Coarse Version of Homotopy Theory
(Axiomatic Structure)

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Abstract

In topology, homotopy theory can be put into an algebraic framework. The most complete such framework is that of a Quillen model Category [[15], [5]].

The usual class of coarse spaces appears to be too small to be a Quillen model category. For example, it lacks a good notion of products. However, there is a weaker notion of a cofibration category due to Baues [[1], [2]].

The aim in this thesis is to look at notions of cofibration category within the world of coarse geometry. In particular, there are several sensible notions of the structure of a coarse version of a cofibration category that we define here.

Later we compare these notions and apply them to computations. To be precise, there are notions of homotopy groups in a Baues cofibration category. So we compare these groups as well for the different structures we have defined, and to the more concrete notion of coarse homotopy groups defined also in [10].

Going further, there is an abstract notion of a cell complex defined in the context of a cofibration category. In the coarse setting, we prove such cell complexes have a more geometric definition, and precisely we prove that a coarse CW-complex is a cell complex.

The ultimate goal of such computations is a version of the Whitehead theorem relating coarse homotopy groups and coarse homotopy equivalences for cell complexes. Abstract versions of the Whitehead theorem are known for cofibration categories [1], so we relate these abstract results to something more geometric.

Another direction of the thesis involves Quillen model categories. As already mentioned, there are obstructions to the class of coarse spaces being a Quillen model category; there is no apparent way to define category-theoretic products of coarse spaces. However, such obvious objections vanish if we add extra spaces to the coarse category. These extra spaces are termed *non-unital coarse spaces* in [9]. We have proved most of Quillen axioms but the existence of limits in one of our categories.

Introduction

Coarse geometry is the study of the very large scale properties of spaces. It comes from the study of metric spaces when the distances between points are great, i.e. small distance does not matter in coarse geometry. For example, a bounded space is equivalent to a point in coarse geometry.

In topology, the concern is with small scale structure. For example, continuity is determined by whether the inverse image of an open set is open or not, and the topology of a space is defined in terms of open sets. On the other hand, in coarse geometry, properties like boundedness do not depend on open sets, but rather on large-scale properties. To study such large scale properties, we define an important notion of a map called a coarse map. Coarse maps play a similar role in coarse geometry to continuous maps in topology.

The notion of a topological space is of course a generalisation of the notion of a metric space. In a topological space, X, certain subsets are termed open, and the collection of open sets is required to satisfy certain axioms. The collection of open sets measures the small scale structure of a space.

The notion of a *coarse space* is analogous generalisation of the notion of a metric space which measures its large-scale structure. In a coarse space X, certain subsets of the product $X \times X$ are termed *entourages* are required to satisfy certain axioms [[17], [11]].

Roe realized that one can define an abstract notion of coarse space; just as the passage from metric space to topological space forgets large scale structure, the passage from metric space to coarse space should forget the small scale information. But an abstract coarse space keeps enough structure to perform the large scale constructions which were perviously done in the metric context. In more detail, in topology, open sets with some requirements construct a topological space.

In other words, the coarse structure on a set X is a collection of subsets of the Cartesian product $X \times X$ with certain properties which allow study of the large scale structure of metric spaces and topological spaces.

Now the definition of a coarse map is superficially similar to the definition of a uniformly continuous map; the main difference is the fact that the quantifiers in the first part of the definition are "the wrong way around" as defined in [16] between metric spaces.

Definition 0.0.1 Let X and Y be metric spaces. We call a map (not necessarily continuous) $f\colon X \to Y$ a *coarse map* if:

- Let $R > 0$. Then there exists $S > 0$ such that $d(x,y) < R$ implies $d(f(x), f(y)) < S$.

• Let $B \subseteq Y$ be bounded. Then the inverse image $f^{-1}[B] \subseteq X$ is bounded.

However, this small difference means that coarse maps are associated with the large-scale geometry of spaces in much the same way as continuous maps are associated with small scale geometry. So a coarse map between coarse spaces can be defined to be a map which respects this structure (e.g., a large-scale Lipschitz map). The small-scale (i.e. the topology) is ignored. This map should send entourages to entourages and the inverse image of bounded sets should be bounded.

Many aspects in topology have an analogous notion in coarse geometry with respect to large scale properties. For example, there is a notion of coarse homotopy which is an analogue of the notion of homotopy in topology. Coarse geometry has a lot of applications in geometric topology, and controlled topology. Further, coarse geometry puts some extra structure on many properties that are defined in topology and examines them under these structures.

Axiomatic homotopy theory is the development of the basic constructions of homotopy theory in an abstract setting, so that they may be applied to other categories with extra structure. And there is, indeed, a wide variety of categories where these techniques are useful.

The usual framework for axiomatic homotopy theory is that of a Quillen model category [15]. One way to proceed might be to define fibrations, cofibrations, and weak equivalences between coarse spaces equipped with extra structure. Unfortunately, the axioms required for a category to be a Quillen model category are, however, quite restrictive and it is difficult to prove the relevant axioms in the coarse setting for the simple reason that direct limits do not in general exist in the category of coarse CW-complexes, and for the category of coarse maps the two last axioms are hard to prove in coarse world.

But Baues introduced a weaker notion of cofibration category in [1] and [2] as a generalization of a Quillen model category. He defined it to be a category with two classes of morphisms called cofibrations and weak equivalences such that specific axioms are satisfied.

The cofibration category is a technical structure that helps us understand homotopy theory even though the axioms are weaker than those of Quillen model category. The axioms are chosen to be:

- Sufficiently strong to permit the basic construction of homotopy theory.
- As weak (and as simple) as possible so that the construction of homotopy theory is available in as many contexts as possible.

In the first chapter, we give background about coarse maps and coarse equivalences between metric spaces, and their properties. Also we give a definition of asymptotically Lipschitz maps, quasi-isometric, and geodesic spaces. Later we define a notion of coarse structure defined on a set X with some examples and we introduce a notion of generalized rays.

The second chapter looked at coarse maps defined between non-unital coarse spaces. We allow our category to have non-unital coarse spaces as this allows us to have push-out diagrams. This requires redefining coarse maps between non-unital coarse spaces, which involves a notion of locally proper maps. We also give a definition of coarse and controlled homotopy for a coarse version of the cylinder.

So the point is to equip our coarse spaces with a small amount of extra structure. To be precise, our extra structure on a space X takes the form of a controlled map $p\colon X \to R$, where R is a generalized ray. This extra structure allows us to define cylinders on coarse spaces and then homotopies.

Later we define coarse (and controlled) path-components and coarse (and controlled) homotopy groups, prove some properties and give some examples. We also prove that the analogous controlled homotopy groups are all trivial.

In the third chapter, we give two examples of coarse categories that have a structure of Baues cofibration category. The first category is the category of controlled maps. The other category is the category of closeness equivalences of coarse maps. We define controlled and coarse cofibrations, controlled and coarse homotopy equivalences in such a way that we turn the categories into a Baues cofibration category.

So for the controlled category, weak equivalences are defined to be controlled homotopy equivalences and cofibrations are the controlled cofibrations. For the coarse category, weak equivalences are the coarse homotopy equivalence classes, and cofibrations are closeness equivalence classes of coarse maps with the homotopy extension property.

The fourth chapter has another coarse category. The objects of this category are non-unital coarse CW-complexes with only finitely many coarse cells in each dimension, and the morphisms are defined to be called coarsely cellular classes. This restriction of this category allows us to turn the category into a Baues cofibration category. We define weak equivalences to be weak coarse homotopy equivalence classes and cofibrations to be classes that have left lifting property with respect to weak equivalence and fibration classes, and the later are defined to be coarse Serre fibration classes.

Further, this structure allows us to prove more general axioms that set up for Quillen model category except finding all non-zero limits in this category.

The fifth chapter is set up for the axiomatic homotopy theory defined in our categories since we have shown they are Baues cofibration category in earlier chapters. We look at the abstractly defined notion of relative homotopy in the coarse example of Baues cofibration category and compare it with the natural geometric notion we have.

The cofibration structure gives us an abstract notion of homotopy groups. We define the coarse homotopy groups π_n^{Qcrs} of a space X in a more geometric way, and compare it with the abstract notion.

The last chapter talks about coarse CW-complexes in coarse example of cofibration category, and we show that every coarse CW-complex is a T-complex in sense of definition (2.2) in [2]. This is the key to proving the Whitehead theorem in coarse geometry.

Finally we could find an equivalence between the category of coarse CW-complexes and the subcategory of the category of coarse spaces.

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Chapter 1

Coarse Geometry

Coarse Geometry is studying spaces regarding only very large scale properties, particularly metric spaces.

Recall a metric space is a set equipped with a distance function $d\colon X \times X \to \mathbb{R}$ which is positive-definite, symmetric, and satisfies the triangle inequality:

$$d(x, y) \leq d(x, z) + d(z, y),$$

where $x, y, z \in X$. When defining continuity, one neglects a great deal of the information contained in the metric d. Only small distances matter when defining continuity.

In coarse geometry our concern is with a dual situation. Instead of focusing on the small scale structure defined by a metric, we will focus on the large scale structure. We will be able to give a precise sense to assertions such as $\mathbb{R}$, $\mathbb{Z}$ are the same or $\mathbb{R}$, $\mathbb{R}^2$ are different on the large scale.

§ 1.1 Coarse Maps and Coarse Equivalence Between Metric Spaces

In the following definition we start to give a notion of the subject of coarse geometry. For more details see [16].

Definition 1.1.1 Let X and Y be metric spaces. We call a map (not necessarily continuous) $f\colon X \to Y$ a *coarse map* if:

- Let $R > 0$. Then there exists $S > 0$ such that $d(x, y) < R$ implies $d(f(x), f(y)) < S$.
- Let $B \subseteq Y$ be bounded. Then the inverse image $f^{-1}[B] \subseteq X$ is bounded.

Any map f satisfying only the first condition is called a *controlled map* (or *coarsely uniform*) .

The above definition shows that coarse maps respect the large scale structure of a space, while they do not necessarily respect smaller-scale structures.

Example 1.1.2 Let $X = Y = \mathbb{N}$. The map $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 7n + 4$ for all $n \in \mathbb{N}$ is coarse. But f defined by $f(n) = 1$ is not coarse (the second condition is failed), and f defined by $f(n) = n^2$ is not coarse either (the first condition is failed).

Remark 1.1.3 Composition of coarse maps gives a coarse map.

Recall that a map between metric (or more generally topological) spaces is termed *proper* if the inverse image of a compact set is compact. The second condition in the above definition is sometimes called *metrically proper* . Thus a coarse map is a controlled and metrically proper map.

Definition 1.1.4 We call two maps from X (a set) into a metric space Y *close* if there is a constant $C > 0$ such that $d(f(x), g(x)) \leq C$ for all $x \in X$.

We call $M \subseteq X \times X$ *an entourage* if the projection maps $\pi_1\colon M \to X$, $\pi_2\colon M \to X$ are close.

Given an entourage $M \subseteq X \times X$ and a subset $A \subseteq X$. We write

$$M[A] = \{y \in X : (x, y) \in M \text{ for some } x \in A\}$$

For a point $x \in X$, we write $M(x) = M[\{x\}]$.

A subset $B \subseteq X$ is said to be *bounded* if the inclusion $B \hookrightarrow X$ is close to a constant map, or equivalently $B = M(x)$ for some an entourage M and some $x \in X$.

A coarse map $f\colon X \to Y$ is called a *coarse equivalence* if there is a coarse map $g\colon X \to Y$ such that the compositions $f \circ g$ and $g \circ f$ are close to the identity maps 1_X and 1_Y respectively.

A controlled map $f\colon X \to Y$ is called a *controlled equivalence* if there is a controlled map $g\colon X \to Y$ such that the compositions $f \circ g$ and $g \circ f$ are close to the identity maps 1_X and 1_Y respectively.

We call two metric spaces X and Y *coarsely equivalent* if a coarse equivalence $f\colon X \to Y$ exists, and we call two metric spaces X and Y *controlledly equivalent* if a controlled equivalence $f\colon X \to Y$ exists. We will show later that coarse equivalence and controlled equivalence are the same.

Example 1.1.5 • Let $\mathbb{R}$ be the real line, with its usual metric, and let $\mathbb{Z}$ be the subspace of integers. Then the inclusion map $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ is a coarse equivalence. Note that i is a coarse map.

Define the map $j\colon \mathbb{R} \to \mathbb{Z}$ by the formula $j(x) = \lfloor x \rfloor$, where we use the symbol $\lfloor x \rfloor$ to denote the highest integer that is less than a real number x. Then j is a coarse map. Observe that $j \circ i = 1_{\mathbb{Z}}$, and given $x \in \mathbb{R}$

$$|i \circ j(x) - 1_{\mathbb{R}}(x)| = |\lfloor x \rfloor - x| \leq 1.$$

Hence the composite $j \circ i$ is close to the identity map $1_{\mathbb{R}}$, and the spaces $\mathbb{R}$ and $\mathbb{Z}$ are coarsely equivalent.

- Every bounded set is coarsely equivalent to a point.
- $\mathbb{R}$, $\mathbb{R}_+ = [0, \infty)$ are not coarsely equivalent.

Proposition 1.1.6 *Let X be a metric space. Then $M \subseteq X \times X$ is an entourage if and only if $M \subseteq \Delta_R$ for some $R > 0$, where*

$$\Delta_R = \{(x, y) \in X \times X \mid d(x, y) < R\}.$$

Proof : Let $M \subseteq X \times X$ be an entourage, that is the projection maps $\pi_1 \colon M \to X$, $\pi_2 \colon M \to X$ are close and this is true if and only if $d(x, y) < R$ for any $(x, y) \in M$ and some $R > 0$. Therefore $M \subseteq \Delta_R$. □

Proposition 1.1.7 *Let X be a metric space, and let S be the set of entourages $M \subseteq X \times X$, then:*

(1) *If $M \in S$ and $M^{'} \subseteq M$, then $M^{'} \in S$.*

(2) *Let $M_1, M_2 \in S$, then $M_1 \cup M_2 \in S$, and*

$$M_1 M_2 = \{(x, z) \mid (x, y) \in M_1, (y, z) \in M_2 \ \ for \ \ some\ y\} \in S.$$

(3) $\Delta_X \in S$.

(4) $\bigcup_{M \in S} M = X \times X$

(5) $M^t = \{(y, x) \mid (x, y) \in M\} \in S$.

Proof :

(1) If $M \in S$ and $M^{'} \subseteq M$, then $M^{'} \subseteq M \subseteq \Delta_R$ which implies that $M^{'} \in S$

(2) If $M_1, M_2 \in S$ such that $M_1 \subseteq \Delta_{R_1}$ and $M_2 \subseteq \Delta_{R_2}$. Then $M_1 \cup M_2 \subseteq \Delta_R$ where $R = max\{R_1, R_2\}$, and by triangle inequality, we have $M_1 M_2 \subseteq \Delta_K$ where $K = R_1 + R_2$ so $M_1 \cup M_2$ and $M_1 M_2$ both are in S.

(3) $\Delta_X \subseteq \Delta_R$ for some $R > 0$.

(4) It is clear that $\bigcup_{M \in S} M \subseteq X \times X$. Now let $(x, y) \in X \times X$, then $M_{x,y} = \{(x, y)\}$ is an entourage which implies $X \times X \subseteq \bigcup_{(x,y) \in X \times X} M_{x,y}$.

(5) This is so since $d(x, y) = d(y, x)$. □

Proposition 1.1.8 *Let X be a metric space such that for any other set S, we have an equivalence relation on the set of maps $Map(S, X)$ called* being close *such that:*

(1) *Let $S = A \cup B$, $f, g\colon S \to X$. Suppose that $f|_A, g|_A\colon A \to X$ are close and $f|_B, g|_B\colon B \to X$ are close. Then $f, g\colon S \to X$ are close.*

(2) *Let $f, g\colon S \to X$ be close, and $h\colon T \to S$ be a map, then $f \circ h, g \circ h\colon T \to X$ are close.*

(3) *Any constant maps $c_1, c_2\colon S \to X$ are close.*

Let $S = A \cup B$. Then the notion of maps $f, g\colon S \to X$ being close satisfies the above axioms.

Proof :

(1) Suppose that $s \in S$, then $s \in A$ or $s \in B$.

Let $s \in A$. Since $f|_A, g|_A\colon A \to X$ are close, then $d(f(s), g(s)) \leq k_1$, for some $k_1 > 0$.

Let $s \in B$. Since $f|_B, g|_B\colon B \to X$ are close, then $d(f(s), g(s)) \leq k_2$, for some $k_2 > 0$. Let $k = max\{k_1, k_2\}$, then for all $s \in S$, $d(f(s), g(s)) \leq k$. Hence $f, g\colon S \to X$ are close.

(2) Let $t \in T$, then $d(f \circ h(t), g \circ h(t)) = d(f(h(t)), g(h(t)))$. Since $h(t) \in S$ and f, g are close, then there exists $c > 0$ such that $d(f \circ h(t), g \circ h(t)) \leq c$ for all $t \in T$. Therefore $f \circ h, g \circ h\colon T \to X$ are close.

(3) It is obvious. □

Definition 1.1.9 Let S be any set. Two maps $f, g\colon S \to X$ are said to be *close* if $\{(f(s), g(s)) \mid s \in S\}$ is an entourage.

Proposition 1.1.10 *Let X be a set, $Map(S, X)$ equipped with a notion of being close satisfying the properties:*

(1) *Let $S = A \cup B$, $f, g\colon S \to X$. Suppose that $f|_A, g|_A\colon A \to X$ are close and $f|_B, g|_B\colon B \to X$ are close. Then $f, g\colon S \to X$ are close.*

(2) *Let $f, g\colon S \to X$ be close, and $h\colon T \to S$ be a map, then $f \circ h, g \circ h\colon T \to X$ are close.*

(3) *Any constant maps $c_1, c_2\colon S \to X$ are close.*

Call a subset $M \subseteq X \times X$ an entourage *if the projections $\pi_1\colon M \to X$, $\pi_2\colon M \to X$ are close. Then the five properties in Proposition 1.1.7 hold.*

Proof :

(1) Let M be an entourage and $M' \subseteq M$ a subset. Consider the inclusion map $h : M' \hookrightarrow M$. Since M is an entourage, the projections $\pi_1\colon M \to X$, $\pi_2\colon M \to X$ are close. It is clear that the maps $\pi_1' = \pi_1 \circ h\colon M' \to X$, $\pi_2' = \pi_2 \circ h\colon M' \to X$ are the projections on M'. Using proposition 1.1.8 (2), we see that π_1', π_2' are close. Hence M' is an entourage.

(2) Let M_1, M_2 be entourages, then $\pi_1|_{M_1}, \pi_2|_{M_1}\colon M_1 \to X$ are close and $\pi_1|_{M_2}, \pi_2|_{M_2}\colon M_2 \to X$ are close. By proposition 1.1.8 (1), $\pi_1, \pi_2\colon M_1 \cup M_2 \to X$ are close. Then $M_1 \cup M_2$ is an entourage.

Now let M_1, M_2 be entourages and suppose there exists $y \in X$ such that $(x,y) \in M_1$ and $(y,z) \in M_2$ where $x, z \in X$ then $(x,z) \in M_1M_2$. The projections $\pi_1, \pi_2\colon M_1 \to X$ are close, and the projections $\pi_1^{'}, \pi_2^{'}\colon M_2 \to X$ are close. But $\pi_2(x,y) = \pi_1^{'}(y,z)$, so then $\pi_1, \pi_2^{'}\colon M_1M_2 \to X$ are close. Therefore M_1M_2 is an entourage.

(3) It is obvious.

(4) Let S be the set of entourages $M \subseteq X \times X$, then $\bigcup_{M\in S} M \subseteq X \times X$. We need to show that $X \times X \subseteq \bigcup_{M\in S} M$.

Let $(x,y) \in X \times X$, and let $M_{x,y} = \{(x,y)\}$, then $M_{x,y} \in S$, and hence $X \times X \subseteq \bigcup_{(x,y)\in X\times X} M_{x,y}$ as required.

(5) By definition of the projection maps, we have $\pi_1|_M = \pi_2|_{M^t}$, and $\pi_2|_M = \pi_1|_{M^t}$. This shows that if M is an entourage, then M^t is an entourage as well. □

Proposition 1.1.11 *Let X be a set with a collection of subsets $M \subseteq X \times X$ termed entourages satisfying the five properties in proposition 1.1.7.*

Consider closeness in definition 1.1.9. Then the properties in proposition 1.1.8 hold.

Proof :

(1) Suppose that $S = A \cup B$, $f, g\colon S \to X$. Let $f|_A, g|_A\colon A \to X$ be close and $f|_B, g|_B\colon B \to X$ be close, then $M_1 = \{(f(a), g(a)) \mid a \in A\}$ is an entourage, and $M_2 = \{(f(b), g(b)) \mid b \in B\}$ is an entourage. Then $M_1 \cup M_2 = \{(f(s), g(s)) \mid s \in A \text{ or } s \in B\} = \{(f(s), g(s)) \mid s \in S\}$ is an entourage. Hence $f, g\colon S \to X$ are close.

(2) Let $f, g\colon S \to X$ be close, and $h\colon T \to S$ be a map, then $\{(f(s), g(s)) \mid s \in S\}$ is an entourage. Suppose that $t \in T$, then $\{(f(h(t)), g(h(t))) \mid t \in T\} = \{(f \circ h(t), g \circ h(t)) \mid t \in T\}$ is an entourage as $h(t) \in S$. Therefore $f \circ h, g \circ h\colon T \to X$ are close.

(3) It is straightforward. □

To identify when spaces are *not* coarsely equivalent we would like some notion of a *coarse invariant.*

Proposition 1.1.12 *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a controlled map. Then the map f is a coarse equivalence if and only if there is a controlled map $g\colon Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are close to the identity maps 1_X and 1_Y respectively.*

Proof : One direction is straightforward. For the other direction, let $f\colon X \to Y$ be a controlled map, and there is a controlled map $g\colon Y \to X$ such that the composites $g\circ f$ and $f\circ g$ are close to the identity maps 1_X and 1_Y respectively. We need to show that f is a coarse equivalence, and this needs to show that the maps f and g are metrically proper. We show it for f. The proof for g is similar.

Let $B \subseteq Y$ be a non-empty bounded subset. Then we have a point $y_0 \in B$, and a constant $R > 0$ such that $d(y, y_0) < R$ for all $y \in B$. Since the map g is controlled, we have $S > 0$ such that $d(g(y), g(y_0)) < S$ for all $y \in B$.

Now, let $x \in f^{-1}[B]$. Then $f(x) \in B$, so $d(g(f(x)), g(y_0)) < S$. But the composite $g \circ f$ is close to the identity 1_X. Hence we have a constant $K > 0$, not depending on our point x such that $d(g(f(x)), x) \leq K$. Thus

$$d(x, g(y_0)) \leq d(x, g(f(x))) + d(g(f(x)), g(y_0)) \leq K + S.$$

We see that the inverse image $f^{-1}[B]$ is bounded. Hence the map f is metrically proper, as required. □

§ 1.2 Geodesics

In this section we introduce a notion of geodesic rays and geodesic spaces which can be found in [3].

Definition 1.2.1 Let X be a metric space, we call an isometric (distance preserving) embedding $\gamma\colon \mathbb{R} \to X$ a *geodesic* . An isometric embedding $\gamma\colon \mathbb{R}_+ \to X$ is called a *geodesic ray* , and an isometric embedding $\gamma\colon [a, b] \to X$ is called a *geodesic segment* joining the points $x = \gamma(a)$ and $y = \gamma(b)$.

Definition 1.2.2 We call a metric space X *geodesic* if any two points can be joined by a geodesic segment.

Example 1.2.3 Let V be a normed vector space. Let $x, y \in V$, $x \neq y$, and set $D = \|x - y\|$. Then we have a geodesic segment $\gamma\colon [0, D] \to V$ joining the points x and y defined by the formula

$$\gamma(t) = x + t\tfrac{y-x}{D}$$

Example 1.2.4 The space $\mathbb{R}^2\backslash\{0\}$ is not a geodesic space as for the points $(1, 1)$ and $(-1, -1)$ there is not any geodesic segment joining them.

Definition 1.2.5 A metric space X is called *proper* if any closed bounded subset of X is compact.

§ 1.3 Asymptotically Lipschitz Maps and Quasi-Isometry

The definitions in this section can be found in [3].

Definition 1.3.1 Let X and Y be metric spaces. A map $f\colon X \to Y$ is called *Lipschitz* if there is a constant $A > 0$ such that

$$d(f(x), f(y)) \leq Ad(x, y)$$

for all $x, y \in X$.

Proposition 1.3.2 *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a Lipschitz map. Then the map f is controlled.*

Further, if the spaces X and Y are proper, and the map f is a proper Lipschitz map, then the map f is coarse.

Proof : Let $R > 0$, and suppose that we have points $x, y \in X$ such that $d(x, y) < R$. Then since f is Lipschitz, we have $d(f(x), f(y)) < AR$ for some fixed $A > 0$. Thus the map f is controlled.

Let $B \subseteq Y$ be bounded. Then since Y is proper we have $\overline{B}$ is compact. Hence, since the map f is proper, the inverse image $f^{-1}[\overline{B}]$ is compact. Therefore $f^{-1}[B] \subseteq f^{-1}[\overline{B}]$ is bounded as required. □

The last part of the above proof shows the following proposition.

Proposition 1.3.3 *Let X and Y be metric spaces, where Y is a proper space. Then any proper map $f\colon X \to Y$ is metrically proper.* □

Example 1.3.4 Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a differentiable map such that we have $A > 0$ with $\|Df_x\| \leq A$ for all $x \in \mathbb{R}^m$. Then f is Lipschitz which implies that the map is controlled.

Proof : Let $x,\ y \in \mathbb{R}^m$. Define $\gamma\colon [0,1] \to \mathbb{R}^m$ by $\gamma(t) = (1-t)x + ty$. Then $f \circ \gamma\colon [0,1] \to \mathbb{R}^n$ is a differentiable map and $f \circ \gamma(0) = f(\gamma(0)) = f(x)$, $f \circ \gamma(1) = f(\gamma(1)) = f(y)$.

By mean value inequality $\|f \circ \gamma(0) - f \circ \gamma(1)\| \leq \|(f \circ \gamma)^{'}(c)\|$ for some $c \in (0,1)$, $(f \circ \gamma)^{'} = (Df)_{\gamma(t)} \circ \gamma^{'}$, so $\|f \circ \gamma(0) - f \circ \gamma(1)\| \leq \|(Df)_{\gamma(t)} \circ \gamma^{'}\| \leq A\|\gamma^{'}(c)\|$ for some $A > 0$, $\gamma^{'}(c) = -x + y = y - x$.

Therefore $\|f \circ \gamma(0) - f \circ \gamma(1)\| \leq A\|x - y\|$, and so $\|f(x) - f(y)\| \leq A\|x - y\|$. Hence f is Lipschitz. □

Definition 1.3.5 Let X and Y be proper metric spaces. We call a map $f\colon X \to Y$ *asymptotically Lipschitz* if there are two constants $A, B \geq 0$ such that

$$d(f(x), f(y)) \leq Ad(x, y) + B$$

for all $x, y \in X$. If we want to keep track of the constants A and B, we call f an (A, B)-asymptotically Lipschitz map.

Proposition 1.3.6 *Let X and Y be metric spaces, and let $f\colon X \to Y$ be an asymptotically Lipschitz map. Then the map f is controlled.*

Further, if the spaces X and Y are proper, and the map f is a proper map, then the map is coarse.

Proof : Similar to the proof of proposition 1.3.2. □

Example 1.3.7 The map $f\colon \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lfloor x \rfloor$ is asymptotically Lipschitz, because if $x, y \in \mathbb{R}$, then

$$|f(x) - f(y)| \leq |f(x) - x| + |x - y| + |y - f(y)| \leq |x - y| + 2.$$

Example 1.3.8 Let $X = \{n^2 : n \in \mathbb{N}\}$ and $Y = \{n^4 : n \in \mathbb{N}\}$. Then we have a metrically proper map $f\colon X \to Y$ defined by $f(x) = x^2$. This map is coarse, but not asymptotically Lipschitz.

To do this, let $x, y \in X$, $R > 0$ such that $|x - y| < R$. Suppose $x < y$. Set $x = m^2$, $y = n^2$, where n, $m \in \mathbb{N}$. Then $n \geq m + 1$, and so

$$n^2 \geq m^2 + 2m + 1, \quad y - x \geq 2m + 1.$$

But $|y - x| < R$, so $R > y - x \geq 2m + 1$, and then $2m + 1 \leq R$ which gives $2\sqrt{x} + 1 \leq R$. Thus

$$x \leq \frac{(R-1)^2}{4}, \quad y \leq \frac{1}{4}(R-1)^2 + R,$$

and

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| \leq R\Big(\frac{(R-1)^2}{4} + \frac{(R-1)^2}{4} + R\Big) = \frac{1}{2}(R^3 + R).$$

Let $S = \frac{1}{2}(R^3 + R)$. Then if $x < y$ and $|x - y| \leq R$, we have $|f(x) - f(y)| \leq S$. The situation is the same when $y < x$, so f is controlled.

It is clear that the inverse image of a bounded subset under the map f is bounded. Hence f is coarse.

Now, suppose that the map f is asymptotically Lipschitz. Then there exist constants $A, B > 0$ such that $|f(x) - f(y)| \leq A|x - y| + B$ for all $x, y \in X$. That is,

$$|x + y||x - y| \leq A|x - y| + B.$$

Suppose $x \neq y$. Then we have

$$x + y \leq A + \frac{B}{|x - y|} \leq A + B.$$

that should be for all $x, y \in X$ where $x \neq y$, which is not always true, so we have a contradiction. Thus the map f is not asymptotically Lipschitz.

The following proposition comes from [4].

Proposition 1.3.9 *Let X be a geodesic metric space, Y be a metric space, and $f\colon X \to Y$ be a controlled map. Then the map f is asymptotically Lipschitz.*

Proof : Consider arbitrary points $x, y \in X$. Then we have a geodesic segment $\gamma\colon [a,b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Let $n = \lfloor b - a \rfloor$ and write

$$x_i = \gamma(a+i), \quad i \in \{0, ..., n\}$$

Then, since γ is a geodesic segment, $d(x_i, x_{i+1}) = d(\gamma(a+i), \gamma(a+i+1)) = d(a+i, a+i+1) = 1$ for all i, and $d(x_n, y) < 1$. Therefore since f is controlled, there exists $S > 0$ such that $d(f(x_i), f(x_{i+1})) < S$ for all i, and $d(f(x_n), f(y)) < S$. By triangle inequality we have

$$\begin{aligned} d(f(x), f(y)) \leq & \sum_{i=0}^{n-1} d(f(x_i), f(x_{i+1})) + d(f(x_n), f(y)) \\ < & (n+1)S \\ \leq & S d(x,y) + S \end{aligned}$$

Hence the map f is asymptotically Lipschitz as needed. □

Definition 1.3.10 Let X and Y be proper metric spaces. We call a map $f\colon X \to Y$ *quasi-isometric embedding* if there are constants $A \geq 1$ and $B \geq 0$ such that

$$\frac{1}{A} d(x,y) - B \leq d(f(x), f(y)) \leq A d(x,y) + B$$

for all $x, y \in X$. If, in addition, we have a constant $C \geq 0$ such that for all $y \in Y$, we have $x \in X$ such that $d(f(x), y) \leq C$, then we call the map f a *quasi-isometry*. We call the spaces X and Y *quasi-isometric* if such a map exists.

Proposition 1.3.11 *A map $f\colon X \to Y$ is a quasi-isometry if and only if it is asymptotically Lipschitz, and there is an asymptotically Lipschitz map $g\colon Y \to X$ such that the composition $g \circ f$ and $f \circ g$ are close to the identity maps 1_X and 1_Y respectively.*

Proof : Let $f\colon X \to Y$ be a quasi-isometry. Then we have constants $A \geq 1$, $B \geq 0$ such that

$$\frac{1}{A} d(x,y) - B \leq d(f(x), f(y)) \leq A d(x,y) + B$$

for all $x, y \in X$. Further, we have a constant C such that for all $y \in Y$, we can find $x \in X$ such that $d(f(x), y) \leq C$.

By definition, the map f is asymptotically Lipschitz. Define a map $g\colon Y \to X$ by choosing an element $g(y^{'}) \in X$ for each point $y^{'} \in Y$ such that $d(fg(y^{'}), y^{'}) \leq$

C. Then, for points $x^{'}, y^{'} \in Y$, we have

$$\begin{aligned}\frac{1}{A}d(g(x^{'}), g(y^{'})) - B \leq & d(fg(x^{'}), fg(y^{'})) \\ \leq & d(fg(x^{'}), x^{'}) + d(x^{'}, y^{'}) + d(fg(y^{'}), y^{'}) \\ \leq & 2C + d(x^{'}, y^{'})\end{aligned}$$

Then we have

$$d(g(x^{'}), g(y^{'})) \leq Ad(x^{'}, y^{'}) + AB + 2AC$$

so the map g is asymptotically Lipschitz.

Let $x \in X$. Then $d(fgf(x), f(x)) \leq C$, so

$$\frac{1}{A}d(gf(x), x) - B \leq C, and \quad d(gf(x), x) \leq AB + AC$$

We see that the composition $g \circ f$ is close to the identity 1_X. By construction, we know that $d(fg(y^{'}), y^{'}) \leq C$ for all $y^{'} \in Y$, so the map $f \circ g$ is close to the identity 1_Y. This completes the first half of the proof.

Conversely, let $f \colon X \to Y$ and $g \colon Y \to X$ be asymptotically Lipschitz maps such that the compositions $f \circ g$ and $g \circ f$ are close the identity maps 1_X and 1_Y respectively. Then we have constants $A, B, C, D, E, F \geq 0$ such that:

- $d(f(x), f(x^{'})) \leq Ad(x, x^{'}) + B$ for all $x, x^{'} \in X$.
- $d(g(y), g(y^{'})) \leq Cd(y, y^{'}) + D$ for all $y, y^{'} \in Y$.
- $d(gf(x), x) \leq E$ for all $x \in X$.
- $d(fg(y), y) \leq F$ for all $y \in Y$. The last inequality tells us that it suffices to prove that the map f is a quasi-isometric embedding.

Let $x, y \in X$. Then by triangle inequality and the third and the second of the above inequalities, we see

$$d(x, y) \leq d(gf(x), gf(y)) + 2E \leq Cd(f(x), f(y)) + D + 2E.$$

Then we have

$$\frac{d(x, y)}{C} - \frac{D + 2E}{C} \leq d(f(x), f(y)) \leq Ad(x, y) + B.$$

Let

$$A^{'} = max(A, C), and \quad B^{'} = max(B, \frac{D + 2E}{C}),$$

Then

$$\frac{d(x, y)}{A^{'}} - B^{'} \leq d(f(x), f(y)) \leq A^{'}d(x, y) + B^{'}.$$

Thus the map f is a quasi-isometry as needed. □

Theorem 1.3.12 *Let X and Y be geodesic metric spaces. Then a map $f \colon X \to Y$ is a quasi-isometry if and only if it is a coarse equivalence.*

Proof : Let $f\colon X \to Y$ be a quasi-isometry, then by the above proposition f is asymptotically Lipschitz, and there is an asymptotically Lipschitz map $g\colon Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are close to the identity maps 1_X and 1_Y respectively.

By proposition 1.3.6 the maps f and g are controlled, and then by proposition 1.1.12 the map f is a coarse equivalence.

Conversely, let $f\colon X \to Y$ be a coarse equivalence. Then by definition there is a coarse map $g\colon Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are close to the identity maps 1_X and 1_Y respectively.

Since the spaces X and Y are geodesic metric spaces, then by proposition 1.3.9 the maps f and g are asymptotically Lipschitz. It follows by the above proposition that the map $f\colon X \to Y$ is a quasi-isometry. □

§ 1.4 The Coarse Category

In topology we define a topological space in terms of open sets. In the large scale we can define an abstract coarse space in terms of entourages. These entourages should satisfy particular axioms that allow us to equip a coarse structure on a set X. In this section we define a notion of a coarse space and we give some examples.

This definition comes from [9] and [11].

Definition 1.4.1 Let X be a set. Then X is called a *coarse space* if it is equipped with a *coarse structure* , defined to be a collection ε of subsets M of $X \times X$ called *entourages* satisfying the following axioms:

(1) If $M \in \varepsilon$ and $M^{'} \subseteq M$, then $M^{'} \in \varepsilon$.

(2) Let $M_1, M_2 \in \varepsilon$, then $M_1 \cup M_2 \in \varepsilon$, and $M_1M_2 = \{(x,z) \mid (x,y) \in M_1, (y,z) \in M_2 \text{ for some y}\} \in \varepsilon$. We call M_1M_2 *the composite* of M_1 and M_2.

(3) $\Delta_X \in \varepsilon$.

(4) $\bigcup_{M\in\varepsilon} M = X \times X$.

(5) If $M \in \varepsilon$, $M^t = \{(y,x) \mid (x,y) \in M\} \in \varepsilon$.

We can use (X, ε) to refer to a coarse space.

A subset M is called *symmetric* if $M = M^t$.

A *non-unital coarse space* is a coarse space defined as above where Δ_X is not necessarily an entourage.

We can generalize definition 1.1.1 as follows;

Definition 1.4.2 Let X and Y be coarse spaces. Then a map $f\colon X \to Y$ is said to be *controlled* or *coarsely uniform* if for every entourage $M \subseteq X \times X$, the image

$$f[M] = \{(f(x), f(y)) : (x, y) \in M\}$$

is an entourage. A controlled map f is called *coarse* if the inverse image of a bounded set is also bounded. A controlled map f is called *rough* if the inverse image of an entourage under the map $f \times f$ is an entourage.

Definition 1.4.3 We call two coarse maps $f, g\colon X \to Y$ *close*, and write $f \sim_{Crs} g$, if the set $\{(f(x), g(x)) \mid x \in X\}$ is an entourage.

We similarly define two controlled maps being close, and denoted by $f \sim_{Crd} g$.

Controlled equivalence and coarse equivalence between coarse spaces are defined as in definition 1.1.4.

We rewrite proposition 1.1.12 as follows;

Proposition 1.4.4 **(1)** *Any coarse equivalence is a controlled equivalence.*

(2) *Any controlled equivalence is a rough map.*

(3) *Any surjective rough map is a coarse equivalence.*

Proof :

(1) It is obvious.

(2) Let $f\colon X \to Y$ be a controlled equivalence, then f is a controlled map and there is a controlled map $g\colon Y \to X$ such $f \circ g$ and $g \circ f$ are close to the identities 1_Y and 1_X respectively.

Let $M \subseteq Y \times Y$ be an entourage, then since g is a controlled map we have $g \times g(M)$ is an entourage, and by assumption we have $F = \{(x, g \circ f(x)) \mid x \in X\}$ is an entourage. Let $(x, y) \in (f \times f)^{-1}(M)$, then $(g \circ f(x), g \circ f(y)) \in g \times g(M)$, and this implies that $(x, y) \in F(g \times g)(M)F^t$.

