

Constructing Monoidal Double Categories of Bimodules



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Abstract

In this thesis we construct a monoidal fibred double category (also known as a proarrow equipment or framed bicategory) of cocommutative Hopf algebras, algebra morphisms, and bimodules in an arbitrary closed symmetric monoidal category $\mathcal{V}$ with sufficient coequalisers. That is, we construct a double category of cocommutative Hopf algebras and bimodules where the data of the strict 1-cells can be recovered from the loose 1-cells. We provide some background on enriched categories, bicategories, and double categories. We will also provide background about Hopf algebras and modules taken in an enriched category $\underline{\mathcal{V}}$ and prove several results about Hopf algebras and modules using intuitive string diagrammatic proofs in the general setting of an enriched category. We introduce the idea of a monoidal bifibration using the example of a functor from a category of representations to the category of groups and then show how this can be used to construct a monoidal fibred double category. We generalise this construction to modules and cocommutative Hopf algebras in a closed symmetric monoidal category $\mathcal{V}$ and explicitly construct a monoidal fibred double category of Hopf algebras and modules. We show this is equivalent to a double category of Hopf algebras and bimodules to explicitly describe the data of the constructed double category.

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Introduction

The aim of this thesis is to use string diagrams to prove results about Hopf algebras and modules in a monoidal category and use the results about modules as well as a result of Shulman [25] to construct fibred monoidal double categories of Hopf algebras and modules.

Double categories are a categorical structure with objects, two different types of morphisms between objects which we call tight and loose 1-cells (or sometimes vertical and horizontal 1-cells), and 2-cells between the 1-cells. Originally introduced by Ehresmann [3] in 1963, double categories are used in situations such as with sets, functions and relations, or groups, group homomorphisms, and bimodules, where there are two different types of morphism on the same object that interact with each other in a way which is measured by the 2-cells. In certain situations, such as those described above, the data of the tight 1-cells can be recovered by the loose 1-cells and such a double category is called a fibred double category. This is an equivalent notion to that of a proarrow equipment, introduced by Wood and Street [29, 30, 27], proarrow equipments take the approach of “equipping” a bicategory with additional 1-cells.

Fibrations are functors satisfying a universal lifting property which allow us to show that certain maps exist. When fibrations were introduced by Grothendieck [6] we were also provided with the equivalent notion (via the Grothendieck construction) of an indexed category which is a pseudo-functor into the 2-category **Cat** of categories allowing an alternative perspective and often an easier way to show that a fibration exists. In 2008 Shulman [25] introduced the notion of a monoidal fibration and later Moeller and Vasilakopoulou [18] fully establish a monoidal Grothendieck construction which

allows us to describe monoidal fibrations and monoidal indexed categories. Shulman [25] described a process using monoidal fibrations which allows us to lift morphisms in monoidal categories which is used to construct the data of a fibred monoidal double category and it is this process by which we construct a fibred monoidal double category of Hopf algebras and modules in this thesis.

The first chapter of this thesis recounts the definitions and some useful results on monoidal categories, enriched categories, and string diagrams which we will use extensively throughout the rest of this thesis. In particular we will define profunctors and coends which will be used to describe modules and their tensor product. We will also introduce a string diagrammatic language for closed monoidal categories.

Chapter 2 introduces the definition of an algebra (also known as a monoid) in a monoidal category. We will also define coalgebras, bialgebras and Hopf algebras in a monoidal category and use string diagrams to prove properties of these structures. In particular we describe a monoidal structure on Hopf algebras and prove properties relating to the antipode of a Hopf algebra. We discuss cocommutative Hopf algebras which often behave like groups and we show that they form a cartesian monoidal category.

In Chapter 3 we discuss modules and representations of algebras using string diagrams for closed monoidal categories to prove that modules and representations are equivalent structures. We define the tensor product of modules over an algebra by a coequaliser, or equivalently a coend, and prove that this tensor product has desirable properties like the fact that for an algebra A and left A -module M that $A \otimes_A M \cong M$. We describe the relationship between left and right modules and bimodules of Hopf algebras which we will use to provide an isomorphism of double categories in the final chapter.

Chapter 4 outlines the definitions of double categories and their functors. This leads into Chapter 5 where we describe the theory of monoidal fibrations due to Shulman [25] and Moeller and Vasilakopoulou [18] and use the motivating example of a monoidal bifibration on the category of group representations $\mathbf{Rep}_K$ to the category of groups $\mathbf{Grp}$ to describe the construction of a fibred monoidal double category of group representations. This

construction is similar to that of Martin [17] where we make different choices for the cleaving of the bifibration $\mathbf{Rep}_K \rightarrow \mathbf{Grp}$ so that it resembles the construction we perform in the final chapter.

In Chapter 6 we prove the main result of this thesis which says that given a closed symmetric monoidal category $\mathcal{V}$ with sufficient coequalisers then you have a monoidal bifibration which allows you to construct a fibred monoidal double category of cocommutative Hopf algebras and modules. We will show that the monoidal double category resulting from this construction is isomorphic to a double category of cocommutative Hopf algebras and bimodules as a way of fully describing the data of the constructed double category.

Chapter 1

Background

In this first chapter we will cover background material on monoidal categories, enriched categories, and string diagrams as well as introducing the notation and conventions which we will use throughout this thesis. An understanding of basic category theory will be assumed, including functors, natural transformations, adjunction, limits and colimits, and the Yoneda Lemma. I recommend either [24] or [13] for readers unfamiliar with these topics while [16] can also be used as a reference for these topics as well as some of the content of this chapter.

We shall start by covering monoidal categories and their functors, symmetric and braided monoidal categories, and monoidal closed categories. The background on monoidal categories will be used to discuss enriched categories, ends and coends, and profunctors. Finally we will finish this chapter by explaining string diagrams for both monoidal and monoidal closed categories. We will discuss bicategories and double categories in Chapter 4.

1.1 Monoidal Categories

1.1.1 Defining Monoidal Categories

Given any two sets A and B we can take their cartesian product $A \times B$ and so we have the notion of being able to take the product of any two objects in the category **Set** of sets and functions. This structure can be described

for sets through a limit. If I have two vector spaces V and W , I can take their tensor product to obtain the vector space $V \otimes W$, but this does not arise from taking a limit in the same way. In general, a monoidal category is the structure which allows us to consider tensor products of objects within a category which we define as follows.

Definition 1.1.1. A **monoidal category** $(\mathcal{C}, \otimes, \mathbb{1})$ consists of the following data:

1. a category $\mathcal{C}$,
2. a functor $- \otimes -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$,
3. an object $\mathbb{1} \in \mathcal{C}$ called the **monoidal unit**,
4. a natural isomorphism

$$\alpha: (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$$

called the **associator**,

5. a pair of natural isomorphisms

$$\ell: \mathbb{1} \otimes - \Rightarrow - \quad r: - \otimes \mathbb{1} \Rightarrow -$$

called the **left unitor** and **right unitor** respectively,

such that for all $a, b, c, d \in \mathcal{C}$, the following diagrams (called the **coherence conditions** for the monoidal category) commute:

1. The **pentagon identity**

$$\begin{array}{ccccc}
 & & (a \otimes (b \otimes c)) \otimes d & & \\
 & \nearrow^{\alpha_{a,b,c} \otimes \text{id}} & & \searrow^{\alpha_{a,b \otimes c,d}} & \\
 ((a \otimes b) \otimes c) \otimes d & & & & a \otimes ((b \otimes c) \otimes d) \\
 \downarrow^{\alpha_{a \otimes b,c,d}} & & & & \downarrow^{\text{id} \otimes \alpha_{b,c,d}} \\
 (a \otimes b) \otimes (c \otimes d) & \xrightarrow{\alpha_{a,b,c \otimes d}} & & & a \otimes (b \otimes (c \otimes d))
 \end{array}$$

2. The **triangle identity**

$$\begin{array}{ccc}
 (a \otimes \mathbb{1}) \otimes b & \xrightarrow{\alpha_{a,\mathbb{1},b}} & a \otimes (\mathbb{1} \otimes b) \\
 \searrow^{r \otimes \text{id}} & & \swarrow^{\text{id} \otimes \ell} \\
 & a \otimes b &
 \end{array}$$

It might not seem clear why we require associators and unitors but in category theory it is best practice to not assume two objects must be equal unless absolutely necessary since they often aren't in practice. This will be clearer with some examples

Example 1.1.2. 1. As discussed above, the category of sets can be given the structure of a monoidal category $(\mathbf{Set}, \times, \{*\})$ where the one element set, $\{*\}$, is the monoidal unit. Notice that for sets A, B , and C that the products $(A \times B) \times C$ and $A \times (B \times C)$ are different, in particular an element of the first product looks like $((a, b), c)$ and an element of the second looks like $(a, (b, c))$. However, there is an obvious natural bijection between these sets which gives us the component for an associator. Similarly the sets $\{*\} \times A$ and A are certainly different sets but the bijection $(*, a) \mapsto a$ gives us the data for the left unitor with the right unitor described similarly. It is often the case that there is an obvious choice of associator and unitors, usually just a rewrit-

ing of the objects as is the case for $(\mathbf{Set}, \times, \{*\})$. In general we will avoid discussing the associators and unitors unless they are particularly relevant or interesting.

2. A category can often be equipped with different monoidal structures. The monoidal category $(\mathbf{Set}, \sqcup, \{*\})$ has the disjoint union of sets as its monoidal structure.
3. Another example of a category where the monoidal structure is given by the categorical product is $(\mathbf{Grp}, \times, \{e\})$ of groups where e is the identity in a group and so $\{e\}$ is the trivial group. A category where the monoidal structure is given by the categorical product is called a **cartesian monoidal category**. In fact as we shall see shortly, any category with products and a terminal object can be given the structure of a monoidal category.
4. As mentioned earlier there is a monoidal category $(\mathbf{Vect}_K, \otimes, K)$ of vector spaces over a fixed field K with the tensor product.
5. The cartesian product in $\mathbf{Vect}_K$ is the direct product and this also gives us a monoidal category $(\mathbf{Vect}_K, \oplus, \{*\})$.
6. The category $\mathbf{Cat}$ of small categories and functors has products and a final object $\{*\}$ and so by Lemma 1.1.3, which we will see in a moment, there is a cartesian monoidal category $(\mathbf{Cat}, \times, *)$ of small categories.
7. An example of a monoidal category where the objects don't have underlying sets is $(\mathbb{R}^+, +, 0)$, where $\mathbb{R}^+$ is the category whose objects are non-negative real numbers and $+\infty$ and where a unique morphism $n \rightarrow m$ exists precisely when $n \geq m$. The monoidal product of two numbers is given by addition with 0 as the monoidal unit. This is an example of a category where the associator and unitors are given by identities. We call such a category a **strict monoidal category**.
8. A commutative monoid M with three (not necessarily unique) invertible elements gives a one object monoidal category $(\{*\}, \otimes, *)$ where

the morphisms are given by elements of M and the tensor product of two morphisms is given by their product in M . The three invertible elements correspond to the unitors and the associator.

9. Any monoid M can be described as a discrete monoidal category where the objects are elements of M , the monoidal product is given by multiplication in the monoid, and the monoidal unit is the unit in the monoid.
10. Given a small category $\mathcal{C}$, the category of endofunctors from $\mathcal{C} \rightarrow \mathcal{C}$ is strict monoidal with the product given by composition and the monoidal unit given by the identity functor.

Lemma 1.1.3. *Let $\mathcal{C}$ be a category with a terminal object $\mathbb{1}$ and for each pair of objects a and b , a product $a \times b$. The category $\mathcal{C}$ forms a monoidal category $(\mathcal{C}, \times, \mathbb{1})$ with canonical associator and unitors. We call such a category a **cartesian monoidal category**.*

The associator and unitors for a cartesian monoidal category are obtained by the unique factorisation and projection maps defined by the product. For a full proof see [16, Prop. III.5.1].

With these examples we have not discussed the coherence diagrams. In practice these almost always hold and checking them is tedious but straightforward. The coherence diagrams are required because there are two different ways to re-bracket four objects and two different ways to apply unitors in the triangle identity so we need to ensure these diagrams commute so that there is a unique way to go from one bracketing to another. A keen reader may ask if there are more complex diagrams where we can cause ambiguity from, for instance, using associators on the product of five objects or more. Fortunately for us, Mac Lane has shown that no such ambiguity arises. We direct readers to [16, Chapter 7.2] for the discussion and proof of this result and we shall not further discuss it here.

A monoidal category is meant to provide a multiplication structure for a category much like how a monoid provides a multiplication structure for a set. One might ask, as we do for monoids, if the multiplication in a monoidal

category is commutative. That is to say if $c \otimes c'$ is a product in a monoidal category $\mathcal{C}$ then is this the same as the product $c' \otimes c$. In the categorical spirit of "the same" meaning isomorphic rather than equal we provide the following definition.

Definition 1.1.4. A **braided monoidal category** is a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ equipped with a natural isomorphism $\beta_{a,b}: a \otimes b \rightarrow b \otimes a$ such that for both β and β^{-1} , the following hexagon diagram commutes for all objects $a, b, c \in \mathcal{C}$.

$$\begin{array}{ccccc}
 & a \otimes (b \otimes c) & \xrightarrow{\beta} & (b \otimes c) \otimes a & \\
 & \nearrow \alpha & & \searrow \alpha & \\
 (a \otimes b) \otimes c & & & & b \otimes (c \otimes a) \\
 & \searrow \beta \otimes \text{id} & & \nearrow \text{id} \otimes \beta & \\
 & (b \otimes a) \otimes c & \xrightarrow{\alpha} & b \otimes (a \otimes c) &
 \end{array}$$

Notice how we call this “commuting” monoidal category braided rather than commutative. This is because when we define isomorphisms between products there is more structure to explore. In particular if we look at the monoidal category $(\mathbf{Set}, \times, \{*\})$ then certainly there is an isomorphism $A \otimes B \rightarrow B \otimes A$ by $(a, b) \mapsto (b, a)$ but if we apply this switch back again we get back to where we started. There is an inherent symmetry to the monoidal product in **Set** which is more structure then merely requiring an isomorphism to exist. This gives us an alternative perspective on what a “commuting” monoidal category might mean.

Definition 1.1.5. A **symmetric monoidal category** is a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ equipped with a natural isomorphism $\sigma_{a,b}: a \otimes b \rightarrow b \otimes a$ such that $\sigma^2 = \text{id}$ and the following hexagon diagram commutes for all objects

$a, b, c \in \mathcal{C}$.

$$\begin{array}{ccccc}
 & a \otimes (b \otimes c) & \xrightarrow{\sigma} & (b \otimes c) \otimes a & \\
 \alpha \nearrow & & & & \searrow \alpha \\
 (a \otimes b) \otimes c & & & & b \otimes (c \otimes a) \\
 \searrow \sigma \otimes \text{id} & & & & \nearrow \text{id} \otimes \sigma \\
 & (b \otimes a) \otimes c & \xrightarrow{\alpha} & b \otimes (a \otimes c) &
 \end{array}$$

Notice that the hexagon for a symmetric monoidal category is the same as that for a braided monoidal category except that since $\sigma = \sigma^{-1}$ we don't need to prove the hexagon commutes for the inverse. It is helpful to realise that braided monoidal categories were actually introduced later than symmetric monoidal categories in a 1985 preprint by Joyal and Street [8], later published in [9]. In particular this means much of the literature on monoidal categories uses symmetric monoidal categories even when braided would suffice. It also helps explain the name “braided”, arising from the fact that taking the braiding β twice induces a change from where you started. For now we will provide some examples of braided and symmetric monoidal categories.

- Example 1.1.6.**
1. The monoidal category $(\mathbf{Set}, \times, \{*\})$ is symmetric monoidal with symmetry given by simply flipping the order we write the elements as described above.
 2. In fact, any cartesian monoidal category is symmetric by the uniqueness of limits.
 3. Similarly the monoidal category $(\mathbf{Set}, \sqcup, \{\})$ is symmetric monoidal, as is any category where the monoidal product is given by the categorical coproduct.
 4. The monoidal category $(\mathbf{Vect}_K, \otimes, K)$ is symmetric despite not being cartesian.

5. The discrete monoidal category arising from a monoid is symmetric if and only if its monoid is commutative.
6. The monoidal category of endofunctors on a small category $\mathcal{C}$ is not symmetric or braided in general.
7. Given an algebra A , there is a monoidal category of A - A -bimodules $A\text{-}\mathbf{Mod}\text{-}A, \otimes_A, A)$ but this is not symmetric or braided. Don't worry if you are unfamiliar with modules as we will discuss them in more detail in Chapter 3.

1.1.2 Monoidal Functors

Now we have introduced monoidal categories, the next step is to ask what the morphisms between monoidal categories are so we have a tool for doing things with them. Functors already provide a morphism between categories so the natural thing to do would be to ask for a functor which preserves the monoidal structure. We again borrow some inspiration from monoids where we would ask that a monoid morphism $f: M \rightarrow N$ preserves the multiplication by requiring $f(m \cdot m') = f(m) \cdot f(m')$ and that applying f to the identity in M gives us the identity in N . However, in true category theoretic fashion, equality is too strong of a notion. What we are really interested in is making sure that if $F: \mathcal{C} \rightarrow \mathcal{D}$ is a morphism of monoidal categories then a product of objects in $\mathcal{C}$ gives us a corresponding product in $\mathcal{D}$. This leads us to the following definition:

Definition 1.1.7. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ be monoidal categories. A **lax monoidal functor** is a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ together with:

1. A morphism

$$\iota: \mathbb{1}_{\mathcal{D}} \rightarrow F(\mathbb{1}_{\mathcal{C}})$$

we call the **unitator**.

2. A natural transformation

$$\omega: F(-) \otimes_{\mathcal{D}} F(-) \Rightarrow F(- \otimes_{\mathcal{C}} -)$$

we call the **multiplicator**,

such that for all objects $a, b, c \in \mathcal{C}$ the following coherence diagrams commute:

1. (associativity)

$$\begin{array}{ccc}
(F(a) \otimes_{\mathcal{D}} F(b)) \otimes_{\mathcal{D}} F(c) & \xrightarrow{\alpha_{\mathcal{D}}} & F(a) \otimes_{\mathcal{D}} (F(b) \otimes_{\mathcal{D}} F(c)) \\
\omega \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \omega \\
F(a \otimes_{\mathcal{C}} b) \otimes_{\mathcal{D}} F(c) & & F(a) \otimes_{\mathcal{D}} F(b \otimes_{\mathcal{C}} c) \\
\omega \downarrow & & \downarrow \omega \\
F((a \otimes_{\mathcal{C}} b) \otimes_{\mathcal{C}} c) & \xrightarrow{F(\alpha_{\mathcal{C}})} & F(a \otimes_{\mathcal{C}} (b \otimes_{\mathcal{C}} c))
\end{array}$$

2. (unitality)

$$\begin{array}{ccc}
\mathbb{1}_{\mathcal{D}} \otimes_{\mathcal{D}} F(a) & \xrightarrow{\ell_{\mathcal{D}}} & F(a) & F(a) \otimes_{\mathcal{D}} \mathbb{1}_{\mathcal{D}} & \xrightarrow{r_{\mathcal{D}}} & F(a) \\
\iota \otimes_{\mathcal{D}} \text{id} \downarrow & & \uparrow F(\ell_{\mathcal{C}}) & \text{id} \otimes_{\mathcal{D}} \iota \downarrow & & \uparrow F(r_{\mathcal{C}}) \\
F(\mathbb{1}_{\mathcal{C}}) \otimes_{\mathcal{D}} F(a) & \xrightarrow{\omega} & F(\mathbb{1}_{\mathcal{C}} \otimes_{\mathcal{C}} a) & F(a) \otimes_{\mathcal{D}} F(\mathbb{1}_{\mathcal{C}}) & \xrightarrow{\omega} & F(a \otimes_{\mathcal{C}} \mathbb{1}_{\mathcal{C}})
\end{array}$$

We use the adjective lax here to describe the fact that unitator and multiplicator only weakly preserve the monoidal structure. There is a dual notion of an **oplax monoidal functor** where the direction of the unitator and multiplicator are reversed and we flip the direction of the arrows in the coherence diagrams in the obvious way. If the unitator and multiplicator are isomorphisms we say we have a **strong monoidal functor**, furthermore if the unitator and multiplicator are identities then we have a **strict monoidal functor** and the associativity and unitality diagrams commute for free. Having a strong monoidal functor means we are preserving the monoidal structure in both directions and so it is natural to ask if we can define an equivalence of monoidal categories. First we need a notion of monoidal natural transformation.

Definition 1.1.8. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be strong monoidal functors. A **monoidal natural transformation** $\eta: F \rightarrow G$ is a natural transformation such that

for each pair of objects a, b in $\mathcal{C}$, the following diagrams commute.

$$\begin{array}{ccc}
 & \mathbb{1}_{\mathcal{D}} & \\
 \iota_F \swarrow & & \searrow \iota_G \\
 F(\mathbb{1}_{\mathcal{C}}) & \xrightarrow{\eta_{\mathbb{1}_{\mathcal{D}}}} & G(\mathbb{1}_{\mathcal{C}})
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(a) \otimes_{\mathcal{D}} F(b) & \xrightarrow{\omega_F} & F(a \otimes_{\mathcal{C}} b) \\
 \eta_a \otimes_{\mathcal{D}} \eta_b \downarrow & & \downarrow \eta_{a \otimes_{\mathcal{C}} b} \\
 G(a) \otimes_{\mathcal{D}} G(b) & \xrightarrow{\omega_G} & G(a \otimes_{\mathcal{C}} b)
 \end{array}$$

A **monoidal natural isomorphism** is a monoidal natural transformation that is also a natural isomorphism.

Now defining when two monoidal categories are equivalent follows naturally.

Definition 1.1.9. Two monoidal categories $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ are **monoidally equivalent** if there exist a pair of strong monoidal functors $F: \mathcal{C} \leftrightarrow \mathcal{D}: G$ such that there exists monoidal natural isomorphisms $\eta: \text{id}_{\mathcal{D}} \rightarrow F \circ G$ and $\theta: G \circ F \rightarrow \text{id}_{\mathcal{C}}$. Furthermore, if F and G provide an isomorphism of the underlying categories then we say that $\mathcal{C}$ and $\mathcal{D}$ are **isomorphic**.

We shall now provide some examples of monoidal functors

Example 1.1.10. 1. The forgetful functor $\mathbf{Grp} \rightarrow \mathbf{Set}$ is strong monoidal.

2. The free functor from $\mathbf{Set} \rightarrow \mathbf{Grp}$ is strong monoidal.

3. Any map of monoids induces a strong monoidal functor on their respective discrete monoidal categories.

4. Multiplication by a number > 1 provides a lax monoidal functor from $\mathbb{R}^+ \rightarrow \mathbb{R}^+$.

Now we have a notion of morphism between monoidal categories we are able to define categories of monoidal functor. We obtain the categories $\text{MonCat}_{\text{lax}}$, $\text{MonCat}_{\text{oplax}}$ and $\text{MonCat}_{\text{strong}}$, each of these categories is itself monoidal, inheriting the monoidal structure from the category $(\mathbf{Cat}, \otimes, *)$, we will prove this result (in more generality) when we talk about algebras and modules in Chapter 3.

1.1.3 Closed monoidal categories

We shall finish this section by discussing closed monoidal categories. The category of sets has the property that functions between two sets A and B itself forms a set $\mathbf{Set}(A, B)$. The question we then ask is whether other categories have this property. In fact many categories have homs which behave like objects within their category. A notable example of these is the category of vector spaces where the set of linear maps between two vector spaces itself forms a vector space. We will give more examples in a moment but first we will provide a definition to describe this structure.

Definition 1.1.11. Let $(\mathcal{V}, \otimes, \mathbb{1}, \beta)$ be a symmetric monoidal category. We say that $\mathcal{V}$ is a **closed monoidal category** if for each object $v \in \mathcal{V}$, the functor $v \otimes -$ has a right adjoint functor $[v, -]$. That is to say there exists an isomorphism

$$\mathcal{V}(a \otimes b, c) \cong \mathcal{V}(b, [a, c]),$$

natural in b and c , for all $a, b, c \in \mathcal{V}$. This generalises to a functor $[-, -]: \mathcal{V}^{\text{op}} \times \mathcal{V} \rightarrow \mathcal{V}$ by making different choices of v for right adjoints to $v \otimes -$ [16, p. 102]. The functor $[-, -]$ is called the **internal hom** and the isomorphism describing the adjunction is called the **tensor-hom** isomorphism.

It is worth noting that it is possible to define an internal hom without the prerequisite of a monoidal structure. However, for a closed category under suitable conditions you recover the monoidal structure [4]. Thus, we shall only be considering closed categories which are monoidal for the purpose of this thesis.

To illustrate the definition of a closed monoidal category we will go back to the example of the category of sets. If you have a function of sets $f: A \times B \rightarrow C$ then you can obtain a function $B \rightarrow \mathbf{Set}(A, C)$ which sends an element $b \in B$ to the function $f(-, b): A \rightarrow C$. Similarly if you have a function $g: B \rightarrow \mathbf{Set}(A, C)$ then you can obtain a function $A \times B \rightarrow C$ by evaluation, sending (a, b) to $g(b)(a)$. The evaluation of g here lends its name to the counit of the tensor-hom adjunction. In particular if $\mathcal{V}$ is a closed monoidal

category, the counit of the adjunction

$$\mathrm{ev}_v: v \otimes [v, -] \Rightarrow -$$

is called the **evaluation** and the unit of the adjunction

$$\mathrm{coev}_v: - \Rightarrow [v, v \otimes -]$$

is called the **coevaluation**. We shall now proceed with some examples.

Example 1.1.12. 1. As discussed above, the category **Set** is closed monoidal with internal hom given by **Set** $(-, -)$.

2. The monoidal category $(\mathbf{Vect}_K, \otimes, K)$ is closed with the vector space $\mathrm{Lin}(V, W)$ of linear maps giving an internal hom.

3. The monoidal category $(\mathbf{Cat}, \times, *)$ is closed with the internal hom between two categories $\mathcal{C}$ and $\mathcal{D}$ given by the category $[\mathcal{C}, \mathcal{D}]$ of functors and natural transformations.

4. The monoidal category $(\mathbb{R}^+, +, 0)$ has internal hom given by $[n, m] := \max(m - n, 0)$, this is called the **truncated difference** and is denoted $m \dot{-} n$.

5. The discrete monoidal category defined by a monoid M is closed if and only if every element has an inverse. That is to say M is a group. Recall that $M(m \cdot n, p)$ contains just the identity if $m \cdot n = p$ and is empty otherwise, hence $M(n, [m, p])$ is empty unless $n = [m, p]$, that is to say $n = m^{-1}p$ and so the internal hom $[m, p]$ is given by $m^{-1}p$.

It is worth noting that even in a monoidal category $\mathcal{V}$ without a symmetry or braiding it is possible to ask if a monoidal product $v \otimes -$ has a right adjoint $[v, -]$. If it does then we say $\mathcal{V}$ is **right closed**, similarly if $- \otimes v$ has a right adjoint we would say $\mathcal{V}$ is **left closed**. It is possible for a category to be both right and left closed but if it is not braided then the internal homs might be different.

1.2 Enriched Categories

Now that we have introduced the necessary background on closed monoidal categories we will use this section to introduce enriched category theory. Many of the definitions in this section are adapted from those given in [11] which is a thorough reference for the topic. The fundamental idea behind enriched categories is that for some categories the hom sets between objects have more structure than that provided by sets. We have already seen this in the previous section where we discussed internal homs, for instance the fact that the linear maps between two vector spaces form a vector space themselves. The following definition extends this notion by saying that instead of a hom *set* between objects we use a different category to provide a **hom object**.

Definition 1.2.1. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category. A $\mathcal{V}$ -category $\underline{\mathcal{C}}$ consists of the following data:

1. a class of objects $\text{ob}(\underline{\mathcal{C}})$,
2. for each pair of objects $a, b \in \text{ob}(\underline{\mathcal{C}})$, a **hom object** $\underline{\mathcal{C}}(a, b) \in \mathcal{V}$,
3. for each triple of objects $a, b, c \in \text{ob}(\underline{\mathcal{C}})$, a **composition morphism**

$$\circ_{a,b,c}: \underline{\mathcal{C}}(a, b) \otimes \underline{\mathcal{C}}(b, c) \rightarrow \underline{\mathcal{C}}(a, c)$$

in $\mathcal{V}$,

4. for each object $a \in \text{ob}(\underline{\mathcal{C}})$, an **identity morphism**

$$\text{id}_a: \mathbb{1} \rightarrow \underline{\mathcal{C}}(a, a)$$

in $\mathcal{V}$.

Such that, for all objects a, b, c , and d , the following diagrams commute

1. (associativity)

$$\begin{array}{ccc}
(\underline{\mathcal{C}}(a, b) \otimes \underline{\mathcal{C}}(b, c)) \otimes \underline{\mathcal{C}}(c, d) & \xrightarrow{\alpha} & \underline{\mathcal{C}}(a, b) \otimes (\underline{\mathcal{C}}(b, c) \otimes \underline{\mathcal{C}}(c, d)) \\
\downarrow \circ_{a,b,c} \otimes \text{id} & & \downarrow \text{id} \otimes \circ_{b,c,d} \\
\underline{\mathcal{C}}(a, c) \otimes \underline{\mathcal{C}}(c, d) & & \underline{\mathcal{C}}(a, b) \otimes \underline{\mathcal{C}}(b, d) \\
\searrow \circ_{a,c,d} & & \swarrow \circ_{a,b,d} \\
& \underline{\mathcal{C}}(a, d) &
\end{array}$$

2. (unitality)

$$\begin{array}{ccccc}
\underline{\mathcal{C}}(a, b) \otimes \mathbb{1} & \xrightarrow{r} & \underline{\mathcal{C}}(a, b) & \xleftarrow{\ell} & \mathbb{1} \otimes \underline{\mathcal{C}}(a, b) \\
\downarrow \text{id} \otimes \text{id}_b & & \parallel & & \downarrow \text{id}_a \otimes \text{id} \\
\underline{\mathcal{C}}(a, b) \otimes \underline{\mathcal{C}}(b, b) & \xrightarrow{\circ_{a,b,b}} & \underline{\mathcal{C}}(a, b) & \xleftarrow{\circ_{a,a,b}} & \underline{\mathcal{C}}(a, a) \otimes \underline{\mathcal{C}}(a, b)
\end{array}$$

To make it easier to distinguish we will use an underline to provide a notational difference between a regular category and an enriched one.

To explain what is happening here it is easiest to compare this definition to a normal category. Both a normal category and enriched category have a class of objects and the hom object for an enriched category is the natural extension of a hom set (or class of morphisms if you are concerned with size issues). For normal categories we would define a composition of morphisms by picking out some morphism $f \in \mathcal{C}(a, b)$ and $g \in \mathcal{C}(b, c)$ and choosing a morphism $g \circ f \in \mathcal{C}(a, c)$ but we can't do this for enriched categories because in general the hom objects might not have elements (we shall see some examples of this in a moment) so we define composition for the whole hom object at once. In a regular category we require every object to have an identity morphism. Again, in an enriched category we can't explicitly pick out an "element" of a hom object but picking a morphism from a set is the same as defining a function $\{*\} \rightarrow \mathcal{C}(a, a)$ with the single element being sent to our chosen function. An identity morphism $\text{id}_a: \mathbb{1} \rightarrow \underline{\mathcal{C}}(a, a)$ for an enriched category has this same purpose. Finally, we require composition of functions in a category to be associative and we require the identity to "do

nothing” under composition which is precisely what the coherence diagrams above are communicating. We shall now proceed with some examples.

Example 1.2.2. 1. Any locally small category is an enriched category over $(\mathbf{Set}, \times, \{*\})$ as described above.

2. As alluded to earlier, we can use internal homs to provide the hom objects in an enriched category. For instance, we have the $\mathbf{Vect}_K$ -category $\underline{\mathbf{Vect}}_K$ where the objects are vector spaces and the hom objects are given by the vector space of linear maps between them. We will explain this in more detail later in Lemma 1.2.7.

3. Enriching over $(\mathbf{Cat}, \times, *)$ defines a structure called a **2-category**. We think of a 2-category as being like a category except that the morphisms have functions between them which we call **2-cells**. A particularly important 2-category is given by letting the objects be categories and the hom objects be the functor categories between them. This means we have categories as objects, functors as the first type of morphism (or **1-cells**) and natural transformations as 2-cells. This defines the 2-category of categories. We discuss 2-categories in more detail in Chapter 4.

4. Any metric space can be described as a $\mathbb{R}^+$ -category. We take the objects of our enriched category to be the points in the metric space and the hom objects are given by the distance between points. The identity morphism tells us that every point is distance 0 from itself and the composition morphism translates to the triangle inequality.

5. For a group G , the category BG with one object, morphisms given by elements of the group and composition given by group multiplication can be viewed as a **Grp**-category $\underline{BG}$.

As with monoidal categories, we need functors between enriched categories as a tool for manipulating them. In a regular category, a functor is defined by a map of objects and a map of morphisms with functoriality and unit conditions. However, the usual definition of functoriality (the property

that $F(f \circ g) = F(f) \circ F(g)$ references elements of the hom objects. It is this structure we are generalising with the following definition.

Definition 1.2.3. Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$ -categories. A $\mathcal{V}$ -functor $F: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ consists of

1. a function of objects $F: \text{ob}(\underline{\mathcal{C}}) \rightarrow \text{ob}(\underline{\mathcal{D}})$,
2. for each pair of objects $a, b \in \text{ob}(\underline{\mathcal{C}})$ a morphism in $\mathcal{V}$

$$F_{a,b}: \underline{\mathcal{C}}(a, b) \rightarrow \underline{\mathcal{D}}(F(a), F(b))$$

such that for all objects $a, b, c \in \text{ob}(\underline{\mathcal{C}})$ the following diagrams commute.

1. (functoriality)

$$\begin{array}{ccc} \underline{\mathcal{C}}(a, b) \otimes \underline{\mathcal{C}}(b, c) & \xrightarrow{\quad \circ \quad} & \underline{\mathcal{C}}(a, c) \\ F_{a,b} \otimes F_{b,c} \downarrow & & \downarrow F_{a,c} \\ \underline{\mathcal{D}}(F(a), F(b)) \otimes \underline{\mathcal{D}}(F(b), F(c)) & \xrightarrow[\quad \circ \quad]{} & \underline{\mathcal{D}}(F(a), F(c)) \end{array}$$

2. (preservation of identity)

$$\begin{array}{ccc} & \mathbb{I} & \\ \text{id}_a \swarrow & & \searrow \text{id}_{F(a)} \\ \underline{\mathcal{C}}(a, a) & \xrightarrow{F_{a,a}} & \underline{\mathcal{D}}(F(a), F(a)) \end{array}$$

We shall now provide some examples.

Example 1.2.4. 1. Any regular functor F of locally small categories can be interpreted as an enriched **Set**-functor. The function $F_{a,b}: \underline{\mathcal{C}}(a, b) \rightarrow \underline{\mathcal{D}}(F(a), F(b))$ is given by $f \mapsto F(f)$ and the commuting diagrams translate immediately to functoriality and the unit condition $F(\text{id}_a) = \text{id}_{F(a)}$.

2. An enriched functor of **Vect**_K-categories is a functor whose map of hom objects is a linear map.

3. A **Cat**-endofunctor F of the category of 2-categories is a map of categories with morphisms between the hom objects (which are functor categories) given by functors.
4. A $\mathbb{R}^+$ -functor F of metric spaces is a map of points in the space such that for points a and b , the distance between a and b is greater than or equal to the distance between $F(a)$ and $F(b)$. This is to say an $\mathbb{R}^+$ -functor is a distance non-increasing map.

We've now defined enriched categories and enriched functors. The natural next question to ask is if we can define enriched natural transformations. Recall that in a regular category, a natural transformation is defined by a collection of morphisms. In a $\mathcal{V}$ -category we can't talk about individual morphisms explicitly so instead use the monoidal unit of $\mathcal{V}$ to pick out a morphism, much like when defining the identity morphism in Definition 1.2.1, by choosing a morphism $\mathbb{1} \rightarrow \underline{\mathcal{C}}(F(a), G(a))$ in $\mathcal{V}$. Recall the motivation for this is that when we take $\mathcal{V} = \mathbf{Set}$ then a function $\{*\} \rightarrow \underline{\mathcal{C}}(F(a), G(a))$ is defined by the morphism which $*$ is sent to. Armed with a method for picking out the components of a natural transformation it remains to describe the naturality condition they must obey.

