate Functional Equation. We first restate the necessary functional equations:

$$\Lambda(s) :=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(1-s), \tag{161}$$

$$\xi(s,\chi) := \left(\frac{\pi}{q}\right)^{-\frac{s+\mathfrak{a}}{2}}\Gamma\left(\frac{s+\mathfrak{a}}{2}\right)L(s,\chi) = \frac{G(\chi)}{i^{\mathfrak{a}}\sqrt{q}}\xi(1-s,\overline{\chi}). \tag{162}$$

If we define

$$\begin{aligned}\Xi_{\alpha,\beta,\gamma,\delta,t}(s,\chi) =&\Lambda\left(\frac{1}{2}+\alpha+s+it\right)\xi\left(\frac{1}{2}+\beta+s+it,\chi\right)\\ &\times\Lambda\left(\frac{1}{2}+\gamma+s-it\right)\xi\left(\frac{1}{2}+\delta+s-it,\overline{\chi}\right)\end{aligned} \tag{163}$$

then by the above two functional equations and the fact that $G(\chi)G(\overline{\chi}) = (-1)^{\mathfrak{a}}q$ we see

$$\Xi_{\alpha,\beta,\gamma,\delta,t}(-s,\chi) = \Xi_{-\gamma,-\delta,-\alpha,-\beta,t}(s,\chi). \tag{164}$$

After expanding equation (163) we group together the zeta and L-functions and group together the gamma factors and write

$$\Xi_{\alpha,\beta,\gamma,\delta,t}(s,\chi) = \zeta_{\alpha,\beta,\gamma,\delta,t}(s,\chi)\Gamma_{\alpha,\beta,\gamma,\delta,t}(s) \tag{165}$$

where

$$\zeta_{\alpha,\beta,\gamma,\delta,t}(s,\chi) = \zeta\left(\frac{1}{2}+\alpha+s+it\right)L\left(\frac{1}{2}+\beta+s+it,\chi\right) \times\zeta\left(\frac{1}{2}+\gamma+s-it\right)L\left(\frac{1}{2}+\delta+s-it,\overline{\chi}\right) \tag{166}$$

and

$$\Gamma_{\alpha,\beta,\gamma,\delta,t}(s) = \pi^{-1-2s-\frac{\alpha+\beta+\gamma+\delta}{2}-\mathfrak{a}}q^{\frac{1}{2}+s+\frac{\beta+\delta}{2}+\mathfrak{a}} \times\Gamma\left(\frac{\frac{1}{2}+\alpha+s+it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\beta+s+it+\mathfrak{a}}{2}\right) \times\Gamma\left(\frac{\frac{1}{2}+\gamma+s-it}{2}\right)\Gamma\left(\frac{\frac{1}{2}+\delta+s-it+\mathfrak{a}}{2}\right). \tag{167}$$

We require an approximate functional equation for $\zeta_{\alpha,\beta,\gamma,\delta,t}(0,\chi)$. By (164), this satisfies the functional equation

$$\zeta_{\alpha,\beta,\gamma,\delta,t}(0,\chi) = \varkappa_{\alpha,\beta,\gamma,\delta,t}\zeta_{-\gamma,-\delta,-\alpha,-\beta,t}(0,\chi)$$

with

$$\varkappa_{\alpha,\beta,\gamma,\delta,t} = \frac{\Gamma_{-\gamma,-\delta,-\alpha,-\beta,t}(0)}{\Gamma_{\alpha,\beta,\gamma,\delta,t}(0)} = \pi^{\alpha+\beta+\gamma+\delta}q^{-(\beta+\delta)}\frac{\Gamma\left(\frac{\frac{1}{2}-\alpha-it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha+it}{2}\right)}\frac{\Gamma\left(\frac{\frac{1}{2}-\beta-it+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\beta+it+\mathfrak{a}}{2}\right)}\frac{\Gamma\left(\frac{\frac{1}{2}-\gamma+it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\gamma-it}{2}\right)}\frac{\Gamma\left(\frac{\frac{1}{2}-\delta+it+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\delta-it+\mathfrak{a}}{2}\right)}. \tag{168}$$

The equivalent of Proposition 1.3.2 is given by the following.

Proposition 2.1.1. *Let $G(s)$ be an even, entire function of rapid decay as $|s|\to\infty$ in any fixed vertical strip $|\Re(s)|\le C$ satisfying $G(0)=1$, and let*

$$V_{\alpha,\beta,\gamma,\delta,t}(x) = \frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}g_{\alpha,\beta,\gamma,\delta}(s,t)x^{-s}ds \tag{169}$$

where

$$g_{\alpha,\beta,\gamma,\delta}(s,t) = \left(\frac{\pi^2}{q}\right)^s \frac{\Gamma_{\alpha,\beta,\gamma,\delta,t}(s)}{\Gamma_{\alpha,\beta,\gamma,\delta,t}(0)}$$

$$(170) \qquad = \frac{\Gamma\left(\frac{\frac{1}{2}+\alpha+s+it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\alpha+it}{2}\right)} \frac{\Gamma\left(\frac{\frac{1}{2}+\beta+s+it+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\beta+it+\mathfrak{a}}{2}\right)} \frac{\Gamma\left(\frac{\frac{1}{2}+\gamma+s-it}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\gamma-it}{2}\right)} \frac{\Gamma\left(\frac{\frac{1}{2}+\delta+s-it+\mathfrak{a}}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\delta-it+\mathfrak{a}}{2}\right)}.$$

Then if all $\alpha, \beta, \gamma, \delta$ have real part less than 1/2, we have

$$\zeta_{\alpha,\beta,\gamma,\delta,t}(0) = \sum_{m,n} \frac{f_{\alpha,\beta}(n,\chi) f_{\gamma,\delta}(m,\overline{\chi})}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{-it} V_{\alpha,\beta,\gamma,\delta,t}\left(\frac{\pi^2 mn}{q}\right)$$

$$(171) \qquad + \varkappa_{\alpha,\beta,\gamma,\delta,t} \sum_{m,n} \frac{f_{-\gamma,-\delta}(n,\chi) f_{-\alpha,-\beta}(m,\overline{\chi})}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{-it}$$

$$\times V_{-\gamma,-\delta,-\alpha,-\beta,t}\left(\frac{\pi^2 mn}{q}\right) + O\left((1+|t|)^{-A}\right).$$

Proof. We start in the familiar way by considering

$$I_1 = \frac{1}{2\pi i}\int_{(1)} \Xi_{\alpha,\beta,\gamma,\delta,t}(s) \frac{G(s)}{s} ds.$$

Moving the line of integration to (-1) we obtain a new integral

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{(-1)} \Xi_{\alpha,\beta,\gamma,\delta,t}(s,\chi) \frac{G(s)}{s} ds \\ &= \frac{1}{2\pi i}\int_{(1)} \Xi_{-\gamma,-\delta,-\alpha,-\beta,t}(s,\chi) \frac{G(s)}{s} ds \end{aligned}$$

where we have made the change of variables $s \mapsto -s$ and used the functional equation (164). Due to the rapid decay of $G(s)$ in the imaginary direction we see that the only residue of the integrand that matters is the one at $s = 0$. The other residues are those occurring at the poles of the zeta functions; these give rise to the equivalent of the R term of Proposition 1.3.2. Therefore, we may write

$$I_1 + I_2 = \Xi_{\alpha,\beta,\gamma,\delta,t}(0) + O((1+|t|)^{-A})$$

and hence

$$\begin{aligned}\zeta_{\alpha,\beta,\gamma,\delta,t}(0,\chi) =&\frac{1}{2\pi i}\int_{(1)}\zeta_{\alpha,\beta,\gamma,\delta,t}(s,\chi)\frac{\Gamma_{\alpha,\beta,\gamma,\delta,t}(s)}{\Gamma_{\alpha,\beta,\gamma,\delta,t}(0)}\frac{G(s)}{s}ds\\ &+\frac{1}{2\pi i}\int_{(1)}\zeta_{-\gamma,-\delta,-\alpha,-\beta,t}(s,\chi)\frac{\Gamma_{-\gamma,-\delta,-\alpha,-\beta,t}(s)}{\Gamma_{\alpha,\beta,\gamma,\delta,t}(0)}\frac{G(s)}{s}ds\\ &+O((1+|t|)^{-A})\\ =&\frac{1}{2\pi i}\int_{(1)}\zeta_{\alpha,\beta,\gamma,\delta,t}(s,\chi)g_{\alpha,\beta,\gamma,\delta}(s,t)\left(\frac{\pi^2}{q}\right)^{-s}\frac{G(s)}{s}ds\\ &+\frac{\varkappa_{\alpha,..}}{2\pi i}\int_{(1)}\zeta_{-\gamma,-\delta,-\alpha,-\beta,t}(s,\chi)g_{-\gamma,-\delta,-\alpha,-\beta,t}(s,t)\left(\frac{\pi^2}{q}\right)^{-s}\frac{G(s)}{s}ds\\ &+O((1+|t|)^{-A}).\end{aligned}$$

Now, on expanding the Dirichlet series we have

$$\zeta_{\alpha,\beta,\gamma,\delta,t}(s,\chi)=\sum_{m,n}\frac{f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(m,\overline{\chi})}{n^{1/2+s+it}m^{1/2+s-it}}$$

and so by reversing the order of integration and summation the result follows. □

We note that by (25) and (26) we have

$$\varkappa_{\alpha,\beta,\gamma,\delta,t}=q^{-\beta-\delta}\left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta}\big(1+O(t^{-1})\big) \tag{172}$$

and

$$g_{\alpha,\beta,\gamma,\delta}(s,t)=\left(\frac{t}{2}\right)^{2s}\big(1+O_s(t^{-1})\big) \tag{173}$$

as $t\to\infty$. The dependence on s in the error term of $g_{\alpha,\beta,\gamma,\delta}(s,t)$ is of polynomial growth, at most, and hence will be negated by the rapid decay of $G(s)$ in any applications we make. Note that $g(s,t)$ is the equivalent of the function $\gamma_L(1/2+it+s)/\gamma_L(1/2+it)$ in the notation of our introductory section on moments. By applying the same argument used there we see that the sums in the above approximate functional equation can be restricted to $mn \ll t^{2+\epsilon}$. This fact will be used later.

It will frequently occur that a function $F_{\alpha,\beta,\gamma,\delta}$, say, arising from the first term of the approximate functional equation has an equivalent $F_{-\gamma,-\delta,-\alpha,-\beta}$ arising from the second. As such, we shall often abbreviate functions of the form $F_{\alpha,\beta,\gamma,\delta}$ to $F_{\boldsymbol{\alpha}}$ and $F_{-\gamma,-\delta,-\alpha,-\beta}$ to $F_{-\boldsymbol{\gamma}}$.

2.1.2. A Formula for the Twisted Second Moment. Applying the approximate functional equation to $I(h,k)$ gives

$$\begin{aligned} &I(h,k)\\ =&\sum_{m,n}\frac{f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(m,\overline{\chi})}{(mn)^{1/2}}\int_{-\infty}^{\infty}\left(\frac{hm}{kn}\right)^{-it}V_{\boldsymbol{\alpha},t}\left(\frac{\pi^2 mn}{q}\right)w(t)dt\\ &+\sum_{m,n}\frac{f_{-\gamma,-\delta}(n,\chi)f_{-\alpha,-\beta}(m,\overline{\chi})}{(mn)^{1/2}}\int_{-\infty}^{\infty}\left(\frac{hm}{kn}\right)^{-it}\varkappa_{\boldsymbol{\alpha},t}V_{-\boldsymbol{\gamma},t}\left(\frac{\pi^2 mn}{q}\right)w(t)dt\\ =&I^{(1)}(h,k)+I^{(2)}(h,k). \end{aligned} \tag{174}$$

By expanding the inner integral and interchanging the orders of integration we have

$$\begin{aligned} I^{(1)}(h,k)=&\sum_{m,n}\frac{f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(m,\overline{\chi})}{(mn)^{1/2}}\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{\pi^2 mn}{q}\right)^{-s}\\ &\times\int_{-\infty}^{\infty}\left(\frac{hm}{kn}\right)^{-it}g_{\boldsymbol{\alpha}}(s,t)w(t)dtds \end{aligned} \tag{175}$$

and similarly

$$\begin{aligned} I^{(2)}(h,k)=&\sum_{m,n}\frac{f_{-\gamma,-\delta}(n,\chi)f_{-\alpha,-\beta}(m,\overline{\chi})}{(mn)^{1/2}}\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{\pi^2 mn}{q}\right)^{-s}\\ &\times\int_{-\infty}^{\infty}\left(\frac{hm}{kn}\right)^{-it}\varkappa_{\boldsymbol{\alpha},t}g_{-\boldsymbol{\gamma}}(s,t)w(t)dtds. \end{aligned} \tag{176}$$

We will now split the sum over m, n into those parts for which $hm=kn$, the diagonal terms, and those for which $hm\neq kn$, the off-diagonals. In what follows we only work with $I^{(1)}(h,k)$ since any result we acquire can be made to apply to $I^{(2)}(h,k)$ by performing the substitutions $\alpha\leftrightarrow-\gamma$, $\beta\leftrightarrow-\delta$ and by inserting $\varkappa_{\boldsymbol{\alpha},t}$ into the integrals over t. This often amounts to multiplying by $q^{-\beta-\delta}(t/2\pi)^{-\alpha-\beta-\gamma-\delta}$ in light of (172) .

2.2. Diagonal Terms

Let $I_D^{(1)}(h,k)$ denote the sum of terms in $I^{(1)}(h,k)$ for which $hm = kn$. Writing $n = hl$ and $m = kl$ with $l \geq 1$ we see

$$I_D^{(1)}(h,k) = \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{\pi^2 hk}{q}\right)^{-s} \times g_{\boldsymbol{\alpha}}(s,t) \sum_{l=1}^{\infty} \frac{f_{\alpha,\beta}(kl,\chi) f_{\gamma,\delta}(hl,\overline{\chi})}{l^{1+2s}} ds dt. \tag{177}$$

Here we have pushed the sum through the integrals but since the shift parameters are small we have absolute convergence and hence this is legal.

Proposition 2.2.1. *Let $Z_{\boldsymbol{\alpha},h,k}(s)$ be given by (125). Then*

$$I_D^{(1)}(h,k) = \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} Z_{\boldsymbol{\alpha},h,k}(0) w(t) dt + J_{\alpha,\gamma}^{(1)} + J_{\beta,\delta}^{(1)} + O\left(\frac{q^{\epsilon} T^{\frac{1}{2}+\epsilon}}{(qhk)^{1/4}}\right) \tag{178}$$

and

$$I_D^{(2)}(h,k) = \frac{1}{q^{\beta+\delta}} \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} Z_{-\boldsymbol{\gamma},h,k}(0) \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} w(t) dt + J_{\alpha,\gamma}^{(2)} + J_{\beta,\delta}^{(2)} + O\left(\frac{q^{\epsilon} T^{\frac{1}{2}+\epsilon}}{(qhk)^{1/4}}\right) \tag{179}$$

where

$$J_{a,b}^{(1)} = q^{-\frac{a+b}{2}} \frac{\mathrm{Res}_{2s=-a-b}(Z_{\boldsymbol{\alpha},h,k}(2s))}{(hk)^{\frac{1}{2}-\frac{a+b}{2}}} \frac{G(-(a+b)/2)}{-(a+b)/2} \int_{-\infty}^{\infty} \left(\frac{t}{2\pi}\right)^{-a-b} w(t) dt \tag{180}$$

and

$$J_{a,b}^{(2)} = \frac{1}{q^{\beta+\delta}} q^{\frac{a+b}{2}} \frac{\mathrm{Res}_{2s=a+b}(Z_{-\boldsymbol{\gamma},h,k}(2s))}{(hk)^{\frac{1}{2}+\frac{a+b}{2}}} \frac{G((a+b)/2)}{(a+b)/2} \times \int_{-\infty}^{\infty} \left(\frac{t}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta+a+b} w(t) dt. \tag{181}$$

PROOF. By the theory of Euler products (see for example [59], section 1.4) we have

$$\sum_{l=1}^{\infty}\frac{f_{\alpha,\beta}(kl,\chi)f_{\gamma,\delta}(hl,\overline{\chi})}{l^{1+s}}=\prod_p\sum_{j\geq 0}\frac{f_{\alpha,\beta}(p^{k_p+j},\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})}{p^{j(1+s)}}$$

$$(182)\quad =\Bigg(\prod_{(p,hk)=1}\sum_{j\geq 0}\frac{f_{\alpha,\beta}(p^{j},\chi)f_{\gamma,\delta}(p^{j},\overline{\chi})}{p^{j(1+s)}}\Bigg)\Bigg(\prod_{p|h}\sum_{j\geq 0}\frac{f_{\alpha,\beta}(p^{j},\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})}{p^{j(1+s)}}\Bigg) \times\Bigg(\prod_{p|k}\sum_{j\geq 0}\frac{f_{\alpha,\beta}(p^{k_p+j},\chi)f_{\gamma,\delta}(p^{j},\overline{\chi})}{p^{j(1+s)}}\Bigg)$$

$$=\Bigg(\sum_{n=1}^{\infty}\frac{f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(n,\overline{\chi})}{n^{1+s}}\Bigg)B_{\boldsymbol{\alpha},h,k}(s).$$

By using a method similar to that used on 1.3.3 of [59] or of that given in section 1.6 of [5] we see the Dirichlet series in parentheses has Euler product

$$(183)\quad \prod_p\left(1-\frac{1}{p^{1+s+\alpha+\gamma}}\right)^{-1}\left(1-\frac{\overline{\chi(p)}}{p^{1+s+\alpha+\delta}}\right)^{-1}\left(1-\frac{\chi(p)}{p^{1+s+\beta+\gamma}}\right)^{-1} \times\left(1-\frac{|\chi(p)|^2}{p^{1+s+\beta+\delta}}\right)^{-1}\left(1-\frac{|\chi(p)|^2}{p^{2+2s+\alpha+\beta+\gamma+\delta}}\right)$$

which equals $A_{\boldsymbol{\alpha}}(s)$. Hence

$$(184)\quad I_D^{(1)}(h,k)=\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}w(t)\frac{1}{2\pi i}\int_{(\epsilon)}\frac{G(s)}{s}\left(\frac{\pi^2hk}{q}\right)^{-s}g_{\boldsymbol{\alpha}}(s,t)Z_{\boldsymbol{\alpha},h,k}(2s)dsdt.$$

On applying the approximation (173) we have

$$(185)\quad I_D^{(1)}(h,k)=\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}w(t)\frac{1}{2\pi i}\int_{(\epsilon)}\frac{G(s)}{s}\left(\frac{qt^2}{4\pi^2hk}\right)^{s}Z_{\boldsymbol{\alpha},h,k}(2s)dsdt+O\left(\frac{(qT)^{\epsilon}}{\sqrt{hk}}\right)$$

where we have used the estimate

$$(186)\quad \int_{-\infty}^{\infty}t^{-1+\epsilon}w(t)dt\ll\int_{T/2}^{4T}t^{-1+\epsilon}|w(t)|dt\ll T^{\epsilon}.$$

Since $G(s)$ is of rapid decay and Z is only of moderate growth we may shift the line of integration to $\Re(s)=-1/4+\epsilon$ (with the shift parameters small) and encounter poles at $s = 0$, $2s = -\alpha - \gamma$ and $2s = -\beta - \delta$. Similarly to before we may estimate the integral on this new line as

$$\ll q^{-1/4+\epsilon}(hk)^{-1/4-\epsilon}T^{\epsilon}\int_{T/2}^{4T} t^{-1/2+\epsilon}|w(t)|dt \ll q^{\epsilon}(qhk)^{-1/4}T^{1/2+\epsilon}. \tag{187}$$

The pole at $s = 0$ gives

$$\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} Z_{\boldsymbol{\alpha},h,k}(0)w(t)dt$$

whilst the two poles at $2s = -\alpha - \gamma$ and $2s = -\beta - \delta$ give rise to $J^{(1)}_{\alpha,\gamma}$ and $J^{(1)}_{\beta,\delta}$ respectively. A similar argument follows for $I^{(2)}_D$. □

We let $I^{(1)}_O(h,k)$ (resp. $I^{(2)}_O(h,k)$) denote the sum of terms in $I^{(1)}(h,k)$ (resp. $I^{(2)}(h,k)$) for which $hm \neq kn$. The goal of the remainder of this chapter is to prove the following.

Proposition 2.2.2. *We have*

$$\begin{aligned} &I^{(1)}_O(h,k) + I^{(2)}_O(h,k) = \\ &\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} w(t)\Bigg(Z_{-\gamma,\beta,-\alpha,\delta,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} + Z_{\alpha,-\delta,\gamma,-\beta,h,k}(0)\left(\frac{qt}{2\pi}\right)^{-\beta-\delta} \\ &+ \mathbf{1}_{q|h}\frac{\chi(k)G(\overline{\chi})}{q^{\delta}}Z'_{-\delta,\beta,\gamma,-\alpha,\frac{h}{q},k}(0,\chi)\left(\frac{t}{2\pi}\right)^{-\alpha-\delta} + \mathbf{1}_{q|k}\frac{\overline{\chi}(h)\overline{G(\overline{\chi})}}{q^{\beta}} \\ &\times Z'_{\alpha,-\gamma,-\beta,\delta,h,\frac{k}{q}}(0,\overline{\chi})\left(\frac{t}{2\pi}\right)^{-\beta-\gamma}\Bigg)dt - J^{(1)}_{\alpha,\gamma} - J^{(1)}_{\beta,\delta} - J^{(2)}_{\alpha,\gamma} - J^{(2)}_{\beta,\delta} + E(T). \end{aligned} \tag{188}$$

where $E(T)$ is the error term of Theorem 1.

By combining this with Proposition 2.2.1 the J terms cancel and we get Theorem 1. To prove Proposition 2.2.2 we first prepare $I^{(1)}_O(h,k)$ for an application of the methods in [**16**]. The results of this application are then given in section 2.4 where we see that $I^{(1)}_O(h,k)$ can be expressed as a sum of four main terms. These terms are then manipulated in sections 2.6 and 2.7 and we find that by combining them with their counterparts in $I^{(2)}_O(h,k)$ we get a cancellation. The remaining terms are, in fact, undercover versions of the terms in Proposition 2.2.2 and the rest of the chapter is devoted to unveiling them.

2.3. The Off-Diagonals: A Dyadic Partition of Unity

As mentioned previously, we may restrict the sums over m, n in (175) and (176) so that $mn \leq T^{2+\epsilon}$ whilst incorporating a negligible error term. Now, let

$$F^*(x,y) = \frac{1}{2\pi i}\int_{(\varepsilon)} \frac{G(s)}{s}\left(\frac{\pi^2 xy}{hkq}\right)^{-s} \frac{1}{T}\int_{-\infty}^{\infty}\left(\frac{x}{y}\right)^{-it} g_{\boldsymbol{\alpha}}(s,t)w(t)dtds \tag{189}$$

and let

$$I_O^{(1)}(h,k) = T\sum_{\substack{m,n\\ hm\neq kn}} \frac{f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(m,\overline{\chi})}{(mn)^{1/2}}F^*(hm,kn) \tag{190}$$

so that

$$I^{(1)}(h,k) = I_D^{(1)}(h,k) + I_O^{(1)}(h,k).$$

We wish to apply the results of [**16**] to $I_O^{(1)}(h,k)$. To do this we follow [**23**] and first apply a dyadic partition of unity for the sums over m and n.

Let $W_0(x)$ be a smooth, nonnegative function with support in $[1,2]$ such that

$$\sum_M W_0(x/M) = 1,$$

where M runs over a sequence of real numbers with $\#\{M : M \leq X\} \ll \log X$. Let

(191)
$$I_{M,N}(h,k) = \frac{T}{\sqrt{MN}}\sum_{\substack{m,n\\ hm\neq kn}} f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(m,\overline{\chi})W\left(\frac{m}{M}\right)W\left(\frac{n}{N}\right)F^*(hm,kn)$$

where

$$W(x) = x^{-1/2}W_0(x).$$

Then

$$\sum_{M,N} I_{M,N}(h,k) = I_O^{(1)}(h,k) \tag{192}$$

and note that by the first remark of this section we may take $MN \le T^{2+\epsilon}$.

We can show that the main contribution to $I_{M,N}(h,k)$ comes from the terms which are close to the diagonal. To demonstrate this we can use integration by parts on the innermost integral of $F^*(x,y)$ and thereby take advantage of the oscillatory factor $(x/y)^{-it}$. This gives

$$\frac{1}{T}\int_{-\infty}^{\infty}(x/y)^{-it}g(s,t)w(t)dt \ll \frac{1}{T|\log(x/y)|^j}\int_{-\infty}^{\infty}\left|g^{(0,j)}(s,t)w^{(j)}(t)\right|dt \ll \frac{P_j(|s|)T^{2\Re(s)}}{T_0^j|\log(x/y)|^j}. \tag{193}$$

for any $j = 0,1,2,\dots$ where P_j is some polynomial. If $|\log(x/y)| \ge T_0^{-1+\epsilon}$ then the above bound can be made arbitrarily small by taking j large. Hence, on writing $hm - kn = r$, we get

$$\begin{aligned} I_{M,N}(h,k) =& \frac{T}{\sqrt{MN}}\sum_{r\neq 0}\sum_{\substack{hm-kn=r\\ |\log(hm/kn)|\ll T_0^{-1+\epsilon}}} f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(m,\overline{\chi}) \\ &\times W\left(\frac{m}{M}\right)W\left(\frac{n}{N}\right)F^*(hm,kn) + O(T^{-A}). \end{aligned} \tag{194}$$

Now, $r/kn \asymp |\log(1+r/kn)| \ll T_0^{-1+\epsilon}$ and therefore $r \ll knT_0^{-1+\epsilon} \ll \sqrt{hkMN}T_0^{-1}T^\epsilon$ since $n \asymp N$ and $hM \asymp kN$. Summarising;

Proposition 2.3.1. *We have*

$$\begin{aligned} I_{M,N}(h,k) =& \frac{T}{\sqrt{MN}}\sum_{0<|r|\ll \frac{\sqrt{hkMN}}{T_0}T^\varepsilon}\sum_{hm-kn=r} f_{\alpha,\beta}(n,\chi)f_{\gamma,\delta}(m,\overline{\chi})F(hm,kn) \\ &+ O(T^{-A}) \end{aligned} \tag{195}$$

where

$$\begin{aligned} F(x,y) =& W\left(\frac{x}{hM}\right)W\left(\frac{y}{kN}\right)\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\left(\frac{\pi^2 xy}{hkq}\right)^{-s} \\ &\times \frac{1}{T}\int_{-\infty}^{\infty}\left(1+\frac{r}{y}\right)^{-it}g(s,t)w(t)dtds. \end{aligned} \tag{196}$$

2.4. The Delta Method

One can show after a short computation that

$$x^i y^j F^{(i,j)}(x,y) \ll \left(1+\frac{x}{X}\right)^{-1}\left(1+\frac{y}{Y}\right)^{-1} P^{i+j}, \quad i,j\geq 0 \tag{197}$$

where $X=hM$, $Y=kN$ and $P=(hkq)^\epsilon T^{1+\epsilon}/T_0$. It should also be noted that F has compact support in the box $[X,2X]\times[Y,2Y]$ due to the support conditions on W. Now, let

$$D_F(h,k,r) = \sum_{\substack{m,n \\ hm-kn=r}} f_{\alpha,\beta}(m,\chi) f_{\gamma,\delta}(n,\overline{\chi}) F(hm,kn). \tag{198}$$

The above observations on F make this sum well suited to an application of the main result of [16]. Following their method we will show that

$$D_F(h,k,r) = \sum_{i,j=1}^{2} \frac{1}{h^{1-a_i}k^{1-b_j}} S_{ij}(h,k,r) \int_{\max(0,r)}^{\infty} x^{-a_i}(x-r)^{-b_j} F(x,x-r)dx$$
$$+ (\text{Error term})$$

where the a_i, b_j are particular shifts and $S_{ij}(h,k,q,r)$ are certain infinite sums.

Before applying the δ-method we first attach to $F(x,y)$ a redundant factor $\phi(x-y-r)$ where $\phi(u)$ is a smooth function supported on $(-U,U)$ such that $\phi(0)=1$ and $\phi^{(i)} \ll U^{-i}$. U will be chosen optimally later. Denote the new function by $F_\phi(x,y) = F(x,y)\phi(x-y-r)$ and note $D_F(h,k,r) = D_{F_\phi}(h,k,r)$. The derivatives of the new function satisfy

$$F_\phi^{(i,j)} \ll \left(\frac{1}{U}+\frac{P}{X}\right)^i \left(\frac{1}{U}+\frac{P}{Y}\right)^j \ll U^{-i-j} \tag{199}$$

provided that $U \leq P^{-1}\min(X,Y)$ which we henceforth assume.

2.4.1. Setting up the δ-method. Throughout this section we closely follow [16]. Let ω be a smooth, even function of compact support in $[\Omega, 2\Omega]$ such that

$$\sum_{d\geq 1} \omega(d) = 1, \ \omega^{(i)} \ll \Omega^{-i-1}, \ i \geq 0. \tag{200}$$

Then the δ symbol, which is equal to 1 for $n = 0$ and 0 for $n \in \mathbb{Z}\backslash\{0\}$, can be given in terms of Ramanujan sums

$$\delta(n) = \sum_{d=1}^{\infty} \Delta_d(n) \sum_{\substack{c=1 \\ (c,d)=1}}^{d} e_d(cn) \tag{201}$$

where

$$\Delta_d(u) = \sum_{m=1}^{\infty} (dm)^{-1} \left(\omega(dm) - \omega\left(\frac{u}{dm}\right)\right). \tag{202}$$

The following Lemma is taken from [16]. It shows that $\Delta_d(u)$ gives a good approximation to the Dirac distribution.

Lemma 2.4.1. *Let $f \in C_0^\infty(\mathbb{R})$ and let $j \geq 1$. Then*

$$\int_{-\infty}^{\infty} f(x)\Delta_d(x)dx = f(0) + O\left(\Omega^{-1}d^j \int_{-\infty}^{\infty} (\Omega^{-j}|f(x)| + \Omega^j|f^{(j)}(x)|)dx\right). \tag{203}$$

Note the derivatives of $\Delta_d(u)$ satisfy

$$\Delta_d^{(i)}(u) \ll (d\Omega)^{-i-1}, \ i > 0. \tag{204}$$

We also have

$$\Delta_d(u) \ll \frac{1}{|u| + d\Omega} + \frac{1}{d\Omega + \Omega^2} \tag{205}$$

(see Lemma 2 of [16]). Now, choosing $\Omega = U^{1/2}$ we see that $\Delta_d(u)$ vanishes if $|u| \leq U$ and $d \geq 2\Omega$. Therefore, using (201),

(206)
$$D_F(h,k,r) = \sum_{m,n} f_{\alpha,\beta}(m,\chi) f_{\gamma,\delta}(n,\overline{\chi}) F_\phi(hm,kn)\delta(hm-kn-r)$$
$$= \sum_{d<2\Omega} \sum_{\substack{c=1 \\ (c,d)=1}}^{d} e_d(-cr) \sum_{m,n} f_{\alpha,\beta}(m,\chi) f_{\gamma,\delta}(n,\overline{\chi}) e_d(chm) e_d(-ckn) F^\sharp(m,n)$$

where $F^\sharp(x,y) = F_\phi(hx,ky)\Delta_d(hx-ky-r)$. We now evaluate the innermost sum using standard techniques.

2.4.2. A Voronoi Summation Formula. We require a formula for the sum

$$\sum_{n=1}^{\infty} f_{\alpha,\beta}(n,\chi) e_d(cn) g(n)$$

where $(c,d) = 1$ and g is a smooth function of rapid decay. For this we apply Mellin transforms to the Dirichlet series

$$E_{\alpha,\beta}(s, c/d, \chi) = \sum_{n=1}^{\infty} \frac{f_{\alpha,\beta}(n,\chi) e_d(cn)}{n^s}. \tag{207}$$

The analytic behaviour of $E_{\alpha,\beta}(s,c/d,\chi)$ is described in [18], [41] albeit without the shifts. Following the methods in these papers one can show that $E_{\alpha,\beta}(s,c/d,\chi)$ admits a meromorphic continuation with at most one pole. It also possesses a functional equation, although not of the usual type. Incorporating the shifts into the arguments of these papers requires little extra effort, in fact, the proofs read almost verbatim. We therefore choose to omit our proofs. The following is adapted from Lemma 1 of [41].

Lemma 2.4.2. *Let $(c,d) = 1$. If $1 < \varrho := (d,q) < q$ then $E_{\alpha,\beta}(s,c/d,\chi)$ is entire. If $\varrho = 1$, then $E_{\alpha,\beta}(s,c/d,\chi)$ is meromorphic with a single simple pole at $s = 1-\alpha$. If $\varrho = q$, then it is meromorphic with a single simple pole at $s = 1-\beta$.*

The residues are given by

$$\operatorname*{Res}_{s=z} E_{\alpha,\beta}(s,c/d,\chi) = \begin{cases} \frac{\chi(d)L(1-\alpha+\beta,\chi)}{d^{1-\alpha+\beta}}, & \text{if } \varrho = 1,\ z = 1-\alpha \\ \frac{\overline{\chi}(c)G(\chi)L(1+\alpha-\beta,\overline{\chi})}{q^{\beta-\alpha}d^{1+\alpha-\beta}}, & \text{if } \varrho = q,\ z = 1-\beta \end{cases}. \tag{208}$$

For the functional equation of $E_{\alpha,\beta}(s,c/d,\chi)$ we follow the methods of Furuya given in [**18**]. The functional equation is written in terms of the Dirichlet series

$$\tilde{E}_{\alpha,\beta}(s,c/d,\chi) = \sum_{n=1}^{\infty} \frac{\sigma_{\alpha,\beta}(n,c/d,\chi)}{n^s} \tag{209}$$

where

$$\sigma_{\alpha,\beta}(n,c/d,\chi) = \sum_{uv=n} u^{-\alpha}v^{-\beta} \sum_{\substack{b=1 \\ b\equiv cu \,(\mathrm{mod}\, d)}}^{d_1} \chi(b)e_{d_1}(bv) \tag{210}$$

and $d_1 = dq/\varrho$, the least common multiple of d and q. Following the proof of Lemma 3 of [**18**] we get

Lemma 2.4.3. *Let*

$$H(s) = H_{\alpha,\beta}(s,d,q) = \frac{(2\pi)^{2s-2+\alpha+\beta}}{d^{2s-1+\alpha+\beta}} \left(\frac{\varrho}{q}\right)^{s+\beta} \Gamma(1-s-\alpha)\Gamma(1-s-\beta) \tag{211}$$

and let

$$\theta(s) = \theta_{\alpha,\beta,\chi}(s) = e^{\pi i\left(s+\frac{\alpha+\beta}{2}\right)} + \chi(-1)e^{-\pi i\left(s+\frac{\alpha+\beta}{2}\right)}. \tag{212}$$

Then

$$E_{\alpha,\beta}(s,c/d,\chi) = H(s)\Big(\theta(-\beta)\tilde{E}_{-\alpha,-\beta}(1-s,\overline{c}/d,\chi) - \theta(s)\tilde{E}_{-\alpha,-\beta}(1-s,-\overline{c}/d,\chi)\Big) \tag{213}$$

where $\overline{c}$ *is the unique solution to the equation* $c\overline{c} \equiv 1 \bmod d$.

A trivial estimate gives that $\sigma_{\alpha,\beta}(n, c/d, \chi) \ll q f_{\Re\alpha,\Re\beta}(n, |\chi|)/\varrho$ and so $\tilde{E}_{\alpha,\beta}(s, c/d, \chi)$ converges absolutely for $\Re s > 1 - \min(\Re\alpha, \Re\beta)$. Also, By Stirling's formula (23) we see

$$\theta(s)H(s) \ll |t|^{1-2\sigma-\Re(\alpha)-\Re(\beta)}. \tag{214}$$

The functional equation therefore implies that $E_{\alpha,\beta}(s, c/d, \chi)$ is polynomially bounded for $\sigma < 0$. By the Phragmén-Lindelöf principle we see that it is polynomially bounded everywhere on $\mathbb{C}$ for which $|t| \geq 1$.

We now have enough to prove the Voronoi summation formula via Mellin transforms; an alternative method can be found in [**29**].

Proposition 2.4.4. *Let $g(x)$ be a smooth, compactly supported function on $\mathbb{R}^+$ and let $(c, d) = 1$. Also, let z be equal to either $1-\alpha$ or $1-\beta$ depending on whether $\varrho = 1$ or $\varrho = q$ respectively and let z be arbitrary in any other case. Then*

$$\begin{aligned}\sum_{n\geq 1} f_{\alpha,\beta}(n, \chi) e_d(cn) g(n) = \left(\operatorname*{Res}_{s=z} E_{\alpha,\beta}(s, c/d, \chi)\right) \int_0^\infty x^{z-1} g(x) dx \\ + \sum_{+-}\sum_{n\geq 1} \sigma_{-\alpha,-\beta}\left(n, \pm\frac{\overline{c}}{d}, \chi\right) g^{\pm}(n)\end{aligned} \tag{215}$$

where

$$g^+(y) = \frac{2\theta(-\beta)}{d}\left(\frac{\varrho}{q}\right)^{1-\frac{\alpha-\beta}{2}} \int_0^\infty g(x) K_{\beta-\alpha}\left(\frac{4\pi\sqrt{\varrho x y/q}}{d}\right)(xy)^{-\frac{\alpha+\beta}{2}} dx \tag{216}$$

and

$$g^-(y) = -\frac{2\pi}{d}\left(\frac{\varrho}{q}\right)^{1-\frac{\alpha-\beta}{2}} \int_0^\infty g(x) B_{\alpha-\beta}\left(\frac{4\pi\sqrt{\varrho x y/q}}{d}\right)(xy)^{-\frac{\alpha+\beta}{2}} dx. \tag{217}$$

Here, $K_\nu(z)$ is the usual Bessel function and $B_\nu(z)$ is defined as

$$B_\nu(z) = \begin{cases} \cos(\frac{\pi}{2}\nu) Y_\nu(z) + \sin(\frac{\pi}{2}\nu) J_\nu(z), & \text{if } \chi \text{ is even} \\ i\cos(\frac{\pi}{2}\nu) J_\nu(z) - i\sin(\frac{\pi}{2}\nu) Y_\nu(z), & \text{if } \chi \text{ is odd} \end{cases} \tag{218}$$

where $Y_\nu(z)$, $J_\nu(z)$ are again the usual Bessel functions.