Therefore $(f \times f)^{-1}(M) \subseteq F(g \times g)(M)F^t$, and by definition of coarse structure we have $(f \times f)^{-1}(M)$ is an entourage. Hence f is a rough map.

(3) Let $f\colon X \to Y$ be a rough map, then by definition f is a controlled map. Let $B \subseteq Y$ be a bounded set, then $B \times B \subseteq Y \times Y$ is an entourage, and since f is rough, we have $f^{-1}[B] \times f^{-1}[B] \subseteq X \times X$ is an entourage which implies that $f^{-1}[B]$ is bounded. Therefore f is a coarse map. For each $y \in Y$ pick some $g(y) \in X$ such that $f(g(y)) = y$ where $g\colon Y \to X$ is a map, and we have $g(y) \in f^{-1}(\{y\})$.

We need to show that g is a coarse map. Let $M \subseteq Y \times Y$ be an entourage, then $g \times g[M] \subseteq f^{-1} \times f^{-1}[M]$ is an entourage. So g is a controlled map.

Let $B \subseteq X$ be a bounded subset, then $f[B]$ is bounded, and $g^{-1}[B] \subseteq f[B]$ is bounded.

Now we have $f(g(y)) = y$ for each $y \in Y$, so $f \circ g = id_Y$. The composite $g(f(x)) \in f^{-1}(\{x\})$, and we need to show that the set $M = \{(g(f(x)), x) \mid x \in X\}$ is an entourage. Since $(f(g(f(x))), f(x)) = (f(x), f(x))$, so $f \times f[M]$ is an entourage which implies that M is an entourage since f is a rough map. □

The above proposition shows that coarse equivalence and controlled equivalence are the same.

The definition of close maps for general coarse spaces is the same for metric spaces in definition 1.1.9.

Note that a subset $B \subseteq X$ is bounded if and only if it takes the form $M(x) = \{y : (x, y) \in M\}$ for some entourage $M \subseteq X \times X$ and point $x \in X$.

We can form the category of all coarse spaces and coarse maps. We call this category the *coarse category*, and we denote it by Crs .

We can similarly form the category of all coarse spaces and controlled maps. We call this category the *controlled category*, and we denote it by Crd .

Definition 1.4.5 Let X be a set and ε a collection of subsets of $X \times X$. The *coarse structure generated by* ε is the minimum coarse structure on X that contains ε.

The following definition comes from [8].

Definition 1.4.6 Let X be a Hausdorff space. A coarse structure on X is said to be *compatible with the topology* if every entourage is contained in an open entourage, and the closure of any bounded set is compact. We call X a *coarse topological space*.

Any coarse topological space is locally compact, and the bounded sets are precisely those which are precompact.

By proposition 1.1.7 we have any proper metric space as one example of a coarse space as follows.

Example 1.4.7 Let (X, d) be a proper metric space. Then d induces a coarse structure on X, which is called *metric structure* such that:

Let $D_r = \{(x, y) \in X \times X \mid d(x, y) < r\}$. Then $E \subseteq X \times X$ is an entourage if $E \subseteq D_r$, for some $r > 0$.

The bounded sets are simply those which are bounded with respect to the metric. If we give the metric space X the metric structure, it is a coarse topological space. That is, the metric structure on a proper metric space is compatible with the topology.

Example 1.4.8 Let X be a coarse topological space, and suppose that X is contained in a Hausdorff topological space $\overline{X}$ as a topologically dense subset. Call the coarse structure already defined on the space X the *ambient coarse structure*. Write $\partial X = \overline{X} \setminus X$, call an open subset $M \subseteq X \times X$ *strongly controlled* if:

- The set M is an entourage with respect to the ambient coarse structure on X.
- Let $\overline{M}$ be the closure of the set M in the space $\overline{X} \times \overline{X}$. Then

$$\overline{M} \cap (\overline{X} \times \partial X \cup \partial X \times \overline{X}) \subseteq \Delta_{\partial X}.$$

Then we define the *continuously controlled coarse structure* with respect to ∂X by saying that the entourages are composites of subsets of strongly controlled open sets.

We write X^d to denote the space X with its ambient coarse structure, and X^{cc} to denote the space X with the new continuously entourage coarse structure.

Definition 1.4.9 Let X and Y be coarse spaces, equipped with collections of entourages ε_X and ε_Y respectively. Then we define the *product* of X and Y to be the Cartesian product $X \times Y$ equipped with the coarse structure defined by forming finite compositions, unions of entourages, and all subsets of entourages in the set

$$\{M \times N : M \in \varepsilon_X, N \in \varepsilon_Y\}.$$

Unfortunately, the above product is not a product in the category-theoretic sense since the projections $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ are not in general coarse maps.

Now we define two different coarse versions of disjoint union of coarse spaces.

The following definition comes from [10].

Definition 1.4.10 Let X and Y be coarse spaces. Then we define the *disjoint union* to be the set $X \sqcup Y$ equipped with the coarse structure given by defining the entourages to be subsets of unions of the form

$$M \cup N \cup (B_X \times B_Y) \cup (B_Y^{'} \times B_X^{'})$$

where $M \subseteq X \times X$ and $N \subseteq Y \times Y$ are entourages, and B_X, $B_X^{'} \subseteq X$ and B_Y, $B_Y^{'} \subseteq Y$ are bounded subsets. We denote this disjoint union by $X \sqcup Y$.

The following result is easy to check.

Proposition 1.4.11 *Let X and Y be coarse spaces, R be a generalised ray. Let $p_X \colon X \to R$ and $p_Y \colon Y \to R$ be controlled maps. Then $X \sqcup Y$ is a coarse space, and the map $p_{X \sqcup Y} \colon X \sqcup Y \to R$ defined by the formula*

$$p_{X \sqcup Y}(x) = \begin{cases} p_X(x) & x \in X \\ p_Y(x) & x \in Y \end{cases}$$

is a controlled map. □

Definition 1.4.12 Let X and Y be coarse spaces. Then we define another type of *disjoint union* to be the set $X \sqcup Y$ equipped with the coarse structure given by defining the entourages to be subsets of unions of the form $M \cup N$ where $M \subseteq X \times X$ and $N \subseteq Y \times Y$ are entourages. We denote this disjoint union by $X \sqcup_\infty Y$.

The space $X \sqcup_\infty Y$ is a non-unital coarse space even when X and Y are unital coarse spaces.

The following is also easy to check.

Proposition 1.4.13 *Let X and Y be coarse spaces, R be a generalised ray. Let $p_X \colon X \to R$ and $p_Y \colon Y \to R$ be controlled maps. Then $X \sqcup_\infty Y$ is a coarse space, and the map $p_{X \sqcup_\infty Y} \colon X \sqcup_\infty Y \to R$ defined by the formula*

$$p_{X \sqcup_\infty Y}(x) = \begin{cases} p_X(x) & x \in X \\ p_Y(x) & x \in Y \end{cases}$$

is a controlled map. □

Chapter 2

Homotopy Groups

Our main work will concern the categories of non-unital coarse spaces to be able to define Baues cofibration structures on the categories we set up. The reason for this restriction is to be able to define a push out diagram in our categories, which is considered to be one of the axioms of Baues cofibration category. In this section we define a notion of locally proper, and coarse maps between non-unital coarse spaces, and a coarse version of abstract homotopy groups.

§ 2.1 Coarse Maps and Generalised Ray

The following definitions are prompted from [9].

Definition 2.1.1 Let X, Y be coarse spaces and $f\colon X \to Y$ a map.

- We call f a *locally proper map* if $f|_{X'}$ is proper whenever $X^{'} \subseteq X$ is a unital coarse subspace, that is, the inverse image of a bounded set $B \subseteq Y$ under the map $f|_{X'}$ is bounded.
- We call f a *coarse map between non-unital coarse spaces* if it is a controlled and locally proper map.
- We call f a *locally effectively proper map* if $f|_{X'}$ is effectively proper whenever $X^{'} \subseteq X$ is a unital coarse subspace, that is, the inverse image of an entourage $M \subseteq Y \times Y$ under the map $(f \times f)|_{X' \times X'}$ is also an entourage.
- We call f a *rough map between non-unital coarse spaces* if it is a controlled and locally effectively proper map.

Any proper map is locally proper, but the converse in not always true as explained in the following example;

Example 2.1.2 Let $X = \mathbb{R}_+ = [0, \infty)$, and $X^{'} = \{0\}$ is a closed subspace of the proper metric space $\mathbb{R}_+$, then $\{0\}$ is itself a coarse space.

There is an obvious *ideal* $\ll \{(0,0)\} \gg_{\mathbb{R}_+}$ of $\varepsilon_{\mathbb{R}_+}$ generated by the diagonal $\{(0,0)\}$ (see definition (3.10.1) in [9]). The space $\mathbb{R}_+$ equipped with the coarse structure $\ll \{(0,0)\} \gg_{\mathbb{R}_+}$ defines a non-unital coarse space.

Now we have the constant map $p\colon \mathbb{R}_+ \to \{0\}$. This is clearly not proper map. However, restrict to the unital coarse subspace $\{0\}$, then $p|_{\{0\}}\colon (\mathbb{R}_+, \ll \{(0,0)\} \gg_{\mathbb{R}_+}) \to \{0\}$ is the identity map which is proper, so p is locally proper.

We define two maps between non-unital coarse spaces being close as follows.

Definition 2.1.3 Let $f, g\colon X \to Y$ be two coarse maps between non-unital coarse spaces. We say that f is *close* to g if for any unital subspace $X^{'} \subseteq X$, we have $f|_{X'}$ is close to $g|_{X'}$ in sense of definition 1.4.3.

We call f a *coarse equivalence between non-unital coarse spaces* if $f|_{X'}$ is a coarse equivalence in sense of definition 1.1.4 whenever $X^{'} \subseteq X$ is a unital coarse subspace.

And we call f a *controlled equivalence between non-unital coarse spaces* if $f|_{X'}$ is a controlled equivalence in sense of definition 1.1.4 whenever $X^{'} \subseteq X$ is a unital coarse subspace.

Proposition 2.1.4 **(1)** *Any coarse equivalence between non-unital coarse spaces is a controlled equivalence between non-unital coarse spaces.*

(2) *Any controlled equivalence between non-unital coarse spaces is a rough map between non-unital coarse spaces.*

(3) *Any rough map between non-unital coarse spaces is a coarse equivalence between non-unital coarse spaces.*

Proof : Straightforward by a similar argument used in proposition 1.4.4. □

Again the above proposition shows that coarse equivalence and controlled equivalence are the same.

The following definition comes from [12].

Definition 2.1.5 Let R be the topological space $[0, \infty)$ equipped with a coarse structure compatible with the topology. We call the space R a *generalised ray* if the following conditions hold.

- Let $M, N \subseteq R \times R$ be entourages. Then the sum

$$M + N = \{(u + x, v + y) \mid (u, v) \in M, (x, y) \in N\}$$

is an entourage.

- Let $M \subseteq R \times R$ be an entourage . Then the set

$$M^s = \{(u,v) \in R \times R \mid x \leq u, v \leq y, (x,y) \in M\}$$

is an entourage.

- Let $N \subseteq R \times R$ be an entourage , and $a \in R$. Then the set

$$a + N = \{(a+x, a+y) \mid (x,y) \in N\}$$

is an entourage.

For example, the space $\mathbb{R}_+$ (with the metric coarse structure) is a generalised ray. The space $[0,\infty)$ equipped with the continuously controlled coarse structure arising from the one point compactification is also a generalised ray.

§ 2.2 Coarse Homotopy

Let X be a topological space. The product $X \times [0,1]$ is called a cylinder on X. We need to define a coarse version of the topological cylinder in order to define a coarse version of homotopy.

The following definition comes from [13].

Definition 2.2.1 Let X be a coarse space, R be a generalised ray, and $p\colon X \to R$ be some controlled map. Then we define the *p-cylinder* of X:

$$I_pX = \{(x,t) \in X \times R \mid t \leq p(x) + 1\}$$

The p-cylinder is a coarse space. We define the projection $p^{'}\colon I_pX \to R$ by the formula $p^{'}(x,t) = p(x)+t$ and we define controlled maps (which are also coarse maps) $i_0, i_1\colon X \to I_pX$ by the formula $i_0(x) = (x,0)$ and $i_1(x) = (x, p(x)+1)$ respectively.

One of our biggest aims in this work is to define a Baues cofibration category on the category of non-unital coarse spaces, and defining our cylinder above not for any general controlled map allows us to make Baues construction successfully and it also works for all our arguments, otherwise we may want to require extra conditions that is not clear whether it preserves the construction. Also not assuming p to be coarse map makes the categories not so big which allows us to do homotopy theory safely.

Definition 2.2.2 Let $f_0, f_1\colon X \to Y$ be controlled maps. A *controlled homotopy* between f_0, f_1 is a controlled map $H\colon I_pX \to Y$ for some controlled map $p\colon X \to R$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$ respectively.

A controlled map $f\colon X \to Y$ is termed *controlled homotopy equivalence* if there is a controlled map $g\colon Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are controlled homotopic to the identities 1_X and 1_Y respectively.

Let $f_0, f_1 \colon X \to Y$ be coarse maps. A *coarse homotopy* between f_0, f_1 is a coarse map $H \colon I_pX \to Y$ for some controlled map $p \colon X \to R$ such that $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$ respectively.

A coarse map $f \colon X \to Y$ is termed *coarse homotopy equivalence* if there is a coarse map $g \colon Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are coarsely homotopic to the identities 1_X and 1_Y respectively.

We say the maps $f_0, f_1 \colon X \to Y$ are *coarsely homotopic between non-unital coarse spaces* if $f_0|_{X'}$ is coarsely homotopic to $f_1|_{X'}$ whenever $X^{'} \subseteq X$ is a unital coarse subspace.

A coarse map $f \colon X \to Y$ is termed a *coarse homotopy equivalence between non-unital coarse spaces* if $f|_{X'} \colon X^{'} \to Y$ is a coarse homotopy equivalence whenever $X^{'} \subseteq X$ is a unital coarse subspace.

It is clear that any coarse homotopy is a controlled homotopy.

Defining coarse homotopy with respect to some controlled (not coarse) map makes the homotopy not very big which again helps in satisfying the axioms of Baues cofibration category of our example and guarantees that the mapping cylinder defined in the next chapter exists.

The proof of the following lemma is found in [6].

Lemma 2.2.3 *Let $f_0, f_1 \colon X \to Y$ be close maps between coarse spaces. If f_0 is a controlled map, then f_1 is a controlled map. If f_0 is a coarse map, then f_1 is a coarse map. If f_0 is a coarse equivalence, then f_1 is a coarse equivalence. If f_0 is a coarse homotopy equivalence, then f_1 is a coarse homotopy equivalence.* □

Example 2.2.4 Let X and Y be coarse spaces, and let $p \colon X \to R$ be a controlled map. Consider two close coarse maps $f_0, f_1 \colon X \to Y$. Then we can define a coarse homotopy $H \colon I_pX \to Y$ between the maps f_0 and f_1 by the formula

$$H(x,t) = \begin{cases} f_0(x) & \text{t<1} \\ f_1(x) & \text{t } \geq \text{1} \end{cases}$$

Similarly we can define controlled homotopy between f_0 and f_1, when f_0 and f_1 are close controlled maps.

Thus, close coarse maps are also coarse homotopic. In particular, any coarse equivalence is a coarse homotopy equivalence which is also a controlled homotopy equivalence since by proposition 1.4.4 and 2.1.4 coarse equivalence and controlled equivalence are the same.

Theorem 2.2.5 *Let X, Y be coarse spaces.*

(1) *Controlled homotopy defines an equivalence relation on the set of all controlled maps from X to Y.*

(2) *Coarse homotopy defines an equivalence relation on the set of all coarse maps from X to Y.*

Before proving this theorem we have to state the following:

The following definition comes from [12].

Definition 2.2.6 We call a union $X = A \cup B$ *coarsely excisive decomposition* if for any entourage $m \subseteq X \times X$, there is an entourage $M \subseteq X \times X$ such that $m[A] \cap m[B] \subseteq M[A \cap B]$.

Lemma 2.2.7 *Let $f \colon X \to Y$ be a map.*

(1) *Let $X = A \cup B$ be a coarsely excisive decomposition such that the restrictions $f|_A$ and $f|_B$ are controlled maps. Then f is a controlled map.*

(2) *Let $X = A \cup B$ be a coarsely excisive decomposition such that the restrictions $f|_A$ and $f|_B$ are coarse maps. Then f is a coarse map.*

Proof :

(1) For any entourage M, we have $M \cup M^t$ is a symmetric entourage, so without loss of generality, let $m \subseteq X \times X$ be a symmetric entourage containing the diagonal. We need to show that the image $f[m]$ is an entourage.

The sets $f[m \cap (A \times A)]$ and $f[m \cap (B \times B)]$ are entourages as $f|_A$ and $f|_B$ are controlled maps.

It is enough to show that $f[m \cap (A \times B)]$ and $f[m \cap (B \times A)]$ are entourages.

For the first case, let $x \in A$ and $y \in B$ such that $(x, y) \in m \cap (A \times B)$. Since $X = A \cup B$ is a coarsely excisive decomposition then there exists an entourage M such that $m[A] \cap m[B] \subseteq M[A \cap B]$, and since m is symmetric we have $A \subseteq m[A]$, $B \subseteq m[B]$ and then $x, y \in m[A] \cap m[B]$ which implies that $x, y \in M[A \cap B]$ for some a symmetric entourage $M \subseteq X \times X$.

Now, we can find $z \in A \cap B$ such that $(x, z) \in M$ and there exists $w \in A \cap B$ such that $(w, y) \in M$, but $z, w \in A \cap B$ so $(z, y) \in M$ such that $(x, z) \in M \cap (A \times A)$ and $(z, y) \in M \cap (B \times B)$ and then $m \cap (A \times B) \subseteq M \cap (A \times A) M \cap (B \times B)$. Therefore $f[m \cap (A \times B)] \subseteq f[M \cap (A \times A)] f[M \cap (B \times B)]$.

Since $f|_A$ and $f|_B$ are controlled maps, then $f[M \cap (A \times A)]$ and $f[M \cap (B \times B)]$ are entourages. Thus $f[m \cap (A \times B)]$ is an entourage.

The second case is similar. This shows that f is a controlled map as required.

(2) First, we show that f is controlled which is by the agrement in (1). Second, let $C \subseteq Y$ be a bounded subset, then $(f|A)^{-1}[C]$ and $(f|B)^{-1}[C]$ are bounded subsets since $f|A$ and $f|B$ are coarse maps, and $f^{-1}(C) \subseteq (f|A)^{-1}[C] \cup (f|B)^{-1}[C]$, so $f^{-1}(C)$ is bounded. Hence f is a coarse map. □

Corollary 2.2.8 *Let $p\colon X \to R$ be some controlled map, $A = \{(x,t) \in X \times R \mid t \leq p(x)\}$, and $B = \{(x,t) \in X \times R \mid t \geq p(x)\}$, $p\colon X \to R$ be a controlled map such that $X \times R = A \cup B$.*

(1) *Suppose that $f\colon X \times R \to Y$ is a map such that the restrictions $f|_A$ and $f|_B$ are controlled maps. Then f is a controlled map.*

(2) *Suppose that $f\colon X \times R \to Y$ is a map such that the restrictions $f|_A$ and $f|_B$ are coarse maps. Then f is a coarse map.*

Proof : To prove (1), it is enough to show that $X \times R = A \cup B$ is a coarsely excisive decomposition. Case (2) is similar.

To prove that $X \times R = A \cup B$ is a coarsely excisive decomposition, let $m \subseteq X \times R \times X \times R$ be an entourage. For any entourage M, we have $M \cup M^t$ is a symmetric entourage so without loss of generality, we suppose that $m = m_1 \times m_2$ where $m_1 \subseteq X \times X$, $m_2 \subseteq R \times R$ are symmetric entourages containing the diagonal.

Let $(z,w) \in X \times R$ be such that $(z,w) \in m[A] \cap m[B]$, so there exists $(x,s) \in A$ and $(y,t) \in B$ such that $((x,s),(z,w)) \in m$, that is, $(x,z) \in m_1$, $(s,w) \in m_2$ and $((z,w),(y,t)) \in m$, that is $(z,y) \in m_1$, $(w,t) \in m_2$. By definition of A and B, we have

$$s \leq p(x), \quad t \geq p(y), \quad \text{and either } \; w \geq p(z), \quad \text{or } \; w \leq p(z).$$

We prove the case when $s \leq p(x)$, $t \geq p(y)$, and $w \geq p(z)$, the other cases are identical.

So we have

$$s \leq p(z) \quad \text{or} \quad s \geq p(z)$$

and

$$t \leq p(z) \quad \text{or} \quad t \geq p(z)$$

By definition 2.1.5 we have

$$s \leq p(z) \quad \text{or} \quad (s, p(z)) \in [p \times p(m_1)]^s,$$

and

$$t \leq p(z) \quad \text{or} \quad (t, p(z)) \in [p \times p(m_1)]^s$$

Hence in either cases and since symmetry is assumed we have $(p(z), s)$, $(p(z), t) \in [p \times p(m_1)]^s$. In all cases we have $(p(z), w) \in [p \times p(m_1)]^s m_2$.

Now let $M = m_1 \times [p \times p(m_1)]^s m_2$ where M depends on m and the controlled map p. Then $((z, p(z)), (z, w)) \in M$, and this means $(z,w) \in M(A \cap B)$, and we are done. □

Proposition 2.2.9 *Let X be a coarse space, and let R be a generalised ray. Let $p, q\colon X \to R$ be controlled maps, then $p+q$ is a controlled map.*

Proof : Let $M \subseteq X \times X$ be an entourage. Then the images $p[M]$, $q[M]$ are entourages. Now

$$(p+q)[M] = \{((p+q)(x),\ (p+q)(y))\ :\ (x,y) \in M\} \subseteq p[M] + q[M]$$

which implies that $(p+q)[M]$ is an entourage. Hence $p+q$ is controlled. $\square$

Proof of Theorem 2.2.5: We will prove (2), and (1) will be similar. Let $f\colon X \to Y$ be a coarse map, then f is coarsely homotopic to itself using a constant coarse homotopy $F\colon I_pX \to Y$ such that $F(x,t) = f(x)$, $x \in X$ where $p\colon X \to R$ is some controlled map. Then $F(x,0) = f(x) = F(x, p(x)+1)$ and since f is a coarse map, then F is also.

Let $f\colon X \to Y$ and $g\colon X \to Y$ be coarse maps such that f is coarsely homotopic to g, so there exists a coarse homotopy $F\colon I_pX \to Y$, where $p\colon X \to R$ is some controlled map such that $F(x,0) = f(x)$, and $F(x, p(x)+1) = g(x)$.

Define a new coarse map $G\colon I_pX \to Y$ by the formula $G(x,t) = F(x, p(x) + 1 - t)$, then G is a coarse homotopy from g to f.

To show this, first we see that $(p+1)$ is a controlled map since for an entourage $M \subseteq X \times X$, then $p+1(M)$ is an entourage as the map p is controlled. Then by definition of a generalised ray, the set

$$\{(p(x)+1-s,\ p(y)+1-t) \mid\ (x,y) \in M,\ 0 \le s \le p(x)+1,\ 0 \le t \le p(y)+1\}$$

is an entourage.

Now since F is a coarse map, it is easy to show that the map G is also. This proves that the equivalence relation is symmetric.

Now we prove that the equivalence relation is transitive. Let $f\colon X \to Y$, $g\colon X \to Y$ and $h\colon X \to Y$ be coarse maps such that f is coarsely homotopic to g and g is coarsely homotopic to h.

So we can define two coarse homotopies $H\colon I_pX \to Y$ and $G\colon I_qX \to Y$ such that $H(x,0) = f(x)$, $H(x,p(x)+1) = g(x) = G(x,0)$, and $G(x,q(x)+1) = h(x)$ for all $x \in X$ where $p\colon X \to R$ and $q\colon X \to R$ are some controlled maps, and $p+q$ is controlled by proposition 2.2.9. Define the map $H+G\colon I_{p+q}X \to Y$ as follow

$$(H{+}G)(x,t) = \begin{cases} H(x,2t) & 0 \le t \le (p(x)+1)/2 \\ G(x, 2t-(p(x)+1)) & (p(x)+1)/2 \le t \le ((p+q)(x)/2)+1 \\ G(x, q(x)+1) & ((p+q)(x)/2)+1 \le t \le (p+q)(x)+1 \end{cases}$$

Note that the maps $K, L\colon I_pX \to I_pX$ defined by $K(x,t) = (x,2t)$ and $L(x,t) = (x, 2t-(p(x)+1))$ are coarse maps, and the maps $((p+q)/2)+1\colon X \to R$ and $(p+1)/2\colon X \to R$ are controlled. Now the map $H+G$ is a coarse homotopy by corollary 2.2.8 (1). $\square$

Lemma 2.2.10 **(1)** *Let* $f\colon X \to Y$ *be a map, and* $i : A \hookrightarrow X$ *be a coarse equivalence. Suppose that the restriction* $f|_A$ *is a controlled map, then* f *is controlled.*

(2) *Let* $f\colon X \to Y$ *be a map, and* $i : A \hookrightarrow X$ *be a coarse equivalence. Suppose that the restriction* $f|_A$ *is a coarse map, then* f *is coarse.*

Proof :

(1) Since i is a coarse equivalence, there exists a coarse map $g\colon X \to A$ such that $g \circ i$ and $i \circ g$ are close to 1_A and 1_X.

We have $f \circ i = f|_A$. Then $f \circ i \circ g = f|_A \circ g$, but $i \circ g$ is close to 1_X and $f \circ 1_X$ is close to f, so then f is close to $f|_A \circ g$, and $f|_A \circ g$ is a controlled map.

Now we need to show that f is controlled.

Let $M = \{(f|_A \circ g(x), f(x)) : x \in X\}$. By closeness M is an entourage.

Let E be an entourage in $X \times X$, then $f \times f(E) \subseteq M^t\{f|_A \circ g(x), f|_A \circ g(y)) : (x, y) \in E\}M$. Since $f|_A \circ g$ is controlled, $\{f|_A \circ g(x), f|_A \circ g(y)) : x, y \in E\}$ is an entourage, and also M^t so the composite is an entourage. This shows that f is a controlled map as required.

(2) The map f is controlled by the same argument in (1). Let $B \subseteq Y$ be bounded, and M the entourage defined in (1) then $M[B]$ is bounded. Since $f|_A \circ g$ is coarse, we have $(f|_A \circ g)^{-1}(M[B])$ is bounded. Let $x \in f^{-1}(B)$, then $f(x) \in B$ and so $f|_A \circ g(x) \in M[B]$ which means $f^{-1}[B]$ is contained in $(f|_A \circ g)^{-1}(M[B])$, so $f^{-1}[B]$ is bounded. Hence f is a coarse map, and we are done. □

Theorem 2.2.11 **(1)** *Let $f_i\colon X \to Y$ and $g_i\colon Y \to Z$ be controlled maps where $i = 0, 1$. If f_0 is controlledly homotopic to f_1 and g_0 is controlledly homotopic to g_1, then $g_0 \circ f_0$ is controlledly homotopic to $g_0 \circ f_1$ and $g_0 \circ f_1$ is contolledly homotopic to $g_1 \circ f_1$. Further, then $g_0 \circ f_0$ is contolledly homotopic to $g_1 \circ f_1$.*

(2) *Let $f_i\colon X \to Y$ and $g_i\colon Y \to Z$ be coarse maps where $i = 0, 1$. If f_0 is coarsely homotopic to f_1 and g_0 is coarsely homotopic to g_1, then $g_0 \circ f_0$ is coarsely homotopic to $g_0 \circ f_1$ and $g_0 \circ f_1$ is coarsely homotopic to $g_1 \circ f_1$. Further, then $g_0 \circ f_0$ is coarsely homotopic to $g_1 \circ f_1$.*

Proof : We prove (2), and (1) is identical. Let $F\colon I_{p_X}X \to Y$ define a coarse homotopy between the coarse maps f_0 and f_1 such that $F(x, 0) = f_0(x)$ and $F(x, p_X(x) + 1) = f_1(x)$ for all $x \in X$ where $p_X\colon X \to R$ is some controlled map, and $G\colon I_{p_Y}Y \to Z$ define a coarse homotopy between the coarse maps g_0 and g_1 such that $G(y, 0) = g_0(y)$ and $G(y, p_Y(x) + 1) = g_1(y)$ for all $y \in Y$ where $p_Y : Y \to R$ is some controlled map.

Define the map $K\colon I_{p_X}X \to Z$ by $K(x, t) = g_0 \circ F(x, t)$. Then K defines a coarse homotopy between $g_0 \circ f_0$ and $g_0 \circ f_1$. Now define the map $H\colon I_{P_Y \circ f_1}X \to Z$ by $H(x, t) = G(f_1(x), t)$, then again H defines a coarse homotopy between $g_0 \circ f_1$ and $g_1 \circ f_1$. By theorem 2.2.5 we have $g_0 \circ f_0$ is coarsely homotopic to $g_1 \circ f_1$. □

Lemma 2.2.12 *The projection map $\pi\colon I_pX \to X$ defined by $\pi(x, t) = x$ where $x \in X$ and $t \in R$ is a coarse homotopy equivalence where $p\colon X \to R$ is some controlled map.*

Proof : Define the inclusion map $i : X \hookrightarrow I_pX$ by the formula $i(x) = (x, 0)$. Then $\pi \circ i = 1_X$, and we can define a coarse homotopy (which is also a controlled homotopy) $H\colon I_{p\circ\pi}(I_pX) \to I_pX$ between the map $i \circ \pi\colon I_pX \to I_pX$ and the identity on I_pX where $i \circ \pi(x,t) = (x, 0)$ as follows;

$$H((x,t),s) = \begin{cases} (x, s+t) & s \leq p(x) \\ (x, 0) & s > p(x) \end{cases}$$

The maps π, i and H are coarse maps, so H defines a coarse homotopy between $i \circ \pi$ and the identity on I_pX. □

Lemma 2.2.13 **(1)** *Let $f\colon X \to Y$ be a controlled map, and $g\colon X \to Y$ be another map that is controlledly homotopic to f, then g is a controlled map.*

(2) *Let $f\colon X \to Y$ be a coarse map, and $g\colon X \to Y$ be another map that is coarsely homotopic to f, then g is a coarse map.*

Proof : We prove (1), and (2) is identical. Suppose that the controlled map f is controlledly homotopic to the map g, so there exists a controlled homotopy $H\colon I_pX \to Y$ such that $H(x, 0) = f(x)$, and $H(x, p(x) + 1) = g(x)$ for all $x \in X$ and some controlled map $p\colon X \to R$. Since H is a controlled map, then the map g is also so. □

Lemma 2.2.14 **(1)** *Let $f\colon X \to Y$ be a map, and $i : A \hookrightarrow X$ be a controlled homotopy equivalence. Suppose that the restriction $f|_A$ is a controlled map, then f is controlled.*

(2) *Let $f\colon X \to Y$ be a map, and $i : A \hookrightarrow X$ be a coarse homotopy equivalence. Suppose that the restriction $f|_A$ is a coarse map, then f is coarse.*

Proof : We prove (1), and (2) is identical. Since i is a controlled homotopy equivalence, there exists a controlled map $g\colon X \to A$ such that $g \circ i$ and $i \circ g$ are controlled homotopic to 1_A and 1_X .

We have $f \circ i = f|_A$. Then $f \circ i \circ g = f|_A \circ g$, but $i \circ g$ is controlled homotopic to 1_X and by theorem 2.2.11 the map $f \circ 1_X$ is controlled homotopic to f, so f is controlled homotopic to $f|_A \circ g$, and since $f|_A \circ g$ is a controlled map, then by lemma 2.2.13 the map f is a controlled map. □

§ 2.3 Controlled and Coarse Path Components

Definition 2.3.1 Let $f\colon X \to Y$ be a controlled map. The controlled equivalence class of a controlled map f under the equivalence relation of controlled homotopy is denoted

$$[f]^{Crd} = \{\text{controlled map } g : X \to Y \mid g \simeq_{Crd} f\}$$

and called *the controlled homotopy class* of f.

The family of all such controlled homotopy classes is denoted by $[X, Y]^{Crd}$.

Definition 2.3.2 Let $f\colon X \to Y$ be a coarse map. The coarse equivalence class of a coarse map f under the equivalence relation of coarse homotopy is denoted

$$[f]^{Crs} = \{\text{coarse map } g : X \to Y \mid g \simeq_{Crs} f\}$$

and called *the coarse homotopy class* of f.

The family of all such coarse homotopy classes is denoted by $[X, Y]^{Crs}$.

We define $\pi_0^{Crd}(X)$ to be the set of all controlled homotopy classes $[f]$ of controlled maps $f\colon R \to X$, and we define $\pi_0^{Crs}(X)$ to be the set of all coarse homotopy classes $[f]$ of coarse maps $f\colon R \to X$. Theses sets for different choices of R are related by natural bijections.

Proposition 2.3.3 *$\pi_0^{Crs}(\mathbb{R}_+)$ has one element which is the identity on $\mathbb{R}_+$.*

Proof : Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a coarse map, we need to show that f is coarsely homotopic to the identity.

Define a map $H\colon I_f\mathbb{R}_+ \to \mathbb{R}_+$ by

$$H(s,t) = \begin{cases} s+t & t \le f(s) \\ s+f(s) & t \ge f(s) \end{cases}$$

Then H is a coarse homotopy between $id_{\mathbb{R}_+}$, $id_{\mathbb{R}_+} + f$ by lemma 2.2.7 and corollary 2.2.8(1). Similarly we define a map $H^{'}\colon I_{id_{\mathbb{R}_+}}\mathbb{R}_+ \to \mathbb{R}$ by

$$H^{'}(s,t) = \begin{cases} t+f(s) & t \le s \\ s+f(s) & t \ge s \end{cases}$$

Then $H^{'}$ is a coarse homotopy between f, $id_{\mathbb{R}_+} + f$ again by lemma 2.2.7 (2) and corollary 2.2.8(2). By theorem 2.2.5, we have $id_{\mathbb{R}_+}$ is coarsely homotopic to f. This shows that $\pi_0^{Crs}(\mathbb{R}_+) = \{1\}$. □

Proposition 2.3.4 **(1)** *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a controlled map, then f is close to some Lipschitz controlled map.*

(2) *Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a coarse map, then f is close to some Lipschitz coarse map.*

Proof : We prove (2), and (1) is identical. Since $\mathbb{R}_+$, $\mathbb{R}$ are geodesic, then by proposition 1.3.9, the map f is asymptotically Libschitz, and so the map $f|_{\mathbb{Z}_+}$ is coarse and asymptotically Lipschitz, that is, we have two constants $A, B \ge 0$ such that

$$\mid f|_{\mathbb{Z}_+}(x) - f|_{\mathbb{Z}_+}(y) \mid \le A \mid x-y \mid + B, \quad x, \quad y \in \quad \mathbb{Z}_+.$$

If $x \ne y$, $\mid x-y \mid \ge 1$ so $B \le B \mid x-y \mid$, then

$$\mid f|_{\mathbb{Z}_+}(x) - f|_{\mathbb{Z}_+}(y) \mid \le A \mid x-y \mid + B \mid x-y \mid = (A+B) \mid x-y \mid$$

which implies that $f|_{\mathbb{Z}_+}$ is a Lipschitz map.

Now we extend the map $f|_{\mathbb{Z}_+}$ piecewise linearly to the map $g\colon \mathbb{R}_+ \to \mathbb{R}$ as follows

$$g(k+t) = tf|_{\mathbb{Z}_+}(k+1) + (1-t)f|_{\mathbb{Z}_+}(k), \quad k \in \mathbb{Z}_+, \quad t \in [0,1]$$

We show that the map g is a coarse, Lipschitz map, and close to the map f.

- The map g is Lipschitz.

  Let $x, y \in \mathbb{R}_+$. Then we have $k_1,\ k_2 \in \mathbb{Z}_+$ such that $x = k_1 + t$, $y = k_2 + s,\ t,\ s \in [0,1]$ then,

  $$\begin{aligned} \mid g(x) - g(y) \mid &= \mid g(k_1+t) - g(k_2+s) \mid \\ &= \mid tf|_{\mathbb{Z}_+}(k_1+1) + (1-t)f|_{\mathbb{Z}_+}(k_1) - (sf|_{\mathbb{Z}_+}(k_2+1) + (1-s)f|_{\mathbb{Z}_+}(k_2)) \mid \\ &= \mid tf|_{\mathbb{Z}_+}(k_1+1) - sf|_{\mathbb{Z}_+}(k_2+1) + (1-t)f|_{\mathbb{Z}_+}(k_1) - (1-s)f|_{\mathbb{Z}_+}(k_2) \mid \\ &= \mid tf|_{\mathbb{Z}_+}(k_1+1) - tf|_{\mathbb{Z}_+}(k_2+1) + tf|_{\mathbb{Z}_+}(k_2+1) - sf|_{\mathbb{Z}_+}(k_2+1) \\ &+ (1-t)f|_{\mathbb{Z}_+}(k_1) - (1-t)f|_{\mathbb{Z}_+}(k_2) + (1-t)f|_{\mathbb{Z}_+}(k_2) - (1-s)f|_{\mathbb{Z}_+}(k_2) \mid \\ &\leq \mid t \mid\mid f|_{\mathbb{Z}_+}(k_1+1) - f|_{\mathbb{Z}_+}(k_2+1) \mid \\ &+ \mid 1-t \mid\mid f|_{\mathbb{Z}_+}(k_1) - f|_{\mathbb{Z}_+}(k_2) \mid \\ &+ \mid t-s \mid\mid f|_{\mathbb{Z}_+}(k_2+1) - f|_{\mathbb{Z}_+}(k_2) \mid \end{aligned}$$

  But $f|_{\mathbb{Z}_+}$ is a Lipschtiz map, so we have $A, B > 0$ such that $\mid f|_{\mathbb{Z}_+}(k_1) - f|_{\mathbb{Z}_+}(k_2) \mid \leq A \mid k_1 - k_2 \mid$, and $\mid f|_{\mathbb{Z}_+}(k_2+1) - f|_{\mathbb{Z}_+}(k_2) \mid \leq B$. So

  $$\begin{aligned} \mid g(x) - g(y) \mid &\leq A \mid t \mid\mid k_1 + 1 - (k_2+1) \mid + A \mid 1-t \mid\mid k_1 - k_2 \mid + B \mid t-s \mid \\ &\leq 2A \mid k_1 - k_2 \mid + B \leq 2A \mid k_1 - k_2 \mid + B \mid k_1 - k_2 \mid \\ &= (2A+B) \mid k_1 - k_2 \mid = (2A+B) \mid x - y \mid \end{aligned}$$

  Hence g is Lipschitz and therefore a continuous map.