Definition 1.2.5. Let $\mathcal{V}$ be a monoidal category and $F, G: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ be $\mathcal{V}$ -functors. A $\mathcal{V}$ -natural transformation $\nu: F \Rightarrow G$ is a collection of morphisms $\nu_a: \mathbb{1} \rightarrow \underline{\mathcal{D}}(F(a), G(a))$ for each object $a \in \underline{\mathcal{C}}$ such that, for all $a, b \in \underline{\mathcal{C}}$ the following diagram commutes.

$$\begin{array}{ccc}
\underline{\mathcal{C}}(a, b) \otimes \mathbb{1} & \xrightarrow{F \otimes \nu_b} & \underline{\mathcal{D}}(F(a), F(b)) \otimes \underline{\mathcal{D}}(F(b), G(b)) \\
\cong \uparrow & & \downarrow \circ \\
\underline{\mathcal{C}}(a, b) & & \underline{\mathcal{D}}(F(a), G(b)) \\
\cong \downarrow & & \uparrow \circ \\
\mathbb{1} \otimes \underline{\mathcal{C}}(a, b) & \xrightarrow{\nu_a \otimes G} & \underline{\mathcal{D}}(F(a), G(a)) \otimes \underline{\mathcal{D}}(G(a), G(b))
\end{array}$$

Since $\mathcal{V}$ -functors have $\mathcal{V}$ -natural transformations between them, for any two $\mathcal{V}$ -categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ we can form a non-enriched category $[\underline{\mathcal{C}}, \underline{\mathcal{D}}]$ of $\mathcal{V}$ -

functors. In particular this means that by enriching in $\mathbf{Cat}$ we can obtain a 2-category $\mathcal{V}\text{-}\mathbf{Cat}$ of $\mathcal{V}$ -categories, $\mathcal{V}$ -functors, and $\mathcal{V}$ -natural transformations. Much like how you can take the product of two categories in $\mathbf{Cat}$ because $\mathbf{Cat}$ is a monoidal category, we will want to be able to take the product of two $\mathcal{V}$ -categories. To be sure that a product of $\mathcal{V}$ -categories is correct we will need define a notion of monoidal 2-category and show that $\mathcal{V}\text{-}\mathbf{Cat}$ has this structure. The data which makes $\mathcal{V}\text{-}\mathbf{Cat}$ a monoidal 2-category is provided in [11, p. 12]. For now, we will just provide a definition of the product of two $\mathcal{V}$ -categories.

Lemma 1.2.6. *Let $\mathcal{V}$ be a symmetric monoidal category and let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$ -categories. There is a $\mathcal{V}$ -category $\underline{\mathcal{C}} \otimes_{\mathcal{V}\text{-}\mathbf{Cat}} \underline{\mathcal{D}}$ with the following data:*

1. *Objects*

$$\text{ob}(\underline{\mathcal{C}} \otimes_{\mathcal{V}\text{-}\mathbf{Cat}} \underline{\mathcal{D}}) := \text{ob}(\underline{\mathcal{C}}) \times \text{ob}(\underline{\mathcal{D}}).$$

2. *Hom objects*

$$\underline{\mathcal{C}} \otimes_{\mathcal{V}\text{-}\mathbf{Cat}} \underline{\mathcal{D}}((c_1, d_1), (c_2, d_2)) := \underline{\mathcal{C}}(c_1, c_2) \otimes \underline{\mathcal{D}}(d_1, d_2).$$

3. *Composition*

$$\begin{aligned} & (\underline{\mathcal{C}}(c_1, c_2) \otimes \underline{\mathcal{D}}(d_1, d_2)) \otimes (\underline{\mathcal{C}}(c_2, c_3) \otimes \underline{\mathcal{D}}(d_2, d_3)) \\ & \xrightarrow{\text{id} \otimes \sigma \otimes \text{id}} (\underline{\mathcal{C}}(c_1, c_2) \otimes \underline{\mathcal{C}}(c_2, c_3)) \otimes (\underline{\mathcal{D}}(d_1, d_2) \otimes \underline{\mathcal{D}}(d_2, d_3)) \\ & \xrightarrow{\circ \otimes \circ} \underline{\mathcal{C}}(c_1, c_3) \otimes \underline{\mathcal{D}}(d_1, d_3). \end{aligned}$$

4. *Identity morphisms*

$$\mathbb{1} \cong \mathbb{1} \otimes \mathbb{1} \xrightarrow{\text{id}_c \otimes \text{id}_d} \underline{\mathcal{C}}(c, c) \otimes \underline{\mathcal{D}}(d, d).$$

Checking the associativity and unitality conditions is straightforward but tedious use of naturality. This can be shown more easily once we introduce string diagrams in Section 1.4

We have seen how the internal hom of vector spaces provides a way to make $\mathbf{Vect}_K$ into a $\mathbf{Vect}_K$ -category. In general any closed monoidal category can be enriched in itself which is what we will discuss for the rest of this section.

Lemma 1.2.7. *Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a closed monoidal category. There exists a $\mathcal{V}$ -category $\underline{\mathcal{V}}$ with the following data:*

1. *Objects $\text{ob}(\underline{\mathcal{V}}) := \text{ob}(\mathcal{V})$.*
2. *Hom objects $\underline{\mathcal{V}}(v_1, v_2) := [v_1, v_2]$ given by the internal hom in $\mathcal{V}$.*
3. *Composition $[v_1, v_2] \otimes [v_2, v_3] \rightarrow [v_1, v_3]$ given by the map*

$$\begin{aligned} [v_1, v_2] \otimes [v_2, v_3] &\xrightarrow{\text{coev}_{v_1}} [v_1, v_1 \otimes ([v_1, v_2] \otimes [v_2, v_3])] \\ &\xrightarrow{[\text{id}, \text{ev}_{v_1} \otimes \text{id}]} [v_1, v_2 \otimes [v_2, v_3]] \\ &\xrightarrow{[\text{id}, \text{ev}_{v_2}]} [v_1, v_3]. \end{aligned}$$

4. *Identity $\mathbb{1} \rightarrow [v, v]$ given by*

$$\mathbb{1} \xrightarrow{\text{coev}_v} [v, v \otimes \mathbb{1}] \xrightarrow{[\text{id}, r]} [v, v].$$

As before, it will be easier to prove $\underline{\mathcal{V}}$ is indeed a $\mathcal{V}$ -category once we have discussed string diagrams in Section 1.4. When $\mathcal{V}$ is symmetric it is possible to extend the definition of a monoidal category to that of a monoidal enriched category but for now we will simply provide a definition of the product on $\underline{\mathcal{V}}$ [11, p. 16].

Lemma 1.2.8. *Let $\mathcal{V}$ be a symmetric closed monoidal category. There exists a $\mathcal{V}$ -functor*

$$-\underline{\otimes}- : \underline{\mathcal{V}} \otimes_{\mathcal{V}\text{-Cat}} \underline{\mathcal{V}} \rightarrow \underline{\mathcal{V}}$$

where the product on objects $a \underline{\otimes} b$ is given by the product $a \otimes b$ in $\mathcal{V}$ and the

product on hom objects is given by

$$\begin{aligned}
[v_1, v_2] \otimes [v_3, v_4] &\xrightarrow{\text{coev}_{v_1} \otimes v_3} [v_1 \otimes v_3, (v_1 \otimes v_3) \otimes ([v_1, v_2] \otimes [v_3, v_4])] \\
&\xrightarrow{[\text{id}, \text{id} \otimes \sigma \otimes \text{id}]} [v_1 \otimes v_3, (v_1 \otimes [v_1, v_2]) \otimes (v_3 \otimes [v_3, v_4])] \\
&\xrightarrow{[\text{id}, \text{ev}_{v_1} \otimes \text{ev}_{v_3}]} [v_1 \otimes v_3, v_2 \otimes v_4].
\end{aligned}$$

1.3 Profunctors, and Ends and Coends

In this section we will introduce profunctors which generalise hom functors, but also relations and bimodules. We will define ends and coends which are used as a way to compose profunctors but also as a generalisation of limits and colimits which allows us to define certain limits and colimits for enriched categories.

1.3.1 Profunctors

In an ordinary (small) category $\mathcal{C}$, there is a hom functor $\mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$. While this is a functor into $\mathbf{Set}$ we might really think of the hom functor as going from $\mathcal{C}$ to itself. Certainly if we have two hom functors $\mathcal{C}(-, c) \times \mathcal{C}(c, -)$ a natural construction is to compose the functions in each hom set via c so really we might think of the hom functor as going from $\mathcal{C}$ to $\mathcal{C}$. The idea of a profunctor generalises this idea.

Definition 1.3.1. A $\mathcal{V}$ -profunctor $T: \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathcal{V}$.

Example 1.3.2. 1. As discussed before, any hom functor (in a locally small category) is a $\mathbf{Set}$ -profunctor. The hom objects in a $\mathcal{V}$ -category are also $\mathcal{V}$ -profunctors which includes the internal hom in a closed monoidal category.

2. Given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ of regular categories we can obtain two different profunctors $\mathcal{D}(\text{id}, F): \mathcal{C} \nrightarrow \mathcal{D}$ and $\mathcal{D}(F, \text{id}): \mathcal{D} \nrightarrow \mathcal{C}$.

3. A relation between two sets S and T can be seen as a function $R: S \times T \rightarrow \{0, 1\}$ between sets where $(s, t) \mapsto 1$ if s is related to t and functions can be described as functors of discrete categories.
4. We will see in Chapter 3 that bimodules can be described as profunctors. For many applications this is the core motivation for using profunctors.

1.3.2 Ends and Coends

Now we have defined what a profunctor is we want to be able to compose them to obtain a new profunctor. We can't just compose profunctors directly like we would for functions or regular functors so we need to define how to do so which is where ends and coends come in. Ends and coends are a generalisation of limits and colimits respectively. We will start by defining ends, with coends being a dual notion. We base our definition for ends and coends on [16, pp. 222–229] with [15] being an excellent reference on the topic of ends and coends. An end is a wedge satisfying a universal property so we will define these first.

Definition 1.3.3. Let $T: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}$ be a functor (or equivalently a $\mathcal{V}$ -profunctor $\mathcal{C} \nrightarrow \mathcal{C}$). A **wedge** of T consists of an object $e \in \mathcal{V}$ and a collection of morphisms $\{\omega_a: e \rightarrow T(a, a)\}_{a \in \text{ob}(\mathcal{C})}$ such that for every morphism $f: a \rightarrow b$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc}
 e & \xrightarrow{\omega_a} & T(a, a) \\
 \omega_b \downarrow & & \downarrow T(\text{id}_a, f) \\
 T(b, b) & \xrightarrow{T(f, \text{id}_b)} & T(a, b)
 \end{array} \tag{1.1}$$

It is most clear what this definition is saying by seeing what a wedge is for a hom functor.

Example 1.3.4. Let T be the hom functor $\mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$. For every morphism $f: a \rightarrow b$, a diagram (1.1) is required by definition to satisfy

$\omega_a \circ \mathcal{C}(\text{id}_a, f) = \mathcal{C}(f, \text{id}_b) \circ \omega_b$. Re-written, that is saying $\omega_a \circ f = f \circ \omega_b$. This is exactly the condition for a collection of morphisms ω to comprise a natural transformation from $\text{id}_{\mathcal{C}}$ to itself. Hence, we can think of this wedge as a each ω_a picking out a set of components of natural transformations from e .

As mentioned above, we define wedges because an end is a wedge satisfying a universal property.

Definition 1.3.5. Let $T(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}$ be a functor. The **end** of T is a wedge (E, Ω) which satisfies the universal property that for any wedge (e, ω) there exists a unique map $e \rightarrow E$ commuting with ω_a and Ω_a for all $a \in \mathcal{C}$. This is to say the end of T is a wedge (E, Ω) such that for any wedge (e, ω) and object a , the following diagram commutes.

$$\begin{array}{ccc} e & & \\ \text{\scriptsize $\exists!$} \downarrow \text{\scriptsize ω_a} & \searrow & \\ E & \xrightarrow{\Omega_a} & T(a, a) \end{array}$$

This universal property provides uniqueness of E up to isomorphism. The object of the end is denoted $\int_{c \in \mathcal{C}} T(c, c)$.

Example 1.3.6. 1. In Example 1.3.4 we have seen that the wedges of a hom functor $\mathcal{C}(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ correspond to maps which pick out components of natural transformations. The universal property ensures that the end $\int_{c \in \mathcal{C}} \mathcal{C}(c, c)$ is given by the set of all natural transformations $\text{NT}(\text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}})$ because if there were natural transformations missing there would be wedges that wouldn't factor through and if any natural transformation appears twice then a set containing just one element corresponding to that natural transformation would give at least two maps into the end rather than a single unique one.

2. Let F and G be functors $\mathcal{C} \rightarrow \mathcal{D}$ then consider the functor

$$T := \mathcal{D}(F^{\text{op}}(-), G(-)): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$$

where F^{op} is the functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ induced by F . Similarly to the example above, the end $\int_{c \in \mathcal{C}} T(c, c)$ is isomorphic to the set of natural transformations $\text{NT}(F, G)$ from F to G .

3. Let $T: \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \mathcal{C}$ be a functor which does nothing in its first component and sends its second component to the image under a functor $F: \mathcal{J} \rightarrow \mathcal{C}$, that is to say $T(a, b) = F(b)$. The definition of the end (E, Ω) of this functor says that for all wedges (e, ω) , and morphisms $f: a \rightarrow b$ in $\mathcal{J}$ the following diagram commutes.

$$\begin{array}{ccccc}
 & e & & & \\
 & \downarrow \omega_a & & & \\
 & E & \xrightarrow{\Omega_a} & F(a) & \\
 \omega_b \swarrow & \downarrow \Omega_b & & \downarrow F(f) & \\
 & F(b) & \xrightarrow{F(\text{id}_b)} & F(b) &
 \end{array}$$

When we identify the objects $F(b)$ we recover the definition of a limit E of some diagram F over an indexing category $\mathcal{J}$. It is in this sense which ends generalise limits.

4. Since ends recover limits then there are cases where an end might not exist since not all categories are complete.

As mentioned at the start of this section, coends are the dual notion of ends. We define a cowedge by reversing the direction of arrows in the definition of a wedge as follows.

Definition 1.3.7. Let $T(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}$ be a functor (or equivalently a $\mathcal{V}$ -profunctor $\mathcal{C} \rightrightarrows \mathcal{C}$). A **cowedge** of T consists of an object $e \in \text{ob}(\mathcal{V})$ and a collection of morphisms $\{\lambda_a: T(a, a) \rightarrow e\}_{a \in \text{ob}(\mathcal{C})}$ such that for every

morphism $f: b \rightarrow a$ in $\mathcal{C}$ the following diagram commutes.

$$\begin{array}{ccc}
 e & \xleftarrow{\lambda_a} & T(a, a) \\
 \lambda_b \uparrow & & \uparrow T(\text{id}_a, f) \\
 T(b, b) & \xleftarrow{T(f, \text{id}_b)} & T(a, b)
 \end{array}$$

Much like with ends, a coend is a universal cowedge.

Definition 1.3.8. Let $T(-, -): \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The **coend** of T is a cowedge (E, Λ) which satisfies the universal property that for any cowedge (e, λ) there exists a unique map $E \rightarrow e$ commuting with λ_a and Λ_a for all $a \in \mathcal{C}$. This is to say the coend of T is a cowedge (E, Λ) such that for any cowedge (e, λ) and object a , the following diagram commutes.

$$\begin{array}{ccc}
 T(a, a) & \xrightarrow{\Lambda_a} & E \\
 & \searrow \lambda_a & \downarrow \exists! \\
 & & e
 \end{array}$$

The object of the coend is denoted $\int^{c \in \mathcal{C}} T(c, c)$.

Example 1.3.9. 1. Similarly to how we can write limits as ends in Example 1.3.6, we can take a colimit of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ by defining $T(a, b) = F(b)$ and taking a coend. In exactly the same way as in the case of limits and ends, we have that the colimit of F is the coend $\int^{c \in \mathcal{C}} T(c, c)$.

2. Let's consider the coend of $\mathbf{Set}(-, -): \mathbf{Set}^{\text{op}} \times \mathbf{Set} \rightarrow \mathbf{Set}$. Since the coend $E := \int^{S \in \mathbf{Set}} \mathbf{Set}(S, S)$ contains the image of every morphism $\Lambda_S: \mathbf{Set}(S, S) \rightarrow E$ we can consider E to be the disjoint union $\bigsqcup_{S \in \mathbf{Set}} \mathbf{Set}(S, S)$ modulo some equivalence relation $\sim$. Our coend diagram exactly says that for every function $f: T \rightarrow S$ we need $f \circ \Lambda_S = \Lambda_T \circ f$ and so we need $(g: S \rightarrow S) \sim (h: T \rightarrow T)$ whenever there exists

a function $f: T \rightarrow S$ such that $gf = fh$. The universal property for a coend tells us E must contain all such elements of $\bigsqcup_{S \in \mathbf{Set}} \mathbf{Set}(S, S) / \sim$.

3. For a group G , consider the one object category BG where the morphisms are group elements composed by multiplication. The coend of $BG(-, -) : BG^{\text{op}} \times BG \rightarrow \mathbf{Set}$ as above is the set $\bigsqcup_{* \in BG} BG(*, *) / \sim$ where $(g: a \rightarrow a) \sim (h: b \rightarrow b)$ if and only if there exists $f: b \rightarrow a$ such that $gf = fh$. For this category however, there is only one object and hence only one hom set given by G . Morphisms are simply group elements of G and so $gf = fh$ implies $h = f^{-1}gf$ which is exactly to say that $g \sim h$ if they share a conjugacy class in G . Hence, the coend $\int^{* \in BG} BG(*, *)$ is the set of conjugacy classes of G .
4. A reason we gave for introducing coends is as a way to compose profunctors. If $T: \mathcal{A} \rightarrow \mathcal{B}$ and $T': \mathcal{B} \rightarrow \mathcal{C}$ are profunctors then the coend $\int^{b \in \mathcal{B}} T(-, b) \times T'(b, -)$ defines a profunctor $\mathcal{A} \rightarrow \mathcal{C}$ (for details see [2, Chapter 7.8]).

Now we have defined ends and coends for regular categories the next step is to find a definition of ends for $\mathcal{V}$ -categories. We define the **opposite** of an enriched category $\underline{\mathcal{C}}^{\text{op}}$ to be the same as $\underline{\mathcal{C}}$ except the homs $\underline{\mathcal{C}}^{\text{op}}(c, c') := \underline{\mathcal{C}}(c', c)$ are given in reverse. In particular if $\mathcal{V}$ is closed and symmetric we can define $\mathcal{V}$ -functors of the form $T: \underline{\mathcal{C}}^{\text{op}} \otimes_{\mathcal{V}\text{-Cat}} \underline{\mathcal{C}} \rightarrow \underline{\mathcal{V}}$ at which point we can ask what the end of such a $\mathcal{V}$ -functor would be. The problem you immediately run into is that none of the morphisms in the commuting diagram for an end can be described explicitly because $\underline{\mathcal{V}}$ is an enriched category. Recall that, as part of the definition, a $\mathcal{V}$ -functor $F: \underline{\mathcal{C}} \rightarrow \underline{\mathcal{V}}$ provides morphisms

$$F_{a,b}: \underline{\mathcal{C}}(a, b) \rightarrow \underline{\mathcal{V}}(F(a), F(b)).$$

In particular this means that if $T: \underline{\mathcal{C}}^{\text{op}} \otimes \underline{\mathcal{C}} \rightarrow \underline{\mathcal{V}}$ is a $\mathcal{V}$ -functor we have two partial $\mathcal{V}$ -functors $T(c_1, \cdot)$ and $T(\cdot, c_2)$ (see [11, pp. 12–13] for the precise definition of a partial functor). By the definition of a $\mathcal{V}$ -functor we hence

obtain two collections of morphisms

$$T(c_1, -): \underline{\mathcal{C}}(c_1, c_2) \rightarrow \underline{\mathcal{V}}(T(c_1, c_1), T(c_1, c_2)) \quad (1.2)$$

and

$$T(-, c_2): \underline{\mathcal{C}}^{\text{op}}(c_2, c_1) \rightarrow \underline{\mathcal{V}}(T(c_2, c_2), T(c_1, c_2)). \quad (1.3)$$

Recall that when $\mathcal{V} = \mathbf{Set}$ this is saying that we are sending a function $f: c_1 \rightarrow c_2$ to the functions $T(\text{id}_{c_1}, f)$ or $T(f, \text{id}_{c_2})$ respectively. These are exactly the maps we want from our commuting square in the definition of a wedge so if we can describe a way to pre-compose these maps with the morphisms ω for a wedge then we would have a way to describe wedges within an enriched category. The issue here is that the wedge morphisms ω_c go from the end E to $T(c, c)$ but the fact that $\mathcal{V}$ is closed and symmetric comes to the rescue here. Recall that $\underline{\mathcal{V}}(c_1, c_2)$ is defined as $[c_1, c_2]$ so by the tensor-hom isomorphism we have

$$\begin{aligned} \mathcal{V}(\underline{\mathcal{C}}(c_1, c_2), [T(c_1, c_1), T(c_1, c_2)]) &\cong \mathcal{V}(T(c_1, c_1) \otimes \underline{\mathcal{C}}(c_1, c_2), T(c_1, c_2)) \\ &\stackrel{\mathcal{V}(\sigma, \text{id})}{\cong} \mathcal{V}(\underline{\mathcal{C}}(c_1, c_2) \otimes T(c_1, c_1), T(c_1, c_2)) \\ &\cong \mathcal{V}(T(c_1, c_1), [\underline{\mathcal{C}}(c_1, c_2), T(c_1, c_2)]), \end{aligned}$$

which makes (1.2) adjoint to a map

$$T(c_1, c_1) \rightarrow [\underline{\mathcal{C}}(c_1, c_2), T(c_1, c_2)],$$

and similarly for (1.3). When $\mathcal{V} = \mathbf{Set}$ we can take the product over all the maps adjoint to (1.2) and (1.3) to provide two parallel maps, allowing us to re-state the definition for an end as being given by the equaliser

$$\int_{c \in \mathcal{C}} T(c, c) \rightarrow \prod_c T(c, c) \rightrightarrows \prod_{c_1, c_2} \mathbf{Set}(\mathcal{C}(c_1, c_2), T(c_1, c_2))$$

where the map $\int_{c \in \mathcal{C}} T(c, c) \rightarrow \prod_c T(c, c)$ is given by the product $\prod_c \Omega_c$. By

a similar argument, the functor $T(c_1, -)$ provides morphisms

$$T(c_1, -): \underline{\mathcal{C}}(c_2, c_1) \rightarrow \underline{\mathcal{V}}(T(c_1, c_2), T(c_1, c_1)) \quad (1.4)$$

and

$$T(-, c_2): \underline{\mathcal{C}}^{\text{op}}(c_1, c_2) \rightarrow \underline{\mathcal{V}}(T(c_1, c_2), T(c_2, c_2)). \quad (1.5)$$

where in **Set**, taking the union of the adjoints of these maps allows us to re-write the definition for coends as being given by the coequaliser

$$\coprod_{c_1, c_2} T(c_1, c_2) \times \mathcal{C}(c_2, c_1) \rightrightarrows \coprod_c T(c, c) \rightarrow \int^c T(c, c).$$

This now allows us to generalise ends and coends to enriched categories.

Definition 1.3.10. Let $T: \underline{\mathcal{C}}^{\text{op}} \otimes_{\mathcal{V}\text{-}\mathbf{Cat}} \underline{\mathcal{C}} \rightarrow \underline{\mathcal{V}}$ be a $\mathcal{V}$ -functor. When $\mathcal{V}$ has sufficient products and equalisers, the end of T is given by the equaliser

$$\int_{c \in \underline{\mathcal{C}}} T(c, c) \rightarrow \prod_c T(c, c) \rightrightarrows \prod_{c_1, c_2} [\underline{\mathcal{C}}(c_1, c_2), T(c_1, c_2)]$$

in $\mathcal{V}$, and when $\mathcal{V}$ has sufficient coproducts and coequalisers, the coend of T is given by the coequaliser

$$\coprod_{c_1, c_2} T(c_1, c_2) \otimes \underline{\mathcal{C}}(c_2, c_1) \rightrightarrows \coprod_c T(c, c) \rightarrow \int^{c \in \underline{\mathcal{C}}} T(c, c).$$

The reader may have noticed that we have only defined ends and coends for $\mathcal{V}$ -functors $T: \underline{\mathcal{C}}^{\text{op}} \otimes_{\mathcal{V}\text{-}\mathbf{Cat}} \underline{\mathcal{C}} \rightarrow \underline{\mathcal{V}}$ rather than more generally functors to $\underline{\mathcal{D}}$ for some arbitrary $\mathcal{V}$ -category $\underline{\mathcal{D}}$. While it is possible to define ends and coends for such functors by introducing the notions of powering and copowering [20] we will only be interested in ends and coends for functors into $\underline{\mathcal{V}}$. It is also possible to define enriched ends and coends via weighted limits which is explored in [15, Chapter 4.3] but this is beyond the scope of this thesis.

To round up this section we will introduce a technical result about coends which we will need later. The following result is often called the Density

formula or co-Yoneda Lemma but we will use the name coined in [15].

Lemma 1.3.11 (ninja Yoneda Lemma). *For a $\mathcal{V}$ -functor $F: \underline{\mathcal{C}}^{\text{op}} \rightarrow \underline{\mathcal{V}}$, there exists a natural equivalence:*

$$F(-) \cong \int^c \mathcal{C}(-, c) \otimes F(c)$$

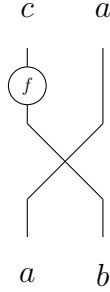
A proof of this result in the non-enriched setting can be found in [15, Prop 2.2.1] and [14]. It is worth noting that the order of the functors $\mathcal{C}(-, -)$ and F are reversed here compared to the same result in [15] and [14] due to a different choice of conventions for the tensor-hom adjunction. A proof for this result in the enriched setting can be found in [19] which uses the same convention as we do here.

1.4 String diagrams

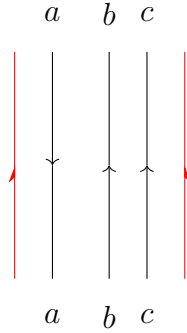
In this final section we will introduce the basics of string diagrams which we use throughout this thesis. String diagrams are a graphical tool used to prove diagrams in monoidal categories commute with the idea being that each string represents an object in a product. Fong and Spivak [5, Chp. 2] provide an introduction to string diagrams. In this thesis we read string diagrams from the bottom to the top and we use crossing strings to denote symmetry. For example, the composition

$$a \otimes b \xrightarrow{\sigma} b \otimes a \xrightarrow{f \otimes \text{id}} c \otimes a$$

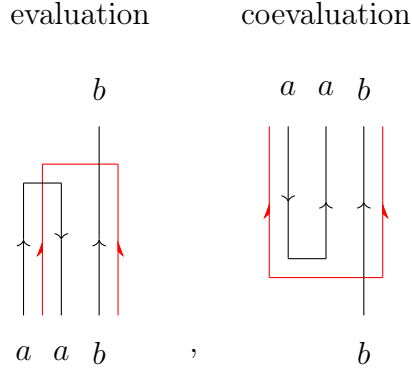
is represented by the following string diagram.



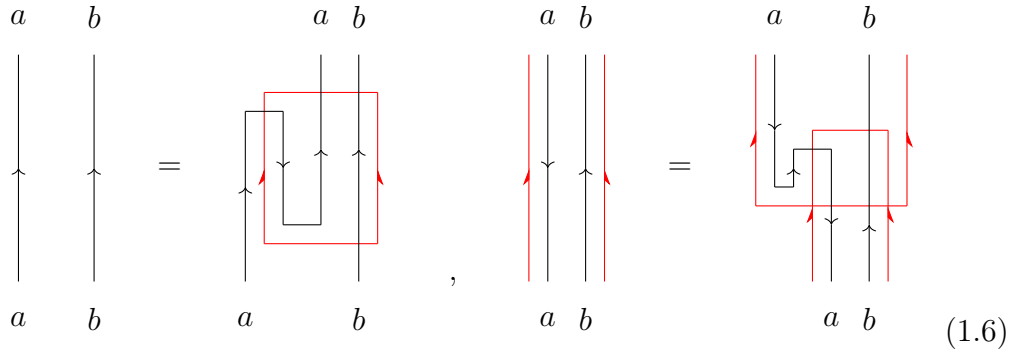
We will often omit labels for the string except for where doing so would cause confusion. As is standard, when a monoidal unit is present we omit its string since the unitors make it isomorphic to not being there. We will also be using string diagrams to represent objects in a closed monoidal category. This requires a description for the internal hom provided and proven to be valid by Reader [23] in a recent preprint. The internal hom is described by writing everything inside the internal hom between red lines where arrows are used to describe whether an object is covariant or contravariant. For example the object $[a, b \otimes c]$ would be described by the following string diagram.



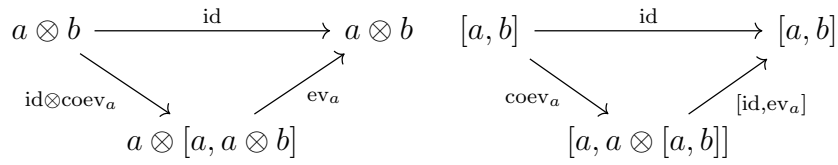
This language allows us to describe evaluation $\text{ev}: a \otimes [a, b] \rightarrow b$ and coevaluation $\text{coev}: b \rightarrow [a \otimes a, b]$ as the following diagrams.



This allows us to represent the triangle identities for the adjunction of evaluation and coevaluation as the equality of the following string diagrams



which are equivalent to the following diagrams commuting.



Finally, naturality is shown in string diagrams by allowing us to move any valid diagram up or down along strings.

Chapter 2

Algebras and Hopf Algebras

In this chapter we will introduce the definitions of an algebra, coalgebra, and Hopf algebra in a monoidal category as well as their morphisms. These definitions, in particular Hopf algebras, are often studied as Hopf algebras in the category of vector spaces over some field [1, 10]. We generalise these definitions to an arbitrary monoidal category and prove some of the important results about algebras and Hopf algebras that still hold in this more general setting.

2.1 Defining Algebras

We will start with the definition of an algebra.

Definition 2.1.1. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category. An **algebra** in $\mathcal{V}$ is a tuple (A, μ, η) consisting of an object $A \in \mathcal{V}$, multiplication $\mu: A \otimes A \rightarrow A$, and unit $\eta: \mathbb{1} \rightarrow A$ such that the following conditions hold.

1. Associativity of multiplication:

$$\begin{array}{ccc} (A \otimes A) \otimes A & \xrightarrow{\cong} & A \otimes (A \otimes A) \\ \mu \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \mu \\ A \otimes A & & A \otimes A \\ & \searrow \mu \quad \swarrow \mu & \\ & A & \end{array}$$

2. Multiplication of units (triangle identity):

$$\begin{array}{ccccc}
 \mathbb{1} \otimes A & \xleftarrow{\cong} & A & \xrightarrow{\cong} & A \otimes \mathbb{1} \\
 \eta \otimes \text{id} \downarrow & & \parallel & & \downarrow \text{id} \otimes \eta \\
 A \otimes A & \xrightarrow{\mu} & A & \xleftarrow{\mu} & A \otimes A
 \end{array}$$

Algebras in $\mathcal{V}$ are also sometimes referred to as a **monoids** in $\mathcal{V}$. Dually, we have the following notion.

Definition 2.1.2. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal category. A **coalgebra** in $\mathcal{V}$ is a tuple (C, Δ, ϵ) consisting of an object $C \in \mathcal{V}$, comultiplication $\Delta: C \rightarrow C \otimes C$, and counit $\epsilon: C \rightarrow \mathbb{1}$ such that the following conditions hold.

1. Co-associativity of comultiplication:

$$\begin{array}{ccc}
 (C \otimes C) \otimes C & \xleftarrow{\cong} & C \otimes (C \otimes C) \\
 \Delta \otimes \text{id} \uparrow & & \uparrow \text{id} \otimes \Delta \\
 C \otimes C & & C \otimes C \\
 \swarrow \Delta & & \searrow \Delta \\
 & C &
 \end{array}$$

2. Comultiplication of counits (triangle identity):

$$\begin{array}{ccccc}
 \mathbb{1} \otimes C & \xrightarrow{\cong} & C & \xleftarrow{\cong} & C \otimes \mathbb{1} \\
 \epsilon \otimes \text{id} \uparrow & & \parallel & & \uparrow \text{id} \otimes \epsilon \\
 C \otimes C & \xleftarrow{\Delta} & C & \xrightarrow{\Delta} & C \otimes C
 \end{array}$$

Similarly to algebras, coalgebras in $\mathcal{V}$ are sometimes referred to as **comonoids** in $\mathcal{V}$.

Example 2.1.3. 1. A monoid is an algebra in $(\mathbf{Set}, \times, \{*\})$. The multiplication is given by the monoid multiplication and the unit picks out the identity in the monoid.

2. Since groups are monoids with extra properties they are also algebras in $(\mathbf{Set}, \times, \{*\})$. Groups and monoids are also coalgebras by the diagonal map $g \mapsto (g, g)$.
3. An algebra in $(\mathbf{Vect}_K, \otimes_K, K)$ recovers the usual definition for an algebra. An important example is a group algebra KG for a group G .
4. In any monoidal category $\mathcal{V}$, the monoidal unit $\mathbb{1}$ is both an algebra and coalgebra with multiplication and comultiplication given by the unitors in $\mathcal{V}$ and the unit and counit given by the identity on $\mathbb{1}$. We call this the **trivial algebra** or **trivial coalgebra** respectively.
5. Given an algebra A in a symmetric monoidal category $\mathcal{V}$ we can define the **opposite algebra** $A^{\text{op}} := (A, \mu \circ \sigma, \eta)$. Similarly for a coalgebra C we can define the **co-opposite coalgebra** $C^{\text{cop}} := (C, \sigma \circ \Delta, \epsilon)$. The opposite algebra of a group G simply flips the order of multiplication. The opposite algebra of G is isomorphic to G by the assignment $g \mapsto g^{-1}$.
6. Since the definition of a coalgebra is given by reversing all the morphisms in the definition of an algebra we can also define a coalgebra in $\mathcal{V}$ as being an algebra in $\mathcal{V}^{\text{op}}$.
7. If $\mathcal{V}$ is a closed monoidal category then $[V, V]$ is an algebra with multiplication given by the composition of the internal hom and the unit given by the identity morphism $\mathbb{1} \rightarrow [V, V]$.
8. We saw in Chapter 1 that given a group G we can define the enriched **Grp**-category BG with one object and morphisms given by elements of G composed by multiplication. This is an algebra in the category of **Grp**-categories. In fact, in any closed symmetric monoidal category $\mathcal{V}$, an algebra $A \in \mathcal{V}$ defines an algebra BA in $\mathcal{V}\text{-Cat}$.

We will be using string diagrams to prove statements about algebras and coalgebras. Recall that we read string diagrams from the bottom up. We

will denote multiplication (left) and comultiplication (right) as follows.



Similarly, recalling that we don't draw strings for the monoidal unit we draw the unit (left) and counit (right) as follows.



We can now easily draw the associativity condition for an algebra as the two diagrams

(2.1)

being equal. Similarly the unit condition is given by

(2.2)

with the conditions on coalgebras being the same except the string diagrams are flipped.

As usual in category theory, once we have defined a new structure the first question is to ask what the morphisms are for this structure. If algebras are supposed to generalise monoids then morphisms between algebras should generalise monoid maps. A function of monoids $f: A \rightarrow B$ is required to preserve the multiplication and preserve the identity. This is to say that

$f(a \cdot a') = f(a) \cdot f(a')$ and $f(e_A) = e_B$. We generalise this to algebras in the following way.

Definition 2.1.4. For two algebras (A, μ, η) and (A', μ', η') in $\mathcal{V}$, a **algebra morphism** is a morphism $f: A \rightarrow A'$ in $\mathcal{V}$ such that f commutes with the multiplication and unit in the sense that the following diagrams commute.