Proof. For simplicity we assume the g is in Schwartz space and that $g(0) = 0$. The general case then follows on taking smooth approximations. We let G denote the Mellin transform of g, that is,

$$G(s) = \int_0^\infty x^{s-1} g(x) dx \tag{219}$$

and note that it is holomorphic in the region $\Re s > -2$ except for a simple pole at $s = 0$ with residue $g(0) = 0$. Applying Mellin inversion and then shifting contours we have

$$\begin{aligned}\sum_{n\geq 1} f_{\alpha,\beta}(n,\chi) e_d(cn) g(n) =& \frac{1}{2\pi i}\int_{(2)} E_{\alpha,\beta}(s, c/d, \chi) G(s) ds \\ =& \left(\operatorname*{Res}_{s=z} E_{\alpha,\beta}(s, c/d, \chi) G(s)\right) + \frac{1}{2\pi i}\int_{(-\frac{1}{4})} E(s) G(s) ds.\end{aligned} \tag{220}$$

Note that interchange of summation and integration in the first line is justified by the absolute convergence of E and that the contour shift is also valid since $G(s)$ decays rapidly whilst $E(s)$ increases at most polynomially as $|\Im s| \to \infty$. Writing $\tilde{E}^{\pm}(s) = \tilde{E}_{-\alpha,-\beta}(s, \pm\overline{c}/d, \chi)$ for short and applying the functional equation (213) gives

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(-\frac{1}{4})} E(s) G(s) ds \\ &\quad = \frac{1}{2\pi i}\int_{(-\frac{1}{4})} H(s)\Big[\theta(-\beta)\tilde{E}^{+}(1-s) - \theta(s)\tilde{E}^{-}(1-s)\Big] G(s) ds \\ &\quad = \frac{1}{2\pi i}\int_{(\frac{5}{4})} H(1-s)\Big[\theta(-\beta)\tilde{E}^{+}(s) - \theta(1-s)\tilde{E}^{-}(s)\Big] G(1-s) ds \\ &\quad = \frac{(2\pi)^{\alpha+\beta}}{d^{1+\alpha+\beta}}\left(\frac{\varrho}{q}\right)^{1+\beta} \sum_{+-}\sum_{n\geq 1} \sigma_{-\alpha,-\beta}(n, \pm\frac{\overline{c}}{d}, \chi) G^{\pm}\left(\frac{4\pi^2 \varrho n}{d^2 q}\right)\end{aligned} \tag{221}$$

where

$$G^{+}(y) = \theta(-\beta)\frac{1}{2\pi i}\int_{(\frac{5}{4})} \Gamma(s-\alpha)\Gamma(s-\beta) G(1-s) y^{-s} ds \tag{222}$$

and

$$G^{-}(y) = -\frac{1}{2\pi i}\int_{(\frac{5}{4})} \theta(1-s)\Gamma(s-\alpha)\Gamma(s-\beta)G(1-s)y^{-s}ds. \tag{223}$$

We note that since the shifts are small, $\tilde{E}_{-\alpha,-\beta}(s)$ is absolutely convergent on the line $\Re s = 5/4$ and hence the interchange of summation and integration is legal. By (219) and the fact that g is Schwartz we have

$$G^{+}(y) = \theta(-\beta)\int_0^\infty g(x)\left(\frac{1}{2\pi i}\int_{(\frac{5}{4})}\Gamma(s-\alpha)\Gamma(s-\beta)(xy)^{-s}ds\right)dx \tag{224}$$

and similarly for $G^{-}(y)$. The result now follows on applying the formulae

$$2K_\nu(z) = \frac{1}{2\pi i}\int_{(c)}\Gamma(s)\Gamma(s-\nu)\left(\frac{z}{2}\right)^{\nu-2s}ds, \tag{225}$$

$$-\pi Y_\nu(z) = \frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)\cos\frac{\pi}{2}(s-\nu)\left(\frac{z}{2}\right)^{-2s}ds, \tag{226}$$

$$\pi J_\nu(z) = \frac{1}{2\pi i}\int_{(c)}\Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu}{2}\right)\sin\frac{\pi}{2}(s-\nu)\left(\frac{z}{2}\right)^{-2s}ds, \tag{227}$$

along with the obvious substitutions for s. □

Before applying this formula we compile some results on $\sigma_{\alpha,\beta}(s,c/d,\chi)$ which will be used later. First, It should be noted that $\sigma_{\alpha,\beta}(s,c/d,\chi)$ is quite similar to $f_{\alpha,\beta}(n,\chi)$ when $\varrho = 1$ or $\varrho = q$. Indeed, we have

$$\sigma_{\alpha,\beta}(n,c/d,\chi) = \begin{cases} \chi(d)e_d(c\overline{q}n)G(\chi)f_{\alpha,\beta}(n,\overline{\chi}) & \text{if } \varrho = 1, \\ \chi(c)e_d(cn)f_{\beta,\alpha}(n,\chi) & \text{if } \varrho = q. \end{cases} \tag{228}$$

The case $\varrho = q$ is easily seen since in this instance $d_1 = d$. This implies a unique solution (mod d_1) to the equation $b \equiv cu \,(\operatorname{mod} d)$. Consequently,

$$\begin{aligned}\sigma_{\alpha,\beta}(n, c/d, \chi) &= \sum_{uv=n} u^{-\alpha} v^{-\beta} \sum_{\substack{b=1 \\ b \equiv cu (\operatorname{mod} d)}}^{d_1} \chi(b) e_{d_1}(bv) \\ &= \sum_{uv=n} u^{-\alpha} v^{-\beta} \chi(cu) e_d(buv) \\ &= \chi(c) e_d(cn) f_{\beta,\alpha}(n, \chi)\end{aligned} \tag{229}$$

For the case $\varrho = 1$ we have the following method which also gives insight into the cases $1 < \varrho < q$. Let $b = j + ql$ where $1 \leq j \leq q$ and $0 \leq l \leq d/\varrho - 1$. Then

$$\sigma_{\alpha,\beta}(n, c/d, \chi) = \sum_{uv=n} u^{-\alpha} v^{-\beta} \sum_{j=1}^{q} \chi(j) e_{d_1}(jv) \sum_{\substack{l=0 \\ ql \equiv cu - j (d)}}^{d/\varrho - 1} e_{d/\varrho}(lv). \tag{230}$$

If we now put $\varrho = 1$ then l is uniquely determined (mod d) by $l \equiv \overline{q}(cu - j)$ (mod d). Therefore, in this case

$$\begin{aligned}\sigma_{\alpha,\beta}(n, c/d, \chi) &= \sum_{uv=n} u^{-\alpha} v^{-\beta} \sum_{j=1}^{q} \chi(j) e_{dq}(jv) e_d(\overline{q}(cu - j)v) \\ &= e_d(c\overline{q}n) \sum_{uv=n} u^{-\alpha} v^{-\beta} \sum_{j=1}^{q} \chi(j) e_q(-rjv) \\ &= e_d(c\overline{q}n) \sum_{uv=n} u^{-\alpha} v^{-\beta} G(-rv, \chi)\end{aligned} \tag{231}$$

where r is the integer such that $q\overline{q} = 1 + rd$. Formula (228) for $\varrho = 1$ now follows on noting $G(-rv, \chi) = \overline{\chi}(-rv) G(\chi)$ and $\overline{\chi}(-r) = \chi(d)$.

In the remaining cases $1 < \varrho < q$ we return to formula (230) and write $Q = q/\varrho$, $D = d/\varrho$. Now, a necessary condition for the existence of a solution to the congruence $ql \equiv cu - j$ (mod d) is that $\varrho | cu - j$. In this case l is uniquely determined (mod D) by $l \equiv \overline{Q}(cu-j)/\varrho$ (mod D). Therefore

$$\begin{aligned} &\sum_{j=1}^{q}\chi(j)e_{d_1}(jv)\sum_{\substack{l=0\\ ql\equiv cu-j(d)}}^{d/\varrho-1} e_{d/\varrho}(lv)\\ &=\frac{1}{\varrho}\sum_{j=1}^{q}\chi(j)e_{d_1}(jv)\sum_{m=1}^{\varrho}e_\varrho(-m(cu-j))e_d(\overline{Q}(cu-j)v) \qquad (232)\\ &=\frac{1}{\varrho}e_d(c\overline{Q}uv)\sum_{m=1}^{\varrho}e_\delta(-mcu)\sum_{j=1}^{q}\chi(j)e_{d_1}\left(j\left(mQD+v(1-Q\overline{Q})\right)\right).\end{aligned}$$

Therefore, for $1<\varrho<q$ we have

$$\sigma_{\alpha,\beta}(n,c/d,\chi)=\frac{1}{\varrho}e_d(nc\overline{Q})G(\chi)\sum_{uv=n}u^{-\alpha}v^{-\beta}\sum_{m=1}^{\varrho}\overline{\chi}(mQ-vw)e_\varrho(-mcu) \qquad (233)$$

where w is defined by the equation $Q\overline{Q}=1+wD$.

2.4.3. Applying Voronoi Summation. Recall that the formula for $D_F(h,k,r)$ was given by

$$D_F(h,k,r)=\sum_{d<2\Omega}\sum_{\substack{c=1\\(c,d)=1}}^{d}e_d(-cr)\sum_{m,n}f_{\alpha,\beta}(m,\chi)f_{\gamma,\delta}(n,\overline{\chi})e_d(chm)e_d(-ckn)F^\sharp(m,n).$$

We first write the fractions ch/d, ck/d in reduced form i.e. as $ch_{(d)}/d_{(h)}$, $ck_{(d)}/d_{(k)}$ where $m_{(n)}=m/(m,n)$. This gives $(ch_{(d)},d_{(h)})=1$, $(ck_{(d)},d_{(k)})=1$ and so we may apply Proposition 2.4.4 to the two innermost sums. Note that the form of the innermost sums will be dependent on $(d_{(h)},q)$ and $(d_{(k)},q)$. So for example, if in the outer sum d is such that $(d_{(h)},q)=1$ and $(d_{(k)},q)=1$, then the inner sum over m takes the form

$$\begin{aligned}&\left(\operatorname*{Res}_{s=1-\alpha}E_{\alpha,\beta}(s,ch_{(d)}/d_{(h)},\chi)\right)\int_0^\infty x^{-\alpha}F^\sharp(x,n)dx+\sum_{+-}\cdots\\ =&\frac{\chi(d_{(h)})L(1-\alpha+\beta,\chi)}{\left(d_{(h)}\right)^{1-\alpha+\beta}}\int_0^\infty x^{-\alpha}F^\sharp(x,n)dx+\sum_{+-}\cdots.\end{aligned}$$

Computing the sum over n in a similar fashion we see that the two innermost sums are given by

$$\frac{\chi(d_{(h)})\overline{\chi}(d_{(k)})L(1-\alpha+\beta,\chi)L(1-\gamma+\delta,\overline{\chi})}{\left(d_{(h)}\right)^{1-\alpha+\beta}\left(d_{(k)}\right)^{1-\gamma+\delta}}\iint_0^\infty x^{-\alpha}y^{-\gamma}F^\sharp(x,y)dxdy+\cdots$$

where the dots represent the three remaining terms (considering $\sum_{+-}\cdots$ as a single term).

In order to gain the full expression for $D_F(h,k,r)$ it should be clear that we must distinguish the d in the outer sum. Accordingly, we partition the positive integers into 9 sets P_{ij}, $1\le i,j\le 3$, subject to the following conditions:

$$d\in\begin{cases}P_{1j} & \text{if } (d_{(h)},q)=1,\\ P_{2j} & \text{if } (d_{(h)},q)=q,\\ P_{3j} & \text{if } 1<(d_{(h)},q)<q\end{cases},\quad d\in\begin{cases}P_{i1} & \text{if } (d_{(k)},q)=1,\\ P_{i2} & \text{if } (d_{(k)},q)=q,\\ P_{i3} & \text{if } 1<(d_{(k)},q)<q\end{cases}. \tag{234}$$

We will later give a description of these sets but for the meanwhile we only make use of the observation that if either $i=2$ or $j=2$ then the elements of the set P_{ij} are divisible by q. This can be seen by writing h, k and $d\in P_{ij}$ as their respective q-parts times non q-parts and then solving the given conditions (the q-part of an integer n is defined as $\prod_{p|q}p^{n_p}$). Let

$$R_i=\begin{cases}\operatorname*{Res}_{s=1-\alpha}E_{\alpha,\beta}(s,ch_{(d)}/d_{(h)},\chi) & \text{if } i=1,\\ \operatorname*{Res}_{s=1-\beta}E_{\alpha,\beta}(s,ch_{(d)}/d_{(h)},\chi) & \text{if } i=2,\\ 0 & \text{if } i=3\end{cases} \tag{235}$$

and

$$R'_j=\begin{cases}\operatorname*{Res}_{s=1-\gamma}E_{\gamma,\delta}(s,ck_{(d)}/d_{(k)},\overline{\chi}) & \text{if } j=1,\\ \operatorname*{Res}_{s=1-\delta}E_{\gamma,\delta}(s,ck_{(d)}/d_{(k)},\overline{\chi}) & \text{if } j=2,\\ 0 & \text{if } j=3.\end{cases} \tag{236}$$

Also, as is evident from Proposition 2.4.4 and Lemma 2.4.2 we must associate the shifts with i, j so let

$$a_i = \begin{cases} \alpha & \text{if } i = 1, \\ \beta & \text{if } i = 2, \\ 0 & \text{if } i = 3, \end{cases} \quad b_j = \begin{cases} \gamma & \text{if } j = 1, \\ \delta & \text{if } j = 2, \\ 0 & \text{if } j = 3. \end{cases} \tag{237}$$

Applying the Voronoi sum formula gives

$$\begin{aligned} D_F(h,k,r) = & \\ \sum_{i,j=1}^{3} \sum_{\substack{d<2\Omega \\ d\in P_{ij}}} \sum_{\substack{c=1 \\ (c,d)=1}}^{d} e_d(-cr) \Bigg\{ & R_i R_j' I_{ij} + \frac{1}{d_{(h)}} A R_j' \sum_{m=1}^{\infty} \frac{\sigma_{-\alpha,-\beta}(m, -\overline{ch_{(d)}}/d_{(h)}, \chi)}{m^{\frac{\alpha+\beta}{2}}} I_h(m) \\ & + \frac{1}{d_{(k)}} B R_i \sum_{n=1}^{\infty} \frac{\sigma_{-\gamma,-\delta}(n, \overline{ck_{(d)}}/d_{(k)}, \overline{\chi})}{n^{\frac{\gamma+\delta}{2}}} I_k(n) \\ + \frac{1}{d_{(h)} d_{(k)}} AB \sum_{m,n=1}^{\infty} & \frac{\sigma_{-\alpha,-\beta}(m, -\overline{ch_{(d)}}/d_{(h)}, \chi) \sigma_{-\gamma,-\delta}(n, \overline{ck_{(d)}}/d_{(k)}, \overline{\chi})}{m^{\frac{\alpha+\beta}{2}} n^{\frac{\gamma+\delta}{2}}} I_{hk}(m,n) + \cdots \Bigg\} \end{aligned} \tag{238}$$

where

$$A = \left(\frac{(d_{(h)}, q)}{q} \right)^{1-\frac{\alpha-\beta}{2}}, \quad B = \left(\frac{(d_{(k)}, q)}{q} \right)^{1-\frac{\gamma-\delta}{2}}$$

and

$$I_{ij} = \iint\limits_0^\infty x^{-a_i} y^{-b_j} F^\sharp(x,y)\,dxdy, \tag{239}$$

$$I_h(m) = -\,2\pi \iint\limits_0^\infty x^{-\frac{\alpha+\beta}{2}} y^{-b_j} B_{\alpha-\beta}\left(\frac{4\pi\sqrt{(d_{(h)},q)mx/q}}{d_{(h)}}\right) F^\sharp(x,y)\,dxdy, \tag{240}$$

$$I_k(n) = -\,2\pi \iint\limits_0^\infty x^{-a_i} y^{-\frac{\gamma+\delta}{2}} B_{\gamma-\delta}\left(\frac{4\pi\sqrt{(d_{(k)},q)ny/q}}{d_{(k)}}\right) F^\sharp(x,y)\,dxdy, \tag{241}$$

$$\begin{aligned} I_{hk}(m,n) =&4\pi^2 \iint\limits_0^\infty x^{-\frac{\alpha+\beta}{2}} y^{-\frac{\gamma+\delta}{2}} B_{\alpha-\beta}\left(\frac{4\pi\sqrt{(d_{(h)},q)mx/q}}{d_{(h)}}\right) \\ &\times B_{\gamma-\delta}\left(\frac{4\pi\sqrt{(d_{(k)},q)ny/q}}{d_{(k)}}\right) F^\sharp(x,y)\,dxdy. \end{aligned} \tag{242}$$

The additional terms of (238) are those involving the K_ν-Bessel function and can be estimated using the same method we use for the ones displayed.

2.4.4. Evaluating the Main Terms.

We have

$$\begin{aligned} I_{ij} =& \iint\limits_0^\infty x^{-a_i} y^{-b_j} F^\sharp(x,y)\,dxdy \\ =&\frac{1}{h^{1-a_i}k^{1-b_j}} \iint\limits_0^\infty x^{-a_i} y^{-b_j} F_\phi(x,y)\Delta_d(x-y-r)\,dxdy \\ =&\frac{1}{h^{1-a_i}k^{1-b_j}} \int_0^\infty \int_{r-x}^\infty x^{-a_i}(x-r+u)^{-b_j} F_\phi(x,x-r+u)\Delta_d(u)\,dudx. \end{aligned} \tag{243}$$

If $r-x \le 0$ then by Lemma 2.4.1 and (199) we see that the inner integral is equal to

$$x^{-a_i}(x-r)^{-b_j} F_\phi(x,x-r) + O((d/\Omega)^A), \quad A \ge 1.$$

If $r - x > 0$ then the integral is $\ll (d/\Omega)^A$ for some $A \geq 1$. Assuming $d \leq \Omega^{1-\epsilon}$ and on taking A large we get

$$I_{ij} = \frac{1}{h^{1-a_i}k^{1-b_j}} \int_{\max(0,r)}^{\infty} x^{-a_i}(x-r)^{-b_j} F(x, x-r) dx + O(\Omega^{-B}) \tag{244}$$

where B is an arbitrary positive constant. By formula (259) below we have $hkI_{ij} \ll XY(X+Y)^{-1}\log\Omega$ valid for all d. Also, by formula (312) below we have the bound

$$\sum_{\substack{c=1\\(c,d)=1}}^{d} \chi(c) e_d(-cr) \ll q^{1/2}(r, d).$$

On applying these in the sum over $d \geq \Omega^{1-\epsilon}$ we get

$$\begin{aligned}\sum_{i,j=1}^{3} \sum_{\substack{d<2\Omega\\d\in P_{ij}}} \sum_{\substack{c=1\\(c,d)=1}}^{d} & e_d(-cr) R_i R'_j I_{ij} \\ = \sum_{i,j=1}^{2} & \frac{1}{h^{1-a_i}k^{1-b_j}} S_{ij}(h,k,r) \int_{\max(0,r)}^{\infty} x^{-a_i}(x-r)^{-b_j} F(x, x-r) dx \\ & + O((hk)^{-1} q^{1/2} XY(X+Y)^{-1}\Omega^{-1+\epsilon})\end{aligned} \tag{245}$$

where

$$S_{ij}(h,k,r) = \sum_{d\in P_{ij}} \sum_{\substack{c=1\\(c,d)=1}}^{d} e_d(-cr) R_i R'_j. \tag{246}$$

These last terms are given explicitly in Proposition 2.5.1 below.

2.4.5. Estimating the Error Terms. Throughout the following analysis we essentially ignore the shift parameters but since they are small this is of no great importance. We first estimate the sums over c. Pushing the sum over c through (238) we encounter sums of the form

$$S_1 = \sum_{\substack{c=1\\(c,d)=1}}^{d} e_d(-cr) R'_j \sigma_{-\alpha,-\beta}(m, -\overline{ch_{(d)}}/d_{(h)}, \chi), \tag{247}$$

$$S_2 = \sum_{\substack{c=1 \\ (c,d)=1}}^{d} e_d(-cr)\sigma_{-\alpha,-\beta}(m, -\overline{ch_{(d)}}/d_{(h)}, \chi)\sigma_{-\gamma,-\beta}(n, \overline{ck_{(d)}}/d_{(k)}, \overline{\chi}) \tag{248}$$

Clearly, we also encounter a slight variant of (247) but this can be estimated using the same method as for the sum displayed. In estimating these we shall make use of Weil's bound for Kloosterman sums

$$S(r,t,d) = \sum_{\substack{c=1 \\ (c,d)=1}}^{d} e_d(cr + \overline{c}t) \ll (r,d)^{1/2} d^{1/2} \tau(d) \tag{249}$$

where τ is the usual divisor function. We will also need an estimate for sums of the form

$$S_\chi(r,t,d) = \sum_{\substack{c=1 \\ (c,d)=1}}^{d} \chi(c) e_d(cr + \overline{c}t). \tag{250}$$

These are similar to Salie sums ([**27**],[**51**]) the difference being that $q|d$ whenever they appear. These are dealt with in [**41**] (see formula (16)) where Müller obtains

$$S_\chi(r,t,d) = S_{\overline{\chi}}(t,r,d) \ll q^{1/2}(r,d)^{1/2} d^{1/2} \tau(d). \tag{251}$$

By inspecting the cases $1 \leq i,j \leq 3$ and using (249), (251) along with the obvious variants of (208), (228), (233) we get

$$S_1 \ll \frac{1}{d_{(k)}} q^{3/2} |L(1,\chi)| (r,d)^{1/2} d^{1/2} \tau(d) \tau(m), \tag{252}$$

$$S_2 \ll q(r,d)^{1/2} d^{1/2} \tau(d) \tau(m) \tau(n). \tag{253}$$

We now estimate the integrals I_h, I_k, I_{hk}. From (204),(199) we have the bound

$$F^{\sharp(i,j)} \ll \frac{a^i b^j}{(d\Omega)^{i+j+1}}. \tag{254}$$

Using this along with the recurrence relations $(z^\nu Y_\nu(z))' = z^\nu Y_{\nu-1}(z)$, $(z^\nu J_\nu(z))' = z^\nu J_{\nu-1}(z)$ an integration by parts argument shows that these integrals are small unless

(255) $$m < \frac{hqX}{(d_{(h)},q)}\Omega^{-2+\epsilon},\ n < \frac{kqY}{(d_{(k)},q)}\Omega^{-2+\epsilon}.$$

For m, n in this range we estimate the integrals using the support conditions on F and the bounds $Y_{\alpha-\beta}(z), J_{\alpha-\beta}(z) \ll z^{-1/2}$ to give

(256) $$I_h(m) \ll \left(\frac{hqd^2}{n(d_{(h)},q)X}\right)^{1/4} \iint$$

(257) $$I_k(n) \ll \left(\frac{kqd^2}{n(d_{(k)},q)Y}\right)^{1/4} \iint$$

(258) $$I_{hk}(m,n) \ll \left(\frac{hkq^2d^4}{mn(d_{(h)},q)(d_{(k)},q)XY}\right)^{1/4} \iint$$

where

(259) $$\begin{aligned}\iint =& \iint_0^\infty |F_\phi(hx,ky)\Delta_d(hx-ky-r)|dxdy\\ =&(hk)^{-1}\iint_0^\infty |F_\phi(x,x-y-r)\Delta_d(y)|dxdy\\ \ll&(hk)^{-1}\frac{XY}{X+Y}\int_{-U}^{U}|\Delta_d(y)|dy\\ \ll&(hk)^{-1}\frac{XY}{X+Y}\log\Omega.\end{aligned}$$

Here, we have used the upper bounds (197) and (205) along with the support conditions on ϕ. Therefore, summing over m, n in the range (255) we have

(260) $$\sum_m \tau(m)|I_h(m)| \ll \frac{d^{1/2}q}{k(d_{(h)},q)}\frac{X^{3/2}Y}{X+Y}\Omega^{-3/2+\epsilon},$$

(261) $$\sum_n \tau(n)|I_k(n)| \ll \frac{d^{1/2}q}{h(d_{(k)},q)}\frac{XY^{3/2}}{X+Y}\Omega^{-3/2+\epsilon},$$

(262) $$\sum_{m,n} \tau(m)\tau(n)|I_{hk}(m,n)| \ll \frac{dq^2}{(d_{(h)},q)(d_{(k)},q)}\frac{(XY)^{3/2}}{X+Y}\Omega^{-3+\epsilon}.$$

Introducing these bounds into (238) along with (252), (253) and summing over d we get an error term of

$$q^{3/2}|L(1,\chi)|\frac{XY}{X+Y}\left(\frac{X^{1/2}}{k}+\frac{Y^{1/2}}{h}\right)\Omega^{-3/2+\epsilon}+q\frac{(XY)^{3/2}}{X+Y}\Omega^{-5/2+\epsilon} \tag{263}$$

where we have used $\sum_{d\leq x}(hk,d) \ll x^{1+\epsilon}$. We now take $U=\Omega^2=P^{-1}(X+Y)^{-1}XY$ and the above becomes

$$q^{3/2}|L(1,\chi)|P^{3/4}\left(\frac{XY}{X+Y}\right)^{1/4+\epsilon}\left(\frac{X^{1/2}}{k}+\frac{Y^{1/2}}{h}\right) + qP^{5/4}(XY)^{1/4+\epsilon}(X+Y)^{1/4}. \tag{264}$$

2.5. Combining Terms and integral manipulations

We now combine the main terms (245) and error term (264) in the formula for $D_F(h,k,r)$ whilst noting that $X \asymp Y \asymp \sqrt{hkMN}$.

Proposition 2.5.1. *Let P_{ij} and a_i, b_j be defined respectively by (234) and (237). Also, let*

$$\Psi_{ij}(h,k,r)=\frac{1}{h^{1-a_i}k^{1-b_j}}S_{ij}(h,k,r)\int_{\max(0,r)}^{\infty}x^{-a_i}(x-r)^{-b_j}F(x,x-r)dx. \tag{265}$$

Then

$$D_F(h,k,r) = \sum_{i,j=1}^{2}\Psi_{ij}(h,k,r)+E^{\flat}(T) \tag{266}$$

where

$$E^{\flat}(T) \ll T^{\epsilon}(hkMN)^{3/8+\epsilon}\Big(q^{3/2+\epsilon}|L(1,\chi)|(T/T_0)^{3/4}+q^{1+\epsilon}(T/T_0)^{5/4}\Big). \tag{267}$$

The terms $S_{ij}(h,k,r)$ *are given by*

$$S_{11}(h,k,r) = L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})\sum_{d\in P_{11}}\frac{c_d(r)\chi(d_{(h)})\overline{\chi}(d_{(k)})}{d_{(h)}^{1-\alpha+\beta}d_{(k)}^{1-\gamma+\delta}}, \tag{268}$$

$$S_{12}(h,k,r) = \frac{\chi(-1)G(\overline{\chi})L_{\alpha,\beta}(\chi)L_{-\gamma,-\delta}(\chi)}{q^{\delta-\gamma}}\sum_{d\in P_{12}}\frac{c_d(r,\chi)\chi(d_{(h)})\chi(k_{(d)})}{d_{(h)}^{1-\alpha+\beta}d_{(k)}^{1+\gamma-\delta}}, \tag{269}$$

$$S_{21}(h,k,r) = \frac{G(\chi)L_{-\alpha,-\beta}(\overline{\chi})L_{\gamma,\delta}(\overline{\chi})}{q^{\beta-\alpha}}\sum_{d\in P_{21}}\frac{c_d(r,\overline{\chi})\overline{\chi}(h_{(d)})\overline{\chi}(d_{(k)})}{d_{(h)}^{1+\alpha-\beta}d_{(k)}^{1-\gamma+\delta}}, \tag{270}$$

$$S_{22}(h,k,r) = \frac{L_{-\alpha,-\beta}(\overline{\chi})L_{-\gamma,-\delta}(\chi)}{q^{-1+\beta-\alpha+\delta-\gamma}}\sum_{d\in P_{22}}\frac{c_d(r,|\chi|^2)\overline{\chi}(h_{(d)})\chi(k_{(d)})}{d_{(h)}^{1+\alpha-\beta}d_{(k)}^{1+\gamma-\delta}}, \tag{271}$$

where $L_{x,y}(\chi) = L(1-x+y,\chi)$,

$$c_d(r,\chi) = \sum_{\substack{c=1\\(c,d)=1}}^{d}\chi(c)e_d(-cr) \tag{272}$$

and $c_d(r)$ *is the usual Ramanujan sum.*

We can now return to Proposition 2.3.1 and apply our formula for $D_F(h,k,r)$. Summing over r gives

$$\begin{aligned} I_{M,N} =& \frac{T}{\sqrt{MN}}\sum_{0\neq r\ll\frac{T^\epsilon\sqrt{hkMN}}{T_0}}\Bigg\{\sum_{i,j=1}^{2}\Psi_{ij}(h,k,r) \\ &+ O\Big(T^\epsilon(hkMN)^{3/8+\epsilon}\big(q^{3/2}|L(1,\chi)|(T/T_0)^{3/4}+q(T/T_0)^{5/4}\big)\Big)\Bigg\} \\ =& \frac{T}{\sqrt{MN}}\sum_{0\neq r\ll\frac{T^\epsilon\sqrt{hkMN}}{T_0}}\sum_{i,j=1}^{2}\Psi_{ij}(h,k,r) \\ &+ O\Big(T^\epsilon(MN)^{3/8+\epsilon}(hk)^{7/8+\epsilon}\big(q^{3/2}|L(1,\chi)|(T/T_0)^{7/4}+q(T/T_0)^{9/4}\big)\Big) \end{aligned} \tag{273}$$

By the usual integration by parts argument (formula (193)) we see that $\Psi_{ij}(h,k,r)$ is small for large r and hence we may freely extend the sum over $r\neq 0$. Summing over M,N we obtain

Proposition 2.5.2.

$$I_O^{(1)}(h,k) = \sum_{M,N}\sum_{r\neq 0}\sum_{i,j=1}^{2} \frac{T}{\sqrt{MN}}\Psi_{ij}(h,k,r) + E(T) \tag{274}$$

where

$$E(T) \ll T^{3/4+\epsilon}(hk)^{7/8+\epsilon}\big(q^{3/2+\epsilon}|L(1,\chi)|(T/T_0)^{7/4} + q^{1+\epsilon}(T/T_0)^{9/4}\big). \tag{275}$$

We now wish to manipulate the integrals in Ψ_{ij}. These are very similar to the integrals of section 6 in [**23**], the only important difference being that we have the presence of $c_d(r,\chi)$ which is not necessarily invariant under the transformation $r \mapsto -r$.

Let

$$I_{ij,\boldsymbol{\alpha}}^{(1)} = \sum_{M,N}\sum_{r\neq 0} \frac{T}{\sqrt{MN}}\Psi_{ij}(h,k,r) \tag{276}$$

so that

$$I_O^{(1)}(h,k) = \sum_{i,j=1}^{2} I_{ij,\boldsymbol{\alpha}}^{(1)} + E(T).$$

Now,

$$I_{ij,\boldsymbol{\alpha}}^{(1)} = \sum_{r\neq 0}\sum_{M,N} \frac{T}{\sqrt{MN}} \frac{1}{h^{1-a_i}k^{1-b_j}} S_{ij}(h,k,r)\int_{\max(0,r)}^{\infty} x^{-a_i}(x-r)^{-b_j}F(x,x-r)dx$$

where

$$\begin{aligned} F(x,y) =& W\left(\frac{x}{hM}\right)W\left(\frac{y}{kN}\right)\frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\left(\frac{\pi^2 xy}{hkq}\right)^{-s} \\ & \times \frac{1}{T}\int_{-\infty}^{\infty}\left(1+\frac{r}{y}\right)^{-it} g(s,t)w(t)dtds. \end{aligned} \tag{277}$$

Using $W(x) = x^{-1/2}W_0(x)$ and recalling $\sum_M W_0(x/M) = 1$, we see that

$$\begin{aligned} I_{ij,\boldsymbol{\alpha}}^{(1)} = & \frac{1}{h^{\frac{1}{2}-a_i}k^{\frac{1}{2}-b_j}}\sum_{r\neq 0} S_{ij}(h,k,r)\int_{\max(0,r)}^{\infty} x^{-\frac{1}{2}-a_i}(x-r)^{-\frac{1}{2}-b_j} \\ & \times \frac{1}{2\pi i}\int_{(\varepsilon)}\frac{G(s)}{s}\left(\frac{\pi^2 x(x-r)}{hkq}\right)^{-s}\int_{-\infty}^{\infty}\left(1-\frac{r}{x}\right)^{it} g(s,t)w(t)dtdsdx \end{aligned}$$

We split the sum over r into two terms; those for which $r > 0$ and those for which $r < 0$. This gives

$$I_{ij,\boldsymbol{\alpha}}^{(1)} = I^+ + I^-$$

where

$$I^{\pm} = \frac{1}{h^{\frac{1}{2}-a_i}k^{\frac{1}{2}-b_j}} \sum_{r=1}^{\infty} S_{ij}(h,k,\pm r)K^{\pm}$$

with

$$K^+ = \int_r^\infty x^{-\frac{1}{2}-a_i}(x-r)^{-\frac{1}{2}-b_j} \frac{1}{2\pi i}\int_{(\varepsilon)} \frac{G(s)}{s}\left(\frac{\pi^2 x(x-r)}{hkq}\right)^{-s} \times \int_{-\infty}^{\infty}\left(1-\frac{r}{x}\right)^{it} g(s,t)w(t)dtdsdx$$

and

$$K^- = \int_0^\infty x^{-\frac{1}{2}-a_i}(x+r)^{-\frac{1}{2}-b_j} \frac{1}{2\pi i}\int_{(\varepsilon)} \frac{G(s)}{s}\left(\frac{\pi^2 x(x+r)}{hkq}\right)^{-s} \times \int_{-\infty}^{\infty}\left(1+\frac{r}{x}\right)^{it} g(s,t)w(t)dtdsdx.$$

Performing the substitution $x \mapsto rx+r$ in K^+ and the substitution $x \mapsto rx$ in K^- gives

$$K^+ = r^{-a_i-b_j}\int_{-\infty}^{\infty} w(t)\frac{1}{2\pi i}\int_{(\varepsilon)} \frac{G(s)}{s}\left(\frac{\pi^2 r^2}{hkq}\right)^{-s} g(s,t) \times \int_0^\infty (x+1)^{-\frac{1}{2}-a_i-s-it}x^{-\frac{1}{2}-b_j-s+it}dxdsdt$$

and

$$K^- = r^{-a_i-b_j}\int_{-\infty}^{\infty} w(t)\frac{1}{2\pi i}\int_{(\varepsilon)} \frac{G(s)}{s}\left(\frac{\pi^2 r^2}{hkq}\right)^{-s} g(s,t) \times \int_0^\infty x^{-\frac{1}{2}-a_i-s-it}(x+1)^{-\frac{1}{2}-b_j-s+it}dxdsdt.$$

Here, we have swapped the orders of integration but as we shall see shortly, the resulting integral is absolutely convergent and so this is allowed. By formula (91) of [23] we have

$$\int_0^\infty (x+1)^{-\frac{1}{2}-a_i-s-it} x^{-\frac{1}{2}-b_j-s+it} dx = \frac{\Gamma(\frac{1}{2}-a_i-s+it)\Gamma(a_i+b_j+2s)}{\Gamma(\frac{1}{2}+b_j+s+it)} \tag{278}$$

and

$$\int_0^\infty x^{-\frac{1}{2}-a_i-s-it}(x+1)^{-\frac{1}{2}-b_j-s+it} dx = \frac{\Gamma(\frac{1}{2}-b_j-s-it)\Gamma(a_i+b_j+2s)}{\Gamma(\frac{1}{2}+a_i+s-it)}. \tag{279}$$

K^+ and K^- are now sufficiently similar to allow for a re-combination of terms. Defining

$$T_{ij}(h,k,r,s,t) = S_{ij}(h,k,r)\frac{\Gamma(\frac{1}{2}-a_i-s+it)}{\Gamma(\frac{1}{2}+b_j+s+it)} + S_{ij}(h,k,-r)\frac{\Gamma(\frac{1}{2}-b_j-s-it)}{\Gamma(\frac{1}{2}+a_i+s-it)}$$

this gives

$$I^{(1)}_{ij,\boldsymbol{\alpha}} = \frac{1}{h^{\frac{1}{2}-a_i}k^{\frac{1}{2}-b_j}} \sum_{r=1}^\infty r^{-a_i-b_j} \int_{-\infty}^\infty w(t) \frac{1}{2\pi i}\int_{(\varepsilon)} \frac{G(s)}{s}\left(\frac{\pi^2 r^2}{hkq}\right)^{-s} g(s,t) \times T_{ij}(h,k,r,s,t)\Gamma(a_i+b_j+2s) ds dt.$$

Note by Stirling's formula

$$T_{ij}(h,k,r,s,t) = \left(S_{ij}(h,k,r)e^{-\frac{\pi i}{2}(a_i+b_j+2s)} + S_{ij}(h,k,-r)e^{\frac{\pi i}{2}(a_i+b_j+2s)}\right) \times t^{-a_i-b_j-2s}\left(1+O\left(\tfrac{1+|s|^2}{t}\right)\right) \tag{280}$$

and so the evaluation of T_{ij} is reduced to the evaluation of $S_{ij}(h,k,r)$ under the transformation $r \mapsto -r$.