- The map g is close to the coarse map f.

  Let $x \in \mathbb{R}_+$ and we write $x = k + t$ where $k \in \mathbb{Z}_+,\ t \in [0,1]$, then

  $$\begin{aligned} \mid g(x) - f(x) \mid &= \mid g(k+t) - f(k+t) \mid \\ &= \mid tf|_{\mathbb{Z}_+}(k+1) + (1-t)f|_{\mathbb{Z}_+}(k) - f(k+t) \mid \\ &\leq \mid t \mid\mid f|_{\mathbb{Z}_+}(k+1) - f|_{\mathbb{Z}_+}(k) \mid + \mid f|_{\mathbb{Z}_+}(k) - f(k+t) \mid \\ &\leq \mid f(k+1) - f(k) \mid + \mid f(k) - f(k+t) \mid \end{aligned}$$

  Since f is asymptotically Lipschitz, so we have $A, B \geq 0$ such that

  $$\begin{aligned} &\mid g(k+t) - f(k+t) \mid \\ &\qquad \leq A \mid k+1-k \mid + B + A \mid k-k+t \mid + B \\ &\qquad \leq 2A + 2B \end{aligned}$$

  Hence the maps f and g are close.

- The map g is a coarse map.

  Straightforward from the first part of proposition 2.2.3. □

Proposition 2.3.5 $\pi_0^{Crs}(\mathbb{R})$ *has two elements.*

Proof : Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a coarse map, then f is close to the coarse map g defined in proposition 2.3.4. This implies that $g^{-1}[0]$ is bounded, so $g^{-1}[0] \subseteq [0,a]$ for some $a > 0$. Now since g is continuous, and by the intermediate value theorem either:

$$g(x) > 0 \ \text{ for all } \ x > a, \quad \text{or} \quad g(x) < 0 \ \text{ for all } \ x > a$$

Then $g\mid_{(a,\infty)}$ is never zero, and since g is continuous which means that $g\mid_{(a,\infty)}$ is always positive or always negative.

Now we need to show that g is coarsely homotopic to $r\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $r(x) = x$ or g is coarsely homotopic to $s\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $s(x) = -x$.

Without loss of generality, let g be always positive. In other words $g(x) > 0$ for all $x > 0$.

First, we show that $r+g$ is a coarse map which so by definition of generalized ray, and it can be shown also as follows.

Let $R > 0$ such that $\mid x-y \mid < R$, then there is $S > 0$ such that

$$\begin{aligned}\mid (r+g)(x) - (r+g)(y) \mid &= \mid r(x) + g(x) - r(y) - g(x) \mid \\ &\leq \mid x-y \mid + \mid g(x) - g(y) \mid \\ &< R + S\end{aligned}$$

Now let $B \subseteq \mathbb{R}_+$ be a bounded set, then we can choose $a > 0$ such that $B \subseteq [0,a]$. Hence

$$(r+g)^{-1}(B) = \{x \in \mathbb{R} : r + g(x) \leq a\} \subseteq \{x \in \mathbb{R} : g(x) \leq a\} = g^{-1}[0,a]$$

So the inverse image $(r+g)^{-1}(B)$ is a bounded set. Therefore $r+g$ is a coarse map.

Now define a map $H\colon I_p\mathbb{R}_+ \to \mathbb{R}$ by

$$H(s,t) = \begin{cases} t + r(s) & t \leq p(s) \\ r(s) + g(s) & t \geq p(s) \end{cases}$$

where $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the identity map. Then H is a coarse homotopy between r, $r+g$ by lemma 2.2.7 and corollary 2.2.8(1). Similarly we define a map $H^{'}\colon I_p\mathbb{R}_+ \to \mathbb{R}$ by

$$H^{'}(s,t) = \begin{cases} t + g(s) & t \leq p(s) \\ r(s) + g(s) & t \geq p(s) \end{cases}$$

Then $H^{'}$ is a coarse homotopy between g, $r+g$ again by lemma 2.2.7 and corollary 2.2.8(1).

By theorem 2.2.5, we have r is coarsely homotopic to g, and hence we have r is coarsely homotopic to f.

Now let $g(x) < 0$ for all $x > 0$. Similarly we show that g is coarsely homotopic to s such that $s(x) = -x$ where $x \in \mathbb{R}_+$. But g is close to f, and so they are coarsely homotopic. Therefore f is coarsely homotopic to r or s, and hence $\pi_0^{Crs}(\mathbb{R})$ has only two elements. □

Proposition 2.3.6 *$\pi_0^{Crs}(\mathbb{R}^2)$ has one element.*

To prove this proposition we need the following propositions.

Proposition 2.3.7 *Let m, $n \in \mathbb{N}$, and let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be a coarse map. Then the map f is close to some coarse Lipschitz map.*

Proof : This is a higher dimension case of proposition 2.3.4, and the proof is similar. □

Proposition 2.3.8 *Let $f\colon \mathbb{R}_+ \to \mathbb{R}^2$ be a coarse map. Then the map f is coarsely homotopic to the map $i\colon \mathbb{R}_+ \to \mathbb{R}^2$ defined by $i(s) = (s, 0)$.*

Proof : By proposition 2.3.7, we can assume without loss of generality that f is Lipschitz. In polar coordinates, we can write f as follows

$$f(s) = (r(s), \theta(s))$$

where the map $\theta\colon \mathbb{R}_+ \to \mathbb{R}$ is bounded.

In polar coordinates the map i is defined by the formula $i(s) = (s, 0)$. It is not hard to prove that i is coarsely homotopic to the map $j\colon \mathbb{R}_+ \to \mathbb{R}^2$ defined by $j(s) = (r(s), 0)$.

Define a map $H\colon I_p\mathbb{R}_+ \to \mathbb{R}^2$ by writing

$$H(s,t) = \begin{cases} (r(s), \frac{t}{r(s)+1}) & \text{t} \leq (r(s)+1)\theta(s) \\ (r(s), \theta(s)) & \text{t} \geq (r(s)+1)\theta(s) \end{cases}$$

where p is the identity on $\mathbb{R}_+$. The map $s \mapsto (r(s)+1)\theta(s)$ is a controlled map, so H is a coarse homotopy between f and j and since coarse homotopy is transitive, so f is coarsely homotopic to i. □

Proving all the above propositions proves proposition 2.3.6.

Example 2.3.9 Let B be a bounded coarse space. There are no coarse maps $R \to B$ so $\pi_0^{Crs}(B) = \emptyset$.

Theorem 2.3.10 *For any coarse space X, $\pi_0^{Crd}(X)$ has one element.*

Proof : Let $f\colon \mathbb{R}_+ \to X$ be a controlled map (not necessarily coarse). Let $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the map $p(x) = x$ for all $x \in \mathbb{R}_+$. Define a map $H\colon I_p\mathbb{R}_+ \to X$ by

$$H(x,t) = \begin{cases} f(t) & t \leq x \\ f(x) & t \geq x \end{cases}$$

for all $x, t \in \mathbb{R}_+$, $0 \leq t \leq p(x)+1$. Then H is a controlled map since f is so, and $H(x,0) = f(0)$ for all $x \in \mathbb{R}_+$ which clearly shows that f is controlledly homotopic to a constant map. Any constant maps are close and so by example 2.2.4 they are controlledly homotopic. □

The above theorem tells us that the set of controlled path components of any space X has one element.

Proposition 2.3.11 *Let $f, g\colon X \to Y$ be coarsely homotopic maps. Then the functorial induced maps $f_*, g_*\colon \pi_0^{Crs}(X) \to \pi_0^{Crs}(Y)$ are equal .*

Proof : Suppose that $f, g\colon X \to Y$ are coarsely homotopic maps, then there is a coarse map $H\colon I_pX \to Y$ such that $H(x,0) = f(x)$ and $H(x, p(x)+1) = g(x)$ for some controlled map $p\colon X \to R$.

Define maps $f_*, g_*\colon \pi_0^{Crs}(X) \to \pi_0^{Crs}(Y)$ by $f_*[h] = [f \circ h]$ and $g_*[h] = [g \circ h]$ where $h\colon R \to X$ is a coarse map, then $f \circ h$ and $g \circ h$ are coarse maps.

Define a map $F\colon I_{p\circ h}R \to Y$ by $F(s,t) = H(h(s),t)$ then F is a coarse map and

$$F(s,0) = f(h(s)), \text{ and } F(s, p(h(s))+1) = g(h(s))$$

Hence $f \circ h$ is coarsely homotopic to $g \circ h$ as required. □

Corollary 2.3.12 *Let $f\colon X \to Y$ be a coarse homotopy equivalence, then*

$$\mid \pi_0^{Crs}(X) \mid = \mid \pi_0^{Crs}(Y) \mid$$

Proof : It is straightforward from the above proposition. □

§ 2.4 Coarse Homotopy Groups

Here we define coarse homotopy groups, and in order to do that we need a notion of basepoint.

Definition 2.4.1 Let X be a coarse space in the category of coarse maps, R a generalised ray. A *basepoint* for X is a coarse map $i_X\colon R \to X$ such that $p_X \circ i_X = id_R$ where $p_X\colon X \to R$ is a controlled map. A coarse space equipped with a basepoint is termed *pointed coarse space*.

If Y is another coarse space with basepoint in the category Crs, then a coarse map $f\colon X \to Y$ is termed *pointed coarse map* if it induced by a basepoint for the space X. That is, $f \circ i_X = i_Y$

We term the category of pointed coarse spaces and pointed coarse maps the *category of pointed coarse maps* . It has an initial object, namely the space R, and we denote this category by Pcrs .

Similarly we define the *category of pointed controlled maps* , and we denote it by Pcrd .

Example 2.4.2 Let B be a bounded coarse space. Then there are no coarse maps $i_B : R \to B$, so B has no coarse basepoint.

Definition 2.4.3 Let X and Y be coarse pointed spaces. Then we write $[X,Y]_R^{Crs}$ to denote the set of coarse homotopy classes of pointed coarse maps from X to Y relative to R. That is, all coarse homotopies are pointed.

The set is equipped with a base element that is defined to be the relative coarse homotopy class of the map

$$X \xrightarrow{p_X} R \xrightarrow{i_Y} Y.$$

And similarly $[X,Y]_R^{Crd}$ is the set of controlled homotopy classes of pointed controlled maps from X to Y relative to R.

Definition 2.4.4 Let X be a coarse pointed space. Let $n > 0$. Then we define *the n-coarse homotopy group* with respect to $R \sqcup R$ to be the set of coarse homotopy classes of pointed coarse maps

$$\pi_n^{Pcrs}(X,R) = [(R \sqcup R)^{n+1}, X]_R^{Crs}$$

where $(R \sqcup R)^{n+1}$ the cone of the n-sphere S^n.

And for controlled case, we have the following definition.

Definition 2.4.5 Let X be a coarse pointed space. Let $n > 0$. Then we define *the n-controlled homotopy group* with respect to $R \sqcup R$ to be the set of controlled homotopy classes of pointed controlled maps

$$\pi_n^{Pcrd}(X,R) = [(R \sqcup R)^{n+1}, X]_R^{Crd}$$

Example 2.4.6 Let B be a bounded coarse space. There are no coarse maps $(R \sqcup R)^n \to B$ so $\pi_n^{Pcrs}(B) = \emptyset$ for any $n > 0$.

Proposition 2.4.7 *Let $n \geq 1$. Then the set $\pi_n^{Pcrd}(X)$ is a group. For $n \geq 2$ then the set $\pi_n^{Pcrd}(X)$ is an abelain group.*

Proof : Straightforward by proposition (3.8) in [10]. □

Similarly, we prove the following proposition.

Proposition 2.4.8 *Let $n \geq 1$. Then the set $\pi_n^{Pcrs}(X)$ is a group. For $n \geq 2$ then the set $\pi_n^{Pcrs}(X)$ is an abelain group.* □

The following result is proved in [10].

Theorem 2.4.9 *The coarse homotopy groups* $\pi_k^{Pcrs}(\mathbb{R}^{n+1})$ *is isomorphic to the basic homotopy groups* $\pi_k(S^n)$*.* □

Example 2.4.10 $\pi_1^{Pcrs}(\mathbb{R})$ is isomorphic to $\pi_1(S^0)$, but $\pi_1(S^0) = \{0\}$ so $\pi_1^{Pcrs}(\mathbb{R})$ is isomorphic to $\{0\}$.

$\pi_1^{Pcrs}(\mathbb{R}^2)$ is isomorphic to $\pi_1(S^1)$, but $\pi_1(S^1) = \mathbb{Z}$ so $\pi_1^{Pcrs}(\mathbb{R})$ is isomorphic to $\mathbb{Z}$.

Proposition 2.4.11 *Let* $f\colon X \to Y$ *be a coarse homotopy equivalence map. Then the functorial induced map* $f_*\colon \pi_n^{Pcrs}(X) \to \pi_n^{Pcrs}(Y)$ *is a bijection when* $n = 0$ *and isomorphism when* $n > 0$*.*

Proof : Similar argument to proposition 2.3.11. □

Theorem 2.4.12 *For any coarse space* X*,* $\pi_n^{Pcrd}(X)$ *has one element.*

Proof : Similar argument to theorem 2.3.10. □

The above theorem tells us that the controlled category is trivial from a homotopy point of view.

Chapter 3

Coarse Examples of Baues Cofibration Categories

In this chapter we define two examples of categories in coarse geometry and try to show they have a structure of Baues cofibration category to be able later to have a coarse version of axiomatic homotopy theory.

§ 3.1 The Controlled Cofibration Category

In this section, we try to show that the category of non-unital coarse spaces and controlled maps Crd satisfies the axioms of a Baues cofibration category. However, as mentioned in chapter 2 the homotopy groups always have one element, so the situation here turns out to be quite trivial. Nonetheless, this category indicates techniques that are useful in more significant examples.

First we recall the definition of Baues cofibration category which is found in [1], [2].

Definition 3.1.1 A *cofibration category* is a category $\mathcal{C}$ with two classes of morphisms, *cof* of cofibrations and *w.e.* of weak equivalences. These are to satisfy:

C1 Composition axiom: Isomorphisms are both cofibrations and weak equivalences. If f and g are in $\mathcal{C}$ such that gf is defined and if two of the three morphisms f, g, and gf are weak equivalences, then so is the third. The composite of cofibrations is a cofibration.

C2 Push out axiom: For a cofibration $i : A \hookrightarrow X$ and a morphism $f \colon A \to Y$ there exists the push out in $\mathcal{C}$

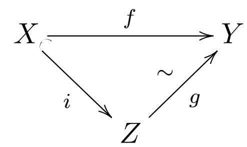
$$\begin{array}{ccc} X & \xrightarrow{f'} & X \cup_A Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ A & \xrightarrow{f} & Y \end{array}$$

and $i^{'}$ is a cofibration. Moreover:

(a) if f is a weak equivalence, so is $f^{'}$,

(b) if i is a weak equivalence, so is $i^{'}$.

C3 Factorization axiom: For any map $f\colon X \to Y$ in $\mathcal{C}$ there exists a commutative diagram



where i is a cofibration and g is a weak equivalence.

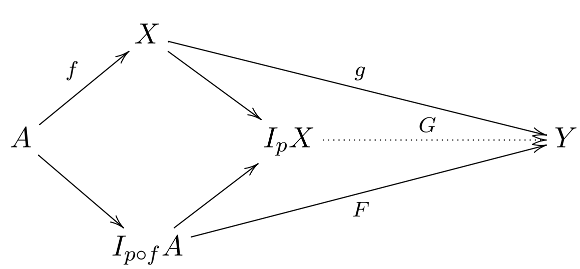
Before stating the last axiom. We need to introduce the following notation which comes from [1].

A map in a cofibration category $\mathcal{C}$ is called a *trivial cofibration* if it is both a weak equivalence and a cofibration. An object S in $\mathcal{C}$ is called a *fibrant model* if each trivial cofibration $i\colon S \to Q$ in $\mathcal{C}$ admits a map $r\colon Q \to S$, where $ri = 1_S$. We call r a *retraction* of i .

C4 Axiom on fibrant models: For each object X in $\mathcal{C}$ there is a trivial cofibration $X \overset{\sim}{\hookrightarrow} SX$ where SX is a model fibrant in $\mathcal{C}$. We call $X \overset{\sim}{\hookrightarrow} SX$ a *fibrant model of* X.

Definition 3.1.2 Let A, X be coarse spaces. A controlled map $f\colon A \to X$ is called a *controlled cofibration* if given a controlled map $g\colon X \to Y$, and a controlled homotopy $F\colon I_{p\circ f}A \to Y$ for some given controlled map $p\colon X \to R$ such that $g(f(a)) = F(a,0)$ for all $a \in A$, $t \in R$, we can find a controlled homotopy $G\colon I_pX \to Y$ such that $g(x) = G(x,0)$ for all $x \in X$, $G(f(a),t) = F(a,t)$ for all $a \in A$, $t \in R$.

This definition is illustrated by the following commutative diagram:



Lemma 3.1.3 *Let X be a coarse space, $p\colon X \to R$ be some controlled map. Let $A \hookrightarrow X$ be an inclusion map in the controlled category Crd, then the inclusion map $j\colon X\times\{0\} \to I_{p_A}A \cup (X\times\{0\})$ is a controlled homotopy equivalence, where $p_A = p\mid_A$.*

Proof : Define a map $\pi\colon I_{p_A}A \cup (X \times \{0\}) \to X \times \{0\}$ by $\pi(x,t) = (x,0)$ for all $x \in A$ or $t = 0$. Then π is a controlled map, and $\pi \circ j = 1_{X\times\{0\}}$. Define a controlled map $p_0\colon X \times \{0\} \to R$ by $p_0(x,0) = p(x)$ for all $x \in X$, so we can define a coarse homotopy (which is also a controlled homotopy) $H\colon I_{p_0\circ\pi}(I_{p_A}A \cup (X \times \{0\})) \to I_{p_A}A \cup (X \times \{0\})$ between the map $j \circ \pi$ and the identity on $I_{p_A}A \cup (X \times \{0\})$ where $j \circ \pi(x,t) = (x,0)$ for all $x \in X$ as follows;

$$H((x,t),s) = \begin{cases} (x, s+t) & s \le p_0 \circ \pi(x,t) \\ (x,0) & s > p_0 \circ \pi(x,t) \end{cases}$$

The maps π, j and H are coarse (and controlled) maps so H defines a coarse homotopy (and a controlled homotopy) between $j \circ \pi$ and the identity on the space $I_{p_A}A \cup (X \times \{0\})$. $\square$

Lemma 3.1.4 *Let X be a coarse space, $p\colon X \to R$ be some controlled map. Let $i : A \hookrightarrow X$ be an inclusion, where $p_A = p\,|_A$. Write*

$$(I_{p_A}A) \cup (X \times \{0\}) = \{(x,t) \in I_pX : x \in A \text{ or } t = 0\}$$

Let $j : (I_{p_A}A) \cup (X \times \{0\}) \hookrightarrow I_pX$ be the inclusion. Then the following are equivalent:

(1) *$i : A \hookrightarrow X$ is a controlled cofibration.*

(2) *Suppose we have a controlled map $f\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$. Then there exists a controlled homotopy $G\colon I_pX \to Y$ such that $G \circ j = f$.*

(3) *There is a controlled homotopy $r\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.*

Proof : First note that since $p_A = p|_X$ and i is the inclusion, then $p_A(a) = p(i(a))$ and p_A is a controlled map.

((1) $\Rightarrow$ (2)) Let $f\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a controlled map. Define the map $f_0\colon X \times \{0\} \to Y$ by $f_0 = f\,|_{X\times\{0\}}$, and $H\colon I_{p_A}A \to Y$ such that $H = f\,|_{I_{p_A}A}$. Then f_0, H are controlled maps, and since i is a controlled cofibration, there exists a controlled homotopy $G\colon I_pX \to Y$ such that $G(i(a),t) = H(a,t)$ for all $a \in A$, $t \in R$ and so then $G \circ j(x,t) = f(x,t)$.

((2) $\Rightarrow$ (3)) Suppose that the map $I\colon (I_{p_A}A) \cup (X \times \{0\}) \to (I_{p_A}A) \cup (X \times \{0\})$ is the identity, so it is a controlled map. By (2) there exists a controlled homotopy $r\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r \circ j = I$, that is, $r(j(x,t)) = r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.

((3) $\Rightarrow$ (2)) Suppose we have a controlled homotopy $r\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.

Let $f\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a controlled map, so we can define a controlled homotopy $G\colon I_pX \to Y$ by writing $G(x,t) = f(r(x,t))$. Then $G \circ j(x,t) = G(x,t) = f(r(x,t)) = f(x,t)$, so $G \circ j = f$.

((2) *and* (3) $\Rightarrow$ (1)) Let $F\colon I_{p_A}A \to Y$ be a controlled homotopy and $g\colon X \to Y$ be a controlled map such that $F(a,0) = g(i(a))$ for all $a \in A$. By (3) we have a controlled homotopy $r\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.

Define a map $f_0\colon X \times \{0\} \to Y$ by $f_0(x,0) = g(x)$. Then f_0 is a controlled map. Let $f\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a map such that $f|_{X\times\{0\}} = f_0$ and $f|_{I_{p_A}A} = F$, we need to show that f is a controlled map. Since the inclusion $X \times \{0\} \hookrightarrow (I_{p_A}A) \cup (X \times \{0\})$ is a controlled homotopy equivalence by lemma 3.1.3 and since the map f_0 is a controlled map, so by lemma 2.2.14 (1), the map f is controlled as required.

Therefore by (2) there exists a controlled homotopy $G\colon I_pX \to Y$ defined by $G(x,t) = f \circ r(x,t)$, and then $G(x,0) = f_0(x,0) = g(x)$, for all $x \in X$, and $G(i(a),t)) = fr(i(a),t) = f(a,t) = F(a,t)$ for all $a \in A$, $t \in R$. $\square$

Our main aim in this section is to prove the following result.

Theorem 3.1.5 *The controlled category Crd is a Baues cofibration category. The weak equivalences are the controlled homotopy equivalences, and the cofibrations are the controlled cofibrations.*

Proving this theorem requires us to prove the following results in order to satisfy the axioms required of a Baues cofibration category.

Proposition 3.1.6 *Let $f\colon X \to Y$ be an isomorphism in the controlled category. Then f is both a controlled cofibration and controlled homotopy equivalence.*

Proof : Since f is an isomorphism, we have $g\colon Y \to X$ a controlled map such that $g \circ f = 1_X$, $f \circ g = 1_Y$. By assumption f is a controlled map, and it is a controlled homotopy equivalence.

Now let $F\colon I_{p_Y \circ f}X \to Z$ be a controlled homotopy where $p_Y\colon Y \to R$ is some controlled map, $h\colon Y \to Z$ a controlled map such that $F(x,0) = h(f(x))$ for all $x \in X$. Now by the above, we have a controlled homotopy $H\colon I_{p_Y}Y \to Y$ such that $H(y,t) = f \circ g(y) = y$.

Define a map $G\colon I_{p_Y}Y \to Z$ by $G(y,t) = F(g(y),t)$. By assumption $p_Y = p_X \circ g$, then $G(f(x),t) = F(g(f(x)),t) = F(x,t)$ for all $x \in X$, $t \in R$, and $G(y,0) = F(g(y),0) = h(f \circ g(y)) = h(y)$ for all $y \in Y$. Hence f is a controlled cofibration. $\square$

Proposition 3.1.7 *Consider two controlled maps $f\colon X \to Y$ and $g\colon Y \to Z$. If any two of the morphisms f, g and gf are controlled homotopy equivalences, then so is the third.*

Proof : Let $f\colon X \to Y$, $g\colon Y \to Z$ be two controlled maps, then $gf\colon X \to Z$ is a controlled map. Suppose that gf, f are controlled homotopy equivalences, then there exist controlled maps $h\colon Y \to X$, $h^{'}\colon Z \to X$ such that $f \circ h$ is controlledly homotopic to 1_Y, $gf \circ h^{'}$ is controlledly homotopic to 1_Z, $h \circ f$ is controlledly homotopic to 1_X, and $h^{'} \circ gf$ is controlledly homotopic to 1_X that is, $h \circ f$ is controlledly homotopic to $h^{'} \circ gf$.

We need to show that there is a controlled map $l\colon Z \to Y$ such that $l \circ g$ is controlledly homotopic to 1_Y, $g \circ l$ is controlledly homotopic to 1_Z that is, $l \circ g$ is controlledly homotopic to $f \circ h$ and $g \circ l$ is controlledly homotopic to $gf \circ h^{'}$.

First, we show that h is controlledly homotopic to $h^{'} \circ g$. We have $h^{'} \circ g \circ f$ is controlledly homotopic to $h \circ f$, and so $h^{'} \circ g \circ f \circ h$ is controlledly homotopic to $h \circ f \circ h$ which implies that h is controlledly homotopic to $h^{'} \circ g$.

Now let $l\colon Z \to Y$ be the map $f \circ h^{'}$, then $l \circ g = f \circ h^{'} \circ g$, but $f \circ h^{'} \circ g$ is controlledly homotopic to $f \circ h$ which implies that $l \circ g$ is controlledly homotopic to the identity on Y.

Similarly, we can show that $g \circ l$ is controlledly homotopic to 1_Z. □

Proposition 3.1.8 *Composition of controlled cofibrations is a controlled cofibration.*

Proof : Straightforward by the definition of controlled cofibration. □

Proposition 3.1.9 *Let X, Y be non-unital coarse spaces, and let $i\colon X \to Y$ be a controlled map that is both a controlled cofibration and a controlled homotopy equivalence. Then there is a controlled map $r\colon Y \to X$ such that $r \circ i = 1_X$.*

Proof : Let $p_Y\colon X \to R$ be some controlled map. Since i is a controlled homotopy equivalence, then there exists a controlled map $g\colon Y \to X$ such that $g \circ i$ and $i \circ g$ are controlledly homotopic to 1_X, and 1_Y respectively.

So we have a controlled homotopy $F\colon I_{p_Y \circ i}X \to X$ such that $F(x,0) = g(i(x))$, $F(x, p_Y \circ i(x)+1) = x$ for all $x \in X$. Since i is a controlled cofibration, then there exists a controlled homotopy $G\colon I_{p_Y}Y \to X$ such that $G(y,0) = g(y)$ for all $y \in Y$, and $G(i(x),t) = F(x,t)$ for all $x \in X$, $t \in R$.

Define a controlled map $r\colon Y \to X$ by the formula $r(y) = G(y, p_Y(y)+1)$. By construction we have $r \circ i(x) = G(i(x), p_Y(i(x))+1) = F(x, p_Y(i(x))+1) = x$ for all $x \in X$. □

Definition 3.1.10 Let X be a non-unital coarse space equipped with an equivalence relation $\sim$ and the *quotient map* $\pi\colon X \to X/\sim$. Then the *quotient space* $X/\sim$ is equipped with the coarse structure formed by defining the set of entourages to be the collection of subsets of sets of the form

$$\{\pi[M] : M \subseteq X \times X \ \text{ is an entourage}\}$$

We will focus on a special case of the quotient space structure.

Definition 3.1.11 Let A, X, and Y be coarse spaces. Suppose that we have controlled maps $i\colon A \to X$ and $f\colon A \to Y$. Then we define

$$X \cup_* Y = X \sqcup Y/\sim$$

where $\sim$ is the equivalence relation defined by $i(a) \sim f(a)$ for all $a \in A$.

Lemma 3.1.12 *Let $i\colon A \to X$, $f\colon A \to Y$ be controlled maps, then the quotient map $\pi\colon X \sqcup Y \to X \cup_* Y$ is a controlled map.*

Proof : By definition of quotient coarse structure, for any $M \subseteq X \sqcup Y \times X \sqcup Y$, $\pi[M] \subseteq (X \cup_* Y) \times (X \cup_* Y)$ is an entourage. □

Theorem 3.1.13 *Let X, Y, and A be coarse spaces, suppose we have controlled maps $i\colon A \to X$, $f\colon A \to Y$. Then we have a push out diagram in the controlled category.*

$$\begin{array}{ccc} X & \xrightarrow{f'} & X \cup_* Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ A & \xrightarrow{f} & Y \end{array}$$

Proof : Let $\pi\colon X \sqcup Y \to X \cup_* Y$ be the quotient map, then by the above lemma it is a controlled map. Now we can factor the maps i', f' as follows

$$Y \xrightarrow{i_Y} X \sqcup Y \xrightarrow{\pi} X \cup_* Y \qquad\qquad X \xrightarrow{i_X} X \sqcup Y \xrightarrow{\pi} X \cup_* Y$$

respectively, where i_Y, i_X are the inclusions. Since π, i_Y, and i_X are controlled maps, then i', f' are also controlled maps.

Let $g_1\colon X \to Z$, $g_2\colon Y \to Z$ be controlled maps where Z be any coarse space and such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g_1} & Z \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle g_2} \\ A & \xrightarrow{f} & Y \end{array}$$

commutes. Then we can define a map $h\colon X \cup_* Y \to Z$ by writing

$$h \circ \pi(x) = \begin{cases} g_1(x) & x \in X \\ g_2(x) & x \in Y \end{cases}$$

We need to show that h is a controlled map. Let $E \subseteq (X \cup_* Y) \times (X \cup_* Y)$ be an entourage, then we can find an entourage $M \subseteq (X \sqcup Y) \times (X \sqcup Y)$ such that $E = \pi[M]$, but $M = M_X \cup M_Y \cup B_X \times B_Y \cup B_Y' \times B_X'$, where $M_X \subseteq X \times X$, $M_Y \subseteq Y \times Y$ are entourages, and B_X, $B_X' \subseteq X$ and B_Y, $B_Y' \subseteq Y$ are bounded subsets.

So then $h \times h(\pi[M]) = h \times h[\pi[M_X] \cup \pi[M_Y] \cup \pi[B_X \times B_Y] \cup \pi[B_Y' \times B_X']] = h \times h[\pi[M_X]] \cup h \times h[\pi[M_Y]] \cup h \times h(\pi[B_X \times B_Y]) \cup h \times h(\pi[B_Y' \times B_X']) =$

$g_1 \times g_1[M_X] \cup g_2 \times g_2[M_Y] \cup g_1 \times g_2[B_X \times B_Y] \cup g_2 \times g_1[B_Y^{'} \times B_X^{'}]$. Since g_1 and g_2 are controlled maps so then $h \times h(\pi[M])$ is an entourage. Therefore h is a controlled map.

Now $h \circ f^{'}(x) = h(\pi(i_X(x))) = h(\pi(x)) = g_1(x)$ for all $x \in X$, and

$h \circ i^{'}(y) = h(\pi(i_Y(y))) = h(\pi(y)) = g_2(y)$ for all $y \in Y$.

To check uniqueness, consider another controlled map $l\colon X \cup_* Y \to Z$ such that $l \circ f^{'} = g_1$, $l \circ i^{'} = g_2$, but $g_1 = h \circ f^{'}$ and $g_2 = h \circ i^{'}$ which implies that $h = l$. □

Proposition 3.1.14 *Let $i : A \hookrightarrow X$ be a controlled cofibration, and $f\colon A \to Y$ be a controlled map. Then we have a push out diagram in the controlled category.*

$$\begin{array}{ccc} X & \xrightarrow{f^{'}} & X \cup_* Y \\ {\scriptstyle i}\big\uparrow & & \big\uparrow{\scriptstyle i^{'}} \\ A & \xrightarrow{f} & Y \end{array}$$

Further, the map $i^{'}$ is a controlled cofibration.

Proof : The above theorem proves the first part. We need to show that $i^{'}$ is a controlled cofibration.

Suppose that $q\colon X \cup_* Y \to R$ is some controlled map, $F\colon I_{q \circ i^{'}} Y \to Z$ is a controlled homotopy, and $g\colon X \cup_* Y \to Z$ is a controlled map such that $F(y, 0) = g(i^{'}(y))$ for all $y \in Y$.

Define a map $G\colon I_{q \circ i^{'} \circ f} A \to I_{q \circ i^{'}} Y$ by $G(a, t) = (f(a), t)$ for all $a \in A$, then G is a controlled map. Since $q \circ i^{'} \circ f = q \circ f^{'} \circ i$, then the cylinders $I_{q \circ i^{'} \circ f} A$ and $I_{q \circ f^{'} \circ i} A$ are coarsely equivalent. The map $F \circ G\colon I_{q \circ f^{'} \circ i} A \to Z$ is a controlled homotopy such that $F \circ G(a, 0) = F(f(a), 0) = g(i^{'}(f(a))) = g(f^{'}(i(a)))$ for all $a \in A$ by the above push out diagram.

By the universal property we have $g^{'} : X \to Z$ defined by $g^{'}(x) = g(f^{'}(x))$ for all $x \in X$, then $g^{'}$ is a controlled map and $F \circ G(a, 0) = g^{'}(i(a))$ for all $a \in A$.

Since the map i is a controlled cofibration, so we have a controlled homotopy $H\colon I_{q \circ f^{'}} X \to Z$ such that $H(x, 0) = g^{'}(x)$ for all $x \in X$, $H(i(a), t) = F \circ G(a, t)$ for all $a \in A$, $t \in R$. We can define a new controlled homotopy $H^{'} : I_q(X \cup_* Y) \to Z$ by writing

$$H^{'}(f^{'}(x), t) = H(x, t), \qquad H^{'}(i^{'}(y), t) = F(y, t).$$

Let $w \in X \cup_* Y$ such that $w = f^{'}(x)$ or $w = i^{'}(y)$ $x \in X$, $y \in Y$, then

$H^{'}(w, 0) = H^{'}(f^{'}(x), 0)$ when $w = f^{'}(x)$, and $H^{'}(i^{'}(x), 0))$ when $w = i^{'}(x)$.

This is equivalent to saying

$$H(x, 0) = g(f^{'}(x)), \text{ and } F(y, 0) = g(i^{'}(y)).$$

Then by the above $H^{'}(w, 0) = g(w)$ for all $w \in X \cup_* Y$, and $H^{'}(i^{'}(y), t) = F(y, t)$ for all $y \in Y$. Hence $i^{'}$ is a controlled cofibration. □

Proposition 3.1.15 *In the following push out diagram in the controlled category.*

$$\begin{array}{ccc} X & \xrightarrow{f'} & X \cup_* Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ A & \xrightarrow{f} & Y \end{array}$$

with i is a controlled homotopy equivalence, then the map i' is a a controlled homotopy equivalence.

Proof : Suppose that i is a controlled homotopy equivalence, then there exists a controlled map $h\colon X \to A$ such that $i \circ h$ is controlledly homotopy to id_X, and $h \circ i$ is controlledly homotopy to id_A. The map i' is defined by $i'(y) = \pi(y)$ for all $y \in Y$.

Now define a map $r\colon X \cup_* Y \to Y$ by

$$r(\pi(y)) = y, \quad y \in Y, \quad \text{and,} \quad r(\pi(x)) = f \circ h$$

A similar argument in theorem 3.1.13 shows that the map r is a controlled map.

The composites $r \circ i'(y) = r(\pi(y)) = y$ for all $y \in Y$, $i' \circ r(\pi(y)) = i'(y) = \pi(y)$, and $i' \circ r(\pi(x)) = i'(f \circ h(x))$, by the push out diagram this implies that $i' \circ f \circ h = f' \circ i \circ h$, but $i \circ h$ is controlledly homotopic to the identity id_X.

By theorem 2.2.11, we have $f' \circ i \circ h$ is controlledly homotopic to f', and therefore $i' \circ r$ is controlledly homotopic to $id_{X \cup_* Y}$. Hence i' is a controlled homotopy equivalence. □

Definition 3.1.16 Let $f\colon X \to Y$ be a controlled map, and $p_X\colon X \to R$ be a controlled map. Then we define *the mapping cylinder* of f, C_f to be the push out $I_{p_X} X \cup_* Y$.

Proposition 3.1.17 *We have a controlled cofibration $i\colon X \to C_f$ and a controlled homotopy equivalence $r\colon C_f \to Y$ such that $f = r \circ i$.*

Proof : Let $\pi\colon I_{p_X} X \sqcup Y \to C_f$ be the quotient map. Then we define the maps $i\colon X \to C_f$, $r\colon C_f \to Y$ by

$$i(x) = \pi(x, 0)$$

$$r(\pi(y)) = y, \quad y \in Y \;\text{ and }\; r(\pi(x,t)) = f(x), \quad x \in X, \quad t \in R$$

Since π is a controlled map, then i is so, and by similar argument in theorem 3.1.13 we show that r is a controlled map.

Define a map $s\colon Y \to C_f$ by $s(y) = \pi \circ i_Y(y)$, for all $y \in Y$, then s is a controlled map, and by proposition 2.2.12, the map $X \hookrightarrow I_{p_X} X$ is a controlled homotopy equivalence. By the push out diagram and proposition 3.1.15, we have the map s a controlled homotopy equivalence. Therefore r is also a controlled homotopy equivalence, and that $f = r \circ i$.

Now we need to prove that $i\colon X \to C_f$ is a controlled cofibration. First, i is a controlled map since it is a composite of two controlled maps. Let $q\colon C_f \to R$ be some controlled map. Suppose we are given a controlled homotopy $F\colon I_{q\circ i}X \to Z$ and a controlled map $g\colon C_f \to Z$ such that $F(x,0) = g(i(x))$ for all $x \in X$.

We can define a map $G\colon I_q C_f \to Z$ by writing $G(\pi(y),t) = g(\pi(y))$ for all $y \in Y$, and

$$G(\pi(x,s),t) = \begin{cases} g(\pi(x, s - \frac{t}{2})) & 0 \le s \le (q(i(x))+1)/2, \quad t \le 2s \\ F(x, t-2s) & 0 \le s \le (q(i(x))+1)/2, \quad t \ge 2s \\ g(\pi(x, q(i(x)) + 1 - s - \frac{t}{2})) & (q(i(x))+1)/2 \le s \le q(i(x))+1, \\ & t \le 2(q(i(x))+1) - 2s \\ F(x, s - q(i(x)) + 1 + \frac{t}{2}) & (q(i(x))+1)/2 \le s \le q(i(x))+1, \\ & t \ge 2(q(i(x))+1) - 2s \end{cases}$$

The maps $(x,s,t) \mapsto g(\pi(x, s-\frac{t}{2}))$, $(x,s,t) \mapsto g(\pi(x, q(i(x)) + 1 - s - \frac{t}{2}))$, $(x,s,t) \mapsto F(x,t-2s)$, $(x,s,t) \mapsto F(x, s - q(i(x)) + 1 + \frac{t}{2})$ are all controlled. Using the same argument as in lemma 2.2.7 (1), the set

$$\{(x,s,t) : 0 \le s \le (q(i(x))+1)/2\} \cup \{(x,s,t) : (q(i(x))+1)/2 \le s \le q(i(x))+1\}$$

is coarsely excisive decomposition.

