Moduli Spaces of Vortices on Surfaces

Castalia Gautam Chaudhuri

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[2] C. G. Chaudhuri, D. Harland and J. M. Speight. *Dynamics of 2-Vortices on Flat Tori*. In preparation.

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Author contributions:

- C. G. Chaudhuri: conceptualisation (equal), methodology and proof (equal), writing — first draft (lead), writing — review/editing (equal).
- D. Harland: conceptualisation (equal), methodology and proof (equal), supervision (equal), writing — first draft (supporting), writing — review/editing (equal).
- J. M. Speight: conceptualisation (equal), methodology and proof (equal), supervision (equal), writing — first draft (supporting), writing — review/editing (equal).

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Dynamics of 2-Vortices on Flat Tori [2]

Author contributions:

- C. G. Chaudhuri: conceptualisation (equal), methodology and proof (lead), software and visualisation (lead), writing — first draft (lead), writing — review/editing (equal).

- D. Harland: conceptualisation (equal), supervision (equal), methodology and proof (supporting), writing — first draft (supporting), writing — review/editing (equal).
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For Sára

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In other words, what is inevitable in a work of art is the style. [...] The most attractive works of art are those which give us the illusion that the artist had no alternatives, so wholly centered is he ***in*** *his style.*

Susan Sontag
On Style
From the collection *"Against Interpretation and Other Essays"*

Abstract

We study the structure and dynamics of Abelian Higgs vortices on compact Riemann surfaces. In particular, we consider vortices close to the dissolving limit, a critical lower bound on the area of the Riemann surface determined by other parameters of the field theory. We partially resolve a conjecture surrounding the pseudovortex approximation of moduli space dynamics in this limit, and study the implications this has in two concrete cases.

We begin by constructing an analytic framework for expressing and comparing the Fubini-Study and vortex metrics on complete linear systems of vortices, going on to show that a rescaled and restricted form of the latter converges uniformly and continuously differentially to the former. We then show that this form of convergence is sufficient to guarantee the convergence of geodesics of the two metric with some fixed initial data, validating the prior result as being sufficient for the study of vortex dynamics.

In the fourth chapter, we turn our attention to vortices on the round sphere which are symmetric around a given axis, considering the behaviour of two species of vortex: one where the cores are evenly distributed among both poles, and one where the cores are confined to one. We show that in the former case, the pseudovortex approximation has bounded error as the vortex charge increases; while in the former case the error in the pseudovortex approximation grows unbounded. This suggests that the validity of the pseudovortex approximation in the large degree limit is deeply tied to the distribution of the vortex cores.

Lastly, we consider 2-vortices on flat tori, applying the geodesic approximation result outlined previously. We interpret geodesic motion of 2-vortices on a certain complete linear system as the relative motion of 2-vortices on a flat torus. We then construct an explicit biholomorphic isometry between this complete linear system and the complex projective plane equipped with the Fubini-Study metric, which allows us to write down explicit geodesics of 2-vortices in terms of their core trajectories. We then completely classify the kinds of relative motion that can be exhibited by these 2-vortices, noting its topological nature. We also construct an explicit expression for pseudovortex Higgs fields in terms of elliptic functions.

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List of Symbols

This lists symbols and conventions we will use throughout the thesis.

Notation

$\diamond$ End of a definition.

$\perp$ When at the right margin, end of a proof by contradiction.

$\langle f \rangle$ Smoothed absolute value, equal to $\sqrt{1+|f|^2}$.

$\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)$ The pseudovortex configuration associated to a vortex (D_A, ϕ) (page 21).

Operators and Maps

$\langle \cdot, \cdot \rangle$ The pointwise real inner product on the space of sections of a metric vector bundle.

$(\!(\cdot, \cdot)\!)$ The L^2 inner product on the space of sections of a metric vector bundle.

$*$ The Hodge star, i.e. the operator such that $\alpha \wedge *\beta = \langle \alpha, \beta \rangle \, \omega^{\Sigma}$. By convention it is complex anti-linear, $*\lambda\beta = \bar{\lambda} * \beta$.

δ The Hodge co-differential, defined on k-forms to be equal to $\delta = - * \mathrm{d} *$.

Δ The Laplace-de Rham operator, equal to $\delta\mathrm{d} + \mathrm{d}\delta$.

AJ The Abel-Jacobi map.

$\wp$ The Weierstrass $\wp$-function [20, (23.2.4)]

Sets, Spaces, and Structures

$\Omega^k(X)$ Real differential k-forms on a manifold X.

$\Omega^{p,q}(X)$ Complex differential (p,q)-forms on a complex manifold X.

$\Omega^0(X; S)$ Smooth sections of the trivial bundle $S \times X$ (if S is some fixed space).

$\Omega^k(X; E)$ Differential k-forms on a manifold X taking values in E (if E is a fixed algebra or a vector bundle).

$\Omega^{p,q}(X; E)$ Complex differential (p,q)-forms on X taking values in E (where E is either a fixed complex algebra or a complex vector bundle).

$\mathcal{A}(L,H)$ The space of Hermitian connections on (L,H).

$\mathcal{G}$ Unitary gauge group, equal to $\Omega^0(\Sigma;\mathrm{U}(1))$.

$\mathrm{Pic}(\Sigma)$, $\mathrm{Pic}^N(\Sigma)$ Resp. the Picard group and the coset corresponding to bundles of degree N.

$\mathrm{Jac}(\Sigma)$ The Jacobian of Σ, i.e. $\mathrm{Pic}^0(\Sigma)$.

$\mathcal{H}^N(\Sigma)$ The set of degree N holomorphic pairs on Σ modulo equivalence.

$\mathrm{Div}(\Sigma), \mathrm{Div}^N(\Sigma)$ Respectively, the Abelian group of all (Weil) divisors on Σ and its Nth grade.

$\mathrm{Div}_+(\Sigma), \mathrm{Div}_+^N(\Sigma)$ Respectively, the Abelian monoid of all effective (Weil) divisors on Σ and its Nth grade.

$X^{(N)}$ N-fold symmetric product of a set X, equal to X^N/S_N.

$\mathcal{C}$, $\mathcal{C}_0$ Respectively, the affine space of all configurations and its underlying model vector space, resp. equal to $\mathcal{A}(L,H)\times\Omega^0(\Sigma;L)$ and $\Omega^1(\Sigma;\mathfrak{u}(1))\times\Omega^0(\Sigma;L)$.

$\mathcal{V}^\varepsilon$, $\mathcal{M}^\varepsilon$ Respectively the vortex *configuration* and *moduli* spaces (pages 16–17).

Abbreviations

- **BPS**: Bogomolny-Prasad-Sommerfield
- **gec.**: gauge equivalence class
- **GL**: Ginzburg-Landau
- **PSC**: prescribed scalar curvature

Chapter 1

Introduction

A topological soliton in a classical field theory is a field configuration with a *topologically* conserved charge, i.e. a quantity that is kept constant by the topology of the domain and the target space of the field. Such charges correspond to a particular algebro-topological invariant, e.g. a homotopy or (co)homology class. Solitons are also particle-like, being described as some variation from the vacuum configuration around a point. The study of topological solitons has proven to be useful in understanding a variety of physical phenomena across a wide range of length and energy scales, from models of cosmic strings in the early universe to approximations of nuclear matter.

This thesis studies the dynamics of topological solitons in the *Abelian Higgs model*, a classical field theory closely linked to the older *Ginzburg-Landau (GL) theory* of superconductivity[1]. It consists a pair of fields: a unitary gauge field, $A_\mu : \mathbb{R}^{2+1} \to \mathfrak{u}(1)$, and a complex Higgs field, $\phi : \mathbb{R}^{2+1} \to \mathbb{C}$; which are coupled by the following Lagrangian [46, p. 181]

$$L = \int -\frac{1}{4}\bar{f}^{\mu\nu} f_{\mu\nu} + \frac{1}{2}\overline{D^\mu \phi} D_\mu \phi - \frac{\lambda}{2}\left(\frac{1}{2}\left(|\phi|^2 - \tau\right)\right)^2 \mathrm{d}x^1\, \mathrm{d}x^2 \,. \tag{1.1}$$

Here $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $D_\mu = \partial_\mu + A_\mu$, and λ and τ are constant positive parameters[2]. The gauge field can be thought of as the usual electromagnetic gauge field and so $f_{\mu\nu}$ is the electromagnetic field strength, whereas the quantity $|\phi|^2$ can be thought of as the expectation value of some bosonic field with a U(1) gauge symmetry.

One notes that ϕ achieves its minimum potential when $|\phi|^2 = \tau$, i.e. the Higgs field has a non-zero vacuum expectation value. Moreover, the space of values it can take to achieve this minimum, the *vacuum manifold*, has a non-vanishing fundamental group. It is this last point which generates topological solitons in the model. Finite-energy field configurations must take on values in the vacuum manifold at spatial infinity; so in addition to the trivial configuration where ϕ takes on a constant value at infinity, it is possible that ϕ could vary in phase at infinity, perhaps even describing a loop in the vacuum manifold.

[1] See e.g. [34, Ch. I.5] or [46, pp. 5–6] for a full history and description.

[2] We use the mostly negative convention for the metric tensor. Greek indices run from 0 to 2, and Roman indices run from 1 to 2.

1.1 Vortices on Riemann Surfaces

While the history of solitons in the Abelian Higgs model and GL-theory dates back to at least 1957 with Abrikosov's original work in [1], we begin our survey of the field with the work of Bogomolny [14] and Taubes [62]. Bogomolny noted that when $\lambda = 1$, so-called *critical coupling*, one can rewrite the spatial component of (1.1) as

$$\begin{aligned} V = \frac{1}{2}\int \left| f_{12} - \frac{\mathbf{i}}{2}\Big(\tau - |\phi|^2\Big)\right|^2 + |(D_1 + \mathbf{i}D_2)\phi|^2 \,\mathrm{d}x^1\,\mathrm{d}x^2 \\ +\tau\int -\mathbf{i}f_{12}\,\mathrm{d}x^1\,\mathrm{d}x^2 - \tau\int \mathrm{d}\big(\overline{\phi}D_i\phi\,\mathrm{d}x^i\big)\,. \end{aligned} \tag{1.2}$$

We can evaluate the second line exactly as $\int \mathrm{d}\big(\overline{\phi}D_i\phi\,\mathrm{d}x^i\big) = 0$ and $\int -\mathbf{i}f_{12}\,\mathrm{d}x^1\,\mathrm{d}x^2 = 2\pi N$ for some $N \in \mathbb{Z}$. It therefore follows that minimisers of the spatial energy satisfy a pair of field equations now known as the *Bogomolny equations*

$$(D_1 + \mathbf{i}D_2)\phi = 0 \tag{1.3}$$

$$f_{12} = \frac{\mathbf{i}}{2}\Big(\tau - |\phi|^2\Big). \tag{1.4}$$

The key property of solutions of the Bogomolny equations is that they are energy minimisers *within their homotopy class*, and thus stable minima of E. These homotopically non-trivial solutions are the topological solitons in the Abelian Higgs model and we call them *vortices*. The associated topological charge, N, is called the *vortex number*. It was shown by Taubes [62] that (non-trivial) solutions to the field equations above all have $N > 0$.

The physical interpretation of vortices is that as they are stable minima of V, they can be thought of as static configurations of the full Lagrangian $L = T - V$. Thinking of vortices as point particles, the existence of static configurations implies that inter-vortex forces vanish for critically coupled vortices at rest. Solutions of the vortex equation were first studied by Taubes [62], where he shows that the zero set of the Higgs field (counted with multiplicity) uniquely determines a vortex up to gauge. Taubes' construction was the first appearance of the *vortex moduli space*, his result is essentially a bijection between the set of charge-N vortices modulo gauge and the symmetric product $\mathbb{C}^{(N)} := \mathbb{C}^N/S_N$. Indeed, much more is true, the former set is a smooth manifold, and it is diffeomorphic to $\mathbb{C}^{(N)}$ (see e.g. [57]).

Shortly after Taubes' work on constructing vortex solutions, Manton [47] articulated the concept of the *moduli space* or *adiabatic approximation* to soliton dynamics[1]. The key realisation was that a path of BPS solitons (of which vortices are an example) could be thought of as a time varying field configuration. Such a configuration is unlikely to be a stationary configuration of the Lagrangian; yet as its spatial component V is constant, the only contribution to the Lagrangian arises from the kinetic term. A slowly varying

[1] We note his work was initially in relation to BPS monopoles. See [61] for a comprehensive review of the adiabatic approximation in various theories.

(or adiabatic) configuration that minimises this kinetic term might then be a reasonable approximation for slowly moving non-static vortices.

This was first applied to the Abelian Higgs model by Strachan [60], who considered integrable vortices on the hyperbolic plane; and then by Samols [59] who refined Strachan's method. They proceeded as follows: under some mild additional assumptions[1], the kinetic term can be written as

$$T = \frac{1}{2}\int |\partial_0 A_i|^2 + |\partial_0 \phi|^2 \, \mathrm{d}x^1 \, \mathrm{d}x^2 \,. \tag{1.5}$$

Viewing the configuration $(A_\mu, \phi) : \mathbb{R}^{2+1} \to \mathfrak{u}(1) \times \mathbb{C}$ as a path in $t \mapsto (A_\mu, \phi)(t)$, the kinetic term is half the L^2-norm of the vector $\frac{\mathrm{d}}{\mathrm{d}t}_{t=0}(A_\mu, \phi) \in \Omega^1(\mathbb{R}^2; \mathfrak{u}(1)) \oplus \Omega^0(\mathbb{R}^2; \mathbb{C})$. This suggests that if we view the space of static configurations of a fixed charge, $\mathcal{M}$, as a Riemannian submanifold of $\Omega^1(\mathbb{R}^2; \mathfrak{u}(1)) \times \Omega^0(\mathbb{R}^2)$ equipped with the L^2-metric, low-velocity geodesic motion on $\mathcal{M}$ yields approximate dynamics for slowly varying Abelian Higgs fields.

Unfortunately, the L^2-metric on $\mathcal{M}$ is difficult to evaluate. Outside the particularly symmetric cases of 1-vortices on either the Euclidean or hyperbolic planes, or 2-vortices on the hyperbolic plane [60], there are no known explicit expressions for the L^2-metric on the entire $\mathcal{M}$. Strachan's work on hyperbolic 2-vortices did however introduce a local formula for the metric, which was extended by Samols to vortices of arbitrary charge on the plane with distinct zeros. This last expression, known as the *Strachan-Samols localisation formula*, was used by Samols to show that the L^2-metric is Kähler and illustrates some key phenomena such as right-angle scattering of vortices. The formula is convenient from a numerical perspective, and has been used to study the Kähler structure of the vortex moduli space (see e.g. [11] for a review). It can also be extended, if we stratify $\mathbb{C}^{(N)}$ by

$$S_\lambda = \left\{ \left\{ \underbrace{z_1, \ldots, z_1}_{\lambda_1 \text{ times}}, \underbrace{z_2, \ldots, z_2}_{\lambda_2 \text{ times}}, \ldots, \underbrace{z_i, \ldots, z_i}_{\lambda_i \text{ times}}, \right\} \,\middle|\, z_i \in \mathbb{C} \text{ distinct} \right\}, \tag{1.6}$$

where $\lambda = (\lambda_1, \ldots, \lambda_i)$ is some integer partition of N, there is a localisation formula for each stratum (see e.g. [49]). More recently, Krusch and Speight [38] considered the case of arbitrary charge vortices on the hyperbolic plane and were able to write down explicit formulae for the L^2-metric on certain geodesic submanifolds.

Vortices on compact manifolds

A natural extension of this body of work is to consider a domain which is an arbitrary compact Riemann surface. In addition to being a direct generalisation of the work of Taubes and others, this has some physical significance as experimental observations of GL-vortices in superconductors show that they organise into a hexagonal *Abrikosov lattice* (see e.g. [1], [31]). Abelian Higgs vortices on a flat equianharmonic torus, $\mathbb{R}^2/\Lambda$, are then

[1] They work in temporal gauge, i.e. $A_0 = 0$ and $\partial^i \partial_0 A_i + \frac{\mathrm{i}}{2}\left(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi}\right)$. Beyond immediate convenience, this choice is significant as it restricts the dynamics to those which are observable i.e. it removes infinitesimal gauge transformations.

natural models for this lattice.

The first to prove the existence of vortices on compact Riemann surfaces was Bradlow [17, 18], who showed that Abelian Higgs vortices exist on arbitrary compact Kähler manifolds if a numerical condition is satisfied[1]. This was a significant generalisation which required a reframing of the original problem; the fields could no longer be global functions and were instead sections of appropriate bundles over the Riemann surface Σ. Bradlow viewed the Higgs field as a section of a degree N Hermitian line bundle, and the gauge field as a unitary connection on this bundle[2]. Yet the moduli space of charge N vortices is still in bijection with the N-fold symmetric product of the underlying domain, with the zeros of the Higgs field uniquely determining a vortex.

This generalisation also uncovered an intriguing property of vortices on compact domains, the so-called *dissolving limit*. This is the name given to the numerical condition previously mentioned, that

$$\varepsilon := \tau \operatorname{Vol} \Sigma - 4\pi N \geq 0. \tag{1.7}$$

While Bradlow's motivation for considering such a condition was not directly physical, one such interpretation of it is that the average energy density of Abelian Higgs vortices, $\frac{2\pi\tau N}{\operatorname{Vol}\Sigma}$, is bounded above by $\frac{1}{2}\tau^2$. The terminology also references the GL-theory, where the Higgs field is interpreted as a wavefunction for a field of Cooper pairs of electrons. The quantity ε described above is equal to the L^2-norm of the Higgs field. Thus when the inequality above is an equality, the Higgs field vanishes, which in the GL-theory corresponds to Cooper pairs delocalising completely.

Bradlow himself motivated the dissolving limit geometrically, likening it to similar conditions encountered in the study of stable vector bundles [64] (see also [56], [10]). Indeed, being above the dissolving limit is a variant of *slope stability*, a criterion on holomorphic vector bundles on Riemann surfaces which implies they cannot be decomposed into simpler bundles. Bradlow and Daskalopoulos [19], [16] and Thaddeus [63] later generalised the stability condition outlined in [17] to higher rank bundles, with Thaddeus observing a similar kind of criticality to the dissolving limit in the rank 2 case.

The vortex metric on compact surfaces

As with vortices on the Euclidean and hyperbolic planes, one can define a metric on the moduli space of vortices on a compact Riemann surface Σ. This has many similarities to the L^2-metric on the plane, including the existence of a Strachan-Samols localisation formula (see e.g. [49]). It is also known to be Kähler, with Baptista [11] explicitly computing the Kähler class of the metric and thus the volume of the moduli space. Yet it is similarly difficult to describe, with no explicit global formula being known outside the

[1] We also note that García-Prada [25] presented an alternate symplectic reduction argument of the same fact shortly afterwards.

[2] Bradlow [17] actually went further and considered a significant generalisation of the Abelian Higgs model where Higgs field is a section of an arbitrary rank Hermitian vector bundle and the gauge field is a unitary connection on this bundle.

usual particularly symmetric cases.

However, the existence of the dissolving limit suggests that the metric might be approximated for extreme values of ε. An example of such a result is work by Nagy [55] who considers vortices far away from the dissolving limit, i.e. $\varepsilon \gg 0$. In this case, the L^2-metric is well approximated by the product metric on $\Sigma^{(N)}$.

In a similar vein, it was conjectured by Baptista and Manton [13] that if $\Sigma = S^2$ is a round sphere and ε was small, the vortex metric should be well approximated by the Fubini-Study metric on $\Sigma^{(N)} \cong \left(\mathbb{P}^1\right)^{(N)} \cong \mathbb{P}^N$. They arrived at this by first approximating vortices on S^2 by *pseudovortices*, fields which solved averaged versions of the Bogomolny equations. The moduli space of pseudovortices is diffeomorphic to the moduli space of vortices, but crucially the associated L^2-metric could be computed explicitly, moreover they found it to be equal to the Fubini-Study metric on $\mathbb{P}^N$.

Significant generalisations of this conjecture were made by Manton and Romão [50] and Rink [58], with the former considering vortices of small charge on curves of positive genus, and the latter considering vortices on hyperelliptic curves. In both cases, they noted that the approximation close to the dissolving limit was tied to the *Abel-Jacobi map*, $\mathrm{AJ} : \Sigma^{(N)} \to \mathrm{Jac}(\Sigma)$, an algebro-geometric object which maps the symmetric product to an associated complex torus called the *Jacobian variety* of Σ. The generalised conjecture relies on the following observation: non-empty fibres of the Abel-Jacobi map are diffeomorphic to (possibly 0-dimensional) complex projective spaces, and $\Sigma^{(N)}$ is stratified by the union of fibres of the same dimension (see e.g. Griffiths and Harris [28, Ch.s 2.2–3]). Essentially, $\Sigma^{(N)}$ is equal to the union of the (possibly empty) regular locus of AJ and a (possibly empty) union of projective spaces. With this, we have the following non-rigorous statement of the conjectures in [58], [50], [13].

Conjecture 1.1.1.
Let Σ be a compact Riemann surface and consider the moduli space of N-vortices on it, $\Sigma^{(N)}$. Note that $\mathrm{Jac}(\Sigma)$ *carries a natural flat metric called the* Bergman metric. *Then close to the dissolving limit:*

- *the restriction of the L^2-metric to the regular locus of the Abel-Jacobi map is well approximated by the pullback of the Bergman metric along* $\mathrm{AJ} : \Sigma^{(N)} \to \mathrm{Jac}(\Sigma)$*, and*
- *the restriction of the L^2-metric to a fibre of the Abel-Jacobi map with positive dimension is well approximated by a natural Fubini-Study metric on that fibre.*

The first rigorous statement of this conjecture was made by García Lara and Speight [23] who studied the case where Σ is diffeomorphic to a sphere and considered vortices of arbitrary charge; in this case, the entire moduli space *is* a single fibre of the Abel-Jacobi map. They were then able to show that the difference between the L^2-metric and the Fubini-Study metric is uniformly bounded in C^0 when viewed as a 2-tensor on the moduli space, proving one particular case of the conjecture.

1.2 Thesis summary

The main aim of this thesis is to resolve the second item of the above conjecture such that rigorous claims about the moduli space dynamics of vortices can be stated and proven. Once this is done, we will then study some consequences of this result, with an eye to the effect on soliton dynamics.

The thesis is structured as follows:

- In chapter 2 we lay out some technical preliminaries, including a rigorous formulation of vortices in the Abelian Higgs model as we shall study them, and a more careful description of their moduli spaces. §2.2 also contains a more careful discussion of some work stated previously.
- In chapter 3 we state and prove the main result of the thesis, that we can approximate the vortex metric by the Fubini-Study metric. It also contains the presentation of some technical estimates required for the proof of the main result. We conclude the chapter with a general application of the main result to approximating geodesics, outlining when such an approximation is valid.
- In chapter 4 we study the validity of our approximation when the vortex number is very large, the so-called *large degree limit*. We show that the moduli space demonstrates some surprising behaviour in this limit, and interpret the effect this has on our approximation.
- In chapter 5 we study the case of 2-vortices on flat tori. This is an application of the results in chapter 3, and contains explicit approximations of 2-vortex dynamics close to the dissolving limit.

We then conclude with some perspectives on the work and some directions for further study in chapter 6.

Also contained in this thesis are two appendices:

- Appendix A, which contains some auxiliary estimates for the analysis performed in chapter 3.
- Appendix B, which contains proofs for two results in chapter 5 which are not directly tied to the content of that chapter.

Chapter 2

Preliminaries

In this chapter we lay out the foundation for the work that follows.

We assume familiarity with foundational ideas from complex, differential, and algebraic geometry, and some knowledge of the language of sheaves. Some good references are Huybrechts [32] (complex geometry and basic sheaf theory), Lee [40] (differential and Riemannian geometry), Griffiths and Harris [28, Ch.s 0, 1.1, 2] (algebraic geometry and sheaf theory).

2.1 Riemann surfaces

2.1.1 Line bundles

The basic setting for this work is a *complex line bundle over a Riemann surface*.

Definition 2.1.1 ([28, p. 15], [32, pp. 52, 66])**.**
A *Riemann surface*, Σ, is a smooth complex manifold of complex dimension 1. A *complex line bundle* over Σ is a smooth vector bundle $L \to \Sigma$ with fibres equal to $\mathbb{C}$. ◇

In this thesis, all Riemann surfaces will be assumed to be compact closed and connected unless explicitly stated otherwise. When fixing a Riemann surface, we also fix a Riemannian metric and a Kähler form on it, written g^Σ and ω^Σ respectively. We collect this data into a tuple, $\left(\Sigma, g^\Sigma, \omega^\Sigma\right)$.

We encourage the reader to keep two examples in mind as they proceed through this thesis:

- Σ diffeomorphic to the complex projective plane, $\mathbb{P}^1 := \left(\mathbb{C}^2 \setminus \{0\}\right)/\mathbb{C}^\times$; and
- Σ diffeomorphic to a complex torus, $\mathbb{C}/\mathbb{Z}\langle 1, \tau_\Sigma\rangle$ for some $\tau_\Sigma \in \mathbb{H}$ in the upper half-plane.

Aside from being the two simplest examples to consider, these form the focus of chapter 4 and chapter 5 respectively. The case of the complex torus is especially noteworthy as it models the Abrikosov lattice.

The basic invariants of use to us are the genus of the Riemann surface and the degree of the line bundle. We note that as we are working in a non-singular setting, both of these invariants coincide with the usual topological definitions (see e.g. [30], [54]).

Remark.
Note that we make no claims about the holomorphic *structure on L. For some results, we will make choices which will* induce *such a structure on L, yet these choices may vary within a given context.*

We will always fix a Hermitian structure on our line bundle.

Definition 2.1.2 ([32, p. 166])**.**
A *Hermitian structure*, H, on $L \to \Sigma$ is a complex valued non-degenerate sesquilinear form on each fibre L_p which depends smoothly on p. We then say that $(L, H) \to \Sigma$ is a *Hermitian line bundle* over Σ.

The *inner product* associated to H is the real valued form $\langle \psi, \xi \rangle_H := \operatorname{Re} H(\psi, \xi)$ given sections $\psi, \xi \in \Omega^0(\Sigma; L)$, and the *pointwise norm* is $|\psi|_H^2 := \langle \psi, \psi \rangle_H$. We shall usually drop the subscript H as which bundle metric we are referring to will often be clear from context.

Note that since we equip Σ with a Riemannian metric, g^Σ, we have a bundle metric on the product $\bigwedge^k T^*\Sigma \otimes L$. This induces a (pointwise) inner product on $\Omega^k(\Sigma; L)$, which we also write as $\langle \cdot, \cdot \rangle$. ◇

Remark.
By sesquilinear*, we mean conjugate linear in the first argument, and linear in the second argument, i.e. given sections $\psi, \xi \in \Omega^0(\Sigma; L)$ and scalars μ, λ, we have $H(\mu\psi, \lambda\xi) = \bar{\mu}\lambda H(\psi, \xi)$.*

We collect the data of a Hermitian line bundle over a Riemann surface into a single expression, $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$. We will usually fix this at the start of a section, and the reader should assume that this is the basic context for all results that follow.

The last object we need to introduce is a *connection*.

Definition 2.1.3 ([32, pp. 173, 176, 182])**.**
A *(Hermitian/unitary) connection* on a line bundle $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ is a $\mathbb{C}$-linear operator

$$D_A : \Omega^k(\Sigma; L) \to \Omega^{k+1}(\Sigma; L),$$

which satisfies the Leibniz rule and respects the Hermitian structure. We write the set of connections as $\mathcal{A}(L, H)$. Given a connection, its *curvature* is the form $F_A \in \Omega^2(\Sigma; \mathfrak{u}(1))$ such that $F_A \cdot \phi = D_A \circ D_A(\phi)$ for all sections $\phi \in \Omega^0(\Sigma; L)$. ◇

Lemma 2.1.4 ([32, p. 173])**.**
Given $D_{A_1}, D_{A_2} \in \mathcal{A}(L,H)$, their difference takes the form

$$(D_{A_1} - D_{A_2})\alpha = \beta \wedge \alpha,$$

for some $\beta \in \Omega^1(\Sigma;\mathfrak{u}(1))$ and where we view $\mathfrak{u}(1)$ acting on L by multiplication. Thus $\mathcal{A}(L,H)$ is a complex affine space modelled on $\Omega^1(\Sigma;\mathfrak{u}(1))$.

Given a connection D_A and $\alpha \in \Omega^1(\Sigma;\mathfrak{u}(1))$, we will sometimes use the affine structure implicitly with the syntactic sugar $D_{A+\alpha} := D_A + \alpha$. We note that as our gauge group is Abelian, we have $F_{A+\alpha} = F_A + \mathrm{d}\alpha$.

As the space of sections of L is an affine space, the product $\mathcal{C} := \mathcal{A}(L,H) \times \Omega^0(\Sigma;L)$ is also an affine space modelled on $\mathcal{C}_0 := \Omega^1(\Sigma;\mathfrak{u}(1)) \times \Omega^0(\Sigma;L)$. It is infinite dimensional, and carries an infinite dimensional group action known as the *gauge action*.

Definition 2.1.5 ([34, Ch. 1])**.**
Let $\mathcal{G} := \Omega^0(\Sigma;\mathrm{U}(1))$ be the *(unitary) gauge group* on Σ. The *gauge action* on pairs $(D_A,\phi) \in \mathcal{C}$ is the group action

$$h \cdot (D_A,\phi) := (D_A - h^{-1}\,\mathrm{d}h\,, h\phi),$$

where $h \in \mathcal{G}$ is a *gauge transformation*. A *global* transformation is one such that $\mathrm{d}h = 0$, i.e. one where $h \equiv e^{\mathrm{i}\theta}$ for some fixed $\theta \in \mathbb{R}$. ◇

At this point, we mention some terminology. We call an element $(D_A,\phi) \in \mathcal{C}$ a *configuration*. We will generally call ϕ the *Higgs field*, but will sometimes refer to it as the *condensate field* when trying to make certain physical interpretations. Lastly, we will generally call $[D_A,\phi] \in \mathcal{C}/\mathcal{G}$ a *gauge equivalence class (gec.)*.

Lastly, we lay out some notation for common operations and spaces.

Definition 2.1.6.
Let $(E,\langle\cdot,\cdot\rangle_E) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ be a metric vector bundle over Σ (e.g. $T^*\Sigma$ with the dual metric or L with $\langle\cdot,\cdot\rangle_H$), then the associated *$L^2$-metric* is defined to be

$$\langle\!\langle \psi,\xi \rangle\!\rangle := \int_\Sigma \langle\psi,\xi\rangle_E \,\mathrm{dVol}_{g^\Sigma},$$

where $\psi,\xi \in \Omega^0(\Sigma;E)$. Thus $\Omega^0(\Sigma;E)$ is understood to be a *real* Hilbert space whenever E is a metric vector bundle. ◇

Although we use the same notation for L^2-metrics across multiple spaces, it will always be clear from context which metric we mean.

Recall that $\mathcal{C}$ is an affine space modelled on $\mathcal{C}_0$, and that the latter is a Hilbert space with

inner product

$$\begin{aligned} (\!(\cdot,\cdot)\!) &: \mathcal{C}_0 \times \mathcal{C}_0 \to \mathbb{R}, \\ (\!(\mathbf{i}a \oplus f, \mathbf{i}b \oplus g)\!) &:= (\!(a,b)\!) + (\!(f,g)\!). \end{aligned} \tag{2.1}$$

As $T\mathcal{C} \cong \mathcal{C}_0 \times \mathcal{C}$ is trivial, the above product gives $\mathcal{C}$ the structure of Hilbert manifold with metric $(\!(\cdot,\cdot)\!)$.

We also make heavy use of the following function norms.

Definition 2.1.7.
Let M be a compact Riemannian manifold with Levi-Civita connection ∇. Given a tensor $\beta \in \Omega^0(M; (T^*M)^{\otimes r})$, recall that we write $|\beta| \in C^\infty(M)$ for the pointwise norm on tensors induced by g.

Let $f \in \Omega^0(M) = C^\infty(M)$. For $k \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{R}_{\geq 1}$ we define

$$\begin{aligned} \|f\|^p_{W^{k,p}} &:= \int_M \sum_{i=0}^{k} \left|\nabla^i f\right|^p \mathrm{dVol}, \\ \|f\|_{C^k} &:= \max\left\{ \left\| \left|\nabla^i f\right| \right\|_{C^0} \;\middle|\; 0 \leq i \leq k \right\}. \end{aligned}$$

The space $W^{k,p}(M)$ is the space of weakly k-differentiable functions which are pth power integrable, by the Meyers-Serrin theorem (see e.g. [27, §7.6]) we identify this with the metric completion of $C^k(M)$ with respect to $\|\cdot\|_{W^{k,p}}$. We then define $H^k := W^{k,2}$ and $L^p := W^{0,p}$. ◇

And finally, a remark on some sign conventions.

Remark.
In this thesis, the Laplacian Δ will always refer to the Laplace-de Rham operator

$$\Delta := \mathrm{d}\delta + \delta\mathrm{d} : \Omega^k(\Sigma) \mapsto \Omega^k(\Sigma),$$

where $\delta = - * \mathrm{d} *$ is the co-differential. Note that this is a positive *definite operator.*

Moreover, in this thesis, the extension of $*$ to complex valued forms is $\mathbb{C}$-anti-linear, i.e. $*\lambda\beta = \bar{\lambda} * \beta$ for $\lambda \in \mathbb{C}$ and $\beta \in \Omega^\bullet(\Sigma)$.

2.1.2 Divisors and the Abel-Jacobi map

We also require some algebraic machinery to motivate our results.

Definition 2.1.8 ([32, pp. 109, 176–7])**.**
Let $L \to \Sigma$ be a complex line bundle, a *holomorphic structure* on L is an operator $\overline{\partial}_L : \Omega^{p,q}(\Sigma; L) \to \Omega^{p,q+1}(\Sigma; L)$ which satisfies the Leibniz rule. Note that for any open

$U \subsetneq \Sigma$, the *sheaf of holomorphic sections of* L

$$\ker \overline{\partial}_L(U) := \left\{ \phi \in \Omega^0(U; L) \;\middle|\; \overline{\partial}_L \phi = 0 \right\},$$

is isomorphic to $\mathcal{O}_\Sigma(U)$. Moreover, since $H^0(\Sigma; L) := \ker \overline{\partial}_L(\Sigma; L) \subset \Omega^0(\Sigma; L)$, it is a (possibly trivial) complex vector space.

If $(L, H) \to \Sigma$ is Hermitian and $D_A \in \mathcal{A}(L, H)$ is a connection on L, we define $D_A^{0,1}$ to be the projection of $D_A : \Omega^0(\Sigma; L) \to \Omega^1(\Sigma; L)$ to $\Omega^{0,1}(\Sigma; L)$. There is a natural extension of the operator to $\Omega^{p,q}(\Sigma; L)$ which satisfies the Leibniz rule, and so we note that $D_A^{0,1}$ induces a holomorphic structure on L. Conversely, given a holomorphic Hermitian line bundle $(L, H, \overline{\partial}_L) \to \Sigma$, there exists a unique (up to gauge) unitary connection, $D_L \in \mathcal{A}(L, H)$, such that $D_L^{0,1} = \overline{\partial}_L$. This is the *Chern connection* associated to $(L, H, \overline{\partial}_L) \to \Sigma$. ◇

Recall that we will not *fix* a holomorphic structure on our line bundle, but that such a structure may be *induced* by a choice of connection on it or that it might vary within a problem. We collect our holomorphic structures into a single object.

Definition 2.1.9 ([32, §2.2 and §3.3], [19])**.**
The *Picard group*, $\mathrm{Pic}(\Sigma)$, is the set of all holomorphic structures on all line bundles over Σ up to isomorphism, the group operation on the Picard group is the tensor product of line bundles. We write the coset corresponding to degree N line bundles as $\mathrm{Pic}^N(\Sigma)$. The 0 coset is an Abelian variety known as the *(algebraic[1]) Jacobian*

$$\mathrm{Jac}_{\mathsf{alg}}(\Sigma) := \mathrm{Pic}^0(\Sigma) = H^{0,1}(\Sigma)/H^1(\Sigma; \mathbb{Z}). \tag{2.2}$$

The last identification above is an application of the exponential sequence.

Given a line bundle of degree $N > 0$, $L \to \Sigma$, a *holomorphic pair* on L is a pair $\left(\overline{\partial}_L, \phi\right)$ where $\overline{\partial}_L$ is a holomorphic structure on L and $\phi \in H^0(\Sigma; L) \setminus \{0\}$ is a holomorphic section. Two holomorphic pairs, $\left(\overline{\partial}_L, \phi\right)$ and $\left(\overline{\partial}'_L, \phi'\right)$ say, are *equivalent* if there is a *complex* gauge transformation $h \in \Omega^0(\Sigma; \mathbb{C}^\times)$ mapping one to the other

$$\left(\overline{\partial}'_L, \phi'\right) = \left(\overline{\partial}_L - h^{-1}\overline{\partial} h, h\phi\right).$$

Note that this includes the scenario where the sections are in *projective equivalence*, i.e. if $\phi = \lambda\phi'$ for some constant $\lambda \in \mathbb{C}^\times$. We write $\mathcal{H}^N(\Sigma)$ for the set of holomorphic pairs up to equivalence on a degree N line bundle over Σ.

Note there is a forgetful map $\mathcal{H}^N(\Sigma) \to \mathrm{Pic}^N(\Sigma)$, $\left[\overline{\partial}_L, \phi\right] \mapsto \left[\overline{\partial}_L\right]$. If we choose an element $\left[\overline{\partial}_{L,0}\right] \in \mathrm{Pic}^1(\Sigma)$, we have a map

$$\begin{aligned} \mathrm{AJ}_{\mathsf{alg}} : \mathcal{H}^N(\Sigma) &\to \mathrm{Jac}_{\mathsf{alg}}(\Sigma), \\ \left[\overline{\partial}_L, \phi\right] &\mapsto \left[\overline{\partial}_L\right] \cdot \left[\overline{\partial}_{L,0}\right]^{-N}. \end{aligned} \tag{2.3}$$

[1] *cf.* the definition of the *analytic* Jacobian (Definition 2.1.14)

This is the *(algebraic) Abel-Jacobi map.* ◇

One notes that the dependence on $\left[\overline{\partial}_{L,0}\right]$ is minor, choosing a different structure amounts to multiplying the image of AJ by some element of $\mathrm{Jac}_{\mathsf{alg}}(\Sigma)$. Indeed, choosing an element $\left[\overline{\partial}_{L,0}\right] \in \mathrm{Pic}^1(\Sigma)$ defines a diffeomorphism $\mathrm{Pic}^N(\Sigma) \to \mathrm{Jac}_{\mathsf{alg}}(\Sigma)$, $\left[\overline{\partial}_L\right] \mapsto \left[\overline{\partial}_L\right] \cdot \left[\overline{\partial}_{L,0}\right]^{-N}$.

We are particularly interested in *complete linear systems* associated to holomorphic line bundles.

Definition 2.1.10 ([32, p. 86])**.**
The *complete linear system* associated to a holomorphic line bundle, $L \to \Sigma$, is the (possibly empty/trivial) space $\mathbb{P}H^0(\Sigma; L)$. We note that if we view L as an element of $\mathrm{Jac}_{\mathsf{alg}}(\Sigma)$, this is precisely the fibre of the Abel-Jacobi map $\mathrm{AJ}_{\mathsf{alg}}^{-1}(L) \subset \mathcal{H}^{\deg L}(\Sigma)$.

Since $H^0(\Sigma; L)$ is a finite dimensional subspace of the Hilbert space $\Omega^0(\Sigma; L)$, it is also a Hilbert space with metric $(\!(\cdot, \cdot)\!)$. The *Fubini-Study* metric on $\mathbb{P}H^0(\Sigma; L)$ is the usual Riemannian metric yielded by quotienting the unit L^2-sphere by the Hopf action (i.e. the circle action on the unit sphere).

Although we will not focus on these, we note that a *linear system* is some projective subspace of a complete linear system. ◇

A useful structural lemma is as follows.

Lemma 2.1.11.
If L is such that

$$\deg L + \chi(\Sigma) > 0, \tag{2.4}$$

where $\chi(\Sigma) = 2 - 2g$ is the Euler characteristic of Σ, then $\mathrm{AJ}_{\mathsf{alg}} : \mathcal{H}^N(\Sigma) \to \mathrm{Jac}_{\mathsf{alg}}(\Sigma)$ has constant fibre $\mathbb{P}^{N-g}$.

Proof. Let L be some degree N complex line bundle which satisfies (2.4) and equip L with a holomorphic structure, we show $H^0(\Sigma; L)$ has dimension $N - g + 1$, from which the fact that $\mathrm{AJ}_{\mathsf{alg}}$ has constant fibre $\mathbb{P}^{N-g}$ follows. We use the Riemann-Roch formula (see e.g. [32, §5.1])

$$h^0(\Sigma; L) - h^1(\Sigma; L) = \deg L + \frac{1}{2}\chi(\Sigma),$$

where $h^i(\Sigma; L) = \dim H^i(\Sigma; L)$. By Serre duality for vector bundles (see e.g. [32, p. 171]), we have $H^1(\Sigma; L) \cong H^0(\Sigma; K_\Sigma \otimes L^\vee)$; here K_Σ is the canonical bundle, for Riemann surfaces this isomorphic to the holomorphic cotangent bundle and has degree $-\chi(\Sigma)$. As $\deg(K_\Sigma \otimes L^\vee) = \deg K_\Sigma - \deg L = -(\deg L + \chi(\Sigma)) < 0$, we see that $h^1(\Sigma; L) = h^0(\Sigma; K_\Sigma \otimes L^\vee) = 0$, so $h^0(\Sigma; L) = \deg L + \frac{1}{2}\chi(\Sigma) = N - g + 1$. □

Indeed, much more is true; if (2.4) is met, $\mathrm{AJ}_{\mathsf{alg}}$ describes an actual $\mathbb{P}^{N-g}$-bundle [51], [63].

Dual to these structures on line bundles are *divisors* on Σ.

Definition 2.1.12 ([32, §2.3])**.**
A *(Weil) divisor* on Σ is a formal linear combination of points on Σ (written additively)

$$D = \sum_{p \in \Sigma} a_p [p],$$

such that the *support* of D, $\{p \in \Sigma \mid a_p \neq 0\}$, is locally finite. As Σ is compact, this implies that all but finitely many of the $a_p \in \mathbb{Z}$ are zero. The set of all divisors on Σ with addition defined pointwise form an Abelian group, $\mathrm{Div}(\Sigma)$.

If $D \in \mathrm{Div}(\Sigma)$ is such that $\forall p \in \Sigma$, $a_p \geq 0$, we say D is *effective*. The set of effective divisors is an Abelian monoid, written $\mathrm{Div}_+(\Sigma)$. The *degree* of a divisor is

$$\deg D := \sum_{p \in \Sigma} a_p \in \mathbb{Z}.$$

Note that the degree yields a grading on both $\mathrm{Div}(\Sigma)$ and $\mathrm{Div}_+(\Sigma)$. We write the Nth grade of these as $\mathrm{Div}^N(\Sigma)$ and $\mathrm{Div}^N_+(\Sigma)$ respectively.

Given some global meromorphic function $f \in \mathcal{K}_\Sigma(\Sigma)$, we define the *principal divisor associated to* f to be the divisor $(f) := \sum_{p \in \Sigma} \mathrm{ord}_p(f)[p]$, where $\mathrm{ord}_p(f)$ is the order of vanishing of f at p. A divisor is said to be *principal* if it is the principal divisor of some meromorphic f. Two divisors are said to be *principally/linearly equivalent* if their difference is a principal divisor. ◇

We will rely on the following correspondence in chapter 5, though we delay an exact specification of the maps until that chapter.