In the cases $i=j$, the sums $S_{ij}(h,k,r)$ are given by

$$S_{11}(h,k,r) = L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi}) \sum_{d\in P_{11}} \frac{c_d(r)\chi(d_{(h)})\overline{\chi}(d_{(k)})}{d_{(h)}^{1-\alpha+\beta} d_{(k)}^{1-\gamma+\delta}}$$

and

$$S_{22}(h,k,r) = \frac{L_{-\alpha,-\beta}(\overline{\chi})L_{-\gamma,-\delta}(\chi)}{q^{-1+\beta-\alpha+\delta-\gamma}} \sum_{d\in P_{22}} \frac{c_d(r,|\chi|^2)\overline{\chi}(h_{(d)})\chi(k_{(d)})}{d_{(h)}^{1+\alpha-\beta} d_{(k)}^{1+\gamma-\delta}}.$$

By Lemma 2.6.1 below, the elements $d \in P_{22}$ are divisible by q and hence

$$c_d(r, |\chi|^2) = \sum_{\substack{c=1 \\ (c,d)=1}}^{d} |\chi(c)|^2 e_d(-cr) = \sum_{\substack{c=1 \\ (c,d)=1}}^{d} e_d(-cr) = c_d(r).$$

Therefore, since $c_d(-r) = c_d(r)$ we have $S_{ij}(h,k,-r) = S_{ij}(h,k,r)$ if $i = j$. In the cases $i \neq j$ the sums in question are given by

$$S_{12}(h,k,r) = \frac{\chi(-1)G(\overline{\chi})L_{\alpha,\beta}(\chi)L_{-\gamma,-\delta}(\chi)}{q^{\delta-\gamma}} \sum_{d \in P_{12}} \frac{c_d(r,\chi)\chi(d_{(h)})\chi(k_{(d)})}{d_{(h)}^{1-\alpha+\beta} d_{(k)}^{1+\gamma-\delta}},$$

and

$$S_{21}(h,k,r) = \frac{G(\chi)L_{-\alpha,-\beta}(\overline{\chi})L_{\gamma,\delta}(\overline{\chi})}{q^{\beta-\alpha}} \sum_{d \in P_{21}} \frac{c_d(r,\overline{\chi})\overline{\chi}(h_{(d)})\overline{\chi}(d_{(k)})}{d_{(h)}^{1+\alpha-\beta} d_{(k)}^{1-\gamma+\delta}}$$

Once again, by Lemma 2.6.1 we see $q|d$ for $d \in P_{ij}$, $i \neq j$. For such d we have

$$c_d(-r,\chi) = \chi(-1) \sum_{\substack{c=1 \\ (c,d)=1}}^{d} \chi(-c)e_d(cr) = \chi(-1)c_d(r,\chi) \tag{281}$$

and consequently $S_{ij}(h,k,-r) = \chi(-1)S_{ij}(h,k,r) = (-1)^{\mathfrak{a}} S_{ij}(h,k,r)$ for $i \neq j$. Inputting the above information into (280) gives

$$T_{ij}(h,k,r,s,t) = S_{ij}(h,k,r)t^{-a_i-b_j-2s} \times \begin{Bmatrix} 2\cos(\frac{\pi}{2}(a_i+b_j+2s)) & \text{if } i=j \\ 2i^{\mathfrak{a}}\cos(\frac{\pi}{2}(a_i+b_j+2s+\mathfrak{a})) & \text{if } i \neq j \end{Bmatrix} \left(1 + O\left(\tfrac{1+|s|^2}{t}\right)\right).$$

We now move the s-line of integration to 1 so that the sums over r converge absolutely allowing us to push them through the integrals. Along with the above formula for T_{ij} we now have

$$\begin{aligned}(I^{(1)}_{ij,\boldsymbol{\alpha}})_{i=j} =& \frac{1}{h^{\frac{1}{2}-a_i}k^{\frac{1}{2}-b_j}} \int_{-\infty}^{\infty} w(t) \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left(\frac{\pi^2}{hkq}\right)^{-s} \sum_{r=1}^{\infty} \frac{S_{ij}(h,k,r)}{r^{a_i+b_j+2s}} \\ &\times 2\cos\left(\tfrac{\pi}{2}(a_i+b_j+2s)\right)\Gamma(a_i+b_j+2s)t^{-a_i-b_j-2s}g(s,t) \\ &\times \left(1 + O\left(\tfrac{1+|s|^2}{t}\right)\right) dsdt. \end{aligned} \tag{282}$$

and

$$(I_{ij,\boldsymbol{\alpha}}^{(1)})_{i\neq j} = \frac{1}{h^{\frac{1}{2}-a_i}k^{\frac{1}{2}-b_j}} \int_{-\infty}^{\infty} w(t)\frac{1}{2\pi i}\int_{(1)} \frac{G(s)}{s}\left(\frac{\pi^2}{hkq}\right)^{-s}\sum_{r=1}^{\infty}\frac{S_{ij}(h,k,r)}{r^{a_i+b_j+2s}}$$
$$\times 2i^{\mathfrak{a}}\cos\left(\tfrac{\pi}{2}(a_i+b_j+2s+\mathfrak{a})\right)\Gamma(a_i+b_j+2s)t^{-a_i-b_j-2s}g(s,t)$$
$$\times\left(1+O\left(\tfrac{1+|s|^2}{t}\right)\right)dsdt. \tag{283}$$

We plan to show that the new sums over r are given by a product of two zeta functions (or L-functions) times a finite Euler product over the primes dividing h and k. It turns out that one of these zeta functions can be paired with the Gamma factors in the line integral allowing us to use the functional equation and hence remove the Gamma factors. A further simplification will occur after using the asymptotic $t^{-2s}g(s,t)\sim 2^{-2s}$ and integrating the error terms over t.

2.6. Some Arithmetical Sums

Let

$$U_{ij}(s) := \sum_{r=1}^{\infty}\frac{S_{ij}(h,k,r)}{r^{a_i+b_j+2s}}. \tag{284}$$

Since the S_{ij} are given as sums over P_{ij}, we first investigate these sets. Recall that for an integer n we define it's q-part by

$$n(q) = \prod_{\substack{p|n\\p|q}} p^{n_p}$$

and it's non-q-part by $n^* := n/n(q)$ so that $(n^*,q)=1$.

Lemma 2.6.1. *We have*

$$P_{11} = \left\{d\in\mathbb{Z}_{\geq 1} : (d,q)=1\right\} \tag{285}$$

and

$$P_{22} = \left\{d\in\mathbb{Z}_{\geq 1} : d = qh(q)k(q)l \text{ where } l\geq 1\right\}. \tag{286}$$

If $q|h$ then

$$P_{12} = \left\{ d \in \mathbb{Z}_{\geq 1} : d = qml \text{ where } l \geq 1\,, (l,q) = 1 \,\text{and}\, m|h(q)/q \right\} \tag{287}$$

otherwise $P_{12} = \emptyset$. Similarly, if $q|k$ then

$$P_{21} = \left\{ d \in \mathbb{Z}_{\geq 1} : d = qnl \text{ where } l \geq 1\,, (l,q) = 1 \,\text{and}\, n|k(q)/q \right\} \tag{288}$$

otherwise $P_{21} = \emptyset$.

Proof. Let $d \in P_{ij}$. We first note that

$$d_{(h)} = \frac{d}{(d,h)} = \frac{d^*}{(d^*,h^*)} \frac{d(q)}{(d(q),h(q))} = d^*_{(h^*)} \cdot d(q)_{(h(q))} \tag{289}$$

and therefore the only influence on $(d_{(h)}, q)$ is due to $(d(q)_{(h(q))}, q)$. This means we can let d^* range freely over the positive integers coprime to q in all of the following.

The conditions defining P_{11} are given by $(d_{(h)}, q) = 1$ and $(d_{(k)}, q) = 1$. Therefore $(d(q)_{(h(q))}, q) = 1$ and $(d(q)_{(k(q))}, q) = 1$. This implies $d(q) = (d(q), h(q)) = (d(q), k(q))$ which is only possible if $d(q) = 1$ since $(h, k) = 1$.

P_{22} is given by the conditions $(d_{(h)}, q) = q$ and $(d_{(k)}, q) = q$. These imply that $d(q) = lq(d(q), h(q)) = mq(d(q), k(q))$ with $l, m \geq 1$ and $(l, q) > 1$, $(m, q) > 1$. Therefore we may write $l = n(d(q), k(q))$, $m = n(d(q), h(q))$ for some n with $(n, q) > 1$. Putting this back into the previous equality gives

$$d(q) = nq(d(q), h(q)k(q)).$$

This is possible if and only if $d(q) = nqh(q)k(q)$ and it is clear that the given n is arbitrary.

For P_{12} the conditions are $(d_{(h)}, q) = 1$, $(d_{(k)}, q) = q$. The first of these implies that d must satisfy $d(q) = (d(q), h(q))$ whilst the second gives that $d(q) = mq(d(q), k(q))$ with $m \geq 1$ and $(m, q) > 1$. Equating these gives

$$d(q) = (d(q), h(q)) = mq(d(q), k(q)).$$

This is possible if and only if $q|h$ hence otherwise P_{12} is empty. Now, if $q|h$ then $k(q)=1$ and therefore

$$mq = (mq, h(q)).$$

This constraint implies that m may only range over the divisors of $h(q)/q$. A similar argument follows for P_{21}.

□

We note that since $(h,k)=1$ at most one of the sets P_{12}, P_{21} is non-empty.

We deal with $U_{11}(s)$ first. By (268) and (285) this reads as

(290)
$$U_{11}(s) = L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})\sum_{r=1}^{\infty}\sum_{\substack{d=1\\(d,q)=1}}^{\infty}\frac{c_d(r)\chi(d_{(h)})\overline{\chi}(d_{(k)})(h,d)^{1-\alpha+\beta}(k,d)^{1-\gamma+\delta}}{d^{2-\alpha+\beta-\gamma+\delta}r^{\alpha+\gamma+2s}}.$$

Proposition 2.6.2. *Let $h=\prod_p p^{h_p}$ and $k=\prod_p p^{k_p}$. Then*

$$\begin{aligned}U_{11}(s) =& L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})\frac{\zeta(\alpha+\gamma+2s)\zeta(1+\beta+\delta+2s)}{\zeta(2-\alpha+\beta-\gamma+\delta)}\\ &\times Q_{11}(s)C_{11,\boldsymbol{\alpha},h,k}(s)\end{aligned} \tag{291}$$

where

$$Q_{11}(s) = Q_{11,\boldsymbol{\alpha},q}(s) = \prod_{p|q}\left(\frac{1-p^{-1-\beta-\delta-2s}}{1-p^{-2+\alpha-\beta+\gamma-\delta}}\right) \tag{292}$$

and

$$C_{11,\boldsymbol{\alpha},h,k}(s) = C_{11,\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi})C_{11,\gamma,\delta,\alpha,\beta,k}(s,\chi) \tag{293}$$

where

$$C_{11,\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi}) = \prod_{\substack{p\nmid q\\ p|h}}\left(\frac{C_{11}^{(0)}(s)-p^{-1}C_{11}^{(1)}(s)+p^{-2}C_{11}^{(2)}(s)}{(1-\overline{\chi}(p)p^{-\alpha-\delta-2s})(1-p^{-2+\alpha-\beta+\gamma-\delta})}\right) \tag{294}$$

with

$$C_{11}^{(0)}(s) = 1 - \overline{\chi}(p)^{h_p+1} p^{-(h_p+1)(\alpha+\delta+2s)} \tag{295}$$

$$C_{11}^{(1)}(s) = (\overline{\chi}(p)p^{\gamma-\delta} + p^{-\beta-\delta-2s})(1 - \overline{\chi}(p)^{h_p} p^{-h_p(\alpha+\delta+2s)}) \tag{296}$$

$$C_{11}^{(2)}(s) = \overline{\chi}(p)p^{-\beta+\gamma-2\delta-2s} - \overline{\chi}(p)^{h_p} p^{-h_p(\alpha+\delta+2s)} p^{\alpha-\beta+\gamma-\delta}. \tag{297}$$

Proof. To simplify things we first define

$$F(a,b,c) = \sum_{r=1}^{\infty} \sum_{\substack{d=1 \\ (d,q)=1}}^{\infty} \frac{c_d(r)\chi(d/(h^*,d))\overline{\chi}(d/(k^*,d))(h^*,d)^a (k^*,d)^b}{d^{a+b} r^{c+1}}$$

so that

$$\frac{U_{11}(s)}{L_{\alpha,\beta}(\chi) L_{\gamma,\delta}(\overline{\chi})} = F(1-\alpha+\beta, 1-\gamma+\delta, -1+\alpha+\gamma+2s).$$

By formula (1.5.5) of [59] we have

$$\sum_{r=1}^{\infty} \frac{c_d(r)}{r^{c+1}} = \zeta(c+1) \sum_{n|d} n^{-c} \mu(d/n)$$

where μ is the mobius function. Summing over r and performing the substitution $n \mapsto d/n$ in the sum over the divisors of d we have

$$\frac{F(a,b,c)}{\zeta(c+1)} = \sum_{\substack{d=1 \\ (d,q)=1}}^{\infty} \frac{\chi(d/(h^*,d))\overline{\chi}(d/(k^*,d))(h^*,d)^a (k^*,d)^b}{d^{a+b+c}} \sum_{n|d} n^c \mu(n).$$

Let $g_c(d) = \sum_{n|d} n^c \mu(n)$. Since the numerator is multiplicative we have

$$\frac{F(a,b,c)}{\zeta(c+1)} = \prod_{p \nmid q} \left(\sum_{m \geq 0} \frac{\chi(p^m/(p^{h_p},p^m))\overline{\chi}(p^m/(p^{k_p},p^m))(p^{h_p},p^m)^a (p^{k_p},p^m)^b}{p^{m(a+b+c)}} g_c(p^m) \right).$$

We now split this product into three parts, the first over the primes $p \nmid hk$ and other two over those for which $p|h$ and $p|k$.

If $p \nmid hk$ then we have factors of the form

$$\sum_{m \geq 0} \frac{|\chi(p^m)|^2 g_c(p^m)}{p^{m(a+b+c)}} = 1 + (1-p^c) \sum_{m \geq 1} \left(\frac{|\chi(p)|^2}{p^{a+b+c}} \right)^m = \frac{1-p^{-a-b}}{1-p^{-a-b-c}}$$

since $p \nmid q$. If $p|h$ then we have factors of the form

$$1 + (1-p^c)\sum_{m\geq 1} \frac{\chi(p^m/(p^{h_p},p^m))\overline{\chi}(p^m)(p^{h_p},p^m)^a}{p^{m(a+b+c)}}.$$

Now

$$\begin{aligned}
& \sum_{m\geq 1} \frac{\chi(p^m/(p^{h_p},p^m))\overline{\chi}(p^m)(p^{h_p},p^m)^a}{p^{m(a+b+c)}} \\
= & \sum_{m=1}^{h_p} \frac{\chi(1)\overline{\chi}(p^m)p^{ma}}{p^{m(a+b+c)}} + \sum_{m=h_p+1}^{\infty} \frac{\chi(p^{m-h_p})\overline{\chi}(p^m)p^{h_pa}}{p^{m(a+b+c)}} \\
= & \frac{\overline{\chi}(p)}{p^{b+c}}\frac{1-\overline{\chi}(p)^{h_p}p^{-h_p(b+c)}}{1-\overline{\chi}(p)p^{-b-c}} + p^{h_pa}\sum_{m\geq 1}\frac{\chi(p^m)\overline{\chi}(p^{m+h_p})}{p^{(a+b+c)(m+h_p)}} \\
= & \frac{\overline{\chi}(p)}{p^{b+c}}\frac{1-\overline{\chi}(p)^{h_p}p^{-h_p(b+c)}}{1-\overline{\chi}(p)p^{-b-c}} + \overline{\chi}(p)^{h_p}p^{-h_p(b+c)}\frac{p^{-a-b-c}}{1-p^{-a-b-c}} \\
= & p^{-b-c}\Big[(\overline{\chi}(p)-\overline{\chi}(p)^{h_p+1}p^{-h_p(b+c)})(1-p^{-a-b-c}) + \overline{\chi}(p)^{h_p} \\
& \times p^{-h_p(b+c)}p^{-a}(1-\overline{\chi}(p)p^{-b-c})\Big]/(1-\overline{\chi}(p)p^{-b-c})(1-p^{-a-b-c}) \\
= & p^{-b-c}\Big[\overline{\chi}(p)-\overline{\chi}(p)^{h_p+1}p^{-h_p(b+c)} - \overline{\chi}(p)p^{-a-b-c} + \overline{\chi}(p)^{h_p}p^{-h_p(b+c)}p^{-a}\Big] \\
& /(1-\overline{\chi}(p)p^{-b-c})(1-p^{-a-b-c})
\end{aligned}$$

The numerator of the local factor is thus given by

$$\begin{aligned}
& (1-\overline{\chi}(p)p^{-b-c})(1-p^{-a-b-c}) + (1-p^c)p^{-b-c} \\
& \times(\overline{\chi}(p)-\overline{\chi}(p)^{h_p+1}p^{-h_p(b+c)} - \overline{\chi}(p)p^{-a-b-c} + \overline{\chi}(p)^{h_p}p^{-h_p(b+c)}p^{-a}) \\
= & 1 - p^{-a-b-c} - \overline{\chi}(p)^{h_p+1}p^{-(h_p+1)(b+c)} + \overline{\chi}(p)^{h_p}p^{-h_p(b+c)}p^{-a} \\
& -\overline{\chi}(p)p^{-b} + \overline{\chi}(p)^{h_p+1}p^{-h_p(b+c)}p^{-b} + \overline{\chi}(p)p^{-a-2b-c} - \overline{\chi}(p)^{h_p}p^{-h_p(b+c)}p^{-a-b} \\
= & (1-\overline{\chi}(p)p^{-b})(1-p^{-a-b-c}) + \overline{\chi}(p)^{h_p}p^{-h_p(b+c)}p^{-b}(\overline{\chi}(p)-p^{-a})(1-p^{-c})
\end{aligned}$$

If $p|k$ then we have the same except $\overline{\chi}$ is replaced by χ and a and b are interchanged. Therefore

$$\frac{F(a,b,c)}{\zeta(c+1)} = \frac{\zeta(a+b+c)}{\zeta(a+b)} \prod_{p|q} \left(\frac{1-p^{-a-b-c}}{1-p^{-a-b}} \right)$$
$$\times \prod_{\substack{p\nmid q\\ p|h}} \left(\frac{(1-\overline{\chi}(p)p^{-b})(1-p^{-a-b-c}) + \overline{\chi}(p)^{h_p} p^{-h_p(b+c)} p^{-b}(\overline{\chi}(p)-p^{-a})(1-p^{-c})}{(1-\overline{\chi}(p)p^{-b-c})(1-p^{-a-b})} \right)$$
$$\times \prod_{\substack{p\nmid q\\ p|k}} \left(\frac{(1-\chi(p)p^{-a})(1-p^{-a-b-c}) + \chi(p)^{k_p} p^{-k_p(a+c)} p^{-a}(\chi(p)-p^{-b})(1-p^{-c})}{(1-\chi(p)p^{-a-c})(1-p^{-a-b})} \right)$$

We can now substitute the values for a, b, c to give the desired result. □

We now turn to $U_{22}(s)$. By (271) and (284) we have

$$\frac{U_{22}(s)q^{-1-\alpha+\beta-\gamma+\delta}}{L_{-\alpha,-\beta}(\overline{\chi})L_{-\gamma,-\delta}(\chi)} = \sum_{r=1}^{\infty} \sum_{d\in P_{22}} \frac{c_d(r)\overline{\chi}(h_{(d)})\chi(k_{(d)})(h,d)^{1+\alpha-\beta}(k,d)^{1+\gamma-\delta}}{d^{2+\alpha-\beta+\gamma-\delta} r^{\beta+\delta+2s}}.$$

Proposition 2.6.3. *Let* $Q_{22}(s) = Q_{11,-\boldsymbol{\gamma},q}(-s)$ *. Then*

$$U_{22}(s) = \frac{L_{-\alpha,-\beta}(\overline{\chi})L_{-\gamma,-\delta}(\chi)}{q^{\beta+\delta+2s}} \frac{\zeta(\beta+\delta+2s)\zeta(1+\alpha+\gamma+2s)}{\zeta(2+\alpha-\beta+\gamma-\delta)} \times Q_{22}(s) C_{22,\boldsymbol{\alpha},h,k}(s) \tag{298}$$

where

$$C_{22,\boldsymbol{\alpha},h,k}(s) = h(q)^{-\beta-\gamma-2s} k(q)^{-\alpha-\delta-2s} C_{22,\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi}) C_{22,\gamma,\delta,\alpha,\beta,k}(s,\chi) \tag{299}$$

and

$$C_{22,\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi}) = \prod_{\substack{p\nmid q\\ p|h}} \left(\frac{C_{22}^{(0)}(s) - p^{-1}C_{22}^{(1)}(s) + p^{-2}C_{22}^{(2)}(s)}{(1-\overline{\chi}(p)p^{\beta+\gamma+2s})(1-p^{-2-\alpha+\beta-\gamma+\delta})} \right) \tag{300}$$

with

$$C_{22}^{(0)}(s) = p^{-h_p(\beta+\gamma+2s)} - \overline{\chi}(p)^{h_p+1} p^{\beta+\gamma+2s} \tag{301}$$

$$C_{22}^{(1)}(s) = (p^{-h_p(\beta+\gamma+2s)} - \overline{\chi}(p)^{h_p})(p^{\beta+\delta+2s} + \overline{\chi}(p)p^{-\alpha+\beta}) \tag{302}$$

$$C_{22}^{(2)}(s) = p^{-h_p(\beta+\gamma+2s)} \overline{\chi}(p) p^{-\alpha+2\beta+\delta+2s} - \overline{\chi}(p)^{h_p} p^{-\alpha+\beta-\gamma+\delta}. \tag{303}$$

PROOF. Similarly to before we define a new function

$$G(a,b,c)=\sum_{r=1}^{\infty}\sum_{d\in P_{22}}\frac{c_d(r)\overline{\chi}(h_{(d)})\chi(k_{(d)})(h,d)^a(k,d)^b}{d^{a+b}r^{c+1}} \tag{304}$$

so that

$$\frac{U_{22}(s)q^{-1-\alpha+\beta-\gamma+\delta}}{L_{-\alpha,-\beta}(\overline{\chi})L_{-\gamma,-\delta}(\chi)}=G(1+\alpha-\beta,1+\gamma-\delta,-1+\beta+\delta+2s).$$

As in Proposition 2.6.2 we first perform the sum over r and get

$$G(a,b,c)=\zeta(c+1)\sum_{d\in P_{22}}\frac{\overline{\chi}(h_{(d)})\chi(k_{(d)})(h,d)^a(k,d)^b}{d^{a+b+c}}g_c(d) \tag{305}$$

where $g_c(d)=\sum_{n|d}n^c\mu(n)$. By Lemma 2.6.1 we may write $d=qh(q)k(q)l$ with $l\geq 1$. This implies that $(h,d)=h(q)\cdot(h^*,l)$ and $(k,d)=k(q)\cdot(k^*,l)$. Therefore

$$G(a,b,c)=A\sum_{l=1}^{\infty}\frac{\overline{\chi}(h^*/(h^*,l))\chi(k^*/(k^*,l))(h^*,l)^a(k^*,l)^b}{l^{a+b+c}}g_c(qh(q)k(q)l) \tag{306}$$

where

$$A=\frac{\zeta(c+1)}{q^{a+b+c}h(q)^{b+c}k(q)^{a+c}}. \tag{307}$$

Writing the Dirichlet series as an Euler product we have

$$\begin{aligned}&\frac{G(a,b,c)}{A}\\ &=\prod_p\sum_{m\geq 0}\frac{\overline{\chi}\left(\frac{p^{h_p^*}}{(p^{h_p^*},p^m)}\right)\chi\left(\frac{p^{k_p^*}}{(p^{k_p^*},p^m)}\right)(p^{h_p^*},p^m)^a(p^{k_p^*},p^m)^b}{p^{m(a+b+c)}}g_c(p^{q_p+h(q)_p+k(q)_p+m}) \qquad (308)\\ &=\prod_{p\nmid q}(\star)\prod_{p|q}(\star).\end{aligned}$$

We deal with product over $p|q$ first. In this case we have $h_p^*=k_p^*=0$ and $q_p\geq 1$. Consequently, we have a local factor of the form

$$\sum_{m\geq 0}\frac{g_c(p^{m+q_p+\cdots})}{p^{m(a+b+c)}}=(1-p^c)\sum_{m\geq 0}p^{-m(a+b+c)}=\frac{1-p^c}{1-p^{-a-b-c}}.$$

If $p \nmid q$ and $p \nmid hk$ then we have a local factor of the form

$$\sum_{m\geq 0} \frac{g_c(p^m)}{p^{m(a+b+c)}} = \frac{1-p^{-a-b}}{1-p^{-a-b-c}}.$$

Finally, if $p \nmid q$ and $p|h$ then we have a local factor of the form

$$\overline{\chi}(p)^{h_p} + (1-p^c)\sum_{m\geq 1} \frac{\overline{\chi}\left(\frac{p^{h_p}}{(p^{h_p},p^m)}\right)(p^{h_p},p^m)^a}{p^{m(a+b+c)}}.$$

Computing this in a similar fashion to proposition 2.6.2 we see the local factor is given by

$$\frac{p^{-h_p(b+c)}(1-p^c)(1-\overline{\chi}(p)p^{-a}) + \overline{\chi}(p)^{h_p}(p^c - p^{-a-b})(1-\overline{\chi}(p)p^b)}{(1-\overline{\chi}(p)p^{b+c})(1-p^{-a-b-c})}.$$

Therefore, similarly to before, we find

$$\begin{aligned} \frac{G(a,b,c)}{A} &= \frac{\zeta(a+b+c)}{\zeta(a+b)} \prod_{p|q} \left(\frac{1-p^c}{1-p^{-a-b}}\right) \\ &\times \prod_{\substack{p\nmid q \\ p|h}} \left(\frac{p^{-h_p(b+c)}(1-p^c)(1-\overline{\chi}(p)p^{-a}) + \overline{\chi}(p)^{h_p}(p^c - p^{-a-b})(1-\overline{\chi}(p)p^b)}{(1-\overline{\chi}(p)p^{b+c})(1-p^{-a-b})}\right) \\ &\times \prod_{\substack{p\nmid q \\ p|h}} \left(\frac{p^{-k_p(a+c)}(1-p^c)(1-\chi(p)p^{-b}) + \chi(p)^{k_p}(p^c - p^{-a-b})(1-\chi(p)p^a)}{(1-\chi(p)p^{a+c})(1-p^{-a-b})}\right) \end{aligned} \tag{309}$$

Inputting the values of a, b, c and a brief computation gives the result. □

At this point we note that there exist certain similarities between U_{11} and U_{22}. Indeed, the equivalent of U_{11} in $I_O^{(2)}$ contains a factor of $q^{-\beta-\delta}$ and has undergone the transformation $\boldsymbol{\alpha} \mapsto -\boldsymbol{\gamma}$. Therefore the L and $\zeta(2+\cdots)^{-1}$ factors match with those of $U_{22}^{(1)}$ as does the Q factor after the transformation $s \mapsto -s$. It is a surprising fact that the finite Euler products $C_{ii,\boldsymbol{\alpha},h,k}(s)$ also possess this symmetry. Indeed, we have

Proposition 2.6.4.

$$h^{\alpha}k^{\gamma}(hk)^{-s}C_{11,\boldsymbol{\alpha},h,k}(-s) = h^{-\delta}k^{-\beta}(hk)^{s}C_{22,-\boldsymbol{\gamma},h,k}(s). \tag{310}$$

By permuting the shifts we also have

$$h^{\beta}k^{\delta}(hk)^{-s}C_{22,\boldsymbol{\alpha},h,k}(-s) = h^{-\gamma}k^{-\alpha}(hk)^{s}C_{11,-\boldsymbol{\gamma},h,k}(s). \tag{311}$$

Proof. Since

$$C_{11,\boldsymbol{\alpha},h,k}(s) = C_{11,\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi})C_{11,\gamma,\delta,\alpha,\beta,k}(s,\chi)$$

and

$$C_{22,-\boldsymbol{\gamma},h,k}(s) = h(q)^{\alpha+\delta-2s}k(q)^{\beta+\gamma-2s}C_{22,-\gamma,-\delta,-\alpha,-\beta,h}(s,\overline{\chi})C_{22,-\alpha,-\beta,-\gamma,-\delta,k}(s,\chi)$$

we only need to prove

$$\left(\frac{h}{h(q)}\right)^{\alpha+\delta-2s} C_{11,\alpha,\beta,\gamma,\delta,h}(-s,\overline{\chi}) = C_{22,-\gamma,-\delta,-\alpha,-\beta,h}(s,\overline{\chi})$$

since the result then follows by symmetry. It suffices to check the formula at each prime dividing h. By inspection of the Euler products we need to show

$$p^{h_p(\alpha+\delta-2s)}C^{(i)}_{11,\alpha,\beta,\gamma,\delta,h}(-s) = C^{(i)}_{22,-\gamma,-\delta,-\alpha,-\beta,h}(s)$$

for $i = 0, 1, 2$ and each of these is immediately apparent when written out. □

We now work with U_{12} and U_{21} which in the above sense are self-similar. First, we need a technical lemma

Lemma 2.6.5. *Let $c_d(r,\chi)$ be given by (272) and suppose $q|d$. Then*

$$c_d(r,\chi) = \overline{G(\overline{\chi})}\sum_{\substack{n|r\\ n|d/q}} \mu\left(\frac{d/q}{n}\right)\chi\left(\frac{d/q}{n}\right)\overline{\chi}\left(\frac{r}{n}\right)n. \tag{312}$$

Under the given conditions this implies

$$\sum_{r=1}^{\infty}\frac{c_d(r,\chi)}{r^s} = \overline{G(\overline{\chi})}L(s,\overline{\chi})\left(\frac{q}{d}\right)^{s-1}\sum_{n|d/q}\mu\left(n\right)\chi\left(n\right)n^{s-1}. \tag{313}$$

Proof. We have

$$\begin{aligned}
c_d(r,\chi) &= \sum_{\substack{n=1\\(n,d)=1}}^{d} \chi(n)e_d(-nr) = \sum_{n=1}^{d}\Bigg(\sum_{\substack{m|n\\m|d}}\mu(m)\Bigg)\chi(n)e_d(-nr)\\
&= \sum_{m|d}\mu(m)\sum_{n=1}^{d/m}\chi(mn)e_d(-mnr)\\
&= \sum_{\substack{m|d\\m\nmid q}}\mu(m)\chi(m)\sum_{n=1}^{d/m}\chi(n)e_{d/m}(-nr)
\end{aligned} \tag{314}$$

where the condition $m \nmid q$ is merely for emphasis. Since $q|d$ we may write $d/m = aq$ for some a say. Now,

$$\begin{aligned}
\sum_{n=1}^{aq}\overline{\chi}(n)e_{aq}(nr) &= \sum_{n=1}^{q}\overline{\chi}(n)e_{aq}(nr)\sum_{k=0}^{a-1}e_a(kr)\\
&= \begin{cases} a\sum_{n=1}^{q}\overline{\chi}(n)e_q(nr/a) & \text{if } a|r,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{315}$$

Since $\sum_{n=1}^{q}\overline{\chi}(n)e_q(nr/a) = \chi(r/a)G(\overline{\chi})$ we have

$$\begin{aligned}
c_d(r,\chi) &= \overline{G(\overline{\chi})}\sum_{\substack{m|d\\ \frac{d}{mq}|r}}\mu(m)\chi(m)\overline{\chi}\left(\frac{r}{d/mq}\right)\frac{d}{mq}\\
&= \overline{G(\overline{\chi})}\sum_{\substack{c|d\\c|qr}}\mu\left(\frac{d}{c}\right)\chi\left(\frac{d}{c}\right)\overline{\chi}\left(\frac{qr}{c}\right)\frac{c}{q}.
\end{aligned} \tag{316}$$

The result now follows on applying the change of variables $c/q \mapsto n$. For the result involving the Dirichlet series we apply the formula to give

$$\begin{aligned}\frac{1}{\overline{G(\overline{\chi})}}\sum_{r=1}^{\infty}\frac{c_d(r,\chi)}{r^s} &= \sum_{r=1}^{\infty}\frac{1}{r^s}\sum_{\substack{n|r\\ n|d/q}}\mu\left(\frac{d/q}{n}\right)\chi\left(\frac{d/q}{n}\right)\overline{\chi}\left(\frac{r}{n}\right)n\\ &= \sum_{n|d/q}\mu\left(\frac{d/q}{n}\right)\chi\left(\frac{d/q}{n}\right)n\sum_{m=1}^{\infty}\frac{\overline{\chi}(m)}{(mn)^s}\\ &= L(s,\overline{\chi})\sum_{n|d/q}\mu\left(\frac{d/q}{n}\right)\chi\left(\frac{d/q}{n}\right)n^{1-s}\\ &= L(s,\overline{\chi})\left(\frac{q}{d}\right)^{s-1}\sum_{n|d/q}\mu\left(n\right)\chi\left(n\right)n^{s-1}.\end{aligned} \tag{317}$$

□

By formula (269) we have

$$U_{12}(s) = \chi(-1)G(\overline{\chi})q^{\gamma-\delta}L_{\alpha,\beta}(\chi)L_{-\gamma,-\delta}(\chi) \times\sum_{r=1}^{\infty}\sum_{d\in P_{12}}\frac{c_d(r,\chi)\chi(d_{(h)})\chi(k_{(d)})(h,d)^{1-\alpha+\beta}(k,d)^{1+\gamma-\delta}}{d^{2-\alpha+\beta+\gamma-\delta}r^{\alpha+\delta+2s}} \tag{318}$$

and by (270) we have

$$U_{21}(s) = G(\chi)q^{\alpha-\beta}L_{-\alpha,-\beta}(\overline{\chi})L_{\gamma,\delta}(\overline{\chi}) \times\sum_{r=1}^{\infty}\sum_{d\in P_{21}}\frac{c_d(r,\overline{\chi})\overline{\chi}(h_{(d)})\overline{\chi}(d_{(k)})(h,d)^{1+\alpha-\beta}(k,d)^{1-\gamma+\delta}}{d^{2+\alpha-\beta-\gamma+\delta}r^{\beta+\gamma+2s}}. \tag{319}$$

If we ignore the factor $\chi(-1)$ then it is fairly clear that $U_{21}(s)$ can be acquired from $U_{12}(s)$ by swapping h with k, replacing χ with $\overline{\chi}$ and by performing the substitutions $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$. Accordingly, we only need work with $U_{12}(s)$.

Proposition 2.6.6. *Suppose $q|h$. Then $U_{12}(s)$ is non-zero and has the form*

$$\begin{aligned} U_{12}(s) =&\chi(-k)L_{\alpha,\beta}(\chi)L_{-\gamma,-\delta}(\chi)\frac{L(\alpha+\delta+2s,\overline{\chi})L(1+\beta+\gamma+2s,\chi)}{L(2-\alpha+\beta+\gamma-\delta,\chi^2)} \\ &\times\Bigg(\sum_{m|h(q)/q}\frac{1}{m^{\alpha+\gamma+2s}}\Bigg)C_{12,\boldsymbol{\alpha},h,k}(s) \end{aligned} \tag{320}$$

where

$$C_{12,\boldsymbol{\alpha},h,k}(s) = C_{12,\alpha,\beta,\gamma,\delta,h}(s)C_{12,\delta,\gamma,\beta,\alpha,k}(s) \tag{321}$$

and

$$C_{12,\alpha,\beta,\gamma,\delta,h}(s) = \prod_{\substack{p\nmid q\\ p|h}}\left(\frac{C_{12}^{(0)}(s)-p^{-1}C_{12}^{(1)}(s)+p^{-2}C_{12}^{(2)}(s)}{(1-p^{-\alpha-\gamma-2s})(1-\chi(p)^2p^{-2+\alpha-\beta-\gamma+\delta})}\right) \tag{322}$$

with

$$C_{12}^{(0)}(s) = 1-p^{-(h_p+1)(\alpha+\gamma+2s)} \tag{323}$$

$$C_{12}^{(1)}(s) = \chi(p)(p^{\delta-\gamma}+p^{-\beta-\gamma-2s})(1-p^{-h_p(\alpha+\gamma+2s)}) \tag{324}$$

$$C_{12}^{(2)}(s) = \chi(p)^2p^{\delta-\beta}(p^{-2(\gamma+s)}-p^{2(\alpha+s)}p^{-(h_p+1)(\alpha+\gamma+2s)}). \tag{325}$$

Proof. Let

$$H(a,b,c) = \chi(-1)G(\overline{\chi})\sum_{r=1}^\infty\sum_{d\in P_{12}}\frac{c_d(r,\chi)\chi(d_{(h)})\chi(k_{(d)})(h,d)^a(k,d)^b}{d^{a+b}r^{c+1}} \tag{326}$$

so that

$$\frac{q^{\delta-\gamma}U_{12}(s)}{L_{\alpha,\beta}(\chi)L_{-\gamma,-\delta}(\chi)} = H(1-\alpha+\beta,1+\gamma-\delta,-1+\alpha+\delta+2s) \tag{327}$$

Using formula (313) and noting that $|G(\overline{\chi})|^2=q$ we have

$$H(a,b,c) = \chi(-1)q^{c+1}L(c+1,\overline{\chi})\sum_{d\in P_{12}}\frac{\chi(d_{(h)})\chi(k_{(d)})(h,d)^a(k,d)^b}{d^{a+b+c}}g_c(d/q,\chi) \tag{328}$$

where

$$g_c(m,\chi) = \sum_{n|m}\mu(n)\chi(n)n^c. \tag{329}$$

By Lemma 2.6.1 we see that $d = qml$ where $(l,q) = 1$ and m is a divisor of $h(q)/q$. Consequently, $(h,d) = (h^*,l)(h(q),qm) = (h^*,l)qm$ and $(k,d) = (k^*,l)$ since $k = k^*$. Therefore

$$\begin{aligned}&\frac{H(a,b,c)}{q^{c+1}L(c+1,\overline{\chi})}\\ =&\frac{1}{q^{b+c}}\sum_{m|h(q)/q}\frac{1}{m^{b+c}}\sum_{\substack{l=1\\(l,q)=1}}^{\infty}\frac{\chi(l/(h^*,l))\chi(k^*/(k^*,l))(h^*,l)^a(k^*,l)^b}{l^{a+b+c}}g_c(ml,\chi)\\ =&\frac{1}{q^{b+c}}\left(\sum_{m|h(q)/q}\frac{1}{m^{b+c}}\right)\sum_{\substack{l=1\\(l,q)=1}}^{\infty}\frac{\chi(l/(h^*,l))\chi(k^*/(k^*,l))(h^*,l)^a(k^*,l)^b}{l^{a+b+c}}g_c(l,\chi)\end{aligned} \tag{330}$$

since $(m,q) > 1$ for $m > 1$. We now express the Dirichlet series as an Euler product.