Now since the maps $g \circ \pi$ and F are both controlled maps on the sets $\{(x,s,t) : 0 \le s \le (q(i(x))+1)/2\}$ and $\{(x,s,t) : (q(i(x))+1)/2 \le s \le q(i(x))+1\}$, so by corollary 2.2.8, the map G is controlled on the set

$$\{(x,s,t) : 0 \le s \le (q(i(x))+1)/2\} \cup \{(x,s,t) : (q(i(x))+1)/2 \le s \le q(i(x))+1\}.$$

It is clear that $F(x,0) = g(\pi(x,0))$ when $t = 2s$ and $t = 2(q(i(x))+1) - 2s$. Hence we have that G is a controlled homotopy as required. □

Using the above results, we can prove theorem 3.1.5 as follows;

Proof of Theorem 3.1.5: Axiom (C1) is proposition 3.1.6, proposition 3.1.7, and proposition 3.1.8. Axiom (C2) is proposition 3.1.9, axiom (C3) is proposition 3.1.14, and the last axiom (C4) is just proposition 3.1.17. □

Proposition 3.1.18 *Let $f\colon A \to Y$ be a controlled homotopy equivalence in the controlled cofibration category. Then in the push out diagram*

$$\begin{array}{ccc} X & \xrightarrow{f'} & X \cup_* Y \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ A & \xrightarrow{f} & Y \end{array}$$

the map $f'\colon X \to X \cup_A Y$ is a controlled homotopy equivalence.

Proof : Suppose that f is a controlled homotopy equivalence, then there is a controlled map $g\colon Y \to A$ such that $g \circ f$ is controlledly homotopic to the identity id_A and $f \circ g$ is controlledly homotopic to the identity id_Y. Let $h\colon X \cup_* Y \to X$ be a map defined by

$$h(f^{'}(x)) = x, \quad \text{for all} \quad x \in X, \quad \text{and}$$

$$h(i^{'}(y)) = i \circ g, \quad \text{for all} \quad y \in Y$$

A similar argument in theorem 3.1.13 shows that h is a controlled map. By definition of h, the composite $h \circ f^{'}$ is controlledly homotopic to id_X. Now the composites

$$f^{'} \circ h(f^{'}(x)) = f^{'}(x), \quad \text{and} \quad f^{'} \circ h(i^{'}(y)) = f^{'} \circ i \circ g(y) = i^{'} \circ f \circ g(y),$$

but by assumption $f \circ g$ is controlledly homotopic to the identity id_Y, so it is enough to show that $i^{'} \circ f \circ g$ is controlledly homotopic to $i^{'}$ which is so by theorem 2.2.11. □

Corollary 3.1.19 *The inclusions $i_0, i_1 : X \hookrightarrow I_pX$ are controlled cofibrations where $p\colon X \to R$ is some controlled map.*

Proof : The cylinder I_pX is the mapping cylinder of the identity map $X \to X$ where $p\colon X \to R$ is some controlled map. It follows by proposition 3.1.17 that the inclusion map i_0 is a controlled cofibration. The map $\kappa\colon I_pX \to I_pX$ defined by the formula

$$\kappa(x, t) = (x, p(x) + 1 - t)$$

is an isomorphism, and the map i_1 is the composition $\kappa \circ i_0$. □

We have now proved an important theorem 3.1.5 in this section which shows that the controlled category Crd has a structure of Baues cofibration category.

Simply if the equivalence classes defined by the equivalence relation are unbounded, then the quotient map is not coarse in general which shows that one can not find a push out diagram in the category of coarse maps Crs. This means that we can not easily give a structure of a Baues cofibration category on this category. However, the above makes us think of another category.

§ 3.2 The Quotient Coarse Cofibration Category

Here we introduce the quotient coarse category as defined by Luu in [9], page (27).

Luu in [9] has proved that the quotient coarse category has all non-zero limits and colimits. This is good as we can prove that the push out diagram exists in this category easily.

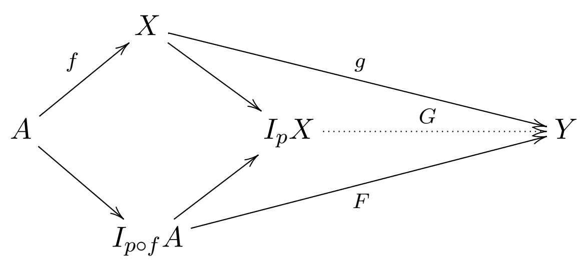
This is a category of non-unital coarse spaces and closeness equivalence classes of coarse maps (between non-unital coarse spaces), and we denote this category by Qcrs. Denote such classes by $[f]\colon X \to Y$ where f is a representative coarse map.

A coarse map $f\colon X \to Y$ is a coarse equivalence in the category of coarse maps if and only if the closeness equivalence class is an isomorphism in the category Qcrs.

The aim in this section is to show that the quotient coarse category has a structure of Baues cofibration category.

Definition 3.2.1 Let A, X be non-unital coarse spaces. A coarse map $f\colon A \to X$ is called a *coarse cofibration* if given a coarse map $g\colon X \to Y$, some controlled map $p\colon X \to R$ and a coarse homotopy $F\colon I_{p\circ f}A \to Y$ such that $g(f(a)) = F(a,0)$ for all $a \in A$, $t \in R$, we can find a coarse homotopy $G\colon I_pX \to Y$ such that $g(x) = G(x,0)$ for all $x \in X$, $G(f(a),t) = F(a,t)$ for all $a \in A$, $t \in R$.

This definition is illustrated by the following commutative diagram:



Lemma 3.2.2 *Let $f, g\colon X \to Y$ be close maps between non-unital coarse spaces. If f is a coarse cofibration, then g is a coarse cofibration. If f is a coarse homotopy equivalence, then g is a coarse homotopy equivalence.*

Proof : First, suppose that f is close to g and g is a coarse cofibration, we need to show that f is a coarse cofibration. Let $p\colon Y \to R$ be some controlled map, $h\colon Y \to Z$ be a coarse map, and $F\colon I_{p\circ f}X \to Z$ be a coarse homotopy such that $F(x,0) = h(f(x))$ for all $x \in X$.

Since f is close to g, so $p \circ f$ is close to $p \circ g$ which implies that the cylinders $I_{p\circ f}X$ and $I_{p\circ g}X$ are coarsely equivalent.

Let $K\colon I_{p\circ g}X \to I_{p\circ f}X$ be a coarse equivalence such that $K(x,0) = (x,0)$ for all $x \in X$.

We also define a map $F^{'}\colon I_{p\circ g}X \to Z$ by

$$F^{'}(x,t) = \begin{cases} h \circ g(x) & t = 0 \\ F \circ K(x,t) & otherwise \end{cases}$$

But $F^{'}$ is close to F, and since F is a coarse map, by lemma 2.2.3 we have $F^{'}$ is a coarse map, and $F^{'}(x,0) = h \circ g(x)$.

Now since g is a coarse cofibration, there is a coarse homotopy $G\colon I_pY \to Z$ such that $G(y,0) = h(y)$ for all $y \in Y$, and $G(g(x),t) = F^{'}(x,t)$ for all $x \in X$, $t \in R$. Define a map $G^{'}: I_pY \to Z$ by

$$G^{'}(y,t) = \begin{cases} h(y) & t = 0 \\ F(x,t) & y = f(x) \\ G(y,t) & otherwise \end{cases}$$

But $G^{'}$ is close to G, and since G is a coarse map, by lemma 2.2.3 we have that $G^{'}$ is a coarse map. Therefore f is a coarse cofibration.

The second statement is straightforward. □

Considering the above lemma, we have the following definition.

Definition 3.2.3 Let X, Y be non-unital coarse spaces. We call a class $[f]\colon A \to X$ a *coarse cofibration class* if the representative coarse map f is a coarse cofibration. A class $[f]\colon A \to X$ is called a *coarse homotopy equivalence class* if the representative coarse map f is a coarse homotopy equivalence as defined in 2.2.2.

Lemma 3.2.4 *Let X be a coarse space, $p\colon X \to R$ be some controlled map. Let $[i] : A \hookrightarrow X$ be an inclusion class in the quotient coarse category, then the inclusion class $[j]\colon X \times \{0\} \to I_{p_A}A \cup (X \times \{0\})$ is a coarse homotopy equivalence class, where $p_A = p\mid_A$*

Proof : By a similar argument to lemma 3.1.3 . □

Lemma 3.2.5 *Let X be a non-unital coarse space, $p\colon X \to R$ be some controlled map. Let $i : A \hookrightarrow X$ be an inclusion, where $p_A = p\mid_A$. Write*

$$(I_{p_A}A) \cup (X \times \{0\}) = \{(x,t) \in I_pX : x \in A \text{ or } t = 0\}$$

Let $j : (I_{p_A}A) \cup (X \times \{0\}) \hookrightarrow I_pX$ be the obvious inclusion. Then the following are equivalent;

(1) *$[i] : A \hookrightarrow X$ is a coarse cofibration class.*

(2) *Suppose we have an equivalence class $[f]\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$, then there exists a coarse homotopy $G\colon I_pX \to Y$ such that $G \circ j = f$.*

(3) *There is a coarse homotopy class $[r]\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.*

Proof : First note that since $p_A = p\mid_A$ and i is an inclusion, then $p_A(a) = p_X(i(a))$ and p_A is a controlled map.

(1) ⇒ (2) Let $[f]\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a closeness equivalence class, then f is a representative coarse map. Define a map $f_0\colon X \times \{0\} \to Y$ by

$f_0 = f\mid_{X\times\{0\}}$, and $H\colon I_{p_A}A \to Y$ such that $H = f\mid_{I_{p_A}A}$. Then f_0, H are coarse maps, and since $[i]$ is a coarse cofibration class, so i is a coarse cofibration. This means that there exists a coarse homotopy $G\colon I_pX \to Y$ such that $G(i(a),t) = H(a,t)$ for all $a \in A$, $t \in R$ and so then $G \circ j(x,t) = f(x,t)$.

(2) ⇒ (3) Suppose that the map $[I]\colon (I_{p_A}A) \cup (X \times \{0\}) \to (I_{p_A}A) \cup (X \times \{0\})$ is the identity class, so I is the representative identity map. By (2) there exists a coarse homotopy class $[r]\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r \circ j = I$ which implies that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$.

(3) ⇒ (2) Suppose we have a coarse homotopy class $[r]\colon I_pX \to (I_{p_A}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\}$.

Let $[f]\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be an equivalence class, then f is the representative coarse map. So we can define a coarse homotopy $G\colon I_pX \to Y$ by writing $G(x,t) = f(r(x,t))$ where $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p_A}A) \cup (X \times \{0\})$. Then $G \circ j(x,t) = f(r(x,t)) = f(x,t)$.

(2) and (3) ⇒ (1) Let $[i] : A \hookrightarrow X$ be a closeness equivalence class, we need to prove that the representative coarse map i is a coarse cofibration. Let $F\colon I_{p\circ i}A \to Y$ be a coarse homotopy and $g\colon X \to Y$ be a coarse map such that $F(a,0) = g(i(a))$ for all $a \in A$. By (3) we have a coarse homotopy $r\colon I_pX \to (I_{p\circ i}A) \cup (X \times \{0\})$ such that $r(x,t) = (x,t)$ for all $(x,t) \in (I_{p\circ i}A) \cup (X \times \{0\})$.

Define a map $f_0\colon X \times \{0\} \to Y$ by $f_0(x,0) = g(x)$. Then f_0 is a coarse map. Let $f\colon (I_{p_A}A) \cup (X \times \{0\}) \to Y$ be a map such that $f|_{X\times\{0\}} = f_0$ and $f|_{I_{p_A}A} = F$. We need to show that f is a coarse map. Since the inclusion $X \times \{0\} \hookrightarrow (I_{p_A}A) \cup (X \times \{0\})$ is a coarse homotopy equivalence by lemma 3.2.4 and the map f_0 is a coarse map, by lemma 2.2.14 (2), the map f is coarse as required.

Therefore by (2) there exists a coarse homotopy $G\colon I_pX \to Y$ defined by $G(x,t) = f \circ r(x,t)$, and then $G(x,0) = f_0(x,0) = g(x)$, for all $x \in X$, and $G(i(a),t)) = f \circ r(i(a),t) = f(a,t) = F(a,t)$ for all $a \in A$, $t \in R$. Therefore i is a coarse cofibration. Hence $[i]$ is a coarse cofibration class. □

Our main aim in this section is to prove the following result which will show that the category Qcrs can be given a structure of Baues cofibration category.

Theorem 3.2.6 *The Quotient coarse category Qcrs is a Baues cofibration category. The weak equivalences are the coarse homotopy equivalence classes, and the cofibrations are the coarse cofibration classes.*

Proving this theorem require us to prove the following results in order to satisfy the axioms required of a Baues cofibration category.

Definition 3.2.7 We call the class $[f]\colon X \to Y$ an *isomorphism class* if the representative coarse map f is a coarse equivalence.

Proposition 3.2.8 *Let* $[f]\colon X \to Y$ *be an isomorphism in the quotient coarse category. Then* $[f]$ *is both a coarse cofibration class and coarse homotopy equivalence class .*

Proof : Let $[f]$ be an isomorphism, then f is the representative coarse map, so we have a coarse map $g\colon Y \to X$ such that $g \circ f$ is close to 1_X, $f \circ g$ is close to 1_Y. Therefore by example 2.2.4 the map f is a coarse homotopy equivalence. Hence $[f]$ is a coarse homotopy equivalence class.

Now let $F\colon I_{p_Y \circ f}X \to Z$ be a coarse homotopy where $p_Y\colon Y \to R$ is some controlled map. Let $h\colon Y \to Z$ be a coarse map such that $F(x,0) = h(f(x))$ for all $x \in X$.

Define a map $G\colon I_{p_Y}Y \to Z$ by $G(y,t) = F(g(y),t)$, then $G(f(x),t) = F(g(f(x)),t)$ for all $x \in X$, $t \in R$, and $G(y,0) = F(g(y),0) = h(f \circ g(y))$ for all $y \in Y$.

But by assumption we have $g \circ f$ is close to 1_X, $f \circ g$ is close to 1_Y which implies that $h \circ f \circ g$ is close to h. Therefore $G\mid_{Y\times\{0\}}$ is close to h and $G\mid_{I_{p_Y f(X)}}$ is close to F.

Now define another map $G^{'}\colon I_{p_Y}Y \to Z$ by

$$G^{'}(y,t) = \begin{cases} h(y) & t = 0 \\ F(x,t) & y = f(x) \\ G(y,t) & otherwise \end{cases}$$

But $G^{'}$ is close to G, and since G is a coarse map so by lemma 2.2.3 we have $G^{'}$ is a coarse map. Therefore f is a coarse cofibration, and hence $[f]$ is a coarse cofibration class. $\square$

Proposition 3.2.9 *Consider two equivalence classes* $[f]\colon X \to Y$ *and* $[g]\colon Y \to Z$*. If any two of the morphisms* $[f]$*,* $[g]$ *and* $[gf]$ *are coarse homotopy equivalence classes, then so is the third.*

Proof : Let $[f]\colon X \to Y$, $[g]\colon Y \to Z$ be two closeness equivalence classes, then $[gf]\colon X \to Z$ is a closeness equivalence class of the coarse map $gf\colon X \to Z$. Suppose that $[gf]$, $[f]$ are coarse homotopy equivalence classes, then f and gf are coarse homotopy equivalence. We need to show that g is a coarse homotopy equivalence.

Since f and gf are coarse homotopy equivalence, then there exist coarse maps $h\colon Y \to X$, $h^{'}\colon Z \to X$ such that $f \circ h$ is coarsely homotopic to 1_Y, $gf \circ h^{'}$ is coarsely homotopic to 1_Z, $h \circ f$ is coarsely homotopic to 1_X, and $h^{'} \circ gf$ is coarsely homotopic to 1_X that is, $h \circ f$ is coarsely homotopic to $h^{'} \circ gf$.

We need to show that there is a coarse map $l\colon Z \to Y$ such that $l \circ g$ is coarsely homotopic to 1_Y, $g \circ l$ is coarsely homotopic to 1_Z that is, $l \circ g$ is coarsely homotopic to $f \circ h$ and $g \circ l$ is coarsely homotopic to $gf \circ h^{'}$.

First, we show that h is coarsely homotopic to $h^{'} \circ g$. We have $h^{'} \circ g \circ f$ is coarsely homotopic to $h \circ f$, and so $h^{'} \circ g \circ f \circ h$ is coarsely homotopic to $h \circ f \circ h$ which implies that h is coarsely homotopic to $h^{'} \circ g$.

Now let $l\colon Z \to Y$ be the map $f \circ h^{'}$, then $l \circ g = f \circ h^{'} \circ g$, but $f \circ h^{'} \circ g$ is coarsely homotopic to $f \circ h$ which implies that $l \circ g$ is coarsely homotopic to the identity on Y.

Similarly, we can show that $g \circ l$ is coarsely homotopic to 1_Z. Hence g is a coarse homotopy equivalence. □

Proposition 3.2.10 *Composition of coarse cofibration classes is a coarse cofibration class.*

Proof : Straightforward by the definition of coarse cofibration class. □

Proposition 3.2.11 *Let X, Y be non-unital coarse spaces, and let $[i]\colon X \to Y$ be an equivalence class that is both a coarse cofibration class and a coarse homotopy equivalence class. Then there is a coarse map $[r]\colon Y \to X$ such that $[r \circ i] = [1_X]$.*

Proof : Let $p_X\colon X \to R$ be some controlled map. Since $[i]$ is a coarse homotopy equivalence class, then i is a coarse homotopy equivalence which means there exists a coarse map $h\colon Y \to X$ such that $h \circ i$ and $i \circ h$ are coarsely homotopic to 1_X, 1_Y respectively.

So we have a coarse homotopy $F\colon I_{p_Y \circ i}X \to X$ such that $F(x,0) = h(i(x))$, $F(x, p_Y \circ i(x) + 1) = x$ for all $x \in X$. Since i is a coarse cofibration, then there exists a coarse homotopy $G\colon I_{p_Y}Y \to X$ such that $G(y,0) = h(y)$ for all $y \in Y$, and $G(i(x),t) = F(x,t)$ for all $x \in X$, $t \in R$.

Define a coarse map $r\colon Y \to X$ by the formula $r(y) = G(y, p_Y(y) + 1)$. By construction we have $r \circ i(x) = G(i(x), p_Y(i(x)) + 1) = F(x, p_Y(i(x)) + 1) = x$ for all $x \in X$. Hence $[r \circ i] = [1_X]$. □

The following definition comes from [9].

Definition 3.2.12 Suppose that (X, ε_X) is a non-unital coarse space, Y is any set, and $f\colon X \to Y$ is any locally proper map.

The *push-forward* coarse structure of ε_X along f is;

$$f_*\varepsilon_X =< \{(f \times f)(F) : F \in \varepsilon_X\} >.$$

The following obvious proposition also comes from [9].

Proposition 3.2.13 *Suppose that (X, ε_X) is a non-unital coarse space, and $f\colon X \to Y$ is any locally proper map. Then $f_*\varepsilon_X$ is the minimum coarse structure on Y which makes f into a coarse map.* □

Definition 3.2.14 Let $f, g\colon A \to X$ be coarse maps between non-unital coarse spaces. The *Coequalizer* of $[f]$, $[g]$ is defined by writing $Coeq([f],[g]) = X$, equipped with the coarse structure

$$\varepsilon_{Coeq([f],[g])} =< \varepsilon_X, f_*\varepsilon_A, g_*\epsilon_A, \{(f \times g)(F) : F \in \varepsilon_A\} >_X,$$

together with the identity map in level of sets $\theta\colon X \to Coeq([f],[g])$ (which is a coarse map).

The proof of the following lemma is found in [9].

Lemma 3.2.15 *Let $f, g\colon A \to X$ be coarse maps between non-unital coarse spaces. The coarse space $Coeq([f],[g])$ is a coequalizer of f and g (in the category-theoretic sense) in the category Qcrs.* $\square$

Definition 3.2.16 Let A, X, and Y be coarse spaces between non-unital coarse spaces. Suppose that we have coarse maps $i\colon A \to X$ and $f\colon A \to Y$. Then we define

$$X \vee_A Y = Coeq([\tilde{i}],[\tilde{f}])$$

where

$$\tilde{i} : A \xrightarrow{i} X \xrightarrow{i_X} X \sqcup_\infty Y, \qquad \tilde{f} : A \xrightarrow{f} Y \xrightarrow{i_Y} X \sqcup_\infty Y$$

are the representative coarse maps and $X \sqcup_\infty Y$ is the disjoint union in definition 1.4.12.

Theorem 3.2.17 *Let X, Y, and A be non-unital coarse spaces, suppose we have equivalence classes of two coarse maps $[i]\colon A \to X$, $[f]\colon A \to Y$. Then we have a push out diagram in the quotient coarse category.*

$$\begin{array}{ccc} X & \xrightarrow{[f']} & X \vee_A Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [i']} \\ A & \xrightarrow{[f]} & Y \end{array}$$

Proof : Define the classes $[\tilde{i}]\colon A \to X \sqcup_\infty Y$, $[\tilde{f}]\colon A \to X \sqcup_\infty Y$ by $[\tilde{i}] = [i_X \circ i]$ and $[\tilde{f}] = [i_Y \circ f]$ where i_X and i_Y are the representative inclusion maps, so the maps $\tilde{i}$ and $\tilde{f}$ are representative coarse maps.

Now the map $\theta\colon X \sqcup_\infty Y \to Coeq([\tilde{i}],[\tilde{f}])$ is the identity map at the level of sets, and clearly it is a coarse map. So we can factor the representative maps i', f' as follows

$$Y \xrightarrow{i_Y} X \sqcup_\infty Y \xrightarrow{\theta} X \vee_A Y \qquad X \xrightarrow{i_X} X \sqcup_\infty Y \xrightarrow{\theta} X \vee_A Y$$

respectively, where i_Y, i_X are the inclusions. Since θ, i_Y, and i_X are coarse maps, then i', f' are also coarse maps.

Let $g\colon X \sqcup_\infty Y \to Z$ be a map defined by writing

$$g(x) = \begin{cases} g_1(x) & x \in X \\ g_2(x) & x \in Y \end{cases}$$

then g is coarse such that $g \circ \tilde{i}$ is close to $g \circ \tilde{f}$. Let $h\colon X \vee_A Y \to Z$ be the same map (as a set map) as g, then clearly $g = h \circ \theta$. Hence $[g] = [h] \circ [\theta]$. We need to show that $[h]$ is a unique coarse map.

First since $g \circ \tilde{i}$ is close to $g \circ \tilde{f}$, then h is controlled. Second, suppose that $B \subseteq Z$ is a bounded subset, and $W \subseteq X \vee_A Y$ is a unital subspace, then we can write $W = \theta(W')$ for some unital subspace $W' \subseteq X \sqcup_\infty Y$.

Now since g is a locally proper map, then $(g\mid_{W'})^{-1}(B)$ is bounded, but $g = h \circ \theta$, and $(g\mid_{W'})^{-1}(B) = (\theta\mid_{W'})^{-1}((h\mid_{\theta(W')})^{-1})(B))$. Since θ is surjective, $h\mid_{W}^{-1}(B)$ is bounded. Therefore h is locally proper. Hence h is a coarse map.

Now $h \circ \tilde{i}(a) = h(\theta(i_y(i(a)))) = h(\theta(i(a)))$, and

$h \circ \tilde{f}(a) = h(\theta(i_y(f(a)))) = h(\theta(f(a)))$ for all $a \in A$.

To check uniqueness, consider another coarse map $l\colon X \vee_A Y \to Z$ such that g is close to $l\circ\theta$. We need to show that l is close to h, so let $E \subseteq X \vee_A Y \times X \vee_A Y$ be an entourage.

Clearly if $E \subseteq X \sqcup_\infty Y \times X \sqcup_\infty Y$, then $(l \times h)(E)$ is an entourage. Now let $E = (\tilde{i} \times \tilde{f})(F)$ for some F an entourage in A. Since the map g is close to $l \circ \theta$, $g \circ \tilde{i}$ is close to $l \circ \theta \circ \tilde{f}$. Therefore

$$h \times l((\tilde{i} \times \tilde{f})(F)) = ((g \circ \tilde{i}) \times (l \circ \theta \circ \tilde{f}))(F)$$

which is an entourage in Z as needed. □

Proposition 3.2.18 *Let $[i]\colon A \to X$ be a coarse cofibration class, and $[f]\colon A \to Y$ be an equivalence class of a coarse map f. Then we have a push out diagram in the Quotient coarse category.*

$$\begin{array}{ccc} X & \xrightarrow{[f']} & X \vee_A Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [i']} \\ A & \xrightarrow{[f]} & Y \end{array}$$

Further, the map $[i']$ is a coarse cofibration class.

Proof : The above theorem proves the first part. We need to show that $[i']$ is a coarse cofibration class, so it enough to show that i' is a coarse cofibration.

Suppose that $F\colon I_{q\circ i'}Y \to Z$ is a coarse homotopy where $q\colon X \vee_A Y \to R$ is some controlled map, and $h\colon X \vee_A Y \to Z$ is a coarse map such that $F(y, 0) = h(i'(y))$ for all $y \in Y$.

Define a map $G\colon I_{q\circ i'\circ f}A \to I_{q\circ i'}Y$ by $G(a, t) = (f(a), t)$ for all $a \in A$, then G is a coarse map. Since $q \circ i' \circ f = q \circ f' \circ i$ by the above push out diagram, the cylinders $I_{q\circ i'\circ f}A$ and $I_{q\circ f'\circ i}A$ are the same. The map $F\circ G\colon I_{q\circ f'\circ i}A \to Z$ is a coarse homotopy such that $F\circ G(a,0) = F(f(a),0) = h(i^{'}(f(a))) = h(f^{'}(i(a)))$ for all $a \in A$ again by the above push out diagram.

By the universal property we have $g^{'}: X \to Z$ defined by $g^{'}(x) = h(f^{'}(x))$ for all $x \in X$. Then $g^{'}$ is a coarse map and $F \circ G(a,0) = g^{'}(i(a))$ for all $a \in A$.

Since $[i]$ is a coarse cofibration class, i is a coarse cofibration which implies that there is a coarse homotopy $H\colon I_{q\circ f'}X \to Z$ such that $H(x,0) = g^{'}(x)$ for all $x \in X$, $H(i(a),t) = F\circ G(a,t)$ for all $a \in A$, $t \in R$. We can define a new coarse homotopy $H^{'}: I_q(X \vee_A Y) \to Z$ by writing

$$H^{'}(\theta(y),t) = F(y,t), \qquad H^{'}(\theta(x),t) = H(x,t), \quad \text{when} \ \ x \in X, \quad \text{and} \ \ y \in Y.$$

Let $w \in X \cup_A Y$. Then $w = \theta(x)$ or $w = \theta(y)$, so we have

$$H^{'}(w,0) = i^{'}(w) \ \text{ if } \ w = \theta(x) \ \text{ or } \ w = \theta(y)$$

Therefore $H^{'}(w,0) = i^{'}(w)$ for all $w \in X \vee_A Y$, $H^{'}(i^{'}(y),t) = F(y,t)$ for all $y \in Y$. Hence $i^{'}$ is a coarse cofibration which implies that $[i^{'}]$ is a coarse cofibration class. □

Proposition 3.2.19 *In the following push out diagram in the quotient coarse category.*

$$\begin{array}{ccc} X & \xrightarrow{[f^{'}]} & X \vee_A Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [i^{'}]} \\ A & \xrightarrow{[f]} & Y \end{array}$$

with $[i]$ a coarse homotopy equivalence class, the map $[i^{'}]$ is a coarse homotopy equivalence class.

Proof : Suppose that $[i]$ is a coarse homotopy equivalence class, then i is a coarse homotopy equivalence. To show that $[i^{'}]$ is a coarse homotopy equivalence class, it is enough to show that $i^{'}$ is a coarse homotopy equivalence. A similar argument to that in proposition 3.1.15 proves that. □

Definition 3.2.20 Let $[f]\colon X \to Y$ be the closeness equivalence class of a coarse map, and $p_X\colon X \to R$ be some controlled map. Then we define *the mapping cylinder* of $[f]$, C_f to be the push out $I_{p_X}X \vee_X Y$ which is defined to be $Coeq([\tilde{f}],[\tilde{i}])$ where $\tilde{f}\colon X \to I_{p_X}X \sqcup_\infty Y$ and $\tilde{i}\colon X \to I_{p_X}X \sqcup_\infty Y$ are coarse maps.

Proposition 3.2.21 *We have a coarse cofibration class $[i]\colon X \to C_f$ and a coarse homotopy equivalence class $[r]\colon C_f \to Y$ such that $[f] = [r\circ i]$.*

Proof : Let $\theta\colon I_{p_X}X \sqcup_\infty Y \to C_f$ be the coequalizer coarse map. Then we define the maps $i\colon X \to C_f$, $r\colon C_f \to Y$ by

$$i(x) = \theta(x,0)$$

$$r(\theta(y)) = y, \quad y \in Y \text{ and } r(\theta(x,t)) = f(x), x \in X, \quad t \in R$$

Since θ is a coarse map, then i is also, and by a similar argument to that in theorem 3.2.17 we show that r is a coarse map.

Define a map $s\colon Y \to C_f$ by $s(y) = \theta \circ i_Y(y)$, for all $y \in Y$, then s is a coarse map, and $r \circ s = 1_Y$ and by proposition 2.2.12, the map $X \hookrightarrow I_{p_X}X$ is a coarse homotopy equivalence. Therefore r is a coarse homotopy equivalence by the push out diagram and proposition 3.2.19. This implies that any coarse map close to r is a coarse homotopy equivalence. Hence $[r]$ is a coarse homotopy equivalence class, and that $f = r \circ i$.

Now we need to prove that $[i]\colon X \to C_f$ is a coarse cofibration class. First, i is a coarse map since it is a composite of two coarse maps, so it is enough to show that i is a coarse cofibration. Let $q\colon C_f \to R$ be a controlled map. Suppose we are given a coarse homotopy $F\colon I_{q\circ g}X \to Z$ and a coarse map $h\colon C_f \to Z$ such that $F(x,0) = h(i(x))$ for all $x \in X$.

We can define a map $G\colon I_qC_f \to Z$ by writing $G(\theta(y),t) = h(\theta(y))$ for all $y \in Y$, and

$$G(\theta(x,s),t) = \begin{cases} h(\theta(x, s-\frac{t}{2})) & 0 \le s \le (q(i(x))+1)/2, \quad t \le 2s \\ F(x, t-2s) & 0 \le s \le (q(i(x))+1)/2, \quad t \ge 2s \\ h(\theta(x, q(i(x))+1-s-\frac{t}{2})) & (q(i(x))+1)/2 \le s \le q(i(x))+1, \\ & t \le 2(q(i(x))+1)-2s \\ F(x, s-q(i(x))+1+\frac{t}{2}) & (q(i(x))+1)/2 \le s \le q(i(x))+1, \\ & t \ge 2(q(i(x))+1)-2s \end{cases}$$

The maps $(x,s,t) \mapsto g(\pi(x,s-\frac{t}{2}))$, $(x,s,t) \mapsto g(\pi(x,q(i(x))+1-s-\frac{t}{2}))$, $(x,s,t) \mapsto F(x,t-2s)$, $(x,s,t) \mapsto F(x,s-q(i(x))+1+\frac{t}{2})$ are all controlled. Using the same argument as in lemma 2.2.7 (2), the set

$$\{(x,s,t) : 0 \le s \le (q(i(x))+1)/2\} \cup \{(x,s,t) : (q(i(x))+1)/2 \le s \le q(i(x))+1\}$$

is coarsely excisive decomposition.

Now since the maps $g \circ \pi$ and F are both controlled maps on the sets $\{(x,s,t) : 0 \le s \le (q(i(x))+1)/2\}$ and $\{(x,s,t) : (q(i(x))+1)/2 \le s \le q(i(x))+1\}$, by corollary 2.2.8 (2), the map G is controlled on the set

$$\{(x,s,t) : 0 \le s \le (q(i(x))+1)/2\} \cup \{(x,s,t) : (q(i(x))+1)/2 \le s \le q(i(x))+1\}$$

It is clear that $F(x,0) = g(\pi(x,0))$ when $t = 2s$ and $t = 2(q(i(x))+1) - 2s$. Hence we have G is a controlled homotopy as required. □

Using the above results, we can prove theorem 3.2.6 as follows.

Proof of Theorem 3.2.6: Axiom (C1) is proposition 3.2.8, proposition 3.2.9, and proposition 3.2.10. Axiom (C2) is proposition 3.2.11. Axiom (C3) is proposition 3.2.18, and the last axiom (C4) is just proposition 3.2.21. □

Proposition 3.2.22 *Let $[f]\colon A \to Y$ be a coarse homotopy equivalence class in the quotient coarse cofibration category, Then in the push out diagram*

$$\begin{array}{ccc} X & \xrightarrow{[f']} & X \vee_A Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [i']} \\ A & \xrightarrow{[f]} & Y \end{array}$$

the map $[f']\colon X \to X \vee_A Y$ is a coarse homotopy equivalence class.

Proof : Similar argument to proposition 3.1.18. □

Corollary 3.2.23 *The inclusions $i_0, i_1 : X \hookrightarrow I_pX$ are coarse cofibrations where $p\colon X \to R$ is some controlled map.*

Proof The same argument used in corollary 3.1.19. □

We have now proved another important theorem 3.2.6 in this section which shows that the Quotient Coarse Category Qcrs has a structure of Baues cofibration category.

Chapter 4

The Category of Coarse CW-Complexes

In this chapter we define another coarse example of a cofibration category. In order to define the structure we use a similar technique to that used to show hat the category of topological spaces is a Quillen model category in [9], and this will show that we nearly have a coarse example of Quillen model category. All axioms are satisfied apart from existence of all limits.

§ 4.1 Coarse CW-Complex

In this section we give an explicit definition of a coarse version of CW-complexes. The following definition is from [10].

Definition 4.1.1 Let X be a subspace of the unit sphere S^{n-1}. Then we define the *open cone* of X to be the metric space

$$\mathcal{C}X = \{\lambda x : \lambda \in \mathbb{R}^{+}, x \in X\} \subseteq \mathbb{R}^{n}.$$

The open cone $\mathcal{C}X$ is a coarse space. The coarse structure is defined by the Euclidean metric on $\mathbb{R}^n$.

The cone of S^{n-1} is the Euclidean space $\mathbb{R}^n$, and the n-cell D^n can be viewed as the upper hemisphere in the cone of S^n, so its cone is $\mathbb{R}^n \times \mathbb{R}_+$.

The following definition comes from [12] and [6].

Definition 4.1.2 Let R be a generalized ray, $n \in \mathbb{N}$. Write

$$S_R^{n-1} = (R \sqcup R)^n, \quad D_R^n = (R \sqcup R)^n \times R$$

We call S_R^{n-1} a *coarse R-sphere of dimension* $n-1$, D_R^n a *coarse R-cell of dimension* n, and the coarse R-sphere $\{(x,0) \in D_R^n : x \in S_R^{n-1}\}$ is called *the boundary* of the coarse R-cell D_R^n, i.e. $\partial D_R^n = S_R^{n-1} \times \{0\}$.

A *coarse cell complex* is defined to be a coarse space (Y, ε) obtained "inductively" by attaching coarse cells to a disjoint union of coarse cells; more formally,

Definition 4.1.3 Suppose that for all $k \in \mathbb{N}$ we have a non-unital coarse space (Y^k, ε_k) and a set I_k. Moreover, for all $k \in \mathbb{N}$ and $i \in I_k$, there is

- a number $n_{k,i} \in \mathbb{N}$,
- a generalized ray $R_{k,i}$, and
- coarse classes $[f_{k,i}]\colon S^{n_{k,i}-1}_{R_{k,i}} \to Y^{k-1}$.

Define $[f_k] = [\coprod_{i\in I_k} f_{k,i}]\colon \coprod_{i\in I_k} S^{n_{k,i}-1}_{R_{k,i}} \to Y^{k-1}$. We construct an object in the category Qcrs defined as follows.

We start with a non-unital coarse space Y^0 to be regarded as a disjoint union of coarse cells of dimension zero; that is, $Y^0 = \coprod_{i\in I_0} D^{n_{0,i}}_{R_{0,i}}$.

Inductively, construct the k-skeleton Y^k from Y^{k-1} by attaching a disjoint union of coarse k-cells via the coarse class $[f_k]$. Then Y^k is the coequalizer space of the disjoint union $Y^{k-1}\coprod(\coprod_{i\in I_k} D^{n_{k,i}}_{R_{k,i}})$ of Y^{k-1} (see definition 3.2.16) with a collection of the disjoint union of coarse k-cells $D^{n_{k,i}}_{R_{k,i}}$ under the equivalence relation $x \sim f_k(x)$ for $x \in \partial D^{n_{k,i}}_{R_{k,i}}$.

One can either stop at a finite stage, and set $Y = Y^n$ for $n < \infty$, or one can continue infinitely by setting Y the colimit $colim Y^n$ which exists in the category Qcrs by theorem (3.7.6) in [9]. We can interpret this space as an increasing union of the space Y, so $Y = \bigcup_{n=0}^{\infty} Y^n$.

Then Y constructed as above will be called a *coarse cell complex* or *coarse CW-complex* .

A coarse CW-complex X with only finitely many cells in each dimension means that each skeleton of X is obtained by attaching a finite union of coarse n-cells.

We define $\delta(Y) = sup\{n_{k,i} : k \in \mathbb{N}, i \in I_k\}$, and $\delta(Y)$ will be called the *cell dimension* of the coarse CW-complex Y.

A *coarse subcomplex* of a coarse CW-complex Y is a subspace A such that Y is obtained from A by attaching coarse cells, where,

$$\varepsilon_A = \varepsilon_Y|_A = \varepsilon_Y \cap \{E : E \subseteq A \times A\},$$

is the coarse structure defined on A.

The *characteristic map class* $[\Phi_R]\colon D^n_R \to A$ for each coarse cell D^n_R which extends the attaching map has image contained in A.

A pair (Y, A) consisting of a coarse CW-complex Y and a coarse subcomplex A will be called a *coarse CW-pair*.

A relative coarse CW-complex is defined to be a pair (X, A) consisting of a coarse space X and a subset A that builds up X by attaching coarse cells, together with a sequence of coarse subspaces satisfying some conditions; more formally,

Definition 4.1.4 Let X be a coarse space, and A be a coarse subspace. We define a *relative coarse CW-complex* to be the pair (X, A) equipped a sequence of coarse subspaces $(X, A)^k$, $k \geq 0$ such that:

(a) the space $(X, A)^0$ is obtained from A by attaching 0-coarse cells;

(b) for $k \geq 1$, $(X, A)^k$ is obtained from $(X, A)^{k-1}$ by attaching a disjoint union of coarse k-cells;

(c) $(X, A) = colim((X, A)^k)$.