Theorem 2.1.13 (Divisor-line bundle correspondence [32, §2.3])**.**
Holomorphic pairs and effective divisors are in bijection via the following correspondence.

Let $[\overline{\partial}_L, \phi] \in \mathcal{H}^N(\Sigma)$. *Note that as* L *trivialises on any contractible holomorphic chart of* Σ, $\phi\big|_U(z) = z^d$ *if* (U, z) *is some sufficiently small holomorphic chart centred at a zero of* ϕ, p. *Define* $\mathrm{ord}_p(\phi) = d$ *if* ϕ *vanishes at* p, *and let it be* 0 *otherwise. Then define a map*

$$\begin{aligned} \mathcal{H}^N(\Sigma) &\to \mathrm{Div}^N_+(\Sigma), \\ [\overline{\partial}_L, \phi] &\mapsto \phi^{-1}(0) = \sum_{p \in \Sigma} \mathrm{ord}_p(\phi)[p]. \end{aligned} \tag{2.5}$$

Indeed, there is a corresponding notion of Jacobian and Abel-Jacobi map for divisors.

Definition 2.1.14 ([22, Ch. III.6], [28, Ch. 2.2])**.**

Define a map

$$\iota : H_1(\Sigma;\mathbb{Z}) \to \left(\Omega^{1,0}(\Sigma)\right)^\vee,$$
$$[\gamma] \mapsto \left(\alpha \mapsto \int_\gamma \alpha\right).$$

The *(analytic) Jacobian* of Σ is the quotient

$$\mathrm{Jac}_{\mathsf{ana}}(\Sigma) := \frac{\left(\Omega^{1,0}(\Sigma)\right)^\vee}{\iota(H_1(\Sigma;\mathbb{Z}))}.$$

Given $p, q \in \Sigma$, write $p \to q$ for a homotopy class of paths from p to q, note that the difference of any two such classes is given by an element of $H_1(\Sigma;\mathbb{Z})$. Fix a point $p \in \Sigma$, the *(analytic) Abel-Jacobi map* is the map

$$\mathrm{AJ}_{\mathsf{ana}} : \mathrm{Div}(\Sigma) \to \mathrm{Jac}_{\mathsf{ana}}(\Sigma),$$
$$[q] \mapsto \left(\alpha \mapsto \int_{p\to q} \alpha\right),$$
$$D = \sum_{q\in\Sigma} a_q[q] \mapsto \sum_{q\in\Sigma} a_q \,\mathrm{AJ}([q]).$$

◇

As with the algebraic Jacobian, the analytic Jacobian is an Abelian variety with the group operation being the sum of linear maps. And as with the algebraic Abel-Jacobi map, the basepoint dependence is minor. Choosing a different basepoint amounts to choosing a different class of curves to integrate along, adding a constant term to the image of the analytic Abel-Jacobi map; on degree zero divisors the map is independent of the choice of basepoint.

In the case of genus 1 curves (complex tori), the Abel-Jacobi map defines an isomorphism $\Sigma \to \mathrm{Jac}(\Sigma)$ [28, p. 238], and the basepoint dependence reflects a choice of 0 in $\Sigma = \mathbb{C}/\Lambda$. If we view $\mathrm{Div}^N_+(\Sigma)$ as the symmetric product $\Sigma^{(N)} := \Sigma^N/S_N$ and the Abel-Jacobi map for complex tori as a function $\Sigma^{(N)} \to \Sigma \cong \mathrm{Jac}(\Sigma)$, it is of the form $\{p_i\} \mapsto \sum_{i=1}^N p_i$.

A key result is that the algebraic and analytic Jacobians and Abel-Jacobi maps are isomorphic, which is a consequence of Abel's theorem and Jacobi inversion. We state the result, but we leave a complete proof to a reference, e.g. [28, Ch. 2.2] or [22, Ch. III.6].

Theorem 2.1.15.
The analytic and algebraic Jacobians are isomorphic. Indeed, for positive N, the following square commutes

$$\begin{array}{ccc} \mathrm{Div}^N_+(\Sigma) & \longleftrightarrow & \mathcal{H}^N(\Sigma) \\ \downarrow{\scriptstyle \mathrm{AJ}_{\mathsf{ana}}} & & \downarrow{\scriptstyle \mathrm{AJ}_{\mathsf{alg}}} \\ \mathrm{Jac}_{\mathsf{ana}}(\Sigma) & \longleftrightarrow & \mathrm{Jac}_{\mathsf{alg}}(\Sigma). \end{array}$$

Here the top map is the bijection between effective divisors and holomorphic pairs and the

bottom map is the aforementioned isomorphism.

Remark.
With this, we shall drop the subscripts on both Jac *and* AJ *as the exact map we mean will be clear from context.*

2.2 The Abelian Higgs model

The main objects of study in this thesis will be vortices in the Abelian Higgs model. Here we outline some basic facts about the theory. For this section, let $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ be fixed degree N Hermitian line bundle.

2.2.1 Vortices

The theory revolves around minimisers of the following functional.

Definition 2.2.1 ([17], [34, p. 53])**.**
Let $\phi \in \Omega^0(\Sigma; L)$ be a section and $D_A \in \mathcal{A}(L, H)$ be a connection. The *Yang-Mills-Higgs energy* is defined to be the functional

$$E[D_A, \phi] := \frac{1}{2}\int_\Sigma |D_A\phi|^2 + |F_A|^2 + \left(\frac{1}{2}\left(|\phi|^2 - \tau\right)\right)^2 \mathrm{dVol}, \tag{2.6}$$

where $\tau > 0$ is some fixed real parameter.

A *vortex* is a gauge equivalence class of pairs, (D_A, ϕ), which minimise the Yang-Mills-Higgs energy. ◇

Beyond the usual kinetic terms, one notes that the potential term is the square of a moment map for the gauge action on Higgs fields. The YMH energy itself can be thought of as the spatial component of the *critically coupled Ginzburg-Landau action* (see e.g. [46, §7.8]). At critical coupling, Ginzburg-Landau vortices (stationary points of the GL-action) exhibit no inter-vortex forces, yielding stable static configurations of GL-vortices. With this framing, a vortex according to the above definition is a static critically coupled GL-vortex.

While we could find minimisers by computing and solving the Euler-Lagrange equations for the YMH energy, one key feature is that it can be written in *Bogomolny form* [14], [62]. i.e. there is non-linear operator, $B : \mathcal{C} \to \Omega^{0,1}(\Sigma; L) \times \Omega^{1,1}(\Sigma; \mathfrak{u}(1))$, and a *topological term*, $2\pi\tau N \in \mathbb{R}$, such that $E[\phi, D_A] = \|B(D_A, \phi)\|^2_{L^2} + 2\pi\tau N$. Minimisers of E therefore have energy $2\pi\tau N$ and are given by the vanishing of B. This ultimately yields the following characterisation in terms of field equations.

Theorem 2.2.2 ([14], see e.g. [46, p. 197], [34, pp. 54–55])**.**
Let $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ be a fixed Hermitian line bundle over a Riemann surface. A pair $(D_A, \phi) \in \mathcal{A}(L, H) \times \Omega^0(\Sigma; L)$ is a vortex if and only if it satisfies the vortex field

equations:

$$D_A^{0,1}\phi = 0, \tag{2.7}$$

$$\mathbf{i}F_A = \frac{1}{2}(\tau - |\phi|^2)\omega^\Sigma. \tag{2.8}$$

Moreover, $E[D_A, \phi] = 2\pi\tau N$.

We note that a vortex defines a holomorphic pair, and in turn, a divisor $\phi^{-1}(0)$. As we shall see shortly, this correspondence is in fact a bijection. We collect all vortices on $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ into a single *moduli space*.

Definition 2.2.3.
Fix $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$. The *configuration space* of vortices, $\mathcal{V} \subset \mathcal{A}(L, H) \times \Omega^0(\Sigma; L)$, is the set of solutions to the vortex equations (2.7)–(2.8). The *moduli space* of vortices is the gauge quotient $\mathcal{M} := \mathcal{V}/\mathcal{G}$. ◇

This is in fact a smooth manifold (see e.g. [57]), yet the smooth structure is not immediately important to us. As mentioned previously, a result of Bradlow [17] allows us to characterise the existence of vortices, and the structure of the moduli space.

Theorem 2.2.4 (Bradlow [17])**.**
Let $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ *be a fixed degree* N *Hermitian line bundle over a Riemann surface. Solutions to the vortex equations* (2.7)–(2.8) *exist if and only if the numerical condition*

$$\varepsilon := \tau \operatorname{Vol}\Sigma - 4\pi N \geq 0, \tag{2.9}$$

is satisfied. If this inequality is strict, i.e. $\varepsilon > 0$, $\mathcal{M}$ *is in bijection with degree* N *effective divisors via the correspondence*

$$[D_A, \phi] \mapsto \phi^{-1}(0). \tag{2.10}$$

Going forward, we decorate $\mathcal{M}$ with ε to make the dependence explicit.

We mention two important points in Bradlow's proof. To show the condition (2.9) is necessary, we integrate $\mathbf{i}F_A - \frac{1}{2}\left(\tau - |\phi|^2\right)\omega^\Sigma = 0$ (2.8) to yield $2\pi N - \frac{1}{2}\left(\tau \operatorname{Vol}\Sigma + \|\phi\|_{L^2}^2\right) = 0$, or $\|\phi\|_{L^2}^2 = \tau \operatorname{Vol}\Sigma - 4\pi N$. Thus Equation 2.9 simply reflects that the L^2-norm of ϕ, $\varepsilon = \|\phi\|_{L^2}^2$, is always non-negative.

The sufficiency argument incorporates the correspondence between vortices and effective divisors. Bradlow originally phrased his result as follows: Let $(L, \overline{\partial}_L, \phi) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ be a degree N *holomorphic* line bundle with a prescribed holomorphic section $\phi \in \ker \overline{\partial}_L \setminus \{0\}$ (i.e. a holomorphic pair). Then if $L \to \Sigma$ satisfies (2.9), there is a unique *τ-Hermitian-Yang-Mills-Higgs metric*, H, such that $[D_L, \phi]$ is a vortex on $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$. Moreover, H does not depend on $\overline{\partial}_L$ or ϕ. Holomorphic pairs are in one-to-one correspondence with effective divisors on Σ, and the claim follows.

Remark.
Bradlow's proof seems to remove the freedom to choose the metric H, yet in our setup we fix H at the outset and make no special assumptions about it. We resolve this apparent conflict by noting that there are two viewpoints to the problem. Choose a background metric H_0 and holomorphic structure $\overline{\partial}_0$ on L. Note that any such metric/holomorphic structure pair, $(H, \overline{\partial})$, defines a unique Chern connection, we say a pair admits solutions to the vortex equations (2.7)–(2.8) if there exists solutions to the vortex equations where the connection is the associated Chern connection of $(H, \overline{\partial})$. One can then either:

- *Keep $\overline{\partial}_0$ fixed and solve for a perturbation of H_0, $H = e^{2u}H$, such that $(H, \overline{\partial}_0)$ admits solutions to the vortex equations.*
- *Or one can keep H_0 fixed and solve for a* complex *gauge transformation of $\overline{\partial}_0$, $\overline{\partial} = \overline{\partial}_0 - \frac{1}{2}\overline{\partial}u$, such that $(H_0, \overline{\partial})$ admits solutions to the vortex equations.*

Bradlow takes the former approach which is analogous to the construction of Hermitian-Einstein metrics on stable vector bundles described by Donaldson-Uhlenbeck-Yau (see e.g. [43, Ch.2–3]). One benefit of this approach is that when trying to construct vortices, the Higgs field data remains fixed and one can easily construct it as a holomorphic section vanishing at a given effective divisor.

We shall take the latter approach as this more closely matches the solitons literature, e.g. Baptista and Manton [13]. For a more detailed exposition of this second viewpoint, see the remark on page 20.

Definition 2.2.5.
Note that for $[D_A, \phi] \in \mathcal{M}^0$, the Higgs field is identically zero. We therefore say that $[D_A, \phi]$ is *dissolved* and call the space $\mathcal{M}^0$ the *dissolved* moduli space. Accordingly, the space $\mathcal{M}^\varepsilon$ for $\varepsilon > 0$ is the *non-dissolved* moduli space, and we call the limit $\varepsilon \to 0$ the *dissolving limit*. Going forward, we shall assume that $\varepsilon > 0$, i.e. $\mathcal{M}^\varepsilon$ is non-dissolved. $\diamond$

As mentioned previously, the choice of terminology here reflects the origins of the theory in the Ginzburg-Landau model of superconductivity (see e.g. [34, Ch. I.5]). In that framing $|\phi|^2$ is an *order parameter* which measures the local density of *Cooper pairs of electrons*, bound pairs of electrons of opposing spin which behave like a single boson. Within a superconductor, it is energetically favourable for free electrons to form these pairs, a process called *condensation*. We can therefore describe the electron density as a single bosonic field (often called the *condensate field*), ϕ. This is similar to a quantum mechanical wavefunction, but it is a classical (i.e. pre-quantum) object and describes classical behaviour.

The Yang-Mills-Higgs energy, and thus the vortex equations, describe the coupling of the Higgs field to an external magnetic field. By the nature of our setup, we see that the total field strength, $\int_\Sigma \mathbf{i}F_A = 2\pi N$, is quantised, which is reflected by the experimental observation that magnetic flux penetrates the superconductor in discrete flux tubes. The

second vortex equation (2.8) couples the magnetic flux and the condensate density, we see that flux is maximised precisely where the condensate density vanishes and vice versa.

With this, the dissolving limit is the observation that a domain with finite volume, be that a superconductor shaped into an actual compact surface or the fundamental domain of a periodic lattice, can only support a finite number of vortices, $N \leq \frac{\tau \operatorname{Vol}\Sigma}{4\pi}$. If one instead fixes a charge N and considers a surface of minimal volume, $\operatorname{Vol}\Sigma = \frac{4\pi N}{\tau}$ (i.e. the dissolved case), we must have that $|\phi|^2 \equiv 0$. Physically, this corresponds to the condensate field delocalising completely, i.e. we can no longer recover any information about the density of Cooper pairs. The only remaining degree of freedom lies in the gauge field which must necessarily be constant strength.

For this section, it is sufficient that we know $\mathcal{M}^\varepsilon$ as a set, yet it would be remiss of us to not mention the smooth structure of vortex moduli spaces.

Theorem 2.2.6 (e.g. [12] or [57])**.**
Note that $\operatorname{Div}^N_+(\Sigma)$ *is in bijection with the* N*-fold symmetric product* $\Sigma^{(N)} := \Sigma^N/S_N$*, and that the latter is a smooth complex manifold. We then have a diffeomorphism* $\mathcal{M}^\varepsilon \cong \Sigma^{(N)}$*.*

Moreover, suppose $\chi(\Sigma) + N = 2 - 2g + N \geq 0$*, where* g *is the genus of* Σ*. Then* $\mathcal{M}^\varepsilon$ *has the structure of a* $\mathbb{P}^{N-g}$*-bundle over* $\operatorname{Jac}(\Sigma)$*.*

For a non-dissolved vortex, we call the divisor associated to it the *vortex core* or the *core locations*.

2.2.2 Pseudovortices

While Bradlow's characterisation of vortices is complete, it is not explicit. Indeed, outside some very symmetric cases, exact solutions to the vortex equations are not known. We overcome this by modelling a vortex as a perturbation of a *known* object; in this case, a configuration called a *pseudovortex*.

Definition 2.2.7 ([13], [23])**.**
A *pseudovortex* is a gauge equivalence class of pairs $(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}) \in \mathcal{C}$ which satisfy:

$$D^{0,1}_{\hat{A}}\hat{\phi} = 0, \tag{2.11}$$

$$\mathbf{i}F_{\hat{A}} = \frac{2\pi N}{\operatorname{Vol}\Sigma}\omega^\Sigma, \tag{2.12}$$

$$\left\|\hat{\phi}\right\|^2_{L^2} = 1. \tag{2.13}$$

The configuration space of pseudovortices is written $\hat{\mathcal{V}}^\varepsilon$, and the moduli space of pseudovortices is written $\hat{\mathcal{M}}^\varepsilon$. ◇

Note that in the dissolved scenario, pseudovortex Higgs fields vanish, yet the underlying holomorphic pair is still non-zero. The first paper to define pseudovortices as written

here is the work by Baptista and Manton [13], where they view pseudovortices as configurations which solved the second vortex equation "on average". This perspective is analytically convenient and allowed them to formulate a conjecture on approximating vortices by pseudovortices, which was later proven by García Lara and Speight [23] (see Theorem 2.2.9). We note that the two spaces are in bijection.

Theorem 2.2.8.
Above the dissolving limit, the moduli space of vortices is in bijection with the moduli space of pseudovortices. Concretely, the map

$$\begin{aligned} \hat{\mathcal{M}}^{\varepsilon} &\to \mathcal{M}^{\varepsilon}, \\ \left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] &\mapsto \left[D_{\hat{A}} + \frac{\mathbf{i}}{2} * \mathrm{d}u\,, \sqrt{\varepsilon} e^{u/2}\hat{\phi}\right], \end{aligned} \tag{2.14}$$

where u is the solution to

$$\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + \varepsilon e^{u}\left|\hat{\phi}\right|^2 = 0, \tag{2.15}$$

is a well-defined bijection.

This result is well known (see e.g. [13], [17]), indeed the proof we provide here follows the work of García Lara and Speight [23]. We mention that this map is actually a diffeomorphism, yet a proof of this fact is not immediately useful. We call (2.15) the *prescribed scalar curvature* equation after Kazdan and Warner [36], who prove the existence unique smooth solutions to it [17], [36].

Proof. For this proof only, we label (2.14) as $F : \hat{\mathcal{M}}^{\varepsilon} \to \mathcal{M}^{\varepsilon}$.

It is not immediately obvious that F is well-defined, i.e. that it does not depend on the choice of gauge on the domain and that it maps pseudovortices to vortices. The former is easy to verify, note that (2.15) is invariant under change of gauge so u does not depend on a choice of gauge. Letting $h \in \mathcal{G}$, we have

$$\begin{aligned} F\left(h \cdot \left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]\right) &= \left[D_{\hat{A}} + \frac{\mathbf{i}}{2} * \mathrm{d}u - h^{-1}\,\mathrm{d}h\,, h\sqrt{\varepsilon} e^{u/2}\hat{\phi}\right] \\ &= \left[D_{\hat{A}} + \frac{\mathbf{i}}{2} * \mathrm{d}u\,, \sqrt{\varepsilon} e^{u/2}\hat{\phi}\right] \\ &= F\left(\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]\right) \end{aligned}$$

and so F does not depend on a choice of gauge.

We show $\operatorname{im} F \subset \mathcal{M}^{\varepsilon}$ by checking that $[D_A, \phi] := F\left(\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]\right)$ satisfies the vortex

equations. We have

$$\begin{aligned} D_A^{0,1}\phi &= D_{\hat{A}}^{0,1}\left(\sqrt{\varepsilon}e^{u/2}\hat{\phi}\right) + \left(\frac{\mathbf{i}}{2} * \mathrm{d}u\,\sqrt{\varepsilon}e^{u/2}\hat{\phi}\right)^{0,1} \\ &= \sqrt{\varepsilon}e^{u/2}D_{\hat{A}}^{0,1}\hat{\phi} + \sqrt{\varepsilon}\left(\overline{\partial}e^{u/2}\right)\hat{\phi} + \sqrt{\varepsilon}\left(\mathbf{i} * \mathrm{d}e^{u/2}\right)^{0,1}\hat{\phi} \\ &= \sqrt{\varepsilon}e^{u/2}D_{\hat{A}}^{0,1}\hat{\phi} + \sqrt{\varepsilon}\left(\overline{\partial}e^{u/2} - \overline{\partial}e^{u/2}\right)\hat{\phi} \\ D_A^{0,1}\phi &= \sqrt{\varepsilon}e^{u/2}D_{\hat{A}}^{0,1}\hat{\phi}. \end{aligned}$$

Which vanishes as $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]$ satisfies $D_{\hat{A}}^{0,1}\hat{\phi} = 0$ (2.11), so $[D_A, \phi]$ satisfies the first vortex equation (2.7). Moreover

$$\begin{aligned} \mathbf{i}F_A &= \mathbf{i}F_{\hat{A}+\frac{\mathbf{i}}{2}*\mathrm{d}u} \\ &= \mathbf{i}F_{\hat{A}} - \frac{1}{2}\,\mathrm{d} * \mathrm{d}u \\ &= \frac{2\pi N}{\operatorname{Vol}\Sigma}\omega^{\Sigma} + \frac{1}{2}(\Delta u)\omega^{\Sigma} \\ &= \frac{1}{2}\left(\frac{4\pi N}{\operatorname{Vol}\Sigma} + \frac{\varepsilon}{\operatorname{Vol}\Sigma} - \varepsilon e^{u}\left|\hat{\phi}\right|^2\right)\omega^{\Sigma} \\ \mathbf{i}F_A &= \frac{1}{2}\left(\tau - \left|\sqrt{\varepsilon}e^{u/2}\hat{\phi}\right|^2\right)\omega^{\Sigma}. \end{aligned}$$

As $\phi = \sqrt{\varepsilon}e^{u/2}\hat{\phi}$, we see that $[D_A, \phi]$ satisfies, $\mathbf{i}F_A = \frac{1}{2}\left(\tau - |\phi|^2\right)\omega^{\Sigma}$ (2.8). We therefore have that $F : \hat{\mathcal{M}}^{\varepsilon} \to \mathcal{M}^{\varepsilon}$ is well-defined.

Indeed, the calculations above also imply that F is bijective. Suppose that $[D_A, \phi] \in \mathcal{M}^{\varepsilon}$ is a vortex, and consider the PDE

$$\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + |\phi|^2 = 0. \tag{2.16}$$

Note that $\left(\left(1, \frac{\varepsilon}{\operatorname{Vol}\Sigma} - |\phi|^2\right)\right) = \int_{\Sigma} \frac{\varepsilon}{\operatorname{Vol}\Sigma} - |\phi|^2\,\mathrm{dVol} = 0$, so by the Fredholm alternative (see e.g. [21, Appx. III], [23]) unique smooth solutions to this PDE exist. If we then *define* $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] = \left[D_A - \frac{\mathbf{i}}{2} * \mathrm{d}u\,, e^{-u/2}\phi\right]$, we see that $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]$ satisfies the pseudovortex equations, thus

$$\begin{aligned} G : \mathcal{M}^{\varepsilon} &\to \hat{\mathcal{M}}^{\varepsilon}, \\ [D_A, \phi] &\mapsto \left[D_A - \frac{\mathbf{i}}{2} * \mathrm{d}u\,, e^{-u/2}\phi\right], \end{aligned}$$

is well-defined (the fact that it does not depend on a choice of gauge is manifest from our prior computation). We notice that if $[D_A, \phi] = F\left(\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]\right)$, (2.16) is exactly the same PDE as (2.15), thus G is a left-inverse for F. Similar reasoning shows that it is a right-inverse and thus F is a bijection. □

Remark.

The pseudovortex-vortex map (2.14) is structurally similar to a gauge transformation and we can indeed interpret it as such. Recall that we fix the complex and Hermitian structure

of a line bundle $(L,H)\to(\Sigma,g^\Sigma,\omega^\Sigma)$ *but allow its holomorphic structure to vary. For any class of holomorphic structures on* L, $[\overline{\partial}_{\hat{A}}]\in \mathrm{Pic}^N(\Sigma)$, *there is a unique (up to unitary gauge) representative,* $\overline{\partial}_{\hat{A}}$, *such that the Chern connection associated to* $(H,\overline{\partial}_{\hat{A}})$, $D_{\hat{A}}$, *is constant curvature [32, p. 186], [35].*

Such a representative yields solutions to the pseudovortex equations. Let $\left[\overline{\partial}_{\hat{A}},\hat{\phi}\right]\in\mathcal{H}^N(\Sigma)$ *be a holomorphic pair, after a constant rescaling of* $\hat{\phi}$, *we can assume* $\left\|\hat{\phi}\right\|_{L^2}=1$. *Then* $\left(D_{\hat{A}},\sqrt{\varepsilon}\hat{\phi}\right)\in\mathcal{C}$ *solves the pseudovortex equations.*

Now note that $\mathbf{i}*\mathrm{d}u=\partial u-\overline{\partial}u$, *so* $D_A^{0,1}=\left(D_{\hat{A}}+\frac{\mathbf{i}}{2}*\mathrm{d}u\right)^{0,1}=\overline{\partial}_{\hat{A}}-\overline{\partial}\frac{u}{2}$. *So the holomorphic pair associated to* $(D_A,\phi)=\left(D_{\hat{A}}+\frac{\mathbf{i}}{2}*\mathrm{d}u\,,e^{u/2}\hat{\phi}\right)$ *is* $\left(D_{\hat{A}}^{0,1},\phi\right)=\left(\overline{\partial}_{\hat{A}}-\overline{\partial}\frac{u}{2},e^{u/2}\hat{\phi}\right)$, *i.e. it is complex gauge equivalent to the pseudovortex holomorphic pair. With this framing, the sets of solutions to the pseudovortex and vortex equations are equivalent under complex gauge transformations. Given a pseudovortex, The solution of the PSC equation* (2.16) *determines a gauge transformation mapping it to the corresponding vortex.*

One immediate feature of pseudovortices is that they approximate vortices in the dissolving limit.

Theorem 2.2.9 ([23])**.**
There exists a constant $C>0$ *which only depends on* $(L,H)\to(\Sigma,g^\Sigma,\omega^\Sigma)$ *such that for all* $\varepsilon>0$, *and for all pseudovortices* $\left[D_{\hat{A}},\sqrt{\varepsilon}\hat{\phi}\right]\in\hat{\mathcal{M}}^\varepsilon$,

$$\begin{aligned}\left\|\frac{1}{\sqrt{\varepsilon}}|\phi|-\left|\hat{\phi}\right|\right\|_{C^0}&\leq C\varepsilon,\\ \left\|F_A-F_{\hat{A}}\right\|_{C^0}&\leq C\varepsilon.\end{aligned}\tag{2.17}$$

Here $[D_A,\phi]\in\mathcal{M}^\varepsilon$ *is the vortex corresponding to* $\left[D_{\hat{A}},\sqrt{\varepsilon}\hat{\phi}\right]$ *(see Theorem 2.2.8).*

García Lara and Speight [23] only prove this for $\Sigma\cong\mathbb{P}^1$, but we note that their proof works in full generality. It is with this approximation in mind that we make the choice of notation below.

Remark.
Going forward the bijection (2.14) shall be implicit, *i.e. if* $[D_A,\phi]$ *is a vortex,* $\left[D_{\hat{A}},\sqrt{\varepsilon}\hat{\phi}\right]$ *is understood to be the corresponding pseudovortex and vice versa.*

Lastly, we extend the definition of the Abel-Jacobi map.

Definition 2.2.10.

The *(vortex) Abel-Jacobi* map is defined to be

$$\begin{aligned} \mathrm{AJ} : \hat{\mathcal{M}}^{\varepsilon} &\to \mathrm{Jac}(\Sigma), \\ \left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] &\mapsto \left[D^{0,1}_{\hat{A}}\right]. \end{aligned} \tag{2.18}$$

We define $\mathrm{AJ} : \mathcal{M}^{\varepsilon} \to \mathrm{Jac}(\Sigma)$ as $\mathrm{AJ}\left([D_A, \phi]\right) = \mathrm{AJ}\left(\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]\right)$.

Fibres of the Abel-Jacobi map are called *complete linear systems*, we write them as e.g. $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}} := \mathrm{AJ}^{-1}\left(\left[D^{0,1}_{\hat{A}}\right]\right) \subset \hat{\mathcal{M}}^{\varepsilon}$. ◇

Note that this is essentially the Abel-Jacobi map on holomorphic pairs. Indeed, if we identify $\mathrm{Jac}(\Sigma)$ with constant curvature unitary connections (see e.g. [35]) Equation 2.18 is exactly the algebraic Abel-Jacobi map $\mathrm{AJ} : \mathcal{H}^N(\Sigma) \to \mathrm{Jac}(\Sigma)$. Thus the vortex Abel-Jacobi map is a gauge theoretic realisation of the analytic Abel-Jacobi map, $\mathrm{AJ} : \mathrm{Div}^N_+(\Sigma) \cong \Sigma^{(N)} \to \mathrm{Jac}(\Sigma)$.

We shall not require smoothness of the entire moduli space of pseudovortices, only of complete linear systems.

Theorem 2.2.11.
Above the dissolving limit, complete linear systems of pseudovortices are smooth manifolds.

We do not give a proof of this as it follows immediately from noting that $\mathrm{AJ}^{-1}\left(\left[D^{0,1}_{\hat{A}}\right]\right)$ is diffeomorphic to $\mathbb{P}(H^0(L))$.

We finish with an important corollary.

Corollary 2.2.12.
The bijection between vortices and pseudovortices preserves fibres of the Abel-Jacobi map.

Proof. Let $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]$ be a pseudovortex, the corresponding vortex takes the form

$$[D_A, \phi] = \left[D_{\hat{A}} + \frac{\mathbf{i}}{2} * \mathrm{d}u\,, \sqrt{\varepsilon} e^{u/2}\hat{\phi}\right],$$

thus $\mathrm{AJ}([D_A, \phi]) = \left[D^{0,1}_{\hat{A}} + \frac{\mathbf{i}}{2}(*\,\mathrm{d}u)^{0,1}\right] = \left[D^{0,1}_{\hat{A}} - \overline{\partial}\frac{u}{2}\right] = \left[D^{0,1}_{\hat{A}}\right] = \mathrm{AJ}\left(\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]\right)$. □

Thus we can speak interchangeably of complete linear systems of vortices and those of pseudovortices.

2.2.3 Metrics on the moduli space of vortices

Finally, we introduce two Riemannian metrics. Recall that $(\!(\cdot,\cdot)\!)$ defines a metric on $\mathcal{V}^{\varepsilon}$ and $\hat{\mathcal{V}}^{\varepsilon}$.

Definition 2.2.13.
Given any smooth manifold $\mathcal{M} = \mathcal{V}/\mathcal{G} \subset \mathcal{C}/\mathcal{G}$, the *$L^2$-metric* on $\mathcal{M}$ is the unique metric on $\mathcal{M}$ such that the gauge quotient $\mathcal{V} \to \mathcal{M}$ is a Riemannian submersion.

The *vortex metric*, g^ε, is the L^2-metric on $\mathcal{M}^\varepsilon$. We write the pullback of the vortex metric along the diffeomorphism $\hat{\mathcal{M}}^\varepsilon \to \mathcal{M}^\varepsilon$ as g^ε also.

Fix a complete linear system $\hat{\mathcal{M}}^\varepsilon_{\hat{A}}$, and let $\tilde{g}^\varepsilon$ be the L^2-metric on $\hat{\mathcal{M}}^\varepsilon_{\hat{A}} \subset \mathcal{C}/\mathcal{G}$. The *Fubini-Study* metric is defined to be $\hat{g}^\varepsilon = \frac{1}{\varepsilon}\tilde{g}^\varepsilon$. ◇

We note that the vortex metric as defined above degenerates as $\varepsilon \to 0$, so we will often consider the rescaled metric $\frac{1}{\varepsilon}g^\varepsilon$, noting that rescaling the metric has no effect on geodesics besides reparamaterising them.

As mentioned in §1.1, geodesics of the vortex metric model the dynamics of slowly moving vortices, yet it is difficult to compute the vortex metric explicitly. Conversely, the Fubini-Study metric on complete linear systems has a well-known form; if one views the space of sections with the L^2-inner product as a complex metric vector space $(H^0(\Sigma; L), \langle\!\langle \cdot, \cdot \rangle\!\rangle)$, the Fubini-Study metric described above is precisely the usual Fubini-Study metric on $\mathbb{P}H^0(\Sigma; L)$.

The results presented in this thesis revolve around the following central question:

Close to the dissolving limit, is the Fubini-Study metric a good approximation for the restriction of the vortex metric to complete linear systems?

The aim of the thesis is to answer this in the affirmative. Specifically, in chapter 3 we shall demonstrate that such an approximation is valid from the perspective of the geodesic approximation. With this in hand, we will examine the approximation more closely in chapter 4 and chapter 5. In the former, we study its validity as the vortex number increases; and in the latter, we apply it to the concrete case of vortices on flat tori.

Chapter 3

The vortex metric in the dissolving limit

In this chapter, we outline the main result of the thesis: that the vortex metric is well approximated on complete linear systems by the Fubini-Study metric. We also discuss an immediate consequence: that in some cases, this allows us to approximate geodesics.

3.1 Approximating the vortex metric

3.1.1 A convenient expression for the vortex metric

In order to state our main result, we require expressions of the Fubini-Study and vortex metrics on complete linear systems of pseudovortices. We begin by defining convenient coordinates for the latter.

Fix a complete linear system of pseudovortices, $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$, of dimension $d > 0$. Choose a representative constant curvature connection $D_{\hat{A}}$ and fix some real L^2-orthonormal basis for $\ker D^{0,1}_{\hat{A}}$, $\{s_0, \mathbf{i}s_0, \ldots, s_d, \mathbf{i}s_d\}$. Then we have an affine chart on $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$

$$\begin{aligned} \mathbb{C}^d &\xrightarrow{\sim} U_0 \subset \hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}, \\ x^i + \mathbf{i}y^i = z^i &\mapsto \left[D_{\hat{A}}, \sqrt{\varepsilon} \langle \mathbf{z} \rangle^{-1} \left(s_0 + \sum_{j=1}^{d} z^j s_j \right) \right]. \end{aligned} \tag{3.1}$$

Where $\langle \mathbf{z} \rangle = \sqrt{1 + \sum_{i=1}^{d} |z^i|^2}$ and i is an index which runs from 1 to d. We note that we can define similar charts and cover all of $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$; for this chapter, we work in the chart U_0, but note that the definitions and statements extend to any other chart after permuting the s_i.

We also have a frame of vector fields on U_0

$$\begin{aligned}\hat{\phi}_{x^i} &= \langle \mathbf{z} \rangle^{-1} s_i - \langle \mathbf{z} \rangle^{-2} \bar{z}^i \hat{\phi}, \\ \hat{\phi}_{y^i} &= \langle \mathbf{z} \rangle^{-1} \mathbf{i} s_i - \langle \mathbf{z} \rangle^{-2} \bar{z}^i \mathbf{i} \hat{\phi},\end{aligned} \tag{3.2}$$

which are L^2-orthogonal to the infinitesimal Hopf action, i.e. the linearisation of the circle action on the unit sphere $S^{2d-1} \subset \mathbb{C}^d$. It therefore follows that the inner products of these vector fields yield the components of the Fubini-Study metric. Indeed, we compute

$$\begin{aligned} h(\hat{\phi}_{x^i}, \hat{\phi}_{x^j}) &:= \left(\!\!\left(\hat{\phi}_{x^i}, \hat{\phi}_{x^j} \right)\!\!\right) + \mathbf{i} \left(\!\!\left(\mathbf{i} \hat{\phi}_{x^i}, \hat{\phi}_{x^j} \right)\!\!\right) \\ &= \langle \mathbf{z} \rangle^{-4} \left(\langle \mathbf{z} \rangle^2 \delta_{ij} - \operatorname{Re} z^i \bar{z}^j \right) + \mathbf{i} \langle \mathbf{z} \rangle^{-4} \operatorname{Im} \bar{z}^i z^j \\ h_{\bar{\imath}j} &= \langle \mathbf{z} \rangle^{-4} \left(\langle \mathbf{z} \rangle^2 \delta_{\bar{\imath}j} - z^i \bar{z}^j \right) \end{aligned} \tag{3.3}$$

which we recognise as the components of the Fubini-Study Hermitian form expressed in affine coordinates. The Hermitian form associated to a Kähler metric uniquely determines the metric (see e.g. [32, p. 48]), thus we can again see that the L^2-inner product is precisely the Fubini-Study metric on complete linear systems.

We wish to write a similar expression for the vortex metric, but before doing so, we verify that the pseudovortex-vortex map is a diffeomorphism on complete linear systems.

Proposition 3.1.1.
On complete linear systems, the pseudovortex-vortex map is a diffeomorphism.

We draw the reader's attention to the construction of the pushforward of the pseudovortex-vortex map given in this proof (3.9).

Proof. From Theorem 2.2.8 and Corollary 2.2.12 we know that the pseudovortex-vortex map is a bijection between complete linear systems. It is then enough to show that the pushforward is everywhere injective and thus a smooth immersion, that it is a global diffeomorphism then follows from the global rank theorem (see e.g. [40, p. 379]). We shall prove this on the level of configurations, i.e. we shall show that the pushforward is a linear injection between horizontal subspaces of the respective tangent spaces of configuration spaces.

Recall the pseudovortex-vortex map (2.14); for this proof only, we call it F.

$$\begin{aligned} F : \hat{\mathcal{M}}^\varepsilon &\to \mathcal{M}^\varepsilon, \\ \left[D_{\hat{A}}, \sqrt{\varepsilon} \hat{\phi} \right] &\mapsto \left[D_{\hat{A}} + \frac{\mathbf{i}}{2} * \mathrm{d}u \,, \sqrt{\varepsilon} e^{u/2} \hat{\phi} \right], \end{aligned}$$

where u is the solution to the PSC equation (2.15)

$$\Delta u - \frac{\varepsilon}{\operatorname{Vol} \Sigma} + \varepsilon e^u \left| \hat{\phi} \right|^2 = 0.$$

We let $\tilde{F} : \hat{\mathcal{V}}^\varepsilon \to \mathcal{V}^\varepsilon$ be the map on configurations.

For $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] \in \hat{\mathcal{M}}^\varepsilon_{\hat{A}}$ and note that this is a smooth point by Theorem 2.2.11, we fix some $\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right) \in \left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]$ and choose some tangent vector

$$v = \mathbf{i}\hat{a} \oplus \sqrt{\varepsilon}\hat{f} \in T_{\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)} \hat{\mathcal{V}}^\varepsilon_{\hat{A}} \subset \Omega^1(\Sigma; \mathfrak{u}(1)) \oplus \Omega^0(\Sigma; L).$$

v must satisfy the linearisation of the equations (2.11)–(2.13)

$$\mathbf{i}\hat{a}^{0,1}\hat{\phi} + D^{0,1}_{\hat{A}}\hat{f} = 0, \tag{3.4}$$

$$\mathbf{i}\,\mathrm{d}\hat{a} = 0, \tag{3.5}$$

$$\left(\!\left(\hat{f}, \hat{\phi}\right)\!\right) = 0. \tag{3.6}$$

And as v tangent to $\hat{\mathcal{V}}^\varepsilon_{\hat{A}}$, $\mathbf{i}\hat{a}$ must be gauge equivalent to 0, i.e. it is exact. It is convenient to choose a gauge where $\mathbf{i}\hat{a} = 0$ and we do so for the rest of this proof. Note that this constraint defines a horizontal subspace of $T_{\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)} \hat{\mathcal{V}}^\varepsilon_{\hat{A}}$, i.e. one complementary to infinitesimal gauge transformations, we write this as

$$H_{\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)} \hat{\mathcal{V}}^\varepsilon_{\hat{A}} := \left\{ 0 \oplus \sqrt{\varepsilon}\hat{f} \in T_{\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)} \hat{\mathcal{V}}^\varepsilon_{\hat{A}} \right\}.$$

Similarly, consider a tangent vector to $(D_A, \phi) := F\left(\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)\right)$

$$\mathbf{i}a \oplus f \in T_{(D_A,\phi)} \mathcal{V}^\varepsilon_A \in \Omega^1(\Sigma; \mathfrak{u}(1)) \oplus \Omega^0(\Sigma; L),$$

which must satisfy the linearisation of the vortex equations (2.7)–(2.8). i.e.

$$\mathbf{i}a^{0,1}\phi + D^{0,1}_A f = 0, \tag{3.7}$$

$$*^{-1}\,\mathrm{d}a = -\langle \phi, f \rangle\,. \tag{3.8}$$

Note that as $\mathbf{i}a \oplus f$ must be tangent to a complete linear system, i.e. $\mathbf{i}a$ cannot be tangent to the Jacobian. We recall that $\mathrm{Jac}(\Sigma) = H^{0,1}(\Sigma)/H^1(\Sigma; \mathbb{Z})$, so we can identify the tangent space of $\mathrm{Jac}(\Sigma)$ with $H^{0,1}(\Sigma)$. Applying the real Hodge decomposition (see e.g. [32, p. 286]), we have $H^{0,1}(\Sigma) \subset \mathcal{H}^1(\Sigma, g^\Sigma)$ where the latter is the space of d-harmonic 1-forms, so we see that $\mathbf{i}a$ must be the sum of an exact and a co-exact form. And again, as infinitesimal gauge fields are exact, we can choose a gauge such that $\mathbf{i}a$ is co-exact only. As with the tangent space of pseudovortices, these constraints define a horizontal subspace of $T_{(D_A,\phi)}\mathcal{V}^\varepsilon_A$ which we write

$$H_{(D_A,\phi)}\mathcal{V}^\varepsilon_A := \left\{ \mathbf{i}a \oplus f \in T_{(D_A,\phi)}\mathcal{V}^\varepsilon_A \;\middle|\; \mathbf{i}a \in \delta\Omega^2(\Sigma; \mathfrak{u}(1)) \right\}.$$

We push v forward via $\tilde{F}$ to yield

$$\tilde{F}_* v = \frac{\mathbf{i}}{2} * \mathrm{d}u_v \oplus \sqrt{\varepsilon} e^{\frac{u}{2}} \left(\frac{1}{2} u_v \hat{\phi} + \hat{f} \right) \tag{3.9}$$

where u_v is the solution to

$$\Delta u_v + \varepsilon e^u \left(\left| \hat{\phi} \right|^2 u_v + 2 \left\langle \hat{f}, \hat{\phi} \right\rangle \right) = 0. \tag{3.10}$$

Existence and uniqueness of solutions to this equation follows from the standard elliptic machinery (see e.g. [21, App. III]). Clearly, $\tilde{F}_*$ is linear.

Letting $\mathbf{i}a \oplus f := \tilde{F}_* v$, we can clearly see that $\mathbf{i}a$ is co-exact, so it remains to check that it satisfies the linearised vortex equations. The second equation is easily verified

$$\begin{aligned} *^{-1} \mathrm{d} \left(\frac{1}{2} * \mathrm{d}u_v \right) &= \frac{1}{2} \Delta u_v \\ &= -\frac{\varepsilon}{2} e^u \left(\left| \hat{\phi} \right|^2 u_v + 2 \left\langle \hat{f}, \hat{\phi} \right\rangle \right) \\ &= - \left\langle \sqrt{\varepsilon} e^{u/2} \hat{\phi}, \sqrt{\varepsilon} e^{u/2} \left(\frac{1}{2} u_v \hat{\phi} + \hat{f} \right) \right\rangle \\ *^{-1} \, \mathrm{d}a &= - \left\langle \phi, f \right\rangle . \end{aligned} \tag{3.11}$$

And for the first, we compute

$$\begin{aligned} \mathbf{i} a^{0,1} \phi &= \left(\frac{\mathbf{i}}{2} * \mathrm{d}u_v \right)^{0,1} \phi \\ &= -\frac{1}{2} \overline{\partial} u_v \phi \\ D_A^{0,1} f &= D_A^{0,1} \left(\frac{1}{2} u_v \phi + \sqrt{\varepsilon} e^{u/2} \hat{f} \right) \\ &= \frac{1}{2} \overline{\partial} u_v \phi + \frac{1}{2} u_v \underbrace{D_A^{0,1} \phi}_{=0} + \sqrt{\varepsilon} D_A^{0,1} \left(e^{u/2} \hat{f} \right). \end{aligned} \tag{3.12}$$

And so

$$\begin{aligned} \mathbf{i} a^{0,1} \phi + D_A^{0,1} f &= \sqrt{\varepsilon} e^{u/2} \frac{1}{2} \overline{\partial} u \hat{f} + \sqrt{\varepsilon} e^{u/2} D_A^{0,1} \hat{f} \\ &= \sqrt{\varepsilon} e^{u/2} \frac{1}{2} \overline{\partial} u \hat{f} + \sqrt{\varepsilon} e^{u/2} \underbrace{D_{\hat{A}}^{0,1} \hat{f}}_{=0} + \sqrt{\varepsilon} e^{u/2} \left(\frac{\mathbf{i}}{2} * \mathrm{d}u \right)^{0,1} \hat{f} \\ &= \sqrt{\varepsilon} e^{u/2} \frac{1}{2} \overline{\partial} u \hat{f} - \sqrt{\varepsilon} e^{u/2} \frac{1}{2} \overline{\partial} u \hat{f} \\ &= 0. \end{aligned} \tag{3.13}$$

So $\tilde{F}_* : H_{\left(D_{\hat{A}}, \sqrt{\varepsilon} \hat{\phi} \right)} \hat{\mathcal{V}}_{\hat{A}}^{\varepsilon} \to H_{(D_A, \phi)} \mathcal{V}_A^{\varepsilon}$ is a linear map.