If $p \nmid hk$ then we have a local factor of the form

$$1 + (1-\chi(p)p^c)\sum_{j\geq 1}\left(\frac{\chi(p)}{p^{a+b+c}}\right)^m = \frac{1-\chi(p)^2p^{-a-b}}{1-\chi(p)p^{-a-b-c}}. \tag{331}$$

If $p|h$ then we have a local factor of the form

$$1 + (1-\chi(p)p^c)\sum_{j\geq 1}\frac{\chi(p^m/(p^{h_p},p^m))(p^{h_p},p^m)^a}{p^{m(a+b+c)}}. \tag{332}$$

Computing this similarly to as in Proposition 2.6.2 we see that the local factor is given by

$$\frac{(1-\chi(p)p^{-b})(1-\chi(p)^{-a-b-c}) - p^{-(h_p+1)(b+c)}(1-\chi(p)p^{-a})(1-\chi(p)p^c)}{(1-p^{-b-c})(1-\chi(p)p^{-a-b-c})}. \tag{333}$$

Finally, if $p|k$ then we have a local factor of the form

$$\begin{aligned}&\chi(p)^{k_p} + (1-\chi(p)p^c)\sum_{j\geq 1}\frac{\chi(p)^m\chi(p^{k_p}/(p^{k_p},p^m))(p^{k_p},p^m)^b}{p^{m(a+b+c)}}\\ =&\chi(p)^{k_p}\left(1 + (1-\chi(p)p^c)\sum_{j\geq 1}\frac{\chi(p^m/(p^{k_p},p^m))(p^{k_p},p^m)^b}{p^{m(a+b+c)}}\right).\end{aligned} \tag{334}$$

Note that the quantity in parentheses is the same as the local factor at $p|h$ with a and b switched. Therefore,

(335)
$$\frac{H(a,b,c)}{L(c+1,\overline{\chi})} = \frac{\chi(k)}{q^{b-1}}\frac{L(a+b+c,\chi)}{L(a+b,\chi^2)}\Bigg(\sum_{m|h(q)/q}\frac{1}{m^{b+c}}\Bigg)$$
$$\times\prod_{\substack{p\nmid q\\ p|h}}\frac{(1-\chi(p)p^{-b})(1-\chi(p)p^{-a-b-c})-p^{-(h_p+1)(b+c)}(1-\chi(p)p^{-a})(1-\chi(p)p^{c})}{(1-p^{-b-c})(1-\chi(p)^2p^{-a-b})}$$
$$\times\prod_{p|k}\frac{(1-\chi(p)p^{-a})(1-\chi(p)p^{-a-b-c})-p^{-(h_p+1)(a+c)}(1-\chi(p)p^{-b})(1-\chi(p)p^{c})}{(1-p^{-a-c})(1-\chi(p)^2p^{-a-b})}.$$

After inputting the values for a,b,c a short computation gives the result. □

We will not write out the equivalent proposition for $U_{21}(s)$ since it is easily acquired from that of $U_{12}(s)$ by multiplying by $\chi(-1)$ and performing the substitutions $h\leftrightarrow k$, $\chi\leftrightarrow\overline{\chi}$ and $\alpha\leftrightarrow\gamma$, $\beta\leftrightarrow\delta$.

To get a functional equation for $C_{12,\boldsymbol{\alpha},h,k}(s)$ we must incorporate the sum over the divisors of $h(q)/q$ as well as an extra factor of q which, happily, makes an appearance in the next section.

Proposition 2.6.7. *We have*

(336)
$$\frac{1}{q^{\alpha+\delta-s}}\Big(\sum_{m|h(q)/q}\frac{1}{m^{\alpha+\gamma-2s}}\Big)h^{\alpha}k^{\delta}(hk)^{-s}C_{12,\boldsymbol{\alpha},h,k}(-s)$$
$$=\frac{1}{q^{\beta+\delta}}\frac{1}{q^{-\beta-\gamma+s}}\Big(\sum_{m|h(q)/q}\frac{1}{m^{-\alpha-\gamma+2s}}\Big)h^{-\gamma}k^{-\beta}(hk)^{s}C_{12,-\boldsymbol{\gamma},h,k}(s).$$

Proof. Since

(337)
$$\sum_{m|h(q)/q}\frac{1}{m^{\alpha+\gamma-2s}} = \left(\frac{h(q)}{q}\right)^{-\alpha-\gamma+2s}\sum_{m|h(q)/q}\frac{1}{m^{-\alpha-\gamma+2s}}$$

we are required to show

(338)
$$\left(\frac{h}{h(q)}\right)^{\alpha+\gamma-2s}k^{\beta+\delta-2s}C_{12,\boldsymbol{\alpha},h,k}(-s) = C_{12,-\boldsymbol{\gamma},h,k}(s).$$

Also, since

$$C_{12,\boldsymbol{\alpha},h,k}(s) = C_{12,\alpha,\beta,\gamma,\delta,h}(s)C_{12,\delta,\gamma,\beta,\alpha,k}(s) \tag{339}$$

it suffices to show

$$\left(\frac{h}{h(q)}\right)^{\alpha+\gamma-2s} C_{12,\boldsymbol{\alpha},h}(-s) = C_{12,-\boldsymbol{\gamma},h}(s) \tag{340}$$

by symmetry. We must therefore check that

$$p^{h_p(\alpha+\gamma-2s)}\frac{C^{(i)}_{12,\boldsymbol{\alpha}}(-s)}{1-p^{-\alpha-\gamma+2s}} = \frac{C^{(i)}_{12,-\boldsymbol{\gamma}}(s)}{1-p^{\alpha+\gamma-2s}} = -\frac{C^{(i)}_{12,-\boldsymbol{\gamma}}(s)}{p^{\alpha+\gamma-2s}(1-p^{-\alpha-\gamma+2s})} \tag{341}$$

for $i = 0, 1, 2$, each of which can easily be verified by inspection. □

The functional equation for $C_{21}(s)$ is acquired from that of $C_{12}(s)$ by performing the necessary substitutions. After re-arranging the factors of q this gives the following.

Proposition 2.6.8.

$$\begin{aligned}&\frac{1}{q^{\beta+\gamma-s}}\Big(\sum_{m|k(q)/q}\frac{1}{m^{\alpha+\gamma-2s}}\Big)h^{\beta}k^{\gamma}(hk)^{-s}C_{21,\boldsymbol{\alpha},h,k}(-s)\\ =&\frac{1}{q^{\beta+\delta}}\frac{1}{q^{-\alpha-\delta+s}}\Big(\sum_{m|k(q)/q}\frac{1}{m^{-\alpha-\gamma+2s}}\Big)h^{-\delta}k^{-\alpha}(hk)^{s}C_{21,-\boldsymbol{\gamma},h,k}(s).\end{aligned} \tag{342}$$

2.7. Application of the Sum Formulae

2.7.1. The Cases $i = j$. Applying Proposition 2.6.2 to (282) we get

$$\begin{aligned}I^{(1)}_{11,\boldsymbol{\alpha}} =&\frac{1}{h^{1/2-\alpha}k^{1/2-\gamma}}\frac{L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})}{\zeta(2-\alpha+\beta-\gamma+\delta)}\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}\left(\frac{hkq}{\pi^2}\right)^s\\ &\times\Gamma(\alpha+\gamma+2s)2\cos\left(\tfrac{\pi}{2}(\alpha+\gamma+2s)\right)\zeta(\alpha+\gamma+2s)\zeta(1+\beta+\delta+2s)\\ &\times Q_{11}(s)C_{11,\boldsymbol{\alpha},h,k}(s)\int_{-\infty}^{\infty}t^{-\alpha-\gamma-2s}g(s,t)w(t)\left(1+O\left(\tfrac{1+|s|^2}{t}\right)\right)dtds\\ =&\frac{1}{\sqrt{hk}}\frac{L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})}{\zeta(2-\alpha+\beta-\gamma+\delta)}\frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}q^s\\ &\times\zeta(1-\alpha-\gamma-2s)\zeta(1+\beta+\delta+2s)Q_{11}(s)h^{\alpha}k^{\gamma}(hk)^sC_{11,\boldsymbol{\alpha},h,k}(s)\\ &\times\int_{-\infty}^{\infty}\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma}\left(\frac{t}{2}\right)^{-2s}g(s,t)w(t)\left(1+O\left(\tfrac{1+|s|^2}{t}\right)\right)dtds \qquad (343)\end{aligned}$$

where we have used the functional equation

$$\begin{aligned}&\pi^{-2s}\zeta(\alpha+\gamma+2s)\Gamma(\alpha+\gamma+2s)2\cos\left(\frac{\pi}{2}(\alpha+\gamma+2s)\right)\\ =&\pi^{\alpha+\gamma}2^{\alpha+\gamma+2s}\zeta(1-\alpha-\gamma-2s).\end{aligned} \qquad (344)$$

Moving the s-line of integration back to ϵ and using the properties of w along with Stirling's approximation for $g(s,t)$ we get

$$\begin{aligned}I^{(1)}_{11,\boldsymbol{\alpha}} =&\frac{1}{\sqrt{hk}}\frac{L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})}{\zeta(2-\alpha+\beta-\gamma+\delta)}\int_{-\infty}^{\infty}\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma}w(t)\frac{1}{2\pi i}\int_{(\epsilon)}\frac{G(s)}{s}q^s\\ &\times\zeta(1-\alpha-\gamma-2s)\zeta(1+\beta+\delta+2s)Q_{11}(s)h^{\alpha}k^{\gamma}(hk)^sC_{11,\boldsymbol{\alpha},h,k}(s)dsdt\\ &+O\left(\frac{1}{\sqrt{hk}}|L(1,\chi)|^2(hkqT)^{\epsilon}\right). \qquad (345)\end{aligned}$$

We note that this error term is of a lower order than $E(T)$. For $i = j = 2$, the same process used in conjunction with Proposition 2.6.3 gives

$$
\begin{aligned}
I^{(1)}_{22,\boldsymbol{\alpha}} =& \frac{1}{\sqrt{hk}} \frac{L_{-\gamma,-\delta}(\chi)L_{-\alpha,-\beta}(\overline{\chi})}{\zeta(2+\alpha-\beta+\gamma-\delta)} \int_{-\infty}^{\infty} \frac{1}{q^{\beta+\delta}} \left(\frac{t}{2\pi}\right)^{-\beta-\delta} w(t) \frac{1}{2\pi i} \int_{(\epsilon)} \frac{G(s)}{s} q^{-s} \\
& \times \zeta(1-\beta-\delta-2s)\zeta(1+\alpha+\gamma+2s) Q_{22}(s) h^{\beta} k^{\delta} (hk)^{s} C_{22,\boldsymbol{\alpha},h,k}(s) ds dt \\
& + O\left(\frac{1}{\sqrt{hk}} |L(1,\chi)|^2 (hkqT)^{\epsilon}\right). \qquad (346)
\end{aligned}
$$

As usual, the formulas for $I^{(2)}_{11,\boldsymbol{\alpha}}$ and $I^{(2)}_{22,\boldsymbol{\alpha}}$ can be acquired by performing the substitutions $\alpha \leftrightarrow -\gamma$, $\beta \leftrightarrow -\delta$ and multiplying by $X_{\boldsymbol{\alpha},t} \sim q^{-\beta-\delta}(t/2\pi)^{-\alpha-\beta-\gamma-\delta}$ in the integrals over t. With this we have enough information to compute the main terms of the off-diagonals.

Proposition 2.7.1. *Let $A_{\alpha,\beta,\gamma,\delta,q}(s)$ be given by formula (126). Then*

$$
\begin{aligned}
& I^{(1)}_{11,\boldsymbol{\alpha}} + I^{(2)}_{22,\boldsymbol{\alpha}} + I^{(1)}_{22,\boldsymbol{\alpha}} + I^{(2)}_{11,\boldsymbol{\alpha}} \\
=& \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} A_{-\gamma,\beta,-\alpha,\delta,q}(0) h^{\alpha} k^{\gamma} C_{11,\boldsymbol{\alpha},h,k}(0) dt \\
& + \frac{1}{q^{\beta+\delta}} \frac{1}{\sqrt{hk}} \int_{-\infty}^{\infty} w(t) \left(\frac{t}{2\pi}\right)^{-\beta-\delta} A_{\alpha,-\delta,\gamma,-\beta,q}(0) h^{\beta} k^{\delta} C_{22,\boldsymbol{\alpha},h,k}(0) dt \\
& - R\left(\tfrac{-\alpha-\gamma}{2}\right) + R\left(\tfrac{-\beta-\delta}{2}\right) + R'\left(\tfrac{-\alpha-\gamma}{2}\right) - R'\left(\tfrac{-\beta-\delta}{2}\right) + E(T) \qquad (347)
\end{aligned}
$$

where

$$
\begin{aligned}
R(b) =& \frac{1}{2} \frac{q^{b}}{\sqrt{hk}} \frac{L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})}{\zeta(2-\alpha+\beta-\gamma+\delta)} \zeta(1-\alpha+\beta-\gamma+\delta) \\
& \times \int_{-\infty}^{\infty} w(t) \left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} \frac{G(b)}{b} h^{\alpha} k^{\gamma} (hk)^{b} Q_{11}(b) C_{11,\boldsymbol{\alpha},h,k}(b) dt \qquad (348)
\end{aligned}
$$

and

$$
\begin{aligned}
R'(b) =& \frac{1}{2} \frac{q^{-b-\beta-\delta}}{\sqrt{hk}} \frac{L_{-\gamma,-\delta}(\chi)L_{-\alpha,-\beta}(\overline{\chi})}{\zeta(2+\alpha-\beta+\gamma-\delta)} \zeta(1+\alpha-\beta+\gamma-\delta) \\
& \times \int_{-\infty}^{\infty} w(t) \left(\frac{t}{2\pi}\right)^{-\beta-\delta} \frac{G(b)}{b} h^{\beta} k^{\delta} (hk)^{b} Q_{22}(b) C_{22,\boldsymbol{\alpha},h,k}(b) dt. \qquad (349)
\end{aligned}
$$

PROOF. We first shift the contour of $I^{(1)}_{11,\boldsymbol{\alpha}}$ to $-\varepsilon$. We encounter poles at $s = -(\alpha+\gamma)/2$ and $s = -(\beta+\delta)/2$ due to the zeta factors and we also encounter a pole at $s = 0$. The poles at $s = -(\alpha+\gamma)/2$ and $s = -(\beta+\delta)/2$ give rise to the terms $-R\left(\frac{-\alpha-\gamma}{2}\right)$ and $R\left(\frac{-\beta-\delta}{2}\right)$ respectively whilst the pole at zero gives the residue

$$\begin{aligned}
&\frac{L(1-\alpha+\beta,\chi)L(1-\gamma+\delta,\overline{\chi})\zeta(1-\alpha-\gamma)\zeta(1+\beta+\delta)}{\zeta(2-\alpha+\beta-\gamma+\delta)}\\
&\times \prod_{p|q}\left(\frac{1-p^{-1-\beta-\delta}}{1-p^{-2+\alpha-\beta+\gamma-\delta}}\right)\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} G(0)h^{\alpha}k^{\gamma}C_{11,\boldsymbol{\alpha},h,k}(0)dt\\
=&\frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\alpha-\gamma} A_{-\gamma,\beta,-\alpha,\delta,q}(0)h^{\alpha}k^{\gamma}C_{11,\boldsymbol{\alpha},h,k}(0)dt
\end{aligned}$$

In the integral on the new line we make the substitution $s \mapsto -s$. Applying the functional equation (310) we see this new integral cancels with $I^{(2)}_{22,\boldsymbol{\alpha}}$ (recall $G(s)$ is even). Using the same process on $I^{(1)}_{22,\boldsymbol{\alpha}}$ along with the functional equation (311) we get the remaining terms. □

2.7.2. The Cases $i \neq j$.

Using the functional equation

$$\begin{aligned}
&\pi^{-2s}\Gamma(a_i+b_j+2s)2i^{\mathfrak{a}}\cos\left(\frac{\pi}{2}(a_i+b_j+2s+\mathfrak{a})\right)L(a_i+b_j+2s,\overline{\chi})\\
=&\pi^{a_i+b_j}\left(\frac{2}{q}\right)^{a_i+b_j+2s}\overline{G(\chi)}L(1-a_i-b_j-2s,\chi)
\end{aligned}\tag{350}$$

and the same procedure as above we get

$$\begin{aligned}
I^{(1)}_{12,\boldsymbol{\alpha}} =&\mathbf{1}_{q|h}\frac{\chi(k)G(\overline{\chi})}{\sqrt{hk}}\frac{L_{\alpha,\beta}(\chi)L_{-\gamma,-\delta}(\chi)}{L(2-\alpha+\beta+\gamma-\delta,\chi^2)}\int_{-\infty}^{\infty}\left(\frac{t}{2\pi}\right)^{-\alpha-\delta} w(t)\frac{1}{2\pi i}\int_{(\epsilon)}\frac{G(s)}{s}\\
&\times q^{-\alpha-\delta-s}\left(\sum_{m|h(q)/q}\frac{1}{m^{\alpha+\gamma+2s}}\right)L(1-\alpha-\delta-2s,\chi)L(1+\beta+\gamma+2s,\chi)\\
&\times h^{\alpha}k^{\delta}(hk)^{s}C_{12,\boldsymbol{\alpha},h,k}(s)dsdt + O\left(\frac{1}{\sqrt{hk}}|L(1,\chi)|^2(hkqT)^{\epsilon}\right)
\end{aligned}\tag{351}$$

where we have used $\chi(-1)\overline{G(\chi)} = G(\overline{\chi})$. Multiplying by $\chi(-1)$ and performing substitutions $h \leftrightarrow k$, $\chi \leftrightarrow \overline{\chi}$ and $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$ we get

$$\begin{aligned} I^{(1)}_{21,\boldsymbol{\alpha}} =& \mathbf{1}_{q|k} \frac{\overline{\chi}(h)\overline{G(\overline{\chi})}}{\sqrt{hk}} \frac{L_{-\alpha,-\beta}(\overline{\chi})L_{\gamma,\delta}(\overline{\chi})}{L(2+\alpha-\beta-\gamma+\delta,\overline{\chi}^2)} \int_{-\infty}^{\infty} \left(\frac{t}{2\pi}\right)^{-\beta-\gamma} w(t) \frac{1}{2\pi i}\int_{(\epsilon)} \frac{G(s)}{s} \\ &\times q^{-\beta-\gamma-s} \left(\sum_{m|k(q)/q} \frac{1}{m^{\alpha+\gamma+2s}} \right) L(1-\beta-\gamma-2s,\overline{\chi}) L(1+\alpha+\delta+2s,\overline{\chi}) \\ &\times h^{\beta}k^{\gamma}(hk)^s C_{21,\boldsymbol{\alpha},h,k}(s)dsdt + O\left(\frac{1}{\sqrt{hk}}|L(1,\chi)|^2(hkqT)^{\epsilon}\right). \end{aligned} \tag{352}$$

The presence of the indicator functions is due to conditions that $U_{ij}(s)$ be non-zero. By using the functional equations (336), (342) and a similar method to that employed in Proposition 2.7.1 we get

Proposition 2.7.2. *Let $A'_{\alpha,\beta,\gamma,\delta}(s,\chi)$ be given by formula (130) and let*

$$M_{\alpha,\gamma,h}(s) = \sum_{m|h(q)/q} \frac{1}{m^{\alpha+\gamma+2s}}. \tag{353}$$

Then

$$\begin{aligned} & I^{(1)}_{12,\boldsymbol{\alpha}} + I^{(2)}_{12,\boldsymbol{\alpha}} + I^{(1)}_{21,\boldsymbol{\alpha}} + I^{(2)}_{21,\boldsymbol{\alpha}} \\ =& \frac{\mathbf{1}_{q|h}\chi(k)G(\overline{\chi})}{q^{\alpha+\delta}\sqrt{hk}} \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\alpha-\delta} A'_{-\delta,\beta,\gamma,-\alpha}(0,\chi)M_{\alpha,\gamma,h}(0) \\ & \times h^{\alpha}k^{\delta}C_{12,\boldsymbol{\alpha},h,k}(0)dt \\ &+ \frac{\mathbf{1}_{q|k}\overline{\chi}(h)\overline{G(\overline{\chi})}}{q^{\beta+\gamma}\sqrt{hk}} \int_{-\infty}^{\infty} w(t)\left(\frac{t}{2\pi}\right)^{-\beta-\gamma} A'_{\alpha,-\gamma,-\beta,\delta}(0,\overline{\chi})M_{\alpha,\gamma,k}(0) \\ & \times h^{\beta}k^{\gamma}C_{21,\boldsymbol{\alpha},h,k}(0)dt + E(T) \end{aligned} \tag{354}$$

We are now almost in a position to prove Proposition 2.2.2. The goal of the remaining sections is to relate the functions $C_{ii,\boldsymbol{\alpha},h,k}$ to $B_{\boldsymbol{\alpha},h,k}$ and $C_{ij,\boldsymbol{\alpha},h,k}$ to $B'_{\boldsymbol{\alpha},h,k}$ (for $i \neq j$). We will then write the main terms of Propositions 2.7.1 and 2.7.2 in terms of $Z_{\boldsymbol{\alpha},h,k}$ and $Z'_{\boldsymbol{\alpha},h,k}$ respectively and we will show that the residue terms of Proposition 2.7.1 cancel with those of Proposition 2.2.1.

2.8. The Functions $B_{\alpha,h,k}(s,\chi)$ and $B'_{\alpha,h,k}(s,\chi)$

We first recall the formula for B;

$$B_{\boldsymbol{\alpha},h,k}(s,\chi) = B_{\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi})B_{\gamma,\delta,\alpha,\beta,k}(s,\chi) \tag{355}$$

where

$$B_{\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi}) = \left(\prod_{p|h}\frac{\sum_{j\geq 0} f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})p^{-j(1+s)}}{\sum_{j\geq 0} f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^j,\overline{\chi})p^{-j(1+s)}}\right) \tag{356}$$

Proposition 2.8.1. *We have*

$$B_{\alpha,\beta,\gamma,\delta,h}(s,\overline{\chi}) = \prod_{p|h}\left(\frac{B^{(0)}(s)-p^{-1}B^{(1)}(s)+p^{-2}B^{(2)}(s)}{(p^{-\gamma}-\overline{\chi}(p)p^{-\delta})(1-|\chi(p)|^2p^{-2-\alpha-\beta-\gamma-\delta-2s})}\right) \tag{357}$$

where

$$B^{(0)}(s) = p^{-\gamma(h_p+1)} - \overline{\chi}(p)^{h_p+1}p^{-\delta(h_p+1)}, \tag{358}$$

$$B^{(1)}(s) = \overline{\chi}(p)p^{-\gamma-\delta}(p^{-\alpha}+\chi(p)p^{-\beta})(p^{-\gamma h_p}-\overline{\chi}(p)^{h_p}p^{-\delta h_p})p^{-s}, \tag{359}$$

$$B^{(2)}(s) = |\chi(p)|^2p^{-\alpha-\beta-\gamma-\delta}(\overline{\chi}(p)p^{-\delta-\gamma h_p}-\overline{\chi}(p)^{h_p}p^{-\gamma-\delta h_p})p^{-2s} \tag{360}$$

Proof. We begin by computing

$$\sum_{j\geq 0} f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})p^{-j(1+s)}.$$

We have

$$\begin{aligned} f_{\alpha,\beta}(p^m,\chi) &= \sum_{n_1n_2=p^m} n_1^{-\alpha}\chi(n_2)n_2^{-\beta} \\ &= \sum_{0\leq j\leq m} p^{-\alpha(m-j)}\chi(p^j)p^{-\beta j} \\ &= \frac{p^{-\alpha(m+1)}-\chi(p)^{m+1}p^{-\beta(m+1)}}{p^{-\alpha}-\chi(p)p^{-\beta}}. \end{aligned} \tag{361}$$

Therefore

$$\sum_{j\geq 0} f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})p^{-j(1+s)}$$
$$=\sum_{j\geq 0}\frac{(p^{-\alpha(j+1)}-\chi(p)^{j+1}p^{-\beta(j+1)})(p^{-\gamma(h_p+j+1)}-\overline{\chi}(p)^{h_p+j+1}p^{-\delta(h_p+j+1)})}{(p^{-\alpha}-\chi(p)p^{-\beta})(p^{-\gamma}-\overline{\chi}(p)p^{-\delta})}p^{-j(s+1)}$$

Expanding the numerator out and performing the summation we have

$$\sum_{j\geq 0} f_{\alpha,\beta}(p^j,\chi)f_{\gamma,\delta}(p^{h_p+j},\overline{\chi})p^{-j(1+s)}$$
$$=\Bigg(\frac{p^{-\alpha-\gamma(h_p+1)}}{1-p^{-1-s-\alpha-\gamma}}-\frac{\chi(p)p^{-\beta-\gamma(h_p+1)}}{1-\chi(p)p^{-1-\beta-\gamma-s}}-\frac{\overline{\chi}(p)^{h_p+1}p^{-\alpha-\delta(h_p+1)}}{1-\overline{\chi}(p)p^{-1-\alpha-\delta-s}}$$
$$+\frac{|\chi(p)|^2\overline{\chi}(p)^{h_p}p^{-\beta-\delta(h_p+1)}}{1-|\chi(p)|^2p^{-1-\beta-\delta-s}}\Bigg)(p^{-\alpha}-\chi(p)p^{-\beta})^{-1}(p^{-\gamma}-\overline{\chi}(p)p^{-\delta})^{-1}$$

which simplifies to

$$\Bigg(\frac{p^{-\gamma(h_p+1)}}{(1-p^{-1-s-\alpha-\gamma})(1-\chi(p)p^{-1-\beta-\gamma-s})}$$
$$-\frac{\overline{\chi}(p)^{h_p+1}p^{-\delta(h_p+1)}}{(1-\overline{\chi}(p)p^{-1-\alpha-\delta-s})(1-|\chi(p)|^2p^{-1-\beta-\delta-s})}\Bigg)(p^{-\gamma}-\overline{\chi}(p)p^{-\delta})^{-1}$$
$$=\left(B^{(0)}(s)-p^{-1}B^{(1)}(s)+p^{-2}B^{(2)}(s)\right)\Bigg((p^{-\gamma}-\overline{\chi}(p)p^{-\delta})(1-p^{-1-s-\alpha-\gamma})$$
$$\times(1-\chi(p)p^{-1-\beta-\gamma-s})(1-\overline{\chi}(p)p^{-1-\alpha-\delta-s})(1-|\chi(p)|^2p^{-1-\beta-\delta-s})\Bigg)^{-1}.$$

Setting $h_p=0$ and dividing the above by the resulting expression gives the result.

□

Recalling the formula for $B'_{\boldsymbol{\alpha},h,k}(s,\chi)$;

$$B'_{\boldsymbol{\alpha},h,k}(s,\chi)=B'_{\alpha,\beta,\gamma,\delta,h}(s,\chi)B'_{\gamma,\delta,\alpha,\beta,k}(s,\chi) \tag{362}$$

where

$$B'_{\alpha,\beta,\gamma,\delta,h}(s,\chi)=\prod_{p|h}\frac{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^{h_p+j})p^{-j(1+s)}}{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j(1+s)}}. \tag{363}$$

Following the same method as for $B_{\boldsymbol{\alpha},h,k}(s,\chi)$ we get

Proposition 2.8.2.

$$B'_{\alpha,\beta,\gamma,\delta,h}(s,\chi) = \prod_{p|h}\left(\frac{B'^{(0)}(s) - p^{-1}B'^{(1)}(s) + p^{-2}B'^{(2)}(s)}{(p^{-\gamma}-p^{-\delta})(1-\chi(p)^2p^{-2-\alpha-\beta-\gamma-\delta-2s})}\right) \tag{364}$$

where

$$B'^{(0)}(s) = p^{-\gamma(h_p+1)} - p^{-\delta(h_p+1)}, \tag{365}$$

$$B'^{(1)}(s) = \chi(p)p^{-\gamma-\delta}(p^{-\alpha}+p^{-\beta})(p^{-\gamma h_p} - p^{-\delta h_p})p^{-s}, \tag{366}$$

$$B'^{(2)}(s) = \chi(p)^2p^{-\alpha-\beta-\gamma-\delta}(p^{-\delta-\gamma h_p} - p^{-\gamma-\delta h_p})p^{-2s}. \tag{367}$$

2.9. Relating Terms

2.9.1. The Cases $i = j$. We now work on the terms in (347), putting them in terms of $Z_{\boldsymbol{\alpha},h,k}(s)$.

Lemma 2.9.1. *We have*

$$h^\alpha k^\gamma C_{11,\boldsymbol{\alpha},h,k}(0) = B_{-\gamma,\beta,-\alpha,\delta,h,k}(0). \tag{368}$$

This implies, by use of the functional equation (311), that

$$h^\beta k^\delta C_{22,\boldsymbol{\alpha},h,k}(0) = B_{\alpha,-\delta,\gamma,-\beta,h,k}(0). \tag{369}$$

Proof. Once again, by symmetry it suffices to show

$$h^\alpha C_{11,\alpha,\beta,\gamma,\delta,h}(0) = B_{-\gamma,\beta,-\alpha,\delta,h}(0). \tag{370}$$

We may first split the product in $B_{-\gamma,\beta,-\alpha,\delta,h}(0)$ over primes $p|q$ and primes $p \nmid q$. If $p|q$ then the local factor is given by $p^{\alpha h_p}$. We may therefore remove a factor of $h(q)^\alpha$ from both sides of (370) and henceforth only consider the products over primes $p \nmid q$. The problem now reduces to showing

$$p^{\alpha h_p}\frac{C^{(i)}_{11,\alpha,\beta,\gamma,\delta,h}(0)}{1-\overline{\chi}(p)p^{-\alpha-\delta}} = \frac{B^{(i)}_{-\gamma,\beta,-\alpha,\delta,h}(0)}{p^\alpha - \overline{\chi}(p)p^{-\delta}}$$

which reduces to showing

$$p^{\alpha h_p}C^{(i)}_{11,\alpha,\beta,\gamma,\delta,h}(0) = p^{-\alpha}B^{(i)}_{-\gamma,\beta,-\alpha,\delta,h}(0)$$

for $i = 0, 1, 2$ which can be checked by inspection. □

We now demonstrate the cancellation of the R terms of Proposition 2.7.1 with the residue terms Proposition 2.2.1. We first work on the R terms with negative coefficient.

Lemma 2.9.2. *We have*

$$R\left(\frac{-\alpha-\gamma}{2}\right) = J^{(1)}_{\alpha,\gamma}. \tag{371}$$

Proof. To prove this it suffices to show

$$\begin{aligned} &\frac{1}{2}\frac{L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})\zeta(1-\alpha+\beta-\gamma+\delta)}{\zeta(2-\alpha+\beta-\gamma+\delta)}\left(\frac{h}{k}\right)^{\frac{\alpha-\gamma}{2}} \\ &\times Q_{11}\left(\frac{-\alpha-\gamma}{2}\right)C_{11,\boldsymbol{\alpha},h,k}\left(\frac{-\alpha-\gamma}{2}\right) \\ =&\frac{\mathrm{Res}_{2s=-\alpha-\gamma}(Z_{\alpha,\beta,\gamma,\delta,h,k}(2s))}{(hk)^{-\frac{\alpha+\gamma}{2}}} \\ =&(hk)^{\frac{\alpha+\gamma}{2}}\mathrm{Res}_{2s=-\alpha-\gamma}(A_{\alpha,\beta,\gamma,\delta,q}(2s))B_{\alpha,\beta,\gamma,\delta,h,k}(-\alpha-\gamma) \end{aligned} \tag{372}$$

which reduces to showing

$$h^{-\gamma}C_{11,\boldsymbol{\alpha},h}\left(\frac{-\alpha-\gamma}{2}\right) = B_{\alpha,\beta,\gamma,\delta,h}(-\alpha-\gamma). \tag{373}$$

Once again we may remove a factor of $h(q)^{-\gamma}$ from both sides and the problem reduces to showing that the following identities hold

$$p^{-\gamma h_p}\frac{C^{(i)}_{11,\alpha,\beta,\gamma,\delta,h}(-(\alpha+\gamma)/2)}{1-\overline{\chi}(p)p^{\gamma-\delta}} = \frac{B^{(i)}_{\alpha,\beta,\gamma,\delta,h}(-\alpha-\gamma)}{p^{-\gamma}-\overline{\chi}(p)p^{-\delta}} \tag{374}$$

for $i = 0, 1, 2$ which can be checked by inspection. □

Lemma 2.9.3. *We have*

$$R'\left(\frac{-\beta-\delta}{2}\right) = J^{(1)}_{\beta,\delta}$$

PROOF. We need to show

$$\begin{aligned}(375)\quad &\frac{1}{2}\frac{L_{-\gamma,-\delta}(\chi)L_{-\alpha,-\beta}(\overline{\chi})\zeta(1+\alpha-\beta+\gamma-\delta)}{\zeta(2+\alpha-\beta+\gamma-\delta)}\left(\frac{h}{k}\right)^{\frac{\beta-\delta}{2}}\\ &\quad\times Q_{22}\left(\frac{-\beta-\delta}{2}\right)C_{22,\boldsymbol{\alpha},h,k}\left(\frac{-\beta-\delta}{2}\right)\\ =&(hk)^{\frac{\beta+\delta}{2}}\mathrm{Res}_{2s=-\beta-\delta}(Z_{\boldsymbol{\alpha},h,k}(2s))\end{aligned}$$

which reduces to showing

$$(376)\quad h^{-\delta}h(q)^{-\gamma+\delta}C_{22,\boldsymbol{\alpha},h}\left(\frac{-\beta-\delta}{2}\right)=B_{\boldsymbol{\alpha},h}(-\beta-\delta).$$

The required identities are thus

$$p^{-\delta h_p}\frac{C^{(i)}_{22,\boldsymbol{\alpha},h}\left(\frac{-\beta-\delta}{2}\right)}{1-\overline{\chi}(p)p^{\gamma-\delta}}=\frac{B^{(i)}_{\alpha,\beta,\gamma,\delta,h}(-\beta-\delta)}{p^{-\gamma}-\overline{\chi}(p)p^{-\delta}}$$

for $i=0,1,2$ each of which can be verified by inspection. □

The cancellation of the residue terms of $I_D^{(2)}$ of proposition 2.2.1 is given by the following.

Lemma 2.9.4. *We have*

$$(377)\quad R\left(\frac{-\beta-\delta}{2}\right)=-J^{(2)}_{\beta,\delta},\qquad R'\left(\frac{-\alpha-\gamma}{2}\right)=-J^{(2)}_{\alpha,\gamma}.$$

PROOF. The first of these requires showing

$$\begin{aligned}(378)\quad &\frac{1}{2}\frac{L_{\alpha,\beta}(\chi)L_{\gamma,\delta}(\overline{\chi})\zeta(1-\alpha+\beta-\gamma+\delta)}{\zeta(2-\alpha+\beta-\gamma+\delta)}h^{\alpha}k^{\gamma}(hk)^{\frac{-\beta-\delta}{2}}\\ &\quad\times Q_{11}\left(\frac{-\beta-\delta}{2}\right)C_{11,\boldsymbol{\alpha},h,k}\left(\frac{-\beta-\delta}{2}\right)\\ =&\frac{\mathrm{Res}_{2s=\beta+\delta}(Z_{-\boldsymbol{\gamma},h,k}(2s))}{(hk)^{\frac{\beta+\delta}{2}}}\end{aligned}$$

which reduces to showing

$$(379)\quad h^{\alpha}C_{11,\boldsymbol{\alpha},h}\left(\frac{-\beta-\delta}{2}\right)=B_{-\boldsymbol{\gamma},h}(\beta+\delta).$$

By (310) we have

$$h^{\alpha}C_{11,\boldsymbol{\alpha},h}\left(\frac{-\beta-\delta}{2}\right) = h^{\beta}h(q)^{\alpha-\beta}C_{22,-\boldsymbol{\gamma},h}\left(\frac{\beta+\delta}{2}\right) \tag{380}$$

but by (376) we have

$$h^{-\delta}h(q)^{-\gamma+\delta}C_{22,\boldsymbol{\alpha},h}\left(\frac{-\beta-\delta}{2}\right) = B_{\boldsymbol{\alpha},h}(-\beta-\delta) \tag{381}$$

and so by permuting the shift parameters we can conclude the result. The result for $R'\left(\frac{-\alpha-\gamma}{2}\right)$ requires

$$h^{\beta}h(q)^{\alpha-\beta}C_{22,\boldsymbol{\alpha},h}\left(\frac{-\alpha-\gamma}{2}\right) = B_{-\boldsymbol{\gamma},h}(\alpha+\gamma) \tag{382}$$

but by (311) we have

$$h^{\beta}h(q)^{\alpha-\beta}C_{22,\boldsymbol{\alpha},h}\left(\frac{-\alpha-\gamma}{2}\right) = h^{\alpha}C_{11,-\boldsymbol{\gamma},h}\left(\frac{\alpha+\gamma}{2}\right). \tag{383}$$

In Lemma 2.9.2 it was shown that

$$h^{-\gamma}C_{11,\boldsymbol{\alpha},h}\left(\frac{-\alpha-\gamma}{2}\right) = B_{\boldsymbol{\alpha},h}(-\alpha-\gamma) \tag{384}$$

and so by permuting the shifts again we have the desired result. □

2.9.2. The Cases $i \neq j$.

Lemma 2.9.5. *Suppose $q|h$. Then*

$$q^{-\alpha}M_{\alpha,\gamma,h}(0)h^{\alpha}k^{\delta}C_{12,\boldsymbol{\alpha},h,k}(0) = B'_{-\delta,\beta,\gamma,-\alpha,h/q,k}(0,\chi) \tag{385}$$

PROOF. We first equate the factors that are given by products over primes $p|q$. By inspection of the Euler product of B' we see that

$$B'_{-\delta,\beta,\gamma,-\alpha,h/q,k}(0,\chi) = \sigma_{\gamma,-\alpha}(h(q)/q)B'_{-\delta,\beta,\gamma,-\alpha,h^*}(0,\chi)B'_{\gamma,-\alpha,-\delta,\beta,k}(0,\chi). \tag{386}$$

But

$$\sigma_{\gamma,-\alpha}(h(q)/q) = \sum_{m|h(q)/q} m^{-\gamma}\left(\frac{h(q)/q}{m}\right)^{\alpha} = \left(\frac{h(q)}{q}\right)^{\alpha} M_{\alpha,\gamma,h}(0) \tag{387}$$

and so we're done. As usual, for the products over primes $p \nmid q$ we must check that the local factors of $(h^*)^\alpha C_{12,\boldsymbol{\alpha},h}(0)$ and $B'_{-\delta,\beta,\gamma,-\alpha,h^*}(0,\chi)$ match. The required identities are thus

$$p^{\alpha h_p} \frac{C^{(i)}_{12,\boldsymbol{\alpha},h}(0)}{1-p^{-\alpha-\gamma}} = \frac{B^{(i)}_{-\delta,\beta,\gamma,-\alpha,h^*}(0,\chi)}{p^{-\gamma}-p^{\alpha}} \tag{388}$$

for $i = 0, 1, 2$ each of which is easily verified by inspection. □

By performing the substitutions $h \leftrightarrow k$, $\chi \leftrightarrow \overline{\chi}$ and $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$ the above equation becomes

$$q^{-\gamma} M_{\alpha,\gamma,k}(0) h^\beta k^\gamma C_{21,\boldsymbol{\alpha},h,k}(0) = B'_{-\beta,\delta,\alpha,-\gamma,k/q,h}(0,\overline{\chi}).$$

On recalling the equation $B'_{\boldsymbol{\alpha},h,k}(0) = B'_{\alpha,\beta,\gamma,\delta,h}(0) B'_{\gamma,\delta,\alpha,\beta,k}(0)$ we acquire the following.

Lemma 2.9.6. *Suppose $q|k$. Then*

$$q^{-\gamma} M_{\alpha,\gamma,k}(0) h^\beta k^\gamma C_{21,\boldsymbol{\alpha},h,k}(0) = B'_{\alpha,-\gamma,-\beta,\delta,h,k/q}(0,\overline{\chi}). \tag{389}$$

Combining the Lemmas of sections 2.9.1 and 2.9.2 with Propositions 2.7.1 and 2.7.2 respectively we get Proposition 2.2.2 and hence Theorem 1.