The space $(X, A)^k$ is called the k-dimensional skeleton of X relative to A. When $A = \emptyset$, the relative coarse CW-complex $(X, \emptyset) = X$ is a coarse CW-complex in the previous sense and its k-dimensional skeleton is X^k.

Definition 4.1.5 A *finite coarse CW-complex* is a coarse space Y obtained by attaching a finite number of coarse cells to a finite disjoint union of generalised rays.

Definition 4.1.6 Let X and Y be coarse CW-complexes, and let X^n denote the coarse n-skeleton of X and Y^n denote the coarse n-skeleton of Y. Let $[i_n]\colon X^n \to X$, and $[j_n]\colon Y^n \to Y$ be the closeness class of the inclusions. A coarse class $[f]\colon X \to Y$ is said to be *coarsely cellular class* if for all n we have a coarse class $[f_n]\colon X^n \to Y^n$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{[f]} & Y \\ {\scriptstyle [i_n]}\downarrow & & \downarrow{\scriptstyle [j_n]} \\ X^n & \xrightarrow[{[f_n]}]{} & Y^n \end{array}$$

commutes.

Similarly, we define a notion of *controlledly cellular class*.

§ 4.2 Some Properties of The Category of Coarse CW-Complexes

Our aim in this section is to construct a new category and try to prove some properties of this category. Having these properties proved allows us to check that the axioms of Baues cofibration category are satisfied, and further we will be even able to have all Quillen model category axioms proved apart from existence of all limits in this category, and proving most Baues axioms will be a particular case. This category will be called the category of coarse CW-complexes.

We term the category of non-unital coarse CW-complexes with only finitely many cells in each dimension and coarsely cellular classes the *category of coarse CW-complexes*, and we denote this category by CWrs .

Similar to definition 2.4.1 we define a notion of *pointed coarse CW-complexes*, and a notion of *pointed coarsely cellular classes*.

Lemma 4.2.1 *Let R be a generalised ray. Then the coarse cellular class $[i]\colon R \to (R \sqcup R)^n \times R$ defined by the formula $i(s) = (0, s)$ is a coarse homotopy equivalence class.*

Proof : It is enough to show that the map i is a coarse homotopy equivalence. This is found in [12], [11]. □

The above lemma shows that D^n_R is coarsely homotopy equivalent to R, and using the same argument we can show that R^n is coarsely homotopy equivalent to R.

Definition 4.2.2 A coarsely cellular class $[f]\colon X \to Y$ in the category CWrs is called a *weak coarse homotopy equivalence class* if the induced class

$$[f_*]\colon \pi_n^{Pcrs}(X) \to \pi_n^{Pcrs}(Y)$$

is a bijection for $n = 0$, and an isomorphism for $n \geq 1$.

Lemma 4.2.3 *Let X be a coarse CW-complex, then the cylinder I_pX is coarsely equivalent to a coarse CW-complex for some controlled map $p\colon X \to R$.*

Proof : Let $p_1\colon R \sqcup R \to R$ be some controlled map. Then the cylinder $I_{p_1}(R \sqcup R)$ is a coarse CW-complex since it is coarsely equivalent to $R^2 \sqcup R^2$.

Now visually $I_{p_2}(R \sqcup R)^2$ is coarsely equivalent to $I_{p_1}(R \sqcup R) \times (R \sqcup R)$ where $p_2\colon (R \sqcup R)^2 \to R$ is a controlled map defined by $p_2(x, 0) = p_1(x)$ for all $x \in R \sqcup R$. But this is coarsely equivalent to $(R^2 \sqcup R^2) \times (R \sqcup R)$ which is coarsely equivalent to $(R^3 \sqcup R^3) \sqcup (R^3 \sqcup R^3)$.

So we have proved that $I_{p_n}(R \sqcup R)^n$ is coarsely equivalence to $\sqcup_{n=1}^{2^{n-1}}(R^{n+1} \sqcup R^{n+1})$ for the cases $n = 1,\ 2$ where $p_n\colon (R \sqcup R)^n \to R$ is some controlled map.

By induction we assume that the statement is true for $n = k$, and we show it is true for $n = k + 1$.

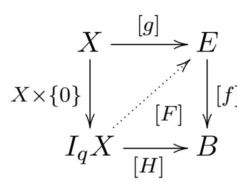
First similar to the case for $n = 2$, the cylinder $I_{p_{k+1}}(R \sqcup R)^{k+1}$ is coarsely equivalent to $I_{p_k}(R \sqcup R)^k \times (R \sqcup R)$ where $p_{k+1}\colon (R \sqcup R)^{k+1} \to R$ is a controlled map defined by $p_{k+1}(x, 0) = p_k(x)$ for all $x \in (R \sqcup R)^k$.

But $I_{p_k}(R \sqcup R)^k$ is coarsely equivalence to $\sqcup_{k=1}^{2^{k-1}}(R^{k+1} \sqcup R^{k+1})$ which implies that $I_{p_{k+1}}(R \sqcup R)^{k+1}$ is coarsely equivalence to $\sqcup_{k=1}^{2^{k-1}}(R^{k+1} \sqcup R^{k+1}) \times (R \sqcup R)$, and the later is coarsely equivalence to $\sqcup_{k=1}^{2^{k-1}}(R^{k+2} \sqcup R^{k+2})$. Therefore the statement is true for all $n \geq 1$.

The above shows that $I_{p_n}(R \sqcup R)^n$ is coarsely equivalent to a coarse CW-complex for all n, and by definition of a coarse CW-complex this implies that the cylinder defined on any coarse CW-complex X is coarsely equivalent to a coarse CW-complex. □

Definition 4.2.4 Let A and X be non-unital coarse CW-complexes with only finitely many cells in each dimension. Let $q\colon X \to R$ be some controlled map, $[i] : A \hookrightarrow X$ an inclusion. Let $[f]\colon E \to B$ be a coarsely cellular class, we say that $[f]$ has the *coarse homotopy lifting property* with respect to X if for any coarse homotopy class $[H]\colon I_qX \to B$, and any coarsely cellular class $[g]\colon X \to E$ such that $[f \circ g] = [H \circ i_X]$, where $[i_X]\colon X \to I_pX$ the obvious inclusion, we have a coarse homotopy class $[F]\colon I_qX \to E$ such that $[F \circ i_X] = [g]$, and $[f \circ F] = [H]$.

The following diagram visualizes the situation:



We give a generalization of the coarse homotopy lifting property and the coarse homotopy extension property.

Definition 4.2.5 [The Coarse Homotopy Lifting Extension Property] Suppose we have a pair of non-unital coarse CW-complexes with only finitely many cells in each dimension (A, X). Let $q\colon X \to R$ be some controlled map, $[i] : A \hookrightarrow X$ be an inclusion. Let $T_q = (X \times \{0\}) \cup I_{q \circ i}A \subseteq I_qX$.

Let $[f]\colon E \to B$ be a coarsely cellular class. We say the triple $(X, A, [f])$ has the *coarse homotopy lifting extension property* if for any coarse homotopy class $[H]\colon I_qX \to B$, and any coarsely cellular class $[g]\colon T_q \to E$ such that $[f \circ g] = [H|_{T_q}]$, we have a coarse homotopy class $[F]\colon I_qX \to E$ such that $[F|_{T_q}] = [g]$, and $[f \circ F] = [H]$.

The coarse homotopy lifting property of X is obtained by taking $A = \emptyset$, so that $(X \times \{0\}) \cup I_{q \circ i}A$ will be $X \times \{0\}$.

The coarse homotopy extension property of (X, A) is obtained by taking $[f]$ to be a constant class.

Proposition 4.2.6 *Let the pairs (X, A) and (X', A') be relatively coarsely homotopy equivalent, and $[f]\colon E \to B$ be a coarsely cellular class. Then $(X, A, [f])$ has the coarse homotopy lifting extension property if and only if the triple $(X', A', [f])$ has the coarse homotopy lifting extension property.*

Proof : It is straightforward. □

Lemma 4.2.7 *The pairs* $(I_pD^n_R, D^n_R \times \{0\})$, $(I_pD^n_R, D^n_R \times \{0\} \cup I_{p\circ j}(\partial D^n_R))$ *are relative coarse homotopy equivalent, where* $[j]\colon \partial D^n_R \to D^n_R$ *is the inclusion.*

Proof : This is obvious since the inclusion

$$[i] : D^n_R \times \{0\} \hookrightarrow D^n_R \times \{0\} \cup I_{p\circ j}(\partial D^n_R)$$

is a coarse homotopy equivalence class by lemma 3.2.4. □

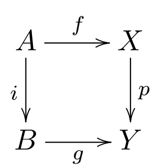
The coarse homotopy lifting extension property for D^k_R is equivalent to the coarse homotopy lifting extension property for $(D^k_R, \partial D^n_R)$ since by proposition 4.2.6 and the above lemma the pairs

$$(I_p(D^n_R), D^n_R \times \{0\}), (I_p(D^n_R), D^n_R \times \{0\} \cup I_{p\circ j}(\partial D^n_R)$$

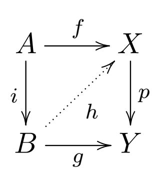
are relatively coarse homotopy equivalent.

The following definition is from [5].

Definition 4.2.8 Given the maps $i\colon A \to B$, $p\colon X \to Y$, if for any maps $f\colon A \to X$, $g\colon B \to Y$ making the diagram



commutes. Suppose there is a map $h\colon B \to X$ such that the resulting diagram



commutes. Then i is said to have a *left lifting property* (LLP) with respect to p, and p is said to have a *right lifting property* (RLP) with respect to i. We call the map h a *lift* .

Definition 4.2.9 Let R be a generalized ray. A coarsely cellular class $[f]\colon X \to Y$ is said to be a *coarse Serre fibration* if for each coarse CW-complex A with only finitely many cells in each dimension, and some controlled map $p\colon A \to R$, the class $[f]$ has the RLP (see definition 4.2.8) with respect to the inclusion $[i] : A \times \{0\} \hookrightarrow I_pA$. That is, a coarsely cellular class $[f]\colon X \to Y$ is a coarse Serre fibration if it satisfies the coarse homotopy lifting property (see definition 4.2.4) with respect to I_pA.

Proposition 4.2.10 *Let R be a generalized ray. Let X be a non-unital coarse CW-complex with only finitely many cells in each dimension such that $\pi_0^{crs}(X) = 0$ with the basepoint inclusion $i_X\colon R \to X$. Let $n \in \mathbb{N}$, then the following are equivalent:*

(1) *For any coarsely cellular class $[f]\colon (R \sqcup R)^n \to X$, the representative map f is coarsely homotopic to the the composition $[i_X \circ p]\colon (R \sqcup R)^n \to X$ where $p\colon (R \sqcup R)^n \to R$ is some controlled map.*

(2) *Given a coarsely cellular class $[f]\colon (R \sqcup R)^n \to X$, we have a coarse class $[F]\colon (R \sqcup R)^n \times R \to X$ such that $F(x,0) = f(x)$ for all $x \in (R \sqcup R)^n$, and the class $[g]\colon (R \sqcup R)^n \to X$ defined by $g(x) = F(0, p(x))$ is coarsely cellular for the usual controlled map p.*

(3) $\pi_n^{Pcrs}(X) = 0$

Proof : (1) $\Leftrightarrow$ (3) From the definition of $\pi_n^{crs}(X)$. The group $\pi_n^{crs}(X) = 0$ if and only if any two coarsely cellular classes $[f], [g]\colon (R \sqcup R)^n \to X$ are the same and that is true if and only if the representative map $f\colon (R \sqcup R)^n \to X$ is coarsely homotopic to $i_X \circ p\colon (R \sqcup R)^n \to X$.

(1) $\Rightarrow$ (2) We prove this in one dimension; the proof in higher dimension is similar. Let $[f]\colon R \sqcup R \to X$ be a coarsely cellular class. By assumption the representative map f is coarse homotopic to the representative map of the coarsely cellular class $[i_X \circ p]\colon R \sqcup R \to X$, so we can define a coarse homotopy $H\colon I_p(R \sqcup R) \to X$ such that

$$H(x,0) = f(x), \quad H(x, p(x)+1) = i_X \circ p(x) \quad x \in R \sqcup R.$$

Define a map $F\colon (R \sqcup R) \times R \to X$ by

$$F(x,t) = \begin{cases} H(x,t) & t \leq p(x)+1 \\ H(x + (t - p(x) - 1)/2, (t + p(x) + 1)/2) & x \geq 0, \quad t \geq p(x)+1 \\ H(x - (t - p(x) - 1)/2, (t + p(x) + 1)/2) & x \leq 0, \quad t \geq p(x)+1 \end{cases}$$

Then F is a coarse map defined for all $(x,t) \in (R \sqcup R) \times R$ and $F(x,0) = H(x,0) = f(x)$ as required.

(2) $\Rightarrow$ (1) Let $[f]\colon (R \sqcup R)^n \to X$ be a coarsely cellular class, by (2) we have a coarse class $[F]\colon (R \sqcup R)^n \times R \to X$ such that $F(x,0) = f(x)$ for all $x \in (R \sqcup R)^n$.

Let $g\colon (R \sqcup R)^n \to X$ be the representative map of the class $[g]$ which defined by $g(x) = F(0, p(x))$, $x \in (R \sqcup R)^n$. Our claim is to show that f is coarsely homotopic to g. Define a map $H\colon I_p(R \sqcup R)^n \to X$ by writing

$$H(x_1, ..., x_n, t) = \begin{cases} F(S(x_1,t), ..., S(x_n,t), t) & t \leq p(x)+1 \\ F(0,t) & t \geq p(x)+1 \end{cases}$$

where

$$S(x,t) = \begin{cases} x-t & 0 \le t \le x \\ 0 & t \ge x \ge 0 \\ x+t & x \le 0, t \le -x \\ 0 & x \le 0, t \ge -x \end{cases}$$

We see that H is a coarse map, and $H(x,0) = F(x,0) = f(x)$, $H(x,p(x)+1)) = F(0,p(x)+1)$, and $F(0,p(x)+1)$ is close to $F(0,p(x)) = g(x)$. So H defines a coarse homotopy between f and g, where $g = h \circ p$, and $[h]\colon R \to X$ is some coarsely cellular class, and that implies that $[f] = [h \circ p]$.

But we need f to be coarsely homotopic to $i_X \circ p$. Clearly this is true since by assumption $\pi_0^{crs}(X) = 0$, which implies that $[h] = [i_X]$, and the result follows. □

The following lemma is proved with the same techniques used in [5].

Lemma 4.2.11 *Let R be a generalized ray, $n \in \mathbb{N}$. Let $[f]\colon X \to Y$ be a coarsely cellular class f between coarse CW-complexes with only finitely many cells in each dimension. Then $[f]$ is a coarse Serre fibration class if and only if it has the RLP with respect to the inclusion $D_R^n \hookrightarrow I_p D_R^n$, $n \ge 0$, where $p\colon (R \sqcup R)^n \times R \to R$ is some controlled map.*

Proof : One direction is clear since D_R^n is a coarse CW-complex.

Conversely, we will prove a general result here of which this lemma is a special case.

Namely, we show that if the coarsely cellular class f has the RLP with respect to the inclusion $D_R^n \times \{0\} \hookrightarrow I_p D_R^n$ then this implies that $[f]$ has the RLP with respect to the inclusion $X \times \{0\} \cup I_{q \circ i} A \hookrightarrow I_q X$, where $q\colon X \to R$ is some controlled map, $[i] : A \hookrightarrow X$ is the inclusion, (X, A) is a non-unital coarse CW-pair.

Then the special case when we take $(B, \emptyset)$ for any coarse CW-pair (X, B) as the CW-pair reduces to this lemma.

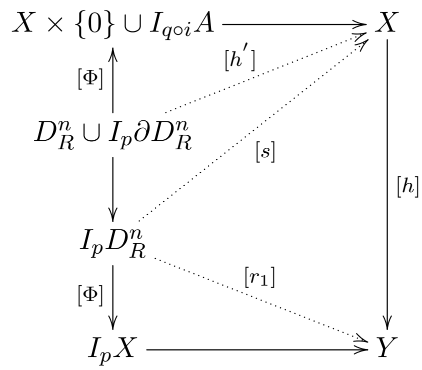
By lemma 4.2.7 we have that the pair

$$(I_p(D_R^n), D_R^n \times \{0\}), (I_p(D_R^n), D_R^n \times \{0\} \cup I_{p \circ j}(\partial D_R^n)$$

are relative coarse homotopy equivalent, which means that the RLP with respect to the two inclusions is equivalent.

We can use induction over the skeletons of X and lift one coarse cell of X at a time. Those lifts reduce to the lifting of coarse cells by composing with the characteristic map class $[\Phi_R]\colon (R \sqcup R)^n \times R \to X$ of a coarse cell.

Then we have the following commutative diagram



where the lower and upper left classes are the restrictions of the characteristic map class, and by the definition of the characteristic map we have $\Phi_1(I_p \partial D^n_R) \subset I_{q \circ i} A$, since $\partial D^n_R = (R \sqcup R)^n \times \{0\}$ which is attached to $k-1$ skeleton.

In other words, by assuming that $[f]$ has the RLP with respect to the inclusion $X \times \{0\} \cup I_{q \circ i} A \hookrightarrow I_q X$ and the construction of X^n, a lift s_{n-1} exists for the $(n-1)$-skeleton X^{n-1} where X^{n-1} is obtained from A by attaching $(n-1)$-coarse cells. By definition of coarse CW-complex, then X^n is obtained from A by attaching n-coarse cells, and since $[i] : X^{n-1} \hookrightarrow X^n$ is the inclusion, the class $[i \circ s_{n-1}]$ defines a lift for the n-skeleton X^n, and we are letting A get bigger in every step of induction.

Continuing in this way, by definition of coarse CW-complex and the inclusion map we have a lift for X, and X is obtained from A by attaching coarse cells. The classes $[h']$, $[r_1]$ are compositions. Then a lift exists for X.

Since $[\Phi]$ is an inclusion, we can construct inductively a lift for whole X with this construction. $\square$

Lemma 4.2.12 *Let R be a generalized ray, $n \in \mathbb{N}$. Let $[f] \colon X \to Y$ be a coarsely cellular class between non-unital coarse CW-complexes with only finitely many coarse cells in each dimension. Then the following conditions are equivalent:*

(1) $[f]$ *is both a coarse Serre fibration class and a weak coarse homotopy equivalence class.*

(2) $[f]$ *has the RLP with respect to the inclusion $A \hookrightarrow B$ where (B, A) is a relative CW- pair, and*

(3) $[f]$ *has the RLP with respect to the inclusions $[J_n] \colon (R \sqcup R)^n \to (R \sqcup R)^n \times R$ defined by $J_n(x) = (x, 0)$ for all $x \in (R \sqcup R)^n$, $n \geq 0$.*

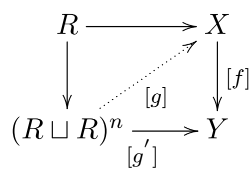
Proof : $(2), (3) \Rightarrow (1))$: By (2) $[f]$ has the RLP with respect to the inclusion $A \hookrightarrow B$ of a relative CW- pair (B, A). In particular $[f]$ has the RLP with repect to $(I_p D^n_R, D^n_R)$. Then by lemma 4.2.11 $[f]$ is a coarse Serre fibration class.

Now we want $[f]$ to be a weak coarse homotopy equivalence class. Let $[g]\colon (R \sqcup R)^n \to X$ be a coarsely cellular class, then $[fg]\colon (R \sqcup R)^n \to Y$ is a coarsely cellular class. We need to show that if fg is a coarse homotopic to the representative map $f \circ i_X \circ p\colon (R \sqcup R)^n \to Y$ where $i_X\colon R \to X$ the basepoint inclusion onto X and $p\colon (R \sqcup R)^n \to R$ is some controlled map, then g is also coarse homotopic to the representative map $i_X \circ p\colon (R \sqcup R)^n \to X$.

But if fg is a coarse homotopic to the representative map $f \circ i_X \circ p$, then by proposition 4.2.10 there exists a coarse class $[F]\colon (R \sqcup R)^n \times R \to Y$ such that $F(x, 0) = fg(x)$ for all $x \in (R \sqcup R)^n$, and from (3) we have a lift $(R \sqcup R)^n \times R \to X$. Therefore there exists a coarse class $[G]\colon (R \sqcup R)^n \times R \to X$ such that $G(x, 0) = g(x)$ for all $x \in (R \sqcup R)^n$, and by proposition 4.2.10 the map g is also coarse homotopic to the representative map $i_X \circ p$. Hence $[f_*]$ is injective.

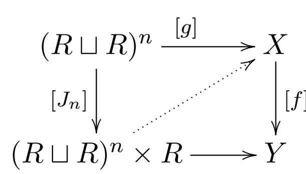
Now for surjectivity, let $[g']\colon (R \sqcup R)^n \to Y$ be a coarsely cellular class, we need to find a coarsely cellular class $[g]\colon (R \sqcup R)^n \to X$ such that fg is coarsely homotopic to g'.

Since $((R \sqcup R)^n, R)$ is a relative CW-pair, a lift $[g]$ exists in the following diagram



and by (2) the class $R \to (R \sqcup R)^n$ has the LLP with respect to $[f]$. Hence $[f_*]$ is surjective, and thus $[f]$ is a weak coarse homotopy equivalence class.

(1) ⇒ (3) Consider the following diagram



where $[f]$ is a coarse Serre fibration and a weak coarse homotopy equivalence class , $[J_n]$ is an inclusion. The composite fg is a coarse homotopic to the representative map $f \circ i_X \circ p$ of the coarse cellular class $[f \circ i_X \circ p]$ defined above since it extends over the space $(R \sqcup R)^n \times R$ in the above diagram and using proposition 4.2.10 we can define a coarse class $[F]\colon (R \sqcup R)^n \times R \to Y$ such that $F(x, 0) = fg(x)$ for all $x \in (R \sqcup R)^n$. Since $[f]$ is a weak coarse homotopy equivalence class, then the map $g\colon (R \sqcup R)^n \to X$ is coarsely homotopic to the representative map $i_X \circ p$ of the coarse cellular class $i_X \circ p$. Therefore again by proposition 4.2.10 there exists a coarse class $[G]\colon (R \sqcup R)^n \times R \to X$ such that $G(x, 0) = g(x)$ for all $x \in (R \sqcup R)^n$. Hence $[f]$ has the RLP with respect to all classes $[J_n]$.

(3) ⇒ (2) Since by (3) the class $[f]$ has the RLP with respect to all inclusions $[J_n]\colon (R \sqcup R)^n \to (R \sqcup R)^n \times R$ defined by $J_n(x) = (x, 0)$ for all $x \in (R \sqcup R)^n$,

$n \geq 0$, just as in the proof of the above lemma we can use induction on the skeleton of $B - A$ and lift by using the characteristic maps;

$$\begin{array}{ccccc} (R \sqcup R)^n & \xrightarrow{[\Phi]} & A & \longrightarrow & X \\ {\scriptstyle [J_n]}\big\downarrow & {\scriptstyle [k]} & \big\downarrow & & \big\downarrow{\scriptstyle [f]} \\ (R \sqcup R)^n \times R & \xrightarrow{[]i} & B & \longrightarrow & Y \end{array}$$

we can use these lifts to get a lift $[h]$ from B to X by taking $[h][] = k \circ i^{-1}$ since $[i]$ is injective.

(2) $\Rightarrow$ (3) Since $((R \sqcup R)^n \times R, (R \sqcup R)^n)$ is a relative CW-pair, then (3) is obvious. $\square$

Remark 4.2.13 Let X_0, X_1, X_2,... $\in Ob(CWrs)$, and $[i_n] : X_n \hookrightarrow X_{n+1}$ are inclusions. Then by definition of coarse CW-complex (see definition 4.1.3), the space $\bigcup_{n=0}^{\infty} X_n$ defines a colimit in the category CWrs and every colimit that comes up in this category takes this form.

Lemma 4.2.14 *Suppose that $X_0 \to X_1 \to X_2 \to ... \to X_n \to ...$ is a directed system of non-unital coarse CW-complexes with only finitely many coarse cells in each dimension such that for each $n \geq 0$, the non-unital coarse pointed CW-coarse complex X_n is a coarse subspace of X_{n+1} and the pair (X_{n+1}, X_n) is a relative non-unital coarse pointed CW-complex. Suppose further that $X = colim_n X_n$ has only finitely many coarse cells of each dimension.*

Let A be a finite coarse CW-complex, then the natural map

$$colim_n Hom_{CWrs}(A, X_n) \xrightarrow{\alpha} Hom_{CWrs}(A, colim_n X_n)$$

is a bijection, where $Hom_{CWrs}(A, X_n)$ is a collection of all morphisms $A \to X_n$ in the category CWrs, and $Hom_{CWrs}(A, colim_n X_n)$ is the set of all morphisms $A \to colim_n X_n$ in the same category CWrs.

Proof : Let $X = colim_n X_n$, then $X = \bigcup_{n=0}^{\infty} X_n$ and we have inclusion classes $[i_{n]} : X_n \hookrightarrow X$. By definition of colimits, the set $colim_n Hom_{CWrs}(A, X_n)$ is equipped with maps $j_n : Hom_{CWrs}(A, X_n) \to colim_n Hom_{CWrs}(A, X_n)$ which satisfy the universal property and where

$$colim_n Hom_{CWrs}(A, X_n) = \bigcup_{n=0}^{\infty} Hom_{CWrs}(A, X_n)$$

Let $[f] \in colim_n Hom_{CWrs}(A, X_n)$. Then $[f] = [j_n(g)]$ for some $[g] \in Hom_{CWrs}(A, X_n)$, and the map α is defined by the formula $\alpha([j_n(g)]) = [i_n \circ g]$.

Since $X_n \subseteq X_{n+1} \subseteq X_{n+2} \subseteq ...$, we also have $[g]$ living in $Hom_{CWrs}(A, X_m)$ for $m \geq n$.

Let $[j^k] : X_n \to X_{n+k}$ be a coarsely cellular class, we need to show that $\alpha([j^k \circ f]) = \alpha([f])$. By definition of coarsely cellular class we have $[i_{n+k} \circ j^k] = [i_n]$.

Then $\alpha([j^k \circ f]) = \alpha([j^k \circ j_n(g)]) = [i_{n+k} \circ j^k \circ g] = [i_n \circ g =]\alpha([f])$. Hence α is well defined.

Now we want to show that α is surjective. Let $[f] \in Hom_{CWrs}(A, colim_n X_n)$, and since A is a finite non-unital coarse CW-complex, then by definition of coarsely cellular class we have $[f] \in colim_n Hom_{CWrs}(A, X_n)$ so α is surjective as required.

Now let $[f]$, $[h] \in colim_n Hom_{CWrs}(A, X_n)$ such that $\alpha([f]) = \alpha([h])$, then $[f] = [j_n(g)]$, $[h] = [j_n(g^{'})]$ for some $[g]$, $[g^{'}] \in Hom_{CWrs}(A, X_n)$ and $[i_n \circ g] = []i_n \circ g^{'}$. Since $[]i_n$ is the inclusion class and $[g]$, $[g^{'}]$ are coarsely cellular classes, $[f] = [j_n(g)] = [j_n(g^{'})] = [h]$. Hence α is injective. $\square$

Definition 4.2.15 An object X of a category of coarse spaces is said to be a *retract* of an object Y if there exist classes $[i]\colon X \to Y$ and $[r]\colon Y \to X$ such that ri is close to id_X.

And a class $[f]$ is a retract of a class $[g]$ if there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{[i]} & Y & \xrightarrow{[r]} & X \\ {\scriptstyle [f]}\downarrow & & {\scriptstyle [g]}\downarrow & & {\scriptstyle [f]}\downarrow \\ X^{'} & \xrightarrow[{[i^{'}]}]{} & Y^{'} & \xrightarrow[{[r^{'}]}]{} & X^{'} \end{array}$$

such that ri is close to id_X, and $r^{'}i^{'}$ is close to $id_{X^{'}}$.

Definition 4.2.16 Let $[f]\colon X \to Y$ be a coarsely cellular class, then $[f]$ is called a *coarse CW-cofibration class* if it has the LLP with respect to a coarse Serre fibration class and a weak coarse homotopy equivalence class.

Theorem 4.2.17 *The category CWrs can be given the structure of Baues cofibration category by defining $[f]\colon X \to Y$ to be*

(1) *a weak equivalence if $[f]$ is a weak coarse homotopy equivalence class.*

(2) *a cofibration if $[f]$ is a coarse CW-cofibration class.*

We can see that the two classes of weak coarse homotopy equivalence and coarse CW-cofibration contain all identity maps and are closed under compositions. We prove this theorem in the next section.

§ 4.3 The Category CWrs is a Baues Cofibration Category

In this section we prove that the category CWrs can be given a structure of Baues cofibration category, and in order to do that we need to satisfy the axioms for the structure of a cofibration category. Actually here we will prove the last four axioms of Quillen model category by a similar technique to that used in [5], and proving the Baues axioms will be a particular case. This shows we nearly have a structure of Quillen model category.

First we need the following proposition.

Proposition 4.3.1 *If* $[f]$ *is a retract of* $[g]$ *and* $[g]$ *is a coarse Serre fibration class, coarse CW-cofibration class, or a weak coarse homotopy equivalence class, then so is* $[f]$*.*

Proof : First, we prove the case when $[g]$ is a weak coarse homotopy equivalence class.

Let $[f]\colon X \to X'$ be a retract of a weak coarse homotopy equivalence class $[g]\colon Y \to Y'$, then $[g_*]\colon \pi_n^{Pcrs}(Y) \to \pi_n^{Pcrs}(Y')$ is an isomorphism, so there exists a coarsely cellular class $[k_*]\colon \pi_n^{Pcrs}(Y') \to \pi_n^{Pcrs}(Y)$ such that $[g_*k_*] = [id_{\pi_n^{Pcrs}(Y')}]$, and $[k_*g_*] = [id_{\pi_n^{Pcrs}(Y)}]$.

Since ri is close to id_X, we have $[r_*i_*] = [id_{\pi_n^{Pcrs}(Y)}]$. Similarly, $[r'_*i'_*] = [id_{\pi_n^{Pcrs}(Y')}]$.

Then $r_*g_*i'_*f_*([\alpha]) = r_*k_*g_*i_*([\alpha]) = r_*i_*([\alpha]) = [\alpha]$, $\alpha \in \pi_n^{Pcrs}(X)$, and $f_*r_*k_*i'_*([\beta]) = r'_*g_*k_*i'_*([\beta]) = r'_*i'_*([\beta]) = [\beta]$, $\beta \in \pi_n^{Pcrs}(X')$. Therefore $r_*k_*i'_*$ is an inverse of h_*, so then $[f_*]$ is an isomorphism. Hence $[f]$ is a weak coarse homotopy equivalence class.

Second, let $[f]$ be a retract of $[g]$, $[g]$ is a coarse Serre fibration class, we want to show that $[f]$ is a coarse Serre fibration class. Let $[S]\colon A \times \{0\} \to X$ be a coarsely cellular class, and $[F]\colon I_pA \to X'$ be a coarse homotopy class where $p\colon A \to R$ is some controlled map such that $F(a,0) = f \circ S(a,0)$ for all $a \in A$, then $[k = i \circ S]\colon A \times \{0\} \to Y$ is a cellular coarse class.

Now define a coarse homotopy class $[K = i' \circ F]\colon I_pA \to Y'$, then $K(a,0) = i' \circ F(a,0) = i' \circ f \circ S(a,0) = g \circ i \circ S(a,0) = g \circ k(a,0)$ for all $a \in A$. Since $[g]$ is coarse Serre fibration class, there exists a coarse homotopy class $[G]\colon I_pA \to Y$ such that $g \circ G = K$, and $G(a,0) = k(a,0)$ for all $a \in A$.

Define a coarse class $[H]\colon I_pA \to X$ such that $H(a,t) = r \circ G(a,t)$, $a \in A$, $t \in R$. Then $f \circ H(a,t) = f \circ r \circ G(a,t) = r' \circ g \circ G(a,t) = r' \circ K(a,t) = F(a,t)$, $a \in A$, $t \in R$, and $H(a,0) = r \circ G(a,0) = r \circ k(a,0) = r \circ i \circ S(a,0)$ for all $a \in A$.

Define another class $[H']\colon I_pA \to X$ by

$$H'(a,t) = \begin{cases} S(a,t) & t = 0 \\ H(a,t) & otherwise \end{cases}$$

But H' is close to H, and since H is a coarse map by lemma 2.2.3 we have H' is a coarse map. Hence $[f]$ is a coarse Serre fibration class.

Finally, let $[f]$ be a retract of $[g]$, $[g]$ is a coarse CW-cofibration class, and we want to show that $[f]$ is a coarse CW-cofibration class. $h' \in [g]$. Let

$[p]\colon E \to B$ be a coarse Serre fibration and weak coarse homotopy equivalence class, and consider a lifting problem in the following diagram

$$\begin{array}{ccc} X & \xrightarrow{[g']} & E \\ {\scriptstyle [f]}\big\downarrow & & \big\downarrow{\scriptstyle [p]} \\ X' & \xrightarrow[{[g]}]{} & B \end{array}$$

Enlarge this to the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{[i]} & Y & \xrightarrow{[g']} & E \\ {\scriptstyle [f]}\big\downarrow & & {\scriptstyle [g]}\big\downarrow & & \big\downarrow{\scriptstyle [p]} \\ X' & \xrightarrow[{[i']}]{} & Y' & \xrightarrow[{[g'']}]{} & B \end{array}$$

Since $[g]$ is coarse CW-cofibartion class, there is a lifting $[H]\colon Y' \to E$ in the above diagram, and since $[f]$ is a retract of $[g]$, the classes $[H]$ and $[i]$ induce the desired lifting $X' \to E$. $\square$

We recall the axioms of a Baues cofibration category in order to prove they are satisfied in the category CWrs.

Axiom C1: (Composition axiom) Isomorphisms are both coarse CW-cofibration classes and weak coarse homotopy equivalence classes. If $[f]$ and $[g]$ are coarsely cellular classes in $CWrs$ such that $[gf]$ is defined and if two of the three morphisms $[f]$,$[g]$, $[gf]$ are weak coarse homotopy equivalence classes, then so is the third. The composite of coarse CW-cofibration classes is a coarse CW-cofibration class.

Proof : Suppose that $[f]\colon X \to Y$ is an isomorphism between non-unital coarse CW-complexes, then we have a coarsely cellular class $g\colon Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y. First we need to show that $[f^0_*]\colon \pi_0^{Pcrs}(X) \to \pi_0^{Pcrs}(Y)$ defined by $f^0_*[\beta] = [f \circ \beta]$ is bijection where $[\beta]\colon R \to X$ is a coarsely cellular class in $\pi_0^{Pcrs}(X)$.

Let $[\beta_1], [\beta_2]\colon R \to X$ be two coarsely cellular classes in $\pi_0^{Pcrs}(X)$ such that $f^0_*[\beta_1] = f^0_*[\beta_2]$, then $[f \circ \beta_1] = [f \circ \beta_2]$ where $[f \circ \beta_1], [f \circ \beta_2]\colon R \to Y$ are two coarsely cellular classes in $\pi_0^{Pcrs}(Y)$. So the representative map $f \circ \beta_1$ is coarsely homotopic to $f \circ \beta_2$. By theorem 2.2.11 we have $g \circ f \circ \beta_1$ is coarsely homotopic to $g \circ f \circ \beta_2$, but by the above $g \circ f$ is close to id_X, and it follows that β_1 is coarsely homotopic to β_2. Hence $[f^0_*]$ is injective.

Now let $[\beta']\colon R \to Y$ be a coarsely cellular class in $\pi_0^{Pcrs}(Y)$, by assumption we have $[g \circ \beta']\colon R \to X$ a coarsely cellular class in $\pi_0^{Pcrs}(X)$ and $f \circ g \circ \beta'$ is close to $id_Y \circ \beta' = \beta'$. Putting $g \circ \beta' = \beta$ shows that $[f^0_*]$ is surjective.

The induced class $[f_*]\colon \pi_n^{Pcrs}(X) \to \pi_n^{Pcrs}(Y)$ is isomorphism for $n \geq 1$ since $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y, we have $[g_* \circ f_*] = [id_{\pi_n^{Crs}(X)}]$

and $[f_* \circ g_*] = [id_{\pi_n^{Crs}(Y)}]$. Hence $[f]$ is a weak coarse homotopy equivalence class.

Now we need to show that $[f]$ is a coarse CW-cofibration class. Given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{[j]} & E \\ {\scriptstyle [f]}\downarrow & & \downarrow{\scriptstyle [p]} \\ Y & \xrightarrow[{[i]}]{} & B \end{array}$$

where $[p]$ is a weak coarse homotopy equivalence class and coarse Serre fibration class, we need to find a coarsely cellular class $[k]\colon Y \to E$ that fills the above diagram. Since $[f]$ is an isomorphism, $[f]$ is a retract of the identity class on Y as follows:

$$\begin{array}{ccccc} X & \xrightarrow{[f]} & Y & \xrightarrow{[g]} & X \\ {\scriptstyle [f]}\downarrow & & \downarrow{\scriptstyle [id]} & & \downarrow{\scriptstyle [f]} \\ Y & \xrightarrow[{[id]}]{} & Y & \xrightarrow[{[id]}]{} & Y \end{array}$$

Since the identity class on Y is a coarse CW-cofibration, by proposition 4.3.1 the map $[f]$ is also so.

Now suppose that $[f]$, $[gf]$ are weak coarse homotopy equivalence classes, so;

$$[f_*]\colon \pi_n^{Pcrs}(X) \to \pi_n^{Pcrs}(Y), \quad [(gf)_*]\colon \pi_n^{Pcrs}(X) \to \pi_n^{Pcrs}(Z)$$

are bijections for $n = 0$ and isomorphisms for $n \geq 1$. It is not hard to see that $[g_*]$ is a bijection for $n = 0$. Notice that $[g_*] = [(gf)_* f_*^{-1}]$ where $[(gf)_*]$, $[]f_*^{-1}$ are isomorphisms, then so is $[g_*]\colon \pi_n^{Pcrs}(Y) \to \pi_n^{Pcrs}(Z)$. □

Axiom C2: (Push out axiom) For a coarse CW-cofibration class $[i] : A \hookrightarrow X$ and a coarsely cellular class $[f]\colon A \to Y$ there exists the push out in $CWrs$

$$\begin{array}{ccc} X & \xrightarrow{[f']} & X \vee_A Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [i']} \\ A & \xrightarrow{[f]} & Y \end{array}$$

and $[i']$ is a coarse CW-cofibration class. Moreover:

(a) if $[f]$ is a weak coarse homotopy equivalence class, so is $[f']$,

(b) if $[i]$ is a weak coarse homotopy equivalence class, so is $[i']$.