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\
 \mu \downarrow & & \downarrow \mu' \\
 A & \xrightarrow{f} & A'
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathbb{1} & \\
 \eta \swarrow & & \searrow \eta' \\
 A & \xrightarrow{f} & A'
 \end{array}$$

These conditions are written in string diagrams as follows.

$$\begin{array}{ccc}
 \begin{array}{c} A' \\ | \\ \bullet \\ / \quad \backslash \\ (f) \quad (f) \\ | \quad | \\ A \quad A \end{array} & = & \begin{array}{c} A' \\ | \\ (f) \\ | \\ \bullet \\ / \quad \backslash \\ A \quad A \end{array}
 \end{array}
 , \quad
 \begin{array}{ccc}
 \begin{array}{c} A' \\ | \\ (f) \\ | \\ \bullet \end{array} & = & \begin{array}{c} A' \\ | \\ \bullet \end{array}
 \end{array}
 \tag{2.3}$$

The category of algebras in $\mathcal{V}$ and algebra morphisms between them is denoted $\text{Alg}(\mathcal{V})$.

We define functions of coalgebras similarly.

Definition 2.1.5. For two coalgebras (C, Δ, ϵ) and (C', Δ', ϵ') in $\mathcal{V}$, a **coalgebra morphism** is a morphism $f: C \rightarrow C'$ in $\mathcal{V}$ such that f commutes with the comultiplication and counit in the sense that the following diagrams commute.

$$\begin{array}{ccc}
 C \otimes C & \xrightarrow{f \otimes f} & C' \otimes C' \\
 \Delta \uparrow & & \uparrow \Delta' \\
 C & \xrightarrow{f} & C'
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathbb{1} & \\
 \epsilon \swarrow & & \nwarrow \epsilon' \\
 C & \xrightarrow{f} & C'
 \end{array}$$

These conditions are written in string diagrams as follows.

$$\begin{array}{c} C' \quad C' \\ \text{f} \quad \text{f} \\ \diagdown \quad \diagup \\ \bullet \\ | \\ C \end{array} = \begin{array}{c} C' \quad C' \\ \diagdown \quad \diagup \\ \bullet \\ | \\ \text{f} \\ | \\ C \end{array}, \quad \begin{array}{c} \bullet \\ | \\ \text{f} \\ | \\ C \end{array} = \begin{array}{c} \bullet \\ | \\ C' \end{array} \quad (2.4)$$

The category of coalgebras in $\mathcal{V}$ and coalgebra morphisms between them is denoted $\text{Coalg}(\mathcal{V})$.

These definitions recover the definitions of monoid morphisms and group homomorphisms. Importantly, the unit for an algebra is an algebra morphism from the trivial algebra and the counit is a coalgebra morphism to the trivial coalgebra. However, to check if the multiplication $A \otimes A \rightarrow A$ is an algebra map we need to define an algebra structure on $A \otimes A$ and similarly for comultiplication. In fact we will go one step further and provide the structure of a monoidal category on $\text{Alg}(\mathcal{V})$ and $\text{Coalg}(\mathcal{V})$.

Lemma 2.1.6. *Let $\mathcal{V}$ be a symmetric monoidal category. The category $\text{Alg}(\mathcal{V})$ of algebras and algebra morphisms in $\mathcal{V}$ is a monoidal category. The product of two algebras A and A' is given by the object $A \otimes A'$ in $\mathcal{V}$, with multiplication*

$$A \otimes A' \otimes A \otimes A' \xrightarrow{\text{id} \otimes \sigma \otimes \text{id}} A \otimes A \otimes A' \otimes A' \xrightarrow{\mu \otimes \mu'} A \otimes A'$$

and unit

$$\mathbb{1} \xrightarrow{\cong} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\eta \otimes \eta'} A \otimes A'.$$

The product of two algebra morphisms f and f' is given by $f \otimes f'$ in $\mathcal{V}$. The monoidal unit is given by the trivial algebra, and the associators and unitors are those from $\mathcal{V}$.

Proof. Firstly the associator and unitors in $\mathcal{V}$ are algebra maps by naturality and since they are associators and unitors in $\mathcal{V}$ we already know they satisfy

the pentagon and triangle identities. To prove $\text{Alg}(\mathcal{V})$ is a monoidal category with the structure given it remains to show the following:

1. $A \otimes A'$ is an algebra,
2. $f \otimes f'$ is an algebra morphism.

Both of these conditions follow almost immediately by naturality of symmetry or the naturality of the unitors in $\mathcal{V}$. $\square$

By flipping our string diagrams above or simply using the fact that coalgebras are monoids in $\mathcal{V}^{\text{op}}$ we obtain the dual result.

Corollary 2.1.7. *Let $\mathcal{V}$ be a symmetric monoidal category. The category $\text{Coalg}(\mathcal{V})$ of coalgebras and coalgebra morphisms in $\mathcal{V}$ is a monoidal category. The product of two coalgebras C and C' is given by the object $C \otimes C'$ in $\mathcal{V}$, with comultiplication*

$$C \otimes C' \xrightarrow{\Delta \otimes \Delta'} C \otimes C \otimes C' \otimes C' \xrightarrow{\text{id} \otimes \sigma \otimes \text{id}} C \otimes C' \otimes C \otimes C'$$

and counit

$$C \otimes C' \xrightarrow{\epsilon \otimes \epsilon'} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\cong} \mathbb{1}.$$

The product of two coalgebra morphisms f and f' is given by $f \otimes f'$ in $\mathcal{V}$. The monoidal unit is given by the trivial coalgebra, and the associators and unitors are the those from $\mathcal{V}$.

Since we need the category $\mathcal{V}$ which we take our algebras in to be symmetric to define the product of two algebras, from this point onwards we will assume $\mathcal{V}$ is always symmetric.

Now we are equipped to go back and answer our earlier question about whether the multiplication maps for an algebra are themselves algebra maps. Since we are required to introduce a twist in our definition for the multiplication map on a product, in general multiplication is not an algebra map since we would need it to commute with symmetry. To fix this we introduce the notion of a commutative algebra.

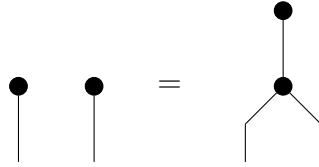
Definition 2.1.8. An algebra A in $\mathcal{V}$ is **commutative** if $\mu \circ \sigma = \mu$. Dually, a coalgebra C in $\mathcal{V}$ is **cocommutative** if $\sigma \circ \Delta = \Delta$.

- Example 2.1.9.**
1. Any commutative monoid is a commutative algebra.
 2. Any abelian group is a commutative algebra in $(\mathbf{Set}, \times, \{*\})$.
 3. Any group or in fact monoid is a cocommutative coalgebra in $(\mathbf{Set}, \times, \{*\})$ when we define the comultiplication as being given by the diagonal map.

We can now prove that allows us to prove the following result.

Lemma 2.1.10. *Let A be an algebra. If A is commutative then multiplication map μ is an algebra morphism.*

Proof. We need to check that μ commutes with the unit and multiplication described in Lemma 2.1.6. First, the condition that μ commutes with the unit is given by the fact that we already know the unit η is an algebra morphism and is described by the following string diagrams being equal.

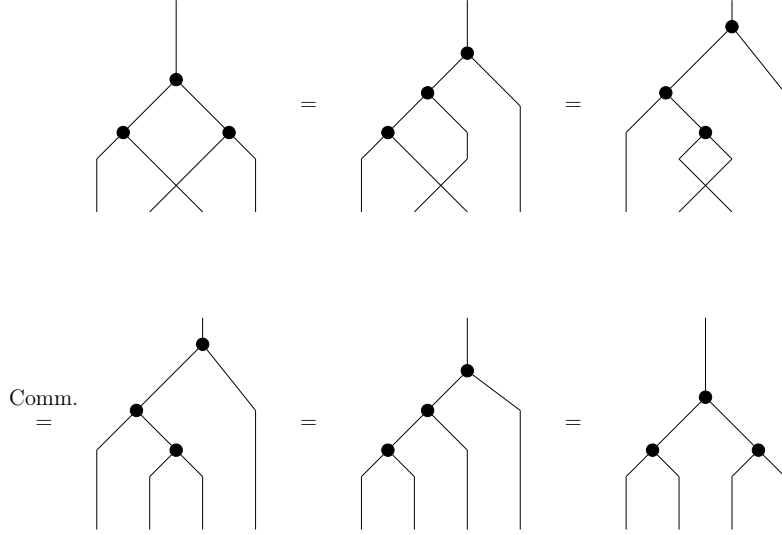


Finally we need to show

$$\begin{array}{ccc}
 A \otimes A \otimes A \otimes A & \xrightarrow{\mu \otimes \mu} & A \otimes A \\
 \mu_{A \otimes A} \downarrow & & \downarrow \mu \\
 A \otimes A & \xrightarrow{\mu} & A
 \end{array}$$

commutes which we demonstrate by applying associativity repeatedly in the

following string diagrams.



□

This of course has a dual result for algebras proven by flipping the diagrams above.

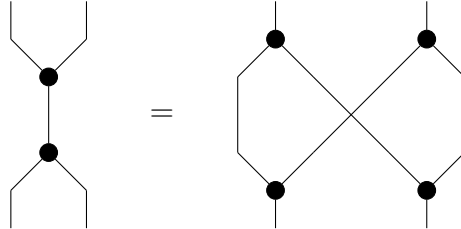
Lemma 2.1.11. *Let C be a coalgebra. If C is cocommutative then the comultiplication map Δ is a coalgebra morphism.*

We have already seen that a group can be interpreted as both an algebra and a coalgebra. It is useful to be able to consider both structures on the same object.

Definition 2.1.12. Let $(\mathcal{V}, \otimes, \mathbb{1}, \sigma)$ be a symmetric monoidal category. A **bialgebra** in $\mathcal{V}$ is a tuple $(B, \mu, \eta, \Delta, \epsilon)$ such that (B, μ, η) is an algebra and (B, Δ, ϵ) is a coalgebra and Δ and ϵ are algebra maps or equivalently μ and η are coalgebra maps. A morphism $f: B \rightarrow B'$ is a **bialgebra morphism** if f is both an algebra and coalgebra morphism.

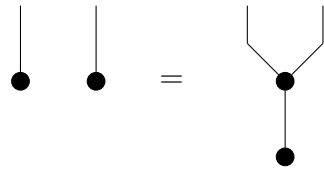
For a bialgebra, the condition that Δ is an algebra map means it satisfies

the multiplication condition



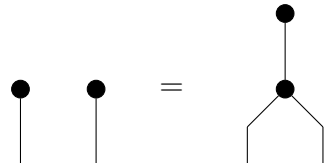
$$(2.5)$$

and the unit condition



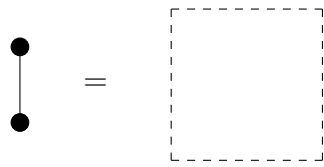
$$(2.6)$$

The condition that ϵ is an algebra map means it satisfies the multiplication condition



$$(2.7)$$

and the unit condition



$$(2.8)$$

We can now see this is equivalent to μ and η being coalgebra maps since (2.5) and (2.7) tell us μ is a coalgebra map and (2.6) and (2.8) tell us η is a coalgebra map.

2.2 Hopf Algebras

The defining property of a group is the existence of an inverse for each element. In this section we will further generalise this notion to that of Hopf algebras. As usual we will start with a definition

Definition 2.2.1. Let $(\mathcal{V}, \otimes, \mathbb{1}, \sigma)$ be a symmetric monoidal category. A **Hopf algebra** in $\mathcal{V}$ is a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ in $\mathcal{V}$ and a morphism $\chi: H \rightarrow H$ called the **antipode** such that the following diagram commutes.

$$\begin{array}{ccccc}
 H \otimes H & \xleftarrow{\Delta} & H & \xrightarrow{\Delta} & H \otimes H \\
 \downarrow \chi \otimes \text{id} & & \downarrow \epsilon & & \downarrow \text{id} \otimes \chi \\
 & & \mathbb{1} & & \\
 & & \downarrow \eta & & \\
 H \otimes H & \xrightarrow{\mu} & H & \xleftarrow{\mu} & H \otimes H
 \end{array}$$

This condition on the antipode can be written as the equivalence of the following string diagrams where we use a circle with a cross to represent the antipode.

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} \tag{2.9}$$

A **Hopf algebra morphism** is just a bialgebra morphism with no explicit condition on the antipode.

Example 2.2.2. 1. As discussed, one way to think of Hopf algebras is as a generalisation of groups. For a group G , the antipode is the map $(-)^{-1}: G \rightarrow G$. The comultiplication sends an element $g \in G$ to (g, g) , applying the inverse on one side gives us (g^{-1}, g) and then multiplying gives us $g^{-1}g = e$. This of course, is the same taking the counit and

then unit for a group which just picks out the identity and so G is a Hopf algebra with antipode given by $(-)^{-1}$.

2. Since a Hopf algebra H is both an algebra and a coalgebra we can define the opposite Hopf algebra H^{op} and co-opposite Hopf algebra H^{cop} .

Notice that in the example of groups, the antipode given by $(-)^{-1}$ is not a group homomorphism but rather flips the order of multiplication. This is true more generally for any antipode. We will prove this fact later in this chapter but first we need to know what it means for a morphism to flip the multiplication.

Definition 2.2.3. For two algebras A and A' in $\mathcal{V}$, an **anti-algebra morphism** is a morphism $f: A \rightarrow A'$ in $\mathcal{V}$ such that the following diagrams commute.

$$\begin{array}{ccc}
 A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\
 \mu \circ \sigma \downarrow & & \downarrow \mu' \\
 A & \xrightarrow{f} & A'
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathbb{1} & \\
 \eta \swarrow & & \searrow \eta' \\
 A & \xrightarrow{f} & A'
 \end{array}$$

The second diagram is just the usual unit condition for an algebra morphism, the first diagram is written in string diagrams as follows.

Similarly an **anti-coalgebra morphism** is a morphism $f: C \rightarrow C'$ satisfy-

ing

$$\begin{array}{ccc}
C \otimes C & \xrightarrow{f \otimes f} & C' \otimes C' \\
\Delta \uparrow & & \uparrow \sigma \circ \Delta' \\
C & \xrightarrow{f} & C'
\end{array}$$

or equivalently the counit condition for a coalgebra morphism and the equality of the following string diagrams.

A **anti-bialgebra morphism** is a morphism which is both an anti-algebra and anti-coalgebra morphism.

Equivalently, we can define an anti-algebra morphism $A \rightarrow A'$ as being an algebra morphism $A^{\text{op}} \rightarrow A'$ and similarly for anti-coalgebra morphisms. We can now prove that the antipode is an anti-coalgebra morphism in general.

Lemma 2.2.4. *For any Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, \chi)$ in $\mathcal{V}$, the antipode is an anti-bialgebra morphism.*

Proof. We will prove that the antipode is an anti-algebra morphism with the proof that χ is an anti-coalgebra morphism being given by flipping the string diagram proofs. We need to show that χ satisfies the unit condition which is to say that

(2.10)

are equal string diagrams. We also need to show that χ satisfies the multi-

plication condition which says that

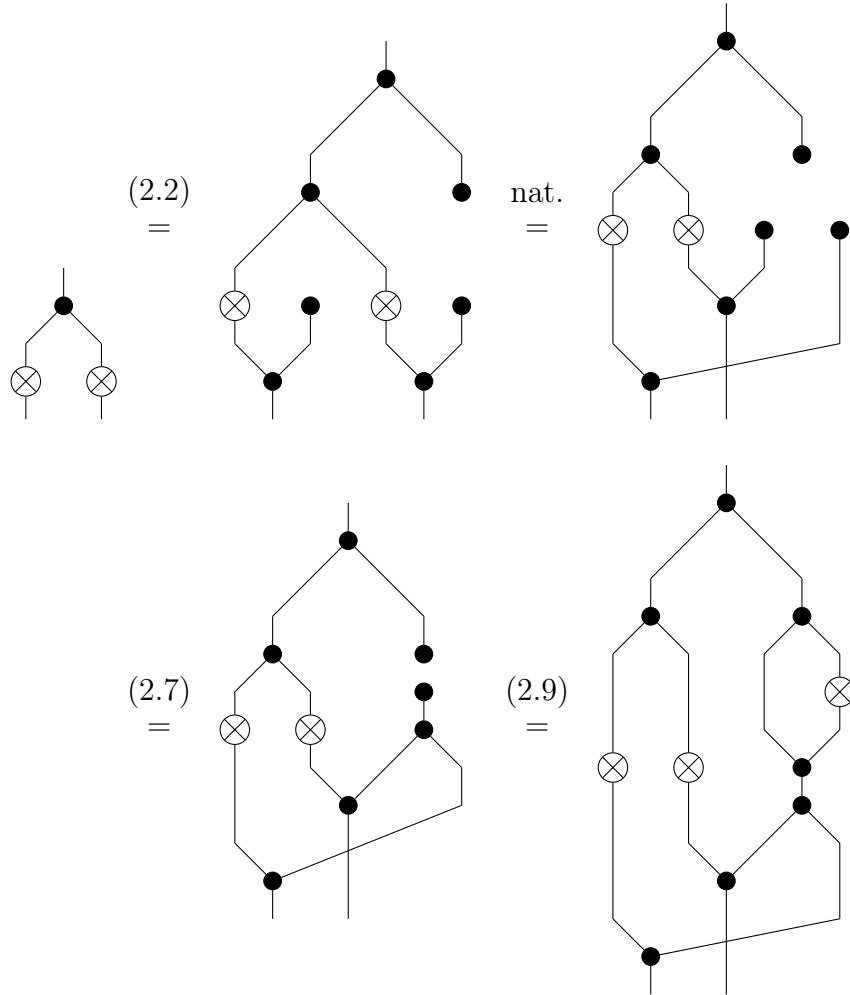
$$\begin{array}{c} \text{---} \\ | \\ \bullet \\ / \quad \backslash \\ \otimes \quad \otimes \\ | \quad | \end{array} = \begin{array}{c} \text{---} \\ | \\ \otimes \\ | \\ \bullet \\ / \quad \backslash \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \quad (2.11)$$

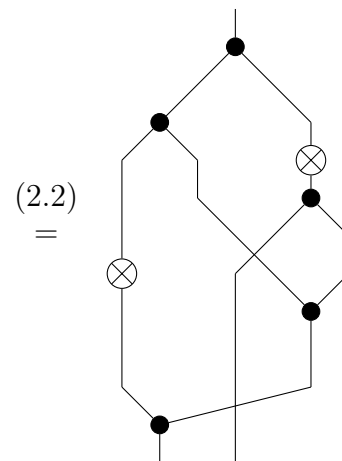
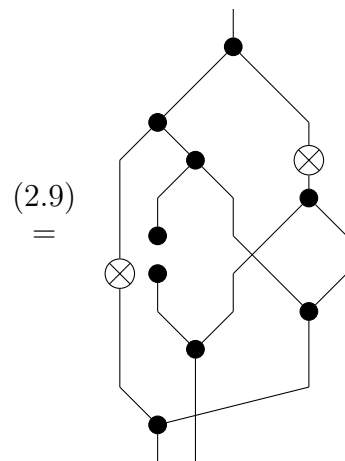
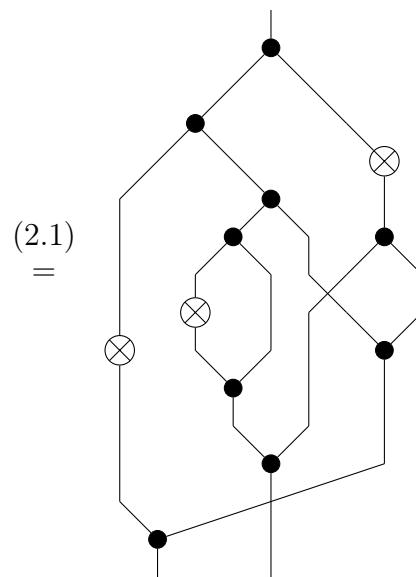
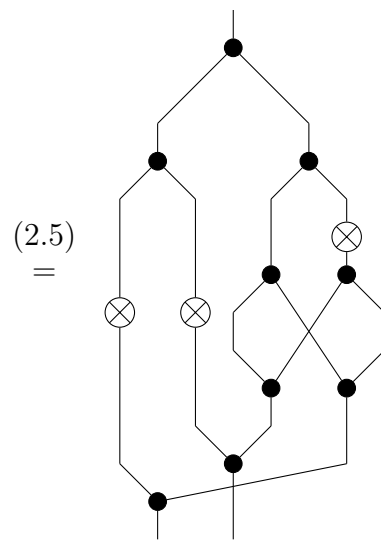
are equal string diagrams. First, we prove the unit condition with the following string diagram proof.

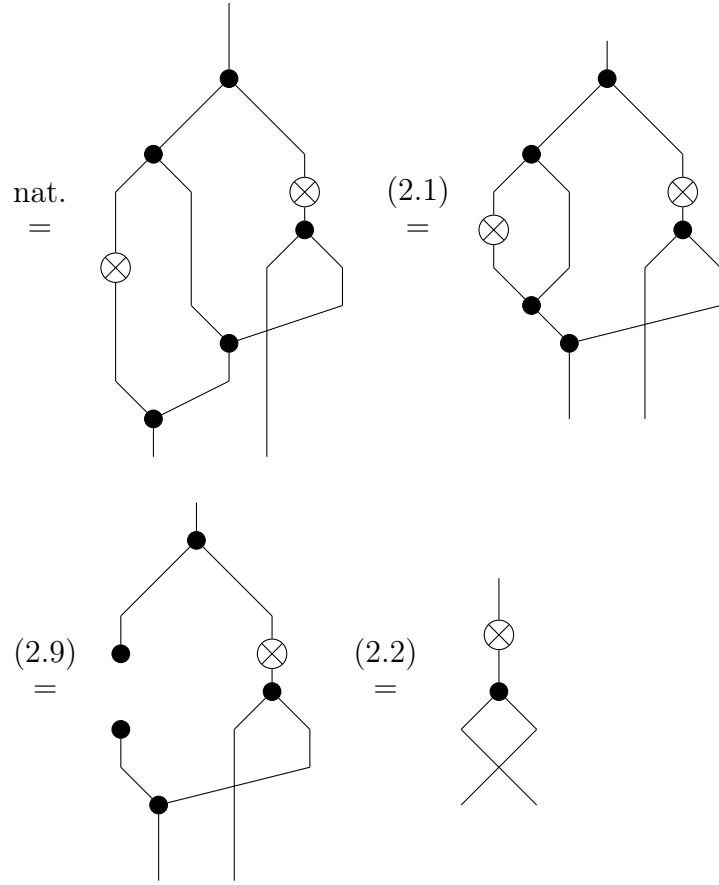
$$\begin{array}{c} \text{---} \\ | \\ \bullet \end{array} \stackrel{(2.8)}{=} \begin{array}{c} \text{---} \\ | \\ \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} \stackrel{(2.9)}{=} \begin{array}{c} \text{---} \\ | \\ \bullet \\ / \quad \backslash \\ \otimes \quad \diagup \\ | \quad | \end{array}$$

$$\stackrel{(2.6)}{=} \begin{array}{c} \text{---} \\ | \\ \bullet \\ / \quad \backslash \\ \otimes \quad | \\ | \quad \bullet \end{array} \stackrel{\text{nat.}}{=} \begin{array}{c} \text{---} \\ | \\ \bullet \\ / \quad \backslash \\ \otimes \quad \bullet \\ | \quad | \end{array} \stackrel{(2.2)}{=} \begin{array}{c} \text{---} \\ | \\ \otimes \\ | \\ \bullet \end{array}$$

Finally, the multiplication condition is proven as follows.





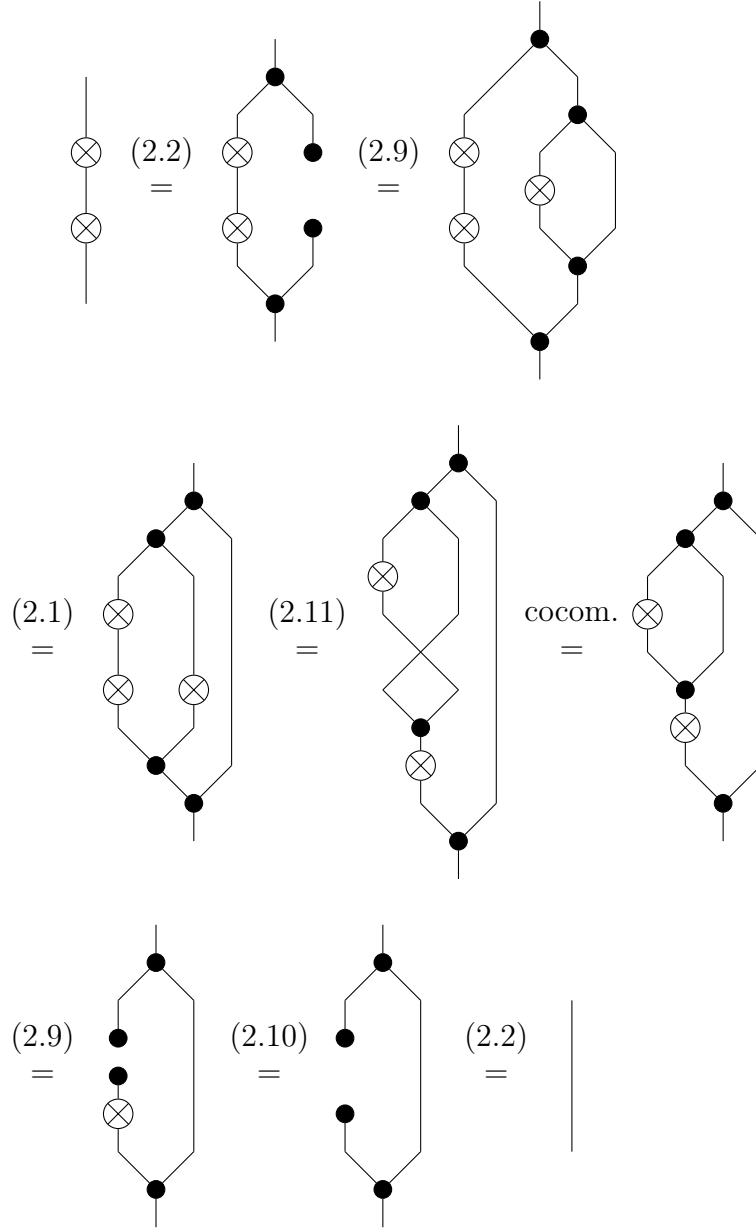


□

Knowing that the antipode is an anti-bialgebra morphism will help simplify certain proofs. The following result uses the fact that the antipode is an anti-bialgebra morphism and generalises the fact that for an element g in a group G , $(g^{-1})^{-1} = g$.

Lemma 2.2.5. *If $(H, \mu, \eta, \Delta, \epsilon, \chi)$ is a cocommutative Hopf algebra then $\chi^2 = \text{id}$.*

Proof. The proof is given by the following string diagrams.



□

By flipping the string diagrams above, we also get the following.

Corollary 2.2.6. *If $(H, \mu, \eta, \Delta, \epsilon, \chi)$ is a commutative Hopf algebra then*

$$\chi^2 = \text{id}.$$

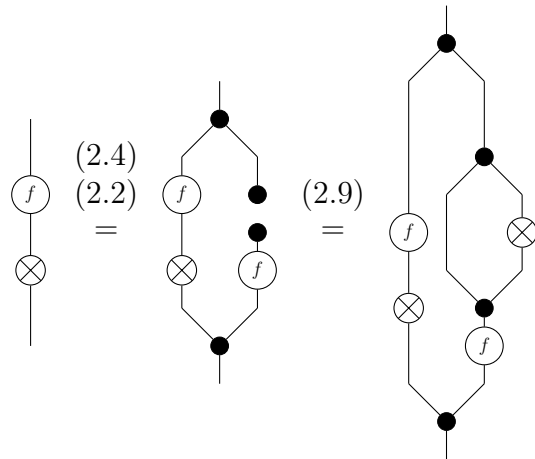
A consequence of this result is that the antipode is a Hopf algebra isomorphism to its co-opposite-opposite Hopf algebra. The co-opposite-opposite Hopf algebra structure is of interest to some texts [10] but for our purposes we will be mostly interested in cocommutative Hopf algebras where taking the co-opposite of a Hopf algebra doesn't do anything. In particular given a cocommutative Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, \chi)$ we write H^{op} to denote the Hopf algebra $(H, \mu \circ \sigma, \eta, \Delta, \epsilon, \chi)$.

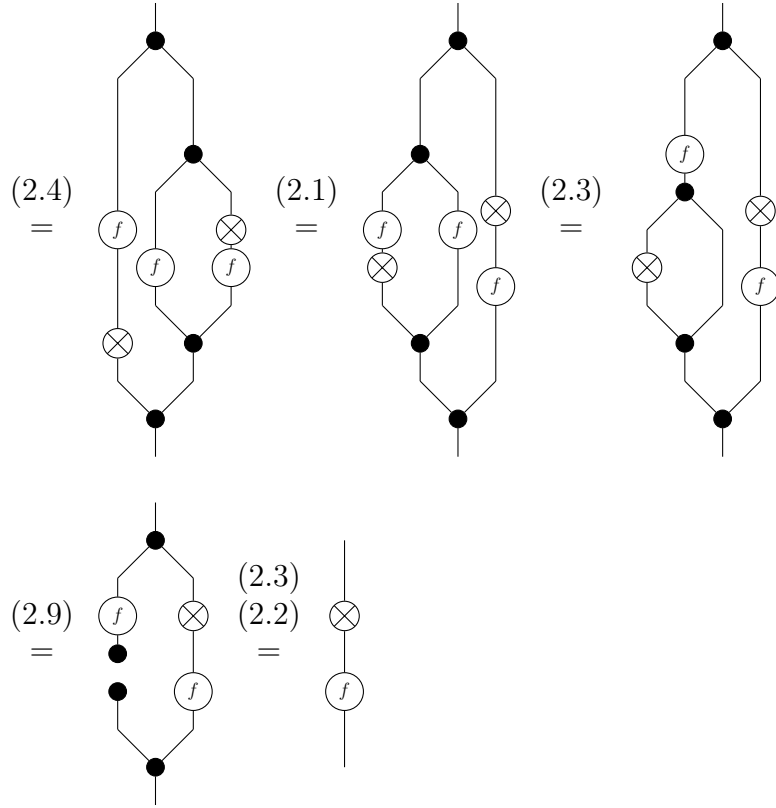
Corollary 2.2.7. *For a cocommutative Hopf algebra H , the antipode is a Hopf algebra isomorphism $\chi: H \rightarrow H^{\text{op}}$.*

One last property of the antipode which will be useful to us generalises the fact that if you have a group homomorphism $f: G \rightarrow G'$ then $f(g^{-1}) = f(g)^{-1}$.

Lemma 2.2.8. *If $f: H \rightarrow H'$ is a morphism of Hopf algebras then $f \circ \chi = \chi' \circ f$.*

Proof. We show this by the equality of the following string diagrams.





□

We shall finish this section by discussing categories of Hopf algebras and their monoidal structure. We have already seen in Lemma 2.1.6 that categories of algebras (and consequently coalgebras) have a monoidal structure. We can easily generalise this to a product of Hopf algebras $H \otimes H'$ where it is straightforward to check that the antipode is given by $\chi \otimes \chi'$. In the category $\text{Hopf}_{\text{cc}}(\mathcal{V})$ of cocommutative Hopf algebras, this product turns out to be cartesian. This follows from a more general fact that category of commutative algebras is co-cartesian and the category of cocommutative coalgebras is cartesian. We will prove the latter by first showing that certain pullbacks exist.

Lemma 2.2.9. *Let $f: G \rightarrow H$ be a morphism of cocommutative coalgebras.*

The category of cocommutative coalgebras has pullback squares as follows.

$$\begin{array}{ccc}
 G \otimes H' & \xrightarrow{f \otimes \text{id}} & H \otimes H' \\
 \text{id} \otimes \epsilon_{H'} \downarrow & \lrcorner & \downarrow \text{id} \otimes \epsilon_{H'} \\
 G & \xrightarrow{f} & H
 \end{array} \quad (2.12)$$

Proof. It is straightforward to check that the product of two cocommutative coalgebras is cocommutative. It remains to verify that (2.12) is indeed a pullback. Let D be a coalgebra with morphisms $a: D \rightarrow G$ and $b: D \rightarrow H \otimes H'$ such that

$$\begin{array}{ccc}
 D & \xrightarrow{b} & H \otimes H' \\
 a \downarrow & & \downarrow \text{id} \otimes \epsilon_{H'} \\
 G & \xrightarrow{f} & H
 \end{array} \quad (2.13)$$

commutes. I claim that

$$D \xrightarrow{\Delta_D} D \otimes D \xrightarrow{a \otimes b} G \otimes H \otimes H' \xrightarrow{\text{id} \otimes \epsilon_H \otimes \text{id}} G \otimes H' \quad (2.14)$$

is the unique morphism which factorises the following diagram

$$\begin{array}{ccccc}
 D & & & & \\
 \text{---} & \text{---} b & & & \\
 \text{---} & \text{---} \exists! & \text{---} & \text{---} & \\
 & G \otimes H' & \xrightarrow{f \otimes \text{id}} & H \otimes H' & \\
 & \text{id} \otimes \epsilon_{H'} \downarrow & \lrcorner & \downarrow \text{id} \otimes \epsilon_{H'} & \\
 & G & \xrightarrow{f} & H & .
 \end{array}$$

First we check that (2.14) does indeed make the above diagram commute.

By the counit property for b , we can see (2.14) factorises a by

and by (2.13) and the comultiplication property for b we have

which shows (2.14) factorises b . It remains to show that (2.14) is unique. Let $e: D \rightarrow G \otimes H'$ be a coalgebra morphism such that

$$\begin{array}{c} G \\ \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \bullet \\ \diagup \quad \diagdown \\ \text{---} \\ e \\ \text{---} \\ D \end{array} = \begin{array}{c} G \\ \text{---} \\ \text{---} \\ a \\ \text{---} \\ D \end{array}, \quad \begin{array}{c} H \quad H' \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ f \\ \text{---} \\ e \\ \text{---} \\ D \end{array} = \begin{array}{c} H \quad H' \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ \text{---} \\ b \\ \text{---} \\ D \end{array} \quad (2.15)$$

then we need to show that $\Delta_{G \otimes H'} \circ e$ is equal to (2.14) which can be seen by the following string diagrams.

(2.15)

(2.4)

□

Corollary 2.2.10. *For a symmetric monoidal category $\mathcal{V}$, the category of cocommutative coalgebras is cartesian monoidal.*

Proof. Firstly the trivial coalgebra is the terminal object in the category of cocommutative coalgebras. This is because if C is a coalgebra and we have a coalgebra morphism $f: C \rightarrow \mathbb{1}$ then by definition it must make the following triangle commute

$$\begin{array}{ccc} & \mathbb{1} & \\ \epsilon \nearrow & & \nwarrow \epsilon' \\ C & \xrightarrow{f} & C' \end{array}$$

and hence $f = \epsilon$. Finally we have all products arising as limits by Lemma 2.2.9 by taking $f = \epsilon_H$ to give the following pullback square.

$$\begin{array}{ccc} H \otimes H' & \xrightarrow{\epsilon_H} & H' \\ \epsilon_{H'} \downarrow & \lrcorner & \downarrow \epsilon_{H'} \\ H & \xrightarrow{\epsilon_H} & \mathbb{1} \end{array}$$

□

In particular since the pullback from Lemma 2.2.9 exists in $\text{Hopf}_{\text{cc}}(\mathcal{V})$ we have the following.

Corollary 2.2.11. *For a symmetric monoidal category $\mathcal{V}$, the category $\text{Hopf}_{\text{cc}}(\mathcal{V})$ of cocommutative Hopf algebras is cartesian monoidal.*

Chapter 3

Modules and Representations

In this chapter we will introduce the definition of a module over an algebra and discuss their relationship to representations both in the traditional sense of a group representation of a vector space but also the more general definition in terms of enriched functors. Many of the definitions and results here are known generalisations [21] of those for Hopf algebras in vector spaces for which we refer to the notes of Baker which we used in the previous chapter [1]. We will prove some results about modules and representations using string diagrams which will later be required to construct monoidal double categories later in Chapter 6.