CHAPTER 3

The Moments Conjecture for the Dedekind Zeta Function of a Galois Extension

In this chapter we extend the hybrid product method of Gonek, Hughes and Keating to the Dedekind zeta function of a Galois extension. We start by deducing the hybrid product itself (Theorem 2). Essentially, this states that there exists a certain product over prime ideals $P_{\mathbb{K}}(s, X)$, and a certain product over non-trivial zeros $Z_{\mathbb{K}}(s, X)$, such that

$$\zeta_{\mathbb{K}}(s) \sim P(s, X)Z(s, X)$$

as $|s| \to \infty$ where the parameter X allows one to mediate between the two products: taking X large (resp. small) gives a majority contribution from $P(s, X)$ (resp. $Z(s, X)$).

After proving the hybrid product we discuss its effect on moments. This begins with the splitting conjecture (Conjecture 1). This claims that the main term of the $2k$th moment of the product should split as the product of the moments. Under this assumption, the problem is reduced to evaluating the moments of P and Z.

We first calculate the $2k$th moment of $P_{\mathbb{K}}(\frac{1}{2} + it, X)$ for Galois extensions. We then conjecture the $2k$th moment of $Z_{\mathbb{K}}(\frac{1}{2} + it, X)$ for Galois extensions using random matrix theory. The majority of this chapter lies in proving this conjecture for $k = 1$ and $\mathbb{K}$ quadratic. Here, our main tool will be the formula of the previous chapter. Finally, we give an alternative derivation of this last conjecture by using a modification of the moments recipe of Conrey et al. The modification again utilises work of the previous chapter.

3.1. The hybrid product

In this section we prove Theorem 2. For this we require a smoothed version of the explicit formula given in Lemma 3.1.3 below. This is based on a result originally due to Bombieri and Hejhal [4]. The proof follows similarly to that of the classical explicit formula and uses the following two Lemmas. As usual, we denote the non-trivial zeros of $\zeta_{\mathbb{K}}(s)$ by ρ.

Lemma 3.1.1. *Suppose $t \neq \gamma$ for any zero $\rho = \beta + i\gamma$, $0 \leq \beta \leq 1$, of $\zeta_{\mathbb{K}}(s)$. Then*

$$\sum_{\rho} \frac{1}{1+(t-\gamma)^2} \ll \log t.$$

This implies $N_1(t) := |\{\gamma : |t-\gamma| < 1\}| \ll \log t$.

Proof. By logarithmic differentiation of the Hadamard product (55) we have

$$\frac{\zeta_{\mathbb{K}}'(s)}{\zeta_{\mathbb{K}}(s)} = B - \frac{1}{s-1} - (r_1+r_2)\frac{\Gamma'(\frac{s+2}{2})}{\Gamma(\frac{s+2}{2})} - r_2\frac{\Gamma'(\frac{s+3}{2})}{\Gamma(\frac{s+3}{2})} + \sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{s-\rho}\right). \tag{390}$$

Using the bound $\Gamma'(\sigma+it)/\Gamma(\sigma+it) \ll \log t$, valid for $|t| \geq 1$, we see

$$\sum_{\rho} \Re\left(\frac{1}{\rho} + \frac{1}{2+it-\rho}\right) \ll \log t + \Re\left(\frac{\zeta_{\mathbb{K}}'(2+it)}{\zeta_{\mathbb{K}}(2+it)}\right) \ll \log t.$$

The first result now follows on noting

$$\sum_{\rho} \frac{1}{1+(t-\gamma)^2} \ll \sum_{\rho} \Re\left(\frac{1}{\rho} + \frac{1}{2+it-\rho}\right).$$

□

Lemma 3.1.2. *For $-1 \leq \sigma \leq 2$ and $t \neq \gamma$ for any zero ρ we have*

$$\frac{\zeta_{\mathbb{K}}'(s)}{\zeta_{\mathbb{K}}(s)} = \sum_{\substack{\rho \\ |t-\gamma|<1}} \frac{1}{s-\rho} + O(\log t)$$

PROOF. Subtracting the logarithmic derivative of $\zeta_{\mathbb{K}}(2+it)$ from that of $\zeta_{\mathbb{K}}(s)$ using (390) gives

$$\begin{aligned}\frac{\zeta_{\mathbb{K}}'(s)}{\zeta_{\mathbb{K}}(s)} &= \sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)+O(\log t)\\ &=\Bigg(\sum_{\substack{\rho\\ |t-\gamma|<1}}+\sum_{\substack{\rho\\ |t-\gamma|\geq 1}}\Bigg)\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)+O(\log t)\\ &=\sum_{\substack{\rho\\ |t-\gamma|<1}}\frac{1}{s-\rho}+O(N_1(t))+O\Bigg(\sum_{\substack{\rho\\ |t-\gamma|\geq 1}}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)\Bigg)+O(\log t)\\ &=\sum_{\substack{\rho\\ |t-\gamma|<1}}\frac{1}{s-\rho}+O\left(\sum_{\rho}\frac{1}{1+(t-\gamma)^2}\right)+O(\log t).\end{aligned}$$

□

Lemma 3.1.3. *Let $u(x)$ be a real, nonnegative smooth function with compact support in $[1,e]$, and let u be normalized so that if*

$$v(t)=\int_t^\infty u(x)dx, \tag{391}$$

then $v(0)=1$. Let

$$\hat{u}(z)=\int_0^\infty u(x)x^{z-1}dx \tag{392}$$

be the Mellin transform of u. Then for s not a zero or a pole of $\zeta_{\mathbb{K}}(s)$ we have

$$\begin{aligned}-\frac{\zeta_{\mathbb{K}}'(s)}{\zeta_{\mathbb{K}}(s)} &= \sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}}\frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s}v(e^{\log\mathfrak{N}(\mathfrak{a})/\log X})-\sum_{\rho}\frac{\hat{u}(1-(s-\rho)\log X)}{s-\rho}\\ &\quad-(r_1+r_2)\sum_{m=1}^{\infty}\frac{\hat{u}(1-(s+2m)\log X)}{s+2m}\\ &\quad-r_2\sum_{j=0}^{\infty}\frac{\hat{u}(1-(s+2j+1)\log X)}{s+2j+1}-\frac{\hat{u}(1-(s-1)\log X)}{s-1}\end{aligned} \tag{393}$$

where $\Lambda(\mathfrak{a})$ is as in (138) and r_1, r_2 are, respectively, the number of real and complex embeddings $\mathbb{K} \to \mathbb{C}$. The sum over primes is finite and the other sums are absolutely convergent.

PROOF. Let $c = \max\{2, 2 - \Re(s)\}$. By absolute convergence we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\zeta_{\mathbb{K}}'(s+z)}{\zeta_{\mathbb{K}}(s+z)}\hat{u}(1+z\log X)\frac{dz}{z} = -\sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\hat{u}(1+z\log X)}{\mathfrak{N}(\mathfrak{a})^z}\frac{dz}{z}$$
$$= -\sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s} v(e^{\log\mathfrak{N}(\mathfrak{a})/\log X}).$$

Let $M_T(d)$ denote the rectangular contour with vertices $(c - iT, c + iT, -d + iT, -d - iT)$, $d > 0$. Then, by the theory of residues we have

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{M_T(d)} \frac{\zeta_{\mathbb{K}}'(s+z)}{\zeta_{\mathbb{K}}(s+z)}\hat{u}(1+z\log X)\frac{dz}{z}\\
=&\frac{\zeta_{\mathbb{K}}'(s)}{\zeta_{\mathbb{K}}(s)} - \sum_{|\gamma|\le T}\frac{\hat{u}(1-(s-\rho)\log X)}{s-\rho} - (r_1+r_2)\sum_{m\le\lfloor d/2\rfloor}\frac{\hat{u}(1-(s+2m)\log X)}{s+2m}\\
&- r_2\sum_{j\le\lfloor (d-1)/2\rfloor}\frac{\hat{u}(1-(s+2j+1)\log X)}{s+2j+1} - \frac{\hat{u}(1-(s-1)\log X)}{s-1}.
\end{aligned}\tag{394}$$

Since

$$\int_{c-iT}^{c+iT} = \int_{M_T(d)} - \int_{c+iT}^{-d+iT} - \int_{-d+iT}^{-d-iT} - \int_{-d-iT}^{c-iT}$$

it remains to show that these other integrals vanish in the limit of large T and d. We first consider the integral over the line $(-d \pm iT)$. Now as long as σ is negative and bounded away from a negative integer we have

$$\frac{\Gamma'(s)}{\Gamma(s)} \ll \log(|s|+1). \tag{395}$$

Hence by logarithmic differentiation of the functional equation of $\zeta_{\mathbb{K}}(s)$ we have

$$\begin{aligned}\frac{\zeta'_{\mathbb{K}}(s)}{\zeta_{\mathbb{K}}(s)} &\ll \frac{\zeta'_{\mathbb{K}}(1-s)}{\zeta_{\mathbb{K}}(1-s)} + \frac{\Gamma'(s)}{\Gamma(s)} \\ &\ll 1 + \log(|s|+1).\end{aligned} \tag{396}$$

As such,

$$\frac{\zeta'_{\mathbb{K}}(z+s)}{\zeta_{\mathbb{K}}(z+s)} \ll \log(|z+s|+1). \tag{397}$$

Hence, if d is a half integer

$$\int_{-d+iT}^{-d-iT} \frac{\zeta'_{\mathbb{K}}(s+z)}{\zeta_{\mathbb{K}}(s+z)} \hat{u}(1+z\log X)\frac{dz}{z} \ll T\frac{\log(|d+s|+1)}{|d|}\frac{\max(u(x))}{(d\log X+1)} \tag{398}$$

and this vanishes as $d \to \infty$ through the half integers.

The behaviours of the other two integrals are equivalent so we only consider the case in the upper half-plane. We split the line $(-d+iT, c+iT)$ at the point $b+iT$ where $b = -1 - \Re(s)$. Then similarly to the above we have

$$\begin{aligned}\int_{-d+iT}^{b+iT} \frac{\zeta'_{\mathbb{K}}(s+z)}{\zeta_{\mathbb{K}}(s+z)} \hat{u}(1+z\log X)\frac{dz}{z} &\ll \frac{\log|T+s|}{T}\int_{-d}^{b} \hat{u}(1+y\log X)dy \\ &\ll_{X,s} \frac{\log T}{T}.\end{aligned} \tag{399}$$

For the integral over the line $(b+iT,\ c+iT)$ we restrict T in such a way that $|T-\gamma|^{-1} \ll \log T$. This is possible since $N_1(T) \ll \log T$. For such T we have

$$\frac{\zeta'_{\mathbb{K}}(\sigma+iT)}{\zeta_{\mathbb{K}}(\sigma+iT)} = \sum_{|T-\gamma|<1} \frac{1}{\sigma+iT-\rho} + O(\log T) \ll N_1(T)\log T \ll \log^2 T.$$

Consequently,

$$\int_{b+iT}^{c+iT} \frac{\zeta'_{\mathbb{K}}(s+z)}{\zeta_{\mathbb{K}}(s+z)} \hat{u}(1+z\log X)\frac{dz}{z} \ll_X \frac{\log^2 T}{T}. \tag{400}$$

If we vary T by a bounded amount then the sum over zeros in (394) incurs $O(\log T)$ extra terms. These terms are all $O_{X,s}(T^{-1})$ so if we want to relax the restriction on T we must take an error of $O(T^{-1}\log T)$. Since this is less than our main error term we can let $T \to \infty$ after d.

The support condition on u implies $v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X}) = 0$ when $\mathfrak{N}(\mathfrak{a}) > X$. Since there are at most n prime ideals above the rational prime p we see the sum over $\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{K}}$ is indeed finite. By integrating $\hat{u}$ by parts l times we have

$$|\hat{u}(z)| \ll \max_x |u^{(l)}(x)| e^{\max(\Re(z)+l,0)}(1+|z|)^{-l}. \tag{401}$$

Therefore, the sums over ρ, j, m in (393) are absolutely convergent. $\square$

Theorem 2. *Let $X \geq 2$ and let l be any fixed positive integer. Let $u(x) = Xf(X\log(x/e)+1)/x$ where f is a smooth, real, nonnegative function of total mass one with support in $[0,1]$. Thus, $u(x)$ is a real, non-negative, smooth function with mass 1 and compact support on $[e^{1-1/X}, e]$. Set*

$$U(z) = \int_0^\infty u(x) E_1(z \log x) dx,$$

where $E_1(z) = \int_z^\infty e^{-w}/w\, dw$. Then for $\sigma \geq 0$ and $|t| \geq 2$ we have

$$\zeta_{\mathbb{K}}(s) = P_{\mathbb{K}}(s,X) Z_{\mathbb{K}}(s,X)\left(1 + O\left(\frac{X^{l+2}}{(|s|\log X)^l}\right) + O(X^{-\sigma}\log X)\right) \tag{402}$$

where

$$P_{\mathbb{K}}(s,X) = \exp\left(\sum_{\substack{\mathfrak{a}\subseteq \mathcal{O}_{\mathbb{K}} \\ \mathfrak{N}(\mathfrak{a}) \leq X}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log \mathfrak{N}(\mathfrak{a})}\right) \tag{403}$$

with

$$\Lambda(\mathfrak{a}) = \begin{cases} \log \mathfrak{N}(\mathfrak{p}) & \text{if } \mathfrak{a} = \mathfrak{p}^m, \\ 0 & \text{otherwise,} \end{cases} \tag{404}$$

and

$$Z_{\mathbb{K}}(s,X) = \exp\left(-\sum_\rho U((s-\rho)\log X)\right), \tag{405}$$

where the sum is over all non-trivial zeros of $\zeta_{\mathbb{K}}(s)$.

PROOF. Let $r_{\mathbb{K}}(n)$ represent the number of ideals of $\mathcal{O}_{\mathbb{K}}$ with norm n. Then

$$\zeta_{\mathbb{K}}(s) = \sum_{n=1}^{\infty} \frac{r_{\mathbb{K}}(n)}{n^s} = 1 + \sum_{n=2}^{\infty} \frac{r_{\mathbb{K}}(n)}{n^s}. \tag{406}$$

By a standard result of Dirichlet series (see Theorem 11.2, [1]) we have $\zeta_{\mathbb{K}}(\sigma+it) \to 1$ as $\sigma \to \infty$ uniformly in t. Integrating (393) along the horizontal line from $s_0 = \sigma_0 + it_0$ to $+\infty$, with $\sigma_0 \geq 0$ and $|t_0| \geq 2$, we get on the left hand side $-\log \zeta_{\mathbb{K}}(s_0)$ if the line does not pass through a zero. If it does, then we define $\log \zeta_{\mathbb{K}}(\sigma + it) = \lim_{\epsilon \to 0^+} \frac{1}{2}\big(\log \zeta_{\mathbb{K}}(\sigma + i(t+\epsilon)) + \log \zeta_{\mathbb{K}}(\sigma + i(t-\epsilon))\big)$. Also, we take the principal branch of the logarithm so that $\lim_{\sigma\to\infty} \log \zeta_{\mathbb{K}}(\sigma + it) = 0$. On the right side we formally push the integrals through the sums. We then encounter integrals of the form

$$\begin{aligned} \int_{s_0}^{\infty} \frac{\hat{u}(1-(s-z)\log X)}{s-z} ds = &\int_0^{\infty} u(x) E_1((s_0 - z)\log X \log x) dx \\ =& U((s_0 - z)\log X). \end{aligned} \tag{407}$$

This equation holds as long as $s_0 - z$ is not real and negative (E_1 has a branch cut along the negative real axis). If it is, then again we define $U((s_0 - z)\log X) = \lim_{\epsilon\to 0^+} \frac{1}{2}\big(U((s_0 - z)\log X + i\epsilon) + U((s_0 - z)\log X - i\epsilon)\big)$. We now have

$$\begin{aligned} \log \zeta_{\mathbb{K}}(s_0) = &\sum_{\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{K}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{s_0} \log \mathfrak{N}(\mathfrak{a})} v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X}) - \sum_{\rho} U((s_0 - z)\log X) \\ &- (r_1 + r_2) \sum_{m=1}^{\infty} U((s_0 + 2m)\log X) \\ &- r_2 \sum_{j=1}^{\infty} U((s_0 + 2j + 1)\log X) - U((s_0 - 1)\log X) \end{aligned} \tag{408}$$

where the interchange of integration and summation is justified by absolute convergence. This equation is valid for all points $\Re(s) \geq 0$ not equal to a zero of $\zeta_{\mathbb{K}}(s)$. We wish to estimate the last three terms of the above.

Suppose that $u(x) = Xf(X\log(x/e) + 1)/x$ where f is smooth, real, nonnegative, of total mass one and with support in $[0,1]$. Then $|u^{(l)}(x)| \ll X^{l+1}$ and therefore, by (401),

$$\hat{u}(s) \ll \frac{e^{\max(\sigma,0)} X^{l+1}}{(1+|s|)^l}. \tag{409}$$

This implies for x real

$$\begin{aligned} U((s_0-x)\log X) &= \int_{s_0}^{\infty} \frac{\hat{u}(1-(s-x)\log X)}{s-x} ds \\ &\ll \frac{X^{l+1}}{\log^l X} \int_{\sigma_0}^{\infty} \frac{X^{\max(x-\sigma,0)}}{|\sigma_0 - x + it_0|^{l+1}} ds \\ &\ll \frac{X^{l+1+\max(x-\sigma_0,0)}}{(|s_0-x|\log X)^l} \end{aligned} \tag{410}$$

since $|t_0| > 2$. Applying this estimate gives

$$\begin{aligned} \log \zeta_{\mathbb{K}}(s_0) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}} & \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{s_0} \log \mathfrak{N}(\mathfrak{a})} v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X}) - \sum_{\rho} U((s_0-z)\log X) \\ & + O\left(\frac{X^{l+2}}{(|s_0|\log X)^l}\right). \end{aligned} \tag{411}$$

On replacing s_0 by s and exponentiating we now have

$$\zeta_{\mathbb{K}}(s) = \tilde{P}_{\mathbb{K}}(s,X) Z_{\mathbb{K}}(s,X)\left(1 + O\left(\frac{X^{l+2}}{(|s|\log X)^l}\right)\right) \tag{412}$$

where

$$\tilde{P}_{\mathbb{K}}(s,X) = \exp\left(\sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^{s}\log \mathfrak{N}(\mathfrak{a})} v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X})\right). \tag{413}$$

We note that this is not too different to $P_{\mathbb{K}}(s,X)$. Indeed, since $v(e^{\log \mathfrak{N}(\mathfrak{a})/\log X}) = 1$ for $\mathfrak{N}(\mathfrak{a}) \leq \ X^{1-1/X}$ we have

$$\begin{aligned}
\tilde{P}_{\mathbb{K}}(s,X) =& P_{\mathbb{K}}(s,X)\exp\bigg(\sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}}\frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s\log\mathfrak{N}(\mathfrak{a})}(v(e^{\log\mathfrak{N}(\mathfrak{a})/\log X})-1)\bigg)\\
=& P_{\mathbb{K}}(s,X)\exp\bigg(\sum_{X^{1-1/X}\leq\mathfrak{N}(\mathfrak{a})\leq X}\frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s\log\mathfrak{N}(\mathfrak{a})}(v(e^{\log\mathfrak{N}(\mathfrak{a})/\log X})-1)\bigg)\\
=& P_{\mathbb{K}}(s,X)\exp\bigg(O\bigg(\sum_{X^{1-1/X}\leq p\leq X}p^{-\sigma}\bigg)\bigg)\\
=& P_{\mathbb{K}}(s,X)\exp\bigg(O\bigg(X^{-\sigma}\log X\bigg)\bigg)\\
=& P_{\mathbb{K}}(s,X)(1+O(X^{-\sigma}\log X),
\end{aligned}$$

where we have again used the fact that at most n prime ideals lie above the rational prime p.

To remove the restriction that s not be a zero of $\zeta_{\mathbb{K}}(s)$, we interpret $\exp(-U(z))$ to be asymptotic to Cz for some constant C as $z \to 0$ so both sides of (402) vanish at the zeros. This interpretation is allowed by the formula

$$E_1(z) = -\gamma - \log z - \sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n n!}, \quad |\arg z| < \pi \tag{414}$$

where γ is the Euler-Mascheroni constant. □

In the same vein as [**20**] we can think of $P_{\mathbb{K}}(s,X)$ as a truncated Euler product and $Z_{\mathbb{K}}(s,X)$ as a truncated Hadamard product. To clarify this, we first assume the Grand Riemann Hypothesis. Denote the non-trivial zeros of $\zeta_{\mathbb{K}}(s)$ by $\rho_n = 1/2 + i\gamma_n$, ordered by their height above the real axis. We set $s = 1/2 + it$ so that

$$Z_{\mathbb{K}}(s,X) = \exp\bigg(-\sum_{\rho_n}U(i(t-\gamma_n)\log X)\bigg).$$

Since the support of u is contained in $[e^{1-1/X}, e]$, which approaches the singleton $\{e\}$,

$$U(z) = \int_0^\infty u(x)E_1(z \log x)dx \approx E_1(z) \sim -\gamma - \log z$$

as $z \to 0$. Hence, for the γ_n close to t we can expect $\exp\big(-U(i(t-\gamma_n)\log X)\big)$ to be roughly equal to $i(t-\gamma_n)e^\gamma \log X$. To see that the γ_n further away do not contribute we first note that $\Re E_1(ix) = -\mathrm{Ci}(|x|)$ for $x \in \mathbb{R}$ where $\mathrm{Ci}(z) = -\int_z^\infty w^{-1} \cos w \, dw$. Therefore,

$$|Z_{\mathbb{K}}(s, X)| \approx \exp\left(\sum_{\rho_n} \mathrm{Ci}(|t-\gamma_n| \log X)\right)$$

and since $\mathrm{Ci}(|x|)$ decays as $x \to \pm\infty$, the only ordinates that contribute are those close to t. In fact, for $|x| > 1$, the function $\mathrm{Ci}(|x|)$ is already close to 0 and is also oscillating. We may therefore assume that the only real contribution to $Z_{\mathbb{K}}(s, X)$ comes from the γ_n such that $|t - \gamma_n| < 1/\log X$. Now, $P_{\mathbb{K}}(s, X)$ is approximately given by $\prod_{\mathfrak{N}(\mathfrak{p}) \le X}(1 - \mathfrak{N}(\mathfrak{p})^{-s})^{-1}$ (see the first line of formula (420) below) and hence

$$\zeta_{\mathbb{K}}(\tfrac{1}{2} + it) \approx \prod_{\mathfrak{N}(\mathfrak{p}) \le X} (1 - \mathfrak{N}(\mathfrak{p})^{-1/2-it})^{-1} \prod_{\substack{\gamma_n \\ |t-\gamma_n| < 1/\log X}} \big(i(t-\gamma_n)e^\gamma \log X\big). \tag{415}$$

If we take X small then $\zeta_{\mathbb{K}}(\frac{1}{2}+it)$ is essentially given by a characteristic polynomial and we recover the model of Keating and Snaith. On the other hand, taking X large the major contribution comes from the primes. To get a contribution from both, and thereby incorporate the primes into the Keating and Snaith model, we take X in an intermediate range.

As an application of the hybrid product to moments, we can make the equivalent of the splitting conjecture of Gonek, Hughes and Keating.

Conjecture 1. *Let* $X, T \to \infty$ *with* $X \ll (\log T)^{2-\epsilon}$*. Then for* $k > -1/2$*, we have*

$$\frac{1}{T}\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim \left(\frac{1}{T}\int_T^{2T} \left|P_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^{2k} dt\right)\left(\frac{1}{T}\int_T^{2T} \left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^{2k} dt\right).$$

The reasoning behind this follows much the same as in [20]. The basic point is that $Z_{\mathbb{K}}(\frac{1}{2}+it, X)$ oscillates faster than $P_{\mathbb{K}}(\frac{1}{2}+it, X)$. Indeed, $P_{\mathbb{K}}(\frac{1}{2}+it, X)$ is approximately given by $\prod_{\mathfrak{N}(\mathfrak{p})\leq X}(1-\mathfrak{N}(\mathfrak{p})^{-1/2-it})^{-1}$ and each term in this product is a periodic function of period $2\pi/\log X$. However, $Z_{\mathbb{K}}(\frac{1}{2}+it, X)$ vanishes at the non-trivial zeros. By Proposition 1.1.6 there are approximately $([\mathbb{K}:\mathbb{Q}]T/2\pi)\log T$ such zeros between 0 and T and hence $Z_{\mathbb{K}}(\frac{1}{2}+it, X)$ oscillates on a scale of $2\pi/([\mathbb{K}:\mathbb{Q}]T\log|t|)$. If $X = o(T)$ then this oscillation is greater than that of $P_{\mathbb{K}}(\frac{1}{2}+it, X)$. It is this separation of scales that suggests they contribute independently to the moments in leading order, which will be proven for $k=1$ and $\mathbb{K}$-quadratic in Theorem 4. It should be noted that this separation does not necessarily occur in lower order terms, and in general mixing can occur. Conjectures for the lower order terms are often better viewed in the light of shifted moments (see [11] for example).

3.2. Moments of the arithmetic factor

In this section we prove Theorem 3. Let $\mathbb{K}$ be a Galois extension with Galois group G and recall that for a rational prime p we have the decomposition

$$p\mathcal{O}_{\mathbb{K}} = \prod_{i=1}^{g} \mathfrak{p}_i^e \tag{416}$$

with

$$\mathfrak{N}(\mathfrak{p}_i) = p^f \tag{417}$$

for some positive integers e, f and g. We then have the identity $efg = n = [\mathbb{K} : \mathbb{Q}]$. The integer g is given by the index of the decomposition group $G_{\mathfrak{p}_i}$ in G for any given $\mathfrak{p}_i$ lying above p.

Theorem 3 *Let $\mathbb{K}$ be a Galois extension of degree n with Galois group $G = \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ and for a given prime $\mathfrak{p}$ let $g_{\mathfrak{p}}$ denote the index of the decomposition group $G_{\mathfrak{p}}$ in G. Let $1/2 \leq c < 1$, $\epsilon > 0$, $k > 0$ and suppose that X and $T \to \infty$ with $X \ll (\log T)^{1/(1-c+\epsilon)}$. Then*

$$\frac{1}{T}\int_T^{2T} \left| P_{\mathbb{K}}\left(\frac{1}{2} + it,\, X\right)\right|^{2k} dt \sim a(k)\psi_{\mathbb{K}}^{nk^2}(e^{\gamma}\log X)^{nk^2} \tag{418}$$

where $\psi_{\mathbb{K}}$ denotes the residue of $\zeta_{\mathbb{K}}(s)$ at $s = 1$ and

$$a(k) = \prod_{\mathfrak{p} \subseteq \mathcal{O}_{\mathbb{K}}} \left(\left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{nk^2} \left(\sum_{m \geq 0} \frac{d_{g_{\mathfrak{p}}k}(\mathfrak{p}^m)^2}{\mathfrak{N}(\mathfrak{p})^m} \right)^{1/g_{\mathfrak{p}}} \right) \tag{419}$$

where $d_k(\mathfrak{p}^m) = d_k(p^m) = \Gamma(m+k)/(m!\Gamma(k))$.

Proof. On raising $P_{\mathbb{K}}(s, X)$ to the kth power we have

$$P_{\mathbb{K}}(s, X)^k = \exp\left(k \sum_{\mathfrak{N}(\mathfrak{a}) \leq X} \frac{\Lambda(\mathfrak{a})}{\mathfrak{N}(\mathfrak{a})^s \log \mathfrak{N}(\mathfrak{a})} \right) = \exp\left(k \sum_m \sum_{\mathfrak{N}(\mathfrak{p})^m \leq X} \frac{1}{m\mathfrak{N}(\mathfrak{p})^{ms}} \right).$$

We wish to write this last sum in the exponential as a sum over rational primes. Let $g_p(e)$ denote the number of prime ideals lying above the rational prime p whose ramification index is e. This is just the number g in the identity $efg = n$; however we need to make its dependence on p and e explicit since we'll be summing over all the possible values it can take. From the identity $efg_p(e) = n$ we see that $g_p(e)$ must be a divisor of n, as must e. Therefore, by summing over all such possibilities and writing $\mathfrak{N}(\mathfrak{p}) = p^f = p^{n/eg_p(e)}$ we acquire

$$P_{\mathbb{K}}(s,X)^k = \exp\left(k\sum_{m}\sum_{g|n} g\sum_{e|\frac{n}{g}} \sum_{\substack{p^{\frac{mn}{eg}}\le X\\ g_p(e)=g}} \frac{1}{mp^{(n/eg)ms}}\right)$$

$$= \prod_{g|n}\prod_{e|\frac{n}{g}}\prod_{\substack{p^{\frac{n}{eg}}\le X\\ g_p(e)=g}} \exp\left(\log(1-p^{-(n/eg)s})^{-gk} - \sum_{\substack{m\\ p^{\frac{mn}{eg}}>X}} \frac{1}{mp^{(n/eg)ms}}\right). \tag{420}$$

The outer products over the divisors of n and n/g are indeed over all divisors with the inner product being empty if no such p match the condition $g_p(e) = g$. We now write the innermost product as the Dirichlet series

$$\sum_{l\in\mathcal{L}_{e,g}(X)} \frac{\beta_{gk}(l)}{l^{(n/eg)s}} \tag{421}$$

where $\mathcal{L}_{e,g}(X) = \{l \in \mathrm{Im}(\mathfrak{N}) : p|l \implies g_p(e) = g \text{ and } p^{n/eg} \le X\}$. We see that $\beta_{gk}(l)$ is a multiplicative function of l, $0 \le \beta_{gk}(l) \le d_{gk}(l)$ for all l and $\beta_{gk}(p^m) = d_{gk}(p^m)$ if $p^m \le X$.

For an integer l, let $l_{e,g}$ denote the greatest factor of l composed of primes p for which $g_p = g$ and whose ramification index is e. Then,

$$P_{\mathbb{K}}(s,X)^k = \prod_{g|n}\prod_{e|\frac{n}{g}}\left(\sum_{l\in\mathcal{L}_{e,g}(X)} \frac{\beta_{gk}(l)}{l^{(n/eg)s}}\right) = \sum_{l\in\mathcal{W}(X)} \frac{\gamma_k(l)}{l^s} \tag{422}$$

where

$$\gamma_k(l) = \prod_{g|n}\prod_{e|\frac{n}{g}} \beta_{gk}(l_{e,g}^{eg/n}) \tag{423}$$

and $\mathcal{W}(X) = \{l \in \mathrm{Im}(\mathfrak{N}) : \mathfrak{N}(\mathfrak{p})|l \implies \mathfrak{N}(\mathfrak{p}) \le X\}$. The product representation of γ is made possible by the fact that for integers l, m belonging to different $\mathcal{L}_{e,g}(X)$, we have $(l,m) = 1$. This would not necessarily be the case for non-Galois extensions. For example, in a cubic extension we may have the factorisation $p\mathcal{O}_{\mathbb{K}} = \mathfrak{p}_1\mathfrak{p}_2$ and hence one of these ideals has norm p, whilst the other has norm p^2. We could then follow the previous reasoning whilst redefining the sets $\mathcal{L}$ with a consideration of this difference. However, we would then lose the coprimality condition.

Since we want to apply the mean value theorem for Dirichlet polynomials we split the sum at T^θ where θ is to be chosen later and obtain

$$P_{\mathbb{K}}(s,X)^k = \sum_{\substack{l\in\mathcal{W}(X)\\ l\le T^\theta}} \frac{\gamma_k(l)}{l^s} + O\Bigg(\sum_{\substack{l\in\mathcal{W}(X)\\ l> T^\theta}} \frac{\gamma_k(l)}{l^s}\Bigg). \tag{424}$$

Now for $\epsilon > 0$ and $\sigma \ge c$ the error term is

$$\begin{aligned}\ll & T^{-\epsilon\theta} \sum_{l\in\mathcal{W}(X)} \frac{\prod_{g|n}\prod_{e|\frac{n}{g}} d_{gk}(l_{e,g}^{eg/n})}{n^{c-\epsilon}} = T^{-\epsilon\theta} \prod_{\mathfrak{N}(\mathfrak{p})\le X} (1-\mathfrak{N}(\mathfrak{p})^{\epsilon-c})^{-k} \\ = & T^{-\epsilon\theta}\exp\Bigg(O\Bigg(k\sum_{\mathfrak{N}(\mathfrak{p})\le X}\mathfrak{N}(\mathfrak{p})^{\epsilon-c}\Bigg)\Bigg) = T^{-\epsilon\theta}\exp\left(O\left(\frac{kX^{1-c+\epsilon}}{(1-c+\epsilon)\log X}\right)\right)\end{aligned}$$

where in the last line we have used the prime ideal theorem (51). If we let $X \asymp (\log T)^{1/(1-c+\epsilon)}$ then this is

$$\ll T^{-\epsilon\theta}\exp\left(O\left(k\frac{\log T}{\log\log T}\right)\right) \ll_k T^{-\epsilon\theta/2} \tag{425}$$

and hence

$$P_{\mathbb{K}}(s,X)^k = \sum_{\substack{l\in\mathcal{W}(X)\\ l\le T^\theta}} \frac{\gamma_k(l)}{l^s} + O_k(T^{-\epsilon\theta/2}). \tag{426}$$

We now let $\theta = 1/2$ and apply the Montgomery-Vaughan mean value theorem to give

$$\begin{aligned}\frac{1}{T}\int_T^{2T}\Bigg|\sum_{\substack{l\in\mathcal{W}(X)\\ l\le T^{1/2}}} \frac{\gamma_k(l)}{l^{\sigma+it}}\Bigg|^2 dt =& (1+O(T^{-1/2}))\sum_{\substack{l\in\mathcal{W}(X)\\ l\le T^{1/2}}} \frac{\gamma_k(l)^2}{l^{2\sigma}} \\ =& (1+O(T^{-1/2}))\Bigg(\sum_{l\in\mathcal{W}(X)} \frac{\gamma_k(l)^2}{l^{2\sigma}} + O(T^{-\epsilon/4})\Bigg) \\ =& (1+O(T^{-\epsilon/4}))\sum_{l\in\mathcal{W}(X)} \frac{\gamma_k(l)^2}{l^{2\sigma}}.\end{aligned} \tag{427}$$

Therefore by (426) and the Cauchy-Schwarz inequality we have

$$\frac{1}{T}\int_T^{2T} |P_{\mathbb{K}}(\sigma+it,X)|^{2k} = (1+O(T^{-\epsilon/4}))\sum_{l\in\mathcal{W}(X)}\frac{\gamma_k(l)^2}{l^{2\sigma}}. \tag{428}$$

We can now re-factorise the above Dirichlet series to give

$$\sum_{l\in\mathcal{W}(X)}\frac{\gamma_k(l)^2}{l^{2\sigma}} = \prod_{g|n}\prod_{e|\frac{n}{g}}\bigg(\sum_{l\in\mathcal{L}_{e,g}(X)}\frac{\beta_{gk}(l)^2}{l^{2(n/eg)\sigma}}\bigg). \tag{429}$$

Since $\beta_k(n)$ is multiplicative and satisfies $0\le\beta_k(n)\le d_k(n)$ and $\beta_k(p^m) = d_k(p^m)$ if $p^m\le X$ we have

$$\prod_{\substack{p^{\frac{n}{eg}}\le X\\ g_p=g}}\sum_{m=0}^{N_p}\frac{d_{gk}(p^m)^2}{p^{2(n/eg)m\sigma}} \le \sum_{l\in\mathcal{L}_{e,g}(X)}\frac{\beta_{gk}(l)^2}{l^{2(n/eg)\sigma}} \le \prod_{\substack{p^{\frac{n}{eg}}\le X\\ g_p=g}}\sum_{m=0}^{\infty}\frac{d_{gk}(p^m)^2}{p^{2(n/eg)m\sigma}}$$

where $N_p=\lfloor\log X/\log p\rfloor$. The ratio of the left side to the right is

$$\begin{aligned}
&\prod_{\substack{p^{\frac{n}{eg}}\le X\\ g_p=g}}\bigg(1-\frac{\sum_{m\ge N_p+1}d_k(p^m)^2p^{-2(n/eg)m\sigma}}{\sum_{m\ge 0}d_k(p^m)^2p^{-2(n/eg)m\sigma}}\bigg)\\
&\quad=\prod_{\substack{p^{\frac{n}{eg}}\le X\\ g_p=g}}\bigg(1+O\bigg(\sum_{m\ge N_p+1}\frac{d_k(p^m)^2}{p^{2(n/eg)m\sigma}}\bigg)\bigg)=\prod_{\substack{p^{\frac{n}{eg}}\le X\\ g_p=g}}\Big(1+O\big(p^{(N_p+1)(\epsilon-2(n/eg)\sigma)}\big)\Big)\\
&\quad=\prod_{\substack{p^{\frac{n}{eg}}\le \sqrt{X}\\ g_p=g}}\bigg(1+O\bigg(p^{\frac{\log X}{\log p}(\epsilon-2(n/eg)\sigma)}\bigg)\bigg)\prod_{\substack{\sqrt{X}<p^{\frac{n}{eg}}\le X\\ g_p=g}}\Big(1+O\big(p^{2(\epsilon-2(n/eg)\sigma)}\big)\Big).
\end{aligned}$$

The error term in the factor of the first product is equal to $O(X^{-2(n/eg)\sigma+\epsilon})$ which is $O(X^{-2\sigma+\epsilon})$ since $n/eg\ge 1$. Therefore, by the prime number theorem the first product is equal to $1+O(X^{-1/2+\epsilon})$. The second product is

$$1+O\bigg(\sum_{\sqrt{X}<p^{\frac{n}{eg}}\le X}p^{-2(n/eg)+\epsilon}\bigg)=1+O\bigg(X^{-c}\sum_{p}p^{-2((n/eg)-c)+\epsilon}\bigg)$$

for some c. Using $n/eg \geq 1$ and taking $c = 1/2 - \epsilon$ we see the second product is also $1 + O(X^{-1/2+\epsilon})$. Therefore,

$$\sum_{l \in \mathcal{L}_{e,g}(X)}^{\infty} \frac{\beta_{gk}(l)^2}{l^{2(n/eg)\sigma}} = (1 + O(X^{-1/2+\epsilon})) \prod_{\substack{p^{\frac{n}{eg}} \leq X \\ g_p = g}} \sum_{m \geq 0} \frac{d_{gk}(p^m)^2}{p^{2m(n/eg)\sigma}}. \tag{430}$$

Now, the product on the right may be divergent as $X \to \infty$. In order to keep the arithmetic information, we factor out the divergent part and write it as

$$\prod_{\substack{p^{\frac{n}{eg}} \leq X \\ g_p = g}} \left(\left(1 - p^{-2(n/eg)\sigma}\right)^{ngk^2} \sum_{m \geq 0} \frac{d_{gk}(p^m)^2}{p^{2m(n/eg)\sigma}} \right) \prod_{\substack{p^{\frac{n}{eg}} \leq X \\ g_p = g}} \left(1 - p^{-2(n/eg)\sigma}\right)^{-ngk^2}. \tag{431}$$

In terms of divergence, the worst case scenario is when $n/eg = 1$. In this case if $g_p = g < n$ then p is ramified and the product is finite. Therefore, we only need consider the case $g = n$ for which the above product equals

$$\begin{aligned} \prod_{\substack{p > X \\ g_p = n}} \left(\left(1 - p^{-2\sigma}\right)^{n^2k^2} \sum_{m \geq 0} \frac{d_{nk}(p^m)^2}{p^{2m\sigma}} \right) &= \prod_{\substack{p > X \\ g_p = n}} \left(1 - n^2k^2p^{-2\sigma} + n^2k^2p^{-2\sigma} + O_k(p^{-4\sigma})\right) \\ &= \prod_{\substack{p > X \\ g_p = n}} \left(1 + O_k(p^{-4\sigma})\right) \\ &= 1 + O_k(X^{-1/2+\epsilon}). \end{aligned} \tag{432}$$

It follows that we can extend the first product in (431) over all primes. Specialising to $\sigma = 1/2$ and using the product representation in (429) we see

$$\sum_{l \in \mathcal{W}(X)} \frac{\gamma_k(l)^2}{l} = a(k) \prod_{\mathfrak{N}(\mathfrak{p}) \leq X} (1 - \mathfrak{N}(\mathfrak{p})^{-1})^{-nk^2} (1 + O_k(X^{-1/2+\epsilon})). \tag{433}$$

By Mertens theorem for number fields (52), we have

$$\prod_{\mathfrak{N}(\mathfrak{p}) \leq X} (1 - \mathfrak{N}(\mathfrak{p})^{-1})^{-nk^2} = \psi_{\mathbb{K}}^{nk^2} (e^{\gamma} \log X)^{nk^2} (1 + O(1/\log^2 X)) \tag{434}$$

and the result follows. ∎

3.3. Support for Conjecture 2

Our support for Conjecture 2 closely follows the arguments of Gonek, Hughes and Keating [20] with one slight modification. Let us first restate the conjecture.