We show $\tilde{F}_*$ is an immersion by showing $\ker \tilde{F}_* = 0$. If $\tilde{F}_* v = 0$, we have $\frac{\mathbf{i}}{2} * \mathrm{d}u_v = 0$

and $\sqrt{\varepsilon}e^{u/2}\left(\frac{1}{2}u_v\hat{\phi}+\hat{f}\right)=0$ (and so $\hat{f}=-\frac{1}{2}u_v\hat{\phi}$). The first implies that $\mathrm{d}u_v=0$; which, by (3.10), implies $u_v\left|\hat{\phi}\right|^2+2\left\langle\hat{f},\hat{\phi}\right\rangle=0$ and so $\left\langle\hat{f},\hat{\phi}\right\rangle=-\frac{1}{2}u_v\left|\hat{\phi}\right|^2$. Since we have $\left(\!\left(\hat{f},\hat{\phi}\right)\!\right)=0$ (3.6) and $\left\|\hat{\phi}\right\|_{L^2}^2=1$ (2.13), we must therefore have $u_v=0$. So $\hat{f}=0$, $v=0\oplus 0$, and we are done. □

In order to write down such an expression for the vortex metric on complete linear systems, we must do slightly more work. The image of (3.9) is unlikely to be L^2-orthogonal to gauge, and so restricting the L^2-inner product to its image will contain contributions from infinitesimal gauge transformations, i.e. it will not be the vortex metric. We therefore need to correct the image of (3.9) to remove these contributions. We do so by imposing a gauge fixing condition.

Lemma 3.1.2.
Let $[D_A,\phi]\in\mathcal{C}/\mathcal{G}$ and let $v=\mathbf{i}a\oplus f\in T_{(D_A,\phi)}\mathcal{C}$. Then v is L^2-orthogonal to the linearised gauge action if and only if

$$\delta a-\langle f,\mathbf{i}\phi\rangle=0. \tag{3.14}$$

Proof. An infinitesimal gauge transformation at (D_A,ϕ) is a vector $-\mathbf{i}\,\mathrm{d}\chi\oplus\mathbf{i}\chi\phi$ for some $\mathbf{i}\chi\in\Omega^0(\Sigma;\mathfrak{u}(1))$. We compute

$$\begin{aligned}\left(\!\left(v,-\mathbf{i}\,\mathrm{d}\chi\oplus\mathbf{i}\chi\phi\right)\!\right)&=\left(\!\left(\mathbf{i}a,-\mathbf{i}\,\mathrm{d}\chi\right)\!\right)+\left(\!\left(f,\mathbf{i}\chi\phi\right)\!\right)\\&=\left(\!\left(-\delta a,\chi\right)\!\right)+\left(\!\left(\langle f,\mathbf{i}\phi\rangle,\chi\right)\!\right)\\&=-\left(\!\left(\delta a-\langle f,\mathbf{i}\phi\rangle,\chi\right)\!\right).\end{aligned}$$

v is L^2-orthogonal to the linearised gauge action if and only if the above vanishes for all χ, which is the case if and only if $\delta a-\langle f,\mathbf{i}\phi\rangle=0$. □

We can now write down an expression for the vortex metric.

Proposition 3.1.3.
On $U_0\subset\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$, the pullback of the vortex metric by the pseudovortex-vortex map is given by $\varepsilon g^{\varepsilon}_{\hat{A}}(\hat{\phi}_a,\hat{\phi}_b)$ where

$$\begin{aligned}g^{\varepsilon}_{\hat{A}}(\hat{\phi}_a,\hat{\phi}_b):=\left(\!\left(\hat{\phi}_a,\hat{\phi}_b\right)\!\right)+\left(\!\left(\tilde{u}\hat{\phi}_a,\hat{\phi}_b\right)\!\right)+\left(\!\left((1+\tilde{u})\hat{\phi},\bar{\eta}_a\eta_b\hat{\phi}+\hat{\phi}_a\bar{\eta}_b+\bar{\eta}_a\hat{\phi}_b\right)\!\right)\\+\frac{1}{\varepsilon}\left(\!\left(\partial\bar{\eta}_a-\overline{\partial}\eta_a,\partial\bar{\eta}_b-\overline{\partial}\eta_b\right)\!\right).\end{aligned} \tag{3.15}$$

Here $\eta_\bullet\in\Omega^0(\Sigma;\mathbb{C})$ is the unique solution to

$$\Delta\eta_\bullet+\varepsilon e^u\left|\hat{\phi}\right|^2\eta_\bullet=-\varepsilon e^u\left(\frac{1}{2}\left\langle\hat{\phi}_\bullet,\hat{\phi}\right\rangle+\mathbf{i}\left\langle\hat{\phi}_\bullet,\mathbf{i}\hat{\phi}\right\rangle\right), \tag{3.16}$$

u solves $\Delta u-\frac{\varepsilon}{\mathrm{Vol}\,\Sigma}+\varepsilon e^u\left|\hat{\phi}\right|^2=0$ (2.15) for $\hat{\phi}\in U_0$, $\tilde{u}=e^u-1$, and the $\hat{\phi}_\bullet$ are frame

fields (3.2).

Proof. As in previous proofs, we write (2.14) as $F : \hat{\mathcal{M}}^\varepsilon \to \mathcal{M}^\varepsilon$, and the map on configurations as $\tilde{F}$.

Given $\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right) \in \hat{\mathcal{V}}^\varepsilon$, note that an infinitesimal gauge transformation at

$$\tilde{F}\left(\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)\right) = \left(D_{\hat{A}} + \frac{\mathbf{i}}{2} * \mathrm{d}u\,, \sqrt{\varepsilon}e^{u/2}\hat{\phi}\right) =: (D_A, \phi),$$

takes the form $-\mathbf{i}\,\mathrm{d}\chi_v \oplus \mathbf{i}\chi_v\sqrt{\varepsilon}e^{u/2}\hat{\phi}$. For $v = 0 \oplus \sqrt{\varepsilon}\hat{f} \in H_{\left(D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right)}\hat{\mathcal{V}}^\varepsilon$, we then see that

$$\tilde{F}_* v - \mathbf{i}\,\mathrm{d}\chi_v \oplus \mathbf{i}\chi_v\sqrt{\varepsilon}e^{u/2}\hat{\phi} = \left(\frac{\mathbf{i}}{2} * \mathrm{d}u_v - \mathbf{i}\,\mathrm{d}\chi_v\right) \oplus \sqrt{\varepsilon}e^{\frac{u}{2}}\left(\frac{1}{2}u_v\hat{\phi} + \mathbf{i}\chi_v\hat{\phi} + \hat{f}\right).$$

Applying Lemma 3.1.2, we see that this is L^2-orthogonal to the linearised gauge action if and only if

$$\delta\left(\frac{1}{2} * \mathrm{d}u_v - \mathrm{d}\chi_v\right) = \left\langle \sqrt{\varepsilon}e^{\frac{u}{2}}\left(\frac{1}{2}u_v\hat{\phi} + \hat{f} + \mathbf{i}\chi_v\hat{\phi}\right), \mathbf{i}\sqrt{\varepsilon}e^{u/2}\hat{\phi}\right\rangle$$

$$\frac{1}{2}\underbrace{\delta * \mathrm{d}u_v}_{=0} - \delta\mathrm{d}\chi_v = \varepsilon e^u\left(\frac{1}{2}u_v\underbrace{\left\langle\hat{\phi}, \mathbf{i}\hat{\phi}\right\rangle}_{=0} + \left\langle\hat{f}, \mathbf{i}\hat{\phi}\right\rangle + \chi_v\left|\hat{\phi}\right|^2\right)$$

i.e.

$$\Delta\chi_v + \varepsilon e^u\left(\chi_v\left|\hat{\phi}\right|^2 + \left\langle\hat{f}, \mathbf{i}\hat{\phi}\right\rangle\right) = 0. \tag{3.17}$$

As with (3.10), existence and uniqueness of solutions to this equation follow from the standard elliptic machinery. Indeed, we note that (3.17) is structurally very similar to (3.10). We let $\eta_v = \frac{1}{2}u_v + \mathbf{i}\chi_v$ and note it solves (3.16)

$$\Delta\eta_v + \varepsilon e^u\left|\hat{\phi}\right|^2\eta_v = -\varepsilon e^u\left(\frac{1}{2}\left\langle\hat{f}, \hat{\phi}\right\rangle + \mathbf{i}\left\langle\hat{f}, \mathbf{i}\hat{\phi}\right\rangle\right).$$

We note that $\frac{\mathbf{i}}{2} * \mathrm{d}u_v - \mathbf{i}\,\mathrm{d}\chi_v = \partial\bar{\eta}_v - \overline{\partial}\eta_v$ and we *define*

$$F_* v := \partial\bar{\eta}_v - \overline{\partial}\eta_v \oplus \sqrt{\varepsilon}e^{u/2}(\bar{\eta}_v\hat{\phi} + \hat{f}). \tag{3.18}$$

Note that the image of this map is L^2-orthogonal to infinitesimal gauge actions by construction. The expression of the vortex metric on $\hat{\mathcal{M}}^\varepsilon_{\hat{A}}$ is then the pullback of the L^2-metric by this differential.

Now suppose $v = 0 \oplus \sqrt{\varepsilon}\hat{f}_v$, $w = 0 \oplus \sqrt{\varepsilon}\hat{f}_w$ are two vectors in $T_{\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right]}\hat{\mathcal{M}}^{\varepsilon}$, then

$$\begin{aligned}
\langle\!\langle F_* v, F_* w \rangle\!\rangle &= \left(\!\left(\sqrt{\varepsilon} e^{u/2}(\bar{\eta}_v \hat{\phi} + \hat{f}_v), \sqrt{\varepsilon} e^{u/2}(\bar{\eta}_w \hat{\phi} + \hat{f}_w) \right)\!\right) + \langle\!\langle \partial\bar{\eta}_v - \overline{\partial}\eta_v, \partial\bar{\eta}_w - \overline{\partial}\eta_w \rangle\!\rangle \\
\frac{1}{\varepsilon}\langle\!\langle F_* v, F_* w \rangle\!\rangle &= \left(\!\left(e^u \bar{\eta}_v \hat{\phi}, \bar{\eta}_w \hat{\phi} \right)\!\right) + \left(\!\left(e^u \bar{\eta}_v \hat{\phi}, \hat{f}_w \right)\!\right) + \left(\!\left(e^u \hat{f}_v, \bar{\eta}_w \hat{\phi} \right)\!\right) + \left(\!\left(e^u \hat{f}_v, \hat{f}_w \right)\!\right) \\
&\quad + \frac{1}{\varepsilon}\langle\!\langle \partial\bar{\eta}_v - \overline{\partial}\eta_v, \partial\bar{\eta}_w - \overline{\partial}\eta_w \rangle\!\rangle \\
&= \left(\!\left((1+\tilde{u})\hat{f}_v, \hat{f}_w \right)\!\right) + \left(\!\left(e^u \hat{\phi}, \bar{\eta}_v \eta_w \hat{\phi} \right)\!\right) + \left(\!\left(e^u \hat{\phi}, \eta_v \hat{f}_w \right)\!\right) + \left(\!\left(e^u \hat{\phi}, \eta_w \hat{f}_v \right)\!\right) \\
&\quad + \frac{1}{\varepsilon}\langle\!\langle \partial\bar{\eta}_v - \overline{\partial}\eta_v, \partial\bar{\eta}_w - \overline{\partial}\eta_w \rangle\!\rangle \\
&= \left(\!\left(\hat{f}_v, \hat{f}_w \right)\!\right) + \left(\!\left(\tilde{u}\hat{f}_v, \hat{f}_w \right)\!\right) + \left(\!\left((1+\tilde{u})\hat{\phi}, \bar{\eta}_v \eta_w \hat{\phi} + \eta_v \hat{f}_w + \eta_w \hat{f}_v \right)\!\right) \\
&\quad + \frac{1}{\varepsilon}\langle\!\langle \partial\bar{\eta}_v - \overline{\partial}\eta_v, \partial\bar{\eta}_w - \overline{\partial}\eta_w \rangle\!\rangle
\end{aligned}$$

where

$$\tilde{u} = e^u - 1. \tag{3.19}$$

If we take $\hat{\phi} \in U_0$, $v = 0 \oplus \sqrt{\varepsilon}\hat{\phi}_a$, and $w = 0 \oplus \sqrt{\varepsilon}\hat{\phi}_b$; the claim in the proposition follows. □

We can now state our main theorem. For the rest of the chapter, we bound ε above by some $\varepsilon_{\max} > 0$. The results that follow depend on $\varepsilon_{\max}$, but this is of no concern as we are interested in the limit $\varepsilon \to 0$ and our estimates will generally be of the form $\|\cdot\|_{C^k} < c\varepsilon$. Indeed, many of the results in §3.2 will generate a maximal ε for which they are valid.

Theorem 3.1.4.
Let $(L, H) \to (\Sigma, g^{\Sigma})$ be some Hermitian line bundle over a Riemann surface of arbitrary genus, and choose some complete linear system such that $\dim \hat{\mathcal{M}}^{\varepsilon}_{\hat{A}} = \dim\ker D^{0,1}_{\hat{A}} - 1 > 0$. On charts, define

$$\alpha(\varepsilon)_{ab} := g^{\varepsilon}_{\hat{A}}(\hat{\phi}_a, \hat{\phi}_b) - \left(\!\left(\hat{\phi}_a, \hat{\phi}_b \right)\!\right), \tag{3.20}$$

and view this as a one-parameter family of symmetric two tensors on the fibre $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$. Then there is some $C_{\hat{A}} > 0$, which only depends on: our initial choice of $(L, H) \to (\Sigma, g^{\Sigma})$, our choice of $\left[D_{\hat{A}}\right]$, and $\varepsilon_{\max}$, such that for all $\varepsilon \in (0, \varepsilon_{\max})$

$$\|\alpha(\varepsilon)\|_{C^1} < C_{\hat{A}}\varepsilon.$$

The remainder of the chapter is structured as follows:

- in §3.1.2 we lay out some technical estimates,
- in §3.1.3 and §3.1.4 respectively we construct C^0 and C^1 estimates for the defect tensor (3.20), this is enough to prove Theorem 3.1.4 which we do in the latter section, and
- in §3.2 we use Theorem 3.1.4 to study the approximation of geodesics on complete

linear systems.

3.1.2 Some technical estimates

We require some technical estimates.

Proposition 3.1.5.
Let $(M^{(n)}, g)$ be a compact Riemannian manifold. In what follows, the constants $C_\bullet$ depend on (M, g) in addition to the quantities in the subscript.

1. *For every $k \geq 0$, we have for all $f, g \in C^\infty(M)$*

$$\|fg\|_{H^k} < \left(2^{k+1} - 1\right)\|f\|_{C^k}\|g\|_{H^k}. \tag{3.21}$$

   In particular, $\|fg\|_{L^2} \leq \|f\|_{C^0}\|g\|_{L^2}$ and $\|fg\|_{H^1} \leq 3\|f\|_{C^1}\|g\|_{H^1}$.

2. *(Sobolev embedding): For every $0 \leq m < k - \frac{n}{p}$, there exists $C_{m,k,p} > 0$ such that for all $f \in C^\infty(M)$*

$$\|f\|_{C^m} < C_{m,k,p}\|f\|_{W^{k,p}}. \tag{3.22}$$

3. *(Elliptic regularity theorem): For every $k \geq 0$, there exists $C_k > 0$ such that for all $f \in C^\infty(M)$*

$$\|f\|_{H^{k+2}} < C_k(\|\Delta f\|_{H^k} + \|f\|_{L^2}). \tag{3.23}$$

   If $\int_M f = 0$, then we can omit the $\|f\|_{L^2}$ term on the RHS.

Aside from the first item which is proven in §A.1 on page 95, proofs for these results can be found in e.g. [21, App. III], [27]. The following pair of corollaries are convenient.

Corollary 3.1.6.
Given $u \in \Omega^0(\Sigma)$, there is a $C > 0$ which only depends on (Σ, g^Σ) such that

$$\|u\|_{C^1} \leq C(\|\Delta u\|_{H^1} + \|u\|_{L^2}). \tag{3.24}$$

Corollary 3.1.7.
Let (Σ, g^Σ) be a compact Riemann surface with $a, b \in C^\infty(\Sigma)$ such that $a \geq 0$. Then if $u \in C^\infty(\Sigma)$ solves the linear elliptic PDE

$$\Delta u + au = b,$$

there exists $C > 0$ depending only on (Σ, g^Σ) such that

$$\|u\|_{C^1} < C((3\|a\|_{C^1} + 1)\|u\|_{H^1} + \|b\|_{H^1}). \tag{3.25}$$

As with the definition of function norms, these results have natural extensions to metric vector bundles. Though in the process of extending these results we introduce a dependence on the bundle metric.

The following pair of lemmas are proven in Appendix A.

Lemma 3.1.8.
In what follows, the constants $C_\bullet$ depend on $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$, and a choice of $[D_{\hat{A}}]$ in addition to the quantities in the subscript. In particular, they do not *depend on position in $\hat{\mathcal{M}}^\varepsilon_{\hat{A}}$ (i.e. the exact choice of $\hat{\phi}$).*

1. *(Pseudovortex regularity) For every $k \geq 0$, there is a $C_k > 0$ such that*

$$\left\|\hat{\phi}\right\|_{C^k} < C_k. \tag{3.26}$$

2. *(Pseudovortex modulus regularity) For every $k \geq 0$, there is a $C_k > 0$, such that*

$$\left\|\left|\hat{\phi}\right|^2\right\|_{C^k} < C_k. \tag{3.27}$$

3. *For every $k > 0$, there is a $C_k > 0$, such that for every affine coordinate chart on $\hat{\mathcal{M}}^\varepsilon_{\hat{A}}$ and every $a \in \{1, \ldots, 2d\}$ we have*

$$\left\|\hat{\phi}_a\right\|^2_{H^k} < C_k. \tag{3.28}$$

4. *For every $k > 0$, there is a $C_k > 0$, such that for every affine coordinate chart on $\hat{\mathcal{M}}^\varepsilon_{\hat{A}}$ and every $a \in \{1, \ldots, 2d\}$ we have*

$$\left\|\left\langle \hat{\phi}_a, \hat{\phi}_b \right\rangle\right\|^2_{H^1} < C_k. \tag{3.29}$$

The proof is given on page 102. The second provides the strategy for finding a C^0-bound on α from its coordinate expression.

Lemma 3.1.9.
Let $\gamma \in C^\infty\left(\mathbb{P}^d; \left(T^*\mathbb{P}^d\right)^{\otimes r}\right)$ be some r-tensor. If in every affine coordinate chart, the components of γ, $\gamma_{i_0,\ldots,i_r}(p)$, are uniformly bounded in C^0 by some $C' > 0$. Then there is a constant $C > 0$ which only depends on r and d such that

$$\|\gamma\|_{C^0} < CC'.$$

In particular, if γ is a two tensor and we wish to find a C^0-estimate for it, it is sufficient to find an estimate $|\gamma_{ab}(p)| < C$ for every affine coordinate chart U_i, every $p \in U_i$, and every choice of indices.

The proof is given on page 107. In essence, this is relying on boundedness properties of the Fubini-Study metric on affine charts. One should compare this result with Lemma 3.1.16 used later on in this chapter.

Finally, we need a pair of results from [23].

Proposition 3.1.10 ([23]).
Let $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] \in \hat{\mathcal{M}}^{\varepsilon}$, *and* u *solve* $\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + \varepsilon e^{u}\left|\hat{\phi}\right|^{2} = 0$ (2.15). *Then there exists a pair of constants,* $C_{\hat{A},1}$ *and* $C_{\hat{A},2}$, *depending only on* $(L,H) \to (\Sigma, g^{\Sigma}, \omega^{\Sigma})$, *and* $D_{\hat{A}}$ *such that* u *and* $\tilde{u} = e^{u} - 1$ *satisfy the bounds*

$$\|u\|_{C^0}, \|\tilde{u}\|_{C^0} < C_{\hat{A},1}, \tag{3.30}$$

and

$$\|u\|_{C^0}, \|\tilde{u}\|_{C^0} < C_{\hat{A},2}\varepsilon, \tag{3.31}$$

García Lara and Speight state the above result only when $\Sigma \cong \mathbb{P}^1$, but their proof is generic and only requires minor modifications to work for general Σ. We note that their estimates do not depend on an upper bound for $\varepsilon_{\max}$ and are valid for all $\varepsilon > 0$.

Lemma 3.1.11 ([23]).
Let (M,g) *be a compact Riemannian manifold with* $a, b \in C^{\infty}(M)$ *such that* $a \geq 0$ *and* $\mu := \frac{1}{\int_M a} > 0$. *Then if* $v \in C^{\infty}(M)$ *solves the linear elliptic PDE*

$$\Delta v + av = b,$$

there exists $C > 0$ *depending only on* (M,g) *such that*

$$\|v\|_{H^1} \leq C\left((1+\mu\|a\|_{L^2})\left(\|b\|_{L^2} + \mu\|a\|_{L^2}\left|\int_M b\right|\right) + \mu\left|\int_M b\right|\right). \tag{3.32}$$

3.1.3 The bound on α

Recall that the symmetric 2-tensor $\alpha = g^{\varepsilon}_{\hat{A}} - \hat{g}^{\varepsilon}_{\hat{A}}$ (3.20) is the difference between the (rescaled) vortex metric and the pseudovortex metric. We begin by constructing a C^0-bound on α. While not strictly necessary as the work of García Lara and Speight [23] does generalise to the higher genus case almost immediately, our construction uses some slightly different estimates and is generally less brittle.

Remark 3.1.12.
For the ensuing proofs it is convenient to follow the convention below:

Within a proof, all constants $c_{\bullet} > 0$ *only depend on* (Σ, g^{Σ}), (L,H), $D_{\hat{A}}$, *and* $\varepsilon_{\max}$. *In particular, they do not depend on* $\hat{\phi}$. *We write* c, c', *etc. for such constants whose scope is restricted to the paragraph.*

Lemma 3.1.13.
There exists $C_{\hat{A}} > 0$ depending only on $(L,H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$, $D_{\hat{A}}$, and $\varepsilon_{\max}$ such that,

$$\|u\|_{C^1} < C\varepsilon \tag{3.33}$$

Proof. Applying the estimate (3.24) we have $\|u\|_{C^1} \leq c_1(\|\Delta u\|_{H^1} + \|u\|_{L^2})$. Thus

$$\begin{aligned}\|u\|_{C^1} &\leq c_1\left(\left\|\frac{\varepsilon}{\operatorname{Vol}\Sigma} - \varepsilon\left|\hat{\phi}\right|^2 e^u\right\|_{H^1} + \|u\|_{L^2}\right)\\ &\leq c_1\left(\left\|\frac{\varepsilon}{\operatorname{Vol}\Sigma}\right\|_{H^1} + 3\varepsilon\left\|\left|\hat{\phi}\right|^2\right\|_{C^1}\|e^u\|_{H^1} + \|u\|_{C^0}\sqrt{\operatorname{Vol}\Sigma}\right).\end{aligned}$$

Now note that $\|u\|_{C^0} \leq c_2\varepsilon$ by (3.31), $\left\|\frac{\varepsilon}{\operatorname{Vol}\Sigma}\right\|_{H^1} = \frac{\varepsilon}{\sqrt{\operatorname{Vol}\Sigma}}$, and $\left\|\left|\hat{\phi}\right|^2\right\|_{C^1} \leq c_3$ by (3.27). Thus

$$\|u\|_{C^1} \leq c_1\left(\frac{1 + c_2\operatorname{Vol}\Sigma}{\sqrt{\operatorname{Vol}\Sigma}} + 3c_3\|e^u\|_{H^1}\right)\varepsilon$$

So it remains to show that $\|e^u\|_{H^1}$ is bounded above by a constant c.

We have,

$$\begin{aligned}\|e^u\|_{H^1}^2 &= \|e^u\|_{L^2}^2 + \|\mathrm{d}(e^u)\|_{L^2}^2\\ &\leq e^{2\|u\|_{C^0}}\operatorname{Vol}\Sigma + \|e^u\,\mathrm{d}u\|_{L^2}^2\\ &\leq e^{2\|u\|_{C^0}}\left(\operatorname{Vol}\Sigma + \|\mathrm{d}u\|_{L^2}^2\right)\\ &\leq e^{2\|u\|_{C^0}}\left(\operatorname{Vol}\Sigma + \|u\|_{L^2}\|\Delta u\|_{L^2}\right)\\ &\leq e^{2\|u\|_{C^0}}\left(\operatorname{Vol}\Sigma + \sqrt{\operatorname{Vol}\Sigma}\|u\|_{C^0}\left\|\frac{\varepsilon}{\operatorname{Vol}\Sigma} - \varepsilon\left|\hat{\phi}\right|^2 e^u\right\|_{L^2}\right)\\ &\leq e^{2\|u\|_{C^0}}\operatorname{Vol}\Sigma\left(1 + \|u\|_{C^0}\left\|\frac{\varepsilon}{\operatorname{Vol}\Sigma} - \varepsilon\left|\hat{\phi}\right|^2 e^u\right\|_{C^0}\right),\end{aligned}$$

which, by repeated application of the estimates (3.30) and (3.27), is bounded above by a constant, say c_4, and we are done. □

We can now find a C^1 estimate on η_a.

Lemma 3.1.14.
Let $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] \in \hat{\mathcal{M}}^\varepsilon$, and let u solve $\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + \varepsilon e^u\left|\hat{\phi}\right|^2 = 0$ (2.15). Then there exists a constant $C_{\hat{A}}$ depending only on (Σ, g^Σ), (L,H), $D_{\hat{A}}$, and $\varepsilon_{\max}$ such that η_a satisfies the bound,

$$\|\eta_a\|_{C^1} < C_{\hat{A}}\varepsilon. \tag{3.34}$$

Where η_a solves Equation 3.16,

$$\Delta\eta_\bullet + \varepsilon e^u\left|\hat{\phi}\right|^2\eta_\bullet = -\varepsilon e^u\left(\frac{1}{2}\left\langle\hat{\phi}_\bullet, \hat{\phi}\right\rangle + \mathbf{i}\left\langle\hat{\phi}_\bullet, \mathbf{i}\hat{\phi}\right\rangle\right).$$

Proof. We require an H^1 bound on η_a to start. Note that the real and imaginary parts of η_a each satisfy the conditions of Lemma 3.1.11. We will use the latter to find H^1 estimates on each part separately, yielding an estimate on the function itself. We only find the estimate for the real part as the process for the imaginary part is almost identical. Applying Lemma 3.1.11, we have

$$\|\mathrm{Re}(\eta_a)\|_{H^1} \leq c_1\left((1+\mu\|a\|_{L^2})\left(\|b\|_{L^2}+\mu\|a\|_{L^2}\left|\int_\Sigma b\right|\right)+\mu\left|\int_\Sigma b\right|\right),$$

where $a = \varepsilon e^u\left|\hat{\phi}\right|^2$, $\mu = \frac{1}{\|a\|_{L^1}}$, $b = \frac{\varepsilon}{2}e^u\left\langle\hat{\phi},\hat{\phi}_a\right\rangle$. We then have

$$\begin{aligned}
\mu\|a\|_{L^2} &= \left\|e^u\left|\hat{\phi}\right|^2\right\|_{L^2}\left(\left\|e^u\left|\hat{\phi}\right|^2\right\|_{L^1}\right)^{-1}\\
&\leq e^{\|u\|_{C^0}}\left\|\left|\hat{\phi}\right|^2\right\|_{L^2}\left(e^{-\|u\|_{C^0}}\left\|\left|\hat{\phi}\right|^2\right\|_{L^1}\right)^{-1}\\
&= e^{2\|u\|_{C^0}}\left\|\left|\hat{\phi}\right|^2\right\|_{L^2}\left\|\left|\hat{\phi}\right|^2\right\|_{L^1}^{-1}.
\end{aligned}$$

Note that $\left\|\left|\hat{\phi}\right|^2\right\|_{L^1} = \int_\Sigma\left|\hat{\phi}\right|^2 = 1$. We have bounds $\left\|\left|\hat{\phi}\right|^2\right\|_{L^2} \leq \left\|\left|\hat{\phi}\right|^2\right\|_{C^0}\mathrm{Vol}\,\Sigma =: c$ (3.27) and $\|u\|_{C^0} \leq c_2$ (3.30). Thus we have the estimate

$$\mu\|a\|_{L^2} \leq ce^{2c_2} =: c_3. \tag{3.35}$$

We can also see that

$$\|b\|_{L^2} \leq \frac{\varepsilon}{2}e^{\|u\|_{C^0}}\left\|\hat{\phi}\right\|_{C^0}\left\|\hat{\phi}_a\right\|_{L^2}.$$

By the above, $e^{\|u\|_{C^0}} \leq e^{c_2}$. By (3.27) and (3.28) respectively we have $\left\|\hat{\phi}\right\|_{C^0}, \left\|\hat{\phi}_a\right\|_{L^2} \leq c$. Thus we see that

$$\|b\|_{L^2} \leq \frac{\varepsilon}{2}e^{c_2}c^2 =: c_4\varepsilon. \tag{3.36}$$

As noted by García Lara and Speight [23], the key term for this interim estimate is $\left|\int_\Sigma b\right|$. We have

$$\begin{aligned}
\left|\int_\Sigma b\right| &= \frac{\varepsilon}{2}\left|\int_\Sigma(1+\tilde{u})\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right|\\
&\leq \frac{\varepsilon}{2}\left(\left|\left(\!\left(\hat{\phi},\hat{\phi}_a\right)\!\right)\right|+\left|\int_\Sigma\tilde{u}\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right|\right).
\end{aligned}$$

Now recall that by construction, $\left(\!\left(\hat{\phi}, \hat{\phi}_a\right)\!\right) = 0$, so

$$\begin{aligned}\left|\int_\Sigma b\right| &\leq \frac{\varepsilon}{2}\left|\int_\Sigma \tilde{u}\left\langle\hat{\phi}, \hat{\phi}_a\right\rangle\right| \\ &\leq \frac{\varepsilon\|\tilde{u}\|_{C^0}}{2}\int_\Sigma\left|\left\langle\hat{\phi}, \hat{\phi}_a\right\rangle\right| \\ &\leq \frac{\varepsilon\|\tilde{u}\|_{C^0}}{2}\left(\!\left(\left|\hat{\phi}\right|, \left|\hat{\phi}_a\right|\right)\!\right) \\ &\leq \frac{\varepsilon\|\tilde{u}\|_{C^0}}{2}\left\|\hat{\phi}\right\|_{L^2}\left\|\hat{\phi}_a\right\|_{L^2}.\end{aligned}$$

Applying (3.31) we have $\|\tilde{u}\|_{C^0} \leq c\varepsilon$; we also have $\left\|\hat{\phi}\right\|_{L^2} \leq \left\|\hat{\phi}\right\|_{C^0}\sqrt{\operatorname{Vol}\Sigma} \leq c'$ from (3.26) and $\left\|\hat{\phi}_a\right\|_{L^2} \leq c''$ from (3.28). Thus

$$\left|\int_\Sigma b\right| \leq \frac{\varepsilon\cdot c\varepsilon}{2}c'c'' =: c_5\varepsilon^2. \tag{3.37}$$

We then see that $\mu\left|\int_\Sigma b\right| \leq c_5\varepsilon\left(\left\|e^u\left|\hat{\phi}\right|^2\right\|_{L^1}\right)^{-1} \leq c_5\varepsilon e^{\|u\|_{C^0}}\left\|\hat{\phi}\right\|_{L^2}^{-1} \leq c_5e^{c_2}\varepsilon =: c_6\varepsilon$. Thus

$$\|\operatorname{Re}(\eta_a)\|_{H^1} \leq c_1\big((1+c_3)\big(c_4\varepsilon + c_3c_5\varepsilon^2\big) + c_6\varepsilon\big).$$

Thus $\|\operatorname{Re}(\eta_a)\|_{H^1} \leq c_7\varepsilon$.

As mentioned, an almost identical procedure yields an H^1 estimate, $\|\operatorname{Im}(\eta_a)\|_{H^1} \leq c_8\varepsilon$. For brevity, we use the single estimate

$$\|\eta_a\|_{H^1} \leq (c_7+c_8)\varepsilon =: c_9\varepsilon. \tag{3.38}$$

We then apply (3.25) to yield

$$\|\eta_a\|_{C^1} \leq c\left(\left(3\left\|\varepsilon e^u\left|\hat{\phi}\right|^2\right\|_{C^1} + 1\right)\|\eta_a\|_{H^1} + \left\|\frac{\varepsilon}{2}e^u\left\langle\hat{\phi}, \hat{\phi}_a\right\rangle + \mathbf{i}\varepsilon e^u\left\langle\mathbf{i}\hat{\phi}, \hat{\phi}_a\right\rangle\right\|_{H^1}\right). \tag{3.39}$$

We have

$$\begin{aligned}\left\|\varepsilon e^u\left|\hat{\phi}\right|^2\right\|_{C^1} &\leq \varepsilon\|e^u\|_{C^1}\left\|\left|\hat{\phi}\right|^2\right\|_{C^1} \\ &\leq \varepsilon\max\left(\|e^u\|_{C^0}, \|e^u\,\mathrm{d}u\|_{C^0}\right)\left\|\left|\hat{\phi}\right|^2\right\|_{C^1} \\ &\leq \varepsilon e^{\|u\|_{C^0}}\max\left(1, \|u\|_{C^1}\right)\left\|\left|\hat{\phi}\right|^2\right\|_{C^1}.\end{aligned}$$

So applying (3.30), (3.33), (3.27) we have the estimate

$$\left\|\varepsilon e^u\left|\hat{\phi}\right|^2\right\|_{C^1} \leq \varepsilon\max\left(1, c\varepsilon\right)c' < \varepsilon_{\max}\max\left(1, c\varepsilon_{\max}\right)c' =: c_{10}. \tag{3.40}$$

And

$$\begin{aligned}\left\|\frac{\varepsilon}{2}e^u\left\langle\hat{\phi},\hat{\phi}_a\right\rangle+\mathbf{i}\varepsilon e^u\left\langle\mathbf{i}\hat{\phi},\hat{\phi}_a\right\rangle\right\|_{H^1}&\leq 3\varepsilon\|e^u\|_{C^1}\left\|\frac{1}{2}\left\langle\hat{\phi},\hat{\phi}_a\right\rangle+\mathbf{i}\left\langle\mathbf{i}\hat{\phi},\hat{\phi}_a\right\rangle\right\|_{H^1}\\&\leq 3\varepsilon\|e^u\|_{C^1}\left(\frac{1}{2}\left\|\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right\|_{H^1}+\left\|\left\langle\mathbf{i}\hat{\phi},\hat{\phi}_a\right\rangle\right\|_{H^1}\right)\\&\leq 9\varepsilon\|e^u\|_{C^1}\left(\frac{1}{2}\left\|\hat{\phi}\right\|_{C^1}\left\|\hat{\phi}_a\right\|_{H^1}+\left\|\hat{\phi}\right\|_{C^1}\left\|\hat{\phi}_a\right\|_{H^1}\right).\end{aligned}$$

We have $\|e^u\|_{C^1}\leq e^{\|u\|_{C^0}}\max\left(1,\|u\|_{C^1}\right)\leq c\max\left(1,c'\varepsilon\right)$ from (3.30), (3.33); and it is clear by (3.26), (3.28) that the terms in parentheses are bounded above by a constant c. So the overall expression is bounded above by $c_{11}\varepsilon$. Putting (3.38), (3.39), (3.40), and the above together, we have

$$\|\eta_a\|_{C^1}<c((3c_{10}+1)c_9+c_{11})\varepsilon,$$

and we are done. □

With this, we can find a C^0 bound on α_{ab}.

Proposition 3.1.15.
There is a constant $C_{\hat{A}}>0$ depending only on (Σ,g^Σ), L, $D_{\hat{A}}$, and $\varepsilon_{\max}$ such that

$$\left|\alpha_{ab}(\hat{\phi})\right|<C_{\hat{A}}\varepsilon,$$

for every $\left[D_{\hat{A}},\sqrt{\varepsilon}\hat{\phi}\right]\in\hat{\mathcal{M}}^\varepsilon_{\hat{A}}$.

Proof. By (3.15) and (3.20), we have

$$\begin{aligned}\alpha_{ab}=&\left(\!\!\left(\tilde{u}\hat{\phi}_a,\hat{\phi}_b\right)\!\!\right)+\left(\!\!\left((1+\tilde{u})\hat{\phi},\bar{\eta}_a\eta_b\hat{\phi}+\hat{\phi}_a\bar{\eta}_b+\bar{\eta}_a\hat{\phi}_b\right)\!\!\right)\\&+\frac{1}{\varepsilon}\left(\!\!\left(\partial\bar{\eta}_a-\overline{\partial}\eta_a,\partial\bar{\eta}_b-\overline{\partial}\eta_b\right)\!\!\right).\end{aligned}\tag{3.41}$$

We construct estimates for each term.

We have

$$\begin{aligned}\left|\left(\!\!\left(\tilde{u}\hat{\phi}_a,\hat{\phi}_b\right)\!\!\right)\right|&\leq\left\|\tilde{u}\hat{\phi}_a\right\|_{L^2}\left\|\hat{\phi}_b\right\|_{L^2}\\&\leq\|\tilde{u}\|_{C^0}\left\|\hat{\phi}_a\right\|_{L^2}\left\|\hat{\phi}_b\right\|_{L^2}\\&\leq\|\tilde{u}\|_{C^0}\frac{1}{2}\left(\left\|\hat{\phi}_a\right\|^2_{L^2}+\left\|\hat{\phi}_b\right\|^2_{L^2}\right)\end{aligned}$$

By (3.31) we see that $\|\tilde{u}\|_{C^0}<c_1\varepsilon$ and from (3.28) we see that $\left\|\hat{\phi}_a\right\|^2_{L^2}<c_2$. Therefore this term is bounded above by $c_3\varepsilon$.

Now note that the second term expands into terms of the form $\left(\!\!\left(f\hat{\phi},g\hat{\psi}\right)\!\!\right)$ where: $f=1$

or $\tilde{u}$; $g = \overline{\eta}_a\eta_b$, $\overline{\eta}_a$, or $\overline{\eta}_b$; and $\hat{\psi} = \hat{\phi}$, $\hat{\phi}_a$, or $\hat{\phi}_b$. We see that

$$\left|\left(\!\left(f\hat{\phi}, g\hat{\psi}\right)\!\right)\right| \leq \|f\|_{C^0}\|g\|_{C^0}\underbrace{\left\|\hat{\phi}\right\|_{L^2}}_{=1}\left\|\hat{\psi}\right\|_{L^2}$$

And similarly to the previous term, we use (3.28), (3.31), and Lemma 3.1.14 to show that terms of this form are bounded above by $c_4\varepsilon$.

For the last term, we find

$$\begin{aligned}\left|\frac{1}{\varepsilon}\left(\!\left(\partial\bar{\eta}_a - \overline{\partial}\eta_a, \partial\bar{\eta}_b - \overline{\partial}\eta_b\right)\!\right)\right| &\leq \frac{1}{\varepsilon}\left\|\partial\bar{\eta}_a - \overline{\partial}\eta_a\right\|_{L^2}\left\|\partial\bar{\eta}_b - \overline{\partial}\eta_b\right\|_{L^2} \\ &\leq \frac{1}{\varepsilon}\left(\|\partial\bar{\eta}_a\|_{L^2} + \left\|\overline{\partial}\eta_a\right\|_{L^2}\right)\left(\|\partial\bar{\eta}_b\|_{L^2} + \left\|\overline{\partial}\eta_b\right\|_{L^2}\right).\end{aligned}$$

We have $\left\|\overline{\partial}\eta_a\right\|_{L^2}, \|\partial\bar{\eta}_a\|_{L^2} \leq \|\mathrm{d}\eta_a\|_{L^2} \leq \|\eta_a\|_{H^1} \leq c_5\varepsilon$ from (3.38), thus the last term is bounded above by $c_6\varepsilon$.

Thus all terms of $\left|\alpha_{ab}(\hat{\phi})\right|$ are bounded above by estimates of the form $c\varepsilon$, thus $\left|\alpha_{ab}(\hat{\phi})\right|$ itself is bounded above by some $c_7\varepsilon$. c_7 does not depend on $\hat{\phi}$, so we set $C_{\hat{A}} := c_7$, and we are done. □

3.1.4 The bound on $\nabla\alpha$

We first clarify some notation. Let (M, g) be a Riemannian manifold covered by charts, U_i, which are all diffeomorphic to $\mathbb{R}^{2d}$ (the example one should bear in mind is the affine chart structure on $(\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}, g^{\mathrm{FS}})$). Suppose further that we have an explicit frame on each chart, $\{e_a \in \mathfrak{X}(U_i)\}$ (these are the $\hat{\phi}_a$). Then given a tensor on M, β, the *a-parametric derivative* of β in a chart is the tensor

$$\beta_{,a} := e_a(\beta) = \frac{\mathrm{d}}{\mathrm{d}t}_{t=0}(\beta \circ \Phi_{e_a,t}).$$

Where $\Phi_{e_a,t}$ is the flow of e_a. With some abuse of notation, if $f(\cdot; p) : X \to \mathbb{R}$ is some smooth function which depends smoothly on $p \in M$, we write $f_{,a}$ for the function $f_{,a}(x; p) := \frac{\mathrm{d}}{\mathrm{d}t}_{t=0} f(x; \Phi_{e_a,t}(p))$. We also use the same notation for parametric derivatives of sections of bundles. In general, if t is some expression which depends smoothly on $p \in M$, we write $(t)_{,a}$ for the a-parametric derivative of t. If we view $f(\cdot; p \in M) : X \to \mathbb{R}$ as a smooth function on $X \times M$, we have a version of the Leibniz integral rule

$$\left(\int_X f\right)_{,a} = \int_X f_{,a}.$$

We note that if $\psi(z), \xi(z) \in \Omega^0(\Sigma; L)$ are two sections which depend on $z \in M$, we have $(\langle\xi, \psi\rangle)_{,a} = \langle\xi_{,a}, \psi\rangle + \langle\xi, \psi_{,a}\rangle$. With this, we can see that $u_{,a} = u_a = \operatorname{Re}\eta_a$.