Proof : We have shown that there is a push out diagram in the category of quotient coarse maps Qcrs in theorem 3.2.17 and since any coarse CW-complex is a coarse space, so the proof is similar also for non-unital coarse CW-complexes with only finitely many coarse cells in each dimension. Therefore it is enough to show that $A \vee_B Y$ is a coarse CW-complex with only finitely many coarse cells in each dimension and $[i^{'}]$ is a coarse CW-cofibration in axiom C2 in definition 3.1.1.

First the push out diagram defined by the coequalizer which is obtained by gluing two non-unital coarse CW-complexes X and Y (where $X \sqcup Y$ is the disjoint union as defined in definition 1.4.12) to each other by identifying the subcomplexes $i(A)$ with $f(A)$ to be close to each other. This argument and definition of wedge sum in [7] clearly produces the same disjoint union of the non-unital coarse CW-complexes X and Y equipped with the coequalizer coarse structure since $[i]$ and $[f]$ are coarsely cellular classes.

Now we need to show that $[i^{'}]$ has the LLP with respect to a weak coarse CW-equivalence class and a coarse Serre fibration class $[p]\colon E \to F$. That is similar to an argument in the above axiom, we consider a commutative daigram

$$\begin{array}{ccc} Y & \xrightarrow{[g]} & E \\ {\scriptstyle [i^{'}]}\downarrow & & \downarrow{\scriptstyle [p]} \\ B \vee_A Y & \xrightarrow[{[g^{'}]}]{} & F \end{array}$$

Enlarge this to the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{[f]} & Y & \xrightarrow{[g]} & E \\ {\scriptstyle [i]}\downarrow & & {\scriptstyle [i^{'}]}\downarrow & & \downarrow{\scriptstyle [p]} \\ X & \xrightarrow[{[f^{'}]}]{} & X \vee_A Y & \xrightarrow[{[g^{'}]}]{} & F \end{array}$$

Since $[i]$ is a coarse CW-cofibration class, so there exists a lifting $[h]\colon X \to E$, by the universal property of push out diagram there exists a unique coarsely cellular class $[h^{'}]\colon X \vee_A Y \to E$ that makes the above diagram commutes. $\square$

Axiom C3: (Factorization axiom) For any coarsely cellular class $[f]\colon X \to Y$ in $CWrs$ there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{[f]} & Y \\ & {\scriptstyle [i]}\searrow \quad \nearrow{\scriptstyle [g]}\,\sim & \\ & Z & \end{array}$$

where $[i]$ is a coarse CW-cofibration class and $[g]$ is a weak coarse homotoy equivalence class.

Axiom C4: (Axiom on fibrant models) For non-unital coarse CW-complexes with only finitely many coarse cells in each dimension X in $CWrs$ there is a trivial cofibration $X \overset{\sim}{\rightarrowtail} SX$ where SX is a model fibrant in $CWrs$.

To prove **axiom C3** and **axiom C4** we need a technical construction beginning with the map we want to factor.

Here we are following the topological technique used by Dwyer and Spalinski in [5] in order to show that the category of topological spaces has the structure of Quillen model category. And to be able to do that we try to benefit from proposition (2.6) in [1] that shows having a model category structure on a category enables us to construct a cofibration category easily even though we do not completely have a structure of Quillen model category.

First, recall a notion of Quillen model category as defined in [5] and [15].

Definition 4.3.2 A *model category* is a category $\mathcal{C}$ with three distinguished classes of maps:

1 weak equivalences ($\xrightarrow{\sim}$),

2 fibrations ($\twoheadrightarrow$), and

3 cofibrations ($\hookrightarrow$)

We require the following axioms.

A1 Finite limits and colimits exist in $\mathcal{C}$.

A2 If f and g are maps in $\mathcal{C}$ such that gf is defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.

A3 If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f.

A4 Given a commutative diagram of the first form in definition 4.2.8, a lift exists in the diagram in either of the following two situations: (i) i is a cofibration and p is both a fibration and weak equivalence, or (ii) i is both a cofibration and weak equivalence and p is a fibration.

A5 Any map f can be factored in two ways: (i) $f = pi$, where i is a cofibration and p is both a fibration and weak equivalence, and (ii) $f = pi$, where i is both a cofibration and weak equivalence, and p is a fibration.

The proof of these axioms is not usually obvious, as we will see in this section. Actually the aim is to try to define a Baues cofibration structure on the category Cwrs, and this is not obvious as well, and in order to do that we prove all Quillen axioms apart from existence of all finite limits. This is good as regarding Baues cofibration category we prove a more general axioms and Baues axioms will be a particular case as will be shown as follows.

The following definition has an identical technique to remark (7.15) in [5].

Definition 4.3.3 Let $\mathfrak{F} = \{[f_i] : A_i \to B_i\}$ be a set of coarsely cellular classes in the category CWrs. Let $[p]\colon X \to Y$ be a coarsely cellular class.

For each $i \in I$, consider the set $\mathcal{D}(i)$ of pairs of coarsely cellular classes $([g],[h])$ that makes the following diagram commute

$$\begin{array}{ccc} A_i & \xrightarrow{[g]} & X \\ {\scriptstyle [f_i]}\big\downarrow & & \big\downarrow{\scriptstyle [p]} \\ B_i & \xrightarrow[{[h]}]{} & Y \end{array} \qquad (1)$$

Then we define the *Gluing Construction* $G^1(\mathfrak{F},[p])$ to be the push out of the diagram;

$$\begin{array}{ccc} \coprod_{i\in I}\coprod_{([g],[h])\in\mathcal{D}(i)} A_i & \longrightarrow & X \\ {\scriptstyle [\coprod_{i\in I} f_i]}\big\downarrow & & \big\downarrow{\scriptstyle [i_1]} \\ \coprod_{i\in I}\coprod_{([g],[h])\in\mathcal{D}(i)} B_i & \longrightarrow & G^1(\mathfrak{F},[p]) \end{array}$$

This push out is defined in the coequalizer sense as in 3.2.14. So we are gluing a copy of B_i to X along A_i for every commuting diagram of the form (1).

Now by universality of push outs, we find a coarsely cellular class $[p_1]$ such that the following diagram commutes;

$$\begin{array}{cccc} \coprod_{i\in I}\coprod_{([g],[h])\in\mathcal{D}(i)} A_i & \longrightarrow & X & \\ {\scriptstyle [\coprod_{i\in I} f_i]}\big\downarrow & & \big\downarrow{\scriptstyle [i_1]} & \searrow{\scriptstyle [p]} \\ \coprod_{i\in I}\coprod_{([g],[h])\in\mathcal{D}(i)} B_i & \longrightarrow & G^1(\mathfrak{F},[p]) & \\ & \searrow & \quad {\scriptstyle [p_1]}\searrow & \\ & & & Y \end{array}$$

Now repeat the process to construct $G^k(\mathfrak{F},[p])$ and a coarsely cellular classes $[p_k]$ from $G^k(\mathfrak{F},[p])$ to Y.

We repeat the gluing construction but now replacing $[p]$ by $[p_1]$, so let

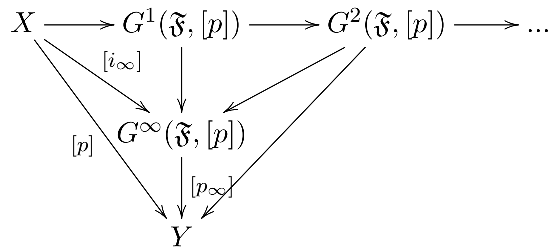
$$G^2(\mathfrak{F},[p]) = G^1(\mathfrak{F},[p_1]), \quad [p_2] = [(p_1)_1]$$

and continue in this way.

More generally $G^k(\mathfrak{F},[p]) = G^1(\mathfrak{F},[p_{k-1}])$ and $[p_k] = [(p_{k-1})_1]$. This results in the following commuting diagram:

$$\begin{array}{ccccccccccc} X & \xrightarrow{[i_1]} & G^1(\mathfrak{F},[p]) & \xrightarrow{[i_2]} & G^2(\mathfrak{F},[p]) & \xrightarrow{[i_3]} & .... & \longrightarrow & G^k(\mathfrak{F},[p]) & \xrightarrow{[i_{k+1}]} & .... \\ {\scriptstyle [p]}\big\downarrow & & {\scriptstyle [p_1]}\big\downarrow & & {\scriptstyle [p_2]}\big\downarrow & & & & {\scriptstyle [p_k]}\big\downarrow & & \\ Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & .... & \xrightarrow{=} & Y & \xrightarrow{=} & .... \end{array}$$

Now let $G^\infty(\mathfrak{F},[p])$, the infinite gluing construction, denote the colimit (see definition 4.2.13) of the upper row. Thus by universality of the colimit, there are natural maps $[i_\infty]\colon X \to G^\infty(\mathfrak{F},[p])$ and $[p_\infty]\colon G^\infty(\mathfrak{F},[p]) \to Y$ such that $[p_\infty i_\infty] = [p]$ as follows



The following proposition is a coarse version of proposition (7.17) in [5].

Proposition 4.3.4 *In the above situation, suppose that for each $i \in I$, the object A_i has the property that,*

$$colim_n Hom_{CWrs}(A_i, G^n(\mathfrak{F},[p])) \xrightarrow{\alpha} Hom_{CWrs}(A_i, colim_n G^n(\mathfrak{F},[p]))$$

is a bijection, where $Hom_{CWrs}(A_i, colim_n G^n(\mathfrak{F},[p])) = Hom_{CWrs}(A_i, G^\infty(\mathfrak{F},[p]))$. Then the map $[p_\infty]$ has the RLP with respect to each of maps in the family $\mathfrak{F}$.

Proof : Consider a commutative diagram that illustrates the lifting problem for any coarsely cellular class in $\mathfrak{F}$ and any coarsely cellular class $[p]$:

$$\begin{array}{ccc} A_i & \xrightarrow{[g]} & G^\infty(\mathfrak{F},[p]) \\ {\scriptstyle [f_i]}\downarrow & & \downarrow{\scriptstyle [p_\infty]} \\ B_i & \xrightarrow[{[k]}]{} & Y \end{array}$$

By lemma 4.2.14 since α is bijection, then there exists an integer k such that for the coarsely cellular class $[g]$, the representative map g is close to the representative composite of a coarsely cellular class $[g^{'}]\colon A_i \to G^k(\mathfrak{F},p)$ with the natural class $G^k(\mathfrak{F},[p]) \to G^\infty(\mathfrak{F},[p])$.

Now we can enlarge the previous commutative diagram to

$$\begin{array}{ccccccc} A_i & \xrightarrow{[g^{'}]} & G^k(\mathfrak{F},[p]) & \xrightarrow{[i_{k+1}]} & G^{k+1}(\mathfrak{F},[p]) & \longrightarrow & G^\infty(\mathfrak{F},[p]) \\ {\scriptstyle [f_i]}\downarrow & & {\scriptstyle [p_k]}\downarrow & & {\scriptstyle [p_{k+1}]}\downarrow & & \downarrow{\scriptstyle [p_\infty]} \\ B_i & \xrightarrow{[k]} & Y & \xrightarrow{=} & Y & \xrightarrow{=} & Y \end{array}$$

where the class $[g]$ is the composite class of the top row. The pair $([g^{'}],[k])$ is contained in the set of classes $\mathcal{D}(i)$ in the construction of $G^{k+1}(\mathfrak{F},[p])$ from $G^k(\mathfrak{F},[p])$. So there is a class from B_i to $G^{k+1}(\mathfrak{F},[p])$ that makes the diagram commute. We can compose this class with the coarsely cellular class $G^{k+1}(\mathfrak{F},[p]) \to G^\infty(\mathfrak{F},[p])$ in the upper row, to get the desired lift in the original square. $\square$

Definition 4.3.5 A *coarse deformation retraction* of a coarse CW-complex X onto a coarse CW-subcomplex A is a coarse map $F\colon I_pX \to X$, where $p\colon X \to R$ is a controlled map such that $F(x,0) = x$, $x \in X$, $F(x, p(x)+1) \in A$, and $F(a,t) = a$ for all $(a,t) \in I_{p \circ i}A$ where $i : A \hookrightarrow X$ is the inclusion.

Lemma 4.3.6 *Every coarsely cellular class* $[p]\colon X \to Y$ *in the category CWrs can be factored as a composite* $[p_\infty i_\infty]$*, where* $[i_\infty]\colon X \to X'$ *is a weak coarse homotopy equivalence class which has the LLP with respect to all coarse Serre fibration classes, and* $[p_\infty]\colon X' \to Y$ *is a coarse Serre fibration class.*

Proof : Let $\mathfrak{F}$ be the set of coarsely cellular classes

$$(R \sqcup R)^n \times R \to I_q((R \sqcup R)^n \times R)$$

where $q\colon (R \sqcup R)^n \times R \to R$ is some controlled map.

Consider the gluing construction $G^1(\mathfrak{F}, [p])$. We obtain $G^1(\mathfrak{F}, [p])$ by gluing many cylinders $I_q((R \sqcup R)^n \times R)$ to X along one end of those cylinders. So it follows that $(G^1(\mathfrak{F}, [p]), X)$ is a non-unital coarse CW-pair.

By the argument in lemma 2.2.12, we see that $I_q((R \sqcup R)^n \times R)$ is coarsely homotopy equivalent to $(R \sqcup R)^n \times R$, and then by lemma 4.2.1 we have $(R \sqcup R)^n \times R$ is coarsely homotopy equivalent to R.

Then $[i_1]\colon X \to G^1(\mathfrak{F}, [p])$ is a coarse deformation retraction, so it is a coarse homotopy equivalence. This class $[i_1]$ is a relative coarse inclusion class and a weak coarse homotopy equivalence class, so it follows from the definition of a coarse Serre fibration class that it has the LLP with respect to all coarse Serre fibration classes.

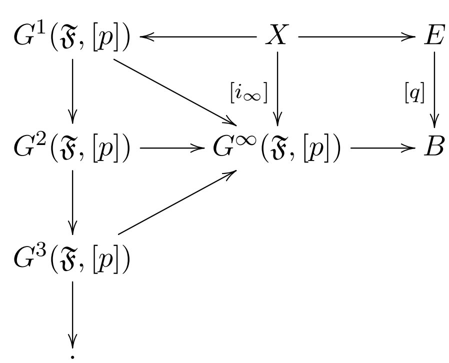
Similarly for each $k \geq 1$, the class $[i_{k+1}]\colon G^k(\mathfrak{F}, [p]) \to G^{k+1}(\mathfrak{F}, [p])$ is a coarse homotopy equivalence class and has the LLP with respect to all coarse Serre fibration classes for each k.

Now consider the factorization;

$$X \xrightarrow{[i_\infty]} G^\infty(\mathfrak{F}, [p]) \xrightarrow{[p_\infty]} Y$$

obtained by the infinite gluing construction.

From the following commutative diagram, we see that $[i_\infty]$ has the LLP with respect to all coarse Serre fibration classes, where $[q]$ is a coarse Serre fibration class;



We can find a lift from $G^1(\mathfrak{F},[p])$ to E, since $[i_1]$ has the LLP with respect to coarse Serre fibration class, but then we can use this to get a lift from $G^2(\mathfrak{F},[p])$ to E, and we can repeat this process to get cellular coarse classes from all $G^n(\mathfrak{F},[p])$ to E. Thus by the universal property of $G^\infty(\mathfrak{F},[p])$, we finally get the wanted lift, from $G^\infty(\mathfrak{F},[p])$ to E.

The proof of proposition 4.3.4 shows that $[p_\infty]$ has the RLP with respect to the coarsely cellular classes in $\mathfrak{F}$ and so by lemma 4.2.11 $[p_\infty]$ is a coarse Serre fibration class.

The problem reduces to show that

$$\mathrm{colim_n Hom_{CWrs}}((R\sqcup R)^i\times R,\mathrm{G^n}(\mathfrak{F},[p]))\xrightarrow{\alpha}\mathrm{Hom_{CWrs}}((R\sqcup R)^i\times R,\mathrm{colim_n G^n}(\mathfrak{F},[p]))$$

is bijection, which is so by lemma 4.2.14.

Now the last thing we need to show is that $[i_\infty]$ is a weak coarse homotopy equivalence class. By lemma 4.2.14, but with $A=(R\sqcup R)^i$ as a finite CW-complex we see that every coarsely cellular class $(R\sqcup R)^i\to G^\infty(\mathfrak{F},[p])$ lies in one of the subsets $G^k(\mathfrak{F},[p])$ for some k, and since all coarsely cellular classes $[i_k]$ are weak coarse homotopy equivalence, so is $[i_\infty]$. $\square$

Here we prove a general result and then our required axioms will be special cases.

Proposition 4.3.7 *Any coarsely cellular class $[f]$ in the category CWrs can be factored in two ways:*

(i) $[f]=[p\circ i]$*, where $[i]$ is a coarse CW-cofibration class and $[p]$ is both a coarse CW-fibration class and weak coarse homotopy equivalence class, and*

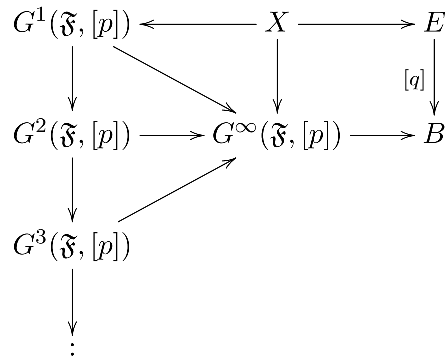
(ii) $[f]=[p\circ i]$*, where $[i]$ is both a coarse CW-cofibration class and weak coarse homotopy equivalence class, and $[p]$ is a coarse Serre fibration class.*

Proof : Part (ii) is an immediate consequence of the above lemma. We see that we can construct a factorization $[p\circ i]$ for every coarsely cellular class $[f]$ in the category CWrs, where $[p]$ is a coarse Serre fibration class and $[i]$ is a weak coarse homotopy equivalence class and has the LLP with respect to all coarse Serre fibration classes (and thus is both a coarse CW-cofibration class and weak coarse homotopy equivalence class).

To prove (i) we use a similar construction. Let $[p]$ be a weak coarsely coarse class in the category Cwrs and let $\mathfrak{F}=\{j_n:(R\sqcup R)^n\hookrightarrow(R\sqcup R)^n\times R\}$.

Now use the Infinite Gluing Construction to find the factorization $[p_\infty\circ i_\infty]$. We see that $G^{n+1}(\mathfrak{F},[p])$ is obtained from $G^n(\mathfrak{F},[p])$ by attaching finitely many n-coarse cells along their boundaries, so $(G^{n+1}(\mathfrak{F},[p]),G^n(\mathfrak{F},[p]))$ is a non-unital relative CW-pair. From lemma 4.2.12 we observe that the classes $[i_{n+1}]\colon G^n(\mathfrak{F},[p])\to G^{n+1}(\mathfrak{F},[p])$ have the LLP with respect to all coarse Serre fibration classes that are also weak coarse homotopy equivalence classes since those have the RLP with respect to inclusion classes of non-unital relative coarse CW-pairs.

Now let $[q]$ in the following diagram be both a weak coarse homotopy equivalence class and coarse Serre fibration class;



Again by induction we can find lifts from $G^k(\mathfrak{F},[p])$ to E for all k and by the universality of the push outs we get a lift from $G^k(\mathfrak{F},[p])$ to E, so $[i_\infty]$ has the LLP with respect to all coarse Serre fibration classes that are also weak coarse homotopy equivalence classes and thus is a coarse CW-cofibration class.

By lemma 4.2.14 and the proof of proposition 4.3.4 we find that $[p_\infty]$ has the RLP with respect to all maps in the set $\mathfrak{F}$ and thus is a coarse Serre fibration class and weak coarse homotopy equivalence class, which is both a coarse Serre fibration class and weak coarse homotopy equivalence class in our category CWrs. □

Proposition 4.3.8 *Given a commutative diagram of the first form in definition 4.2.8, a lift exists in the diagram in either of the following two situations:*

1 $[i]$ is a coarse CW-cofibration class and $[p]$ is both a coarse Serre fibration class and a weak coarse homotopy equivalence class, or

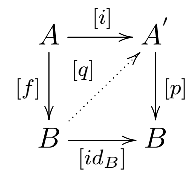
2 $[i]$ is both a coarse cofibration class and weak coarse homotopy equivalence class and $[p]$ is a coarse Serre fibration class.

Proof : (i) is obvious as by definition the coarse CW-cofibrations have the RLP with respect to coarse Serre fibration class and weak coarse homotopy equivalence classes.

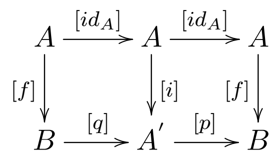
Now for (ii). Suppose that $[f]\colon A \to B$ is both a coarse CW-cofibration and weak coarse homotopy equivalence class. By lemma 4.3.6 we can factor $[f]$ as $[p \circ i]$, where $[p]$ is a coarse Serre fibration class and $[i]$ a weak coarse homotopy equivalence class that has the LLP with respect to all coarse Serre fibration class.

We want to show that $[f]$ is a retract of $[i]$, since this lifting property is closed under taking of retracts. So if we manage to show this, then $[f]$ has the LLP with respect to all coarse Serre fibration classes and we will be done.

We can find a lift $[q]\colon B \to A'$ that makes the following diagram commute;



since $[f]$ is a coarse CW-cofibration class and $[p]$ is both a coarse Serre fibration class and weak coarse homotopy equivalence class, where $[p]$ is a weak coarse homotopy equivalence class because $[i]$ and $[f]$ are. So we find the following commutative diagram;

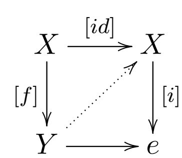


The composition $[p \circ q]$ is equal to the identity class on B by the first diagram, so then $[f]$ is a retract of $[i]$.

By the argument used in proposition 4.3.1, we show that the coarsely cellular classes which have the LLP with respect to all coarse Serre fibration classes is closed under retracts. It follows that $[p]$ has the LLP with respect to all coarse Serre fibration classes because $[i]$ does. $\square$

Proof of C3: It follows directly from proposition 4.3.7. $\square$

Proof of C4: For any coarse CW-cofibration class and a weak coarse homotopy equivalence class $[f]\colon X \to Y$ by proposition 4.3.8 we have the following commutative diagram



where e is a fibrant object. This implies a retraction is found. $\square$

Proof of Theorem 4.2.17: This is just by proving the above axioms. $\square$

We have now proved the important theorem 4.2.17 which shows that the category of coarse CW-complexes CWrs has a structure of Baues cofibration category.

Observe that in order to prove the last two axioms in theorem 4.2.17 we proved all the axioms of Quillen model category defined in 4.3.2 but not the first one, as we do not know if the category of coarse CW- complexes has all non-zero limits. However we could define one colimit which is enough for our work. As we said before proving those axioms of Quillen allows us easily to prove the last two axioms of Baues cofibration category.

Chapter 5

Axiomatic Coarse Homotopy Groups

§ 5.1 Relative Coarse Homotopy

Cofibration categories carry an abstract notion of relative homotopy. There is a more intuitive version of relative homotopy in the quotient coarse category. In this section we define and compare these two notions.

The following definition generalized from [1].

Definition 5.1.1 Let $Qcrs$ be the quotient coarse cofibration category defined in chapter 3. Then we define ***Pair(****Qcrs****)*** to be the category in which objects are morphisms $[h_X]\colon Y \to X$ in $Qcrs$, the morphisms are the pairs $([f],[f^{'}])\colon h_A \to h_X$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{[f^{'}]} & Y \\ {\scriptstyle [h_A]}\downarrow & & \downarrow{\scriptstyle [h_X]} \\ A & \xrightarrow{[f]} & X \end{array}$$

commutes in $Qcrs$.

The morphism $([f],[f^{'}])$ is a coarse homotopy equivalence class if $[f]$, $[f^{'}]$ are coarse homotopy equivalence classes, and $([f],[f^{'}])$ is a coarse cofibration class if $[f^{'}]$ and $([f],[h_X])\colon A \vee_B Y \to X$ are coarse cofibration classes in $Qcrs$.

We call $([f],[f^{'}])$ a *push out* if the diagram is a pushout diagram with $[h_A]$: $B \rightarrowtail A$ a coarse cofibration class.

The proof of the following theorem is found in lemma ((1.5), chapter (II), [1]).

Theorem 5.1.2 *The category **Pair(**Qcrs**)** with coarse cofibration classes and coarse homotopy equivalence classes as in the previous definition is a Baues cofibration category.*

An object $[h_A]\colon B \to A$ *is fibrant in* ***Pair(****Qcrs****)*** *if and only if* B *and* A *are fibrant in* *Qcrs*. □

The following definition comes from [1], and [2].

Definition 5.1.3 A *based object* in a cofibration category $\mathcal{C}$ is a cofibrant object X (that is, $* \to X$ is a cofibration) with a map $p\colon X \to *$ from X to the initial object termed the *trivial map*. This defines the trivial map $i_U \circ p$: $X \to * \to U$ for all objects U in $\mathcal{C}$ representing $i_U \circ p \in [X, U]$.

A map $f\colon A \to B$ between based objects is based if $pf = p$.

Definition 5.1.4 We term the category of non-unital pointed coarse spaces and closeness equivalence classes of pointed coarse maps the *pointed quotient coarse category*. It has an initial object, namely the space R, and we denote this category by PQcrs .

In this category for later requirement, we need to know that our basepoint inclusion in a space we consider is a coarse cofibration, that is, all objects are cofibrant. For this point to be be true for spaces of interest, we need to check at least we have the following result:

Lemma 5.1.5 *The inclusion* $i : R \hookrightarrow R^n$ *is a coarse cofibration.*

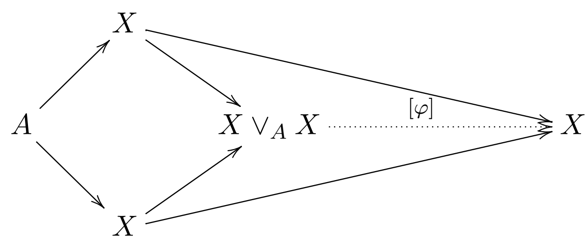
Proof : The inclusion i is a coarse homotopy equivalence by example (3.9) in [10]. Now by lemma 2.2.13 (2) and a similar argument to that used in proposition 3.2.8 shows that i is a coarse cofibration. □

Proposition 5.1.6 *The category PQcrs is a Baues cofibration category. The weak equivalences are coarse homotopy equivalence classes relative to* R*, and cofibrations are pointed coarse cofibration classes.*

Proof : By definition (1.4) of chapter III in [2], the category PQcrs is a subcategory of the category **Pair(***Qcrs***)**. Objects are the the non-unital pointed coarse spaces, and the maps are the pointed coarse classes. Weak equivalences and cofibrations in the category *Qcrs* yield the structure of Baues cofibration category for the category PQcrs. □

So we have the category PQcrs is a Baues cofibration category which has weak equivalences to be coarse homotopy equivalence classes relative to R in Qcrs and cofibrations are defined to be the pointed coarse cofibration classes. The cofibrant objects X in PQcrs are the coarse cofibration classes $R \hookrightarrow X$.

Definition 5.1.7 Let $[i]\colon A \to X$ be a coarse cofibration class . The *folding class* $[\varphi]\colon X \vee_A X \to X$ defined in Qcrs by the commutative diagram



where the maps in the left square all are coarse cofibration classes.

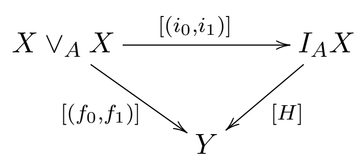
In our pointed quotient coarse cofibration category, the folding class $[\varphi]$ exists by the factorization axiom and is defined by $[\varphi \circ \theta] = [id]$, where $\theta\colon X \sqcup_\infty X \to X \vee_A X$ is the coequalizer map as defined in 3.2.14.

By the axiom of factorization, then $[\varphi]$ can be written as a composite:

$$X \vee_A X \xrightarrow{[i']} Z \xrightarrow{[r]} X$$

We call $Z = I_A X$, together with $[i']$ is a coarse cofibration class, $[r]$ a coarse homotopy equivalence class, a *relative cylinder* for the pair (X, A).

Definition 5.1.8 Let $[f_0], [f_1]\colon X \to Y$ be morphisms in the quotient coarse cofibration category Qcrs. Suppose we have a coarse cofibration class $[i]\colon A \to X$ such that $[f_0 \circ i] = [f_1 \circ i]$. Then we say that the maps $[f_0]$, $[f_1]$ are *strongly coarse homotopic relative to* A on the relative cylinder $I_A X$ if there is a commutative diagram



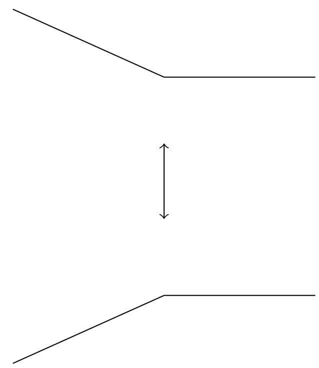
such that $[H \circ i_0] = [f_0]$, and $[H \circ i_1] = [f_1]$.

By proposition (2.2), chapter (II) in [1], the notion of strong coarse homotopy is independent of the choice of relative cylinders.

Thus for the quotient coarse category, by our work in chapter 3 we have the mapping cylinder $I_p(X \vee_A X) \vee_X X$ (defined by the coequalizer) where $p\colon X \vee_A X \to R$ is some controlled map.

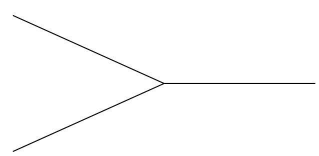
By definition of mapping cylinder we can choose this mapping cylinder to be our relative cylinder. We consider the disjoint union $R \sqcup R$ to be the line $(-\infty, \infty)$ equipped with the metric coarse structure. Initially, we can not view the above space $I_A X$ explicitly in abstract general picture, but the following example will give a picture of what the space look like.

The space $I_R(R \sqcup R)$. First the space $(R \sqcup R) \vee_R (R \sqcup R)$ can be viewed as follows:



where the distance in the left hand side is infinite and in the hand right side is finite.

Since this space is pointed, the easy way of showing a picture of the space $I_R(R \sqcup R)$ is to define a coarse equivalence between the space $(R \sqcup R) \vee_R (R \sqcup R)$ equipped with the coequalizer coarse structure and the *glued coarse space* (we use the notation "glued" to not be confused with the quotient space defined in chapter 3) $((R \sqcup R) \sqcup_\infty (R \sqcup R))/R$, with the later space pictured as:



If we consider the disjoint union $R \sqcup R$ to be the line $(-\infty, \infty) = \mathbb{R}_+ \sqcup \mathbb{R}_-$ equipped with the metric coarse structure, where $\mathbb{R}_+ = \{x : x \geq 0\}$ and $\mathbb{R}_+ = \{x : x \leq 0\}$, and the ray R to be the line $\mathbb{R}_+$. So the glued space has two apart different copies of $\mathbb{R}_-$ and one copy of $\mathbb{R}_+$ while the other space still have two apart different copies of $\mathbb{R}_-$ and two copies of $\mathbb{R}_+$ within finite distance as explained in the above pictures

Lemma 5.1.9 *There is a coarse equivalence between the spaces* $(R \sqcup R) \vee_R (R \sqcup R)$ *and the coarse space* $((R \sqcup R) \sqcup_\infty (R \sqcup R))/R$.

Proof : Define a map $f\colon ((R \sqcup R) \sqcup_\infty (R \sqcup R))/R \to (R \sqcup R) \vee_R (R \sqcup R)$ by writing

$$f(x_1) = x_1 \quad \textit{and} \quad f(x_2) = x_2$$

if x_1 and x_2 are in different copies of $\mathbb{R}_-$, and

$$f(x) = x$$

where $x \in \mathbb{R}_+$. That is, the map f defines the inclusion for any $x \in \mathbb{R}_+$

It is clear that this map is a coarse map.

Now define another map $g\colon (R \sqcup R) \vee_R (R \sqcup R) \to ((R \sqcup R) \sqcup_\infty (R \sqcup R))/R$ by writing

$$g(x_1) = x_1 \quad \text{and} \quad g(x_2) = x_2$$

if x_1 and x_2 are in different copies of $\mathbb{R}_-$, and

$$g(x_1) = g(x_2) = x_1$$

if x_1 and x_2 are from different copies of $\mathbb{R}_+$.

This is also a coarse map which clearly sends entourages to entourages and the inverse image of a bounded set under the map g restricted to any unital coarse subspace of $(R \sqcup R) \vee_R (R \sqcup R)$ is a bounded set.

The composite $g \circ f = id_{((R \sqcup R) \sqcup_\infty (R \sqcup R))/R}$ and the composite $f \circ g$ is close to the identity $id_{(R \sqcup R) \vee_R (R \sqcup R)}$ as follows.

$f \circ g(x_1) = x_1$ and $f \circ g(x_2) = x_2$ if x_1 and x_2 are in different copies of $\mathbb{R}_-$, so $f \circ g = id_{(R \sqcup R) \vee_R (R \sqcup R)}$ in this case.

$f \circ g(x_1) = f(x_2) = x_2$ where x_1 and x_2 are in different copies of $\mathbb{R}_+$.

By definition of the coequalizer coarse structure, the two copies of $\mathbb{R}_+$, which are apart, are within finite distance as in the picture. This implies that $d(f \circ g(x_1), x_1) = d(x_2, x_1) < c$ for some $c > 0$, so the composite is close to the identity. Hence the above spaces are coarsely equivalent. □

The above implies that the space $I_R(R \sqcup R)$ is coarsely equivalent to the space $(I_R(R \sqcup R))_{Glue}$ (We use "Glue" for the glued coarse structure) which can be viewed as follows:

$\{(x,t) \in (R \sqcup R)^2 : -|x| - 1 \le t \le |x| + 1\}/\sim$ such that $(s,t) \sim (s,-t)$ for all $s \in R$, $-|x| - 1 \le t \le |x| + 1$ which is equivalent to the following picture:

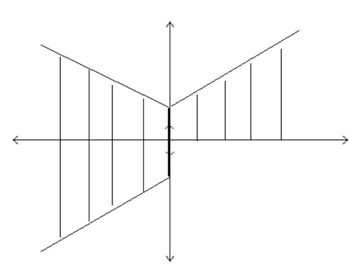


Figure 5.1: $(I_R(R \sqcup R))_{Glue}$.

Definition 5.1.10 Suppose we have a closeness equivalence class $[i]\colon A \to X$. A *relative coarse homotopy* is a coarse homotopy $F\colon I_pX \to Y$ such that the map $t \to F(x,t)$ is constant if $x = g(a)$ for any $g \in [i]$ and some point $a \in A$.

If $F\colon I_pX \to Y$ is a relative coarse homotopy, the closeness equivalence classes $[f_0]\colon X \to Y$ and $[f_1]\colon X \to Y$ are said to be *coarsely homotopic relative to* A if representative maps f_0 and f_1 are defined by the formulae

$$f_0(x) = F(x, 0), \quad f_1(x) = F(x, p(x) + 1)$$

respectively.

Lemma 5.1.11 *The notion of relative coarse homotopy between closeness equivalence classes of coarse maps is an equivalence relation.*

Proof : By the same method used in proof of theorem 2.2.5. □

Lemma 5.1.12 *Let* X *be a non-unital coarse space, and* $p\colon X \to R$ *be some controlled map. Let* $[i] : A \hookrightarrow X$ *be a coarse cofibration class. Then the induced class* $[i_*] : I_{p\circ i}A \hookrightarrow I_pX$ *defined by the formula* $i_*(a,t) = (i(a), t)$ *is a coarse cofibration class.*

Proof : It is enough to show that the representative map i_* is a coarse cofibration. Let $q\colon I_pX \to R$ be the controlled map defined by the formula $q(x,t) = p(x) + t$. By lemma 3.2.5, it suffices to show that the inclusion class $[j_*] : I_{q\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\}) \hookrightarrow I_q(I_pX)$ has a retraction, that is, there exists a coarse homotopy $[r_*]\colon I_q(I_pX) \to I_{q\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\})$ such that $r_* \circ j_* = 1_{I_{p\circ i}A}$.

Since $[i] : A \hookrightarrow X$ is a coarse cofibration class, then using lemma 3.2.5, there is a coarse class $[r]\colon I_pX \to (I_{p\circ i}A) \cup (X \times \{0\})$ such that $r \circ j = 1_{(I_{p\circ i}A)\cup(X\times\{0\})}$.

We define the class $[r_*]$ by writing $r_*(x,t) = (r(x), t)$, then r_* is a representative coarse map, and $r_* \circ j_*(x,t) = r_*(j(x), t) = (r(j(x)), t) = (x, t)$ for all $x \in I_{q\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\})$ as required. □

Lemma 5.1.13 *Suppose that we have a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{[g]} & Y \\ {\scriptstyle [i]}\uparrow & & \uparrow{\scriptstyle [j]} \\ A & \xrightarrow{[f]} & B \end{array}$$

such that the inclusions $[i]$, $[j]$ *are coarse cofibration classes. Then we have a canonical coarse cofibration class* $[k]\colon I_{p\circ i}A \vee_A B \to I_pX \vee_X Y$ *such that the following diagram*

$$\begin{array}{ccccc} X & \longrightarrow & I_pX \vee_X Y & \longrightarrow & Y \\ {\scriptstyle [i]}\uparrow & & {\scriptstyle [k]}\uparrow & & \uparrow{\scriptstyle [j]} \\ A & \longrightarrow & I_{p\circ i}A \vee_A B & \longrightarrow & B \end{array}$$

commutes.

Proof : First we have the diagram

$$\begin{array}{ccccc} X & \longrightarrow & I_pX \vee_X Y & \longrightarrow & Y \\ {\scriptstyle i}\uparrow & & {\scriptstyle k}\uparrow & & {\scriptstyle j}\uparrow \\ A & \longrightarrow & I_{p\circ i}A \vee_A B & \longrightarrow & B \end{array}$$

by the factorization axiom in the quotient coarse cofibration category. Similar to the argument in the last lemma we need to find a retraction, that is; to show that there exists a coarse homotopy class $[h_*]\colon I_{q_1}(I_pX \vee_X Y) \to I_{q_1\circ k}(I_{p\circ i}A \vee_A B) \cup ((I_pX \cup_X Y) \times \{0\})$.