Conjecture 2. *Let $\mathbb{K}$ be a Galois extension of degree n. Suppose that $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then for $k > -1/2$ we have*

$$\frac{1}{T}\int_T^{2T} \left| Z_{\mathbb{K}}\left(\frac{1}{2}+it, X\right)\right|^{2k} dt \sim (e^\gamma \log X)^{-nk^2} \prod_\chi \frac{G(\chi(1)k+1)^2}{G(2\chi(1)k+1)} \Big(\log \big(q(\chi) T^{d_\chi}\big)\Big)^{\chi(1)^2k^2} \tag{435}$$

where the product is over the irreducible characters of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$*, G is the Barnes G-function, $q(\chi)$ is the conductor of $L(s,\chi,\mathbb{K}/\mathbb{Q})$ and d_χ is its degree.*

By (75), we have the factorisation

$$\zeta_{\mathbb{K}}(s) = \prod_\chi L(s,\chi,\mathbb{K}/\mathbb{Q})^{\chi(1)} \tag{436}$$

where the product is over the non-equivalent irreducible characters of $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ and $L(s,\chi,\mathbb{K}/\mathbb{Q})$ is the Artin L-function attached to χ. For each character χ, the associated L-function satisfies the functional equation

$$\Lambda(s,\chi) := q(\chi)^{s/2}\gamma(s,\chi)L(s,\chi) = W(\chi)\Lambda(1-s,\overline{\chi}) \tag{437}$$

where $W(\chi)$ is some complex number of modulus one and $q(\chi)$ is the conductor, for which we do not require an explicit expression. The gamma factor is given by

$$\gamma(s,\chi) = \pi^{-sd_\chi/2}\prod_{j=1}^{d_\chi}\Gamma\left(\frac{s+\mu_j}{2}\right) \tag{438}$$

with μ_j equal to 0 or 1. If we assume the Artin conjecture then $L(s,\chi)$ is an entire function for all non-trivial χ. If χ is the trivial character then $L(s,\chi)$ equals the Dedekind zeta function of the base field, which in our case is $\zeta(s)$. Under this assumption, Artin L-functions are in the Selberg class. Therefore, by Proposition 1.1.6 the mean density of zeros of $L(\beta + it, \chi)$, $0 \leq \beta \leq 1$, is given by

$$\frac{1}{\pi} \log \left(q(\chi) \left(\frac{t}{2\pi} \right)^{d_\chi} \right) = \frac{1}{\pi} \mathcal{L}_\chi(t), \tag{439}$$

say. For each $L(s, \chi)$ in the product of equation (436), we associate to its zeros $\gamma_n(\chi)$ at height T, a unitary matrix $U(N(\chi))$ of size $N(\chi) = \lfloor \mathcal{L}_\chi(T) \rfloor$ chosen with respect to Haar measure, which we denote $d\mu(\chi)$. After rescaling, the zeros $\gamma_n(\chi)$ are conjectured [**49**] to share the same distribution as the eigenangles $\theta_n(\chi)$ of $U(N(\chi))$ when chosen with $d\mu(\chi)$.

In addition to the previous assumptions, we now also assume the extended Riemann hypothesis. Let $Z_{\mathbb{K}}(s, X)$ be given by (139). Since $\Re E_1(ix) = -\mathrm{Ci}(|x|)$ for $x \in \mathbb{R}$, where

$$\mathrm{Ci}(z) = -\int_z^\infty \frac{\cos w}{w} dw, \tag{440}$$

we see that

$$\begin{aligned} &\frac{1}{T} \int_T^{2T} \left| Z_{\mathbb{K}} \left(\frac{1}{2} + it, X \right) \right|^{2k} dt \\ &\quad = \frac{1}{T} \int_T^{2T} \prod_{\gamma_n} \exp \left(2k \int_1^e u(y) \mathrm{Ci}(|t - \gamma_n| \log y \log X) \right) dy dt \\ &\quad = \frac{1}{T} \int_T^{2T} \prod_{\chi} \prod_{\gamma_n(\chi)} \exp \left(2k\chi(1) \int_1^e u(y) \mathrm{Ci}(|t - \gamma_n(\chi)| \log y \log X) \right) dy dt \end{aligned} \tag{441}$$

where $u(y)$ is a smooth, non-negative function supported on $[e^{1-1/X}, e]$ and of total mass one. Since the zeros are conjectured to share the same distribution as the eigenangles, we now replace the zeros with the eigenangles and argue that the above should be modeled by

$$\mathbb{E} \Bigg[\prod_{\chi} \prod_{n=1}^{N(\chi)} \phi(k\chi(1), \theta_n(\chi)) \Bigg] \tag{442}$$

where

$$\phi(m,\theta) = \exp\left(2m\int_1^e u(y)\mathrm{Ci}(|\theta|\log y\log X)\right) \tag{443}$$

and the expectation is taken with respect to the product measure $\prod_\chi d\mu(\chi)$. We now assume that the matrices $U(N(\chi))$ can be chosen independently for any two distinct χ. This corresponds to a 'superposition' of ensembles; the behaviour of which is also shared by the distribution of zeros of a product of distinct L-functions [34]. With this assumption, the expectation factorises as

$$\prod_\chi \mathbb{E}\left[\prod_{n=1}^{N(\chi)} \phi(k\chi(1),\theta_n(\chi))\right]. \tag{444}$$

In [20] it is shown (Theorem 4) that for $k > -1/2$ and $X \geq 2$,

$$\mathbb{E}\left[\prod_{j=1}^{N}\phi(m,\theta_j)\right] \sim \frac{G(m+1)^2}{G(2m+1)}\left(\frac{N}{e^\gamma\log X}\right)^{m^2}\left(1+O_m\left(\frac{1}{\log X}\right)\right). \tag{445}$$

Therefore, by forming the product over χ and using $\sum_\chi \chi(1)^2 = |\mathrm{Gal}(\mathbb{K}/\mathbb{Q})|= n$ we are led to conjecture 2.

After combining this with our formula for the moments of $P_{\mathbb{K}}(\frac{1}{2}+it,X)$ via the splitting conjecture we gain the full conjecture for the moments of $\zeta_{\mathbb{K}}(\frac{1}{2}+it)$. Note that after using $\sum_\chi \chi(1)^2 = |\mathrm{Gal}(\mathbb{K}/\mathbb{Q})|= n$ the resulting expression in our conjecture is $\sim c\log^{nk^2} T$ for some determinable constant c. Now, in the paper [9], Conrey and Farmer express the idea that the mean square of $\zeta(s)^k$ should be a multiple of the sum $\sum_{n\leq T} d_k(n)^2 n^{-1}$, and that this multiple is the measure of how many Dirichlet polynomials are needed to capture the full moment. Their reasoning is based on a combination of the Montgomery-Vaughan mean value Theorem and the form of the sixth and eighth moment conjectures given in [15]. Assuming this idea applies to other L-functions, we note the aforementioned result of Chandrasekharan and Narasimhan [7]. They showed that for a Galois extension of degree n,

$$\sum_{m \le T} r_{\mathbb{K}}(m)^2 \sim cT \log^{n-1} T, \tag{446}$$

where $r_{\mathbb{K}}(m)$ is the number of integral ideals of norm m and c is some constant. Applying partial summation we thus gain a result which supports our conjecture, at least in the case $k = 1$ (we note the results of [7] should easily extend to general k, and remain consistent with our conjecture). Alternatively, one could view our conjecture as adding support to the idea of Conrey and Farmer.

3.4. The second moment of $Z_{\mathbb{K}}$ for quadratic extensions

In this section we prove Theorem 4. For the most part, the remainder of this chapter is concerned with quadratic extensions so we first restate some of the useful facts. As mentioned in the introduction, $\zeta_{\mathbb{K}}(s) = \zeta(s)L(s,\chi)$ where χ is the Kronecker character. The modulus q of χ is given explicitly in terms of the discriminant $d_{\mathbb{K}}$ by formula (78). We shall have occasion to work with more general (complex) characters χ mod $q > 1$ when the arguments in question work in such generalities, although at some points we may specialise to the Kronecker character without mention. We also recall the splitting of primes in quadratic extensions:

$$\begin{aligned}
p \text{ is split} \,(\chi(p) = 1) \quad &: \quad p\mathcal{O}_{\mathbb{K}} = \mathfrak{p}_1\mathfrak{p}_2 \quad \Longrightarrow \quad \mathfrak{N}(\mathfrak{p}_1) = \mathfrak{N}(\mathfrak{p}_2) = p \\
p \text{ is inert} \,(\chi(p) = -1) \quad &: \quad p\mathcal{O}_{\mathbb{K}} = \mathfrak{p}_1 \quad \Longrightarrow \quad \mathfrak{N}(\mathfrak{p}_1) = p^2 \\
p \text{ is ramified} \,(\chi(p) = 0) \quad &: \quad p\mathcal{O}_{\mathbb{K}} = \mathfrak{p}_1^2 \quad \Longrightarrow \quad \mathfrak{N}(\mathfrak{p}_1) = p.
\end{aligned}$$

At some points we shall use the notation $p_{\mathrm{s}}, p_{\mathrm{i}}, p_{\mathrm{r}}$ to denote split, inert and ramified primes respectively.

3.4.1. The setup.

Our aim is to show the following.

Theorem 4. *Let $\mathbb{K}$ be a quadratic number field and suppose $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then*

$$\frac{1}{T}\int_{T}^{2T} \left|Z_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)\right|^2 dt \sim \frac{\log T \cdot \log qT}{\left(e^{\gamma}\log X\right)^2}. \tag{447}$$

Since $\zeta_{\mathbb{K}}(1/2+it)P_{\mathbb{K}}(1/2+it, X) = Z_{\mathbb{K}}(1/2+it, X)(1+o(1))$ for $t \in [T, 2T]$, it is enough to show that

$$\frac{1}{T}\int_{T}^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right) P_{\mathbb{K}}\left(\tfrac{1}{2}+it, X\right)^{-1}\right|^2 dt \sim \frac{\log T \cdot \log qT}{\left(e^{\gamma}\log X\right)^2}. \tag{448}$$

To evaluate the left hand side we first express $P_{\mathbb{K}}(1/2+it)^{-1}$ as a Dirichlet polynomial and then apply our formula for the twisted second moment. The means to do this are given by the following sequence of Lemmas.

Lemma 3.4.1. *Let*

$$Q_{\mathrm{s}}(s, X) = \prod_{\substack{p \le \sqrt{X} \\ p \text{ split}}} \left(1 - p^{-s}\right) \prod_{\substack{\sqrt{X} < p \le X \\ p \text{ split}}} \left(1 - p^{-s} + \frac{1}{2}p^{-2s}\right) \tag{449}$$

and define $Q_{\mathrm{i}}(s, X)$ and $Q_{\mathrm{r}}(s, X)$ as the same products except over the inert and ramified primes respectively. Then for X sufficiently large, we have

$$P_{\mathbb{K}}(s, X)^{-1} = Q_{\mathrm{s}}(s, X)^2 Q_{\mathrm{i}}(2s, \sqrt{X}) Q_{\mathrm{r}}(s, X)\left(1 + O\left(\frac{1}{\log X}\right)\right) \tag{450}$$

and this holds uniformly for $\sigma \ge 1/2$.

PROOF. First, note

$$
\begin{aligned}
P_{\mathbb{K}}(s,X)^{-1} &= \exp\left(-\sum_{\mathfrak{N}(\mathfrak{p})^m\le X}\frac{1}{m\mathfrak{N}(\mathfrak{p})^{ms}}\right)\\
&= \prod_{\substack{p\le X\\ p \text{ split}}}\exp\left(-2\sum_{1\le m\le\left\lfloor\frac{\log X}{\log p}\right\rfloor}\frac{1}{mp^{ms}}\right)\prod_{\substack{p^2\le X\\ p \text{ inert}}}\exp\left(-\sum_{1\le m\le\left\lfloor\frac{\log X}{2\log p}\right\rfloor}\frac{1}{mp^{2ms}}\right)\\
&\quad\times\prod_{\substack{p\le X\\ p \text{ ramified}}}\exp\left(-\sum_{1\le m\le\left\lfloor\frac{\log X}{\log p}\right\rfloor}\frac{1}{mp^{ms}}\right)
\end{aligned}
\tag{451}
$$

and so it suffices to consider just one of these products. Let A be a subset of the primes and let $N_p = \lfloor \log X/\log p\rfloor$. Since $N_p = 1$ if $\sqrt{X} < p \le X$ we have

$$
\begin{aligned}
\prod_{\substack{p\le X\\ p\in A}}\exp\left(-\sum_{1\le m\le N_p}\frac{1}{mp^{ms}}\right) &= \prod_{\substack{p\le \sqrt{X}\\ p\in A}}\exp\left(\log(1-p^{-s})+\sum_{m>N_p}\frac{1}{mp^{ms}}\right)\\
&\quad\times\prod_{\substack{\sqrt{X}<p\le X\\ p\in A}}\exp(-p^{-s}).
\end{aligned}
$$

Now, on noting that $N_p+1 > \log X/\log p$ we have for $\sigma \ge 1/2$;

$$
\exp\left(\sum_{p\le\sqrt{X}}\sum_{m>N_p}\frac{1}{mp^{ms}}\right) \ll \exp\left(\sum_{p\le\sqrt{X}}\frac{1}{p^{\sigma(N_p+1)}}\right) \ll \exp\left(X^{-1/2}\sum_{p\le\sqrt{X}}1\right)
\tag{452}
$$

and this is $\ll 1 + O(1/\log X)$ by the prime number theorem. Also,

$$
\begin{aligned}
&\prod_{\sqrt{X}<p\le X}\left(1-p^{-s}+\frac{1}{2!}p^{-2s}-\frac{1}{3!}p^{-3s}+O(p^{-4\sigma})\right)\\
&= \prod_{\sqrt{X}<p\le X}\left(1-p^{-s}+\frac{1}{2}p^{-2s}\right)\left(1+O(p^{-3\sigma})\right)\\
&= \prod_{\sqrt{X}<p\le X}\left(1-p^{-s}+\frac{1}{2}p^{-2s}\right)\left(1+O\left(\frac{1}{\log X}\right)\right)
\end{aligned}
\tag{453}
$$

and so we're done. □

Lemma 3.4.2. *We have*

$$P_{\mathbb{K}}\left(\frac{1}{2}+it, X\right)^{-1} = \left(1+O\left(\frac{1}{\log X}\right)\right)\sum_{n\in\mathcal{W}(X)}\frac{\alpha(n)}{n^{1/2+it}} \tag{454}$$

where $\mathcal{W}(X) = \{n \in \mathrm{Im}(\mathfrak{N}) : \mathfrak{N}(\mathfrak{p})|n \implies \mathfrak{N}(\mathfrak{p}) \le X\}$ *and the behaviour of* α *at primes is determined by*

(455)

$$\alpha(p_{\mathrm{s}}^j) = \begin{cases} -2 & \text{if } j=1,\ p_{\mathrm{s}} \le X, \\ 1 & \text{if } j=2,\ p_{\mathrm{s}} \le \sqrt{X}, \\ 2 & \text{if } j=2,\ \sqrt{X} < p_{\mathrm{s}} \le X, \\ 0 & \text{if } j \ge 3, \end{cases} \qquad \alpha(p_{\mathrm{i}}^{2j}) = \begin{cases} -1 & \text{if } j=1,\ p_{\mathrm{i}}^2 \le X, \\ 0 & \text{if } j=2,\ p_{\mathrm{i}}^2 \le \sqrt{X}, \\ \frac{1}{2} & \text{if } j=2,\ \sqrt{X} < p_{\mathrm{i}}^2 \le X, \\ 0 & \text{if } j \ge 3 \end{cases}$$

and

$$\alpha(p_{\mathrm{r}}^j) = \begin{cases} -1 & \text{if } j=1,\ p_{\mathrm{r}} \le X, \\ 0 & \text{if } j=2,\ p_{\mathrm{r}} \le \sqrt{X}, \\ \frac{1}{2} & \text{if } j=2,\ \sqrt{X} < p_{\mathrm{r}} \le X, \\ 0 & \text{if } j \ge 3. \end{cases} \tag{456}$$

We also have the bound $\alpha(n) \ll d(n)$ *for all* $n \in \mathcal{W}(X)$.

Proof. We first note that the square of the product over split primes in (450) is given by

$$\begin{aligned} Q_{\mathrm{s}}(s,X)^2 &= \prod_{\substack{p\le\sqrt{X} \\ p \text{ split}}}\left(1-2p^{-s}+p^{-2s}\right)\prod_{\substack{\sqrt{X}<p\le X \\ p \text{ split}}}\left(1-2p^{-s}+2p^{-2s}+O(p^{-3\sigma})\right) \\ &= \prod_{\substack{p\le\sqrt{X} \\ p \text{ split}}}\left(1-2p^{-s}+p^{-2s}\right)\prod_{\substack{\sqrt{X}<p\le X \\ p \text{ split}}}\left(1-2p^{-s}+2p^{-2s}\right) \\ &\quad \times\left(1+O\left(\frac{1}{\log X}\right)\right). \\ &= R_{\mathrm{s}}(s,X)\left(1+O\left(\frac{1}{\log X}\right)\right), \end{aligned} \tag{457}$$

say. On writing

$$R_{\mathrm{s}}(s,X)Q_{\mathrm{i}}(2s,\sqrt{X})Q_{\mathrm{r}}(s,X)=\sum_{n\in\mathcal{W}(X)}\frac{\alpha(n)}{n^{1/2+it}} \tag{458}$$

we can read off the behaviour of α at the primes from the Euler products. □

Lemma 3.4.3. *Let $\theta>0$. Then*

$$\begin{aligned}&\frac{1}{T}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)P_{\mathbb{K}}\left(\tfrac12+it,X\right)^{-1}\right|^2dt\\&=\left(1+O\left(\frac{1}{\log X}\right)\right)\frac{1}{T}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)\sum_{\substack{n\in\mathcal{W}(X)\\n\le T^\theta}}\frac{\alpha(n)}{n^{1/2+it}}\right|^2dt.\end{aligned} \tag{459}$$

Proof. First, we write

$$Q_{\mathbb{K}}\left(\tfrac12+it\right)=\sum_{\substack{n\in\mathcal{W}(X)\\n\le T^\theta}}\frac{\alpha(n)}{n^{1/2+it}}+O\Bigg(\sum_{\substack{n\in\mathcal{W}(X)\\n> T^\theta}}\frac{\alpha(n)}{n^{1/2+it}}\Bigg). \tag{460}$$

We can show, by using the bound $\alpha(n)\ll d(n)$ and a similar reasoning to that used between (424) and (426), that if $X\ll\log^{2-\epsilon}T$ then the error term is $\ll T^{-\epsilon\theta/10}$. Rewriting (460) as $Q_{\mathbb{K}}(1/2+it)=\sum+O(T^{-\epsilon\theta/10})$ we see

$$\begin{aligned}&\frac{1}{T}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)Q_{\mathbb{K}}\left(\tfrac12+it\right)\right|^2dt\\&=\frac{1}{T}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)\sum\right|^2dt+O\left(\frac{1}{T^{1+\epsilon\theta/10}}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)\right|^2\left|\sum\right|dt\right)\\&\qquad+O\left(\frac{1}{T^{1+\epsilon\theta/5}}\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)\right|^2dt\right).\end{aligned} \tag{461}$$

The final term is $\ll T^{-\epsilon\theta/10}$ by Motohashi's result (79). Using the Cauchy-Schwarz inequality we can show that the second term is

$$\begin{aligned}&\ll\frac{1}{T^{1+\epsilon\theta/10}}\left(\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)\sum\right|^2dt\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)\right|^2dt\right)^{1/2}\\&\ll\frac{1}{T^{1/2+\epsilon\theta/20}}\left(\int_T^{2T}\left|\zeta_{\mathbb{K}}\left(\tfrac12+it\right)\sum\right|^2dt\right)^{1/2}\end{aligned} \tag{462}$$

and the result follows. □

We are now required to show that for $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$,

$$\frac{1}{T}\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right) \sum_{\substack{n\in\mathcal{W}(X)\\ n\le T^\theta}} \frac{\alpha(n)}{n^{1/2+it}}\right|^2 = \frac{\log T \cdot \log qT}{(e^\gamma \log X)^2}\left(1+O\left(\frac{1}{\log X}\right)\right). \tag{463}$$

So take a Dirichlet polynomial $M(s)=\sum_{n\le T^\theta} a(n)n^{-s}$ with $\theta \le 1/11-\epsilon$ and let $w(t)$ satisfy the conditions of Theorem 1. Then, upon expanding, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^2 \left|M\left(\tfrac{1}{2}+it\right)\right|^2 w(t)dt \\ = \sum_{h,k\le T^\theta} \frac{a(h)\overline{a(k)}}{\sqrt{hk}} \lim_{\alpha,\beta,\gamma,\delta\to 0} I(h_{(k)},k_{(h)})\end{aligned} \tag{464}$$

where I is given by (154) and $h_{(k)} = h/(h,k)$. In order to evaluate this inner limit we express $Z_{\alpha,\beta,\gamma,\delta,h,k}(0)$ as a Laurent series and express the other terms as Taylor series. In doing this, the only real difficulty lies in calculating the derivatives of $B_{\alpha,\beta,\gamma,\delta,h,k}(0)$. For our purposes, which is to work over X-smooth numbers, we only need upper bounds on these derivatives.

Proposition 3.4.4. *Let $M(s)=\sum_{n\le T^\theta} a(n)n^{-s}$ with $\theta\le 1/11-\epsilon$. Then,*

$$\begin{aligned}\frac{1}{T}\int_T^{2T} \left|\zeta_{\mathbb{K}}\left(\tfrac{1}{2}+it\right)\right|^2 \left|M\left(\tfrac{1}{2}+it\right)\right|^2 dt \\ = \sum_{h,k\le T^\theta} \frac{a(h)\overline{a(k)}}{hk}(h,k)\left[\sum_{n=0}^{2} c_n(h,k,T) + O\left(T^{-\frac{1}{4}+\epsilon}(h_k k_h)^{11/8+\epsilon}\right)\right]\end{aligned} \tag{465}$$

where the leading order term is given by

$$\begin{aligned}c_2(h,k,T) = \frac{6}{\pi^2}L(1,\chi)^2 \prod_{p\mid d_{\mathbb{K}}}\left(1+\frac{1}{p}\right)^{-1} \\ \times \delta(h_{(k)})\delta(k_{(h)})\left[\log T\cdot \log qT + O(\log T \log \tfrac{hk}{(h,k)^2})\right]\end{aligned} \tag{466}$$

with

$$\delta(m) = \begin{cases} \prod_{\substack{p|m \\ p \text{ split}}} \left(1 + m_p \frac{1-p^{-1}}{1+p^{-1}}\right) & \textit{if } m_{\text{inert}} \textit{ is square} \\ 0 & \textit{otherwise} \end{cases} \tag{467}$$

where m_p *is the highest power of* p *dividing* m *and* m_{inert} *is the greatest factor of* m *composed solely of inert primes. For the lower order terms we have*

$$c_1(h, k, T) \ll \delta(h_{(k)})\delta(k_{(h)}) \log T \log\log \tfrac{hk}{(h,k)^2} \tag{468}$$

and

$$\begin{aligned} c_0(h, k, T) =& c_0'(h, k, T) + \mathbf{1}_{q|h_{(k)}} \chi(k_{(h)}) G(\chi) Z'_{0,0,0,0,\frac{h_{(k)}}{q},k_{(h)}}(0, \chi) \\ &+ \mathbf{1}_{q|k_{(h)}} \chi(h_{(k)}) G(\chi) Z'_{0,0,0,0,h_{(k)},\frac{k_{(h)}}{q}}(0, \chi) \end{aligned} \tag{469}$$

with

$$c_0'(h, k, T) \ll \delta(h_{(k)})\delta(k_{(h)}) (\log\log \tfrac{hk}{(h,k)^2})^2. \tag{470}$$

The Z' *terms may be written as*

$$Z'_{0,0,0,0,m,n}(0, \chi) = \frac{L(1, \chi)^4}{L(2, \chi^2)} \delta'(m)\delta'(n) \tag{471}$$

where

$$\delta'(m) = \prod_{\substack{p|m \\ p \textit{ split}}} \left(1 + m_p \frac{p-1}{p+1}\right) \prod_{\substack{p|m \\ p \textit{ inert}}} \left(1 + m_p \frac{p+1}{p-1}\right). \tag{472}$$

PROOF. To simplify things we work with $I(h,k)$ instead of $I(h_{(k)},k_{(h)})$ and we let $\beta=\alpha$ and $\gamma=\delta=0$. Then, by Theorem 1,

$$\begin{aligned}
I(h,k) =& \int_{-\infty}^{\infty}\left(\frac{h}{k}\right)^{-it}\zeta_{\mathbb{K}}\left(\frac{1}{2}+\alpha+it\right)\zeta_{\mathbb{K}}\left(\frac{1}{2}-it\right)w(t)dt \\
=& \frac{1}{\sqrt{hk}}\int_{-\infty}^{\infty}w(t)\Bigg(Z_{\alpha,\alpha,0,0,h,k}(0)+Z_{0,0,-\alpha,-\alpha,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\alpha}\left(\frac{qt}{2\pi}\right)^{-\alpha} \\
&+Z_{0,\alpha,-\alpha,0,h,k}(0)\left(\frac{t}{2\pi}\right)^{-\alpha}+Z_{\alpha,0,0,-\alpha,h,k}(0)\left(\frac{qt}{2\pi}\right)^{-\alpha} \\
&+\mathbf{1}_{q|h}\chi(k)G(\chi)Z'_{0,\alpha,0,-\alpha,\frac{h}{q},k}(0,\chi)\left(\frac{t}{2\pi}\right)^{-\alpha} \\
&+\mathbf{1}_{q|k}\chi(h)G(\chi)Z'_{\alpha,0,-\alpha,0,h,\frac{k}{q}}(0,\chi)\left(\frac{qt}{2\pi}\right)^{-\alpha}\Bigg)dt+E(T)
\end{aligned} \tag{473}$$

where $E(T)\ll q^{1+\epsilon}T^{3/4+\epsilon}(hk)^{7/8+\epsilon}(T/T_0)^{9/4}$. The various Z terms of the integrand are given by

$$\begin{aligned}
Z_{\alpha,\alpha,0,0,h,k}(0) &= \frac{\zeta(1+\alpha)^2L(1+\alpha,\chi)^2}{\zeta(2+2\alpha)}\prod_{p|q}\left(1+\frac{1}{p^{1+\alpha}}\right)^{-1}B_{\alpha,\alpha,0,0,h,k}(0), \\
Z_{0,0,-\alpha,-\alpha,h,k}(0) &= \frac{\zeta(1-\alpha)^2L(1-\alpha,\chi)^2}{\zeta(2-2\alpha)}\prod_{p|q}\left(1+\frac{1}{p^{1-\alpha}}\right)^{-1}B_{0,0,-\alpha,-\alpha,h,k}(0), \\
Z_{0,\alpha,-\alpha,0,h,k}(0) &= \frac{\zeta(1+\alpha)\zeta(1-\alpha)L(1,\chi)^2}{\zeta(2)}\prod_{p|q}\left(\frac{1-p^{-1-\alpha}}{1-p^{-2}}\right)B_{0,\alpha,-\alpha,0,h,k}(0), \\
Z_{\alpha,0,0,-\alpha,h,k}(0) &= \frac{\zeta(1+\alpha)\zeta(1-\alpha)L(1,\chi)^2}{\zeta(2)}\prod_{p|q}\left(\frac{1-p^{-1+\alpha}}{1-p^{-2}}\right)B_{\alpha,0,0,-\alpha,h,k}(0), \\
Z'_{0,\alpha,0,-\alpha,\frac{h}{q},k}(0,\chi) &= \frac{L(1,\chi)^2L(1-\alpha,\chi)L(1+\alpha,\chi)}{L(2,\chi^2)}B'_{0,\alpha,0,-\alpha,\frac{h}{q},k}(0,\chi), \\
Z'_{\alpha,0,-\alpha,0,h,\frac{k}{q}}(0,\chi) &= \frac{L(1,\chi)^2L(1+\alpha,\chi)L(1-\alpha,\chi)}{L(2,\chi^2)}B'_{0,\alpha,0,-\alpha,h,\frac{k}{q}}(0,\chi)
\end{aligned}$$

where

$$B_{\alpha,\beta,\gamma,\delta,h,k}(0,\chi)=\prod_{p|hk}\frac{\sum_{j\geq 0}f_{\alpha,\beta}(p^{k_p+j},\chi)f_{\gamma,\delta}(p^{h_p+j},\chi)p^{-j}}{\sum_{j\geq 0}f_{\alpha,\beta}(p^{j},\chi)f_{\gamma,\delta}(p^{j},\chi)p^{-j}}$$

and

$$B'_{\alpha,\beta,\gamma,\delta,h,k}(0,\chi) = \prod_{p|hk} \frac{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^{k_p+j})\sigma_{\gamma,\delta}(p^{h_p+j})p^{-j}}{\sum_{j\geq 0}\chi(p^j)\sigma_{\alpha,\beta}(p^j)\sigma_{\gamma,\delta}(p^j)p^{-j}}$$

with

$$f_{\alpha,\beta}(n,\chi) = \sum_{n_1n_2=n} n_1^{-\alpha}n_2^{-\beta}\chi(n_2), \qquad \sigma_{\alpha,\beta}(n) = \sum_{n_1n_2=n} n_1^{-\alpha}n_2^{-\beta}.$$

As previously mentioned, we want to expand the Z terms as Laurent series (we can just let the variables in the Z' terms tend to zero since these terms are holomorphic). The expansion of the B terms is facilitated by the following observation. Since

$$f_{\alpha,\alpha}(p^m,\chi) = \sum_{d|p^m}(p^m/d)^{-\alpha}d^{-\alpha}\chi(d) = p^{-\alpha m}\sum_{d|p^m}\chi(d) = p^{-\alpha m}F_\chi(p^m),$$

say, we have that

$$\begin{aligned} B_{\alpha,\alpha,0,0,h,k}(0) =& \prod_{p|hk} \frac{\sum_{j\geq 0} f_{\alpha,\alpha}(p^{k_p+j},\chi)f_{0,0}(p^{h_p+j},\chi)p^{-j}}{\sum_{j\geq 0} f_{\alpha,\alpha}(p^j,\chi)f_{0,0}(p^j,\chi)p^{-j}} \\ =& k^{-\alpha}\prod_{p|hk} \frac{\sum_{j\geq 0} F_\chi(p^{k_p+j})F_\chi(p^{h_p+j})p^{-j(1+\alpha)}}{\sum_{j\geq 0} F_\chi(p^j)^2p^{-j(1+\alpha)}} \qquad (474)\\ =& k^{-\alpha}C_{h,k}(\alpha), \end{aligned}$$

say. Note $B_{0,0,-\alpha,-\alpha,h,k}(0) = h^\alpha C_{h,k}(-\alpha)$. For the two remaining B terms we cannot fully extract the powers of h or k. However they are similar enough that we only need deal with a single function. Let $D_{h,k}(\alpha) := B_{\alpha,0,0,-\alpha,h,k}(0)$. Then since

$$f_{\alpha,0}(p^m,\chi) = p^{-\alpha m}f_{0,-\alpha}(p^m,\chi),$$

we have $D_{h,k}(\alpha) = h^\alpha k^{-\alpha}B_{0,-\alpha,\alpha,0}(0)$ and so $B_{0,\alpha,-\alpha,0,h,k}(0) = h^\alpha k^{-\alpha}D_{h,k}(-\alpha)$.

In expanding the $Z_{\alpha,\alpha,0,0,h,k}(0)$ term we write $\zeta(1+\alpha) = 1/\alpha+\gamma_0+\alpha\gamma_1+O(\alpha^2)$ and

$$\frac{L(1+\alpha,\chi)^2}{\zeta(2+2\alpha)}\prod_{p|q}\left(1+\frac{1}{p^{1+\alpha}}\right)^{-1} = c_0+\alpha c_1+O(\alpha^2), \quad c_0 = \frac{L(1,\chi)^2}{\zeta(2)}\prod_{p|q}\left(1+\frac{1}{p}\right)^{-1}$$

for some constants γ_i, c_i. We write $B_{\alpha,\alpha,0,0,h,k}(0) = k^{-\alpha}C_{h,k}(\alpha)$ which in turn we write as

$$\left(1-\alpha\log k+\frac{\alpha^2}{2}\log^2 k+O(\alpha^3)\right)\left(C_{h,k}(0)+\alpha C'_{h,k}(0)+\frac{\alpha^2}{2}C''_{h,k}(0)+O(\alpha^3)\right)$$

where the $'$ in the C terms denotes differentiation with respect to α. This should cause no confusion with meaning of the Z' terms since we never differentiate these. The other Z terms are treated similarly. Also, we write $Z'_{0,\alpha,0,-\alpha,h/q,k}(0,\chi) = Z'_{0,0,0,0,h/q,k}(0,\chi)+O(\alpha)$ again with a similar treatment for the other Z' term. Finally, we let $\mathcal{L}=\log(t/2\pi)$ and $\mathcal{Q}=\log(qt/2\pi)$. Then, the integrand of (473) is given by

$$\begin{aligned}
&\left(\frac{1}{\alpha}+\gamma_0+\cdots\right)^2\left(c_0+\cdots\right)\left(1-\alpha\log k+\frac{\alpha^2}{2}\log^2 k+\cdots\right)\\
&\quad\times\left(C_{h,k}(0)+\alpha C'_{h,k}(0)+\frac{\alpha^2}{2}C''_{h,k}(0)+\cdots\right)\\
&+\left(-\frac{1}{\alpha}+\gamma_0+\cdots\right)^2\left(c_0+\cdots\right)\left(1+\alpha\log h+\frac{\alpha^2}{2}\log^2 h+\cdots\right)\\
&\quad\times\left(C_{h,k}(0)-\alpha C'_{h,k}(0)+\frac{\alpha^2}{2}C''_{h,k}(0)-\cdots\right)\left(1-\alpha\mathcal{L}+\frac{\alpha^2}{2}\mathcal{L}^2+\cdots\right)\\
&\quad\times\left(1-\alpha\mathcal{Q}+\frac{\alpha^2}{2}\mathcal{Q}^2+\cdots\right)\\
&+\left(\frac{1}{\alpha}+\gamma_0+\cdots\right)\left(-\frac{1}{\alpha}+\gamma_0+\cdots\right)\left(c_0+\cdots\right)\left(1+\alpha\log\frac{h}{k}+\frac{\alpha^2}{2}\log^2\frac{h}{k}+\cdots\right)\\
&\quad\times\left(D_{h,k}(0)-\alpha D'_{h,k}(0)+\frac{\alpha^2}{2}D''_{h,k}(0)-\cdots\right)\left(1-\alpha\mathcal{L}+\frac{\alpha^2}{2}\mathcal{L}^2+\cdots\right)\\
&+\left(\frac{1}{\alpha}+\gamma_0+\cdots\right)\left(-\frac{1}{\alpha}+\gamma_0+\cdots\right)\left(c_0+\cdots\right)\\
&\quad\times\left(D_{h,k}(0)+\alpha D'_{h,k}(0)+\frac{\alpha^2}{2}D''_{h,k}(0)+\cdots\right)\left(1-\alpha\mathcal{Q}+\frac{\alpha^2}{2}\mathcal{Q}^2+\cdots\right)\\
&+\mathbf{1}_{q|h}\chi(k)G(\chi)Z'_{0,0,0,0,h/q,k}(0,\chi)+\mathbf{1}_{q|k}\chi(h)G(\chi)Z'_{0,0,0,0,h,k/q}(0,\chi)+\cdots
\end{aligned}$$

We note that $\log hk \ll \log T$ and, as we shall see later, $D_{h,k}^{(i)}(0) \ll D_{h,k}(0)\log^i hk$, $C_{h,k}^{(i)}(0) \ll C_{h,k}(0)(\log\log hk)^i$. Collecting like terms this becomes

$$\begin{aligned} c_0\Bigg[C_{h,k}(0)\mathcal{L}\mathcal{Q} - \mathcal{L}\Bigg(C_{h,k}(0)\log h - D_{h,k}(0)\log\frac{h}{k} + D'_{h,k}(0)\Bigg) \\ - \mathcal{Q}\Bigg(C_{h,k}(0)\log h - D'_{h,k}(0)\Bigg) + \frac{1}{2}\Bigg(C_{h,k}(0)(\log^2 k + \log^2 h) \\ - D_{h,k}(0)\log^2\frac{h}{k} + 2D'_{h,k}(0)\log\frac{h}{k} - 2D''_{h,k}(0)\Bigg)\Bigg] + \cdots \\ = c_0 D_{h,k}(0)\Bigg[\mathcal{L}\mathcal{Q} - \mathcal{L}\log hk + \log h\log k + \frac{D'_{h,k}(0)}{D_{h,k}(0)}\log\frac{h}{k} - \frac{D''_{h,k}(0)}{D_{h,k}(0)}\Bigg] + \cdots \end{aligned}$$

where we have used $C_{h,k}(0) = D_{h,k}(0)$ and $\mathcal{Q} = \mathcal{L} + \log q$. The dots represent the lower order (in t) terms and they are all seen to be holomorphic in α. We can therefore let $\alpha \to 0$. The above expression will give rise to the leading term (466) once we show $D_{h,k}(0) = \delta(h)\delta(k)$ and $D_{h,k}^{(i)}(0) \ll D_{h,k}(0)\log^i hk$. The estimates for the lower order terms (468) and (469) will follow after showing $C_{h,k}^{(i)}(0) \ll C_{h,k}(0)(\log\log hk)^i$. We prove this latter estimate first.