The following lemma proves useful, though we relegate the proof to §A.3 (Lemma A.3).

Lemma 3.1.16.

Let $\gamma \in C^\infty\left(\mathbb{P}^d; \left(T^*\mathbb{P}^d\right)^{\otimes r}\right)$ *be some* r*-tensor. If in every affine coordinate chart, the components of* γ*,* $\gamma_{i_1\cdots i_r, i_{r+1}}(p)$*, are uniformly bounded in* C^1 *by some* $C' > 0$*. Then there is a constant* $C > 0$ *which only depends on* $\deg L - 1$ *such that*

$$\|\nabla\gamma\|_{C^0} < CC'.$$

In particular, if γ *is a two tensor and we wish to find a* C^0*-estimate for* $\nabla\gamma$*, it is sufficient to find an estimate* $\left|\gamma_{ab,c}(p)\right| < C_{\hat{A}}$ *for every affine coordinate chart* U_i*, every* $p \in U_i$*, and every choice of indices.*

As with Lemma 3.1.9, one should view this as exploiting boundedness properties of the Fubini-Study metric when it is expressed in affine coordinates; in this case, boundedness of the associated Christoffel symbols in addition to boundedness of the metric itself.

With this, it is clear that our goal is to prove the following.

Proposition 3.1.17.
There is a constant $C_{\hat{A}} > 0$ *depending only on* $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$*,* $D_{\hat{A}}$*, and* $\varepsilon_{\max}$ *such that*

$$\left|\alpha_{ab,c}(\hat{\phi})\right| < C_{\hat{A}}\varepsilon,$$

for every $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] \in \hat{\mathcal{M}}^\varepsilon_{\hat{A}}$.

This result will require some intermediary lemmas.

Lemma 3.1.18.
Let $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] \in \hat{\mathcal{M}}^\varepsilon$*, and let* u *solve* $\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + \varepsilon e^u\left|\hat{\phi}\right|^2 = 0$ (2.15)*. Then there exist constants* $C_{\hat{A},i}$ *depending only on* (Σ, g^Σ)*,* (L, H)*,* $D_{\hat{A}}$*, and* $\varepsilon_{\max}$ *such that*

- $\tilde{u}_{,a} = (e^u - 1)_{,a}$ *satisfies the bound*

$$\|\tilde{u}_{,a}\|_{C^0} < C_{\hat{A},1}\varepsilon. \tag{3.42}$$

- $\hat{\phi}_{a,b}$ *satisfies the bound*

$$\left\|\hat{\phi}_{a,b}\right\|_{H^1} < C_{\hat{A},2}. \tag{3.43}$$

Proof. For the first result, we have $\tilde{u}_{,a} = e^u u_a$. Then $\|\tilde{u}_{,a}\|_{C^0} \leq e^{\|u\|_{C^0}}\|u_a\|_{C^0}$. The estimate then follows from (3.30) and (3.34) (as $u_a = \operatorname{Re}\eta_a$).

We give the proof of the estimate (3.43) in §A.2 (Lemma A.2.8 on page 106). □

Lemma 3.1.19.
Let $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] \in \hat{\mathcal{M}}^\varepsilon$*, and let* u *solve* $\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + \varepsilon e^u\left|\hat{\phi}\right|^2 = 0$ (2.15)*. Then there exists a*

constant $C_{\hat{A}}$ depending only on (Σ, g^{Σ}), (L, H), $D_{\hat{A}}$, and $\varepsilon_{\max}$ such that $\eta_{a,b}$ satisfies the bound

$$\|\eta_{a,b}\|_{C^1} < C_{\hat{A}}\varepsilon. \tag{3.44}$$

Proof. The construction of this estimate is structurally similar to that of (3.34); we first define $\eta_{a,b}$. Recall that η_a is the unique solution to the PDE (3.16)

$$\Delta\eta_a + \varepsilon e^u\left|\hat{\phi}\right|^2\eta_a = -\varepsilon e^u\left(\frac{1}{2}\left\langle\hat{\phi},\hat{\phi}_a\right\rangle + \mathbf{i}\left\langle\mathbf{i}\hat{\phi},\hat{\phi}_a\right\rangle\right).$$

It then follows that $\eta_{a,b}$ is the unique solution to the PDE

$$\Delta\eta_{a,b} + \varepsilon e^u\left|\hat{\phi}\right|^2\eta_{a,b} = -\varepsilon\left(e^u\left|\hat{\phi}\right|^2\right)_{,b}\eta_a - \varepsilon\left(e^u\left(\frac{1}{2}\left\langle\hat{\phi},\hat{\phi}_a\right\rangle + \mathbf{i}\left\langle\mathbf{i}\hat{\phi},\hat{\phi}_a\right\rangle\right)\right)_{,b}. \tag{3.45}$$

Sobolev embedding and the elliptic regularity theorem then imply that

$$\|\eta_{a,b}\|_{C^1} \le c\|\eta_{a,b}\|_{H^3} \le c'\left(\|\Delta\eta_{a,b}\|_{H^1} + \|\eta_{a,b}\|_{L^2}\right),$$

so we require an H^1-estimate (which yields an L^2-estimate) on $\eta_{a,b}$ to begin, this is provided by (3.32). As in the construction of (3.34), we need to find H^1-estimates on the real and complex parts of $\eta_{a,b}$ separately. The constructions of the two bounds are similar, so we only bound the real part. Applying Lemma 3.1.11, we have

$$\|\mathrm{Re}(\eta_{a,b})\|_{H^1} \le c_1\left((1+\mu\|a\|_{L^2})\left(\|b\|_{L^2} + \mu\|a\|_{L^2}\left|\int_\Sigma b\right|\right) + \mu\left|\int_\Sigma b\right|\right),$$

where $a = \varepsilon e^u\left|\hat{\phi}\right|^2$, $\mu = \frac{1}{\|a\|_{L^1}}$, and $b = -\varepsilon\left(e^u\left|\hat{\phi}\right|^2\right)_{,b}\mathrm{Re}(\eta_a) - \varepsilon\left(e^u\left(\frac{1}{2}\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right)\right)_{,b}$.

From (3.35), we can see that $\mu\|a\|_{L^2} \le c_2$. Moreover, the construction of (3.38) shows that if we can find estimates $\|b\|_{L^2} \le f(\varepsilon)\varepsilon$ and $\left|\int_\Sigma b\right| \le g(\varepsilon)\varepsilon^2$ for $f, g \in \mathbb{R}[\varepsilon]$, we have an estimate $\left\|\eta_{a,b}\right\|_{H^1} \le c''\varepsilon$.

We begin by expanding $\frac{-b}{\varepsilon}$ and bounding $\left|\int_\Sigma \frac{-b}{\varepsilon}\right|$,

$$\begin{aligned}
\frac{-b}{\varepsilon} &= \left(e^u\left|\hat{\phi}\right|^2\right)_{,b}\mathrm{Re}(\eta_a) + \left\{e^u\left(\frac{1}{2}\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right)\right\}_{,b} \\
&= \left(u_b\left|\hat{\phi}\right|^2 + 2\left\langle\hat{\phi},\hat{\phi}_b\right\rangle\right)e^u\,\mathrm{Re}(\eta_a) + \frac{1}{2}\left\{(1+\tilde{u})\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right\}_{,b} \\
\left|\int_\Sigma \frac{-b}{\varepsilon}\right| &= \left|\int_\Sigma \left(u_b\left|\hat{\phi}\right|^2 + 2\left\langle\hat{\phi},\hat{\phi}_b\right\rangle\right)e^u\,\mathrm{Re}(\eta_a) + \frac{1}{2}\int_\Sigma\left\{(1+\tilde{u})\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right\}_{,b}\right| \\
&\le \underbrace{\int_\Sigma\left|\left(u_b\left|\hat{\phi}\right|^2 + 2\left\langle\hat{\phi},\hat{\phi}_b\right\rangle\right)\right|}_{=:t_1}\|e^u\,\mathrm{Re}(\eta_a)\|_{C^0} + \frac{1}{2}\underbrace{\left|\left\{\int_\Sigma(1+\tilde{u})\left\langle\hat{\phi},\hat{\phi}_a\right\rangle\right\}_{,b}\right|}_{=:t_2}.
\end{aligned}$$

At this point, it is convenient to bound t_1 and t_2 separately. For the first we have

$$\begin{aligned} t_1 &\leq \int_\Sigma |u_b| \left|\hat{\phi}\right|^2 + 2\int_\Sigma \left|\left\langle \hat{\phi}, \hat{\phi}_b \right\rangle\right| \\ &\leq \|u_b\|_{C^0} \left\|\hat{\phi}\right\|_{L^2}^2 + 2\left(\left(\left|\hat{\phi}\right|, \left|\hat{\phi}_b\right|\right)\right) \\ &\leq \|u_b\|_{C^0} \left\|\hat{\phi}\right\|_{L^2}^2 + 2\left\|\hat{\phi}\right\|_{L^2} \left\|\hat{\phi}_b\right\|_{L^2}. \end{aligned}$$

We have the following estimates: $\left\|\hat{\phi}\right\|_{L^2} = 1$, $\|u_b\|_{C^0} \leq \|\eta_b\|_{C^0} \leq c'\varepsilon$ (3.34), and $\left\|\hat{\phi}_b\right\|_{L^2} \leq c''$ (3.28). So $t_1 \leq c_3\varepsilon + c_4$. Then as $\|\eta_a\|_{C^0} \leq c\varepsilon$ (3.34), $e^{\|u\|_{C^0}} \leq c'$ (3.30), we have $t_1 \|e^u \operatorname{Re}(\eta_a)\|_{C^0} \leq (c_3\varepsilon + c_4)cc'\varepsilon \leq (c_5\varepsilon + c_6)\varepsilon$.

For the second we have

$$\begin{aligned} t_2 &= \left| \left\{ \underbrace{\left(\left(\hat{\phi}, \hat{\phi}_a\right)\right)}_{=0} \right\}_{,b} + \int_\Sigma \left\{ \tilde{u} \left\langle \hat{\phi}, \hat{\phi}_a \right\rangle \right\}_{,b} \right| \\ &= \left| \int_\Sigma \left(\tilde{u}_{,b} \left\langle \hat{\phi}, \hat{\phi}_a \right\rangle + \tilde{u}\left(\left\langle \hat{\phi}_b, \hat{\phi}_a \right\rangle + \left\langle \hat{\phi}, \hat{\phi}_{a,b} \right\rangle \right) \right) \right| \\ &\leq \|\tilde{u}_{,b}\|_{C^0} \left\|\hat{\phi}\right\|_{L^2} \left\|\hat{\phi}_a\right\|_{L^2} + \|\tilde{u}\|_{C^0} \left(\left\|\hat{\phi}_b\right\|_{L^2} \left\|\hat{\phi}_a\right\|_{L^2} + \left\|\hat{\phi}\right\|_{L^2} \left\|\hat{\phi}_{a,b}\right\|_{L^2} \right). \end{aligned}$$

We have $\|\tilde{u}\|_{C^0} \leq c\varepsilon$ (3.31), $\left\|\tilde{u}_{,b}\right\|_{C^0} \leq c'\varepsilon$ (3.42), $\left\|\hat{\phi}\right\|_{L^2} \leq \left\|\hat{\phi}\right\|_{C^0} \sqrt{\operatorname{Vol}\Sigma} \leq c''$ (3.26), $\left\|\hat{\phi}_a\right\|_{L^2} \leq c'''$ (3.28), and $\left\|\hat{\phi}_{a,b}\right\|_{L^2} \leq c''''$ (3.43). Thus we have $t_2 \leq c_7\varepsilon$. Finally, this yields $\left|\int_\Sigma b\right| = \left|\int_\Sigma \frac{-b}{\varepsilon}\right|\varepsilon \leq \left(c_5\varepsilon + c_6 + \frac{1}{2}c_7\right)\varepsilon^2 =: (c_8 + c_9\varepsilon)\varepsilon^2$.

Similarly, we have $\left\|\frac{-b}{\varepsilon}\right\|_{L^2} \leq \left\|\frac{-b}{\varepsilon}\right\|_{H^1}$ and

$$\begin{aligned} \left\|\frac{-b}{\varepsilon}\right\|_{H^1} &\leq 3\left\| u_b \left|\hat{\phi}\right|^2 + 2\left\langle \hat{\phi}, \hat{\phi}_b \right\rangle \right\|_{H^1} \|e^u \operatorname{Re}(\eta_a)\|_{C^1} + \frac{1}{2}\left\| \left\{ e^u \left\langle \hat{\phi}, \hat{\phi}_a \right\rangle \right\}_{,b} \right\|_{H^1} \\ &\leq 3\left(3\|u_b\|_{C^1} \left\| \left|\hat{\phi}\right|^2 \right\|_{H^1} + 2\left\| \left\langle \hat{\phi}, \hat{\phi}_b \right\rangle \right\|_{H^1} \right) \|e^u \operatorname{Re}(\eta_a)\|_{C^1} \\ &\quad + \frac{1}{2}\left\| e^u \left(u_b \left\langle \hat{\phi}, \hat{\phi}_a \right\rangle + \left\langle \hat{\phi}_b, \hat{\phi}_a \right\rangle + \left\langle \hat{\phi}, \hat{\phi}_{a,b} \right\rangle \right) \right\|_{H^1} \\ &\leq 3\left(3\|u_b\|_{C^1} \left\| \left|\hat{\phi}\right|^2 \right\|_{H^1} + 6\left\|\hat{\phi}\right\|_{C^1} \left\|\hat{\phi}_b\right\|_{H^1} \right) \|e^u\|_{C^1} \|\operatorname{Re}(\eta_a)\|_{C^1} \\ &\quad + \frac{3}{2}\|e^u\|_{C^1} \left(9\|u_b\|_{C^1} \left\|\hat{\phi}\right\|_{C^1} \left\|\hat{\phi}_a\right\|_{H^1} + \left\| \left\langle \hat{\phi}_b, \hat{\phi}_a \right\rangle \right\|_{H^1} + 3\left\|\hat{\phi}\right\|_{C^1} \left\|\hat{\phi}_{a,b}\right\|_{H^1} \right). \end{aligned}$$

As before, repeated application of the estimates (3.30), (3.34), (3.26), (3.28), (3.43) yield the bound $\left\|\frac{-b}{\varepsilon}\right\|_{H^1} \leq c$ and thus $\|b\|_{L^2} \leq c_{10}\varepsilon$. As we also have $\left|\int_\Sigma b\right| \leq (c_8 + c_9\varepsilon)\varepsilon^2$, we conclude via Lemma 3.1.11 that $\left\|\operatorname{Re}\eta_{a,b}\right\|_{H^1} \leq c_{11}\varepsilon$.

From (3.45) and (3.25), we have $\left\|\operatorname{Re}\eta_{a,b}\right\|_{C^1} \leq \left((3\|a\|_{C^1} + 1)\left\|\operatorname{Re}\eta_{a,b}\right\|_{H^1} + \|b\|_{H^1} \right)$. By the above, we have $\|b\|_{H^1} \leq c\varepsilon$ and $\left\|\operatorname{Re}\eta_{a,b}\right\|_{H^1} \leq c_{11}\varepsilon$; and from (3.40) we have $\|a\|_{C^1} \leq \max(1, c\varepsilon)c'\varepsilon$. We therefore conclude that $\left\|\operatorname{Re}\eta_{a,b}\right\|_{C^1} < c\varepsilon$ and we are done. □

Proof. *Proposition 3.1.17.* Writing out $\alpha_{ab,c}$ in full would be cumbersome, so we group the terms into families and consider the parametric derivatives of those. Recall that we have

$$\alpha_{ab} = \Big(\!\Big(\tilde{u}\hat{\phi}_a, \hat{\phi}_b\Big)\!\Big) + \Big(\!\Big(\hat{\phi}, (1+\tilde{u})\Big(\bar{\eta}_a\eta_b\hat{\phi} + \hat{\phi}_a\bar{\eta}_b + \bar{\eta}_a\hat{\phi}_b\Big)\Big)\!\Big) + \frac{1}{\varepsilon}\big(\!\big(\partial\bar{\eta}_a - \overline{\partial}\eta_a, \partial\bar{\eta}_b - \overline{\partial}\eta_b\big)\!\big).$$

Expanding out the metric fully, we see that each term is either of the form

- $(\!(\psi, f\xi)\!)$ where f is a finite product of elements of $\{\tilde{u}, \eta_a, \bar{\eta}_a\}$ and $\psi, \xi \in \Big\{\hat{\phi}, \hat{\phi}_a\Big\}$, or
- $\frac{1}{\varepsilon}(\!(\psi, \xi)\!)$ where $\psi, \xi \in \big\{\partial\bar{\eta}_a, \overline{\partial}\eta_a\big\}$.

We suppress displaying all combinations of indices but note that as α is a covariant two-tensor, each term must have two covariant indices. It follows that $\alpha_{ab,c}$ has terms of the form

- $(\!(\psi, f\xi)\!)$ where f is a finite product of elements of $\Big\{\tilde{u}, \tilde{u}_{,a}, \eta_a, \eta_{a,b}, \bar{\eta}_a, \bar{\eta}_{a,b}\Big\}$ and $\psi, \xi \in \Big\{\hat{\phi}, \hat{\phi}_a, \hat{\phi}_{a,b}\Big\}$, or
- $\frac{1}{\varepsilon}(\!(\psi, \xi)\!)$ where $\psi, \xi \in \Big\{\partial\bar{\eta}_a, \partial\bar{\eta}_{a,b}, \overline{\partial}\eta_a, \overline{\partial}\eta_{a,b}\Big\}$.

Again, we suppress all combinations of indices, but note that each term must have three covariant indices, exactly one of which must be a parametric derivative.

We can see that

$$\begin{aligned}|(\!(\psi, f\xi)\!)| &\leq \|f\|_{C^0}\|\psi\|_{L^2}\|\xi\|_{L^2} \\ &\leq \frac{1}{2}\|f\|_{C^0}\Big(\|\psi\|_{L^2}^2 + \|\xi\|_{L^2}^2\Big).\end{aligned}$$

For $\psi \in \Big\{\hat{\phi}, \hat{\phi}_a, \hat{\phi}_{a,b}\Big\}$ we have $\|\psi\|_{L^2} \leq c$ by either (3.26), (3.28), or (3.43). For f a finite product of $\Big\{\tilde{u}, \tilde{u}_{,a}, \eta_a, \eta_{a,b}, \bar{\eta}_a, \bar{\eta}_{a,b}\Big\}$ we have $\|f\|_{C^0} \leq c'\varepsilon$ by either (3.42), (3.34), or (3.1.19). Thus $|(\!(\psi, f\xi)\!)| \leq c''\varepsilon$.

We also see that $\big|\frac{1}{\varepsilon}(\!(\psi, \xi)\!)\big| \leq \frac{1}{\varepsilon}\|\psi\|_{L^2}\|\xi\|_{L^2} \leq \frac{1}{\varepsilon}\operatorname{Vol}\Sigma\|\psi\|_{C^0}\|\xi\|_{C^0}$. Since we have $\psi, \xi \in \Big\{\partial\bar{\eta}_a, \partial\bar{\eta}_{a,b}, \overline{\partial}\eta_a, \overline{\partial}\eta_{a,b}\Big\}$, we see that $\|\psi\|_{C^0}, \|\xi\|_{C^0} \leq c\varepsilon$ by either (3.34) or (3.1.19). This yields an estimate $\big|\frac{1}{\varepsilon}(\!(\psi, \xi)\!)\big| \leq c'\varepsilon$.

Thus all terms of $\alpha_{ab,c}$ are bounded above by $c\varepsilon$, and we are done. □

We can now prove our main result.

Proof. *Theorem 3.1.4.* By Lemma 3.1.8 and Lemma 3.1.16, it suffices to find C^0-estimates on charts of α and $\nabla\alpha$ of the required format. These are given by Proposition 3.1.15 and Proposition 3.1.17 respectively and we are done. □

3.2 Approximating geodesics

While an intriguing result in its own right, the main utility of Theorem 3.1.4 is to study the behaviour of moduli space dynamics. In particular, we would like to understand the behaviour of geodesics of the vortex metric close to the dissolving limit. We do this with the following result

Theorem 3.2.1.
Suppose $\mathcal{M}^{\varepsilon}_{\hat{A}}$ is a complete linear system which is totally geodesic*, i.e. any geodesic on $(\mathcal{M}^{\varepsilon}_{\hat{A}}, g^{\varepsilon}_{\hat{A}})$ with the restricted and rescaled vortex metric is a geodesic of $(\mathcal{M}^{\varepsilon}, g^{\varepsilon})$. Note that this property is independent of ε.*

Let $\gamma_0 : [0, t_1] \to \hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$ be a geodesic of the Fubini-Study metric with initial velocity $v_0 \in T_{p_0}\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}} \cong T_{p_0}\mathcal{M}^{\varepsilon}_{\hat{A}}$. Then there is some $\varepsilon_0 > 0$, and some continuous function

$$\gamma_\bullet : [0, \varepsilon_0) \times [0, t_1] \to \hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}, \tag{3.46}$$

with the following properties:

- *$\gamma_\bullet$ is continuous, and*
- *for every $0 \leq \varepsilon < \varepsilon_0$, $\gamma_\varepsilon : [0, t_1] \to \hat{\mathcal{M}}^{\varepsilon}_{\hat{A}} \cong \mathcal{M}^{\varepsilon}_{\hat{A}}$ is the geodesic of $(\mathcal{M}^{\varepsilon}, g^{\varepsilon})$ with initial velocity v_0 at p_0.*

To prove this, we require the following theorem from Hartman [29, pp. 93–5].

Theorem 3.2.2.
Let $f(t, y, z)$ be continuous on an open (t, y, z)-set $E \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^e$ with the property that for every $(t_0, y_0, z) \in E$, the initial value problem

$$y' = f(t, y, z), \qquad y(t_0) = y_0,$$

has a unique solution $y(t) = \eta(t, t_0, y_0, z)$, defined on a maximal interval $\omega_-(t_0, y_0, z) < t < \omega_+(t_0, y_0, z)$. Then ω_+ is lower semicontinuous and ω_- is upper semicontinuous on E, and $\eta(t, t_0, y_0, z)$ is continuous on the set

$$\{(t, t_0, y_0, z) \mid \omega_-(t_0, y_0, z) < t < \omega_+(t_0, y_0, z),\ (t_0, y_0, z) \in E\}.$$

Hartman takes lower semi-continuity of ω_+ at some (t_0, y_0, z_0) to mean that if $t_1 < \omega_+(t_0, y_0, z_0)$, then $t_1 \leq \omega_+(t_2, y_2, z_2)$ for all (t_2, y_2, z_2) in some open neighbourhood of (t_0, y_0, z_0). Upper semi-continuity of ω_- is defined analogously, yet this is not important to us as we do not consider backwards time existence for our geodesics.

Proof. Theorem 3.2.1. Consider some dimension n complete linear system, $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$, and let $g(0)$ be the Fubini-Study metric on it. For $\varepsilon \in (0, \varepsilon_{\max}]$, we let $g(\varepsilon) = g^{\varepsilon}$ (3.15) be the

rescaled pullback of the vortex metric. By Theorem 3.1.4, there is some $C > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_{\max}$, $\|g(\varepsilon) - g(0)\|_{C^1} < C\varepsilon$.

Recall that geodesics on some arbitrary Riemannian manifold (M, g) are integral curves of the vector field

$$\begin{aligned} G &: TM \to TTM, \\ G(p, v) &= v^i \partial_{p^i} + v^j v^k \Gamma^i{}_{jk} \partial_{v^i}, \end{aligned}$$

where $\Gamma^i{}_{jk}$ are the Christoffel symbols of g. We then define

$$\begin{aligned} G_\bullet &: [0, \varepsilon_{\max}) \times T\hat{\mathcal{M}}^\varepsilon_{\hat{A}} \to TT\hat{\mathcal{M}}^\varepsilon_{\hat{A}}, \\ G_\varepsilon(p, v) &\mapsto v^i \partial_{p^i} + v^j v^k \Gamma(\varepsilon)^i{}_{jk} \partial_{v^i}, \end{aligned} \tag{3.47}$$

where $\Gamma(\varepsilon)^i{}_{jk}$ are the Christoffel symbols associated to $g(\varepsilon)$.

We show that this defines a continuously varying one-parameter family of smooth vector fields. Combined with the fact that we have short time existence and uniqueness for each integral curve, we can then apply Theorem 3.2.2, yielding the required function $\gamma_\bullet$.

Given that we fix our initial time and initial conditions across all parameter values, we can actually treat ω_+ in Theorem 3.2.2 as a function $\omega_+(\varepsilon)$. Global existence and uniqueness of geodesics of the Fubini-Study metric then implies we have $t_1 < \omega_+(0)$ for all $t_1 > 0$, so for all ε in some open neighbourhood of 0, we have $\omega_+(\varepsilon) \geq t_1$, i.e. we have the existence and continuity of $\gamma_\bullet$ on the region $[0, \varepsilon_0) \times [0, t_1]$. The fact that $\hat{\mathcal{M}}^\varepsilon_{\hat{A}}$ is totally geodesic then implies that each curve, γ_ε, is the required geodesic and we are done.

To show that $G_\bullet$ is continuous, it suffices to show that the $\Gamma(\varepsilon)^i{}_{jk}$ are continuous in ε. We have

$$\Gamma(\varepsilon)^i{}_{jk} = \frac{1}{2} g(\varepsilon)^{im} \big(g(\varepsilon)_{mk,j} + g(\varepsilon)_{mj,k} - g(\varepsilon)_{jk,m} \big),$$

so from (3.15) it is clear that $\Gamma(\varepsilon)^i{}_{jk}$ is continuous for $\varepsilon > 0$. For $\varepsilon = 0$, we compute

$$\begin{aligned} \big| \Gamma(\varepsilon)_{mjk} - \Gamma(0)_{mjk} \big| &\leq \frac{1}{2} \big(\big| g(\varepsilon)_{mk,j} - g(0)_{mk,j} \big| + \big| g(\varepsilon)_{mj,k} - g(0)_{mj,k} \big| \big) \\ &\quad + \frac{1}{2} \big| g(\varepsilon)_{jk,m} - g(0)_{jk,m} \big| \\ &\leq \frac{3}{2} \| g(\varepsilon) - g(0) \|_{C^1} \\ &< \frac{3}{2} C\varepsilon. \end{aligned}$$

We also require an estimate on $g(\varepsilon)^{-1} - g(0)^{-1}$. Recall we have $\alpha(\varepsilon) := g(\varepsilon) - g(0)$. We have

$$\begin{aligned} g(\varepsilon)^{ij} - g(0)^{ij} &= -g(\varepsilon)^{ik} (g(\varepsilon)_{kl} - g(0)_{kl}) g(0)^{lj} \\ \left| g(\varepsilon)^{-1} - g(0)^{-1} \right|^2 &= g(0)_{ir} g(0)_{ju} \Big(g(\varepsilon)^{ik} \alpha(\varepsilon)_{kl} g(0)^{lj} \Big) \big(g(\varepsilon)^{rs} \alpha(\varepsilon)_{st} g(0)^{tu} \big) \\ &= g(0)_{ri} g(\varepsilon)^{ik} \alpha(\varepsilon)_{kl} g(0)^{lt} \alpha(\varepsilon)_{ts} g(\varepsilon)^{sr}. \end{aligned}$$

Henceforth we shall compute pointwise tensor norm quantities in a $g(0)$-orthonormal frame. Such a frame can always be constructed at any point, and this choice does not affect the validity of estimates of coordinate-free objects. With this framing, we see that

$$\left|a^i{}_j b^j{}_i\right| \leq \sum_{i,j} \left|a^i{}_j\right| \left|b^j{}_i\right| \leq \sum_{i,j} \|a\|_{C^0} \|b\|_{C^0} \leq n^2 \|a\|_{C^0} \|b\|_{C^0}. \tag{3.48}$$

So

$$\left|g(\varepsilon)^{-1} - g(0)^{-1}\right|^2 \leq n^6 \|g(0)\|_{C^0} \left\|g(0)^{-1}\right\|_{C^0} \left\|g(\varepsilon)^{-1}\right\|_{C^0}^2 \|\alpha(\varepsilon)\|_{C^0}^2$$

$$\left|g(\varepsilon)^{-1} - g(0)^{-1}\right| \leq n^3 \underbrace{\sqrt{\|g(0)\|_{C^0} \left\|g(0)^{-1}\right\|_{C^0}}}_{=:C'} \left\|g(\varepsilon)^{-1}\right\|_{C^0} \|\alpha(\varepsilon)\|_{C^0}.$$

We see n, $\|g(0)\|_{C^0}$, and $\left\|g(0)^{-1}\right\|_{C^0}$ are finite constants which are independent of ε. As mentioned previously, we have an estimate $\|\alpha(\varepsilon)\|_{C^0} < C\varepsilon$, this yields the estimate

$$\left\|g(\varepsilon)^{-1} - g(0)^{-1}\right\|_{C^0} < n^3 C C' \left\|g(\varepsilon)^{-1}\right\|_{C^0} \varepsilon. \tag{3.49}$$

It is therefore enough to show that $\left\|g(\varepsilon)^{-1}\right\|_{C^0}$ is bounded above by some constant which is independent of ε.

Now recall that we are working (pointwise) in a $g(0)$-orthonormal frame, thus $g(\varepsilon)\big|_p$ is symmetric in this frame and thus diagonalisable by orthogonal matrices. WLOG we assume our initial frame is such that $g(\varepsilon)\big|_p$ is diagonal, thus $g(\varepsilon)^{-1}\big|_p$ is also diagonal. Now note that for any tensor such that $a\big|_p$ is diagonal in this frame, $|a|^2\big|_p = \sum_i \left(a_{ii}\big|_p\right)^2$. So supposing $g(\varepsilon)\big|_p = \operatorname{diag}(a_1(\varepsilon), a_2(\varepsilon), \ldots, a_n(\varepsilon))$, we have $\left|g(\varepsilon)^{-1}\right|\big|_p \leq \sum_i |a_i(\varepsilon)|^{-1}$. Now as $g(\varepsilon) - g(0)\big|_p = \operatorname{diag}(a_1(\varepsilon)-1, \ldots a_n(\varepsilon)-1)$, we also have the estimate $|g(\varepsilon) - g(0)|^2\big|_p = \sum_i (a_i(\varepsilon) - 1)^2 \leq C^2\varepsilon^2$. So for every i, $|a_i(\varepsilon) - 1| < C\varepsilon$ and so $(1 + C\varepsilon)^{-1} < (a_i(\varepsilon))^{-1} < (1 - C\varepsilon)^{-1}$. So if $\varepsilon < \frac{1}{2}C^{-1} =: \varepsilon_1$, we have $\left|g(\varepsilon)^{-1}\right|\big|_p \leq n(1 - C\varepsilon)^{-1} < 2n$. Since our initial choice of p was arbitrary and the expression we have bounded is coordinate free, this yields a global estimate on $g(\varepsilon)^{-1}$, $\left\|g(\varepsilon)^{-1}\right\|_{C^0} < 2n$. So if we restrict ourselves to $0 < \varepsilon < \varepsilon_1$ and apply this global estimate to (3.49) we have

$$\left\|g(\varepsilon)^{-1} - g(0)^{-1}\right\|_{C^0} \leq C''\varepsilon,$$

where C'' is a fixed constant which does not depend on ε.

So for $0 < \varepsilon < \varepsilon_1$ we then have

$$\begin{aligned}
\left|\Gamma(\varepsilon)^i{}_{jk} - \Gamma(0)^i{}_{jk}\right| &\leq \left|g(\varepsilon)^{il}\left(\Gamma(\varepsilon)_{ljk} - \Gamma(0)_{ljk}\right)\right| + \left|\left(g(\varepsilon)^{il} - g(0)^{il}\right)\Gamma(0)_{ljk}\right| \\
&\leq n\left\|g(\varepsilon)^{-1}\right\|_{C^0}\left\|\Gamma(\varepsilon)_{ljk} - \Gamma(0)_{ljk}\right\|_{C^0} \\
&+ n\left\|g(\varepsilon)^{-1} - g(0)^{-1}\right\|_{C^0}\underbrace{\left\|\Gamma(0)_{ijk}\right\|_{C^0}}_{=:C_3} \\
&\leq 3n^2C\varepsilon + rC''C_3\varepsilon
\end{aligned}$$

so $\Gamma(\varepsilon)^i{}_{jk}$ is also continuous at 0. Thus $\Gamma(\varepsilon)^i{}_{jk}$ is continuous on $\varepsilon \geq 0$, and so $G_\bullet$ is also continuous over the same range and we are done. □

We note that Theorem 3.2.1 only applies to totally geodesic submanifolds of the moduli space. While it is by no means the only construction of such submanifolds, we consider the following method (see e.g. [40, p. 262]).

Lemma 3.2.3.
Let (M, g) be a Riemannian manifold and let T be some non-trivial isometry of M.

Then every connected component of the fixed point set of T,

$$M^T = \{p \in M \mid T(p) = p\},$$

is a smoothly embedded totally geodesic submanifold of M.

We can apply this to our particular case of line bundles over Riemann surfaces with the following theorem.

Theorem 3.2.4.
Let $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ be some Hermitian line bundle over a Riemannian surface and suppose G is some non-trivial subgroup of $\mathrm{Isom}(\Sigma)$ *with the further property that $T^*H = H$ for every $T \in G$. If G is a Lie group, we assume it is smoothly embedded.*

Then G defines a subgroup of $\mathrm{Isom}(\mathcal{M}^\varepsilon)$ *and*

$$(\mathcal{M}^\varepsilon)^G = \{p \in \mathcal{M}^\varepsilon \mid \forall T \in G, T(p) = p\},$$

is a totally geodesic submanifold of the moduli space.

We do not give a proof here, but note that this results from general considerations about symmetries of the energy functional/Lagrangian (see e.g. [46, Ch. 4.4–5]).

In general, an arbitrary complete linear system is unlikely to be a fixed point set of some isometry of the moduli space. Yet there are classes of Riemann surfaces which either have large isometry groups, or those which have distinguished isometries from their construction. Two examples of the latter are elliptic and hyperelliptic curves. In both cases

the curve is equipped with a (hyper)elliptic involution; under the additional assumption that the curve is flat, this involution is an isometry which has certain complete linear systems as fixed point sets. With an appropriate choice of Hermitian metric, we then have that the complete linear system in question is a totally geodesic submanifold.

We consider the case of 2-vortices on elliptic curves in chapter 5, explicitly constructing approximations to geodesics and the Hermitian metric. The case of hyperelliptic curves was studied by Rink [58], who constructs many (not necessarily complete) linear systems.

Chapter 4

Vortices in the large degree limit

Having shown that in the dissolving limit the vortex metric is well approximated by the Fubini-Study metric on complete linear systems, it is natural to then ask whether that approximation is valid in other parametric limits. In this chapter we study one such limit, when the degree of the line bundle is very large. The original motivation for studying this limit comes from statistical mechanics where one considers a vortex gas [48], with the spectrum of the Laplacian on the vortex moduli space being tied to the quantum statistical mechanics of vortices [45].

We limit ourselves to *axially symmetric vortices* on a round sphere, i.e. vortices where $|\phi|^2$ is invariant under rotations of $\Sigma \cong \mathbb{P}^1$ around a fixed axis. The cores of axially symmetric vortices (where ϕ vanishes) must therefore be at either pole of the axis. We shall focus on two particular species: *unipolar* vortices, where the cores are concentrated at a single pole of the axis; and *bipolar* vortices, where the cores are equally distributed at both poles of the axis. The main object of study for this chapter is the PSC equation, $\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + \varepsilon e^u \left|\hat{\phi}\right|^2 = 0$ (2.15); we recall that this defines the gauge transformation mapping a pseudovortex to a vortex (see Theorem 2.2.8), and is deeply tied to the validity of the pseudovortex approximation (Theorem 3.1.4).

4.1 Axially symmetric pseudovortices

For this chapter, let $(\Sigma, g^\Sigma, \omega^\Sigma)$ be the unit round 2-sphere, as usual, L is a degree N line bundle. If z and w are stereographic coordinates defined, respectively, on the complement of the north and south poles, the transition map takes the form

$$\phi_S(z) \mapsto z^{-N}\phi_S(z) = w^N \phi_S(w) = \phi_N(w).$$

Here, ϕ_S is trivialisation of a section on the south chart ($\mathbb{P}^1 \setminus \{[0:1]\}$) and ϕ_N is that on the complementary north chart.

We let H be the unique Hermitian metric which is invariant under rotations of Σ. We can write out an expression for H as follows. In the $N = 1$ case, we can see that such a metric

must be of the form $H_z(\phi,\psi) = \left(\frac{1}{1+|z|^2}\right)\bar{\phi}(z)\psi(z)$. We find the expression for higher N by taking powers of H, yielding

$$H_z(\phi,\psi) = \left(\frac{1}{1+|z|^2}\right)^N \bar{\phi}(z)\psi(z). \tag{4.1}$$

A convenient coordinate system for the following results are cylindrical coordinates: i.e. coordinates (x,φ) where $x \in [-1,1]$ is the height coordinate and $\varphi \in S^1$ is the azimuthal angle. We can write

$$z(\varphi,x) = \sqrt{\frac{1+x}{1-x}}e^{\mathbf{i}\varphi},$$

and so H then becomes

$$H_{(x,\varphi)}(\phi,\psi) := \left(\frac{1-x}{2}\right)^N \bar{\phi}(x,\varphi)\psi(x,\varphi). \tag{4.2}$$

Note that $\mathrm{Jac}(\Sigma)$ is trivial, so the Higgs field uniquely determines points in the moduli space; indeed, in these coordinates we can write down explicit expressions for the Higgs fields of some pseudovortices. As mentioned previously, we restrict our attention to axially symmetric pseudovortices, those which vanish with degree $N-k$ at $x=-1$ and degree k at $x=1$. The ultimate utility of imposing this axial symmetry is that with it, we can reduce the defining PDE of the pseudovortex-vortex diffeomorphism, $\Delta u - \frac{\varepsilon}{\mathrm{Vol}\,\Sigma} + \varepsilon e^u\left|\hat{\phi}\right|^2 = 0$ (2.15), to an ODE in x.

We define

$$\hat{\phi}_{N,k}(z) := \sqrt{\alpha_{N,k}}z^{N-k} = \sqrt{\alpha_{N,k}}\left(\sqrt{\frac{1+x}{1-x}}\right)^{N-k} e^{\mathbf{i}(N-k)\varphi}, \tag{4.3}$$

where $\alpha_{N,k} > 0$ is a normalisation constant which ensures that $\left\|\hat{\phi}_{N,k}\right\|_{L^2} = 1$. This section is clearly holomorphic and vanishes with degree $N-k$ at the south pole. In the complementary coordinate w, we have $\hat{\phi}_{N,k}(w) = \sqrt{\alpha_{N,k}}w^{k-N}w^N = \sqrt{\alpha_{N,k}}w^k$ and so it vanishes with degree k at the north pole. Thus this is the Higgs field of the N-pseudovortex which vanishes with degree $N-k$ at the south pole and degree k at the north pole.

We compute $\alpha_{N,k}$ as follows. Using $\left\|\hat{\phi}\right\|_{L^2} = 1$ (2.13) and (4.2), we have

$$\begin{aligned}
\left\|\hat{\phi}_{N,k}\right\|_{L^2}^2 &= 2\pi\int_{x=-1}^{1}\left(\frac{1-x}{2}\right)^N\left|\hat{\phi}_{N,k}(x,\varphi)\right|^2\mathrm{d}x \\
1 &= 2\pi\alpha_{N,k}\int_{x=-1}^{1}\left(\frac{1-x}{2}\right)^N\left(\frac{1+x}{1-x}\right)^{N-k}\mathrm{d}x \\
\frac{1}{2\pi\alpha_{N,k}} &= \int_{x=-1}^{1}\left(\frac{1+x}{2}\right)^N\left(\frac{1-x}{1+x}\right)^k\mathrm{d}x\,.
\end{aligned}$$

We make a further restriction to the two simplest cases: $k=0$, and $N=2k$. It is

immediate that $\frac{1}{2\pi\alpha_{N,0}} = \frac{2}{N+1}$, thus $\left|\hat{\phi}_{N,0}(x,\varphi)\right|_H^2 = \frac{N+1}{4\pi}\left(\frac{1+x}{2}\right)^N$. We also see that

$$\frac{1}{2\pi\alpha_{2k,k}} = \int_{x=-1}^{1}\left(\frac{1-x^2}{4}\right)^k \mathrm{d}x =: I_k.$$

This has a combinatorial expression. We first rewrite this integral

$$\begin{aligned} I_k &= 2\int_{x=0}^{1}\left(\frac{1-x^2}{4}\right)^k \mathrm{d}x \\ &= 2\int_{t=\frac{1}{4}}^{0} t^k\left(-\frac{2\,\mathrm{d}t}{x}\right) \\ &= \int_{t=0}^{\frac{1}{4}} \frac{4t^k}{\sqrt{1-4t}}\,\mathrm{d}t\,. \end{aligned}$$

Integrating by parts

$$\begin{aligned} I_k &= \int_{t=0}^{\frac{1}{4}} \mathrm{d}\Big(-2t^k\sqrt{1-4t}\Big) - \int_{t=0}^{\frac{1}{4}} \big(-2\sqrt{1-4t}\big)kt^{k-1}\,\mathrm{d}t \\ &= 0 + 2k\int_{t=0}^{\frac{1}{4}} \frac{(1-4t)t^{k-1}}{\sqrt{1-4t}}\,\mathrm{d}t \\ &= \frac{k}{2}\int_{t=0}^{\frac{1}{4}} \frac{4t^{k-1}}{\sqrt{1-4t}}\,\mathrm{d}t - 2k\int_{t=0}^{\frac{1}{4}} \frac{4t^k}{\sqrt{1-4t}}\,\mathrm{d}t \\ I_k &= \frac{k}{2}I_{k-1} - 2kI_k. \end{aligned}$$

Hence $(2k+1)I_k = \frac{k}{2}I_{k-1}$. It is easier to solve the recurrence relation for $\alpha_{2k,k} = \frac{I_k^{-1}}{2\pi}$, i.e.

$$\alpha_{2k,k} = \frac{2(2k+1)}{k}\alpha_{2k-2,k-1}, \tag{4.4}$$

we can compute I_1 explicitly to yield $\alpha_{2,1} = \frac{3}{2\pi}$ (and thus $\alpha_{0,0} = \frac{1}{4\pi}$ formally). Since $\frac{2(2k+1)}{k} = \frac{(2k+1)(2k)}{k\cdot k}$, we have

$$\begin{aligned} \alpha_{2k,k} &= \frac{(2k+1)!}{(k!)^2}\alpha_{0,0} \\ &= \frac{1}{4\pi}(2k+1)\binom{2k}{k}. \end{aligned}$$

With this, we can write down the field strengths of the two Higgs fields explicitly.

$$f_N(x) := \left|\hat{\phi}_{N,0}(x)\right|^2 = \frac{N+1}{4\pi}\left(\frac{1+x}{2}\right)^N, \tag{4.5}$$

$$f_k(x) := \left|\hat{\phi}_{2k,k}(x)\right|^2 = \frac{2k+1}{4\pi}\binom{2k}{k}\left(\frac{1-x^2}{4}\right)^k. \tag{4.6}$$

We can write the Higgs fields of the corresponding vortices in terms of these pseudovortex

Higgs fields by considering solutions to the PSC equation $\Delta u - \frac{\varepsilon}{4\pi} + \varepsilon e^u \left|\hat{\phi}\right|^2 = 0$ (2.15). Concretely, if we let u_N and u_k solve

$$\Delta u_N - \frac{\varepsilon}{4\pi} + \varepsilon e^{u_N} f_N(x) = 0, \tag{4.7}$$

$$\Delta u_k - \frac{\varepsilon}{4\pi} + \varepsilon e^{u_k} f_k(x) = 0. \tag{4.8}$$

Then $|\phi_{N,0}|^2 = \varepsilon e^{u_N} \left|\hat{\phi}_{N,0}\right|^2$ is the field strength of the *vortex* Higgs field when there are N coincident vortices at the south pole (the *unipolar* case), and $|\phi_{2k,k}|^2 = \varepsilon e^{u_k} \left|\hat{\phi}_{2k,k}\right|^2$ is that for k coincident vortices at each pole (the *bipolar* case).