3.1 Defining Modules and their Morphisms

Definition 3.1.1. Given an algebra A in a monoidal category $\mathcal{V}$, a **left A -module** (M, ρ, A) is an object $M \in \mathcal{V}$ and a **left action** $\rho: A \otimes M \rightarrow M$ such that the following diagrams commute.

$$\begin{array}{ccc}
 A \otimes A \otimes M & \xrightarrow{\text{id} \otimes \rho} & A \otimes M \\
 \mu \otimes \text{id} \downarrow & & \downarrow \rho \\
 A \otimes M & \xrightarrow{\rho} & M
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{1} \otimes M & \xrightarrow{\eta \otimes \text{id}} & A \otimes M \\
 & \searrow \cong & \downarrow \rho \\
 & & M
 \end{array}$$

Equivalently, in string diagrams this is saying the following where we use a blue line to represent the module and a dot to represent the module action.

$$(3.1)$$

Similarly a **right A -module** (N, ρ, A) is an object $N \in \mathcal{V}$ and a **right action** $\rho: N \otimes A \rightarrow N$ satisfying similar conditions to those above. Finally, a **A - A' -bimodule** (M, ρ, A, A') is an object M and an action $\rho: A \otimes M \otimes A' \rightarrow M$ satisfying the following conditions.

$$(3.2)$$

- Example 3.1.2.**
1. In $\mathbf{Vect}_K$ we can take modules over group algebras to obtain a group action on a vector space.
 2. In a monoidal category $\mathcal{V}$ we can take modules over any Hopf algebra in $\mathcal{V}$. We will be looking at these more later.
 3. Any algebra A is an A - A' -bimodule by the algebra multiplication.

In a closed symmetric monoidal category $\mathcal{V}$ we have already seen that the internal homs $[v, v]$ are algebras under composition. Given an algebra $A \in \mathcal{V}$ then a right A -module is given by an action $\rho: A \otimes M \rightarrow M$. Since $\mathcal{V}$ is closed we have a natural isomorphism

$$\mathcal{V}(M \otimes A, M) \cong \mathcal{V}(A, [M, M]).$$

It turns out that when we restrict ourselves to looking at only morphisms

$M \otimes A \rightarrow M$ which define a module action then the corresponding morphisms $A \rightarrow [M, M]$ are algebra morphisms in $\mathcal{V}$. We will prove this fact here with string diagrams.

Lemma 3.1.3. *Let (M, ρ, A) be a right module in a closed symmetric monoidal category $\mathcal{V}$. The morphism*

$$\hat{\rho}: A \rightarrow [M, M]$$

corresponding to ρ under the tensor hom adjunction is an algebra morphism. Similarly, any algebra morphism $\hat{\rho}: A \rightarrow [M, M]$ corresponds under the tensor hom adjunction to a right A -module (M, ρ, A) .

Proof. We need to check that given a right A -module (M, ρ, A) , the multiplication and unit conditions for a $\hat{\rho}$ to be an algebra morphism are satisfied. Conversely we need to check that given an algebra morphism $\hat{\rho}$, that (M, ρ, A) is indeed a module. First we assume that (M, ρ, A) is an A -module and prove the multiplication condition

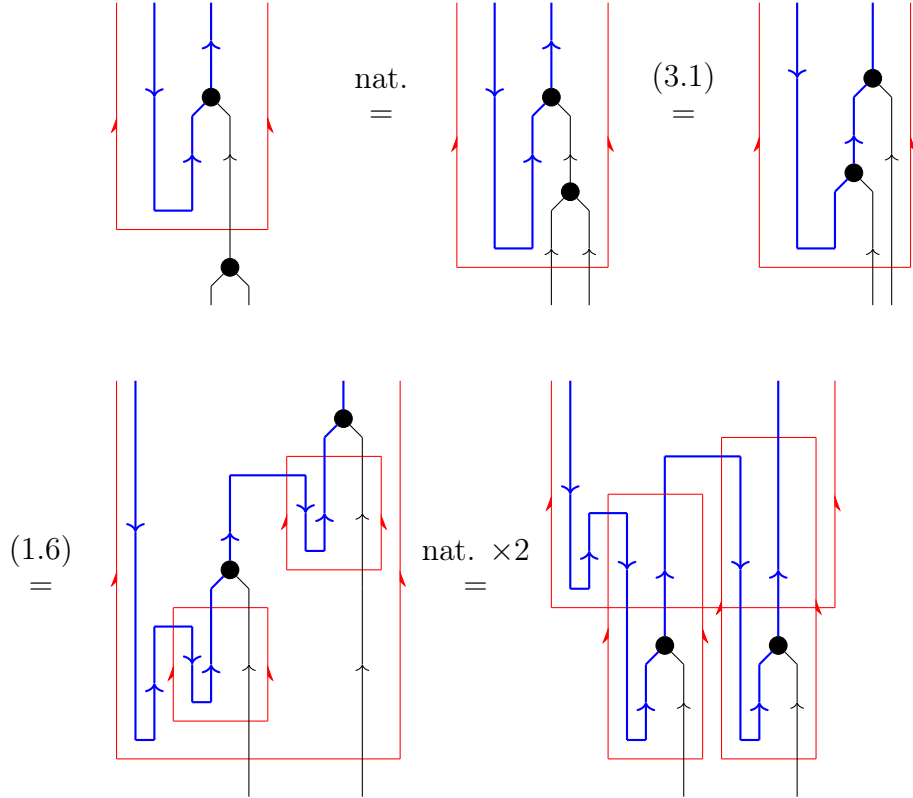
$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu} & A \\ \hat{\rho} \otimes \hat{\rho} \downarrow & & \downarrow \hat{\rho} \\ [M, M] \otimes [M, M] & \xrightarrow{\circ} & [M, M] \end{array}$$

for $\hat{\rho}$ holds. Expanding the definitions for $\hat{\rho}$, the multiplication condition

requires that

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\mu} & A \\
\text{coev} \otimes \text{coev} \downarrow & & \downarrow \text{coev} \\
[M, M \otimes A] \otimes [M, M \otimes A] & & [M, M \otimes A] \\
[\text{id}, \rho] \otimes [\text{id}, \rho] \downarrow & & \downarrow [\text{id}, \rho] \\
[M, M] \otimes [M, M] & & \\
\text{coev} \downarrow & & \\
[M, M \otimes [M, M]] \otimes [M, M] & & \\
[\text{id}, \text{id} \otimes \text{ev}] \downarrow & & \\
[M, M \otimes [M, M]] & \xrightarrow{[\text{id}, \text{ev}]} & [M, M]
\end{array}$$

commutes. We verify this fact with the following string diagram proof.



The unit condition for $\hat{\rho}$ is more simple, requiring that

$$\begin{array}{ccc} & \mathbb{1} & \\ \eta \swarrow & & \searrow \text{id}_M \\ A & \xrightarrow{\hat{\rho}} & [M, M] \end{array}$$

commutes which upon expanding our definitions requires us to prove that

$$\begin{array}{ccc} A & \xleftarrow{\eta} & \mathbb{1} \\ \text{coev} \downarrow & & \downarrow \text{coev} \\ [M, M \otimes A] & \xrightarrow{[\text{id}, \rho]} & [M, M] \end{array}$$

commutes which we show by the following string diagram proof.

$$\begin{array}{c} \text{nat.} \\ = \end{array} \begin{array}{c} \text{(3.1)} \\ = \end{array}$$

Now instead assume that $\hat{\rho}: A \rightarrow [M, M]$ is an algebra morphism we will show that (M, ρ, A) is a module. This means we need to show the module conditions

$$\begin{array}{ccc} M \otimes A \otimes A & \xrightarrow{\rho \otimes \text{id}} & M \otimes A \\ \text{id} \otimes \mu \downarrow & & \downarrow \rho \\ M \otimes A & \xrightarrow{\rho} & M \end{array} \quad \begin{array}{ccc} M \otimes \mathbb{1} & \xrightarrow{\text{id} \otimes \eta} & M \otimes A \\ & \searrow \cong & \downarrow \rho \\ & & M \end{array}$$

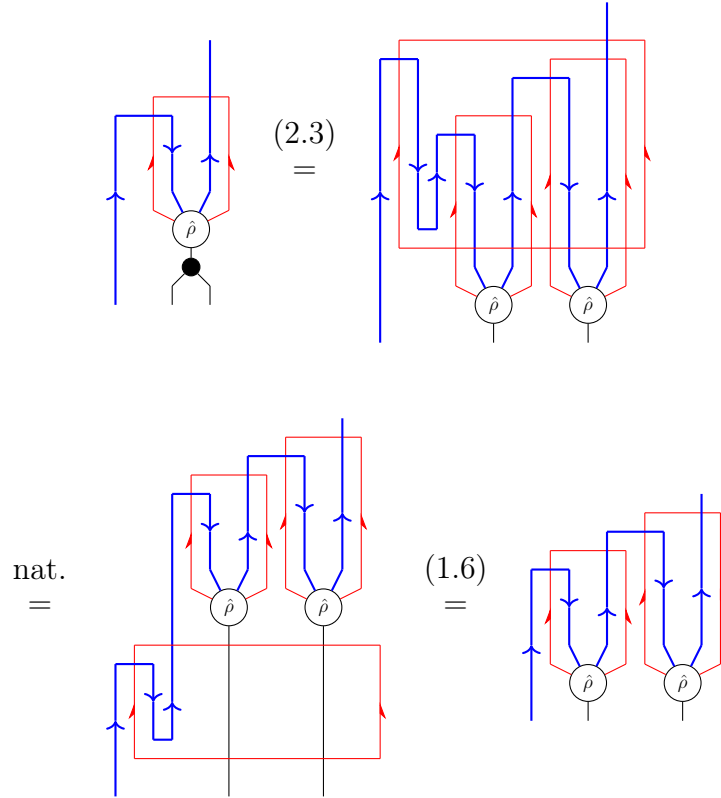
are satisfied. Recall that ρ is defined in terms of $\hat{\rho}$ as

$$M \otimes A \xrightarrow{\text{id} \otimes \hat{\rho}} M \otimes [M, M] \xrightarrow{\text{ev}} M.$$

The multiplication condition is therefore requiring we prove that

$$\begin{array}{ccccc}
 M \otimes A \otimes A & \xrightarrow{\text{id} \otimes \hat{\rho} \otimes \text{id}} & M \otimes [M, M] \otimes A & \xrightarrow{\text{ev}} & M \otimes A \\
 \downarrow \text{id} \otimes \mu & & & & \downarrow \text{id} \otimes \hat{\rho} \\
 & & & & M \otimes [M, M] \\
 & & & & \downarrow \text{ev} \\
 M \otimes A & \xrightarrow{\text{id} \otimes \hat{\rho}} & M \otimes [M, M] & \xrightarrow{\text{ev}} & M
 \end{array}$$

commutes using only the fact that $\hat{\rho}$ is an algebra map which we show with string diagrams as follows.



Finally the unit condition translates to requiring that

$$\begin{array}{ccc}
 M \otimes \mathbb{1} & \xrightarrow{\text{id} \otimes \eta} & M \otimes A \\
 & \searrow \cong & \downarrow \text{id} \otimes \hat{\rho} \\
 & & M \otimes [M, M] \\
 & & \downarrow \text{ev} \\
 & & M
 \end{array}$$

commutes which we again show with a string diagram proof.

□

Now we are equipped with some examples of modules, the obvious thing to do from a category theory perspective is to define morphisms of modules.

Definition 3.1.4. Let (M, ρ, A) and (M', ρ', A') be left modules. A **left module morphism** is a pair (f, θ) consisting of a morphism $\theta: M \rightarrow M'$ and an algebra morphism $f: H \rightarrow H'$ such that the following diagram commutes.

$$\begin{array}{ccc}
 A \otimes M & \xrightarrow{f \otimes \theta} & A' \otimes M' \\
 \rho \downarrow & & \downarrow \rho' \\
 M & \xrightarrow{\theta} & M'
 \end{array}$$

We write this in string diagrams as the following.

(3.3)

A **right module morphism** is a pair (θ, f) satisfying a similar commuting diagram to that above and similarly one can define **bimodule morphisms** which are a triple (f, θ, f') again satisfying a similar condition. We write the algebra morphism on the right to signify this is a right module morphism. In either case I will simply use θ to denote a module morphism when the algebra morphism is clear from context.

We denote the category of left modules of cocommutative Hopf algebras and left module morphisms in $\mathcal{V}$ as $\mathbf{Mod}_{\mathcal{V}}$. It is normal to also consider categories of modules over a fixed algebra as these categories have important properties which we will see later. We define such categories as follows.

Definition 3.1.5. Let A be an algebra. We write $A\text{-}\mathbf{Mod}$ to denote the category of left A -modules and module morphisms of the form (id, θ) . Similarly we write $\mathbf{Mod}\text{-}A$ to denote the category of right A -modules and module morphisms of the form (θ, id) and $A\text{-}\mathbf{Mod}\text{-}A'$ to denote the category of A - A' -bimodules and module morphisms $(\text{id}, \theta, \text{id})$.

When we take modules in a symmetric monoidal category we can define opposite algebras. We can see that any left module corresponds to a right module over the opposite algebra and vice versa.

Lemma 3.1.6. *If $\mathcal{V}$ is a symmetric monoidal category and A is an algebra in $\mathcal{V}$ then we have an equivalence of categories*

$$\mathbf{Mod}\text{-}A \cong A^{\text{op}}\text{-}\mathbf{Mod}.$$

Proof. We define a functor $F: \mathbf{Mod}\text{-}A \rightarrow A^{\text{op}}\text{-}\mathbf{Mod}$ which sends a right A -module (M, ρ, A) to a left A^{op} -module M with action given by

$$A \otimes M \xrightarrow{\sigma} M \otimes A \xrightarrow{\rho} M$$

which we will denote ρ^{op} . To check this is indeed a left A^{op} -module we first

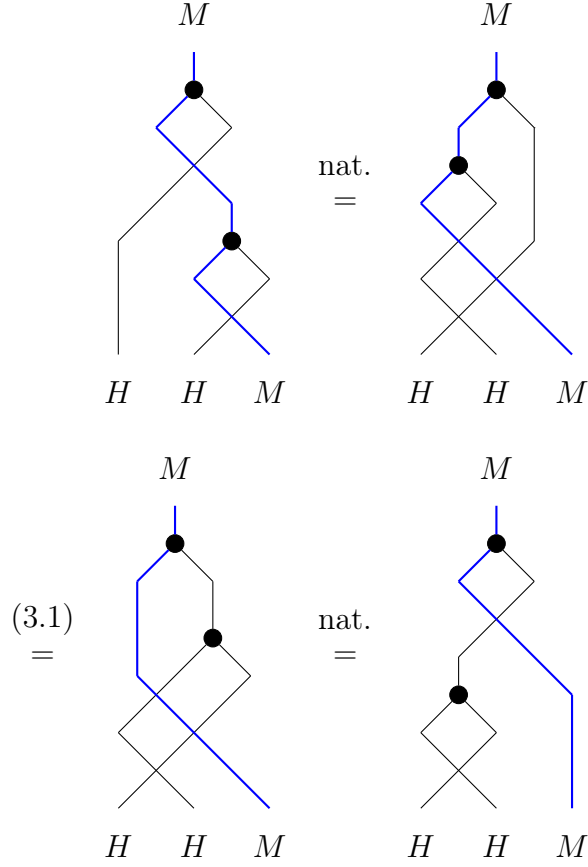
check the multiplication and action commute which is to say

$$\begin{array}{ccc}
 A \otimes A \otimes M & \xrightarrow{\rho^{\text{op}} \otimes \text{id}} & A \otimes M \\
 (\mu \circ \sigma) \otimes \text{id} \downarrow & & \downarrow \rho^{\text{op}} \\
 A \otimes M & \xrightarrow{\rho^{\text{op}}} & M
 \end{array}$$

commutes. Expanding our definitions this means the square

$$\begin{array}{ccccc}
 A \otimes A \otimes M & \xrightarrow{\text{id} \otimes \sigma} & A \otimes M \otimes A & \xrightarrow{\rho \otimes \text{id}} & A \otimes M \\
 \sigma \otimes \text{id} \downarrow & & & & \downarrow \sigma \\
 A \otimes A \otimes M & & & & M \otimes A \\
 \mu \otimes \text{id} \downarrow & & & & \downarrow \rho \\
 A \otimes M & \xrightarrow{\sigma} & M \otimes A & \xrightarrow{\rho} & M
 \end{array}$$

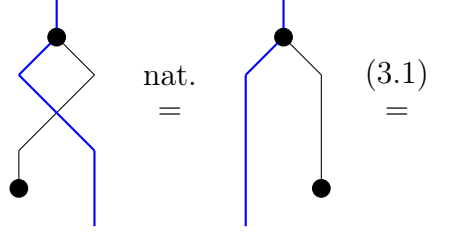
commutes which can be seen using the following string diagrams.



The other condition we need to check for M to be a well defined left A^{op} -module is the unit condition described by the commutativity of following diagram.

$$\begin{array}{ccc}
 \mathbb{1} \otimes M & \xrightarrow{\eta \otimes \text{id}} & A \otimes M \\
 & \searrow \cong & \downarrow \rho^{\text{op}} \\
 & & M
 \end{array}$$

This commutes by the following string diagram proof.



Conversely, F has an inverse which sends a left A^{op} -module (M, ρ, A^{op}) to a right A -module M with action

$$M \otimes A \xrightarrow{\sigma} A \otimes M \xrightarrow{\rho} M$$

which satisfies the definition of a right module by a proof similar to that above. These assignments of modules are inverse since $\sigma^2 = \text{id}$.

Finally, F sends a left A^{op} -module morphism (id, θ) to the right A -module morphism (θ, id) and with the inverse of F doing the opposite. Clearly these assignments of morphisms are inverse and functoriality follows immediately. It remains to show that (id, θ) is a well defined left A^{op} -module morphism given that (θ, id) is a valid right A -module morphism with the converse following similarly. We require the following diagram commutes

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\text{id} \otimes \theta} & A \otimes M' \\ \rho^{\text{op}} \downarrow & & \downarrow \rho'^{\text{op}} \\ M & \xrightarrow{\theta} & M' \end{array}$$

which is easy to see is true by naturality of σ by expanding our definitions as follows.

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\text{id} \otimes \theta} & A \otimes M' \\ \sigma \downarrow & & \sigma \downarrow \\ M \otimes A & \xrightarrow{\theta \otimes \text{id}} & M' \otimes A \\ \rho \downarrow & & \downarrow \rho' \\ M & \xrightarrow{\theta} & M' \end{array}$$

□

If instead of an algebra, we take a Hopf algebra H then we need cocommutativity to define the opposite H^{op} in which case the above result still holds.

Corollary 3.1.7. *If a Hopf algebra H is cocommutative then $\mathbf{Mod}\text{-}H \cong H^{\text{op}}\text{-}\mathbf{Mod}$.*

We can also use symmetry to show a category of bimodules is isomorphic to some category of right modules.

Lemma 3.1.8. *If $\mathcal{V}$ is a symmetric monoidal category and A and A' are algebras in $\mathcal{V}$ then we have an equivalence of categories*

$$A\text{-}\mathbf{Mod}\text{-}A' \cong \mathbf{Mod}\text{-}A^{\text{op}} \otimes A'.$$

Proof. We define a functor $F: A\text{-}\mathbf{Mod}\text{-}A' \rightarrow \mathbf{Mod}\text{-}A^{\text{op}} \otimes A'$ which sends a A - A' -bimodule M with action ρ we define to the right $A^{\text{op}} \otimes A'$ -module M with action

$$M \otimes A \otimes A' \xrightarrow{\sigma \otimes \text{id}} A \otimes M \otimes A' \xrightarrow{\rho} M.$$

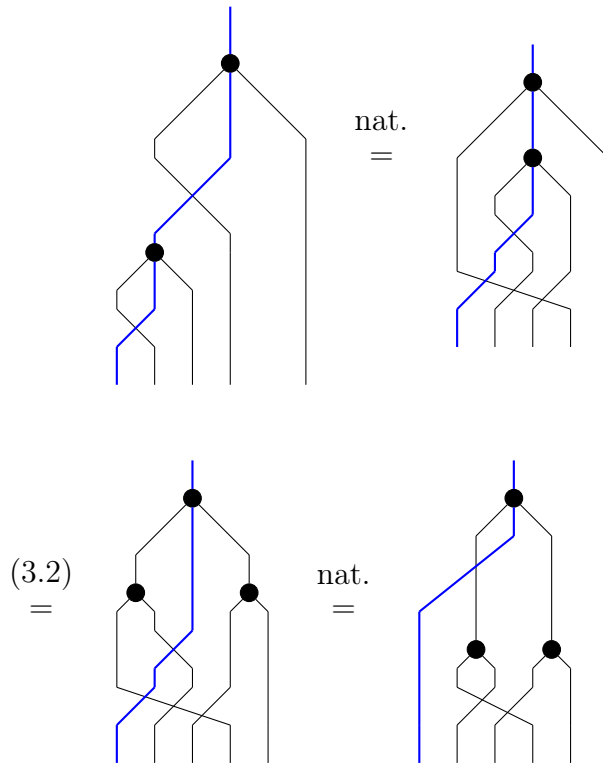
For this to be well defined we need the following diagrams to commute

$$\begin{array}{ccccc}
M \otimes A \otimes A' \otimes A \otimes A' & \xrightarrow{\sigma \otimes \text{id}} & A \otimes M \otimes A' \otimes A \otimes A' & \xrightarrow{\rho \otimes \text{id}} & M \otimes A \otimes A' \\
\text{id} \otimes \text{id} \otimes \sigma \otimes \text{id} \downarrow & & & & \downarrow \sigma \otimes \text{id} \\
M \otimes A \otimes A \otimes A' \otimes A' & & & & A \otimes M \otimes A' \\
\text{id} \otimes \sigma \otimes \text{id} \otimes \text{id} \downarrow & & & & \downarrow \rho \\
M \otimes A \otimes A \otimes A' \otimes A' & & & & \\
\text{id} \otimes \mu \otimes \mu' \downarrow & & & & \\
M \otimes A \otimes A' & \xrightarrow{\sigma \otimes \text{id}} & A \otimes M \otimes A' & \xrightarrow{\rho} & M
\end{array}$$

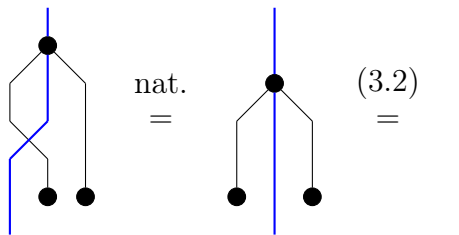
$$\begin{array}{ccccc}
M \otimes 1 & \xrightarrow{\cong} & M \otimes 1 \otimes 1 & \xrightarrow{\text{id} \otimes \eta \otimes \eta'} & M \otimes A \otimes A' \\
& & & & \downarrow \sigma \otimes \text{id} \\
& & & & A \otimes M \otimes A' \\
& \searrow \cong & & & \downarrow \rho \\
& & & & M
\end{array}$$

which we check by the following string diagram proofs:

1. Multiplication condition



2. Unit Condition



Conversely, the inverse to F sends a right $A^{\text{op}} \otimes A'$ -module M to a A - A' -bimodule by symmetry with a proof similar to that above. Finally F sends a bimodule morphism $(\text{id}, \theta, \text{id})$ to the right $A^{\text{op}} \otimes A'$ -module morphism (θ, id) and vice versa. $\square$

By Lemma 3.1.6 we also have the following.

Corollary 3.1.9. *If $\mathcal{V}$ is a symmetric monoidal category and A and A' are algebras in $\mathcal{V}$ then we have an equivalence of categories*

$$A\text{-}\mathbf{Mod}\text{-}A' \cong A \otimes A'^{\text{op}}\text{-}\mathbf{Mod}.$$

If instead we take Hopf algebras H and H' then we need cocommutativity to define the opposite H^{op} in which case the above results still hold.

Corollary 3.1.10. *Let H and H' be cocommutative Hopf algebras then*

$$H\text{-}\mathbf{Mod}\text{-}H' \cong \mathbf{Mod}\text{-}H^{\text{op}} \otimes H' \cong H \otimes H'^{\text{op}}\text{-}\mathbf{Mod}.$$

3.2 Relationship with Representations

A traditional representation theory course or textbook [28] would likely define a group representation as follows.

Definition 3.2.1. Let G be a group. A G -representation is a vector space V and a group homomorphism $\hat{\rho}: G \rightarrow \text{GL}(V)$. We denote this as the triple $(V, \hat{\rho}, G)$.

We have seen that given a group G we can consider modules in $\mathbf{Vect}_K$ over the group algebra KG . If V is a right KG -module then we can define a G -representation $\hat{\rho}: G \rightarrow \text{GL}(V)$ by $\hat{\rho}(g) = (-) \cdot g$. In fact it is well known these structures are equivalent to the extent that often the terms “module” and “representation” are used interchangeably [22]. The reason we have this correspondence is because $\mathbf{Vect}_K$ is a closed monoidal category and hence we have a natural isomorphism

$$\mathbf{Vect}_K(V \otimes KG, V) \cong \mathbf{Vect}_K(KG, [V, V]).$$

In this section we will generalise the idea of group representations to representations of algebras in an arbitrary closed symmetric monoidal category $\mathcal{V}$ and discuss their relationship to modules. We define a representation of an algebra in a monoidal category as follows.

Definition 3.2.2. Let $\mathcal{V}$ be a closed monoidal category and let A be an algebra in $\mathcal{V}$ and BA be the corresponding algebra in $\mathcal{V}\text{-Cat}$. An A -**representation** is a $\mathcal{V}$ -functor $BA \rightarrow \underline{\mathcal{V}}$.

It is easiest to see what this definition is saying by seeing how this generalises Definition 3.2.1.

Example 3.2.3. Let G be a group and $\hat{\rho}: G \rightarrow \text{GL}(V)$ be a G -representation in the sense of Definition 3.2.1. We define a $\mathbf{Vect}_K$ -functor $\hat{V}: BKG \rightarrow \mathbf{Vect}_K$ by sending the one object $* \in BKG$ to $V \in \mathbf{Vect}_K$ and by defining the map of hom objects $BKG(*, *) = KG$ to $\mathbf{Vect}_K(V, V)$ by $g \mapsto \hat{\rho}(g)$. This constructs a KG -representation in the sense of Definition 3.2.2.

In fact these definitions are equivalent since if $\hat{V}$ is a KG -representation then the map of hom objects $KG \rightarrow \mathbf{Vect}_K(V, V)$ has to be an algebra morphism by the definition of composition in KG . In particular, the image of the group G has to be a group. Since the invertible elements of $\mathbf{Vect}_K(V, V)$ are exactly those in $\text{GL}(V)$ then an algebra morphism $KG \rightarrow \mathbf{Vect}_K(V, V)$ is the same data as a group homomorphism $G \rightarrow \text{GL}(V)$.

By extension it would seem that the correct notion of morphism of A -representations “should” be a $\mathcal{V}$ -natural transformation. When we translate what a $\mathcal{V}$ -natural transformation means to the language of group representations we arrive at the following notion.

Definition 3.2.4. Let $(V, \hat{\rho}, G)$ and $(W, \hat{\tau}, H)$ be group representations. An **intertwining map** $(\hat{\theta}, f)$ consists of a linear map $\hat{\theta}: V \rightarrow W$ and a group homomorphism $f: G \rightarrow H$ such that the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} V & \xrightarrow{\hat{\theta}} & W \\ \hat{\rho}(g) \downarrow & & \downarrow \hat{\tau}(f(g)) \\ V & \xrightarrow{\hat{\theta}} & W \end{array}$$

The reason we introduce the definition of a representation is precisely because of the relationship they have with modules, allowing us to prove properties of modules more easily. We make explicit the relationship with the following result.

Theorem 3.2.5. *Given a closed symmetric monoidal category $\mathcal{V}$, an algebra A and its corresponding algebra BA in $\mathcal{V}\text{-Cat}$. The category $\mathbf{Mod}\text{-}A$ of right A -modules and right module morphisms (id, θ) in $\mathcal{V}$ is equivalent to the category of $\mathcal{V}$ -functors $[BA, \underline{\mathcal{V}}]$ and $\mathcal{V}$ -natural transformations.*

Proof. We define a functor

$$(\hat{-}): \mathbf{Mod}\text{-}A \rightarrow [BA, \underline{\mathcal{V}}]$$

as follows.

1. A module (M, ρ, A) is sent to the functor $\hat{M}$ such that:

- (a) The single object $* \in BA$ is sent to $M \in \underline{\mathcal{V}}$
- (b) The morphism on the single hom object

$$\hat{M}: BA(*, *) \rightarrow \underline{\mathcal{V}}(\hat{M}(*), \hat{M}(*))$$

is given by the algebra map $\hat{\rho}: A \rightarrow [M, M]$ defined in Lemma 3.1.3.

2. A module morphism $\theta: M \rightarrow M'$ is sent to the natural transformation $\hat{M} \Rightarrow \hat{M}'$ with one component $\hat{\theta}: \mathbb{1} \rightarrow \underline{\mathcal{V}}(M, M')$ provided by the tensor hom adjunction

$$\mathcal{V}(M \otimes \mathbb{1}, M') \cong \mathcal{V}(\mathbb{1}, [M, M']).$$

Since the data provided above is given by the tensor hom isomorphism then we can work backwards to provide a functor which is inverse to $(\hat{-})$. It remains to show the assignments in both directions are well defined. Since composition and the identities for $BA(*, *)$ and $[M, M]$ are given by the

algebra structures on each object then the conditions for showing $\hat{M}$ is a $\mathcal{V}$ -functor given that M is a module are the same as showing $\hat{\rho}$ is an algebra morphism given that ρ is a module action and vice versa. Hence the assignments $M \leftrightarrow \hat{M}$ are well defined by Lemma 3.1.3.

It remains to show that given a module morphism $\theta: M \rightarrow M'$ that $\hat{\theta}$ is a well defined $\mathcal{V}$ -natural transformation and vice versa. First assume that θ is a module morphism then we need to show that

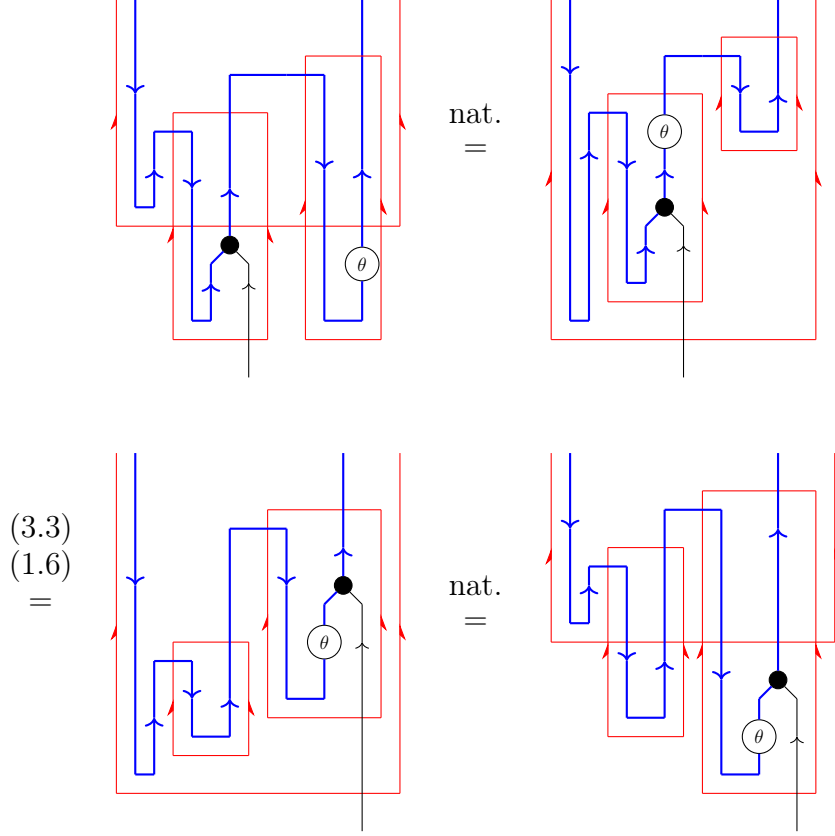
$$\begin{array}{ccccc}
 & & BA(*, *) & & \\
 & \swarrow \cong & & \searrow \cong & \\
 BA(*, *) \otimes \mathbb{1} & & & & \mathbb{1} \otimes BA(*, *) \\
 \downarrow \hat{\rho} \otimes \hat{\theta} & & & & \downarrow \hat{\theta} \otimes \hat{\rho}' \\
 \underline{\mathcal{V}}(M, M) \otimes \underline{\mathcal{V}}(M, M') & & & & \underline{\mathcal{V}}(M, M') \otimes \underline{\mathcal{V}}(M', M') \\
 \searrow \circ_M & & \underline{\mathcal{V}}(M, M') & \swarrow \circ_{M'} & \\
 & & & &
 \end{array}$$

commutes. Expanding out our definitions this means we need to show the

following diagram commutes.

$$\begin{array}{ccccc}
& & A & & \\
& \swarrow \cong & & \searrow \cong & \\
A \otimes \mathbb{1} & & & & \mathbb{1} \otimes A \\
\downarrow \text{coev} \otimes \text{coev} & & & & \downarrow \text{coev} \otimes \text{coev} \\
[M, A \otimes M] \otimes [M, \mathbb{1} \otimes M] & & & & [M, \mathbb{1} \otimes M] \otimes [M', A \otimes M'] \\
\downarrow \cong & & & & \downarrow \cong \\
[M, A \otimes M] \otimes [M, M] & & & & [M, M] \otimes [M', A \otimes M'] \\
\downarrow [\text{id}, \rho] \otimes [\text{id}, \theta] & & & & \downarrow [\text{id}, \theta] \otimes [\text{id}, \rho'] \\
[M, M] \otimes [M, M'] & & & & [M, M'] \otimes [M', M'] \\
\downarrow \text{coev} & & & & \downarrow \text{coev} \\
[M, M \otimes [M, M] \otimes [M, M']] & & & & [M, M \otimes [M, M'] \otimes [M', M']] \\
\downarrow [\text{id}, \text{ev} \otimes \text{id}] & & & & \downarrow [\text{id}, \text{ev} \otimes \text{id}] \\
[M, M \otimes [M, M']] & & & & [M, M' \otimes [M', M']] \\
& \searrow [\text{id}, \text{ev}] & [M, M'] & \swarrow [\text{id}, \text{ev}] &
\end{array}$$

We prove this diagram commutes with the following string diagrams.



Conversely, if we assume $\hat{\theta}$ is a natural transformation then we need to prove that θ is a module morphism which is to say that

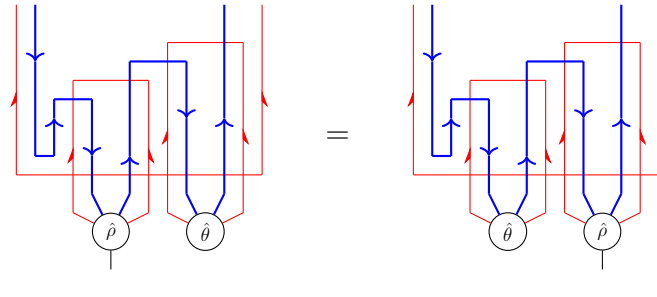
$$\begin{array}{ccc}
 M \otimes A & \xrightarrow{\theta \otimes \text{id}} & M' \otimes A \\
 \rho \downarrow & & \downarrow \rho' \\
 M & \xrightarrow{\theta} & M'
 \end{array}$$

commutes. Again expanding the definitions for θ and ρ in terms of $\hat{\theta}$ and $\hat{\rho}$

this requires that the following diagram commutes.