Let

$$G_{h,k}(\alpha,p) = \sum_{j\geq 0} F_\chi(p^{k_p+j})F_\chi(p^{h_p+j})p^{-j(1+\alpha)}$$

where $F_\chi(p^m) = \sum_{d|p^m}\chi(d)$ and let $G(\alpha,p) = G_{1,1}(\alpha,p)$ so that

$$C_{h,k}(\alpha) = \prod_{p|hk}\frac{G_{h,k}(\alpha,p)}{G(\alpha,p)}.$$

By logarithmic differentiation we have

$$C'_{h,k}(\alpha) = C_{h,k}(\alpha)\sum_{p|hk}\left[\frac{G'_{h,k}(\alpha,p)}{G_{h,k}(\alpha,p)} - \frac{G'(\alpha,p)}{G(\alpha,p)}\right]$$

and differentiating again gives

$$C''_{h,k}(\alpha) = C_{h,k}(\alpha)\Bigg\{\Bigg(\sum_{p|hk}\left[\frac{G'_{h,k}(\alpha,p)}{G_{h,k}(\alpha,p)} - \frac{G'(\alpha,p)}{G(\alpha,p)}\right]\Bigg)^2 + \sum_{p|hk}\left[\frac{G'_{h,k}(\alpha,p)}{G_{h,k}(\alpha,p)} - \frac{G'(\alpha,p)}{G(\alpha,p)}\right]'\Bigg\}.$$

Now, since

$$F_\chi(p^j) = \begin{cases} j+1 & \text{if } \chi(p) = 1, \\ 1 & \text{if } \chi(p) = -1 \text{ and } j \text{ is even}, \\ 0 & \text{if } \chi(p) = -1 \text{ and } j \text{ is odd}, \\ 1 & \text{if } \chi(p) = 0 \end{cases} \tag{475}$$

we have

$$\begin{aligned} -\frac{G'_{h,k}(\alpha,p)}{G_{h,k}(\alpha,p)} &= \log p \frac{\sum_{j\geq 0} jF_\chi(p^{k_p+j})F_\chi(p^{h_p+j})p^{-j(1+\alpha)}}{\sum_{j\geq 0} F_\chi(p^{k_p+j})F_\chi(p^{h_p+j})p^{-j(1+\alpha)}} \\ &\ll \log p \frac{\sum_{j\geq 0} jp^{-j(1+\alpha)}}{\sum_{j\geq 0} p^{-j(1+\alpha)}} \\ &\ll \frac{\log p}{p} \end{aligned}$$

where in the last line we have used $\sum_{j\geq 0} j^n x^{-j} \ll 1/x$ for $n \geq 1$. Similarly, we find $G'(\alpha,p)/G(\alpha,p) \ll p^{-1}\log p$ and $G''_{h,k}(\alpha,p)/G_{h,k}(\alpha,p) \ll p^{-1}\log^2 p$. The upper bound $C^{(i)}_{h,k}(0) \ll C_{h,k}(0)(\log\log hk)^i$ now follows if we can show

$$\sum_{p|hk} \frac{\log^i p}{p} \ll (\log\log hk)^i.$$

To see this last inequality, first note that the largest values of the sum occur successively at the primorials i.e. when $hk = \prod_{p\leq x} p$ for some x. In this case we have

$$\sum_{p|hk} \frac{\log^i p}{p} = \sum_{p\leq x} \frac{\log^i p}{p} = \log^i x + O(\log^{i-1} x)$$

where we have used the well known result $\sum_{p\leq x} 1/p = \log\log x + O(1)$ and partial summation. Since $hk = \exp(\sum_{p\leq x}\log p) = \exp(x + O(1))$ we have $x \ll \log hk$ and we're done.

Let

$$H_{h,k}(\alpha,p) = \sum_{j\geq 0} f_{\alpha,0}(p^{k_p+j},\chi)f_{0,-\alpha}(p^{h_p+j},\chi)p^{-j}$$

and $H(\alpha, p) = H_{1,1}(\alpha, p)$ so that

$$D_{h,k}(\alpha) = \prod_{p|hk} \frac{H_{h,k}(\alpha, p)}{H(\alpha, p)}.$$

As in the case of $C_{h,k}(\alpha)$, the upper bounds of $D^{(i)}_{h,k}(\alpha)$ are determined by those of $\sum_{p|hk} H^{(i)}_{h,k}(\alpha, p)/H_{h,k}(\alpha, p)$. Let

$$f^{(i,k)}_{\alpha,-\beta}(n, \chi) = \frac{\partial^i}{\partial \alpha^i} \frac{\partial^k}{\partial \beta^k} f_{\alpha,-\beta}(n, \chi) = \sum_{n_1 n_2 = n} (-\log n_1)^i (\log n_2)^k n_1^{-\alpha} n_2^{\beta} \chi(n_2).$$

Then

$$H'_{h,k}(\alpha, p) = \sum_{j \geq 0} \bigg[f^{(1,0)}_{\alpha,0}(p^{k_p+j}, \chi) f_{0,-\alpha}(p^{h_p+j}, \chi) \\ + f_{\alpha,0}(p^{k_p+j}, \chi) f^{(0,1)}_{0,-\alpha}(p^{h_p+j}, \chi) \bigg] p^{-j}$$

and

$$H''_{h,k}(\alpha, p) = \sum_{j \geq 0} \bigg[f^{(2,0)}_{\alpha,0}(p^{k_p+j}, \chi) f_{0,-\alpha}(p^{h_p+j}, \chi) \\ + 2 f^{(1,0)}_{\alpha,0}(p^{k_p+j}, \chi) f^{(0,1)}_{0,-\alpha}(p^{h_p+j}, \chi) + f_{\alpha,0}(p^{k_p+j}, \chi) f^{(0,2)}_{0,-\alpha}(p^{h_p+j}, \chi) \bigg] p^{-j}.$$

Since $f^{(i,k)}_{\alpha,-\beta}(n, \chi) \ll (\log n)^{i+k} f_{\alpha,-\beta}(n, \chi)$ we see

$$H^{(i)}_{h,k}(\alpha, p) \ll \big((h_p + k_p) \log p \big)^i H_{h,k}(\alpha, p)$$

and therefore

$$\frac{D^{(i)}_{h,k}(\alpha)}{D_{h,k}(\alpha)} \ll \left(\sum_{p|hk} \frac{H'_{h,k}(\alpha, p)}{H_{h,k}(\alpha, p)} \right)^i + \sum_{p|hk} \frac{H^{(i)}_{h,k}(\alpha, p)}{H_{h,k}(\alpha, p)} \ll \log^i hk.$$

It remains to evaluate

$$D_{h,k}(0) = \prod_{p|h} \frac{\sum_{j \geq 0} F_\chi(p^j) F_\chi(p^{h_p+j}) p^{-j}}{\sum_{j \geq 0} F_\chi(p^j)^2 p^{-j}} \prod_{p|k} \frac{\sum_{j \geq 0} F_\chi(p^{k_p+j}) F_\chi(p^j) p^{-j}}{\sum_{j \geq 0} F_\chi(p^j)^2 p^{-j}}.$$

Let us work with the product over $p|h$. We split the product over primes for which $\chi(p) = \pm 1, 0$ and apply formula (475). If $\chi(p) = 0$ then $F_\chi(p^m) = 1$ regardless of

m and so the product equals 1. Now suppose $\chi(p) = -1$ and h_p is even. Then $F_\chi(p^{h_p+j})$ equals 1 for even j and equals 0 for odd j. Since $F_\chi(p^j)$ shares the same behaviour with respect to j, the product equals 1 again. If however h_p is odd i.e. h contains a non-square inert prime, then j and $h_p + j$ cannot both be even and the product is zero. Finally, if $\chi(p) = 1$ then

$$\begin{aligned}\frac{\sum_{j\ge 0} F_\chi(p^j)F_\chi(p^{h_p+j})p^{-j}}{\sum_{j\ge 0} F_\chi(p^j)^2 p^{-j}} =& \frac{\sum_{j\ge 0}(j+1)(h_p+j+1)p^{-j}}{\sum_{j\ge 0}(j+1)^2p^{-j}}\\ =&1 + h_p\frac{\sum_{j\ge 0}(j+1)p^{-j}}{\sum_{j\ge 0}(j+1)^2p^{-j}}\\ =&1 + h_p\frac{(1-p^{-1})^{-2}}{(1+p^{-1})(1-p^{-1})^{-3}} = 1 + h_p\frac{1-p^{-1}}{1+p^{-1}}\end{aligned}$$

and it follows that $D_{h,k}(0) = \delta(h)\delta(k)$. A similar calculation gives the formula for $\delta'(h)$ in (472). We now use $w(t)$ to take smooth approximations to the characteristic function of the interval $[T, 2T]$ with $T_0 = T^{1-\epsilon}$. Upon integrating the result follows. □

3.4.2. Evaluating the main term.

Proposition 3.4.5. *Let $c_2(h,k,T)$ be given by (466) and let $\alpha(n)$ be defined as in Lemma 3.4.2. Suppose $X, T \to \infty$ with $X \ll (\log T)^{2-\epsilon}$. Then*

$$\sum_{\substack{h,k\le T^\theta\\ h,k\in\mathcal{W}(X)}} \frac{\alpha(h)\alpha(k)c_2(h,k,T)}{hk}(h,k) = (1+o(1))\frac{\log T\cdot\log qT}{(e^\gamma\log X)^2}. \tag{476}$$

Proof. Inputting the formula for $c_2(h,k,T)$ we see that we are required to show

(477)

$$\begin{aligned}S_0 :=& \sum_{\substack{h,k\le T^\theta\\ h,k\in\mathcal{W}(X)}} \frac{\alpha(h)\alpha(k)\delta(h_{(k)})\delta(k_{(h)})}{hk}(h,k)\Big[\log T\cdot\log qT + O(\log T\log\tfrac{hk}{(h,k)^2})\Big]\\ =&(1+o(1))\frac{\pi^2}{6}L(1,\chi)^{-2}\prod_{p\mid d_{\mathbb{K}}}\left(1+\frac{1}{p}\right)\frac{\log T\cdot\log qT}{(e^\gamma\log X)^2}.\end{aligned}$$

We first group together the terms for which $(h,k)=g$. Replacing h by hg and k by kg we obtain

$$S_0 = \sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}} \frac{\alpha(kg)\delta(k)}{k} \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}} \frac{\alpha(hg)\delta(h)}{h} \times \Big[\log T\cdot \log qT + O(\log T\log hk)\Big] \tag{478}$$

where $Y=T^\theta$. Let us first estimate the error term. We have

$$\begin{aligned}
&\sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}} \frac{\alpha(kg)\delta(k)}{k} \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}} \frac{\alpha(hg)\delta(h)}{h}\log{(hk)}\\
&\ll \sum_{g\in\mathcal{L}(X)} \frac{d(g)^2}{g} \sum_{h,k\in\mathcal{L}(X)} \frac{d(k)^2d(h)^2}{hk}\log hk \qquad (479)\\
&\ll \sum_{g\in\mathcal{L}(X)} \frac{d(g)^2}{g} \Bigg(\sum_{m\in\mathcal{L}(X)} \frac{d(m)^2\log m}{m}\Bigg)^2.
\end{aligned}$$

Writing $f(\sigma)=\sum_{m\in\mathcal{L}(X)} d(m)^2m^{-\sigma}$ the inner sum is $-f'(1)$. Since $f(\sigma)=\prod_{p\le X}(1-p^{-\sigma})^{-4}(1-p^{-2\sigma})$ we see $f'(1)\ll f(1)\sum_{p\le X}\log p/(p-1)\ll \log^5 X$ and hence the above sum is $\ll \log^{14} X$. We can now turn to the main term and consider

$$S := \sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}} \frac{\alpha(kg)\delta(k)}{k} \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}} \frac{\alpha(hg)\delta(h)}{h}. \tag{480}$$

We define the function $\mu' : \mathrm{Im}(\mathfrak{N}) \to \mathbb{C}$, $\mathfrak{N}(\mathfrak{a}) \mapsto \mu(\mathfrak{a})$ where μ is the extension of the usual möbius function to ideals given by

$$\mu(\mathfrak{a}) = \begin{cases} 1 & \text{if } \mathfrak{a}=\mathcal{O}_{\mathbb{K}},\\ (-1)^r & \text{if } \mathfrak{a}=\mathfrak{p}_1\mathfrak{p}_2\ldots\mathfrak{p}_r,\\ 0 & \text{otherwise.}\end{cases} \tag{481}$$

So basically; for split and ramified primes $\mu'(p) = -1$ and $\mu'(p^j) = 0$ for $j \geq 2$; for inert primes $\mu(p^2) = -1$ and $\mu(p^{2j}) = 0$ for $j \geq 2$, and μ' is multiplicative. Similarly to the usual möbius function we now have

$$\sum_{\substack{d|h\\ d|k\\ d\in \mathrm{Im}(\mathfrak{N})}} \mu'(d) = \begin{cases} 1 & \text{if } (h,k)=1 \\ 0 & \text{otherwise} \end{cases} \tag{482}$$

for $h, k \in \mathrm{Im}(\mathfrak{N})$. Substituting this into the sum over h in S we see

$$\begin{aligned} S &= \sum_{\substack{g\leq Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\leq Y/g\\ k\in\mathcal{W}(X)}} \frac{\alpha(kg)\delta(k)}{k} \sum_{\substack{h\leq Y/g\\ h\in\mathcal{W}(X)}} \left(\sum_{\substack{d|h\\ d|k\\ d\in \mathrm{Im}(\mathfrak{N})}} \mu'(d)\right) \frac{\alpha(hg)\delta(h)}{h} \\ &= \sum_{\substack{g\leq Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{l\leq Y/g\\ l\in\mathcal{W}(X)}} \frac{\mu'(l)}{l^2} \left(\sum_{\substack{m\leq Y/gl\\ m\in\mathcal{W}(X)}} \frac{\alpha(glm)\delta(lm)}{m} \right)^2. \end{aligned} \tag{483}$$

Manipulating the sums in this way allows us to avoid the rather technical and lengthy calculations involved in [20].

We wish to extend these sums over all $\mathcal{W}(X)$ and this requires some estimates. These will follow in a similar fashion to that found between (424) and (426). Throughout we use $\alpha(m), \delta(m) \ll d(m)$ and $d(mn) \leq d(m)d(n)$. First, let b be positive and small, then

$$\begin{aligned} \frac{1}{d(g)d(l)^2} \sum_{\substack{m>Y/lg\\ m\in\mathcal{W}(X)}} \frac{\alpha(glm)\delta(lm)}{m} &\ll \sum_{\substack{m>Y/lg\\ m\in\mathcal{W}(X)}} \frac{d(m)^2}{m} \ll \left(\frac{Y}{lg}\right)^{-b} \sum_{m\in\mathcal{W}(X)} \frac{d(m)^2}{m^{1-b}} \\ &\ll \left(\frac{Y}{lg}\right)^{-b} \prod_{p\leq X} \left(1-p^{-1+b}\right)^{-4} \left(1-p^{-2(1-b)}\right) \\ &\ll \left(\frac{Y}{lg}\right)^{-b} e^{8X^b/\log X} \ll (lg)^b T^{-b\theta/2}. \end{aligned} \tag{484}$$

Second,

$$\sum_{m\in\mathcal{W}(X)} \frac{\alpha(glm)\delta(lm)}{m} \ll d(g)d(l)^2 \sum_{m\in\mathcal{W}(X)} \frac{d(m)^2}{m} \ll d(g)d(l)^2 \log^4 X. \tag{485}$$

From these it follows that the square of the sum over m in (483) is

$$\left(\sum_{m\in\mathcal{W}(X)}\frac{\alpha(glm)\delta(lm)}{m}\right)^2+O\left(d(g)^2d(l)^4(lg)^{2b}T^{-b\theta/4}\right). \tag{486}$$

Similarly we find

$$\sum_{l\in\mathcal{W}(X)}\frac{\mu'(l)d(l)^4}{l^{2-2b}}\ll 1,\qquad \sum_{\substack{l>Y/g\\ l\in\mathcal{W}(X)}}\frac{\mu'(l)d(l)^4}{l^{2-2b}}\ll g^cT^{-c\theta}, \tag{487}$$

for some small $c>0$, and

$$\sum_{g\in\mathcal{W}(X)}\frac{d(g)^2}{g^{1-2b-c}}\ll T^{\epsilon},\qquad \sum_{\substack{g>Y\\ g\in\mathcal{W}(X)}}\frac{d(g)^2}{g^{1-2b-c}}\ll T^{-d\theta} \tag{488}$$

for some small $d>0$. The above estimates give

$$\begin{aligned} S=&\Bigg(\sum_{g\in\mathcal{W}(X)}-\sum_{\substack{g>Y\\ g\in\mathcal{W}(X)}}\Bigg)\frac{1}{g}\Bigg(\sum_{l\in\mathcal{W}(X)}-\sum_{\substack{l>Y/g\\ l\in\mathcal{W}(X)}}\Bigg)\frac{\mu'(l)}{l^2}\\ &\times\left[\left(\sum_{m\in\mathcal{W}(X)}\frac{\alpha(glm)\delta(lm)}{m}\right)^2+O\left(d(g)^2d(l)^4(lg)^{2b}T^{-b\theta/4}\right)\right]\\ =&(1+o(1))\sum_{g\in\mathcal{W}(X)}\frac{1}{g}\sum_{l\in\mathcal{W}(X)}\frac{\mu'(l)}{l^2}\left(\sum_{m\in\mathcal{W}(X)}\frac{\alpha(glm)\delta(lm)}{m}\right)^2. \end{aligned} \tag{489}$$

Now, since all coefficients in S are multiplicative we may expand the sum into an Euler product:

$$S=(1+o(1))\prod_{\substack{p\le X\\ p\text{ split}}}G(p)\prod_{\substack{p\le\sqrt{X}\\ p\text{ inert}}}G(p^2)\prod_{\substack{p\le X\\ p\text{ ramified}}}G(p) \tag{490}$$

with

$$G(p)=\sum_{i,j,u,v\ge 0}\frac{\mu'(p^j)\alpha(p^{i+j+u})\alpha(p^{i+j+v})\delta(p^{j+u})\delta(p^{j+v})}{p^{i+2j+u+v}}. \tag{491}$$

Performing the various sums whilst using the support conditions of α and μ' we see

$$G(p) = 1 + \frac{2\alpha(p)\delta(p) + \alpha(p)^2}{p} + \frac{2\alpha(p^2)\delta(p^2) + \alpha(p^2)^2 + 2\alpha(p)\alpha(p^2)\delta(p)}{p^2}. \tag{492}$$

Recall that for a split prime p we have $\delta(p^r) = 1 + r(p-1)/(p+1)$ and hence $\delta(p) = 2p/(p+1)$ and $\delta(p^2) = 2\delta(p) - 1$. We also have $\alpha(p) = -2$ for all $p \leq X$, $\alpha(p^2) = 1$ for $p \leq \sqrt{X}$ and $\alpha(p^2) = 2$ for $\sqrt{X} < p \leq X$. A straightforward calculation now gives

$$\begin{aligned} \prod_{\substack{p \leq X \\ p \text{ split}}} G(p) =& \prod_{\substack{p \leq \sqrt{X} \\ p \text{ split}}} \left(\frac{(1-1/p)^4}{1-1/p^2} \right) \prod_{\substack{\sqrt{X} < p \leq X \\ p \text{ split}}} \left(\frac{(1-1/p)^4}{1-1/p^2} + O\left(\frac{1}{p^2}\right) \right) \\ =& \prod_{\substack{p \leq X \\ p \text{ split}}} \left(\frac{(1-1/p)^4}{1-1/p^2} \right) \prod_{\substack{\sqrt{X} < p \leq X \\ p \text{ split}}} \left(1 + O\left(\frac{1}{p^2}\right) \right) \\ =& (1+o(1)) \prod_{\substack{p \leq X \\ p \text{ split}}} \left(1 - \frac{1}{p}\right)^4 \prod_{p \text{ split}} \left(1 - \frac{1}{p^2}\right)^{-1}. \end{aligned} \tag{493}$$

In evaluating the remaining products in (490) we note that α behaves the same on square inert primes as it does on ramified primes. The same goes for δ since the number 1 varies little. We describe the ramified case since the inert case is simply handled by replacing p with p^2.

For a ramified prime p we have $\delta(p) = \delta(p^2) = 1$, $\alpha(p) = -1$ for all $p \leq X$, $\alpha(p^2) = 0$ for $p \leq \sqrt{X}$ and $\alpha(p^2) = 1/2$ for $\sqrt{X} < p \leq X$. With this information we see

$$\begin{aligned}\prod_{\substack{p\le X\\ p \text{ ramified}}} G(p) &= \prod_{\substack{p\le\sqrt{X}\\ p \text{ ramified}}}\left(1-\frac{1}{p}\right)\prod_{\substack{\sqrt{X}<p\le X\\ p \text{ ramified}}}\left(1-\frac{1}{p}+\frac{1}{4p^2}\right)\\ &= \prod_{\substack{p\le X\\ p \text{ ramified}}}\left(1-\frac{1}{p}\right)\prod_{\substack{\sqrt{X}<p\le X\\ p \text{ ramified}}}\left(1+O\left(\frac{1}{p^2}\right)\right)\\ &=(1+o(1))\prod_{\substack{p\le X\\ p \text{ ramified}}}\left(1-\frac{1}{p}\right).\\ &=(1+o(1))\prod_{\substack{p\le X\\ p \text{ ramified}}}\left(1-\frac{1}{p}\right)^2\prod_{p \text{ ramified}}\left(1+\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)^{-1}.\end{aligned} \tag{494}$$

In extending this last product over all ramified primes we have used the fact that a prime is ramified if and only if it divides $d_{\mathbb{K}}$ and hence $\sum_{X<p|d_{\mathbb{K}}} 1/p = o(1)$. Similarly, for inert primes we find

$$\begin{aligned}\prod_{\substack{p\le\sqrt{X}\\ p \text{ inert}}} G(p^2) &=(1+o(1))\prod_{\substack{p\le\sqrt{X}\\ p \text{ inert}}}\left(1-\frac{1}{p^2}\right)\\ &=(1+o(1))\prod_{\substack{p\le\sqrt{X}\\ p \text{ inert}}}\left(1-\frac{1}{p^2}\right)^2\prod_{p \text{ inert}}\left(1-\frac{1}{p^2}\right)^{-1}\end{aligned} \tag{495}$$

Collecting the infinite products in (493), (494) and (495) we acquire the factor

$$\frac{\pi^2}{6}\prod_{p|d_{\mathbb{K}}}\left(1+\frac{1}{p}\right). \tag{496}$$

The remaining terms are then given by

$$(1+o(1))\prod_{\mathfrak{N}(\mathfrak{p})\le X}\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^2 = (1+o(1))(L(1,\chi)e^{\gamma}\log X)^{-2} \tag{497}$$

where we have used (52). □

3.4.3. Estimating the lower order terms. By virtue of the upper bounds (468), (470) and Proposition 3.4.5 we are only required to evaluate the sum of the 'big O' and Z' terms of formula (465). For the 'big O' term we have

$$(498)\qquad \begin{aligned} &T^{-\frac{1}{4}+\epsilon} \sum_{\substack{h,k\le T^\theta\\ h,k\in\mathcal{W}(X)}} \frac{\alpha(h)\alpha(k)(h,k)}{hk}\left(\frac{hk}{(h,k)^2}\right)^{11/8+\epsilon}\\ \ll &T^{-\frac{1}{4}+\epsilon}\left(\sum_{n\le T^\theta} d(n)n^{3/8+\epsilon}\right)^2 \ll T^{\frac{11}{4}\theta-\frac{1}{4}+\epsilon}\end{aligned}$$

and so taking $\theta \le 1/11-\epsilon$ the error term is $o(1)$.

We now estimate the sums involving the Z' terms. By (469) and (471) we see that we must consider sums of the form

$$(499)\qquad \begin{aligned} S' :=& \sum_{\substack{h,k\le Y\\ h,k\in\mathcal{W}(X)}} \frac{\alpha(h)\alpha(k)}{hk}(h,k)\mathbf{1}_{q|h_{(k)}}\chi(k_{(h)})\delta'(h_{(k)}/q)\delta'(k_{(h)})\\ =& \sum_{\substack{g\le Y\\ g\in\mathcal{W}(X)}} \frac{1}{g} \sum_{\substack{k\le Y/g\\ k\in\mathcal{W}(X)}} \frac{\chi(k)\alpha(kg)\delta'(k)}{k} \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}} \mathbf{1}_{q|h}\frac{\alpha(hg)\delta'(h/q)}{h}.\end{aligned}$$

where $Y = T^\theta$. The innermost sum is given by

$$(500)\qquad \sum_{\substack{h\le Y/qg\\ qh\in\mathcal{W}(X)\\ (qh,k)=1}} \frac{\alpha(qhg)\delta'(h)}{qh} \ll \sum_{\substack{h\le Y/g\\ h\in\mathcal{W}(X)\\ (h,k)=1}} \frac{\alpha(hg)\delta'(h)}{h}$$

where we have used $|\alpha(qm)| \le \alpha(m)$ which follows from (78) and the definition of α. We deduce that S' is $\ll$ a sum of the form (480) with δ replaced by δ'. Using the bound $\delta'(n) \le d(n^2)$ we may follow the analysis of Proposition 3.4.5 to see that

$$(501)\qquad S' \ll (1+o(1)) \prod_{\substack{p\le X\\ p \text{ split}}} G'(p) \prod_{\substack{p\le\sqrt{X}\\ p \text{ inert}}} G'(p^2) \prod_{\substack{p\le X\\ p \text{ ramified}}} G'(p)$$

where

$$G'(p) = 1 + \frac{2\alpha(p)\delta'(p) + \alpha(p)^2}{p} + \frac{2\alpha(p^2)\delta'(p^2) + \alpha(p^2)^2 + 2\alpha(p)\alpha(p^2)\delta'(p)}{p^2}. \tag{502}$$

For split and ramified primes we have $\delta'(p^r) = \delta(p^r)$ and so we only need evaluate G at the inert primes. For inert p we have $\delta(p^2) = 1 + 2(p+1)/(p-1) \le 5$ and hence

$$G'(p^2) = 1 + O\left(\frac{1}{p^2}\right) \tag{503}$$

For the sake of argument we write

$$\prod_{\substack{p \le \sqrt{X} \\ p \text{ inert}}} G'(p^2) = (1 + o(1)) \prod_{\substack{p \le \sqrt{X} \\ p \text{ inert}}} \left(1 - \frac{1}{p^2}\right)^2 \tag{504}$$

and then combine this with the products over split and ramified primes given by (493) and (494). This gives

$$S' \ll (\log X)^{-2}. \tag{505}$$

3.5. Conjecture 4 via the moments recipe

In this section we modify the recipe given in [10] to reproduce our main conjecture in the case of quadratic extensions.

Conjecture 4. *Let $\mathbb{K}$ be a quadratic extension. Then*

$$\frac{1}{T}\int_T^{2T} |\zeta_{\mathbb{K}}(\tfrac{1}{2}+it)|^{2k}dt \sim a(k)L(1,\chi)^{2k^2}\left(\frac{G(k+1)^2}{G(2k+1)}\right)^2 (\log T\cdot\log qT)^{k^2} \tag{506}$$

where q is the modulus of the character χ in the equation $\zeta_{\mathbb{K}}(s)=\zeta(s)L(s,\chi)$, G is the Barnes G-function and

$$a(k)=\prod_{\mathfrak{p}\subseteq\mathcal{O}_{\mathbb{K}}}\left(\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{2k^2}\left(\sum_{m\geq 0}\frac{d_{g_{\mathfrak{p}}k}(\mathfrak{p}^m)^2}{\mathfrak{N}(\mathfrak{p})^m}\right)^{1/g_{\mathfrak{p}}}\right). \tag{507}$$

The recipe in question is concerned with primitive L-functions, so cannot be applied directly to our situation without some modification. Our modifications are based on Theorem 1 and are in keeping with the reasoning of the original recipe. Let us first describe the process as it appears in [10] with the Riemann zeta function as the example.

Consider the shifted product

$$Z(s,\boldsymbol{\alpha})=\zeta(s+\alpha_1)\cdots\zeta(s+\alpha_k)\zeta(1-s-\alpha_{k+1})\cdots\zeta(1-s-\alpha_{2k}) \tag{508}$$

We first replace each occurrence of ζ with its approximate functional equation

$$\zeta(s)=\sum_m\frac{1}{m^s}+\varkappa(s)\sum_n\frac{1}{n^{1-s}},\quad \varkappa(s)=\pi^{\frac{1}{2}-s}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)} \tag{509}$$

and multiply out the the expression to give a sum of 2^{2k} terms. We throw away any terms that do not have an equal amount of $\varkappa(s+\alpha_i)$ and $\varkappa(1-s-\alpha_j)$ factors, the reason being that these terms are oscillatory. Indeed, by (25) we have

$$\begin{aligned}\varkappa(s+\beta_1)\cdots\varkappa(s+\beta_J)\varkappa(1-s-\gamma_1)\cdots\varkappa(1-s-\gamma_K)&\\ \sim\left(\frac{t}{2\pi e}\right)^{-i(J-K)}e^{i(J-K)\pi/4}\left(\frac{t}{2\pi}\right)^{-\sum\beta_j+\sum\gamma_k}&\end{aligned} \tag{510}$$

which is oscillating unless $J = K$. In each of the remaining $\binom{2k}{k}$ terms we retain only the diagonal from the sum, which we then extend over all positive integers. If we denote the resulting expression by $M(s, \boldsymbol{\alpha})$ then the conjecture is

$$\int_{-\infty}^{\infty} Z\left(\frac{1}{2} + it, \boldsymbol{\alpha}\right) w(t)dt \sim \int_{-\infty}^{\infty} M\left(\frac{1}{2} + it, \boldsymbol{\alpha}\right) w(t)dt \tag{511}$$

for any reasonable function $w(t)$.

To describe a typical term of $M(s, \boldsymbol{\alpha})$ let us first define the prototypical diagonal sum

$$R(\sigma, \alpha_1, \ldots, \alpha_{2k}) = \sum_{m_1 \cdots m_k = n_1 \cdots n_k} \frac{1}{m_1^{\sigma+\alpha_1} \cdots m_k^{\sigma+\alpha_k} n_1^{1-\sigma-\alpha_{k+1}} \cdots n_1^{1-\sigma-\alpha_{2k}}}. \tag{512}$$

This is in fact the term acquired by taking the first sum of the approximate functional equation in each ζ-factor of $Z(s, \boldsymbol{\alpha})$. If, for example, we were to take the second sum in $\zeta(s + \alpha_1)$ and the second sum in $\zeta(1 - s - \alpha_{k+1})$ whilst taking the first in the rest we would acquire the term

$$\left(\frac{t}{2\pi}\right)^{-\alpha_1+\alpha_{k+1}} R(\sigma, \alpha_{k+1}, \alpha_2, \ldots, \alpha_k, \alpha_1, \alpha_{k+2}, \ldots, \alpha_{2k}). \tag{513}$$

It is then clear that the full expression will be a sum over permutations $\tau \in S_{2k}$, and that any permutation other than the identity will swap elements of $\{\alpha_1, \ldots, \alpha_k\}$ with elements of $\{\alpha_{k+1}, \ldots, \alpha_{2k}\}$ in the R terms. Since R is symmetric in the first k variables and also in the second, we may reorder the entries such that the subscripts of the first k are in increasing order, as are the last k. We thus see that the full expression will be a sum over the $\binom{2k}{k}$ permutations $\tau \in S_{2k}$ such that

$$\tau(1) < \ldots < \tau(k), \;\; \tau(k+1) < \ldots < \tau(2k). \tag{514}$$

Denote the set of such permutations by Ξ. A typical term now takes the form

$$\left(\frac{t}{2\pi}\right)^{(-\alpha_1 - \cdots - \alpha_k + \alpha_{k+1} + \cdots + \alpha_{2k})/2} W(s, \boldsymbol{\alpha}, \tau) \tag{515}$$

with $\tau \in \Xi$ and where

$$W(s,\boldsymbol{\alpha},\tau) = \left(\frac{t}{2\pi}\right)^{(\alpha_{\tau(1)}+\cdots+\alpha_{\tau(k)}-\alpha_{\tau(k+1)}-\cdots-\alpha_{\tau(2k)})/2} \times R(\sigma,\alpha_{\tau(1)},\ldots,\alpha_{\tau(k)},\alpha_{\tau(k+1)},\ldots,\alpha_{\tau(2k)}). \tag{516}$$

Combining all terms we have

$$M(s,\boldsymbol{\alpha}) = \left(\frac{t}{2\pi}\right)^{(-\alpha_1-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})/2} \sum_{\tau\in\Xi} W(s,\boldsymbol{\alpha},\tau). \tag{517}$$

To recover the kth-moment conjecture for the Riemann zeta function we first extract the polar behaviour of R. This gives

$$R(\sigma,\alpha_1,\ldots,\alpha_{2k}) = A_k(\sigma,\alpha_1,\ldots,\alpha_{2k}) \prod_{i,j=1}^{k} \zeta(1+\alpha_i-\alpha_{k+j}) \tag{518}$$

where A_k is some Euler product that is absolutely convergent for $\sigma > 1/4$. Now, in [**10**] it is shown (Lemma 2.5.1) that the sum over permutations in (517) can be written as a contour integral. We reproduce this result here since we shall have use for it later.

Lemma 3.5.1 ([**10**])**.** *Suppose $F(a;b) = F(a_1,\ldots,a_k;b_1,\ldots,b_k)$ is a function of $2k$ variables which is symmetric with respect to the first k and also symmetric with respect to the second set of k variables. Suppose also that F is regular near $(0,\ldots,0)$. Suppose further that $f(s)$ has a simple pole of residue 1 at $s=0$ but is otherwise analytic in a neighbourhood about $s=0$. Let*

$$K(a_1,\ldots,a_k;b_1,\ldots,b_k) = F(a_1,\ldots,a_k;b_1,\ldots,b_k) \prod_{i,j=1}^{k} f(a_i-b_j). \tag{519}$$

If for all $1 \le i,j \le k$, $\alpha_i - \alpha_{k+j}$ is contained in the region of analyticity of $f(s)$ then

$$\begin{aligned} &\sum_{\tau\in\Xi} K(\alpha_{\tau(1)},\ldots,\alpha_{\tau(k)};\alpha_{\tau(k+1)},\ldots,\alpha_{\tau(2k)}) \\ &= \frac{(-1)^k}{k!^2(2\pi i)^{2k}} \oint\cdots\oint K(z_1,\ldots,z_k,z_{k+1},\ldots,z_{2k}) \frac{\Delta^2(z_1,\ldots,z_{2k})}{\prod_{i,j=1}^{k}(z_i-\alpha_j)} dz_1\cdots dz_{2k} \end{aligned} \tag{520}$$

where Δ is the vandermonde determinant.

By the above Lemma and (517), (518) we see that $M(1/2+it,\mathbf{0})$ is given by

$$\begin{aligned}&\frac{(-1)^k}{k!^2(2\pi i)^{2k}}\oint\cdots\oint A_k(1/2,z_1,\ldots,z_{2k})\prod_{i,j=1}^{k}\zeta(1+z_i-z_{k+j})\\&\times\frac{\Delta^2(z_1,\ldots,z_{2k})}{\prod_{j=1}^{2k}z_j^{2k}}\exp\left(\frac{1}{2}\log(t/2\pi)\sum_{j=1}^{k}z_j-z_{k+j}\right)dz_1\cdots dz_{2k}\\&=A_k(1/2,0,\ldots,0)\log^{k^2}\left(\frac{t}{2\pi}\right)(1+O((\log t)^{-1}))\frac{(-1)^k}{2^{k^2}k!^2(2\pi i)^{2k}}\\&\times\oint\cdots\oint\frac{\Delta^2(z_1,\ldots,z_{2k})}{\left(\prod_{i,j=1}(z_i-z_{k+j})\right)\prod_{j=1}^{2k}z_j^{2k}}e^{\sum_{j=1}^{k}z_j-z_{k+j}}dz_1\cdots dz_{2k},\end{aligned} \tag{521}$$

after a change of variables. It is then shown that $A_k(1/2,0,\ldots,0)=a(k)$, where

$$a(k)=\prod_p\left(1-\frac{1}{p}\right)^{k^2}\sum_{m=1}^{\infty}\frac{d_k(p^m)^2}{p^m}.$$

This is of course equal to our arithmetic factor (142) when $\mathbb{K}=\mathbb{Q}$. The conjecture is then completed by showing that

$$\frac{(-1)^k}{2^{k^2}k!^2(2\pi i)^{2k}}\oint\cdots\oint\frac{\Delta^2(z_1,\ldots,z_{2k})e^{\sum_{j=1}^{k}z_j-z_{k+j}}}{\left(\prod_{i,j=1}(z_i-z_{k+j})\right)\prod_{j=1}^{2k}z_j^{2k}}dz_1\cdots dz_{2k}=\frac{G(k+1)^2}{G(2k+1)}. \tag{522}$$

We now turn our attention to the shifted product

$$Z(s,\boldsymbol{\alpha},\boldsymbol{\beta})=Z_\zeta(s,\boldsymbol{\alpha})Z_L(s,\boldsymbol{\beta}) \tag{523}$$

where

$$Z_\zeta(s,\boldsymbol{\alpha})=\zeta(s+\alpha_1)\cdots\zeta(s+\alpha_k)\zeta(1-s-\alpha_{k+1})\cdots\zeta(1-s-\alpha_{2k}) \tag{524}$$

and

$$Z_L(s,\boldsymbol{\beta})=L(s+\beta_1,\chi)\cdots L(s+\beta_k,\chi)L(1-s-\beta_{k+1},\overline{\chi})\cdots L(1-s-\beta_{2k},\overline{\chi}). \tag{525}$$

As before, we plan to substitute the respective approximate functional equations, which we now write as

$$\zeta(s) = \sum_m \frac{1}{m^s} + \varkappa_\zeta(s) \sum_n \frac{1}{n^{1-s}}, \tag{526}$$

$$L(s,\chi) = \sum_m \frac{\chi(m)}{m^s} + \varkappa_L(s) \sum_n \frac{\overline{\chi}(n)}{n^{1-s}}. \tag{527}$$

We have

$$\varkappa_L(s) = \frac{G(\chi)}{i^{\mathfrak{a}}\sqrt{q}} \left(\frac{\pi}{q}\right)^{-\frac{1}{2}+s} \frac{\Gamma((1-s+\mathfrak{a})/2)}{\Gamma((s+\mathfrak{a})/2)} \tag{528}$$

and by (25) this is

$$\varkappa_L(s) = \frac{G(\chi)}{i^{\mathfrak{a}}\sqrt{q}} \left(\frac{qt}{2\pi}\right)^{\frac{1}{2}-s} e^{it+i\pi/4} \left(1 + O\left(\frac{1}{t}\right)\right). \tag{529}$$

If we now follow the recipe and treat the L-functions as if they were zeta functions, then after expanding and throwing away the terms with an unequal amount of $\varkappa_{\zeta,L}(s+\gamma_i)$ and $\varkappa_{\zeta,L}(1-s-\delta_j)$ factors, we are still left with some terms that have a factor of q^{-it}. Since this is oscillating we modify the recipe to throw these terms away also. We note with this modification the recipe reproduces Theorem 1, which adds some justification.