From this description of vortex Higgs fields, it is immediate that the behaviour of u_N and u_k govern how good the pseudovortex approximation is. Concretely, the C^0-norms of u_N and u_k respectively determine the rate of convergence of unipolar and bipolar vortices to pseudovortices in the dissolving limit.

4.2 Bipolar vortices

The behaviour in the bipolar case is straightforward. We shall show that u_k is bounded independently of k, indicating that bipolar vortices converge to their corresponding pseudovortices at some fixed rate, regardless of degree.

Proposition 4.2.1.
There exists a constant $C > 0$ such that for all $k \in \mathbb{Z}^+$, $\|u_k\|_{C^0} < C$.

Proof. We will construct families of bounded sub- and super-solutions to (4.8) from which boundedness of the solutions follows. First, consider the related Poisson problem

$$\Delta v - \frac{\varepsilon}{4\pi} + \varepsilon f_k = 0. \tag{4.9}$$

The integral of $\frac{\varepsilon}{4\pi} - \varepsilon f_k$ vanishes, so unique solutions to this problem exist up to an additive constant.

Let v_k be the unique solution to (4.9) such that $v_k(0) = 0$. Furthermore, let $v_k^- = v_k - \max(v_k) \leq 0$, and $v_k^+ = v_k - \min(v_k) \geq 0$. We compute $\Delta v_k^\pm - \frac{\varepsilon}{4\pi} + \varepsilon e^{v_k^\pm} f_k = \varepsilon\left(e^{v_k^\pm} - 1\right) f_k$, so v_k^+ is a super-solution of (4.8) and v_k^- is a sub-solution, thus

$$\min(v_k) - \max(v_k) \leq v_k^- \leq u_k \leq v_k^+ \leq \max(v_k) - \min(v_k).$$

So it suffices to show that v_k is bounded above and below independently of k.

Now note that for an axially symmetric f we have

$$\Delta f = -\partial_x\big((1-x^2)\partial_x f\big) = 2x\partial_x f + \big(x^2-1\big)\partial_x^2 f,$$

it is therefore convenient to view the problem from the perspective of ODEs. We have $\Delta v_k\big|_{x_0} = 2x_0\,\partial_x v_k\big|_{x_0} + (x_0^2-1)\,\partial_x^2 v_k\big|_{x_0} = \frac{\varepsilon}{4\pi}(1-4\pi f_k(x_0))$. This implies that if v_k is critical at x_0, $\partial_x^2 v_k\big|_{x_0}$ has the same sign as

$$g_k(x_0) := 4\pi f_k(x_0) - 1.$$

Since f_k is smooth and symmetric about $x=0$; v_k must be smooth and symmetric about $x=0$, it is therefore critical *at* $x=0$. Note that $4\pi f_1(0) = 3\cdot\binom{2}{1}\cdot 4^{-1} > 1$, and for $k\geq 1$ we have $f_{k+1}(0)/f_k(0) = \frac{2k+3}{2k+1}\cdot\binom{2k+2}{k+1}\cdot\binom{2k}{k}^{-1}\cdot\frac{4^{-k-1}}{4^{-k}} = \frac{(2k+3)(2k+2)}{4(k+1)^2} = 1+\frac{1}{2(k+1)} > 1$, thus $4\pi f_k(0) > 1$ for $k\geq 1$. Thus $\partial_x^2 v_k\big|_0 > 0$ and v_k attains a local minimum at $x=0$. We can now show that $v_k \geq 0$, indeed we show that v_k cannot attain a local minimum anywhere other than at $x=0$.

Suppose for a contradiction that v_k attains a local minimum at some $b\in(0,1]$. If $b\neq 1$, then as $x=0$ is also a minimum, it attains a local maximum at some $a\in(0,b)$. Thus we have $0<a<b\leq 1$ such that $\partial_x^2 v_k\big|_a < 0$ and $\partial_x^2 v_k\big|_b > 0$. As v_k is critical at $a\neq 1$, we have $g_k(a) < 0$. Now since $b\neq 1$, we also have $g_k(b) > 0$ which is an immediate contradiction as $g_k(x)$ is monotonically decreasing for $x\in[0,1]$.

If instead we have $b=1$, the contradiction is slightly more subtle. For any $c\in(a,1)$, we have $I(c) := \int_c^1 \frac{\varepsilon}{4\pi} - \varepsilon f_k\,\mathrm{d}x = -\int_c^1 \mathrm{d}\big((1-x^2)\partial_x v_k\big) = \big(1-c^2\big)\partial_x v_k\big|_c$. If v_k attains a local minimum at $x=1$, then for c sufficiently close to 1 we have $(1-c^2)\partial_x v_k\big|_c < 0$ and so $I(c) < 0$.

We compute

$$\begin{aligned}
I(c) &= \frac{c\varepsilon}{4\pi} - \varepsilon\int_c^1 f_k\,\mathrm{d}x \\
c - \frac{4\pi}{\varepsilon}I(c) &= 4\pi\int_c^1 \frac{2k+1}{4\pi}\binom{2k}{k}\left(\frac{1-x^2}{4}\right)^k \mathrm{d}x \\
&\leq (2k+1)\binom{2k}{k}\int_c^1 \frac{1-x^2}{4}\,\mathrm{d}x \\
&\leq (2k+1)\binom{2k}{k}\frac{1}{12}\big(c^3-3c+2\big) \\
I(c) &\geq \frac{\varepsilon}{4\pi}\left(c-(c-1)^2\binom{2k}{k}\frac{(2k+1)(c+2)}{12}\right).
\end{aligned}$$

Which implies that $I(c) > 0$ for c sufficiently close to 1 which is a contradiction, and so v_k cannot attain a local minimum at $x=1$. Thus v_k cannot attain a local minimum anywhere aside from $x=0$, and so $v_k\geq 0$. $\perp$

Let $w^+(x) = \frac{\varepsilon}{4\pi}\log(1+|x|)$, we will show that $w^+\geq v_k$, bounding v_k above independently of k and completing the proof. $w^+ - v_k$ is continuous on $[-1,1]$, so it attains its extrema and it is smooth away from $x=0$. Suppose $w^+ - v_k$ is critical at $x_0\in[-1,1]\setminus\{0\}$, we

have

$$-\partial_x\big((1-x^2)\partial_x(w^+ - v_k)\big)\Big|_{x=x_0} = \frac{\varepsilon}{4\pi} - \frac{\varepsilon}{4\pi} + \varepsilon f_k$$
$$-\partial_x^2(w^+ - v_k)\Big|_{x=x_0} = \frac{\varepsilon\left|\hat{\phi}_{2k,k}(x_0)\right|^2}{1-x_0^2}.$$

From (4.6), we note that for any $k \geq 1$, $\frac{\left|\hat{\phi}_{2k,k}(x)\right|^2}{1-x^2}$ is a smooth, non-negative, and bounded function from. Indeed, if $w^+ - v_k$ is critical at some $x_0 \neq \pm 1, 0$, $-\partial_x^2(w^+ - v_k)\big|_{x=x_0} > 0$ so this critical point must be a maximum. So $w^+ - v_k$ must attain a global minimum at one of ± 1 or 0. If the minimum is at 0, we are done since $w^+(0) - v_k(0) = 0$, so $w^+ - v_k \geq 0$. We show that it cannot attain a minimum at $x = \pm 1$.

The argument is essentially the same as showing that v_k cannot attain a minimum at $x = 1$. We show that for sufficiently small c, $\int_c^1 -\partial_x\big((1-x^2)(w^+ - v_k)\big)\,\mathrm{d}x > 0$. We compute

$$\int_c^1 -\partial_x\big((1-x^2)(w^+ - v_k)\big)\,\mathrm{d}x = \varepsilon\int_c^1 f_k\,\mathrm{d}x$$
$$\geq 0.$$

As with v_k, this forbids $w^+ - v_k$ from attaining a negative minimum at $x = 1$. □

We compute u_k when $\varepsilon = 1$ numerically in order to demonstrate the limiting behaviour. Note that in the figure below, the independent variable is the polar angle from the nadir, $\theta = \arccos(-x)$. As shown in Proposition 4.2.1, we see that u_k is bounded both above and

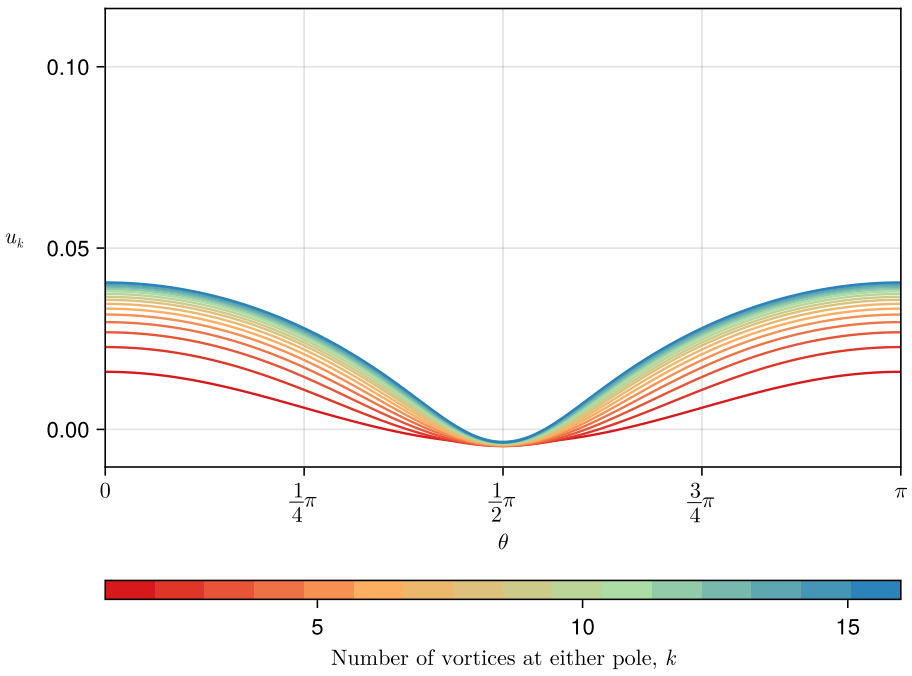


Figure 4.1: u_k with $\varepsilon = 1$ against the polar angle from the nadir, θ

below. Indeed, viewing $e^{u_k(x)}$ as the quotient $\frac{|\phi_{2k,k}(x)|^2}{|\hat{\phi}_{2k,k}(x)|^2}$, we see that in the bipolar case for $N \leq 32$, the pseudovortex and vortex Higgs fields disagree by a factor of no more than $\exp(0.05) \approx 1.051$. Indeed, from the proof above we have $\|u_k\|_{C^0} \leq w^+(1) = \frac{\varepsilon}{4\pi}\log(2)$, implying that the pseudovortex and vortex Higgs fields disagree by a factor of no more than $2^{\frac{\varepsilon}{4\pi}}$. We note that the difference is greatest towards the zeros of the fields, indicating that the approximation is weakest around the (pseudo)vortex cores.

4.3 Unipolar vortices

Unipolar vortices are different as the solutions to the PSC in the unipolar case are not bounded as the degree increases.

Proposition 4.3.1.
For all ε such that $0 < \varepsilon < 1$, $\|u_N\|_{C^0}$ is unbounded as $N \to \infty$.

This implies that while unipolar vortices still converge uniformly to the corresponding pseudovortex in the dissolving limit, the rate of convergence worsens as the degree increases.

To prove the result, we shall require the following lemma, though for overall legibility we delay its proof to the end of this section.

Lemma 4.3.2.
For $N \in \mathbb{Z}$ such that $N > 4$, let ϵ_N be such that $4\pi f_N(\epsilon_N) = \frac{N+1}{N^2}$. Consider the family of continuous functions on $[-1, 1]$

$$v_N(x) := \begin{cases} -c\varepsilon\log(1-\epsilon_N) + 2c\varepsilon\log(1-x) & : x \leq \epsilon_N \\ c\varepsilon\log(1-\epsilon_N) + N\log(\epsilon_N) - N\log x & : x > \epsilon_N. \end{cases}$$

Where $c, \varepsilon > 0$ are constants.

Then:

1. *$V_N(\varphi, x) = v_N(x)$ is a C^0 function on S^2 which is smooth for $x \neq \epsilon_N$,*
2. *$v_N(\epsilon_N)$ is unbounded below as $N \to \infty$,*
3. *v_N is unbounded above as $N \to \infty$, and*
4. *for every c, ε such that $c < \frac{1}{8\pi}$ and $\varepsilon < 1$, there is some $N_\varepsilon > 4$ such that for every $N \geq N_\varepsilon$, v_N is a sub-solution of,*

$$Q_N(u) := -\partial_x\big((1-x^2)\partial_x u\big) - \frac{\varepsilon}{4\pi} + \varepsilon e^u f_N,$$

on $[-1, 1] \setminus \{\epsilon_N\}$. i.e. $Q_N(v_N) < 0$ on $[-1, 1] \setminus \{\epsilon_N\}$.

Proof. *Proposition 4.3.1.* Let $\varepsilon \in (0,1)$, let $c \in \left(0, \frac{1}{8\pi}\right)$, and let v_N, ϵ_N, and N_ε be as in Lemma 4.3.2. Let u_N be the solution to (4.7)

$$\Delta u_N - \frac{\varepsilon}{4\pi} + \varepsilon e^{u_N} f_N = 0.$$

Suppose $N \geq N_\varepsilon$. Noting that u_N is constant in φ, we shall also write u_N for the function $x \mapsto u_N(0,x)$. We can therefore rewrite (4.7) as an ODE

$$Q_N(u) := -\partial_x\big((1-x^2)\partial_x u\big) - \frac{\varepsilon}{4\pi} + \varepsilon e^{u_N} f_N = 0.$$

We first show that for all $N \geq N_\varepsilon$, $u_N - v_N$ does not attain a negative minimum on the set $[-1, \epsilon_N) \cup (\epsilon_N, 1]$.

Suppose for a contradiction that $u_N - v_N$ does attain a negative minimum at some $x_0 \in [-1,1] \setminus \{\epsilon_N\}$. We tackle the case where $x_0 \neq \pm 1$ first, if this is the case we can conclude $u_N(x_0) - v_N(x_0) < 0$ and $\partial_x^2(u_N - v_N)\big|_{x_0} \geq 0$. By monotonicity of exp, there is some $C > 0$ such that

$$e^{u_N(x_0)} - e^{v_N(x_0)} = C(u_N(x_0) - v_N(x_0)) < 0. \tag{4.10}$$

As x_0 is a minimum of $u_N - v_N$, we also have

$$\begin{aligned} -\partial_x\big((1-x^2)\partial_x(u_N - v_N)\big)\Big|_{x_0} &= \big(1-x_0^2\big)\left(-\partial_x^2(u_N - v_N)\Big|_{x_0}\right) \\ &\leq 0. \end{aligned}$$

Thus

$$\begin{aligned} (Q_N(u_N) - Q_N(v_N))\Big|_{x_0} &= -\partial_x\big((1-x^2)\partial_x(u_N - v_N)\big)\Big|_{x_0} \\ &\quad + \varepsilon\Big(e^{u_N(x_0)} - e^{v_N(x_0)}\Big) f_N(x_0) \\ &< 0. \end{aligned}$$

And so $Q_N(v_N)\big|_{x_0} > 0$, yet this contradicts the fact that v_N is a sub-solution of $Q_N(u) = 0$ on $[-1,1] \setminus \{\epsilon_N\}$ and so x_0 cannot be a negative minimum of $u_N - v_N$.

If $x_0 = \pm 1$, we proceed similarly to the proof of Proposition 4.2.1. We note that it must be an isolated minimum as otherwise we apply the previous case. For $x_0 = 1$, let $x_1 \in (\epsilon_N, 1)$. We compute

$$\begin{aligned} I(x_1) &:= \int_{x_1}^1 -\partial_x\big((1-x^2)\partial_x(u_N - v_N)\big)\,\mathrm{d}x \\ &= -\int_{x_1}^1 \mathrm{d}\big((1-x^2)\partial_x(u_N - v_N)\big) \\ &= (1-x_1^2)\partial_x(u_N - v_N)\big|_{x_1}. \end{aligned}$$

As $x_0 = 1$ is a negative minimum of $u_N - v_N$, for x_1 sufficiently close to 1 we have that

$u_N - v_N$ is negative on $[x_1, 1]$, and $\partial_x(u_N - v_N)\big|_{x_1} < 0$; by the latter $I(x_1) < 0$. However

$$I(x_1) = \int_{x_1}^{1} Q_N(u_N - v_N) - \varepsilon(e^{u_N} - e^{v_N}) f_N \, \mathrm{d}x \,.$$

We have that $Q_N(u_N - v_N) \geq 0$ as v_N is a sub-solution of Q_N, and $(e^{u_N} - e^{v_N}) < 0$ on $[x_1, 1]$ by (4.10). Thus $I(c) > 0$ which is a contradiction.

For $x_0 = -1$, let $x_1 \in (-1, \epsilon_N)$. We compute

$$\begin{aligned} I'(x_1) &:= \int_{-1}^{x_1} -\partial_x\big((1-x^2)\partial_x(u_N - v_N)\big) \, \mathrm{d}x \\ &= -\int_{-1}^{x_1} \mathrm{d}\big((1-x^2)\partial_x(u_N - v_N)\big) \\ &= -(1-x_1^2)\partial_x(u_N - v_N)\big|_{x_1} . \end{aligned}$$

As $x_0 = -1$ is a negative minimum of $u_N - v_N$, for x_1 sufficiently close to -1 we have that $u_N - v_N$ is negative on $[-1, x_1]$, and $\partial_x(u_N - v_N)\big|_{x_1} > 0$; by the latter $I'(x_1) < 0$. However

$$I'(x_1) = \int_{-1}^{x_1} Q_N(u_N - v_N) - \varepsilon(e^{u_N} - e^{v_N}) f_N \, \mathrm{d}x \,.$$

We have that $Q_N(u_N - v_N) \geq 0$ as v_N is a sub-solution of Q_N, and $(e^{u_N} - e^{v_N}) < 0$ on $[-1, x_1]$ by (4.10). Thus $I'(c) > 0$ which is a contradiction. $\perp$

As $u_N - v_N$ is continuous and $[-1, 1]$ is compact, the former attains its minimum. If the minimum is non-negative, $u_N \geq v_N$; as the latter is unbounded above as $N \to \infty$, u_N is unbounded above as $N \to \infty$, and we are done. If the minimum is negative, it must be attained at ϵ_N and so $u_N(\epsilon_N) < v_N(\epsilon_N)$. By construction, $v_N(\epsilon_N)$ is unbounded below as $N \to \infty$, so u_N is unbounded below as $N \to \infty$. Either way, u_N is unbounded and we are done. $\square$

We note that this proof does not indicate whether u_N is unbounded above or below, yet numerical evidence indicates that the former is likely the case. In Figure 4.2, we numerically compute u_n when $\varepsilon = 1$. The figure indicates that $u_N(1)$ rapidly converges to some global minimum in the large degree limit while $u_N(-1)$ is unbounded above. As in the previous case, we see that u_N is maximised close to the vortex cores, again suggesting that the pseudovortex approximation is weakest around the cores.

Taken with the earlier result on the boundedness of the u_k, this presents an interesting picture in terms of the structure of the two moduli spaces. The two results indicate that there are certain points in the (pseudo)vortex moduli spaces (e.g. those defined by bipolar vortices) which remain at a bounded difference from one another in the large degree limit, yet there other points (e.g. those defined by unipolar vortices) which have an unbounded difference in the large degree limit.

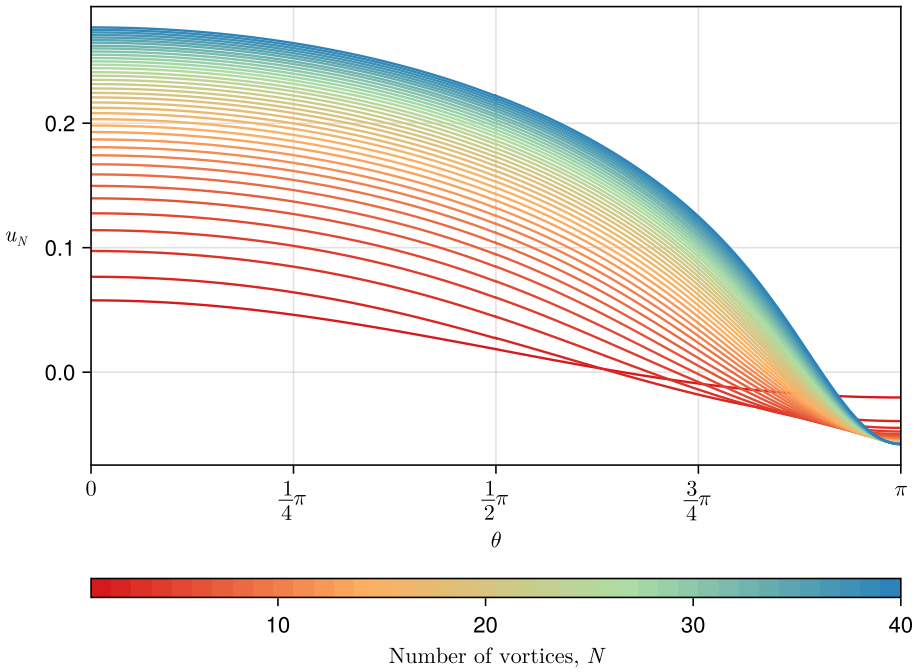


Figure 4.2: u_N with $\varepsilon = 1$ against the polar angle from the nadir, θ

We conclude with the proof of Lemma 4.3.2.

Proof. Lemma 4.3.2. $v_N(x)$ is smooth for $x \neq \epsilon_N$ and $V_N(\varphi, x) := v_N(x)$ is constant in φ; thus V_N is smooth away from the poles and the line of latitude of height ϵ_N. To show smoothness at the poles, we express it in stereographic coordinates in a neighbourhood around each pole. In a neighbourhood around the south pole, we have

$$V_N(z) = -c\varepsilon \log(1-\epsilon_N) - 2c\varepsilon \log\left(\frac{|z|^2+1}{2}\right),$$

which is clearly smooth at $z = 0$. Similarly, in a neighbourhood around the north pole we have

$$V_N(w) = -c\varepsilon \log(1-\epsilon_N) + N\log(\epsilon_N) - N\log\left(1-|w|^2\right) + N\log\left(1+|w|^2\right),$$

which is again smooth at $w = 0$.

We now state some facts about ϵ_N as these will be necessary for the rest of the proof, most of these follow from standard computations in real analysis, so we will omit proofs. First since $4\pi f_N(\epsilon_N) = (N+1)\left(\frac{1+\epsilon_N}{2}\right)^N = \frac{N+1}{N^2}$ we have $\epsilon_N = 2\left(\frac{1}{N^2}\right)^{\frac{1}{N}} - 1$. It is convenient to let $y = N^{-1}$ and consider it to be a continuous variable $y \in \left(0, \frac{1}{4}\right)$. We have $\epsilon_{y^{-1}} = 2y^{2y} - 1$, and so

$$\lim_{N\to\infty} \epsilon_N = -1 + 2\lim_{y\downarrow 0} y^{2y} = 1.$$

We also see that for $y \in \left(0, \frac{1}{4}\right)$, $y^{2y} \in \left(\frac{1}{2}, 1\right)$. Thus $\forall N > 4$, $\epsilon_N \in (0,1)$.

From the previous paragraph, it is clear that $v_N(\epsilon_N) = c\varepsilon\log(1-\epsilon_N)$ is unbounded below as a sequence in N. Moreover, we note that v_N is monotonically decreasing, so it achieves its maximum at $x = -1$. We have

$$\begin{aligned}\max v_N &= -c\varepsilon\log(1-\epsilon_N) + 2c\varepsilon\log(2)\\ &= c\varepsilon\log\left(\frac{4}{1-\epsilon_N}\right).\end{aligned}$$

So by the properties of ϵ_N described in the previous paragraph, $\max v_N$ is unbounded above as a sequence in N.

It remains to show our sub-solution hypothesis. For any $N > 4$, we compute

$$-\partial_x\left((1-x^2)\partial_x v_N\right) = \begin{cases} 2c\varepsilon & : x \leq \epsilon_N \\ -N\left(1+\frac{1}{x^2}\right) & : x > \epsilon_N. \end{cases}$$

So for x such that $0 < \epsilon_N \leq x \leq 1$ we have

$$\begin{aligned}
\max Q_N(v_N) &\leq \max\left(-N\left(1+\frac{1}{x^2}\right)\right) - \frac{\varepsilon}{4\pi} + \varepsilon \max\left(e^{v_N} f_N\right) \\
&\leq -2N + \frac{\varepsilon}{4\pi}\left(-1 + 4\pi \max\left((1-\epsilon_N)^{c\varepsilon}\left(\frac{\epsilon_N}{x}\right)^N \left(\frac{N+1}{4\pi}\right)\left(\frac{1+x}{2}\right)^N\right)\right) \\
&\leq -2N + \frac{\varepsilon}{4\pi}\left(-1 + (1-\epsilon_N)^{c\varepsilon}\left(\frac{\epsilon_N}{2}\right)^N (N+1)\max\left(\frac{1+x}{x}\right)^N\right) \\
&\leq -2N + \frac{\varepsilon}{4\pi}\left(-1 + (1-\epsilon_N)^{c\varepsilon}\left(\frac{\epsilon_N}{2}\right)^N (N+1)\left(\epsilon_N^{-1}+1\right)^N\right) \\
&\leq -2N + \frac{\varepsilon}{4\pi}\left(-1 + (1-\epsilon_N)^{c\varepsilon}(N+1)\left(\frac{1}{2}(\epsilon_N+1)\right)^N\right) \\
&\leq -2N + \frac{\varepsilon}{4\pi}\left(-1 + (1-\epsilon_N)^{c\varepsilon}\left(\frac{N+1}{N^2}\right)\right)
\end{aligned}$$

Recall $\lim_{N\to\infty}(1-\epsilon_N) = \lim_{N\to\infty}\frac{N+1}{N^2} = 0$. So for any fixed $c, \varepsilon > 0$, there is some $M_\varepsilon > 4$ such that for all $N \geq M_\varepsilon$, the product $(1-\epsilon_N)^{c\varepsilon}\left(\frac{N+1}{N^2}\right)$ is less than 1. When this is the case, the overall expression is clearly negative.

We now note that f_N is monotonically increasing, so on $(-1, \epsilon_N)$ it achieves its maximum $f(\epsilon_N) = \frac{N+1}{N^2}$. Thus for $x : -1 \leq x \leq \epsilon_N < 1$, we have

$$\begin{aligned}
\max Q_N(v_N) &\leq -\frac{\varepsilon}{4\pi} + 2c\varepsilon + e^{\max v_N} \max f_N \\
&\leq -\varepsilon\left(\frac{1}{4\pi} - 2c\right) + \left(\frac{4}{1-\epsilon_N}\right)^{\varepsilon} \cdot \frac{N+1}{N^2} \\
&\leq -\varepsilon\left(\frac{1}{4\pi} - 2c\right) + 4^{\varepsilon} \cdot \frac{N+1}{N^2(1-\epsilon_N)^{\varepsilon}}
\end{aligned}$$

Thus given $c \in (0, \frac{1}{8\pi})$ and $\varepsilon \in (0,1)$, we need to find $N_\varepsilon \geq M_\varepsilon$ such that for all $N \geq N_\varepsilon$

$$\left(N^{-2} + N^{-1}\right)(1-\epsilon_N)^{-\varepsilon} \leq \varepsilon\left(\frac{1}{4\pi} - 2c\right) \cdot 4^{-\varepsilon},$$

and we are done. Let $y^{-1} = N$ (so $y \in \left(0, \frac{1}{4}\right)$), we then have

$$\begin{aligned}
\left(N^{-2} + N^{-1}\right)(1-\epsilon_N)^{-\varepsilon} &= \left(y^2 + y\right)\left(1-\epsilon_{y^{-1}}\right)^{-\varepsilon} \\
&= 2^{-\varepsilon}\left(y^2+y\right)\left(1-y^{2y}\right)^{-\varepsilon},
\end{aligned}$$

so our inequality becomes $2^{-\varepsilon}\left(y^2+y\right)\left(1-y^{2y}\right)^{-\varepsilon} \leq 2\left(\frac{1}{4\pi} - 2c\right) \cdot 2^{-2\varepsilon}$, or

$$(y+1)\left(\frac{y^{\frac{1}{\varepsilon}}}{1-y^{2y}}\right)^{\varepsilon} \leq 2\left(\frac{1}{4\pi} - 2c\right) \cdot 2^{-\varepsilon}. \tag{4.11}$$

We wish to find some $\frac{1}{N_\varepsilon} \leq \frac{1}{M_\varepsilon}$ such that the above inequality holds for all $y < \frac{1}{N_\varepsilon}$, possibly introducing constraints on c and ε in the process.

Now note that the LHS is always positive and the RHS is positive if and only if $c < \frac{1}{8\pi}$, so the latter is certainly necessary if (4.11) is to hold. We also note that the LHS is continuous in y, so to detect whether the inequality is ever valid for a given choice of $\varepsilon > 0$, it is enough to check whether the limit of the LHS as $y \to 0$ is 0.

Consider $z \in \left[\frac{1}{2}, 1\right]$, the first Taylor approximation of $-\log z$ about $z = 1$ is

$$-\log z = (1-z) + \frac{1}{2}\left(\frac{1-z}{\xi_L}\right)^2$$

for some $\xi_L \in [z, 1]$. Thus $-\log z \geq 1 - z$. Moreover

$$\begin{aligned} -\frac{1}{2}\log z &= \frac{1}{2}(1-z) + \left(\frac{1}{2}(1-z)\right)^2 \xi_L^{-2} \\ &= (1-z)\left(\frac{1}{2} + \frac{\xi_L^{-2}(1-z)}{4}\right). \end{aligned}$$

We see that $0 < \frac{1}{2} + \frac{\xi_L^{-2}(1-z)}{4} < 1$ so $-\frac{1}{2}\log z \leq 1 - z$. Thus $-\frac{1}{2}\log z \leq 1 - z \leq -\log z$ for $z \in \left[\frac{1}{2}, 1\right]$ and so

$$-\frac{1}{\log y^{2y}} \leq \frac{1}{1-y^{2y}} \leq -\frac{2}{\log y^{2y}}$$

for $y^{2y} \in \left[\frac{1}{2}, 1\right]$ (which follows from our restriction that $y \in \left(0, \frac{1}{4}\right)$). Thus

$$\begin{aligned} \frac{y^{\frac{1}{\varepsilon}}}{1-y^{2y}} &\leq -\frac{y^{\frac{1}{\varepsilon}}}{y\log y} \\ &\leq y^{\frac{1}{\varepsilon}-1}(-\log y)^{-1}. \end{aligned}$$

It is clear that the $y \to 0$ limit of the RHS is 0 if and only if $\varepsilon \in (0, 1)$ and we are done. □

Chapter 5

2-vortices on flat tori

After $\mathbb{P}^1$, the next simplest class of Riemann surfaces are the complex tori. As mentioned previously, vortices on complex tori have a physical interpretation as models of the Abrikosov lattice, the configuration of GL-vortices observed in laboratory experiments, so understanding the dynamics of vortices on such surfaces is of direct physical importance.

In this chapter, we apply the results of chapter 3 to vortices on flat complex tori. After some preparatory remarks, we construct a model for a complete linear system of 2-vortices on a flat torus, i.e. an explicit isometry between the former and $\mathbb{P}^1$ equipped with the Fubini-Study metric. We will also justify an interpretation of geodesics on the latter space as approximations of relative 2-vortex dynamics close to the dissolving limit. We then study these geodesics and completely classify them, describing their behaviour. We will also derive explicit pseudovortex Higgs fields, which can be viewed as explicit approximations of vortex Higgs fields in light of Theorem 2.2.9.

5.1 Rationale

For this chapter it is convenient to view (pseudo)vortices as being determined by their core locations i.e. as points in $\mathrm{Div}^N_+(\Sigma) \cong \Sigma^{(N)}$.

Remark 5.1.1.
Throughout this chapter and in Appendix B, τ will be a fixed complex number in the upper half plane and not the (real) parameter of the Abelian Higgs model. When we need to refer to the latter, we use $\tau_0 > 0$.

Let $\omega_1, \omega_3 \in \mathbb{C}^\times$ be such that $\tau = \omega_3/\omega_1$, let $\omega_2 := \omega_1 + \omega_3$ and consider the lattice

$$\Lambda := \{2a\omega_1 + 2b\omega_3 \mid a, b \in \mathbb{Z}\}.$$

We let $\Sigma := \mathbb{C}/\Lambda$ be a complex torus. We shall write points on Σ as complex numbers leaving the equivalence implicit, i.e. we write $0 \in \Sigma$ instead of $[0]$. Note that Σ inherits

the structure of a complex Abelian variety and a quotient metric, g^Σ, from the ambient $\mathbb{C}$. We write points in $\Sigma^{(N)}$ as N-subsets of Σ, $\{p_1, \dots, p_N\} \subset \Sigma$ with these being implicitly unordered, though we will write them with multiplicity (e.g. we write $(z, z) \in \Sigma^{(2)}$).

Recall that as Σ is genus 1, $\mathrm{Jac}(\Sigma)$ is biholomorphic to Σ and that the Abel-Jacobi map takes the form $\mathrm{AJ} : \Sigma^{(N)} \to \Sigma, (p_i) \mapsto \sum_{i=1}^N p_i$. The moduli space dynamics of 1-vortices on a homogeneous surface is given by free motion of the vortex core on the surface and is well understood. We therefore consider 2-vortices on Σ.

The Abel-Jacobi map takes the form

$$\begin{aligned} &\mathrm{AJ} : \Sigma^{(2)} \to \Sigma, \\ &\{p, q\} \mapsto p + q. \end{aligned} \tag{5.1}$$

From a Riemann-Roch computation (see Lemma 2.1.11), we can immediately see that this map has constant fibre $\mathbb{P}^1$. As mentioned previously, it is actually a $\mathbb{P}^1$-bundle over Σ [51], [63]. We note that this bundle is non-trivial.

Lemma 5.1.2.
The cohomology rings (with coefficients in $\mathbb{Z}$) of $\Sigma^{(2)}$ and $\mathbb{P}^1 \times \Sigma$ are not-isomorphic and thus $\mathrm{AJ} : \Sigma^{(2)} \to \Sigma$ cannot be the trivial bundle.

A proof of this fact is given in §B.1 on page 111.

Now consider the involution $\Sigma \to \Sigma, p \mapsto -p$. As Σ is a flat torus, this is an isometry of Σ. It yields an involution of the symmetric product

$$\begin{aligned} &\Sigma^{(2)} \to \Sigma^{(2)}, \\ &\{p, q\} \mapsto \{-p, -q\}. \end{aligned} \tag{5.2}$$

This has the fixed point set $\mathrm{AJ}^{-1}(0) = \{\{p, q\} \mid p + q = 0\}$, i.e. a complete linear system.

We determine the corresponding element of $\mathrm{Pic}^2(\Sigma)$ as follows. Let $L \to \Sigma$ be some degree 2 line bundle and note that it trivialises on some small open neighbourhood around 0, V, and on $U = \Sigma \setminus \{0\}$. Consider the Weierstrass $\wp$-function [20, (23.2.4)], $\wp$. For sufficiently small V, $\wp(z)$ is a meromorphic function on V with no zeros and a single pole of order two at 0. It then follows that $\tau_{UV}(z) := \wp(z)$ is a holomorphic transition function.

Holomorphic sections of L with respect to this structure are then determined by doubly periodic holomorphic functions on U, i.e. elliptic functions. Let $\phi : U \to \mathbb{C}$ be such a section and note that it cannot be given by an arbitrary elliptic function; as it is a holomorphic section, it cannot have poles in U and must have a single pole of order 2 at 0 in order to define a valid holomorphic section over V. By the general theory of elliptic curves (see e.g. [39, p. 7]), we have that the sums of the locations of the poles of ϕ must equal the sums of the locations of the zeros of ϕ (counted with multiplicity). i.e. we must have that ϕ vanishes at precisely two points (p, q) such that $p + q = 0$, i.e. the

effective divisor associated to ϕ lies in $\mathrm{AJ}^{-1}(0)$. Thus by Theorem 2.1.13, this holomorphic structure on L is the element of $\mathrm{Pic}^2(\Sigma)$ which corresponds to $0 \in \Sigma \cong \mathrm{Jac}(\Sigma) \cong \mathrm{Pic}^2(\Sigma)$.

We define a Hermitian metric, H, to be such that the associated Chern connection is constant curvature. From the construction of the Chern connection (see e.g. [32, p. 177]), finding such a metric is equivalent to solving the Poisson equation on Σ which has unique solutions, so we see that this metric is unique.

We can write out an explicit expression for H.

Proposition 5.1.3.
Let $U := \Sigma \setminus \{0\}$ and consider the trivialisation of L over U, i.e. fix a gauge such that the Chern connection has the expression $D_L^{0,1} = \frac{\mathrm{d}}{\mathrm{d}\bar{z}}$. Then the Hermitian metric H has the local expression

$$H(\phi,\psi)\Big|_z = \exp(h(z))\overline{\phi_U(z)}\psi_U(z), \tag{5.3}$$

where h is given by,

$$h(z) := 4\log\left|\theta_1\left(\frac{z}{2\omega_1}\right)\right| - \frac{4\pi\left(\mathrm{im}\left(\frac{z}{2\omega_1}\right)\right)^2}{\mathrm{im}\,\tau}. \tag{5.4}$$

Here θ_1 is the Jacobi theta function [20, (20.2.1)][1].

Proof. We note that there is some real function $h(z)$, such that the metric H has the local expression $H(\phi,\psi)\big|_z := e^{h(z)}\overline{\phi_U(z)}\psi_U(z)$. Here $\phi, \psi \in \ker D_L^{0,1}$.

As D_L is the Chern connection associated to H and $\overline{\partial}$, we see that the curvature satisfies $\Lambda F_L = \Lambda\overline{\partial}\partial h = -\frac{\mathbf{i}}{2}\Delta h$. Thus $\Delta h = -\frac{8\pi}{\mathrm{Vol}\,\Sigma}$, i.e. h is a Green's function for the Laplacian on U_2. Lin and Wang [41] show that such functions are of the form of (5.4) and we are done. □

With this, we have that if $\{x,-x\} \in \mathrm{AJ}^{-1}(0)$, the pointwise norm of the holomorphic section vanishing at $(x,-x)$ with unit norm (i.e. the pseudovortex vanishing at $(x,-x)$) has the expression

$$\left|\hat{\phi}\right|^2\Big|_U(z) = \alpha(x)\exp(h(z))\left|\frac{\sigma(z-x)\sigma(z+x)}{(\sigma(z))^2}\right|^2 \tag{5.5}$$

where $\sigma(z)$ is the Weierstrass sigma function [20, (23.2.6)], and $\alpha(x) > 0$ is a normalisation constant ensuring $\left\|\hat{\phi}\right\|^2 = 1$.

We let $\hat{\mathcal{M}}_0^\varepsilon$ be the complete linear system of pseudovortices corresponding to $\mathrm{AJ}^{-1}(0)$ and $\mathcal{M}_0^\varepsilon$ the respective complete linear system of vortices. By construction, we see that H is

[1] We use the convention where θ_1 is quasi-periodic on the lattice spanned by 1 and τ. This is equivalent to rescaling the argument of the DLMF convention [20, (20.2.1)] by π i.e. using $\theta_1(z) := \tilde{\theta}_1(\pi z)$ where $\tilde{\theta}_1$ is the DLMF convention.

invariant under the involution $p \mapsto -p$, so by Theorem 3.2.4, we have that $\mathcal{M}_0^\varepsilon \subset \mathcal{M}^\varepsilon$ is a totally geodesic submanifold. Theorem 3.2.1 then applies, and we see that pseudovortex geodesics approximate vortex geodesics close to the dissolving limit.

For this approximation to be of any use, we require an explicit isometry

$$\Phi : (\mathrm{AJ}^{-1}(0), \hat{g}_0^\varepsilon) \to \left(\mathbb{P}^1, g^{\mathrm{FS}} := 2\,\langle z\rangle^{-2}\,\mathrm{d}z\,\mathrm{d}\bar{z}\right). \tag{5.6}$$

And as we shall see in §5.2, such an isometry is unique up to conjugation by an element of $\mathbb{P}\,\mathrm{SU}(2)$, moreover this isometry be determined explicitly. We can then pull back geodesics of $\left(\mathbb{P}^1, g^{\mathrm{FS}}\right)$ (which are great circles) to yield core trajectories on Σ.

Before continuing, we note that particular approximation has a direct physical interpretation. In this case, the vortex metric decomposes into a convenient form.

Proposition 5.1.4.
Equip the open set $U := \Sigma^{(2)}\backslash\{\{z, z\} \mid z \in \Sigma\}$ *with local coordinates* $\left\{z^1, z^2\right\}$*, where* z^i *takes values in the open fundamental polygon of* Σ *centred at 0. Furthermore, let* $\zeta^1 := z^1 - z^2$ *and* $\zeta^2 := z^1 + z^2$*.*

Then on U*, the vortex metric is equal to*

$$g^\varepsilon\big|_U = \alpha(\zeta^1, \bar{\zeta}^1)\,\mathrm{d}\zeta^1\,\mathrm{d}\bar{\zeta}^1 + \tau_0\pi\,\mathrm{d}\zeta^2\,\mathrm{d}\bar{\zeta}^2\,, \tag{5.7}$$

for some function α*.*

We prove this in §B.2 on page 112. The significance of this result is that if we view ζ^1 as a fibre coordinate and ζ^2 as a coordinate on $\mathrm{Jac}(\Sigma)$, we see that geodesic motion on the base decouples from motion on the fibre. This yields an interpretation where the motion along $\hat{\mathcal{M}}_0^\varepsilon$ is *relative motion*, which we approximate using Φ, and the motion along the base is *drift motion*. Moreover, the drift dynamics are equivalent to free motion on $\left(\Sigma, \tau_0\pi g^\Sigma\right)$ and are thus well understood.

5.2 Constructing the Isometry on Fibres

In this section we construct $\Phi : \left(\hat{\mathcal{M}}_0^\varepsilon, \hat{g}_0^\varepsilon\right) \to \left(\mathbb{P}^1, g^{\mathrm{FS}}\right)$ (5.6) explicitly, this is done by exploiting isometries of our base space Σ. In particular, consider the action of $K_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ on Σ by isometries,

$$\begin{aligned} K_4 &\to \mathrm{Aut}(\Sigma), \\ (1,0) &\mapsto (z \mapsto z + \omega_1), \\ (0,1) &\mapsto (z \mapsto z + \omega_3). \end{aligned}$$

We see this naturally extends to a K_4 action by isometries on $\Sigma^{(2)}$ and we see that fibres of AJ are preserved by this action since $\mathrm{AJ}(\{x + \omega_i, y + \omega_i\}) = x + y + 2\omega_i = \mathrm{AJ}(\{x, y\})$. This then induces an action on $\hat{\mathcal{M}}_0^\varepsilon \cong \mathbb{P}^1$ by isometries. In the lemma below we write

out this action on $\mathbb{P}^1$ explicitly, indeed we show that there is only one such action (up to conjugation/choice of coordinates on $\mathbb{P}^1$).