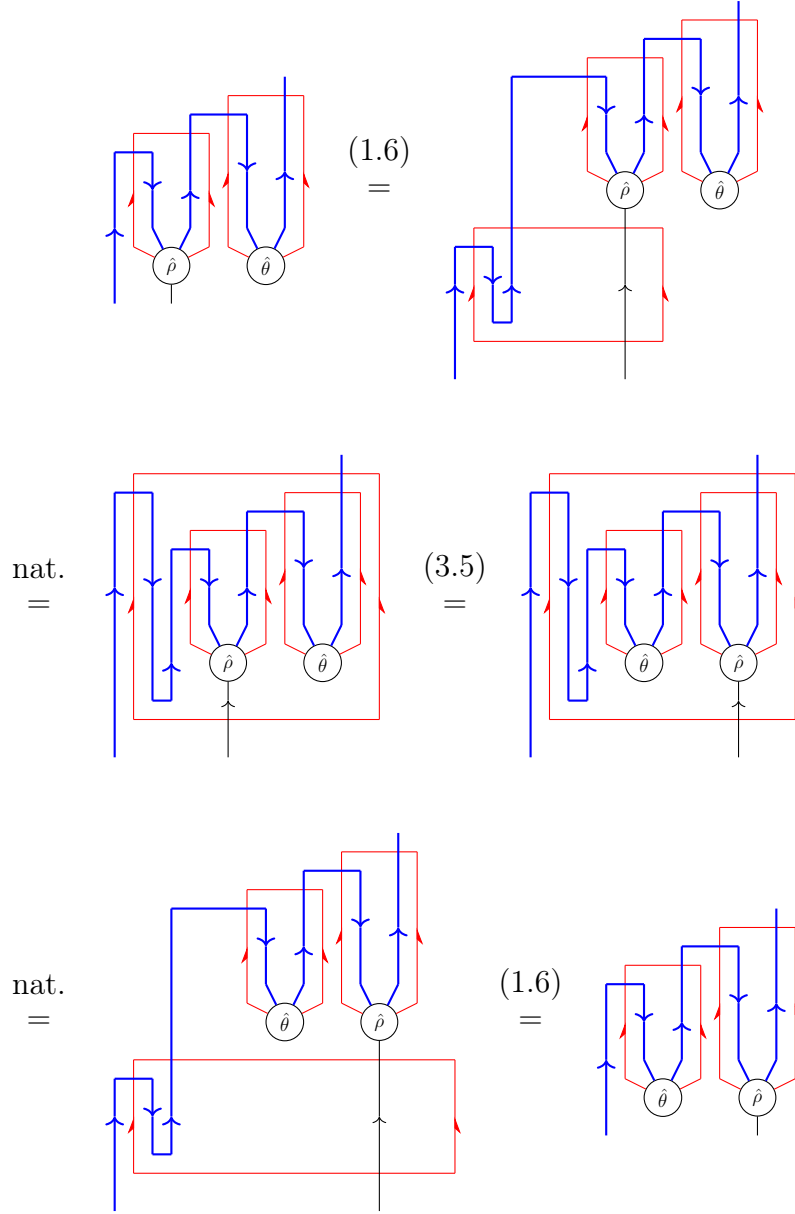
$$\begin{array}{ccccc}
M \otimes A & \xrightarrow{\hat{\theta}} & M \otimes [M, M'] \otimes A & \xrightarrow{\text{ev}} & M' \otimes A \\
\hat{\rho} \downarrow & & & & \downarrow \hat{\rho}' \\
M \otimes [M, M] & & & & M' \otimes [M', M'] \\
\text{ev} \downarrow & & & & \downarrow \text{ev} \\
M & \xrightarrow{\hat{\theta}} & M \otimes [M, M'] & \xrightarrow{\text{ev}} & M'
\end{array} \tag{3.4}$$

The condition which says that $\hat{\theta}$ is a natural transformation is written in string diagrams as



$$\tag{3.5}$$

which we use to show that (3.4) commutes with the following string diagrams.



□

In particular this result holds for any Hopf algebra H .

Corollary 3.2.6. *For any Hopf algebra H , let $U(H)$ be the underlying algebra of H then we have $\mathbf{Mod}\text{-}U(H) \cong [BU(H), \underline{\mathcal{V}}]$.*

Furthermore, the relationship between left and right modules (Lemma 3.1.6) gives us the following result.

Corollary 3.2.7. *For any algebra A in a symmetric closed monoidal category $\mathcal{V}$, the category $A\text{-}\mathbf{Mod}$ of left A -modules and left module morphisms (id, θ) is equivalent to the category $[BA^{\text{op}}, \underline{\mathcal{V}}]$ of $\mathcal{V}$ -functors and $\mathcal{V}$ -natural transformations.*

Proof. Clearly $B(A^{\text{op}}) \cong (BA)^{\text{op}}$. Hence, by Lemma 3.1.6 we have $A\text{-}\mathbf{Mod} \cong \mathbf{Mod}\text{-}A^{\text{op}}$ and by Lemma 3.2.5 $\mathbf{Mod}\text{-}A^{\text{op}} \cong [BA^{\text{op}}, \underline{\mathcal{V}}]$. $\square$

Corollary 3.2.8. *For any algebras A and A' in a symmetric closed monoidal category $\mathcal{V}$, the category $A\text{-}\mathbf{Mod}\text{-}A'$ of $A\text{-}A'$ -bimodules is equivalent to the category of $\mathcal{V}$ -functors $[BA^{\text{op}} \otimes BA', \underline{\mathcal{V}}]$.*

Proof. Since $B(A^{\text{op}} \otimes A') \cong BA^{\text{op}} \otimes BA'$ we have

$$A\text{-}\mathbf{Mod}\text{-}A' \cong \mathbf{Mod}\text{-}A^{\text{op}} \otimes A' \cong [BA^{\text{op}} \otimes BA', \underline{\mathcal{V}}]$$

by Lemma 3.1.8 and Lemma 3.2.5 respectively. $\square$

In particular, if H is a cocommutative Hopf algebra then we can define H^{op} and as a result we have $H\text{-}\mathbf{Mod} \cong [BH^{\text{op}}, \underline{\mathcal{V}}]$. Given another Hopf algebra H' we have $H\text{-}\mathbf{Mod}\text{-}H' \cong [BH^{\text{op}} \otimes BH', \underline{\mathcal{V}}]$.

3.3 Tensor Products of Modules

We will finish this chapter by introducing two different ways to take tensor products on modules. We mentioned one of these products in Chapter 1 when discussing monoidal categories, the category $A\text{-}\mathbf{Mod}\text{-}A$ of bimodules with product $\otimes_A$ seen to be “modding out” the A action between two bimodules. This product does not work however for arbitrary modules over different algebras so we arrive at the following definition taking inspiration from Lemma 2.1.6.

Lemma 3.3.1. *The category $\mathbf{Mod}_{\mathcal{V}}$ of left modules over cocommutative algebras is a monoidal category with the product given by the product in $\mathcal{V}$ and $\mathrm{Hopf}_{\mathrm{cc}}(\mathcal{V})$ with the module action defined by first taking symmetry on the middle terms as follows.*

$$(M, \rho, H) \otimes (M', \rho', H') := (M \otimes M', (\rho \otimes \rho') \circ (\mathrm{id} \otimes \sigma \otimes \mathrm{id}), H \otimes H')$$

The product on morphisms is given piecewise by the product in $\mathcal{V}$

$$(f, \theta) \otimes (f', \theta') := (f \otimes f', \theta \otimes \theta').$$

The monoidal unit is $(\mathbb{1}, \ell, \mathbb{1})$ with multiplication given by the left (or equivalently right) unitor of the monoidal product in $\mathcal{V}$. The associator and unitors are those from $\mathcal{V}$.

In fact more generally you can take the product of any two modules over any algebra but we will mostly be interested in the category $\mathbf{Mod}_{\mathcal{V}}$ for the purpose of this thesis.

Given that Theorem 3.2.5 tells us that we can describe modules as functors, it is natural to ask if this translates to a product of functors. Taking right modules (M, ρ, A) and (N, τ, A') and considering their corresponding functors $\hat{M}$ and $\hat{N}$, the product $\otimes_{\mathcal{V}\text{-}\mathbf{Cat}}$ in $\mathcal{V}\text{-}\mathbf{Cat}$ gives us a functor

$$\hat{M} \otimes_{\mathcal{V}\text{-}\mathbf{Cat}} \hat{N}: BA \otimes BA' \rightarrow \underline{\mathcal{V}} \otimes \underline{\mathcal{V}}$$

which is not in the correct form for a functor corresponding to a module. We can fix this however by noticing as before that $BA \otimes BA' \cong B(A \otimes A')$ and post-composing our functor above with the product $\underline{\mathcal{V}} \otimes \underline{\mathcal{V}} \rightarrow \underline{\mathcal{V}}$ defined in Lemma 1.2.8.

Definition 3.3.2. Let $\hat{M}$ be an A -representation and $\hat{N}$ be an A' -representation in a symmetric closed monoidal category $\mathcal{V}$. The product of $\hat{M}$ and $\hat{N}$ is given by

$$\hat{M} \otimes \hat{N} := B(A \otimes A') \xrightarrow{\hat{M} \otimes_{\mathcal{V}\text{-}\mathbf{Cat}} \hat{N}} \underline{\mathcal{V}} \otimes \underline{\mathcal{V}} \xrightarrow{- \otimes -} \underline{\mathcal{V}}. \quad (3.6)$$

It remains to check if this is compatible with the product of modules

defined in Lemma 3.3.1.

Lemma 3.3.3. *Let (M, ρ, A) and (N, τ, A') be right modules. The functor $\widehat{M \otimes N}$ corresponding to the module $M \otimes N$ is naturally isomorphic to the tensor product $\hat{M} \otimes \hat{N}$.*

Proof. By Lemma 3.2.5 it is sufficient to show that $\widehat{M \otimes N}$ and $\hat{M} \otimes \hat{N}$ are equivalent as their corresponding modules. Since both functors correspond to $A \otimes A'$ -modules on $M \otimes N$ we only need to show that the module action is the same in both cases. For $\widehat{M \otimes N}$ the module action is given by $(\rho \otimes \tau) \circ (\text{id} \otimes \sigma \otimes \text{id})$. The module action defined by $\hat{M} \otimes \hat{N}$ is the image under the tensor hom isomorphism of $\hat{\rho} \otimes \hat{\tau}$ composed with the map

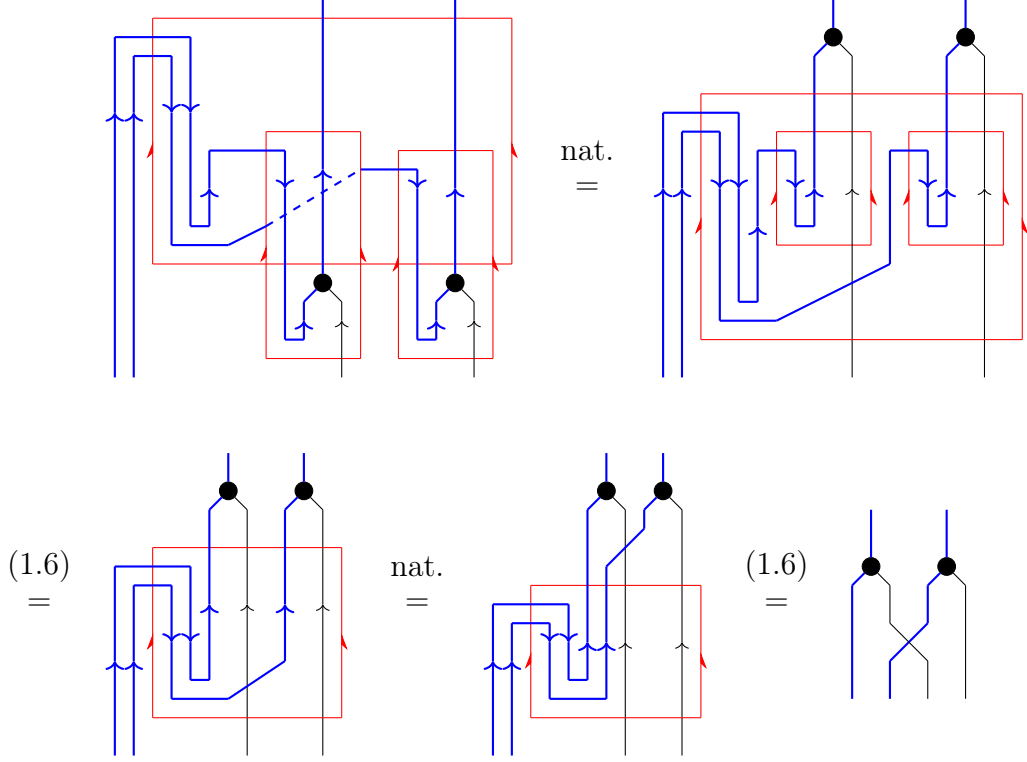
$$[M, M] \otimes [N, N] \rightarrow [M \otimes N, M \otimes N]$$

defined in Lemma 1.2.8. Unpacking this gives us the following composition:

$$\begin{aligned} M \otimes N \otimes H \otimes H' &\xrightarrow{\text{id} \otimes \text{coev} \otimes \text{coev}} M \otimes N \otimes [M, M \otimes H] \otimes [N, N \otimes H'] \\ &\xrightarrow{\text{id} \otimes [\text{id}, \rho] \otimes [\text{id}, \tau]} M \otimes N \otimes [M, M] \otimes [N, N] \\ &\xrightarrow{\text{id} \otimes \text{coev}} M \otimes N \otimes [M \otimes N, M \otimes N \otimes [M, M] \otimes [N, N]] \\ &\xrightarrow{\text{id} \otimes [\text{id}, \text{id} \otimes \sigma \otimes \text{id}]} M \otimes N \otimes [M \otimes N, M \otimes [M, M] \otimes N \otimes [N, N]] \\ &\xrightarrow{\text{id} \otimes [\text{id}, \text{ev} \otimes \text{ev}]} M \otimes N \otimes [M \otimes N, M \otimes N] \\ &\xrightarrow{[\text{id}, \text{ev}]} M \otimes N \end{aligned}$$

We show the composition above is equal to $(\rho \otimes \tau) \circ (\text{id} \otimes \sigma \otimes \text{id})$ by the

following string diagrams.



□

As mentioned earlier, there are two products on modules we will be interested in. The other one is the tensor product over an algebra which is defined as follows.

Definition 3.3.4. Let $\mathcal{V}$ be a symmetric monoidal category with coequalisers. Let A be an algebra in $\mathcal{V}$, let (M, ρ, A) be a right A -module, and let (N, τ, A) be a left A -module in $\mathcal{V}$. The **tensor product of M and N over A** is the $\mathcal{V}$ object $M \otimes_A N$ defined by the following coequaliser in $\mathcal{V}$.

$$M \otimes A \otimes N \xrightarrow[\tau \otimes \text{id}]{\text{id} \otimes \rho} M \otimes N \longrightarrow M \otimes_A N$$

Since we are using the module structure to define the tensor product then the resulting object isn't a module. However, if we take two bimodules we

can take the tensor product over only one of their algebras.

Definition 3.3.5. Let $\mathcal{V}$ be a symmetric monoidal category with coequalisers. Let (M, ρ) be a A - A' -bimodule and let (N, τ) be a A' - A'' -bimodule in $\mathcal{V}$ then we can define the tensor product $M \otimes_{A'} N$ as in Definition 3.3.4 with action defined by the unique map factorising the coequalisers below.

$$\begin{array}{ccccc}
A \otimes M \otimes A' \otimes N \otimes A'' & \xrightarrow[\tau \circ \eta_A']{\rho \circ \eta_A} & A \otimes M \otimes N \otimes A'' & \longrightarrow & A \otimes M \otimes_{A'} N \otimes A'' \\
(\rho \otimes \tau) \circ (\eta_{A'} \otimes \eta_{A'}) \downarrow & & \downarrow (\rho \otimes \tau) \circ (\eta_{A'} \otimes \eta_{A'}) & & \downarrow !\exists \\
M \otimes A' \otimes N & \xrightarrow[\tau \circ \eta_A']{\rho \circ \eta_A} & M \otimes N & \longrightarrow & M \otimes_{A'} N
\end{array}$$

If we want to define tensor products for representations we now have an issue because representations are functors of enriched categories and we haven't defined colimits for enriched categories. Recall however that in Example 1.3.9 we saw that coends generalise colimits and if M and N are right and left A -modules respectively then $\hat{M} \otimes \hat{N}$ is a functor $BA^{\text{op}} \otimes BA \rightarrow \underline{\mathcal{V}}$. In particular this means when $\mathcal{V}$ has sufficient coequalisers we can take the coend of $\hat{M} \otimes \hat{N}$ and in fact this gives us the correct notion of the tensor product for representations in the following sense.

Lemma 3.3.6. Let (M, ρ) and (N, τ) be right and left A -modules respectively with corresponding $\mathcal{V}$ -functors $\hat{M}: BA \rightarrow \underline{\mathcal{V}}$ and $\hat{N}: BA^{\text{op}} \rightarrow \underline{\mathcal{V}}$. The coend

$$\int^* \hat{M}(*) \otimes \hat{N}(*)$$

is equivalent to the tensor product $M \otimes_A N$. Furthermore, if M and N are bimodules with corresponding functors $\hat{M}: BA^{\text{op}} \otimes BA' \rightarrow \underline{\mathcal{V}}$ and $\hat{N}: BA'^{\text{op}} \otimes BA'' \rightarrow \underline{\mathcal{V}}$, then the coend

$$\int^* \hat{M}(-, *) \otimes \hat{N}(*, -)$$

is a $\mathcal{V}$ -functor $BA^{\text{op}} \otimes BA'' \rightarrow \underline{\mathcal{V}}$ which corresponds to the A - A'' bimodule $M \otimes_{A'} N$ defined in Definition 3.3.5.

Proof. First in the case that M and N are one sided modules then since BA

has only one object, the coend

$$\int^* \hat{M}(*) \otimes \hat{N}(*)$$

is defined as the coequaliser of

$$\hat{M}(*_2) \otimes \hat{N}(*_1) \otimes BA^{\text{op}}(*_2, *_1) \rightrightarrows \hat{M}(*) \otimes \hat{N}(*)$$

or more simply

$$M \otimes N \otimes A \rightrightarrows M \otimes N. \quad (3.7)$$

The parallel arrows of the above equaliser are the adjuncts of the morphisms $A \rightarrow [M \otimes N, M \otimes N]$ defined by the partial functors $\hat{M}(*) \otimes \hat{N}(-)$ and $\hat{M}(-) \otimes \hat{N}(*)$. Since the unit η is the identity morphism for $*$ in both BA and BA^{op} , the partial functors are defined on homs by composing either $\text{id} \otimes \eta$ or $\eta \otimes \text{id}$ with $\hat{M} \otimes \hat{N}$ as defined by (3.6). The adjoint of $(\hat{M} \otimes \hat{N}) \circ (\text{id} \otimes \eta)$ is precisely the map

$$M \otimes N \otimes H \xrightarrow{\text{id} \otimes \eta} M \otimes N \otimes A \otimes A \xrightarrow{\hat{M} \otimes \hat{N}} M \otimes N \otimes [M \otimes N, M \otimes N] \xrightarrow{\text{ev}} M \otimes N$$

which we showed in the proof of Lemma 3.3.3 is equal to

$$M \otimes N \otimes A \xrightarrow{\text{id} \otimes \eta} M \otimes N \otimes A \otimes A \xrightarrow{\text{id} \otimes \sigma \otimes \text{id}} M \otimes A \otimes N \otimes A \xrightarrow{\rho \otimes \tau^{\text{op}}} M \otimes N.$$

By naturality of symmetry this is equal to

$$M \otimes N \otimes A \xrightarrow{\text{id} \otimes \sigma} M \otimes A \otimes N \xrightarrow{\text{id} \otimes \eta} M \otimes A \otimes N \otimes A \xrightarrow{\rho \otimes \tau^{\text{op}}} M \otimes N$$

which in turn is equal to $(\rho \otimes \text{id}) \circ (\text{id} \otimes \sigma)$ by the definition of a module. Similarly, the other parallel arrow of the equaliser above is equal to $(\text{id} \otimes \tau^{\text{op}}) \circ (\text{id} \otimes \sigma) = (\text{id} \otimes \tau)$ and since σ is an isomorphism then the coequaliser of (3.7) is precisely the coequaliser given in Definition 3.3.4.

If M and N are instead bimodules then Kelly [11, p. 28] tells us that taking the coend provides us with a unique functor $\hat{M} \otimes_{A'} \hat{N}: BA^{\text{op}} \otimes BA'' \rightarrow \underline{\mathcal{V}}$ which is natural in $(*, *) \in BA^{\text{op}} \otimes BA''$ as well as in BA' . Since composition

in BA, BA' and BA'' is defined by multiplication, Kelly's result tells us that the coend defines a unique functor $\hat{M} \otimes_{A'} \hat{N}$ which corresponds uniquely to a bimodule whose action respects the A and A'' multiplication which of course the bimodule given in Definition 3.3.4 does. $\square$

We will spend the rest of this chapter investigating and proving properties of these two products.

Lemma 3.3.7. *Let (M, A) and (N, A') be right modules and let (P, A) and (Q, A') be left modules then there exists a canonical isomorphism*

$$(M \otimes N) \otimes_{A \otimes A'} (P \otimes Q) \cong (M \otimes_A P) \otimes (N \otimes_{A'} Q). \quad (3.8)$$

Proof. This follows simply from factoring symmetry through the coequalisers

$$\begin{array}{ccccc} M \otimes N \otimes A \otimes A' \otimes P \otimes Q & \rightrightarrows & M \otimes N \otimes P \otimes Q & \rightarrow & (M \otimes N) \otimes_{A \otimes A'} (P \otimes Q) \\ \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow \exists! \\ M \otimes A \otimes P \otimes N \otimes A' \otimes Q & \rightrightarrows & M \otimes P \otimes N \otimes Q & \rightarrow & (M \otimes_A P) \otimes (N \otimes_{A'} Q) \end{array}$$

which define each side of (3.8). $\square$

Furthermore, if M, N, P , and Q are bimodules then the isomorphism defined above is a module morphism since it is defined by the universal property of a coequaliser.

If the tensor product over an algebra A is supposed to be thought of as modding out by the A action then we would expect $A \otimes_A M \cong M$, providing a monoidal unit for $\otimes_A$. This does indeed turn out to be the case.

Lemma 3.3.8. *Let (M, ρ, A) be a left A -module in a symmetric closed monoidal category $\mathcal{V}$. Let A be considered as an A - A -bimodule by its multiplication μ then the tensor product $A \otimes_A M$ is isomorphic to M as an A -module.*

Proof. Let $\hat{M}$ denote the $\mathcal{V}$ -functor corresponding to M . The functor corresponding to A as a bimodule is exactly the hom functor $BA(-, -): BA^{\text{op}} \times BA \rightarrow \underline{\mathcal{V}}$ (see for instance, [11, Eqn 1.29]). By Lemma 3.3.6, the tensor

product $A \otimes_A M$ over A is defined by the co-end

$$\int^* BA(-, *) \otimes \hat{M}(*).$$

By Lemma 1.3.11 we therefore have a natural isomorphism

$$\hat{M}(-) \cong \int^* BA(-, *) \otimes \hat{M}(*) = \widehat{A \otimes_A M}.$$

Finally, by Corollary 3.2.7, naturality precisely tells us this isomorphism is an A -module isomorphism. $\square$

In particular the universal property of the coend tells us that the isomorphism $A \otimes_A M \cong M$ is unique and since the module action ρ is a module morphism which commutes with both the multiplication and action $\mu, \rho: A \otimes A \otimes M \rightarrow A \otimes M$ then the isomorphism $A \otimes_A M \cong M$ is given by the factorisation by ρ of the coequaliser defining $A \otimes_A M$ as follows.

$$\begin{array}{ccc} & & A \otimes_A M \\ & \nearrow & \downarrow \\ A \otimes A \otimes M & \rightrightarrows & A \otimes M \\ & \searrow \rho & \downarrow \\ & & M \end{array} \quad \begin{array}{c} \\ \\ \exists \cong \\ \end{array}$$

By the commuting property for the unit and module action we can see that the inverse to this morphism is given by taking the unit $M \rightarrow A \otimes M$ and composing with the coequaliser map $A \otimes M \rightarrow A \otimes_A M$.

Finally, we can use Lemma 3.3.8 to change the left action of a module by precomposing the action by some algebra morphism on both sides.

Lemma 3.3.9. *Let (M, ρ, A') be a left A' -module and $f: A \rightarrow A'$ be a Hopf algebra morphism. Define ${}_f A'$ to be A' considered as a left A -module by $\mu' \circ (f \otimes \text{id})$ and as a right A' -module by multiplication in A' . Define ${}_f M$ to be M considered as a left A -module by $\rho \circ (f \otimes \text{id})$ then we have an isomorphism of modules*

$${}_f A' \otimes_{A'} M \cong {}_f M.$$

Proof. By Lemma 3.3.8 we know ${}_f A' \otimes_{A'} M \cong {}_f M$ as objects in $\mathcal{V}$ so it remains to show this is a A -module morphism. That is to say the following diagram commutes

$$\begin{array}{ccc} A \otimes A' \otimes_{A'} M & \xrightarrow{\text{id} \otimes \cong} & A \otimes M \\ \mu' \circ (f \otimes \text{id}) \downarrow & & \downarrow \rho \circ (f \otimes \text{id}) \\ A' \otimes_{A'} M & \xrightarrow{\cong} & M \end{array}$$

which we verify by using the fact that $A' \otimes_{A'} M \cong M$ is a A' -module morphism in the bottom square of the following diagram.

$$\begin{array}{ccc} A \otimes A' \otimes_{A'} M & \xrightarrow{\text{id} \otimes \cong} & A \otimes M \\ f \otimes \text{id} \downarrow & & \downarrow f \otimes \text{id} \\ A' \otimes A' \otimes_{A'} M & \xrightarrow{\text{id} \otimes \cong} & A' \otimes M \\ \mu' \downarrow & & \downarrow \rho \\ A' \otimes_{A'} M & \xrightarrow{\cong} & M \end{array}$$

□

Chapter 4

2-Categories, Bicategories, and Double Categories

In this chapter we will provide the definitions of double categories and their morphisms required for the final two chapters as well as an example we will use again in Chapter 6.

Definition 4.0.1. A **double category** $\mathbb{D}$ consists of categories $\mathbb{D}_0$ and $\mathbb{D}_1$ with the following data.

1. A unit functor $U: \mathbb{D}_0 \rightarrow \mathbb{D}_1$.
2. Source and target functors $L, R: \mathbb{D}_1 \rightrightarrows \mathbb{D}_0$.
3. A weak composition functor $\odot: \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \rightarrow \mathbb{D}_1$ over the pullback along L and R .
4. Natural isomorphisms satisfying the usual coherence conditions.
 - (a) Associator $\alpha: (a \odot b) \odot c \cong a \odot (b \odot c)$.
 - (b) Left and right unitors $\ell: U(x) \odot a \cong a$ and $\rho: a \odot U(x) \cong a$.

Such that $L(U(x)) = R(U(x)) = x$, $L(a \odot b) = L(a)$, and $R(a \odot b) = R(b)$.

We call the objects of $\mathbb{D}_0$ the **0-cells** and the morphisms of $\mathbb{D}_0$ are called **tight 1-cells** after their strict composition. We call the objects of $\mathbb{D}_1$ the **loose 1-cells** and the morphisms of $\mathbb{D}_1$ are called **2-cells**.

Example 4.0.2. For a closed symmetric monoidal category $\mathcal{V}$, the double category $\mathbb{B}iMod_{\mathcal{V}}$ has the following data.

- $\mathbb{D}_0$ is $\text{Hopf}_{cc}(\mathcal{V})$ so 0-cells are cocommutative Hopf algebras and tight 1-cells are Hopf algebra morphisms.
- $\mathbb{D}_1$ is the category of bimodules of cocommutative Hopf algebras and bimodule morphisms.
- The loose 1-cell picked out by the unit functor U for a Hopf algebra H is $U(H) = H$ with multiplication by H providing the the left and right actions.
- Composition of bimodules $(M, H, H') \odot (N, H', H'')$ over H' is by the tensor product $\otimes_{H'}$.
- The associators and unitors of composition are the same as for the tensor product of bimodules.

We will prove $\mathbb{B}iMod_{\mathcal{V}}$ is a double category in Chapter 6 by constructing another double category and showing it is equivalent.

Definition 4.0.3 ([26, Def 2.9]). A double category $\mathbb{D}$ is a **monoidal double category** if the following additional properties are satisfied.

1. Both $\mathbb{D}_0$ and $\mathbb{D}_1$ are monoidal categories.
2. If $\mathbb{1}$ is the monoidal unit of $\mathbb{D}_0$ then $U(\mathbb{1})$ is the monoidal unit of $\mathbb{D}_1$.
3. The functors L , R , and U are strict monoidal.
4. There is a **composition interchange** isomorphism

$$\iota: (a \otimes b) \odot (c \otimes d) \cong (a \odot b) \otimes (c \odot d)$$

satisfying axioms [26, Def 2.9].

Example 4.0.4. The double category $\mathbb{B}iMod_{\mathcal{V}}$ is monoidal. We have already seen in Chapter 2 how to define a product on $\text{Hopf}_{cc}(\mathcal{V})$ and in Chapter 3

how to define a product of bimodules and their morphisms by using the tensor product in $\mathcal{V}$. The unit functor U as well as the source and target functors L and R are a strict monoidal functor by definition. The composition interchange is given by Lemma 3.3.7.

Definition 4.0.5. Let $\mathbb{D}$ and $\mathbb{D}'$ be double categories. A **strong double functor** $F: \mathbb{D} \rightarrow \mathbb{D}'$ consists of:

1. Two functors $F_0: \mathbb{D}_0 \rightarrow \mathbb{D}'_0$ and $F_1: \mathbb{D}_1 \rightarrow \mathbb{D}'_1$ such that $L' \circ F_1 = F_0 \circ L$ and $R' \circ F_1 = F_0 \circ R$.
2. Composition and unit natural isomorphisms $F_c: F_1(-) \odot F_1(-) \rightarrow F_1(- \odot -)$ and $F_u: U'(F_0(-)) \rightarrow F_1(U(-))$ which, for all a, b , and c in $\mathbb{D}_1$, satisfy the following coherence axioms.

(a) The following hexagon commutes (associativity compatibility):

$$\begin{array}{ccccc}
 & F_1(a) \odot (F_1(b) \odot F_1(c)) & & & \\
 & \swarrow \alpha' & & \searrow \text{id} \odot F_c & \\
 (F_1(a) \odot F_1(b)) \odot F_1(c) & & & & F_1(a) \odot F_1(b \odot c) \\
 \downarrow F_c \odot \text{id} & & & & \downarrow F_c \\
 F_1(a \odot b) \odot F_1(c) & & & & F_1(a \odot (b \odot c)) \\
 & \searrow F_c & & \swarrow F_1(\alpha) & \\
 & F_1((a \odot b) \odot c) & & &
 \end{array}$$

(b) The following left and right unit identities commute (where $a: x \rightarrow y$ is a loose 1-cell).

$$\begin{array}{ccccc}
 F_1(a) \odot U'(F_0(y)) & \xrightarrow{\text{id} \odot F_u} & F_1(a) \odot F_1(U(y)) & \xrightarrow{F_c} & F_1(a \odot U(y)) \\
 \downarrow \rho & & & \swarrow F_1(\rho) & \\
 & & F_1(a) & &
 \end{array}$$

$$\begin{array}{ccccc}
F_1(U(x) \odot a) & \xleftarrow{F_c} & F_1(U(x)) \odot F_1(a) & \xleftarrow{F_u \odot \text{id}} & U'(F_0(y)) \odot F_1(a) \\
& & & & \downarrow \ell \\
& \searrow F_1(\ell) & & & F_1(a)
\end{array}$$

Definition 4.0.6. A **strong monoidal double functor** is a strong double functor $F: \mathbb{D} \rightarrow \mathbb{D}'$ such that:

1. The functors F_0 and F_1 are strong monoidal.
2. The natural isomorphisms F_c and F_u are monoidal natural transformations.

Chapter 5

Fibrations and a Fibred Monoidal Double Category of Group Representations

The aim of this chapter is to introduce the notion of fibred double categories (also known as proarrow equipments) and use a result of Shulman [25] to construct a fibred double category of group representations. Ultimately this chapter exists to provide an example of the main result of this thesis which takes a closed monoidal category and constructs a fibred double category of its Hopf algebras and bimodules.

We will use Shulman's result which constructs a double category from a fibration with certain properties. We will use groups and group representations to motivate the definition of a fibration as arising from the notions of restriction and extension. We will discuss the Grothendieck construction and its generalisation for monoidal fibrations, in particular we will introduce the notions of internal and external products, and external homs. We will introduce the Beck-Chevalley condition which is a necessary piece of machinery for constructing fibred double categories. Finally we will prove the necessary results to construct a fibred double category of representations.

5.1 Restrictions, Extensions, and Fibrations

5.1.1 The Fibration on Group Representations

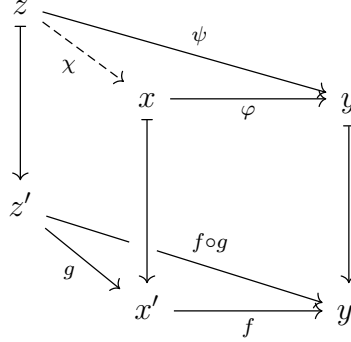
Given a group representation $(V, \hat{\rho}, H)$ and a group homomorphism $f: G \rightarrow H$, a common tool in representation theory is the idea of taking the **restriction** of V along f . This is defined by simply precomposing $\hat{\rho}$ with f to give a new representation $(V, \hat{\rho} \circ f, G)$ which we simply denote as ${}_fV$.

Alternatively, we can obtain a H -representation via f from a G -representation $(W, \hat{\tau}, G)$ by **extension** (although this is often called induction) which gives us an H -representation on $KH_f \otimes_{KG} W$. We will not focus on the uses for restrictions and extensions in representation theory here but they are often discussed in textbooks on representation theory [7, 28]. Restriction and extension can be thought of as functors which we will see in the next section but they can also be described instead by a lifting property taking group homomorphisms to intertwining maps between representations.

To make this concrete, take the category $\mathbf{Rep}_K$ of group representations in $\mathbf{Vect}_K$ and their intertwining maps and construct a functor $\Omega_{\mathbf{Rep}}: \mathbf{Rep}_K \rightarrow \mathbf{Grp}$ which sends a representation $(V, \hat{\rho}, G)$ to the group G and sends an intertwining map $(\hat{\theta}, f)$ (as defined by Definition 3.2.4) to the group homomorphism f . The lifting property which restriction satisfies is the following.

Definition 5.1.1. Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a functor and let $\varphi: x \rightarrow y$ and $f: x' \rightarrow y'$ be morphisms in $\mathcal{A}$ and $\mathcal{B}$ respectively such that $\Omega(\varphi) = f$. We say that φ is a **cartesian arrow over f** if for every pair of morphisms $\psi: z \rightarrow y$ and $g: z' \rightarrow x'$ such that $\Omega(\psi) = f \circ g$ there exists a unique morphism $\chi: z \rightarrow x$ such that $\Omega(\chi) = g$ and $\varphi \circ \chi = \psi$. We call this the **factorisation property** of

a cartesian arrow.



We say that Ω is a **fibration** if for every morphism $f: x' \rightarrow y'$ in $\mathcal{B}$ and object $y \in \mathcal{A}$ such that $\Omega(y) = y'$, there exists a cartesian arrow $\varphi_{f,y}$ over f with codomain y . We call $\mathcal{B}$ the **base** of the fibration.

The definition of a fibration does not specify uniqueness of cartesian arrows so it is useful to make a choice.

Definition 5.1.2. Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a fibration. A **cleaving** of F is a choice of unique cartesian arrow $\varphi_{f,y}$ for each morphism $f: x' \rightarrow y' \in \mathcal{B}$ and object $y \in \mathcal{A}$ over y' .

When taken at face value this seems like a somewhat obtuse definition. The core idea is that a fibration behaves like a functor from a category $\mathcal{A}$ equipped with structure from a category $\mathcal{B}$ which then forgets that this structure was supposed to be doing something. For instance, as described earlier, the functor $\Omega_{\text{Rep}}: \mathbf{Rep}_K \rightarrow \mathbf{Grp}$ sends a representation to the group that is acting on it. That is, we are forgetting that a group is providing an action on a vector space.

Fibrations are prevalent objects in category theory (see for instance [25, Chp. 3] for some examples). For now however, we will only be interested group representations, in particular, restriction provides us with cartesian arrows over any group homomorphism.

Lemma 5.1.3. *Let $(V, \hat{\rho}, H')$ be a group representation and $f: H \rightarrow H'$ be a group homomorphism. The intertwining map*

$$(\text{id}, f): {}_f V \rightarrow V$$

is a cartesian arrow where ${}_fV$ has action provided by precomposition of $\hat{\rho}$ with f . In particular $\Omega_{Rep}: \mathbf{Rep}_K \rightarrow \mathbf{Grp}$ is a fibration.