One way of arriving at the resultant expression is is to apply the first step of the recipe to $Z_\zeta(s,\boldsymbol{\alpha})$ and $Z_L(s,\boldsymbol{\beta})$ separately. This procedure prevents the occurrence of the extra oscillatory terms without throwing away any other terms unnecessarily. When applying this step to Z_L one can use the fact that $G(\chi)G(\overline{\chi}) = (-1)^{\mathfrak{a}} q$ to provide some cancellation. We then form the product to gain a sum of $\binom{2k}{k}^2$ terms and retain only the diagonals, as before. Extending the sums over all positive integers we then denote the resulting expression by $M(s,\boldsymbol{\alpha},\boldsymbol{\beta})$ and conjecture that

$$\int_{-\infty}^{\infty} Z\left(\frac{1}{2}+it,\boldsymbol{\alpha},\boldsymbol{\beta}\right) w(t)dt \sim \int_{-\infty}^{\infty} M\left(\frac{1}{2}+it,\boldsymbol{\alpha},\boldsymbol{\beta}\right) w(t)dt. \tag{530}$$

Applying the modified recipe and using a similar reasoning given in the case of the Riemann zeta function above, we see that

$$\begin{aligned} M(s,\boldsymbol{\alpha},\boldsymbol{\beta}) = &\left(\frac{t}{2\pi}\right)^{(-\alpha_1-\cdots-\alpha_k+\alpha_{k+1}+\cdots+\alpha_{2k})/2} \left(\frac{qt}{2\pi}\right)^{(-\beta_1-\cdots-\beta_k+\beta_{k+1}+\cdots+\beta_{2k})/2} \\ &\times \sum_{\tau,\tau'\in\Xi} W(s,\boldsymbol{\alpha},\boldsymbol{\beta},\tau,\tau') \end{aligned} \tag{531}$$

where

$$\begin{aligned} W(s,\boldsymbol{\alpha},\boldsymbol{\beta},\tau,\tau') = &\left(\frac{t}{2\pi}\right)^{(\alpha_{\tau(1)}+\cdots+\alpha_{\tau(k)}-\alpha_{\tau(k+1)}-\cdots-\alpha_{\tau(2k)})/2} \\ &\times \left(\frac{qt}{2\pi}\right)^{(\beta_{\tau'(1)}+\cdots+\beta_{\tau'(k)}-\beta_{\tau'(k+1)}-\cdots-\beta_{\tau'(2k)})/2} \\ &\times S(\sigma;\alpha_{\tau(1)},\ldots,\alpha_{\tau(2k)};\beta_{\tau'(1)},\ldots,\beta_{\tau'(2k)}) \end{aligned} \tag{532}$$

with

$$\begin{aligned} &S(\sigma;\alpha_1,\ldots,\alpha_{2k};\beta_1,\ldots,\beta_{2k}) \\ &= \sum_{\substack{m_1\cdots m_k m_1'\cdots m_k' = \\ n_1\cdots n_k n_1'\cdots n_k'}} \chi(m_1')\cdots\chi(m_k')\overline{\chi(n_1')\cdots\chi(n_k')}\Big[m_1^{\sigma+\alpha_1}\cdots m_k^{\sigma+\alpha_k} \\ &\quad \times {m_1'}^{\sigma+\beta_1}\cdots {m_k'}^{\sigma+\beta_k} n_1^{1-\sigma-\alpha_{k+1}}\cdots n_k^{1-\sigma-\alpha_{2k}} {n_1'}^{1-\sigma-\beta_{k+1}}\cdots {n_k'}^{1-\sigma-\beta_{2k}}\Big]^{-1}. \end{aligned} \tag{533}$$

Since the condition $m_1\cdots m_k m_1'\cdots m_k' = n_1\cdots n_k n_1'\cdots n_k'$ is multiplicative we have

$$\begin{aligned} &S(\sigma;\alpha_1,\ldots,\alpha_{2k};\beta_1,\ldots,\beta_{2k}) \\ =&\prod_p \sum_{\substack{\sum_{j=1}^k e_j+e_j' = \\ \sum_{j=1}^k e_{j+k}+e_{j+k}'}} \chi(p^{e_1'})\cdots\chi(p^{e_k'})\overline{\chi(p^{e_{k+1}'})\cdots\chi(p^{e_{2k}'})}\Big[p^{e_1(\sigma+\alpha_1)}\cdots p^{e_k(\sigma+\alpha_k)} \\ &\times p^{e_1'(\sigma+\beta_1)}\cdots p^{e_k'(\sigma+\beta_k)} p^{e_{k+1}(1-\sigma-\alpha_{k+1})}\cdots p^{e_{2k}(1-\sigma-\alpha_{2k})} \\ &\times p^{e_{k+1}'(1-\sigma-\beta_{k+1})}\cdots p^{e_{2k}'(1-\sigma-\beta_{2k})}\Big]^{-1} \end{aligned}$$

$$
\begin{aligned}
=&A_k(\sigma,\boldsymbol{\alpha},\boldsymbol{\beta})\prod_{i,j=1}^{k}\zeta(1+\alpha_i-\alpha_{k+j})L(1+\beta_i-\beta_{k+j},|\chi|^2)\\
&\times L(1+\beta_i-\alpha_{j+k},\chi)L(1+\alpha_i-\beta_{j+k},\overline{\chi})\\
=&A_k(\sigma,\boldsymbol{\alpha},\boldsymbol{\beta})\prod_{i,j=1}^{k}\zeta(1+\alpha_i-\alpha_{k+j})\zeta(1+\beta_i-\beta_{k+j})\\
&\times L(1+\beta_i-\alpha_{j+k},\chi)L(1+\alpha_i-\beta_{j+k},\overline{\chi})\left(\prod_{p|q}\left(1-p^{-1-\beta_i+\beta_{k+j}}\right)\right)
\end{aligned}
$$

where A_k is an Euler product that is absolutely convergent for $\sigma > 1/4$. For $\sigma = 1/2$ we have the following explicit expression for A_k:

$$
\begin{aligned}
A_k(1/2,\boldsymbol{\alpha},\boldsymbol{\beta}) =& \prod_p\prod_{i,j=1}^{k}(1-p^{-1-\alpha_i+\alpha_{j+k}})(1-|\chi(p)|^2p^{-1-\beta_i+\beta_{j+k}})\\
&\times(1-\chi(p)p^{-1-\beta_i+\alpha_{j+k}})(1-\overline{\chi}(p)p^{-1-\alpha_i+\beta_{j+k}})B_p(\boldsymbol{\alpha},\boldsymbol{\beta})
\end{aligned}
\tag{534}
$$

where

(535)

$$\begin{aligned}
B_p(\boldsymbol{\alpha},\boldsymbol{\beta}) &= \sum_{\substack{\sum_{j=1}^k e_j+e'_j= \\ \sum_{j=1}^k e_{j+k}+e'_{j+k}}} \frac{\chi(p^{e'_1})\cdots\overline{\chi(p^{e'_{2k}})}}{p^{e_1(1/2+\alpha_1)}\cdots p^{e'_{2k}(1/2-\beta_{2k})}} \\
&= \int_0^1 \sum_{\substack{e_1,\dots,e_{2k}\\ e'_1,\dots,e'_{2k}}} \frac{\chi(p^{e'_1})\cdots\overline{\chi(p^{e'_{2k}})}}{p^{e_1(1/2+\alpha_1)}\cdots p^{e'_{2k}(1/2-\beta_{2k})}} \\
&\quad \times e\left(\left(\sum_{j=1}^k e_j+e'_j-\sum_{j=1}^k e_{j+k}+e'_{j+k}\right)\theta\right)d\theta \\
&= \int_0^1 \prod_{j=1}^k \sum_{e_j=0}^\infty \frac{1}{p^{e_j(1/2+\alpha_j)}} e(e_j\theta) \prod_{j=1}^k \sum_{e_{j+k}=0}^\infty \frac{1}{p^{e_{j+k}(1/2-\alpha_{j+k})}} e(-e_{j+k}\theta) \\
&\quad \times \prod_{j=1}^k \sum_{e'_j=0}^\infty \frac{\chi(p^{e'_j})}{p^{e'_j(1/2+\beta_j)}} e(e'_j\theta) \prod_{j=1}^k \sum_{e'_{j+k}=0}^\infty \frac{\overline{\chi}(p^{e'_{j+k}})}{p^{e'_{j+k}(1/2-\beta_{j+k})}} e(-e'_{j+k}\theta)d\theta \\
&= \int_0^1 \prod_{j=1}^k \zeta_p\left(\frac{e(\theta)}{p^{1/2+\alpha_j}}\right)\zeta_p\left(\frac{e(-\theta)}{p^{1/2\alpha_{k+j}}}\right) L_p\left(\frac{e(\theta)}{p^{1/2+\beta_j}}\right)\overline{L}_p\left(\frac{e(-\theta)}{p^{1/2-\beta_{j+k}}}\right)d\theta
\end{aligned}$$

with $\zeta_p(x)=(1-x)^{-1}$, $L_p(x)=(1-\chi(p)x)^{-1}$ and $\overline{L}_p(x)=(1-\overline{\chi}(p)x)^{-1}$.

Now, denote the holomorphic part of $S(1/2,\boldsymbol{\alpha},\boldsymbol{\beta})$ by

(536)

$$\begin{aligned}
A'_k(1/2,\boldsymbol{\alpha},\boldsymbol{\beta}) =& A_k(1/2,\boldsymbol{\alpha},\boldsymbol{\beta}) \prod_{i,j=1}^k L(1+\beta_i-\alpha_{j+k},\chi)L(1+\alpha_i-\beta_{j+k},\overline{\chi}) \\
&\times\left(\prod_{p|q}\left(1-p^{-1-\beta_i+\beta_{k+j}}\right)\right).
\end{aligned}$$

Applying Lemma 3.5.1 twice to (531) we see that $M(1/2+it,\mathbf{0},\mathbf{0})$ is given by

(537)

$$\begin{aligned}
&\left(\frac{(-1)^k}{k!^2(2\pi i)^{2k}}\right)^2 \oint\cdots\oint A'_k(1/2,u_1,\dots,u_{2k},v_1,\dots,v_{2k}) \\
&\times\prod_{i,j=1}^k \zeta(1+u_i-u_{k+j})\zeta(1+v_i-v_{k+j})\frac{\Delta^2(u_1,\dots,u_{2k})}{\prod_{j=1}^{2k}u_j^{2k}}\frac{\Delta^2(v_1,\dots,v_{2k})}{\prod_{j=1}^{2k}v_j^{2k}} \\
&\times e^{\frac12\mathcal{L}\sum_{j=1}^k u_j-u_{k+j}} e^{\frac12\mathcal{Q}\sum_{j=1}^k v_j-v_{k+j}} du_1\cdots du_{2k}dv_1\cdots dv_{2k}
\end{aligned}$$

where $\mathcal{L} = \log(t/2\pi)$ and $\mathcal{Q} = \log(qt/2\pi)$. Since $A_k(1/2, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is holomorphic in the neighbourhood of $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mathbf{0}, \mathbf{0})$ after a change of variables this becomes

$$\begin{aligned}
&\left(\frac{(-1)^k}{k!^2(2\pi i)^{2k}}\right)^2 \oint \cdots \oint A_k'\left(1/2, \frac{u_1}{\mathcal{L}/2}, \ldots, \frac{u_{2k}}{\mathcal{L}/2}, \frac{v_1}{\mathcal{Q}/2}, \ldots, \frac{v_{2k}}{\mathcal{Q}/2}\right) \\
&\times \prod_{i,j=1}^{k} \zeta\left(1 + \frac{u_i - u_{k+j}}{\mathcal{L}/2}\right) \zeta\left(1 + \frac{v_i - v_{k+j}}{\mathcal{Q}/2}\right) \frac{\Delta^2(u_1, \ldots, u_{2k})}{\prod_{j=1}^{2k} u_j^{2k}} \\
&\times \frac{\Delta^2(v_1, \ldots, v_{2k})}{\prod_{j=1}^{2k} v_j^{2k}} e^{\sum_{j=1}^k u_j - u_{k+j}} e^{\sum_{j=1}^k v_j - v_{k+j}} du_1 \cdots du_{2k} dv_1 \cdots dv_{2k}. \\
&= A_k'(1/2, \mathbf{0}, \mathbf{0}) \mathcal{L}^{k^2} \mathcal{Q}^{k^2} \left(1 + O\left(\frac{1}{\mathcal{L}}\right)\right) \left(\frac{(-1)^k}{2^{k^2} k!^2 (2\pi i)^{2k}}\right)^2 \oint \cdots \oint \\
&\times \frac{\Delta^2(u_1, \ldots, u_{2k})}{\prod_{i,j=1}^k (u_i - u_{k+j}) \prod_{j=1}^{2k} u_j^{2k}} \frac{\Delta^2(v_1, \ldots, v_{2k})}{\prod_{i,j=1}^k (v_i - v_{k+j}) \prod_{j=1}^{2k} v_j^{2k}} \\
&\times e^{\sum_{j=1}^k u_j - u_{k+j}} e^{\sum_{j=1}^k v_j - v_{k+j}} du_1 \cdots du_{2k} dv_1 \cdots dv_{2k}. \\
&\sim A_k'(1/2, \mathbf{0}, \mathbf{0}) \mathcal{L}^{k^2} \mathcal{L}_q^{k^2} \left(\frac{(-1)^k}{2^{k^2} k!^2 (2\pi i)^{2k}} \oint \cdots \oint \right. \\
&\left. \times \frac{\Delta^2(u_1, \ldots, u_{2k})}{\prod_{i,j=1}^k (u_i - u_{k+j}) \prod_{j=1}^{2k} u_j^{2k}} e^{\sum_{j=1}^k u_j - u_{k+j}} du_1 \cdots du_{2k} \right)^2.
\end{aligned} \tag{538}$$

By (522) the quantity within parentheses is given by $G(k+1)^2/G(2k+1)$. It only remains to show that $A_k'(1/2, \mathbf{0}, \mathbf{0}) = a(k) L(1, \chi)^{2k^2}$ where $a(k)$ is given by (419). Since,

$$A_k'(1/2, \mathbf{0}, \mathbf{0}) = A_k(1/2, \mathbf{0}, \mathbf{0}) L(1, \chi)^{2k^2} \prod_{p|q} (1 - p^{-1})^{k^2} \tag{539}$$

we only need show that $a(k)$ is equal to the quantity

$$\begin{aligned} b(k) :=& A_k\left(1/2, \mathbf{0}, \mathbf{0}\right) \prod_{p|q} (1-p^{-1})^{k^2} \\ =& \prod_p \left[(1-p^{-1})(1-|\chi(p)|^2 p^{-1})(1-\chi(p)p^{-1})(1-\overline{\chi}(p)p^{-1})\right]^{k^2} \\ & \times B_p(\mathbf{0}, \mathbf{0}) \prod_{p|q}(1-p^{-1})^{k^2} \\ =& \prod_p \left[(1-p^{-1})(1-\chi(p)p^{-1})\right]^{2k^2} B_p(\mathbf{0}, \mathbf{0}). \end{aligned} \tag{540}$$

In the case of quadratic extensions, $a(k)$ is the product of the following three factors:

$$\prod_{p \text{ split}} \left(1 - \frac{1}{p}\right)^{4k^2} \sum_{m=1}^{\infty} \frac{d_{2k}(p^m)^2}{p^m}, \tag{541}$$

$$\prod_{p \text{ inert}} \left(1 - \frac{1}{p^2}\right)^{2k^2} \sum_{m=1}^{\infty} \frac{d_k(p^m)^2}{p^{2m}}, \tag{542}$$

$$\prod_{p \text{ ramified}} \left(1 - \frac{1}{p}\right)^{2k^2} \sum_{m=1}^{\infty} \frac{d_k(p^m)^2}{p^m}. \tag{543}$$

Now, since $\chi(p) = 1$ for split primes, the relevant factor in $b(k)$ is given by

$$\prod_{p \text{ split}} \left(1 - \frac{1}{p}\right)^{4k^2} \int_0^1 \left(1 - \frac{e(\theta)}{p^{1/2}}\right)^{-2k} \left(1 - \frac{e(-\theta)}{p^{1/2}}\right)^{-2k} d\theta. \tag{544}$$

Since k is an integer we can expand the integrand into a double series. Upon integration this is easily seen to be equal to the sum in (541) after using

$$\binom{m+2k-1}{2k-1} = \binom{m+2k-1}{m} = d_{2k}(p^m). \tag{545}$$

For inert primes we have $\chi(p) = -1$ and so the relevant factor is

$$\prod_{p \text{ inert}} \left(1 - \frac{1}{p^2}\right)^{2k^2} \int_0^1 \left(1 - \frac{e(2\theta)}{p}\right)^{-k} \left(1 - \frac{e(-2\theta)}{p}\right)^{-k} d\theta, \tag{546}$$

which is again easily seen to be equal to (542). Finally, for ramified primes, or equivalently the primes dividing q, we have $\chi(p) = 0$. Therefore, this factor in $b(k)$ is given by

$$\prod_{p \text{ ramified}} \left(1 - \frac{1}{p}\right)^{2k^2} \int_0^1 \left(1 - \frac{e(\theta)}{p^{1/2}}\right)^{-k} \left(1 - \frac{e(-\theta)}{p^{1/2}}\right)^{-k} d\theta \tag{547}$$

which equals (543).

CHAPTER 4

Moments of general non-primitive L-functions

In this short chapter we suggest how the main ideas of this thesis can be applied to general non-primitive L-functions. These are functions of the form

$$L(s) = \prod_{j=1}^{m} L_j(s)^{e_j}$$

where the $L_j(s)$ are distinct, primitive members of the Selberg class $\mathcal{S}$ and e_j are some positive integers. For each $L_j(s)$ we assume a functional equation of the form

$$\Lambda_{L_j}(s) = \gamma_{L_j}(s) L_j(s) = \epsilon \overline{\Lambda}_{L_j}(1-s) \tag{548}$$

where

$$\gamma_{L_j}(s) = Q_j^{s/2} \prod_{i=1}^{d_j} \Gamma(\tfrac{1}{2}s + \mu_{i,j}). \tag{549}$$

with the set $\{\mu_{i,j}\}$ stable under complex conjugation for each j. Note we have set $\lambda_{i,j} = 1/2$. This condition, along with the condition on $\{\mu_{i,j}\}$, is required to apply the moments recipe. Our conjecture takes the following form.

Conjecture 5. *let $\alpha_{L,k}(n)$ be the Dirichlet coefficients of $L(s)^k$. Then for $k > -1/2$ we have*

$$\frac{1}{T}\int_0^T \left|L\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim a_L(k) \prod_{j=1}^{m} \frac{G^2(e_j k+1)}{G(2e_j k+1)} \left(\log(Q_j T^{d_j})\right)^{(e_j k)^2} \tag{550}$$

where

$$a_L(k) = \prod_p \left(1-\frac{1}{p}\right)^{n_L k^2} \sum_{n=0}^{\infty} \frac{|\alpha_{L,k}(p^n)|^2}{p^n} \tag{551}$$

and $n_L = \sum_{j=1}^m e_j^2$.

A key point in both derivations of Conjecture 4 was that, aside from the arithmetic factor, the leading term in the moment of $\zeta(1/2+it)L(1/2+it,\chi)$ was given by the product of the leading terms of the moments of $\zeta(1/2+it)$ and $L(1/2+it,\chi)$. We believe this should be the case for general non-primitive L-functions too. Indeed, by applying our modified moments recipe to non-primitive L-functions this idea becomes more apparent.

The recipe for general non-primitive L-functions goes as follows. For each $L_j(s)$ in the product $L(s) = \prod_{j=1}^m L_j(s)^{e_j}$ we have the approximate functional equation

$$L_j(s) = \sum_n \frac{\alpha_{L_j}(m)}{m^s} + \varkappa_{L_j}(s) \sum_n \frac{\overline{\alpha_{L_j}(n)}}{n^s} \tag{552}$$

where

$$\varkappa_{L_j}(s) = \frac{\overline{\gamma}_{L_j}(1-s)}{\gamma_{L_j}(s)} = Q_j^{1/2-s} \prod_{i=1}^{d_j} \frac{\Gamma((1-s)/2+\overline{\mu_{i,j}})}{\Gamma(s/2+\mu_{i,j})}. \tag{553}$$

Similarly to before, if we apply the original recipe we encounter terms of the form $(Q_j Q_{j'})^{-it}$ which are oscillating. We can prevent the occurrence of these terms by applying the first step of the recipe to each $L_j(s)$ separately. We then continue as in the original recipe up to the point where we write the expression as a contour integral. It should be clear that the only element of this expression that is dependent on all integration variables is what we would probably notate as $A_k(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m)$. Once again, this would be holomorphic in the neighbourhood of $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m) = (\mathbf{0}, \ldots, \mathbf{0})$ allowing us to use a similar change of variables to that used in (538). After this, the expression would factorise and upon applying (522) we acquire the main term in the form

$$A_k(\mathbf{0}, \ldots, \mathbf{0}) \prod_{j=1}^m \frac{G^2(e_j k+1)}{G(2e_j k+1)} \left(\log(Q_j T^{d_j})\right)^{(e_j k)^2}.$$

Instead of directly evaluating $A_k(\mathbf{0}, \ldots, \mathbf{0})$ we derive the arithmetic factor (551) via a somewhat simpler method below.

In terms of the random matrix theory, let us assume that we have a hybrid product for $L(s)$. Since the $L_j(s)$ are distinct their zeros are uncorrelated [34], and so their associated matrices should act independently. Hence, when the moment of the product over zeros is considered as an expectation, it will factorise. Let us now provide some examples.

As we have already seen, the factorisation phenomenon occurs when considering $\zeta(s)L(s,\chi)$, at least when χ is the Kronecker character. We can restate Conjecture 4 in the more descriptive form

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)^k L\left(\tfrac{1}{2}+it,\chi\right)^k\right|^2 dt \sim a_{\zeta L}(k)\frac{G(k+1)^2}{G(2k+1)}\log^{k^2} T\cdot\frac{G(k+1)^2}{G(2k+1)}\log^{k^2} qT, \tag{554}$$

with

$$a_{\zeta L}(k)=\prod_p\left(1-\frac{1}{p}\right)^{2k^2}\sum_{m\geq 0}\frac{|F_{\chi,k}(p^m)|^2}{p^m}, \tag{555}$$

$$F_{\chi,k}(n)=\sum_{n_1n_2=n} d_k(n_1)d_k(n_2)\chi(n_2). \tag{556}$$

We note that $a_{\zeta L}(k)$ is indeed equal to $a(k)L(1,\chi)^{2k^2}$. To see this, we first split the product

$$L(1,\chi)^{2k^2}\prod_{\mathfrak{p}\subset\mathcal{O}_{\mathbb{K}}}\left(1-\frac{1}{\mathfrak{N}(\mathfrak{p})}\right)^{2k^2}$$

over split, inert and ramified primes. Inputting the relevant behaviour of χ we see that this product equals $\prod_p(1-p^{-1})^{2k^2}$. Also, by considering $F_{\chi,k}(n)$ as the coefficients of the Dirichlet series for $\zeta(s)^kL(s,\chi)^k$, we see that

$$\sum_{m\geq 0}\frac{|F_{\chi,k}(p^m)|^2}{p^m}=\begin{cases}\sum_{m\geq 0} d_{2k}(p^m)^2p^{-m} & \text{if } \chi(p)=1,\\ \sum_{m\geq 0} d_k(p^m)^2p^{-2m} & \text{if } \chi(p)=-1,\\ \sum_{m\geq 0} d_k(p^m)^2p^{-m} & \text{if } \chi(p)=0.\end{cases} \tag{557}$$

Here, it may be helpful to note

$$\zeta(s)^k L(s,\chi)^k = \prod_p \left(1 - \frac{1}{p^s} - \frac{\chi(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-k}.$$

As another example, we state a result to appear in a forthcoming joint paper between the author and Caroline Turnage-Butterbaugh. Here it is established, by an application of Theorem 1, that

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac12+it\right)L\left(\tfrac12+it,\chi\right)\sum_{n\le T^\theta}\frac{1}{n^{1/2+it}}\right|^2 dt \sim a_{\zeta^2L}(1)\log^4 T\cdot\log qT\left(\frac{4\theta^3-3\theta^4}{12}\right) \tag{558}$$

where

$$a_{\zeta^2L}(1) = \prod_p \left(1-\frac1p\right)^5 \sum_{m\ge 0}\frac{|H_\chi(p^m)|^2}{p^m}, \qquad H_\chi(n) = \sum_{n_1n_2=n} d(n_1)\chi(n_2), \tag{559}$$

and $\theta < 1/11 - \epsilon$. It is expected that Theorem 1 remains valid for $\theta = 1$, in which case the above relation reads as

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac12+it\right)^2 L\left(\tfrac12+it,\chi\right)\right|^2 dt \sim \frac{a_{\zeta^2L}(1)}{12}\log^4 T\cdot\log qT = a_{\zeta^2L}(1)\cdot\frac{G(3)^2}{G(5)}\log^4 T\cdot\frac{G(2)^2}{G(3)}\log qT. \tag{560}$$

Clearly, this is consistent with our conjecture in the case $L_1(s) = \zeta(s)^2$, $L_2(s) = L(s,\chi)$ and $k = 1$. Guided by these examples we are led to Conjecture 5 which, after ignoring the conductors, we restate as

$$\frac{1}{T}\int_0^T \left|L\left(\tfrac12+it\right)\right|^{2k} dt \sim \frac{a_L(k)g_L(k)}{\Gamma(n_Lk^2+1)}\log^{n_Lk^2} T \tag{561}$$

where $n_L = \sum_{j=1}^m e_j^2$,

$$g_L(k) = \Gamma(n_Lk^2+1)\prod_{j=1}^m \frac{G^2(e_jk+1)}{G(2e_jk+1)} d_j^{(e_jk)^2}, \tag{562}$$

and

$$a_L(k) = \prod_p \left(1 - \frac{1}{p}\right)^{n_L k^2} \sum_{n=0}^{\infty} \frac{|\alpha_{L,k}(p^n)|^2}{p^n}. \tag{563}$$

We remark that for integral k,

$$g_L(k) = \binom{n_L k^2}{(e_1 k)^2, \dots, (e_m k)^2} \prod_{j=1}^{m} g(e_j k) d_j^{(e_j k)^2} \tag{564}$$

where the first factor is the multinomial coefficient and the function g is defined by $g(n)/n^2! = G(n+1)^2/G(2n+1)$. It is shown in [9] that $g(n)$ is an integer, and hence $g_L(k)$ is an integer for integral k.

Let us cast this conjecture in the light of some of the Selberg's conjectures. First, we note that the integer n_L is the same integer appearing in Selberg's 'regularity of distribution' conjecture:

$$\sum_{p \le x} \frac{|\alpha_L(p)|^2}{p} = n_L \log\log x + O(1), \tag{565}$$

as will be shown below. This is not so surprising since one expects the mean square of $L(1/2+it)$ to be asymptotic to a multiple of the sum $\sum_{n \le T} |\alpha_L(n)|^2 n^{-1}$. The implication of (565) is that this sum is in fact $\sim (a_L(1)/n_L!) \log^{n_L} T$.

We can outline a verification of this last assertion allowing for integral $k \ge 1$. We assume the following two conjectures of Selberg [53]: For primitive $F \in \mathcal{S}$ we have

$$\sum_{p \le x} \frac{|\alpha_F(p)|^2}{p} = \log\log x + O(1), \tag{566}$$

and for two distinct and primitive $F, G \in \mathcal{S}$ we have

$$\sum_{p \le x} \frac{\alpha_F(p)\overline{\alpha_G(p)}}{p} = O(1). \tag{567}$$

We also require that the functions

$$M_j(s) = \sum_{n=1}^{\infty} \frac{|\alpha_{L_j}(n)|^2}{n^s} \tag{568}$$

behave 'reasonably', in particular, that they posses an analytic continuation.

Now, given the factorisation $L(s) = \prod_{j=1}^{m} L_j(s)^{e_j}$ into primitive functions we have

$$\alpha_{L,k}(p) = k\sum_{j=1}^{m} e_j \alpha_{L_j}(p), \tag{569}$$

since the coefficients $\alpha_{L_j}(n)$ are multiplicative. Therefore,

$$\begin{aligned}\sum_{p\le x}\frac{|\alpha_{L,k}(p)|^2}{p} =&\sum_{p\le x} k^2\left(\sum_{j=1}^{m} e_j^2|\alpha_{L_j}(p)|^2 + \sum_{i\ne j} e_i e_j \alpha_{L_i}(p)\overline{\alpha_{L_j}(p)}\right)p^{-1}\\ =& n_L k^2\log\log x + O(1).\end{aligned} \tag{570}$$

If $M(s) = \sum |\alpha_{L,k}(n)|^2 n^{-s}$, then the above equation implies a factorisation of the form

$$M(s) = U_k(s)\prod_{j=1}^{m} M_j(s)^{(e_j k)^2} \tag{571}$$

where $U_k(s)$ is some Euler product that is absolutely convergent for $\sigma > 1/2$. Therefore, we may analytically continue $M(s)$ to $\sigma > 1/2$. Also, by applying partial summation to (570) we see

$$\sum_p \frac{|\alpha_{L,k}(p)|^2}{p^{s+1}} = n_L k^2\int_2^\infty \frac{dx}{x^{s+1}\log x} + \cdots = -n_L k^2 \log s + \cdots, \tag{572}$$

for small $\sigma > 0$. If we write

$$\begin{aligned} M(s+1) =& \prod_p\left(1 + \frac{|\alpha_{L,k}(p)|^2}{p^{s+1}} + \frac{|\alpha_{L,k}(p^2)|^2}{p^{2(s+1)}} + \cdots\right)\\ =& \prod_p\left(\exp\left(\frac{|\alpha_{L,k}(p)|^2}{p^{s+1}}\right) + E_k(p,s)\right)\\ =& \exp\left(\sum_p \frac{|\alpha_{L,k}(p)|^2}{p^{s+1}}\right)\prod_p\left(1+F_k(p,s)\right),\end{aligned} \tag{573}$$

where $E_k(p,s)$ and $F_k(p,s)$ are both $\ll p^{-2(\sigma+1)+\epsilon}$, we see that $M(s+1)$ has a pole of order $n_L k^2$ at $s = 0$. It is shown in [14] that on the assumption of Selberg's conjectures, if $F \in \mathcal{S}$ has a pole of order m at $s = 1$ then $\zeta(s)^m$ divides $F(s)$.

Consequently, the residue of $M(s+1)$ at $s=0$ is given by $a_L(k)$. The usual argument involving Perron's formula now gives

$$\sum_{n\le T}\frac{|\alpha_{L,k}(n)|^2}{n}\sim\frac{a_L(k)}{(n_Lk^2)!}\log^{n_Lk^2}T. \tag{574}$$

In order to emphasise the factorisation property of our conjecture we can use a new notation and absorb the factor of $(n_Lk^2)!$. For a primitive L-function L_j of degree d_j we let

$$f_{L_j}(k)=\frac{G(k+1)^2}{G(2k+1)}d_j^{k^2} \tag{575}$$

and for a general non-primitive L-function we just let

$$f_L(k)=\frac{g_L(k)}{\Gamma(n_Lk^2+1)}. \tag{576}$$

Then, according to our conjecture, for an L-function of the form $L(s)=\prod_{j=1}^m L_j(s)^{e_j}$ with the L_j primitive we have

$$f_L(k)=\prod_{j=1}^m f_{L_j}(e_jk). \tag{577}$$

It is of interest that the arithmetic factor does not possess such a complete factorisation. Let $\alpha_{L_j,e_j}(n)$ denote the coefficients of $L_j(s)^{e_j}$ and note $\alpha_{L_j,e_j}(p)=e_j\alpha_{L_j}(p)$. As mentioned previously we have

$$\begin{aligned}|\alpha_{L,k}(p)|^2=&k^2\left(\sum_{j=1}^m e_j^2|\alpha_{L_j}(p)|^2+\sum_{i\ne j}e_ie_j\alpha_{L_i}(p)\overline{\alpha_{L_j}(p)}\right)\\ =&\sum_{j=1}^m|\alpha_{L_j,e_jk}(p)|^2+k^2\sum_{i\ne j}e_ie_j\alpha_{L_i}(p)\overline{\alpha_{L_j}(p)}\end{aligned} \tag{578}$$

and so

$$\sum_{p\le x}\frac{|\alpha_{L,k}(p)|^2}{p}=\sum_{j=1}^m\sum_{p\le x}\frac{|\alpha_{L_j,e_jk}(p)|^2}{p}+O(1) \tag{579}$$

by orthogonality. Therefore, we may write

$$
\begin{aligned}
a_L(k) &= \prod_p \left(1-\frac{1}{p}\right)^{n_L k^2} \sum_{n=0}^{\infty} \frac{|\alpha_{L,k}(p^n)|^2}{p^n} \\
&= \prod_p \left[\prod_{j=1}^{m} \left(1-\frac{1}{p}\right)^{e_j^2 k^2}\right] \left(1 + \sum_{j=1}^{m} \frac{|\alpha_{L_j,e_j k}(p)|^2}{p} + \sum_{n=1}^{\infty} \frac{c_{L,k}(p^n)}{p^n}\right) \\
&= \prod_p \left[\prod_{j=1}^{m} \left(1-\frac{1}{p}\right)^{e_j^2 k^2} \sum_{n=0}^{\infty} \frac{|\alpha_{L_j,e_j k}(p^n)|^2}{p^n}\right] \sum_{n=0}^{\infty} \frac{c'_{L,k}(p^n)}{p^n} \\
&= \left[\prod_{j=1}^{m} a_{L_j}(e_j k)\right] \prod_p \sum_{n=0}^{\infty} \frac{c'_{L,k}(p^n)}{p^n}
\end{aligned}
\tag{580}
$$

for some coefficients $c_{L,k}$ and $c'_{L,k}$ dependent on the L_j. Note

$$
\sum_{p \le x} \frac{c'_{L,k}(p^n)}{p^n} = O(1)
$$

for $n \ge 1$ and so the product

$$
b_L(k) := \prod_p \sum_{n=0}^{\infty} \frac{c'_{L,k}(p^n)}{p^n}
\tag{581}
$$

is convergent. We can now re-phrase our conjecture in the form

$$
\frac{1}{T}\int_0^T \left|L\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim b_L(k) \prod_{j=1}^{m} \frac{1}{T}\int_0^T \left|L_j\left(\tfrac{1}{2}+it\right)^{e_j}\right|^{2k} dt.
\tag{582}
$$

Here we can clearly see the independence between the L-functions in the geometric sense. Indeed, as we have already noted

$$
f_L(k) = \prod_{j=1}^{m} f_{L_j}(e_j k),
$$

which can be thought of as a consequence of the independence, or uncorrelatedness, of the zeros. However the presence of $b_L(k)$ suggests a lack of arithmetic independence between the $L_j(s)$. In fact, it can be thought of as a measure of their arithmetic dependence in the following sense. Note that one of the larger contributions to $b_L(k)$ is given by the sum

$$\sum_p \frac{\alpha_{L_i}(p)\overline{\alpha_{L_j}(p)}}{p} \tag{583}$$

with $i \neq j$. If $\alpha_{L_j}(p)$ approximates $\alpha_{L_i}(p)$ frequently enough then this sum becomes large since

$$\sum_p \frac{|\alpha_{L_i}(p)|^2}{p} \tag{584}$$

is unbounded. Consequently, the greater the arithmetic similarities between any two L-functions L_i and L_j, the larger $b_L(k)$ is; or put another way, the smaller $b_L(k)$ is, the more distinct the L_j are in an arithmetic sense.

Notation

- Our complex variables are most frequently denoted $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$.
- Line integrals of the form $\int_{c-i\infty}^{c+i\infty}$ with $c \in \mathbb{R}$, are abbreviated to $\int_{(c)}$.
- We represent very small quantities by ϵ. These may not be the same at each occurrence in an equation although they may be notated as such. The same goes for very large quantities, which we usually denote A.
- Primes are always, and exclusively, denoted by p.
- $e_d(x) := e^{2\pi ix/d}$ and $e(x) := e_1(x)$.
- All Dirichlet characters considered are primitive. We denote them by χ and their moduli are denoted by q. The nth Gauss sum associated to χ is given by $G(n, \chi) = \sum_{m=1}^{q} \chi(m)e_q(nm)$ and we write the usual Gauss sum as $G(\chi) = G(1, \chi)$. Two useful facts are

$$G(n, \chi) = \overline{\chi}(n)G(\chi) \text{ for } n \in \mathbb{Z}, \tag{585}$$

  and $|G(\chi)|^2 = q$. These may be used implicitly in some arguments.
- For a positive integer h we express its prime decomposition as $\prod_{p|h} p^{h_p}$ so that h_p is the highest power of p dividing h. This will mostly be used in Chapter 2.
- $n_{(m)} := n/(n, m)$ where (n, m) is the greatest common divisor of n and m.
- We denote ideals in gothic type and prime ideals are always, and exclusively, denoted by $\mathfrak{p}$.

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