Lemma 5.2.1.
Up to conjugation, there is a unique faithful action of K_4 on $\mathbb{P}^1$ by isometries (i.e. an essentially unique non-trivial $\mathbb{P}\,\mathrm{SU}(2) \cong \mathrm{SO}(3)$ action).

For this result, we note that a *stem extension of G by H* is a short exact sequence of groups

$$0 \longrightarrow H \longrightarrow E \longrightarrow G \longrightarrow 0 \ ,$$

where the image of $H \to E$ lies in $Z(E) \cap [E, E]$, the intersection of the centre and commutator of E.

Proof. By a theorem of Schur, we note that projective representations of a group G, $\rho : G \to \mathbb{P}\,\mathrm{GL}(n, \mathbb{C})$, are in bijection with linear representations of a Schur covering group (see e.g. [65, §6.6]), which is a stem extension of G by $H^2(G; \mathbb{C}^\times)$. The covering group is not necessarily unique, but isomorphism classes of projective representations of G are in bijection with isomorphism classes of linear representations of any choice of cover. This is captured by the commutative diagram below

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \rho'} & \longleftrightarrow & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{GL}(n, \mathbb{C}) & \longrightarrow & \mathbb{P}\,\mathrm{GL}(n, \mathbb{C}) & \longrightarrow & 0
\end{array}$$

where E is a choice of covering group and ρ' is a linear representation.

We find the stem extensions of K_4 as follows. We first compute $H^2(K_4; \mathbb{C}^\times) \cong \mathbb{Z}_2$ (by e.g. the bar resolution [65, §6.5]), and then let E be a stem extension of K_4 by $\mathbb{Z}_2$. First, note that E cannot be abelian as otherwise we would have $[E, E] = 0$ and thus $\mathbb{Z}_2 \to E$ would not be injective. Noting that E must be of order 8 restricts us to:

- $E \cong D_4$ (we let σ and ρ be the rotation and reflection generators respectively), or
- $E \cong Q_8$, the quaternion group (we let $\pm 1, \pm i, \pm j, \pm k$ be the standard generators).

In either case we have:

- $Z(D_4) \cap [D_4, D_4] = \{1, \sigma^2\} \cong \mathbb{Z}_2$, and
- $Z(Q_8) \cap [Q_8, Q_8] = \{1, -1\} \cong \mathbb{Z}_2$.

So either group can be a stem extension with a canonical choice of $\mathbb{Z}_2 \to E$. We choose $E \cong Q_8$.

There is a unique-up-to-conjugation $\mathrm{SU}(2)$-representation of Q_8 given by

$$-1 \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, i \mapsto \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix},$$
$$j \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k \mapsto \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}.$$

Which in turn yields an essentially unique $\mathbb{P}\,\mathrm{SU}(2)$-representation of K_4 given by

$$\rho_\bullet : K_4 \to \mathbb{P}\,\mathrm{SU}(2)$$
$$(1,0) \mapsto \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, (0,1) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

□

Henceforth, when we speak of *the* K_4 action on $\mathbb{P}^1$, we mean $\rho_\bullet$. As fractional linear transformations, these are

$$\rho_{(1,0)}(z) = -z,$$
$$\rho_{(0,1)}(z) = -\frac{1}{z}.$$

We wish to construct Φ explicitly, so we require a concrete model for $\hat{\mathcal{M}}_0^\varepsilon$. Recall that $\hat{\mathcal{M}}_0^\varepsilon = \mathrm{AJ}^{-1}(0)$, and as $\mathrm{AJ}(\{p,q\}) = p+q$ we have $\hat{\mathcal{M}}_0^\varepsilon = \{\{z,-z\} \mid z \in \Sigma\}$. We know from previous computations (e.g. Equation 2.4) that this is just $\mathbb{P}^1$, but we also have a direct topological proof of this fact. Consider a fundamental domain of Σ, a point $\{z,-z\} \in \Sigma^{(2)}$ is fully determined by the point in the pair which lies in the left half of the domain (as this half domain is itself a fundamental domain for the isometry $z \mapsto -z$).

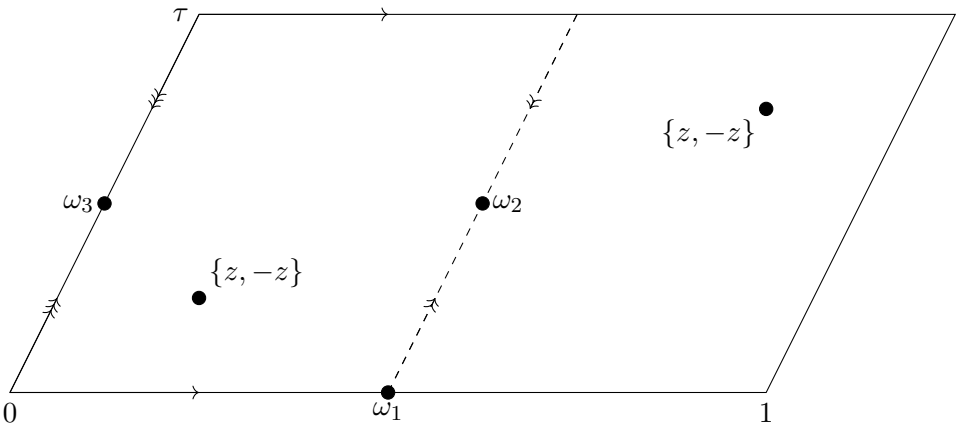


Figure 5.1: A fundamental domain of Σ showing a point $\{z,-z\}$ and half edge identifications.

This leads to identifications of the half edges, shown in the figure above, which show that $\hat{\mathcal{M}}_0^\varepsilon$ is certainly homeomorphic to $\mathbb{P}^1$. We note that elements of $\hat{\mathcal{M}}_0^\varepsilon$ are an unordered pair of *distinct* points of Σ except for the half periods $\{\omega_i, \omega_i\}$ and $\{0,0\}$.

The following theorem then constructs Φ explicitly.

Theorem 5.2.2.
The biholomorphic isometry $\left(\hat{\mathcal{M}}_0^\varepsilon, \frac{1}{\varepsilon}\hat{g}_0^\varepsilon\right) \to \left(\mathbb{P}^1, g^{\mathrm{FS}}\right)$ *is of the form*

$$\begin{aligned} \Phi : \left(\hat{\mathcal{M}}_0^\varepsilon, \frac{1}{\varepsilon}\hat{g}_0^\varepsilon\right) &\to \left(\mathbb{P}^1, g^{\mathrm{FS}}\right), \\ \{z, -z\} &\mapsto M(\wp(z)). \end{aligned} \tag{5.8}$$

Here, M is a Möbius transformation which is uniquely determined by the lattice and is given explicitly in Equation 5.11.

Proof. First, we note that as $\hat{\mathcal{M}}_0^\varepsilon = \{\{z, -z\} \mid z \in \Sigma\}$, biholomorphisms $f : \hat{\mathcal{M}}_0^\varepsilon \to \mathbb{P}^1$ are in bijection with holomorphic even maps $\tilde{f} : \Sigma \to \mathbb{P}^1$ of degree 2, i.e. even degree 2 elliptic functions. Concretely, given f we define $\tilde{f}(z) := f(\{z, -z\})$ and given $\tilde{f}$ we define $f(\{z, -z\}) := \tilde{f}(z)$. Going forward, we will freely switch between the two perspectives.

We consider elliptic functions as taking values in $\mathbb{P}^1$ and recall that all even elliptic functions are rational functions of the Weierstrass $\wp$-function. Moreover, since $\wp$ is degree 2, all degree 2 even elliptic functions are of the form $\tilde{f} : z \mapsto M(\wp(z))$ for some $M \in \mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$, thus f is of the form of (5.8). We note that such an f is an isometry only if it commutes with the K_4-actions by isometries defined on $\Sigma^{(2)}$ and $\mathbb{P}^1$, i.e. for all $\sigma \in K_4$, $\{x, y\} \in \hat{\mathcal{M}}_0^\varepsilon$

$$f(\sigma \cdot \{x, y\}) = \sigma \cdot f(\{x, y\}).$$

Which implies that $\tilde{f}$ is also K_4-equivariant. We show that this property uniquely determines M up to conjugation by an element of $\mathbb{P}\,\mathrm{SU}(2)$.

Defining $e_i := \wp(\omega_i)$, we note that the e_i are distinct and $\{e_i\}$ is the set of critical values of $\wp$, thus $\{f_i := M(e_i)\}$ is the set of critical values of $\tilde{f}$. We define the *cross ratio* of four points $z_i \in \mathbb{P}^1$ to be

$$(z_0, z_1; z_2, z_3) := \frac{(z_2 - z_0)(z_3 - z_1)}{(z_3 - z_0)(z_2 - z_1)}. \tag{5.9}$$

Recall that the action of $\mathbb{P}\,\mathrm{GL}(2, \mathbb{C})$ on $\mathbb{P}^1$ preserves cross ratios, thus $(e_0, e_3; e_1, e_2) = (f_0, f_3; f_1, f_2)$. Note that the former ratio is precisely the value of the modular lambda function, $\lambda(\tau)$ [20, (23.15.6)].

K_4 acts faithfully and transitively on $\{\omega_i\}$, and so by equivariance of $\tilde{f}$, we must have that $\rho_\bullet$ acts faithfully and transitively on $\{f_i\}$. Thus we must have that $\{f_i\} = \left\{b, -b, \frac{1}{b}, -\frac{1}{b}\right\}$ for some $b \in \mathbb{P}^1$ with trivial $\rho_\bullet$-stabiliser (i.e. b not equal to 0, ± 1, $\pm\mathbf{i}$, or ∞). Evaluating the cross-ratio yields $(f_0, f_3; f_1, f_2) = \left(b, -b, \frac{1}{b}, -\frac{1}{b}\right) = \frac{(1-b^2)^2}{(1+b^2)^2}$, thus b is a root of the quartic $P(X) = \lambda(\tau)\left(1 + X^2\right)^2 - \left(1 - X^2\right)^2$. We compute $P(\rho_{(1,0)}(X)) = P(X)$ and $P(\rho_{(0,1)}(X)) = P(X)/X^4$, from which we see that if b is a root of P, then the $\rho_\bullet$-orbit of b consists of roots of P (and thus all possible roots of P).

Now choose a root

$$b = \sqrt{\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}}}. \tag{5.10}$$

The Möbius transformation mapping $e_0 = \infty \mapsto b$, $e_1 \mapsto \frac{1}{b}$, and $e_3 \mapsto -b$, is then

$$M(z) = \left(z \mapsto \frac{bz + b\sqrt{\lambda(\tau)}}{z - \sqrt{\lambda(\tau)}}\right) \circ \left(z \mapsto \frac{z - e_3}{e_1 - e_3}\right). \tag{5.11}$$

Now suppose we made a different choice of root of P, b' say, then there is some $\sigma \in K_4$ such that $\rho_\sigma(b) = b'$. Moreover, if M' maps $e_0 \mapsto b'$, $e_1 \mapsto \frac{1}{b'}$, $e_3 \mapsto -b'$ we have $M' = \rho_\sigma \circ M$. Since ρ_σ is just a π-rotation of $\mathbb{P}^1$, choosing a different root of P amounts to choosing a different coordinate for $\mathbb{P}^1$, and so M is essentially unique. Thus we let $\tilde{\Phi} := M \circ \wp$ and see that

$$\Phi(\{z, -z\}) := M(\wp(z)),$$

is the unique map with the properties hypothesised. □

We draw attention to $b \in \mathbb{P}^1$ defined in (5.10) and the $f_i = M(e_i) = \tilde{\Phi}(\omega_i)$ as they are used in future proofs; note that they only depend on the lattice. The f_i have immediate geometric importance as they are the ramification values of $\tilde{\Phi}$.

5.3 Relative Dynamics of Vortices

Having constructed an explicit isometry $\Phi : \hat{\mathcal{M}}_0^\varepsilon \to \mathbb{P}^1$, we can easily construct geodesics on $\hat{\mathcal{M}}_0^\varepsilon$ by taking Φ^{-1} of some geodesic $\gamma : S^1 \to \left(\mathbb{P}^1, g^{\mathrm{FS}}\right)$. Recalling the metric decomposition of Proposition 5.1.4, we interpret these geodesics as approximations of the relative motion of 2-vortices on Σ. Note that as all geodesics on $\left(\mathbb{P}^1, g^{\mathrm{FS}}\right)$ are closed, it is clear that this motion is always periodic.

We first characterise all geodesics of $\left(\mathbb{P}^1, g^{\mathrm{FS}}\right)$. View S^1 as the unit circle in the complex plane with the usual orientation. Given $p \in \mathbb{P}^1$, we define $\hat{p}$ to be the directed great circle normal to p,

$$\begin{aligned} \hat{p} &: S^1 \to \mathbb{P}^1, \\ z &\mapsto \frac{pz - 1}{z + \overline{p}}. \end{aligned} \tag{5.12}$$

Note that $\hat{p}$ is the restriction of a Möbius transformation to the unit circle and that every geodesic in $\left(\mathbb{P}^1, g^{\mathrm{FS}}\right)$ is of this form. We collect some results about this construction, though we do not provide proofs as these are consequences of some rudimentary spherical geometry.

Lemma 5.3.1.
Let $p, q \in \mathbb{P}^1$, then:

- $p \in \operatorname{im} \hat{q}$ *if and only if* $\operatorname{im} \hat{p} \ni q$.

- $\operatorname{im}\hat{p}$ *and* $\operatorname{im}\hat{q}$ *intersect tangentially if and only if either* $p = q$*, or* p *and* q *are antipodal i.e.* $p\bar{q} = -1$*; in either case, their images are equal.*
- *If* $T \in \mathbb{P}\,\mathrm{SU}(2;\mathbb{C})$ *and* $q = Tp$*,* $\hat{q} = T\hat{p}$*.*

We extend the characterisation of geodesics to $\hat{\mathcal{M}}_0^\varepsilon$ using Φ. Concretely, given a geodesic $\gamma : S^1 \to (\hat{\mathcal{M}}_0^\varepsilon, \frac{1}{\varepsilon}\hat{g}_\varepsilon^\varepsilon)$, we let $p_\gamma \in \mathbb{P}^1$ be the unique point such that $\hat{p}_\gamma = \Phi \circ \gamma$. Recall that points in $\Sigma^{(2)}$ correspond to core locations of 2-vortices, thus a geodesic γ models low-energy vortex dynamics by specifying the core locations as a function of time.

In Figure 5.2 we show some sample $p, \hat{p}$ pairs on $\mathbb{P}^1$ when Σ is a square torus. The figure shows the stereographic projection of $\mathbb{P}^1$ away from ∞, on it we plot the points -1, $\exp\left(\frac{5}{8}\pi\right)$, and $1 - \sqrt{2}$ in orange, blue, and purple respectively, the lines are the corresponding $\hat{p}$. Figure 5.3 then shows $\Phi^{-1} \circ \hat{p}$ in appropriate colours; in both images, the transition from high to low opacity indicates position along the geodesic, an arrow marks the start of the geodesic. The 40 points along the geodesic are evenly spaced to illustrate the changing velocity of the vortex core.

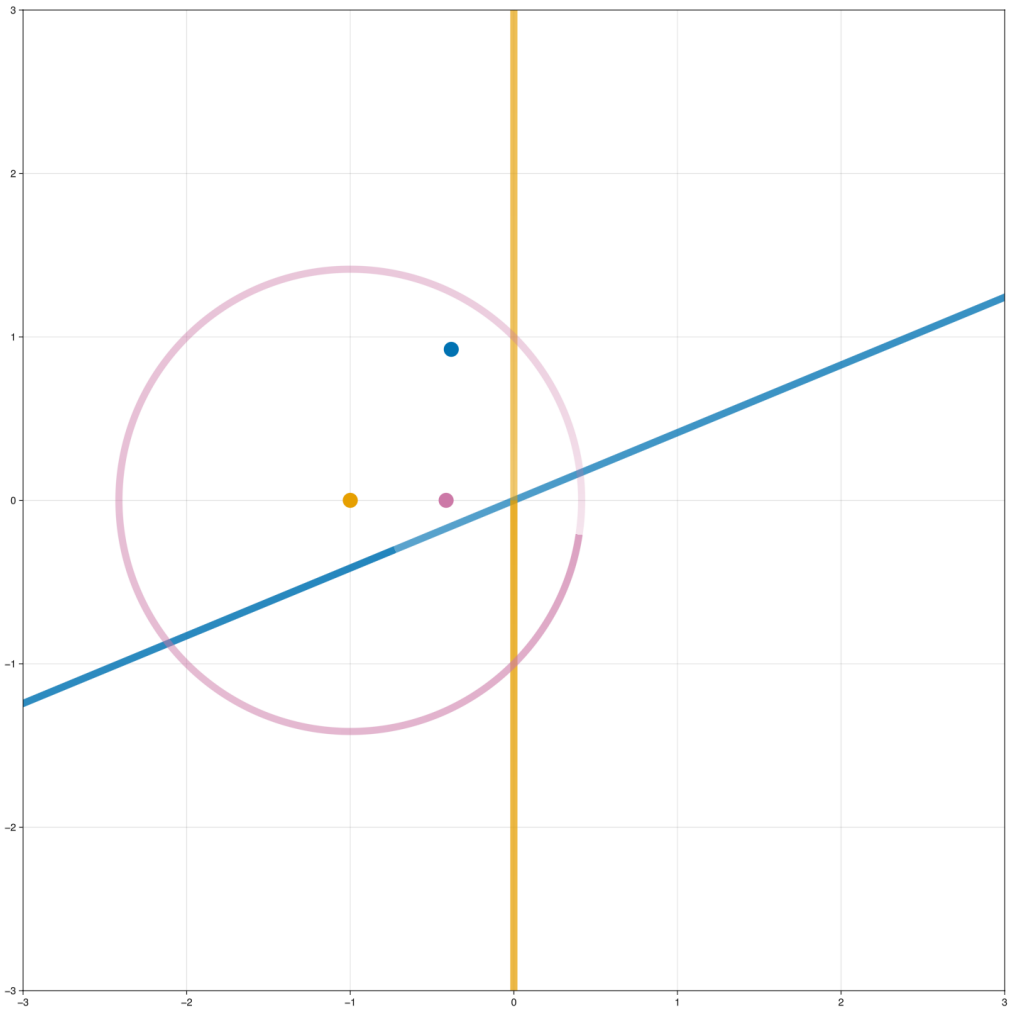


Figure 5.2: Stereographic projection of $\mathbb{P}^1$ showing sample $p, \hat{p}$ pairs for the square torus.

The purple geodesic illustrates the phenomenon of *vortex scattering*, where the cores ap-

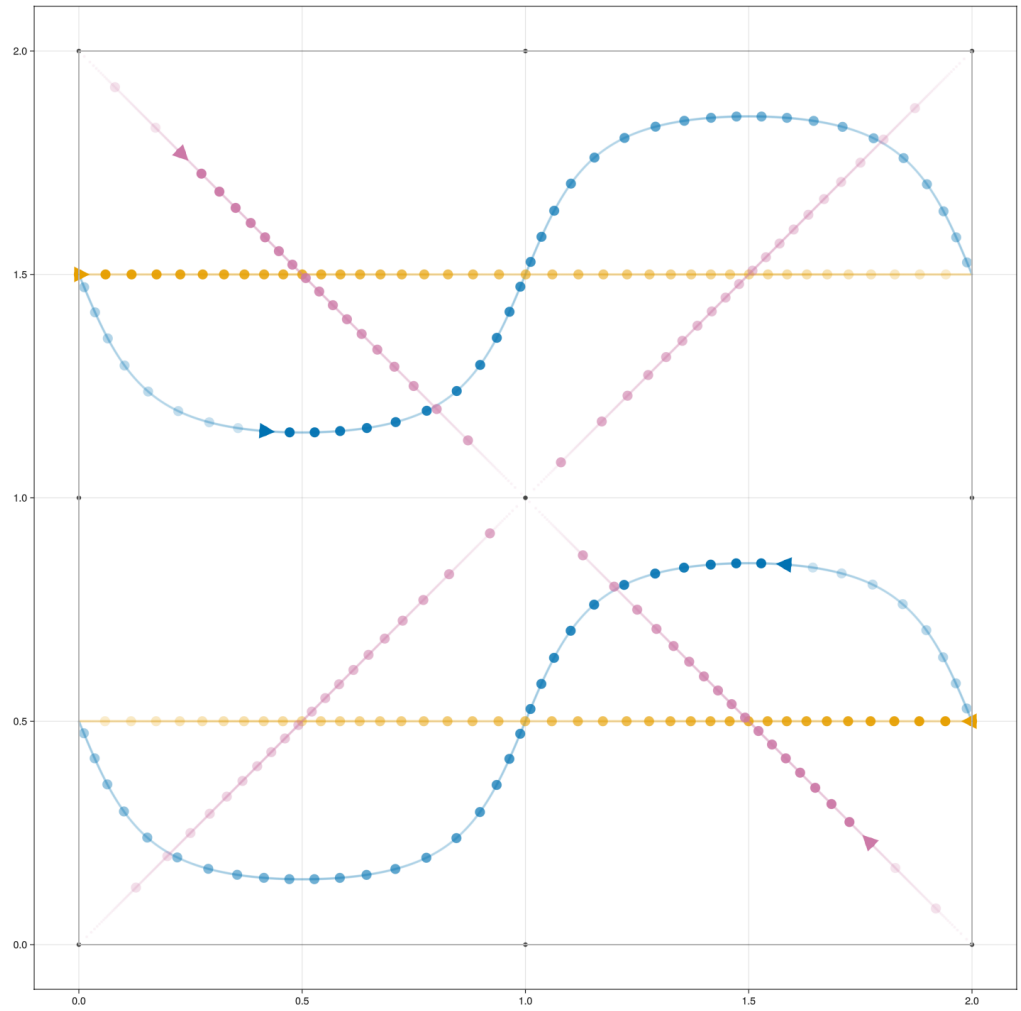


Figure 5.3: Geodesics on Σ corresponding to the $\hat{p}$ in Figure 5.2.

proach one another at right-angles, merge, and then separate again at right angles. A curious feature of the relative dynamics is that scattering can only occur at certain points on Σ. To see this, note that for $\gamma : S^1 \to \hat{\mathcal{M}}_0^\varepsilon$ we either have $\gamma(z_0) = \{x, -x\}$ or $\gamma(z_0) = \{w, w\}$ for $z_0 \in S^1$, the latter case corresponding to the scattering of vortex cores. But recall we only have $\gamma(z_0) = \{w, w\} \in \hat{\mathcal{M}}_0^\varepsilon$ if w is either 0 or a half-period of Σ. So the only points where scattering can occur are $\{\omega_i, \omega_i\}$ for $\omega_i \in \{\omega_0 = 0, \omega_1, \omega_3, \omega_2 = \omega_1 + \omega_3\}$. We say a geodesic $\gamma : S^1 \to \hat{\mathcal{M}}_0^\varepsilon$ is *scattering* if there is some $z_0 \in S^1$ such that $\gamma(z_0) = \{\omega_i, \omega_i\}$ for some i. Accordingly, a geodesic is *non-scattering* if the image of γ consists of two distinct points for all $z_0 \in S^1$.

The following lemma distinguishes the two cases. Recall that we define $f_i := \tilde{\Phi}(\omega_i)$ and have $f_0 = b := \sqrt{\frac{1-\sqrt{\lambda(\tau)}}{1+\sqrt{\lambda(\tau)}}}$.

Lemma 5.3.2.
A geodesic $\gamma : S^1 \to \hat{\mathcal{M}}_0^\varepsilon$ is scattering if and only if p_γ lies in the set,

$$\chi_\Lambda := \bigcup_{i=0}^{3} \operatorname{im} \hat{f}_i. \tag{5.13}$$

Furthermore, χ_Λ consists of either:

- *four great circles intersecting transversely and 3-wise disjointly (i.e. every triple intersection of the $\operatorname{im} \hat{f}_i$ is empty), or*
- *two great circles intersecting transversely.*

The latter case occurs if and only if $\lambda(\tau)$ is real.

Proof. $\gamma(z_0) = \{\omega_i\}$, if and only if $\hat{p}_\gamma(z_0) = f_i$, which is equivalent to $p_\gamma \in \operatorname{im} \hat{f}_i \subset \chi_\Lambda$.

For the second claim, suppose $i \neq j$, then $\operatorname{im} f_i$ and $\operatorname{im} f_j$ intersect tangentially if and only if f_i and f_j are antipodal (the f_i are distinct so f_i and f_j cannot be equal). We now show that the following are equivalent:

1. three of the $\operatorname{im} \hat{f}_i$ intersect at a single point.
2. there is a single $p \in \mathbb{P}^1$ such that $\{f_0, f_1, f_2, f_3\} \subset \operatorname{im} \hat{p}$,
3. $\lambda(\tau) = (f_0, f_3; f_1, f_2)$ is real.
4. two of the $\operatorname{im} \hat{f}_i$ intersect tangentially.
5. χ_Λ is the union of two great circles intersecting transversely.

From which it is clear that either the $\operatorname{im} \hat{f}_i$ intersect transversely and 3-wise disjointly, or they form two great circles intersecting transversely, with the latter case occurring if and only if $\lambda(\tau)$ is real.

$5 \implies 1$ is clear and for $2 \iff 3$ we refer to [2, p. 79]. To show $1 \implies 2$, suppose $p \in \operatorname{im} \hat{f}_i \cap \operatorname{im} \hat{f}_j \cap \operatorname{im} \hat{f}_k$ for distinct i, j, k, then $\{f_i, f_j, f_k\} \subset \operatorname{im} \hat{p}$. Let f_l be the critical value not mentioned. As the critical values are distinct, f_i is not antipodal to at least one of f_j or f_k, WLOG assume this is f_j. We then let $\sigma \in K_4$ be such that $\rho_\sigma : f_i \mapsto f_j, f_k \mapsto f_l$, and see that $\{f_i, f_j, f_l\} \subset \operatorname{im} \widehat{\rho_\sigma p}$. But since f_i and f_j are not antipodal, $\operatorname{im} \hat{p}$ is the unique great circle passing through them both so $\operatorname{im} \widehat{\rho_\sigma p} = \operatorname{im} \hat{p}$ and indeed $\{f_0, f_1, f_2, f_3\} \subset \operatorname{im} \hat{p}$.

For $2 \implies 4$, recall that great circles in $\mathbb{P}^1$ are either circles in the plane or lines through the origin after stereographic projection away from infinity. Thus if the f_i are contained in some $\operatorname{im} \hat{p}$, they are either the vertices of a cyclic quadrilateral in the plane, or they are collinear. Now $\{f_0, f_2, f_3, f_1\} = \left\{b, -\frac{1}{b}, -b, \frac{1}{b}\right\}$ are either the vertices (written cyclically)

of a parallelogram with centre 0, or four collinear points on either $\mathbb{R}$ or $\mathbf{i}\mathbb{R}$. In the latter two cases it is clear that f_0 is antipodal to f_2 or f_1 respectively. In the former, we note that the cyclic parallelogram $\{f_0, f_1, f_3, f_1\}$ is a rectangle; moreover, as it is centred at 0, we must have $|b| = \frac{1}{|b|}$ and so $|b| = 1$. It is therefore clear that $f_0 = b$ and $f_3 = -b$ are antipodal.

For $4 \implies 5$, suppose WLOG that $\operatorname{im} \hat{f}_0$ and $\operatorname{im} \hat{f}_j$ intersect tangentially, i.e. $f_0 \bar{f}_j = -1$. Examining each case of $f_j \in \left\{f_1 = \frac{1}{b}, f_2 = -\frac{1}{b}, f_3 = -b\right\}$ shows that $f_0 \bar{f}_j = -1$ implies that the other two critical values must satisfy $f_k \bar{f}_l = -1$ and thus the great circles they define intersect tangentially. Since any point has a unique antipode and the critical values are distinct, the great circles $\operatorname{im} \hat{f}_0 = \operatorname{im} \hat{f}_j$ and $\operatorname{im} \hat{f}_k = \operatorname{im} \hat{f}_l$ cannot be equal and so must intersect transversely. □

We call χ_Λ the *critical locus*. In Figure 5.4 we plot the ramification values of $\tilde{\Phi}$ and the critical locus for the square torus, the coloured points are the same as in Figure 5.2. In addition to the clear distinction between the scattering and non-scattering behaviour, note how the blue point approaches the critical locus, with the corresponding geodesic being close to scattering.

The critical locus for the square torus is clearly of the second type (two transversely intersecting circles), reflecting the fact that $\lambda(\mathbf{i}) = \frac{1}{2}$. We note in passing that the space $\mathbb{P}^1 \setminus \chi_\Lambda$ has four connected components. We can contrast this with the critical locus for the equianharmonic torus shown in Figure 5.5. In this scenario, the critical locus is of the first type, and $\mathbb{P}^1 \setminus \chi_\Lambda$ has 14 connected components.

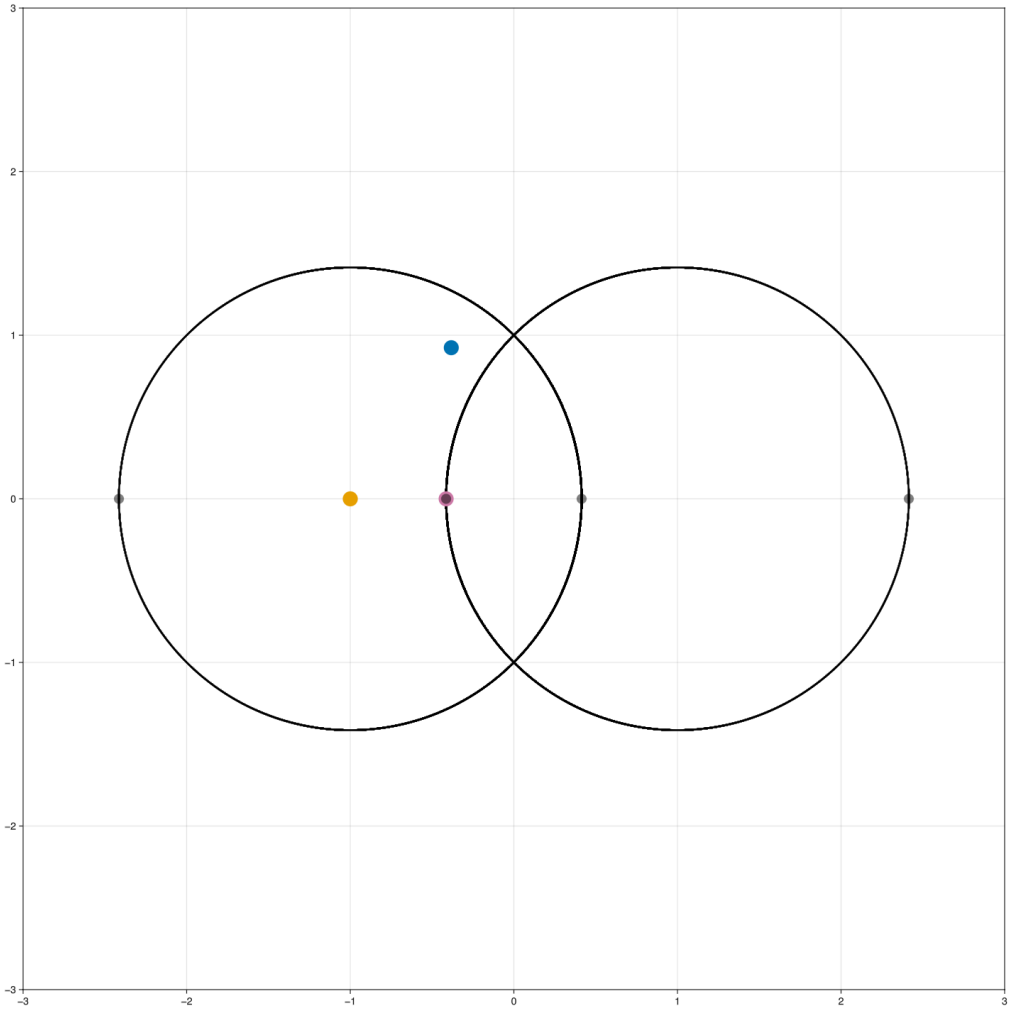


Figure 5.4: Stereographic projection of $\mathbb{P}^1$ showing the ramification values $\{f_i\}$ in gray and critical locus χ_Λ (black) for the square torus. Points corresponding to sample geodesics are also plotted.

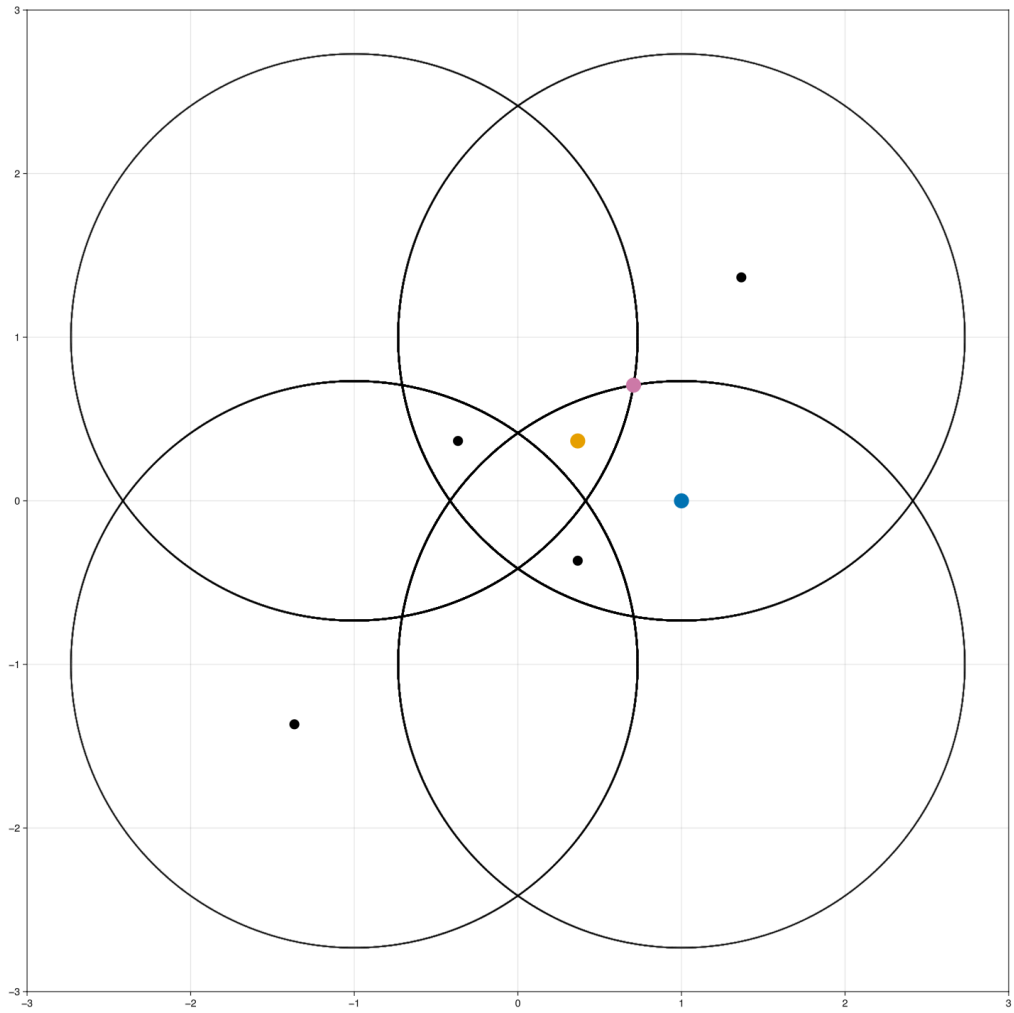


Figure 5.5: Stereographic projection of $\mathbb{P}^1$ showing the $\{f_i\}$ (black points) and χ_Λ (black lines) for the equianharmonic lattice. Points corresponding to sample geodesics are also plotted.

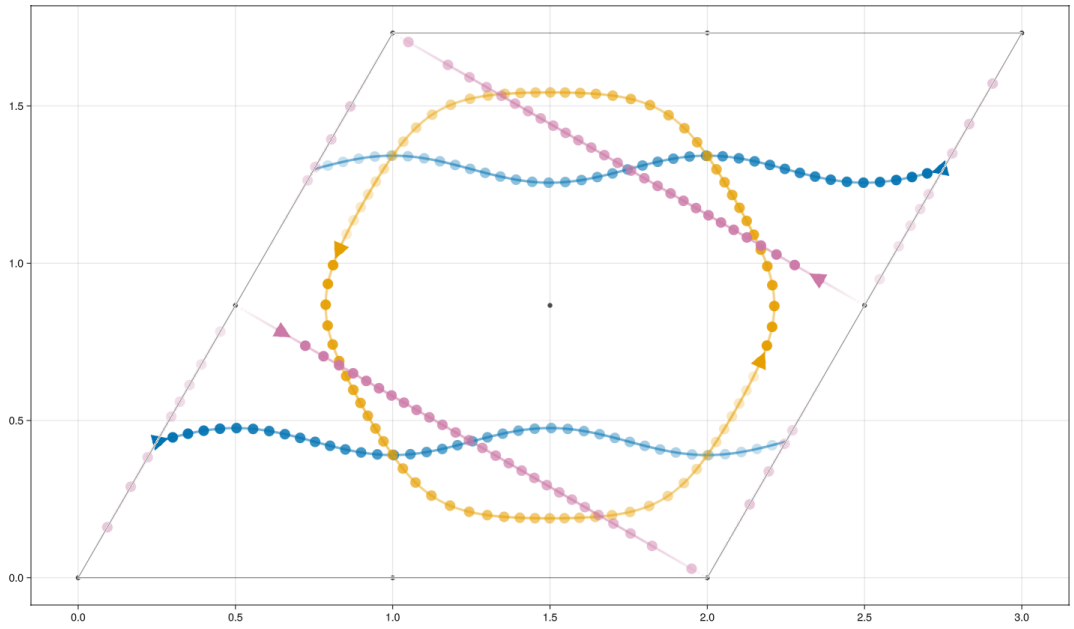


Figure 5.6: Geodesics on Σ corresponding to the coloured points in Figure 5.5.

Indeed, the following lemma shows that the four-circle critical locus is generic amongst all tori. We refer to [15, p. 118] for a proof.

Lemma 5.3.3.
$\lambda(\tau) \in \mathbb{R}$ if and only if Σ is a rectangular torus. Moreover, the real locus of λ, i.e.

$$\{\tau \in \mathbb{C} \mid \operatorname{im}\tau > 0, \lambda(\tau) \in \mathbb{R}\},$$

is nowhere dense in the upper half plane, and so non-rectangular tori are generic amongst all tori.

The complement of the critical locus holds topological significance beyond identifying non-scattering geodesics. Note that in Figure 5.3 the blue and orange geodesics are homotopic, and that the points defining them lie in the same connected component of $\mathbb{P}^1 \setminus \chi_\Lambda$ (Figure 5.4). Indeed, this is the case for all simple closed geodesics on $\hat{\mathcal{M}}_0^\varepsilon \setminus \{\{\omega_i, \omega_i\}\}$.

Lemma 5.3.4.
Connected components of $\mathbb{P}^1 \setminus \chi_\Lambda$ are in bijection with homotopy classes of non-scattering geodesics, i.e. simple closed geodesics on $\mathbb{P}^1 \setminus \{f_i\} \cong \hat{\mathcal{M}}_0^\varepsilon \setminus \{\{\omega_i, \omega_i\}\}$.

To prove this, we require a combinatorial description of $\mathbb{P}^1 \setminus \chi_\Lambda$. Define $B(p) \subset \mathbb{P}^1$ to be the component of $\mathbb{P}^1 \setminus \operatorname{im}\hat{p}$ which contains p. Intuitively, this is the open hemisphere centred at p. Viewing $\hat{p}$ as an oriented geodesic, we note that this is always the component to the left of $\hat{p}$. We can then classify connected components of $\mathbb{P}^1 \setminus \chi_\Lambda$ as follows.

Lemma 5.3.5.
Let $A \subset \{f_i\}$ be a proper non-empty subset that does not contain a pair of antipodes, and let $\mathcal{A}_\Lambda$ be the set of all such sets. Then the map,

$$\begin{aligned} \mathbb{P}^1 \setminus \chi_\Lambda &\to \mathcal{A}_\Lambda, \\ p &\mapsto B(p) \cap \{f_i\} \end{aligned}$$

is locally constant and defines a bijection $\pi_0(\mathbb{P}^1 \setminus \chi_\Lambda) \to \mathcal{A}_\Lambda, [p] \mapsto B(p) \cap \{f_i\}$.

Furthermore, if Σ is rectangular, the number of connected components of $\mathbb{P}^1 \setminus \chi_\Lambda$ is 4, otherwise it is 14.

Proof. For this proof, it is useful to note that for $p, q \in \mathbb{P}^1$ we have the duality $p \in B(q)$ if and only if $B(p) \ni q$; and for subsets $A_1, A_2 \subset \mathbb{P}^1$, $A_1 \subset \bigcap_{q \in A_2} B(q)$ if and only if $\bigcap_{p \in A_1} B(p) \supset A_2$.

We first show that given $p \in X := \mathbb{P}^1 \setminus \chi_\Lambda$, $A_p := B(p) \cap \{f_i\}$ is a proper non-empty subset that does not contain a pair of antipodes, i.e. $A_p \in \mathcal{A}_\Lambda$. The latter condition is immediate as an open hemisphere cannot contain a pair of antipodes. For the former two, we show

that the $\{f_i\}$ are not contained in a single hemisphere. If this is the case, then for any $p \in \mathbb{P}^1$, A_p cannot be empty nor can it be equal to $\{f_i\}$ and thus $B(p) \cap \{f_i\} \in \mathcal{A}_\Lambda$.

View the $\{f_i\}$ as points on $S^2 \subset \mathbb{R}^3$, note that if they lie in a single hemisphere, the mean of their positions (in $\mathbb{R}^3$) is non-zero. However, $\{f_i\}$ is a K_4-orbit, so the average of their positions must be invariant under the K_4 action (i.e. rotations by π by the x, y, and z-axes). Yet, this can never be the case if the average of their positions is non-zero, and thus they cannot lie in a single hemisphere.

Now we show that given $p, q \in X$, $A_p = A_q$ if and only if p and q lie in the same connected component of X; which at once shows that $F : \pi_0(X) \to \mathcal{A}_\Lambda, [p] \mapsto A_p$ is an injective function. We have the following chain of equivalences:

$$\begin{aligned} A_p = A_q &\iff \forall i, (f_i \in A_p \iff f_i \in A_q) \\ &\iff \forall i, (B(f_i) \ni p \iff B(f_i) \ni q) \\ &\iff \forall i, p \text{ and } q \text{ are connected by a regular arc in } \mathbb{P}^1 \setminus \operatorname{im} \hat{f}_i. \end{aligned}$$

The last assertion is clearly true if p and q lie in the same arc component of X since X is a subspace of $\mathbb{P}^1 \setminus \operatorname{im} \hat{f}_i$ for every i. If p and q lie in different arc components of X, then there is some i such that every arc connecting p and q intersects $\operatorname{im} \hat{f}_i$, this is precisely the negation of the last assertion. Thus $A_p = A_q$ if and only if p and q lie in the same arc component of X, and F as defined above is a well defined injective function.

For $A \in \mathcal{A}_\Lambda$, consider $p \in \bigcap_{f \in A} B(f) \cap \bigcap_{f \notin A} B\left(-\frac{1}{f}\right)$. Then $F([p]) = A$, and thus F is surjective.