Proof. Let $g: L \rightarrow H$ be a group homomorphism and let $(\hat{\theta}, f \circ g)$ be an intertwining map $(W, \hat{\tau}, L) \rightarrow (V, \hat{\rho}, H)$. We need to prove that there exists a unique map $(\chi, g): W \rightarrow V$ commuting with $(\hat{\theta}, f \circ g)$ and (id, f) as follows.

$$\begin{array}{ccc} W & \xrightarrow{(\chi, g)} & V \\ & \searrow (\hat{\theta}, f \circ g) & \downarrow (\text{id}, f) \\ & & V \end{array}$$

Clearly the only such map is $(\hat{\theta}, g)$. □

We provide some useful facts about fibrations courtesy of Shulman [25, Prop 3.4].

Proposition 5.1.4. *Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a fibration then we have the following.*

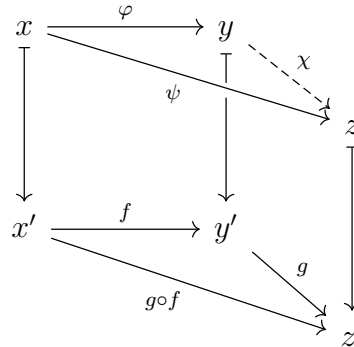
1. *The composite of cartesian arrows is cartesian.*
2. *If ϕ and ψ are cartesian arrows over $f \circ g$ and f respectively then the unique factorisation χ of ψ through ϕ is cartesian over g .*
3. *Cartesian lifts are unique up to isomorphism. That is to say that if $\phi: b \rightarrow c$ and $\psi: b' \rightarrow c$ are both cartesian lifts of f then there is a unique isomorphism $b \cong b'$ commuting with ϕ and ψ .*
4. *Any isomorphism in $\mathcal{A}$ is cartesian.*
5. *If f is an isomorphism in $\mathcal{B}$ then any cartesian lift of f is an isomorphism.*

For the sake of consistency with Chapter 6 (and extensions in a moment), instead of ${}_fV$ we will make an isomorphic choice of cleaving such that the restriction along $f: H \rightarrow H'$ of a representation V is ${}_fKH' \otimes_{KH'} V$, where the action is given by $h \mapsto f(h) \cdot \text{id} \otimes_{KH'} \text{id}_V$. When we write $\otimes_{KH'}$ here we mean the tensor product defined by Lemma 3.3.6, although in this case this

definition recovers the usual tensor product over a group where $h' \otimes_{KH'} v = e \otimes_{KH'} \hat{\rho}(h')v$. Clearly ${}_fKH' \otimes_{KH'} V \cong {}_fV$ as a representation but this also follows from Lemma 3.3.9. Since isomorphisms are cartesian arrows by Proposition 5.1.4, then the composite of (id, f) with this isomorphism gives us a cartesian arrow ${}_fKH' \otimes_{KH'} V \rightarrow V$. When we need a choice of cleaving for the fibration Ω_{Rep} then these lifts are the choice we will make.

The dual notion to restriction is extension. This generalises to the following dual notion of a fibration.

Definition 5.1.5. Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a functor and let $\varphi: x \rightarrow y$ and $f: x' \rightarrow y'$ be morphisms in $\mathcal{A}$ and $\mathcal{B}$ respectively such that $\Omega(\varphi) = f$. We say that φ is an **opcartesian arrow over f** if for every pair of morphisms $\psi: x \rightarrow z$ and $g: y' \rightarrow z'$ such that $\Omega(\psi) = g \circ f$ there exists a unique morphism $\chi: y \rightarrow z$ such that $\Omega(\chi) = g$ and $\chi \circ \varphi = \psi$.



Furthermore, we say that Ω is an **opfibration** if for every morphism $f: x' \rightarrow y'$ in $\mathcal{B}$ and object $x \in \mathcal{A}$ such that $\Omega(x) = x'$, there exists an opcartesian arrow $\varphi_{f,x}$ over f with domain x .

More simply, a functor $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ is an **opfibration** if the functor $\Omega^{\text{op}}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$ is a fibration.

If a functor is both a fibration and an opfibration then we call it a **bi-fibration**. It is worth mentioning that homotopy theorists often refer to fibrations as “cartesian fibrations” and opfibrations as “cocartesian fibrations”. This is not the language we will be using here, in particular this distinction is important to make clear for when we discuss cartesian monoidal fibrations

later.

We will now show that extensions provide an opfibration structure on the forgetful functor $\Omega_{\text{Rep}}: \mathbf{Rep}_K \rightarrow \mathbf{Grp}$.

Lemma 5.1.6. *Let $(V, \hat{\rho}, H)$ be a group representation and $f: H \rightarrow H'$ be a group homomorphism. The intertwining map*

$$V \xrightarrow{\cong} KH \otimes_{KH} V \xrightarrow{(f \otimes_{KH} \text{id}, f)} KH'_f \otimes_{KH} V$$

into the extension of f on V is an opcartesian arrow over f . In particular Ω_{Rep} is an opfibration.

Proof. Let $g: H' \rightarrow L$ be a group homomorphism, W be an L -representation with action $\hat{\tau}$, and $(\varphi, g \circ f): V \rightarrow W$ be an intertwining map. We need to prove that there exists a unique map (φ', g) such that the following diagram commutes.

$$\begin{array}{ccccc}
 V & \xrightarrow{\cong} & KH \otimes_{KH} V & \xrightarrow{f \otimes_{KH} \text{id}} & KH'_f \otimes_{KH} V \\
 \downarrow & & \searrow \varphi & & \downarrow \text{ } \quad \searrow \varphi' \text{ (dashed)} \\
 & & & & W \\
 H & \xrightarrow{f} & H' & \xrightarrow{g} & L \\
 & \searrow g \circ f & & & \downarrow \\
 & & & & L
 \end{array} \tag{5.1}$$

I claim that (φ', g) is given by the composition

$$KH'_f \otimes_{KH} V \xrightarrow{(g \otimes_{KL} \varphi, g)} KL \otimes_{KL} W \cong W. \tag{5.2}$$

Certainly (5.2) makes (5.1) commute so it remains to show this map is unique. Assume that $(\theta, g): KH'_f \otimes_{KH} V \rightarrow W$ is an intertwining map such that (5.1) commutes. The map $f \otimes_{KH} \text{id} \circ \cong: V \rightarrow KH'_f \otimes_{KH} V$ sends an element $v \in V$ to $e_{H'} \otimes_{KH} v$ which θ must then send to $\varphi(v)$ to make (5.1) commute.

Since θ is an intertwining map then

$$\theta(h' \otimes_{KH} v) = \hat{\tau}(g(h'))\theta(e \otimes_{KH} v) = \hat{\tau}(g(h'))\varphi(v)$$

which is to say that θ sends any element of $KH'_f \otimes_{KH} V$ to the same place as (5.2). Since θ agrees with (5.2) on elements then they are the same map. $\square$

5.1.2 Restriction and Extension as Functors

Given a fibration $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ and a choice of cleaving, then for every morphism $f: b \rightarrow b'$ in $\mathcal{B}$ we are provided with an object above b for each object above b' . In fact, in the case of representations, if $f: H \rightarrow H'$ is a group homomorphism we obtain a functor $f^*: H'\text{-Rep} \rightarrow H\text{-Rep}$ called the **restriction functor** which sends a H' -representation to its restriction. For any functor $F: \mathcal{A} \rightarrow \mathcal{B}$ we have **fibre categories** over every object in $b \in \mathcal{B}$, which have as objects, every object in $\mathcal{A}$ sent to b , and as morphisms every morphism in $\mathcal{A}$ which is sent to the identity on b . In general, if we have a fibration and cleaving then for every morphism in the base of the fibration we obtain a functor between these fibre categories. If we have two morphisms $f: b \rightarrow b'$ and $g: b' \rightarrow b''$ in the base of a fibration, then in general the functors $f^* \circ g^*$ and $(g \circ f)^*$ are not equal but the factorisation property of fibrations tells us they are isomorphic.

Similarly, since an opfibration $\Omega: \mathcal{A} \rightarrow \mathcal{B}$, is just a fibration $\Omega^{\text{op}}: \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}^{\text{op}}$ then for any morphism $f: b \rightarrow b' \in \mathcal{B}$ we obtain (up to a choice of cleaving) an **extension functor** from the fibre category over b to the fibre category over b' which we denote $f_!$. In particular for our fibration Ω_{Rep} , a group homomorphism $f: H \rightarrow H'$ gives us the extension functor

$$f_! := KH'_f \otimes_{KH} (-): H\text{-Rep} \rightarrow H'\text{-Rep}$$

and restriction functor

$$f^* := {}_fKH' \otimes_{KH'} (-): H'\text{-Rep} \rightarrow H\text{-Rep}.$$

Furthermore, for any group homomorphism f we obtain an adjunction $f_! \dashv f^*$ and this holds more generally for any bifibration. By the definition of a fibration, for any representation V we have a morphism $V \rightarrow f^*f_!(V)$ given by the unique factorisation in the following diagram.

$$\begin{array}{ccccc}
 V & & & & \\
 \downarrow & \searrow & & \searrow & \\
 & f^*f_!(V) & \xrightarrow{\quad} & f_!(V) & \\
 & \downarrow & & \downarrow & \\
 H & \searrow \text{id} & & H & \xrightarrow{f} & H' \\
 & & & & &
 \end{array}$$

Similarly, we obtain a morphism $f_!f^*(V) \rightarrow V$ by the following factorisation.

$$\begin{array}{ccccc}
 f^*(V) & \xrightarrow{\quad} & f_!f^*(V) & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & & & V \\
 H & \xrightarrow{f} & H' & \searrow \text{id} & \\
 & \searrow & & & \downarrow \\
 & & & & H'
 \end{array}$$

These factorisations provide the components for the unit and counit respectively for the adjunction $f_! \dashv f^*$ and we can check by Lemma 5.1.3 and Lemma 5.1.6 that for representations $(V, \hat{\rho}, H)$ and $(W, \hat{\tau}, H')$, the components of the unit are given by

$$V \xrightarrow{\cong} KH \otimes_{KH} KH \otimes_{KH} V \xrightarrow{f \otimes_f f \otimes_{KH} \text{id}} {}_fKH' \otimes_{KH'} KH'_f \otimes_{KH} V \quad (5.3)$$

which sends v to $e \otimes_{KH'} e \otimes_{KH} v$. Similarly the components of the counit are

given by

$$KH'_f \otimes_{KH} {}_f KH' \otimes_{KH'} W \xrightarrow{\text{id} \otimes {}_f \text{id} \otimes_{KH'} \text{id}} KH' \otimes_{KH'} KH' \otimes_{KH'} W \xrightarrow{\cong} W \quad (5.4)$$

which sends $h \otimes_{KH} h' \otimes_{KH'} w$ to $\hat{\tau}(h \cdot h')(w)$.

5.1.3 Fibred Double Categories

Now we have an understanding of fibrations we are able to expand the definition of a double category in Chapter 4 and properly define what we mean by a fibred double category.

Definition 5.1.7. A **fibred double category** is a double category $\mathbb{D}$ such that the functor

$$\langle L, R \rangle : \mathbb{D}_1 \rightarrow \mathbb{D}_0 \times \mathbb{D}_0$$

is a fibration.

One of the main reasons we care about fibred double categories is that if you have a 2-cell

$$\begin{array}{ccc} a & \xrightarrow{M} & b \\ f \downarrow & \Downarrow & \downarrow g \\ c & \xrightarrow{N} & d \end{array} \quad (5.5)$$

in a double category $\mathbb{D}$ then you can take the cartesian lift of (f, g) into N to give you a 2-cell

$$\begin{array}{ccc} a & \xrightarrow{f^* N g^*} & b \\ f \downarrow & \Downarrow & \downarrow g \\ c & \xrightarrow{N} & d \end{array} \quad (5.6)$$

which when we take the unique factorisation of (5.5) through (5.6) gives us the 2-cell

$$\begin{array}{ccc} a & \xrightarrow{M} & b \\ \parallel & \Downarrow & \parallel \\ a & \xrightarrow{f^* N g^*} & b \end{array}$$

which carries the same information as (5.5) by simply composing it with (5.6). In particular this means the data of the tight 1-cells is captured entirely by the loose 1-cells, which is to say that the horizontal bicategory of $\mathbb{D}$ has the same information as $\mathbb{D}$. In particular, given any tight 1-cell $f: a \rightarrow b$ in $\mathbb{D}_0$ then taking the cartesian lift of the pair (f, id) along the identity loose 1-cell $U_b: b \rightarrow b$ gives us a loose 1-cell which we denote ${}_f b: a \rightarrow b$ as well as a cartesian 2-cell

$$\begin{array}{ccc} a & \xrightarrow{{}_f b} & b \\ f \downarrow & \Downarrow & \parallel \\ b & \xrightarrow{U_b} & b \end{array}$$

which we can think about as saying that ${}_f b$ is “the same” as f . For a more comprehensive explanation for the correspondence of a fibred double category and its horizontal bicategory see [25, Chp. C].

It is worth mentioning that because fibred double categories can be viewed under lots of different lenses they have acquired a few different names. In [25] fibred double categories are referred to as as “framed bicategories” but a common term used in the literature, originating from [29], is a **proarrow equipment** as we are seen to be “equipping” the 1-cells in a 2-category (or more generally a bicategory) with adjoint 1-cells or “proarrows” in a bicategory.

As previously discussed, the aim of the rest of this thesis is to construct fibred double categories of bimodules. The construction we use is a result of [25, Thm. 14.2] which takes as input a bifibration with certain properties. We will provide this construction now and explain what the required properties of this bifibration are over the course of the rest of this chapter and prove that the forgetful functor $\Omega_{\text{Rep}}: \mathbf{Rep}_K \rightarrow \mathbf{Grp}$ has these properties required as we proceed. In the next chapter we will then extend this idea to fibred double categories of Hopf algebras and bimodules.

Theorem 5.1.8 (Shulman). *Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a bifibration such that $\mathcal{A}$ is monoidal with product $\otimes$ and unit $\mathbb{1}$ and such that $\mathcal{B}$ is cartesian monoidal with diagonal maps Δ_b and projection maps π_b . If Ω is*

1. *cartesian monoidal (Definition 5.2.1),*

2. *internally closed (Definition 5.2.5),*

3. *satisfies the weak Beck-Chevalley condition (Definition 5.2.8),*

then there is a fibred double category $\mathbb{F}\mathbf{r}(\Omega)$ defined as follows.

1. $\mathbb{F}\mathbf{r}(\Omega)_0 = \mathcal{B}$.

2. $\mathbb{F}\mathbf{r}(\Omega)_1$, L , and R are defined by the following pullback square

$$\begin{array}{ccc} \mathbb{F}\mathbf{r}(\Omega)_1 & \longrightarrow & \mathcal{A} \\ (L,R) \downarrow & \lrcorner & \downarrow \Omega \\ \mathcal{B} \times \mathcal{B} & \xrightarrow{\times} & \mathcal{B} \end{array}$$

which is to say $\mathbb{F}\mathbf{r}(\Omega)_0 = \mathcal{B}$ consists of loose 1-cells which are objects of $\mathcal{A}$ which are sent by Ω to a product of two objects in $\mathcal{B}$ and the 2-cells are the morphisms in $\mathcal{A}$ of such objects.

3. *The loose composition over an object $b \in \mathcal{B}$ is the composition of the restriction and extension functors*

$$- \odot - := (\pi_b)_! \Delta_b^*(- \otimes -).$$

4. *The loose unit of a is the composition of the restriction and extension functors*

$$U_a := (\Delta_a)_! \pi_a^*(\mathbb{1}).$$

We already know what a bifibration is, we will discuss the the remaining conditions in the next section.

5.2 Products and the Beck-Chevalley Condition

The new properties we need to discuss to use Theorem 5.1.8 are the Beck-Chevalley condition and a bifibration being cartesian monoidal and internally closed. We will start with cartesian monoidal.

5.2.1 Monoidal Fibrations

The basic idea of a monoidal fibration is that we have a fibration which is a monoidal functor preserving the fibration structure.

Definition 5.2.1. Given monoidal categories $\mathcal{A}$ and $\mathcal{B}$, a **monoidal fibration** is a fibration $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ such that Ω is a strict monoidal functor and the tensor product of cartesian arrows in $\mathcal{A}$ is always a cartesian arrow.

Furthermore, a **cartesian** monoidal fibration is a monoidal fibration where the base category is cartesian monoidal. A **monoidal opfibration** is a opfibration Ω where Ω^{op} is a monoidal fibration. A **monoidal bifibration** is a functor which is both a monoidal fibration and a monoidal opfibration.

Much like how we can obtain restriction and extension functors from a given bifibration, the reverse is also true. In particular if the restriction functors are monoidal then and furthermore that taking the restriction functor of a pair of morphisms is preserved by the monoidal product then we can see that the fibration which they define is a monoidal fibration with the following result. This will make it significantly easier to show Ω_{Rep} is a monoidal fibration.

Lemma 5.2.2. *Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a monoidal functor which is a fibration. The following are equivalent:*

1. *The product of any two cartesian arrows in $\mathcal{A}$ is cartesian.*
2. *For any cartesian arrows*

$$\theta: f^*M \rightarrow M,$$

$$\theta': g^*M \rightarrow N,$$

and

$$\varphi: (f \otimes g)^*(M \otimes N) \rightarrow M \otimes N$$

from a choice of cleaving, there exists an isomorphism

$$f^*M \otimes g^*N \xrightarrow{\iota} (f \otimes g)^*(M \otimes N)$$

such that the following triangle commutes.

$$\begin{array}{ccc}
 f^*M \otimes g^*N & & \\
 \downarrow \iota & \searrow \theta \otimes \theta' & \\
 (f \otimes g)^*(M \otimes N) & \xrightarrow{\varphi} & M \otimes N
 \end{array}$$

Proof. First assume (1), then given cartesian arrows

$$\theta: f^*M \rightarrow M,$$

$$\theta': g^*M \rightarrow N,$$

and

$$\varphi: (f \otimes g)^*(M \otimes N) \rightarrow M \otimes N$$

then by assumption, $\theta \otimes \theta'$ is cartesian and hence by Proposition 5.1.4 we have a canonical isomorphism

$$f^*M \otimes g^*N \cong (f \otimes g)^*(M \otimes N)$$

which commutes with $\theta \otimes \theta'$ and φ .

Conversely, assume (2) and we will prove (1). By Proposition 5.1.4, it is sufficient to show that, given cartesian arrows $\theta: f^*(M) \rightarrow M$ and $\theta': g^*(N) \rightarrow N$ over f and g from a given choice of cleaving, that $\theta \otimes \theta'$ is cartesian. For any morphism $\psi: f^*M \otimes g^*N \rightarrow L$, we have a commuting triangle

$$\begin{array}{ccccc}
 f^*M \otimes g^*N & & & & \\
 \downarrow \iota & \searrow \theta \otimes \theta' & \searrow \psi & & \\
 (f \otimes g)^*(M \otimes N) & \xrightarrow{\varphi} & M \otimes N & \dashrightarrow & L \\
 & & & \exists! &
 \end{array}$$

and hence $\theta \otimes \theta'$ is cartesian since we can define any unique factorisation of ψ through $\theta \otimes \theta'$ by taking the unique factorisation through φ and then composing with the isomorphism $(f \otimes g)^*(M \otimes N) \xrightarrow{\cong} f^*M \otimes g^*N$. $\square$

We immediately obtain a dual result for opfibrations.

Corollary 5.2.3. *Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a monoidal functor which is an opfibration. The following are equivalent:*

1. *The product of any two opcartesian arrows in $\mathcal{A}$ is cartesian.*
2. *For any opcartesian arrows*

$$\theta: M \rightarrow f_!M,$$

$$\theta': M \rightarrow g_!N,$$

and

$$\varphi: M \otimes N \rightarrow (f \otimes g)_!(M \otimes N)$$

from a choice of cleaving, there exists an isomorphism

$$f_!M \otimes g_!N \cong (f \otimes g)_!(M \otimes N)$$

such that the following triangle commutes.

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\theta \otimes \theta'} & f_!M \otimes g_!N \\ & \searrow \varphi & \cong \downarrow \\ & & (f \otimes g)_!(M \otimes N) \end{array}$$

To define $\Omega_{\text{Rep}}: \mathbf{Rep}_K \rightarrow \mathbf{Grp}$ as a cartesian monoidal bifibration we need to define monoidal structure on $\mathbf{Rep}_K$ and $\mathbf{Grp}$. We already know that $\mathbf{Grp}$ is a cartesian monoidal category so we need to find an appropriate monoidal structure on $\mathbf{Rep}_K$, show that Ω_{Rep} preserves the monoidal structure, and show that the product of cartesian and opcartesian arrows are cartesian and opcartesian respectively. We have already seen in Definition 3.3.2 that we can define a product of representations described as functors. This translates to the following product of group representations,

$$(V, \hat{\rho}, G) \otimes (W, \hat{\tau}, H) := (V \otimes_K W, \hat{\rho} \otimes \hat{\tau}, G \times H)$$

where $\hat{\rho} \otimes \hat{\tau}$ is defined by

$$\hat{\rho} \otimes \hat{\tau}(g, h)(v \otimes_K w) = \hat{\rho}(g)(v) \otimes_K \hat{\tau}(h)(w).$$

We can define a product on intertwining maps by

$$(\theta, f) \otimes (\theta', f') := (\theta \otimes_K \theta', f \times f')$$

and by Lemma 3.3.1 and Lemma 3.3.3, this product makes $\mathbf{Rep}_K$ a monoidal category with monoidal unit

$$(K, \ell, \{e\})$$

where ℓ is the left unitor in $\mathbf{Vect}_K$. Clearly Ω_{Rep} preserves this monoidal structure since we just forget to the group and group homomorphisms where the product is given by the cartesian product in $\mathbf{Grp}$. The last thing to check to see that Ω_{Rep} is a cartesian monoidal bifibration is that the product in $\mathbf{Rep}_K$ preserves cartesian and opcartesian morphisms.

Lemma 5.2.4. *The bifibration $\Omega_{\text{Rep}}: \mathbf{Rep}_K \rightarrow \mathbf{Grp}$ is a cartesian monoidal bifibration.*

Proof. We have seen that Ω_{Rep} is a strict monoidal functor so it remains to show that the product of cartesian and opcartesian morphisms are cartesian and opcartesian respectively. First we will show that the product of cartesian arrows is cartesian. Consider the cleaving defined in Lemma 5.1.3 where given a H' -representation M , a group homomorphism $f: H \rightarrow H'$ and is lifted to the cartesian morphism $(\text{id}, f): {}_f M \rightarrow M$. By Lemma 5.2.2 we only need to show that for any group homomorphisms $f: H \rightarrow H'$ and $g: L \rightarrow L'$ that there exists an isomorphism which makes the following triangle commute

$$\begin{array}{ccc} {}_f M \otimes_K {}_g N & & \\ \cong \downarrow & \searrow (\text{id}, f \times g) & \\ {}_{f \times g} M \otimes_K N & \xrightarrow{(\text{id}, f \times g)} & M \otimes_K N \end{array}$$

Of course in this case $fM \otimes_K gN = f \times g M \otimes_K N$ which makes the triangle commute.

Finally we will show the product in $\mathbf{Rep}_K$ preserves opcartesian morphisms. By Corollary 5.2.3 it remains to show that for group homomorphisms $f: H \rightarrow H'$ and $g: L \rightarrow L'$, a H -representation M and L -representation N , there exists opcartesian arrows θ , θ' , φ , and an isomorphism such that the following diagram commutes.

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\theta \otimes \theta'} & f_! M \otimes g_! N \\ & \searrow \varphi & \cong \downarrow \\ & & (f \otimes g)_!(M \otimes N) \end{array}$$

Taking the choice of cleaving provided by Lemma 5.1.6 then

$$f_! \otimes_K g_!(M \otimes_K N) = (KH'_f \otimes_{KH} M) \otimes_K (KL'_g \otimes_{KL} N)$$

and

$$(f \times g)_!(M \otimes_K N) = K(H' \times L')_{f \times g} \otimes_{K(H \times L)} (M \otimes_K N)$$

are isomorphic by

$$(h' \otimes_{KH} m) \otimes_K (l' \otimes_{KL} n) \mapsto (h', l') \otimes_{K(H \times L)} (m \otimes_K n)$$

which commutes with our choice of cleaving. $\square$

5.2.2 Internal product and internal closure

Earlier we mentioned that a fibration gives rise to fibre categories taken as sub-categories of the domain of our fibration. In the case of a cartesian monoidal fibration these fibre categories inherit a monoidal structure from the base of the fibration. In particular for the case of $\Omega_{\mathbf{Rep}}$ this means that for each group G , the fibre category $G\text{-}\mathbf{Rep}$ is canonically a monoidal category and the restriction and extension functors are monoidal with respect to this product.

For a general cartesian monoidal fibration $\Omega: \mathcal{A} \rightarrow \mathcal{B}$, the fact that $\mathcal{B}$

is cartesian monoidal means we have diagonal maps $\Delta_b: b \rightarrow b \times b$ and projection maps $\pi_b: b \rightarrow \mathbb{1}_B$. With this information [25, Thm. 12.7] tells us that the fibre category over $b \in \mathcal{B}$ is a monoidal category with monoidal product

$$(-) \boxtimes_b (-) := \Delta_b^*((-) \otimes_{\mathcal{A}} (-)),$$

with monoidal unit

$$\mathbb{1}_b := \pi_b^*(\mathbb{1}_{\mathcal{A}}).$$

We call this monoidal structure the **internal product** of the fibration. For Ω_{Rep} and a group G we obtain a product of two G -representations V and W given by

$$\Delta_G K(G \times G) \otimes_{K(G \times G)} (V \otimes_K W)$$

and the monoidal unit with respect to this product is

$$\pi_G K \otimes_K K.$$

For ease of calculation we will instead use the product on $G\text{-Rep}$ defined by

$$V \boxtimes_G W := \Delta_G(V \otimes_K W) \tag{5.7}$$

with monoidal unit

$$\pi_G K \tag{5.8}$$

which is an isomorphic monoidal structure to that above with structure maps obtained by [25, Thm. 12.7].

In particular the fact that given a group homomorphism $f: H \rightarrow G$, the restriction functor f^* is strong monoidal comes from the unique isomorphism factorising the cartesian arrows over $\Delta_G \circ f$ and $f \times f \circ \Delta_H$ and so in fact by using the product (5.7) as our internal product, the strong monoidal structure on f^* is just equality

$$\Delta_H(fV \otimes_K fW) = f(\Delta_G(V \otimes_K W))$$

and hence the restriction functor of a group homomorphism $f: H \rightarrow G$ is in

fact a strict monoidal functor from the fibre over G to the fibre over H . From now on, when we refer to the internal product for $\Omega_{\mathbf{Rep}}$ we will be referring to the product given by (5.7) and (5.8).

Since the tensor product $\otimes_K$ for vector spaces is symmetric it follows that the internal product is also symmetric. Now we have defined the internal product, we can now discuss the next condition we need to use Theorem 5.1.8.

Definition 5.2.5. Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a cartesian monoidal fibration. We say that Ω is **internally closed** if for every fibre category over an object $b \in \mathcal{B}$, the internal product $\boxtimes_b$ is both left and right closed and the restriction functors are all closed monoidal functors.

In particular if $\boxtimes_b$ is symmetric this means we have internal hom functors which we denote by $[a, a']_b$ and for every morphism $f: b \rightarrow b'$ in $\mathcal{B}$, the canonical natural transformation with components

$$f^*([a, a']_{b'}) \rightarrow [f^*(a), f^*(a')]_b$$

is an isomorphism.

Since the internal product in the category of G -representations is given on objects by the tensor product of vector spaces we would expect the internal hom in $G\text{-}\mathbf{Rep}$ to resemble the internal hom in $\mathbf{Vect}_K$ which is given by linear maps.

Lemma 5.2.6. *For a group G , the fibre category $G\text{-}\mathbf{Rep}$ is closed with respect to the internal product. The internal hom of two G -representations $(V, \hat{\rho}_V)$ and $(W, \hat{\rho}_W)$ is given by*

$$[V, W]_G := \text{Lin}(V, W)$$

where $g \in G$ acts on a linear map $\theta: V \rightarrow W$ by

$$\hat{\tau}(g): \theta(-) \mapsto \hat{\rho}_W(g)\theta(\hat{\rho}_V(g^{-1})(-)).$$

Proof. Since $\mathbf{Vect}_K$ is closed monoidal with internal hom given by $\text{Lin}(-, -)$

it is sufficient to prove that given a linear map $\varphi: V \otimes_K W \rightarrow U$, the corresponding linear map under the tensor hom adjunction given by

$$\begin{aligned}\varphi^\sharp: W &\rightarrow [V, U]_G \\ w &\mapsto \varphi(- \otimes_K w)\end{aligned}$$

is an intertwining map. Let $\hat{\rho}_V$, $\hat{\rho}_W$, and $\hat{\rho}_U$ denote the G -actions on V , W , and U respectively then we need to show the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} W & \xrightarrow{\varphi^\sharp} & [V, U]_G \\ \hat{\rho}_W(g) \downarrow & & \downarrow \hat{\tau}(g) \\ W & \xrightarrow{\varphi^\sharp} & [V, U]_G \end{array}$$

The anticlockwise composition of the above diagram sends an element $w \in W$ to the linear map $\varphi(- \otimes_K \hat{\rho}_W(g)w)$ and the clockwise composition sends w to

$$\hat{\rho}_U(g)\varphi((\hat{\rho}_V(g^{-1})(-)) \otimes_K w).$$

Since φ is an intertwining map, this is equal to

$$\varphi((\hat{\rho}_V(g)\hat{\rho}_V(g^{-1})(-)) \otimes_K \hat{\rho}_W(g)w) = \varphi(- \otimes_K \hat{\rho}_W(g)w)$$

and hence the diagram commutes as required. $\square$

Now all that remains to show that Ω_{Rep} is internally closed is to prove that the restriction functors respect the closed structure.

Lemma 5.2.7. *Let $f: H \rightarrow H'$ be a group homomorphism and let V and W be H' -representations. The map*

$$f^*[V, W]_{H'} \rightarrow [f^*V, f^*W]_H \tag{5.9}$$

defined by the strong monoidal structure on f^* is an isomorphism. Hence, Ω_{Rep} is internally closed.

Proof. The map (5.9) is given explicitly by

$$\begin{aligned} f^*[V, W]_{H'} &\xrightarrow{\text{coev}} [f^*V, f^*V \boxtimes_H f^*[V, W]_{H'}] \\ &\xrightarrow{\cong} [f^*V, f^*(V \boxtimes_{H'} [V, W]_{H'})] \\ &\xrightarrow{\text{ev}} [f^*V, f^*W] \end{aligned}$$

The strong monoidal structure for f^* is given by the equality

$$\Delta_H(fV \otimes_K f[V, W]_{H'}) = f(\Delta_{H'}(V \otimes_K [V, W]_{H'}))$$

and hence (5.9) is just coevaluation composed with evaluation and so is an isomorphism (in fact an equality) by the naturality of ev and coev and the triangle identities for the tensor hom adjunction. $\square$

5.2.3 The Beck-Chevalley Condition

The final condition we need to use Theorem 5.1.8 is for Ω_{Rep} to satisfy the weak Beck-Chevalley condition which says the following.

Definition 5.2.8. Let $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ be a bifibration and let

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ g \downarrow & \lrcorner & \downarrow k \\ c & \xrightarrow{h} & d \end{array}$$

be a pullback in $\mathcal{B}$. By lifting this square to $\mathcal{A}$ we obtain an isomorphism $f^*h^* \xrightarrow{\xi} g^*k^*$ and since restriction and extension are adjoint we obtain a morphism

$$g_!f^* \xrightarrow{\eta} g_!f^*k^*k_! \xrightarrow{\xi} g_!g^*h^*k_! \xrightarrow{\epsilon} h^*k_!$$

which we call the **Beck-Chevalley morphism**. If the Beck-Chevalley morphism is an isomorphism we say this pullback square is a **Beck-Chevalley square**. If every pullback in $\mathcal{B}$ is a Beck-Chevalley square then we say that Ω satisfies the **strong Beck-Chevalley condition**. If $\mathcal{B}$ has all products then we say the bifibration $\Omega: \mathcal{A} \rightarrow \mathcal{B}$ satisfies the **weak Beck-Chevalley**

condition if every pullback square where one of the legs (h or k above) is a projection from a product is a Beck-Chevalley square.

For the reader further interested in the Beck-Chevalley condition and related results see [17, Chp. 3]. The importance of the Beck-Chevalley condition for Theorem 5.1.8 can be seen in [25, Chp. 13, 16, 17]. The main idea is that the Beck-Chevalley condition is useful as it allows us to switch the order of extension and restriction functors.

For now we will just be interested in showing the weak Beck-Chevalley condition holds for Ω_{Rep} . We will proceed by describing what the Beck-Chevalley morphism is actually doing for an arbitrary square as this will make it easier to see why the weak Beck-Chevalley condition holds.

Lemma 5.2.9. *Let*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & \lrcorner & \downarrow p \\ C & \xrightarrow{q} & D \end{array} \quad (5.10)$$

*be a pullback square in **Grp**. The Beck-Chevalley morphism for (5.10) is given by*

$$q \otimes_{p \circ f} p: KC_g \otimes_{KA} fKB \otimes_{KB} (-) \rightarrow {}_qKD \otimes_{KD} KD_p \otimes_{KB} (-).$$

Proof. We already know what the unit and counit of the restriction-extension adjunction are from (5.3) and (5.4) and hence (omitting the $\otimes_{KB}(-)$ for readability) we have that the Beck-Chevalley morphism for (5.10) is given by

$$\begin{aligned} KC_g \otimes_{KA} fKB &\xrightarrow{\cong} KC_g \otimes_{KA} fKB \otimes_{KB} KB \otimes_{KB} KB \\ &\xrightarrow{\text{id} \otimes_{KB} p \otimes_p p} KC_g \otimes_{KA} fKB \otimes_{KB} {}_pKD \otimes_{KD} KD_p \\ &\xrightarrow{\cong} KC_g \otimes_{KA} gKC \otimes_{KC} {}_qKD \otimes_{KD} KD_p \\ &\xrightarrow{\text{id} \otimes_g \text{id}} KC \otimes_{KC} KC \otimes_{KC} {}_qKD \otimes_{KD} KD_p \\ &\xrightarrow{\cong} {}_qKD \otimes_{KD} KD_p. \end{aligned}$$

We can check this morphism sends an element $c \otimes_{KA} b$ of $KC_g \otimes_{KA} fKB$ to

by following what happens under each map above as follows,

$$\begin{aligned}
c \otimes_{KA} b &\mapsto c \otimes_{KA} b \otimes_{KB} e \otimes_{KD} e = c \otimes_{KA} e \otimes_{KB} e \otimes_{KB} b \\
&\mapsto c \otimes_{KA} e \otimes_{KB} e \otimes_{KD} p(b) \\
&\mapsto c \otimes_{KA} e \otimes_{KC} e \otimes_{KD} p(b) \\
&\mapsto c \otimes_{KC} e \otimes_{KC} e \otimes_{KD} p(b) = e \otimes_{KC} e \otimes_{KC} q(c) \otimes_{KD} p(b) \\
&\mapsto q(c) \otimes_{KD} p(b).
\end{aligned}$$

Hence, the Beck-Chevalley morphism is given by $q \otimes_{p \circ f} p$ as required. $\square$

It is now straightforward to show that the weak Beck-Chevalley condition holds.

Lemma 5.2.10. *The bifibration Ω_{Rep} satisfies the weak Beck-Chevalley condition.*

Proof. Recall (for instance as a consequence of Lemma 2.12) that a pullback in **Grp** where one of the legs is a projection, is always of the form

$$\begin{array}{ccc}
G \otimes H' & \xrightarrow{f \times \text{id}} & H \otimes H' \\
\pi_{H'} \downarrow & \lrcorner & \downarrow \pi_{H'} \\
G & \xrightarrow{f} & H
\end{array} \tag{5.11}$$

which, by Lemma 5.2.9, has Beck-Chevalley morphism given by

$$f \otimes_{K(G \times H')} \pi_{H'} : KG_{\pi_{H'}} \otimes_{K(G \times H')} f \otimes \text{id} K(H \otimes H') \rightarrow {}_f K H \otimes_{KH} K H_{\pi_{H'}}. \tag{5.12}$$

Notice that an element $g \otimes_{K(G \times H')} (h, h')$ in the domain of (5.12) can be re-written as $e \otimes_{K(G \times H')} (f(g)h, e)$ and so (5.12) has inverse given by

$$h_1 \otimes_{KH} h_2 \mapsto e \otimes_{K(G \times H')} (h_1 h_2, e).$$

We can define an inverse to the Beck-Chevalley morphism for the transpose of (5.11) similarly. $\square$

5.3 Constructing the Fibred Double Category of Representations

Finally we are ready to construct a fibred double category of representations using Theorem 5.1.8.