Let $q_1\colon I_pX \vee_X Y \to R$ is some controlled map. By the above lemma since $[j]$ is a coarse cofibration class, we have the induced class $[i_*] : I_{p\circ i}A \hookrightarrow I_pX$ a coarse cofibration class. So the induced classes

$$[j_*^1]\colon (I_{q_1\circ f'\circ j}B) \cup (Y \times \{0\}) \to I_{q_1\circ f'}Y \ \textit{and}\ [j_*^2]\colon I_{q_1\circ i'\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\}) \to I_{q_1\circ i'}(I_pX)$$

have retractions

$$[r_*^1]\colon I_{q_1\circ f'}Y \to (I_{q_1\circ f'\circ j}B) \cup (Y \times \{0\}) \ \textit{and}\ [r_*^2]\colon I_{q_1\circ i'}(I_pX) \to I_{q_1\circ i'\circ i_*}(I_{p\circ i}A) \cup (I_pX \times \{0\})$$

where $f'\colon I_pX \to I_pX \vee_X Y$ and $i'\colon Y \to I_pX \vee_X Y$ are coarse maps defined by $f'(x,t) = \theta(x,t)$ for any $x \in X$, and $i'(y) = \theta(y)$ for $y \in Y$ (the map θ is the coequalizer map).

Define the class $[h_*]$ by the formula;

$$h_*(\theta(y),t) = r_*^1(\theta(y),t), \quad y \in Y$$

$$h_*(\theta(x,s),t) = r_*^2(\theta(x,s),t), \quad (x,s) \in I_pX$$

Then $[h_*]$ is the required retraction, and we are done. $\square$

Theorem 5.1.14 *Let $[i] : A \hookrightarrow X$ be a coarse cofibration class, and suppose that we have coarse classes $[f_0],[f_1]\colon X \to Y$ such that $[f_0 \circ i] = [f_1 \circ i]$.*

Suppose that the classes $[f_0]$, $[f_1]$ are strongly coarse homotopic relative to A, then $[f_0]$, $[f_1]$ are also relatively coarse homotopic.

Proof : Suppose we have a commutative diagram of the form

$$\begin{array}{ccc} X \vee_A X & \xrightarrow{[i']} & I_AX \\ & {\scriptstyle [(f_0,f_1)]}\searrow \quad \swarrow{\scriptstyle [H]} & \\ & Y & \end{array}$$

Let I_AX' be the quotient space $I_AX/\sim$, where the equivalence relation $\sim$ is defined by writing

$$(i(a),s) \sim (i(a),t) \quad \text{whenever} \quad -p(i(a)) - 1 \leq s,t \leq p((a)) + 1$$

We need to prove that the spaces $I_A X$ and $I_A X'$ are coarsely homotopy equivalent.

First, note that by the following push out diagram

$$\begin{array}{ccc} X & \longrightarrow & X \vee_A X \\ {\scriptstyle [i]}\uparrow & & \uparrow \\ A & \xrightarrow{\ i\ } & X \end{array}$$

the obvious class $X \to X \vee_A X$ is a coarse cofibration class. Hence the composite class $A \to X \to X \vee_A X$ is also a coarse cofibration class, so by lemma 5.1.13 and 5.1.12 we have a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & I_A X & \longrightarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & I_{p \circ i} A & \longrightarrow & A \end{array}$$

where the vertical arrows are all coarse cofibration class.

Now the space $I_A X'$ is obtained by a push out diagram

$$\begin{array}{ccc} A & \longrightarrow & I_A X' \\ \uparrow & & \uparrow \\ I_{p \circ i} A & \longrightarrow & I_A X \end{array}$$

The class $I_{p \circ i} A \to A$ is certainly a coarse homotopy equivalence class, so by proposition 3.2.22 the class $I_A X \to I_A X'$ is also a coarse homotopy equivalence class, and we are done. □

§ 5.2 The Quotient Coarse Homotopy Groups

In this section we define a notion of the quotient coarse homotopy groups in the pointed quotient coarse category as defined in 5.1.4 which we proved to be a Baues cofibration category in 5.1.6.

There is an abstract notion of homotopy groups in a Baues cofibration category equipped with a small amount of extra structure. The quotient coarse homotopy groups we are going to define here are slightly different from the abstract homotopy groups in Baues cofibration category.

The point is that the quotient coarse homotopy groups are constructed via the notion of strong coarse homotopy defined earlier. Because of computability, our groups use the usual definition of a coarse homotopy. In general, our definition is the same as the abstract definition.

Definition 5.2.1 Let X and Y be non-unital pointed coarse spaces. Then we write $[X, Y]_R$ to denote the set of strong coarse homotopy classes of pointed coarse maps from X to Y relative to R.

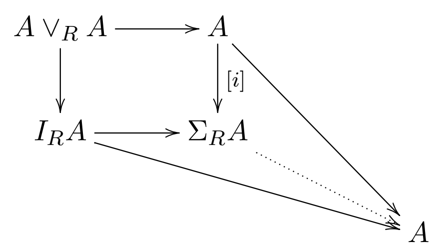
Note that we have a canonical base (trivial) element of the set $[X, Y]_R$ defined by the strong coarse homotopy class of the pointed coarse map relative to R

$$X \xrightarrow{p_X} R \xrightarrow{i_Y} Y$$

where p_X is some controlled map, and i_Y is the basepoint in the space Y.

If $i\colon R \to A$ is the basepoint inclusion, we set $I_R A = I_{i[R]} A$.

Definition 5.2.2 For a given based object A in the category PQcrs, we define the *torus* $\Sigma_R A$, where $R \subseteq A$ by the push out diagram



Here the space $\Sigma_R A$ is a based object.

To draw an explicit picture, we take $A = R \sqcup R$ as an example and the torus can be seen by lemma 5.1.9 to be coarsely equivalent to the following space

$$(\Sigma_R(R \sqcup R))_{Glue} = \{(x,t) \in (R \sqcup R)^2 \mid -|x| - 1 \leq t \leq |x| + 1\}/\sim$$

where $(s,t) \sim (s,-t)$ for all $s \in R$ and $-s-1 \leq t \leq s+1$, and $(x, |x|+1) \sim (x, -|x|-1)$ for all $x \in R \sqcup R$. Geometrically it can be viewed as follows

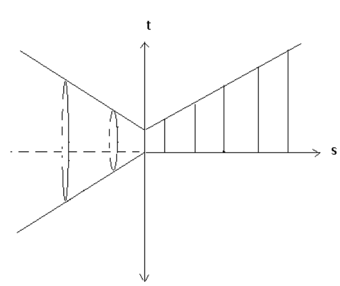


Figure 5.2: $(\Sigma_R(R \sqcup R))_{Glue}$.

Definition 5.2.3 Given a based object A in the category PQcrs and let $[\varphi]\colon A \vee_R A \to A$ be the folding class. Then *the suspension*, ΣA, is defined by the commutative diagram

$$\begin{array}{ccccc} & & A & \longrightarrow & R \\ & \nearrow & \uparrow & & \uparrow \\ I_R A & \longrightarrow & \Sigma_R A & \longrightarrow & \Sigma A \\ \uparrow & & \uparrow & & \uparrow \\ A \vee_R A & \xrightarrow{[\varphi]} & A & \xrightarrow{[p]} & R \end{array}$$

where the lower two squares are push out diagrams. Here the spaces $I_R A$, $\Sigma_R A$, and ΣA are based objects.

Explicitly, and by lemma 5.1.9 we define the suspension of our above example to be coarsely equivalent to the following space again;

$$(\Sigma(R \sqcup R))_{Glue} = \{(x,t) \in (R \sqcup R)^2 \mid -|x| - 1 \leq t \leq |x| + 1\} / \sim$$

where $(s,t) \sim (s,-t)$ for all $s \in R$, $-s-1 \leq t \leq s+1$, $(x, |x|+1) \sim (x, -|x|-1)$ for all $x \in A$, and $(x, |x|+1) \sim (-x, |x|+1)$ for all $x \in R$.

The above space will be seen as the following:

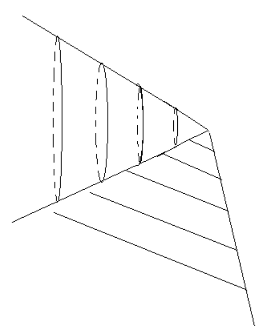


Figure 5.3: $(\Sigma(R \sqcup R))_{Glue}$.

By lemma 2.2.12, since $I_p R$ is coarsely homotopy equivalent to R, and any bounded subset is coarsely equivalent to a point, our suspension is coarsely homotopic to the space in the following figure, which is isomorphic to the space $(R \sqcup R)^2$:

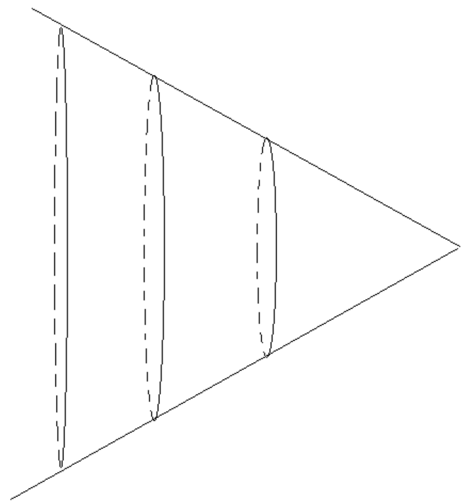


Figure 5.4: $\Sigma(R \sqcup R)_{Glue} \simeq (R \sqcup R)^2$

This implies by the above that the space $\Sigma(R \sqcup R)$ equipped with the Co-equalizer coarse structure is coarsely homotopic equivalent to the coarse sphere $S^1_R = (R \sqcup R)^2$.

The suspension ΣA depends on the choice of the coarse map p. Since ΣA is a based object, we can define inductively

$$\Sigma^n A = \Sigma(\Sigma^{n-1} A), \quad n \geq 1, \quad \Sigma^0 A = A.$$

Theorem 5.2.4 *$\Sigma(R \sqcup R)^k$ is coarsely homotopy equivalent to $(R \sqcup R)^{k+1}$ for $k \geq 1$.*

Proof : First let $k = 1$. Then the statement is true by the above calculation.

Now let $k = 2$. Then $(R \sqcup R)^3 = (R \sqcup R)^2 \times (R \sqcup R)$. By the above this is coarsely homotopy equivalent to $\Sigma(R \sqcup R) \times R \sqcup R$. It is enough to show that $\Sigma(R \sqcup R) \times R \sqcup R$ is coarsely homotopic equivalent to $\Sigma(R \sqcup R)^2$.

We have $I_R(R \sqcup R)^2 \simeq I_p(R \sqcup R)^2 \sqcup I_p(R \sqcup R)^2/\sim$ such that $(a, 0, t) \sim (a, 0, -t)$, $a \in R$, $p\colon (R \sqcup R)^2 \to R$ is some controlled map defined by $p(x,t) = \|(x,t)\|$, x, $t \in R \sqcup R$. By lemma 2.2.12, $I_p(R \sqcup R)^2$ is coarsely homotopy equivalent to $(R \sqcup R)^2$.

Define a map $q\colon R \sqcup R \to R$ by $q(x) = p(j(x)) = p(x, 0) = |x|$ where $j\colon (R \sqcup R) \to (R \sqcup R)^2$ is the inclusion, then q is a controlled map, and again by lemma 2.2.12, $I_q(R \sqcup R)$ is coarsely homotopy equivalent to $(R \sqcup R)$. Therefore $I_p(R \sqcup R)^2$ is coarsely homotopy equivalent to $I_q(R \sqcup R) \times (R \sqcup R)$.

Define a map $f\colon I_q(R\sqcup R)\times(R\sqcup R)\to I_p(R\sqcup R)^2$ as follows

$$f(x,t,y)=\begin{cases}(x,y,t) & t\leq p(x,y)+1\\ (x,y,p(x,y)+1) & t\geq p(x,y)+1\end{cases}$$

We have another map $g\colon I_p(R\sqcup R)^2\to I_q(R\sqcup R)\times(R\sqcup R)$ defined as follows

$$g(x,y,t)=\begin{cases}(x,t,y) & t\leq q(x)+1\\ (x,q(x)+1,y) & t\geq q(x)+1\end{cases}$$

Then f, g are coarse maps, also we have

$$f\circ g(x,y,t)=\begin{cases}(x,y,t) & t\leq q(x)+1\\ (x,y,q(x)+1) & q(x)+1\leq t\leq p(x,y)+1\\ (x,y,p(x,y)+1) & t\geq p(x,y)+1\end{cases}$$

and

$$g\circ f(x,t,y)=\begin{cases}(x,t,y) & t\leq q(x)+1\\ (x,q(x)+1,y) & q(x)+1\leq t\leq p(x,y)+1\\ (x,q(x)+1,y) & t\geq p(x,y)+1\end{cases}$$

It is very easy to verify that $g\circ f$ and $f\circ g$ are coarsely homotopy equivalent to the identities.

Consider the space

$$I_R(R\sqcup R)\times(R\sqcup R)=(I_q(R\sqcup R)\sqcup I_q(R\sqcup R)/\sim)\times(R\sqcup R)$$

where $(a,t)\sim(a,-t)$, $a\in R$. This can be written as:

$$(I_q(R\sqcup R)\times(R\sqcup R)\sqcup I_q(R\sqcup R)\times(R\sqcup R))/\sim$$

where $(a,t,0)\sim(a,-t,0)$, $a\in R$

The above shows that $I_R(R\sqcup R)\times(R\sqcup R)$ is coarsely homotopy equivalent to $I_p(R\sqcup R)^2\sqcup I_p(R\sqcup R)^2/\sim$.

where $(a,0,t)\sim(a,0,-t)$, $a\in R$, and the later space is exactly $I_R(R\sqcup R)^2$. Hence $I_R(R\sqcup R)\times(R\sqcup R)$ is coarsely homotopy equivalent to $I_R(R\sqcup R)^2$.

Now, look at the space

$$\Sigma_R(R\sqcup R)\times(R\sqcup R)=(I_R(R\sqcup R)\sqcup(R\sqcup R)/\sim\times(R\sqcup R)$$

where $(x,q(x)+1)\sim(x,-q(x)-1)$, $x\in R\sqcup R$. This is equal to

$$I_R(R\sqcup R)\times(R\sqcup R)\sqcup(R\sqcup R)^2/\sim$$

where $(x,q(x)+1,y)\sim(x,-q(x)-1,y)$, $x,y\in R\sqcup R$

Again the above shows that $\Sigma_R(R\sqcup R)\times(R\sqcup R)$ is coarsely homotopy equivalent to $I_R(R\sqcup R)^2\sqcup(R\sqcup R)^2/\sim$ where $(x,y,p(x,y)+1)\sim(x,,y-p(x,y)-1)$, $x,y\in R\sqcup R$, and this is exactly $\Sigma_R(R\sqcup R)^2$. Therefore

$$\Sigma_R(R\sqcup R)\times(R\sqcup R)\simeq\Sigma_R(R\sqcup R)^2$$

Finally, consider

$$\Sigma(R \sqcup R) \times (R \sqcup R) = (\Sigma_R(R \sqcup R) \sqcup R/\sim) \times (R \sqcup R)$$

where $(x, q(x)+1)/\sim (y, q(y)+1)$ if $q(x) = q(y)$, $x, y \in R \sqcup R$. This is equal to

$$\Sigma_R(R \sqcup R) \times (R \sqcup R) \sqcup R \times (R \sqcup R)/\sim$$

where $(x, q(x)+1, z)/\sim (y, q(y)+1, z)$ if $q(x) = q(y)$, x, y, and $z \in R \sqcup R$.

By the same technique used in lemma 4.2.1 we can prove that $R \times (R \sqcup R)$ is coarsely homotopic to R. All that shows

$$\Sigma(R \sqcup R) \times (R \sqcup R) \simeq \Sigma_R(R \sqcup R)^2 \sqcup R/\sim$$

where $(x, z, p(x,z)+1) \sim (y, z, p(y,z)+1)$ if $p(x,z) = p(y,z)$, x, y, and $z \in R \sqcup R$. This is exactly $\Sigma(R \sqcup R)^2$. Therefore

$$\Sigma(R \sqcup R) \times (R \sqcup R) \simeq \Sigma(R \sqcup R)^2$$

Similarly we can prove that $\Sigma(R \sqcup R)^k \times (R \sqcup R)$ is coarsely homotopic equivalent to $\Sigma(R \sqcup R)^{k+1}$ for $k > 2$.

In this stage we have proved that $(R \sqcup R)^3$ is coarsely homotopic equivalent to $\Sigma(R \sqcup R)^2$. Therefore the statement $(R \sqcup R)^n$ is coarsely homotopic to $\Sigma(R \sqcup R)^{n-1})$ is true for $n = 1$, 2, and 3.

Now suppose that $(R \sqcup R)^n$ is coarsely homotopic equivalent to $\Sigma(R \sqcup R)^{n-1}$ for $n = k$. We need to prove the statement is true for $n = k+1$.

By the above it is easy to see that:

$$(R \sqcup R)^{k+1} = (R \sqcup R)^k \times (R \sqcup R) \simeq \Sigma(R \sqcup R)^{k-1} \times (R \sqcup R) \simeq \Sigma(R \sqcup R)^k$$

Hence by induction the statement is true for all n, and we are done. □

Corollary 5.2.5 $\Sigma(R \sqcup R)^k$ *is coarsely homotopy equivalent to* $\Sigma^k(R \sqcup R)$ *for* $k \geq 0$.

Proof : $\Sigma^k(R \sqcup R) = \Sigma^{k-1}(\Sigma(R \sqcup R))$. By the above theorem the later is coarsely homotopy equivalent to $\Sigma^{k-1}(R \sqcup R)^2 = \Sigma^{k-2}(\Sigma(R \sqcup R)^2)$. Again by the above theorem we have $\Sigma^{k-2}(\Sigma(R \sqcup R)^2)$ is coarsely homotopy equivalent to $\Sigma^{k-2}((R \sqcup R)^3)$.

We continue in this way for $k-2$ more processes, we have $\Sigma^k(R \sqcup R)$ is coarsely homotopy equivalent to $\Sigma^k(R \sqcup R)$. □

Definition 5.2.6 Let A and X be non-unital pointed coarse spaces. Let $n \geq 0$. Then we define the *n-th coarse homotopy group* with respect to A to be the set of coarse homotopy classes of pointed coarse maps $\Sigma^n A \to X$ relative to R, and denoted by $\pi_n^A(X)$ where

$$\pi_n^A(X) = [\Sigma^n A, X]_R$$

Proposition 5.2.7 *If $A = R \sqcup R$, then $\pi_n^{R\sqcup R}(X)$ is isomorphic to the group $\pi_n^{Pcrs}(X, R)$ in definition 2.4.4.*

Proof : Straightforward by theorem 5.2.4. □

Definition 5.2.8 Let A and X be non-unital pointed coarse spaces. Let $n \geq 0$. Then we define the *n-th strong coarse homotopy group* to be the set of strong coarse homotopy classes of pointed coarse maps $\Sigma^n A \to X$ relative to R

$$\pi_n^{A,Strong}(X) = [\Sigma^n A, X]_R^{strong}$$

In particular if $A = R \sqcup R$, we have $\pi_0^{R\sqcup R,Strong}(X) = [R \sqcup R, X]_R^{Strong} = [R, X]^{strong}$ so we define the set $\pi_0^{R\sqcup R,Strong}(X)$ to be the set of strong coarse homotopy classes (not relative homotopy classes) of coarse maps $R \to X$, and we define the higher coarse homotopy groups by writing $\pi_n^{R\sqcup R,Strong}(X) = [\Sigma^n(R \sqcup R), X]_R^{strong}$.

Corollary 5.2.9 *We have a well defined surjective homomorphism*

$$\alpha \colon \pi_n^{A,Strong}(X) \to \pi_n^A(X)$$

.

Proof : Let $f \colon \Sigma^n A \to X$ be a pointed coarse map. By theorem 5.1.14, if $[f]_{Strong} = [g]_{Strong}$ then $[f] = [g]$. So we have a homomorphism

$$\alpha \colon \pi_n^{A,Strong}(X) \to \pi_n^A(X)$$

defined by $\alpha([f]_{Strong}) = [f]$, and α is clearly surjective. □

Definition 5.2.10 Let $[i] : A \hookrightarrow X$ be a coarse cofibration class. We write $[I_R A, X]_R^0$ to denote the set of relative coarse homotopy classes of coarse maps $F \colon I_R A \to X$ such that the map F restricts to the base element

$$A \xrightarrow{p_A} R \xrightarrow{i_X} X$$

at the ends of the cylinder.

There is a canonical map from the set $[\Sigma A, X]_R$ to the set $[I_R A, X]_R^0$ arising from the maps on the top row in the push-out diagram in the category PQcrs

$$\begin{array}{ccccc} I_R A & \longrightarrow & \Sigma_R A & \longrightarrow & \Sigma A \\ \uparrow & & \uparrow & & \uparrow \\ A \vee_R A & \xrightarrow{\varphi} & A & \xrightarrow{p} & A \end{array}$$

used to define the suspension.

Proposition 5.2.11 *The above canonical map $[\Sigma A, X]_R \to [I_R A, X]_R^0$ is a bijection.*

Proof : By construction of abstract cofibration categories, every object in the pointed quotient coarse category is both cofibrant and based. It follows that the quotient map $I_R A \to \Sigma_R A$ induces a bijection $[\Sigma_R A, X]_R \to [I_R A, X]^0_R$ from results in section (2) of chapter (II) of [1].

Also from [1], sections 5 and 6 of chapter (II), we have $[I_R A, X]^0_R$ is a group, and by proposition (2.11(b)) in [1], the quotient map $\sigma\colon \Sigma_R A \to \Sigma A$ yields a bijection $[\Sigma A, X]_R =^{\sigma^*} [\Sigma_R A, X]_R$. Since the composite of bijections is a bijection, we are done. □

The abstract proof of the following proposition can be found in [1].

Proposition 5.2.12 *Let $n \geq 1$. Then the set $\pi_n^{A,Strong}(X)$ is a group. The operation is defined by composition of strong coarse homotopies using the last proposition. The identity element is the strong coarse homotopy class of closeness class of the base map*

$$I_R A \xrightarrow{p_{I_R A}} R \xrightarrow{i_X} X$$

Further. For $n \geq 2$, the strong coarse homotopy group $\pi_n^{A,Strong}(X)$ is abelian. □

Theorem 5.2.13 *Let X be a non-unital pointed coarse space. Then there is a surjective homomorphism $\beta\colon \pi_n^{R\sqcup R,Strong}(X) \to [S^n_R, X]_R$.*

Proof : First, by corollary 5.2.9, we have a surjective homomorphism

$$\alpha\colon \pi_n^{R\sqcup R,Strong}(X) \to \pi_n^{R\sqcup R}(X),$$

and by proposition 5.2.7 we have $\pi_n^{R\sqcup R}(X)$ is isomorphic to the group $[S^n_R, X]_R$. So it is enough to show that $\Sigma^n(R \sqcup R)$ is coarsely homotopic to $(R \sqcup R)^{n+1}$ which is so by theorem 5.2.4, and we are done . □

Chapter 6

Properties of The Coarse Cofibration Category

§ 6.1 Coarse CW-Complexes in The Coarse Cofibration Category

In proposition 5.1.6, chapter 5 we proved that the pointed quotient coarse category PQcrs is a Baues cofibration category, which implies that we have a push out in this category. This can be defined as follows;

Definition 6.1.1 Let X , Y be pointed coarse spaces with basepoint inclusions $i_X : R \to X$, $i_Y : R \to Y$ respectively. Then we define the *glued coarse space* (we use the notation glued to not be confused with the quotient space defined in chapter 3) by writing

$$X \cup_R Y = X \sqcup_\infty Y / \sim$$

where $\sim$ is the equivalence relation defined by $i_X(t) \sim i_Y(t)$ for all $t \in R$. The space $X \cup_R Y$ is equipped with the coarse structure described in the following proposition.

Proposition 6.1.2 *The above glued space* $X \cup_R Y$ *is defined in the category PQcrs by the following push out diagram;*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \cup_R Y \\ {\scriptstyle i_X}\uparrow & & \uparrow{\scriptstyle f'} \\ R & \xrightarrow[\ i_Y\ ]{} & Y \end{array}$$

Proof : First, let $\pi \colon X \sqcup_\infty Y \to X \cup_R Y$ be the *glued map* defined similarly to the quotient map in definition 3.1.10, and the glued space $X \cup_R Y$ equipped with the coarse structure formed by defining the set of entourages to be the collection of subsets of sets of the form

$$\{\pi[M] : M \subseteq X \sqcup_\infty Y \times X \sqcup_\infty Y \ \text{ is an entourage}\}$$

By a similar argument to that in lemma 3.1.12, the glued map is controlled, and so by a similar argument to that in theorem 3.1.13 the map f is controlled.

Now let $B \subseteq X \cup_R Y$ be a bounded subset, then $B = \pi(B_1 \cup B_2)$, where $B_1 \subseteq X$, $B_2 \subseteq Y$ are bounded. The set

$$f^{-1}(B) = B_1 \cup (i_X(i_Y^{-1}[B_2 \cap i_Y[R]]))$$

is bounded.

Therefore the map f is a coarse map. Similarly, we show that the map f' is a coarse map. Again by a similar argument to that of theorem 3.1.13 we satisfy the universal property, and we are done. □

By definition (2.2) chapter (I) in [2] the sum $X \vee_R Y$ of the pointed coarse maps $i_X\colon R \to X$ and $i_Y\colon R \to Y$ is defined by the push out in the category Qcrs as defined in 3.2.18 in chapter 3. In the particular pointed category PQcrs and regarding lemma 5.1.9 we see a coarse equivalence between the pointed glued space and the pointed coequalizer space.

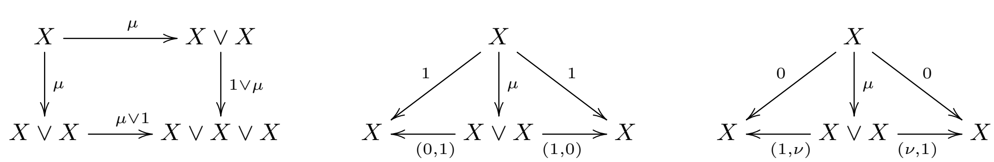
The above proposition shows that the *sum* or the *coproduct* in PQcrs is the pointed non-unital coarse space $X \cup_R Y$.

The following definitions come from [1], and [2].

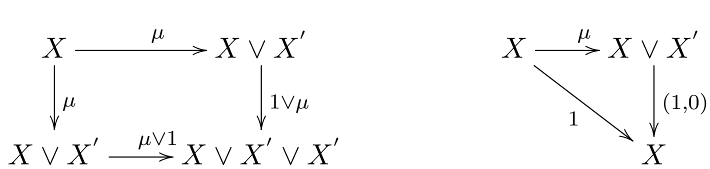
Definition 6.1.3 A *theory* is a category $\mathcal{C}$ with an initial object $*$ and finite sums where the sum of objects X and Y is denoted by $X \vee Y$. We consider $*$ as the empty sum. A map between theories is a functor $F\colon T \to T'$ which preserves sums. This is an *equivalence of theories* if there is a map $G\colon T' \to T$ with FG and GF natural isomorphic to the corresponding identity functors.

Definition 6.1.4 Let T be a theory. A *based object* in T is an object X endowed with a map $0_X = 0\colon X \to *$. This map defines for all objects Y in T the zero map $0 : X \to * \to Y$.

A *cogroup* $X = (X, 0, \mu, \nu)$ in T is a based object $(X, 0)$ together with a comultiplication $\mu_X = \mu\colon X \to X \vee X$ and a map $\nu_X = \nu\colon X \to X$ such that the following diagrams commute



Definition 6.1.5 Let T be a theory. A *coaction* $X = (X, X', \mu)$ in T is an object X together with a map $\mu_X = \mu\colon X \to X \vee X'$ where X' is a cogroup such that the following diagrams commute



Clearly, each cogroup X yields a coaction with $X' = X$.

Definition 6.1.6 A *theory of cogroups* is a theory G for which each object X in G is endowed with the structure of a cogroup which is compatible with sums; that is, the cogroup structure of a sum $X \vee Y$ is given by the cogroup structures of X and Y respectively by $0_{X\vee Y} = (0_X, 0_Y)$, $\nu_{X\vee Y} = \nu_X \vee \nu_Y$, and

$$\mu_{X\vee Y} : X \vee Y \overset{\mu_X \vee \mu_Y}{\longrightarrow} (X \vee X) \vee (Y \vee Y) = (X \vee Y) \vee (X \vee Y).$$

Definition 6.1.7 A *theory of coactions* is a theory T for which each object X in G is endowed with a cogroup object X' and a coaction $\mu_X \colon X \to X \vee X'$ in T.

This structure of coactions on X is compatible with sums; that is, for $X \vee Y$ the coaction $\mu_{X\vee Y}$ is the composite

$$\mu_{X\vee Y} : X \vee Y \overset{\mu_X \vee \mu_Y}{\longrightarrow} (X \vee X') \vee (Y \vee Y') = (X \vee Y) \vee (X' \vee Y')$$

where $(X \vee Y)' = X' \vee Y'$.

Each coaction μ_X has the following *affine property*: For all objects Y and all maps $f, g \colon X \to Y$ in T, there exists a unique map $\alpha \colon X' \to Y$ with $g = f + \alpha$. That is, there is a unique map $(f, \alpha) \colon X \vee X' \to Y$ with $(f, \alpha)\mu_X = g$

Example 6.1.8 The cofibration category PQcrs has an initial object R and finite sum $X \cup_R Y$ as defined in 6.1.2, so this category defines a theory T. In addition, any object A is a based object equipped with a controlled map $p \colon A \to R$, where A can be one of the following coarse spaces: R, $R \sqcup R$, $R \sqcup R \cup_R R \sqcup R$, $R \sqcup R \cup_R R \sqcup R \cup_R R \sqcup R$,....

Here we generalize Baues's definition in [2] to the following definition:

Definition 6.1.9 Let PQcrs be the coarse cofibration category defined in chapter 5.

Given a based object, we define the *cone* $\Sigma_R A$ and the *suspension* ΣA in PQcrs by the push out diagram

$$\begin{array}{ccccc} I_R A & \longrightarrow & \Sigma_R A & \overset{[\pi_0]}{\longrightarrow} & \Sigma A \\ \uparrow{\scriptstyle [(i_0,i_1)]} & & \uparrow{\scriptstyle [i_0]} & & \uparrow \\ A \cup_R A & \overset{\varphi}{\longrightarrow} & A & \overset{[p]}{\longrightarrow} & R \end{array}$$

Here $\Sigma_R A$ and ΣA are based objects defined earlier by the map $I_R A \to A \to R$, and since ΣA is a based object, we define inductively,

$$\Sigma^n A = \Sigma(\Sigma^{n-1} A), \quad n \geq 1, \quad \Sigma^0 A = A.$$

The *homotopy category* hPQcrs of the pointed cofibration category PQcrs is the category whose objects are non-unital pointed coarse spaces, and the morphisms in hPQcrs between two spaces X and Y are given by the equivalence classes of all coarse maps $X \to Y$ with respect to the relation of coarse homotopy. So the morphism sets in the category $PQcrs/\simeq$ are the sets $[\Sigma^n A, X]_R$ where these sets defines the groups $\pi_n^{A,Strong}(X)$.

Let $susp(R) \subseteq PQcrs/\simeq$ be the full subcategory consisting of all suspensions ΣA in PQcrs as defined in 5.2.3. A suspension has a cogroup structure $\mu\colon \Sigma A \to \Sigma A \cup_R \Sigma A$ in $PQcrs/\simeq$ given by $p_{\Sigma A \cup_R \Sigma A} = p \circ \varphi$ where $\varphi\colon \Sigma A \cup_R \Sigma A \to \Sigma A$ is the folding map and $p\colon \Sigma A \to R$ a controlled map and $\nu_{\Sigma A \cup_R \Sigma A}\colon \Sigma A \cup_R \Sigma A \to \Sigma A \cup_R \Sigma A$ where $\nu_{\Sigma A}\colon \Sigma A \to \Sigma A$ is a coarse map such that the diagram

$$\begin{array}{ccc} \Sigma A & \xrightarrow{[\mu]} & \Sigma A \cup_R \Sigma A \\ \downarrow{\scriptstyle [\mu]} & & \downarrow{\scriptstyle [1 \vee \mu]} \\ \Sigma A \cup_R \Sigma A & \xrightarrow{[\mu \vee 1]} & \Sigma A \cup_R (\Sigma A \cup_R \Sigma A) \end{array}$$

commutes.

So then $susp(R)$ is a theory of cogroups. And since a theory of cogroups is an example of theory of coaction, then $susp(R)$ is a theory of coaction.

Coarse Principal Cofibration:

The following definition comes from [2].

Definition 6.1.10 Let PQcrs be the cofibration category. Let B be a based object in PQcrs, $Y \in ob(PQcrs)$, we define for a coarse map $g\colon B \to Y$, the *mapping cone* $\mathcal{C}_g$ by the push out diagram

$$\begin{array}{ccc} \Sigma_R B & \xrightarrow{[\pi_g]} & \mathcal{C}_g = \Sigma_R B \cup_B Y \\ \uparrow{\scriptstyle [i_0]} & & \uparrow{\scriptstyle [i_g]} \\ B & \xrightarrow{[g]} & Y \end{array}$$

where $\Sigma_R B$ as defined before and $[i_0]$ is a coarse cofibration class, so that $[i_g]$ is a coarse cofibration class, and for a based object B we have g a based map, then $\mathcal{C}_g$ is based by $q = (p_{\Sigma_R B}, p)\colon \mathcal{C}_g \to R$ where $p_{\Sigma_R B}\colon \Sigma_R B \to R$, $p\colon Y \to R$ are controlled maps. In this case i_g is a based map.

Definition 6.1.11 We call a coarse cofibration class $Y \hookrightarrow \overline{Y}$ a *coarse principal cofibration class* with attaching map $[g] \in [B, Y]$ if there is a coarse class

$$B \xrightarrow{[g]} SY \xleftarrow{\sim} Y$$

in the category $PQcrs$ represents $[g]$ where SY is a fibrant in PQcrs together with a coarse homotopy equivalence

$$\mathcal{C}_g = \mathcal{C}B \cup_g SY \xrightarrow{\sim} S\overline{Y}$$

under Y.

At this point If the object Y is fibrant, then we can choose $SY = Y$.

In particular, $[i_g]$ is a coarse principle cofibration class.

The following definition comes from [2].

Definition 6.1.12 Let $\mathcal{C}$ be a cofibration category with initial object $*$. A *homotopy cogroup* in $\mathcal{C}$ is a cofibrant object A in $\mathcal{C}$ which is a cogroup $(A, 0_A, \mu, \nu)$ in the homotopy category $HO(\mathcal{C})$ such that the map $0_A\colon A \to *$ in $HO(\mathcal{C})$ can be represented by a map $0_A\colon A \to *$ in $\mathcal{C}$ such that $\mu\colon A \to A \vee A$ can be represented by $\mu\colon A \to S(A \vee A)$ in $\mathcal{C}$ where

$$(i_1, \mu) : A \vee A \xrightarrow{\sim} S(A \vee A)$$

is a weak equivalence in $\mathcal{C}$.

A *homotopy coaction* in $\mathcal{C}$ is a cofibrant X in $\mathcal{C}$ which has the structure (X, A, μ) of a coaction in $HO(\mathcal{C})$. Here A is a homotopy cogroup and $\mu\colon X \to X \vee A \in HO(\mathcal{C})$ can be represented by a map $\mu\colon X \to S(X \vee A)$ in $\mathcal{C}$ such that

$$(i_1, \mu) : X \vee X \xrightarrow{\sim} S(X \vee A)$$

is a weak equivalence in $\mathcal{C}$.

Proposition 6.1.13 *Each suspension ΣA in PQcrs is a homotopy cogroup and the coarse principal cofibration (X, R) in PQcrs with attaching map $[f] \in [Q, R]$ yields a homotopy coaction $(X, \Sigma Q, \mu)$, where Q is a based object, and μ is the coaction*

$$\mu \in [\mathcal{C}_g, \mathcal{C}_g \cup_R \Sigma Q]^R = [X, X \cup_R \Sigma Q]^R$$

Proof : See proposition (5.2), chapter III in [2]. □

Remark 6.1.14 Let T be a theory of coactions, the homotopy category of a cofibration category $\mathcal{C}$ with initial object $*$ given by $\mathrm{HO}(\mathcal{C}_c)$ or $\mathcal{C}_{cf}/\simeq$, and we have the equivalence of categories

$$S\colon HO(\mathcal{C}_c) \to \mathcal{C}_{cf}/\simeq$$

which carries X to a fibrant model SX with $X \overset{\sim}{\hookrightarrow} SX$. Sums $X \vee Y$ exist in $\mathcal{C}_c$ and $X \vee Y$ is also a sum in $\mathrm{HO}(\mathcal{C}_c)$ so that $S(X \vee Y)$ is the sum of SX and SY in $\mathcal{C}_{cf}/\simeq$ where $\mathcal{C}_c$ is the full subcategory of $\mathcal{C}$ consisting of cofibrant objects in $\mathcal{C}$, and $\mathcal{C}_{cf}$ is the full subcategory of $\mathcal{C}$ consisting of cofibrant and fibrant objects in $\mathcal{C}$.

The following definition comes from [2].

Definition 6.1.15 Let T be a theory of coactions, A *cofibration category under* T is a cofibration category $\mathcal{C}$ together with a full embedding of categories

$$T \subset HO(\mathcal{C}_c) \sim \mathcal{C}_{cf}/\simeq$$

which carries sums in T to sums in $HO(\mathcal{C}_c)$ such that objects in T are homotopy coactions in $\mathcal{C}$. Given an object X in T, we denote the corresponding object in $\mathcal{C}_{cf}$ as well by X.

Remark 6.1.16 The coarse homotopy category is given by $HO(PQcrs_c)$ or by $PQcrs_{cf}/\simeq$ and that we have the equivalence of categories

$$HO(PQcrs_c) \longrightarrow PQcrs_{cf}/\simeq$$

which carries X to a fibrant model SX with $X \overset{\sim}{\to} SX$.

Sums $X \cup_R Y$ exist in $PQcrs_c$ and also in $HO(PQcrs_c)$ so $S(X \cup_R Y)$ is the sum of SX and SY in $PQcrs_{cf}/\simeq$.

So for our cofibration category $PQcrs$ with initial object R, we obtain canonically full subcategories

$$susp(R) \subset cone(R) \subset HO(PQcrs_c)$$

where $susp(R)$ is the homotopy category of suspension in $PQcrs$ consisting of all suspensions ΣB where B is a based object in $PQcrs$ such that $B = R \rightarrowtail B \overset{p}{\to} R$ where ΣB is a homotopy cogroup in $PQcrs_c$ by proposition 6.1.13.

If B and B' are based objects then $B \cup_R B'$ is also, and we have $\Sigma(B \cup_R B') = \Sigma B \cup_R \Sigma B'$, then again proposition 6.1.13 shows that $susp(R)$ is a theory of cogroups.

Let $\mathcal{D}$ be the disjoint union $R \sqcup R \sqcup R \sqcup ...$ in PQcrs, and let

$$susp(R, \mathcal{D}) \subset susp(R)$$

be the subcategory of suspension ΣX (see 5.2.3) of the disjoint union $X = R \sqcup R \sqcup R \sqcup ...$ in PQcrs. Then ΣX is the glued coarse space defined by the push-out diagram in proposition 6.1.2 of one dimensional coarse spheres, and $susp(R, \mathcal{D})$ is a subtheory of $susp(R)$.