For the last assertion, note that by Lemma 5.3.3 and the proof of Lemma 5.3.2 the following statements are equivalent:

- Σ is rectangular,
- $\{f_i\}$ contains a pair of antipodal points, and
- $\{f_i\}$ consists of two pairs of antipodal points, $\left\{p, q, -\frac{1}{p}, -\frac{1}{q}\right\}$, such that $p \neq q$.

If Σ is rectangular

$$\mathcal{A}_\Lambda = \left\{ \{p, q\}, \left\{p, -\frac{1}{\bar{q}}\right\}, \left\{-\frac{1}{\bar{p}}, q\right\}, \left\{-\frac{1}{\bar{p}}, -\frac{1}{\bar{q}}\right\} \right\},$$

and thus $|\mathcal{A}_\Lambda| = |\pi_0(X)| = 4$. Otherwise, no two f_i are antipodal, and $\mathcal{A}_\Lambda$ is the set of proper non-empty subsets of $\{f_i\}$. Thus $|\mathcal{A}_\Lambda| = |\pi_0(X)| = 14$. □

Proof. Lemma 5.3.4. It is clear that for arbitrary $p \in X := \mathbb{P}^1 \setminus \chi_\Lambda$, $\hat{p}$ is a simple closed geodesic in the space $Y := \mathbb{P}^1 \setminus \{f_i\}$ and all simple closed geodesics of Y arise in this manner. It is therefore enough to show that $p, q \in X$ lie in the same connected component if and only if $\hat{p}$ and $\hat{q}$ are homotopic as maps $S^1 \to Y$. We begin by noting that since χ_Λ is closed, X is an open subset of a smooth manifold and so connected components are

connected by regular arcs. Thus suppose p and q lie in the same connected component of X and let $\gamma : [0,1] \to X$ be a regular arc between them. Define

$$\hat{\gamma} : S^1 \times [0,1] \to Y,$$
$$(z,t) \mapsto \widehat{\gamma(t)}(z).$$

Note that $\hat{\gamma}(z,t)$ is a rational function of z with coefficients that depend smoothly on t, it is therefore continuous and thus a homotopy between $\hat{p} = \hat{\gamma}\big|_{t=0}$ and $\hat{q} = \hat{\gamma}\big|_{t=1}$.

Now suppose p and q lie in different connected components, and let

$$A_p = B(p) \cap \{f_i\},$$
$$A_q = B(q) \cap \{f_i\}.$$

By the previous lemma, $A_p \neq A_q$ and we are in one of two cases:

- neither of A_p or A_q is a subset of the other, or
- one of A_p or A_q is a proper subset of the other (WLOG suppose $A_p \subsetneq A_q$).

In the first case, pick $r \in A_p \setminus A_q$ and $s \in A_q \setminus A_p$ and consider $\mathbb{P}^1 \setminus \{r,s\}$. This is topologically a twice punctured sphere, so after a homeomorphism we may view r as the north pole of $\mathbb{P}^1$ and s as the south pole. Using Lemma 5.3.5 again, we have $p \in B(r)$ and thus $B(p) \ni r$ i.e. r lies to the left of $\hat{p}$. By a similar logic, we see that s must lie to the right of $\hat{p}$, and $\hat{p}$ winds around the twice punctured sphere with winding number 1. By the exact same method, we can see that $\hat{q}$ winds around the twice punctured sphere with winding number -1. Thus $\hat{p}$ and $\hat{q}$ cannot be homotopic as maps into the twice punctured sphere, and so they cannot be homotopic as maps into $Y \subset \mathbb{P}^1 \setminus \{r,s\}$.

In the second case, pick $r \notin A_q$ and $s \in A_q \setminus A_p$ and consider $\mathbb{P}^1 \setminus \{r,s\}$. We again see that $\hat{p}$ and $\hat{q}$ cannot be homotopic as maps $S^1 \to \mathbb{P}^1 \setminus \{r,s\}$ as $\hat{q}$ has a non-zero winding number and $\hat{p}$ is contractible. And thus they cannot be homotopic as maps into Y. □

The classification of simple closed geodesics on $\hat{\mathcal{M}}_0^\varepsilon \setminus \{\{\omega_i\}\}$ can be further refined. Considering the plot of geodesics on the equianharmonic lattice (Figure 5.6), we see that non-scattering geodesics fall into two categories. Either the paths of the vortex cores are *disjoint* (the blue trajectory), or the vortex cores switch positions after a single period and the paths of the cores are *connected* (the orange trajectory).

Looking at a single component of a disjoint geodesic, we see that the unramified two-to-one cover $\tilde{\Phi} : \Sigma \setminus \{\omega_i\} \to \mathbb{P}^1 \setminus \{f_i\}$ maps it to a simple closed geodesic on $\mathbb{P}^1 \setminus \{f_i\}$, whereas $\tilde{\Phi}$ of a connected geodesic fails to be simple. It is then clear that the disjoint geodesics are precisely those whose homotopy class lies in the image of the pushforward $\tilde{\Phi}_* : \pi_1(\Sigma \setminus \{\omega_i\}) \to \pi_1(\mathbb{P}^1 \setminus \{f_i\})$, which is an index two subgroup of $\pi^1(\mathbb{P}^1 \setminus \{f_i\}) \cong \pi_1\left(\hat{\mathcal{M}}_0^\varepsilon \setminus \{\{\omega_i\}\}\right)$. Empirically, we find that these homotopy classes correspond to the $[p] \in \pi_0(\mathbb{P}^1 \setminus \chi_\Lambda)$ such that $B(p)$ contains precisely two critical values of $\tilde{\Phi}$; implying that

all non-scattering geodesics on rectangular tori are disjoint.

Chapter 6

Further directions and conclusion

We review the results proven in this thesis, discussing their features and direct generalisations/extensions. We then conclude with a brief survey of the current research landscape for vortices, considering some new directions for study.

6.1 Review of results

In this thesis, we stated and proved a rigorous estimate of the difference between the vortex metric restricted to a complete linear system and the Fubini-Study metric on that complete linear system, showing that the two converge uniformly in C^1 in the dissolving limit. In addition to partially resolving a longstanding conjecture of Baptista and Manton [13], Manton and Romão [50], and Rink [58]; we showed that this result was sufficient to guarantee the convergence of geodesics in the dissolving limit, implying that moduli space dynamics along totally geodesic complete linear systems in the dissolving limit is well approximated by geodesic motion of the Fubini-Study metric. We then studied this approximation in two cases:

- Σ equal to the round unit sphere, where we considered two species of vortices in the large degree limit, and discussed the validity of our approximation in this limit, and
- Σ equal to a flat complex torus, where we were able to construct explicit approximations to vortex dynamics, classifying them completely. This approximation was interpreted as modelling the relative dynamics of 2-vortices on flat tori.

The main result of the thesis discussed in chapter 3 relies on a concrete pseudovortex-vortex map (2.14), which is in turn deeply tied to the prescribed scalar curvature equation (2.15) and the Abel-Jacobi map. In Proposition 3.1.1, we show that the pseudovortex-vortex map is a diffeomorphism on complete linear systems, but much more is true. It is known that the pseudovortex-vortex map is a diffeomorphism between the two moduli spaces (see e.g. [23]), and the results in this thesis show that it respects the stratification of the two spaces by complete linear systems. If we consider the moduli space as the space of divisors, $\mathrm{Div}^N_+(\Sigma)$, then much is known about this stratification. The strata are parametrised by

the Jacobian, giving rise to a *Picard bundle*, $\mathrm{AJ} : \mathrm{Div}^N_+(\Sigma) \to \mathrm{Jac}(\Sigma)$ [51]. Indeed, one can compute via Riemannian-Roch that if

$$\deg(L) + \chi(g) \geq 0, \tag{6.1}$$

then the Picard bundle is a fibre bundle with constant fibre $\mathbb{P}^{N-g}$.

This suggests some immediate generalisations of the metric approximation result. As it stands, the result is only valid on a single fibre, and we are unable to compare distinct fibres even if they are diffeomorphic; one might ask whether we can do more if we are in the case where (6.1) holds, perhaps approximating the vortex metric on the entire moduli space by some metric which is the Fubini-Study metric on the fibre. Two first steps towards this would be a generalisation of Proposition 3.1.1 to consider the pseudovortex-vortex map on some family of complete linear systems; and extending Theorem 3.1.4 so that in the cases where the degree criterion (6.1) holds, the two metrics still converge uniformly along complete linear systems, but with a convergence rate that is uniform across the entire moduli space.

The former is essentially an exercise in complex geometry; while not presented here, tangent spaces of vortices in the general sense are well understood (see e.g. [57]), and there are clear links to the algebraic understanding of the Picard bundle which can aid in the construction of results. The latter is more subtle, it is not immediately obvious how to remove the dependency on the choice of complete linear system in the proof of Theorem 3.1.4, but we believe that the tools and estimates constructed here should be sufficiently flexible to remove this dependency. We certainly believe that the method of proof should not require significant changes for this extension.

One obstacle to the final goal of approximating the vortex metric on the entire moduli space is that it is not clear what the choice of metric along $\mathrm{Jac}(\Sigma)$ should be. Manton and Romão [50] consider a flat metric known as the *Bergman metric* in their work, but the expression of the Bergman metric in terms of pseudovortex data is unclear. Furthermore, the physical significance of the degree criterion (6.1) is also unknown. While the algebraic motivation behind it is quite clear, we do not currently know of any physical reason as to why the geometry of the vortex moduli space should simplify when (6.1) is met. Although it is not crucial for the development of results, physical motivation is important as a guiding light, so the lack of an obvious physical reason for the criterion causes some hesitancy.

Another immediate extension is improving the regularity of convergence to C^2. This has the benefit of guaranteeing convergence of the curvature of the moduli space, a property that influences the quantisation of vortices, but an immediate roadblock is the tractability of the computations required. Already, the expression of the first coordinate derivative of the defect tensor $\alpha(\varepsilon)$ (3.20) contains on the order of 30 distinct terms before gathering into families. To prove convergence in C^2, we would require estimating the second coordinate derivative of $\alpha(\varepsilon)$, yielding an expression which could possibly have an order of magnitude more terms. Moreover, we would require estimates of increasing regularity on these terms,

again lengthening the proof. So while it is technically *possible* that the machinery presented here is suitable for proving convergence in C^2, we believe that such a proof is not tractable with these tools and thus a different approach is required.

The study of the large degree limit in chapter 4 offers comparatively few opportunities for direct generalisation. The essential conclusion of that chapter is that the convergence result from chapter 3 as we have stated here is ill suited for considering the large degree limit, so there is little reason to consider this limit further from the perspective of vortices. However, one possible improvement would be to make the proofs more pliable and to better situate the theory. The proof of unboundedness in the unipolar case Proposition 4.3.1 is very brittle and strange. From the proof alone, it is not immediately clear what quality of unipolar pseudovortices leads to the solution of the PSC equation $\Delta u - \frac{c}{\operatorname{Vol}\Sigma} + ce^u f = 0$ (2.15) being unbounded. We therefore believe that a proper understanding of these results will come from studying the PSC equation on spheres more generally, considering a general f with unit L^1-norm and a sphere which is not necessarily round.

Finally, there are several directions in which to generalise the results of chapter 5. The two most obvious directions are to consider higher degree vortices or Riemann surfaces of higher genus, yet both come with complications. In either case, the metric decoupling argument Proposition 5.1.4 does not directly generalise, so the splitting of the dynamics into relative and drift dynamics is not as clear. Yet this is not an insurmountable issue, it is possible that generalising the metric approximation result along the lines suggested earlier will present a clearer view of the dynamics for concrete Riemann surfaces resolving the above issue. Another obstacle is finding complete linear systems which are totally geodesic. Yet this issue benefits greatly from better algebraic understanding of complete linear systems and encoding the problem in computer algebra. If the existence of totally geodesic complete linear systems on constant curvature Riemann surfaces can be reduced to an algebraic criterion, it is possible that a computer search of e.g. the *The L-functions and Modular Forms Database* [42] could be performed yielding a large number of surface geometries which can be studied in this manner. Some initial studies have been performed as to whether these results can be extended to hyperelliptic curves, in particular special hyperelliptic curves that are ramified covers of elliptic curves. We expect that certain (not necessarily complete) linear systems on these curves are fixed by a symmetry of the curve and thus totally geodesic, indeed Rink [58] constructs many such linear systems on general hyperelliptic curves. It might then be possible to compute the restriction of the Fubini-Study metric on these linear systems, yielding computable approximations of geodesics.

6.2 The current research landscape for vortices

Vortices in the Abelian Higgs model are a source of continued interest, with extensions including considering vortices on surfaces with boundary [24] and considering fields which take values in some non-linear target e.g. [26]. For this section, we focus on two themes which have seen recent academic interest, vortex shape modes, and gravitating vortices.

[8]. The theory outlined in this thesis only considers the dynamics of vortices which meet the Bogomolnyi bound, but it has long been known that there are excited states of vortices which have an energy above the Bogomolnyi bound which is not radiated away [9]. In the 1-vortex case, this results in the Higgs field oscillating with a specific frequency; and in recent years there is a growing body of work studying the existence and structure of these excitations [3]–[6], [26], [33], [37], [53].

What this work shows is that these normal modes have a significant effect on the dynamics of vortices, requiring corrections to be made to the moduli space dynamics. It is therefore clear that any further development of the moduli space approximation must take these modes into account. A current gap in the research literature is the study of normal modes for compact Riemann surfaces. This is likely to be a model of interest, as unlike the non-compact case, the second order problem which governs the existence of these shape modes (see e.g. [6]) will have a discrete spectrum. Indeed, heuristic calculations suggest that it might be possible to determine the spectrum exactly at the dissolving limit, raising the possibility of some spectral convergence result for excited BPS vortices close to the dissolving limit.

The second area of interest is coupling vortices to the Riemannian metric on Σ, i.e. considering gravitating vortices [7]. This has immediate physical importance as a model of cosmic strings in the early universe, yet they are also of intrinsic mathematical interest as a relatively new variation of the Abelian Higgs model [8]. The existence of gravitating vortices has been demonstrated for surfaces of positive genus [7], and it is known that it exhibits a type of criticality distinct from the dissolving limit. However, there are still many open questions regarding the structure of the space of solutions. For example, it is not known whether the space of solutions admits a moduli space structure. While resolving this question in full generality is probably intractable, understanding it in some restricted e.g. for certain tori is of interest.

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Appendix A

Auxiliary estimates for Chapter 3

This appendix collects various estimates which we use in chapter 3. It is broken into three parts:

- §A.1: a product estimate for Sobolev spaces,
- §A.2: estimates on pseudovortex Higgs fields, and
- §A.3: estimates on C^k-norms on tensors on projective space.

It is useful to recall the definition of the function norms (Definition 2.1.7). Let (M, g^M) be a compact Riemannian manifold with Levi-Civita connection ∇. And given some smooth function $f \in C^\infty(M)$, let $\nabla^i f \in \Omega^0(M; (T^*M)^{\otimes i})$ be its i-fold total covariant derivative. Note that g^M induces a pointwise norm on tensors, written $|\cdot|$.

Let $f \in \Omega^0(M) = C^\infty(M)$. For $k \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{R}_{\geq 1}$ we define

$$\|f\|^p_{W^{k,p}} := \int_M \sum_{i=0}^k \left|\nabla^i f\right|^p \mathrm{dVol},$$
$$\|f\|_{C^k} := \max\left\{ \left\| \left|\nabla^i f\right| \right\|_{C^0} \mid 0 \leq i \leq k \right\}.$$

It is convenient to define $\|\cdot\|_{H^k} := \|\cdot\|_{W^{k,2}}$ and $\|\cdot\|_{L^p} := \|\cdot\|_{W^{0,p}}$.

As before, we note that if $W^{k,p}(M)$ is the space of weakly k-differentiable functions which are pth power integrable, the Meyers-Serrin theorem (see e.g. [27, §7.6]) states that we can identify this space with the metric completion of $C^k(M)$ with respect to $\|\cdot\|_{W^{k,p}}$. That being said, in this chapter we work entirely in the smooth category, i.e. all functions and sections are assumed to be smooth. We also note there are natural extensions of these definitions to metric vector bundles.

A.1 A product estimate for Sobolev spaces

We have the following useful bound on H^k-norms of a product,

Lemma A.1.1.
Let (M, g^M) be a compact Riemannian manifold with Levi-Civita connection ∇. Suppose $f, g \in C^\infty(M)$, then for each $k \geq 0$,

$$\|fg\|_{H^k} \leq \left(2^{k+1} - 1\right)\|f\|_{C^k}\|g\|_{H^k}$$

This estimate is clearly non-sharp, yet it is useful for low orders of differentiability.

Proof. Suppose f, g are smooth functions on M, then in some local coordinate chart we have (by repeated application of the Leibniz rule)

$$\begin{aligned}(\nabla^s(fg))_{i_1\cdots i_s} &= \sum_{J\subset\{1,\ldots,s\}} \left(\nabla^{|J|}f\right)_{(i_j)_{j\in J}} \left(\nabla^{s-|J|}g\right)_{(i_j)_{j\notin J}} \\ &= \sum_{t=0}^{s} \sum_{J\subset\{1,\ldots,s\},|J|=t} \left(\nabla^t f\right)_{(i_j)_{j\in J}} \left(\nabla^{s-t}g\right)_{(i_j)_{j\notin J}}.\end{aligned}$$

Now note the terms of the interior summation are all the possible permutations of the indices of the s-tensor $\nabla^t f \otimes \nabla^{s-t} g$. In local coordinates, the pointwise norm of an s-tensor α is given by $\sqrt{g^{i_1 j_1}\cdots g^{i_s j_s}\alpha_{i_1\cdots i_s}\alpha_{j_1\cdots j_s}}$; this is invariant under a permutation of the indices of α, so we see the pointwise norm of every interior summand in the equation above is equal to $\left|\nabla^t f \otimes \nabla^{s-t} g\right| = \left|\nabla^t f\right| \cdot \left|\nabla^{s-t} g\right|$, This yields the pointwise *coordinate free* estimate

$$|\nabla^s(fg)| \leq \sum_{t=0}^{s} \binom{s}{t} \left|\nabla^t f\right| \cdot \left|\nabla^{s-t} g\right|.$$

Via the triangle inequality and the fact that $\|fg\|_{L^2} \leq \|f\|_{C^0}\|g\|_{L^2}$ we then have

$$\begin{aligned}\left\||\nabla^s(fg)|\right\|_{L^2} &= \left\|\sum_{t=0}^{s} \binom{s}{t} \left|\nabla^t f\right| \cdot \left|\nabla^{s-t} g\right|\right\|_{L^2} \\ &\leq \sum_{t=0}^{s} \binom{s}{t} \left\|\left|\nabla^t f\right| \cdot \left|\nabla^{s-t} g\right|\right\|_{L^2} \\ &\leq \sum_{t=0}^{s} \binom{s}{t} \left\|\left|\nabla^t f\right|\right\|_{C^0} \left\|\left|\nabla^{s-t} g\right|\right\|_{L^2} \\ &\leq \sum_{t=0}^{s} \binom{s}{t} \|f\|_{C^t}\|g\|_{H^{s-t}} \\ &\leq \|f\|_{C^s}\|g\|_{H^s} \sum_{t=0}^{s} \binom{s}{t} \\ \left\||\nabla^s(fg)|\right\|_{L^2} &\leq 2^s\|f\|_{C^s}\|g\|_{H^s}.\end{aligned}$$

The third last and penultimate inequalities follow from the definitions given at the start

of this Appendix. We can then compute

$$\begin{aligned}
\|fg\|_{H^k}^2 &= \sum_{s=0}^{k} \||\nabla^s(fg)|\|_{L^2}^2 \\
\|fg\|_{H^k} &\leq \sum_{s=0}^{k} \||\nabla^s(fg)|\|_{L^2} \\
&\leq \sum_{s=0}^{k} 2^s \|f\|_{C^s} \|g\|_{H^s} \\
&\leq \left(\sum_{s=0}^{k} 2^s\right) \|f\|_{C^k} \|g\|_{H^k} \\
\|fg\|_{H^k} &\leq \left(2^{k+1} - 1\right) \|f\|_{C^k} \|g\|_{H^k},
\end{aligned}$$

and we are done. □

For us, the two most useful forms of this estimate are the standard estimate $\|fg\|_{L^2} \leq \|f\|_{C^0}\|g\|_{L^2}$, and also $\|fg\|_{H^1} \leq 3\|f\|_{C^1}\|g\|_{H^1}$, we will use both frequently. The same proof also generalises to when f and g are sections of a metric vector bundle and we wish to bound the H^k-norm of $\langle f, g\rangle$.

A.2 Estimates on Higgs fields

We require multiple estimates on pseudovortex Higgs fields which are independent of position in the moduli space. These are tackled in much the same way: we first show that the quantities to estimate depend continuously on *both* position on the moduli space and position on the Riemann surface, we can then remove the dependence on the Riemann surface using Corollary A.2.4 from the section below, yielding the estimate via compactness of the moduli space.

A.2.1 C^0 norms on product spaces

We recall the definition of the compact-open topology on function spaces (see e.g. [52, Ch. 6.2]).

Definition A.2.1.
Let X, Y be topological spaces and write $C^0(X, Y)$ for the set of continuous functions from X to Y. Given $K \subset X$ compact, $U \subset Y$ open, write $W(K, U)$ for the $f \in C^0(X, Y)$ such that $f(K) \subset U$. The set of finite intersections of the $W(K, U)$ (including null intersections)

$$\left\{ \bigcap_{i=1}^{n} W(K_i, U_i) \subset C^0(X, Y) \;\middle|\; K_i \subset X \text{ compact }, U_i \subset Y \text{ open} \right\},$$

is a basis for the *compact-open topology* on $C^0(X, Y)$ (i.e. the $W(K, U)$ form a subbase for the compact-open topology). ◇

Going forward, we shall always equip $C^0(X, Y)$ with the compact-open topology.

Lemma A.2.2.
Let X be a compact Hausdorff space, then the C^0-norm when viewed as a function

$$\|\cdot\|_{C^0} : C^0(X, \mathbb{R}) \to \mathbb{R}_{\geq 0},$$

is continuous.

Proof. Let $f \in C^0(X, \mathbb{R})$ and let $(a, b) \ni \|f\|_{C^0}$, we shall construct an open set $U \ni f$ such that $\{\|g\|_{C^0} \mid g \in U\} \subset (a, b)$, as f and $(a, b) \ni \|f\|_{C^0}$ are arbitrary, this then implies that the function $g \mapsto \|g\|_{C^0}$ is continuous.

Choose some sub-interval (a', b') such that $[a', b'] \subset (a, b)$ and $\|f\|_{C^0} \in (a', b')$, then let K be the closure of $f^{-1}((-b', -a') \cup (a', b'))$. K is non-empty as compactness of X assures f attains its extrema, at least one of which lies in $\{-\|f\|_{C^0}, \|f\|_{C^0}\} \subset (-b', -a') \cup (a', b')$. We then see that

$$f \in W(K, (-b, -a) \cup (a, b)) \cap W(X, (-b, b)) =: U.$$

So for every $g \in U$: there exists $x \in K$ such that $g(x) \in (-b, -a) \cup (a, b)$ and so $\|g\|_{C^0} > a$; and for any $x \in X$, we have $g(x) \in (-b, b)$ so $\|g\|_{C^0} < b$. Thus $\{\|g\|_{C^0} \mid g \in U\} \subset (a, b)$ and thus $\|\cdot\|_{C^0}$ is continuous. □

The key proposition for this section is the following.

Proposition A.2.3.
Let X, Y, Z be locally compact Hausdorff (LCH) spaces. Then there is a well defined association map

$$\alpha : C^0(X \times Y, Z) \to C^0(X, C^0(Y, Z)),$$
$$(\alpha f(x))(y) := f(x, y).$$

i.e. given a continuous function $f : X \times Y \to Z$, the function $x \mapsto f(x, \cdot)$ is continuous.

Writing $\alpha f(x)$ might be unfamiliar to some readers, so for this section only we offer the alternative notation $f_x : Y \to Z, y \mapsto f(x, y)$, one can also think of this as the partial application $f(x, \cdot)$.

Proof. For this proof, given a topological space S, $U_{(S)}$ will be an open subset of S.

Let $f \in C^0(X \times Y, Z)$; given any $x \in X$, it is clear that $\alpha f(x) = f_x : Y \to Z$ is continuous, so α maps continuous functions $C^0(X \times Y, Z)$ to (not necessarily continuous) functions $X \to C^0(Y, Z)$. We show that αf is continuous in $x \in X$, i.e. α maps $C^0(X \times Y, Z)$ to $C^0(X, C^0(Y, Z))$.

Fix $x \in X$ and consider $f_x \in C^0(Y,Z)$, let $W(K,U_{(Z)}) \ni f_x$. As before, to show that $\alpha f = (x' \mapsto f_{x'})$ is continuous in X, it suffices to find $U_{(X)} \ni x$ such that

$$\left\{ f_{x'} = \alpha f(x') \;\middle|\; x' \in U_{(X)} \right\} \subset W(K,U_{(Z)}).$$

First note that since $f_x \in W(K,U_{(Z)})$, we have $f(\{x\} \times K) \subset U_{(Z)}$. We let $U_{(X\times Y)} := f^{-1}(U_{(Z)})$ and note that $\{x\} \times K \subset U_{(X\times Y)}$. We cover $U_{(X\times Y)}$ by basic opens

$$\left\{ V \times W \subset U_{(X\times Y)} \;\middle|\; V \subset X \text{ open }, W \subset Y \text{ open } \right\}.$$

This family also covers $\{x\}\times K$, so by compactness, we have a finite subcover of $\{x\}\times K$ by basic opens, $\left\{ V_i \times W_i \subset U_{(X\times Y)} \;\middle|\; V_i \subset X \text{ open }, W_i \subset Y \text{ open } \right\}$. We discard any $V_i \times W_i$ that are disjoint from $\{x\} \times K$, so we then have $x \in \bigcap_i V_i =: U_{(X)}$. We set $U_{(Y)} := \bigcup_i W_i$ (which contains K) and note that we have

$$\{x\} \times K \subset U_{(X)} \times U_{(Y)} \subset U_{(X\times Y)}.$$

Now let $x' \in U_{(X)}$, we verify the continuous function $f_{x'} = \alpha f(x') : Y \to Z$ is in $W(K,U_{(Z)})$. We have, by definition, that for $y \in K$, $f_{x'}(y) = f(x',y)$, but since $(x',y) \in U_{(X)} \times K \subset U_{(X)} \times U_{(Y)} \subset U_{(X\times Y)}$, we must have $f_{x'}(y) = f(x',y) \in U_{(Z)}$ as $U_{(X\times Y)}$ is the pre-image of $U_{(Z)}$ by f. Thus we have $\left\{ f_{x'} = \alpha f(x') \;\middle|\; x' \in U_{(X)} \right\} \subset W(K,U_{(Z)})$, i.e. for an arbitrary x and arbitrary open $W(K,U_{(Z)}) \ni \alpha f(x)$, we have found an open set containing x whose image under αf is contained in $W(K,U_{(Z)})$. Continuity of αf follows. □

We then use this result to show the following.

Corollary A.2.4.
Let X, Y be LCH spaces and suppose further that Y is compact. Then the map

$$\beta : C^0(X \times Y; \mathbb{R}) \to C^0(X, \mathbb{R}_{\geq 0}),$$
$$\beta f(x) := \|\alpha f(x)\|_{C^0} = \|f_x\|_{C^0}$$

is well-defined.

One should think about this map as a partial C^0-norm along the compact fibres of $X \times Y$. Note that we are *not* making the claim that β is a continuous map on function spaces, merely that βf is continuous for any continuous $f : X \times Y \to \mathbb{R}$.

Proof. Let $f \in C^0(X \times Y, \mathbb{R})$, then by Proposition A.2.3, $\alpha f \in C^0(X, C^0(Y,\mathbb{R}))$. Now view the C^0-norm on $C^0(Y,\mathbb{R})$ as a function $\|.\|_{C^0} : C^0(Y,\mathbb{R}) \to \mathbb{R}$. We can see that

$$\begin{aligned} \beta f(x) &= \|\alpha f(x)\|_{C^0} \\ &= (\|\cdot\|_{C^0} \circ (\alpha f))(x) \\ \beta f &= \|\cdot\|_{C^0} \circ (\alpha f). \end{aligned}$$

$\alpha f : X \to C^0(Y, \mathbb{R})$ is continuous in $x \in X$, and by Lemma A.2.2, $\|\cdot\|_{C^0} : C^0(Y, \mathbb{R}) \to \mathbb{R}$ is continuous in $g \in C^0(Y, \mathbb{R})$. Thus $\beta f : X \to \mathbb{R}$ is continuous and β is well defined. □

The map β is crucial for the ensuing estimates as it allows us to transfer continuity in both moduli space and Σ coordinates to continuity of the C^0-norm in just the moduli space coordinate. In what follows, one should remember that β (and α) application binds more tightly than function application, i.e. βf is a function and that one should read $\beta f(x)$ as $(\beta f)(x)$.

We state some useful corollaries, in all of these we let X, Y be LCH and suppose that Y is compact. We omit proofs as these are mainly syntactic consequences that we wish to bring to the forefront of the reader's mind. The first is the main motivation for introducing β.

Corollary A.2.5.
Suppose we have an estimate

$$\beta f(x) \leq C,$$

for some $C \geq 0$ not depending on $x \in X$. Then we have the following estimate for all $x \in X$

$$\|\alpha f(x)\|_{C^0} = \|f_x\|_{C^0} \leq C.$$

The second allows us to perform some algebraic manipulations in the construction of later estimates.

Corollary A.2.6.
We have

$$\beta(af + g)(x) \leq |a(x)|\beta f(x) + \beta g(x),$$

and

$$\beta(fg)(x) \leq (\beta f(x))(\beta g(x)),$$

where $a \in C^0(X, \mathbb{R})$, and $f, g \in C^0(X \times Y, \mathbb{R})$. If $f = g$, we have equality.

And the last links β to Sobolev norms.

Corollary A.2.7.
Suppose that Y is Riemannian. Then for every $x \in X$,

$$\|\alpha f(x)\|_{L^p} = \|f_x\|_{L^p} \leq \beta f(x)(\operatorname{Vol} Y)^{1/p}.$$

A.2.2 Estimates on pseudovortex Higgs fields

We recall some definitions. Let $(L, H) \to (\Sigma, g^\Sigma)$ be a fixed Hermitian line bundle over a Riemann surface with degree $\deg L = N > 0$. Recall that if we let $D_{\hat{A}}$ be a fixed constant curvature unitary connection on L (which is equivalent to choosing a holomorphic structure on L), $\ker D^{0,1}_{\hat{A}}$ is a finite dimensional complex vector space. Suppose that

$\dim\ker D^{0,1}_{\hat{A}} = d+1 > 0$ and let $\{s_0, \mathbf{i}s_0, \dots, s_d, \mathbf{i}s_d\}$ be an L^2-orthonormal $\mathbb{R}$-basis for $\ker D^{0,1}_{\hat{A}}$. We let $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}} = \mathbb{P}\ker D^{0,1}_{\hat{A}} \cong \mathbb{P}^d$ and note that the L^2-structure on the space of sections of L determines an explicit model for $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$ as the quotient of the unit L^2-sphere.

Define

$$\begin{aligned} \hat{\phi} : \mathbb{C}^d \times \Sigma &\to L, \\ (\mathbf{z}, x) &\mapsto \langle \mathbf{z} \rangle^{-1} \left(s_0(x) + \sum_{j=1}^{d} z^j s_j(x) \right), \end{aligned} \tag{A.1}$$

where $\mathbf{z} = \left(z^i\right)$ and $\langle \mathbf{z} \rangle := \sqrt{1 + \sum_{i=1}^{d} |z^i|^2}$. A key property to note is that $\langle \mathbf{z} \rangle^{-1}$ is continuous and $\langle \mathbf{z} \rangle^{-1} < \min\left\{1, |\mathbf{z}|^{-1}\right\}$ for all $\mathbf{z} \in \mathbb{C}^d$. Note that $\alpha\hat{\phi}(\mathbf{z}) \in \ker D^{0,1}_{\hat{A}}$, indeed $\alpha\hat{\phi}$ defines an explicit chart $\mathbb{C}^d \to U_0 \subset \mathbb{P}\ker D^{0,1}_{\hat{A}}$. Other charts can be defined by permuting the basis vectors in the definition above.

We define vector fields on this chart

$$\begin{aligned} \hat{\phi}_{x^i} &= \langle \mathbf{z} \rangle^{-1} s_i - \langle \mathbf{z} \rangle^{-2} \bar{z}^i \hat{\phi}, \\ \hat{\phi}_{y^i} &= \langle \mathbf{z} \rangle^{-1} \mathbf{i} s_i - \langle \mathbf{z} \rangle^{-2} \bar{z}^i \mathbf{i} \hat{\phi}. \end{aligned}$$

And we let $a \in \{1, \dots, 2d\}$ be an index and define

$$\hat{\phi}_a := \begin{cases} \hat{\phi}_{x^a} & : a \leq d, \\ \hat{\phi}_{y^{a-d}} & : a > d \end{cases}.$$

Again, these definitions extend to any chart. The first set of estimates we wish to construct are as follows.

Lemma 3.1.8.
In what follows, the constants $C_\bullet$ depend on $(L, H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$, and a choice of $\left[D_{\hat{A}}\right]$ in addition to the quantities in the subscript. In particular, they do not *depend on position in $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$ (i.e. the exact choice of $\hat{\phi}$).*

1. *(Pseudovortex regularity) For every $k \geq 0$, there is a $C_k > 0$ such that*

$$\left\| \hat{\phi} \right\|_{C^k} < C_k. \tag{3.26}$$

2. *(Pseudovortex modulus regularity) For every $k \geq 0$, there is a $C_k > 0$, such that*

$$\left\| \left| \hat{\phi} \right|^2 \right\|_{C^k} < C_k. \tag{3.27}$$

3. *For every $k > 0$, there is a $C_k > 0$, such that for every affine coordinate chart on*

$\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$ and every $a \in \{1, \dots, 2d\}$ we have

$$\left\| \hat{\phi}_a \right\|^2_{H^k} < C_k. \tag{3.28}$$

4. *For every $k > 0$, there is a $C_k > 0$, such that for every affine coordinate chart on $\hat{\mathcal{M}}^{\varepsilon}_{\hat{A}}$ and every $a \in \{1, \dots, 2d\}$ we have*

$$\left\| \left\langle \hat{\phi}_a, \hat{\phi}_b \right\rangle \right\|^2_{H^1} < C_k. \tag{3.29}$$

Proof. Consider $\hat{\phi}$ as a function $\mathbb{C}^d \times \Sigma \to L$ using the definition on charts given in (A.1). We then have

$$\begin{aligned} \nabla^i \hat{\phi} &= \nabla^i \left(\langle \mathbf{z} \rangle^{-1} \left(s_0 + \sum_{j=1}^{d} z^j s_j \right) \right) \\ &= \langle \mathbf{z} \rangle^{-1} \left(\nabla^i s_0 + \sum_{j=1}^{d} z^j \nabla^i s_j \right). \end{aligned}$$

One should view the $\nabla^i s_j$, and thus $\nabla^i \hat{\phi}$, as sections of the bundle $(T^*\Sigma)^{\otimes i} \otimes L \to \Sigma$. The pointwise norm $\left|\nabla^i s_j\right|$ is then the contraction of this tensor with the bundle metric on L, $\langle \cdot, \cdot \rangle$, and the i-fold contraction with the dual metric on Σ, $\left(g^{\Sigma}\right)^{-1}$. While we always work in the chart U_0, our arguments extend to any other chart by virtue of the fact that changing the chart amounts to permuting the basis vectors.

We have

$$\begin{aligned} \left|\nabla^i \hat{\phi}\right|^2 &= \langle \mathbf{z} \rangle^{-2} \left| \nabla^i s_0 + \sum_{j=1}^{d} z^j \nabla^i s_j \right|^2 \\ &= \langle \mathbf{z} \rangle^{-2} \left|\nabla^i s_0\right|^2 + 2 \langle \mathbf{z} \rangle^{-2} \sum_{j=1}^{d} \left\langle \nabla^i s_0, z^j \nabla^i s_j \right\rangle + \langle \mathbf{z} \rangle^{-2} \sum_{j,k=1}^{d} \left\langle z^j \nabla^i s_j, z^k \nabla^i s_k \right\rangle \\ &= \langle \mathbf{z} \rangle^{-2} \left|\nabla^i s_0\right|^2 + 2 \langle \mathbf{z} \rangle^{-2} \sum_{j=1}^{d} \left(\operatorname{Re} z^j \left\langle \nabla^i s_0, \nabla^i s_j \right\rangle + \operatorname{Im} z^j \left\langle \nabla^i s_0, \mathbf{i} \nabla^i s_j \right\rangle \right) \\ &+ \langle \mathbf{z} \rangle^{-2} \sum_{j,k=1}^{d} \left(\operatorname{Re} \left(\bar{z}^j z^k \right) \left\langle \nabla^i s_j, \nabla^i s_k \right\rangle + \operatorname{Im} \left(\bar{z}^j z^k \right) \left\langle \nabla^i s_j, \mathbf{i} \nabla^i s_k \right\rangle \right). \end{aligned}$$

$\left|\nabla^i \hat{\phi}\right|^2 : \mathbb{C}^d \times \Sigma \to \mathbb{R}_{\geq 0}$ is continuous as it is the product and sum of continuous functions on $\mathbb{C}^d \times \Sigma$, so by Corollary A.2.4 $\beta \left|\nabla^i \hat{\phi}\right| : \mathbb{C}^d \to \mathbb{R}_{\geq 0}$ is also continuous. Thus

$$\begin{aligned} \beta \left|\nabla^i \hat{\phi}\right|^2 &\leq \langle \mathbf{z} \rangle^{-2} \beta \left|\nabla^i s_0\right|^2 + \sum_{j=1}^{d} 2 \langle \mathbf{z} \rangle^{-2} \left(\left|\operatorname{Re} z^j\right| \beta \left| \left\langle \nabla^i s_0, \nabla^i s_j \right\rangle \right| + \left|\operatorname{Im} z^j\right| \beta \left| \left\langle \nabla^i s_0, \mathbf{i} \nabla^i s_j \right\rangle \right| \right) \\ &+ \sum_{j,k=1}^{d} \langle \mathbf{z} \rangle^{-2} \left(\left| \operatorname{Re} \left(\bar{z}^j z^k \right) \right| \beta \left| \left\langle \nabla^i s_j, \nabla^i s_k \right\rangle \right| + \left| \operatorname{Im} \left(\bar{z}^j z^k \right) \right| \beta \left| \left\langle \nabla^i s_j, \mathbf{i} \nabla^i s_k \right\rangle \right| \right). \end{aligned}$$

Consider the terms of the form βf in the expression above (e.g. $\beta\left|\nabla^i s_0\right|^2$, $\beta\left|\left\langle\nabla^i s_0, \nabla^i s_j\right\rangle\right|$, etc.); the f in question are smooth functions on Σ which do not depend on $\mathbf{z}$, and so they are each bounded by some constant, independent of $\mathbf{z}$. We note this constant only depends on $(L,H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$, $\left[D_{\hat{A}}\right]$, and the choice of orthonormal basis on $\ker D_{\hat{A}}^2$. This last dependence is minor, any two orthonormal bases are related by some special unitary transformation which changes e.g. $\beta\left|\nabla^i s_0\right|^2$ by some bounded constant so we can assume that the βf are bounded by some C which only depends on $(L,H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ and $\left[D_{\hat{A}}\right]$.

Moreover, the coefficients of the βf terms are bounded above by one of the following: $\langle\mathbf{z}\rangle^{-2}$, $\langle\mathbf{z}\rangle^{-2}|\mathbf{z}|$, or $\langle\mathbf{z}\rangle^{-2}|\mathbf{z}|^2$; note that all of these are bounded above by 1. Writing $\beta\left|\nabla^i\hat{\phi}\right|^2 = \sum_r a_r \beta f_r$ for the above sum, we have $\left(\beta\left|\nabla^i\hat{\phi}\right|\right)^2 = \beta\left|\nabla^i\hat{\phi}\right|^2 = \sum_r a_r\beta f_r$ (Corollary A.2.6), and so $\left\|\nabla^i\hat{\phi}(\mathbf{z})\right\|_{C^0}^2 = \sum_r a_r\beta f_r \leq \sum_r C$, so $\left\|\nabla^i\hat{\phi}(\mathbf{z})\right\|_{C^0}^2 \leq C_i$ for some constant $C_i > 0$ which is independent of $\mathbf{z}$ and only depends on $(L,H) \to (\Sigma, g^\Sigma, \omega^\Sigma)$ and $\left[D_{\hat{A}}\right]$.

So for $\mathbf{z} \in \mathbb{C}^d$, $\left\|\alpha\hat{\phi}(\mathbf{z})\right\|_{C^k} = \max\left\{\left\|\nabla^i\alpha\hat{\phi}\right\|_{C^0} \leq \sqrt{C_i} \;\middle|\; 0 \leq i \leq k\right\} =: C_{U_0}$. This argument extends to any chart, so let C_k be the maximum of the constants C_{U_i}, it then follows that $\left\|\hat{\phi}\right\|_{C^k} \leq C_k$.

We proceed in much the same way for the second estimate, we have

$$\begin{aligned}
\left|\hat{\phi}\right|^2 &= \langle\mathbf{z}\rangle^{-2}\left|s_0 + \sum_{j=1}^d z^j s_j\right|^2 \\
&= \langle\mathbf{z}\rangle^{-2}|s_0|^2 + 2\langle\mathbf{z}\rangle^{-2}\sum_{j=1}^d\left\langle s_0, z^j s_j\right\rangle + \langle\mathbf{z}\rangle^{-2}\sum_{j,k=1}^d\left\langle z^j s_j, z^k s_k\right\rangle \\
&= \langle\mathbf{z}\rangle^{-2}|s_0|^2 + 2\langle\mathbf{z}\rangle^{-2}\sum_{j=1}^d\left(\operatorname{Re} z^j\langle s_0, s_j\rangle + \operatorname{Im} z^j\langle s_0, \mathbf{i}s_j\rangle\right) \\
&\quad + \langle\mathbf{z}\rangle^{-2}\sum_{j,k=1}^d\left(\operatorname{Re}\left(\bar{z}^j z^k\right)\langle s_j, s_k\rangle + \operatorname{Im}\left(\bar{z}^j z^k\right)\langle s_j, \mathbf{i}s_k\rangle\right) \\
\nabla^i\left|\hat{\phi}\right|^2 &= \langle\mathbf{z}\rangle^{-2}\nabla^i|s_0|^2 + 2\langle\mathbf{z}\rangle^{-2}\sum_{j=1}^d\left(\operatorname{Re} z^j\nabla^i\langle s_0, s_j\rangle + \operatorname{Im} z^j\nabla^i\langle s_0, \mathbf{i}s_j\rangle\right) \\
&\quad + \langle\mathbf{z}\rangle^{-2}\sum_{j,k=1}^d\left(\operatorname{Re}\left(\bar{z}^j z^k\right)\nabla^i\langle s_j, s_k\rangle + \operatorname{Im}\left(\bar{z}^j z^k\right)\nabla^i\langle s_j, \mathbf{i}s_k\rangle\right).
\end{aligned}$$

As before, we see that $\nabla^i\left|\hat{\phi}\right|^2 : \mathbb{C}^d \times \Sigma \to \mathbb{R}$ is continuous and so an almost identical procedure to the previous estimate yields an estimate $\left\|\left|\hat{\phi}\right|^2\right\|_{C^k} \leq C_k$.