Corollary 5.3.1. *There is a monoidal fibred double category $\text{Fr}(\Omega_{\text{Rep}})$ with the following data:*

- 0-cells are groups.
- Tight 1-cells are group homomorphisms.
- Loose 1-cells are group representations $(V, \rho, G \times H)$.
- A 2-cell along group homomorphisms f and g is an intertwining map of group representations and the group homomorphism $f \times g$.
- The loose unit of a group G is $K(G \times G)_{\Delta_G} \otimes_{KG} \pi_G K$
- The composition of loose 1-cells $(V, \rho, G \times H) \odot (W, \tau, H \times L)$ is given by

$$K(G \times L)_{\pi_H} \otimes_{K(G \times H \times L)} \Delta_H K(G \times H \times H \times L) \otimes_{K(G \times H \times H \times L)} (V \otimes_K W).$$

- The horizontal composition of following 2-cells

$$\begin{array}{ccc} G & \xrightarrow{V} & H \\ f \downarrow & \Downarrow \varphi & \downarrow g \\ J & \xrightarrow{W} & L \end{array} \quad \begin{array}{ccc} H & \xrightarrow{M} & P \\ g \downarrow & \Downarrow \theta & \downarrow h \\ L & \xrightarrow{N} & Q \end{array}$$

is the map

$$\begin{aligned} & (f \times h) \otimes (f \times g \times g \times h) \otimes (\varphi \otimes \theta): \\ & K(G \times P)_{\pi_H} \otimes_{K(G \times H \times P)} \Delta_H K(G \times H \times H \times P) \otimes_{K(G \times H \times H \times P)} (V \otimes_K M) \\ & \rightarrow \\ & K(J \times Q)_{\pi_L} \otimes_{K(J \times J \times Q)} \Delta_L K(J \times L \times L \times Q) \otimes_{K(G \times H \times H \times L)} (V \otimes_K W). \end{aligned}$$

Chapter 6

Constructing Monoidal Fibred Double Categories of Hopf Algebras and Modules

In the last chapter we introduced fibrations and showed how use them to construct a fibred double category of group representations. Group algebras and group representations are Hopf algebras and modules in the category of vector spaces. In this chapter we will extend this construction to Hopf algebras and modules in an arbitrary closed monoidal category.

6.1 A Fibration for Hopf Algebras

The aim now is to construct a fibration from some category of modules to some category of Hopf algebras such that the conditions required to use Theorem 5.1.8 are satisfied. First, we will need to take Hopf algebras in a symmetric monoidal category $\mathcal{V}$ so that we have products of algebras and modules. Furthermore, we will also need this category $\mathcal{V}$ to be closed so we can use the correspondence between modules and representations described in Chapter 2 (but also so that our fibre categories are monoidal closed). It is important for our category $\mathcal{V}$ to have coequalisers as they are required to define the tensor product of bimodules. Finally if we are defining a fibration

Definition 6.1.1. Let $\mathcal{V}$ be a symmetric closed monoidal category with coequalisers. Let $\text{Hopf}_{\text{cc}}(\mathcal{V})$ denote the category of cocommutative Hopf algebras and Hopf algebra morphisms in $\mathcal{V}$ and let $\mathbf{Mod}_{\mathcal{V}}$ denote the category of left modules in $\mathcal{V}$ over Hopf algebras in $\text{Hopf}_{\text{cc}}(\mathcal{V})$. We define a functor

[illegible]

$$f^* := {}_f H' \otimes_{H'} (-)$$
$$f_! := H'_f \otimes_H (-)$$

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Lemma 6.1.2. *Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be Hopf algebra morphisms, let N be a left A -module and M be a left C -module. Let $(g \circ f, \theta): N \rightarrow M$ and $(g, \varphi): B_f \otimes_A N \rightarrow C \otimes_C M$ be left Hopf algebra module morphisms. If the triangle*

$$\begin{array}{ccc} A \otimes_A N & \xrightarrow{f \otimes_A \text{id}} & B_f \otimes_A N \\ & \searrow (g \circ f) \otimes_{g \circ f} \theta & \downarrow \varphi \\ & & C \otimes_C M \end{array}$$

lying over the triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \circ f & \downarrow g \\ & & C \end{array}$$

commutes, then $\varphi = g \otimes \theta$.

Proof. We will build this result from commuting diagrams. Firstly we use the fact that the unit morphism of a Hopf algebra corresponds to the isomorphism $N \cong A \otimes_A N$ of Lemma 3.3.8, the unit property for Hopf algebra morphisms, as well as functoriality of coequalisers, to show the inside of the following square commutes and hence the outside square commutes.

$$\begin{array}{ccccc} C \otimes_C M & \xrightarrow{\cong} & C \otimes_C C \otimes_C M & & \\ & \swarrow & \uparrow & \searrow & \\ & C \otimes M & \xrightarrow{\text{id} \otimes \eta_C \otimes \text{id}} & C \otimes C \otimes M & \\ & \uparrow g \otimes \theta & & \uparrow g \otimes g \circ f \otimes \theta & \\ B_f \otimes_A N & \xrightarrow{\text{id} \otimes \eta_A \otimes \text{id}} & B \otimes A \otimes N & \xrightarrow{\quad} & B \otimes_A A \otimes_A N \\ & \swarrow & \uparrow & \searrow & \\ & B \otimes N & \xrightarrow{\text{id} \otimes \eta_A \otimes \text{id}} & B \otimes A \otimes N & \\ & \uparrow g \otimes_{g \circ f} \theta & & \uparrow g \otimes_{g \circ f} g \circ f \otimes_{g \circ f} \theta & \end{array}$$

We glue the square above to the following diagram

$$\begin{array}{ccc}
C \otimes_C C \otimes_C M & \xrightarrow{\cong} & C \otimes_C M \\
\uparrow g \otimes g \circ f \otimes \theta & \nwarrow g \otimes \varphi & \uparrow \varphi \\
& B \otimes_A B \otimes_A N & \\
& \downarrow \text{id} \otimes f \otimes \text{id} & \\
& B \otimes_B B \otimes_A N & \\
& \searrow \cong & \\
B \otimes_A A \otimes_A N & \xrightarrow{\cong} & B \otimes_C N
\end{array}$$

where the left triangle commutes by assumption, the top right triangle commutes by the fact that (g, φ) is a module map and because the isomorphisms are induced by the module action. Finally, the bottom triangle commutes by the definition of the right action of B by A and B respectively so gluing our two commuting diagrams together we have $\varphi = g \otimes \theta$ as required. $\square$

We can now show that $\Omega_{\mathbf{Mod}_{\mathcal{V}}} : \mathbf{Mod}_{\mathcal{V}} \rightarrow \mathbf{Hopf}_{\text{cc}}(\mathcal{V})$ is a bifibration.

Lemma 6.1.3. *The functor $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ is a bifibration.*

Proof. First we will show $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ is a fibration by finding a cartesian arrow for each Hopf algebra morphism. Take a Hopf algebra morphism $f: H \rightarrow H'$ and module (M, ρ, H') . Let $f^*(M) := {}_f H' \otimes_{H'} M$ and then define a cartesian arrow by the composite

$$f^*(M) = {}_f H' \otimes_{H'} M \xrightarrow{(f, \text{id})} H' \otimes_{H'} M \xrightarrow{\cong} M$$

with the isomorphism given by Lemma 3.3.8. It remains to show this map is indeed cartesian. Certainly if $g: L \rightarrow H$ is a Hopf algebra map and $(f \circ g, \theta): N \rightarrow M$ is a module map then

$$N \cong L \otimes_L N \xrightarrow{(g, (f \circ g) \otimes f \circ g \theta)} {}_f H' \otimes_{H'} M$$

does factorise θ through our cartesian arrow but it is simpler to show the

uniqueness property of a cartesian factorisation by showing (f, id) is cartesian and using the fact that isomorphisms are always cartesian to ignore the composite with $H' \otimes_{H'} M \cong M$. This is to say, for any module (N, τ, L) and module morphism $(f \circ g, \theta): N \rightarrow H' \otimes_{H'} M$, there exists a unique map (g, θ') such that the following diagram commutes.

$$\begin{array}{ccc} N & & \\ \downarrow (g, \theta') & \searrow (f \circ g, \theta) & \\ {}_f H' \otimes_{H'} M & \xrightarrow{(f, \text{id})} & H' \otimes_{H'} M \end{array}$$

By definition, the only possible θ' which commutes in this diagram is $\theta' = \theta$ and hence (f, id) is cartesian and $\Omega_{\mathbf{Mod}_V}$ is a fibration.

Finally we show that $\Omega_{\mathbf{Mod}_V}$ is an opfibration by showing every Hopf algebra morphism lifts to an opcartesian arrow. Take a Hopf algebra morphism $f: H \rightarrow H'$ and module (N, τ, H) . Let $f_!(N) := H'_f \otimes_H N$ and then define an opcartesian arrow by the composite

$$N \xrightarrow{\cong} H \otimes_H N \xrightarrow{(f, f \otimes_H \text{id})} H'_f \otimes_H N = f_!(N)$$

with the isomorphism given by Lemma 3.3.8. It remains to show this map is indeed opcartesian which is to say, for any module (M, ρ, L) and module morphism $(g \circ f, \varphi): N \rightarrow M$, there exists a unique map (g, φ') such that the following diagram commutes.

$$\begin{array}{ccc} N & \xrightarrow{\cong} H \otimes_H N & \xrightarrow{(f, f \otimes_H \text{id})} H'_f \otimes_H N \\ & \searrow (g \circ f, \varphi) & \downarrow (g, \varphi') \\ & & M \end{array}$$

A map (g, φ') which makes this triangle commute is given by the composition

$$H'_f \otimes_H N \xrightarrow{(g, g) \otimes_{g \circ f} (g \circ f, \varphi)} L \otimes_L M \cong M$$

and uniqueness follows immediately from Lemma 6.1.2. $\square$

Now we have restriction and extension functors we know we have an

adjunction $f_! \vdash f^*$. As in the previous chapter we can calculate the unit and counit of this adjunction by unique factorisation. In particular let $f: H \rightarrow H'$ be a Hopf algebra morphism then the component of the unit for a H -module N is calculated with the unique factorisation provided in Lemma 6.1.3 as

$$N \xrightarrow{\cong} H \otimes_H N \xrightarrow{(\text{id}, f \otimes_f ((f \otimes_H \text{id}) \circ \cong))} {}_f H' \otimes_{H'} H'_f \otimes_H N$$

which can be written more simply as

$$N \xrightarrow{\cong} H \otimes_H N \cong H \otimes_H H \otimes_H N \xrightarrow{f \otimes_f f \otimes_H \text{id}} {}_f H' \otimes_{H'} H'_f \otimes_H N. \quad (6.1)$$

Similarly, for a H' -module M , the component of the counit $f_! f^* \Rightarrow \text{id}$ at M is given by

$$H'_f \otimes_H {}_f H' \otimes_{H'} M \xrightarrow{\text{id} \otimes_f \text{id} \otimes_{H'} \text{id}} H' \otimes_{H'} H' \otimes_{H'} M \cong M. \quad (6.2)$$

6.2 Monoidal Structure and the Beck-Chevalley Condition

6.2.1 Cartesian Monoidal Structure on $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$

The next step for our construction is to provide a cartesian monoidal structure for our fibration. We have already established that $\text{Hopf}_{\text{cc}}(\mathcal{V})$ is a cartesian monoidal category by Corollary 2.2.11 and that $\mathbf{Mod}_{\mathcal{V}}$ is a monoidal category by Lemma 3.3.1. By definition of the product in $\mathbf{Mod}_{\mathcal{V}}$ we immediately get the fact that $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ is a strict monoidal functor so to show that $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ is a cartesian monoidal bifibration it remains to show that it preserves cartesian and opcartesian arrows.

Lemma 6.2.1. *The product in $\mathbf{Mod}_{\mathcal{V}}$ preserves cartesian and opcartesian arrows. In particular $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ is a cartesian monoidal bifibration.*

Proof. By Lemma 5.2.2 we only need to show that there exists isomorphisms $(f^* \otimes g^*)(M \otimes N) \cong (f \otimes g)^*(M \otimes N)$ and $(f_! \otimes g_!)(M \otimes N) \cong (f \otimes g)_!(M \otimes N)$

such that the following diagrams commute where we use cart and opcart respectively to denote the (op)cartesian arrows above f and g .

$$\begin{array}{ccc}
f^*M \otimes g^*N & & \\
\cong \downarrow & \searrow \text{cart} \otimes \text{cart} & \\
(f \otimes g)^*(M \otimes N) & \xrightarrow{\text{cart}} & M \otimes N, \\
\\
M \otimes N & \xrightarrow{\text{opcart} \otimes \text{opcart}} & f_!M \otimes g_!N \\
& \searrow \text{opcart} & \downarrow \cong \\
& & (f \otimes g)_!(M \otimes N).
\end{array}$$

In both cases the isomorphism which makes these triangles commute is the tensor product interchange isomorphism given by Lemma 3.3.7. The reason this isomorphism makes each diagram commute is because the interchange isomorphism is defined by factoring symmetry through the coequaliser defining the tensor product and hence by naturality of symmetry the diagrams above both commute. $\square$

6.2.2 The Beck-Chevalley Condition for $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$

As in the previous chapter, the next step to being able to use $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ to construct a double category is to check the Beck-Chevalley condition. Much like how we used Lemma 5.2.9 in the last chapter, it will be easier to show the Beck-Chevalley condition if we can find a convenient way to describe the Beck-Chevalley morphism.

Lemma 6.2.2. *Let*

$$\begin{array}{ccc}
A & \xrightarrow{f} & C \\
g \downarrow & & \downarrow k \\
B & \xrightarrow{h} & D
\end{array}$$

be a commuting square in the category $\text{Hopf}_{\text{cc}}(\mathcal{V})$ of cocommutative Hopf

algebras in $\mathcal{V}$. The Beck-Chevalley map

$$B_g \otimes_A fC \otimes_C (-) = g_! f^* \xrightarrow{\eta} g_! f^* k^* k_! \xrightarrow{\cong} g_! g^* h^* k_! \xrightarrow{\epsilon} h^* k_! = {}_h D \otimes_D D_k \otimes_C (-)$$

is given by the map of B - C -bimodules

$$h \otimes_{k \circ f} k: B_g \otimes_A fC \Rightarrow {}_h D \otimes_D D_k.$$

Proof. By functoriality of $\otimes_C$ we can omit the $\otimes_C(-)$ and instead just consider the module map $B_g \otimes_A fC \rightarrow {}_h D \otimes_D D_k$. We will prove the Beck-Chevalley morphism given in Definition 5.2.8 is given by $h \otimes_{k \circ f} k$ by composing both morphisms with the isomorphism ${}_h D \otimes_D D_k \cong {}_h D_k$ and showing the map $B \otimes C \rightarrow D$ which induces the Beck-Chevalley map via the coequaliser map defining the tensor product $B_g \otimes_A fC$ is given by $\mu_D \circ (h \otimes k)$.

First we will write out the map $B \otimes C \rightarrow D$ which induces the Beck-Chevalley map under the coequaliser maps. We described the unit for the restriction-extension adjunction in Equation (6.1) as the map

$$B_g \otimes_A fC \xrightarrow{\eta} B_g \otimes_A fC \otimes_C kD \otimes_D D_k$$

which is induced by unique factorisation of the coequaliser of

$$B \otimes C \xrightarrow{\eta_C} B \otimes C \otimes C \xrightarrow{\eta_C} B \otimes C \otimes C \otimes C \xrightarrow{\text{id} \otimes k \otimes k} B \otimes C \otimes D \otimes D.$$

Similarly, Equation (6.2) tells us that the counit

$$B_g \otimes_A gB \otimes_B {}_h D \otimes_D D_k \xrightarrow{\epsilon} {}_h D \otimes_D D_k$$

is induced by

$$B \otimes B \otimes D \otimes D \xrightarrow{\mu_B} B \otimes D \otimes D \xrightarrow{h} D \otimes D \otimes D \xrightarrow{\mu_D} D \otimes D.$$

The isomorphism ${}_h D \otimes_D D_k \cong {}_h D_k$ is induced by μ_D . Finally the isomorphism

$$B_g \otimes_A fC \otimes_C kD \otimes_D D_k \cong B_g \otimes_A gB \otimes_B {}_h D \otimes_D D_k$$

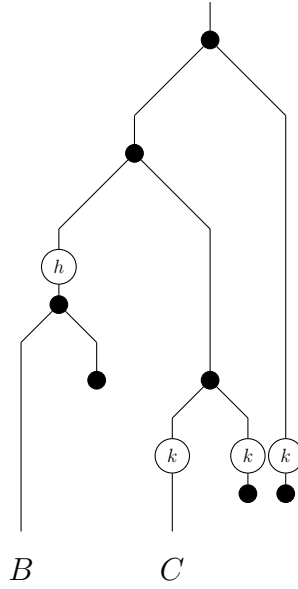
is given by the composition $f^*k^* \cong (k \circ f)^* = (h \circ g)^* \cong g^*h^*$ which is given by the tensor product of the modules

$$\begin{aligned} B_g \otimes_A fC \otimes_C kD \otimes_D D_k &\cong B_g \otimes_A k \circ f D \otimes_D D_k \\ &= B_g \otimes_A h \circ g D \otimes_D D_k \cong B_g \otimes_A gB \otimes_B hD \otimes_D D_k \end{aligned}$$

which is a map induced by

$$B \otimes C \otimes D \otimes D \xrightarrow{\text{id} \otimes k \otimes \text{id}} B \otimes D \otimes D \otimes D \xrightarrow{\mu_D} B \otimes D \otimes D \xrightarrow{\eta_B} B \otimes B \otimes D \otimes D.$$

Written out in string diagrams, the full composition of the maps above which induce the map $B_g \otimes_A fC \rightarrow {}_hD_k$ is given by



which is equivalent to $\mu_D \circ (h \otimes k)$ by the unit condition of an algebra as required. $\square$

We are nearly ready to show the Beck-Chevalley condition holds but first we will need the following bookkeeping results.

Lemma 6.2.3. *Let H and H' be Hopf algebras and let M be a left H -module. The left $H \otimes H'$ -modules ${}_{\text{id} \otimes \epsilon_{H'}}M$ and $M \otimes ({}_{\epsilon_{H'}}\mathbb{1})$ are isomorphic.*

Similarly, let N be a right H -module. The right $H \otimes H'$ -modules $N_{\text{id} \otimes \epsilon_{H'}}$ and $N \otimes \mathbb{1}_{\epsilon_{H'}}$ are isomorphic.

Proof. The isomorphism $\text{id} \otimes \epsilon_{H'} M \cong M \otimes (\epsilon_{H'} \mathbb{1})$ is given by the right unitor which satisfies the commuting diagrams for modules by naturality of symmetry with respect to the counit for H' with the result for right modules proven similarly using the left unitor. $\square$

Since we could only prove the weak Beck-Chevalley condition in the previous chapter we would expect the same to be true in the case of Hopf algebras. We need to show that pullbacks along projections are Beck-Chevalley squares. Recall by Lemma 2.2.9 that pullbacks along projections in $\text{Hopf}_{\text{cc}}(\mathcal{V})$ are of the form

$$\begin{array}{ccc} G \otimes H' & \xrightarrow{f \otimes \text{id}} & H \otimes H' \\ \text{id} \otimes \epsilon_{H'} \downarrow & \lrcorner & \downarrow \text{id} \otimes \epsilon_{H'} \\ G & \xrightarrow{f} & H \end{array}$$

for some Hopf algebra morphism f . We use this to show the weak Beck-Chevalley condition for $\Omega_{\text{Mod}_{\mathcal{V}}}$ as follows.

Lemma 6.2.4. *The bifibration $\Omega_{\text{Mod}_{\mathcal{V}}}$ is weakly Beck-Chevalley. That is, all pullbacks in $\text{Hopf}_{\text{cc}}(\mathcal{V})$ of the form*

$$\begin{array}{ccc} G \otimes H' & \xrightarrow{\text{id} \otimes \epsilon_{H'}} & G \\ f \otimes \text{id} \downarrow & \lrcorner & \downarrow f \\ H \otimes H' & \xrightarrow{\text{id} \otimes \epsilon_{H'}} & H \end{array} \quad (6.3)$$

or

$$\begin{array}{ccc} G \otimes H' & \xrightarrow{f \otimes \text{id}} & H \otimes H' \\ \text{id} \otimes \epsilon_{H'} \downarrow & \lrcorner & \downarrow \text{id} \otimes \epsilon_{H'} \\ G & \xrightarrow{f} & H \end{array} \quad (6.4)$$

are Beck-Chevalley squares as given by Definition 5.2.8.

Proof. We will show both (6.3) and (6.4) are Beck-Chevalley squares by using Lemma 6.2.2 to describe the Beck-Chevalley transformation and verifying that this is an isomorphism.

First, for (6.3), Lemma 6.2.2 tells us the Beck-Chevalley map is given by

$$(\text{id} \otimes \epsilon_{H'}) \otimes_{f \otimes \epsilon_{H'}} f: (H \otimes H')_{f \otimes \text{id}} \otimes_{G \otimes H'} \text{id} \otimes \epsilon_{H'} G \rightarrow \text{id} \otimes \epsilon_{H'} H \otimes_H H_f$$

which is defined by factoring $\text{id} \otimes \epsilon_{H'} \otimes f$ through the coequaliser for the tensor product. Post composing the Beck-Chevalley morphism with $\text{id} \otimes \epsilon_{H'} H \otimes_H H_f \cong \text{id} \otimes \epsilon_{H'} H_f$ we obtain a map which factors $\mu_H \circ (\text{id} \otimes \epsilon_{H'} \otimes f)$ through the coequaliser for the tensor product and is an isomorphism if and only if the Beck-Chevalley morphism is. Notice that $\mu_H \circ (\text{id} \otimes \epsilon_{H'} \otimes f)$ is the same as the composition

$$\begin{aligned} H \otimes H' \otimes G &\cong H \otimes H' \otimes G \otimes \mathbb{1} \xrightarrow{\text{id} \otimes \sigma \otimes \text{id}} H \otimes G \otimes H' \otimes \mathbb{1} \\ &\xrightarrow{\text{id} \otimes \epsilon_{H'}} H \otimes G \otimes \mathbb{1} \xrightarrow{\text{id} \otimes f} H \otimes H \otimes \mathbb{1} \\ &\xrightarrow{\mu_H} H \otimes \mathbb{1} \cong H. \end{aligned}$$

Recall that by Lemma 6.2.3 the unitor lifts to an isomorphism $\text{id} \otimes \epsilon_{H'} G \cong G \otimes_{\epsilon_{H'}} \mathbb{1}$ and by Lemma 3.3.7 we obtain interchange for the tensor product by lifting $\text{id} \otimes \sigma \otimes \text{id}$. Since $\epsilon_{H'}$ and $\mu_H \circ (\text{id} \otimes f)$ define the action on $\epsilon_{H'} \mathbb{1}$ and H_f respectively these lift to isomorphisms too by Lemma 3.3.9. Putting this together we lift the composition above to obtain

$$\begin{aligned} H_f \otimes H' \otimes_{G \otimes H'} \text{id} \otimes \epsilon_{H'} G &\cong H_f \otimes H' \otimes_{G \otimes H'} G \otimes_{\epsilon_{H'}} \mathbb{1} \\ &\cong H_f \otimes_G G \otimes_{H'} \otimes_{H'} \epsilon_{H'} \mathbb{1} \\ &\cong H_f \otimes_G G \otimes_{\epsilon_{H'}} \mathbb{1} \cong H_f \otimes_{\epsilon_{H'}} \mathbb{1} \\ &\cong \text{id} \otimes \epsilon_{H'} H_f. \end{aligned}$$

Since $(H \otimes H')_{f \otimes \text{id}} = H_f \otimes H'$, this isomorphism lifts a map equal to $\mu_H \circ (\text{id} \otimes \epsilon_{H'} \otimes f)$ and hence (6.3) is a Beck-Chevalley square as required.

We will show (6.4) is Beck-Chevalley by a similar argument. The Beck-Chevalley morphism for (6.4) given by Lemma 6.2.2 is

$$f \otimes_{f \otimes \epsilon_{H'}} (\text{id} \otimes \epsilon_{H'}): G_{\text{id} \otimes \epsilon_{H'}} \otimes_{G \otimes H'} f \otimes \text{id} (H \otimes H') \rightarrow f H \otimes H_{\text{id} \otimes \epsilon_{H'}}$$

given by lifting $f \otimes \text{id} \otimes \epsilon_{H'}$ through the coequaliser. Composing this with μ_H as before gives a map equal to the composition

$$\begin{aligned} G \otimes H \otimes H &\cong G \otimes \mathbb{1} \otimes H \otimes H' \xrightarrow{\text{id} \otimes \sigma \otimes \text{id}} G \otimes H \otimes \mathbb{1} \otimes H' \\ &\xrightarrow{\text{id} \otimes \epsilon_{H'}} G \otimes H \otimes \mathbb{1} \xrightarrow{f \otimes \text{id}} H \otimes H \otimes \mathbb{1} \xrightarrow{\mu_H} H \otimes \mathbb{1} \\ &\cong H. \end{aligned}$$

For the same reasons as before, this composition lifts to the following chain of isomorphisms

$$\begin{aligned} G_{\text{id} \otimes \epsilon_{H'}} \otimes_{G \otimes H'} fH \otimes H' &\cong G \otimes \mathbb{1}_{\epsilon_{H'}} \otimes_{G \otimes H'} fH \otimes H' \\ &\cong G \otimes_G fH \otimes \mathbb{1}_{\epsilon_{H'}} \otimes_{H'} H' \cong G \otimes_G fH \otimes \mathbb{1}_{\epsilon_{H'}} \\ &\cong fH \otimes \mathbb{1}_{\epsilon_{H'}} \\ &\cong fH_{\text{id} \otimes \epsilon_{H'}} \end{aligned}$$

which is the same as the Beck-Chevalley morphism for (6.4) composed with $fH \otimes H_{\text{id} \otimes \epsilon_{H'}} \cong fH_{\text{id} \otimes \epsilon_{H'}}$ and hence (6.4) is a Beck-Chevalley square. $\square$

6.2.3 Internal Product and Closure of $\Omega_{\mathbf{Mod}_V}$

The final ingredient necessary to construct our double category from $\Omega_{\mathbf{Mod}_V}$ is a description of an internal product and to show that our fibration is internally closed with respect to this product. As in the previous chapter, [25, Thm 12.7] provides a definition for the internal product for each fibre category of a fibration. For $\Omega_{\mathbf{Mod}_V}$, Shulman's definition tells us that, over a Hopf algebra H , the fibre category $H\text{-}\mathbf{Mod}$ is monoidal with the product of two modules (M, ρ) and (N, τ) given by

$$\Delta_H(H \otimes H) \otimes_{H \otimes H} (M \otimes N)$$

with unit

$$\epsilon_H \mathbb{1} \otimes \mathbb{1}.$$

As in the previous chapter, it will be easier instead to define the internal product for $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ as the isomorphic product

$$(M, \rho, H) \boxtimes_H (N, \tau, H) := (M \otimes N, (\rho \otimes \tau) \circ \sigma \circ \Delta_H, H) \quad (6.5)$$

and the monoidal unit as

$$(\mathbb{1}, \epsilon_H, H). \quad (6.6)$$

As before the structure maps which make $H\text{-}\mathbf{Mod}$ a monoidal category are given by the proof of Theorem 12.7 in Shulman's paper [25, Thm 12.7] composed with the isomorphisms from Lemma 3.3.9 which make the above definitions equivalent. By the monoidal Grothendieck construction we immediately have the following result

Corollary 6.2.5. *The restriction and extension functors for $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ are respectively strong and oplax monoidal with respect to the internal product (6.5).*

Another similarity to the previous chapter is that the way we have defined the internal product means that since we require $\mathcal{V}$ to be a symmetric monoidal then since the internal product is the same on objects as the product in $\mathcal{V}$ and we are only dealing with cocommutative Hopf algebras, then the internal product is also symmetric. This means that when we now prove that $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ is internally closed then we know that the left and right internal homs are isomorphic. Also, since the internal product is the same as that in $\mathcal{V}$ on objects, the internal hom in $\mathcal{V}$ provides an obvious candidate for the internal hom in the fibre categories $H\text{-}\mathbf{Mod}$.

Lemma 6.2.6. *The fibration $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ is internally closed with internal hom of modules (M, ρ) and (N, τ) given by*

$$[M, N]_H := \underline{\mathcal{V}}(M, N)$$

where the H action is defined by

$$\begin{aligned} H \otimes [M, N]_H &\xrightarrow{\Delta_H} H \otimes H \otimes [M, N]_H \\ &\xrightarrow{\text{coev}} [M, M \otimes H \otimes H \otimes [M, N]_H]_H \end{aligned}$$

$$\begin{aligned}
& \xrightarrow{\sigma} [M, H \otimes H \otimes M \otimes [M, N]_H]_H \\
& \xrightarrow{\chi_H} [M, H \otimes H^{\text{op}} \otimes M \otimes [M, N]_H]_H \\
& \xrightarrow{\rho} [M, H \otimes M \otimes [M, N]_H]_H \\
& \xrightarrow{\text{ev}} [M, H \otimes N]_H \\
& \xrightarrow{\tau} [M, N]_H.
\end{aligned}$$

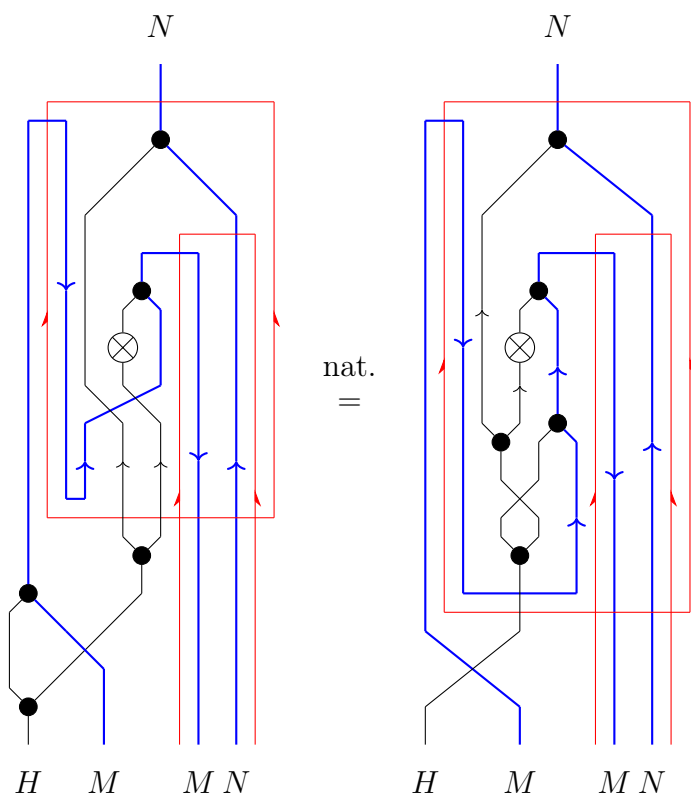
That is to say, the fibre categories $H\text{-}\mathbf{Mod}$ are closed for each H and the restriction functors f^* are closed functors.

Proof. First we will show that each fibre category $H\text{-}\mathbf{Mod}$ is closed. This is to say that for all modules M , N , and P there is an isomorphism of sets

$$H\text{-}\mathbf{Mod}(M \boxtimes_H N, P) \cong H\text{-}\mathbf{Mod}(N, [M, P]_H).$$

As objects of $\mathcal{V}$ we have that $M \boxtimes_H N = M \otimes N$ and $[M, P]_H = \underline{\mathcal{V}}(M, N)$ so given a map $(\varphi, \text{id}): M \boxtimes_H N \rightarrow P$ we obtain a map $(\theta, \text{id}): N \rightarrow [M, P]_H$ by precomposition with the coevaluation map. Similarly we can go from θ to φ by postcomposition with evaluation. Therefore, it is sufficient to show that evaluation and coevaluation are H -module maps. First we show that

evaluation is a H -module morphism with the following string diagram proof.



(3.1)

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N

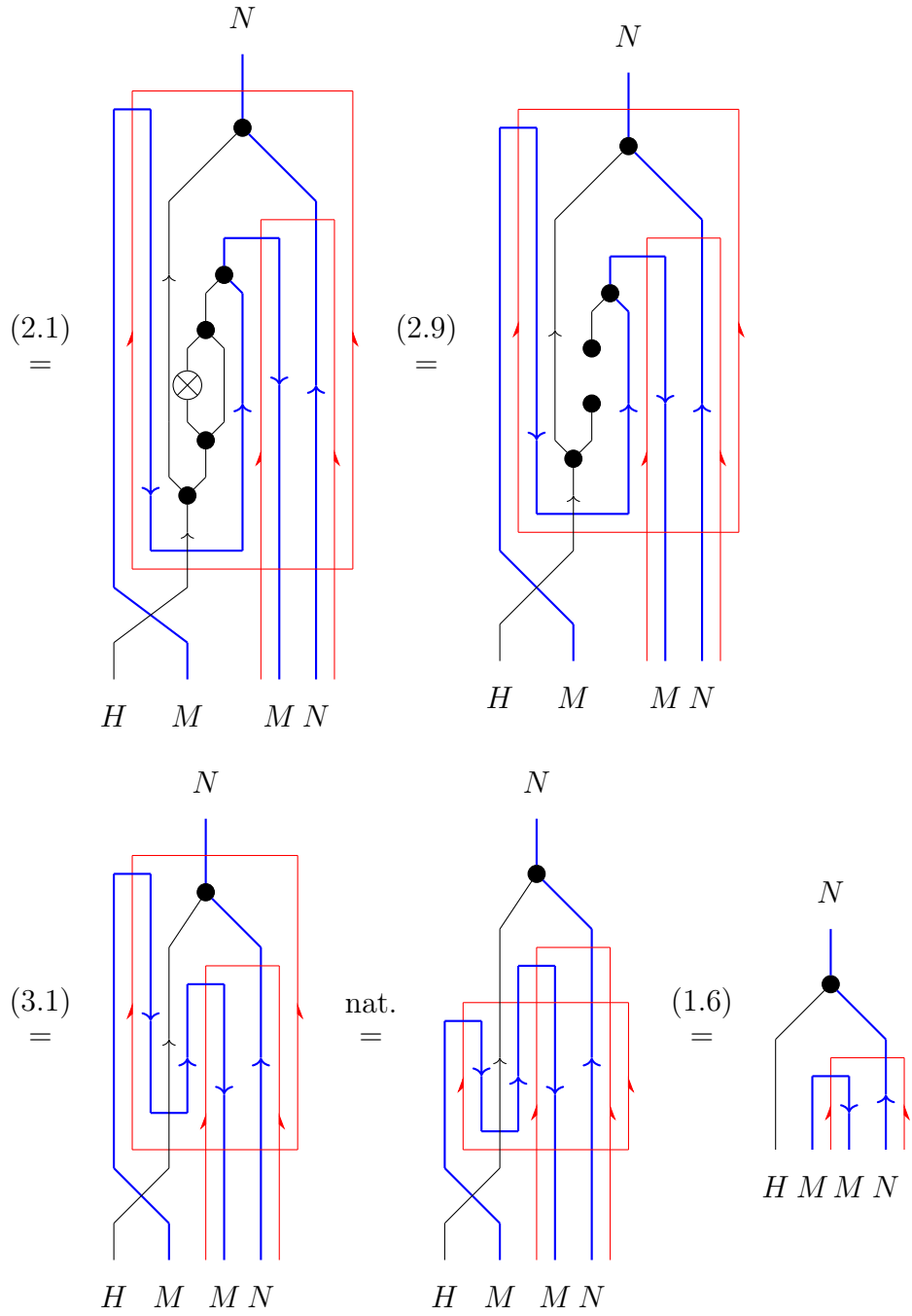
H M MN

Cocomm.