For each based object B in PQcrs as above, we may have controlled maps $g\colon B \to R$ which do not coincide with the trivial controlled map $p\colon B \to R$. In this case we obtain the mapping cone, termed $R-cone$, $\mathcal{C}_g = R \cup_g \mathcal{C}B$, by the following push out diagram

$$\begin{array}{ccc} \Sigma_R B & \longrightarrow & \mathcal{C}_g = \Sigma_R B \cup_B R \\ \uparrow & & \uparrow \\ B & \xrightarrow{\quad g \quad} & R \end{array}$$

where the cone $\mathcal{C}B$ is defined by the trivial controlled map p.

If $g = p$, then $\mathcal{C}_g = \Sigma B$.

And if we use a cylinder object $I_R B$ of B we obtain C_g by the push-out diagram in PQcrs as follows

$$\begin{array}{ccc} I_R B & \longrightarrow & \mathcal{C}_g = I_R B \cup_{B \cup_R B} R \\ {\scriptstyle [(i_0, i_1)]} \uparrow & & \uparrow \\ B \cup_R B & \xrightarrow{[(g,p)]} & R \end{array}$$

Hence $\mathcal{C}_g$ may also be considered to be a *double mapping cylinder* in PQcrs in which one of the gluing maps is specified to play the role of the trivial map $p\colon B \to R$. One needs this specification to define the coaction map $\mu\colon \mathcal{C}_g \to \mathcal{C}_g \cup_R \Sigma B$ in $HO(PQcrs_c)$. This is a homotopy coaction by proposition 6.1.13. Moreover, if $(g, g')\colon B \cup_R B' \to R$ is defined on a sum of based objects then $\mathcal{C}_{(g,g')} = \mathcal{C}_g \cup_R \mathcal{C}_{g'}$.

Let $cone(R)$ be the homotopy category of $R - cones$ in PQcrs. This is the full subcategory in $HO(PQcrs_c)$ consisting of $R - cones$. Then we have the inclusions of full subcategories

$$susp(R) \subset cone(R) \subset HO(PQcrs_c)$$

By definition of $\mathcal{C}_g$ above and μ, $\mathcal{C}_{(g,g')}$, we see that $cone(R)$ is actually a theory of coactions, and by definition of theory of coactions, then the cogroups in $cone(R)$ are exactly the suspensions in $susp(R)$.

By proposition 6.1.13, PQcrs is a cofibration category under the theory of coactions $cone(R)$.

The following definition comes from [2].

Definition 6.1.17 Let $\mathcal{C}$ be a cofibration category. Then $Fil_0(\mathcal{C})$ is the following category of *filtered objects* in $\mathcal{C}$. Objects are diagrams

$$A_{\geq 0} = (A_0 \to A_1 \to ...A_n \to A_{n+1} \to ...)$$

of maps $i\colon A_n \to A_{n+1}$ in $\mathcal{C}$, $n \geq 0$. A morphism $f\colon A_{\geq 0} \to B_{\geq 0}$ is a sequence of maps $f_n\colon A_n \to B_n$ with $if_n = f_{n+1}i$. We say that f is weak equivalence if each f_n is a weak equivalence in $\mathcal{C}$. Moreover f is a cofibration if each map $(f_{n+1}, f_n)\colon (A_{n+1}, A_n) \to (B_{n+1}, B_n)$ is a cofibration in $Pair(\mathcal{C})$. We have the full inclusion of categories

$$\mathcal{C} \subseteq Fil_0(\mathcal{C})$$

which carries $A \in \mathcal{C}$ to the constant filtered object with $A_n = A$ for $n \geq 0$ and $i = 1_A$. The initial object of $Fil_0(\mathcal{C})$ is the constant filtered object given by $*$ in $\mathcal{C}$. Moreover, we say that $X_{\geq 0}$ is of dimension $\leq n$ if $X_m = X_n$ for $m \geq n$ and $i\colon X_m \to X_{m+1}$ is the identity.

Let $Fil_1(\mathcal{C}) \subset Fil_0(\mathcal{C})$ be the full subcategory of objects $X_{\geq 0}$ with X_0 the initial object of $\mathcal{C}$. We write $X_{\geq 1} \in Fil_1(\mathcal{C})$ where $X_{\geq 1} = (X_1 \to X_2 \to ...)$ is given by $X_{\geq 0} = (* \to X^1 \to X_2 \to ...)$.

Lemma 6.1.18 *The category $Fil_0(\mathcal{C})$ with weak equivalences and cofibration as in the above definition is a Baues cofibration category.*

An object $A_{\geq 0}$ is fibrant if and only if all objects A_i, $i \geq 0$, are fibrant in $\mathcal{C}$. Moreover $A_{\geq 0}$ is cofibrant of A_0 is cofibrant in $\mathcal{C}$ and all $i\colon A_n \to A_{n+1}$ are cofibrations in $\mathcal{C}$.

Proof : see [1], and [2]. □

The following definition is found in [2].

Definition 6.1.19 In the category $Fil_0(\mathcal{C})$, we consider two notions of homotopies. Given a cofibrant object $A_{\geq 0}$, the cylinder

$$IA_{\geq 0} = (IA_0 \subset IA_1 \subset ...)$$

consists of a sequence of cylinders IA_n in $\mathcal{C}$, $n \geq 0$.

Two maps $f, g\colon A_{\geq 0} \to U_{\geq 0}$ are 0-homotopic if there exists a map $H\colon IA_{\geq 0} \to U_{\geq 0}$ in $Fil_0(\mathcal{C})$ with $Hi_0 = f$, $Hi_1 = g$.

We call such a homotopy a *0-homotopy* $H : f \simeq^0 g$. Let

$$i\colon U_{\geq 0} \to s^{-1}U_{\geq 0} \qquad (*)$$

be the canonical shift map in $Fil_0(\mathcal{C})$. Here we set $(s^{-1}U_{\geq 0})_n = U_{n+1}$ for $n \geq 0$, so that $s^{-1}U_{\geq 0} = (U_1 \to U_2 \to ...)$. Then (*) in degree n is the map $i\colon U_n \to U_{n+1}$.

The maps f, g are 1-*homotopic* $H : f \simeq^1 g$; if there exists a 0-homotopy $if \simeq^0 ig$. We define the *cylinder object* for 1-homotopies $\overline{I}A_{\geq 0} \in Fil_0(\mathcal{C})$ by

$$\begin{cases} (\overline{I}A_{\geq 0})_0 = A_0 \vee A_0 \\ (\overline{I}A_{\geq 0})_n = A_n \cup IA_{n-1} \cup A_n \quad for \quad n \geq 1 \end{cases}$$

where the right hand side is the push out of $A_n \vee A_n \leftarrow A_{n-1} \vee A_{n-1} \to IA_{n-1}$. Hence we have the cofibration $A_{\geq 0} \vee A_{\geq 0} \rightarrowtail \overline{I}A_{\geq 0}$ and a 1-homotopy $H : f \simeq^1 g$ is the same as a map $H\colon \overline{I}A_{\geq 0} \to U_{\geq 0}$ with $Hi_0 = f$, $Hi_1 = g$.

Example 6.1.20 Let $\mathcal{C} = Qcrs$, and if $A_{\geq 0}$ is the filtration of the coarse skeleta of the coarse CW-complex with only finitely many coarse cells in each dimension A in $Qcrs$, then $\overline{I}A_{\geq 0}$ is the filtration of coarse skeleta of the p-cylinder I_pA where $p\colon A \to R$ is a controlled map. Further if $\mathcal{C} = PQcrs$ is the category of non-unital pointed coarse spaces, and if $A_{\geq 1}$ is the filtration of the coarse skeleta $A^1 \subset A^2 \subset ...$ of a reduced coarse CW-complex with only finitely many coarse cells in each dimension A with $A^0 = R$, then $\overline{I}A_{\geq 1}$ is the filtration of coarse skeleta of the reduced cylinder I_RA.

The following definition is found in [2].

Definition 6.1.21 A cofibrant object $A_{\geq 0}$ in $Fil_0(\mathcal{C})$ is said to have the *limit property* if the colimit $A = Colim(A_{\geq 0})$ and $IA = Colim(IA_{\geq 0})$ exist, and if IA is a cylinder object for A. That is, A is a cofibrant object in $\mathcal{C}$ and the maps

$$A_{\geq 0} \vee A_{\geq 0} \xrightarrow{(i_0,i_1)} IA_{\geq 0} \xrightarrow{p} A_{\geq 0}$$

in $Fil_0(\mathcal{C})$ induce maps on colimits

$$A \vee A \xrightarrow{(i_0,i_1)} IA \xrightarrow{p} A$$

where (i_0, i_1) is a cofibration, and p is a weak equivalence in $\mathcal{C}$. It is clear that each finite dimensional object $A_{\geq j}$ in $Fil_0(\mathcal{C})$ has this limit property. Moreover the object $\overline{I}A_{\geq 0}$ in definition 6.1.19 satisfies

$$colim(IA_{\geq 0}) = colim(\overline{I}A_{\geq 0}).$$

The following definition is found in [2].

Definition 6.1.22 Let $\mathcal{C}$ be a cofibration category under T. A *complex* or a *T-complex* in $\mathcal{C}$ is a cofibrant object

$$X_{\geq 1} = (X_1 \subset X_2 \subset ...)$$

in $Fil_0(\mathcal{C})$ with the following properties. The object X_1 is an object in T and the pair (X_{n+1}, X_n), $n \geq 1$ is a principle cofibration with attaching map

$$\partial_{n+1} \in [\Sigma^{n+1} A_{n+1}, X_n].$$

Here A_{n+1} is a cogroup in T for $n \geq 1$. In particular $\partial_X = \partial_2 \in [A_2, X_1]$ is given by a map in T which represents an object ∂_X in the category Coef of coefficients that has objects $\partial_X \colon X'' \to X$ in T, where X'' is a cogroup, and morphism $\{f\} \colon \partial_X \to \partial_Y$ is the ∂-equivalence class of a ∂-compatible map $f \colon X \to Y$ (see definition (4.1), chapter (I) in [2]).

And X_2 is given by the mapping cone of ∂_X. Let

$$Complex \subset Fil_0(\mathcal{C})_c$$

be the full subcategory consisting of T-complexes $X_{\geq 1} = (X_{\geq 1}, A_{\geq 1}, \partial_{\geq 2})$. Here A_1 is the cogroup associated to the coaction on X_1. We write Complex= Complex (T). We also call a T-complex a *reduced complex*.

Theorem 6.1.23 *Each reduced coarse CW-complex with only finitely many coarse cells in each dimension X in the category $\mathcal{C} = PQcrs$ yields a filtered object $X^{\geq 0} = (X^1 \subset X^2 \subset ...)$ given by the skeleton X^n of X. This filtered object is a T-complex in the sense of the above definition.*

Proof : Let PQcrs be the quotient coarse cofibration category of non-unital pointed coarse spaces, and $PQcrs/\simeq$ the homtopy category, and $\mathcal{D}$ defined as before to be the disjoint union $R \sqcup R \sqcup ....$ in PQcrs, and let

$$susp(R, \mathcal{D}) \subset susp(R)$$

be the subcategory of suspension ΣX (see 5.2.3) of the disjoint union $X = R \sqcup R \sqcup ...$ in PQcrs. Then ΣX is the glued space defined by the push-out diagram in proposition 6.1.2 of one dimensional coarse spheres.

Weak equivalences are coarse homotopy equivalence classes in Qcrs and coarse cofibrations are pointed coarse cofibration classes which are coarse maps satisfying the homotopy extension property in Qcrs. If the objects in $PQcrs$ are fibrant, and the cofibrant objects X in $PQcrs$ are also termed well pointed coarse spaces when the classes $R \hookrightarrow X$ are coarse cofibration classes in Qcrs. Let

$$T_R = susp(R, \mathcal{D}) \subset (PQcrs)_c/\simeq,$$

be the full subcategory consisting of the glued spaces defined by the push-out diagram in proposition 6.1.2of coarse 1-spheres, namely $S^1_R \cup_R S^1_R \cup_R S^1_R \cup ....$

Then T_R is a theory of cogroups, and PQcrs is a cofibration category under T_R.

A coarse CW-complex with only finitely many coarse cells in each dimension X is *reduced* if the 0-skeleton $X^0 = R$ is the base point.

The skeleton X^1 is the glued space defined by the push-out diagram in proposition 6.1.2 of coarse 1-spheres, so $X^1 \in T_R$ and there exists a map $g\colon \Sigma^{n-1}A \to X^n$ in PQcrs with $A \in T_R$ such that X^{n+1} is coarse homotopy equivalent under X^n to the mapping cone $\mathcal{C}_g$.

If the reduced coarse CW-complex with only finitely many coarse cells in each dimension X has the property that (all attaching maps $\alpha\colon S^n_R \to X^n$ of $(n+1)$-coarse cells in X carry the basepoint of S^n_R to $R = X^0$), then the structure of X as a coarse T- complex is well defined.

So then for each a coarse CW-complex with only finitely many coarse cells X^n in Qcrs with $X^0 = R$, then $X = \{X^n, f_n\}$ is a coarse complex with only finitely many coarse cells in each dimension in PQcrs with attaching map $f_n \in [A_n, X^{n-1}]$ where $A_n = S^{n-1}_R \cup_R S^{n-1}_R \cup_R S^{n-1}_R \cup_R S^{n-1}_R \cup_R ...$ is the glued space defined by the push-out diagram in proposition 6.1.2 of one dimensional coarse spheres. In particular, A_1 is not a suspension, but the spaces A_i $(i > 1)$ are suspensions.

A reduced coarse CW-complex with only finitely many coarse cells in each dimension has the limit property in the category PQcrs as explained in 6.1.24.

Also one can show easily that a coarse subcomplex of a reduced coarse CW-complex with only finitely many coarse cells in each dimension is also subcomplex in the sense of definition (2.4), chapter IV in [2]. □

Proposition 6.1.24 *Any reduced coarse CW-complex with only finitely many coarse cells in each dimension $X_{\geq 0}$ in the category PQcrs has the limit property.*

Proof : By definition of coarse CW-complex and the sequential colimit we have $X = Colim(X^{\geq 0})$ exists and defines a reduced coarse CW-complex with only finitely many coarse cells.

Now by proposition (2.5) chapter IV in [2], then the cylinder $\overline{I}X^{\geq}$ is a reduced coarse CW-complex with only finitely many coarse cells with $X^{\geq 0} \cup_R X^{\geq 0}$ is a coarse CW-complex of $\overline{I}X^{\geq}$ and again $\overline{I}X = Colim(\overline{I}X^{\geq 0})$ exists and defines a reduced coarse CW-complex with only finitely many coarse cells.

Now the maps

$$X^{\geq 0} \cup_R X^{\geq 0} \xrightarrow{(i_0, i_1)} \overline{I}X^{\geq 0} \xrightarrow{p} X^{\geq 0}$$

in the category $PQcrs$ induce maps on colimits

$$X \cup_R X \xrightarrow{(i_0, i_1)} \overline{I}X \xrightarrow{p} X$$

where (i_0, i_1) is a coarse cofibration, and p is a coarse homotopy equivalence in PQcrs, and we are done. □

§ 6.2 The Whitehead Theorem

We obtain here a coarse version of the classical Whitehead theorem as corollary of an abstract result in Baues cofibration categories. This result implies an equivalence between two of our cofibration categories.

This leads to recall the following definitions from [2]:

Definition 6.2.1 We say that a map $f \colon X_{\geq 1} \to Y_{\geq 1}$ in $Fil_1(\mathcal{C})_{cf}$ is a *lifting map* if for all T-complexes $K_{\geq 1}$ with subcomplexes (see definition (2.4), chapter IV in [2]) $L_{\geq 1}$ and commutative diagrams

$$\begin{array}{ccc} L_{\geq 1} & \xrightarrow{b} & X_{\geq 1} \\ \downarrow j & & \downarrow f \\ K_{\geq 1} & \xrightarrow{a} & Y_{\geq 1} \end{array}$$

in $Fil_1(\mathcal{C})_c$ (where j is the inclusion) there exists a map

$$d \colon K_{\geq 1} \to X_{\geq 1} \in Fil_1(\mathcal{C})_c$$

with $dj = b$ and 1-homotopy $fd \simeq^1 a$ relative $L_{\geq 1}$. This map is termed a *lift* of the diagram.

Definition 6.2.2 A map $f \colon X_{\geq 1} \to Y_{\geq 1}$ in $Fil_1(\mathcal{C})_{cf}$ is a T*-equivalence* if for all T-complexes $K_{\geq 1}$ the induced map

$$f_* \colon [K_{\geq 1}, X_{\geq 1}]/\simeq^1 \to [K_{\geq 1}, Y_{\geq 1}]/\simeq^1$$

is a bijection. The above sets define the sets of 1-homotopy classes. Moreover f is a *weak T-equivalence* if for all cogroups A and objects Z in T and $n \geq 1$ the induced maps f_* below are bijections, where im = image.

$$\mathrm{im}\{[Z, Y_1] \xrightarrow{i_*} [Z, Y_2]\} \xrightarrow{f_*} \mathrm{im}\{[Z, X_1] \xrightarrow{i_*} [Z, X_2]\}$$

$$\mathrm{im}\{\pi_n^A Y_{n+1} \to \pi_n^A Y_{n+2}\} \xrightarrow{f_*} \mathrm{im}\{\pi_n^A X_{n+1} \to \pi_n^A X_{n+1}\}$$

By an argument in [2] it is easy to show that a T-equivalence is a weak T-equivalence.

The proof of the following proposition is found in [2].

Proposition 6.2.3 *A lifting map f is a T-equivalence.* □

This implies that f is a weak T-equivalence.

Definition 6.2.4 We say that $X_{\geq 1} \in Fil_1(\mathcal{C})_{cf}$ is *T-good* if for all cogroups A in T and $n \geq 1$, the groups $\pi_n^A(X_{n+2}, X_{n+1}) = 0$ are trivial and $\pi_n^A(X_1) \to \pi_n^A(X_2)$ is surjective.

The following proposition is found in [2].

Proposition 6.2.5 *A weak T-equivalence between T-good objects in $Fil_1(\mathcal{C})_{cf}$ is also a lifting map.* □

An argument in [2] and the above proposition proves the following result.

Theorem 6.2.6 *Let $\mathcal{C}$ be a cofibration category under T and $f\colon X_{\geq 1} \to Y_{\geq 1}$ be a map between T-good objects in $Fil_1(\mathcal{C})_{cf}$. Then the following are equivalent:*

1 *The map f is lifting map.*

2 *The map f is T-equivalence.*

3 *The map f is weak T-equivalence.* □

The proof of the following proposition is found in [2].

Proposition 6.2.7 *Let $f\colon X_{\geq 1} \to Y_{\geq 1}$ be a map between T-complexes which is a T-equivalence. Then f is a 1-homotopy equivalence (that is an isomorphism in the category **Complex**$/\simeq^1$).* □

As a corollary of the last two propositions we get the following generalization of the classical Whitehead theorem.

General Whitehead Theorem (I) . Let $X_{\geq 1}$, $Y_{\geq 1}$ be T-complexes which are T-good. Then a map $f\colon X_{\geq 1} \to Y_{\geq 1}$ is a weak T-equivalence if and only if f is a 1-homotopy equivalence. □

Cellular Approximation

We consider the coarse version of the classical *coarse cellular approximation theorem* of reduced coarse CW-complexes with only finitely many coarse cells in each dimension in the category PQcrs that has been proved in [14] for a particular case as follows:

Before stating the coarse version of the classical coarse cellular approximation, we need the following definition.

Definition 6.2.8 Let X be a coarse space. We call X is *coarsely path connected* if $\pi_0^{Pcrs}(X) = \{0\}$.

The following definition comes from

Definition 6.2.9 Let X, Y be coarse CW-complexes. A coarse map $f\colon X \to Y$ is called *cellular* if $f(X^n) \subseteq Y^n$ for each n.

The proof of the following theorem is found in [14].

Theorem 6.2.10 *If X and Y are coarse CW-complexes with only finitely many coarse cells in each dimension, Y is coarsely path connected, and $f\colon X \to Y$ is a coarse map such that $f\,|_K$ is a coarse cellular map for some subcomplex K of X (possibly empty), then there exists a coarse cellular map $g\colon X \to Y$ such that $g\,|_K = f\,|_K$ and g is coarse homotopic to f relative K.* $\square$

Note that all of our coarse CW-complexes are *full* in the sense of [14].

Compare with theorem 6.1.23. A coarsely cellular map f between pointed non-unital reduced coarse CW-complexes with only finitely many coarse cells in each dimension in the category PQcrs is equivalent to a filtered map $f^{\geq 1}\colon X^{\geq 1} \to Y^{\geq 1}$ with $lim(f^{\geq 1}) = f$.

Now consider the following commutative diagram of reduced coarse CW-complexes as defined above;

$$\begin{array}{ccc} L^{\geq 1} & \xrightarrow{g|_L} & X^{\geq 1} \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle p} \\ K^{\geq 1} & \xrightarrow{g} & X \end{array}$$

where $g\colon K^{\geq 1} \to X$ is a coarsely cellular map in PQcrs which restricted to a subcomplex L of K is coarsely homotopic to a cellular map $f\colon K \to X$ relative L. Here X is constant coarse CW-complex and p is the canonical map given in degree n by the inclusion $X^n \subset X$. The filtered map g in the diagram is given in degree n by the composite $K^n \subset K \xrightarrow{g} X$.

The cellular approximation theorem is equivalent to the existence of a lift of the above diagram with $fj = g|_L$ and $pf \simeq^1 g$ relative $L^{\geq 1}$.

By the notion of lifting map defined in 6.2.1, we have the following theorem.

Theorem 6.2.11 ***(Coarse Cellular Approximation Theorem.)*** *Let X be a reduced coarse CW-complex with only finitely many coarse cells in each dimension and which is coarsely path connected. Then $p\colon X^{\geq 1} \to X$ in the previous diagram is a lifting map.*

Proof : Straightforward from theorem 6.2.10, and definition of lifting map. □

This lifting map is defined as in definition 6.2.1 on the category PQcrs under $T = susp(R, D)$ as defined before; compare theorem 6.1.23.

Theorem 6.2.11 leads to the following definition that comes from [2].

Definition 6.2.12 Let $\mathcal{C}$ be a cofibration category under T. We call a cofibrant and fibrant object X in $\mathcal{C}$ *weakly cellular* if there exists a T-complex $X_{\geq 1}$ and a lifting map

$$p\colon X_{\geq 1} \to X$$

where X is the constant filtered object given by X. Moreover X is a *cellular* if there exists a lifting map as p for which $X_{\geq 1}$ has the limit property (see definition 6.1.21), and the induced map

$$p\colon lim(X_{\geq 1)} \to X$$

is a weak equivalence in $\mathcal{C}$.

Let **Cell** $\subset$ **Well** $\subset \mathcal{C}_{cf}$ be full subcategories where **Cell** consists of cellular objects and **Well** consists of weakly cellular objects.

Remark 6.2.13 A reduced full coarse CW-complex X with only finitely many coarse cells in each dimension and which is coarsely path connected has the property that

$$X = colim(X^{\geq 1})$$

is the colimit of the filtered object $X^{\geq 1}$ in the category PQcrs. Then each reduced coarse CW-complex with only finitely many coarse cells in each dimension and which is coarsely path connected is cellular by the limit property and since the cellular approximation holds for this particular coarse CW-complex.

The notion of cellular object in a cofibration category under T is the appropriate generalization of the classical notion of coarse CW-complex in coarse geometry that satisfies specific conditions.

By an argument in [2] Baues proved the following theorem:

Theorem 6.2.14 ***General Whitehead Theorem (II):*** *Let $\mathcal{C}$ be a cofibration category under T and let X and Y be cellular objects in $\mathcal{C}$. Then $f\colon X \to Y$ is a homotopy equivalence in $C_{cf}/\simeq$ if and only if for all cogroups A and objects Z in T and $n \geq 1$ the induced maps*

$$f_*\colon \pi_n^A(X) \to \pi_n^A(Y)$$

$$f_*\colon [Z,X] \to [Z,Y]$$

are bijections. □

Now we show that we could have a coarse version of the Whitehead Theorem as follows;

Let $\mathcal{C} = PQcrs$, $T = susp(R,D)$ be as defined early and the group $\pi_n^{PQcrs}(X,R)$ for a non-unital coarse space X defined similarly to the group $\pi_n^{Pcrs}(X,R)$ in definition (2.4.4). Then we obtain the following special case of the Whitehead theorem:

Theorem 6.2.15 *Let $f\colon X \to Y$ be a map between pointed non-unital coarse CW-complexes and which is coarsely path connected in PQcrs. Then f is a coarse homotopy equivalence if and only if f induces isomorphisms*

$$f_*\colon \pi_n^{PQcrs}(X,R \to \pi_n^{PQcrs}(Y,R)$$

for $n \geq 1$. □

Since all objects of $susp(R,D)$ are the glued spaces defined by the push-out diagram in proposition 6.1.2 of coarse 1-spheres we can see that the previous isomorphisms are equivalent to the corresponding condition in theorem 6.2.14. It is clear that all reduced coarse CW-complexes with only finitely many coarse cells in each dimension and which are coarsely path connected are cellular objects by theorem 6.2.11.

§ 6.3 Equivalence of Cofibration Categories

In this section we will show that there is an equivalence between a subcategory of the category Qcrs and the category of coarse CW-complexes that is a generalization of Quillen equivalence (see [15]), and denoted *cofibration equivalence*.

The following definition is found in [5].

Definition 6.3.1 Let $F\colon C \to D$ and $G\colon D \to C$ be a pair of functors. An *adjunction* from F to G is a collection of isomorphisms

$$\alpha_{X,Y} : Hom_D(F(X),Y) \cong Hom_C(X,G(Y)),\ X \in Ob(C),\ Y \in Ob(D)$$

natural in X and Y , i.e., a collection which gives a natural equivalence (2.2) in [5] between the two indicated Hom-functors $C^{op} \times D \to Set$ (see 2.4 in [5]). If such an adjunction exists we write

$$F : C \Leftrightarrow D : G$$

and say that F and G are *adjoint functors* or that (F,G) is an *adjoint pair*, the functor F being the *left adjoint* of G and G the *right adjoint* of F. Any two left adjoints of G (resp. right adjoints of F) are canonically naturally equivalent, so we speak of the left adjoint or right adjoint of a functor (if such a left or right adjoint exists).

If $f\colon F(X) \to Y$ (resp. $g\colon X \to G(Y)$), we denote its image under the bijection $\alpha_{X,Y}$ by $f^{'}: X \to G(Y)$ (resp. $g^{'}: F(X) \to Y$).

The following definition is the same definition of Quillen equivalence as that found in [15] for model categories, but here we consider the equivalence between cofibration categories.

Definition 6.3.2 Given two cofibration categories $\mathcal{C}$ and $\mathcal{D}$. A *cofibration adjunction* is a pair $(F, G) : \mathcal{C} \leftrightarrows \mathcal{D}$ of adjoint functors with F left adjoint to G such that F preserves cofibrations and weak equivalences or, equivalently G preserves weak equivalences.

In such an adjunction F is called the *left functor* , and G is called the *right functor* .

If (F, G) is a cofibration adjunction as above such that $F(c) \to d$ is a weak equivalence in $\mathcal{D}$ if and only if $c \to G(d)$ is a weak equivalence in $\mathcal{C}$, then (F, G) is called a *cofibration equivalence* of the cofibration categories $\mathcal{C}$ and $\mathcal{D}$.

Now we will show that there is no a cofibration equivalence between the controlled category and the quotient coarse category or the coarse CW-complexes category .

Theorem 6.3.3 *The category Crd is not cofibration equivalent to the category Qcrs nor the category CWrs.*

Proof : It is known that having Quillen equivalence between model categories implies having the same homotopy groups. The same still works for cofibration equivalence between cofibration categories, and that of course means that not having the same homotopy groups implies no cofibration exists between categories. This is the case for our categories.

The homotopy group for any coarse space in the category Crd has always one element while the homotopy group in the category Qcrs does have more than one element for at least one example as shown in example 2.4.10. That is, the homotopy groups for both categories are different which implies that no cofibration equivalence can be found between both categories Qcrs, CWrs and the category Crd. □

Now consider the full subcategory of PQcrs that has objects to be all pointed non-unital reduced coarse CW-complexes which are coarsely path connected, and morphisms to be pointed coarsely cellular classes. Then it can be easily proved that this category is a cofibration category, and we denote it by PCWQcrs, and a similar argument to theorem 6.2.15 shows that the Whitehead theorem is satisfied in this subcategory as well.

By a similar argument to that found in proposition 5.1.6 that shows the category PQcrs is a cofibration category, we are able to show that the pointed coarse CW-complexes category PCWrs that has objects to be pointed non-unital coarse CW-complexes which are coarsely path connected and morphisms to be pointed coarsely cellular classes is a cofibration category as follows.

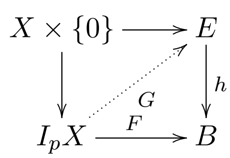
Proposition 6.3.4 *The category PCWrs is a Baues cofibration category. The weak equivalences are weak coarse homotopy equivalence classes relative to R, and cofibrations are pointed coarse CW-cofibration classes.*

Proof : By definition (1.4) chapter III in [2], the category PCWrs is a subcategory of the category **Pair(**$CWrs$**)**. Objects are the the non-unital pointed coarse CW-complexes, and the morphisms are the pointed coarsely cellular classes. Weak equivalences and cofibrations in the category $CWrs$) yield the structure of Baues cofibration category for the category PCWrs since by lemma ((1.5), chapter (II), [1]) **Pair(**$CWrs$**)** is a cofibration category. □

Now we want to show that there is a cofibration equivalence between the cofibration categories PCWQcrs and PCWrs. This implies that the homotopy theory of both categories is the same. (See [15]).

Definition 6.3.5 Let E, B be coarse spaces. A coarse map $h\colon E \to B$ is called a *coarse fibration* if given coarse maps $f\colon X \to E$, and a coarse homotopy $F\colon I_pX \to B$ such that $h(f(x)) = F(x,0)$ for all $x \in X$, $t \in R$ where $p\colon X \to R$ is some controlled map, we can find a coarse homotopy $G\colon I_pX \to E$ such that $f(x) = G(x,0)$ for all $x \in X$, $hG = F$.

This definition is illustrated by the following commutative diagram:

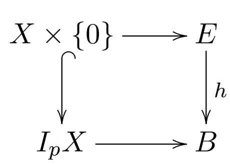


Clearly any coarse fibration is a coarse Serre fibration.

The following proposition is a coarse version of theorem (9) in [18].

Proposition 6.3.6 *Let $f\colon A \to X$ be a coarse cofibration in the category PCWQcrs, then f has the LLP with respect to a coarse fibtation and a coarse homotopy equivalence (we may assume f is an inclusion).*

Proof : Let $h\colon E \to B$ be a pointed coarse map such that the map f has the LLP with respect to the map h. Applying this to the maps



we see that h must be a coarse fibration. The pair $(B, \emptyset)$ is a cofibrant pair, and a retraction $s\colon B \to E$ of h is obtained as a lifting of

$$\begin{array}{ccc} \emptyset & \hookrightarrow & E \\ \downarrow & & \downarrow h \\ B & \xrightarrow[id_B]{} & B \end{array}$$

Finally, let $q\colon E \to R$ be some controlled map where $F\colon I_q E \to E$ is a lifting of the maps

$$\begin{array}{ccc} \{(e,0),(e,q(e)+1) : e \in E\} & \xrightarrow{f''} & E \\ \downarrow & & \downarrow h \\ I_q E & \xrightarrow[f']{} & B \end{array}$$

with $f''(e,0) = sh(e)$, $f''(e,q(e)+1) = e$, $f'(e,t) = h(e)$. Then F defines a coarse homotopy between sh and the identity on E. Hence h is a coarse homotopy equivalence as required. □

Proposition 6.3.7 *If f is a retract of g and g is a coarse fibration, coarse cofibration, or a coarse homotopy equivalence, then so is f.*

Proof : Suppose that a map $f\colon X \to X'$ is a retract of the map $g\colon Y \to Y'$, that is, there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ f\downarrow & & g\downarrow & & f\downarrow \\ X' & \xrightarrow[i']{} & Y' & \xrightarrow[r']{} & X' \end{array}$$

in which ri is close to id_X, $r'i'$ is close to $id_{X'}$.

First, we prove the case when g is a coarse homotopy equivalence, so there is a coarse map $h\colon Y' \to Y$ such that $g \circ h$ is coarse homotopic to $id_{Y'}$ and $h \circ g$ is coarse homotopic to id_Y. Now we have by the above diagram $rhi'f = rhgi$ which by assumption implies that $rhi'f$ is coarsely homotopic to id_X, and similarly we have $frhi'$ is coarsely homotopic to $id_{X'}$. Hence f is a coarse homotopy equivalence.

Second, we prove the case when g is coarse cofibration. Suppose that we are given a coarse map $k\colon X' \to Z$, and a coarse homotopy $F\colon I_{p\circ f}X \to Z$ where $p\colon X' \to R$ is some given controlled map such that $F(x,0) = k(f(x))$. By the above diagram we can define a controlled map $q\colon Y' \to R$ by $q = p \circ r'$, a coarse map $k'\colon Y' \to Z$ by $k' = k \circ r'$, and another coarse map $F'\colon I_{p\circ f\circ r}Y \to I_{p\circ f}X$ by $F'(y,t) = (r(y),t)$ for all $y \in Y$.

Now define a map $G\colon I_{p\circ f\circ r}Y \to Z$ by $G(y,t) = F \circ F^{'}(y,t)$. Then G is a coarse homotopy map, and $G(y,0) = F \circ F^{'}(y,0) = F(r(y),0) = k \circ f(r(y)) = k \circ r^{'} \circ g(y) = k^{'}(g(y))$. Since g is a coarse cofibration, there exists a coarse homotopy $G^{'} : I_{p\circ r^{'}}Y^{'} \to Z$ such that $G^{'}(y^{'},0) = k^{'}(y^{'})$ for all $y^{'} \in Y^{'}$, and $G^{'}(g(y),t) = G(y,t)$.

Define a map $H\colon I_pX^{'} \to Z$ by $H(x^{'},t) = G^{'}(i^{'}(x^{'}),t)$, then H is a coarse homotopy and $H(x^{'},0) = G^{'}(i^{'}(x^{'}),0) = k^{'}(i^{'}(x^{'})) = k\circ r^{'}\circ i^{'}(x^{'})$ and $H(f(x),t) = G^{'}(i^{'}(f(x),t) = G^{'}(g(i(x)),t) = G(i(x),t) = F \circ F^{'}(i(x),t) = F(r(i(x)),t)$.

Define another map $H^{'} : I_pX^{'} \to Z$ by

$$H^{'}(x^{'},t) = \begin{cases} k(x^{'}) & t = 0 \\ F(x,t) & x^{'} = f(x) \\ H(x^{'},t) & \text{otherwise} \end{cases}$$

But $H^{'}$ is close to G, and since H is a coarse map by lemma 2.2.3 we have $H^{'}$ is a coarse map. Then f is a coarse cofibration.

The third case is similar to the second. □

The above proposition is still true in the category PCWQcrs.

The key idea of the following proposition is similar to proposition (3.13) in [5].

Proposition 6.3.8 *The coarse cofibrations in the category Qcrs, PQcrs, and PCWQcrs are the coarse maps which have the LLP with respect to coarse fibrations and coarse homotopy equivalences.*

Proof : We prove the case for the catgeory Qcrs, and the other cases are identical.

Suppose that $f\colon K \to L$ a coarse map that has the LLP with respect to all coarse fibrations and coarse homotopy equivalences. Since the category Qcrs is a cofibration category, the factorization axiom allows us to factor the map f as $K \hookrightarrow L^{'} \overset{\sim}{\to} L$, where $i : K \hookrightarrow L^{'}$ is a coarse cofibration and $r\colon L^{'} \to L$ is a coarse homotopy equivalence. By assumption there exists a lifting $g\colon L \to L^{'}$ in the following diagram

$$\begin{array}{ccc} K & \xrightarrow{i} & L^{'} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r} \\ L & \xrightarrow[id]{} & L \end{array}$$

This implies that f is a retract of i as follows

$$\begin{array}{ccccc} K & \xrightarrow{id} & K & \xrightarrow{id} & K \\ {\scriptstyle f}\downarrow & & {\scriptstyle i}\downarrow & & {\scriptstyle f}\downarrow \\ L & \xrightarrow[g]{} & L^{'} & \xrightarrow[r]{} & L \end{array}$$

Since i is a coarse cofibration, so by proposition 6.3.7, then f is a coarse cofibration. □

One of our main aims in this chapter is to prove the following theorem.

Theorem 6.3.9 *There is a cofibration equivalence between the cofibration categories PCWQcrs and PCWrs.*

Proof : Define a pair of of Functors $(F, G) : PCWQcrs \leftrightarrows PCWrs$ to be the identity functors in the sense of sending objects and morphisms to themselves, so we need to prove first if F preserves coarse cofibrations and coarse homotopy equivalences.

But it is easy to show that any coarse homotopy equivalence class is a weak coarse homotopy equivalence class, and proposition 6.3.6 shows that any coarse cofibration cla$[f]$ in the category PCWQcrs has the LLP with respect to coarse fibration classes and coarse homotopy equivalence classes which implies easily that f has the LLP with respect to coarse Serre fibration classes and weak coarse homotopy equivalence classes, and that is the definition of coarse CW-cofibration class in the category PCWrs. Therefore the pair (F, G) is a cofibration adjunction.

Now let $[h]\colon F(c) \to d$ be a weak coarse homotopy equivalence class in the category PCWrs. Define a map $\alpha\colon Hom_{PCWrs}(F(c), d) \to Hom_{PCWQcrs}(c, G(d))$ by $\alpha(h(F(c)) = g(c)$ where g is the map $g\colon c \to G(d)$ in the category PCWQcrs defined by $g(c) = G(d)$,and since the functors F, and G are the identities, so the map α sends the weak coarse homotopy equivalence class $[h]$ to itself. Now theorem 6.2.15 shows that any weak coarse homotopy equivalence class is a coarse homotopy equivalence class which means that the map g is a coarse homotopy equivalence class.

Since coarse homotopy equivalence classes imply weak coarse homotopy equivalence classes and the functors are the idenities, then any coarse homotopy equivalence class $[g]\colon c \to G(d)$ in the category PCWQcrs defined by $g(c) = G(d)$ implies easily that $[h]\colon F(c) \to d$ is a weak coarse homotopy equivalence class in the category PCWrs. Hence the categories PCWQcrs and PCWrs are cofibration equivalent. □

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