We have

$$\nabla^k\hat{\phi}_a = \langle\mathbf{z}\rangle^{-1}\nabla^k s_a - \langle\mathbf{z}\rangle^{-2}\bar{z}^a\lambda_a\nabla^k\hat{\phi},$$

where if $a \leq d$, $\lambda_a = 1$; and if $\alpha > d$, $s_a = \mathbf{i}s_{a-d}$, $z^a = z^{a-d}$, and $\lambda_a = \mathbf{i}$. Again, continuity of this expression in both $\mathbf{z}$ and $p \in \Sigma$ is clear.

And so

$$\begin{aligned}
\left|\nabla^k \hat{\phi}_a\right|^2 &= \langle \mathbf{z}\rangle^{-2} \left|\nabla^k s_a\right|^2 + \langle \mathbf{z}\rangle^{-4} \left|\bar{z}^a \lambda_a \nabla^k \hat{\phi}\right|^2 \\
&\quad + 2\langle \mathbf{z}\rangle^{-3} \left\langle \nabla^k s_a, \bar{z}^a \lambda_a \nabla^k \hat{\phi}\right\rangle \\
&= \langle \mathbf{z}\rangle^{-2} \left|\nabla^k s_a\right|^2 + \langle \mathbf{z}\rangle^{-4} |\bar{z}^a|^2 \left|\nabla^k \hat{\phi}\right|^2 \\
&\quad + 2\langle \mathbf{z}\rangle^{-4} \left(\left\langle \nabla^k s_a, \bar{z}^a \lambda_a \nabla^k s_0 \right\rangle + \sum_{j=1}^{d} \left\langle \nabla^k s_a, z^j \nabla^k s_j \right\rangle \right) \\
&= \langle \mathbf{z}\rangle^{-2} \left|\nabla^k s_a\right|^2 + \langle \mathbf{z}\rangle^{-4} |\bar{z}^a|^2 \left|\nabla^k \hat{\phi}\right|^2 \\
&\quad + 2\langle \mathbf{z}\rangle^{-4} \left(\operatorname{Re} \bar{z}^a \left\langle \nabla^k s_a, \lambda_a \nabla^k s_0 \right\rangle + \operatorname{Im} \bar{z}^a \left\langle \nabla^k s_a, \mathbf{i}\lambda_a \nabla^k s_0 \right\rangle \right) \\
&\quad + 2\langle \mathbf{z}\rangle^{-4} \sum_{j=1}^{d} \left(\operatorname{Re} z^j \left\langle \nabla^k s_a, \nabla^k s_j \right\rangle + \operatorname{Im} z^j \left\langle \nabla^k s_a, \mathbf{i}\nabla^k s_j \right\rangle \right) \\
\beta\left|\nabla^k \hat{\phi}_a\right|^2 &\leq \langle \mathbf{z}\rangle^{-2} \beta\left|\nabla^k s_a\right|^2 + \langle \mathbf{z}\rangle^{-4} |\bar{z}^a|^2 \beta\left|\nabla^k \hat{\phi}\right|^2 \\
&\quad + 2\langle \mathbf{z}\rangle^{-4} \left(|\operatorname{Re} \bar{z}^a| \beta \left\langle \nabla^k s_a, \lambda_a \nabla^k s_0 \right\rangle + |\operatorname{Im} \bar{z}^a| \beta \left\langle \nabla^k s_a, \mathbf{i}\lambda_a \nabla^k s_0 \right\rangle \right) \\
&\quad + 2\langle \mathbf{z}\rangle^{-4} \sum_{j=1}^{d} \left(\left|\operatorname{Re} z^j\right| \beta \left\langle \nabla^k s_a, \nabla^k s_j \right\rangle + \left|\operatorname{Im} z^j\right| \beta \left\langle \nabla^k s_a, \mathbf{i}\nabla^k s_j \right\rangle \right).
\end{aligned}$$

Consider terms of the form βf in the expression above, they are all bounded by a constant which is independent of $\mathbf{z}$, either by inspection or the proof of prior estimates. The coefficients of these terms are bounded above by $\langle \mathbf{z}\rangle^{-2}$, $\langle \mathbf{z}\rangle^{-4} |z^a|$, or $\langle \mathbf{z}\rangle^{-4} |z^a|^2$; all of which are bounded above by 1. By Corollary A.2.7, we have $\left\|\nabla^i \hat{\phi}_a(\mathbf{z}, \cdot)\right\|_{L^2}^2 \leq \beta\left|\nabla^i \hat{\phi}_a\right|^2 \operatorname{Vol} \Sigma$, so it then follows that $\left\|\nabla^i \hat{\phi}_a(\mathbf{z}, \cdot)\right\|_{L^2}^2$ is bounded above by a constant which is independent of $\mathbf{z}$. Since $\left\|\hat{\phi}_a\right\|_{H^k}^2 = \sum_{i=0}^{k} \left\|\nabla^i \hat{\phi}_a(\mathbf{z}, \cdot)\right\|_{L^2}^2$, the third estimate follows.

For the last estimate, we proceed as follows

$$\begin{aligned}
\left\langle \hat{\phi}_a, \hat{\phi}_b \right\rangle &= \left\langle \langle \mathbf{z}\rangle^{-1} s_a - \langle \mathbf{z}\rangle^{-2} \bar{z}^a \lambda_a \hat{\phi}, \langle \mathbf{z}\rangle^{-1} s_b - \langle \mathbf{z}\rangle^{-2} \bar{z}^b \lambda_b \hat{\phi} \right\rangle \\
&= \langle \mathbf{z}\rangle^{-2} \langle s_a, s_b \rangle + \langle \mathbf{z}\rangle^{-4} \left\langle \hat{\phi}, z^a \bar{z}^b \bar{\lambda}^a \lambda^b \hat{\phi} \right\rangle \\
&\quad - \langle \mathbf{z}\rangle^{-3} \left(\left\langle s_a, \bar{z}^b \lambda_b \hat{\phi} \right\rangle + \left\langle \bar{z}^a \lambda_a \hat{\phi}, s_b \right\rangle \right) \\
&= \langle \mathbf{z}\rangle^{-2} \langle s_a, s_b \rangle + \langle \mathbf{z}\rangle^{-4} \left(\operatorname{Re}(\bar{z}^a z^b \bar{\lambda}^a \lambda^b) \left|\hat{\phi}\right|^2 + \operatorname{Im}(\bar{z}^a z^b \bar{\lambda}^a \lambda^b) \left\langle \hat{\phi}, \mathbf{i}\hat{\phi} \right\rangle \right) \\
&\quad - \langle \mathbf{z}\rangle^{-3} \left(\operatorname{Re} \bar{z}^b \lambda_b \left\langle s_a, \hat{\phi} \right\rangle + \operatorname{Im} \bar{z}^b \lambda_b \left\langle s_a, \mathbf{i}\hat{\phi} \right\rangle \right) \\
&\quad - \langle \mathbf{z}\rangle^{-3} \left(\operatorname{Re} z^a \bar{\lambda}_a \left\langle \hat{\phi}, s_b \right\rangle + \operatorname{Im} z^a \bar{\lambda}_a \left\langle \hat{\phi}, \mathbf{i}s_b \right\rangle \right).
\end{aligned}$$

At this point it is convenient to start inspecting individual terms. We have

$$\begin{aligned}
\left\langle \hat{\phi}, \mathbf{i}\hat{\phi} \right\rangle &= \langle \mathbf{z} \rangle^{-2} \left\langle s_0 + \sum_{i=1}^{d} z^i s_i, \mathbf{i}s_0 + \mathbf{i} \sum_{j=1}^{d} z^j s_j \right\rangle \\
&= \langle \mathbf{z} \rangle^{-2} \left(\langle s_0, \mathbf{i}s_0 \rangle + \sum_{i=1}^{d} \left(\langle s_0, \mathbf{i}z^i s_i \rangle + \langle -\mathbf{i}z^i s_i, s_0 \rangle \right) + \sum_{i,j=1}^{d} \langle s_i, \mathbf{i}\bar{z}^i z^j s_j \rangle \right) \\
&= \langle \mathbf{z} \rangle^{-2} \left(\langle s_0, \mathbf{i}s_0 \rangle + \sum_{i,j=1}^{d} \left(\operatorname{Re}\left(\bar{z}^i z^j\right) \langle s_i, \mathbf{i}s_j \rangle - \operatorname{Im}\left(\bar{z}^i z^j\right) \langle s_i, \mathbf{i}s_j \rangle \right) \right) \\
\beta \left| \nabla^k \left\langle \hat{\phi}, \mathbf{i}\hat{\phi} \right\rangle \right| &\leq \langle \mathbf{z} \rangle^{-2} \beta \left| \nabla^k \langle s_0, \mathbf{i}s_0 \rangle \right| + \langle \mathbf{z} \rangle^{-2} \sum_{i,j=1}^{d} \left| \operatorname{Re} \bar{z}^i z^j \right| \beta \left| \nabla^k \langle s_i, \mathbf{i}s_j \rangle \right| \\
&\quad + \langle \mathbf{z} \rangle^{-2} \sum_{i,j=1}^{d} \left| \operatorname{Im} \bar{z}^i z^j \right| \beta \left| \nabla^k \langle s_i, \mathbf{i}s_j \rangle \right|.
\end{aligned}$$

In the above, the terms of the form βf are bounded above by a constant independently of $\mathbf{z} \in \mathbb{C}^d$, the same being true for their coefficients. Thus the entire expression is bounded above by a constant independent of $\mathbf{z} \in \mathbb{C}^d$. Secondly

$$\begin{aligned}
\left\langle s_a, \hat{\phi} \right\rangle &= \left\langle s_a, \langle \mathbf{z} \rangle^{-1} \left(s_0 + \sum_{i=1}^{d} z^i s_i \right) \right\rangle \\
&= \langle \mathbf{z} \rangle^{-1} \left(\langle s_a, s_0 \rangle + \sum_{i=1}^{d} \langle s_a, z^i s_i \rangle \right) \\
&= \langle \mathbf{z} \rangle^{-1} \left(\langle s_a, s_0 \rangle + \sum_{i=1}^{d} \left(\operatorname{Re} z^i \langle s_a, s_i \rangle + \operatorname{Im} z^i \langle s_a, \mathbf{i}s_i \rangle \right) \right) \\
\beta \left| \nabla^k \left\langle s_a, \hat{\phi} \right\rangle \right| &\leq \langle \mathbf{z} \rangle^{-1} \beta \left| \nabla^k \langle s_a, s_0 \rangle \right| + \langle \mathbf{z} \rangle^{-1} \sum_{i=1}^{d} \left(\left| \operatorname{Re} z^i \right| \beta \left| \nabla^k \langle s_a, s_i \rangle \right| \right) \\
&\quad + \langle \mathbf{z} \rangle^{-1} \sum_{i=1}^{d} \left(\left| \operatorname{Im} z^i \right| \beta \left| \nabla^k \langle s_a, \mathbf{i}s_i \rangle \right| \right)
\end{aligned}$$

Again, this is bounded by a constant independently of $\mathbf{z} \in \mathbb{C}^d$. A similar argument shows the same for $\left\langle s_a, \mathbf{i}\hat{\phi} \right\rangle$. So, we have

$$\begin{aligned}
\beta \left| \nabla^i \left\langle \hat{\phi}_a, \hat{\phi}_b \right\rangle \right| &\leq \langle \mathbf{z} \rangle^{-2} \beta \left| \nabla^i \langle s_a, s_b \rangle \right| \\
&\quad + \langle \mathbf{z} \rangle^{-4} \left(\left| \operatorname{Re}(\bar{z}^a z^b) \right| \beta \left| \nabla^i \left| \hat{\phi} \right|^2 \right| + \left| \operatorname{Im}(\bar{z}^a z^b) \right| \beta \left| \nabla^i \left\langle \hat{\phi}, \mathbf{i}\hat{\phi} \right\rangle \right| \right) \\
&\quad - \langle \mathbf{z} \rangle^{-3} \left(\left| \operatorname{Re} \bar{z}^b \lambda_b \right| \beta \left| \nabla^i \left\langle s_a, \hat{\phi} \right\rangle \right| + \left| \operatorname{Im} \bar{z}^b \lambda_b \right| \beta \left| \nabla^i \left\langle s_a, \mathbf{i}\hat{\phi} \right\rangle \right| \right) \\
&\quad - \langle \mathbf{z} \rangle^{-3} \left(\left| \operatorname{Re} z^a \bar{\lambda}_a \right| \beta \left| \nabla^i \left\langle s_b, \hat{\phi} \right\rangle \right| + \left| \operatorname{Im} z^a \bar{\lambda}_a \right| \beta \left| \nabla^i \left\langle s_b, \mathbf{i}\hat{\phi} \right\rangle \right| \right).
\end{aligned}$$

And like the previous estimate, the terms of the form βf are bounded above by a constant independent of $\mathbf{z} \in \mathbb{C}^d$, and the same is true for the coefficients of those terms. The stated

estimate then holds. □

Finally, we need to prove the following.

Lemma A.2.8.
Let $\left[D_{\hat{A}}, \sqrt{\varepsilon}\hat{\phi}\right] \in \hat{\mathcal{M}}^{\varepsilon}$*, and let* u *solve* $\Delta u - \frac{\varepsilon}{\operatorname{Vol}\Sigma} + \varepsilon e^{u}\left|\hat{\phi}\right|^2 = 0$ (2.15)*. Then there exist a constant* $C_{\hat{A}}$ *depending only on* (Σ, g^{Σ})*,* (L, H)*, and* $D_{\hat{A}}$ *such that* $\hat{\phi}_{a,b}$ *satisfies the bound*

$$\left\|\hat{\phi}_{a,b}\right\|_{H^1} < C_{\hat{A}}. \tag{A.2}$$

Proof. We compute

$$\begin{aligned}
\hat{\phi}_{a,b} &= \left(\langle \mathbf{z}\rangle^{-1} s_a - \langle \mathbf{z}\rangle^{-2} \bar{z}^a \lambda_a \hat{\phi}\right)_{,b} \\
&= \left(\langle \mathbf{z}\rangle^{-1}\right)_{,b} s_a - \left(\langle \mathbf{z}\rangle^{-2} \bar{z}^a\right)_{,b} \lambda_a \hat{\phi} - \langle \mathbf{z}\rangle^{-2} \bar{z}^a \lambda_a \hat{\phi}_b \\
&= -\langle \mathbf{z}\rangle^{-2} \langle \mathbf{z}\rangle_{,b} s_a - \left(-2\langle \mathbf{z}\rangle^{-3} \langle \mathbf{z}\rangle_{,b} \bar{z}^a + \langle \mathbf{z}\rangle^{-2} \bar{z}^a_{,b}\right)\lambda_a \hat{\phi} - \langle \mathbf{z}\rangle^{-2} \bar{z}^a \lambda_a \hat{\phi}_b \\
\nabla^i \hat{\phi}_{a,b} &= -\frac{1}{2}\langle \mathbf{z}\rangle^{-3} \left(\sum_{j=1}^{2} \left|z^j\right|^2\right)_{,b} \nabla^i s_a + \left(\langle \mathbf{z}\rangle^{-4} \left(\sum_{j=1}^{2} \left|z^j\right|^2\right)_{,b} \bar{z}^a - \langle \mathbf{z}\rangle^{-2} \bar{z}^a_{,b}\right) \lambda_a \nabla^i \hat{\phi} \\
&\quad - \langle \mathbf{z}\rangle^{-2} \bar{z}^a \lambda_a \nabla^i \hat{\phi}_b \\
\beta\left|\nabla^i \hat{\phi}_{a,b}\right| &\leq \frac{1}{2}\langle \mathbf{z}\rangle^{-3} \left|\left(\sum_{j=1}^{2} \left|z^j\right|^2\right)_{,b}\right| \beta\left|\nabla^i s_a\right| \\
&\quad + \left(\langle \mathbf{z}\rangle^{-4} \left|\left(\sum_{j=1}^{2} \left|z^j\right|^2\right)_{,b} \bar{z}^a\right| + \langle \mathbf{z}\rangle^{-2} \left|\bar{z}^a_{,b}\right|\right) \beta\left|\nabla^i \hat{\phi}\right| \\
&\quad + \langle \mathbf{z}\rangle^{-2} |\bar{z}^a| \beta\left|\nabla^i \hat{\phi}_b\right|
\end{aligned}$$

Terms of the form βf are bounded by previously constructed estimates and their coefficients are clearly bounded above by 1. Thus as with prior proofs, the estimate follows. □

A.3 Estimates on projective space

The last set of estimates concern tensors on projective space.

Lemma 3.1.9.
Let $\gamma \in C^\infty\Big(\mathbb{P}^d; \big(T^*\mathbb{P}^d\big)^{\otimes r}\Big)$ be some r-tensor. If in every affine coordinate chart, the components of γ, $\gamma_{i_0,\dots,i_r}(p)$, are uniformly bounded in C^0 by some $C' > 0$. Then there is a constant $C > 0$ which only depends on r and d such that

$$\|\gamma\|_{C^0} < CC'.$$

In particular, if γ is a two tensor and we wish to find a C^0-estimate for it, it is sufficient to find an estimate $|\gamma_{ab}(p)| < C$ for every affine coordinate chart U_i, every $p \in U_i$, and every choice of indices.

This result is essentially a reflection of the fact that any point on $\mathbb{P}^d$ has an expression in some affine coordinate chart where it is contained in the unit L^∞-ball in $\mathbb{C}^d$. The dual of the Fubini-Study metric, which governs the pointwise norm of a covariant tensor, is bounded above on this set.

Proof. Let g^{ij} be the dual of the Fubini-Study metric on some affine chart of $\mathbb{P}^d$, the choice of chart will be clear from context. Let γ be an r-tensor as in the hypothesis where we write $\gamma = \gamma_I \, \mathrm{d}z^I$ in some affine coordinate chart, U_ℓ, with affine coordinate $\mathbf{z} = (z^i)$. Then

$$\begin{aligned}
|\gamma(\mathbf{z})|^2 &= \sum_{|I|,|J|=r} \gamma_I(\mathbf{z})\gamma_J(\mathbf{z}) \left(\prod_{k=1}^{r} g^{i_k j_k}(\mathbf{z}) \right) \\
&\leq \sum_{|I|,|J|=r} \big(C'\big)^2 \big(\max\big\{ \big| g^{i_k j_k}(\mathbf{z}) \big| \;\big|\; 1 \leq k \leq r \big\}\big)^r \\
&\leq \big(C'\big)^2 (2d)^{2r} \big(\max\big\{ \big| g^{ij}(\mathbf{z}) \big| \;\big|\; 1 \leq i,j \leq 2d \big\}\big)^r
\end{aligned}$$

Thus we have

$$|\gamma(\mathbf{z})| \leq C'(2d)^r \big(\max\big\{ \big| g^{ij}(\mathbf{z}) \big| \;\big|\; 1 \leq i,j \leq 2d \big\}\big)^{r/2}.$$

Now note that for a point $p \in \mathbb{P}^d$, the above estimate holds simultaneously in every affine coordinate chart in which it is contained. We can suppose, after perhaps permuting the coordinates, that p has homogeneous coordinates

$$p = \big[Z^0 : Z^1 : \cdots : Z^a : 0 : \cdots : 0 \big]$$

where $Z^0, \dots, Z^a \neq 0$. For $0 \leq \ell \leq a$, let

$$\mathbf{z}_\ell = \left(\frac{Z^0}{Z^\ell}, \dots, \frac{Z^{\ell-1}}{Z^\ell}, \frac{Z^{\ell+1}}{Z^\ell}, \dots, \frac{Z^a}{Z^\ell}, 0, \dots, 0 \right) \in \mathbb{C}^d.$$

We then have

$$|\gamma(p)| \leq C'(2d)^r \left(\min\left\{\max\left\{\left|g^{ij}(\mathbf{z}_\ell)\right| \;\middle|\; 1 \leq i,j \leq 2d\right\} \;\middle|\; 0 \leq \ell \leq a\right\}\right)^{r/2}.$$

So it suffices to bound $\min\left\{\max\left\{\left|g^{ij}(\mathbf{z}_\ell)\right| \;\middle|\; 1 \leq i,j \leq 2d\right\} \;\middle|\; 0 \leq \ell \leq a\right\}$ by a constant.

Now let ℓ be such that $\left|Z^\ell\right|$ is maximal amongst the $\left|Z^i\right|$, then every component of $\mathbf{z}_\ell$ has modulus ≤ 1, and so $\langle \mathbf{z}_\ell \rangle \leq \sqrt{1+a} \leq \sqrt{1+d}$. Recall that we can write the Hermitian form associated to the Fubini-Study metric in affine coordinates as $h_{\bar{\imath}j} = \langle \mathbf{z} \rangle^{-4} \left(\langle \mathbf{z} \rangle^2 \delta_{\bar{\imath}j} - z^i \bar{z}^j \right)$. We compute

$$\begin{aligned}
\langle \mathbf{z} \rangle^2 h_{\bar{\imath}j} \left(\delta^{j\bar{k}} + z^j \bar{z}^k \right) &= \left(\delta_{\bar{\imath}j} - \langle \mathbf{z} \rangle^{-2} z^i \bar{z}^j \right) \left(\delta^{j\bar{k}} + z^j \bar{z}^k \right) \\
&= \delta_{\bar{\imath}}{}^{\bar{k}} + \delta_{\bar{\imath}j} z^j \bar{z}^k - \langle \mathbf{z} \rangle^{-2} z^i \bar{z}^j \delta^{j\bar{k}} - \sum_j \langle \mathbf{z} \rangle^{-2} z^i \left| z^j \right|^2 \bar{z}^k \\
&= \delta_{\bar{\imath}}{}^{\bar{k}} + \left(1 - \langle \mathbf{z} \rangle^{-2} - \langle \mathbf{z} \rangle^{-2} \sum_j \left| z^j \right|^2 \right) z^i \bar{z}^k \\
&= \delta_{\bar{\imath}}{}^{\bar{k}} + \left(1 - \langle \mathbf{z} \rangle^{-2} - \langle \mathbf{z} \rangle^{-2} \left(\langle \mathbf{z} \rangle^2 - 1 \right) \right) z^i \bar{z}^k \\
&= \delta_{\bar{\imath}}{}^{\bar{k}}.
\end{aligned}$$

So $\left(\langle \mathbf{z} \rangle^2 h_{\bar{\imath}j} \right)^{-1} = \delta^{i\bar{\jmath}} + z^i \bar{z}^j$. It then follows that $h^{i\bar{\jmath}} = \langle \mathbf{z} \rangle^2 \left(\delta^{i\bar{\jmath}} + z^i \bar{z}^j \right)$.

Now recall that on the level of forms $H(X,Y) = g(X,Y) + \mathbf{i} g(JX,Y)$ where J is the almost complex structure on $T\mathbb{P}^d$. Both g and H are non-degenerate so they define isomorphisms $g^\flat : T\mathbb{P}^d \to T^*\mathbb{P}^d$ and $H^\flat : T\mathbb{P}^d \to \left(T^{1,0}\mathbb{P}^d \right)^*$, concretely $g^\flat(X) := g(X, \cdot)$ and $H^\flat(X) := H(X, \cdot)$. We then see that $H^\flat = g^\flat + \mathbf{i} g^\flat \circ J$, or $g^\flat = \operatorname{Re} H^\flat$. Now for $\alpha, \beta \in \Omega^{1,0}(\mathbb{P}^d)$, let $X = \left(H^\flat \right)^{-1}(\alpha)$ and $Y = \left(H^\flat \right)^{-1}(\beta)$, note that $\operatorname{Re} \alpha = g^\flat(X)$ and $\operatorname{Re} \beta = g^\flat(Y)$. We compute $H^\flat(X)(Y) = H(X,Y) =: H^{-1}(\alpha, \beta)$ and $g^{-1}(\operatorname{Re}\alpha, \operatorname{Re}\beta) = g(X,Y) = g^\flat(X)(Y) = \operatorname{Re} H^\flat(X)(Y)$, and so $g^{-1}(\operatorname{Re}\alpha, \operatorname{Re}\beta) = \operatorname{Re} H^{-1}(\alpha, \beta)$. Since $\mathrm{d}x^i = \operatorname{Re} \mathrm{d}z^i$ and $\mathrm{d}y^i = \operatorname{Re}(-\mathbf{i}\,\mathrm{d}z^i)$, it follows that the absolute values of: $g^{-1}(\mathrm{d}x^i, \mathrm{d}x^j)$, $g^{-1}(\mathrm{d}y^i, \mathrm{d}y^j)$, $g^{-1}(\mathrm{d}x^i, \mathrm{d}y^j)$, and $g^{-1}(\mathrm{d}y^i, \mathrm{d}x^j)$, are all bounded above by $\left| H^{-1}(\mathrm{d}z^i, \mathrm{d}z^j) \right|$. With some abuse of notation, we combine these into the single estimate $\left| g^{ij} \right| \leq \left| h^{i\bar{\jmath}} \right|$.

We then compute

$$\begin{aligned}
\left| g^{ij}(\mathbf{z}_\ell) \right| &\leq \langle \mathbf{z}_\ell \rangle^2 \left(1 + \left| z^i \bar{z}^j \right| \right) \\
&\leq (1+d)(1+1) \\
&\leq 2(1+d),
\end{aligned}$$

and we are done. □

The second estimate is a consequence of the first.

Lemma 3.1.16.
Let $\gamma \in C^\infty\left(\mathbb{P}^d; \left(T^*\mathbb{P}^d \right)^{\otimes r} \right)$ be some r-tensor. If in every affine coordinate chart, the components of γ, $\gamma_{i_1 \cdots i_r, i_{r+1}}(p)$, are uniformly bounded in C^1 by some $C' > 0$. Then there

is a constant $C > 0$ *which only depends on* $\deg L - 1$ *such that*

$$\|\nabla\gamma\|_{C^0} < CC'.$$

In particular, if γ *is a two tensor and we wish to find a* C^0*-estimate for* $\nabla\gamma$*, it is sufficient to find an estimate* $\left|\gamma_{ab,c}(p)\right| < C_{\hat{A}}$ *for every affine coordinate chart* U_i*, every* $p \in U_i$*, and every choice of indices.*

Proof. In coordinates, we have that $(\nabla\gamma)_{i_1\cdots i_k j} = \gamma_{i_1\cdots i_k,j} - \sum_{s=1}^{k} \gamma_{i_1\cdots p\cdots i_k}\Gamma^{p}{}_{ji_s}$, so it is enough to show that the components of the Christoffel symbols of the Fubini-Study metric are bounded above uniformly. The result then follows from Lemma 3.1.9.

Again, this problem becomes simpler if we switch to the holomorphic framing. From Chapter 4.A of Huybrechts [32], for a Kähler manifold (X, g^X), we have an isomorphism between the real and holomorphic tangent bundles

$$\begin{aligned} \xi : (TX, J) &\to T^{1,0}X. \\ u_p &\mapsto \frac{1}{2}(u - \mathbf{i}J(u)), \end{aligned}$$

where $J \in \mathrm{Aut}(TX)$ is the almost-complex structure. Moreover, this isomorphism induces a correspondence between the Chern and Levi-Civita connections, in particular the (real) Christoffel symbols are just ± 1 times the real and imaginary parts of the Chern connection coefficients. In particular, the absolute value of any real Christoffel symbol is bounded above by the modulus of the corresponding Chern connection coefficient.

As in the proof of the previous lemma, we can restrict our attention to some $\mathbf{z} \in \mathbb{C}^d$ in the unit L^∞-ball in some affine coordinate chart. Locally, the Chern connection is given by $\mathrm{d} + H^{-1}\partial H$ [32, p. 178]. By the previous proof, we know that the components of H^{-1} are bounded in this chart by some fixed constant which only depends on the dimension of the projective space, so it remains to bound the components of ∂H.

We compute

$$\begin{aligned} \partial_i h_{\bar{k}j} &= \partial_i\left(\langle\mathbf{z}\rangle^{-2}\,\delta_{\bar{k}j} - \langle\mathbf{z}\rangle^{-4}\,z^k\bar{z}^j\right) \\ &= \delta_{\bar{k}j}\partial_i\left(\langle\mathbf{z}\rangle^2\right)^{-1} - z^k\bar{z}^j\partial_i\left(\langle\mathbf{z}\rangle^2\right)^{-2} - \langle\mathbf{z}\rangle^{-4}\left(\partial_i z^k\right)\bar{z}^j \\ &= -\delta_{\bar{k}j}\,\langle\mathbf{z}\rangle^{-4}\,\partial_i\,\langle\mathbf{z}\rangle^2 + 2z^k\bar{z}^j\,\langle\mathbf{z}\rangle^{-6}\,\partial_i\,\langle\mathbf{z}\rangle^2 - \langle\mathbf{z}\rangle^{-4}\,\delta_i{}^k\bar{z}^j \\ &= \left(-\delta_{\bar{k}j}\,\langle\mathbf{z}\rangle^{-4} + 2\,\langle\mathbf{z}\rangle^{-6}\,z^k\bar{z}^j\right)\bar{z}^i - \langle\mathbf{z}\rangle^{-4}\,\delta_i{}^k\bar{z}^j \\ &= \langle\mathbf{z}\rangle^{-4}\left(-\delta_{\bar{k}j}\bar{z}^i + 2\,\langle\mathbf{z}\rangle^{-2}\,z^k\bar{z}^j\bar{z}^i - \delta_i{}^k\bar{z}^j\right) \\ \left|\partial_i h_{\bar{k}j}\right| &\leq \langle\mathbf{z}\rangle^{-4}\left(\left|\bar{z}^i\right| + 2\,\langle\mathbf{z}\rangle^{-2}\left|z^k\right|\left|\bar{z}^j\right|\left|\bar{z}^i\right| + \left|\bar{z}^j\right|\right) \end{aligned}$$

Recall we have that $\mathbf{z} \in \mathbb{C}^d$ lies in the unit L^∞-ball, i.e. all components have modulus ≤ 1. Moreover, we can also see that $1 \leq \langle\mathbf{z}\rangle$ so $\langle\mathbf{z}\rangle^{-1} \leq 1$. It therefore follows that $\left|\partial_i h_{\bar{k}j}\right| \leq 4$

and we are done. □

Appendix B

Auxiliary results for Chapter 5

B.1 Non-triviality of the Abel-Jacobi fibration

By $\mathbb{Z}[X^k]$ for $k \in \mathbb{Z}^+$ we mean the polynomial ring with coefficients in $\mathbb{Z}$ and free variable X^k, we equip it with the usual grading. In what follows the tensor product is always that of graded rings. We also define

$$\bigwedge [X, Y] := (\mathbb{Z}[X] \otimes \mathbb{Z}[Y])/(XY - YX).$$

Lemma 5.1.2.
The cohomology rings (with coefficients in $\mathbb{Z}$) of $\Sigma^{(2)}$ and $\mathbb{P}^1 \times \Sigma$ are not-isomorphic and thus $\mathrm{AJ} : \Sigma^{(2)} \to \Sigma$ cannot be the trivial bundle.

Proof. From a result of MacDonald [44] and the Künneth formula (e.g. [30, p. 216]) respectively, we have

$$H^\bullet(\Sigma^{(2)}; \mathbb{Z}) \cong \left(\mathbb{Z}[X^2] \otimes \bigwedge [Y_1, Y_2]\right) / (X^4 - Y_1 Y_2 X^2),$$
$$H^\bullet(\mathbb{P}^1 \times \Sigma; \mathbb{Z}) \cong \mathbb{Z}[B^2]/(B^4) \otimes \bigwedge [A_1, A_2]\,.$$

We show that any homomorphism of graded rings,

$$\phi : H^\bullet(\mathbb{P}^1 \times \Sigma; \mathbb{Z}) \to H^\bullet(\Sigma^{(2)}; \mathbb{Z}),$$

cannot be both injective and surjective. Suppose that

$$\phi : A_i \mapsto \sum_j {a_i}^j Y_j,$$
$$B^2 \mapsto b^1 X^2 + b^2 Y_1 Y_2,$$

for some choice of ${a_i}^j, b^i \in \mathbb{Z}$.

A necessary condition for ϕ to be a well defined homomorphism is that $\phi(B^2)^2 = 0$ which implies $(b^1)^2 + 2b^1b^2 = 0$ and thus $b^1 = 0$ or $b^1 = -2b^2$. If we further suppose that ϕ is injective, we must have:

- $\phi(A) \neq 0$ where $\deg A = 1$ thus $\det a_i{}^j \neq 0$, and
- $\phi(AB^2) \neq 0 \implies b^1 \neq 0$ and $b^1 = -2b^2$.

But any injective map fails to be surjective. If, in addition, we suppose ϕ is surjective, we must have $\det\left(a_i{}^j\right) = 1$. Now suppose we have $a, b \in \mathbb{Z}$ such that $\phi(bB^2 + aA_1A_2) = X^2$, then,

$$\phi(bB^2 + aA_1A_2) = bb^1X^2 + bb^2Y_1Y_2 + a\det\left(a_i{}^j\right)Y_1Y_2.$$

So we must have $bb^1 = 1$ and $bb^2 + a = 0$. By the second injectivity criterion, the former equation implies $-2bb^2 = 1$, which in turn implies $2a = 1$, contradicting $a \in \mathbb{Z}$. $\perp$

Thus $\phi : H^\bullet(\mathbb{P}^1 \times \Sigma; \mathbb{Z}) \to H^\bullet(\Sigma^{(2)}; \mathbb{Z})$ cannot be both injective and surjective and thus cannot be an isomorphism. $\square$

B.2 Metric decoupling

The intuition behind this result is that the abelian group structure on $\Sigma \cong \mathbb{C}/\Lambda$ allows us to construct a 'centre of mass' of a given vortex configuration. We can then model 2-vortex dynamics as the dynamics *of* this centre of mass, and relative dynamics between the two vortices. This proposition shows that for 2-vortices, the two kinds of motion decouple. It is then enough to understand each in isolation as we can recover the full dynamics by simply adding centre of mass motion to the relative position of the cores.

The statement of this result and ensuing proof is due to J.M.Speight following the statement of Strachan-Samols localisation due to Manton and Nasir [49].

Proposition 5.1.4.
Equip the open set $U := \Sigma^{(2)}\backslash\{\{z, z\} \mid z \in \Sigma\}$ *with local coordinates* $\left\{z^1, z^2\right\}$*, where* z^i *takes values in the open fundamental polygon of* Σ *centred at 0. Furthermore, let* $\zeta^1 := z^1 - z^2$ *and* $\zeta^2 := z^1 + z^2$*.*

Then on U*, the vortex metric is equal to*

$$g^\varepsilon\big|_U = \alpha(\zeta^1, \bar{\zeta}^1)\,\mathrm{d}\zeta^1\,\mathrm{d}\bar{\zeta}^1 + \tau_0\pi\,\mathrm{d}\zeta^2\,\mathrm{d}\bar{\zeta}^2\,, \tag{5.7}$$

for some function α*.*

Proof. We define $\partial_{\bar{\imath}} := \frac{\partial}{\partial \bar{z}^i}$ and $\mathrm{d}z^{\bar{\jmath}} := \mathrm{d}\bar{z}^j$. We shall suppress summation symbols throughout with Roman indices (and their barred variants) taking values in $\{1, 2\}$. We first apply the Strachan-Samols localisation formula as stated in Manton and Nasir [49]

$$g^\varepsilon\big|_U = \tau_0\pi(\delta_{i\bar{\jmath}} + 2\partial_{\bar{\jmath}}b_i)\,\mathrm{d}z^i\,\mathrm{d}z^{\bar{\jmath}}\,.$$

Here, the b_i are the linear coefficients of the Taylor expansion of $\log|\phi|_H^2$ in local coordinates,

$$\log|\phi|_H^2(z) = \sum_{i=1}^{2}\Big(\log\big|z-z^i\big|^2 + a_i\Big) + \frac{1}{2}\big(b_i(z-z^i) + b_{\bar\imath}(\bar z - \bar z^i)\big) + O\Big(\big(z-z^i\big)^2\Big) + O\Big(\big(\bar z-\bar z^i\big)^2\Big) + O\big(\big(z-z^i\big)\big(\bar z-\bar z^i\big)\big).$$

From this, we see that they have the following properties:

- conjugation symmetry: $\overline{\partial_j b_i} = \partial_{\bar\imath} b_j$.
- permutation symmetry: $b_2(z^1,z^2,\bar z^1,\bar z^2) = b_1(z^2,z^1,\bar z^2,\bar z^1)$.
- translation invariance: there are complex valued functions β_i such that

$$b_i(z^1,z^2,\bar z^1,\bar z^2) = \beta_i(z^1-z^2,\bar z^1-\bar z^2).$$

- rotation equivariance: $b_i(-z^1,-z^2,-\bar z^1,-\bar z^2) = -b_i(z^1,z^2,\bar z^1,\bar z^2)$.

From these properties, we conclude

$$\begin{aligned} b_2(z^1,z^2,\bar z^1,\bar z^2) &= b_1(z^2,z^1,\bar z^2,\bar z^1)\\ &= \beta_1(z^2-z^1,\bar z^2-\bar z^1)\\ &= b_1(-z^1,-z^2,-\bar z^1,-\bar z^2)\\ &= -b_1(z^1,z^2,\bar z^1,\bar z^2). \end{aligned}$$

Thus $\partial_i b_1 = -\partial_i b_2$ and $\partial_{\bar\imath} b_1 = -\partial_{\bar\imath} b_2$. Moreover, from translation symmetry we have $(\partial_{\bar 1}+\partial_{\bar 2})b_i = 0$ and $(\partial_{\bar 1}-\partial_{\bar 2})b_1 = (\partial_{\bar 2}-\partial_{\bar 1})b_2 = \alpha(z^1-z^2,\bar z^1-\bar z^2)$ for some function α.

Let $T^i{}_j := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and noting that $\mathrm{d}z^i = \frac{1}{2}T^i{}_j\,\mathrm{d}\zeta^j$, we have

$$g_\varepsilon^\varepsilon\big|_U = \frac{\tau_0\pi}{2}(\delta_{i\bar\jmath} + 2\partial_{\bar\jmath}b_i)T^i{}_k\,\mathrm{d}\zeta^k\,T^{\bar\jmath}{}_{\bar l}\,\mathrm{d}\zeta^{\bar l}\,.$$

Noting that $T^i{}_j = \delta^i{}_j + \epsilon^i{}_j$, we expand $T^i{}_k\,T^{\bar\jmath}{}_{\bar l} = \delta^i{}_k\delta^{\bar\jmath}{}_{\bar l} + \Big(\delta^i{}_k\epsilon^{\bar\jmath}{}_{\bar l} + \epsilon^i{}_k\delta^{\bar\jmath}{}_{\bar l}\Big) + \epsilon^i{}_k\epsilon^{\bar\jmath}{}_{\bar l}$. Thus

$$\begin{aligned} \delta_{i\bar\jmath}T^i{}_k\,T^{\bar\jmath}{}_{\bar l} &= 2\delta_{k\bar l},\\ (\partial_{\bar\jmath}b_i)T^i{}_k\,T^{\bar\jmath}{}_{\bar l} &= (\partial_{\bar l}b_k) + (\partial_{\bar\jmath}b_i)\Big(\delta^i{}_k\epsilon^{\bar\jmath}{}_{\bar l} + \epsilon^i{}_k\delta^{\bar\jmath}{}_{\bar l}\Big) + (\partial_{\bar\jmath}b_i)\epsilon^i{}_k\epsilon^{\bar\jmath}{}_{\bar l}. \end{aligned}$$

We simplify the second equation as follows, consider the first and last terms,

$$\begin{aligned}(\partial_{\bar{l}} b_k) + (\partial_{\bar{\jmath}} b_i)\epsilon^i{}_k \epsilon^{\bar{\jmath}}{}_{\bar{l}} &= \begin{bmatrix} \partial_{\bar{1}} b_1 & -\partial_{\bar{1}} b_1 \\ \partial_{\bar{2}} b_1 & -\partial_{\bar{2}} b_1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_{\bar{1}} b_1 & -\partial_{\bar{1}} b_1 \\ \partial_{\bar{2}} b_1 & -\partial_{\bar{2}} b_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \partial_{\bar{1}} b_1 & -\partial_{\bar{1}} b_1 \\ \partial_{\bar{2}} b_1 & -\partial_{\bar{2}} b_1 \end{bmatrix} + \begin{bmatrix} -\partial_{\bar{2}} b_1 & -\partial_{\bar{2}} b_1 \\ \partial_{\bar{1}} b_1 & \partial_{\bar{1}} b_1 \end{bmatrix} \\ &= \begin{bmatrix} (\partial_{\bar{1}} - \partial_{\bar{2}}) b_1 & 0 \\ 0 & (\partial_{\bar{1}} - \partial_{\bar{2}}) b_1 \end{bmatrix} \\ &= \alpha \delta_{k\bar{l}}.\end{aligned}$$

And the middle term,

$$\begin{aligned}\partial_{\bar{\jmath}} b_i \left(\delta^i{}_k \epsilon^{\bar{\jmath}}{}_{\bar{l}} + \epsilon^i{}_k \delta^{\bar{\jmath}}{}_{\bar{l}} \right) &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \partial_{\bar{1}} b_1 & -\partial_{\bar{1}} b_1 \\ \partial_{\bar{2}} b_1 & -\partial_{\bar{2}} b_1 \end{bmatrix} + + \begin{bmatrix} \partial_{\bar{1}} b_1 & -\partial_{\bar{1}} b_1 \\ \partial_{\bar{2}} b_1 & -\partial_{\bar{2}} b_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\partial_{\bar{2}} b_1 & \partial_{\bar{2}} b_1 \\ \partial_{\bar{1}} b_1 & -\partial_{\bar{1}} b_1 \end{bmatrix} + \begin{bmatrix} \partial_{\bar{1}} b_1 & \partial_{\bar{1}} b_1 \\ \partial_{\bar{2}} b_1 & \partial_{\bar{2}} b_1 \end{bmatrix} \\ &= \begin{bmatrix} (\partial_{\bar{1}} - \partial_{\bar{2}}) b_1 & 0 \\ 0 & (\partial_{\bar{2}} - \partial_{\bar{1}}) b_1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}.\end{aligned}$$

So we have $(\partial_{\bar{\jmath}} b_i) T^i{}_k T^{\bar{\jmath}}{}_{\bar{l}} = \begin{bmatrix} 2\alpha & 0 \\ 0 & 0 \end{bmatrix}$. Thus

$$\begin{aligned}g^{\varepsilon}_{\varepsilon}\big|_U &= \frac{\tau_0 \pi}{2} \left(\delta_{i\bar{\jmath}} T^i{}_k T^{\bar{\jmath}}{}_{\bar{l}} + 2 \partial_{\bar{\jmath}} b_i T^i{}_k T^{\bar{\jmath}}{}_{\bar{l}} \right) \mathrm{d}\zeta^k \, \mathrm{d}\zeta^{\bar{l}} \\ &= \frac{\tau_0 \pi}{2} (2\delta_{k\bar{l}} + 4\alpha \delta_{1k} \delta_{\bar{1}\bar{l}}) \, \mathrm{d}\zeta^k \, \mathrm{d}\zeta^{\bar{l}} \\ &= \tau_0 \pi (1 + 2\alpha) \, \mathrm{d}\zeta^1 \, \mathrm{d}\zeta^{\bar{1}} + \tau_0 \pi \, \mathrm{d}\zeta^2 \, \mathrm{d}\zeta^{\bar{2}}\end{aligned}$$

and we are done. □

*I would prefer to centre **desire** and **will** when talking about transition. […] It's taking your life in your hands and shaping it yourself. I think that's beautiful.*

Abigail Thorn
Have we got it wrong on dysphoria?
2022-06-16