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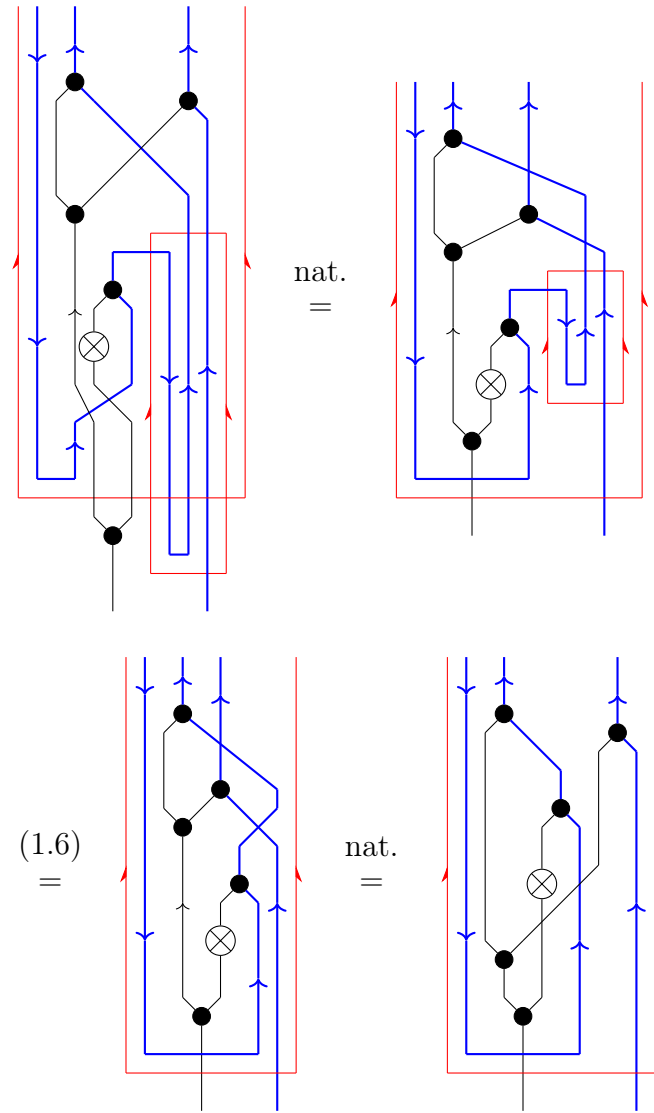
N

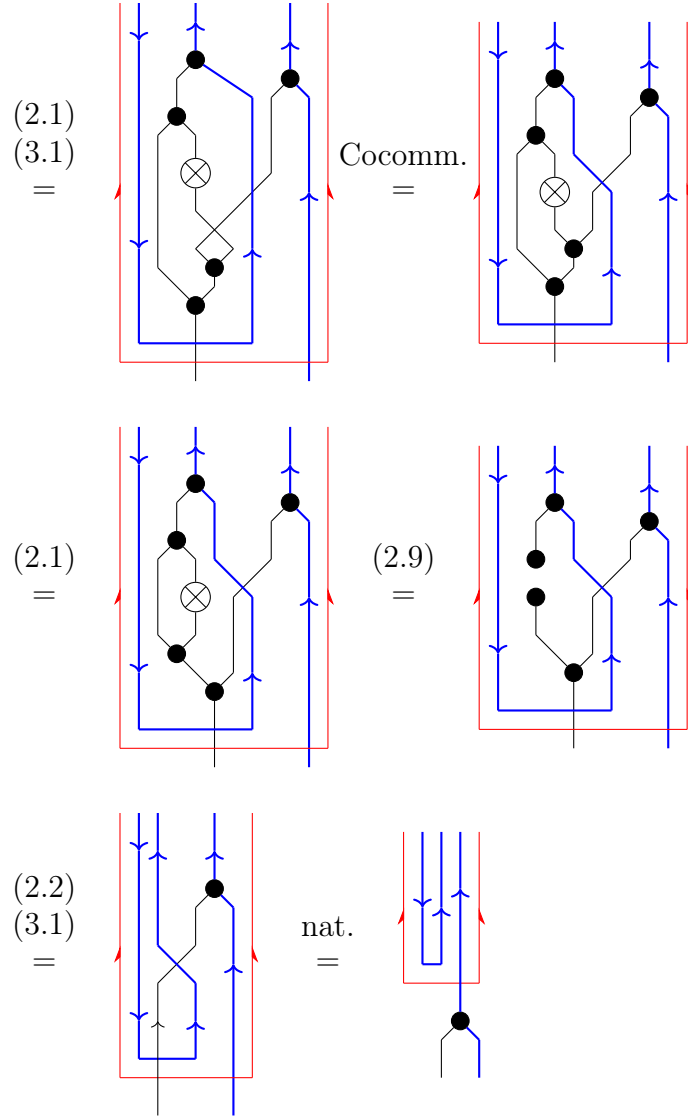
H M MN



Next, we show that coevaluation is a H -module morphism with string dia-

grams as follows.





Finally, it remains to show that the restriction functors are closed. That is to say that for a Hopf algebra map $f: H \rightarrow H'$, the map

$$f^*[M, N]_{H'} \rightarrow [f^*M, f^*N]_H \quad (6.7)$$

defined via the strong monoidal structure on f^* is an isomorphism. The strong monoidal structure on f^* is given by unique factorisation which defines

(6.7) as the composite

$$\begin{aligned}
{}_f H' \otimes_{H'} [M, N]_{H'} &\xrightarrow{\text{coev}} [{}_f H' \otimes_{H'} M, ({}_f H' \otimes_{H'} M) \boxtimes_{H'} ({}_f H' \otimes'_H [M, N]_{H'})]_H \\
&\cong [{}_f H' \otimes_{H'} M, {}_f M \boxtimes_{H'} {}_f [M, N]_{H'}]_H \\
&= [{}_f H' \otimes_{H'} M, {}_f (M \boxtimes_{H'} [M, N]_{H'})]_H \\
&\cong [{}_f H' \otimes_{H'} M, {}_f H' \otimes_{H'} (M \boxtimes_{H'} [M, N]_{H'})]_H \\
&\xrightarrow{\text{ev}} [{}_f H' \otimes_{H'} M, {}_f H' \otimes_{H'} N]_H.
\end{aligned}$$

Recall that a Hopf algebra morphism f commutes with Δ_H by definition and commutes with the antipode by Lemma 2.2.8 and hence ${}_f [M, N]_{H'} = [{}_f M, {}_f N]_H$. We use this fact and naturality of (co)evaluation to obtain the following commuting diagram

$$\begin{array}{ccccc}
{}_f^* [M, N]_{H'} & \xrightarrow{\text{coev}} & [{}_f^* M, ({}_f^* M) \boxtimes_{H'} ({}_f^* [M, N]_{H'})]_H & \xrightarrow{\cong} & [{}_f^* M, {}_f M \boxtimes_{H'} {}_f [M, N]_{H'}]_H \\
\cong \downarrow & & \downarrow \cong & & \downarrow = \\
{}_f [M, N]_{H'} & \xrightarrow{\text{coev}} & [{}_f M, {}_f M \boxtimes_{H'} {}_f [M, N]_{H'}]_H & \xrightarrow{=} & [{}_f M, {}_f (M \boxtimes_{H'} [M, N]_{H'})]_H \xrightarrow{\cong} [{}_f^* M, {}_f (M \boxtimes_{H'} [M, N]_{H'})]_H \\
= \downarrow & & \downarrow = & & \downarrow \cong \\
[{}_f M, {}_f N]_H & \xrightarrow{\text{coev}} & [{}_f M, {}_f M \boxtimes_{H'} [{}_f M, {}_f N]_{H'}]_H & \xrightarrow{\text{ev}} & [{}_f M, {}_f N]_H \\
\cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
[{}_f^* M, {}_f^* N]_H & \xrightarrow{\text{coev}} & [{}_f^* M, {}_f^* M \boxtimes_{H'} [{}_f^* M, {}_f^* N]_{H'}]_H & \xrightarrow{\text{ev}} & [{}_f^* M, {}_f^* N]_H \xrightarrow{=} [{}_f^* M, {}_f^* N]_H
\end{array}$$

where (6.7) is given by the clockwise composition and the anticlockwise composition is an isomorphism by the triangle identities for (co)evaluation. $\square$

6.3 Constructing Fibred Double Categories of Bimodules

Finally we are ready to construct a monoidal fibred double category using Theorem 5.1.8 and $\Omega_{\mathbf{Mod}_V}$. We will first provide a description of the double category constructed by Theorem 5.1.8 and then we will spend the rest of the chapter proving this double category is isomorphic to the double category $\mathbb{BiMod}_V$ since the latter has a nicer description, and proving this equivalence provides a way to understand exactly what the data of the constructed double category is. Before we give the description of our double category it is worth

being reminded the conditions placed on $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ and why they are needed for the following result. First we need our monoidal category $\mathcal{V}$ to be closed symmetric monoidal for almost all of the required definitions. We need $\mathcal{V}$ to have pullbacks so that we can define the tensor product over a Hopf algebra. Finally, we need our category $\text{Hopf}_{\text{cc}}(\mathcal{V})$ of Hopf algebras in $\mathcal{V}$ to be cocommutative so that $\text{Hopf}_{\text{cc}}(\mathcal{V})$ is a cartesian monoidal category, so that the antipode χ of any Hopf algebra is a Hopf algebra morphism (and in fact an involution), and finally we will need cocommutativity to allow us to define the opposite Hopf algebra and hence obtain results such as the fact that $\mathbf{Mod}\text{-}H \cong H^{\text{op}}\text{-}\mathbf{Mod} \cong [BH, \underline{\mathcal{V}}]$. With that out of the way we shall describe the double category constructed from $\Omega_{\mathbf{Mod}_{\mathcal{V}}}$ as follows.

Corollary 6.3.1. *There is a monoidal fibred double category $\mathbb{F}\mathbf{r}(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$ with the following data:*

- 0-cells are cocommutative Hopf algebras.
- Tight 1-cells are Hopf algebra morphisms.
- Loose 1-cells $H \rightarrow H'$ are left $H \otimes H'$ -modules.
- 2-cells are module morphisms of the form $(f \otimes g, \theta)$ for Hopf algebra morphisms f and g .
- The loose unit for a Hopf algebra H is $(H \otimes H)_{\Delta_H} \otimes_H \epsilon_H \mathbb{1} \otimes_{\mathbb{1}} \mathbb{1} \cong (H \otimes H)_{\Delta_H} \otimes_H \epsilon_H \mathbb{1}$.
- The composition of loose 1-cells $(H \otimes G, M) \odot (G \otimes D, N)$ is given by

$$(H \otimes D)_{\epsilon_G} \otimes_{H \otimes G \otimes D} \Delta_G (H \otimes G \otimes G \otimes D) \otimes_{H \otimes G \otimes G \otimes D} (M \otimes N).$$

- The horizontal composition of following 2-cells

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ f \downarrow & \Downarrow \varphi & \downarrow g \\ C & \xrightarrow{N} & D \end{array} \quad \begin{array}{ccc} B & \xrightarrow{P} & G \\ g \downarrow & \Downarrow \theta & \downarrow h \\ D & \xrightarrow{Q} & H \end{array}$$

is the map

$$\begin{aligned}
& (f \otimes h) \otimes (f \otimes g \otimes g \otimes h) \otimes (\varphi \otimes \theta): \\
& (A \otimes G)_{\epsilon_B} \otimes_{A \otimes B \otimes G} \Delta_B (A \otimes B \otimes B \otimes G) \otimes_{A \otimes B \otimes B \otimes G} (M \otimes P) \\
& \rightarrow \\
& (C \otimes H)_{\epsilon_D} \otimes_{C \otimes D \otimes H} \Delta_D (C \otimes D \otimes D \otimes H) \otimes_{C \otimes D \otimes D \otimes H} (N \otimes Q).
\end{aligned}$$

- The composition interchange isomorphism is given by σ^* where σ is the symmetry morphism required to swap the Hopf algebras written in the front of loose 1-cell composition with the tensor product.

To show $\mathbb{F}r(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$ is equivalent to $\mathbf{BiMod}_{\mathcal{V}}$ we will need to construct an isomorphism between its units and composition. First we will provide an isomorphism for its unit.

Lemma 6.3.2. *Let H be a Hopf algebra in a closed symmetric monoidal category $\mathcal{V}$. The left $H \otimes H$ -module $(H \otimes H)_{\Delta_H} \otimes_H \epsilon_H \mathbb{1} \otimes_1 \mathbb{1}$ is isomorphic to H as a $H \otimes H$ -module which we denote H_{χ_H} with action given by*

$$H \otimes H \otimes H \xrightarrow{\text{id} \otimes \chi \otimes \text{id}} H \otimes H \otimes H \xrightarrow{\text{id} \otimes \sigma} H \otimes H \otimes H \xrightarrow{\mu \circ \mu} H.$$

Proof. We will define an isomorphism

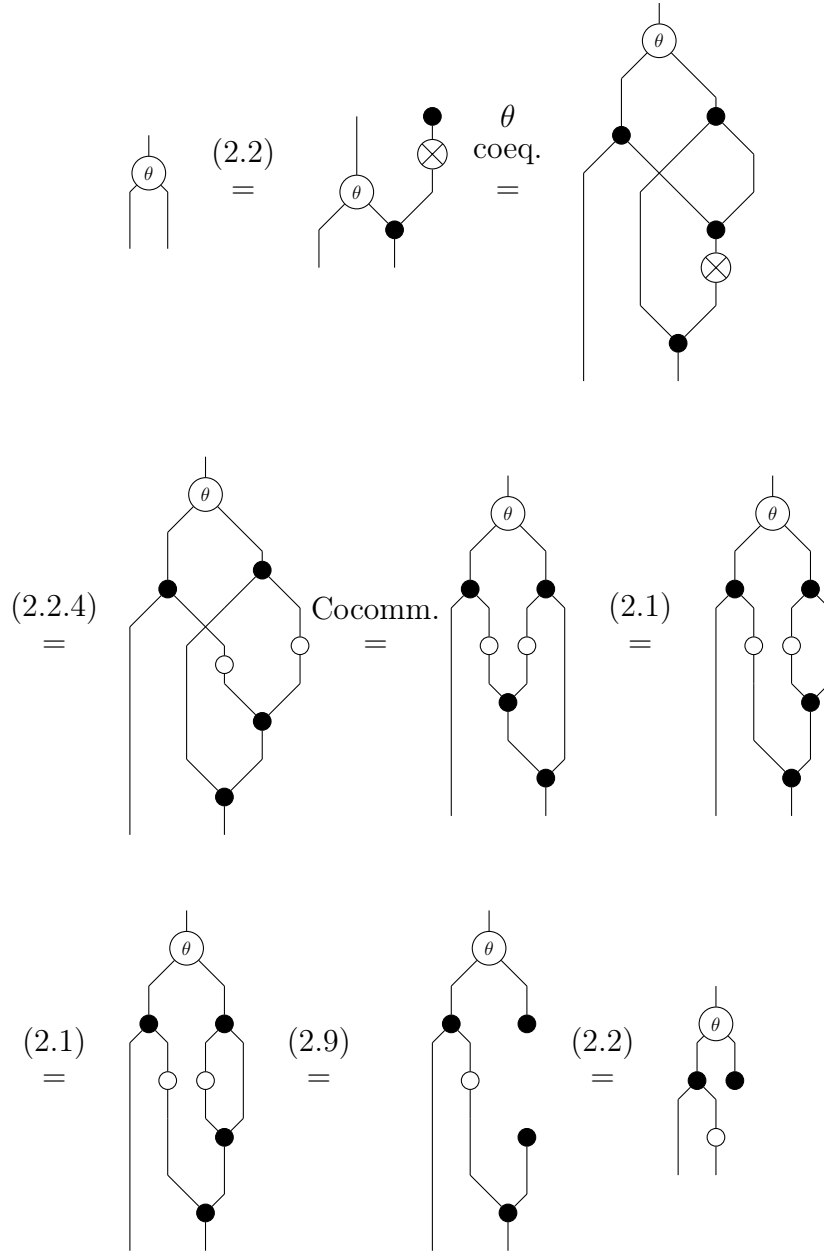
$$(H \otimes H)_{\Delta_H} \otimes_H \epsilon_H \mathbb{1} \otimes_1 \mathbb{1} \cong H_{\chi_H}$$

by showing that H is given by the same coequaliser which defines the above tensor product over $\otimes_H$ and then showing the action commutes. Ignoring the $\otimes_1 \mathbb{1}$ in the above expression, I claim the following is a coequaliser in $\mathcal{V}$.

$$H \otimes H \otimes H \otimes \mathbb{1} \xrightarrow[\text{id} \otimes \text{id} \otimes \epsilon_H]{(\mu \otimes \mu) \circ \sigma \circ \Delta_H} H \otimes H \otimes \mathbb{1} \xrightarrow{\mu \circ (\text{id} \otimes \chi)} H_{\chi_H}$$

Let $H \otimes H \otimes \mathbb{1} \xrightarrow{\theta} M$ be a morphism which coequalises the above diagram then we need to show there exists a unique map $H \rightarrow M$ which commutes with θ . I claim this map is given by $H \xrightarrow{\text{id} \otimes \eta_H} H \otimes H \otimes \mathbb{1} \xrightarrow{\theta} M$. First we need to show

this map does indeed commute with θ and then we need to show it is unique. Showing this commutes with θ is saying we need $\theta = \theta \circ (\text{id} \otimes \eta_H) \circ \mu \circ (\text{id} \otimes \chi)$ which we show with a string diagram proof as follows.



To show uniqueness we need to show that for any morphism $\psi: H \otimes H \otimes \mathbb{1} \rightarrow$

M which satisfies $\theta = \psi \circ \mu \circ (\text{id} \otimes \chi)$ then $\psi = \theta \circ (\text{id} \otimes \eta_H)$ which follows immediately from the unit property of the antipode and multiplication.

Finally we can see immediately that the isomorphism

$$(H \otimes H)_{\Delta_H} \otimes_H \epsilon_H \mathbb{1} \otimes_{\mathbb{1}} \mathbb{1} \cong H_{\chi_H}$$

defined by the unique factorisation defined above is a module morphism since it is given by a composition of module morphisms. $\square$

With an understanding of the units of $\text{Fr}(\Omega_{\mathbf{Mod}_V})$ we will next construct an isomorphism which simplifies its loose composition.

Lemma 6.3.3. *Let $(M, \rho, H \otimes G)$ and $(N, \tau, G \otimes D)$ be left modules. Let ${}_{\chi_G}N$ denote the left $G^{\text{op}} \otimes D$ -module N with action given by $\tau \circ (\chi_G \otimes \text{id})$. Finally let $\theta: M \otimes N \rightarrow M \otimes_G {}_{\chi_G}N$ denote the map which coequalises the following diagram.*

$$G \otimes M \otimes N \xrightarrow[\rho \circ \eta_H]{\tau \circ \chi_G \circ \eta_D \circ \sigma} M \otimes N \xrightarrow{\theta} M \otimes_G {}_{\chi_G}N. \quad (6.8)$$

Then $M \otimes_G {}_{\chi_G}N$ is also the coequaliser of

$$H \otimes D \otimes H \otimes G \otimes D \otimes M \otimes N \xrightarrow[\rho \otimes \tau \circ \Delta_G \circ \sigma]{\mu \otimes \mu \circ \sigma \circ \epsilon_G} H \otimes D \otimes M \otimes N \quad (6.9)$$

by

$$(H \otimes D) \otimes (M \otimes N) \xrightarrow{\sigma \circ \eta_G \otimes \eta_G} (H \otimes G \otimes M) \otimes (G \otimes D \otimes N) \xrightarrow{\rho \otimes \tau} M \otimes N \xrightarrow{\theta} M \otimes_G {}_{\chi_G}N. \quad (6.10)$$

Hence,

$$M \otimes_G {}_{\chi_G}N \cong M \odot N.$$

Proof. Let $\varphi: H \otimes D \otimes M \otimes N \rightarrow Q$ be a map which coequalises (6.9) then we can split the proof into three parts. First we show that $\varphi \circ (\eta_H \otimes \eta_D)$ coequalises (6.8) and hence we have a unique map $\psi: M \otimes_G {}_{\chi_G}N \rightarrow Q$

commuting with θ and $\varphi \circ (\eta_H \otimes \eta_D)$ as follows

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\theta} & M \otimes_G \chi_G N \\ & \searrow \varphi & \downarrow \psi \\ & & Q. \end{array}$$

Second, we show ψ post composed with (6.10) is equal to φ by using the fact that $\psi \circ \theta = \varphi \circ (\eta_H \otimes \eta_D)$ and showing that

$$\varphi = \varphi \circ (\eta_H \otimes \eta_D) \circ (\rho \otimes \tau) \circ \sigma \circ (\eta_G \otimes \eta_G).$$

Finally we show ψ is unique in that its composition with (6.10) is equal to φ .

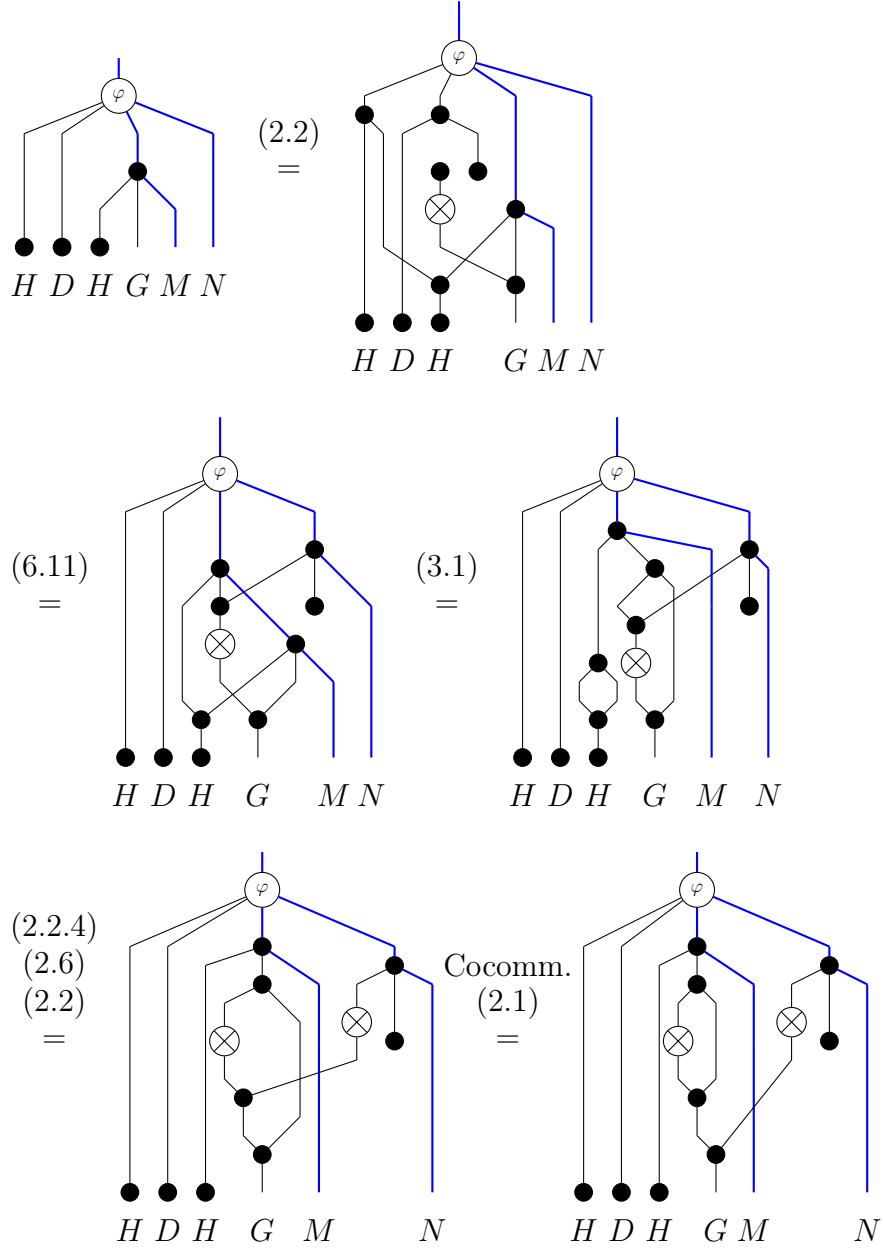
For the first step we have by definition that φ commutes with (6.9) which is captured by the following equality of string diagrams.

The diagram shows an equality between two string diagrams. Both diagrams have a top node labeled φ and a bottom row of labels $H D H G D M N$. The left diagram shows φ connected to a node that is the composition of η_H and η_D , which then connects to the bottom labels. The right diagram shows φ connected to a node that is the composition of χ_G and η_G , which then connects to the bottom labels. The diagrams are connected by an equals sign.

(6.11)

We use this to prove $\varphi \circ (\eta_H \otimes \eta_D)$ commutes with (6.8) by the following

string diagrams.



$$\begin{array}{ccc}
(2.9) & & (3.1) \\
= & & (2.2) \\
= & & =
\end{array}$$

For the next step we need to show

$$\varphi = \varphi \circ (\eta_H \otimes \eta_D) \circ (\rho \otimes \tau) \circ \sigma \circ (\eta_G \otimes \eta_G)$$

which do with the following string diagrams.

$$\begin{array}{ccc}
& & (2.6) \\
& & = \\
& & \\
(6.11) & & (2.2) \\
= & & (2.8) \\
= & & =
\end{array}$$

The final step is to prove uniqueness which is to say that if we have a map $\lambda: M \otimes_G \chi_G N \rightarrow Q$ which commutes with φ and (6.10) then it is equal to ψ . This follows immediately since by the previous step in the proof as it means λ commutes with θ and $\varphi \circ (\eta_H \otimes \eta_D)$ and so is unique by the unique

factorisation of (6.8). $\square$

To be able to construct an isomorphism $\mathbb{B}\text{Mod}_{\mathcal{V}}$ to $\mathbb{F}\text{r}(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$ we will need to be able to describe the unitor and associator of $\mathbb{F}\text{r}(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$. These are provided for us by Shulman [25, Prop 17.1] and we will proceed first by describing the associator in terms of the composition in $\mathbb{B}\text{Mod}_{\mathcal{V}}$.

Lemma 6.3.4. *Let $(M, A \otimes B)$, $(N, B \otimes C)$, $(Q, C \otimes D)$ be left modules. The associator*

$$M \otimes_B \chi_B (N \otimes_C \chi_C Q) \rightarrow (M \otimes_B \chi_B N) \otimes_C \chi_C Q \quad (6.12)$$

defined by factorisation of coequalisers commutes with the isomorphism defined in Lemma 6.3.3 in the following diagram.

$$\begin{array}{ccc} M \odot (N \odot Q) & \xrightarrow{\cong} & M \otimes_B \chi_B (N \otimes_C \chi_C Q) \\ \alpha_{\odot} \downarrow & & \downarrow \\ (M \odot N) \odot Q & \xrightarrow{\cong} & (M \otimes_B \chi_B N) \otimes_C \chi_C Q \end{array}$$

Proof. The associator for the composition in $\mathbb{F}\text{r}(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$ is defined in [25, Prop 17.1] by unique factorisation of the associator for the tensor product through cartesian and opcartesian arrows respectively in the following diagram.

$$\begin{array}{ccccc} (M \otimes N) \otimes Q & \xleftarrow{\text{cart}} & \Delta_{BC}^* (M \otimes N) \otimes Q & \xrightarrow{\text{opcart}} & M \odot (N \odot Q) \\ \alpha_{\otimes} \downarrow & & \cong \downarrow & & \downarrow \alpha_{\odot} \\ M \otimes (N \otimes Q) & \xleftarrow{\text{cart}} & \Delta_{BC}^* M \otimes (N \otimes Q) & \xrightarrow{\text{opcart}} & (M \odot N) \odot Q \end{array} \quad (6.13)$$

The cartesian arrows are just those from our choice of cleaving over Δ_{BC} so the middle isomorphism is just given by $\text{id} \otimes \alpha_{\otimes}$. The aim here is to prove that by taking the composition of the opcartesian arrows in (6.13) with the isomorphism provided by Lemma 6.3.3 provides us with a unique

factorisation

$$\begin{array}{ccccc}
\Delta_{BC}^*((M \otimes N) \otimes Q) & \xrightarrow{\text{opcart}} & (M \odot N) \odot Q & \xrightarrow{\cong} & (M \otimes_B \chi_B N) \otimes_C \chi_C Q \\
\alpha_{\otimes} \downarrow & & \downarrow \alpha_{\odot} & & \downarrow \\
\Delta_{BC}^*(M \otimes (N \otimes Q)) & \xrightarrow{\text{opcart}} & M \odot (N \odot Q) & \xrightarrow{\cong} & M \otimes_B \chi_B (N \otimes_C \chi_C Q)
\end{array} \tag{6.14}$$

which is equal to the associator (6.12) for the tensor product over an algebra.

To do this we need to understand what the opcartesian arrow

$$\Delta_{BC}^*(M \otimes N) \otimes Q \rightarrow (M \odot N) \odot Q$$

is. We recover this from the proof of [25, Prop 17.1] as the the following composition

$$\begin{aligned}
\Delta_{BC}^*(M \otimes N) \otimes Q & \xrightarrow{\cong} \Delta_C^* \Delta_B^*(M \otimes N) \otimes Q \\
& \rightarrow (\epsilon_B)_! \Delta_C^* \Delta_B^*(M \otimes N) \otimes Q \\
& \xrightarrow{\xi} \Delta_C^*(\epsilon_B)_! \Delta_B^*(M \otimes N) \otimes Q \\
& \rightarrow (\epsilon_C)_! \Delta_C^*(\epsilon_B)_! \Delta_B^*(M \otimes N) \otimes Q \\
& = (M \odot N) \odot Q
\end{aligned}$$

where ξ is the Beck-Chevalley morphism for Δ_C and ϵ_B . These maps are all defined by factoring morphisms through coequalisers which define the tensor product over an algebra. These morphisms which factor through coequalisers to give the above map is the following composition where we omit $\otimes$ and $(M \otimes N) \otimes Q$ at the end and just write algebras next to each other for clarity.

$$\begin{aligned}
ABBCCD & \xrightarrow{\eta_{ABCCD}} ABCCD \otimes ABBCCD \\
& \xrightarrow{\eta_{ACD}} ACD \otimes ABCCD \otimes ABBCCD \\
& \xrightarrow{\Delta_C \otimes \epsilon_B} ACCD \otimes ACCD \otimes ABBCCD \\
& \xrightarrow{\eta_{AD}} AD \otimes ACCD \otimes ACCD \otimes ABBCCD
\end{aligned} \tag{6.15}$$

By Lemma 6.3.3 the morphism

$$AD \otimes ACCD \otimes ACCD \otimes ABBCCD \otimes (M \otimes N) \otimes Q \rightarrow (M \odot N) \odot Q$$

out of the coequaliser and then composing with the isomorphism

$$(M \odot N) \odot Q \cong (M \otimes_B \chi_B N) \otimes_C \chi_C Q$$

is the same as taking $\eta_{BB} \otimes \eta_{CC}$ and then taking the action of all the algebras into M , N , and Q which when composed with (6.15) means the net effect is that we have just multiplied the original $ABBCCD$ into $(M \otimes N) \otimes Q$. The same is true for the opcartesian arrow

$$\Delta_{BC}^*(M \otimes (N \otimes Q)) \rightarrow M \odot (N \odot Q)$$

and hence the unique morphism which makes (6.14) commutes is precisely the unique factorisation of $\alpha_\otimes$ through the coequalisers defining $(M \otimes_B \chi_B N) \otimes_C \chi_C Q$ and $M \otimes_B \chi_B (N \otimes_C \chi_C Q)$. $\square$

Finally we need a result to let us describe the unitor in $\mathbb{F}r(\Omega_{\text{Rep}})$. It is known that in a monoidal category, if there are two candidates for units and unitors on the same product then they are equivalent unit structures [12, Prop 1.7]. The idea is similar to proving that a monoid has a unique identity element and proving that two loose units for a double category are equivalent can be done in an identical way to the result for monoidal structures provided by [12, Prop 1.7]. As such, to show that two unit structures are equivalent for different products it suffices to show that there exists a pair of equivalent unit structures for the two products. We will do this for the left unitor with the case for the right unitor being identical.

Lemma 6.3.5. *Let M be a left $H \otimes H'$ -module. The left unitor $\ell_\odot: M \rightarrow (\Delta_H)_! \epsilon_H^* \mathbb{1} \odot M$ in $\mathbb{F}r(\Omega_{\text{Mod}_V})$ is equivalent to the left unitor $M \rightarrow H_{\chi_H} \otimes_H \chi_H M$ defined the isomorphism provided in Lemma 3.3.8 by factoring η_H through the coequaliser defining $\otimes_H$.*

Proof. By [12, Prop 1.7] we only need to show that there exists a commuting

triangle

$$\begin{array}{ccc}
 M & & \\
 \ell_{\odot} \downarrow & \searrow \cong & \\
 (\Delta_H)_! \epsilon_H^* \mathbb{1} \odot M & \xrightarrow{\cong} & H_{\chi_H} \otimes_H \chi_H M
 \end{array}$$

of isomorphisms. We can define the isomorphism

$$(\Delta_H)_! \epsilon_H^* \mathbb{1} \odot M \rightarrow H_{\chi_H} \otimes_H \chi_H M$$

by the composition of the isomorphisms from Lemma 6.3.2 and Lemma 6.3.3 and then the isomorphism $M \rightarrow H_{\chi_H} \otimes_H \chi_H M$ which makes the above triangle commute is uniquely provided by the others. $\square$

We finally have all the components we need to prove that $\mathbb{F}r(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$ is isomorphic to $\mathbb{B}i\mathbf{Mod}_{\mathcal{V}}$.

Theorem 6.3.6. *The monoidal double categories $\mathbb{F}r(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$ and $\mathbb{B}i\mathbf{Mod}_{\mathcal{V}}$ are isomorphic.*

Proof. We will proceed by defining monoidal double functors $F: \mathbb{F}r(\Omega_{\mathbf{Mod}_{\mathcal{V}}}) \rightleftarrows \mathbb{B}i\mathbf{Mod}_{\mathcal{V}}: F^{-1}$, showing they are inverse and then checking they both satisfy the definition of monoidal double functor. Both $\mathbb{F}r(\Omega_{\mathbf{Mod}_{\mathcal{V}}})$ and $\mathbb{B}i\mathbf{Mod}_{\mathcal{V}}$ have the same objects and tight 1-cells so we define F_0 and F_0^{-1} as the identity on $\mathbf{Hopf}_{cc}(\mathcal{V})$. We define F_1 on objects as sending a module $(M, \rho, G \otimes H)$ to the left module $(M, \rho \circ (\text{id} \otimes \chi_H), G \otimes H)$ and then using Corollary 3.1.10 to obtain a G - H -bimodule M , which we denote M_{χ_H} , is invertible because χ_H is an isomorphism by Corollary 2.2.6. F_1 sends a module morphism $(\theta, f \otimes f')$ to a bimodule morphism (θ, f, f') which is allowed because algebra morphisms commute with the antipode by Lemma 2.2.8. Finally we define the unitator and compositor F_u and F_c as the isomorphisms of Lemma 6.3.2 and Lemma 6.3.3 respectively. In particular for modules $(M, A \otimes B)$ and $(N, B \otimes C)$ then F_c is given by

$$F_c: M_{\chi_B} \otimes_G N_{\chi_C} = (M \otimes_G \chi_B N)_{\chi_C} \xrightarrow{\cong} (M \odot N)_{\chi_C}$$

and F_u is given by

$$F_u: H = {}_{\chi_H}({}_{\chi_H}H) \xrightarrow{\cong} {}_{\chi_H}((\Delta_H)_! \epsilon_H^* \mathbb{1}).$$

The monoidal structure on these two categories are the same (given by the product in $\mathcal{V}$ so it remains to show F_c and F_u satisfy the hexagon and unit identities. Since F_c is just given by the isomorphism of Lemma 6.3.3 then the hexagon commuting is exactly the condition of Lemma 6.12. Finally the unit condition is exactly Lemma 6.3.5 and so we are done. $\square$

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